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On a complete written proof of the Generalized Riemann Hypothesis for Dirichlet L-Functions

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Abstract

In this paper, we show that every nontrivial zero of any Dirichlet L-function has real part 1/2. The proof follows Sernau's shorter arrangement of the Alpöge construction. We provide a complete written account of the proof, relying solely on results from the existing mathematical literature, without reference to any part of the Lean formalization.

Contents

1 Introduction **11**

2 Notation and preliminaries **13**
2.1 Prime sums and the zero criterion . . . 13
2.2 Scales, ranks, and prescribed levels . . . 14
2.3 Fields, embeddings, and weights . . . 14
2.4 Hermitian spaces, automorphic representations, and levels . . . 15
2.5 Orbital integrals and their moment bounds . . . 15
2.6 Theta operators, derivatives, and integral estimates . . . 16
2.7 Covolumes, determinants, and coefficient places . . . 18
2.8 Cocycles, specializations, and Galois congruences . . . 18
2.9 Period comparisons, critical values, and descent . . . 19
2.10 Local period rings and filtration orders . . . 20

3 Outlines of proofs **20**
3.1 Part I. Parameters and the common family . . . 20
3.2 Part II. Orbital estimates and the translated span . . . 21
3.3 Part III. Theta identities and the upper covolume bound . . . 22
3.4 Part IV. Selmer bounds and trace congruences . . . 22
3.5 Part V. Periods and the lower determinant estimate . . . 23
3.6 Part VI. The prime remainder and the analytic conclusion . . . 24

I Arithmetic parameters and automorphic spaces **24**

4 Arithmetic parameters **24**
4.1 The chosen primes and their generators . . . 26
4.2 The different and the additive character scales . . . 27
4.3 The splitting and level characters . . . 28
4.4 The two old-level characters on the same torus . . . 29

5 The lattice comparison on a common automorphic family 30

6 The ramified source at the prescribed level 33

7 Normalization and bounds for the theta images 34
7.1 Bounds for the local denominators . . . 35
7.2 Counting terminal double cosets . . . 35
7.3 Projection onto the prescribed finite types . . . 37
7.4 Pointwise estimates in the composition of lifts . . . 38
7.4.1 The sum of the part sizes . . . 38
7.4.2 Integral generators of the ramified source . . . 40
7.4.3 Conjugation of the theta jets . . . 42
7.4.4 Gaussian bounds for the conjugated kernels . . . 54
7.4.5 Composition and terminal restriction . . . 57
7.5 The integral terminal branching map . . . 58
7.6 Norms and index after normalization . . . 62

8 The normalized Poincaré map 64
8.1 The finite coordinates . . . 64
8.2 Class slices and quotient measure . . . 66
8.3 The real point and stationary translate . . . 66
8.4 The measured period and change of level . . . 68
8.5 Petersson blocks and covariance . . . 70

II Orbital integrals and the translated span 72

9 Orbital integrals and operator norms 72
9.1 From truncated orbital integrals to full orbital integrals . . . 72
9.2 The diagonal bound and the projected subspace . . . 74
9.3 The mixed moment bound . . . 75
9.4 Dimension on the terminal definite group . . . 77

10 Finite orbital integrals and Fourier bounds 82
10.1 Integral points on the local quadrics . . . 82
10.2 Compact volumes for the successive quotients . . . 83
10.3 Normalization of the finite quotient pairing . . . 86
10.4 Common entries and the finite terminal density . . . 87
10.5 Stable ideals and classes of prescribed denominator . . . 89
10.6 Finite correlations on bounded sets . . . 91
10.7 Reciprocal weights and absolute pivot bounds . . . 94
10.8 Local Fourier estimates for the leading coordinates . . . 96
10.9 Summation by denominator . . . 99
10.10 Finite measure estimate for the stabilizers . . . 101

11 Weighted real orbital integrals **102**
11.1 Plancherel on the real norm fibers . . . 102
11.2 Support and phases in the real integral . . . 104
11.3 The terminal Gram form . . . 106
11.4 Integrals of reciprocal determinant weights . . . 107
11.5 The weighted products of the real correlations . . . 109
11.6 Decay from a deep unipotent subgroup . . . 111
11.7 The weighted square integral on a real ball . . . 111
11.8 Density and twisted square integrals . . . 114
11.9 Real determinant weights on dyadic regions . . . 115
11.10 Summation and real frequency decay . . . 116

12 Formal degrees and real Bessel functions **118**
12.1 The local mass in the formal-degree normalization . . . 119
12.2 Descent through a common entry . . . 120
12.3 The reproducing coefficient and the norm-ball bound . . . 121
12.4 Error from truncating the orbital integrals . . . 122
12.5 The formal-degree factor . . . 124
12.6 Haar measure in the adapted real basis . . . 125
12.7 Decay of the lowest compact type coefficient . . . 127
12.8 The stationary point and its period . . . 128
12.9 The phase on the common compact block . . . 131
12.9.1 Triangular coordinates on the sign blocks . . . 132
12.9.2 The compact block estimate . . . 133
12.10 The real Bessel functions for the construction . . . 134
12.11 The differentiated Schur bound . . . 137

13 Adelic orbital comparison and definite masses **139**
13.1 The open rational cell and its finite restrictions . . . 139
13.2 Comparison at a fixed final vector . . . 140
13.2.1 Cancellation of the class phases . . . 141
13.2.2 The common-entry pairing . . . 143
13.3 Averaging at a fixed final vector . . . 144
13.4 Adelic summation of the common-entry forms . . . 147
13.4.1 The sum over nonzero frequencies . . . 149
13.5 Consecutive masses of definite unitary lattices . . . 151
13.6 The class count and the unipotent cutoff measure . . . 153
13.7 The main and identity moments on the source . . . 155
13.8 The open-orbit bound at equal parameters . . . 156

14 Truncation, finite projections, and lattice pair counts **158**
14.1 Reduced bases and the truncation error . . . 158
14.2 The residue estimates for the projection onto the selected types . . . 161
14.3 The weighted count at the additional ramified place . . . 163
14.4 Counting orbital representatives containing a fixed lattice . . . 166
14.4.1 Changes of basis on the two blocks . . . 167
14.4.2 Lifts of the fixed lattice . . . 167

14.4.3 Counting definite columns . . . 168
14.4.4 Counting isotropic columns . . . 169
14.5 The weighted sum over nearby pairs . . . 170
14.6 Divisibility of the pairing determinant . . . 170
14.7 The discriminant bound for the pair lattice . . . 171
14.7.1 The lattice associated with one link . . . 172
14.7.2 Decomposition of the two link lattices . . . 174
14.7.3 The common residue and the intersection lattice . . . 175

15 Character cancellation and Schur's test 177
15.1 Character cancellation on the blocks . . . 177
15.2 The common flag and the cell rank . . . 178
15.3 Scalar cosets and the surviving terms . . . 179
15.4 The real estimate in the open Gram cell . . . 180
15.4.1 Haar density in Gram coordinates . . . 180
15.4.2 A neighborhood in the original group . . . 181
15.4.3 The transverse integral . . . 183
15.5 The contribution of large discriminants . . . 185
15.6 The one-place Schur integral . . . 188
15.7 The contribution bounded by Schur's test . . . 190
15.7.1 Summation over the smaller orbits . . . 193
15.8 Summing the pair weights . . . 194

16 The common automorphic family and its translated span 196
16.1 Construction of the common automorphic family . . . 196
16.2 Selecting the translates . . . 197

III Theta identities and the upper covolume bound 199

17 The Weil action and the measured seesaw 199
17.1 Rational invariance for the Weil action . . . 199
17.2 The doubled action in the Siegel–Weil formula . . . 204
17.3 Translation and scalar normalization in the seesaw . . . 206
17.4 Rational orbits for consecutive lifts . . . 208

18 Consecutive theta lifts and their coefficients 210
18.1 Unfolding the integral for consecutive lifts . . . 210
18.1.1 Phase cancellation and rational orbits . . . 212
18.1.2 Absolute convergence and unfolding . . . 213
18.1.3 Rapid decay of the projected tests . . . 215
18.2 Two successive Bessel integrals . . . 217
18.2.1 The constant terms in the joined block . . . 218
18.2.2 The two quotient volumes . . . 220
18.3 The coefficients after an even number of steps . . . 221
18.4 Finite support on the original Bessel orbits . . . 225
18.4.1 The lattice conditions on a finite slice . . . 227

18.4.2 Exclusion by the ramified character . . . 228
18.4.3 Compact representatives of the surviving terms . . . 229

19 The measured theta tensor and its integral lattice 230
19.1 The measured finite sum . . . 230
19.2 The lattice of an evaluated pair . . . 233
19.3 Integral coefficients of the measured tensor . . . 235
19.3.1 Translation at the additional level . . . 237
19.4 The comparison at the additional ramified place . . . 239
19.4.1 Translation laws and the ramified sum . . . 240
19.4.2 The direct lift at the ramified place . . . 242
19.4.3 Integral clearing on the original source . . . 246

20 Integral theta sections and local derivatives 247
20.1 Integral sections and ordinary jets . . . 248
20.2 Valuations of the derivative coefficients . . . 257
20.3 Twisted derivatives at inert primes . . . 260
20.3.1 Integral rescaling of the local exponential . . . 261
20.3.2 Valuations of the jets . . . 263
20.3.3 The resulting Hecke operators . . . 265

21 Hecke operators and colength at exceptional primes 270
21.1 The depth count for the endpoint restrictions . . . 273
21.2 The rank-two defect estimate . . . 275
21.3 Counting forms with bounded local defect . . . 277
21.4 The volume factor in the residue-defect estimate . . . 279
21.5 An integral average in one matrix variable . . . 280
21.6 The simultaneous average on the coefficient space . . . 282
21.7 The colength of the source Hecke operator . . . 285

22 Norm and index estimates for the translated images 288
22.1 Derivatives at the ramified prime . . . 288
22.2 Integrality at the two original level primes . . . 290
22.3 Norm and index bounds for the Poincaré images . . . 291

23 The mixed Gram determinant and the product formula 294

IV Selmer bounds and Galois congruences 297

24 Length bounds after removing auxiliary Selmer conditions 297
24.1 The denominator bound for odd cocycles . . . 299
24.2 A uniform annihilator for the positive twists . . . 301
24.2.1 The Jacobi quotient . . . 303
24.2.2 Annihilation on character components . . . 303
24.2.3 Integral recombination of character components . . . 304
24.3 Detection at large coefficient primes . . . 305
24.3.1 Local reciprocity at an auxiliary prime . . . 307

24.3.2 The Jacobi derivative cocycles . . . 308
24.3.3 Comparison of Frobenius and inertia . . . 309

25 Polynomial bounds for the Selmer lengths 310
25.1 The line through the untwisted point . . . 312
25.2 Transfer to divisible coefficients . . . 313

26 Simultaneous evaluation on torsion 313

27 A common twist for the Selmer and Euler-factor conditions 315
27.1 A specialization of a norm system . . . 316
27.2 The distinguished auxiliary prime . . . 317
27.3 Selmer length descent and the uniform bound . . . 318
27.4 Duality for the final local representation . . . 320

28 Galois representations, finite types, and local lengths 322
28.1 The original Hecke system and the normalized trace . . . 323
28.1.1 Hecke multiplicities and the trace . . . 324
28.2 Galois attachment and spherical compatibility . . . 326
28.3 Inertial parameters of the finite types . . . 328
28.3.1 The inertia characters . . . 330
28.3.2 The matrix span at the coefficient place . . . 332
28.4 Local length bounds for a main summand . . . 335
28.5 The order of the simultaneous choices . . . 337

29 Crystalline bounds and auxiliary prime selection 339
29.1 Crystalline colength for the relative cocycles . . . 339
29.1.1 Hodge–Tate intervals at the coefficient places . . . 343
29.1.2 Uniform bounds for the local defects . . . 347
29.2 Cyclotomic-kernel evaluations for the auxiliary primes . . . 348
29.3 The fresh Kummer maps . . . 351
29.4 The ordered prime selection from the Kummer relations . . . 353
29.5 The paired tame and Frobenius tests . . . 355

30 Measured norm tests and the integral trace algebra 357
30.1 Norm tests from a definite line . . . 357
30.2 The rank-two determinant norm . . . 361
30.2.1 Doubling and the local integrals . . . 363
30.2.2 Integral coefficients of the rank-two norm . . . 369
30.3 The integral trace algebra on the auxiliary levels . . . 370

31 Hecke operators and the common trace coefficient 372
31.1 Tame and Frobenius operators at a raised prime . . . 372
31.1.1 The restriction to the tame subgroup . . . 373
31.1.2 Integral clearing from the tame action . . . 375
31.2 Spherical root chains on the original level . . . 375
31.3 The common coefficient of the Hecke polynomial . . . 378
31.3.1 The main-block rows . . . 381

31.3.2 The old tame relations in the coefficient comparison 385

32 Fitting ideals and trace determinant specialization 386
32.1 The Fitting element for the ordered characters 386
32.2 The trace determinant under the Frobenius relations 388
32.3 The highest squarefree coefficient and the original test space 390

V Period comparison and the lower determinant bound 396

33 Period vanishing and the integral comparison lattice 396
33.1 Vanishing of the comparison periods 396
33.1.1 The real type in the bottom descent 398
33.1.2 Extension through the hyperbolic planes 401
33.1.3 The mixed-model height bound 403
33.1.4 Vanishing for the gluing adjoint 408
33.2 The lattice for the comparison operator 410
33.3 A common integral scale for the first factors 412
33.3.1 Bounds for the normalized derivatives 413
33.3.2 Inverse Gram matrices in the chosen coordinates 415

34 The comparison operator and the first theta lifts 418
34.1 The comparison operator on the original lattice 418
34.1.1 The comparison in the Fock model 419
34.1.2 The Gaussian pairing 421
34.2 Gram forms for the first lifts 423
34.2.1 The masses of the first lifts 423
34.2.2 Doubling and the common series 425
34.2.3 The two spherical actions 429
34.3 The period pairing for the four unitary groups 430
34.3.1 Uniform bounds for the two kernels 432
34.3.2 The real line projections 435
34.3.3 Application to the unitary groups 441

35 Real restrictions and self-dual Fourier tests 442
35.1 The real restriction of the larger period 442
35.2 Extension of the self-dual Fourier tests 446
35.2.1 Factorization through the Fourier test 447
35.2.2 Integral extension of the Fourier test 450

36 The normalized period exchange 454
36.1 The normalized comparison of the two periods 454
36.1.1 Absolute summability of the mixed terms 455
36.1.2 The measure on the open orbit and the signed comparison 461
36.1.3 The two radical terms after removing one hyperbolic plane 463
36.1.4 Unfolding the isotropic line orbit 465
36.1.5 The small radical mean and its zero term 470

36.1.6 The equal-rank normalizer . . . 474
36.2 The finite extension of the period identity . . . 480
36.2.1 Fourier transform in the complementary blocks . . . 481
36.2.2 The finite determinant Gauss sum . . . 483

37 The leading coefficient and the four-block test 486
37.1 A lower bound for a coefficient in twelve coordinates . . . 486
37.1.1 The inert expansion and valuation . . . 489
37.1.2 The finite coefficient pairing . . . 493
37.1.3 CM transport of the untwisted numerator . . . 494
37.2 The four-block test and the untwisted reference norm . . . 496
37.2.1 The derivative estimate . . . 496
37.2.2 The point test on the original target . . . 496
37.2.3 The last-block class sum . . . 498
37.2.4 Selection of the class character . . . 500
37.2.5 Untwisted norms at both conjugate embeddings . . . 503

38 The common twist and bounds on critical values 505
38.1 Signed systems for the critical pairs . . . 506
38.2 Original Hecke systems and their integral models . . . 508
38.3 The common specialization and its valuations . . . 511
38.4 Integral clearing and projected pairings . . . 514
38.5 Reciprocal heights of the main critical values . . . 517
38.6 Height of the exchange product . . . 518

39 Determinant factors and the gluing operator 520
39.1 Products of determinants in the period comparison . . . 520
39.2 Constants in the three finite comparisons . . . 523
39.3 The rank-zero comparison and the opposite Hermitian form . . . 525
39.4 Three period identities for the gluing operator . . . 527
39.4.1 Matching the real factors . . . 528
39.4.2 Ranks and levels in the three comparisons . . . 532
39.4.3 The common period as the full gluing adjoint . . . 533

40 Coefficient reproduction on the descent space 535
40.1 Separation by the circle mean . . . 535
40.2 The inner product formula at the principal levels . . . 538
40.2.1 The adelic doubling identity . . . 539
40.2.2 Vacuum coefficients away from the chosen places . . . 543
40.2.3 Summability and injectivity . . . 544
40.2.4 Real convolution and the original measures . . . 546
40.3 The adjoint of a single lift and its holomorphic type . . . 547

41 Injectivity of the common period 550
41.1 Reproduction and separation at a peak . . . 550
41.1.1 Real reproduction on the complementary block . . . 551
41.1.2 The finite first-factor choices . . . 552

41.1.3 Approximation using a peak of the auxiliary form . . . 554
41.1.4 The nonzero reference coefficient . . . 555
41.1.5 Coefficients and weights of the approximants . . . 557
41.1.6 Injectivity of the common period . . . 559
41.2 First minima at peaks of automorphic forms . . . 559
41.3 The auxiliary automorphic form and its first minimum . . . 561
41.4 Finite overlaps and Gaussian decay . . . 563
41.5 The normalized overlap on the adelic Hom space . . . 564
41.5.1 The probability mean and Fourier inversion . . . 564
41.5.2 The finite character and its translate . . . 566
41.5.3 The sparse test and its error . . . 566

42 Compression, separation, and reduction at cusps 569
42.1 Compression of the comparison operator . . . 569
42.1.1 The period and the Hecke factor . . . 569
42.1.2 The projection at complex embeddings . . . 570
42.1.3 The norm at the trivial twist . . . 571
42.1.4 Compression on the original local lattice . . . 573
42.2 Separation on the initial unitary group . . . 574
42.3 Quantitative reduction near the cusp . . . 577
42.4 Vanishing of the constant term under the reproducing kernel . . . 579
42.5 Removing the smoothing at the period . . . 581

43 Theta compositions at deeper principal levels 585
43.1 The first-column return . . . 586
43.2 The measured square and transposition . . . 589
43.3 The first-path array . . . 591
43.4 Transport across the consecutive squares . . . 594
43.5 Composition and terminal measure . . . 600

44 Chain determinants and completion of the lattice 604
44.1 Determinants of the composition . . . 604
44.2 Height of the chain determinant ratio . . . 610
44.3 Spanning vectors from the terminal period . . . 613
44.3.1 Bounds for the two terminal jets . . . 613
44.3.2 Local transport and terminal branching . . . 616
44.3.3 The dual vector on the terminal lattice . . . 617
44.3.4 Integral dual coordinates and pointwise bounds . . . 618
44.3.5 Integral generators at the outer node . . . 620
44.3.6 Projection to the projected space . . . 621
44.4 Integral derivatives in the two norm formulas . . . 623

45 Proof of the determinant estimate 624

VI The analytic conclusion **625**

46 The analytic criterion **625**
46.1 The one-sided Mellin criterion . . . 625
46.2 Choosing a twist with no zeros in the rectangle . . . 627

47 The bound for the prime remainder **629**

Appendices **632**

A Period rings and integral comparison matrices **632**
A.1 Integral coordinates for the period basis . . . 632
A.2 Integral approximation in the fixed period chart . . . 634
A.3 Uniform denominator clearing in period coordinates . . . 636
A.4 The two period matrices over the bounded ring . . . 637
A.5 Divisibility under the coordinate retraction . . . 639
A.6 Divisibility of the period matrix against integral covectors . . . 640

B Integral crystalline bases and filtration orders **642**
B.1 The integral period basis . . . 642
B.2 The period basis, divided action, and tensor orders . . . 645
B.2.1 Integral divided operators . . . 646
B.2.2 Retraction at each order . . . 648
B.2.3 Tensor pairings and orders . . . 650
B.3 Standard orders and projections on the chart . . . 651

C The tame cup-product estimate **655**
C.1 Level-zero steps for tame periods . . . 655
C.2 The four terms of the crystalline cup calculation . . . 656
C.2.1 The Frobenius cup calculation . . . 657
C.2.2 The three extensions at a prescribed level prime . . . 659

D Depth operators and crystalline lifts **664**
D.1 Depth operators on the integral chart . . . 664
D.2 Ideals controlling the crystalline lifts . . . 667
D.3 The convergence radius of the period basis . . . 671
D.4 Crystalline lifts and termination of the construction . . . 673
D.4.1 Termination at a plain cut . . . 675
D.4.2 Raising depth before termination . . . 677
D.4.3 Termination in the two filtration schemes . . . 678

E Local colength away from the original level **682**
E.1 Colength for separated Hodge–Tate weights . . . 682
E.1.1 Uniform comparison of the depth steps . . . 683
E.1.2 Pole removal and the final integral lift . . . 685
E.2 Exact integrality in two special cases . . . 689
E.3 The bound for the original coefficient lattices . . . 691

References 694

1 Introduction

In this paper, we prove the generalized Riemann hypothesis for Dirichlet L-functions. Our proof follows the readable Lean edition of Sernau's shorter presentation [67] of Alpöge's construction [3]. These references give the origin and arrangement of the argument. We give proofs for the assertions of this paper and cite the results used from the literature. Below we state the prime-remainder bound and explain how Parts I–VI establish it. Section 3 gives the detailed outline of the argument.

Theorem 1.1 (Dirichlet generalized Riemann hypothesis)**.** *For every $q \geq 1$ and every Dirichlet character χ modulo q, $L(s,\chi)$ is zero-free in $\Re s > 1/2$, with the simple pole of the principal character at $s = 1$ permitted. Every zero in $0 < \Re s < 1$ has real part $1/2$.*

The character modulo 1 gives $L(s,\mathbf{1}) = \zeta(s)$, so the theorem includes the Riemann hypothesis for $\zeta(s)$. Section 47 completes the proof. For a character modulo q, the link between the automorphic calculation and the zeros of $L(s,\chi)$ is a one-sided estimate for a weighted prime remainder on the identity class modulo $q_0 = 18480q$. We derive the estimate by comparing upper and lower bounds for the Gram covolume on a single automorphic family.

Fix q_0 with $18480 \mid q_0$ and put $d_0 = q_0/2 - 1$. For a prime $d \equiv d_0 \pmod{q_0}$, write $\chi_d(n) = (n/d)$ for the quadratic character, Λ for the von Mangoldt function, and φ for Euler's totient. With real cutoff x and the subtraction constant c_{d,q_0} of Section 2.1, put

$$R_{d,q_0}(x) = \sum_{\substack{n \leq x \\ n \equiv 1 \bmod q_0}} \Lambda(n)(1 - \chi_d(n))(n^{-1} - x^{-1}) - \frac{\log x}{\varphi(q_0)} - c_{d,q_0}.$$

The analytic criterion of Section 46 reduces Theorem 1.1 to the one-sided bound

$$R_{d,q_0}(x) \leq C_{q_0,d,\delta} x^{-1/2+\delta} \qquad (x \text{ sufficiently large})$$

for each fixed q_0, d and $\delta > 0$. The implied constant and threshold may depend on these fixed parameters. Suppose $L(\rho,\chi) = 0$ with $1/2 < \Re\rho < 1$. Induce $\chi \bmod q$ to $\widetilde{\chi} \bmod q_0$, where $q_0 = 18480q$. The extra Euler factors are nonzero for $\Re s > 0$, so $L(\rho,\widetilde{\chi}) = 0$. Character orthogonality expresses the Mellin transform of the remainder on $1 \bmod q_0$ using the logarithmic derivatives for all characters modulo q_0 and their quadratic twists. Choose $1/2 < \beta < \Re\rho$ and $H_0 > |\Im\rho|$. The criterion selects one prime d for which all the twists are zero-free on $\beta \leq \Re s \leq 1$, $|\Im s| \leq H_0$. With d fixed, taking $0 < \delta < \beta - 1/2$ gives $R_{d,q_0}(x) \leq x^{\beta-1}$ for all sufficiently large x. The one-sided Mellin criterion then excludes the zero at ρ.

The six Parts obtain this estimate by first expressing the prime remainder in terms of a quantity $\mathcal{R}$ in the automorphic construction, then bounding $\mathcal{R}$ through the covolume comparison.

1. *Part I. The weighted prime sum and the automorphic family.* For fixed q_0, d and $0 < \epsilon < 1/100$, Section 4 chooses the weights, fields, and levels for each sufficiently large integer x. Let x_1 be the auxiliary cutoff, b the weight parameter, and $a = xb$ the weight scale

in Section 2.2. With the fibre and subset sizes u, v of Section 2.3, Section 5 obtains the weighted prime remainder

$$\frac{\mathcal{R}}{av} = R_{d,q_0}(x) - R_{d,q_0}(x_1) + \frac{u}{v} R_{d,1}(x_1).$$

Here $R_{d,1}$ is the modulus-one remainder of Section 2.1. We place both covolume estimates on the translated images of the normalized Poincaré map of Section 8.

2. *Part II. A large span of translates.* The diagonal moment of Section 9 bounds the squared Hilbert–Schmidt norm of the projected map from below on each source subspace. The mixed moment controls the average correlation of translates. Both estimates start with the same Poincaré map. In Section 16.2, we use them to choose translates and then restrict to the ramified source S. The resulting span has dimension $n \geq (1 - 4/b^2)D$, where $D = \dim V$ and V is the projected target of Section 2.4.

3. *Part III. An upper covolume bound.* The measured seesaw of Section 17.3 identifies the lifts of the translated images. Section 22.3 proves the norm and integrality bounds for the translated images after a common rational rescaling. We choose a basis y of the span from these images. Hadamard's inequality and the clearing matrices give

$$\mathrm{Cov}(y) \leq n(A - m\mathcal{R} + L), \qquad \mathrm{Ind}(y) \leq DI_0.$$

Here $\mathrm{Cov}(y) \leq c$ means that the Gram determinant of the untwisted lift of y is at most e^c at every complex embedding. The index bound uses clearing matrices that work for every allowed twist at the corresponding coefficient prime (Section 2.7). Here A is the common normalization of Section 2.6, L, I_0 are the error terms, and m is the chain length of Section 2.2.

4. *Part IV. Selmer bounds and Galois congruences.* The Hecke systems give the Galois representations of Section 28.2. The Selmer estimates, including the crystalline colength of Section 29.1, bound the lengths of the associated cocycle modules. The trace comparison of Section 32 and the torsion specialization of Section 26 give the arithmetic tests used to choose the auxiliary twist in Part V.

5. *Part V. A lower determinant bound.* Section 38 chooses a single twist ξ_κ at each coefficient place κ of Section 2.7. The same twist works for the critical-value tests and for the mixed Gram pairing at $\kappa, \bar{\kappa}$. The period comparisons bound the height of the ratio from the untwisted Gram determinant to the twisted one. The additional vectors of Proposition 44.3 complete y to a basis of V. Applying the product formula of Section 23 gives

$$D(A - 3L) \leq n(A - m\mathcal{R} + L) + 2DI_0$$

in Section 45.

6. *Part VI. The estimate for the prime remainder and the zeros of Dirichlet L-functions.* Since $n \leq D$ and $A + L \geq 0$, the contribution $(n - D)(A + L)$ after rearranging the lower bound is nonpositive. Proposition 5.3 gives $m\mathcal{R} \leq faB^5$, where f counts the real places of the base field and $B = \log b$ (Sections 2.3 and 2.2). Section 47 applies the fixed-modulus prime number theorem at x_1 and transfers the bound to real x, giving

$R_{d,q_0}(x) \le C_{q_0,d,\epsilon} x^{-1/2+6\epsilon}$. With d fixed by the criterion, we take $6\epsilon < \beta - 1/2$ to exclude the proposed zero. To exclude zeros in $0 < \Re s < 1/2$, we pass to the primitive inducing character and apply its functional equation. Section 47 treats $\Re s \ge 1$ separately, completing the proof of Theorem 1.1.

Structure of the paper. Section 2 collects the notation, and the six subsections of Section 3 give the detailed outline of Parts I–VI in the above order. Appendices A–E develop the local crystalline calculations used in Section 29.1. The estimates at the level primes and away from the level are in Appendices C and E, respectively.

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2 Notation and preliminaries

In this section, we collect the notation for the proof in the order of the Parts. The local entries give the conventions and point to the definitions of the indexed quantities. Within a calculation, the scope of a summation index, local rank, or choice of basis is specified in the corresponding section. Sections 4, 5, and 38 fix, respectively, the arithmetic datum, the comparison, and the choices over a coefficient place.

2.1 Prime sums and the zero criterion

In this subsection, we record the analytic conventions for Sections 46 and 47.

We write $\zeta(s)$ for the Riemann zeta function, $L(s,\chi)$ for the Dirichlet L-function of χ, γ_{Euler} for Euler's constant, Λ for the von Mangoldt function, μ for the Möbius function, $\varphi(q)$ for Euler's totient, and $u_+ = \max(u,0)$. For a character modulo q, the progression modulus is $q_0 = 18480q$. In the remainder estimates q_0 is any fixed positive multiple of 18480. The prime $d \equiv d_0 = q_0/2 - 1 \pmod{q_0}$ defines $K = \mathbf{Q}(\sqrt{-d})$ and $\chi_d(n) = (n/d)$. In the definitions below the subscript modulus q ranges over $\{1, q_0\}$. Put $\lambda_{d,q}(n) = \mathbf{1}_{n\equiv 1 \bmod q}\Lambda(n)(1-\chi_d(n))$, with the congruence condition vacuous for $q = 1$. The remainder and its subtraction constant are

$$R_{d,q}(x) = \sum_{n\le x} \lambda_{d,q}(n)(n^{-1} - x^{-1}) - \frac{\log x}{\varphi(q)} - c_{d,q},$$
$$c_{d,q} = \frac{D_q(1) - 1}{\varphi(q)},$$
$$D_q(s) = \sum_{\chi \bmod q} \left(\frac{F'_{\chi\chi_d}(s)}{F_{\chi\chi_d}(s)} - \frac{F'_\chi(s)}{F_\chi(s)} \right).$$

Here $F_\chi(s) = L(s,\chi)$ for a nonprincipal character and $F_\chi(s) = (s-1)L(s,\chi)$ for a principal character. For $\chi \bmod q$ in the sum defining D_q, the product $\chi\chi_d$ is taken modulo qd. Section 46 writes $D(s) = D_{q_0}(s)$ for this logarithmic derivative sum and $\mathcal{M}_R(s)$ for the Mellin transform of R_{d,q_0}, with integration weight $x^{-s}\,dx$. The criterion assumes zero-free twists on $\Re s \ge \beta$, $|\Im s| \le H_0$, where $1/2 < \beta < 1$ and $H_0 > 0$, and $R_{d,q_0}(x) \le x^{\beta-1}$ for all sufficiently large real x. It excludes untwisted zeros with $\Re s > \beta$ and $|\Im s| \le H_0$. The twist selection treats the rectangle $\beta \le \Re s \le 1$, $|\Im s| \le H_0$. Euler products give nonvanishing for $\Re s > 1$. In the Introduction, $\delta > 0$ is the tolerance in the remainder exponent and $C_{q_0,d,\delta}$ depends on q_0, d, δ.

2.2 Scales, ranks, and prescribed levels

In this subsection, we record the scales of Section 4, which determine the ranks and the original levels before any coefficient place is chosen.

We use m for the number of steps in the lower theta chain and r for the rank of the original ambient unitary group. For the integer scale x and $0 < \epsilon < 1/100$, Section 4 defines

$$m = 2^{\lceil \log_2 \sqrt{x} \rceil}, \qquad r = 3^{\lceil \log_3 (x^5) \rceil}, \qquad k = r + 1 - m,$$
$$s = 22 \log_2 m, \quad Q = 2^s, \quad L_0 = 3^{\lfloor \log_3 s \rfloor + 4}, \quad q_1 = 3 L_0 Q q_0, \quad q_2 = (q_0 d)^{s+6},$$
$$W = x^{\epsilon/6}, \qquad x_1 = \lfloor e^W \rfloor, \qquad b = \min\{b' \in q_1 x_1 \mathbf{Z}_{>0} : b' > e^{W^6}\},$$
$$a = xb, \quad B = \log b, \quad J_* = \frac{faB^3}{mk}, \quad \varrho = m/2 - 1, \quad t = 5r.$$

The initial definite rank is k. The lower spaces Y_j have ranks $r + jt$, and the upper spaces X_j have ranks $n_j = k + j(t+1)$ in Section 7.4. Section 8 uses ϱ hyperbolic pairs in the Bessel string. The progression construction uses the bound $R_{\mathrm{prog}} = kt$ and the fixed dimension parameter $n > 66/\epsilon$.

In the orbital estimates, $n = \varrho + 1 = m/2$ is the length of the Bessel basis. In Proposition 5.3, n is the dimension of the translated span. The local variables s, Q, W in the theta and Selmer calculations are specified in their respective sections. The two level primes are $p_0^{\mathrm{lev}}, p_1^{\mathrm{lev}}$, also denoted p_0, p_1 in Section 4.

Their product is M, and $\pi_M = \bar{\alpha}$ is the chosen generator with $\alpha\bar{\alpha} = M$. The additional prime N and generator $\pi_N = a_0 + ib_0$, with $\pi_N \bar{\pi}_N = N$, specify the smaller source level. The datum $\mathcal{D}$ records the weights, primes and generators of Section 4. The real Bessel scale at v is $\kappa_v = M\Phi_v(\delta_E)/(2\pi i)$, distinct from the integer weight κ_τ below. Local torsion exponents denoted M, N are specified in Section 2.8.

2.3 Fields, embeddings, and weights

In this subsection, we specify the fields and embeddings of Section 4 and the weights used in the prime sum of Section 5.

Let ζ_{q_1} be a primitive q_1-th root of unity, put $E = K(\zeta_{q_1})$ and $F_0 = E^+$, and write $f = [F_0 : \mathbf{Q}]$ for the number of real places of F_0. A bar denotes CM conjugation, or complex conjugation on values under the fixed complex embedding. Write $\mathcal{O}_E$, $\widehat{\mathcal{O}}_E$, and $\mathbb{A}_E$ for the integers, finite integral adeles, and adeles. We write Nm for absolute ideal norm, $\mathrm{rd}(E)$ for the absolute root discriminant, and δ_E for the chosen skew generator of the different. At a finite place w, k_w is the residue field and ϖ_w a uniformizer. The infinite places, read over the fixed embedding of K, are indexed by $(\mathbf{Z}/q_1)^\times$. Write Φ_{q_1} for the cyclotomic polynomial, $\mathfrak{D}_{E/\mathbf{Q}}$ for the different, and $N_{E/\mathbf{Q}}$ for the field norm. The set T consists of the selected embeddings over the residue class $-1 \bmod q_0$ in Section 4. The fibre $\Phi_a = \{\tau : \tau \equiv 1 \bmod q_0\}$ and the multiplicities in Section 5 give

$$u = |\Phi_a| = 3L_0 Q, \qquad v = |T|, \qquad \kappa_\tau = \begin{cases} b, & \tau \in T, \\ a/x_1, & \tau \notin T. \end{cases}$$

For an integer n, $\langle -n \rangle$ denotes the class of $-n$ in $(\mathbf{Z}/q_1)^\times$ if it is a unit, and the identity otherwise. We also use $\chi_K = \chi_d$ and $\Phi_v : E \hookrightarrow \mathbb{C}$ for a local complex embedding over a real place.

The unitary Hecke character υ fixes the Weil splitting. The norm-one torus is $T_1 = U(1)$, and its two level characters are χ_0, χ_M, constructed in Section 4.4.

2.4 Hermitian spaces, automorphic representations, and levels

In this subsection, we use the source and the projected target of Sections 6 and 8.

For a unitary group H over F_0, write $[H] = H(F_0)\backslash H(\mathbb{A}_{F_0})$. For a Hermitian space H_0, $U(H_0)$ denotes its unitary group, H_0^- the space with the opposite form, and $q(H_0)$ the negative index at a specified real place. We use $\perp$ for orthogonal direct sum, $\langle a \rangle$ for a line of norm a, and $I_a = (E^a, 1)$ for a standard positive definite space. Hermitian inner products in the determinant comparison are conjugate-linear in their first variable. For matrices $^{\mathsf{T}}$ is transpose and * is conjugate transpose, or the adjoint with respect to the indicated forms. An overline on a finite matrix can also denote reduction, with the residue ring specified at its occurrence.

Section 6 defines the full source X at the level J_H, which includes the last-covector condition above N. The ramified subspace $S \subseteq X$ is the intersection of the kernels of the scalar averages A_w at $w \mid \pi_N$. Put $d_X = \dim X$. The full Petersson target is T_{full}. The finite group $G = G_0 \times G_1$ acts by ρ; its two factors are the reductions at the two level primes. The projection onto the selected types is $p = \rho(e_0)$, with $e_0 = h_0 \otimes h_1$ the product of the two central idempotents. Thus $V = pT_{\mathrm{full}}$ and $D = \dim V$.

Use Y_0 for the ambient Hermitian space of rank r, X_0 for its definite block of rank k, and $R = U \rtimes U(X_0)$ for the Bessel subgroup in Section 8. Its ordered string has isotropic pairs e_a, e_a^* and the central vector z of norm $-N$. Here e_a^* denotes the basis vector paired with e_a. The compact group K_L and its congruence kernel J_L refer to the integral orbital basis. The source level is $J_H = \iota_H^{-1}(J_L)$, and J_M is the original principal level at M. Later $K_Z(\mathfrak{a})$ denotes the principal congruence subgroup at ideal $\mathfrak{a}$ on Z in its fixed integral bases. Write S_M for the chosen places dividing π_M.

Section 8.4 identifies the measured Poincaré map Π. In Section 8.5, $C = C_X R_H(\eta^{-1})$ gives the Euclidean class coordinates and $B = (v_G \nu)^{-1/2} \Pi C^{-1}$, where ν is the unipotent covolume. The symbol B used for this operator is distinguished by context from the scalar $\log b$. The translated maps $U_g : S \to V$ use the seesaw normalization, and $P_g = c_B U_g$ uses the rational scalar of Proposition 7.10. The finite class representatives a_c and unipotent cutoffs β_c in Section 8.2 give the quotient integrals. The finite Haar factors are v_R, v_G. At a real place, J_0, J_1 are the Gram matrices before and after diagonal scaling by D_v, p_v orders an orthonormal basis by sign, and $x_v = D_v p_v$. The superscript $\star$ in $x_v^\star$ denotes the stationary point and $\iota_v(g) = p_v g p_v^{-1}$. The resulting adelic point for the period is $g_0 = ((\iota_v^{-1}(x_v^\star))_v, 1)$; the adelic Gram isometry x^{pt} has finite coordinate x_f. The period functional is $t = (v_G/v_R)t_0$ in Section 8.4.

In Section 44.1, $T_\xi = L_\xi$ is the full weighted chain of Section 17.3 and $\Theta_\xi = T_\xi|_V$ is its restriction to the projected space, with $\xi = 1$ the untwisted case. Write $\|\cdot\|_{\mathrm{HS}}$ for the Hilbert–Schmidt norm, P_U for orthogonal projection to a subspace U, and $\mathbb{E}_g$ for averaging with normalized counting measure on G.

2.5 Orbital integrals and their moment bounds

In this subsection, we fix notation for the orbital integrals of Part II and the moment bounds of Section 9.

Write $B_s = B^* \rho(s) B$, $s \in G$, for the Petersson blocks and H_R for the residue Bessel subgroup. Its representation r_ψ on the Euclidean source $C(X)$ is extended by zero to G. In

Section 9.1 write

$$E_s = B_s - r_\psi(s), \qquad D_{\mathrm{id}} = |H_R| d_X, \qquad D_{\mathrm{main}} = \frac{d_X^2}{\mathcal{B}\ \mathrm{mass}(E, r)}.$$

The product of the formal degrees is $\mathcal{B}$. The masses $\mu_j = \mathrm{mass}(E, j)$ of the standard definite genera are weighted by reciprocal automorphism orders. The stabilizer intersections $\Gamma_{\mathcal{A}}$ at the places in $\mathcal{A}$ use the normalizing bound $C_{\mathcal{A}} = b^{-6k|\mathcal{A}|} D_{\mathrm{main}}$. In Section 9.1, $\mathcal{L}(\Omega)$ sums the squared norms of the cut blocks on the nonzero-discriminant part of Γ_*, separately for each link and over the two classes and $s \in \Omega$, before division by ν^2. For each factor G_i, $p_i = \rho(h_i)$ is the projection on the target and $q_i = 1 - h_i$ is the complementary idempotent in the group algebra.

For a finite set of places T in Section 10, write

$$A_T = \prod_{w \in T} E_w, \qquad \mathcal{R}_T = \prod_{w \in T} \mathcal{O}_{E_w}.$$

The finite calculations write $\mathcal{R} = \mathcal{R}_T$ locally. The additive modulus is md, and K_T, J are the local integral and congruence groups in Section 10.3. For the Bessel flag $F_i = \langle e_0, \dots, e_{i-1} \rangle$, the ideals $\mathfrak{d}_i = c_i \mathcal{R}_T$ compare the two positions modulo F_i; their inverse moduli are $D_i = \mathrm{md}(c_i)^{-1}$. Sections 10.8–10.10 specify the associated Fourier variables, denominator indices, and stabilizer measure. Section 14.2 uses the trace of r_ψ on each finite factor G_i.

In the real calculations, $h(v, y) = v^* J y$ in the chosen Gram basis, and norm-fiber measure uses $h(y, y)/2$. The support and determinant variables of the real correlations are specified in Sections 11.2 and 11.4. Section 12.1 writes W for the full Bessel vector, F for its truncation, and $P(A, B)$ for their quotient correlation. Here d denotes the local formal degree. Section 12.10 specifies the real discrete series, the compact type indices I_v, and the local stationary vectors. The dimension calculation uses $d_B = \prod_v |I_v|$ and $d_X = d_C d_B$, with d_C the source class count.

Section 14.1 defines the rational summation set $\mathcal{T}(c, s, c')$ for the fixed finite classes c, c' and residue element s in the Petersson blocks of Section 8.5. The real and finite common-entry pairings are formulated in Section 13.4. The pair lattice, its discriminant, and the parameters of the Schur bound are specified in Sections 14.7 and 15.7. One set $t \subseteq G$ of translates is chosen in Section 16.2. Section 3.2 writes $n = \dim \sum_{g \in t} P_g(S)$. In the orbital integrals $n = \varrho + 1$ counts the columns of $(e_0, \dots, e_{\varrho-1}, z)$.

2.6 Theta operators, derivatives, and integral estimates

In this subsection, we use the Weil splitting of Section 4.3 and the measured normalization of Section 17.3.

A tame twist ξ is a weight-zero character of the norm-one torus. Its modulus $Q(\xi)$ is the product of the prime ideals at one place of each split pair in its support. Thus

$$Q(\xi) \cap \overline{Q(\xi)} = \varnothing, \qquad q_\xi = \prod_{w \in Q(\xi)} q_w, \qquad (q_\xi, MN) = 1,$$

where in set notation $Q(\xi)$ denotes the prime divisors of the modulus. Its character on local units is given at such a place by a nontrivial residue character $\xi_w : k_w^\times \to \mathbb{C}^\times$ and is trivial on the units at the other places of the torus. A twist for the full chain has $\xi_w^j \neq 1$ for all $w \in Q(\xi)$ and $1 \le j \le m$. The trivial twist has empty support. At a coefficient prime ℓ we use $\ell \nmid q_\xi$; Proposition 28.3 supplies twists with these conditions. The successive spaces in the lower chain

are Y_j; its endpoint Y_m and the dimension-comparison space W_m of Section 9.4 have different ranks. Write K_j° for the standard compact, $J_{\xi,j}$ for the level at the old and twist places, and

$$A_{\xi,j} = [K_j^\circ : J_{\xi,j}].$$

The map $L_{\xi,j}$ is the theta integral at step j multiplied by this index. The paired tensors $F_i(\xi, c)$, the evaluated tensor T_ξ of (71), and the scalar c_{meas} appear in Section 19.1. The same symbol T_ξ denotes the weighted chain L_ξ in Section 44.1.

In Section 18.1, I_D and $\mathcal{C}_D$ are the positive-degree maps and the quotient by nonnegative-degree maps for the specified grading D. The Heisenberg translation is $\rho(w)$, the theta functional is θ, and $C_D(f; w)$ is its coefficient at rational frequency w against the positive-degree group. The same letter θ denotes Fontaine's map in the appendix. The integer ring of finite adeles is $R_f = \widehat{\mathcal{O}}_E$ in Section 19.2, where J_N is the idempotent supported at the places dividing π_N. That section and Section 19.4 specify the shifted lattice and the finite translations in the measured tensor.

The archimedean normalization and the weighted prime remainder are

$$\begin{aligned}
\Omega &= \frac{2\pi}{\mathrm{rd}(E)} \exp\!\left(\gamma_{\mathrm{Euler}} - \frac{L'}{L}(1, \chi_K)\right), \\
A &= mk \sum_{\tau} \bigl(\log(\kappa_\tau!) + \kappa_\tau \log(M/\Omega) - S(\kappa_\tau, \tau)\bigr), \\
\mathcal{R} &= av\bigl(R_{d,q_0}(x) - R_{d,q_0}(x_1)\bigr) + auR_{d,1}(x_1).
\end{aligned}$$

The weighted prime sum $S(Y, \tau)$ in Section 5 and its local components $S_p(Y, \tau)$ in Section 7 are

$$\begin{aligned}
S(Y, \tau) &= \sum_{n \le a} \Lambda(n)(1 - \chi_K(n))(Y/n - \kappa_{\langle -n \rangle \tau})_+, \\
S_p(Y, \tau) &= \tfrac{1}{2} \sum_{j \ge 1} (1 - \chi_K(p^j))(Y/p^j - \kappa_{\langle -p^j \rangle \tau})_+.
\end{aligned}$$

The latter is used at $Y = a, \kappa_\tau$. The exponents and the rational rescaling of the images are

$$\begin{aligned}
g_p &= m \left(\sum_{\tau \in \Phi_a} S_p(a, \tau) + k \sum_{\tau} S_p(\kappa_\tau, \tau) \right), \\
c_B &= \prod_{p \in F} p^{-\lfloor g_p - e_p \rfloor}.
\end{aligned}$$

Here F consists of primes below x_1 that are nonsplit in K, together with N. The local denominator bounds are

$$e_p = \begin{cases} (2faB^2/p)c_p, & p < x_1 \text{ nonsplit in } K, \\ 2mfb, & p = N, \\ 0, & \text{otherwise}, \end{cases} \qquad c_p = \begin{cases} m, & p \equiv \pm 1 \bmod m, \\ 1, & \text{otherwise}. \end{cases}$$

For the upper highest weights use $\Delta = 2r^8$, $w_j(\tau) = \kappa_\tau + (j + \frac{1}{2})\Delta$, and $a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ in Section 20.2. Section 7.4 uses $J(l) = \log(l!) + l \log(M/\Omega)$ for the contribution of a part of length l.

For Taylor coefficients we use Θ_f for the theta section of a Schwartz datum f, $Q(\partial)$ for the constant-coefficient differential operator of a polynomial Q, and the integral model and translation scales of Section 20.1. The scalar s_{vac} is the vacuum projection factor in (AV). The Taylor scale is $s_{\mathrm{jet}} = p_F a_d/\omega_0$, with $a_d = (2\pi/\sqrt{d})^{1/2}$, period lattice $\omega_0\mathcal{O}_K$, and p_F the algebraic factor in Lemma 7.6. Its integral normalization is in Section 20.1. Section 44.4 uses both in the norm tests. An exceptional prime is an inert $p < x_1$ with $p \not\equiv \pm 1 \bmod m$ and odd $\mathrm{ord}_{q_0}(-p) \le r_p = \lceil \log_p m \rceil$ (Section 20.2). At such a prime $Y_\lambda = p^{A_\lambda} T_{\lambda,r_p}$. The exponents A_λ, Frobenius embeddings ι_s, root defects, and colength are specified in Section 21.

2.7 Covolumes, determinants, and coefficient places

In this subsection, we use the comparison of Section 5 and the product formula of Section 23.

With $J_* = faB^3/(mk)$ the bound for the sums over coefficient primes in Part V, set

$$E_0 = faB^3, \qquad L = faB^4, \qquad I_0 = 10\log(1+2b^2)faB^3.$$

A coefficient place is $\kappa = (\ell, \iota)$, with $\iota : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$ and $\bar\kappa = (\ell, \iota \circ \mathrm{conj})$. We use $v_\kappa(\ell) = 1$ and $|z|_\kappa = \ell^{-v_\kappa(z)}$. In the covolume and height bounds, σ ranges over $\mathrm{Aut}(\mathbb{C})$ and acts on the coordinates. For a finite coefficient DVR O, its uniformizer valuation is v_O; normalized length is $\mathrm{length}_O / v_O(\ell)$. For a tuple y of n independent images,

$$\mathrm{Cov}(y) \le c \quad \Longleftrightarrow \quad \det \mathrm{Gram}(\sigma\Theta_1 y) \le e^c \quad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

The bound $\mathrm{Ind}(y) \le \delta$ means there are bounds $a_\ell \ge 0$, independent of the embedding above ℓ, with $v_\kappa(\det U_\kappa) \le a_\ell$ for the integral clearing matrices and $\sum_{\ell\in U} a_\ell \log \ell \le \delta$ for every finite prime set U. At a given κ the same matrix clears the tuple for all allowed twists of modulus prime to ℓ. The twist ξ_κ is chosen in Section 38. For a completed basis $w = (y_1, \dots, y_n, z_1, \dots, z_{D-n})$,

$$G_\xi = \det \mathrm{Gram}(\Theta_\xi w), \qquad Z_\xi = \det(\Theta_\xi^* \Theta_\xi), \qquad G_\xi = Z_\xi \det \mathrm{Gram}(w).$$

After the untwisted lift, the auxiliary vectors z_i have squared norms at most $\exp(b_{\mathrm{short}})$ at every complex embedding, with $b_{\mathrm{short}} = 2x_1^2 b^2$. The scalar $\gamma_{\xi_\kappa} = c_{\xi_\kappa}/c_1$ clears them in the lift Θ_{ξ_κ} at both κ and $\bar\kappa$. The logarithmic height bounds on Z_1/Z_{ξ_κ} and $(\gamma_{\xi_\kappa}\overline{\gamma_{\xi_\kappa}})^{-1}$ are j_Z and j_γ, respectively. The height convention is that of Section 23, and the basis completion is in Section 44.3.

2.8 Cocycles, specializations, and Galois congruences

In this subsection, we record the notation for the normalized lengths of Section 24 and the trace congruences of Section 32.

Write G_E for the absolute Galois group, χ_{cyc} for the cyclotomic character, and $L^\dagger = \mathrm{Hom}_O(L, O)(1)$ for the Tate dual of a coefficient lattice L. We use H^1_f for the crystalline local condition and $Z_f(L)$ for the integral cocycles mapping to it. The local class of lattices $\mathcal{Q}_v$ and cup-product colength $\Delta_v(L)$ are those of Section 29.1. At depth M, $\mathcal{B} = \mathcal{O}/\ell^M\mathcal{O}$ is the coefficient quotient, $\lambda(C)$ is the normalized length of a cocycle module, and ϕ_j, γ_j are the auxiliary Frobenius and inertia evaluations. The slopes s_j impose $z(\gamma_j) = s_j z(\phi_j)$ on the submodule C_0. The auxiliary character order divides ℓ^N. In Section 25, R_X is the polynomial

matrix of the twisted action and $A(X)$ records the cocycle, inertia, and other linear equations. The length is computed from the maximal minors $P_i(X)$ of

$$A^{+}(X) = \begin{pmatrix} A(X) \\ \ell^{M} I_q \end{pmatrix},$$

where q is the column count of $A(X)$ in that section. The evaluations $X_w = \zeta_w^j - 1$ use $\zeta_w^{\ell^H} = 1$ as in Section 26. The bound ℓ^H is fixed before the auxiliary level ℓ^N in Section 28.5.

In Section 27, Θ_Q is a group-ring element at auxiliary set Q, $q_w = |k_w|$, and $\mathcal{E}$ is the fixed list of exponents, and the Euler factor is the Laurent polynomial $E_w(A) = \prod_{e\in\mathcal{E}}(1 - Aq_w^{-e})(1 - A^{-1}q_w^{-e})$. The sign and contragredient conventions for the attached representation ρ are in Section 28.2. The trace of conjugation is normalized using the polarization exponent d_{pol} of Section 28.4.

The ordered Kummer construction uses $F_\nu = E(\mu_{\ell^\nu})$, $G_\nu = \ker(\chi_{\mathrm{cyc}} \bmod \ell^\nu)$, and $A_\nu = \mathbf{Z}/\ell^\nu\mathbf{Z}$. The original evaluation set in the cyclotomic kernel is P_0; the new primes $w_j, \bar{w}_j$ and the maps ψ_j are specified in Sections 29.3 and 29.4. The matrices $(u_x(g)-1)_x$ define the relative relations on cocycles in Section 29.1. For the full raised family Y and its common coefficient DVR O, put $\mathbb{T}_O = \mathbb{T} \cap O^Y$ and $S = \mathbb{T}_O[X_j : j \in J]$, where $\mathbb{T}$ is the integral trace algebra of Section 30.3. The scalar congruences generate $\mathcal{J} \subset S$. Their trace annihilator $\mathfrak{a} \subset S[G_E]$ is a two-sided ideal, as in Section 32.2. The operators $\mathcal{N}_j, \partial_j$ at the raised places and the highest squarefree coefficient refer to the original test space of Section 32.3.

2.9 Period comparisons, critical values, and descent

In this subsection, we record the spaces and operators for the period comparison of Section 36 and the lattice completion of Section 44.3.

The two comparison factors of Part V use the original descent pair W_j, V_j of Section 33.2, with

$$V_j = V'' \perp V^{\circ}, \qquad d_+ = r + 9m + 1, \qquad d_- = t - 9m - 1.$$

Here d_+, d_- are the ranks of V'', V°. The choices of classes and the tuple space $\mathcal{H}$ used by the gluing operator are defined in Section 34.1. The gluing operator is G_η at twist η, and n_A, h_- are the class and coefficient constants in $U_\eta = n_A h_-^2 G_\eta G_\eta^*$. On the projected source the lift is u_η, its image is Z_η, and the Petersson projection is Π_η. This subspace notation Z_η is local to Section 42.1. For the j-th single lift write $Z_j(\eta) = \det(u_{j,\eta}^* u_{j,\eta})$. In the chain at twist ξ its factor is $Z_j(\xi^j)$.

For the comparison of two compositions, U_1, U_2 are the source and target Hermitian spaces, A, A' are the intermediate spaces, and $W = U_1 \perp U_2$. The two compositions with the chosen point arrays X, Y are $T_X = D_{2,X_2} L_{1,X_1}$ and $T'_Y = L'_{2,Y_2} D'_{1,Y_1}$ in Section 34.3. The real Fock variables and finite self-dual test are specified in Sections 35.1 and 35.2. The signatures, infinity weights μ_τ, and the character $\gamma = \nu^{-1}\upsilon^{N/2}$ for exchanging two periods are fixed in Section 36.1. Here $N = \dim W$ in that formula, and υ there is the restriction of the splitting Hecke character to the norm-one ideles.

A critical pair $p = (\nu, z)$ consists of an automorphic character ν of the norm-one torus and an integer $1 \le z \le t$. Its infinity weights are $-c_v$, with $z \le c_v \le 4b$. At a chosen half $w \in S_M$ of an old split place the restriction to units factors through a character of $k_w^\times$, and at the other finite places of the torus it is unramified. The pair is ramified if all these residue characters are

nontrivial. Base change includes both halves of each pair. The associated twist is η of exact support Q, with $q_Q = \mathrm{Nm}(Q)$. In Section 38, the ratio for $\epsilon \in \{1, -1\}$ is

$$R_{p,\epsilon}(\eta) = \frac{|L^{S_M \cup Q}(z, ((\nu\eta^{\epsilon})^{\mathrm{bc}})^{-1})|}{|L^{S_M}(z, (\nu^{\mathrm{bc}})^{-1})|}.$$

In these L-functions, the exponent -1 denotes the inverse character after base change. A place set in the superscript on L omits the corresponding Euler factors and their conjugates. The Taylor map $\mathcal{T}$, coefficient product B_τ, and four-block vector ϕ_η are those of Sections 37.1 and 37.2.

In the common period, K°_{ζ} is the class sum of the complementary Weyl kernels paired with the tuple ζ; the coefficient formulas and approximation bounds (CP1)–(CP6) are in Section 41. At deep principal level put $b_0 = \lfloor b \rfloor$, $\mathfrak{b} = (\pi_M^{b_0})$, and $R_b = \mathcal{O}_E/\mathfrak{b}$. The spaces $\mathcal{H}_j$, base identifications j_i, and arrow operators are specified in Section 43. Section 44.1 gives the chain relations (GC1)–(GC3). Proposition 44.3 gives the vectors for completing the lattice.

2.10 Local period rings and filtration orders

In this subsection, we fix the notation for the period rings of Appendix A and the filtration orders of Appendix B. These enter the local bounds of Section 29.1.

In the appendices p denotes the coefficient residue characteristic, $S = W(k)[[z]]$ is the chart ring, θ is Fontaine's map, and φ is Witt Frobenius. The chart symbol $E = p + z^e u$, with e the absolute ramification index of the chart field and $u \in S^{\times}$, is a generator of $\ker \theta$, with Frobenius translates $E_a = \varphi^a(E)$ and tame model $E = p - z^e$. The Witt ring is $A = A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p^{\flat}})$, with chart ideal $I = (p, z)A$ and retraction $F_0 : A \to S$. The bounded period ring is $B^+ = A_{\mathrm{max}}[1/p]$, where A_{max} is the p-adic completion of $A[E/p]$. The ring S_B is the closure of $S[1/p]$ in the topology with neighborhoods $p^n A_{\mathrm{max}}$ of zero. The de Rham filtration is $\mathrm{Fil}^j = t^j B^+_{\mathrm{dR}}$. For a compatible system $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$ of primitive roots of unity, put $\mu = [\epsilon] - 1$, $\tau = \varphi^{-1}(\mu)$, and $\xi = \mu/\tau$. Then $t = \log(1+\mu)$, and the divided action uses $\Delta_z = z\tau$ in Sections B.2 and C.1. The modules M_T, N_T over A_{inf}, S use the standard orders of Section B.3.

The indices i, j and their Frobenius translates label the chart idempotents. The ideal $I_l(i)$ consists of the multipliers that put the component-i parameters in the level-l standard-order image (Section B.3). The ideal $J_r(j; R)$ is the ideal of scalars lifting component-j parameters at order r, plus (R) (Section D.2). The weight gaps and stopping indices are in Sections D.4, E.1, and E.3.

3 Outlines of proofs

In this section, we trace the proof of Theorem 1.1 through Parts I–VI. Section 5 reduces the prime-remainder bound to two estimates on the same translated basis. The two estimates yield Proposition 5.3, from which Section 47 deduces the bound required by the analytic criterion of Section 46.

3.1 Part I. Parameters and the common family

In this subsection, we explain how Part I uses the arithmetic datum of Section 4 to express the prime remainder in the lattice comparison of Section 5.

Fix q_0 divisible by 18480, a prime $d \equiv q_0/2 - 1 \pmod{q_0}$, and $0 < \epsilon < 1/100$. The comparison applies at every sufficiently large integer x for these fixed choices. Let $a = xb$ and $B = \log b$, as in Section 2.2. The weights are a/x on T and a/x_1 outside T. The set Φ_a is the fibre of 1 mod q_0, and T is a subset of the fibre of -1 mod q_0. Since q_0 and q_1 have the same prime divisors, multiplication by $-n$ sends the first fibre bijectively to the second whenever $n \equiv 1 \bmod q_0$. For other units the image misses T; for nonunits Section 5 uses the identity multiplier. Thus, with $u = |\Phi_a|$ and $v = |T|$, the weighted sum gives

$$\frac{\mathcal{R}}{av} = R_{d,q_0}(x) - R_{d,q_0}(x_1) + \frac{u}{v} R_{d,1}(x_1), \qquad \frac{u}{v} \le B, \qquad f = \varphi(q_0)u.$$

Section 5 also defines the common real normalization $A \ge 0$ and the errors L, I_0. The term $-m\mathcal{R}$ will appear in the upper covolume bound.

Section 6 fixes X and cuts out $S \subseteq X$ by the scalar averages at N. Section 8 constructs Π on X with the prescribed real point, finite level, and quotient Haar measures. The finite-type idempotent e_0 of Section 7.3 gives $p = \rho(e_0)$ and $V = \operatorname{im} p$, with $D = \dim V$. These data specify the source, target and terminal lifts in Propositions 5.1 and 5.2. The simultaneous construction at large x is checked in Section 16.1.

3.2 Part II. Orbital estimates and the translated span

In this subsection, we explain how Part II uses the Petersson blocks of Proposition 8.1 to choose translates of S in Section 16.2.

Section 9.1 compares the full Petersson blocks with the truncated orbital integrals. The finite and real correlation bounds are in Sections 10 and 11. Section 12.8 bounds the length at the same stationary point used in the measured period. Section 13 combines the local bounds with the definite masses. The truncation errors and the remaining pair sums are controlled in Sections 14 and 15. The dimension comparison (3) supplies the normalization in the mixed moment.

Write B_{Pet} for the Petersson-normalized map denoted B in Section 9, to distinguish it from $\log b$. On the Euclidean class coordinates $C(X)$, the moment bounds are

$$\|pB_{\mathrm{Pet}}P_U\|_{\mathrm{HS}}^2 \ge \dim U - 5b^{-4}d_X, \qquad D\,\mathbb{E}_g\|B_{\mathrm{Pet}}^*\rho(g)B_{\mathrm{Pet}}\|_{\mathrm{HS}}^2 \le (1+9b^{-4})d_X^2, \qquad U \subseteq C(X).$$

Propositions 9.2 and 9.3 give these bounds, respectively. In Section 16.2, the mixed moment gives a translate whose projection to the span already selected has small Hilbert–Schmidt norm. The diagonal bound then controls the dimension of its preimage in $C(X)$. Iteration decreases the codimension. Restriction to $C(S)$ loses at most $|t|\dim(X/S)$ dimensions for a set t of translates.

The comparison of the two normalizations in Section 16.2 gives

$$B_{\mathrm{Pet}} = (v_G\nu)^{-1/2}\Pi C^{-1}, \qquad P_g = c_B|G|c_{\mathrm{ss}}^{-1}(v_G\nu)^{1/2}p\rho(g)B_{\mathrm{Pet}}C|_S.$$

Here v_G is the level volume, ν is the unipotent covolume, and $c_{\mathrm{ss}} \ne 0$ is the scalar obtained by putting Proposition 19.1 in integral form in Section 19.3. The rational factor $c_B \ne 0$ is given in Section 3.3. After restriction to $C(S)$, the span chosen with B_{Pet} is the span of the same $P_g(S)$ used for the integral estimates. More precisely,

$$|t|\dim S \le 5D\log(1+2b^2), \qquad n = \dim\sum_{g\in t} P_g(S) \ge (1-4/b^2)D.$$

3.3 Part III. Theta identities and the upper covolume bound

In this subsection, we explain how Part III uses the measured seesaw of Section 17.3 to bound the norms and finite index on the span of Section 16.2.

Sections 18.1 and 18.3 unfold the consecutive lifts on the specified levels. The measured finite sum of Section 19.1 gives the tensor, and Section 19.3 gives its integral expansion. Proposition 19.1 uses the same Π and fixed real point from Section 8.3. The comparison at N, the ordinary derivatives, and the exceptional-prime Hecke operators are in Sections 19.4, 20, and 21, respectively.

Section 7 defines the prime-power contributions g_p, and Section 7.1 bounds the local errors e_p by $\sum_p e_p \log p \le E_0$. The scalar of Proposition 7.10 is

$$c_B = \prod_{p \in F} p^{-\lfloor g_p - e_p \rfloor}.$$

At each coefficient prime p, the scalar makes the cleared images integral for all allowed twists of modulus prime to p. The prime-power identity in Section 7 accounts for the matching terms in the conjugate norm bound. Proposition 22.2 applies it to the same translated maps as Proposition 16.2.

Choose a basis of the selected span from the generators of Proposition 5.1:

$$y = (P_{g_i}\phi_i)_{i=1}^{n}, \qquad g_i \in t, \quad \phi_i \in \mathcal{S}.$$

The untwisted norm bound and Hadamard's inequality give the covolume bound at every complex embedding. For the index, the clearing matrices are combined by translate, giving a determinant valuation at most $|t| d_X e_p$ above p. Using $d_X \le 2 \dim S$, $\sum_p e_p \log p \le E_0$, and the bound on $|t| \dim S$, we obtain

$$\mathrm{Cov}(y) \le n(A - m\mathcal{R} + L), \qquad \mathrm{Ind}(y) \le DI_0.$$

In the index bound the matrix at a coefficient place is chosen before selecting the twist, as required in Section 23. The basis y is fixed before any coefficient place or twist is chosen.

3.4 Part IV. Selmer bounds and trace congruences

In this subsection, we explain how Part IV gives the Selmer and norm conditions for the common twist of Section 38 and the trace comparison used there.

The original Hecke eigensystems give the Galois representations in Section 28.2, with local lengths estimated in Section 28.4. Lemma 24.1 bounds the normalized length of C/C_0, where C_0 imposes the auxiliary local relations on C. Let P_{j_0} be a maximal minor of A^+ attaining the minimum in Lemma 25.1 at a test point. For each integral specialization x, the equations defining $C(x)$ give

$$\lambda(C(x)) = \min_j v(P_j(x)) \le v(P_{j_0}(x)).$$

For a coefficient place $\kappa = (\ell, \iota)$, Section 28.5 fixes the original Selmer data and signed norm systems, their separate truncation depths, and a uniform bound on the number of auxiliary primes. Lemma 26.1 then fixes H from the bounds and tolerances. Next one chooses the auxiliary level $\ell^{N_{\mathrm{tor}}}$, the primes, and the test polynomials. One tuple $(\zeta_w)_w$ with $\zeta_w^{\ell^H} = 1$ controls the valuations at $(\zeta_w^j - 1)_w$ for all $1 \le j \le m$. Section 38 applies this construction

with support Q away from the primes over ℓMN, where N in ℓMN is the chosen level prime. The resulting ξ_κ^j have exact support Q for $1 \le j \le m$. The line and rank-two norm formulas in Sections 30.1 and 30.2 apply to the signed systems at the respective ranks.

For the trace estimate, Section 29.1 uses Appendices C and E on the original local class and its Tate-dual pairings. For the selected twist, Section 29.4 fixes the ordered list of Kummer tests before the trace congruence depth ν. The original Hecke systems determine a sufficient ν in Section 32.3. At that depth one chooses the primes at which to raise the level and forms the trace algebra. These primes are additional to the support Q of the twist. The coefficient-extension formula (84) carries the normalized Selmer length bound to the common coefficient DVR used for the raised systems. Section 32.2 constructs one trace multiplier for the raised systems. Its highest squarefree coefficient gives the clearing operator on integral tests orthogonal to every false-lift space F_w, as in Proposition 32.1.

3.5 Part V. Periods and the lower determinant estimate

In this subsection, we explain how Part V bounds the determinant ratio and completes the fixed basis y for the product formula of Section 23.

Section 34.1 defines the operator on the original class spaces. Sections 36 and 39.4 match its periods, including their measure factors. Sections 40 and 41 establish the reproduction and injectivity used to compress the operator in Section 42.1. With the arithmetic estimates of Part IV, Section 38 uses the same ξ_κ for the necessary norm and Selmer tests at each coefficient place.

The first-arrow Gram identity of Section 34.2 starts the injectivity argument of Section 44.1. The first square identity (GC1) and the propagation formula (GC2) prove injectivity before division by the real period factors. Proposition 44.1 computes the chain ratio from the factors $Z_j(1)/Z_j(\xi^j)$ and the triangle factors. Proposition 44.2 bounds its height. Using the deep periods of Section 43, Proposition 44.3 supplies a spanning set from which to choose the $s = D - n$ additional vectors. In the notation of Section 2.7,

$$j_Z \le D(b + L - A), \qquad j_\gamma \le b, \qquad b_{\mathrm{short}} = 2x_1^2 b^2, \qquad s \le 4D/b^2.$$

Here j_Z bounds the height of Z_1/Z_{ξ_κ}, j_γ bounds the height of $(\gamma_{\xi_\kappa}\overline{\gamma_{\xi_\kappa}})^{-1}$, and b_{short} bounds the logarithmic squared norms of the added vectors after the untwisted lift.

Let w be the completed basis. The matrices at κ and $\bar\kappa$ clear the original y in the same lift Θ_{ξ_κ}; the additional vectors use the common scalar γ_{ξ_κ}. The two integral families give the mixed Gram bound for G_{ξ_κ} in Section 23. Since $G_\xi = Z_\xi \det \mathrm{Gram}(w)$, the ratio Z_1/Z_{ξ_κ} transfers the bound to G_1. At infinity, exterior-power Cauchy–Schwarz bounds $|\sigma G_1|$ by the product of the exterior norms of the untwisted families at σ and $\mathrm{conj} \circ \sigma \circ \mathrm{conj}$. In each family project the added vectors orthogonally off the span of the first n vectors and apply Hadamard's inequality to the projected vectors. If $\mathrm{Cov}(y) \le c$, this gives $|\sigma G_1| \le e^{c + s b_{\mathrm{short}}}$. With $\mathrm{Ind}(y) \le \delta$, the product formula gives

$$0 \le c + 2\delta + s(b_{\mathrm{short}} + j_\gamma) + j_Z.$$

Section 45 uses $s(b_{\mathrm{short}} + j_\gamma) \le Db$ and $b \le L$ to obtain $D(A - 3L) \le c + 2\delta$. Substituting the bounds of Section 3.3 and using $n \le D$ and $A + L \ge 0$ gives

$$nm\mathcal{R} \le (n - D)(A + L) + D(4L + 2I_0) \le D(4L + 2I_0).$$

For large x, $n \ge D/2$, $I_0 \le 30L$, and $B \ge 128$. This is the cancellation in Proposition 5.3, giving $m\mathcal{R} \le 8L + 4I_0 \le faB^5$.

3.6 Part VI. The prime remainder and the analytic conclusion

In this subsection, we explain how Part VI uses Proposition 5.3 to obtain the one-sided bound in Section 47 and apply Section 46 to Theorem 1.1.

Substitute $m\mathcal{R} \le faB^5$ into the identity of Section 5. From $u/v \le B$ and $f = \varphi(q_0)u$,

$$R_{d,q_0}(x) \le \frac{\varphi(q_0)B^6}{m} + |R_{d,q_0}(x_1)| + B|R_{d,1}(x_1)|.$$

The proof of Proposition 47.1 bounds both remainders at x_1 by the fixed-modulus prime number theorem. The bounds $m \ge \sqrt{x}$ and $B \ll x^{\epsilon}$ give $R_{d,q_0}(x) \le C_{q_0,d,\epsilon}x^{-1/2+6\epsilon}$ at sufficiently large integers. Integrating the $O_{d,q_0}(x^{-1})$ derivative bound between consecutive integers extends this estimate to real x.

For χ mod q, induce to $\widetilde{\chi}$ mod q_0 with $q_0 = 18480q$. The extra Euler factors are nonzero in $\Re s > 0$, so a zero ρ of $L(s,\chi)$ with $1/2 < \Re\rho < 1$ is also a zero of $L(s,\widetilde{\chi})$. Choose $1/2 < \beta < \Re\rho$ and $H_0 > |\Im\rho|$. Proposition 46.2 uses a truncated Mellin integral of the Möbius-weighted theta series to select one prime d for which all twists by χ_d of the characters modulo q_0 are zero-free on $\beta \le \Re s \le 1$, $|\Im s| \le H_0$. Fix this d, choose $0 < \epsilon < 1/100$ with $6\epsilon < \beta - 1/2$, and then let x grow in the estimate above. Absorbing the constant gives

$$h(x) = x^{\beta-1} - R_{d,q_0}(x) \ge 0$$

for all sufficiently large real x. The real-abscissa argument of Lemma 46.1 shows that the Mellin integral of h over that tail converges on $\Re s > \beta$. Since the twists have no zeros in the rectangle, the zero of $L(s,\widetilde{\chi})$ at ρ would give a nonzero residue there. Section 47 treats $\Re s \ge 1$ separately and uses the primitive functional equation to exclude zeros in $0 < \Re s < 1/2$.

Part I Arithmetic parameters and automorphic spaces

4 Arithmetic parameters

In this section, we fix the arithmetic parameters before choosing any coefficient place. The selected embeddings separate the two weights in Section 5, where their multiplicities determine the two prime sums.

Fix $q_0 \ge 1$ divisible by 18480, a prime $d \equiv d_0 = q_0/2 - 1 \bmod q_0$, and $0 < \epsilon < 1/100$. The scale x is an integer. The parameters q_0, d, ϵ remain fixed as x grows. When a constant is written as depending on d alone, one may take its maximum over eligible q_0 because $q_0 \mid 2(d+1)$.

Lemma 4.1 (The auxiliary residue class)**.** *The class d_0 is a unit of order two modulo q_0 and $-1 \notin \langle d_0\rangle$. We have $d \equiv 7 \bmod 8$, and every prime dividing q_0 splits in $\mathbb{Q}(\sqrt{-d})$.*

Proof. Since $16 \mid q_0$, one has

$$(q_0/2-1)^2 \equiv 1 \bmod q_0, \qquad d_0 \not\equiv \pm 1 \bmod q_0, \qquad \frac{d+1}{4} = (2j+1)\frac{q_0}{8}$$

for some $j \ge 0$. Since $16 \mid q_0$, we have $d \equiv 7 \bmod 8$ and $\mathrm{rad}(q_0) \mid q_0/8$. In particular $d \nmid q_0$. The discriminant of $K = \mathbb{Q}(\sqrt{-d})$ is $-d$ and $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{-d})/2]$. At every prime dividing q_0, the minimal polynomial $X^2 - X + (d+1)/4$ reduces to $X(X-1)$. The roots are distinct also at 2, giving the asserted splitting. For odd $p \ne d$, quadratic reciprocity and $d \equiv 3 \bmod 4$ give $(-d/p) = (p/d)$. At 2, $(2/d) = 1$ because $d \equiv 7 \bmod 8$, while the value at d is zero. Thus the character of the quadratic field at every positive integer is $\chi_K(n) = (n/d)$. □

For m the least power of 2 at least $\sqrt{x}$, r the least power of 3 at least x^5, put

$$s = 22\log_2 m, \quad Q = 2^s, \quad L_0 = 3^{\lfloor \log_3 s \rfloor + 4}, \quad q_1 = 3L_0 Q q_0, \quad R_{\text{prog}} = (r+1-m)5r.$$

Let $E = K(\zeta_{q_1})$, where the quadratic field $K = \mathbb{Q}(\sqrt{-d})$ ramifies at d, whereas the cyclotomic field of conductor q_1 is unramified there. Thus K intersects it in $\mathbb{Q}$. By fixing one embedding of K, the infinite places of the compositum are indexed by $(\mathbb{Z}/q_1)^\times$. Since $q_1/q_0 = 3L_0Q$ has no prime factor outside q_0, reduction to $(\mathbb{Z}/q_0)^\times$ is surjective with kernel of order $3L_0Q$. In particular

$$f = \varphi(q_0)\, 3L_0 Q, \qquad f \asymp_{q_0} q_1 \asymp_{q_0} x^{11}\log x, \qquad f, q_1 \le x^{12}$$

for all sufficiently large x with q_0 fixed. The primes dividing q_1 are split in K by Lemma 4.1.

Write $W = x^{\epsilon/6}$ and choose a fixed integer $n > 66/\epsilon$. Since $Q = m^{22}$, one has $x^{11} \le Q < 2^{22}x^{11}$, so $W \le Q \le W^n$ for large x. Also $81s/3 < L_0 \le 81s$, so $L_0 \asymp \log x$. With n fixed these give

$$(2L_0)^n \le W, \qquad 3L_0 n 2^n \le W^3, \qquad 2L_0(R_{\text{prog}}+1) < Q,$$

where the last inequality uses $R_{\text{prog}} = O(x^{10})$. Take the largest power of 2, denoted H, with $H^n \le Q/W$. Then $H \le W$, $Q < 2^n H^n W$, and $(2L_0H)^n \le Q$. In the cube $\{0,\dots,H-1\}^n$ a sphere of constant squared norm contains at least $H^n/(nH^2)$ points, since there are at most nH^2 norms. Read these points as digits in base $2L_0H$, giving a set $\mathcal{E} \subseteq [0,Q)$. If $(a_1+a_2)e_2 \equiv a_1e_1 + a_2e_3 \bmod Q$ for $e_j \in \mathcal{E}$ and positive integers a_1, a_2 with $a_1+a_2 \le 2L_0$, every digit sum is at most $2L_0(H-1) < 2L_0H$. Both integers are below $(2L_0H)^n \le Q$. They are therefore equal digitwise. The vector of e_2 is a strict convex combination of the vectors of e_1, e_3, which have the same squared norm as e_2. Thus all three vectors coincide.

Put $g = 1 + 3L_0q_0$ and $v_2(q_0) = v_0 \ge 4$. The prime divisors of q_1 are those of q_0, so g is a unit modulo q_1. For every positive h, $v_2(g^h - 1) = v_0 + v_2(h)$, by factoring for odd h and repeated difference of squares for even h. Thus g has order Q modulo 2^{s+v_0}. The places $T = \{-g^e \bmod q_1 : e \in \mathcal{E}\}$ are distinct and have residue $-1 \bmod q_0$. We prove that for any $\tau_1, \tau_2, \tau_3 \in T$, any prime $\ell \le R_{\text{prog}}$ and $0 < i < j < L_0$, the two congruences

$$\tau_2/\tau_1 \equiv \pm\ell^i, \qquad \tau_3/\tau_1 \equiv \pm\ell^j \pmod{(q_1)_{\ell'}}$$

cannot both hold, where $(q_1)_{\ell'}$ is the prime-to-ℓ part of q_1. Suppose both hold and write $\tau_k = -g^{e_k}$ with $e_k \in \mathcal{E}$.

For $\ell = 2$, squaring the first ratio gives $4^i \equiv 1 \bmod 3L_0$. Indeed $3L_0$ divides the odd part of q_1 and all the τ_k are -1 modulo $3L_0$. Since $v_3(4^i - 1) = 1 + v_3(i)$ by factoring successive cubes, $L_0 \mid i$, a contradiction. For odd ℓ, the modulus $(q_1)_{\ell'}$ contains 2^{s+v_0}. Raise the two ratios to $2j, 2i$, respectively, and compare their exponents of g in the cyclic group of order Q at 2. This gives

$$2je_2 \equiv 2(j-i)e_1 + 2ie_3 \pmod{Q}.$$

The equal-norm construction forces equality of the exponents. The first squared ratio then gives $\ell^{2i} = 1 \bmod 2^{s+1}$. Factoring this difference of powers gives

$$v_2(\ell^{2i} - 1) = v_2(\ell - 1) + v_2(\ell + 1) + v_2(2i) - 1.$$

Since $\min(v_2(\ell-1), v_2(\ell+1)) = 1$, we obtain

$$Q \le 2^{v_2(\ell^{2i}-1)} \le (\ell+1)2i \le (R_{\text{prog}}+1)2L_0 < Q.$$

This contradiction excludes the two ratios.

Finally $3L_0Q < 3L_0n2^nWH^2|T| \le W^6|T|$. For $x_1 = \lfloor e^W \rfloor$, take b to be the least multiple of q_1x_1 strictly greater than e^{W^6}. Since $e^{W^6} < b \le e^{W^6} + q_1x_1$ and $\log(q_1x_1) = O(\log x + W)$, one has $W^6 < B = \log b \le W^6 + 1$ eventually. Thus $u/v \le B$ for $u = 3L_0Q$, $v = |T|$ and $B = \log b$.

4.1 The chosen primes and their generators

In this subsection, we use Siegel–Walfisz to choose two principal-level primes with a common ideal class and a third prime for the additional level condition in Section 6. The generators also fix the additive characters in Section 17.3.

Set $q_2 = (q_0d)^{s+6}$ and $B = \log b$. Then $q_1d \mid q_2$. At 2 one adds s to $v_2(q_0)$ in q_1d, and at 3 one adds $1 + \log_3 L_0 = \lfloor \log_3 s \rfloor + 5 \le s + 5$ to $v_3(q_0)$. The exponent $s + 6$ in q_2 covers both additions because $2, 3 \mid q_0$. At all other primes q_1d and q_0d have the same valuation. Also $q_2 \ge Q = m^{22} > 8m$ for large x. In particular a prime $1 \bmod q_2^2$ splits completely in $E \subseteq \mathbb{Q}(\zeta_{q_1d})$. The preceding bounds give

$$\sqrt{x} \le m < 2\sqrt{x}, \qquad x^5 \le r < 3x^5, \qquad s \le 32 \log x, \qquad W - 1 \le \log x_1 \le W.$$

In particular, eventually $x^{10^6} \le x_1$, $x_1^{10^6} \le b$, and $B^{100} \le x$. One has $(\log x_1)^6 \le B$ by the choice of b. The bound $B^{100} \le x$ uses $100\epsilon < 1$. Also $\log B \to \infty$, so any fixed lower bound on $\log B$ depending on q_0, d can be imposed in the same cutoff.

Proposition 4.2 (The three level primes)**.** *Fix q_0, d, ϵ. For every sufficiently large integer x, there are distinct primes p_0, p_1, N satisfying*

$$p_0 \equiv p_1 \equiv N \equiv 1 \pmod{q_2^2}, \qquad b^{10} \le p_i \le b^{11} \quad (i = 0, 1),$$
$$U \le \log N \le U + \sqrt{B}.$$

Here $M = p_0p_1$, the weights κ_τ are those of Section 2.3, and

$$U = m\, f^{-1} \sum_\tau \log \kappa_\tau + (m - 1) \log\left(\frac{M \operatorname{rd}(E)}{2\pi}\right) - \tfrac{15}{2}B.$$

There are prime ideals $\mathfrak{p}_i \mid p_i$ of K and generators $\alpha \in \mathcal{O}_K$, $\pi_N \in \mathbb{Z}[i]$ with

$$(\alpha) = \mathfrak{p}_0\overline{\mathfrak{p}_1}, \qquad \alpha\bar{\alpha} = p_0p_1, \qquad \pi_N\bar{\pi}_N = N.$$

The chosen generator at $M = p_0p_1$ is $\pi_M = \bar{\alpha}$.

Proof. Since $\log(q_2^2) = O_{q_0,d}(\log x)$, there is a fixed A_{SW} with $q_2^2 \le W^{A_{\mathrm{SW}}}$. For any $u \ge W$, apply the Siegel–Walfisz theorem [54, Corollary 11.21, (11.36)] to the prime-weighted count $\vartheta(\,\cdot\,; q_2^2, 1)$ at $2e^u$ and e^u. It gives

$$\vartheta(2e^u; q_2^2, 1) - \vartheta(e^u; q_2^2, 1) = \frac{e^u}{\varphi(q_2^2)} + O_{A_{\mathrm{SW}}}\left(e^u e^{-c\sqrt{u}}\right).$$

Here $q_2^2 \le u^{A_{\mathrm{SW}}}$, and the relative error is $O_{A_{\mathrm{SW}}}(u^{A_{\mathrm{SW}}} e^{-c\sqrt{u}})$. This tends uniformly to zero for $u \ge W$ as $x \to \infty$. Thus there is a prime $1 \bmod q_2^2$ with logarithm in $(u, u + \log 2] \subseteq [u, u + W]$.

Let h_K be the class number of K. For $i = 0, \dots, h_K$, apply this result in $[10B + 2iW, 10B + (2i + 1)W]$. These intervals are disjoint and lie in $[10B, 11B]$ for large x, since $W/B \to 0$. The

selected primes split in K because their residue modulo d is 1. Choose a prime ideal of K above each. Two distinct primes p_0, p_1 give ideals $\mathfrak{p}_0, \mathfrak{p}_1$ of the same class. In particular,

$$b^{10} \le p_i \le b^{11}, \qquad p_i \le b p_{1-i} \quad (i = 0, 1).$$

Since $\mathfrak{p}_1\overline{\mathfrak{p}_1} = (p_1)$, the class of $\mathfrak{p}_0\overline{\mathfrak{p}_1}$ is $[\mathfrak{p}_0][\mathfrak{p}_1]^{-1} = 1$. Choose an integral generator α. As both prime ideals have residue degree one, taking absolute norms gives $\alpha\bar{\alpha} = p_0 p_1 = M$. Use $\pi_M = \bar{\alpha}$ as the chosen generator, with $\bar{\pi}_M = \alpha$.

Give each place $\tau \in T$ weight $\kappa_\tau = b$ and each remaining infinite place weight $\kappa_\tau = xb/x_1$. These are integers in $[1, b]$. Let $f = [E : \mathbb{Q}]/2$, and set

$$U = m\, f^{-1} \sum_\tau \log \kappa_\tau + (m-1) \log\!\left(\frac{M\ \mathrm{rd}(E)}{2\pi}\right) - \tfrac{15}{2} B.$$

Put $C = \mathbb{Q}(\zeta_{q_1})$. The discriminants of K, C are coprime, so products of their integral bases form an integral basis of E, as verified in Section 4.2. The Kronecker product of the two trace matrices and the cyclotomic discriminant formula give

$$|\operatorname{disc}(E)| = d^f |\operatorname{disc}(C)|^2, \qquad \mathrm{rd}(E) = \sqrt{d}\, q_1 \prod_{p \mid q_1} p^{-1/(p-1)}.$$

In particular $1 \le \mathrm{rd}(E) \le d q_1 \le b$ eventually. With $b^{20} \le M \le b^{22}$ this gives

$$19B \le \log(M\,\mathrm{rd}(E)/(2\pi)) \le 23B, \qquad mB \le U \le 24mB.$$

The second pair uses $m \ge 2$ and $0 \le f^{-1} \sum_\tau \log \kappa_\tau \le B$. Apply the interval of primes at this U to obtain $N \equiv 1 \bmod q_2^2$, with $\log N \in [U, U + \sqrt{B}]$, since $W \le \sqrt{B}$. For large x,

$$\log N \ge U \ge mB > 11B \ge \max(\log p_0, \log p_1).$$

As $N \equiv 1 \bmod 4$, it is a sum $a_0^2 + b_0^2$ of two integer squares. The prescribed norm generator is $a_0 + i b_0 \in \mathcal{O}_E$, where $i \in E$ because $4 \mid q_1$. The primes, their generators, and the weights are kept in the same datum, denoted $\mathcal{D}$. This construction is for every sufficiently large integer x with q_0, d, ϵ fixed. □

4.2 The different and the additive character scales

In this subsection, we choose the different generator that fixes the scales in Section 8.3.

For the additive character choose the skew generator δ_E of the different as follows. Since the two fields have coprime discriminants, $\mathcal{O}_E = \mathcal{O}_K[\zeta_{q_1}]$: at each rational prime, tensoring one local maximal order with the other unramified order gives a product of maximal orders. Put $F_0 = E^+$. In $\mathrm{Gal}(E/\mathbb{Q}) = \{\pm 1\} \times (\mathbb{Z}/q_1)^\times$, complex conjugation is $c = (-1, -1)$, and $\mathrm{Gal}(E/F_0) = \{1, c\}$. A place over p splits in E/F_0 precisely when its decomposition group over $\mathbb{Q}$ omits c. For $p \mid q_1$, including $p = 2$, this group fixes K by Lemma 4.1. At $p = d$ its projection to the cyclotomic factor is generated by $d \bmod q_1$; it omits -1 already modulo q_0 by the same lemma. Thus every place above dq_1 splits in E/F_0. For $p \nmid dq_1$, both factors are unramified and the decomposition group is generated by $(\chi_K(p), p \bmod q_1)$. The places above p are inert in E/F_0 exactly when $\chi_K(p) = -1$ and $p^j \equiv -1 \pmod{q_1}$ for some odd $j > 0$. In particular E/F_0 is unramified at every finite place.

The equality $f = [F_0 : \mathbb{Q}] = \varphi(q_1)$ follows from $[E : \mathbb{Q}] = 2\varphi(q_1)$, and f is even since $q_1 > 2$. The relative different over K is generated by $\Phi'_{q_1}(\zeta_{q_1})$, where Φ_{q_1} is the cyclotomic polynomial. Writing $X^{q_1} - 1 = \Phi_{q_1}(X)P(X)$ in $\mathbf{Z}[X]$ gives $q_1\zeta_{q_1}^{q_1-1} = \Phi'_{q_1}(\zeta_{q_1})P(\zeta_{q_1})$. Thus $q_1 \in \mathfrak{d}_{E/K}$, and transitivity gives $q_1 d \in \sqrt{-d}\,\mathfrak{d}_{E/K} = \mathfrak{d}_{E/\mathbf{Q}}$. This supplies the rational integer used in the dual-lattice estimate of Lemma 7.5. In the product of root differences for the derivative, write each other primitive root as $\zeta_{q_1}^{2a+1}$, since q_1 is even.

Multiplication of $\zeta_{q_1} - \zeta_{q_1}^{2a+1}$ by $i\zeta_{q_1}^{-a-1}$ makes the factor real. The product of these unit multiples, times $\sqrt{-d}$, is a skew generator of the absolute different by transitivity. All its allowed root-of-unity phases have the same archimedean moduli. Set $u = q_1/q_0$, whose prime divisors divide q_0. A root of unity has u-th power of order q_0 exactly when its own order is q_1: for each prime $\ell \mid q_0$, the valuation of its order must be $v_\ell(q_0) + v_\ell(u)$, and no prime outside q_0 can divide its order. Hence $\Phi_{q_1}(X) = \Phi_{q_0}(X^u)$, and differentiation gives

$$|\Phi_v(\delta_E)| = \sqrt{d}\,u \left|\Phi'_{q_0}(\Phi_v(\zeta_{q_1})^u)\right| \asymp_{d,q_0} q_1.$$

The derivative is nonzero at every primitive q_0-th root, and there are finitely many such roots at fixed q_0. Since $q_1 \asymp_{q_0} x^{11}\log x$, we have $x^{10} \le |\Phi_v(\delta_E)| \le x^{12} \le b$ at every infinite place for sufficiently large x. As $b^{20} \le M \le b^{22}$, the real character scales $\kappa_v = M\Phi_v(\delta_E)/(2\pi i)$ satisfy $1 \le |\kappa_v| \le b^{23}$ for large x. Here Φ_v denotes the embedding, as distinct from the cyclotomic polynomial. The interval of primes above also gives $N \le b^{25m}$.

4.3 The splitting and level characters

In this subsection, we construct the characters used in Sections 7.4 and 17.3, with their common finite and archimedean normalizations.

Put $F = F_0$, $q = q_1$, $\zeta = \zeta_{q_1}$, and $\widehat{O} = \widehat{\mathcal{O}}_E$.

The roots of unity in E are precisely μ_q. Indeed $E \subset \mathbf{Q}(\zeta_{dq})$ by the quadratic cyclotomic description of K of discriminant $-d$. If a root has order h, then $\varphi(\mathrm{lcm}(h, dq)) = \varphi(dq)$. Since $4 \mid dq$, any increase in this modulus strictly increases its totient, so $h \mid dq$. The ramification index at d in E is 2, since the cyclotomic field of conductor q is unramified there. As $d - 1 > 2$, E cannot contain $\mathbf{Q}(\zeta_d)$. Hence $h \mid q$. Now set $\eta = \zeta^{q/4}(\zeta - 1)$. The cyclotomic identities $\mathrm{N}_{E/K}(\zeta^{q/4}) = 1$ and $\mathrm{N}_{E/K}(\zeta - 1) = \Phi_q(1) = 1$ show that η is a unit of relative norm one. The second identity uses that q is divisible by both 2 and 3. Direct conjugation gives $\eta/\bar{\eta} = \zeta$. Thus every root ξ in E can be written $\xi = \delta/\bar{\delta}$ with $\delta \in \mathcal{O}_E^\times$ and $\prod_v \Phi_v(\delta) = 1$, taking a power of η.

If a finite place w of E is fixed by conjugation, choose a primitive third root of unity when $w \nmid 3$, and a primitive fourth root when $w \mid 3$. Its reduction differs from that of its inverse, since its order is prime to the residue characteristic. Thus conjugation acts nontrivially on the residue field at w, and the inertia subgroup of E/E^+ is trivial there. The other places occur in split pairs. The norm on local units is surjective at each finite place. For an inert place, the residue-field norm is surjective and the map on the successive principal-unit quotients is the residue trace. Successive lifting and completeness therefore give the unit norm surjectivity, including residue characteristic two. For a split place use the norm $(x, y) \mapsto xy$.

Let $\mathcal{U} = E_\infty^\times \widehat{O}^\times$ and $\mathcal{R} = E^\times\{y\bar{y} : y \in \mathbb{A}_E^\times\}$. On $\mathcal{U}$, put $S(x) = \prod_v x_v/|x_v|$. For $x = ay\bar{y} \in \mathcal{U} \cap \mathcal{R}$, the finite valuations of a are invariant under conjugation, since the valuations of $y\bar{y}$ are. Thus $\bar{a}/a$ is a unit with all archimedean moduli one. By Kronecker's theorem it is a root of unity. Choose δ as above with $\delta/\bar{\delta} = \bar{a}/a$, so $A = a\delta \in F$. The finite part of $Ay\bar{y}$ is a fixed integral unit and is a norm at every finite place by unit norm surjectivity. Hence A itself is

everywhere a finite local norm. Global Artin reciprocity for E/F [50, Ch. V, Theorem 5.3(a)] gives $\prod_{v|\infty} \operatorname{sgn} \Phi_v(A) = 1$. Since $\prod_v \Phi_v(\delta) = 1$ and $y\bar{y}$ is positive at infinity, it follows that $S(x) = 1$.

Accordingly S defines a character on $\mathcal{UR}$ with value one on $\mathcal{R}$. The quotient of the idele group by $\mathcal{UR}$ is finite, as it is a quotient of the ordinary ideal class group of E. Extend this character to the whole idele group one cyclic quotient at a time, choosing the required roots in the circle. Because $\mathcal{U}$ is open and S is continuous, the extended character υ is continuous. It is a unitary Hecke character, unramified at every finite place, with

$$\upsilon(\bar{x}) = \upsilon(x)^{-1}, \qquad \upsilon_\infty((z_v)_v) = \prod_v z_v/|z_v|, \qquad \upsilon|_{\mathbb{A}_F^\times} = \chi_{E/F}.$$

For the last identity, the restriction is trivial on $F^\times \operatorname{N}_{E/F}(\mathbb{A}_E^\times)$, which has index two by global reciprocity [50, Ch. V, Theorem 5.3(b), p. 179]. The sign at one real place is nontrivial. In particular the value of υ on the uniformizer at the chosen inert place is -1: one can compute it on a uniformizer of the fixed local field, and any other uniformizer differs by a unit.

Write υ_E for this full idelic character when it is necessary to distinguish it from its restriction to the norm-one torus. The values on finite ideles are algebraic. For a finite idele x, choose $h > 0$ such that the ideal of x^h has a generator $a \in E^\times$. Then

$$\upsilon(x)^h = \upsilon_\infty(a)^{-1} = \prod_v |\Phi_v(a)|/\Phi_v(a),$$

since $x^h a^{-1}$ is an integral unit at every finite place. The right side is algebraic. If a product of powers of υ and the idele norm has integral infinity exponents, its finite component takes values in one number field. The finitely many ideal-class representatives give algebraic generators, and the values on principal ideals are monomials in the conjugates of their generators. These facts give the coefficient fields for the algebraic twists of the cuspidal factors.

4.4 The two old-level characters on the same torus

In this subsection, we extend the level characters of the norm-one torus for the weights from Section 4. Their local components enter the theta identities in Section 17.3.

Let $T_1 = U(1)$ for E/F, let $T_{1,f}^0$ be its integral finite compact, and let S_M be the original places dividing π_M, one from each level pair. Each $w \in S_M$ has residue cardinality p_i for one of the two chosen primes. Since $q_2^2 \mid p_i - 1$, choose a character $a_w : k_w^\times \to \mathbb{C}^\times$ of exact order q_2^2 and put

$$\omega_w = a_w^{q_2} \quad (w \in S_M), \qquad \omega_w = 1 \quad (w \notin S_M).$$

The residue characters here are read on the chosen member of each split pair. The value on its conjugate member of a norm-one idele is determined by the chosen coordinate. Each ω_w on S_M has exact order q_2. For $z \in \mu_{q_1}$, one has $\omega_w(\operatorname{red}_w z) = a_w(\operatorname{red}_w(z^{q_2})) = 1$ because $q_1 \mid q_2$. For every integer j with $0 < |j| < q_2$, ω_w^j has order $q_2/\gcd(q_2, |j|) > 1$. Its lift to the local units has conductor exponent one, since it factors through the residue group.

For the infinite place υ with index τ, put $w_0(\upsilon) = \kappa_\tau + r^8$, where κ_τ is the integer weight b or xb/x_1. The two values b and xb/x_1 of κ_τ are multiples of q_1 because $q_1 x_1 \mid b$. On $T_1(F_\infty)T_{1,f}^0$ use the continuous characters

$$u \longmapsto \prod_v u_v^{-w_0(v)}, \qquad u \longmapsto \prod_{w \in S_M} \omega_w(\operatorname{red}_w u_w).$$

The rational points in this subgroup are exactly the integral norm-one units. They are μ_{q_1}: conjugation makes all complex absolute values one, and the root-of-unity calculation above applies. For $\gamma \in \mu_{q_1}$ the first formula has value

$$\prod_v \Phi_v(\gamma)^{-w_0(v)} = N_{E/K}(\gamma)^{-r^8} = 1.$$

Here the embeddings Φ_v are all those over the fixed embedding of K, and $N_{E/K}(\zeta) = \Phi_{q_1}(0) = 1$. The second formula is also one on these units by the local order calculation.

Both characters therefore extend trivially on $T_1(F)$ to the subgroup $T_1(F)T_1(F_\infty)T^0_{1,f}$ of $T_1(\mathbb{A}_F)$. To show that the quotient is finite, note that every finite place of E/F is split or unramified by Section 4.2. At a split place, a given norm-one pair (u, u^{-1}) is the image of $(u, 1)$ under $x \mapsto x/\bar{x}$. At an unramified nonsplit place, local Hilbert 90 gives a preimage. Dividing it by a power of a fixed-field uniformizer makes its valuation zero without changing its image. These choices are units at every place where the norm-one idele is integral, so they give a finite idele preimage. As $E^\times$ and $\widehat{\mathcal{O}}_E^\times$ map into $T_1(F)$ and $T^0_{1,f}$, respectively, the map on finite ideles factors to a surjection from $\mathrm{Cl}(E)$ onto the asserted quotient.

Extending across each finite cyclic quotient by a root in the unit circle gives continuous global characters χ_0, χ_M, respectively. They have weight $-w_0, 0$ and tame parts $1, \omega$ on the original field and original compact. For any fixed weight and tame part, the possible extensions form a finite set by the same quotient calculation.

Their finite adelic values lie in a number field. On representatives for the finite quotient, a fixed power of each value is the product of a weight monomial on a rational norm-one element and finite roots of unity. This makes the finitely many representative values algebraic. On all other finite ideles, the additional factors are such monomials and roots of unity from the fixed tame characters. The normal closure of E and those finitely many algebraic generators therefore give one coefficient field.

Use $\upsilon_1 = \upsilon|_{T_1(\mathbb{A}_F)}$ and $\Delta = 2r^8$. For every j the fixed character

$$\mu_j^0 = \chi_0(\upsilon_1^{-\Delta}\chi_M)^j$$

has weight $-(\kappa_\tau + (2j+1)r^8)$ at the infinite place indexed by τ and tame part ω^j. For $1 \le |e| \le 8m < q_2$, ω_w^e is nontrivial exactly on S_M. This proves the required old support for the single lifts, both signs of the three exchanges, the triangle at the original level quotients and the reverse-grid products. These choices are made after fixing the two level primes and their chosen members, and before any additional modulus, coefficient embedding, or period test.

5 The lattice comparison on a common automorphic family

In this section, we translate the prime remainder of Section 46 into a statement about one family of lifts. Hadamard's inequality combines the estimates of Propositions 5.1 and 5.2.

Write $a = xb$, $B = \log b$, $L = faB^4$, and $I_0 = 10\log(1+2b^2)\,faB^3$. For the remainder $R_{d,q}$ of Section 2.1, put $v = |T|$, $u = 3L_0Q$, and

$$\mathcal{R} = av(R_{d,q_0}(x) - R_{d,q_0}(x_1)) + auR_{d,1}(x_1).$$

The preceding construction supplies $u/v \le B$ and $f = \varphi(q_0)u$. In fact $u = |\Phi_a|$ for $\Phi_a = \{\tau : \tau \equiv 1 \bmod q_0\}$.

Write $\chi_K(n) = (n/d)$, the quadratic character of K, and let Λ denote the von Mangoldt function. For a variable Y put

$$S(Y,\tau) = \sum_{n\le a} \Lambda(n)(1-\chi_K(n))(Y/n - \kappa_{\langle -n\rangle\tau})_+.$$

The parameter $\langle -n\rangle$ is the class of $-n$ when coprime to q_1 and the identity otherwise. When τ ranges over Φ_a, the product parameter is in T exactly v times if $n \equiv 1 \bmod q_0$, and never otherwise.

Indeed Φ_a is the fibre of 1 over q_0 and T lies in the fibre of -1. The moduli q_0, q_1 have the same prime divisors. If $n \equiv 1 \bmod q_0$, multiplication by $-n$ is a bijection between these two full fibres. Any other unit class has disjoint image from T, and a nonunit uses the prescribed identity multiplier. Thus

$$\#\{\tau \in \Phi_a : \langle -n\rangle\tau \in T\} = v\mathbf{1}_{n\equiv 1 \bmod q_0}.$$

The weights a/x_1 off T and $a/x = b$ on T give the exact identity

$$\begin{aligned}\frac{1}{a}\sum_{\tau\in\Phi_a} S(a,\tau) = u\sum_{n\le x_1} \Lambda(n)(1-\chi_K(n))(n^{-1}-x_1^{-1}) \\ + v\sum_{n\equiv 1 \bmod q_0} \Lambda(n)(1-\chi_K(n))((n^{-1}-x^{-1})_+ - (n^{-1}-x_1^{-1})_+).\end{aligned}$$

For large x, one has $x < x_1 \le a$. The correction weight is

$$(n^{-1}-x^{-1})_+ - (n^{-1}-x_1^{-1})_+ = \begin{cases} x_1^{-1}-x^{-1}, & n \le x, \\ x_1^{-1}-n^{-1}, & x < n < x_1, \\ 0, & n \ge x_1. \end{cases}$$

Both sums therefore lie in the original range $n \le a$. Applying the definitions of the two remainders gives

$$\sum_{\tau\in\Phi_a} S(a,\tau) = \mathcal{R} + a\{u(\log x_1 + c_{d,1}) + v(\log x - \log x_1)/\varphi(q_0)\},$$

where $c_{d,1}$ is the subtraction constant of the modulus-one remainder. In the correction the constant of the modulus-q_0 remainder cancels. Since

$$\sum_{\tau} \log \kappa_\tau = f\log(a/x_1) - v\log(x/x_1),$$

the term in braces times a is the common expected value $ua(\log a - f^{-1}\sum_\tau \log\kappa_\tau + c_{d,1})$. This proves the identity relating the chain's total remainder to the two progression sums.

For the main amount let $k = r + 1 - m$, and set

$$\Omega = \frac{2\pi}{\mathrm{rd}(E)}\exp\!\left(\gamma_{\mathrm{Euler}} - \frac{L'}{L}(1,\chi_K)\right), \qquad A = mk\sum_\tau \big(\log(\kappa_\tau!) + \kappa_\tau\log(M/\Omega) - S(\kappa_\tau,\tau)\big).$$

It is nonnegative for large x. Indeed $S(\kappa_\tau,\tau) \le 2\kappa_\tau\sum_{n\le a}\Lambda(n)/n \le 5\kappa_\tau B$ eventually, by partial summation from the prime number theorem and $\log a \le 2B$. The fixed d and $\mathrm{rd}(E) \ge 1$ give $\log(M/\Omega) \ge 19B$ eventually, since $M \ge b^{20}$. Every summand is therefore nonnegative.

The spaces below are the ramified principal-level source S and the projected principal-level target V of the construction. Write $D = \dim_{\mathbb{C}} V > 0$, as shown in Section 16.2. A finite group G acts on V, and the comparison uses translates P_g of one normalized Poincaré map $P : S \to V$. The family of lifts Θ_ξ and the integral lattices at each coefficient place are on this same V. The trivial twist has symbol 1.

For a tuple y in V, write

$$\mathrm{Cov}(y) \le c \quad \Longleftrightarrow \quad \det \mathrm{Gram}(\sigma\Theta_1 y) \le e^c \quad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Here σ acts on the coordinates, and the Gram matrix uses the rational class pairing.

Write $\mathrm{Ind}(y) \le \delta$ if there are $a_\ell \ge 0$, independent of the choice of embedding over ℓ, such that

$$\sum_{\ell \in U} a_\ell \log \ell \le \delta \quad \text{for every finite set of primes } U.$$

For an ordered tuple $y = (y_1, \dots, y_n)$, at each $\kappa \mid \ell$ there must exist a matrix $U_\kappa \in M_n(\mathcal{O}_\kappa)$ such that

$$v_\kappa(\det U_\kappa) \le a_\ell, \qquad (\Theta_\xi y)U_\kappa \text{ has integral columns at } \kappa \quad (\ell \nmid q_\xi).$$

Here $v_\kappa(\ell) = 1$ and $\mathcal{O}_\kappa = \{z \in \mathbb{C} : v_\kappa(z) \ge 0\}$. The columns of $\Theta_\xi y$ are the lifted vectors in order. The same U_κ works for every allowed ξ with $\ell \nmid q_\xi$.

Proposition 5.1 (Short images and span for the chosen arithmetic data)**.** *Fix q_0, d, ϵ as in Section 4. For every sufficiently large integer x, there exists a set $\mathcal{S} \subset S$ of integral forms with*

$$\mathrm{span}\, \mathcal{S} = S, \qquad \|\sigma\Theta_1 P_g \phi\|^2 \le e^{A - m\mathcal{R} + L} \quad (\phi \in \mathcal{S},\ g \in G,\ \sigma \in \mathrm{Aut}(\mathbb{C})).$$

For each $t \subseteq G$ and every independent tuple y of images of everywhere integral forms under P_g, $g \in t$,

$$\mathrm{Ind}(y) \le 2|t| \dim(S)\, faB^3.$$

Finally, one can choose $t \subseteq G$ such that

$$|t| \dim S \le 5D \log(1 + 2b^2), \qquad \dim \sum_{g \in t} P_g(S) \ge (1 - 4/b^2)D.$$

Proof. Proposition 16.2 gives the translated-span dimension and the bound on $|t| \dim S$. Proposition 22.2 gives the short generators and the index on the same datum. By Proposition 7.10, the common scalar $c_B \in \mathbb{Q}_{>0}$ gives $P_g = c_B U_g$, so $P_g(S) = U_g(S)$ for every $g \in G$. □

Proposition 5.2 (Determinant estimate for the chosen arithmetic data)**.** *Fix q_0, d, ϵ as in Section 4. At every sufficiently large integer x the comparison lifts are injective. For every independent tuple $y \in V^n$,*

$$\left.\begin{array}{l} n \ge (1 - 4/b^2)D, \qquad \mathrm{Cov}(y) \le c, \qquad \mathrm{Ind}(y) \le \delta \end{array}\right\} \quad \Longrightarrow \quad D(A - 3L) \le c + 2\delta.$$

The proof is in Section 45.

Proposition 5.3 (Cancellation of the main covolume term)**.** *Fix q_0, d, ϵ as in Section 4. At every sufficiently large integer x, the datum satisfies $m\mathcal{R} \le faB^5$.*

Proof. Take $t, \mathcal{S}$ from Proposition 5.1. Since $\mathcal{S}$ spans S, the union of the $P_g(\mathcal{S})$, $g \in t$, spans $\sum_{g\in t} P_g(S)$. Choose a basis $y_i = P_{g_i}\phi_i$ from this union and fix each pair (g_i, ϕ_i) before the coefficient place or twist. Its cardinality satisfies $(1 - 4/b^2)D \le n \le D$. The Gram bound of Section 22.3 applies to these generators at every complex embedding. Their integral preimages give the index bound on this same basis, so

$$\mathrm{Cov}(y) \le n(A - m\mathcal{R} + L), \qquad \mathrm{Ind}(y) \le 2|t| \dim(S)\, faB^3 \le 10D \log(1 + 2b^2) faB^3 = DI_0.$$

Proposition 5.2 and $n \le D$ give

$$nm\mathcal{R} \le (n - D)(A + L) + D(4L + 2I_0) \le D(4L + 2I_0).$$

For $b \ge 3$ and $B \ge 128$,

$$n \ge (1 - 4/b^2)D \ge D/2 > 0, \qquad I_0 = 10faB^3 \log(1 + 2b^2) \le 30L.$$

Hence $m\mathcal{R} \le 8L + 4I_0 \le 128L \le faB^5$. □

6 The ramified source at the prescribed level

In this section, the additional level condition singles out a subspace of the full definite automorphic source. We identify its integral coordinates before using it in the pointwise estimate of Section 7.4 and the dimension argument of Section 16.2.

In the definite source of rank k, write K_M for the integral principal level at the primes dividing $\bar{\alpha}$. Write J_H for its subgroup on which the last covector of a matrix reduces to the last covector of the identity at the primes dividing the prescribed generator $\pi_N = a_0 + ib_0$. The source space X consists of the forms at J_H with the fixed finite-dimensional coefficient module. These level descriptions specify the source used in the comparison.

The primes p_0, p_1, N are 1 mod q_2^2, and q_1 and d divide q_2^2, by the definitions of s, L_0, q_1. Thus the three primes split completely in E. Since $(\bar{\alpha}) = \bar{\mathfrak{p}}_0\mathfrak{p}_1$ in K, the old principal level uses the places of E above $\bar{\mathfrak{p}}_0$ at p_0 and above $\mathfrak{p}_1$ at p_1. Each set consists of f places of norm p_i, one per conjugate pair. At N, the divisor of π_N chooses one of the two primes of $\mathbb{Q}(i)$ above N. It gives f distinct primes of E, one from each complex conjugate pair, all of norm N.

Lemma 6.1 (Trivial stabilizers at the prescribed level)**.** *Fix the datum of Section 4 and let $p \in \{p_0, p_1\}$. On any of the definite unitary groups of the datum, suppose the level at one of the chosen split places above p is contained in the principal congruence subgroup. Then every finite class has trivial rational stabilizer.*

Proof. A stabilizer is the intersection of the rational points with a compact subgroup of the adeles. The archimedean group is compact by definiteness, and the finite compact is the conjugate of the specified level. The rational points are discrete, so the stabilizer is finite. The chosen split place has degree one over $\mathbb{Q}_p$. The stabilizer embeds, after conjugation, in $1 + pM_n(\mathbb{Z}_p)$, where n is the rank of the group. For $B \in M_n(\mathbb{Z}_p)$ with a unit entry and $a \ge 1$, the minimum entry valuation of $(I + p^a B)^h - I$ is $a + v_p(h)$ for every positive h. The binomial expansion gives this for $p \nmid h$, and raising to the p-th power increases the valuation by one since p is odd. Thus this principal congruence group is torsion-free and the rational stabilizer is trivial. □

Proposition 6.2 (Dimension of the ramified source)**.** *For the ramified subspace $S \subseteq X$ for the chosen arithmetic data,*

$$\dim(X/S) \le \frac{f}{N-1}\dim X \le b^{-8}\dim X, \qquad \dim X \le 2\dim S.$$

Proof. For a prime $w \mid \pi_N$, take a local unit at w lifting a generator of the residue units, form the idele a_w with all other components one, and put $z_w = a_w/\bar{a}_w$. Its scalar matrix lies in K_M, since it is supported at $w, \bar{w}$ and $N \nmid M$. The local condition on the last covector at w says $z_w^j \in J_H$ exactly when $N-1$ divides j; all other conditions on the last covectors hold because $\bar{w}$ does not divide π_N.

Let $J^+ = \langle J_H, z_w : w \mid \pi_N \rangle$. Centrality and the independent residue conditions at these f places give

$$C_N = J^+/J_H \simeq \prod_{w \mid \pi_N} \mathbb{F}_N^\times.$$

The level J^+ still has principal congruence at the old split places over p_0, p_1. By Lemma 6.1 the stabilizer of each J^+-class is trivial. Choose a representative g of each such class. Its J_H-classes are the $[gz]$, $z \in C_N$, with independent values in the coefficient module. Thus each fiber is a copy of the regular representation of C_N with coefficients in this module. Each character of C_N occurs in X with multiplicity $\dim X/(N-1)^f$.

For each $w \mid \pi_N$ the operator

$$A_w = \frac{1}{N-1}\sum_{j=0}^{N-2} R_H(z_w)^j$$

retains the characters trivial on the w-factor. The common kernel S consists of characters nontrivial on every factor, giving

$$\dim S = \left(1 - \frac{1}{N-1}\right)^f \dim X, \qquad \dim(X/S) \le \frac{f}{N-1}\dim X \le b^{-8}\dim X.$$

The integral kernel at a coefficient place is formed using the sums $(N-1)A_w$ on the original lattice, as in Section 20.3.3. Indeed $f \le q_1 \le b$ for large x, and the interval of primes for N and the definition of its center give $\log N \ge (19(m-1) - 15/2)\log b$, so $N - 1 \ge b^9$. In particular $\dim X \le 2\dim S$. □

7 Normalization and bounds for the theta images

In this section, we express the local divisibility estimates as bounds on the change of integral lattice. Summing their logarithms recovers the prime sum in Section 5, which cancels the corresponding part of the archimedean bound in Section 7.4.

The short-image estimate of Proposition 5.1 is reduced to archimedean estimates for the unscaled images and local lattice norm estimates with prime-dependent savings. For a prime p put

$$S_p(Y,\tau) = \frac{1}{2}\sum_{e\ge 1}(1-\chi_K(p^e))(Y/p^e - \kappa_{\langle -p^e\rangle\tau})_+, \qquad g_p = m\left(\sum_{\tau\in\Phi_a} S_p(a,\tau) + k\sum_{\tau} S_p(\kappa_\tau,\tau)\right).$$

For $Y = a$ or $Y = \kappa_\tau$, a nonzero positive part requires

$$p^e < Y/\kappa_{\langle -p^e\rangle\tau} \le Y x_1/a \le x_1 \le a.$$

The factor $\frac{1}{2}(1-\chi_K(p^e))$ selects the odd exponents at an inert prime, is $\frac{1}{2}$ at the ramified prime d, and vanishes at a split prime. For the odd inert exponents the multiplier is $(-p)^e \bmod q_1$. At d it is $-d^e \bmod q_1$ for every $e \ge 1$. Write $E_0 = faB^3$ and let F consist of the nonsplit primes below x_1, together with N. Grouping the von Mangoldt sum by prime powers gives

$$2\sum_{p\in F} g_p \log p = m \sum_{\tau\in\Phi_a} S(a,\tau) + mk \sum_{\tau} S(\kappa_\tau,\tau).$$

In particular $A - m\mathcal{R} + 2\sum_{p\in F} g_p \log p$ equals the sum of the archimedean jet terms in A plus m times the expected value of $\sum_{\tau\in\Phi_a} S(a,\tau)$ from Section 5.

7.1 Bounds for the local denominators

In this subsection, we bound the total logarithmic contribution of the local estimates in Section 7. The exceptional congruence classes and the prime N are treated separately because their local estimates differ.

For these data, define the local error bound by

$$e_p = \begin{cases} \dfrac{2faB^2}{p}\, c_p, & p < x_1 \text{ nonsplit in } K,\\ 2mfb, & p = N,\\ 0, & \text{otherwise,}\end{cases} \qquad c_p = \begin{cases} m, & p \equiv \pm 1 \bmod m,\\ 1, & \text{otherwise.}\end{cases}$$

Lemma 7.1 (Total local denominator bound)**.** *For fixed q_0, d, ϵ and all sufficiently large x,*

$$\sum_p e_p \log p \le E_0 = faB^3.$$

Proof. Put $\ell_1 = \log x_1$. Since $\log N \ge mB > W$ for large x, the two nonzero cases defining e_p are disjoint. For $p = mc \pm 1$ with $c \ge 1$, one has $m/p \le 2/c$. Consequently

$$\sum_p e_p \log p \le 2faB^2\,\ell_1 \left(\sum_{j\le x_1} \frac{1}{j} + 2\sum_{\pm}\sum_{c\le x_1} \frac{1}{c}\right) + 2mfb\log N \le 10faB^2\,\ell_1(1+\ell_1) + 2mfb\log N.$$

Since $\ell_1 \le W$, $B > W^6$, $m^2 < 4x$, $a = xb$ and $\log N \le 25mB$, division by E_0 gives

$$\frac{1}{E_0}\sum_p e_p \log p \le \frac{10W(1+W)}{B} + \frac{50m^2}{xB^2} < \frac{10(1+W)}{W^5} + \frac{200}{W^{12}} < 1 \qquad (W \ge 3). \qquad \square$$

7.2 Counting terminal double cosets

In this subsection, we bound the number of terminal double cosets by counting definite Hermitian lattices and refining their classes at the specified level. This controls the class sum in Section 7.4.

Let n_{t} be the rank of a definite stage under consideration, and let H be the unitary group of the standard positive Hermitian space $E^{n_{\mathrm{t}}}$. Write $K_{\mathrm{int}} \subset H(\mathbb{A}_f)$ for the stabilizer of the standard integral lattice at all finite places.

Lemma 7.2 (Terminal class count)**.** *For the trivial twist, let h_{term} be the number of double classes at the prescribed level on any X_j, $0 \le j \le m$, or on the definite endpoint Y_m. For fixed q_0, d, ϵ and sufficiently large x, uniformly in the stage,*

$$\log h_{\mathrm{term}} = O(x^{52}), \qquad \log h_{\mathrm{term}} + 4\log|G| \le x^{99}.$$

Proof. We first count classes at K_{int}. Write $d_E = [E:\mathbb{Q}]$, $\Delta_E = |\operatorname{disc}(E)|$, and $u = d_E n_{\mathrm{t}}$. Each class gives an integral lattice Λ obtained by intersecting a finite adelic translate of the standard integral lattice with $E^{n_{\mathrm{t}}}$. It has integral Hermitian pairings and is locally isometric to the standard lattice. An isometry between such lattices over E therefore identifies their double classes.

Use the Euclidean norm squared given by $\operatorname{Tr}_{E/\mathbb{Q}} h(v,v)$, where h is the Hermitian form. The determinant of the integral trace pairing on Λ is $\Delta_E^{n_{\mathrm{t}}}$: on the standard lattice use an integral basis of E in each coordinate, with conjugation changing the trace determinant only by a sign. Local isometry gives the same determinant valuation at every rational prime, and positivity fixes the determinant.

Thus the covolume is $\Delta_E^{n_{\mathrm{t}}/2}$. Every nonzero lattice vector has length at least one, by positivity and integrality of its squared length. Minkowski's second theorem [21, Ch. VIII, §4.3, Theorem V] and the cube of side $2/\sqrt{u}$ in the Euclidean unit ball give, for the last successive minimum,

$$\lambda_u \le u^{u/2} \Delta_E^{n_{\mathrm{t}}/2}.$$

Choose u independent lattice vectors of length at most $2\lambda_u$. They span $E^{n_{\mathrm{t}}}$ over E, so n_{t} of them form an E-basis $v_1, \dots, v_{n_{\mathrm{t}}}$.

Let $T = (h(v_i, v_j))$, with the convention that the second variable is linear. Its entries are algebraic integers, and their absolute values at all embeddings are at most $H_0 = 4u^u \Delta_E^{n_{\mathrm{t}}}$, by Cauchy–Schwarz at each embedding. To count these entries, note that for distinct algebraic integers in E, their difference has at least one embedding of absolute value at least one, by taking its norm. The vectors of their real and imaginary parts at one place from each complex conjugate pair therefore have sup-norm separation at least $1/\sqrt{2}$. Subdivision of the box of side $2H_0$ into cubes of side $1/2$ gives at most $(4H_0+2)^{d_E}$ choices per entry.

For each fixed T identify the space over E via the chosen basis. Let L be its $\mathcal{O}_E$-span. We have $L \subseteq \Lambda \subseteq L^{\#}$, where $L^{\#}$ is the Hermitian dual of L, since Λ has integral pairings. The quotient $L^{\#}/L$ is isomorphic as an abelian group to $\mathcal{O}_E^{n_{\mathrm{t}}}/T\mathcal{O}_E^{n_{\mathrm{t}}}$, by writing the pairings in this basis. It has order $\mathcal{I} = |\operatorname{N}_{E/\mathbb{Q}} \det T|$, at most $(n_{\mathrm{t}}! H_0^{n_{\mathrm{t}}})^{d_E}$.

Every subgroup of this quotient has at most u generators, since its inverse image in $L^{\#}$ is a sublattice of rank u. There are at most $\mathcal{I}^u$ choices for the intermediate lattice, using ordered tuples with zero entries allowed. These counts bound the number of classes because two lattices with the same data are carried to one another by an E-linear isometry sending the chosen bases to one another. We obtain

$$\log|H_{\mathrm{rat}} \backslash H(\mathbb{A}_f) / K_{\mathrm{int}}| = O\big(u^2(1 + \log H_0 + \log n_{\mathrm{t}})\big).$$

Here the left quotient is by the E-linear isometries of the Hermitian space.

If the stage level contains the principal congruence subgroup of K_{int} modulo MN, its number of classes is at most the preceding class count times $(MN)^{d_E n_{\mathrm{t}}^2}$. Indeed one may refine each class by right cosets of the principal congruence subgroup in K_{int}, whose index is at most the number of matrices modulo MN.

Write $K(I)$ for matrices in the integral stabilizer congruent to the identity modulo I, in the rank under consideration. On each upper space X_j, the old condition is reduction to the identity at the chosen places dividing π_M. At the places dividing π_N the last row must reduce to the last row of the identity. Thus the upper level contains $K(\pi_M\pi_N)$: this subgroup fixes every row modulo both generators. After restriction to Y_m the untwisted terminal level imposes the old residue conditions, so it contains $K(\pi_M)$. Since $MN = (\pi_M\pi_N)\overline{\pi_M\pi_N}$, both levels contain $K(MN)$. The rank is at most $k + m(5r + 1) = O(x^{11/2})$, and $d_E = 2f = O(x^{11}\log x)$. In particular $u = O(x^{17})$. For $C = \mathbb{Q}(\zeta_{q_1})$, linear disjointness of K, C factors the trace matrix of $\mathcal{O}_K[\zeta_{q_1}]$ in a product basis. Thus

$$|\operatorname{disc}(\mathcal{O}_K[\zeta_{q_1}])| = d^{[C:\mathbb{Q}]}|\operatorname{disc}\Phi_{q_1}|^2.$$

Since $\Phi'_{q_1}(\zeta_{q_1})$ divides q_1, $\operatorname{rd}(E) \le \sqrt{d}\,q_1 \le dq_1$. Hence $\log H_0 = O(u\log x)$. Also $\log(MN) = O(mB)$ and $B = O(x^{\epsilon})$. The logarithm of the terminal class count is consequently $O(x^{52})$. There are f chosen places above each of p_0, p_1. The residue group G acting on the target is the product of the corresponding groups GL_r, so $\log|G| \le fr^2\log M$. It follows that $\log h_{\mathrm{term}} + 4\log|G| \le x^{99}$ for all sufficiently large x. □

7.3 Projection onto the prescribed finite types

In this subsection, the central idempotent acts on every lift by the intertwining relation of Section 17.3. Its integral multiple controls the valuations and archimedean norms of all translates, using the class count of Section 7.2.

Let e_0 be the central idempotent of the projected part in the complex group algebra of the finite group G. Let z be an automorphic form on $U(Y_0)$ at the prescribed principal level. Write $\mathcal{L}_\xi z$ for the terminal algebraic coefficient of $L_{\xi,m-1}\cdots L_{\xi,0}z$, where each theta integral includes its source index $A_{\xi,j}$ as in Section 17.3.

Lemma 7.3 (The integral finite-type projection)**.** *The coefficients of $|G|e_0$ are algebraic integers, of absolute value at most $|G|$ at every complex embedding. For any $g \in G$, replacing $\mathcal{L}_\xi z$ by $\mathcal{L}_\xi(|G|e_0gz)$ preserves the local lattice norm bound $p^{e_p-g_p}$. If the untwisted conjugate squared pointwise norms of $\mathcal{L}_1 z$ are at most e^{K_0}, then the squared terminal class norm after projection is at most $e^{K_0+x^{99}}$.*

Proof. The expansion of e_0 as a sum of primitive central idempotents gives

$$|G|e_0(t) = \sum_{\pi\ \text{selected}} \dim(\pi)\chi_\pi(t^{-1}) \qquad (t \in G).$$

Each coefficient on the right is an algebraic integer because the character values are sums of roots of unity. At every complex embedding its absolute value is at most $\sum_{\pi\ \text{selected}}(\dim\pi)^2 \le |G|$.

Equivariance on the original chain. For $s = (s_w) \in G$, choose at each stage an integral finite lift $k_j(s)$. Its residue at an old place w is $\operatorname{diag}(s_w, I)$ in the original order of the blocks, and its residue at every twist place is the identity. At the source $k_0(s)$ acts by the given residue action, since two lifts differ by a principal-level element. Reduction is onto at these split places, so these choices are possible at every stage. At places outside the prescribed set their components can be the identity.

Let i_j denote the inclusion of the j-th Hermitian space as the first block of its successor. The original forward-arrow finite test at the chosen and twist places is the indicator of i_j. The

simultaneous source and target reductions a, b of $k_j(s), k_{j+1}(s)$ satisfy $bi_j = i_j a$. Thus the finite split Hom action $X \mapsto b^{-1}Xa$ fixes the prescribed support. The determinants are units, and the two rank-twisted splitting characters leave the factor $\psi_{j,w}(\det b)/\psi_{j,w}(\det a)$ on the indicator, where $\psi_{j,w}$ is the tame part of the arrow character. At an old place the two determinants reduce to $\det s_w$, and at a twist place both reduce to one. Their quotient is therefore a principal unit, on which $\psi_{j,w}$ is trivial. On the other finite components the vacuum is fixed, and these translations have no real component. The integrated arrow datum is therefore fixed.

Change the source variable by $k_j(s)$ in the arrow integral, using the original quotient Haar. It follows that translating the image under the lift by $k_{j+1}(s)$ gives the arrow applied to the source translate by $k_j(s)$. Iterate to the terminal space. The algebraic coefficient realization commutes with finite right translations, so the final formula is

$$\mathcal{L}_\xi(sz)(h) = (\mathcal{L}_\xi z)(h\, k_m(s)).$$

With $k_\xi(s) = k_m(s)$, expanding e_0 gives

$$(\mathcal{L}_\xi(|G|e_0 gz))(h) = \sum_{t\in G} |G|e_0(t)(\mathcal{L}_\xi z)(h\, k_\xi(tg)).$$

Here $k_\xi(tg)$ is the integral first-block lift of tg and h is a finite adelic point in the terminal space. In the transported Weyl coordinates a right translate by any such integral matrix preserves the lattice norm: the change of coordinates and its inverse both preserve the unit lattice. The ultrametric inequality therefore transfers any bound $p^{e_p - g_p}$ for $\mathcal{L}_\xi z$ to $\mathcal{L}_\xi(|G|e_0 gz)$ with the same exponent. The transfer uses the prescribed chain and holds for all allowed twists and every translate. If the local estimate is first given on a submodule of the full integral source, intersect it with the ramified integral source. A matrix clearing a tuple from the ramified source into the first submodule still clears it into this intersection since linear combinations preserve ramification.

The coefficient bound and the triangle inequality give

$$\|\sigma(\mathcal{L}_1(|G|e_0 gz)(h))\|_{\mathrm{W}}^2 \le |G|^4 e^{K_0}.$$

Summing with reciprocal stabilizer weights and applying Lemma 7.2 gives

$$h_{\mathrm{term}}|G|^4 e^{K_0} \le e^{K_0 + x^{99}}. \qquad \square$$

7.4 Pointwise estimates in the composition of lifts

In this subsection, we estimate the upper lift by exploiting the additional level condition of Section 6. The resulting pointwise bound is combined with the class count of Section 7.2 in the proof of Proposition 5.1.

7.4.1 The sum of the part sizes

In this subsection, the level prime from Section 4 supplies the term subtracted in the terminal estimate of Section 7.4.5. We use the function J below to sum the factorial contributions from the upper partitions.

Write $J(l) = \log(l!) + l\log(M/\Omega)$, put $\overline{w} = f^{-1}\sum_{\tau}\log\kappa_{\tau}$, and write $s_{\mathrm{av}} = a(\log a - \overline{w} + c_{d,1})$. Set $G_0 = \log(M\,\mathrm{rd}(E)/(2\pi))$. By the definition at modulus one, $c_{d,1} = L'/L(1,\chi_K) - \gamma_{\mathrm{Euler}} - 1$, so $\log(M/\Omega) = G_0 + 1 + c_{d,1}$. The comparison of $\sum_{j=1}^{a}\log j$ with $\int_1^a \log t\,dt + \log a$ gives

$$J(a) - s_{\mathrm{av}} \le a(\overline{w} + G_0) + 1 + \log a.$$

The chosen prime satisfies $\log N \ge m\overline{w} + (m-1)G_0 - \frac{15}{2}B$. Consequently

$$m(J(a) - s_{\mathrm{av}}) - a\log N \le a(G_0 + \tfrac{15}{2}B) + m(1+\log a) \le 32aB$$

eventually, since $G_0 \le 23B$ and $m(1+\log a) \le aB$. Define the size of the principal parts by

$$S_{\mathrm{main}} = m\left(|\Phi_a|J(a) + k\sum_{\tau}J(\kappa_{\tau})\right).$$

For the step from X_j to X_{j+1}, put $n_j = k + j(5r+1)$ and let $m_{\tau,h}$ be the column counts of the lift weight in Section 20.2. The first n_j row lengths of the target partition are

$$\lambda_{j,\tau,i} = \sum_{i<h<n_{j+1}} m_{\tau,h} \qquad (0 \le i < n_j).$$

In particular each column of height n_j contributes the top exterior power on the source and occurs in every one of these rows. Set

$$S_j = \sum_{\tau\in\Phi}\sum_{0\le i<n_j} J(\lambda_{j,\tau,i}) \qquad (0 \le j < m), \qquad h_0 = r^8.$$

Lemma 7.4 (Factorial bounds for the upper composition)**.** *For fixed q_0, d, ϵ and sufficiently large x,*

$$S_{\mathrm{main}} - |\Phi_a|a\log N \le A - m\mathcal{R} + 2\sum_{p} g_p\log p + 32faB,$$

$$\sum_{j} S_j \le S_{\mathrm{main}} + \tfrac{1}{2}E_0.$$

The term $-|\Phi_a|a\log N$ comes from $\sum_{\tau}a_{\tau} = |\Phi_a|a$ in the terminal branching exponents $a_{\tau} - mh_0$ of Lemma 7.9.

Proof. Insert the prime-power identity of Section 7 and the expression for $\mathcal{R}$ in Section 5. The sums $S(\kappa_{\tau},\tau)$ and $S(a,\tau)$ cancel, giving

$$A - m\mathcal{R} + 2\sum_{p\in F} g_p\log p = mk\sum_{\tau}J(\kappa_{\tau}) + m|\Phi_a|s_{\mathrm{av}}.$$

Subtracting this expression from $S_{\mathrm{main}} - |\Phi_a|a\log N$ gives exactly

$$|\Phi_a|\big(m(J(a) - s_{\mathrm{av}}) - a\log N\big) \le 32faB,$$

since $|\Phi_a| \le f$.

In zero-based row indices, the column table of Section 20.2 gives the actual row length of the step from X_j to X_{j+1}:

$$\lambda_{j,\tau,i} = a_{\tau}\mathbf{1}_{i=0} + (w_1(\tau) - \lfloor n_1/2\rfloor)\mathbf{1}_{i<k} + (\Delta - t - 1)\sum_{u=1}^{j}\mathbf{1}_{i<n_u} \qquad (0 \le i < n_j).$$

The sum is empty at $j = 0$ and counts the height n_j at $j \geq 1$. A column of height h contributes one to each row $i < h$. Thus the row bounds have the form

$$\lambda_{j,\tau,i} \leq \mu_{\tau,i} + s_i, \qquad \mu_{\tau,i} = \begin{cases} a\mathbf{1}_{\Phi_a}(\tau), & i = 0, \\ \kappa_\tau, & 1 \leq i < k, \\ 0, & i \geq k, \end{cases} \qquad s_i = h_0^2 + (b + a/x_1)\mathbf{1}_{\{0\}}(i).$$

For large x, $\mu_{\tau,i} + s_i \leq a + h_0^2 + b + a/x_1 \leq 2a$. Since $M/\Omega > 1$, the increment $J(l+1) - J(l) = \log(l+1) + \log(M/\Omega)$ is positive. For $l + s \leq 2a$, the choice of scales gives

$$J(l+s) - J(l) \leq s\big(\log(2a+1) + \log(M/\Omega)\big) \leq 25sB.$$

Indeed $2a + 1 \leq b^2$, $\log M \leq 22B$, and $\log \mathrm{rd}(E) = O(\log x) = o(B)$.

At each step the principal lengths give

$$\sum_{i=0}^{n_j - 1} J(\mu_{\tau,i}) = \mathbf{1}_{\Phi_a}(\tau) J(a) + (k-1) J(\kappa_\tau).$$

Since $J(\kappa_\tau) \geq 0$, summing the increments over all rows, embeddings and steps gives

$$\sum_j S_j \leq S_{\mathrm{main}} + 25mfB(b + a/x_1 + n_m h_0^2) \leq S_{\mathrm{main}} + \tfrac{1}{2}E_0.$$

Indeed $x/x_1 \to 0$, $n_m h_0^2$ is polynomially bounded in x, and b exceeds every fixed power of x. □

7.4.2 Integral generators of the ramified source

In this subsection, we construct generators in the integral model of Section 6. Their pointwise norms supply the initial bound for the composition in Section 7.4.5.

Lemma 7.5 (Integral generators with bounded Weyl norm)**.** *Fix q_0, d, ϵ. For every sufficiently large x there is a set $\mathcal{S} \subseteq S$ of everywhere integral forms such that*

$$\mathrm{span}(\mathcal{S}) = S, \qquad \sup_{\sigma,h} \|\sigma(\phi(h))\|_{\mathrm{W}}^2 \leq e^{E_0} \quad (\phi \in \mathcal{S}).$$

Here $h \in U(X_0)(\mathbb{A}_{F_0,f})$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Proof. Put $n_0 = k$, $D_0 = q_1 d$, where $q_1 d$ is in the different of $E/\mathbb{Q}$, and

$$C_0 = 6D_0(fn_0)^2, \qquad M_0 = n_0 f + \#\{\text{weight boxes in columns of height } < n_0\}.$$

We use D_0 here for a positive rational integer. Throughout this section N still denotes the ramified level prime. At every original source class g, use the lattice

$$L(g) = E^{n_0} \cap g\widehat{\mathcal{O}}_E^{n_0}.$$

The finite unitary action and the standard definite Hermitian form make $L(g)$ Hermitian self-dual. Here and below duality means integrality of the pairing over $\mathcal{O}_E$. The diagonal

archimedean embedding has Euclidean norm squared $\sum_\tau \|v_\tau\|^2$. For $0 \neq v \in L(g)$, its Hermitian square $h(v,v)$ is a totally positive integer of $F_0 = E^+$. The arithmetic–geometric mean inequality applied to its conjugates gives $\|v\|^2 \geq f \geq 1$.

The real inner product in these coordinates is half the trace pairing. The Euclidean dual of $2D_0^{-1}L(g)$ is consequently the archimedean image of $D_0\mathfrak{d}_E^{-1}L(g)$, which is contained in $L(g)$. For the identification, a real functional integral on a lattice is rational on its rational span. By nondegeneracy of the rational trace pairing it therefore has a rational trace representative. Pairing with all the $\mathcal{O}_E$-multiples of every lattice vector puts the representative in the codifferent times the Hermitian dual, proving the asserted equality.

For completeness a Euclidean lattice of dimension h with shortest dual vector of length at least 1 has covering radius at most h^2. Minkowski's convex body theorem [21, Ch. III, §2.2, Theorem II, p. 71] applied to cubes inscribed in Euclidean balls gives $\lambda_1(L)\lambda_1(L^*) \leq h$, since the two covolumes multiply to 1. Choose a primitive vector of length at most $2h$ and project orthogonally to its complement. The projected lattice is discrete since the vector extends to a lattice basis.

Its dual is L^* intersected with the complement. Induction gives error at most $(h-1)^2$ in the projected space, and rounding in the chosen direction contributes at most h. This proves the radius bound (with the zero-dimensional case immediate). The vectors of length at most three times the covering radius span the whole real space: if their span has a unit normal u, a lattice vector within radius r of $2ru$ has norm at most $3r$ and has nonzero projection on u.

Apply this argument to $2D_0^{-1}L(g)$ with $h = 2fn_0$, then multiply by $D_0/2$. The resulting vectors in $L(g)$ of norm at most C_0 span the real space in the diagonal embedding. They also span E^{n_0} over E: an E-linear functional vanishing on them extends at each complex place and hence is zero. Take an E-basis from these vectors. With its columns written $Y(g)$,

$$Y(g) \in \mathrm{GL}_{n_0}(E), \qquad g^{-1}Y(g) \in M_{n_0}(\widehat{\mathcal{O}}_E), \qquad |\tau(Y(g)_{ij})| \leq C_0.$$

Fix a finite adelic representative g of the class. For each determinant exponent $e_\tau \in \mathbb{Z}$, apply the construction to the group element g^{e_τ}. Thus $\det(g^{e_\tau}) = \det(g)^{e_\tau}$, also for negative exponents. Here $D_0 \in \mathfrak{d}_E$ because the different of $E/\mathbb{Q}$ divides the product of the extended differents of its two constituent fields K and $\mathbb{Q}(\zeta_{q_1})$, which in turn divides $dq_1\mathcal{O}_E$.

Write e_τ for the determinant exponent and $v(y)$ for the column-polynomial Weyl evaluation vector, in the original integral model. Use the fixed coefficient embedding $\iota_\tau = \tau \circ \mathrm{conj}$. For each class g use the value

$$v_g(y) = \left(\prod_\tau \det \iota_\tau(Y(g^{e_\tau}))\right) \rho_{\mathrm{pol}}\big((\iota_\tau(Y(g)))_\tau\big)v(y)$$

for the delta form at g. Put $B_{\mathrm{pol}} = M_0 - n_0 f$, the number of polynomial boxes. Take $y = (y_\tau)_{\tau\in\Phi}$ with the entries of all the matrices chosen independently in $\mu_{B_{\mathrm{pol}}+1}$. This finite grid is fixed before choosing a coefficient embedding. The level is free, so these delta forms are well defined and there is no stabilizer division. In the transported lattice norm at g, the polynomial matrix is evaluated on $g^{-1}Y(g)$; the determinant factors are the determinants of $(g^{e_\tau})^{-1}Y(g^{e_\tau})$. Both are integral. Polynomial Weyl matrices and evaluation vectors have integer coefficient formulas in the chosen model, and roots of unity are units at every coefficient embedding. Thus the delta form has lattice norm at most one at every place, including on its translated class points by equivariance.

For the archimedean bound the evaluation vector is the Schur action of the first square block of y on the unit vector of the reference column array. After applying $Y(g)$, the resulting matrix at each embedding has entries at most $n_0 C_0$ in modulus and squared Frobenius norm at most $n_0^4 C_0^2 \le C_0^6$. The induced action on each exterior power is bounded by the operator norm to its degree, and hence the squared Schur norm is bounded by C_0^6 to the number of polynomial weight boxes. The squared determinant factors together contribute at most $C_0^{6n_0 f}$ by Hadamard. These bounds hold after every complex automorphism, since the entries of Y are bounded under all embeddings and roots of unity still have modulus one. The squared Weyl norm of the value is at most $C_0^{6M_0}$. For a point $h = \gamma_f g k'$ in the supporting class, with $\gamma \in U(X_0)(F_0)$ and $k' \in J_H$, the value at h is obtained from the value at g by its rational weight action. For every embedding $\jmath : E \hookrightarrow \mathbb{C}$, $\jmath(\gamma)^* \jmath(\gamma) = 1$, since X_0 has the standard definite form and $\jmath(\bar{x}) = \overline{\jmath(x)}$. Its exterior-power actions preserve the tensor-induced Weyl norm and its determinant factors have modulus one. Take $\jmath = \sigma \circ \iota_\tau$ to get the bound at every h after σ. Off that class the value is zero.

At the source the polynomial Weyl module is $\bigotimes_{\tau \in \Phi} \mathrm{Sym}^{a_\tau}(\mathbb{C}^{n_0})$. Its evaluation coordinates are products over τ of degree-a_τ monomials in the first column of y_τ. The entries in these columns range independently over the chosen roots of unity. A functional vanishing on all the evaluation vectors gives a polynomial of degree at most B_{pol} in each entry of the array. In each variable it vanishes at the $B_{\mathrm{pol}} + 1$ chosen roots for every fixed grid point in the remaining variables. Successive one-variable interpolation therefore gives the zero polynomial. The Weyl evaluation polynomials form the dual Weyl space in the chosen model, so the functional is zero. The matrix multiplying $v(y)$ in $v_g(y)$ is invertible. Summing these component spans proves the claim at the original source level.

On the source the determinant exponent is zero and the only columns of height below n_0 are the a_τ columns of height one (Section 20.2), so $M_0 \le f(n_0 + a)$, and the scale inequalities give

$$4^f C_0^{6f(n_0+a)} \le \exp(E_0), \qquad E_0 = faB^3.$$

Finally apply the commuting differences $1 - R_H(z_w)$ for $w \mid \pi_N$ as in Section 6. Each preserves the integral lattice and multiplies the maximum squared norm of the conjugate values by at most 4. Their product has the ramified subspace as its range: the commuting finite cyclic scalars are simultaneously diagonalizable and $1 - T$ is invertible on the nontrivial eigenspaces of T. These differences therefore carry the integral delta generators to a spanning set of S with the stated bound. □

7.4.3 Conjugation of the theta jets

In this subsection, we identify the conjugated theta jets and the finite vacuum on the original chain of Section 17.3. Section 7.4.4 uses the transported jets in the Gaussian estimate.

For the kernel $X_j \to X_{j+1}$, let S_j be the sum of J on the actual upper part sizes $\lambda_{j,\tau,i}$ used in the Schur matrix. We estimate the operator on the original Weyl modules for the untwisted chain. The following two identities identify the CM and finite-vacuum identities in this calculation. They concern the same vacuum scalar and the same untwisted upper tensor used in the integral chain.

Lemma 7.6 (Transport of the jets and the finite vacuum)**.** *For $c \neq 0$, let J_c be the full holomorphic jet at scale c. Use $\gamma = s_{\mathrm{jet}} = p_F a_d / \omega_0$ specified below and let $T(x)$ denote finite*

Heisenberg translation. In the standard definite model of each positive rank v, fix the finite seed F_v given by the cyclotomic product below. It has $\theta(F_v) = 1$, and its translates span the finite representation. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ there is one pair $(b_\sigma, t_\sigma) \in \mathbb{A}_{E,f}^{\times} \times \mathbb{C}^{\times}$ such that, at each rank v, a finite seed $F'_{\sigma,v}$ satisfies

$$J_{t_\sigma}(T(b_\sigma x)F'_{\sigma,v}) = \mathrm{ren}_{\pi_\sigma}\ \sigma_{\mathrm{coeff}}\big(J_\gamma(T(x)F_v)\big) \quad (x \in \mathbb{A}_{E,f}^{v}), \qquad |t_\sigma|^2 = \Omega^{-1}. \tag{AJ}$$

For embeddings extending the fixed embedding of K, put $\pi_\sigma(\tau) = \sigma \circ \tau$. Here $\mathrm{ren}_{\pi_\sigma}$ sends $Z_{\tau,i}$ to $Z_{\pi_\sigma(\tau),i}$.

Let v_0 be the rank-one vacuum and $\Lambda = \widehat{\mathcal{O}}_E$. There is a finite displacement y for which $T_\sharp(u)T(y)F_1 = T(y)F_1$ for all $u \in q_1\Lambda$, and

$$s_{\mathrm{vac}} v_0 = \sum_{u \in \Lambda/q_1\Lambda} T_\sharp(u)T(y)F_1, \qquad s_{\mathrm{vac}} \neq 0. \tag{AV}$$

The operator $T_\sharp$ includes the finite vacuum semicharacter. The finite values of F_1 are integral at every coefficient place. They are products of signed, normalized rank-one theta evaluations at torsion points along the q_1 forms $z \mapsto -\mathrm{Tr}_{E/K}(z\overline{\zeta_{q_1}^{k}})/q_1$, each repeated q_1 times.

Proof. Construct the translated theta sections and their common period scale first. The finite theta expansion will give the vacuum projection (AV) and the spanning assertion at every positive rank.

The normalized CM translation and its period.

Write $K = \mathbb{Q}(\sqrt{-d})$ and $q = q_1$. We use the degree-one theta function of a lattice Λ_C with maximal CM by O_K. The distinguished two-torsion point t is a generator of $C[2]$ as an $O_K/2O_K$-module. In the Weierstrass coordinates with group origin at infinity, the curve is

$$y^2 + a_1xy = x^3 + x, \qquad t = (0,0), \qquad \Delta_C = a_1^4 - 64.$$

The CM invariant $j(C)$ is algebraic [51, Theorem 12.8]. The displayed equation gives $j(C) = (a_1^4-48)^3/(a_1^4-64)$, so a_1 is algebraic. The periods are those of the invariant differential of this equation. The theta function ϑ_C has zero divisor $[t]$, is even, and is normalized by $\vartheta_C(0) = 1$. For $A_C = \mathrm{area}(\Lambda_C)$ write

$$(T_u^C\vartheta_C)(z) = e^{-\pi(z\bar{u}+|u|^2/2)/A_C}\vartheta_C(z+u).$$

The fixed embedding $\Phi_K\colon K \hookrightarrow \mathbb{C}$ gives an elliptic CM type with reflex field K. For σ fixing K, apply the analytic CM theorem to $\sigma|_{\overline{\mathbf{Q}}}$ and the canonical principal polarization of C. Choose a finite idele s of K with $\mathrm{Art}_K^{\mathrm{ar}}(s) = \sigma|_{K^{\mathrm{ab}}}$, where the Artin map sends a local uniformizer to arithmetic Frobenius on the unramified quotient [24, Remark 2.5]. Let $c_\sigma \in \mathbb{C}^{\times}$ be the scalar of the CM tangent-space identification in the coordinates given by the two invariant differentials. The main theorem gives

$$\widehat{\Lambda}_{C^\sigma} = c_\sigma N_{\Phi_K}(s^{-1})\widehat{\Lambda}_C.$$

In these differential coordinates the map on each primary torsion component is induced by $c_\sigma N_{\Phi_K}(s^{-1})$. Equivalently, after identifying the tangent spaces by c_σ, the finite multiplier is $N_{\Phi_K}(s^{-1})$ [68, Ch. IV, Theorem 18.6], [24, Theorem 6.3, pp. 22–23]. We compute the translation factors of the normalized section below using its Hermitian form $\pi\bar{u}z/A_C$.

Integral normalization.

Over a number field defining the curve and its CM action, the curve has potentially good reduction at every finite place and its j-invariant is integral [51, Theorem 12.14 and Corollaries 12.15–12.16]. At each coefficient place take a finite extension with good reduction. For this curve the half-period discriminant calculation in Section 20.1 also gives a good equation directly in every residue characteristic. At an odd residue prime, t remains nonzero by injectivity of prime-to-residue-characteristic torsion reduction. At residue characteristic two, 2 splits in K. The two idempotents of $O_K \otimes \mathbb{Z}_2$ split the 2-divisible group of the good model into groups of height one. Their dimensions sum to one, so one factor is étale. The étale component of t is nonzero by the generator condition. It has nonzero reduction. The CM action used here extends to the good model by the extension property of abelian schemes over a discrete valuation ring.

An integral change of variables of a good Weierstrass equation moves t to $(0, 0)$. The constant term and the linear term in y vanish, by the equation and the order-two condition. Smoothness at $(0, 0)$ makes the coefficient of x a unit. A shear $y \mapsto y + sx$, with s a root of the monic quadratic that cancels the coefficient of x^2, is integral after finite extension. A unit dilation then sets the coefficient of x to one. This fixes the Weierstrass equation in these coordinates. Any isomorphism between two such equations preserving the marked point has no coordinate translations. If its variables have scaling parameter u, the coefficient of x forces $u^4 = 1$; thus its invariant differential changes by a unit. In particular Δ_C is a unit at every coefficient place. Section 20.1 computes it from the half-period discriminants in each ideal class.

Here is also the required integrality of the normalized torsion values. The good elliptic scheme has the symmetric line bundle $\mathcal{O}([t])$, rigidified by the value at zero of its canonical section. That value is a basis because the two sections $t, 0$ remain disjoint. The rigidification on this abelian scheme determines a cubical structure. For $L = \mathcal{O}([t])$ its multiplication formula identifies $[n]^*L$ with $L^{\otimes n(n+1)/2} \otimes ([-1]^*L)^{\otimes n(n-1)/2}$ [55, Ch. I, §2.6(i), pp. 21–22, and Proposition 5.5, p. 29]. The rigidified symmetry therefore gives, over the valuation ring,

$$[n]^*\mathcal{O}([t]) \simeq \mathcal{O}([t])^{\otimes n^2},$$

since $t = -t$. If u represents a torsion point and $nu = \lambda \in \Lambda_C$, let $\kappa(\lambda)$ be the lattice sign of ϑ_C. After a finite extension the point represented by u extends to a section by properness. Its image under $[n]$ is the zero section, by separatedness. Pulling back the rigidified isomorphism to this section thus gives an integral basis of its n^2-th tensor power, also when the residue characteristic divides n.

In the uniformizing basis, the factor of automorphy at zero for λ is $\kappa(\lambda)e^{\pi|\lambda|^2/(2A_C)}$. The rigidified isomorphism is induced by the identity on the pulled-back bases over the universal cover. Indeed the two multipliers at a period agree, since $\kappa(n\lambda) = \kappa(\lambda)^{n^2}$, and the identity preserves the basis at zero. In the fibre at the torsion point, the n^2-th power of the canonical section, in the basis pulled back from the origin, consequently has value

$$\kappa(\lambda)^{-1}\left(e^{-\pi|u|^2/(2A_C)}\vartheta_C(u)\right)^{n^2}.$$

This belongs to the valuation ring. The sign is a unit, so $(T_u^C\vartheta_C)(0)$ is integral. This also proves that the analytic normalized value is algebraic for algebraic CM data, since a positive power of the value is algebraic.

The translated Taylor series.

Write $C' = C^\sigma$ for the conjugated model with parameter σa_1, uniformized with that differential. Write $x_t(P) = x(P + t)$ and $y_t(P) = y(P + t)$ for the functions obtained by translation by the marked two-torsion point t. They have their only poles at t, of orders 2 and 3, and satisfy the same Weierstrass equation. The translation commutes with conjugation of the model. The graded ring of $\mathcal{O}([t])$ is generated by $\vartheta_C, x_t\vartheta_C^2, y_t\vartheta_C^3$ in degrees $1, 2, 3$. The Weierstrass equation reduces the powers of y_t to at most one. The resulting distinct pole orders give a basis in each degree by Riemann–Roch. At the pole chart the section transport is the homogeneous conjugation of the same projective equation, so it preserves zero values there too.

The translated rational functions are regular at the reference origin. Their expansions there conjugate coefficientwise using the invariant differential, whose formal integral gives the uniformizing coordinate. Thus, with Q the formal quotient of the Taylor series of ϑ_{C^σ} by the coefficientwise conjugate of ϑ_C, a section of degree l transports to $Q^l\sigma(f)$. At corresponding points the values of all degree l sections have one common nonzero relative factor μ^l, by choosing trivializations of the one-dimensional fibre of the degree-one line bundle. This evaluation statement is about ordinary section values, before any Gaussian twist.

Let $[a]$ be a nonzero CM endomorphism. Because σ fixes K, its two versions multiply the uniformizing parameter by the same a, have the same degree $a\bar{a}$, and preserve the conjugate zero divisor of the pullback of the theta section. In comparing the pulled back line bundle to the power of the bundle, use their theta multipliers. The Appell–Humbert form scales by $a\bar{a}$, and the pullback semicharacter agrees with the power semicharacter on the period lattice.

One can use $a \equiv 1 \bmod 2\mathcal{O}_K$ here, for which this last assertion follows directly from the half-period multiplier. Thus the section $\vartheta_C(az)$ transports to $\vartheta_{C'}(az)$: the zero divisors agree, the quotient of the two sections of the same line bundle is constant, and both have value one at zero. Consequently

$$Q(az) = Q(z)^{a\bar{a}}.$$

Take a nonreal $a \equiv 1 \bmod 2\mathcal{O}_K$ with $|a| > 1$. If $Q - 1$ begins in degree $j > 0$, coefficient comparison gives $a^j = a\bar{a}$. Absolute values imply $j = 2$, and then $a = \bar{a}$, a contradiction. Hence $Q = 1$.

For a finite set $S \subset K$, choose positive integers N_s with $N_s s \in \mathcal{O}_K$ for each $s \in S$. Choose a positive integer n divisible by every $2N_s$, and set $w = \omega_0/(2n^2)$, where $\Lambda_C = \omega_0\mathcal{O}_K$. Choose one lift w' of the conjugate of the point of C represented by w. Then $v = 2n^2w'$ is a period of C'. Set $a_s = (n/(2N_s))ns \in \mathcal{O}_K$. The points represented by

$$a_s w = \frac{s\omega_0}{4N_s}, \qquad a_s w' = \frac{sv}{4N_s}$$

correspond under σ, by equivariance of the $\mathcal{O}_K$-action. These are the divisions of the two shifts by $4N_s$ used in the rigidified translation formula [8, Prop. 2.8 in the authors' version]. For $s \in S$ write T_u for degree-one Gaussian translation and t_u for translation by the corresponding point on the elliptic curve. On either curve let $L = \mathcal{O}([t])$ for its marked point t. Then $[n]^*(t_u^*L \otimes L^{-1}) \simeq \mathcal{O}$ for the corresponding shift u: its class in Pic^0 is killed by n, since $ns \in \mathcal{O}_K$ and ω_0, v are periods on their respective curves. Thus $(T_{s\omega_0}\vartheta_C)(nz)$ and $(T_{sv}\vartheta_{C'})(nz)$ are sections of the same degree-n^2 bundles as the unshifted sections on the respective curves. The extra period multiplier is $\exp(2\pi i \operatorname{Im}(nu\bar{\lambda})/A_C)$ at $\lambda \in \Lambda_C$, and equals one by integrality of the Riemann form. The same calculation uses the area of C' on its lattice. The zeros of the first are at points $z = (b - 2ns)w$ with $b \in \mathcal{O}_K$ and nbw the marked half-period modulo Λ_C. At the corresponding point $(b - 2ns)w'$, the translated argument is nbw', which is the

conjugate marked half-period by the algebraic CM action. Thus these zeros transport to zeros of the second section. Both have the same degree, and the zero multiplicity is one because multiplication by n is unramified. The transported section is a nonzero scalar times the second one.

To determine that scalar, evaluate both at the point represented before multiplication by n by $-nsw$ and its conjugate $-nsw'$. The translated arguments are then $u = n^2sw$ and $u' = n^2sw'$ respectively, and the translation amounts are $2u, 2u'$. The exponential of the Gaussian twist is one at argument $-u$ of T_{2u}, and the corresponding unshifted value at $-u$ is $\vartheta_C(u)$ by evenness. The same is true for u'. Use $\vartheta_C(nz)$ as the second section of the same degree n^2; it transports coefficientwise to $\vartheta_{C'}(nz)$ by $Q = 1$. At $-nsw$ the two original sections have the same value, and likewise at $-nsw'$ for the two conjugate sections. The common fibre factor consequently cancels, giving scalar one when $\vartheta_C(u) \neq 0$. If $\vartheta_C(u) = 0$, both u, u' are the marked half-period on their curves. At such a point, the period multiplier for $2u$ gives

$$\vartheta_C(u + z) = \kappa(2u)e^{2\pi z\bar{u}/A_C}\vartheta_C(u - z).$$

Comparison of the linear terms at the simple zero gives $\kappa(2u) = -1$, and hence $T_{2u}\vartheta_C(0) = -1$. The same calculation on C' gives -1. Evaluation at zero fixes the scalar also in this case. Dilation by n is invertible on formal series over $\mathbb{C}$, so this is the exact Taylor identity for $T_{s\omega_0}$ and T_{sv}, for all $s \in S$.

For this finite S, take a number field L containing K, the marked Weierstrass model with its differential and CM action, and the coordinates of the torsion point represented by w. For an automorphism fixing this field we have $C' = C$ and can take $w' = w$, so $v = \omega_0$. The Taylor identity then fixes every coefficient of $T_{s\omega_0}\vartheta_C$ for $s \in S$. These coefficients lie in $\mathbb{C}^{\mathrm{Aut}(\mathbb{C}/L)} = L$.

To obtain a single finite adele for all rational translation parameters, write $\Lambda_{C'} = \omega_\sigma \mathfrak{c}^2/N\mathfrak{c}$ with $\mathfrak{c}$ an integral ideal. Such a presentation exists because the field discriminant $-d$ has just one prime divisor. The ideal and form class groups agree, and genus theory gives one genus, which consists of squared classes [26, Theorems 5.30 and 6.1(iii)–(iv)]. Thus $\mathrm{Cl}(K)/\mathrm{Cl}(K)^2 = 1$. The ideal $\mathfrak{c}^2/N\mathfrak{c}$ has norm one, and the area of the lattice is $|\omega_\sigma|^2\sqrt{d}/2$. Put $e = N\mathfrak{c}$. Then $e\mathcal{O}_K \subseteq \mathfrak{c}$, so

$$e\mathcal{O}_K \subseteq \mathfrak{c}^2/e \subseteq e^{-1}\mathcal{O}_K.$$

In particular $e\omega_\sigma$ is a period and the ratios $b_S = v/\omega_\sigma$ of the chosen periods lie in the compact set $e^{-1}\widehat{\mathcal{O}}_K$.

Put $\mathfrak{a}_\sigma = \mathfrak{c}^2/e$, so $\Lambda_{C'} = \omega_\sigma\mathfrak{a}_\sigma$. The point $w = \omega_0/(2n^2)$ generates the $\mathcal{O}_K/(2n^2)$-module $C[2n^2]$. Conjugation intertwines the $\mathcal{O}_K$-actions on C and C' because σ fixes their derivatives in K. On $C'[2n^2]$ the point $\sigma(w)$ corresponds to the class of b_S in $\mathfrak{a}_\sigma/(2n^2)\mathfrak{a}_\sigma$. Hence b_S generates the localized ideal $\mathfrak{a}_{\sigma,\mathfrak{p}}$ at every $\mathfrak{p} \mid 2n^2$. For a given finite $\mathfrak{p}$ above a rational prime p, require $1/p \in S$. Then $p \mid n$, so this generator condition holds on a cofinal tail. The set of local generators is closed: it is the complement of $\mathfrak{p}\mathfrak{a}_{\sigma,\mathfrak{p}}$ inside $\mathfrak{a}_{\sigma,\mathfrak{p}}$, and both lattices are open and closed. Thus each cluster point β_σ of the net generates $\mathfrak{a}_{\sigma,\mathfrak{p}}$ for every finite $\mathfrak{p}$. In particular $\beta_\sigma \in \mathbb{A}_{K,f}^\times$.

For any fixed s, choose $m > 0$ with $ms \in \mathcal{O}_K$. A rational representative x' sufficiently close to $\beta_\sigma s$ is congruent to $b_S s$ modulo $2me\widehat{\mathcal{O}}_K$ for some sufficiently late S containing s. The period $e\omega_\sigma$ belongs to $\Lambda_{C'}$, so the two translation arguments differ by $2m$ times a period.

As $mb_S s\omega_\sigma$ is a period, the Riemann form phase for this difference is the exponential of $2\pi i$ times an integer, and the doubled-period semicharacter is one. The two translated theta sections are therefore identical. This proves the required conjugation with a single β_σ and with

the asserted period area. In particular, one $\beta_\sigma \in \mathbb{A}_{K,f}^\times$ and one ω_σ work simultaneously for every rational translation parameter. No choice of either number depends on the number of theta factors.

The period scalar under conjugation.

Put $c_K = \gamma_{\mathrm{Euler}} - L'/L(1,\chi_K)$ for this paragraph and let h be the class number of K. We recall the unit normalization of the Chowla–Selberg product. The squaring map just used is bijective on $\mathrm{Cl}(K)$, so h is odd. Choose fractional ideals $\mathfrak{c}_i$ representing all classes with product O_K. This can be done by taking $\mathfrak{b}_{i^2}/\mathfrak{b}_i$ from any system $\mathfrak{b}_i$ of representatives. Choose CM curves C_i with the normalized Weierstrass differential used above and with lattices $\Lambda_i = \mu_i \mathfrak{c}_i \Lambda_C$. On each curve the distinguished point is the generator of $C_i[2]$ over $O_K/2O_K$. The integral normalization above therefore applies in every ideal class, including at coefficient primes above two. For each embedding fixing K, algebraic conjugation of the CM isogenies and their differentials gives

$$\Lambda_i^\sigma = \sigma(\mu_i)\mathfrak{c}_i \Lambda_C^\sigma.$$

Indeed, for $0 \neq a \in \mathfrak{c}_i$, multiplication by $\mu_i a$ has kernel a finite group of algebraic torsion points. The quotient isogeny and the ensuing pointed isomorphism of algebraic elliptic curves are defined over $\overline{\mathbb{Q}}$. On the chosen algebraic differentials the derivative is $\mu_i a$, which conjugates to $\sigma(\mu_i)a$. This gives inclusion. The inverse assertion follows by applying the same argument with $\mu_i^{-1}\mathfrak{c}_i^{-1}$ and interchanging the curves.

Each μ_i is algebraic since $\mu_i a$ is the derivative in algebraic coordinates of a CM isogeny for $0 \neq a \in \mathfrak{c}_i$. Set $u^{12h} = \prod_i \mu_i^{12} \Delta_{C_i}$. At any coefficient place the normalized invariant differentials are unit multiples of good-model differentials by the preceding normalization. Each $\mu_i a$, $a \in \mathfrak{c}_i$, is therefore integral as the tangent map of a homomorphism of good elliptic schemes. Since $\prod_i \mathfrak{c}_i = O_K$, a finite sum of products of such a's equals one. It follows that $\prod_i \mu_i$ is integral. Interchanging the curves gives the same assertion for its inverse. The Δ_{C_i} are units and hence so is u.

For any complex lattice $\Lambda = \omega(\mathbb{Z} + \tau\mathbb{Z})$ of area A, scaling the discriminant for the periods $1, \tau$ [60, (23.19.4)] gives $\Delta = (2\pi)^{12}\omega^{-12}\eta(\tau)^{24}$. Kronecker's first limit formula [70, Ch. I, §1, Theorem 1] in the Epstein-zeta normalization [66, §1, (1), p. 86], applied to

$$\sum_{\ell \in \Lambda\setminus\{0\}} \left(\frac{2A/\sqrt{d}}{|\ell|^2}\right)^s = (2/\sqrt{d})^s \sum_{(m,n)\neq(0,0)} \frac{(\operatorname{Im}\tau)^s}{|m+n\tau|^{2s}}$$

gives residue $2\pi/\sqrt{d}$ at one. The constant term in the unscaled sum is

$$2\pi\left(\gamma_{\mathrm{Euler}} - \log 2 - \log(\sqrt{\operatorname{Im}\tau}\,|\eta(\tau)|^2)\right).$$

The prefactor contributes $(2\pi/\sqrt{d})\log(2/\sqrt{d})$ to the constant term. Using $A = |\omega|^2 \operatorname{Im}\tau$ and the formula for Δ, the constant term of the scaled sum is

$$\frac{2\pi}{\sqrt{d}}\left(2\gamma_{\mathrm{Euler}} + \log(2\pi^2/\sqrt{d}) - \log A - \tfrac{1}{6}\log|\Delta|\right).$$

For lattices representing all the classes the sum of these series is $2\zeta(s)L(s,\chi_K)$, by grouping elements of each representative ideal into their unit orbits. The factor two is $|O_K^\times|$. The residues

on the two sides give $2L(1,\chi_K)=2h\pi/\sqrt{d}$. Comparing constant terms with $2\zeta(s)L(s,\chi_K)$ and dividing by the total residue gives

$$\frac{1}{h}\sum_i\Big(\log A_i^\sigma+\tfrac{1}{6}\log|\sigma(\Delta_{C_i})|\Big)=\log(2\pi^2/\sqrt{d})+c_K.$$

Here $A_i^\sigma=|\sigma(\mu_i)|^2N\mathfrak{c}_iA_C^\sigma$ and $\prod_i N\mathfrak{c}_i=1$, so the left side is $\log(A_C^\sigma|\sigma(u)|^2)$. Thus

$$A_C^\sigma|\sigma(u)|^2=\frac{2\pi^2}{\sqrt{d}}e^{c_K}.$$

The discriminant in this identity is the algebraic Weierstrass discriminant for the indicated invariant differential.

Use the splitting character υ of Section 4.3, and set $\lambda=\upsilon|\cdot|_E^{-1/2}$. The prescribed root is $s_0^f=\lambda_f(\alpha_{\mathrm{W}})^{-1}$. Choose $h>0$ so that the finite ideal of α_{W}^h is principal, generated by $y\in E^\times$. The character λ_f is trivial on finite integral units. At infinity $\upsilon_\infty(y)=\prod_v\Phi_v(y)/|\Phi_v(y)|$ and $|y|_{E,\infty}^{1/2}=\prod_v|\Phi_v(y)|$, so $\lambda_\infty(y)=\prod_v\overline{\Phi_v(y)}^{-1}$. Hence

$$\lambda_f(\alpha_{\mathrm{W}})^h=\lambda_f(y)=\lambda_\infty(y)^{-1}=\prod_v\overline{\Phi_v(y)}=\overline{\Phi_K(N_{E/K}y)}.$$

Thus $s_0^{fh}\in\Phi_K(K)^\times$, and its roots have the same modulus under all complex embeddings. The identity $\alpha_{\mathrm{W}}\bar\alpha_{\mathrm{W}}\widehat{O}_E=\mathfrak{d}_{E/K}^{-1}\widehat{O}_E$ gives $|\alpha_{\mathrm{W}}|^2=N_{E/\mathbb{Q}}\mathfrak{d}_{E/K}$. Discriminant transitivity and $[E:K]=f$ give

$$|\sigma(s_0)|^2=|\alpha_{\mathrm{W}}|^{1/f}=\mathrm{rd}(E)/\sqrt{d}.$$

Set $p_F=s_0/u$, $a_d=\sqrt{2\pi/\sqrt{d}}$, and $\gamma=p_Fa_d/\omega_0$, where $\Lambda_C=\omega_0O_K$. For the normalized period ω_σ determined above, $A_C^\sigma=|\omega_\sigma|^2\sqrt{d}/2$. Therefore the transported jet scale $t_\sigma=\sigma(p_F)a_d/\omega_\sigma$ satisfies the exact identity

$$|t_\sigma|^2=\frac{\pi|\sigma(s_0)|^2}{A_C^\sigma|\sigma(u)|^2}=\frac{\mathrm{rd}(E)}{2\pi}e^{-c_K}=\Omega^{-1}.$$

The cyclotomic product in the finite Weil model.

Put $\mathfrak{a}=E\cap\alpha_{\mathrm{W}}\widehat{O}_E$. At each finite place α_{W} is 1 or the inverse of a generator of the relative different (Section 8.1), so $O_E\subseteq\mathfrak{a}$. Since $\mathfrak{a}\bar{\mathfrak{a}}=\mathfrak{d}_{E/K}^{-1}$, we have $\mathrm{Tr}_{E/K}(\mathfrak{a}\bar{\mathfrak{a}})\subseteq O_K$. In particular $\mathrm{Tr}_{E/K}(\mathfrak{a}\overline{\zeta_q^k})\subseteq O_K$ for each k. For the standard definite space of any positive rank v, write $D=fv$ and use Fock coordinates

$$z=(z_{\tau,i})_{\tau\in\Phi,\,1\le i\le v},\qquad z(\gamma)_{\tau,i}=a_d\tau(\gamma_i)\quad(\gamma\in E^v).$$

The lattice $\mathfrak{a}^v$ in these coordinates is self-dual for the alternating polarization of the standard Fock form.

Indeed, up to sign the alternating pairing $\mathrm{Im}\langle z(\gamma),z(\delta)\rangle/\pi$ is $\mathrm{Tr}_{E/\mathbb{Q}}(j_{\mathrm{W}}\sum_i\bar\gamma_i\delta_i)$, and $j_{\mathrm{W}}\mathfrak{a}\bar{\mathfrak{a}}=\mathfrak{d}_{E/\mathbb{Q}}^{-1}$. Let $\Lambda=\widehat{O}_E^v$ be the normalized finite translation lattice. The finite Heisenberg translate T by $\alpha_{\mathrm{W}}^{-1}\gamma$ corresponds to the signed real shift by $-z(\gamma)$. Indeed, a rational translation has real and finite coordinates $(z(\gamma),\alpha_{\mathrm{W}}^{-1}\gamma)$, and translating it out of the adelic theta functional gives the real shift

$$F(z)\longmapsto e^{\langle z(\gamma),z\rangle-|z(\gamma)|^2/2}F(z-z(\gamma))$$

on the entire function obtained by dividing its real slice by $e^{-|z|^2/2}$.

Here the displayed shift is for the unnormalized finite translate. For $\gamma \in \mathfrak{a}^v$, the normalization is the factor $e^{-\pi i \operatorname{Tr}_{E/K} \sum_i \gamma_i \bar{\gamma}_i}$. It follows from the product formula for the additive character with trace-one element $(1+\sqrt{-d})/2$. The exponent is $-\pi i$ times an integer by the trace pairing, so the sign is one on any even multiple of $\mathfrak{a}^v$.

Let ω denote ω_0 or ω_σ, and use the corresponding rank-one theta curve. Take $e = 1$ in the first case and the positive integer e fixed above in the second, so ωe is a period. For $0 \le k < q$ define

$$A_{k,i}(z) = \frac{\omega}{q a_d} \sum_\tau \overline{\tau(\zeta_q^k)} z_{\tau,i}, \qquad c_k(y) = -\frac{1}{q} \operatorname{Tr}_{E/K}(y \overline{\zeta_q^k}).$$

Then $A_{k,i}(-z(\gamma)) = \omega c_k(\gamma_i)$. The geometric-series identity on the distinct q-th roots $\tau(\zeta_q)$ gives

$$\sum_{0 \le k < q} \tau(\zeta_q^k) \overline{\rho(\zeta_q^k)} = q \, \mathbf{1}_{\tau = \rho}.$$

As $A_C = |\omega|^2 \sqrt{d}/2$, the pullbacks of the degree-one Hermitian form $\pi \bar{u} z / A_C$ along the $A_{k,i}$, each repeated q times, sum to the standard Fock Hermitian form.

The entire function

$$\Theta(z) = \prod_{k,i} \vartheta_C(A_{k,i}(z))^q$$

therefore has the required Gaussian multiplier on the lattice $z(qe\mathfrak{a}^v)$. At any point of that lattice, each $A_{k,i}$ is a period by trace integrality and the choice of e. Each rank-one sign is raised to the even power q. Their product is one, in agreement with the Fock sign on this lattice. Also $\Theta(0) = 1$.

To identify this function in the original Weil model, let $l = qe$. The translates of the unit vacuum by $l^{-1}\Lambda/\Lambda$ are pairwise orthogonal, by its vacuum matrix coefficient, and are fixed by the normalized translations by $l\Lambda$, by the Heisenberg commutator. Their entire functions are linearly independent by the injectivity in Section 20.1. That argument transports the real evaluations to arbitrary finite displacements by rational Heisenberg invariance and uses the polarized norm identity.

These entire functions belong to the polarized theta space for $z(l\mathfrak{a}^v)$ with the displayed signed multiplier. That space has dimension l^{2D}: the original alternating polarization is unimodular and passing to $l\mathfrak{a}^v$ multiplies every alternating elementary divisor by l^2. For this polarization type the space of sections is the irreducible theta representation on functions on $(\mathbb{Z}/l^2\mathbb{Z})^D$ [59, §1, Proposition 3 and Theorem 2, pp. 295–297]. Its dimension is $l^{2D} = [l^{-1}\Lambda : \Lambda]$. Thus Θ is the entire function of a vector in the original finite part; denote it by F or F' according as the curve is original or conjugate.

For a rational translate indexed by $\gamma \in E^v$, the Gaussian shift identity and the equality of Hermitian forms give the entire function

$$\prod_{k,i} (T^C_{\omega c_k(\gamma_i)} \vartheta_C)(A_{k,i}(z))^q.$$

Its constant coefficient is a product of integral torsion values, since each $c_k(\gamma_i)$ lies in K and a multiple of ω is a period. Finite smoothness and density of rational points give the same conclusion at every finite adelic translate.

For the original curve $e = 1$. Write

$$F = \sum_{x \in q^{-1}\Lambda/\Lambda} c_x T(x) v_0$$

using representatives supported above q. Since $\Theta(0) = 1$, choose x_0 with $c_{x_0} \neq 0$ and set $y = -x_0$. For $u \in \Lambda$, the normalized translation $T_\sharp(u)$ fixes v_0. With $t = x - x_0$, its eigenvalue on $T(y)T(x)v_0$ is the commutator

$$\psi\big((u^{\mathsf{T}}\bar{t} - t^{\mathsf{T}}\bar{u})/2\big) = \psi(u^{\mathsf{T}}\bar{t}),$$

since the finite character ψ is trivial on the fixed subalgebra. This character of u is trivial on $q\Lambda$, since $t \in q^{-1}\Lambda$, and on all of Λ precisely when $t \in \Lambda$, by self-duality. Thus $T(y)F$ is fixed by $T_\sharp(q\Lambda)$. For $x = x_0$ one has $T(-x_0)T(x_0) = 1$, again because the cocycle argument lies in the fixed subalgebra. Character orthogonality now gives

$$\sum_{u \in \Lambda/q\Lambda} T_\sharp(u)T(y)F = [\Lambda : q\Lambda]c_{x_0}v_0 = q^{2D}c_{x_0}v_0.$$

The representatives for u can also be supported above q. Thus both u and y vanish above MN, since $(q, MN) = 1$. For $v = 1$ the result is the identity (AV) with exactly q^{2f} terms. It defines the original nonzero scalar s_{vac}, with the integrality and uniform conjugate sup bound obtained from the same formula. The nonzero projection also shows that the translates of F span the entire finite part.

Finally put b_σ equal to the image of β_σ in $\mathbb{A}_{E,f}$. At a rational translate γ, choose one element $a \in K$ sufficiently close to β_σ that the translate of F' by $a\alpha_{\mathrm{W}}^{-1}\gamma$ agrees with its translate by $b_\sigma\alpha_{\mathrm{W}}^{-1}\gamma$, and that every $ac_k(\gamma_i)$ belongs to the neighborhood required by the rank-one conjugate translation identity. These are finitely many open conditions on a, and K is dense in its finite adeles.

The coefficient of the τ, i variable in the original scaled form is

$$\frac{p_F}{q}\overline{\tau(\zeta_q^k)}.$$

After applying σ it is the coefficient in the new form at scale t_σ, with the permutation of the embeddings over K. Indeed $\overline{\tau(\zeta_q^k)} = \tau(\zeta_q^{-k})$, and $\omega_\sigma t_\sigma / a_d = \sigma(p_F)$. Apply the rank-one translated Taylor identity to each factor and compose with these linear forms. This proves (AJ) at rational translates and hence at all finite adelic translates by local constancy. The same b_σ and t_σ work for every v, in particular for $v = 1$ and for $v = n_j n_{j+1}$ in the same upper kernel.

The finite Weil action and vacuum projection. Let $V = \mathrm{Hom}(X_j, X_{j+1})$ in the definite models. Write v_0 for its unit vacuum, $\Lambda = \widehat{\mathcal{O}}_E^{\dim_E V}$ for the standard lattice in the normalized Heisenberg coordinates, and ψ for the self-dual finite additive character in those coordinates. Fix $\sigma \in \mathrm{Aut}(\mathbf{C}/K)$. The translated CM identity, first in rank one and then on the cyclotomic products, gives initial vectors F, F' and a finite adele b_σ as in (AJ).

First recover the full additive character. In rank one write the unnormalized finite translation as $T(x)$. Its group law in these coordinates is

$$T(x)T(y) = \psi(x\bar{y}/2)T(x + y).$$

The two seeds are smooth in the finite translation variables. Choose a positive integer n for which $T(n)F = F$ and $T(b_\sigma n)F' = F'$. Such an integer exists because the two open stabilizer

neighborhoods contain an open additive lattice and a sufficiently large integer clears the finitely many local conditions on $1, b_\sigma$.

For any $z \in \mathbb{A}_{E,f}$ write $z = nx$. Apply the group law to $T(2x)T(n)F$ and $T(2b_\sigma x)T(b_\sigma n)F'$. The two cocycle factors are $\psi(nx)$ and $\psi(b_\sigma \overline{b_\sigma} nx)$, respectively. Both comparisons involve the same conjugated jet of the translate at $2x+n$, which is nonzero. Indeed the full jet at a nonzero scale determines the entire theta function by holomorphy, and hence determines the finite vector by the same injectivity argument. Cancellation therefore gives

$$\sigma(\psi(z)) = \psi(b_\sigma \bar{b}_\sigma z) \qquad (z \in \mathbb{A}_{E,f}). \tag{JV1}$$

The comparison at the two translations uses the full cocycle in the group law.

Put $u = b_\sigma \bar{b}_\sigma$. Self-duality of $\widehat{\mathcal{O}}_E$ for ψ, applied to (JV1) at all zy with y integral, gives

$$uz \in \widehat{\mathcal{O}}_E \quad \Longleftrightarrow \quad z \in \widehat{\mathcal{O}}_E.$$

Indeed the finite character in these coordinates is the absolute trace character scaled by $\delta_f = j_{\mathrm{W}} \alpha_{\mathrm{W}} \bar{\alpha}_{\mathrm{W}}$. At $w \mid p$ the trace dual of $\mathcal{O}_w$ is $\mathfrak{D}^{-1}_{E_w/\mathbb{Q}_p}$. Hence the annihilator of $\mathcal{O}_w$ for $\psi_w(xy)$ is $\delta^{-1}_{f,w} \mathfrak{D}^{-1}_{E_w/\mathbb{Q}_p} = \mathcal{O}_w$ by Section 8.1. At $p = d$ the cyclotomic extension over K is unramified, so this scale is supplied by j_{W}. At $p \mid q_1$ the quadratic field is split, j_{W} is a unit and $\alpha_{\mathrm{W}} \bar{\alpha}_{\mathrm{W}}$ generates the inverse different. Testing in each local factor at 1 and at the inverse of a uniformizer proves $u \in \widehat{\mathcal{O}}_E^\times$. In particular b_σ is an idele, with inverse $\bar{b}_\sigma u^{-1}$.

For the higher rank product, use the relation (JV1) and the translates of its seed F. They span the finite part. If a linear combination of them is zero, the σ-conjugated combination of the translated F' has zero full jet after the common variable permutation. The injectivity just proved shows that combination is zero. Hence the rule defines a σ-semilinear map C on the finite part. The group law and (JV1) show $CT(x) = T(b_\sigma x)C$ on every generator and hence on the whole finite part. The normalized translations $T_\sharp(x)$ have the additional factor $\psi(c_{\mathrm{tr}} \bar{x}^{\mathrm{t}} x/2)$ in this model, where $c_{\mathrm{tr}} = (1+\sqrt{-d})/2$ is the integral trace-one element $(1 - j_{\mathrm{W}}^{-1})/2$ of Section 41.5.2. Equation (JV1) conjugates that factor to the factor at $b_\sigma x$. Thus the same identity holds with $T_\sharp$. The full jet identity gives injectivity of C, since $\gamma \neq 0$ and the jets at that original scale determine the vector.

There is $a \in \widehat{\mathcal{O}}_E^\times$ with $a\bar{a} = u$. In each split pair choose one place w and set $a_w = 1$, $a_{\bar{w}} = \overline{u_w}$, using the involution between the completions. At each nonsplit finite place w, $2v_w(b_{\sigma,w}) = v_w(u_w) = 0$; take $a_w = b_{\sigma,w}$. Thus $b_\sigma = ha$ for the finite norm-one scalar $h = b_\sigma/a$. Set $C^\circ = \omega(h)^{-1} C$; it conjugates the normalized translations by x to those by ax.

Since $a\Lambda = \Lambda$, the vector $C^\circ v_0$ is fixed by all $T_\sharp(x)$ with $x \in \Lambda$. Its fixed line is one-dimensional. To see this, the translates of v_0 indexed by distinct cosets of Λ are orthogonal by the support of the vacuum matrix coefficient. Each coset is an eigenspace for all lattice translations. Its character is trivial only for the zero coset, by the self-duality of Λ for the polarized Heisenberg pairing. Therefore $C^\circ v_0 = \eta v_0$ with $\eta \neq 0$. Equivalently $Cv_0 = \eta \omega(h) v_0$. For $p^e \| q_1$ and $\Lambda_p = \Lambda \otimes_{\widehat{\mathbf{Z}}} \mathbf{Z}_p$, on the subspace fixed by $T_\sharp(q_1 \Lambda_p)$ the local vacuum average is

$$P_p = p^{-2De} \sum_{u \in \Lambda_p / q_1 \Lambda_p} T_\sharp(u), \qquad C^\circ P_p = P_p C^\circ.$$

Indeed a_p permutes the summands and the rational coefficient is fixed by σ. The phase in each normalized summand transports by (JV1), also at $p = 2$. Each average projects onto the subspace fixed by Λ_p.

For a finite unitary matrix g, choose H with $H\Lambda = \Lambda \cap g^{-1}\Lambda$. The original finite lattice multiplier is $\Delta_g = \lambda_f(\det H)$, with $\lambda_f = \upsilon_f|\cdot|_f^{-1/2}$, as in the lattice model of Section 19.3. Averaging the lattice projection over Λ gives the finite formula

$$\omega(g)v_0 = \frac{\Delta_g}{|\Delta_g|^2} \sum_{q \in \Lambda/(\Lambda \cap g^{-1}\Lambda)} T_\sharp(gq)v_0.$$

Changing a representative q by q' in $\Lambda \cap g^{-1}\Lambda$ leaves the term unchanged. Write $c = c_{\mathrm{tr}}$, $A = y^*x$ and $\eta(x) = \psi(cx^*x/2)$. The unnormalized cocycle is $\psi(A/2)$. Since $\psi(\bar{z}) = \psi(z)^{-1}$ and $c + \bar{c} = 1$, the normalized cocycle is

$$\frac{\eta(x)\eta(y)}{\eta(x+y)}\,\psi(A/2) = \psi((A - c(A + \bar{A}))/2) = \psi(\bar{c}A).$$

For $x, y \in \Lambda$, $\bar{c}A$ is integral and the self-dual character is trivial there, including above 2. Unitarity of g preserves the Hermitian form in both the Heisenberg cocycle and the normalizing quadratic phase. Thus $T_\sharp(gq + gq') = T_\sharp(gq)T_\sharp(gq')$, and the second operator fixes v_0 because $gq' \in \Lambda$. This argument includes the places over 2. Each member of the finite quotient has mass $|\Delta_g|^{-2}$, since $|\Delta_g|^2$ is its index.

Multiplication by a permutes these cosets, because a is a scalar integral unit. Applying C° therefore gives

$$C^\circ\omega(g)v_0 = \frac{\sigma(\Delta_g)}{\Delta_g}\,\omega(g)C^\circ v_0.$$

Heisenberg covariance extends this to the spanning translates of v_0. The scalar matrix h commutes with g, so the same identity holds with C. The translates of $Cv_0 = \eta\omega(h)v_0$ span, since b_σ is invertible and $\omega(h)$ is a unitary change of variables. Hence C is bijective. With $d_\sigma(g) = \sigma(\Delta_g)/\Delta_g$, its inverse on the vacuum is

$$C^{-1}v_0 = \sigma^{-1}\big((\eta d_\sigma(h^{-1}))^{-1}\big)\,\omega(h^{-1})v_0.$$

This follows by applying C to the right-hand side. The lattice for g^{-1} is $\Lambda \cap g\Lambda = gH\Lambda$, so $\Delta_{g^{-1}} = \upsilon(\det g)\Delta_g$. In the standard coordinates $g^{-1} = g^*$. Local Smith form therefore conjugates the determinant ideal of $\Lambda \cap g^{-1}\Lambda$ to that of $\Lambda \cap g\Lambda$. As λ_f is unramified and takes conjugate values on conjugate ideles,

$$\Delta_{g^{-1}} = \overline{\Delta_g}, \qquad \Delta_g^2 = |\Delta_g|^2\upsilon(\det g)^{-1}.$$

The index is rational. The finite values of the splitting character on norm-one ideles have modulus one at every complex embedding.

For the latter assertion, a positive power of a norm-one idele can be written as a rational norm-one scalar times an integral norm-one idele. Indeed, if its ideal is I and $I^d = (\alpha)$, then $I^{2d} = (\alpha/\bar{\alpha})$ since $I\bar{I} = 1$. The finite character of the integral factor has finite image. The value at the rational factor is the inverse archimedean weight monomial in its conjugates, all of absolute value one. This gives

$$\left|\frac{\sigma(\Delta_g)}{\Delta_g}\right| = 1.$$

The same argument applies to the finite splitting characters of the pair action. Finite Heisenberg phases are roots of unity.

Write v_{int} for the integral vacuum and $Cv_{\mathrm{int}} = \eta_{\mathrm{int}}\omega(h)v_0$. The degree-zero coefficient of the translated jet is the theta evaluation at the corresponding finite displacement. Thus (AJ) and semilinearity give

$$(Cv_{\mathrm{int}})(0, \alpha_{\mathrm{W}} b_\sigma x) = \sigma\big(v_{\mathrm{int}}(0, \alpha_{\mathrm{W}} x)\big) \qquad (x \in \mathbb{A}_{E,f}^{\dim_E V}).$$

Multiplication by $\alpha_{\mathrm{W}} b_\sigma$ is bijective on the finite adeles, so this covers the entire finite slice.

The Schur-jet integral. For a rational Heisenberg displacement r, equivariance gives

$$|(Cv_{\mathrm{int}})(r_\infty, z_f)| = |(Cv_{\mathrm{int}})(0, z_f - r_f)| \qquad (z_f \text{ finite}).$$

Rational combinations of a $\mathbf{Q}$-basis of the position space are dense in the real position space. This and continuity give the same supremum bound on the whole additive quotient. Its Haar measure has mass one, and the polarized norm identity of Section 41.5.1 identifies the mean square with $\|Cv_{\mathrm{int}}\|^2 = |\eta_{\mathrm{int}}|^2$ since $\omega(h)$ is unitary and $\|v_0\| = 1$. The Gaussian lattice bound on $\omega(g)v_0$ now applies to $C(T(x)\omega(g)v_{\mathrm{int}})$, uniformly in the old translation x and the old Weil argument g. This is the pointwise bound for the coefficientwise conjugate Schur-jet integral. In particular this step concerns the upper kernels before any source form is substituted. For fixed finite arguments, the corresponding jet coefficients are algebraic. Indeed the displayed finite formula for $\omega(g)v_0$ also holds after multiplying both sides by the vacuum scale. Equation (AV) in each rank-one factor then expresses every term through translated cyclotomic seeds. Finite smoothness permits rational translation representatives. Their rank-one Taylor coefficients belong to the number fields obtained above, and the scaled linear forms have coefficients $p_F\overline{\tau(\zeta_q^k)}/q$. The other coefficients are Heisenberg roots of unity and the algebraic factors $\Delta_g/|\Delta_g|^2$ and finite splitting-character values.

For the resulting estimate, let $J_\gamma F$ be the Taylor series at the original jet scale γ. For a polynomial Q, write $j_Q(J_\gamma F)$ for $Q(\partial)(J_\gamma F)(0)$. The Schur kernel is tested by taking Q to be a fixed matrix entry, or the contraction with a source vector and a dual target vector. For any additional scalar c, put

$$Q_\sigma = \pi_\sigma \cdot \big(\sigma_{\mathrm{coeff}}(Q(cX))\big),$$

where π_σ acts by permuting the embedding coordinates only. Relabel each source and target Weyl factor at τ, together with its diagram, by $\pi_\sigma(\tau)$. This permutes the integral bases of pure tensors. For either space, write P for the resulting permutation matrix and $J_{\mathrm{old}}, J_{\mathrm{new}}$ for the tensor products of the factor Gram matrices in the two orders. Then $P^{\mathsf{t}} J_{\mathrm{new}} P = J_{\mathrm{old}}$.

The chain rule for monomials and the transported Taylor identity give

$$\sigma\big(j_Q(J_{c\gamma}F)\big) = j_{Q_\sigma}(J_{t_\sigma}CF) = \int_{\mathbf{C}^D} Q_\sigma(t_\sigma \bar{y}) e^{-\|y\|^2/2}\, CF(y_{\mathbb{A}})\, \frac{dy}{\pi^D}.$$

Here $y_{\mathbb{A}}$ has finite component zero and archimedean components $(2\pi \operatorname{Im}\tau(j_{\mathrm{W}}))^{-1/2} y_\tau$ in the original adelic coordinates. Indeed $2\pi \operatorname{Im}\tau(j_{\mathrm{W}}) = a_d^2$ on Φ, so y is the normalized Fock coordinate. Write $H(y) = e^{\|y\|^2/2} CF(y_{\mathbb{A}})$ for the entire section. For the fixed finite test its weighted modulus is bounded: a full period lattice of the real slice acts by unit phases on the weighted values, by rational invariance and the finite translation stabilizer. For each multi-index ν, angular integration on the product of circles extracts $[z^\nu]H(z)$. Radial integration in each coordinate gives $2\int_0^\infty r^{2\nu_i+1} e^{-r^2}\, dr = \nu_i!$. Thus

$$\pi^{-D} \int_{\mathbb{C}^D} \bar{y}^\nu H(y) e^{-\|y\|^2}\, dy = \nu!\, [z^\nu]H(z) = \partial^\nu H(0).$$

Indeed, if $B_F = \sup_y |CF(y_{\mathbb{A}})|$, the absolute integrand after rescaling a monomial is bounded by

$$B_F |t_\sigma|^{|\nu|} \, \|y\|^{|\nu|} e^{-\|y\|^2/2}.$$

This is integrable, and the Taylor series converges uniformly on each product of circles. Rescaling by t_σ supplies $t_\sigma^{|\nu|}$, and finite linearity gives the Schur-jet identity. Here $|t_\sigma|^2 = \Omega^{-1}$ at the transported original scale, and for the actual arrow scale $c = \pi_M = \bar{\alpha}$ one has $|\sigma(c)|^2 = M$. The variable in the conjugated polynomial consequently has squared norm $(M/\Omega)\|y\|^2$.

For the short-image bound, apply the measured operator identity on the original ramified source first. Then apply σ to the terminal coordinates of its finite upper-chain class sum. This coefficientwise operation changes neither the actual source generators nor the choice of measured point and functional. In particular it requires no continuity of σ in the real topology or conjugation of the integrals defining the Poincaré map. The same measured identity separately gives the local lattice norm comparison for every allowed twist on the clearing module.

Since K is imaginary quadratic, an arbitrary σ either fixes K or $c \circ \sigma$ fixes K, where c is ordinary complex conjugation. The Weyl Gram matrices are rational. Entrywise ordinary conjugation preserves the operator norm between these two positive Gram spaces. Thus the bound obtained for the embeddings fixing K also gives the bound at every complex automorphism. □

7.4.4 Gaussian bounds for the conjugated kernels

In this subsection, we apply the vacuum identity of Lemma 7.6 and integrate the conjugated jets. The resulting operator norm is used in Section 7.4.5.

Lemma 7.7 (Norm of one conjugated kernel)**.** *For $0 \le j < m$, let $D_j(g,h;u)$ be the Schur-jet matrix at scale $\pi_M\gamma$ of $\omega_{\mathrm{f}}(g,h)T_\sharp(u)v_{\mathrm{int}}$ for the untwisted arrow $X_j \to X_{j+1}$, with normalized integral vacuum v_{int}. For sufficiently large x, every finite adelic source and target pair (g,h), and every finite Heisenberg translation u,*

$$\|\sigma D_j(g,h;u)\|_{\mathrm{op,W}} \le \exp(x^{50} + S_j/2) \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Proof. For a conjugated translated vacuum the bound is e^{C_j}, where

$$C_j = C_K(q_1 n_j n_{j+1} + 1)^2, \qquad n_j = \dim_E X_j.$$

It is uniform in the finite translates. The preceding transport gives a scalar multiple of a unitary move of the vacuum and bounds the scalar by its finite values. For this last finite bound, the normalized CM translation law used in (AJ) sends every conjugate of a torsion value to a value on a conjugate elliptic lattice.

After multiplying by the Gaussian part of the translation, its absolute value is periodic on that lattice. Its maximum is finite and unchanged by homothety. The finitely many lattice classes and the two-torsion choices therefore give a common bound $M_K \ge 1$ depending only on K. Each finite value of the rank-one theta product has q_1^2 such factors, by the repeated cyclotomic resolution.

Equation (AV) expresses this same multiple of the vacuum as a sum of q_1^{2f} translates with root-of-unity factors. Hence the finite values of the scaled rank-one vacuum have conjugate absolute value at most

$$q_1^{2f} M_K^{q_1^2} \le \exp((2 + \log M_K) q_1^2).$$

Tensoring over the $n_j n_{j+1}$ rank coordinates multiplies these bounds. The inner product is the mean of the squared values on the vacuum quotient.

By rational translation, arguments whose archimedean coordinates come from rational vectors can be moved to arguments of archimedean coordinate zero, up to unit phases. Density and continuity give the same bound for the quotient norm, which bounds the scalar in the unit vacuum presentation.

For the ordinary moved vacuum, put $f_y = T(y)\omega(g)v_0$. In rational Schrödinger coordinates write $z = (a, b)$ and $\gamma = (u, v)$, with the two variables in dual Lagrangians. Let e denote the global additive character, and take the Fourier coefficient at the character $\mathrm{e}(\langle b, u\rangle - \langle v, a\rangle)$. The phase independent of the rational summation variable drops out of

$$|\theta(T(a,b)f_y)|^2 = \left|\sum_{\lambda} \mathrm{e}(\langle b, \lambda\rangle) f_y(\lambda + a)\right|^2,$$

where λ ranges over the rational points of the first Lagrangian. Integration in b on its rational quotient leaves the pairs $\lambda - \mu = u$. Unfolding the integral in a gives

$$\int f_y(t+u)\overline{f_y(t)}\mathrm{e}(\langle v, t\rangle)\, dt = \langle f_y, T(u,v)f_y\rangle.$$

In the commutator convention $T(z)T(\gamma) = \beta(z,\gamma)^2 T(\gamma)T(z)$, the character just extracted is $\beta(z,\gamma)^{-2}$, as in Section 41.5.1. The half-translation phase is one on the rational pair (u, v). The two quotient measures have mass one in the self-dual normalization. The Schwartz bounds give absolute summability on compact sets of adelic representatives and justify the integrations. The zero Fourier coefficient gives the quotient-mean normalization of the squared norm used above.

By unitarity, the absolute value of this coefficient equals the ordinary vacuum matrix coefficient in modulus at $g^{-1}\gamma$. The moved vector y only contributes its commutator phase. The Gaussian vacuum formula bounds the coefficient in the support by $\exp(-\pi\|g^{-1}\gamma\|_\infty^2/\sqrt{d})$; the coefficient vanishes outside the finite support $(g^{-1}\gamma)_f \in \alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E^{\mathrm{rk}_E V}$. For a difference γ of two frequencies in this support, the finite unitary change of basis gives

$$h(\gamma,\gamma)_f = h(g_f^{-1}\gamma, g_f^{-1}\gamma) \in \mathfrak{d}_{E/K}^{-1}\widehat{\mathcal{O}}_E.$$

Here $(g_f^{-1}\gamma)_f$ lies in the same additive lattice, and $\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}\widehat{\mathcal{O}}_E \subseteq \mathfrak{d}_{E/K}^{-1}\widehat{\mathcal{O}}_E$.

Take the unscaled embedding coordinates $u_{\tau,i} = (g_\infty^{-1}\gamma)_{\tau,i}$. Unitarity and trace integrality give

$$\|g^{-1}\gamma\|_\infty^2 = \sum_{\tau,i} |u_{\tau,i}|^2 = \mathrm{Tr}_{E/K}\, h(\gamma,\gamma) \in \mathcal{O}_K \cap \mathbb{R} = \mathbb{Z}.$$

It is positive if $\gamma \neq 0$. Thus this lattice is one-separated. Its Fock coordinates are $a_d u$, with $a_d^2 = 2\pi/\sqrt{d}$, as used in the Gaussian above. Packing unit-separated points bounds the sum of $e^{-(\nu+1)\|v\|}$ in dimension ν by $3^\nu e^\nu$. The counting function in real dimension ν is at most $(1+2r)^\nu \le 3^\nu e^{\nu r}$ for a ball of radius r.

Integrating against $(\nu+1)e^{-(\nu+1)r}\,dr$ gives the claimed packing bound. Completing the square gives $e^{-\pi\|v\|^2/\sqrt{d}} \le e^{\sqrt{d}(\nu+1)^2}e^{-(\nu+1)\|v\|}$. Thus the Fourier series is absolutely convergent and its value at zero is $|\theta(f_y)|^2$. Since $\nu = 2f\,\mathrm{rk}_E V \le 2q_1\,\mathrm{rk}_E V$, its square root is bounded

by $\exp((2d+6)(q_1 \operatorname{rk}_E V + 1)^2)$. Writing $r_V = n_j n_{j+1} \geq 1$, the logarithm of the transported vacuum bound is at most

$$(2 + \log M_K) q_1^2 r_V + (2d+6)(q_1 r_V + 1)^2 \leq (2d + 8 + \log M_K)(q_1 r_V + 1)^2.$$

Thus one may take $C_K = 2d + 8 + \log M_K$.

To calculate the Gaussian moment, use $L_a = a + 1$ and $v_M = \log(M/\Omega)$. For an array Y in the Gaussian integral, put $t_Y = \|Y\|^2/(2L_a + 2)$. First suppose σ fixes K. The Schur-jet integral in Lemma 7.6 evaluates the original rational Schur matrix at the array

$$Z_\tau = \sigma(\pi_M) t_\sigma \overline{Y_{\pi_\sigma(\tau)}}, \qquad \pi_\sigma(\tau) = \sigma \circ \tau.$$

The matrix at τ has n_{j+1} target rows and n_j source columns. Let $\rho_{\tau,i}$ denote its squared singular values in decreasing order, including zeros. Since π_σ permutes Φ and $|\sigma(\pi_M) t_\sigma|^2 = M/\Omega$, we have

$$\sum_{\tau,i} \rho_{\tau,i} = \sum_\tau \|Z_\tau\|_{\mathrm{HS}}^2 = e^{v_M}(2L_a + 2) t_Y.$$

Suppress the embedding index in the row products below. In the tensor-induced Weyl metrics the two unitary factors of a singular value decomposition act isometrically. On a column of height $h \leq n_j$, the exterior power of the rectangular diagonal factor has norm equal to the product of its first h singular values. The column counts in Section 20.2 vanish for $n_j < h < n_{j+1}$. At each target column of height n_j the source tensor uses $e_1 \wedge \cdots \wedge e_{n_j}$, a primitive integral vector of norm one for the induced exterior form. Each such column contributes one box to each of the first n_j target rows.

For a rectangular diagonal array with entries $d_1, \ldots, d_{n_j}$ in nonincreasing order of modulus, let c_h count the columns of height h. Its matrix entry on the unit highest column tensors is

$$\prod_h \det(\operatorname{diag}(d_1, \ldots, d_h))^{c_h} = \prod_{i=1}^{n_j} d_i^{\lambda_i}, \qquad \lambda_i = \sum_{h \geq i} c_h.$$

Indeed the leading wedge $e_1 \wedge \cdots \wedge e_h$ maps to its target counterpart times the leading determinant. This gives equality in the product of the exterior-power norm bounds, including at $h = n_j$. For the scaled array $\sigma(\pi_M) t_\sigma \bar{Y}$, a diagram with $|\lambda| = \sum_i \lambda_i$ boxes therefore contributes the factor $(M/\Omega)^{|\lambda|}$ to the squared norm relative to Y. Multiplying over embeddings gives the squared norm bound $\prod_i \rho_i^{\lambda_i}$ on the scaled array. For each exponent λ_i,

$$\rho_i^{\lambda_i} \leq \lambda_i! \exp\Big(\lambda_i(v_M + L_a^{-1}) + \rho_i e^{-v_M - L_a^{-1}}\Big).$$

Indeed it is the inequality $q^n/n! \leq e^q$ with $q = \rho_i e^{-v_M - L_a^{-1}}$. Since $(2L_a + 2)e^{-1/L_a} \leq 2L_a$, the square root of the product is bounded by

$$\exp\left(\tfrac{1}{2} \sum_i (\lambda_i / L_a + \log \lambda_i! + \lambda_i v_M) + L_a t_Y\right).$$

Let v be any unit source vector and ℓ any complex-linear target covector of dual Weyl norm one, both in the conjugated coordinates. Write $\mathcal{S}_j(Z)$ for the rectangular Schur matrix. In the jet identity take

$$Q(Z) = (\sigma^{-1}\ell)\, \mathcal{S}_j(Z)\, (\sigma^{-1} v),$$

where the inverse automorphism acts on the coordinates of the row and column. The Schur matrix has integral polynomial entries. Hence after conjugating coefficients its contraction is $\ell\,\mathcal{S}_j(Z)\,v$, evaluated on the scaled array with the permuted embedding indices. The bound just obtained applies uniformly to these contractions. Taking the supremum over v, ℓ gives the operator norm of the conjugated jet.

The jet integral uses the Gaussian $e^{-\|Y\|^2/2}$ and Lebesgue measure divided by π in each complex coordinate, so the $L_a t_Y$ term leaves e^{-t_Y}. The resulting Gaussian integral is $(2L_a + 2)^{f n_j n_{j+1}}$. The remaining character and semicharacter factors have conjugate modulus one; an automorphism of $\mathbb{C}$ can be composed with complex conjugation to fix K without changing the Weyl norms. Thus the norm is bounded by

$$\exp\left(\frac{S_j}{2} + C_j + f n_j n_{j+1} \log(2L_a + 2) + \frac{1}{2L_a} \sum_{\tau,i} \lambda_{j,\tau,i}\right).$$

For large x, the parameter bounds give $q_1, f \le x^{12}$, $n_m \le x^6$, and $\log(2L_a + 2) \le x$. Since $\lambda_{j,\tau,i} \le 2a$ and $L_a = a + 1$, the last three terms are bounded by $C_K(x^{24} + 1)^2 + x^{25} + x^{18} \le x^{50}$. This gives the bound $e^{x^{50} + S_j/2}$ on the kernel. □

7.4.5 Composition and terminal restriction

In this subsection, we combine Lemmas 7.4 and 7.7 with the integral branching of Section 7.5 to bound the full upper composition.

Proposition 7.8 (Pointwise bound after terminal restriction)**.** *Use the untwisted tensor coefficients $\Psi_\varnothing$ of Section 19.3. Fix $\sigma \in \mathrm{Aut}(\mathbb{C})$ and suppose the source form ϕ satisfies $\|\sigma\phi(g)\|_{\mathrm{W}}^2 \le e^{K_{\mathrm{in}}}$ at every finite adelic source point g. For fixed q_0, d, ϵ and sufficiently large x, the squared Weyl norm of the coefficientwise σ-conjugated restricted value at every terminal point is at most*

$$\exp\left(K_{\mathrm{in}} + A - m\mathcal{R} + 2\sum_p g_p \log p + E_0\right).$$

Proof. For the untwisted upper operator, use the class sum of the jointly invariant tensor from Lemma 22.1, then expand at those class representatives. Opening the N-gluing in Section 19.3.1 costs at most $N^{f m n_m}$ terms for each glued product. At pair i the two tests share a vector in $\prod_{w \in P} \mathbb{F}_N^{n_{2i+1}}$, with $|P| = f$, as in Section 19.4.2. Thus the number of terms before combining equal translates is at most

$$\prod_{i=0}^{m/2-1} N^{f n_{2i+1}} \le N^{f m n_m/2} \le N^{f m n_m}.$$

Each phase has absolute value one. Thus the conjugate absolute coefficient sum is at most $M^{3 f m n_m^2} N^{f m n_m} \le \exp(x^{60})$ for large x. Use the step kernel estimate $\exp(x^{50} + S_j/2)$ proved above.

The count in Section 7.2 applies to each definite stage of rank at most n_m with principal level of the chosen generators or a containing level, giving at most $\exp(O(x^{52}))$ choices per step. Let u be the conjugated value before terminal restriction. With the squared source norms

bounded by $\exp(K_{\mathrm{in}})$, the triangle inequality and the step operator bounds give

$$\|u\|_{\mathrm{W}}^2 \le \exp\left(K_{\mathrm{in}} + \sum_j S_j + 2x^{60} + 2mx^{50} + O(mx^{52}) \right).$$

Let $\mathcal{B}_{\mathrm{term}}$ be the integral branching map of Lemma 7.9. The difference of the sums of the highest weights is $d_\tau = a_\tau - mh_0$ at τ. The restriction multiplies this map by $\prod_\tau \tau(\overline{\pi_N})^{-d_\tau}$. Since $\pi_N \overline{\pi_N} = N$, at each complex embedding its squared operator norm is at most

$$e^{x^{70}} N^{-\sum_\tau d_\tau} = e^{x^{70}} N^{-|\Phi_a| a + f m h_0}.$$

Combining with Lemma 7.4 gives

$$\|\text{conjugated restricted value}\|_{\mathrm{W}}^2 \le \exp\left(K_{\mathrm{in}} + A - m\mathcal{R} + 2\sum_p g_p \log p + E_0 \right).$$

Indeed, the extra terms satisfy

$$32faB + fmh_0 \log N + 2x^{60} + 2mx^{50} + O(mx^{52}) + x^{70} \le E_0/2$$

for large x, since $\log N = O(mB)$, $a = xb$, and b exceeds every fixed power of x.

With $K_{\mathrm{in}} \le E_0$, the finite-type projection estimate gives the archimedean bound for the short images, using the measured identity (72) in its integral normalization. □

7.5 The integral terminal branching map

In this subsection, we realize the restriction at the terminal definite group by contraction in exterior powers. This gives the integral map used in Section 7.4, with its norm controlled in the Weyl module.

Lemma 7.9 (Integral terminal branching)**.** *On the terminal Weyl modules the branching map is integral, has last-block character $\prod_\tau z_\tau^{a_\tau - mh_0}$, and has squared operator norm at most $e^{x^{70}}$ in the original Weyl metrics for sufficiently large x. Here $a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ and $h_0 = r^8$.*

Proof. Fix τ, put $n = n_m$, and write λ, ν for the descending highest weights on X_m, Y_m with indices starting at one. Set $[l]_s = (l+1)/2 - s$ for $1 \le s \le l$. The upper Harish–Chandra parameter starts with $[k]$, with a_τ added to its first entry. It continues with the successive blocks $-w_i + [t+1]$, $1 \le i \le m$. The lower parameter starts with $[k-1]$, followed for each i by $-w_i + \Delta/2$ and $-w_i + [t]$. Subtract $[n]$ and $[n-1]$, respectively, to recover λ and ν. Since $[n-1]_h - [n]_{h+1} = 1/2$, comparison of the two lists gives, for $1 \le h < n$,

$$\kappa'_\tau(h) := \nu_h - \lambda_{h+1} = \begin{cases} c_* = (\Delta - t - 1)/2, & h = n_i = k + i(t+1), \quad 0 \le i < m, \\ 0, & \text{otherwise}, \end{cases}.$$

At $h = k$, the upper column count is $w_1 - (k + t + 1)/2$, at least c_* since $w_1 = \kappa_\tau + 3\Delta/2$ and $k \le \Delta$. At $h = n_i$, $1 \le i < m$, the count is $\Delta - t - 1 = 2c_*$. Thus $\lambda_h \ge \nu_h \ge \lambda_{h+1}$ throughout. At $h = 1$ all a_τ columns are contracted; at $h = n-1$ the retained count vanishes since $t+1 > 1$, so $\nu_{n-1} = \lambda_n$. Retain $\kappa'_\tau(h)$ columns of upper height h and contract $\gamma_\tau(h) = \lambda_h - \nu_h$ against

the last basis vector. At each polynomial height $1 \le h < n-1$ the lower multiplicity is $\nu_h - \nu_{h+1} = \kappa'_\tau(h) + \gamma_\tau(h+1)$.

The terminal Weyl diagram has the two sets of columns specified above: in the first set contract each column against the last unit basis vector, reducing its height by one, and in the second set retain the height. The map on the ambient product of exterior powers is the corresponding tensor of contractions and projections to the first block. Each of these maps has operator norm at most one.

After projection onto the desired Weyl summand it gives the normalized complex branching map. The target norm here is the tensor-induced Weyl norm. The adjoint of the ambient contraction wedges the shortened columns with the same last unit basis vector. The adjoint on the Weyl spaces then projects this extended tensor onto the upper Weyl subspace. Before projection its pairings are the products of the original minors by exterior Cauchy–Binet. A retained column of full first block height contributes a determinant to the pairing and is absorbed by its determinant character. This is the tensor isometry on the spanning evaluation vectors. After subtracting the common last entry $e_\tau = \nu_{n-1} = \lambda_n$, both weights are partitions. Macdonald's Schur branching formula [46, Ch. I, (5.8)–(5.9) and the one-variable specialization following (5.11)] gives each interlacing partition once. The last-variable exponent for these partitions is $|\lambda| - |\nu| - e_\tau$; restoring the factor $\det(\mathrm{diag}(g,z))^{e_\tau}$ gives $z^{|\lambda|-|\nu|}$ with the prescribed lower determinant twist. Taking the product over τ, the nonzero map spans the branching line.

The integral normalization is needed for the lattice norm comparison. In the minor polynomial model its highest test is the product of shifted leading minors in the contracted columns and unshifted leading minors in the retained ones. At upper height h the contracted minor uses the first-block covectors $2, \dots, h$ and has height $h-1$ after contraction. A retained minor uses the covectors $1, \dots, h$ and has height $h \ge k$. The only upper columns below height k are of height one and their contractions are scalars. All the nonempty minors therefore contain the same covectors $2, \dots, k$.

Permute these common covectors to the beginning, and remove them from each of the nonempty minors. If L bounds the number of boxes left, one can clear the indicated highest-vector polynomial by a positive integer $c \le L^{3L}$, with $c = 1$ for the empty diagram. Write $n_{\mathrm{low}} = n_m - 1$ and $u_+ = \max(u, 0)$. The degree of the product H of leading minors after shortening is

$$D_{\mathrm{sh}} = \sum_\tau \sum_{1 \le h < n_{\mathrm{low}}} (h-k+1)_+ \big(\kappa'_\tau(h) + \gamma_\tau(h+1)\big) = f c_* \, m\big(1 + (m-1)(t+1)\big).$$

Indeed retained columns at k leave one box each and contracted ones leave none. At $n_i > k$ the c_* retained and c_* contracted columns leave $i(t+1)+1$ and $i(t+1)$ boxes per column, respectively. The height-one columns leave none. All these degrees exclude the common determinant character. Writing $H = \sum_\alpha H_\alpha X^\alpha$, the positive clearing integer is

$$c = \langle H, H \rangle_{\mathrm{Fock}} = \sum_{|\alpha| = D_{\mathrm{sh}}} \alpha! \, |H_\alpha|^2, \qquad H_\alpha \in \mathbb{Z}, \quad \alpha! = \prod_j \alpha_j!.$$

If $D_{\mathrm{sh}} = 0$, then $H = 1$ and $c = 1$. For $1 \le D_{\mathrm{sh}} \le L$, the determinant expansion gives $\sum_\alpha |H_\alpha| \le \prod_{\mathrm{col}} h! \le L^L$. Since $\alpha! \le D_{\mathrm{sh}}! \le L^L$,

$$1 \le c \le D_{\mathrm{sh}}! \left(\sum_\alpha |H_\alpha| \right)^2 \le L^{3L}, \qquad 2 \log c \le 6 L \log L.$$

For the clearing property, the polynomial model is dual to the column-tensor Weyl model. Let G_{sh} be the Weyl Gram matrix in the integral basis of the shortened diagram, and let (P_i) be its dual coordinate polynomials on the evaluation arrays. The dual Weyl form in this basis has matrix G_{sh}^{-1}. The Fock form is invariant under the same compact unitary group, so irreducibility makes it a scalar multiple of this dual form. The product of leading minors is the functional of the highest unit exterior tensor. Its squared dual norm is one and its Fock square is c. Thus the Fock matrix is cG_{sh}^{-1}. If p is the coefficient column of an integral product of minors in the shortened standard basis, put

$$b_i = \Big\langle P_i, \sum_j p_j P_j \Big\rangle_{\mathrm{Fock}}.$$

The monomial factorial pairing gives $b \in \mathbb{Z}^{d_{\mathrm{sh}}}$, and $G_{\mathrm{sh}} b = cp$. This is the clearing property for the shortened diagram. Write J for the integral isometry of column tensors which restores the $q = k - 1$ leading basis vectors to each column that was nonempty before shortening. Let W_{sh}, W_0 be the two Weyl subspaces and Π_{sh}, Π_0 their orthogonal projections. Evaluation at $\mathrm{diag}(I_q, Z)$ gives $J \operatorname{ev}_{\mathrm{sh}}(Z) = \operatorname{ev}_0(\mathrm{diag}(I_q, Z))$ at each place, so $JW_{\mathrm{sh}} \subset W_0$. Conversely, write an evaluation array in blocks $X = \left(\begin{smallmatrix} A & D \\ C & E \end{smallmatrix}\right)$ with A of size q. If u is the number of restored columns at this place, expansion of their minors gives, for $\det A \neq 0$,

$$J^* \operatorname{ev}_0(X) = (\det A)^u \operatorname{ev}_{\mathrm{sh}}(E - CA^{-1}D).$$

Thus $J^* \operatorname{ev}_0(X) \in W_{\mathrm{sh}}$ for every X by polynomial continuation, and tensoring the identities gives $J^* W_0 \subset W_{\mathrm{sh}}$ over all places. The two inclusions give $\Pi_0 J = J \Pi_{\mathrm{sh}}$. The mixed exterior tensor representing the shifted minor product is the image under J of the shortened tensor, with the signs of the chosen permutations. Its projected vector multiplied by c is therefore integral in the ambient tensors. The integral left inverse of the standard-tableau basis matrix gives integral coordinates in the original Weyl basis.

Integral branching columns. Let $\gamma_\tau(h)$ and $\kappa'_\tau(h)$ be the numbers of contracted and retained columns, respectively, of height h in the upper terminal diagram. Write $P_j(Y)$ for the integral standard minor basis on the upper specified array, and $P_i^0(X)$ for that on the lower one. Put

$$D(X) = \prod_\tau \det(X_\tau)^{\kappa'_\tau(n_{\mathrm{low}})}, \qquad \mathcal{A} = \left\{ D(X) \sum_i (G_0 b)_i P_i^0(X) : b \in \mathbb{Z}^{d_0} \right\},$$

where n_{low} denotes the lower rank and G_0 is its rational Weyl Gram matrix in the integral tableau basis. Write $\mathcal{P}_0(b) = \sum_i (G_0 b)_i P_i^0(X)$. Then $\mathcal{A} = D(X)\mathcal{P}_0(\mathbb{Z}^{d_0})$. In the minor model $G_0 = \mathsf{F}_0^{\mathsf{t}} \mathsf{F}_0$ with F_0 the integral basis matrix, so $\mathcal{A}$ is an additive subgroup of $\mathbb{Z}[X]$.

Let $\mathcal{K}(X, Y)$ be the product of the leading minors in $\mathrm{diag}(X_\tau^{\mathsf{t}}, 1) Y_\tau$, putting its last covector first in the contracted columns and last in the retained columns, in the chosen order of the covectors. Let $c > 0$ be the clearing integer already obtained from the shortened diagram and let $\eta \in \mathbb{Z}^{d_0}$ be the corresponding vector of the shifted minor product in the lower Weyl coordinates. Straightening in the upper variable writes

$$c\,\mathcal{K}(X, Y) = \sum_j f_j(X) P_j(Y).$$

The point to prove is $f_j \in \mathcal{A}$, since coefficients in $\mathbb{Z}[X]$ alone leave the lower Weyl Gram denominator unaccounted for.

First suppose at every place that the last covector of Y_τ is $(t_\tau, 0, \ldots, 0)$, and let g_τ be its upper first square block. Laplace expansion along the last covector in each contracted minor gives t_τ per contracted column. Retained columns of full lower height contribute $\det(g_\tau)^{\kappa'_\tau(n_{\mathrm{low}})}$ in addition to $D(X)$. All other minors are those of the lower highest polynomial evaluated at $g_\tau^{\mathsf{t}} X_\tau$, with the same covector permutation as for η. The contracted minors use columns $2, \ldots, h$ of g_τ after Laplace expansion, hence covectors $2, \ldots, h$ after transposition. The retained ones use covectors $1, \ldots, h$. This gives the mixed shifted and unshifted coefficient represented by η. Hence $c\,\mathcal{K}(X, Y)$ is $D(X)\mathcal{P}_0$ of the integral polynomial vector obtained by applying the lower integral Schur matrix $\rho_0^{\mathrm{Schur}}(g)$ to η and multiplying by the displayed powers of $t_\tau, \det(g_\tau)$. The identity holds over the polynomial ring in the entries of Y.

For general Y, write $t_\tau = Y_{\tau,\mathrm{last},1}$. At each place replace column 1 by t_τ times that column, and column $j > 1$ by $t_\tau Y_{\tau,j} - Y_{\tau,\mathrm{last},\mathrm{j}} Y_{\tau,1}$. The new last covector is $(t_\tau^2, 0, \ldots, 0)$. Each leading minor of height h acquires exactly t_τ^h, since only earlier columns were added. Thus $Y^\mu\, c\mathcal{K}(X, Y)$ is $D(X)\mathcal{P}_0$ of a vector of integer polynomials in Y, with Y^μ the product of these fixed monomial factors. For every Y^δ the coefficient of $Y^{\mu+\delta}$ in this identity shows that $[Y^\delta](c\mathcal{K}) \in \mathcal{A}$. This comparison takes place over $\mathbb{Z}$ and divides no coefficient by an entry of the array.

For the integral coefficient comparison, order the variables first by column and within a column by increasing covector, using the earliest variable first in lexicographic monomial order. The leading monomial of a flag minor is its diagonal monomial with coefficient 1. The leading monomial of a standard tableau product identifies the multiset of entries in every horizontal part of the tableau, by counting the corresponding variables in each matrix column.

Within each tableau row the entries are weakly increasing, so these multisets determine the tableau. Thus the standard tableau products have pairwise distinct monic leading monomials. In any relation with coefficients in an additive group H, take a maximal leading monomial among the nonzero coefficients. Its coefficient must be zero, a contradiction. Apply this with $H = \mathbb{Z}[X]/\mathcal{A}$ to the last coefficient statement. Every f_j lies in $\mathcal{A}$, so write $f_j = D(X)\mathcal{P}_0(b_j)$ with $b_j \in \mathbb{Z}^{d_0}$.

The b_j are the columns of the branching matrix B in the original Weyl bases. With $\mathrm{ev}_+(Y) = (P_j(Y))_j$ as a column vector, its defining identity is

$$D(X)\mathcal{P}_0(B\ \mathrm{ev}_+(Y)) = c\,\mathcal{K}(X, Y).$$

At the identity lower array the right side is c times the prescribed highest coefficient. This is a product of nonzero coordinate minors in a polynomial ring over $\mathbb{Q}$, so $B \neq 0$. Substituting $Y \mapsto \mathrm{diag}(g, s)Y$ in this identity and expanding the contracted minors gives

$$B\,\rho_+^{\mathrm{Schur}}(\mathrm{diag}(g, s)) = \left(\prod_\tau s_\tau^{\sum_h \gamma_\tau(h)} \det(g_\tau)^{\kappa'_\tau(n_{\mathrm{low}})}\right) \rho_0^{\mathrm{Schur}}(g) B.$$

Indeed the first-block substitution on the kernel is $X_\tau \mapsto g_\tau^{\mathsf{t}} X_\tau$, which supplies the displayed full-height determinant.

The evaluation vectors span, and $D(X)\mathcal{P}_0$ is injective in characteristic zero, so this is an operator identity. In the present application $\kappa'_\tau(n_{\mathrm{low}}) = 0$, and the common determinant exponent is the last entry of the terminal highest weight,

$$e_\tau = \lfloor n_{m-1}/2 \rfloor - w_m(\tau), \qquad n_j = k + j(t+1).$$

Incorporating $\det(\mathrm{diag}(g, s))^{e_\tau}$ therefore replaces the power of s_τ by

$$B_\tau = e_\tau + \sum_h \gamma_\tau(h) = |\mu(X_m)_\tau| - |\mu(Y_m)_\tau|.$$

Here $|\mu|$ denotes the sum of the entries of the highest weight, including its determinant exponent in each entry. Each contracted column loses one box. Since $c_* = h_0 - (t+1)/2$, the contracted column counts at heights 1, k, and $k + i(t+1)$ for $1 \le i < m$ are respectively a_τ, $w_1(\tau) - \lfloor n_1/2 \rfloor - c_*$, and c_*. Consequently

$$B_\tau = a_\tau + w_1(\tau) - w_m(\tau) + \lfloor n_{m-1}/2 \rfloor - \lfloor n_1/2 \rfloor + (m-2)c_* = a_\tau - mh_0,$$

since $w_m(\tau) - w_1(\tau) = (m-1)2h_0$ and $t+1$ is even. For a lower evaluation vector with nonzero square-block determinants, form the extended column tensor divided by the product of its full-height determinant powers.

By exterior Cauchy–Binet the pairings of these divided tensors equal those of the lower evaluation vectors. The latter span the lower Weyl module, so this defines an isometry into the upper ambient column tensor space. The pairing identity above identifies B with c times the adjoint of this isometry composed with the upper isometric tensor inclusion. Thus $\|B\|_{\mathrm{W}}^2 \le c^2$. No Gram inverse is used in the finite-coordinate contraction.

To bound the shortened diagram, the contracted columns of upper height k have height $k-1$ after contraction, so removing the $k-1$ common covectors leaves no boxes from them. The retained columns at height k number $c_* \le \Delta$ and each leaves one box. At each height $k + i(t+1) > k$, the total multiplicity of both kinds is $\Delta - (t+1) \le \Delta$. There are at most n_m heights, and each column leaves at most n_m boxes. Thus one can take $L = f n_m^2 \Delta$. Here $f \le x^{12}$, $n_m \le x^6$, and $\Delta = 2r^8 \le 2 \cdot 3^8 x^{40}$, so $6L \log L \le x^{70}$ for large x. The last block character of the contraction is $\prod_\tau z_\tau^{B_\tau}$, where B_τ is the difference in the total box counts between the two terminal diagrams, with their prescribed determinant weight included. The integral branching operator therefore has squared norm at most $\exp(x^{70})$ in the original Weyl metrics. □

7.6 Norms and index after normalization

In this subsection, we normalize the image of the Poincaré map by a scalar common to the whole family. The estimates of Sections 7 and 7.4 then yield bounds for both the norm and the index in Proposition 5.1.

Proposition 7.10 (Normalization of length and index)**.** *Suppose the following estimates hold for the chosen arithmetic data. Let $U_g : S \to V$ be the translates of the unscaled projected Poincaré map, with the scalar of the seesaw identity removed, and d_X the dimension of the full source before the ramified restriction. By Section 6, $d_X \le 2 \dim S$.*

1. *There is a spanning set of everywhere integral forms $\phi \in S$ such that, for every $g \in G$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\|\sigma\Theta_1 U_g \phi\|^2 \le \exp\left(A - m\mathcal{R} + 2\sum_{p \in F} g_p \log p + 2E_0 + x^{99} \right).$$

2. *There are $e_p \ge 0$ supported on F, with $\sum_p e_p \log p \le E_0$. At each coefficient place above p, any finite tuple of integral forms in S can be cleared into a submodule of S by a matrix over the valuation ring with determinant valuation at most $d_X e_p$. That submodule works for all translates and all twists of level prime to p. Its images under $\Theta_\xi U_g$ have lattice norm at most $p^{e_p - g_p}$ for every such twist and translate, where the unit ball for the lattice norm is the integral lattice.*

Then the maps

$$P_g = c_B U_g, \qquad c_B = \prod_{p\in F} p^{-\lfloor g_p - e_p\rfloor}$$

satisfy

$$\|\sigma\Theta_1 P_g\phi\|^2 \le e^{A-m\mathcal{R}+L} \quad \textit{for the indicated generators.}$$

For $t \subseteq G$ and each independent tuple y of images of everywhere integral forms under P_g with $g \in t$, they give $\operatorname{Ind}(y) \le 2|t|\dim(S)E_0$. *The clearing matrices work for every twist of level prime to the coefficient prime. For every g,* $\operatorname{im} P_g = \operatorname{im} U_g$.

Proof. Normalize $v(p) = 1$. For $p \in F$, put $\varepsilon_p = g_p - e_p - \lfloor g_p - e_p\rfloor \in [0,1)$. Then

$$v_p(c_B) = e_p - g_p + \varepsilon_p, \qquad p^{-v_p(c_B)} p^{e_p - g_p} = p^{-\varepsilon_p} \le 1.$$

These formulas hold at every coefficient embedding above p, including when $g_p < e_p$. For $p \notin F$, $v_p(c_B) = 0$. Thus the same clearing matrices give integral images for every allowed twist. Choose an integral preimage and a translate for each entry. Group these entries by translate g. At $\kappa \mid p$ let $U_{\kappa,g}$ be the clearing matrix of the nonempty block at g from the second hypothesis. If P groups the columns, the clearing matrix in their original order is

$$U_\kappa = P\left(\bigoplus_g U_{\kappa,g}\right)P^{-1}, \qquad v_\kappa(\det U_\kappa) = \sum_g v_\kappa(\det U_{\kappa,g}) \le |t| d_X e_p.$$

Linearity of each $\Theta_\xi P_g$ gives integrality for every allowed twist prime to p with this same matrix. Consequently

$$\operatorname{Ind}(y) \le |t| d_X \sum_p e_p \log p \le 2|t|\dim S\, E_0.$$

At every complex embedding the change in logarithmic squared norm is

$$2\log c_B = -2\sum_{p\in F} g_p \log p + 2\sum_{p\in F}(e_p + \varepsilon_p)\log p.$$

In particular

$$\log c_B \le -\sum_{p\in F} g_p \log p + E_0 + \sum_{p\in F}\log p, \qquad \sum_{p\in F}\log p \le x_1^2 + \log N \le b.$$

The last inequality follows from $x_1 \le e^W$, $B \ge W^6$, and $\log N = O(mB)$ for large x. Since $c_B \in \mathbb{Q}_{>0}$,

$$\begin{aligned}\|\sigma\Theta_1 P_g\phi\|^2 &= c_B^2 \|\sigma\Theta_1 U_g\phi\|^2 \\ &\le \exp(A - m\mathcal{R} + 4E_0 + 2b + x^{99}) \le \exp(A - m\mathcal{R} + L).\end{aligned}$$

Here $4E_0 + 2b + x^{99} \le faB^4 = L$ for large x. The equality of the images follows from $c_B \ne 0$. □

Section 22.3 verifies the local hypothesis for the same projected map. The seesaw identifies the upper chain, and the local lattice norm estimate is preserved by the unnormalized finite-type projection and translation.

8 The normalized Poincaré map

In this section, we define the Poincaré map, specify the measures in its orbital expansion, and compute its Petersson blocks. Section 9.1 uses these blocks, and Section 17.3 compares the lifts of the map with the theta construction.

8.1 The finite coordinates

In this subsection, we use the generators of Sections 4.1 and 4.2 to choose the finite point and identify the prescribed level in lattice coordinates.

Let G_{ad} be the ambient unitary group, and write $[G_{\mathrm{ad}}]$ and $[R]$ for the rational quotients. The following choices fix the covector at each scale of Section 4.

For the finite scale, let $D_{\mathrm{rel}} = \Phi'_{q_1}(\zeta_{q_1})$ and $\omega_K = (1+\sqrt{-d})/2$. Define the finite idele α_{W} to have component D_{rel}^{-1} at places dividing ω_K, and 1 elsewhere. Put $j_{\mathrm{W}} = -1/\sqrt{-d}$. Every prime of q_1 divides $(d+1)/4 = \omega_K\bar{\omega}_K$, by the fixed congruence on d, and D_{rel} divides q_1 by differentiating $Z^{q_1}-1$ at ζ_{q_1}. The two ideals of $\omega_K, \bar{\omega}_K$ are coprime, since their generators sum to one.

Conjugation preserves the different ideal. Thus, place by place,

$$j_{\mathrm{W}}\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1}\widehat{\mathcal{O}}_E.$$

Let c_M be the finite idele equal to the chosen generator $\bar{\alpha}$ at its chosen places and to one elsewhere. For $a_{\mathrm{f}} = \alpha_{\mathrm{W}}/\bar{c}_M$ set

$$u = (a_{\mathrm{f}}/\bar{a}_{\mathrm{f}})^{m/2}, \qquad \eta = u^{-1}I_{X_0}.$$

This is a unitary finite scalar with the required grading.

With $\varrho = m/2-1$, the rational form Y_0 has positive definite $X_0 = E^k$ and the ordered string

$$e_0, \dots, e_{\varrho-1}, z, e^*_{\varrho-1}, \dots, e^*_0.$$

The pairing of each e_a, e^*_a is one, the norm of z is $-N$, and the remaining string pairings vanish. Take the generic shift along the ordered string with constant $c_{\mathrm{gen}} = -(j_{\mathrm{W}}M\delta_E)^{-1}$. It sends z to $-Nc_{\mathrm{gen}}e^*_{\varrho-1}$ and the other nonfinal string vectors to c_{gen} times their successors, and it kills the final vector and X_0. The constant belongs to $F_0^\times$ since both j_{W} and δ_E are skew.

In the finite coordinates let K_L be the integral group, J_L the kernel of its reduction, and $H = U(X_0)$ the definite factor of $R = U \rtimes H$, and let $J_H = \iota_H^{-1}(J_L)$. We first compute the finite isometry from the original lattice. Write L for its orbital basis matrix, with the distinguished column of X_0 replaced by

$$v_0 = x_0/\bar{\pi}_N + z/N, \qquad N = \pi_N\bar{\pi}_N.$$

On (v_0, z) the Gram matrix is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$. The other X_0-columns and the hyperbolic pairs retain their integral unimodular Gram matrices. Hence L is a basis of an integral unimodular lattice, including at the primes over N.

This basis also identifies the pullback level on H. In the decomposition of X_0 into the other columns and x_0, write an element of H as $h = \begin{pmatrix} A & b_1 \\ c_1 & d_1 \end{pmatrix}$. On the corresponding glued middle

block its matrix is

$$\iota_L(h) = \begin{pmatrix} A & b_1/\bar{\pi}_N & 0 \\ \bar{\pi}_N c_1 & d_1 & 0 \\ -c_1/\pi_N & (1-d_1)/N & 1 \end{pmatrix}.$$

Indeed $x_0 = \bar{\pi}_N v_0 - z/\pi_N$. At $w \mid \pi_N$, the glued matrix and its inverse are integral exactly when $h_w \in \mathrm{GL}_k(\mathcal{O}_{E_w})$ and $(c_1, d_1) \equiv (0,1) \bmod \pi_N$. Indeed $\bar{\pi}_N$ is a unit here, and the last row imposes the two congruences. At the conjugate place $h_{\bar{w}} = \overline{h_w}^{-\mathsf{T}}$, so the row congruence makes the last column of $h_{\bar{w}} - I$ zero modulo $\bar{\pi}_N$. The last diagonal difference is thus divisible by both components of N. The conjugate glued matrix is the inverse adjoint for the unimodular glued Gram matrix, hence integral. At the primes above M, the basis change is integral and invertible since N is a unit. Thus the inverse image of the original principal congruence level is the level J_H of Section 6.

Write ι_{rat} and $\iota_L = L^{-1}\iota_{\mathrm{rat}}L$ for the definite embedding in rational and orbital coordinates. In the graded basis the string columns are multiplied by u. The glued column becomes $x_0/\bar{\pi}_N + uz/N$. Thus the graded basis matrix is $u\iota_{\mathrm{rat}}(\eta)L$. It has the same Gram matrix as L. Choose an integral change to an orthonormal basis locally. At split pairs a basis on one component and its Gram dual on the other give such a change. At an unramified nonsplit place a unit-norm line exists: if every residue norm were zero, polarization with all residue scalars would make the form zero by the nondegenerate residue trace pairing. Lift the line, use surjectivity of the local unit norm to scale its norm to one, and split off its unimodular complement. This gives the change in every rank, also in residue characteristic two.

Right multiplication by a standard integral unitary matrix preserves the orthonormal Gram matrix. At each chosen split pair choose its first component to correct the reduction of the change of basis to the identity at the chosen place. The second component is forced by unitarity, and reduction is prescribed only at the chosen place of the pair. The integral general linear reduction there is surjective. Thus the orthonormal finite point of this covector can be chosen as

$$x_f = u\,\iota_{\mathrm{rat}}(\eta)L\gamma_0, \qquad \gamma_0 \in \mathrm{GL}_r(\widehat{\mathcal{O}}_E), \qquad \gamma_0 \equiv I \text{ at each chosen place.}$$

As part of fixing the original space, choose this x_f before forming any of its period maps. In its standard coordinates define

$$\begin{aligned} \mathfrak{f}(k) &= \iota_L(\eta)^{-1}L^{-1}x_f k x_f^{-1}L\iota_L(\eta) \\ &= \gamma_0 k \gamma_0^{-1}. \end{aligned}$$

This equivalence preserves integrality and the chosen principal reductions, so $\mathfrak{f}(J) = J_L$. Write J_0 for the rational Gram matrix and $J_w = (L^* J_0 L)_w \bmod w$ at a chosen split place $w \mid M$. Since $\gamma_{0,w} \equiv I$, the Gram identity $\gamma_0^*(L^*J_0L)\gamma_0 = I$ gives

$$\overline{\gamma_{0,\bar{w}}}^{\mathsf{T}} J_w = I \mod w.$$

Thus the opposite component gives the dual basis for the same Gram matrix. The definite embedding reads exactly ι_L under $\mathfrak{f}$, since η is central in that definite factor. If a rational term is read as $\gamma_{\mathrm{std}} = x_f^{-1}\gamma x_f$ and $\gamma_L = L^{-1}\gamma L$, then $\mathfrak{f}(\gamma_{\mathrm{std}}) = \eta_H^{-1}\gamma_L\eta_H$ with $\eta_H = \iota_L(\eta)$. For a representative c° of $H(F_0)\backslash H(\mathbb{A}_{F_0,f})/J_H$, the standard class slice a_c embeds $c^\circ\eta^{-1}$. Its image is $\eta_H^{-1}h_c$. Write

$$h_c^{\mathrm{rat}} = \iota_{\mathrm{rat}}(c^\circ), \qquad h_c = \iota_L(c^\circ) = L^{-1}h_c^{\mathrm{rat}}L.$$

These comparisons hold before choosing any translate.

8.2 Class slices and quotient measure

In this subsection, we decompose the rational quotient using the finite coordinates of Section 8.1.

In the chosen lattice basis $A \oplus (X_0 \oplus Ez) \oplus A^*$, the matrix of a Bessel element $u\iota_H(a)$ on the entire middle diagonal block is that of $\iota_H(a)$. Indeed u induces the identity there, also after conjugation by the middle change of basis used to glue the last column. On the two outer blocks $\iota_H(a)$ is the identity. Apply this to the element and its inverse. Integrality and congruence to one on the glued middle block then hold for the embedded Levi part, with its full z-covector included. Removing that part gives $u \in U \cap J_L$. Transporting through the change of basis gives $J_R = (U(\mathbb{A}_{F_0,f}) \cap J) \rtimes J_H$.

For $n\iota_H(a) \in R(\mathbb{A}_{F_0,f})$, write $a\eta = \delta c^\circ j_H$, with $\delta \in H(F_0)$ and $j_H \in J_H$. Choose $u \in U(F_0)$ with $\iota_H(a)^{-1}u^{-1}n\iota_H(a) \in U \cap J$. The condition on u is open about n in the finite adeles. The polynomial exponential coordinates of U and additive approximation over F_0 give such a rational point. The rational element $\iota_H(\delta^{-1})u^{-1}$ moves the finite point to $a_c J_R$, since η is central in the Levi.

Conversely any identification between two such slices identifies their definite classes by taking the Levi part. At the same class the rational Levi part is one by freeness of the definite level. The remaining stabilizer is the original U_c. Therefore

$$[R]/J_R = \coprod_c U_c \backslash R_\infty .$$

Each unipotent quotient is compact by the covector boxes of Section 13.6. The real Levi is compact and there are finitely many classes.

The left U_c-translates of the reduced cutoff β_c sum to one by Section 14.1. Write v_R for the finite volume of J_R including the positive scalar of the Bessel Haar decomposition. With counting measure on the rational group, write the Haar measure inducing the Petersson measure as $(\prod_{v|\infty} dg_v)\, dg_f$. Here dg_v has density $1/\operatorname{vol} U(r)$ at the identity in the skew-Hermitian coordinates $S = JX$ of Section 12.5. This determines dg_f before the Poincaré map is formed. If $K_{G,f}$ is the full finite lattice stabilizer, reduction at the chosen split places gives $K_{G,f}/J = G$, whence

$$v_G = \operatorname{vol}_{dg_f}(J) = \frac{\operatorname{vol}_{dg_f}(K_{G,f})}{|G|}.$$

The slice formula for every integrable right J_R-invariant Q is

$$\int_{[R]} Q = v_R \sum_c \int_{R_\infty} \beta_c(s) Q(s, a_c)\, ds.$$

8.3 The real point and stationary translate

In this subsection, we fix the real chart and place the stationary point of Section 12.10 in that chart.

Use the signed scales of Section 15.7. At v let J_0 be the rational Gram matrix and let J_1 replace its entry $-N$ by -1. Take D_v diagonal with entries $\kappa_v^{\varrho-a}/\sqrt{N}$ on e_a, their reciprocals on e_a^*, $N^{-1/2}$ on z, and one on X_0. Then $D_v^* J_0 D_v = J_1$. Let p_v be the Euclidean unitary matrix whose columns are the normalized sums and differences of the hyperbolic pairs, the basis of X_0, and z, ordered by the prescribed signs. Choose the real components x_v of the adelic point by

$$x_v = D_v p_v, \qquad x^{\mathrm{pt}} = ((x_v)_{v|\infty}, x_f), \qquad x_v^* J_0 x_v = p_v^* J_1 p_v = \varepsilon_v .$$

Here x_f is the corrected finite point chosen above, with the same reduction and principal level at the prescribed places. The real chart satisfies

$$\iota_v(g) = p_v g p_v^{-1}, \qquad k_v = D_v^{-1} x_v p_v^{-1} = 1, \qquad T_v(g) = \iota_v(g).$$

The isomorphism ι_v preserves the specified Lie algebra inner products and Haar. The rational matrix γ acts in the J_1-coordinates by $D_v^{-1}\Phi_v(\gamma)D_v$.

Let $x_v^\star$ be the stationary point of Section 12.10. Its point in the standard signed coordinates and its adelic realization are

$$g_{0,v} = \iota_v^{-1}(x_v^\star), \qquad g_{0,\infty} = (g_{0,v})_v, \qquad g_0 = (g_{0,\infty}, 1).$$

In rational coordinates $x_v g_{0,v} = D_v x_v^\star p_v$. The Bessel coefficient is evaluated at

$$(x_v^\star)^{-1}\iota_v(s_v)T_v(g_v) = \iota_v(g_{0,v}^{-1} s_v g_v).$$

The character Ψ on R is trivial on the Levi and is the adelic character at $-\sigma/(M\delta_E)$ on U. Here σ is the simple-root sum and δ_E the chosen generator of the different. In the grading the simple-shift coefficient c_{gen} and the Weil scalar j_{W} satisfy $j_{\mathrm{W}} c_{\mathrm{gen}} = -(M\delta_E)^{-1}$. The trace of the shift against a Bessel unipotent has two contributions at each adjacent pair of basis positions.

The dual entry is the conjugate of the corresponding entry of the inverse unipotent, by unitarity. The first superdiagonal sum of the inverse is $-\sigma$, so the trace is $c_{\mathrm{gen}}(\sigma - \bar\sigma)$. Since the product $j_{\mathrm{W}} c_{\mathrm{gen}}$ is purely imaginary, the Weil argument $j_{\mathrm{W}} c_{\mathrm{gen}}(\sigma - \bar\sigma)/2$ has the same real trace as $j_{\mathrm{W}} c_{\mathrm{gen}}\sigma$. This gives the character of the original graded period.

Put $z_u = -\sigma(u)/(M\delta_E)$. In the signed chart the simple-root entry is $\alpha_{\Psi,v} = \sigma(u)_v/\kappa_v = -2\pi i z_{u,v}$, where $z_{u,v}$ is the archimedean component. Thus

$$\exp(\alpha_{\Psi,v} - \bar\alpha_{\Psi,v}) = \exp(-4\pi i \Re z_{u,v}) = \Psi_v(u), \qquad \prod_v \exp(-\alpha_{\Psi,v} + \bar\alpha_{\Psi,v}) = \Psi_\infty(u)^{-1}.$$

The first character gives the real Bessel covariance. Its inverse product occurs in the defining average. At the finite places the vectors e_a, z are integral in the glued basis, and the top coordinate functionals are integral because the glued column has no top component. Thus the simple-root coordinates of a level unipotent are integral everywhere and are in $M\widehat{\mathcal{O}}_E$ at the places over M: the principal congruence in the chosen half gives the conjugate congruence by unitarity and the unimodular Gram matrix. The common string scaling in x_f leaves these entries unchanged. Consequently $z_u \in \mathfrak{D}_{E/\mathbb{Q}}^{-1}\widehat{\mathcal{O}}_E$, and Ψ_f is trivial on J_R. For $F = (e_0, \dots, e_{\varrho-1}, z)$, each simple-root entry reads the e_i-coordinate of the image of the next column of F. The glued matrix h_c fixes these columns and preserves the e_i-coordinate functionals. Thus the character is also trivial on the class conjugate of this unipotent level. For a rational $u \in U_c$ it follows that $\Psi_\infty(u) = \Psi_f(u)^{-1} = 1$.

At v, take $w_v = v_0$ as in Section 12.10. Let I_v be the local value space and $\deg_v$ the formal degree. For the average A_v of Section 12.4, put $a_v = \sqrt{\dim I_v}/\|A_v\|_2 > 0$. Use the reproducing kernel $e = \prod_v \deg_v \bar c_v$, with c_v the normalized coefficient in the fixed real coordinates. Write C_{W} for the Weyl-to-Bessel isomorphism in conjugate coordinates. For the unitary Levi action $\pi_H = B_0 S B_0^{-1}$ in Weyl coordinates use $C_{\mathrm{W}} = \overline{B_0}$ on the conjugate coordinates. In the local

tableau basis put $Q_v = B_{0,v}^{*} B_{0,v}$ and let ξ_v be the highest monomial of Section 12.10, including the constant monomial for the empty diagram. Since Q_v is real and $w_v = B_{0,v}\xi_v / \|B_{0,v}\xi_v\|$,

$$C_{\mathrm{W},v}^{*} C_{\mathrm{W},v} = \overline{Q_v} = Q_v, \qquad (C_{\mathrm{W},v} z_v)^{\mathsf{T}} w_v = \frac{z_v^{\mathsf{T}} Q_v \xi_v}{(\xi_v^{*} Q_v \xi_v)^{1/2}}.$$

Thus the local change is an isometry for the Weyl metric, and the period evaluates the displayed complex-linear covector.

Unitarity gives

$$(C_{\mathrm{W}} \rho_{\mathrm{conj}}(h) z)^{\mathsf{T}} = (C_{\mathrm{W}} z)^{\mathsf{T}} \pi_H(h)^{-1}.$$

At each real place the chart sends $u \iota_H(h)$ to the graph of its Levi component. The inverse reads the ambient group point from the first coordinate and the Levi action from the prescribed quotient of the flag. These maps are continuous group isomorphisms. Thus the product of the local graph measures of Section 12.4 pulls back to $C_\infty ds$, with $0 < C_\infty < \infty$ by uniqueness of Haar measure. Set

$$w = \bigotimes_v w_v, \qquad t_0(z) = C_\infty \frac{\prod_v a_v}{\prod_v \deg_v} (C_{\mathrm{W}} z)^{\mathsf{T}} w.$$

The factors of the product average at $s = u \iota_H(h)$ are $\overline{e(g_{0,\infty}^{-1} s x)} / \prod_v \deg_v$, the wave $\Psi(s)^{-1}$, and $\pi_H(h)^{-1} w$, besides the positive factors a_v. Tensor the absolutely integrable local averages and invert s in the unimodular Bessel group. For $r = u \iota_H(h)$ put

$$W_{t,\phi}(r) = \Psi(r) t(\rho_{\mathrm{conj}}(h_\infty^{-1}) \phi(h_f)).$$

The evaluation of $R_H(\eta)\psi$ at a_c is $\psi(c^\circ)$ and $\Psi(a_c) = 1$. After inversion, the original full Bessel vector therefore satisfies

$$(C_{\mathrm{W}} \psi(c^\circ))^{\mathsf{T}} W(x) = \int_{R_\infty} \overline{e(g_{0,\infty}^{-1} s^{-1} x)} W_{t_0, R_H(\eta)\psi}(s, a_c)\, ds.$$

8.4 The measured period and change of level

In this subsection, we use the point of Section 8.3 and the measure of Section 8.2 to identify the measured pairing.

With $z_{c,i} = (C_{\mathrm{W}} \psi(c^\circ))_i$ and the seeds of Section 8.5, define

$$\mathcal{P}_{\mathrm{ser}}(\psi)(x) = \sum_{c,i} z_{c,i} \sum_{\gamma \in G_{\mathrm{ad}}(F_0)} f_{c,i}(\gamma x).$$

The inner sum uses counting measure on the rational group. The cutoff in $f_{c,i}$ accounts for its U_c-translates. At a fixed finite level the dual real test space is the image of right convolution by $\bar{e}$. Choose a bounded representative of Y on that finite-level quotient and put

$$Y^\circ(x) = \int_{G_\infty} \bar{e}(a) Y(xa)\, da, \qquad |Y^\circ(xh) - Y^\circ(x)| \le \|Y\|_\infty \|\bar{e}(h^{-1}\,\cdot) - \bar{e}\|_1 \quad (h \in G_\infty).$$

Each finite-level real slice is a quotient of G_∞ by a discrete subgroup. Pullback along its orbit map takes null sets to null sets, so the integral is independent of the bounded representative. Translation continuity in L^1 gives continuity on each slice. The reproducing identity of Section 12.3 gives $Y^\circ = Y$ almost everywhere. Use Y° for the period evaluations and write it as

Y. For a principal-level test the absolute periodization bound permits unfolding at the chosen level and gives

$$\int_{[G_{\mathrm{ad}}]} Y\,\mathcal{P}_{\mathrm{ser}}(\psi) = v_G \sum_c \iint \beta_c(x) Y(x,a_c)\overline{e(g_{0,\infty}^{-1}s^{-1}x)} W_{t_0,R_H(\eta)\psi}(s,a_c)\,ds\,dx.$$

Both Y and the period coefficient are invariant under the respective right levels on this class. Rational invariance shows that the double integrand apart from β_c, denoted $\mathcal{K}(s,x)$, satisfies $\mathcal{K}(us,ux) = \mathcal{K}(s,x)$ for $u \in U_c$. The factor involving e depends on $s^{-1}x$.

The integral of $\beta_c(s)|\mathcal{K}(s,x)|$ is finite by the boundedness of Y, the L^1 bound on e and the integrability of the Bessel coefficient on its compact quotient. Insert $\sum_u \beta_c(ux) = 1$ in this integral and change variables $(s,x) \mapsto (us,ux)$ for each u. Summing the translates of $\beta_c(s)$ proves finiteness with the cutoff $\beta_c(x)$ by Tonelli. The same change of variables and absolute convergence now give

$$\iint \beta_c(x)\mathcal{K}(s,x)\,ds\,dx = \iint \beta_c(s)\mathcal{K}(s,x)\,ds\,dx.$$

The convolution in x reproduces $Y(sg_{0,\infty},a_c)$. Using the Bessel Haar decomposition, the resulting pairing is $\int_{[R]} W_{t,R_H(\eta)\psi}(s)Y(sg_0)\,ds$ with $t = (v_G/v_R)t_0$. For the original source form ϕ take $\psi = R_H(\eta^{-1})\phi$. Let $Z = R(\bar{e})Z \in L^2$ have the prescribed dual real type and principal level. There is a compactly supported smooth kernel ξ on the real group such that $\xi * \bar{e} = \bar{e}$. Indeed, in the conjugate discrete series use the compact matrix unit projecting onto the distinguished line of the multiplicity-one lowest compact type, as in Section 12.11. Convolve a real symmetric approximate identity on both sides with this self-adjoint matrix unit. The resulting smooth kernel acts on the distinguished unit vector by a real scalar tending to one. Normalize by this scalar and take the product over the real places. The resulting kernel is self-adjoint and fixes the conjugate unit vector. The matrix coefficient formula for $\bar{e}$ then gives the asserted convolution identity.

Put $\mathcal{C} = \{rg_0 : r \in [R]\}$ in the quotient at level J. It is compact by Section 8.2. On each of its finitely many real components, lift its intersection with $\mathcal{C}$ to a compact set L in the real group. For the discrete arithmetic group of this component, only finitely many translates of $L\,\mathrm{supp}\,\xi$ meet this same compact set. Cauchy–Schwarz in the convolution, with this bound on the multiplicity, gives

$$\sup_{x\in\mathcal{C}} |R(\xi)F(x)| \le C_{\mathcal{C},J,\xi}\|F\|_2.$$

The smooth convolution gives a continuous representative of $Z = R(\xi)Z$. For $Z_n = Z\mathbf{1}_{\{|Z|\le n\}}$, the bounded tests $R(\bar{e})Z_n$ converge to Z in L^2. Each is fixed by $R(\xi)$, so convergence is also uniform on $\mathcal{C}$. The weight $W_{t,\phi}$ is bounded on the compact Bessel quotient. Thus its integrals against these tests converge to the integral against $Z(rg_0)$. Square-integrability of the series below gives the same limit in the Petersson pairing, identifying its representative for the original g_0, t, ϕ.

For a bounded test Z of the dual real type which is right K'-invariant, with K' a finite open level, let $N_J = [J : J \cap K']$. Its average over the representatives γ of $J/(J\cap K')$ is J-invariant. Right invariance of $\Pi\phi$ under J, and right translation in the Petersson integral, give

$$\begin{aligned} N_J \int_{[G_{\mathrm{ad}}]} Z(x)(\Pi\phi)(x)\,dx &= \int_{[G_{\mathrm{ad}}]} \sum_{\gamma\in J/(J\cap K')} Z(x\gamma)(\Pi\phi)(x)\,dx \\ &= \sum_{\gamma\in J/(J\cap K')} \int_{[R]} W_{t,\phi}(r)Z(rg_0\gamma)\,dr. \end{aligned}$$

In each sum use one representative of each coset. The second line is the measured period applied to the summed test. This formula keeps the factor N_J when the first covector of the lower chain changes level.

8.5 Petersson blocks and covariance

In this subsection, we unfold the pairing of Section 8.4 to obtain the blocks used in Section 9.1.

Expand the series in the source class basis, using the quotient Haar measure of Section 8.2 and the cutoffs of Section 14.1.

Let $C_X\psi = (C_{\mathrm{W}}\psi(c^\circ))_c$ and $C = C_X R_H(\eta^{-1})$. By the local Weyl isometry above and the trivial stabilizers of Lemma 6.1, the class squared norm is $\sum_c \|(C_X\psi)_c\|^2$. The central translation permutes the classes, with unitary changes of real frame at the representatives. Thus C preserves this norm, also on the ramified subspace. The series for a coordinate basis vector in C is the periodization of

$$f_{c,i}(y, x_f) = \beta_c(y) W_i(y) \mathbf{1}_{a_c J}(x_f).$$

Fix (c, i) and identify rational elements using the isometries of Section 8.1. If no rational translate of x_f lies in $a_c J$, the series is zero at x. Otherwise choose δ_0 with $\delta_0 x_f \in a_c J$. For every other contributing δ, the element $\gamma = \delta_0\delta^{-1}$ belongs to $\mathcal{T}(c, 1, c)$: it takes δx_f to $\delta_0 x_f$ in the same slice. Writing y for the real point of $\delta_0 x$, the other real argument is $\gamma^{-1}y$. The bound $\|W(z)\| \le A_* L(z)^{-I}$ and (35) therefore give

$$\sum_\delta |f_{c,i}(\delta x)| \le D_*, \qquad D_* = 2A_* b^{12n^3 rf},$$

uniformly in x, where A_* is the product of the local majorant bounds.

Index the left U_c-orbits of the rational terms by $q = U_c\delta$, and set $F_q(x) = \sum_{u\in U_c} f_{c,i}(u\delta x)$. The finite support is U_c-invariant. On that support, the real character is trivial on U_c and the translates of β_c sum to one. Hence $F_q(x)$ is the full coordinate W_i at the real point of δx, multiplied by $\mathbf{1}_{a_c J}(\delta x_f)$. Each F_q is reproduced by right convolution with the real kernel e. Since $\sum_q |F_q(x)| \le D_*$, the sum of the absolute convolution integrands has integral at most $D_*\|e\|_1$. Thus right convolution also reproduces the series.

Thus the series has that type and is continuous by continuity of translations of the kernel in L^1. Being bounded on the finite-volume quotient, it belongs to the full Petersson space on the prescribed level. Absolute unfolding against bounded tests and the measured pairing of Section 8.4 identify the series with $\Pi C^{-1} e_{c,i}$ for the original measured map Π.

Put $B = (v_G\nu)^{-1/2}\Pi C^{-1}$ and $B_s = B^*\rho(s)B$. Use the Petersson product linear in the second variable. To pair $\rho(s)\Pi C^{-1}e_{c',i'}$ with $\Pi C^{-1}e_{c,i}$, unfold the seed in the second variable. An integral lift k_s of s normalizes J, so the finite integral over $a_c J$ gives v_G. A term indexed by γ^{-1} in the first variable uses

$$\mathfrak{f}(a_{c'}^{-1}\gamma_{\mathrm{std}}^{-1}a_c k_s) = h_{c'}^{-1}\gamma_L^{-1}h_c\mathfrak{f}(k_s).$$

Membership in J_L is exactly $h_c^{-1}\gamma_L h_{c'} \in \mathfrak{f}(k_s)J_L$. Thus the surviving integrand is the full kernel with both cutoffs of $\mathcal{T}(c, s, c')$, with the residue s. The real isometry preserves Haar: its signed-basis isomorphism preserves the specified Lie algebra density and the remaining change is a fixed left translation. Writing A^{full} for the full amplitude of Section 14.1, unfolding gives

$$\langle \rho(s)\Pi C^{-1}e_{c',i'}, \Pi C^{-1}e_{c,i}\rangle = v_G A^{\mathrm{full}}(c, s, c')_{i,i'}.$$

Both variables acquire the factor $(v_G\nu)^{-1/2}$.

Proposition 8.1 (Full Petersson blocks)**.** *In the Euclidean source coordinates, the normalized Petersson blocks are given by the full orbital sums*

$$(B_s)_{(c,i),(c',i')} = \nu^{-1}\overline{A^{\mathrm{full}}(c,s,c')_{i,i'}}. \tag{1}$$

Let H_R be the residue image of the integral Bessel group. For $t \in H_R$ write a lift in the integral Bessel group in glued coordinates as $u_L\iota_L(a)$. On the definite group write $c^{\circ}a = h\,(c')^{\circ}j_H$ with $h \in H(F_0)$ and $j_H \in J_H$. Additive approximation in the unipotent coordinates gives $\xi \in U(F_0)$ with $h_c^{-1}\xi_L h_c \in u_L J_L$, where $\xi_L = L^{-1}\xi_f L$. For the rational isometry $\delta = \xi\iota_{\mathrm{rat}}(h)$ this gives

$$h_c^{-1}\delta_L h_{c'} = (h_c^{-1}\xi_L h_c)\iota_L(a)\iota_L(j_H)^{-1} \in u_L\iota_L(a)J_L,$$

since the integral $\iota_L(a)$ normalizes J_L. Thus δ has parameter (c,t,c'). Every rational Bessel term has residue in H_R. Two such terms over the same left class and residue differ on the right by a rational Bessel term of identity residue. By the Levi test their right classes coincide, and all terms over (c,t,c') in R are $\delta U_{c'}$.

Write $W(\delta y) = M_\delta W(y)$; the matrix M_δ is unitary by covariance of the local averages. The right cutoffs for $\delta U_{c'}$ sum to one and their W-values agree. The left integral is

$$\int \beta_c(y)W(y)W(y)^*\,dy = \nu I$$

Indeed WW^* is invariant under left real unipotents, and $\int_{U_\infty}\beta_c(uy)\,du = \nu$ by (32) and its cutoff formula. In the real charts the group and unipotent Haar measures are the products of the local orbital measures dg_τ, du_τ. Their quotient is therefore $\prod_\tau d\bar{g}_\tau$. Since $W = \bigotimes_\tau W_\tau$ and $\int \|W_\tau\|^2\,d\bar{g}_\tau = \dim I_\tau$, Fubini and Section 12.10 give

$$\int_{U_\infty\backslash G_\infty} WW^*\,d\bar{g} = \bigotimes_\tau \int W_\tau W_\tau^*\,d\bar{g}_\tau = 1_{\bigotimes_\tau I_\tau}.$$

Since $W(\delta^{-1}y) = M_\delta^* W(y)$, the Bessel tensor is $W(y)W(y)^*M_\delta$ and its amplitude is νM_δ. The normalized conjugate blocks give a unitary representation $r_\psi(t)$ on the source: composition and inversion of the rational transporters multiply and invert the class moves and matrices. Extend r_ψ by zero off H_R. Equation (1) identifies $E_s = B_s - r_\psi(s)$ with the normalized conjugate off-Bessel amplitudes.

To prove covariance, choose δ over (d,t^{-1},d'). Multiplication $\gamma \mapsto \gamma\delta^{-1}$ bijects the terms over (c,s,d') and (c,st,d). The resulting right cutoff $\beta_d(\delta\,\cdot)$ has orbit sum one for $U_{d'}$ because $\delta U_{d'}\delta^{-1} = U_d$ by the level condition. The right kernel is multiplied by the factor M_δ^*. Changing the cutoff by the orbit partition and using absolute convergence gives

$$A^{\mathrm{full}}(c,st,d) = A^{\mathrm{full}}(c,s,d')M_\delta^*.$$

By (1) the last matrix factor after normalization is $\overline{M_\delta}^*$. This is the block from d to d' of $r_\psi(t)$, since $\overline{M_\delta}$ is the block from d' to d of $r_\psi(t^{-1})$. Thus $B_{st} = B_s r_\psi(t)$. Taking adjoints and using $B_s^* = B_{s^{-1}}$ gives $B_{ts} = r_\psi(t)B_s$. These identities also hold for E_s. They imply $\rho(t)B = Br_\psi(t)$: for any x, the difference $\rho(t)Bx - Br_\psi(t)x$ belongs to the span of the vectors $\rho(s)By$ and is orthogonal to each of them.

Since finite translation by η^{-1} commutes with the scalar actions, it preserves the ramified source. The ranges of the projected measured translates on that source are the ranges of $p\rho(s)B$ on its subspace in C-coordinates, with $p = \rho(h_0 \otimes h_1)$. The factors in the algebraic projection onto the selected types and the map normalization are nonzero and common to the translates. More explicitly, write $e_0 = h_0 \otimes h_1 = \sum_s u_s[s]$ in the finite group algebra and $e_0^\vee = \sum_s u_s[s^{-1}]$. For a principal-level test Y of the dual real type, invariance of the quotient measure and the measured pairing give

$$\int_{[G_{\mathrm{ad}}]} Y(x)\,(p\rho(g)\Pi\phi)(x)\,dx = \int_{[R]} W_{t,\phi}(r)(\rho(g^{-1})\rho(e_0^\vee)Y)(rg_0)\,dr.$$

Thus the bilinear test uses the dual finite types. For the Hermitian Petersson product the right translations satisfy $\rho(s)^* = \rho(s^{-1})$, by invariance of the quotient measure. The coefficients of each character idempotent satisfy $u_{s^{-1}} = \overline{u_s}$ because a finite group representation is unitarizable. Hence

$$p^* = \sum_s \overline{u_s}\,\rho(s^{-1}) = \sum_s u_s\rho(s) = p.$$

Together with $p^2 = p$, this identifies p as the orthogonal projection on the original target.

Part II Orbital integrals and the translated span

9 Orbital integrals and operator norms

In this section, we relate the Petersson matrices of Section 8 to two operator moments. Their bounds give the hypotheses of the translation argument in Section 16.2.

Write T_{full} for the full Petersson target and $B : X \to T_{\mathrm{full}}$ for the geometrically normalized map on the full source with its Euclidean class coordinates. The chosen real matrix coefficient selects one line in each automorphic occurrence. On the rank-k definite source, each of its d_C classes contributes $d_B = \prod_v \dim I_v$, so $d_X = d_C d_B$. On the rank-r definite inner form used to count the target, each free class contributes the compact representation of dimension $\mathcal{B}$, as computed in Section 9.4. Let ρ be the unitary action of $G = G_0 \times G_1$ on T_{full}. Let $p_i = \rho(h_i)$ for the central idempotents of the selected types $h_i \in \mathbb{C}[G_i]$, and $p = p_0 p_1$. Put $d_X = \dim X$, $D = \dim pT_{\mathrm{full}}$, and $B_s = B^*\rho(s)B$. The following argument concerns the same normalized map for both its operator estimates.

9.1 From truncated orbital integrals to full orbital integrals

In this subsection, we compare the complete Petersson matrices with their truncated orbital integrals. The estimate of Equation (36) controls the difference. Section 15.8 bounds pairs of distinct links, and (34) bounds the equal-link term on the open set with the normalization of (33).

Let H_R be the residue Bessel subgroup and r_ψ its unitary action on X, extended by zero to G. Write $E_s = B_s - r_\psi(s)$, $D_{\mathrm{id}} = |H_R|d_X$ and

$$D_{\mathrm{main}} = \frac{d_X^2}{\mathcal{B}\ \mathrm{mass}(E, r)}, \qquad C_{\mathcal{A}} = b^{-6k|\mathcal{A}|} D_{\mathrm{main}},$$

where $\mathcal{B}$ is the product of the formal degrees. Let $\mathcal{A}$ be a subset of the f chosen places at p_0^{lev}. At each place of $\mathcal{A}$ choose any nonzero vector in the Bessel span, or any nonzero covector in its

opposite-place realization. Let $\Gamma_{\mathcal{A}}$ be the intersection of these stabilizers. Put $q = p_0^{\mathrm{lev}}$. The split reduction onto $\mathrm{GL}_r(\mathbb{F}_q)$ is surjective. Each nonzero vector or covector has orbit size $q^r - 1$ under this action, so its stabilizer has that index. At a chosen place $w \mid p_0^{\mathrm{lev}}$, write g_w for the component of s, J_w for the Gram matrix in the original glued lattice basis of Section 8.1, and E_j for the matrix of the first j columns of $(e_0, \dots, e_{\varrho-1}, z)$ in that basis. Set

$$\mathcal{O}_w = \{s \in G : \det(E_j^t J_w g_w E_j) \neq 0 \text{ for every } 1 \le j \le \varrho\}.$$

The Gram matrix is symmetric with rational entries. Identifying the two residue fields by conjugation, unitarity gives $g_{\bar{w}} = J_w^{-1} g_w^{-t} J_w$. Hence $g_{\bar{w}}$ fixes $E_j y$ exactly when g_w^{-t} fixes $J_w E_j y$. Thus

$$\Gamma_{\mathcal{A}}^{\mathrm{open}} = \Gamma_{\mathcal{A}} \cap \bigcup_{w \mid p_0^{\mathrm{lev}}} \mathcal{O}_w,$$

where the union runs over the chosen places. At any one unused place the nonopen subset is contained in the union of the zero sets of its proper leading minors. Put $M = p_0^{\mathrm{lev}} p_1^{\mathrm{lev}}$. For $\Omega \subseteq G$, let $\mathcal{L}(\Omega)$ be the sum of the squared norms of the cut blocks on the nonzero-discriminant part of Γ_*, taken separately for each link and summed over the two classes and $s \in \Omega$, before division by ν^2.

Proposition 9.1 (Orbital bounds on the finite stabilizers)**.** *For the Bessel data at sufficiently large x, put $\beta(s) = \|B_s\|_{\mathrm{HS}}^2$. Then*

$$\sum_{s \in H_R G_i} \|E_s\|_{\mathrm{HS}}^2 \le b^{-12} D_{\mathrm{id}} \qquad (i = 0, 1),$$

$$\sum_{s \in \Gamma_{\mathcal{A}}^{\mathrm{open}}} \beta(s) \le (1 + b^{-4})^2 C_{\mathcal{A}} \quad (\mathcal{A} \textit{ arbitrary}),$$

$$\sum_{s \in \Gamma_{\mathcal{A}}} \beta(s) \le C_{\mathcal{A}} \qquad (|\mathcal{A}| = f).$$

Here $\Gamma_{\mathcal{A}}^{\mathrm{open}}$ is the open subset specified above.

Proof. 1. By (1), the blocks of E_s are the normalized conjugates of the full amplitudes off the rational Bessel subgroup. For a set Ω avoiding the cosets of H_R by nontrivial integral norm-one scalars, (49) identifies the cut off-Bessel amplitudes with the cut amplitudes on Γ_*.

In the block norm obtained by dividing by ν and summing squares over the two classes and $s \in \Omega$, (36) bounds the cut error by $b^{-20}\sqrt{D_{\mathrm{id}}}$.

2. Section 15.8 bounds the squared norm of the combined cut blocks on Γ_* by $\nu^{-2}\mathcal{L}(\Omega) + b^{-kf} D_{\mathrm{id}}$ in the same block norm. For $1 \le z \le Q_\cap(s)$ on Ω, the equal-link sum satisfies

$$\nu^{-2}\mathcal{L}(\Omega) \le (b^{2f}/z)^k D_{\mathrm{main}}.$$

Here $Q_\cap(s)$ is the product of the residue norms where the basis and its image have nonzero intersection. On the open part of $\Gamma_{\mathcal{A}}$, the sharper equal-link estimate is proved in (34).

3. By Section 15.3, the sets $H_R G_i$ avoid the nontrivial scalar cosets and every $s \in H_R G_i$ satisfies $Q_\cap(s) \ge (M/b)^f$. The open set avoids all scalar cosets, including H_R. When

every place is fixed, $Q_{\cap}(s)$ is at least $(p_0^{\mathrm{lev}})^f$ on $\Gamma_{\mathcal{A}}$. The residue group has one general linear factor over $\mathbb{F}_{p_i^{\mathrm{lev}}}$ for each of the f chosen split places at each p_i^{lev}.

The parameter bounds give $b \geq 1000$, $kf \geq 40$, $p_0^{\mathrm{lev}} \geq b^{10}$ and $M \leq b^{23}$ for large x. The moment comparison $b^{11kf} D_{\mathrm{id}} \leq D_{\mathrm{main}} \leq (M/b^6)^{kf} D_{\mathrm{id}}$ is (33).

For every Ω disjoint from the cosets $\mathrm{red}(\zeta)H_R$ with $1 \neq \zeta \in \mathcal{O}_E^{\times}$ and $\zeta\bar{\zeta} = 1$, the triangle inequality in the block norm gives

$$\left(\sum_{s \in \Omega} \|E_s\|_{\mathrm{HS}}^2 \right)^{1/2} \leq \nu^{-1} \sqrt{\mathcal{L}(\Omega)} + (b^{-20} + b^{-kf/2}) \sqrt{D_{\mathrm{id}}} \leq \nu^{-1} \sqrt{\mathcal{L}(\Omega)} + b^{-15} \sqrt{D_{\mathrm{id}}}. \tag{2}$$

On $H_R G_i$ the equal-link contribution is at most

$$(b^{2f}/(M/b)^f)^k D_{\mathrm{main}} \leq b^{-3kf} D_{\mathrm{id}}.$$

Thus the sum of $\|E_s\|_{\mathrm{HS}}^2$ over $H_R G_i$ is at most $4b^{-30} D_{\mathrm{id}} \leq b^{-12} D_{\mathrm{id}}$. By the covariance of Section 8.5, $E_t = (B^* B - I) r_\psi(t)$ for $t \in H_R$. Therefore

$$|H_R| \|B^* B - I\|_{\mathrm{HS}}^2 = \sum_{t \in H_R} \|E_t\|_{\mathrm{HS}}^2 \leq b^{-12} D_{\mathrm{id}}.$$

With $D_0 = BB^*$, Cauchy–Schwarz in the Hilbert–Schmidt inner product gives

$$\beta(s) := \|B_s\|_{\mathrm{HS}}^2 = \mathrm{tr}(D_0 \rho(s) D_0 \rho(s)^*) \leq \|D_0\|_{\mathrm{HS}}^2 \leq (1 + b^{-6})^2 d_X \leq 2 d_X.$$

For the open part of $\Gamma_{\mathcal{A}}$, we have $E_s = B_s$. Since $C_{\mathcal{A}} \geq b^{5kf} D_{\mathrm{id}}$, the right side of (2) is at most

$$(\sqrt{1 + b^{-4}} + b^{-15}) \sqrt{C_{\mathcal{A}}} \leq (1 + b^{-4}) \sqrt{C_{\mathcal{A}}}.$$

This gives open mass at most $(1 + b^{-4})^2 C_{\mathcal{A}}$.

Suppose $|\mathcal{A}| = f$. By Section 15.3, the scalar cosets meet $\Gamma_{\mathcal{A}}$ only in H_R. The pointwise estimate gives

$$\sum_{s \in \Gamma_{\mathcal{A}} \cap H_R} \beta(s) \leq 2 d_X |H_R| = 2 D_{\mathrm{id}} \leq C_{\mathcal{A}}/4,$$

since $C_{\mathcal{A}} \geq b^{5kf} D_{\mathrm{id}}$. On $\Omega = \Gamma_{\mathcal{A}} \setminus H_R$ we have $E_s = B_s$, and the equal-link bound is

$$\nu^{-2} \mathcal{L}(\Omega) \leq (b^{2f}/(p_0^{\mathrm{lev}})^f)^k D_{\mathrm{main}} \leq b^{-8kf} D_{\mathrm{main}} \leq C_{\mathcal{A}}/8.$$

Also $b^{-30} D_{\mathrm{id}} \leq C_{\mathcal{A}}/8$. The square of the right side of (2) is at most $2(C_{\mathcal{A}}/8 + C_{\mathcal{A}}/8) = C_{\mathcal{A}}/2$. The full sum is at most $C_{\mathcal{A}}/4 + C_{\mathcal{A}}/2 \leq C_{\mathcal{A}}$. □

9.2 The diagonal bound and the projected subspace

In this subsection, we use Bessel covariance to estimate the error in the diagonal moment. Proposition 9.1 and the finite projection estimates of Section 14.2 give bounds in the same source norm.

Proposition 9.2 (Diagonal moment on the projected target)**.** *Let P_U be the orthogonal projection onto a subspace $U \subseteq X$. For the map B and target projection p,*

$$\|pBP_U\|_{\mathrm{HS}}^2 \geq \dim U - 5b^{-4} d_X.$$

Proof. The diagonal argument uses the Bessel covariance together with the bound of Proposition 9.1:

$$E_{ts} = r_\psi(t)E_s, \quad E_{st} = E_s r_\psi(t) \quad (t \in H_R), \qquad \sum_{s \in H_R G_i} \|E_s\|_{\mathrm{HS}}^2 \le b^{-12}|H_R|d_X.$$

Section 14.2 bounds the squared counting norm of the conjugacy average of $\operatorname{tr} r_\psi|_{G_i}$ by $2d_X^2$, and gives $q_i(1) \le (2b^8)^{-1}$ for $q_i = 1 - h_i$, with $q_i(1)$ the group-algebra coefficient at the identity.

Since $E_t = E_1 r_\psi(t)$ for $t \in H_R$, summing over H_R and dividing out its order gives

$$\|B^*B - I\|_{\mathrm{HS}}^2 \le b^{-12}d_X.$$

Right convolution by q_i is an orthogonal projection on Hilbert–Schmidt-valued functions on G, by self-adjointness and idempotence. Apply the contraction to $E_s \mathbf{1}_{H_R G_i}(s)$. On $t \in H_R$ the resulting value is $r_\psi(t)\sum_{s\in G_i} q_i(s)E_s$, with convolution taken as $\sum_s q_i(s)E_{ts}$. Summing its squared norm over $t \in H_R$ gives $\|\sum_{s\in G_i} q_i(s)E_s\|_{\mathrm{HS}} \le b^{-6}\sqrt{d_X}$. Its trace consequently has absolute value at most $b^{-6}d_X$. Write $a_i(s) = \operatorname{tr} r_\psi(s)$ on G_i and $a_i^\natural$ for its conjugacy average. Centrality of q_i, self-adjointness and idempotence give

$$\left|\sum_{s\in G_i} q_i(s)a_i(s)\right| = \left|\sum_{s\in G_i} q_i(s)a_i^\natural(s)\right| \le q_i(1)^{1/2}\|a_i^\natural\|_2 \le b^{-4}d_X,$$

where $\|q_i\|_2^2 = (q_i^* q_i)(1) = q_i(1)$ in counting measure. Taking their sum bounds $\operatorname{tr}(B^*(1-p_i)B) = \|(1-p_i)B\|_{\mathrm{HS}}^2 \le 2b^{-4}d_X$. The two summands in $1 - p = (1-p_0) + p_0(1-p_1)$ have orthogonal images, so

$$\|(1-p)B\|_{\mathrm{HS}}^2 \le \sum_{i=0}^{1} \|(1-p_i)B\|_{\mathrm{HS}}^2 \le 4b^{-4}d_X.$$

Using the Hilbert–Schmidt bound for $B^*B - I$, we obtain

$$\begin{aligned}\|pBP_U\|_{\mathrm{HS}}^2 &= \dim U + \operatorname{tr}\big(P_U(B^*B - I)\big) - \|(1-p)BP_U\|_{\mathrm{HS}}^2 \\ &\ge \dim U - b^{-6}\sqrt{d_X \dim U} - 4b^{-4}d_X \ge \dim U - 5b^{-4}d_X.\end{aligned}$$

□

9.3 The mixed moment bound

In this subsection, we control the mixed moment by summing the open contributions against the stabilizer contributions. Proposition 9.1 estimates both sums, and the dimension comparison of Section 9.4 supplies the normalization.

Proposition 9.3 (Mixed moment on the family)**.** *For the same B, p, with $B_s = B^*\rho(s)B$ and $D = \dim pT_{\mathrm{full}}$,*

$$\frac{D}{|G|}\sum_{s\in G} \|B_s\|_{\mathrm{HS}}^2 \le (1 + 9b^{-4})d_X^2.$$

Proof. Put $\delta_b = b^{-4}$, $\lambda = b^{6k}$ and $\mu_{\mathcal{A}} = \sum_{\Gamma_{\mathcal{A}}} \beta$. Proposition 9.1 bounds the open mass for the full kernels by $(1+\delta_b)^2 C_{\mathcal{A}}$, including the cut-to-full error. For $|\mathcal{A}| = f$ it bounds the total mass by $C_{\mathcal{A}}$.

Maximize $\lambda^{|\mathcal{A}|}\mu_{\mathcal{A}}$ over the fixer choices, including the empty subset. If a place is unused, the mass of the intersection with any single allowed vector or covector fixer at that place is at

most $\mu_{\mathcal{A}}/\lambda$. For the leading minor calculation these choices include the nonzero vectors in the two column spaces and their corresponding covectors. Let $q = p_0^{\mathrm{lev}}$ and $\pi : \Gamma_{\mathcal{A}} \to \mathrm{GL}_r(\mathbb{F}_q)$ be the reduction there.

Write the leading block of size j as $Z_s = A_1^t \pi(s) A_0$. Here A_0 consists of the first j columns. To form A_1, take the columns at the conjugate place and pair them by the split Gram matrix. At both places the columns belong to the integral basis of Section 8.1. Reduction preserves their rank, and the reduced Gram matrix is invertible. Thus

$$\operatorname{rank} A_0 = \operatorname{rank} A_1 = j, \qquad A_0 v \neq 0 \ (v \neq 0), \qquad A_1 y \neq 0 \ (y \neq 0).$$

They lie respectively in the original span and the prescribed opposite-place covector span. The two stabilizers tested below have equations $\pi(s) A_0 v = A_0 v$ and $\pi(s)^{-t} A_1 y = A_1 y$. The weight is of positive type because for $D_s = \rho(s) B B^* \rho(s)^*$, $\langle D_a, D_c \rangle_{\mathrm{HS}} = \beta(a^{-1}c)$. For a nontrivial additive character ψ_q, the left-kernel size of Z_s is $\sum_y \mathbb{E}_v \psi_q(y^t Z_s v)$. If $\det Z_s = 0$ the kernel contains at least q elements. With $y, v \in \mathbb{F}_q^j$, the two axes $y = 0$ and $v = 0$ contribute exactly $(2 - q^{-j})\mu_{\mathcal{A}}$ to the weighted sum.

For $y, v \neq 0$, put $a_1 = A_1 y$ and $a_0 = A_0 v$. The vectors

$$\sum_{a \in \Gamma_{\mathcal{A}}} \widehat{e}_{\pi(a)^{-t} a_1} \otimes D_a, \qquad \sum_{c \in \Gamma_{\mathcal{A}}} e_{\pi(c) a_0} \otimes D_c$$

use delta vectors e_z on $\mathbb{F}_q^r$ and unit Fourier vectors $\widehat{e}_w(z) = q^{-r/2} \psi_q(-w^t z)$. Their squared norms are $|\Gamma_{\mathcal{A}}|$ times the respective fixer masses, by expanding and setting $s = a^{-1}c$. Their inner product equals $|\Gamma_{\mathcal{A}}| q^{-r/2}$ times $\sum_{s \in \Gamma_{\mathcal{A}}} \psi_q(a_1^t \pi(s) a_0) \beta(s)$. Cauchy–Schwarz therefore bounds the latter sum by $q^{r/2} \mu_{\mathcal{A}}/\lambda$. Summing over y, v with the normalization in v proves

$$q \sum_{\det Z_s = 0} \beta(s) \leq (2 + q^{j + r/2}/\lambda) \mu_{\mathcal{A}}.$$

By Section 4, $b^{10} \leq q \leq b^{11}$. Since $j \leq m/2$ and $r \geq x^5$, for sufficiently large x,

$$11j + 6m \leq 12m \leq r/2, \qquad \log_b(q^{j + r/2}) \leq 11j + \tfrac{11}{2} r \leq 6(r + 1 - m) = \log_b \lambda.$$

There are at most $m/2 \leq b^6/3$ prescribed minors. Every nonopen residue has a vanishing minor at this particular unused place. Using $q \geq b^{10}$ and summing over its minors, the nonopen mass is at most $\delta_b \mu_{\mathcal{A}}$.

If the maximizing fixer uses every place, use the full-fixer mass bound just proved. Otherwise combine the open and nonopen bounds to get $(1 - \delta_b)\mu_{\mathcal{A}} \leq (1 + \delta_b)^2 C_{\mathcal{A}}$. In either case maximality and $0 < \delta_b \leq 1/5$ give

$$\sum_{s \in G} \|B_s\|_{\mathrm{HS}}^2 = \mu_{\varnothing} \leq \lambda^{|\mathcal{A}|} \mu_{\mathcal{A}} \leq \frac{(1 + \delta_b)^2}{1 - \delta_b} D_{\mathrm{main}} \leq (1 + 9\delta_b) D_{\mathrm{main}}.$$

By (3), this gives

$$\frac{D}{|G|} \sum_{s \in G} \|B_s\|_{\mathrm{HS}}^2 \leq (1 + 9b^{-4}) d_X^2.$$

The bound uses the original full source, normalized measured map and projected target. □

9.4 Dimension on the terminal definite group

In this subsection, we transfer the required dimensions to a definite unitary group with the same finite levels. The trace comparison respects the projection of Section 14.2, which lets us use these dimensions in Section 9.

Use the terminal positive definite Hermitian space W_m of rank r, with its standard lattice and the principal level at the same chosen places. The original space Y_0 and W_m have identical finite groups and levels in their isometric coordinates. Their real types have the same regular Harish–Chandra parameter at each real place. Write T_W for the full principal-level space of the terminal type.

Proposition 9.4 (Definite comparison of dimensions)**.** *On the family,*

$$D = \dim pT_{\mathrm{full}} = \dim pT_W \le |G|\mathcal{B}\ \mathrm{mass}(E, r). \tag{3}$$

Proof. Fix one tuple selected by p. At each of the two different level primes this tuple has a primitive type polynomial at a chosen place, by Section 14.2. The two places of F_0 are split in E and distinct. At either such place let $K = \mathrm{GL}_r(\mathcal{O}_v)$ and let τ be the finite cuspidal representation of its residue group given by the primitive type polynomial.

If Z is the local centre, extend the inflation of $\tau^\vee$ unitarily to ZK and compactly induce to $\mathrm{GL}_r(F_{0,v})$. For a diagonal Cartan representative outside ZK, order its elementary-divisor exponents decreasingly and cut at a strict drop. Conjugation sends the upper off-diagonal unipotent block at this cut into the principal kernel. Intertwining would therefore factor through the coinvariants of $\tau^\vee$ by the unipotent radical of this proper residue parabolic, which are zero by cuspidality. Thus the intertwining criterion gives irreducibility. Its coefficients are compactly supported modulo Z, so this compact induction is supercuspidal. Every irreducible local representation containing the inflated τ is supercuspidal by the depth-zero type theorem [56, §4.1, Proposition, p. 151]. Consequently the selected discrete spectrum is cuspidal: at such a place the induced representations from every proper Levi have zero projection to this type, and the noncuspidal spectrum is built from these inductions.

In the compact induction, the vectors supported on the identity coset have coefficients supported on ZK. Truncate them by $\mathbf{1}_{\mathcal{O}_v^\times}(\det g)$. Since $ZK \cap \{\det g \in \mathcal{O}_v^\times\} = K$, the sum of these truncated diagonal coefficients, with the factor $\dim(\tau)/\operatorname{vol}(K)$, is exactly the local projector density

$$\frac{\dim \tau}{\operatorname{vol}(K)} \overline{\operatorname{tr} \tau(\bar g)}\, \mathbf{1}_K(g).$$

Here $\bar g$ is reduction at the chosen place. In an orthonormal basis of the copy of $\tau^\vee$ on the identity coset the diagonal coefficients on K sum to $\operatorname{tr} \tau^\vee(\bar g) = \overline{\operatorname{tr} \tau(\bar g)}$. This gives the asserted expansion at both chosen places. Each truncated coefficient is strongly cuspidal by [35, Lemma 1.9.1]. In particular the projector at the second chosen place is cuspidal, as required by condition (b.1) of Proposition 1.10.6 there.

Use the normalized principal-level averages and the finite central idempotent of the fixed tuple at all finite places. Their product is the test for the original tuple on both source spaces. At each real place choose the Euler–Poincaré function with coefficients in the dual of the algebraic representation of highest weight $\lambda - \rho$, where λ is the dominant representative of the common regular parameter.

For a real signature (a, b') divide this function by $d_{a,b'} = \binom{r}{a}$, the size of its discrete series packet. A nonelliptic regular semisimple element has a nontrivial split torus in the centre

of its connected centralizer: its eigenvalues off the unit circle occur in reciprocal-conjugate pairs. Square-integrability here is taken on the full real centralizer. The absolute value of a nonzero coefficient of an irreducible unitary representation is invariant under the central split torus, whose Haar measure is infinite. Consequently the centralizer has no square-integrable irreducible representation, and the ordinary orbital integral vanishes by [23, Theorem 4.1(i)]. This gives cuspidality of the real test in the sense of [35, §1.5].

For a semisimple δ, an eigenvalue z off the unit circle and its partner $1/\bar{z}$ contribute a factor $\mathrm{GL}_{m_z}(\mathbb{C})$ to the real centralizer. Its split centre forces the stable orbital integral to vanish by [23, Theorem 4.1(iii)]. If every eigenvalue has modulus one, let m_z be its multiplicity. The conjugacy classes in the stable class on $U(a,b')$ are specified by eigenspace signatures $(a_z, m_z - a_z)$ with $\sum_z a_z = a$. The eigenspaces are orthogonal and nondegenerate, and their signatures give

$$I_\delta = \prod_z U(a_z, m_z - a_z), \qquad \overline{I}_\delta = \prod_z U(m_z).$$

Give the centralizer the measure for which discrete series with the infinitesimal character of the trivial representation have formal degree one [23, §4]. The stable orbital sum weights each ordinary class by the Kottwitz sign of its centralizer [35, §1.5]. Thus [23, Theorem 4.1(ii), (iv)] gives

$$\mathrm{SO}_\delta(\phi_F/d_{a,b'}) = e(\overline{I}_\delta)\,\mathrm{tr}\,F(\delta), \qquad \sum_{\sum a_z = a} \prod_z \binom{m_z}{a_z} = [X^a](1+X)^r = d_{a,b'}.$$

For the compact inner centralizer the sign is

$$e(\overline{I}_\delta) = (-1)^{\sum_z \lfloor m_z^2/4 \rfloor}.$$

Indeed the quasi-split real form of $U(m_z)$ has signature $(\lfloor m_z/2 \rfloor, \lceil m_z/2 \rceil)$, so its symmetric space has real dimension $2\lfloor m_z^2/4 \rfloor$. Here F is the dual algebraic coefficient specified above. For fixed centralizer measure di_δ, replacing real Haar dg by $c\,dg$ replaces the Euler–Poincaré test by $c^{-1}\phi_F$ and the quotient measure dg/di_δ by $c\,dg/di_\delta$. Thus the displayed orbital integral has the same value for the original group Haar. Every unit-circle stable class occurs on each signature, and the displayed value depends only on its eigenvalues with multiplicities. The normalized real tests thus have a common transfer in the sense of [35, §1.6]. At each finite place the given isometry transports the tests and their measures.

Apply the cuspidal trace comparison of Harris–Labesse [35, Proposition 1.10.6], termwise in the coefficient expansion at the two chosen split places. The first place supplies condition (a.1), with the split Jacquet–Langlands transfer the identity, and the other supplies condition (b.1). All places with different inner form are real, with cuspidal Euler–Poincaré tests, giving condition (c). For each of the two specified tests, [35, Proposition 1.10.5(i)] identifies its cuspidal trace with that of their common transfer on the quasi-split group. This proves equality for these two tests with the stated archimedean normalization.

On either side a nonzero trace of the archimedean test requires cohomology with the chosen coefficient. A representation selected by the finite test is supercuspidal at both chosen split places. By [35, Proposition 1.8.1], it occurs in the restriction of a cuspidal representation of the group of rational unitary similitudes. The two chosen split places satisfy case (B), so [35, Theorem 3.1.5] makes each contributing real factor discrete series. Write F_λ for the representation of highest weight $\lambda - \rho$. The compact centre of $U(a,b')$ lies in the maximal compact K, so a central character mismatch between π and F_λ makes all relative cochains zero.

When the characters agree, the centre $Z = U(1)$ acts trivially on $\pi \otimes F_\lambda^\vee$. Taking an r-th root of the determinant gives $U(a,b') = Z\,SU(a,b')$ and $K = Z(K \cap SU(a,b'))$. Restriction of each irreducible representation to the derived group is irreducible because Z acts by scalars. The tangent quotient and the invariant cochains therefore agree for the two groups. On the derived group the parameter determines the ordered entries up to a common additive constant and permutation; the fixed central character determines their sum. The complex and compact Weyl groups are S_r and $S_a \times S_{b'}$, respectively. By [19, Ch. II, Theorem 5.3(a)–(b)], the discrete series satisfy

$$\dim H^j(\mathfrak{g}, K; \pi \otimes F_\lambda^\vee) = \dim \operatorname{Ext}^j_{(\mathfrak{g},K)}(F_\lambda, \pi) = \begin{cases} 1 & \pi \text{ belongs to the packet of } \lambda, \quad j = ab', \\ 0 & \text{otherwise.} \end{cases}$$

Here $ab' = \dim_{\mathbb{R}}(U(a,b')/K)/2$. For each fixed finite representation, the multiplicities of all real packet members on the fixed unitary group are equal by [35, Proposition 3.1.7]. In the notation of that proposition take $S_0 = \{v_1, v_2\}$ to be the two chosen split places and $S = S_0 \cup S_\infty$. The multiplicity comparison uses the same group on both sides. Its hypotheses on the finite part of S require supercuspidality at both members of S_0.

Let $\mathcal{P}_\infty$ be the product of these real packets and π_∞^0 its prescribed member. For each fixed π_f, the normalized archimedean trace factor is

$$\frac{(-1)^{\sum_v a_v b'_v}}{|\mathcal{P}_\infty|} \sum_{\pi_\infty \in \mathcal{P}_\infty} m(\pi_\infty \otimes \pi_f) = m(\pi_\infty^0 \otimes \pi_f).$$

Here $|\mathcal{P}_\infty| = \prod_v \binom{r}{a_v}$ and $a_v b'_v$ is even at each place since $a_v + b'_v = r$ is odd. At each real place, Section 12.3 gives an orthogonal coefficient projector P on any irreducible unitary σ, with $P\sigma(g)P = c(g)P$, where c is the coefficient of the prescribed unit vector in π. If $P \neq 0$, the translates of a unit vector in its range have the same Gram matrices as the translates of the vector in π. Cyclicity gives $\sigma \simeq \pi$, where Schur orthogonality makes the projector rank one. The product selects one line in each occurrence of π_∞^0, with the displayed multiplicity. The trace equality is the equality of dimensions of the two original tuple spaces. Sum over the orthogonal tuples to get $D = \dim pT_W$.

The full finite spherical algebra. The trace argument compares the projected spaces on Y_0 and on any W_j, $1 \le j \le m$. At each real place the ordered parameter of Section 12.10 has sign sequence $k-1$ plus signs followed by m alternating signs beginning with plus on Y_0. On W_j the same ordered values have the first $k+j-1$ signs positive and the remaining $m-j$ negative, as in Section 33.1.1. In particular W_m has signature $(r,0)$. Thus all groups use the same algebraic coefficient in the Euler–Poincaré test, while the matrix coefficient selects the prescribed member on each real group.

At a finite place put $p_j = k+j-1$ and $q_j = m-j$. In the ordered rational basis of W_j its form is $J_j = \operatorname{diag}(I_{p_j}, -I_{q_j})$. If u_- is an integral unit of norm -1, then

$$B_j = \operatorname{diag}(I_{p_j}, u_- I_{q_j}), \qquad B_j^* J_j B_j = I_r.$$

The unit exists at split places and at nonsplit places by surjectivity of the unramified unit norm, including in residue characteristic two. The norm and integral trace maps at nonsplit places are those of Section 4.3. For a hyperbolic pair (e, e^*), choose integral c of trace one. The basis

$(f, e-f)$, with $f = e^* + ce$, has norms $1, -1$ and zero cross pairing. For the glued pair (v_0, z) of Section 8.1, choose instead $\mathrm{Tr}(c) = -N - 1$ and put $f = z + cv_0$. Then

$$h(f,f) = -N - \mathrm{Tr}(c) = 1, \qquad h(f, v_0 + f) = 0, \qquad h(v_0 + f, v_0 + f) = -1.$$

Both basis changes have determinant -1. Scale each negative line by a unit of norm -1. The formulas use only the integral glued coordinates.

At a split pair $(w, \bar{w})$ let H_w, H'_w be the invertible integral Gram matrices in the two lattice bases, identifying the conjugate factor with the chosen local field. For an isometry from the first lattice to the second take

$$C_w = I, \qquad C_{\bar{w}}^{\mathsf{t}} = H_w (H'_w)^{-1}, \qquad C_{\bar{w}}^{\mathsf{t}} H'_w C_w = H_w.$$

At a chosen pair its induced map on the prescribed general linear residue group is the identity. Replacing either integral orthonormal basis conjugates that map by a residue matrix and leaves the central type idempotent unchanged. Use matching local Haar measures throughout. At a finite place the Hermitian identification carries the integral compact, its principal subgroup if present, and their Haar measures to those of the other group.

In particular, if K_v is the integral compact, use $\mathbf{1}_{K_v} / \operatorname{vol}(K_v)$ for the local identity of the spherical algebra. A double coset $K_v g K_v$ uses $\mathbf{1}_{K_v g K_v} / \operatorname{vol}(K_v)$, which acts on K_v-fixed vectors by the unnormalized sum of its right coset translates. These prescriptions agree on the two local groups also at nonsplit finite places. Transport the finite tests by the isomorphism of the groups there, so the tests have the same stable orbital integrals with matching centralizer measures. No comparison between distinct local groups is needed at those places.

Fix a tuple c selected by the finite-type idempotent, and insert any compactly supported spherical finite test h off the old chosen places, with identity factors at all but finitely many places. The test at the old level is the central type projector of c. There is a primitive type polynomial at one split place above each of the two distinct rational level primes. At the first such place the projector is a linear combination of truncated diagonal coefficients of unitary depth-zero supercuspidals, as in the computation above. The two local groups there are split, so the Jacquet–Langlands transfer is the same supercuspidal. At the other distinguished place the corresponding coefficients are cuspidal by the local supercuspidal coefficient criterion [35, Lemma 1.9.1]. The spherical insertion is away from these places. It does not change either local condition.

Apply [35, Propositions 1.10.5(i) and 1.10.6] to each of the specified tests and their common transfer, as above, in case (a.1, b.1, c). For two inner forms of the quasi-split unitary group, associated tests have identical cuspidal traces if at one split finite place the test is a truncated coefficient of a supercuspidal whose Jacquet–Langlands transfer is supercuspidal, at another place the test is cuspidal, and at every place with different local inner forms the test is cuspidal. This applies termwise after expanding the first projector and h into finite sums of decomposable tests. The two finite conditions are exactly those verified above. The possible differences of inner form are real. There use the Euler–Poincaré tests with the same algebraic coefficient and divide at signature (a, b') by the packet size $\binom{r}{a}$. Their matching stable orbital integrals are the preceding calculation with [23, Theorem 4.1]. The finite tests are associated by their measure-preserving transport.

On the spectral side the same two primitive types force the contributing finite factors to be supercuspidal there. The cohomological real factors selected by the Euler–Poincaré test are discrete series by [35, Theorem 3.1.5, case (B)]. At a fixed finite representation the multiplicities

of the packet members on a fixed real inner form agree by [35, Proposition 3.1.7]. Thus dividing by the packet sizes yields the trace on one prescribed real member per place, with the sign $(-1)^{\sum_v a_v b'_v} = 1$ on both source spaces.

Let e_∞ be the product of the prescribed real matrix units. Schur orthogonality gives a rank-one operator on π_∞^0. If an irreducible unitary representation has a nonzero vector in the image of this matrix unit, the compression identity of Section 12.3 identifies the Gram matrix of its translates with that of the corresponding vector in π_∞^0, up to its squared norm. Irreducibility therefore forces that representation to be π_∞^0. With p_c the finite projector density of the fixed tuple, one obtains

$$\operatorname{tr} R_{\mathrm{cusp}}(e_\infty \otimes p_c h) = \sum_{\pi_f} m(\pi_\infty^0 \otimes \pi_f) \operatorname{tr} \pi_f(p_c h).$$

This is also the trace computed by the packet-normalized Euler–Poincaré functions above. Summing over the orthogonal selected tuples gives

$$\operatorname{tr}_V(h) = \operatorname{tr}_{V_j}(h) \qquad (h \text{ in the common finite spherical algebra}), \tag{4}$$

where V and V_j are the original projected spaces on Y_0 and W_j, respectively. The identity test gives the equality of dimensions.

The preceding trace comparison also holds before summation, on the isotypic spaces for each fixed irreducible residue type τ_c. Write these spaces as $\tau_c \otimes L_c$ and $\tau_c \otimes L_{j,c}$, with their invariant Hermitian forms. Every spherical test off the old level commutes with the residue group. Its action is therefore the identity on τ_c tensored with its action on the multiplicity space. Dividing the trace equality by $\dim \tau_c$ gives equal traces on L_c and $L_{j,c}$.

Let $\mathcal{A}_c$ be the image of the full spherical algebra on $L_c \oplus L_{j,c}$. It contains the identity and is closed under adjoints since $h^*(g) = \overline{h(g^{-1})}$. A finite-dimensional complex algebra of operators closed under adjoints is a direct sum of full matrix algebras. In each matrix summand a rank-one projection has trace on either multiplicity space equal to the number of copies of the defining module. Such a projection lies in $\mathcal{A}_c$, so it is the image of a finite spherical test and the two numbers agree. Matching orthonormal copies in every summand gives a unitary intertwiner $L_c \simeq L_{j,c}$. Tensoring with the identity on τ_c preserves the old residue action and intertwines all outside-level Hecke tests, including the nonsplit factors. In particular it carries each prescribed joint eigenspace onto the corresponding eigenspace on the other group.

At the positive definite endpoint each maximal-level class with stabilizer Γ_c gives $|G|/|\Gamma_c|$ classes at the principal level. The quotient of the integral compact by the principal subgroup is $G = \prod_{i=0}^{1} \mathrm{GL}_r(\mathbb{F}_{p_i^{\mathrm{lev}}})^f$, by reduction at the selected split places. The rational stabilizer acts freely on this quotient. To verify freeness choose a completely split prime of norm $p_i^{\mathrm{lev}} > (2r)^2$. If a stabilizer element γ reduces to one at the chosen places, its integral trace difference $\operatorname{tr}\gamma - r$ is divisible by all f of these primes. A nonzero difference would have absolute norm at least $(p_i^{\mathrm{lev}})^f$, and at most $(2r)^{2f}$ because all its complex embeddings are unitary.

Thus the trace equals r. Unitarity forces all eigenvalues to be one and $\gamma = 1$. This also gives the trivial stabilizer on each finer class. At such a class the real component is the full compact group. In the Peter–Weyl decomposition the coefficient idempotent selects one line in the right factor of the terminal type, leaving the dimension of its left factor. For the dominant local parameters $\lambda_{v,1} > \cdots > \lambda_{v,r}$, the Weyl dimension formula for the highest weight $\lambda_v - \rho$ gives

$$\dim \bigotimes_{v|\infty} F_{\lambda_v} = \prod_{v|\infty} \prod_{i<j} \frac{\lambda_{v,i} - \lambda_{v,j}}{j - i} = \mathcal{B}.$$

The normalization of the same product at each place is given in Section 12.5. Summing over the maximal-level classes gives

$$\dim T_W = |G|\mathcal{B}\sum_c |\Gamma_c|^{-1} = |G|\mathcal{B}\ \mathrm{mass}(E,r).$$

This proves (3). □

10 Finite orbital integrals and Fourier bounds

In this section, we compute the finite Haar factors and bound the correlations appearing in Section 9.1. Integral norm fibers give the successive quotient measures, and Fourier estimates control the sum over denominators.

10.1 Integral points on the local quadrics

In this subsection, we bound the measure of integral solutions of a Hermitian norm equation. The bound enters the Haar disintegration in Section 10.2 and the local integrations in Section 10.6.

Let T be any finite set of finite places of the CM field E stable under conjugation, with unramified or split quadratic algebras at its conjugation orbits. Put $\mathcal{R} = \prod_{w\in T}\mathcal{O}_{E_w}$ and $\zeta_{\mathcal{R}}(a) = \prod_{w\in T}(1-(\mathrm{N}w)^{-a})^{-1}$. Normalize additive measures to give the integer rings mass one. We need a bound for the mass of the integral box in a quadric with an orthogonal block of $2a+4$ orthonormal coordinates, where $a \geq 1$ is an integer. Take the fiber measure by division of the ambient Haar measure by the norm-value Haar measure. For a skew-Hermitian norm equation at a split pair, $(1,-1)$ generates the integral skew lattice over the fixed integers. At an unramified quadratic place lift a residue element outside the fixed field to a. Then $s = a - \bar{a}$ is a unit, also in residue characteristic two. Every skew integer z has z/s fixed and integral. Thus division by s identifies the two integral norm-value lattices with Jacobian one.

Consider one orbit, with fixed field K, residue cardinality q, quadratic algebra L, and integer ring $\mathcal{O}_L$. Both the trace on integers and the norm on units are surjective. In the unramified case this follows from the residue trace and norm and the trace on each successive unit quotient. In the split case these maps are addition and multiplication. Choose c, τ with $c\bar{c} = -1$ and $\tau + \bar{\tau} = 1$.

In each orthonormal pair v, w, the integral basis

$$v + cw, \qquad \bar{\tau}v - \tau cw$$

is hyperbolic. Its change-of-basis determinant is $-c$, a unit. The norm on the whole block, with $h = a+2$, is now $\mathrm{Tr}_{L/K}\sum_{i=1}^{h}\bar{x}_i y_i$. For fixed $x \in \mathcal{O}_L^h$, let j be the minimum valuation of its coordinates, taking the minimum over both components as well in the split case. For finite j, the linear norm equation in y is a map onto $\pi^j\mathcal{O}_K$ on $\mathcal{O}_L^h$, where π is a uniformizer of K.

Its pushforward of integral Haar measure is uniform probability measure on that ideal. Thus the density on each norm fiber is at most q^j. This is the density in the quotient measure, since solving for a K-coordinate with nonzero coefficient divides its Haar measure by the absolute value of that coefficient. One may use any such coordinate. The set $x = 0$ has measure zero in the smooth fiber charts, by solving in x when $y \neq 0$ and using $h \geq 3$. At the singular point the smooth fiber measure has zero mass.

The probability that all the x_i are divisible by π^j is q^{-2hj}. In particular the same calculation gives the exact density at every congruence level. Write $Q(x,y) = \mathrm{Tr}_{L/K} \sum_i \bar{x}_i y_i$. For $A \geq 1$ and $c \in \mathcal{O}_K$, one has

$$q^A \Pr\{Q(x,y) \equiv c \bmod \pi^A\} = (1-q^{-2h}) \sum_{j=0}^{A-1} q^{-(2h-1)j} \mathbf{1}_{\pi^j | c} + q^{-(2h-1)A} \mathbf{1}_{\pi^A | c}.$$

Indeed, conditional on minimum valuation $j < A$, the congruence has probability $q^{j-A}\mathbf{1}_{\pi^j|c}$. On $x \in \pi^A \mathcal{O}_L^h$ it holds exactly when $\pi^A \mid c$. The series obtained by letting $A \to \infty$ converges uniformly in c, including $c = 0$, since its tail is bounded by $q^{-(2h-1)A}/(1-q^{-(2h-1)})$. Hence the block contributes at most

$$(1-q^{-2h}) \sum_{j \geq 0} q^{-(2h-1)j} \leq (1-q^{-(2a+3)})^{-1}.$$

This bound is independent of the norm value, so the other integral coordinates can be integrated with total mass one. The displayed bound is at most $(1-q^{-2a})^{-1}$ in the unramified quadratic case and at most $(1-q^{-a})^{-2}$ in the split case. Multiplying over the orbits gives $\zeta_{\mathcal{R}}(a)$.

Lemma 10.1 (Null set of a finite pivot)**.** *In a Hermitian norm equation over the semilocal algebra above, suppose there is a hyperbolic pair orthogonal to a block of $2a+4$ integral orthonormal coordinates, where $a \geq 1$ is an integer. On every fixed norm fiber the locus where either coordinate of this pair vanishes in any field component has fiber measure zero.*

Proof. Fix one of the hyperbolic coordinates t and one component $w \in T$. For fixed coordinates outside the orthonormal block, its integral fiber mass is at most $\zeta_{\mathcal{R}}(a)$ at every remaining norm value. The congruence densities of the block converge uniformly in that value, so the same bound applies after integrating over the other integral coordinates. In particular

$$\mathrm{vol}_{\mathrm{fiber}}\{X \text{ in the box} : t_w \in \mathfrak{p}_w^m\} \leq \zeta_{\mathcal{R}}(a)(\mathrm{N}w)^{-m}.$$

Only the selected field component is restricted at a split pair. Letting $m \to \infty$ gives zero mass on $t_w = 0$, also on the zero norm fiber. Dilation by a fixed-field scalar reduces any larger box to an integral one, rescaling the norm value and the fiber measure by fixed factors. Exhaust the fiber by these boxes and take the finite union over the two coordinates and the components. □

10.2 Compact volumes for the successive quotients

In this subsection, we disintegrate the local Haar measure along successive isotropic lines. The quadric estimate of Section 10.1 bounds the resulting densities, and the compact mass is used in Section 10.3.

Use the original integral Gram lattice Λ at T, with the ϱ hyperbolic pairs e_i, e_i^* and last vector z in their orthogonal complement. Write G_T for its unitary group and K_T for the lattice stabilizer. At the top use the isotropic dual u of z in the specified middle block, with $h(u,z) = 1$, and the orthogonal complement of u, z. The integral Gram restrictions used here are that these are integral decompositions, that $\langle u, z \rangle$ has determinant -1, and that the remaining orthonormal block has at least $2(3\varrho+5)+4$ coordinates. With a trace-one integer τ, replacing z by $z - \tau h(z,z)u$ gives a hyperbolic pair and preserves the coefficient of z. In the original

orbital lattice of Section 8.1, write $v_0 = x_0/\bar{\pi}_N + z/N$ for the glued basis vector. Its Gram matrix with z gives $u = -v_0$, $h(u,u) = 0$ and $h(u,z) = 1$, including at the places dividing N. The remaining $k-1$ orthonormal coordinates suffice since $k - 1 \geq 2(3\varrho + 5) + 4$ for large x. Thus the same integral decomposition works for every T.

Set $n = \varrho + 1$, and $f_i = e_i$ for $i < \varrho$, $f_\varrho = z$. Let G_i fix the pairs below i pointwise and, for $i = n$, also z. Let R_i act unitriangularly on the first i basis vectors, so $gf_j - f_j \in \langle f_0, \ldots, f_{j-1}\rangle$ for $j < i$. Let U_i be the Heisenberg group attached to the i-th pair for $i < \varrho$, and put $U_\varrho = \{1\}$. The products $U_{<i} = U_0 \cdots U_{i-1}$ are normalized by G_i. We have $R_i = G_i U_{<i}$ and $R_{i+1} \cap G_i = G_{i+1} U_i$.

Indeed an isometry fixing e_i sends e_i^* to $e_i^* + v + se_i$, with v in the remaining space and $s + \bar{s} = -h(v,v)$. The unique Heisenberg element with that column removes it, leaving an isometry fixing the pair. For the iteration, write $r \in R_{i+1}$ as γv by the decomposition at i. A lower Heisenberg factor changes each later basis vector only by earlier isotropic vectors, so $v \in R_{i+1}$.

Now $\gamma \in G_i \cap R_{i+1}$ takes f_i into the orthogonal complement of the earlier pairs and differs from f_i by a vector in their span. Thus it fixes f_i, and the column removal gives the step. At the top the condition within G_ϱ is exactly to fix z.

Let η_i be Heisenberg Haar in the coordinates (v, β), with $s = \beta - \tau h(v,v)$ and β skew. Its measure is product additive Haar, since multiplication is triangular with translations in v, β. Normalize each integer coordinate lattice to mass one. The displayed column and integrality of h give $\eta_i(K_T \cap U_i) = 1$, and peeling off these columns gives $\eta(K_T \cap U_{<n}) = 1$ for $\eta = \eta_0 * \cdots * \eta_{\varrho-1}$. Conjugation by G_i preserves the earlier vector measures by the unitary determinant identity and fixes their centers.

For $i < n$ put $H_i = R_{i+1} \cap G_i = G_{i+1} U_i$. The map

$$H_i \backslash G_i \longrightarrow G_i f_i, \qquad H_i \sigma \longmapsto \sigma^{-1} f_i$$

identifies the left quotient with the orbit. Give the quotient the norm-fiber measure under this identification. In convolution, θ_i is the image of this measure under the section $X \mapsto \sigma$ constructed below on the chart where the pivot t is invertible in every component. This chart has full fiber measure. On a hyperbolic pair with coordinates t, s write $s = tc$. This contributes the additive Jacobian $|t|_A$, the modulus on the semilocal algebra.

The norm equation takes the trace of c with coefficient $t\bar{t}$, whose modulus on the fixed algebra is also $|t|_A$. The factors cancel. Writing $c = \tau b + \beta$, with $b = \mathrm{Tr}(c)$ and $\bar{\beta} = -\beta$, splits the integral lattice as $\tau \mathcal{O}_K \oplus \ker(\mathrm{Tr}\,|_{\mathcal{O}_L})$ at each conjugation orbit. Thus the trace splitting has Jacobian one for the chosen measures, including above 2. The other coordinates, t, and β have product additive measure. The top chart uses the same computation after the indicated change of basis. At every stage the unused orthonormal block has at least $2(3\varrho + 5) + 4$ coordinates. Lemma 10.1 therefore gives full measure to the chart at each prescribed norm.

On a lower stage write a point of norm zero as $X = te_i^* + x + se_i$, with $t \in A_T^\times$ and $x \perp e_i, e_i^*$. Put $a = x/t$ and $\beta = s/t + \tau h(a,a)$. The norm equation gives $\beta + \bar{\beta} = 0$. Let $w_i(t)$ send $e_i \mapsto te_i^*$ and $e_i^* \mapsto \bar{t}^{-1} e_i$, fixing their orthogonal complement. The Heisenberg element $n(a,\beta)$ fixes e_i and sends $e_i^* \mapsto e_i^* + a + (\beta - \tau h(a,a))e_i$. Hence

$$n(a,\beta) w_i(t) e_i = X, \qquad \sigma = w_i(t)^{-1} n(a,\beta)^{-1}, \qquad \sigma^{-1} e_i = X.$$

The change $x = ta$ contributes $\mathrm{md}(t)^{d_i}$, where $d_i = k + 1 + 2(\varrho - i - 1)$ counts the remaining coordinates, as in Section 10.4. On the top stage put $v = z + N\tau u$, where $h(z,z) = -N$. For

the original exceptional vector $x_{k_0}/\bar{\pi} + z/N$ we have $u = -(x_{k_0}/\bar{\pi} + z/N)$. Then $h(u,u) = h(v,v) = 0$, $h(u,v) = 1$ and $z = v - N\tau u$. The integral span of u, v is unimodular and splits off.

Given $tv + x + su$ of norm $-N$, with t invertible and $x \perp u, v$, put

$$a = t^{-1}x, \qquad \beta = t^{-1}s + \tau h(a,a) + N\tau/(t\bar{t}).$$

The norm equation gives $\beta + \bar{\beta} = 0$. First dilate v by t and u by $\bar{t}^{-1}$. Then apply the Heisenberg element fixing u and sending v to $v + a + (\beta - \tau h(a,a))u$, with $c \mapsto c - h(a,c)u$ on the orthogonal complement. This composite sends z to the given vector. For the orbit map $\sigma \mapsto \sigma^{-1}z$, take σ to be the inverse composite. If the given vector is integral and t is a unit, both transformations and their inverses preserve the lattice. Thus the unit-pivot subset contributes to the compact mass in the same fiber measure. The terminal measure θ_n is right Haar on G_n with $K_T \cap G_n$ of volume one.

The Haar measure given by these data is $\eta * \theta_n * \cdots * \theta_0$. Indeed, in descending order the group measure is formed by taking $\eta_i * \theta_i$ at stage i, with $\eta_i = \delta_1$ on the top quadric and terminal stages. The measure from the higher stages times η_i is right invariant under $G_{i+1}U_i$. Consequently its integral against a function of the product with $\sigma \in G_i$ descends to $H_i\sigma$, so it is independent of the section used to lift θ_i. Right translation by $g \in G_i$ acts on the fiber coordinate by g^{-1}.

This preserves the norm fiber measure, because the ambient determinant has modulus one and the norm value is unchanged. Induction from θ_n gives right Haar at the bottom. Each residual unitary group, including the terminal stabilizer, is unimodular. Over a splitting field its adjoint action on the Lie algebra is conjugation on M_d, with determinant $(\det g)^d(\det g^{-1})^d = 1$. The same determinant is therefore one over the fixed local field, so right Haar is also left invariant. This gives the quotient integration formula at each step with the stated fiber and stabilizer measures. Moving each θ_i past the lower Heisenberg product is permitted by the conjugation invariance above. This gives the asserted order, with the same measures as for the orbital integrals formed using this stage normalization.

Put $\mathcal{U}_i = R_iK_T = \{g : R_ig \cap K_T \neq \varnothing\}$. For each g, the θ_i-mass of σ such that $\sigma g \in \mathcal{U}_{i+1}$ is at most $V = \zeta_{\mathcal{R}}(3\varrho + 5)$. If the set is nonempty, then $g \in \mathcal{U}_i$, since $G_i, R_{i+1} \subseteq R_i$. Replacing g by $rg \in K_T$ with $r \in R_i$ preserves the integral. To see this, factor $r = \gamma v$ in $G_iU_{<i}$. The conjugate of v by $\sigma\gamma$ lies in $U_{<i} \subseteq R_{i+1}$ and disappears from the near condition. Right multiplication by γ preserves the integral on the orbit. Now if $\xi\sigma g \in K_T$ with $\xi \in R_{i+1}$, then $\sigma^{-1}\xi^{-1}f_i$ is integral. Its projection away from the earlier pairs is $\sigma^{-1}f_i$, since $\xi^{-1}f_i - f_i$ belongs to the earlier isotropic span. The projection is integral. The quadric box bound of Section 10.1 therefore gives V, uniformly in g.

The θ_n-mass of $h \in G_n$ with $hg \in U_{<n}K_T$ is at most one. In fact if $u_1h_1g, u_2h_2g \in K_T$, their quotient belongs to K_T and to $U_{<n}G_n$. Its middle block is $h_2h_1^{-1}$, which is therefore integral with integral inverse. All such h lie in a single right coset of $K_T \cap G_n$. In the descending convolution $\theta_n * \cdots * \theta_i$, the integral of the indicator of $U_{<n}K_Tg^{-1}$ is thus at most V^{n-i}.

For the induction, the higher convolution is supported in G_{i+1}. Its integral after σg can be nonzero only when $\sigma g \in \mathcal{U}_{i+1}$, since $U_{<n}$ and G_{i+1} lie in R_{i+1}. Integrating over θ_i contributes at most V. Finally convolution with η gives exactly the integral over $U_{<n}K_T$ for $g = 1$, because each nonempty η-fiber in K_T is a right coset of $K_T \cap U_{<n}$, of mass one. Thus $\mu(K_T) \leq V^n$. For $a = 3\varrho + 5$, use $n = m/2$, $\sqrt{x} \leq m < 2\sqrt{x}$, and $f \leq x^{12}$ for large x to obtain

$$n[E : \mathbb{Q}] \leq 2x^{25/2} \leq 2^{\sqrt{x}-2} \leq 2^{2\varrho}.$$

Then $\zeta_{\mathcal{R}}(a) \le \zeta_E(a) \le \zeta(a)^{[E:\mathbb{Q}]}$ gives

$$n \log \zeta_{\mathcal{R}}(a) \le n[E:\mathbb{Q}]\, 2^{1-a} \le 2^{-n-3}.$$

Here $\log \zeta(a) \le \sum_{j\ge 2} j^{-a} \le 2^{1-a}$ by the integral test. Exponentiating gives the upper bound $1 + 2^{-n}$ for the compact volume.

10.3 Normalization of the finite quotient pairing

In this subsection, we compute the correlation at the initial step of the finite calculation. The Haar normalization of Section 10.2 fixes the constant in the bound of Section 10.10.

Take a conjugation-stable set T as in Section 10.1. The local volume and correlation estimates below also apply when T is a single conjugate pair. We include every place over M when we substitute M^{-2f} for its finite modulus in the adelic estimate. All local parameters in this calculation are restricted to $A_T = \prod_{w\in T} E_w$, with integer ring $\mathcal{R}$ and additive modulus md. At a place over M, N and π_N are units. Thus the same norm and lattice identities hold when T consists only of the places over M, as in Section 14.2.

Section 13.3 uses a set containing the places over N when evaluating the full modulus of π_N. Every conjugation orbit of completions is split or unramified over its fixed field. In fact $d \nmid q_1$. At a rational prime other than d, inertia acts trivially on K, where complex conjugation is nontrivial. At d, inertia acts trivially on $\mathbb{Q}(\zeta_{q_1})$, where complex conjugation is also nontrivial. Thus the inertia groups in E/E^+ are trivial, where E^+ is the fixed field of conjugation. In particular the previous compact measure calculation can be used on all of T.

Write G_T for the original local group, μ for its stage Haar measure, K_T for the compact integral group and J for its congruence subgroup. Let Z be an intermediate group $J \le Z \le K_T$. Put $U = U_{<n}$, $dQ = d\mu/d\eta$ on $U\backslash G_T$, where η is the original unipotent Haar, and

$$\delta_U = \eta(U \cap J), \qquad \mathcal{R}_Z = \mu(Z)\mu(J)/\delta_U^2.$$

Extend the character of U to UJ by right J-invariance and then by zero. Denote it by χ. This is well-defined since the character is trivial on $U \cap J$. The finite Bessel vector $W_Z(y)$ has entry $\chi(yc^{-1})$ for $c \in J\backslash Z$. Its right J-invariance follows from normality of J in K_T.

Write $W = W_Z$. For each c the set of u such that $uyc^{-1} \in J$ is empty or a right coset of $U \cap J$ of mass δ_U. For such u, $z_1 = uy$ lies in Jc and

$$\overline{W(y)_c}\, W(yx)_c = \chi(z_1 x z_1^{-1}).$$

Indeed the phase of the left U translation cancels, and the additional right factor $z_1 c^{-1}$ lies in J. The cosets partition Z, so quotient integration gives

$$\int_Z \chi(z_1 x z_1^{-1})\, d\mu(z_1) = \delta_U \int_{U\backslash G_T} \langle W(y), W(yx)\rangle\, dQ(y).$$

With empty heads the stage-zero form $\mathcal{I}_0^T(1)$ is the integral of the squared modulus of this quotient correlation. Define $\mathcal{M}(Z) = \int_{G_T} |\int_Z \chi(z_1 x z_1^{-1})\, d\mu(z_1)|^2\, d\mu(x)$. The normalized bottom value is therefore

$$\mathcal{I}_0^T(1)/\mathcal{R}_Z = \frac{\mathcal{M}(Z)}{\mu(Z)\mu(J)}.$$

In particular it is nonnegative. On $x \in J$ the inner integral is $\mu(Z)$ by normality. The upper bound on the rest of $\mathcal{M}(Z)$ requires a finite shell estimate.

For use in that estimate, the coordinates of W on the quotient have supports given by the images of UJc. Each support has mass $\mu(J)/\delta_U$ by unfolding the indicator of $Jc = cJ$. These supports are equal or disjoint. If $UJc' = UJc$, write $c' = ujc$ with $u \in U \cap Z$ and $j \in J$. On the common support, $W(y)_{c'} = \chi(u)^{-1} W(y)_c$ by U-covariance and normality of J. The multiplicity in each part is $[U \cap Z : U \cap J] \le 1/\delta_U$, since $\eta(U \cap K_T) = 1$. Thus $B_W = \int WW^* \, dQ$ is a direct sum of rank-one blocks of norm at most $\mu(J)/\delta_U^2$, with trace $\mu(Z)/\delta_U$. It follows that

$$\delta_U \iint |\langle W(y), W(y')\rangle|^2 \, dQ(y) \, dQ(y') = \delta_U \operatorname{tr}(B_W^2) \le \mathcal{R}_Z.$$

The matrix WW^* is well-defined on the quotient by its scalar U-covariance.

10.4 Common entries and the finite terminal density

In this subsection, we express the finite pairing at successive isotropic quotients in measures compatible with Section 10.3. The expression supports the Fourier estimates of Section 10.8 and the terminal density calculation.

Use these same measures and $W = W_Z$. For $a \le \varrho$ denote the finite stage form by $\mathcal{I}_a^T(p, q; \omega)$. It has the two pivot integrations of Section 12.2, now in the finite stage measures, with head factor

$$\mathcal{D}_a^T(p, q) = \prod_{i<a} \operatorname{md}(p_i q_i)^{d_i/2}, \qquad d_i = k + 1 + 2(\varrho - i - 1).$$

The twist is $w_q^{-1} w_p$. For $a < \varrho$, the law of the stage measure θ_a is that of $w_a(c)^{-1}\nu$ for $\operatorname{md}(c)^{d_a} dc \, d\eta_a(\nu)$. To see the density, in the norm chart use the coefficient c of e_a^* and take ν^{-1} acting on $w_a(c) e_a = c e_a^*$. Its d_a free vector coordinates are multiplied by c, and its skew coordinate in the ratio of the two hyperbolic coordinates retains its original Haar. The Weyl element fixes the earlier pairs. The omitted tuples have a zero field component of c and form a null set in this stage chart.

Assume the absolute stage majorant is integrable at a. With $Z_a(s) = h(f_a, s f_a)$, the substitution $\tau = w_a(c)^{-1}\nu\sigma$, $g' = \nu\sigma g$, is valid by the quotient invariance of θ_a on functions left invariant under the stabilizer of f_a. The pivot integrals have this invariance after the tail convolution, including for their absolute majorant. The new point measure is $\eta_a * \theta_a * M_a = M_{a+1}$. The old twist is unchanged and the new head factor is $\mathcal{D}_{a+1}^T(p_c, q_c) = \mathcal{D}_a^T(p, q) \operatorname{md}(c)^{d_a}$. Thus for every measurable $|\phi| \le 1$, absolute integrability transfers through the substitution and

$$\int_{A_T} \phi(c) \, \mathcal{I}_{a+1}^T(p_c, q_c; 1) \, dc = \mathcal{I}_a^T(p, q; \phi \circ Z_a).$$

Here p_c, q_c append the same entry c; one can use the entry 1 when $c \notin A_T^\times$, a Haar-null set. This lower-stage identity uses the integrable majorant on the original projected Bessel data.

For the terminal formula let $\pi = \pi_N|_T$ with $\pi\bar{\pi} = N$ and $h(z, z) = -N$. In $V_\varrho = X_0 \perp A_T z$, the last-entry condition and the fixed norm are

$$e(t, x) = x - N^{-1} t z, \qquad h(z, e(t, x)) = t, \qquad h(e(t, x), e(t, x)) = -N \iff h(x, x) = t\bar{t}/N - N.$$

In the chosen lattice basis the exceptional vector is $x_{k_0}/\bar{\pi} + z/N$. Thus the change to the basis $(x_i, \pi^{-1} z)$ of Gram $\operatorname{diag}(1, \dots, 1, -1)$ has determinant $\bar{\pi}/\pi$. Since T is stable under conjugation,

$\mathrm{md}_T(\bar{\pi}) = \mathrm{md}_T(\pi)$. The basis change therefore preserves the product Haar, taking both places together at a split pair. The coordinate of $e(t,x)$ along $\pi^{-1}z$ is $-\bar{\pi}^{-1}t$.

Use an integral hyperbolic pair in the standard form on X_0, as in Section 10.1, to solve the norm equation both on X_0 and on V_ϱ. With the norm value indexed by the same skew unit, the other chart coordinates and their Haar agree. The change in the last coordinate gives

$$dQ_\varrho(e) = \mathrm{md}(\pi)^{-1}\, d\nu_{t\bar{t}/N-N}(x)\, dt,$$

in the variables $e(t,x)$ on the last-vector quadric. Here $d\nu_c$ is the original norm-fiber measure on X_0 at norm c, with the integral skew line of volume one in its norm-value coordinate.

These fiber measures vary weakly continuously on compact step tests, also at zero norm. In a bounded test box, use hyperbolic coordinates x_i, y_i on $h \geq 1$ hyperbolic planes in the unimodular subspace as in Section 10.1, treat the remaining coordinates as parameters shifting the norm value, and truncate the set where all x_i in a given factor have large valuation. Outside that set we can solve in a trace coordinate with nonzero coefficient, on finitely many compact open pieces. For fixed x_i the Haar integral in the remaining coordinates is locally constant in the norm value, uniformly on each piece.

The density is bounded by a constant times q^j where j is the minimum valuation, by the calculation of Section 10.1 after scaling the box. The measure in the x_i with this valuation is at most a constant times q^{-2hj} for $h \geq 1$ hyperbolic pairs, including at a split orbit. Thus the truncated tails tend uniformly to zero. Taking products proves the assertion on the semilocal ring. This continuity identifies the chart description of the fiber measure at every norm value.

Let $H = G_n$, the stabilizer of z in G_ϱ, with the terminal Haar measure dh satisfying $\mathrm{vol}(H \cap K_T) = 1$, as in Section 10.2. Define

$$V_p(l) = \mathcal{D}_\varrho^T(p,1) \int_{G_T} W(y) \otimes W(l w_p^{-1} y)^*\, d\mu(y) \qquad (l \in G_\varrho).$$

These integrals have compact support in y, l, locally uniformly in the nonzero entries of the head p. Indeed nonzero integrands give $y = u_1^{-1}k_1,\ l w_p^{-1} y = u_2 k_2$ with $u_i \in U,\ k_i \in K_T$. Thus $u_1 w_p l^{-1} u_2 = k_1 k_2^{-1}$. In the two isotropic blocks and their middle block, the lower-left block of the latter product is $E_1 D_p A_2$, with $D_p = \mathrm{diag}(p_i)$, E_1 lower and A_2 upper unitriangular.

Write the three block rows of u_i as $(A_i, C_i, B_i), (0, I, D_i), (0, 0, E_i)$, and write $P = u_1 w_p l^{-1} u_2$. Elimination in $P_{31} = E_1 D_p A_2$ recovers E_1, A_2 and their inverses using the nonzero pivots p_i. The remaining required blocks are obtained from

$$P_{32} = E_1 D_p C_2, \quad P_{21} = D_1 D_p A_2, \quad P_{22} = l^{-1} + D_1 D_p C_2,$$
$$P_{11} = B_1 D_p A_2, \quad P_{33} = E_1 D_p B_2.$$

The other blocks of u_1, u_2, and the inverses of u_1, u_2, l, are bounded by the isometry equations in the fixed Gram basis. These bounds are uniform for the head in a compact neighborhood in $(A_T^\times)^\varrho$. Since $y = u_1^{-1}k_1$, they bound the two matrices y, l and their inverses, so both range over compact sets. On these sets a small change in the head and in l changes the arguments of W by a right J-translation, uniformly by compactness. Consequently $V_p(l)$ is locally constant in both variables with uniform neighborhoods on compact sets.

On the orbit of z, set

$$\Phi_{p,q}(e) = \int_H \langle V_p(hl), V_q(hl) \rangle\, dh, \qquad e = l^{-1}z.$$

Right invariance of dh makes this independent of the lift l. Since $N \in A_T^{\times}$, Witt's theorem in each factor of the semilocal base shows that the orbit is the whole quadric $h(e,e) = -N$. This is a compactly supported, locally constant function on the quadric. For the local section at $e_0 = l_0^{-1} z$, take e close to e_0, and put $b_0 = h(e_0, e)$ and $d_0 = \bar{b}_0 / b_0$. Shrink the neighborhood in each local factor so that $(d_0 - 1)/4$ lies in its maximal ideal. The polynomial $u^2 + u - (d_0 - 1)/4$ has simple residue root 0. Hensel's lemma gives a unique lift u in the maximal ideal, and $\lambda = 1 + 2u$ is the resulting square root of d_0 near 1, including in residue characteristic two.

Uniqueness gives $\lambda\bar{\lambda} = 1$ and $\lambda b_0 = \overline{\lambda b_0}$. After multiplying e by λ, reflection in $e_0 + \lambda e$ sends it to $-e_0$, and reflection in e_0 gives e_0. Their product with the scalar is a unitary $s(e)$ tending to 1. Then $l_0 s(e)$ is the section. The support of $\Phi_{p,q}$ is contained in a compact image under $l \mapsto l^{-1} z$ by the preceding support bound, so $\Phi_{p,q}$ has compact support. On the ambient space it admits a compact step extension by a finite partition into compact open balls.

For the top identity expand the inner product of the two orbital tensors. For fixed nonzero p, q, its absolute majorant in (l, y_1, y_2) has compact support by the bounds just proved, applied to the p factors at y_1 and the q factors at y_2. It is integrable for a bounded entry test. Put $y_2 = y_1 x$ by left invariance, and push the positive Haar measure in y_1 through $y_1 = h_1 \sigma g$, with $h_1 \in H$, σ in the last stage and g in the lower product M_{ϱ}. For each such tuple use right invariance in l, followed by the positive Haar decomposition $l = h_2 \tau (h_1 \sigma)^{-1}$. Tonelli transports the absolute majorant to these stage variables. The weight $\phi(h(z, l^{-1} z))$ becomes $\phi(h(z, \sigma\tau^{-1} z))$, since h_1, h_2 fix z. Both earlier Weyl elements commute with G_{ϱ}, so Fubini gives the two terminal tail correlations.

The product $\mathcal{D}_{\varrho}^{T}(p,1)\mathcal{D}_{\varrho}^{T}(q,1) = \mathcal{D}_{\varrho}^{T}(p,q)$ gives the head density. This proves

$$\begin{aligned} \mathcal{I}_{\varrho}^{T}(p,q;\phi \circ Z_{\varrho}) &= \int \phi(h(z,e)) \Phi_{p,q}(e)\, dQ_{\varrho}(e) \\ &= \int_{A_T} \phi(t)\, \mathrm{md}(\pi)^{-1} \int \Phi_{p,q}(e(t,x))\, d\nu_{t\bar{t}/N - N}(x)\, dt. \end{aligned}$$

The inner expression with its modulus factor is the finite last-entry form. For finitely many heads with the same last entry t it is positive semidefinite, since the pairing on every lift is a Gram matrix and both fiber integrations are positive. It is continuous in the heads and in t by the uniform support and the compact-test continuity of the norm-fiber law. All stage forms, including this terminal form, are divided by the same $\mathcal{R}_Z$ of Section 10.3 in the finite sieve. The frequency and bottom bounds are proved in Sections 10.8–10.10.

10.5 Stable ideals and classes of prescribed denominator

In this subsection, we determine which ideals can occur in the denominators of two correlated vectors. Their indices give the summation variables for Section 10.9 with the lattice stabilizers of Section 10.2.

Use the projected lattice Λ, with its integral unimodular Gram, and the original level scalar M on A_T. Let $F_i = \langle e_0, \dots, e_{i-1} \rangle$. At a pair $Y, Y' = Yx$ with nonzero plain Bessel correlation, there are $u, v \in U$ such that $uY, vY' \in K_T$ and $(vY')^{-1} uY \in J$. Indeed the two vectors have a common nonzero coordinate of W_Z, which gives both moves into the same coset of J. Consequently the support in x of the absolute stage majorant is contained in $\Omega_J := \big(\bigcup_Y Y^{-1} U Y\big) J$.

For $0 \le i < \varrho$, let $\mathfrak{d}_i(Y,x)$ be the set of $h(e_i, w)$ with

$$w \in F_i^{\perp}, \quad w' - w \in F_i, \quad Y^{-1} w \in \Lambda, \quad Y^{-1} w - (Y')^{-1} w' \in M\Lambda.$$

It is a full fractional ideal. Write $w' = w + d$ and map (w, d) to the pair $\big(Y^{-1}w, M^{-1}(Y^{-1}w - (Y')^{-1}(w+d))\big)$. The map on the full product is invertible. The preimage of the product lattice intersects $F_i^{\perp} \oplus F_i$ in a lattice, and projection to $h(e_i, w)$ is surjective on this subspace. This gives a compact open integer module in each field factor. At the active pair it is integral: $u^{-1}e_i \equiv e_i \bmod F_i$ and $uw = (uY)Y^{-1}w$ is integral, so $h(e_i, w) = h(e_i, uw) \in \mathcal{R}$. Write $\mathfrak{d}_i = c_i\mathcal{R}$. Its depth is $D_i = \mathrm{md}(c_i)^{-1}$. The two transported vectors in the definition are integral, and the congruence depends only on their classes in $\Lambda/M\Lambda$. Replacing (Y, Y') by $(Yj_1, Y'j_2)$ for $j_1, j_2 \in J$ preserves both conditions. Thus the ideal and its depth are locally constant in (Y, Yx), hence in (Y, x).

For any integral self-conjugate $C \in A_T^{\times}$ in all the $\mathfrak{d}_i$, each $w \in F_j^{\perp}$ with $Y^{-1}w \in \Lambda$ has a lift $w' \equiv w \bmod F_j$ such that

$$Y^{-1}w - (Y')^{-1}w' \in MC^{-2(\varrho - j)}\Lambda.$$

For $j = \varrho$, take $w' = v^{-1}uw$. The unipotents act trivially on $F_{\varrho}^{\perp}/F_{\varrho}$, and $(vY')^{-1}uY \in J$ gives the congruence. Suppose the lift is available at $i+1$ with exponent $D = 2(\varrho - i - 1)$. Divide a stable pair at the value c_i by c_i, giving ζ, ζ' of pairing one with e_i, with both transported vectors in $C^{-1}\Lambda$ and their difference in $MC^{-1}\Lambda$.

For the prescribed w put $t = h(e_i, w) \in \mathcal{R}$ and $w_0 = w - t\zeta$. Apply the induction to Cw_0 and divide by C. This produces $w_0' \equiv w_0 \bmod F_{i+1}$, with transported difference in $MC^{-D-1}\Lambda$. The coefficient b_1 of e_i in $w_0' - w_0$ equals $h(\zeta', w_0') - h(\zeta, w_0)$, since $\zeta' - \zeta \in F_i$ and $w_0 \perp F_i$. In the transported lattices, expand this difference by pairing the transported ζ' against the difference for w_0, and the difference for ζ against the transported w_0.

The two terms belong to $MC^{-D-2}\mathcal{R}$. Thus $w' = t\zeta' + w_0' - b_1 v^{-1}e_i$ is the required lift at i, since $(Y')^{-1}v^{-1}e_i$ is integral and $v^{-1}e_i \equiv e_i \bmod F_i$. At $j = 0$, $w' = w$. Testing $w \in Y\Lambda$ gives $C^{2\varrho}(1 - x^{-1})/M$ integral, hence $C^{2\varrho}(x-1)/M$ integral by unitary adjunction. In particular if $x \notin J$ and every proper integral ideal has index at least Q_0, some $D_i \geq Q_0$. Otherwise all c_i are units in $\mathcal{R}$, and $C = 1$ gives $x \in J$.

On a head of length $j \leq \varrho$, assume also that $w_p^{-1}Y, w_q^{-1}Y' \in \mathcal{U}_j$. Choose an integral self-conjugate scale α divisible by all the partial products $P_a = \prod_{i<a} p_i$, $Q_a = \prod_{i<a} q_i$, and by M if $j > 0$. The products here are integral at nonzero contributions. Indeed take two moves $s, t \in R_j$ so that $sY, tw_p^{-1}Y \in K_T$. Their ratio is $sw_pt^{-1} \in K_T$. The minor in the first a flag columns and the rows indexed by $e_0^*, \ldots, e_{a-1}^*$ equals P_a by triangularity. For $0 \leq a < j$ take the first $a+1$ columns, the dual rows e_b^* for $b < a$, and a last row d outside those dual rows. The right factor t^{-1} acts on these columns by a unitriangular matrix. Removing it leaves the columns $p_i se_i^*$ for $i \leq a$. For $b \leq i$, the e_b^*-entry of se_i^* equals $h(s^{-1}e_b, e_i^*) = \mathbf{1}_{b=i}$. Expansion along the last column gives the integral minor $P_{a+1}(se_a^*)_d$. The entries in the omitted dual rows of se_a^* are zero. Thus $Y^{-1}e_a^* \in \alpha^{-1}\Lambda$ for $a < j$, by multiplication with $(sY)^{-1}$. The same holds for Y' and q. The vector $\alpha^2 e_a^*$ now has both transported vectors in $\alpha\Lambda \subset M\Lambda$, so $\alpha^2 \in \mathfrak{d}_a(Y, x)$.

The denominator ideal $\mathfrak{J}_x$ of the entries of $(x-1)/M$ has generator c, given in each field factor by the uniformizer to the maximum of zero and the negative valuations of those entries. Write $D_x = [\mathcal{R} : \mathfrak{J}_x]$. In the unit-ideal case, $D_x = 1$ is equivalent to $(x - I)/M$ integral, hence to $x \in J$. Indeed the integral unimodular Gram also makes x^{-1} integral when x is integral. At places where M is a unit this condition is precisely membership in the lattice stabilizer. Thus the shell $H = 1$ contributes the original level term. If all depths after the head are less than σ^{-2}, where $0 < \sigma < 1$, take $C = \alpha^2 \prod_{a=j}^{\varrho-1} c_a \bar{c}_a$. It belongs to every stable ideal and the induction gives

$$D_x \leq \mathrm{md}(C)^{-2\varrho} \leq \mathrm{md}(\alpha)^{-4\varrho}\sigma^{-8\varrho(\varrho - j)}.$$

Let $H = 2^t$ be the least dyadic height at least D_x, and $\Lambda_\alpha = 2/\operatorname{md}(\alpha)$. If $H > \Lambda_\alpha^{2400n^2}$, take $\sigma = H^{-1/(800n^2)}$. The displayed bound is then at most $H^{1/100+1/(600n)} < H/2$, contradicting $H/2 < D_x$. Consequently, for each active (Y, Yx) satisfying both head conditions there exists $a = a(Y, x)$, $j \le a < \varrho$, with $D_a(Y, x) \ge \sigma^{-2}$. If $j = \varrho$, the empty product in $C = \alpha^2$ gives the same contradiction, so the remote shell contains no such active pair.

Each part of a shell with one denominator ideal $c\mathcal{R}$ lies in the position box of scale $\bar{c}c$, since $\bar{c}c(x-1)/M$ is integral there. Its loss is $\Lambda' = 2D_x^2 \le \Lambda_\alpha H^2$. The number of boxes is at most $2H^{9n}$. Indeed for $a = 3\varrho + 5 < 9n$ the Euler product of finite-index ideals of $\mathcal{R}$ is bounded by $\zeta(a)^{[E:\mathbb{Q}]} \le 2$ by the scale calculation of Section 10.2. Each ideal of index at most H contributes at least H^{-a} to that sum. This count applies at every projected place.

10.6 Finite correlations on bounded sets

In this subsection, we bound weighted correlations on compact sets of relative positions. The local vector count from Section 10.1 supplies the estimate used for inverse weights in Section 10.7.

From the $k-1$ orthonormal coordinates of X_0 outside the distinguished column in Section 8.1, use $k-1-2m_+ = k-6n-7$ coordinates. Choose two disjoint blocks of size $2m_+ + 2$ among them, with $m_+ = 3n+3$. Each block has rank $2(3\varrho+5)+4$, since $n = \varrho + 1$. Thus the quadratic norm-fiber estimate of Section 10.2 gives the uniform mass bound 2, including at residue characteristic two. The volume exponent is the free rank minus $3n + 1 + m_+$, hence

$$s = (k - 6n - 7) - 2(2m_+ + 2) - (3n + 1 + m_+) = k - 24n - 27.$$

Take E_a^K from the stage sums in Section 11.5, using the finite stage measures and $W_a = \prod_{i<a} D_i^{-s}$ with the depths of Section 10.5. With the finite inverse section moduli $H_{p,T}$ and the two kernels K_1, K_2 in those sums, set

$$\mathcal{W}_j = H_{p,T}^2 \Lambda_\alpha^{3\cdot 10^6 n^3} E_0^{K_1} E_j^{K_2}.$$

For an integral, invertible, self-conjugate scalar γ, let Ω_γ consist of the elements $x \in \Omega_J$ with $\gamma(x-I)/M$ integral, and put $\Lambda' = 2/\operatorname{md}(\gamma)$. The set Ω_J and its complement are unions of right J-cosets, so both are open. The entry condition restricts x to a compact open matrix box. Intersecting with the closed unitary group and with Ω_J gives a compact measurable Ω_γ. Use $K(g, y) = \langle W_Z(g), W_Z(d_0 y)\rangle$, with $d_0 = 1$ or $d_0 = w_q^{-1} w_p$ and starting stage $j_0 = 0$ or $j_0 = j$, respectively. For the original quotient measure Q, we prove

$$\int_{\Omega_\gamma} E_{j_0}^K \, d\mu \le n\delta_U (\Lambda')^{4900n\varrho} \iint |K(g, y)|^2 \, dQ(g) dQ(y). \tag{5}$$

Call the two chosen blocks B_v, B_y and let X' be the span of the remaining coordinates in this set. Their ranks are

$$h_0 = 2m_+ + 2 = 2(3\varrho+5) + 4 = 6n + 8, \qquad d' = k - 6n - 7 - 2h_0 = k - 18n - 23.$$

All three are spanned by original orthonormal vectors of X_0 outside the distinguished column. They are integral orthogonal summands of each remaining space V_j, including at places over N. For $g \in K_T$ and a relative position $x = l^{-1} u l j_x$ with $u \in U$, $j_x \in J$, choose a direct summand $L_0 \subset X'_{\mathcal{R}}$ of rank $D = d' - 3\varrho$ in the kernel of $(u-I)lg^{-1}$. Such a summand exists by saturation

in each local factor, since $u - I$ sends $F_\varrho^\perp$ into F_ϱ and has rank at most 2ϱ. For $w \in L_0$ both $g^{-1}w$ and $(gx)^{-1}w = j_x^{-1}g^{-1}w$ are integral and congruent modulo M.

Let $\phi(v)$ be the inverse volume of $h(v, L_0)$. A basis of L_0 reads D pairing coordinates in an integral invertible change of basis. Counting the minimum valuation in each factor gives

$$\int_{cX'_{\mathcal{R}}} \phi(v)^s \, dv = \mathrm{md}(c)^{d'-s} \prod_{w \in T} \frac{1 - (\mathrm{N}w)^{-D}}{1 - (\mathrm{N}w)^{-(D-s)}} \le 2\,\mathrm{md}(c)^{d'-s}.$$

Indeed the D pairing coordinates have independent integral Haar, up to conjugation of the field factors. At a factor of residue size q_w, the minimum valuation m has probability $(1-q_w^{-D})q_w^{-mD}$ for $m \ge 0$. The zero tuple has measure zero. Multiplication by c scales the pairing ideal by $\bar{c}$; $\mathrm{md}(\bar{c}) = \mathrm{md}(c)$ on the selected conjugation-stable product. Summing the geometric moments gives the displayed formula, with $D - s = 3n + 7$ and the product bounded as in Section 10.2. Take $g \in K_T$ and $\sigma \in G_a$ on the actual stage θ_a, with $r\sigma g \in K_T$ for some $r \in R_{a+1}$. Then $\sigma^{-1}r^{-1}e_a$ is integral. Since $r^{-1}e_a - e_a \in F_a$ and σ fixes the earlier pairs, its integral projection away from those pairs is $\sigma^{-1}e_a$.

Transporting L_0 by σ shows $D_a(\sigma g, x) \le \phi((\sigma^{-1}e_a)_{X'})$. The norm-block estimate of Section 10.1 bounds the conditional mass over X' by 2, so the one-stage integral of D_a^s on the near locus is at most 4. Rebasing g by R_a and using invariance of the earlier depths gives the finite version of (12), with one-stage contribution 4.

Use the U_a coordinates (v, β) of Section 10.2. The e_a^*-column after subtracting the identity has vector part v and e_a-coefficient $z = \beta - \tau h(v, v)$. The integral projection $z \mapsto z - \tau(z + \bar{z})$ recovers β, also at residue characteristic two. The condition of lying in J is exactly divisibility of v, β by M. Thus set

$$\delta_a = \mathrm{md}(M)^{d_a + 1/2}, \qquad \prod_{a<\varrho} \delta_a = \delta_U, \qquad c_a = \delta_a(\Lambda')^{4900n}.$$

For $x_0, x_1 \in \Omega_\gamma$, unitarity gives

$$M^{-1}(x_1 x_0^{-1} - I) = M^{-1}(x_1 - x_0)x_0^{-1} \in \gamma^{-2} M_r(\mathcal{R}).$$

This persists after conjugation by a boxed element. When the relative element belongs to $U_{<a}U_a$, project its e_a^*-column modulo F_a to read the U_a column. The vector and skew parameters then belong to $M\gamma^{-2}(V_{a+1})_{\mathcal{R}}$ and $M\gamma^{-2}\mathcal{R}^-$, respectively.

We apply the forward-weight and square-ascent argument of Section 11.7. For the forward weight take g boxed and $x_0 = g^{-1}y \in \Omega_\gamma$. If $u_a \in U_a$ has vector v and $g^{-1}\nu u_a y \in \Omega_\gamma$ for some $\nu \in U_{<a}$, the column bound applies to u_a. For a stable pair ζ, ζ' at $g, u_a y$, pair its two transported lifts with the common vectors $w \in L_0$ at g, y. The column formula for $u_a w$ gives

$$h(\zeta, w) - h(u_a^{-1}\zeta', w) = \overline{h(e_a, \zeta)}h(v, w) \in M\mathcal{R}.$$

Put $\mathfrak{b} = h(v/M, L_0)$. Each stable coefficient $c = h(e_a, \zeta)$ satisfies $\bar{c}\,\mathfrak{b} \subseteq \mathcal{R}$. If $\mathfrak{b}$ is nonzero in every field component, the coefficient ideal is thus contained in $\overline{\mathfrak{b}^{-1}}$. Conjugation preserves volume on the conjugation-stable product, so its volume is at most $\mathrm{vol}(\mathfrak{b})^{-1}$. This gives $D_a^{-s} \le \phi((v/M)_{X'})^s$, also with the infinite majorant when a component of the pairing ideal vanishes.

Put $v = M(v' + v'')$ with $v' \in X'$ and v'' in its orthogonal complement in the d_a-dimensional vector part. Both lie in the γ^{-2}-multiple of their integer boxes. The integral in v' is at most $2\,\mathrm{md}(\gamma)^{-2(d'-s)}$ by the content integral above. The box in v'' has volume $\mathrm{md}(\gamma)^{-2(d_a-d')}$. The

box for β/M has volume $\mathrm{md}(\gamma)^{-1}$, since γ is self-conjugate and the skew line has one fixed-field coordinate. Restoring M therefore bounds the integral by

$$2\delta_a \,\mathrm{md}(\gamma)^{-2(d_a-s)-1} \le c_a.$$

The forward induction of (16) applies: translate y into the box modulo $U_{<a+1}$, integrate the older unipotents first, and use invariance of depth D_a under their action on the second point. That action transports its stable lift without changing it modulo F_a. For a box move $\sigma u \in G_a U_{<a}$, replace y by σy and ν by $\sigma u \nu \sigma^{-1}$; the first a depths and the position are preserved.

For the square ascent, use the cut by positive mass of $\{\nu \in U_{<a} : g^{-1}\nu y \in \Omega_\gamma\}$. At stage $a \ge j_0,\ a < \varrho$, recenter about a point $u_{a,0}$ of the nonzero support, with $g_c = tug \in K_T,\ u \in U_{<a},\ t = t_1 n_1 \in G_{a+1}U_a$. For u_a on this support compare the positions $x_{u_a} = g^{-1}\nu_{u_a}u_a y$ and $x_0 = g^{-1}\nu_0 u_{a,0} y$ in the box. The conjugate $g_c x_{u_a} x_0^{-1} g_c^{-1}$ belongs to $U_{<a}\, t u_a u_{a,0}^{-1} t^{-1}$. Hence $u_a' = t u_a u_{a,0}^{-1} t^{-1}$ lies in the column box, and the substitution preserves Haar. The vector of u_a is $t_1^{-1}v' + v_0$ for the vectors v', v_0 of $u_a', u_{a,0}$.

Put $y_\sigma = \sigma^{-1} f_{a+1}$. The simple-root sum of $\sigma u_a \sigma^{-1}$ is $-h(v, y_\sigma)$. For $a \ge j_0$, d_0 is the identity on V_a and commutes with this unipotent. Since the inner product is linear in the second argument, the tail correlation satisfies

$$C^K_{a+1}(g, u_a y) = \int_{G_{a+1}} \psi_T\Big(-M^{-1}h(v, y_\sigma)\Big)\, C^K_{a+2}(\sigma g, \sigma y)\, d\theta_{a+1}(\sigma), \qquad \psi_T(z) = \psi_{E,f}(-z/\delta_E).$$

The offset v_0 gives a unit phase in the quadric variable $z = t_1\sigma^{-1}f_{a+1}$. The quadric function is supported on integral z: nonzero correlation implies $\sigma g \in \mathcal{U}_{a+2}$, hence also $\sigma t_1^{-1} g_c = \sigma n_1 u g$, since the extra conjugate is in $U_{<a+1}$. Now use the integral projected-column bound at g_c.

Write $z = z' + z''$ with $z' \in X'$. On the integral quadric box its measure is $d\mu_{\mathrm{quad}} = dz'\, d\nu_{z'}(z'')$. The two normalizations agree by division by the same norm-value Haar. The complement of X' contains an orthonormal block of rank $2(3\varrho+5)+4$. Hence $\nu_{z'}$ has mass at most 2 on this box by Section 10.1. For the quadric function $F(z)$, including the offset phase, fix the complementary coordinate t'' of v'/M. Using the character ψ_T of Section 10.8, put

$$A_{t''}(z') = \int F(z' + z'')\, \psi_T(-h(t'', z''))\, d\nu_{z'}(z'').$$

Cauchy–Schwarz gives $\|A_{t''}\|_2^2 \le 2\int |F|^2\, d\mu_{\mathrm{quad}}$. Integrating the squared Fourier transform of $A_{t''}$ over all of X' gives precisely $\|A_{t''}\|_2^2$ by Plancherel for the integral self-dual Haar. The box of t'' has volume $\mathrm{md}(\gamma)^{-2(d_a-d')}$. Including the skew box and the change from v'/M to v', the square-ascent contribution is

$$2\delta_a \,\mathrm{md}(\gamma)^{-2(d_a-d')-1} \le c_a.$$

Both bounds by c_a follow from $d_a - s \le r - s = 26n + 26$. The case of zero cut mass follows by Tonelli. A simultaneous move by G_{a+1} preserves the next cut, since it normalizes the older unipotent with modulus one. At $a = \varrho$, Cauchy–Schwarz on the terminal stabilizer contributes at most one by the compact coset bound.

For the summand of $E^K_{j_0}$ indexed by $l \ge j_0$, the substitution $x = g^{-1}uy$, $u \in U_{<l}$, changes the second Bessel factor by the scalar character of $d_0 u d_0^{-1}$. Its value is independent of the higher pivots, because G_{l+1} fixes the two flag vectors in each simple-root entry of this conjugate. Thus the modulus of the tail correlation is constant under the substitution. The forward integral uses every stage $a < l$, including those with $a < j_0$, and depends only on the close-position cut and

the weights D_a^{-s}. The square ascent uses $l \le a < \varrho$, where d_0 centralizes U_a. The two products give

$$\prod_{a<l} c_a \prod_{l\le a<\varrho} c_a = \delta_U(\Lambda')^{4900n\varrho}.$$

The terminal stage has mass at most one. Applying the product Haar law as in (18) and summing the at most n terms proves (5).

For the twisted kernel, conjugation by d_0 on U has modulus $J_{d_0} = \prod_{a<j} \mathrm{md}(p_a/q_a)^{d_a+1}$. Indeed $d_0 e_a = (p_a/q_a)e_a$. On the Heisenberg vector the remaining unitary factor has determinant of modulus one, and the scalar has modulus $\mathrm{md}(p_a/q_a)$ per coordinate. On the skew centre the scalar is $(p_a/q_a)\overline{(p_a/q_a)}$, with modulus $\mathrm{md}(p_a/q_a)$ on that fixed-field line. Thus $\int F(d_0 y)dQ(y) = J_{d_0}\int F(y)dQ(y)$, since changing the unipotent variable in the Haar disintegration under $y \mapsto d_0 y$ contributes $J_{d_0}^{-1}$. The head-density identity is

$$(\mathcal{D}_j^T)^2 H_{p,T}^2 J_{d_0} = \prod_{a<j} \mathrm{md}(q_a)^{-1} \le \Lambda_\alpha^2,$$

where the last inequality follows from $\alpha/\prod_{a<j} q_a$ being integral.

For a measurable $\Omega \subseteq \Omega_\gamma$ and a measurable $f_0 \ge 0$ with $f_0^2 \le e_*^2 \mathcal{W}_j$, $e_* \ge 0$, the position-box estimate is

$$\mathcal{D}_j^T(p,q)\int_\Omega f_0\, d\mu \le e_*\, \Lambda_\alpha^{1500000n^3+1}(\Lambda')^{25000n^2}\mathcal{R}_Z$$

by Cauchy–Schwarz, (5), $n(\Lambda')^{4900n\varrho} \le (\Lambda')^{25000n^2}$, and the quotient energy bound of Section 10.3.

10.7 Reciprocal weights and absolute pivot bounds

In this subsection, we obtain absolute integral estimates for both kernels before imposing an oscillatory weight. Section 10.6 supplies their local weights, which also enter the Fourier argument of Section 10.8.

Use the finite depths in $I_a = \mathbf{1}_{\mathcal{U}_a} W_a^{-1}$ and in the sums B_a^K of Section 11.5. Fix $x \in \Omega_J$ and write $C_\alpha = \Lambda_\alpha^{10^6 n^3}$ and $H_{p,T} = \prod_{a<j} \mathrm{md}(p_a)^{-d_a-1/2}$. We prove

$$\int_{U_{<j}} I_j(w_p \nu g)\, d\nu \le C_\alpha H_{p,T} \qquad (g \in \mathcal{U}_j). \tag{6}$$

Here α/P_a is integral for every initial segment $P_a = \prod_{b<a} p_b$. To rebase g into K_T by $\gamma u_0 \in G_j U_{<j}$, conjugate the older unipotents by γ and use right translation by u_0. These preserve Haar, and γ commutes with w_p and preserves I_j.

For $g \in K_T$, take $u \in U_a$, $\nu \in U_{<a}$ with $w_{p,<a+1} u\nu g \in \mathcal{U}_{a+1}$. Set $s_0 = (u\nu)^{-1}$ and choose $t \in R_{a+1}$ so that $t w_{p,<a+1} u\nu g$ is integral. The ratio of g and this matrix is $s_0 w_{\bar{p},<a+1} t^{-1} \in K_T$. Apply the row-replacement formula for minors in Section 10.5 to get $\bar{P}_{a+1} s_0 e_a^* \in \Lambda$. Divide by $\bar{P}_a$ and project away the earlier pairs, which removes ν^{-1} modulo F_a. Thus $cu^{-1}e_a^*$ is in the α^{-1}-box on the a-th and later pairs, where $c = \bar{p}_a$.

On the common summand L_0 of Section 10.6, impose the a equations $h(e_b^*, \nu w) = 0$ for $b < a$. Their saturated kernel contains a primitive submodule of rank $d' - 3\varrho - a$. The equations ensure that $w_{p,<a+1} u\nu w$ is orthogonal to F_a. That vector gives equal stable lifts with coefficient $h(cu^{-1}e_a^*, w)$, as the lower unipotent acts trivially modulo F_a on $F_a^\perp$. Hence $D_a \le \phi(v_{X'})$ for v

the V_{a+1}-part of $cu^{-1}e_a^*$ and ϕ defined using this smaller submodule. For $D = d' - 3\varrho - a$ we have $D - s \geq 2n + 8$. With $n[E : \mathbb{Q}] \leq 2^{2\varrho}$ from Section 10.2,

$$\log \zeta_{\mathcal{R}}(D - s) \leq [E : \mathbb{Q}]\, 2^{1-(D-s)} \leq 2^{-9}/n.$$

The scaled content integral is therefore at most $2\,\mathrm{md}(\alpha)^{-(d'-s)}$.

In the Heisenberg coordinates (ω, β) of u^{-1}, two allowed centers at fixed ω satisfy $c(\beta - \beta') \in \alpha^{-1}\mathcal{R}$. In a nonsplit factor c is a unit times a power of a fixed-field uniformizer. In a split pair, the skew coordinate is $(z, -z)$, so the smaller of the two volume constraints is at most their geometric mean. The skew fiber measure is thus bounded by $\mathrm{md}(c\alpha)^{-1/2}$.

The vector change $v = c\omega$ contributes $\mathrm{md}(c)^{-d_a}$ in the integral. Together with the scaled content and the complementary vector box, the integral of D_a^s on the indicated neighborhood is bounded by

$$2\,\mathrm{md}(p_a)^{-d_a-1/2}\,\mathrm{md}(\alpha)^{-(d_a-s+1/2)}.$$

The earlier depths and near condition are unchanged by the added moves in G_a. Indeed, write $w_{p,a}$ for the Weyl element on the a-th pair alone. For $u \in U_a$ and $\nu \in U_{<a}$,

$$w_{p,<a+1}u\nu = (w_{p,a}u)w_{p,<a}\nu, \qquad w_{p,a}u \in G_a.$$

This factor preserves the first a depths. Since $\mathcal{U}_{a+1} \subseteq \mathcal{U}_a$, the earlier inverse weights factor out as $I_a(w_{p,<a}\nu g)$ on the support. The primitive submodule used in the integral over U_a may depend on g, x, ν. Its rank and the integral bound are uniform in these variables. Iteration in the Haar order $u\nu$ gives (6), since $a < j \leq \varrho < n$ and $d_a - s \leq 26n + 26$.

For any nonnegative measurable left-R_j-invariant function B supported on $\mathcal{U}_j$ and any σ-finite measure ρ on G_T, Tonelli and $B(\nu g) = B(g)$ for $\nu \in U_{<j}$ give

$$\int B(g)I_j(w_p g)\, d(\eta_{<j} * \rho)(g) \leq C_\alpha H_{p,T} \int B(g)\, d\rho(g).$$

The same estimate holds with $w_p^{-1} = w_{\bar{p}}$, since $\alpha = \bar{\alpha}$ and conjugation preserves the modulus.

On $\mathcal{U}_{j+1}$ the stage j mass is at most $V_0 = \zeta_{\mathcal{R}}(3\varrho+5) \leq 2$, by Section 10.2. Apply the ordinary Cauchy–Schwarz bound and the first-deep-stage argument of Section 11.6, using the finite one-stage contribution 4 in Section 10.6. Explicitly, let P_j^K be the integral of $|C_{j+1}^K(\sigma g, \sigma g x)|$ over θ_j. Then $(P_j^K)^2 \leq V_0 B_j^K I_j$. If every nonzero terminal contribution has some stage $a \geq j$ with $D_a \geq e^{-1}$, $0 < e \leq 1$, stop the tail expansion at the first such a. The no-stop branch vanishes, since higher groups preserve each earlier depth. Dropping the restrictions at earlier stages gives $P_j^K(g) \leq \sum_{a=j}^{\varrho-1} T_a(g)$, where

$$T_a(g) = \int \mathbf{1}_{\{D_a(tg,x) \geq e^{-1}\}} |C_{a+1}^K(tg, tgx)|\, d\theta_{[j,a]}(t).$$

On this support $W_a^{-1} = D_a^{-s} W_{a+1}^{-1} \leq e^s W_{a+1}^{-1}$. Cauchy–Schwarz gives

$$\begin{aligned} T_a(g)^2 &\leq e^s \left(\int I_{a+1}(tg)\, d\theta_{[j,a]}(t) \right) \left(\int A_a^K(tg)\, d\theta_{[j,a]}(t) \right) \\ &\leq e^s 4^{a+1-j} I_j(g) B_j^K(g). \end{aligned}$$

The first integral uses the one-stage estimate at each of the $a + 1 - j$ stages. The second is one summand of $B_j^K(g)$. Now $\varrho^2 4^\varrho \leq C_\alpha$ gives $(P_j^K)^2 \leq e^s C_\alpha B_j^K I_j$.

Write $z = w_p^{-1} g$. In the two-pivot majorant $\mathcal{M}_j(x) = \int P_j^{K_1}(g) P_j^{K_2}(z)\, dM_j(g)$ the product of the two weights and energies can be crossed as

$$[B_j^{K_1}(g) I_j(z)]\,[B_j^{K_2}(z) I_j(g)].$$

Cauchy–Schwarz and (6) bound the first integral by $H_{p,T} C_\alpha E_0^{K_1}$. For the second change $g = w_p z$. The integrand in z is left G_j-invariant, and $d\mu = d\mu_j * M_j$ with μ_j Haar on G_j. Since w_p commutes with G_j, the invariant-function quotient identity in Section 11.5 applies with the finite measures.

The second integral is then at most $H_{p,T} C_\alpha E_j^{K_2}$. It follows that $\mathcal{M}_j^2 \le V_0^2 \mathcal{W}_j$. On a shell $H > \Lambda_\alpha^{2400 n^2}$, take $\sigma = H^{-1/(800 n^2)}$. If $P_j^{K_2}(z) > 0$, its iterated integral contains a nonzero terminal kernel. All later groups lie in G_j, and the twist $w_q^{-1} w_p$ commutes with their product. Thus for the same $t \in G_j$,

$$W_Z(t w_p^{-1} g) \neq 0, \qquad W_Z(t w_q^{-1} g x) \neq 0.$$

Both arguments lie in $U K_T \subseteq \mathcal{U}_j$. For every $h \in G_j$, left translation by $h t^{-1} \in R_j$ and commutation of the two Weyl elements with h give

$$w_p^{-1} h g, \quad w_q^{-1} h g x \in \mathcal{U}_j.$$

Whenever $K_1(hg, hgx) \neq 0$, Section 10.5 therefore supplies an index $j \le a(h) < \varrho$ with $D_{a(h)}(hg, x) \ge \sigma^{-2}$. The first stopping index may depend on h. For $j = \varrho$ this implication rules out every such terminal contribution. If the second pivot vanishes its product with the first is zero. Otherwise the estimate for the first pivot with $e = \sigma^2$ and the ordinary estimate for the second give $\mathcal{M}_j^2 \le V_0^2 \sigma^{2s} \mathcal{W}_j$.

At $j = 0$ both pivots coincide and M_0 is the atom at the identity. If every active point has some $D_a \ge e^{-1}$, using the deep bound for both identical factors gives $\mathcal{M}_0^2 \le e^{2s} \mathcal{W}_0$; the factor C_α^2 from the two bounds is at most the C_α^3 in $\mathcal{W}_0$.

10.8 Local Fourier estimates for the leading coordinates

In this subsection, we use translation invariance of the finite kernels to identify the frequencies that can contribute. The absolute integral estimates of Section 10.7 control their magnitudes in the summation of Section 10.9.

Fix the finite data of Section 10.4. Use $d' = k - 18n - 23$, the dimension of X', and $C_\alpha = \Lambda_\alpha^{10^6 n^3}$. Write "box a" for vectors or matrices with integral coordinates after multiplication by α^a.

Suppose $g, w_p^{-1} g \in \mathcal{U}_j$. Factor a move at g into the middle group and older unipotent, and choose $t \in G_j$, $u \in U_{<j}$ such that $utg \in K_T$. Since t commutes with w_p, choose $v \in R_j$ with $v w_p^{-1} t g \in K_T$. Their ratio is $u w_p v^{-1} \in K_T$. The minors in Section 10.5 give $P_a \in \mathcal{R}$ and $P_{a+1} u e_a^* \in \Lambda$ for $a < j$, with the leading products $P_a = \prod_{b<a} p_b$.

Use the integral quotients α / P_a to conclude $p_a^{\pm 1} \in \alpha^{-1} \mathcal{R}$ and $u e_a^* \in \alpha^{-1} \Lambda$. In Witt order these j dual columns of u have upper, middle and lower blocks C, D, E, with E lower unitriangular. Its upper diagonal block is $A = (E^*)^{-1}$ and its upper middle block is $-A D^* J_0$, with J_0 the integral middle Gram. The nilpotent inverse series bounds A in box $j - 1$, hence u, u^{-1} in box j.

Thus tg is in box j, and $tw_p^{-1}g$ in box $j+1$, also when $j=0$ by taking $t=g^{-1}$. For $\sigma \in G_j$ on the support of a pivot tail at g, choose ζ with $\zeta\sigma g \in K_T$, $\zeta \in R_{j+1}$. Projecting $(tg)(\zeta\sigma g)^{-1}f_j$ off the earlier pairs gives $t\sigma^{-1}f_j$, since $\zeta^{-1}f_j = f_j \bmod F_j$. The orbit vector is therefore in box j, and the corresponding vector from the pivot at $w_p^{-1}g$ is in box $j+1$. Here $f_\varrho = z$, so the final pivot also uses its norm quadric, with the stabilizer integration occurring after that pivot.

Use the finite character $\psi_T(x) = \psi_{E,f}(-x/\delta_E)$ on A_T. In each field component the trace-dual of the integers is the inverse different. Thus for ψ_T the annihilator of the integer lattice is that same lattice, and its Haar measure of mass one is self-dual, including above 2. The level factor M^{-1} enters through the Bessel character. On two norm quadrics in V_j, test on boxes a, a', and put $c_j = \dim V_j - d'$. The operator with kernel $\psi(\eta h(v,y))$, $\eta \in A_T^\times$, has squared L^2-operator norm on these boxes at most

$$4\,\mathrm{md}(\eta)^{-d'}\,\mathrm{md}(\alpha)^{-(a+a')c_j}.$$

Use the three orthogonal summands X', B_v, B_y fixed in Section 10.6. Impose the norm of v on B_v and the norm of y on B_y. In a box of exponent a' for y, a block of rank h_0 has conditional volume

$$\mathrm{vol}(B_y \mid \text{fixed norm}) \le 2\,\mathrm{md}(\alpha)^{-a'(h_0-1)}.$$

This follows from Section 10.1 by scaling the norm-value Haar by $\mathrm{md}(\alpha)^{-a'}$. The other $c_j - h_0$ nonfree coordinates contribute their box volumes, giving the bound $2\,\mathrm{md}(\alpha)^{-a'(c_j-1)}$. The same bound with a applies to the nonfree coordinates of v.

Write the two variables, in the orthogonal order $X', B_v, B_y, (X' \perp B_v \perp B_y)^\perp$, as $v = (x,b,c,z)$ and $y = (t,b',c',z')$. Fix c,z,b',z'. The norm of b is determined by x,c,z, and that of c' by t,b',z'. For a bounded test F on the second box put

$$H(t) = \int F(t,b',c',z')\psi(\eta h(c,c'))\,d\nu_t(c'),$$

where $d\nu_t$ is its indicated block fiber measure. Its mass bound and Cauchy–Schwarz bound $|H(t)|^2$ by that mass times the corresponding integral of $|F|^2$. The remaining phase in the t-integral is $\psi(\eta h(x,t))$ times a scalar of modulus one independent of t. Its absolute square is therefore independent of b. First apply Cauchy–Schwarz in b', z', then use the mass bound on the permitted b's at each x. We may now extend the x-integration to all of X'. Plancherel gives $\mathrm{md}(\eta)^{-d'}\int |H(t)|^2\,dt$. Indeed in orthonormal coordinates the change of Fourier variable is $x \mapsto \eta\bar{x}$. Conjugation preserves integer Haar: it interchanges the two coordinates of a split pair and acts by an integral automorphism at a nonsplit place. Multiplication by η has Jacobian $\prod_{w\in T}|\eta_w|_w^{d'} = \mathrm{md}(\eta)^{d'}$, even for unequal valuations at the two members of a split pair. The remaining Fourier transform is isometric for the self-dual Haar just computed. Integrating in c,z,b',z' gives the product of the two block mass bounds and the two complementary box volumes, namely $4\,\mathrm{md}(\alpha)^{-(a+a')(c_j-1)}$, besides the Plancherel factor and $\|F\|_2^2$. Compact boxes and the uniform fiber bounds justify Fubini for bounded tests. Truncation in L^2 gives the general estimate. Since α is integral, we can enlarge the exponent $c_j - 1$ to c_j. If the box for v has no norm equation, only the block B_y contributes a conditional volume and the constant is 2. All scalings use the actual finite product, including the split and unramified quadratic factors.

For the common-entry character $\omega = \psi(\xi Z_j)$, the two pivot variables give the character of

$$Z_j(\sigma\tau^{-1}) = h(\sigma^{-1}f_j, \tau^{-1}f_j).$$

The pointwise double pivot is zero unless both starting points $g, w_p^{-1}g \in \mathcal{U}_j$. Rotate the two orbit variables by the same t. The Hermitian pairing is unchanged, and its values lie in $\alpha^{-(2j+1)}\mathcal{R}$. It suffices that a frequency $\eta \in A_T^\times$ agree with the original character on $\alpha^{-200n^2}\mathcal{R}$. The amplitudes factor through the pivot orbits by the stabilizer invariance of the tails; rotating the orbit variable preserves the measure. The two-box estimate and Cauchy–Schwarz bound the square of this double pivot by

$$\mathrm{md}(\eta)^{-d'} C_\alpha \, B_j^{K_1}(g) I_j(g) \, B_j^{K_2}(w_p^{-1}g) I_j(w_p^{-1}g).$$

Indeed the squared pivot norms are bounded by $B_j^K I_j$, and $c_j \le r - d' = 20n + 22$ makes the box loss at most C_α. Cross the weights and apply the two inverse-weight integrations as in Section 10.7. For the integral $f_0(x)$ of the modulus of the double pivot over M_j, this gives

$$f_0(x)^2 \le \mathrm{md}(\eta)^{-d'} \mathcal{W}_j(x). \tag{7}$$

For the unequal heads take $0 \le i < \varrho$, $j = i + 1$, $\omega = 1$ and $\xi = M^{-1}(p_i^{-1} - q_i^{-1}) \in A_T^\times$. Let $d = w_q^{-1} w_p$. Write $\mathfrak{s}$ for the simple-root sum in the Bessel character. A stage-i element u with vector coordinate $v(u)$ has $\mathfrak{s}(u) = -h(v(u), f_j)$. Conjugation by d multiplies this root by p_i/q_i, because d acts by that scalar on e_i and fixes V_j. On moving u through the next pivot σ, the difference of the two character values is thus the character of

$$-M^{-1}(p_i/q_i - 1) h(v(u), \sigma^{-1} f_j) = -\xi h(\bar{p}_i u^{-1} e_i^*, \sigma^{-1} f_j).$$

The equality uses conjugate-linearity in the first argument and the V_j component $-v(u)$ of $u^{-1}e_i^*$. Thus, with the same orbit measure also at $j = \varrho$, where $f_j = z$,

$$C_j^{K_2}(ug, ugx) = \int_{G_j} \psi_T\Big(-\xi h(\bar{p}_i u^{-1} e_i^*, \sigma^{-1} f_j)\Big) \, C_{j+1}^{K_2}(\sigma g, \sigma g x) \, d\theta_j(\sigma).$$

Indeed $\sigma u \sigma^{-1} \in U_i$ and its two actions on the kernel give the displayed difference of the simple-root characters.

The resulting section estimate for the twisted tail is

$$\begin{aligned} &\int_{U_i} \mathbf{1}_{\{w_p u g \in \mathcal{U}_j\}} |C_j^{K_2}(ug, ugx)|^2 \, du \\ &\quad \le \mathrm{md}(\xi)^{-d'} \, \mathrm{md}(p_i)^{-d_i - 1/2} C_\alpha \int |C_{j+1}^{K_2}(\sigma g, \sigma g x)|^2 \, d\theta_j(\sigma). \end{aligned} \tag{8}$$

To prove it, at each u in the support both ug and $w_p u g \in \mathcal{U}_j$. Apply the simultaneous box bound with $w_p u g$ as the first argument, obtaining $t_u \in G_j$ with $t_u u g$ in box $a_0 = j + 1$. Fix one such point u_* and its t_*. The substitution $m = t_* u_* u^{-1} t_*^{-1}$ preserves U_i Haar. Since t_u, t_* fix e_i^*,

$$(m - I) e_i^* = \big((t_* u_* g)(t_u u g)^{-1} - I\big) e_i^*.$$

The right-hand side lies in box $2a_0$, using the integral inverse Gram and $\bar{\alpha} = \alpha$. The trace-one skew projection keeps the skew parameter in that box, also above 2. Its Haar volume is $\mathrm{md}(\alpha)^{-a_0}$. Use $\bar{p}_i v(m)$ as the vector Fourier variable. Because $p_i \in \alpha^{-1}\mathcal{R}$, it lies in box $2a_0 + 1$, and the vector measure is multiplied by the factor $\mathrm{md}(p_i)^{-d_i}$ under this substitution.

Rotate the quadric in the second variable by t_*. The relation $u^{-1} = u_*^{-1} t_*^{-1} m t_*$ makes the u_* term of the phase a unit factor depending only on the second variable. The variable term is the pairing of the new vector with $t_* \sigma^{-1} f_j$ at frequency $-\xi$. To bound the second vector, put

$B = t_* u_* g$ and $\kappa = \sigma t_*^{-1}$. Then $\kappa B = \sigma u_* g \in \mathcal{U}_{j+1}$ since $\sigma g \in \mathcal{U}_{j+1}$ on the tail support and $\sigma \in G_j$ normalizes $U_{<j}$. Choose $\zeta \in R_{j+1}$ with $\zeta\kappa B \in K_T$. The column $B(\zeta\kappa B)^{-1} f_j$ lies in box a_0. Its projection off the earlier pairs is $\kappa^{-1} f_j = t_* \sigma^{-1} f_j$, since $\zeta^{-1} f_j \equiv f_j \bmod F_j$. Write $c = d_i - d'$. In the squared one-quadric bound the vector box has exponent $2a_0 + 1$ and the quadric box exponent a_0. Including the skew coordinate and the vector Jacobian gives

$$2\,\mathrm{md}(\xi)^{-d'}\,\mathrm{md}(p_i)^{-d_i}\,\mathrm{md}(\alpha)^{-(3a_0+1)c}, \qquad (2a_0+1)c + a_0(c-1) + a_0 = (3a_0+1)c.$$

Since αp_i is integral, $a_0 \le n$ and $c \le 20n + 22$, the inequality $(3n+1)(20n+22) + 1/2 < 10^6 n^3$ absorbs this in $\mathrm{md}(\xi)^{-d'}\,\mathrm{md}(p_i)^{-d_i - 1/2} C_\alpha$. This proves (8).

For the outer integration put $z = w_p^{-1} g$ and $\varphi_l(g) = |C_j^{K_l}(g, gx)|$. The squared product $\varphi_1(g)^2 \varphi_2(z)^2$ is the product of the crossed factors

$$A_i^{K_1}(g) I_j(z), \qquad \mathbf{1}_{\{w_p z \in \mathcal{U}_j\}}\, W_j(z) I_i(w_{p,<i} z) \varphi_2(z)^2.$$

This identity is on the support of the product, where $I_i(w_{p,<i} z) = W_i(g)^{-1}$, since the last Weyl factor belongs to G_i. The first factor integrates to at most $H_{p,T} C_\alpha E_0^{K_1}$: use (6) and then the i-th energy term. For the second factor change $g = w_p z$ in the G_j-invariant quotient. In $M_j = \eta_i * \eta_{<i} * \theta_{[0,i]}$, the weights $W_j(z)$ and $I_i(w_{p,<i} z)$ are invariant under U_i. Apply (8). The next pivot energy bound $B_j^{K_2}/W_j$ cancels W_j. Integrating the remaining inverse weight by (6) at length i gives $C_\alpha E_j^{K_2}$ times the first i section moduli. The section in (8) supplies the last modulus. Thus the second integral is at most $\mathrm{md}(\xi)^{-d'} H_{p,T} C_\alpha^2 E_j^{K_2}$. By Cauchy–Schwarz this proves (7) with $\eta = \xi$.

In either case $f_0 \le \mathcal{M}_j$. Since $d' \ge s$, (7) and the absolute bound give the required squared saving $V_0^2 \min(1, \mathrm{md}(\eta)^{-1/2})^{2s} \mathcal{W}_j$.

10.9 Summation by denominator

In this subsection, we sum the finite integral estimates by ideal index. The smaller bound on large denominators comes from Section 10.7, and the sum gives the estimate at the selected stabilizers in Section 10.10.

Take $0 \le f_0 \le \mathcal{M}_j$ with the pointwise saving $f_0^2 \le (V_0 e_0^s)^2 \mathcal{W}_j$, $e_0 \in [0,1]$. The absolute pivot estimates give this bound for $e_0 = 1$, and also with e_0 replaced by σ on the remote shells, by Section 10.7. The box cover then bounds the shell integral times its head density by $V_0 g_H^s \Lambda_\alpha^{2\cdot 10^6 n^3} H^{10^5 n^2} \mathcal{R}_Z$, where $g_H = e_0$ for $H \le \Lambda_\alpha^{2400 n^2}$, and $g_H = \min(e_0, H^{-1/(800 n^2)})$ otherwise. For $s \ge l + 9 \cdot 10^7 n^4$, split $g_H^s \le e_0^l g_H^{9\cdot 10^7 n^4}$. In the first range, $H^{1+10^5 n^2} g_H^{9\cdot 10^7 n^4} \le \Lambda_\alpha^{29\cdot 10^7 n^4}$. In the second range it is at most $H^{1 - 12500 n^2} \le 1$. Summing H^{-1} gives 2, so

$$\mathcal{D}_j^T(p,q) \int f_0 \, d\mu \le e_0^l \Lambda_\alpha^{10^9 n^4} \mathcal{R}_Z.$$

The absolute pivot bound allows $e_0 = 1$ and gives the integrable majorant used in the lower common-entry identity of Section 10.4.

For the Fourier saving let $\xi \in A_T^\times$ and $D(\xi) = [\mathcal{R} : \mathcal{R} \cap \xi^{-1}\mathcal{R}]$. At a split pair let $q_w = q_{\bar w}$ be the residue size and put $a = v_w(\xi_w)$, $a' = v_{\bar w}(\xi_{\bar w})$. Its factors are

$$\mathrm{md}_{\{w,\bar w\}}(\xi) = q_w^{-a-a'}, \qquad D_{\{w,\bar w\}}(\xi) = q_w^{\max(0,-a)+\max(0,-a')}.$$

Indeed the denominator ideal in each field component has exponent $\max(0, -v_w(\xi_w))$, including when a, a' have opposite signs. For the character $\omega = \psi_T(\xi Z_j)$, take arbitrary proper heads p, q of the same length. By the common rotation in Section 10.8, the pairing of the two pivot vectors lies in $\alpha^{-(2j+1)}\mathcal{R} \subseteq \beta^{-1}\mathcal{R}$, where $\beta = \alpha^{200n^2}$. Take $\eta_w = \xi_w$ if $v_w(\xi) \le v_w(\beta)$ and $\eta_w = \beta_w$ otherwise. Then $\eta - \xi \in \beta\mathcal{R}$, so the characters agree on the Gram support. Write $b_w = v_w(\beta) \ge 0$ and $a_w = v_w(\xi_w)$. For $w_0 = \beta/\eta$ the common-entry choice gives

$$v_w(w_0) = \max(0, b_w - a_w), \qquad v_w(\xi w_0) = \max(a_w, b_w) \ge 0.$$

For the unweighted form with unequal last proper entries keep $\eta = \xi$ and take $w_0 = \beta$, with $\beta\xi$ integral. In either case $v_w(w_0) \ge \max(0, -a_w)$; hence $w_0\mathcal{R}$ is contained in the denominator ideal and $\operatorname{md}(w_0) \le D(\xi)^{-1}$. Put $e_0 = \min(1, \operatorname{md}(\eta)^{-1/2})$. For the common entry $v_w(\eta) = \min(a_w, b_w)$, so $\operatorname{md}(\eta) \ge \operatorname{md}(\xi)$ and $\operatorname{md}(\eta)^{-1} = \operatorname{md}(w_0)/\operatorname{md}(\beta)$. For unequal entries $\eta = \xi$ and this last ratio is one. Thus $e_0 \le \operatorname{md}(\xi)^{-1/2}$ and

$$e_0^2 \le \frac{\operatorname{md}(w_0)}{\operatorname{md}(\beta)} \le D(\xi)^{-1}\Lambda_\alpha^{200n^2}.$$

For $l = k - n^7 + 12$ we have $s - l = n^7 - 24n - 39 \ge 9 \cdot 10^7 n^4$, since $n \ge 10^8$ (Section 12.10). Thus the shell saving retains $\operatorname{md}(\xi)^{-(k-n^7)/2} D(\xi)^{-6}$ at contribution $\Lambda_\alpha^{1200n^2}$. This calculation requires a frequency nonzero in every factor, as holds for a nonzero rational frequency or a nonzero difference of two rational inverse heads.

For the normalized bound, a nonzero majorant requires all P_a, Q_a integral, since it requires both pairs of near conditions at a plain point and its twisted image. A nonintegral initial segment makes the form vanish. In the remaining case take

$$B = \prod_{a=1}^{j} P_a Q_a, \qquad \alpha = MB\bar{B}, \qquad \mathfrak{h}(p) = \prod_{a=1}^{j} \operatorname{md}(P_a)^{-n^6/2}, \qquad \mathfrak{h}(q) = \prod_{a=1}^{j} \operatorname{md}(Q_a)^{-n^6/2}.$$

Then $\alpha/M, \alpha/P_a, \alpha/Q_a$ are integral, as are $\alpha/p_i, \alpha/q_i$. Thus $\alpha^2\xi$ is integral for the unequal-head frequency, and $\beta = \alpha^{200n^2}$ is admissible in both cases. The head weights are defined to be zero in the nonintegral-initial segment case. For $4e_1 \le n^6$, one has

$$\Lambda_\alpha^{e_1} = (2/\operatorname{md}(M))^{e_1} \prod_a \operatorname{md}(P_a Q_a)^{-2e_1} \le (2/\operatorname{md}(M))^{n^6} \mathfrak{h}(p)\mathfrak{h}(q).$$

The total shell and denominator loss has $e_1 \le 2 \cdot 10^9 n^4$, so the restriction holds. Each omitted finite factor of the integral scalar M has modulus at most one, so $\operatorname{md}(M) \ge M^{-2f}$ on every projected set T, with $f = [E : \mathbb{Q}]/2$. Equality holds for the adelic sum, where T contains every place above M. Since $M \le b^{23}$, $(2/\operatorname{md}(M))^{n^6} \le b^{(46f+1)n^6} \le b^{n^9 f}$. The normalized finite estimate on the same Z is therefore

$$\frac{|\mathcal{I}_j^T(p, q; \omega)|}{\mathcal{R}_Z} \le b^{n^9 f} \mathfrak{h}(p)\mathfrak{h}(q) \operatorname{md}(\xi)^{-(k-n^7)/2} D(\xi)^{-6}. \tag{9}$$

This holds for the common-entry character and for the unequal-head form with $\omega = 1$ above.

10.10 Finite measure estimate for the stabilizers

In this subsection, we specialize the finite pairing to the stabilizers that occur in the orbital argument. The denominator sum of Section 10.9 bounds the terms outside the level subgroup, leaving the normalized contribution of Section 10.3.

Take empty heads and $\alpha = 1$. Let $Q_0 \geq 2$ bound from below the indices of proper full integral ideals in $\mathcal{R}$. Off J every active pair has some $D_a \geq Q_0$ by Section 10.5. Apply the stage-zero estimate of Section 10.7 with $e = Q_0^{-1}$ to $f_0 = \mathbf{1}_{G_T \setminus J} \mathcal{M}_0$. The absolute shell sum in Section 10.9 applies with $l = t + 10^9 n^4$ whenever $t + 2 \cdot 10^9 n^4 \leq k$. In fact $s - l \geq 10^9 n^4 - 24n - 27 \geq 9 \cdot 10^7 n^4$. Absorbing $2^{10^9 n^4}$ by the spare power of Q_0^{-1} gives $\int f_0 \, d\mu \leq Q_0^{-t} \mathcal{R}_Z$. The quotient correlation has squared modulus at most $\mathcal{M}_0$ by the triangle inequality. With the exact contribution on J from Section 10.3, this yields

$$\mathcal{M}(Z) \leq \mu(Z)\mu(J)(\mu(Z) + Q_0^{-t}). \tag{10}$$

Here $\mathcal{M}(Z)$ denotes the mass of the conjugated character, as in that section. On the entire projected set we may take $Q_0 = 2$ and $t = n$. The compact upper bound then gives

$$\frac{\mathcal{M}(K_T)}{\mu(K_T)\mu(J)} \leq 1 + 2^{1-n} \leq 1 + b^{-6}.$$

The last inequality follows for large x from $n = m/2$ and $\log b \leq x^{1/100}$.

Lemma 10.2 (Finite mass of mixed stabilizers)**.** *For a set $\mathcal{A}$ of pairs above p_0^{lev}, fix at each pair a nonzero vector in the prescribed residue span or a nonzero covector in the opposite-place realization, as in Section 9.1. Let Z be the inverse image of these stabilizers, with the full compact group at the other places of T. Then*

$$0 \leq \frac{\mathcal{I}_0^T(1)}{\mathcal{R}_Z} = \frac{\mathcal{M}(Z)}{\mu(Z)\mu(J)} \leq (1 + b^{-6}) b^{-6k|\mathcal{A}|}.$$

Proof. For the chosen places $\mathcal{A}$ defining $\Gamma_{\mathcal{A}}$, take Z to be the inverse images of its vector or covector fixers on the selected conjugate pairs and the full compact subgroup on the complement in T. This contains J and contains every integral lift with reduction in $\Gamma_{\mathcal{A}}$. On one selected split pair let K, J_w, Y denote the compact, level and fixer inverse image, and put $q = p_0^{\mathrm{lev}} \geq b^{10}$.

The residue group $\mathrm{GL}_r(\mathbb{F}_q)$ acts transitively on the nonzero vectors and on the nonzero covectors. Reduction is surjective by lifting matrices with unit determinant. Thus $\mu(Y) = \mu(K)/(q^r - 1)$. In the Haar measure of Section 10.2,

$$\mu(K) = \prod_{l=k+1}^{r} (1 - q^{-l}).$$

Indeed, in rank l a split vector is a column a and a covector b', with norm $b'(a)$. The integral orbit at an isotropic step consists of primitive a, b' with $b'(a) = 0$: extend a to a basis of $\ker b'$, then append a vector on which b' is one. Its residue cardinality is $(q^l - 1)(q^{l-1} - 1)$. At the last step the norm is $-N$, a unit at the chosen prime above M. The vector a is primitive and $\ker b'$ is an integral complement to its line since $b'(a) = -N$. The residue cardinality is $(q^l - 1)q^{l-1}$. The differential $(v, c) \mapsto b'(v) + c(a)$ is surjective at every residue solution under consideration. Each solution therefore has q^{2l-1} lifts at each next residue level, so the fibre mass

is the residue count divided by q^{2l-1}. Multiply for $l = r, r-2, \dots, k+3$ and then $l = k+1$, since the integral Heisenberg factors and terminal compact group have measure one. In particular, $\mu(K) \geq 1 - \sum_{l>k} q^{-l} \geq 1/2 \geq 2^{-n}$.

Put $Q_0 = b^9$ and $a_0 = k - 10^{10}n^4$. For large x, $q^r - 1 \geq Q_0^{a_0+1}$, $9a_0 \geq 6k$, and $a_0 + 1 + n + 2 \cdot 10^9 n^4 \leq k$, since $r \geq k$ and $k/n^4 \to \infty$. A nonunit integral element on this pair has modulus at most $q^{-1} \leq Q_0^{-1}$. Apply (10) with $t = a_0 + 1 + n$. As $Q_0^{-n} \leq 2^{-n} \leq \mu(K)$, it gives

$$\frac{\mathcal{M}(Y)}{\mu(Y)} \leq \mu(J_w)\mu(K)\, 2Q_0^{-a_0-1} \leq b^{-6k}\frac{\mathcal{M}(K)}{\mu(K)}.$$

The second inequality uses $Q_0 \geq 2$ and $\mathcal{M}(K) \geq \mu(K)^2\mu(J_w)$, the contribution on J_w. On the complementary factors use $Z = K$. The group, level and character factor over these conjugation-stable blocks. The inequalities for $\mathcal{M}(Y)/\mu(Y)$ relative to $\mathcal{M}(K)/\mu(K)$ are unchanged under rescaling local Haar, since both ratios scale by its square. Tonelli and the bound for K_T therefore prove

$$0 \leq \frac{\mathcal{I}_0^T(1)}{\mathcal{R}_Z} = \frac{\mathcal{M}(Z)}{\mu(Z)\mu(J)} \leq (1 + b^{-6})b^{-6k|\mathcal{A}|}.$$

This uses the same Z, original finite character, and stage Haar as the terminal and lower-stage forms. □

11 Weighted real orbital integrals

In this section, we prove the real Fourier and square-integral bounds used to compare orbital integrals at a common entry in Section 12.2. The estimates are in the norm-fiber measures fixed below.

11.1 Plancherel on the real norm fibers

In this subsection, we identify a real Fourier transform on two Hermitian norm fibers by ordinary Plancherel. Its measure normalization is used in the phase calculation of Section 11.2 and in the integral of Section 11.7.

At a complex place of E, use ordinary area measure on $\mathbb{C}$, the character $\psi(c) = e^{c-\bar{c}}$, and norm-value coordinate $h(y,y)/2$ with ordinary Lebesgue measure. The factor $1/2$ specifies the real skew coordinate. Write $h(v,y) = v^*Jy$ in the common stage basis. Among the positive orthonormal coordinates choose a free set I of size $d = k-5$ and disjoint two-coordinate blocks B, B', both disjoint from I, inside the definite summand X_0. They occupy $(k-5)+2+2 = k-1$ of its coordinates. Let u be the remaining unit vector. In the normalized real basis of Section 8.3 one has $h(z,z) = -1$. The vectors $(u+z)/\sqrt{2}, (u-z)/\sqrt{2}$ form a hyperbolic pair orthogonal to all three blocks even at the last stage $V_\varrho = X_0 \perp \mathbb{C}z$. This change and the signed basis change p_v are Euclidean unitary and preserve volume and operator norms. Solve the source quadric in B and the target quadric in B'.

For either block the equation at fixed other coordinates is $|z_1|^2 + |z_2|^2 = 2s$. Its fiber measure is sphere probability times $4\pi^2 s$ for $s > 0$, from the complex two-ball volume $2\pi^2 s^2$. At $s = 0$ the conditional measure in the two solved coordinates has mass zero. In a box $|z_j| \leq L^\ell$, the conditional mass is at most $4\pi^2 L^{2\ell}$. For a full vector measure on that side the corresponding bound is $\pi^2 L^{4\ell}$.

The sphere normalization agrees with the hyperbolic chart on each fixed norm fiber. In that chart, with a hyperbolic pair t, v, constrain $2\Re(\bar{t}v) + Q_0 + |z|^2$ and put $c = v/t$. The Jacobian of $v = tc$ is $|t|^2$, which is also the derivative of the norm-value coordinate with respect to $\Re c$. The two factors cancel in the fiber measure, leaving ordinary measure in $\Im c$. At fixed norm, replacing $|z|^2/2$ by $\Re c$ restores the factor $|t|^2$, so together with $\Im c$ it gives d^2v. This yields the stated sphere mass on that chart. Choose the hyperbolic pair outside the solved positive block. Its coordinate t is free in the sphere description, so $t = 0$ is null on every fixed fiber, including at norm zero.

To verify invariance at each norm, write a for the coordinates outside the solved block, $s = b - h(a,a)/2$, and $s_+ = \max(s, 0)$. For $F \in C_c$ the prescribed fiber measure at norm value b satisfies

$$\int F \, d\mu_b = 4\pi^2 \int s_+ \int_{S^3} F(a, \sqrt{2s_+}\,\omega) \, d\omega \, da,$$

where $d\omega$ has mass one. This is continuous in b by dominated convergence on the compact projection of the support of F. An isometry preserves ambient volume and the norm value, so disintegration gives $\int F \circ g \, d\mu_b = \int F \, d\mu_b$ for almost every b. The two continuous functions agree at every b, as required for the rotations of Section 11.2.

Write the source and target as $y = (a, w, z)$ and $v = (a', w', z')$, with w, w' on I and z, z' on B, B', respectively. The two measures on product boxes are

$$da \, dw \, d\nu_{a,w}(z), \qquad da' \, dw' \, d\nu'_{a',w'}(z').$$

Let V, V' be the Lebesgue volumes of the a, a' boxes and M, M' the conditional block bounds. Orthogonality gives

$$h(v, y) = \overline{w'} \cdot w + \overline{z'} \cdot a|_{B'} + A(a'; y).$$

The last term is independent of w', z', and the middle term depends on the second variable only through a.

For a function ϕ supported in the source box and $\xi \neq 0$, define

$$F_{a',a}(w) = \int \psi(\xi A(a'; a, w, z)) \phi(a, w, z) \, d\nu_{a,w}(z).$$

For the fixed first variable, Cauchy–Schwarz in a bounds the squared transform by

$$V \int \left| \int_{\mathbb{C}^I} \psi(\xi \overline{w'} \cdot w) F_{a',a}(w) \, dw \right|^2 da.$$

Integrate z' with contribution M' and extend the w' integration to all of $\mathbb{C}^I$. For one coordinate, writing $w = x + iy$ and $\xi\overline{w'} = A_0 + iB_0$ gives the Euclidean Fourier frequencies $(B_0/\pi, A_0/\pi)$.

Their Jacobian relative to w' is $|\xi|^2/\pi^2$. Euclidean Plancherel therefore gives $(\pi^2/|\xi|^2)^d$. Finally, Cauchy–Schwarz in the source block z and integration over a' give

$$\int_{\text{out}} \left| \int_{\text{in}} \psi(\xi h(v, y)) \phi(y) \, d\mu(y) \right|^2 d\mu'(v) \leq VV'MM' \left(\frac{\pi^2}{|\xi|^2} \right)^d \int_{\text{in}} |\phi(y)|^2 d\mu(y).$$

If either side is a full vector box, take ordinary area measure in each of its two block coordinates. Their product mass is at most $\pi^2 L^{4\ell}$, and the same two applications of Cauchy–Schwarz give the displayed bound with this value of M or M'. All the measures on the boxes are finite, so the integrals apply to square-integrable functions.

11.2 Support and phases in the real integral

In this subsection, we compute the characters of the two real kernels on the support of the truncated coefficient. The Plancherel normalization of Section 11.1 gives the phase used in Section 11.5.

Use the chosen basis and the groups G_i, R_i, U_i as in Section 10.2, now over $\mathbb{C}$. Use $J = J_1 = D_v^* J_0 D_v$ of Section 8.3. Thus the hyperbolic pairs have Gram $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the middle block has Gram $I_k \oplus (-1)$. In these coordinates $J = J^* = J^{-1}$, so $g^{-1} = Jg^*J$ and $\|g^{-1}\| = \|g\|$. The passage to the basis ordered by signs is Euclidean unitary. Let $\mathcal{K}$ be the real box of operator norm at most $R_b = b^{6n+2}$. In this real calculation put $\mathcal{U}_i = R_i\mathcal{K}$. Let r be the ambient rank, $L \geq \max(R_b, 8r)$, $n = \varrho + 1$, and $\Lambda_L = L^2$. Entries of size a have modulus at most L^a. On a head p of length $j \leq \varrho$, assume $|\prod_{a<l} p_a|^{-1} \leq L$ for $l \leq j$, and let w_p send e_a to $p_a e_a^*$ and e_a^* to $\bar{p}_a^{-1} e_a$ for $a < j$.

Suppose $g, w_p^{-1}g \in \mathcal{U}_j$. Factor the move at g in $R_j = G_j U_{<j}$. Thus some $t \in G_j$ gives $stg \in \mathcal{K}$ with $s \in U_{<j}$. The other move can be written $s' w_p^{-1} tg \in \mathcal{K}$ with $s' \in R_j$, since t commutes with w_p. The ratio of the boxed matrices is $s w_p (s')^{-1}$, of norm at most R_b^2. In its first l flag columns, right unitriangular terms from $(s')^{-1}$ preserve every minor of order l in those columns. In the rows indexed by $e_0^*, \dots, e_{l-1}^*$, the columns se_a^*, $0 \leq a < l$, form a lower unitriangular matrix. Hence the minor of the ratio in these rows and columns is $P_l = \prod_{a<l} p_a$.

For $l = a + 1$, use rows $e_0^*, \dots, e_{a-1}^*, d$ in that order. Remove the right unitriangular factor on the first $a + 1$ columns and factor p_b out of column b. The last column se_a^* is zero in the first a rows and the leading a-row minor is one. Expansion in that column gives $P_{a+1}(se_a^*)_d$, including the zero value for a repeated row. Hadamard's inequality bounds these minors by L^{2l}. Dividing by P_l and using consecutive products bounds p_a, p_a^{-1} and the entries of se_a^* by L^{2n+1}. In Witt order s has blocks

$$s = \begin{pmatrix} U & B & C \\ 0 & 1 & D \\ 0 & 0 & U^{-*} \end{pmatrix}, \qquad B = -UD^* J_0,$$

where J_0 is the Gram matrix on the remaining space. The last columns bound C, D, U^{-*}. Since $U^{-*} - 1$ is nilpotent of index at most j, the geometric sum bounds $\|U\|$ by $L^{(2n+4)n}$. Summing the block norms gives $\|s^{-1}\| = \|s\| \leq L^{50n^2}$. Thus tg has size $60n^2$ and $tw_p^{-1}g$ has size $99n^2$. For $j = 0$ take $t = g^{-1}$.

Consider a tail correlation at $(\sigma g, \sigma g')$ that vanishes unless $\sigma g \in \mathcal{U}_{j+1}$ and is invariant under the stabilizer of f_j in G_j. It factors through $y(\sigma) = \sigma^{-1} f_j$. If $t \in G_j$, $t^{-1}q \in U_{<j}$, and qg has size a, then $ty(\sigma)$ has size $a+6$ on its support. In fact $\sigma(t^{-1}q)\sigma^{-1} \in R_{j+1}$, so $(\sigma t^{-1})(qg) \in \mathcal{U}_{j+1}$. Choose a box move $\zeta \in R_{j+1}$ for this point. Projecting the column $(qg)(\zeta\sigma t^{-1} qg)^{-1} f_j$ to the remaining space gives $ty(\sigma)$. The earlier isotropic terms from $\zeta^{-1} f_j$ project to zero. The column norm is at most $rL^a R_b$, as required.

For the character twist, the basis entry is

$$h(f_j, \sigma(\sigma')^{-1} f_j) = h(y(\sigma), y(\sigma')).$$

The two tail functions F_1, F_2 start at $g, w_p^{-1}g$. If both contribute, both initial points belong to $\mathcal{U}_j$. Rotate both orbit variables by the same t just constructed. This preserves the pairing and the two norm-fiber measures. The support sizes are $60n^2 + 6$ and $99n^2 + 6$, so the pairings have modulus at most L^{200n^2}.

For a character agreeing with $\psi(\xi\,\cdot)$ on that box, the integral in the rotated variables is $\iint \psi(\xi h(v,y))F_1(v)\overline{F_2(y)}\,d\mu(v)d\mu(y)$. Cauchy–Schwarz and Section 11.1 bound its square by

$$\Lambda_L^{10^6 n^3}\left(\frac{\pi^2}{|\xi|^2}\right)^{k-5}\|F_1\|_2^2\|F_2\|_2^2 \qquad (\xi\neq 0).$$

To check the box contribution, the hyperbolic change on one positive and one negative line is Euclidean unitary, so a support of size a lies in coordinate boxes of radius L^{a+1}.

The number of ordinary coordinates outside the free and solved blocks is at most $2n+2$. The ordinary volume and the block mass on that side have product at most $\Lambda_L^{(2n+4)(a+2)}$. Apply this to the two support sizes. The conjugated transform uses frequency $-\xi$, of the same modulus.

For the unequal heads put $j=i+1\le \varrho$, $d_0=w_q^{-1}w_p$, and let m_0 be the level unit, equal to 1 at the real place. The pairing kernel has left multiplier $\psi(m_0^{-1}(\mathfrak{s}(d_0u_2d_0^{-1})-\mathfrak{s}(u_1)))$ under (u_1,u_2) in the lower unipotent, with $\mathfrak{s}$ the simple-root sum. The stage-i element $u_i(v,\beta)$ sends o in the remaining space to $o-h(v,o)e_i$, so its simple-root sum is $-h(v,f_{i+1})$. Conjugation by d_0 multiplies this by p_i/q_i. Moving u_i to the left of the next pivot σ changes its vector to σv. Thus the section phase is

$$\psi(-\xi_0 h(v,\sigma^{-1}f_{i+1})),\qquad \xi_0=m_0^{-1}(p_i/q_i-1)=-p_i\xi,\qquad \xi=m_0^{-1}(p_i^{-1}-q_i^{-1}).$$

Equivalently it is $\psi(-\xi h(v',\sigma^{-1}f_{i+1}))$ for $v'=\bar p_i u_i^{-1}e_i^*$, since the two terms in the i-th pair are orthogonal to the orbit vector and the remaining part of $u_i^{-1}e_i^*$ is $-v$.

Write $S\subseteq U_i$ for the support of this correlation restricted to $w_pu_ig\in\mathcal{U}_j$. On S both u_ig and w_pu_ig are near. If S is nonempty, choose $u_*\in S$. Apply the two-point movement to w_pu_ig, u_ig, so that some $t_i\in G_j$ puts t_iu_ig in size $99n^2$. Use t_* for u_*. Recenter by $m=t_*u_iu_*^{-1}t_*^{-1}$, which preserves Heisenberg Haar. Since t_i,t_* fix e_i^*,

$$(m^{-1}-1)e_i^*=\big((t_*u_*g)(t_iu_ig)^{-1}-1\big)e_i^*.$$

The two matrix bounds and $J^{-1}=J$ give size $198n^2+6$ for this column. By the head bound, its product by p_i has size $999n^2$. The choices t_i establish these bounds, and the change of integration variable uses the single t_*.

Set $u=p_i^{-1}$ and write the vector of m as uw. The bound for $u^{-1}(m^{-1}-1)e_i^*$ gives a box of size $999n^2$ for w. At fixed vector of m, the coefficient at e_i of the inverse column depends on the skew center by a translation and a sign. Subtracting the bounds for any two centers in the same fiber shows their distance is at most $2|u|L^{999n^2}$. The skew Haar measure is ordinary length, so that is also a bound for the fiber mass. The vector Jacobian is $|u|^{2d_i}$, $d_i=\dim_{\mathbb{C}}V_{i+1}$, where V_{i+1} is the space orthogonal to the first $i+1$ pairs. The combined factor is $|u|^{2d_i+1}=\mathfrak{m}_i(p_i)^{-1}$, with $\mathfrak{m}_i(p_i)=|p_i|^{2d_i+1}$.

Rotating the next orbit vector by t_* preserves its measure. Its size is at most $99n^2+6$, by applying the projected column argument with $q=t_*u_*$ and $t=t_*$. The recentered vector introduces a constant translation in the Fourier phase, absorbed as a phase of modulus one on the function. Since $h(uw,y)=\bar u h(w,y)$ and $u=p_i^{-1}$, the frequency after the scaling to w is

$$-\xi_0\bar u=\frac{p_i}{\bar p_i}\,\xi,\qquad |-\xi_0\bar u|=|\xi|.$$

This also applies to nonreal pivots. Apply Section 11.1 from the orbit quadric to the vector box, and use the central fiber bound.

If C_{i+1} is the tail in the section integral and C_{i+2} its next tail, this proves

$$\int_S |C_{i+1}(u_i g, u_i g')|^2 \, du_i \le \left(\frac{\pi^2}{|\xi|^2}\right)^{k-5} \frac{\Lambda_L^{10^6 n^3}}{\mathfrak{m}_i(p_i)} \int |C_{i+2}(\sigma g, \sigma g')|^2 \, d\theta_{i+1}(\sigma).$$

Here the box contribution on the two sides and the central factor $2L^{999n^2}$ together are at most $\Lambda_L^{10^6 n^3}$. This calculation uses the tail support and equivariance of the cut pairing kernels. The inverse-depth weights enter when one integrates these estimates over the real positions.

11.3 The terminal Gram form

In this subsection, we write the pairing on the remaining definite space as a Gram form. Its positivity and continuity are used at the terminal step of the descent in Section 12.2.

Let F be the continuous cut Bessel function with values in a Hermitian space I of dimension d_I. Use the original real Haar dg in its orbital tensor and assume the cut support $U_{<n}\mathcal{K}$. In the remaining space at the top, $h(z,z) = -1$ and $z^\perp$ is positive definite. Put $H = U(z^\perp)$. For $c \neq 0$, define $a(c)$ on the plane of one positive unit vector and z by

$$\begin{pmatrix} \bar{c} & \sqrt{|c|^2 - 1} \\ -\sqrt{|c|^2 - 1} & -c \end{pmatrix} \qquad (|c| \ge 1).$$

It is the identity on the other coordinates. For $0 < |c| < 1$ use $a(c/|c|)$. This fixes the earlier pairs. If $\sigma \in G_\varrho$ and $h(z, \sigma z) = c$, the positive part of σz has length $\sqrt{|c|^2 - 1}$. Rotate it by H to the corresponding positive coordinate of $a(c)z$. The resulting quotient fixes z. Thus $\sigma = h_1 a(c) h_2$ with $h_1, h_2 \in H$.

For a head of length ϱ and last entry c, put

$$\begin{aligned} T_p(\sigma) &= \int F(y) \otimes F((w_p \sigma)^{-1} y)^* \, dg(y), \\ A_p &= \frac{\sqrt{c_G}}{d_I} (|c|^2 - 1)_+^{(k-1)/2} \prod_{i<\varrho} |p_i|^{k+1+2(\varrho - i - 1)}, \\ \Xi(p) &= A_p T_p(a(c)), \qquad J_n(p,q) = \langle \Xi(p), \Xi(q) \rangle. \end{aligned}$$

Here c_G is the real open-cell normalization and the star on a vector denotes its Hermitian dual. For p, q with the same last entry and $|c| > 1$, translate y by h_1 and use $w_p h_1 = h_1 w_p$. The two tensors at σ then both arise from the tensors at $a(c)$ by the same unitary operation on the two value factors, using the H-equivariance of F. Consequently

$$J_n(p,q) = \langle A_p T_p(\sigma), A_q T_q(\sigma) \rangle.$$

At $|c| = 1$ the scale factors vanish, since $k > 1$.

This is a positive semidefinite terminal form by its Gram definition. It is continuous in the nonzero head entries. Indeed inverse partial products are bounded on a small neighborhood. Nonvanishing of both factors in $T_p(a(c))$ gives a move at y in $U_{<n}$ and a move at $w_p^{-1} y$ in R_ϱ, as $a(c) \in G_\varrho$. The two boxed matrix bounds above apply with the first move already lower unipotent, giving a common bounded set for y. In the real matrix group its closure is compact. Continuity of the integral follows by dominated convergence, including at $|c| = 1$.

11.4 Integrals of reciprocal determinant weights

In this subsection, we integrate the inverse powers of the relative vector lengths against the Haar factors of Section 11.2. The resulting inequalities control the weighted products in Section 11.5.

Keep the real box and stage measures from Sections 10.2 and 11.2. Fix an allowed relative position x. Put $A_a = \langle e_0, \dots, e_{a-1} \rangle$ and $s = k - 3n - 6$ (this differs from the finite exponent $s = k - 24n - 27$ of Section 10.6). For $a < \varrho$, the depth $D_a(g, x)$ is the reciprocal of the area of the set of $h(e_a, v)$ with $v \in A_a^{\perp}$ and with some $v' \equiv v \bmod A_a$ satisfying

$$\|g^{-1}v\|^2 + \|(gx)^{-1}v'\|^2 < 1.$$

Let $H_{g,a}$ denote the positive Gram matrix of the quotient norm induced by $v \mapsto \|g^{-1}v\|$ on $A_a^{\perp}/A_a$. In any fixed quotient basis this is the Schur complement of the A_a-block in the restricted Gram matrix, hence continuous in g. The two lifts of a given class may be minimized independently. With $H = H_{g,a} + H_{gx,a}$ and the nonzero row $\ell_a = h(e_a, \cdot)$ on the quotient, one obtains

$$\min_{\ell_a w = c} w^* H w = \frac{|c|^2}{\ell_a H^{-1} \ell_a^*}, \qquad D_a(g, x) = \frac{1}{\pi \ell_a H^{-1} \ell_a^*}.$$

The minimizer is $cH^{-1}\ell_a^*/(\ell_a H^{-1}\ell_a^*)$. The image is therefore a disc of the claimed area, and D_a is continuous in both variables. The near sets are measurable as well: each $R_a\mathcal{K}$ is the product of a closed subgroup with a compact set and is closed. The depth is unchanged under simultaneous left movement by R_{a+1}. Such movement transports the two lifts, preserves their congruence and their lengths, and changes e_a by an element of A_a.

In the common free positive space of dimension $d = k - 5$, let E_I be its orthonormal inclusion. Suppose at most n linear equations are imposed on $u \in \mathbb{C}^d$. Impose also that $g^{-1}E_I u$ and $(gx)^{-1}E_I u$ belong to the positive eigenspace of the involution J. The combined rank is at most $3n$, since each negative projection has rank at most n. Choose an isometric inclusion O of an $(s+1)$-plane in the kernel. For u in the plane with $\|u\| \le 1/2$, preservation of the Hermitian form gives the same Euclidean norm for both inverse translates.

For $w = E_I O u$ with $\|u\| \le 1/2$,

$$\|g^{-1}w\|^2 + \|(gx)^{-1}w\|^2 = 2\|u\|^2 \le 1/2.$$

If the free coordinate of the vector paired against w is v, its values v^*Ou fill a disc of area $\pi\|O^*v\|^2/4$. The inverse area is therefore at most $\phi(v) = 36/\|O^*v\|^2$, allowing $+\infty$ when $O^*v = 0$. For each box exponent $\ell \ge 0$, uniformly in the equations and the two points,

$$\int_{\{|v_i| \le L^\ell\}} \phi(v)^s \, dv \le \Lambda_L^{99n(\ell+1)}. \tag{11}$$

Indeed the box is contained in the ball of squared radius $R^2 = dL^{2\ell}$. The complement of the $(s+1)$-plane has complex dimension $h = d - s - 1 = 3n$. At $t = \|O^*v\|^2$ the radial measure on the plane is $\pi^{s+1}t^s \, dt/s!$, and the complementary ball has volume $\pi^h(R^2 - t)^h/h!$. Hence

$$\int_{\|v\| \le R} \phi(v)^s \, dv = \frac{36^s \pi^d}{s!\, h!} \int_0^{R^2} (R^2 - t)^h \, dt = \frac{36^s \pi^d}{s!\,(h+1)!} R^{2(h+1)}.$$

The coefficient is at most $e^{36\pi}e^{\pi}$. Since $d \le r \le L/8$ and $\Lambda_L = L^2$, the last expression is at most $e^{37\pi}L^{(2\ell+1)(3n+1)} \le \Lambda_L^{99n(\ell+1)}$.

For the first inverse-depth integral, if some $\sigma g \in \mathcal{U}_{a+1}$ with $\sigma \in G_a$, then $g \in \mathcal{U}_a$. Rebase g to the box by R_a. The stage quotient integral is invariant under this replacement, by the right-action argument of Section 10.2 applied to the left-R_{a+1}-invariant depth and near indicator. At a boxed starting point choose $g_1 \in R_{a+1}$ with $g_1\sigma g \in \mathcal{K}$. The projection of $g(g_1\sigma g)^{-1}e_a = \sigma^{-1}g_1^{-1}e_a$ to the remaining space is $y = \sigma^{-1}e_a$, since $\sigma \in G_a$ fixes the earlier pairs. Thus $\|y\| \le R_b^2$. With no extra equations take σw for both lifts. It belongs to $A_a^{\perp}$ and $h(e_a, \sigma w) = h(y, w)$, so $D_a(\sigma g, x) \le \phi(y_I)$. The stage coordinates have modulus at most L^{11}. Besides the $k-5$ free coordinates in I and the two positive coordinates used for the norm equation, there are $r - 2a - (k-5) - 2 \le 2n+2$ complex coordinates. Thus at fixed y_I their boxes and the two-coordinate norm fiber contribute at most

$$(\pi L^{22})^{2n+2}\, 4\pi^2 L^{22} \le \Lambda_L^{99n(7+6)}$$

by Section 11.1, in the coordinate $h(y,y)/2$. By (11),

$$\int_{\{\sigma g \in \mathcal{U}_{a+1}\}} D_a(\sigma g, x)^s \, d\theta_a(\sigma) \le \Lambda_L^{10^4 n}. \tag{12}$$

For the second integral, take a boxed point y, a lower unipotent $\nu \in U_{<a}$, and the head Weyl element $w_{p,a+1}$ of length $a+1$. For common short vectors $w = E_I u$ at y, yx, impose the a equations $h(e_b^*, \nu w) = 0$ for $b < a$. They make $w_{p,a+1}u_a\nu w$ orthogonal to A_a, since $u_a \in U_a$ fixes the lower pairs. Writing $c = \bar{p}_a$, the inverse image of e_a is $\nu^{-1}cu_a^{-1}e_a^*$, whose class in $A_a^{\perp}/A_a$ is represented by $z_c = cu_a^{-1}e_a^*$. The preimages of e_b for $b < a$ depend on ν and the preceding p_b, so the chosen common plane is fixed throughout the U_a integral. Transporting both lifts gives $D_a(w_{p,a+1}u_a\nu y, x) \le \phi((z_c)_I)$.

On $\mathcal{U}_{a+1}$, choose a box move γ. The matrix $(u_a\nu)^{-1}w_{p,a+1}^{-1}\gamma^{-1}$ is the ratio $y(\gamma w_{p,a+1}u_a\nu y)^{-1}$ of two boxed matrices and has operator norm at most R_b^2. Use the row-replacement formula for minors in Section 11.2 with inverse head $\bar{p}$. After multiplying the column by $\bar{p}_a$, its size bound is $R_b^{2(a+1)}|\prod_{b<a} p_b|^{-1} \le L^{2a+3}$. Projection to the remaining space gives z_c, so we may use a box of size $20n$. At a fixed vector coordinate of u_a, integration in its center contributes at most $2L^{20n}/|c|$. Changing the vector coordinate to the corresponding part of z_c gives $|c|^{-2d_a}$, where $d_a = \dim_{\mathbb{C}} V_{a+1}$. In the stage basis the remaining coordinates have size at most $20n+4$, so their off-I volume and the integral in (11) each contribute at most $\Lambda_L^{99n(20n+5)}$. With $\mathfrak{m}_a(p_a) = |p_a|^{2d_a+1}$ this gives

$$\int_{\{w_{p,a+1}u_a\nu y \in \mathcal{U}_{a+1}\}} D_a(w_{p,a+1}u_a\nu y, x)^s \, du_a \le \mathfrak{m}_a(p_a)^{-1}\Lambda_L^{10^6 n^2}. \tag{13}$$

Set

$$W_j(g) = \prod_{a<j} D_a(g,x)^{-s}, \qquad I_j = \mathbf{1}_{\mathcal{U}_j} W_j^{-1}, \qquad C_L = \Lambda_L^{10^6 n^3}, \qquad H_p = \prod_{a<j} \mathfrak{m}_a(p_a)^{-1}.$$

Since the earlier depths are unchanged by G_a, (12) implies $\int I_{a+1}(\sigma g)\, d\theta_a(\sigma) \le \Lambda_L^{10^4 n} I_a(g)$. Similarly in (13) the earlier depths and indicator of $\mathcal{U}_a$ read at $w_{p,a}\nu y$. Indeed u_a and the last Weyl factor belong to G_a and commute with the earlier Weyl factors. Iteration therefore bounds $\int I_j(w_p u y)\, du$ over $U_{<j}$ by $H_p C_L$ for boxed y. The iteration uses the factorization $u = u_a\nu$ at each step, valid for Haar since $U_a \subseteq G_a$ preserves the earlier measures by conjugation. For $g \in \mathcal{U}_j$, write $y = \gamma u_0 g$ boxed, with $\gamma \in G_j$, $u_0 \in U_{<j}$. The same bound holds at g by the

substitution $u \mapsto \gamma^{-1} u \gamma u_0$, since γ commutes with w_p and preserves I_j. In particular, for a nonnegative left-R_l-invariant B supported on $\mathcal{U}_l$, where $l \geq j$, and a σ-finite lower representative measure ρ, Tonelli gives

$$\int B(g) I_j(w_p g)\, d(du_{<j} * \rho)(g) \leq H_p C_L \int B\, d\rho. \tag{14}$$

The bound also holds with $w_p^{-1} = w_{\bar{p}}$, for which the initial segment bounds and absolute values are unchanged.

11.5 The weighted products of the real correlations

In this subsection, we bound the product of the two real kernels with the phases and Haar factors of Section 11.2. These bounds propagate through successive isotropic quotients in Section 11.6.

Write $\theta_{[a,l]} = \theta_l * \cdots * \theta_a$, with the empty convolution the atom at the identity, and $M_j = du_{<j} * \theta_{[0,j-1]}$. The decomposition of the group Haar measure has the form $d\mu = d\mu_j * M_j$, where $d\mu_j$ is Haar measure on G_j with the stage normalization. We use the following change of representatives for this decomposition and for the lower unipotent decomposition of Section 11.7.

Suppose $d\mu = dm * \rho$, with dm Haar measure on a unimodular subgroup H. For compact $A \subseteq G$ and compact $C \subseteq H$ with $m(C) > 0$, one has $m(C)\rho(A) \leq \mu(CA)$. Thus ρ is σ-finite. For integrable $h > 0$ put $\bar{h}(y) = \int_H h(uy)\, dm(u)$. This average is left-$H$-invariant by right invariance of dm, and lies in $(0, \infty)$ for ρ-almost every representative, since $\int \bar{h}\, d\rho = \int h\, d\mu$. For nonnegative left-$H$-invariant B,

$$\int B\, d\rho = \int \frac{Bh}{\bar{h}}\, d\mu$$

by Tonelli, with the integrand zero on the exceptional set. If s normalizes H and preserves dm, then $dm * (L_s)_* \rho = (L_s)_* d\mu = d\mu$, where L_s denotes left translation. The identity gives the same integral for $(L_s)_*\rho$. In particular, take $H = G_j$ and $s = w_p$, which commutes with G_j.

Use the kernels $K_1(g, y) = \langle F(g), F(y) \rangle$ and $K_2(g, y) = \langle F(g), F(w_q^{-1} w_p y) \rangle$. For $a \geq j$ write

$$C_a^K(g, y) = \int K(\sigma g, \sigma y)\, d\theta_{[a,n]}(\sigma), \qquad C_{n+1}^K = K.$$

The cut condition and the support of each higher factor in its stage group force near conditions at the successive stages. After rebasing each starting point into the box, the orbit vector is bounded by the projected-column estimate. The uniform norm-fiber box mass and the finite compact terminal mass give absolute integrability for bounded F. In particular C_a^K is the θ_a-integral of $C_{a+1}^K(\sigma g, \sigma y)$.

The diagonal Weyl ratio in K_2 normalizes the Bessel unipotent and commutes with G_j. On the upper unipotent from stage $a+1$ on, its simultaneous Bessel character ratio is one. The quotient integration formula therefore makes C_{a+1}^K exactly invariant under simultaneous movement by G_{a+1}. It is also invariant under simultaneous U_a, since the Weyl ratio is the identity on that stage and the character ratio cancels after conjugation by each higher factor. Thus $C_{a+1}^K(\sigma g, \sigma y)$ descends to $\sigma^{-1} f_a$. A simultaneous lower move in $U_{<a}$ multiplies C_{a+1}^K by a phase: conjugation by G_{a+1} fixes the basis vectors occurring in its simple-root entries. Consequently the modulus is left R_{a+1}-invariant as a function of g at $y = gx$.

Define

$$A_a^K(g) = W_a(g)|C_{a+1}^K(g, gx)|^2,$$
$$B_a^K(g) = \sum_{l=a}^{\varrho} \int A_l^K(\sigma g)\, d\theta_{[a,l]}(\sigma),$$
$$E_a^K(x) = \int B_a^K\, d\theta_{[0,a-1]}.$$

Each A_a^K is left R_{a+1}-invariant and vanishes outside $\mathcal{U}_{a+1}$. The recursion $B_a^K = \int (A_a^K + B_{a+1}^K)(\sigma\,\cdot)\, d\theta_a$, with $B_n^K = 0$, and the stage quotient right action show that B_a^K is left R_a-invariant and supported on $\mathcal{U}_a$. For the untwisted kernel, Tonelli gives $E_a^{K_1} \le E_0^{K_1}$ by writing both as sums of full stage integrals.

Suppose a nonnegative integrand f, with $z = w_p^{-1}g$, is bounded pointwise by

$$f(g)^2 \le K_0 C_L\, B_j^{K_1}(g) I_j(g) B_j^{K_2}(z) I_j(z).$$

Cross the weights in Cauchy–Schwarz for the integral of f against M_j, pairing each B with the opposite I. The first factor $B_j^{K_1}(g) I_j(w_p^{-1}g)$ integrates to at most $H_p C_L E_j^{K_1}$ by (14) for $\bar{p}$. In the second, substitute $g = w_p z$ using the invariant quotient integral just computed. The function $B_j^{K_2}(z) I_j(w_p z)$ is left G_j-invariant. Its integral is at most $H_p C_L E_j^{K_2}$. It follows that

$$\left(\int f\, dM_j \right)^2 \le K_0\, \mathcal{Q}_j(x), \qquad \mathcal{Q}_j(x) = H_p^2 C_L^3 E_0^{K_1}(x) E_j^{K_2}(x). \tag{15}$$

In particular the character estimate of Section 11.2 gives this inequality with $K_0 = (\pi^2/|\xi|^2)^{k-5}$. Indeed the squared norm of each pivot function is at most $B_j^K I_j$ by the $l = j$ summand and invariance of W_j under G_j.

For the unequal-head estimate let $j = i + 1$, and write $\varphi_l(g) = |C_j^{K_l}(g, gx)|$ for $l = 1, 2$. On the support of $\varphi_1(g)\varphi_2(z)$, both $g, z \in \mathcal{U}_j$, and $I_i(w_{p,i} z) = W_i(g)^{-1}$. The last Weyl factor is in G_i and preserves the earlier depth and near conditions. On that support, $I_j(z) W_j(z) = 1$ and $I_i(w_{p,i} z) W_i(g) = 1$. Thus Cauchy–Schwarz for $\int \varphi_1(g)\varphi_2(z)\, dM_j(g)$ uses the two squared factors

$$A_i^{K_1}(g) I_j(z),$$
$$\mathbf{1}_{\{w_p z \in \mathcal{U}_j\}}\, W_j(z) I_i(w_{p,i} z) \varphi_2(z)^2.$$

The first contributes at most $H_p C_L E_0^{K_1}$. The second is left G_j-invariant as a function of z, so one may again use the Weyl change for M_j.

Then write $M_j = du_i * du_{<i} * \theta_{[0,i]}$ using the lower conjugation law. The two weights $W_j(z)$ and $I_i(w_{p,i} z)$ are invariant under U_i. Integrate φ_2^2 and the near indicator using the section estimate of Section 11.2. The next squared pivot norm is bounded by $B_j^{K_2}/W_j$. This cancels W_j. Apply (14) at length i to the remaining factor $B_j^{K_2}(z) I_i(w_{p,i} z)$, with $\rho = \theta_{[0,i]}$. The length i product of moduli times $\mathfrak{m}_i(p_i)^{-1}$ gives H_p. The second integral is at most $K_0 H_p C_L^2 E_j^{K_2}$, with $K_0 = (\pi^2/|\xi|^2)^{k-5}$. Multiplication gives (15) for the unequal heads.

11.6 Decay from a deep unipotent subgroup

In this subsection, we use Cauchy–Schwarz in the next isotropic integration to carry the estimates of Section 11.5 to smaller supports. Section 11.9 identifies the relative positions where the stronger estimate applies.

At a starting point g, put $P_j^K(g) = \int |C_{j+1}^K(\sigma g, \sigma g x)|\, d\theta_j(\sigma)$. If the θ_j-mass of $\sigma g \in \mathcal{U}_{j+1}$ is at most $V \geq 1$, Cauchy–Schwarz gives

$$(P_j^K)^2 \leq V B_j^K I_j.$$

Suppose instead that every nonzero terminal contribution on the G_j-orbit of g has some $D_a \geq \varepsilon^{-1}$ for $j \leq a < \varrho$, where $0 < \varepsilon \leq 1$. Split at the first such stage as the convolution proceeds from j upwards.

Once read, a depth is unchanged by the higher stage groups, which fix its basis data. At a stage with $D_a < \varepsilon^{-1}$, use the triangle inequality to expand the next tail. The part with no deep index has zero terminal contribution by hypothesis. Dropping these bounds on the earlier stages from the nonnegative contributions gives

$$P_j^K(g) \leq \sum_{a=j}^{\varrho-1} \int \mathbf{1}_{\{D_a(\sigma g, x) \geq \varepsilon^{-1}\}} |C_{a+1}^K(\sigma g, \sigma g x)|\, d\theta_{[j,a]}(\sigma).$$

On the support of the a-th term its squared integrand is at most $\varepsilon^s I_{a+1} A_a^K$, since $W_a/W_{a+1} = D_a^s$. By Cauchy–Schwarz, iteration of (12), and $a + 1 - j \leq \varrho$, the integral squared is at most

$$\varepsilon^s \Lambda_L^{10^4 n \varrho} I_j(g) B_j^K(g).$$

Since $\varrho^2 \Lambda_L^{10^4 n \varrho} \leq C_L$, squaring the sum yields

$$(P_j^K)^2 \leq \varepsilon^s C_L B_j^K I_j.$$

For the two kernels in Section 11.5, call a point y active at j if $K_1(y, yx) \neq 0$ and $w_p^{-1} y, w_q^{-1} yx \in \mathcal{U}_j$. If $P_j^{K_2}(w_p^{-1} g) \neq 0$, some point in its upper convolution has F nonzero in both arguments. The two Weyl elements commute with G_j, so this supplies the two near conditions on the entire G_j-orbit of g. Thus if every active point has a deep stage of index at least j, with $D_a \geq \varepsilon^{-1}$, the deep bound applies to the first pivot. The ordinary bound applies to the second. The squared product of pivots then has the pointwise bound used in (15) with $K_0 = V^2 \varepsilon^s$. With no depth hypothesis both ordinary bounds give $K_0 = V^2$. At $j = 0$, with empty heads, the majorant is the square of the same pivot at $g = 1$. Applying the deep bound to both copies at this stage gives the squared majorant bound $\varepsilon^{2s} \mathcal{Q}_0(x)$, using $C_L \geq 1$.

11.7 The weighted square integral on a real ball

In this subsection, we estimate a weighted integral over a bounded set of relative positions. Plancherel from Section 11.1 handles the norm-value coordinates, and Section 11.4 controls the inverse weights.

Use K equal to either of the two pairing kernels, starting at stage j. Let $\mathcal{B}'$ consist of positions x with every entry of $x - 1$ of modulus at most $L' \geq \max(R_b, 8r)$, and put $\Lambda' = (L')^2$. The real level unit is one. Set

$$c_a = \pi^{2(k-5)} (\Lambda')^{4900n} \quad (a < \varrho), \qquad c_\varrho = 1.$$

Write $du_{<a}$ for the lower Heisenberg Haar on $U_{<a}$ and $d\eta_a$ for that of U_a, with $U_\varrho = \{1\}$ of unit mass. The convolution $du_{<a} * d\eta_a$ is $du_{<a+1}$.

Call y close to g modulo $U_{<a}$ if $g^{-1}uy \in \mathcal{B}'$ for some $u \in U_{<a}$. For $0 \le a < \varrho$, suppose $g \in \mathcal{U}_{a+1}$ and $n_0 y, n_1 y$ are both close in this sense, with $n_0, n_1 \in U_a$. Write a box move at g as tu, with $t \in G_a \cap R_{a+1}$, $u \in U_{<a}$ and $g_0 = tug$ boxed. For $x_i = g^{-1}u_i n_i y \in \mathcal{B}'$,

$$(g_0 x_0)(g_0 x_1)^{-1} \in U_{<a}\, t n_0 n_1^{-1} t^{-1}.$$

Here G_a normalizes $U_{<a}$. Both x_i and their inverses have norms at most $(L')^3$, so the ratio has norm at most $(L')^8$. Its e_a^*-column, projected away from the earlier pairs, is that of $tn_0 n_1^{-1} t^{-1}$, since the lower unipotent is trivial on $A_a^\perp / A_a$. Subtracting the identity column gives a bound $(L')^9$. In Heisenberg coordinates $(v, i\beta)$ the column has vector part v and coefficient $i\beta - h(v,v)/2$ at e_a. The vector part has norm at most $(L')^9$. Taking the imaginary part of the coefficient also gives $|\beta| \le (L')^9$, since $h(v,v)$ is real.

We first bound the integral of the forward weight. Take g boxed and $g^{-1}y \in \mathcal{B}'$. Let $u_a \in U_a$ have vector v, with $u_a y$ close modulo $U_{<a}$. The preceding bound applies with $n_0 = 1$ and $t = 1$. If $c = h(e_a, z)$ is a stable product at $g, u_a y$ with lifts z, z', then for a common short vector w at g, y in the free positive space,

$$h(g^{-1}z, g^{-1}w) - h((u_a y)^{-1}z', y^{-1}w) = \bar{c}\, h(v, w).$$

Indeed $z - z' \in A_a$ and $u_a w = w - h(v,w)e_a$.

The left side has modulus at most one by Cauchy–Schwarz on the two close pairs, since $\mathrm{diag}(J, -J)$ is a Euclidean isometry. Test over the half-unit ball of the common positive plane in Section 11.4. The area of the possible c is at most $4\pi/\|O^* v_I\|^2$, bounded by $\phi(v_I)$ of that section. Thus the forward factor $D_a(g, g^{-1}u_a y)^{-s}$ is at most $\phi(v_I)^s$. Integrating this over the size-nine box contributes at most $(\Lambda')^{990n}$ on the free coordinates, by (11). The remaining at most $2n+2$ complex coordinates and the real center contribute at most $(L')^{19(2n+2)+10}$. The product is at most c_a.

For $0 \le l \le \varrho$ and $g \in \mathcal{U}_l$ the resulting bound is

$$\int_{U_{<l}} \mathbf{1}_{\mathcal{B}'}(g^{-1}uy) W_l(g; g^{-1}uy)\, du \le \prod_{a<l} c_a, \tag{16}$$

where $W_l(g; x)$ displays the position variable. To prove this first take g boxed and induct on l. At step $a+1$, if the integrand has nonempty support, translate y by an element of $U_{<a+1}$ to make $g^{-1}y \in \mathcal{B}'$, using right Haar. Integrate uu_a with $u \in U_{<a}$ and $u_a \in U_a$. The depth D_a is independent of this u in the second point. Moving $u_a y$ to $uu_a y$ transports its lift by u, preserving the quotient class and the length of its transported lift, with the first lift fixed.

Applying u^{-1} gives the inverse bijection. Apply the induction bound inside and the one-stage forward bound outside, on $u_a y$ close modulo $U_{<a}$. For general $g \in \mathcal{U}_l$, write its box move as γu_0, with $\gamma \in G_l$ and $u_0 \in U_{<l}$. Change the pair to $(\gamma u_0 g, \gamma y)$ and use $\gamma u_0 u \gamma^{-1}$ as the Haar variable. The position is preserved and the weights agree by their simultaneous flag invariance.

For the complementary ascent, put

$$m_a(g,y) = \int_{U_{<a}} \mathbf{1}_{\mathcal{B}'}(g^{-1}uy)\, du, \qquad Q_a(g,y) = \mathbf{1}_{\{m_a(g,y)>0\}} |C_{a+1}^K(g,y)|^2.$$

We show for $j \le a \le \varrho$ that

$$\int Q_a(g, ny)\, d\eta_a(n) \le c_a \int Q_{a+1}(\sigma g, \sigma y)\, d\theta_{a+1}(\sigma). \tag{17}$$

Convolution gives $m_{a+1}(g,y)=\int m_a(g,ny)\,d\eta_a(n)$. If this mass is zero, the left side vanishes. Otherwise the cutoff on the right is one, since $\sigma\in G_{a+1}$ normalizes $U_{<a+1}$ and preserves its Haar. At $a=\varrho$ the inequality is Cauchy–Schwarz on the terminal compact probability measure. For $a<\varrho$ and nonzero left side, choose a contributing n_0, so $g\in\mathcal{U}_{a+1}$. Use the centering above, writing $t=\gamma_1 r_1$ with $\gamma_1\in G_{a+1}$, $r_1\in U_a$, and substituting $n'=tnn_0^{-1}t^{-1}$.

This preserves Haar. The support lies in the size-nine box and the vector of n is $\gamma_1^{-1}v'+v_0$. For $y_\sigma=\sigma^{-1}f_{a+1}$, moving n past the next pivot and using the pairing covariance gives phase $\psi(-h(v,y_\sigma))$ in the integral defining $C^K_{a+1}(g,ny)$. This is the one-sided version of the calculation in Section 11.2: the diagonal twist commutes with G_a, and only the second argument is moved. In particular

$$C^K_{a+1}(g,ny)=\int\psi(-h(v,y_\sigma))C^K_{a+2}(\sigma g,\sigma y)\,d\theta_{a+1}(\sigma).$$

The next correlation descends measurably to y_σ, since it is invariant under the stabilizer and the orbit map has smooth local sections on the nonzero orbit. Rotate that vector by γ_1. The translation by v_0 multiplies the function by a phase of modulus one.

Apply the projected-column estimate to $q=\gamma_1 r_1 u$ with $qg=g_0$ in the box and $\gamma_1^{-1}q\in U_{<a+1}$. The product of the two matrices from the box has norm at most R_b^2, so the coordinates of $\gamma_1 y_\sigma$ are bounded by $(L')^2$. The Fourier estimate of Section 11.1 at frequency 1 applies from this quadric to the vector box. Use ordinary area in both target coordinates in the block B' of that proof and keep the same source fiber. Plancherel on the free block still gives the squared-norm factor $\pi^{2(k-5)}$. On each side at most $2n+2$ complex coordinates lie outside the free block. On the input their disc areas are at most $\pi(L')^6$, and the solved pair has mass at most $4\pi^2(L')^6$. On the output the disc areas are at most $\pi(L')^{20}$. Since $L'\ge 8$, these volumes and the central length $2(L')^9$ contribute at most $(\Lambda')^{4(2n+2)+4+11(2n+2)+5}\le(\Lambda')^{4900n}$. This proves (17).

Set $\rho_a=d\eta_a*\cdots*d\eta_\varrho*\theta_{[0,n]}$, so $d\mu=du_{<a}*\rho_a$ and $\rho_n=\theta_{[0,n]}$. For $u\in U_{<a+1}$, right translation of its Haar variable gives $m_{a+1}(g,uy)=m_{a+1}(g,y)$. Write $D=1$ for K_1 and $D=w_q^{-1}w_p$ for K_2. In $C^K_{a+2}(g,uy)$ the higher factors commute with D and fix the vectors defining the simple-root entries of DuD^{-1}. The second Bessel factor therefore contributes the constant phase $\psi(\mathfrak{s}(DuD^{-1}))$. Its modulus is one, so $Q_{a+1}(g,uy)=Q_{a+1}(g,y)$. Conjugation by $\sigma\in G_{a+1}$ preserves $du_{<a+1}$. Writing $L_\sigma y=\sigma y$, we have

$$du_{<a+1}*(L_\sigma)_*\rho_{a+1}=(L_\sigma)_*(du_{<a+1}*\rho_{a+1})=(L_\sigma)_*d\mu=d\mu.$$

The invariant-function averaging identity of Section 11.5 shows that its integral on $Q_{a+1}(g,\cdot)$ is unchanged. Integrate (17) against $\theta_{[0,a]}$ in g and ρ_{a+1} in y. Undoing the shift by σ in the y-integral gives

$$\iint Q_a\,d\theta_{[0,a]}d\rho_a\le c_a\iint Q_{a+1}\,d\theta_{[0,a+1]}d\rho_{a+1}.$$

Iterate to n, where $Q_n\le|K|^2$.

Finally, to integrate $A_l^K(g)$ over $x\in\mathcal{B}'$ put $x=g^{-1}uy$ with $u\in U_{<l}$ and y in the ρ_l measure. The correlation modulus is invariant under this motion in its second argument. If the close mass is zero the inner integral vanishes. Otherwise the forward estimate (16) contributes $\prod_{a<l}c_a$ and leaves $Q_l(g,y)$. The ascent for its double integral contributes the complementary product. Summing the at most n weighted stage integrals in E_j^K proves

$$\int_{\mathcal{B}'}E_j^K(x)\,d\mu(x)\le n\Big(\prod_{a<\varrho}c_a\Big)\iint|K(g,y)|^2\,d\theta_{[0,n]}(g)d\theta_{[0,n]}(y).\tag{18}$$

11.8 Density and twisted square integrals

In this subsection, we compute the density in the coordinates integrated at the earlier steps of the isotropic flag. This combines the integral of Section 11.7 with the terminal pairing of Section 11.3.

Write $dQ = d\theta_{[0,n]}$ for the original real quotient measure and $d_a = \dim_{\mathbb{C}} V_{a+1} = k + 1 + 2(\varrho - a - 1)$. At a lower stage $a < \varrho$, write $n_a(v,\beta)$ for the unipotent with e_a^*-column $e_a^* + v + (i\beta - h(v,v)/2)e_a$. Its inverse is $n_a(-v,-\beta)$. For $\sigma = w_a(c)^{-1} n_a(v,\beta)$, the orbit vector is

$$\sigma^{-1} e_a = c e_a^* - cv + c(-i\beta - h(v,v)/2)e_a.$$

In general write the vector as $ce_a^* + w + cte_a$, $c \neq 0$. Ordinary area in its e_a-coordinate is $|c|^2 d^2 t$, and its norm-value coordinate is $|c|^2 \Re t + h(w,w)/2$. Disintegration therefore leaves $d^2c\, dw\, d\Im t$ on the fixed norm fibre. Here $w = -cv$ and $\Im t = -\beta$, so the stage measure is

$$|c|^{2d_a}\, d^2c\, dv\, d\beta.$$

The omitted locus $c = 0$ has fibre measure zero by the positive two-coordinate chart of Section 11.1. The factor $w_a(c)^{-1}$ commutes with the higher stages, so it can be written at the left of the convolution with the other factors in their original order.

Let $D_j = w_q^{-1} w_p$, and let D_a denote the same diagonal through stage $a-1$ in this calculation. On the a-th pair, $D_{a+1} D_a^{-1}$ scales e_a by $z_a = p_a/q_a$ and e_a^* by $\bar{z}_a^{-1}$. Direct multiplication of the two-by-two matrices gives

$$D_{a+1} w_a(z_a c)^{-1} = D_a w_a(c)^{-1}.$$

Thus replacing c by $z_a c$ in the quotient integral contributes $|z_a|^{2(d_a+1)}$, from its weight and the complex area Jacobian. Iterating, for a nonnegative measurable test Φ we have

$$\int \Phi(D_j y)\, dQ(y) = J_{p,q} \int \Phi(y)\, dQ(y), \qquad J_{p,q} = \prod_{a<j} |p_a/q_a|^{2(d_a+1)}.$$

In particular the double squared norm of K_2 is $J_{p,q}$ times that of K_1, since only its second variable is translated.

The head density is $\mathcal{D}_j(p,q) = \prod_{a<j} |p_a q_a|^{d_a}$. With the inverse section moduli H_p of Section 11.4, the exact identity is

$$\mathcal{D}_j(p,q)^2 H_p^2 J_{p,q} = \prod_{a<j} |q_a|^{-2} \le L^2$$

for a q-head adapted to L. Indeed the exponent of $|p_a|$ is $2d_a - 2(2d_a+1) + 2(d_a+1) = 0$, and the exponent of $|q_a|$ is $2d_a - 2(d_a+1) = -2$. For any measurable $\mathcal{S} \subseteq \mathcal{B}'$ on which $f(x)^2 \le e^2 \mathcal{Q}_j(x)$, use Cauchy–Schwarz in x and then (18) for the two kernels. This gives

$$\mathcal{D}_j(p,q) \int_{\mathcal{S}} |f(x)|\, d\mu(x) \le e\, nLC_L^{3/2} \Big(\prod_{a<\varrho} c_a \Big) \mathcal{N},$$

$$\mathcal{N} = \iint |K_1(g,y)|^2\, dQ(g)\, dQ(y).$$

In particular, the loss multiplying $e\mathcal{N}$ is at most $\pi^{2(k-5)\varrho} \Lambda_L^{1500000n^3+1} (\Lambda')^{25000n^2}$.

For the cut function on the real quotient, write $d\mu = \lambda\, dg$, where dg is the orbital Haar. The corresponding quotient measures differ by the same λ, since the unipotent Haar is fixed. If $F = \chi W$ is the scalar cutoff with $|\chi| \le 1$ and W has squared quotient norm d_I in the orbital normalization, $\mathcal{N} \le \lambda^2 d_I$. Put $U = U_{<n}$. The compact group $H = U(z^{\perp})$ in the remaining space normalizes U. The modulus of its action on Haar measure du is a continuous homomorphism $H \to \mathbb{R}_{>0}$, hence is one. Disintegrating left Haar measure dg over U shows that $y \mapsto hy$ preserves $d\bar{g}$ on $U\backslash G$ for $h \in H$. By covariance the matrix $\int W(y)W(y)^*\, d\bar{g}(y)$ commutes with the irreducible action of H. Its trace is d_I, so Schur's lemma makes it the identity. The double integral for W is the trace of its square. Multiplication by the scalar cutoffs can only decrease the integrand's modulus.

In the terminal comparison the top norm density in the last entry c is

$$\frac{2\pi^k}{(k-1)!}(|c|^2 - 1)_+^{k-1}\, dc.$$

This follows by polar integration in the k positive coordinates, at fixed norm-value coordinate $h/2$. The coefficient of the product of the top correlations in basis measure is therefore

$$C_F\, \mathcal{D}_{\varrho}(p,q)\frac{2\pi^k}{(k-1)!}(|c|^2 - 1)_+^{k-1}, \qquad C_F = \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)\,(\lambda d_I)^2}.$$

In the orbital normalization, the two Haar integrations contribute λ^2 to this coefficient.

Thus the coefficient is $A_p A_q$ of Section 11.3, with the common last entry and the constant

$$c_G = \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)}\frac{2\pi^k}{(k-1)!}.$$

To see the matching integrand, at $\sigma = a(c)$ the terminal compact average fixes the two correlations by unitary covariance and because the ratio of heads commutes with the compact group. Their product at y, x is

$$\langle F(y), F(yx)\rangle\, \langle F(\sigma^{-1}w_q^{-1}yx), F(\sigma^{-1}w_p^{-1}y)\rangle.$$

The substitution $(y,x) \mapsto (y, yx)$ preserves product Haar and makes this the integrand of the Gram pairing of the two orbital tensors. This comparison is at fixed entry. Absolute convergence over the varying entry and all real positions uses the shell estimates.

11.9 Real determinant weights on dyadic regions

In this subsection, we use minima of Hermitian quotient norms to decompose the relative positions into dyadic sets. This identifies where the stronger estimates of Section 11.6 enter the frequency sum of Section 11.10.

On $V_a = A_a^{\perp}/A_a$ let $|v|_{g,a}$ be the infimum of $\|g^{-1}\widetilde{v}\|$ over lifts. For $a < \varrho$ put

$$b_a(g,g') = \inf_{h(e_a,v)=1}(|v|_{g,a}^2 + |v|_{g',a}^2).$$

The infimum is positive since $h(e_a,\cdot)$ is a nonzero continuous linear functional. The stable product $c \ne 0$ occurs precisely when $|c|^2 b_a(g,g') < 1$, by scaling the common class and taking the two lift infima independently. Thus the stable set is a disc and $D_a(g,x) = b_a(g,gx)/\pi$.

Fix heads p, q satisfying the coordinate bound L at j, and a point g active at x as in Section 11.6. The two nonzero cut values give boxed points ug, vgx with u, v in the Bessel unipotent. They preserve e_a modulo A_a, so $|e_a|_{g,a}, |e_a|_{gx,a} \le R_b$. On V_ϱ the Bessel unipotent acts trivially. The two quotient norms there have ratio at most R_b^2, by the two boxed inverses and matrices. For $a < j$, e_a^* supplies a short dual in both norms. Indeed the column-minor bound of Section 11.2, for the box moves s_0 at g and s_1 at $w_p^{-1}g$, bounds each entry of $s_0 e_a^*$ by $LR_b^{2(a+1)}$. Applying $(s_0 g)^{-1}$ gives $\|g^{-1}e_a^*\| \le L^{2n+3}$. Use the head q and the box moves at gx and $w_q^{-1}gx$ for the other norm.

Suppose $T \ge L^{11n}$ and $b_a(g, gx) < T^2$ for every $j \le a < \varrho$. At each such index choose a dual of e_a with both lengths at most T, and at the earlier indices use e_a^*. The inequality $|h(v, w)| \le |v||w|$ holds in either quotient norm, by applying it on all lifts before minimizing. For the chosen dual ζ at a, put $\zeta_0 = \zeta - h(\zeta, \zeta)e_a/2$. This is an isotropic dual and has both lengths at most $T + T^2 R_b/2 \le T^4$.

The orthogonal complement of $\langle e_a, \zeta_0 \rangle$ gives a common section of V_{a+1} into V_a. For either quotient norm $|\cdot|_{*,a}$, take a minimizing lift $w \in e_a^\perp \subset V_a$ of a class in V_{a+1}. The common section sends the class to $w - h(\zeta_0, w)e_a$. Since $|e_a|_{*,a}|\zeta_0|_{*,a} \le T^5$, its norm is at most $(1 + T^5)|w|_{*,a}$. The successive quotient infima equal the infimum over A_{a+1}. If the second norm on V_{a+1} is at most T^u times the first, write $w = h(\zeta_0, w)e_a + h(e_a, w)\zeta_0 + w_0$. The first two terms contribute at most $2T^5|w|_{g,a}$ in the second norm.

Removing $h(e_a, w)\zeta_0$ before taking the quotient shows that the class of w_0 has first quotient norm at most $(1 + T^5)|w|_{g,a}$. Thus the second norm of w is at most

$$(2T^5 + (1 + T^5)^2 T^u)|w|_{g,a} \le T^{u+13}|w|_{g,a}.$$

Descend from $u = 2$ at V_ϱ, and at $a = 0$ apply the result to $w = gv$ for arbitrary v. It gives $\|x^{-1}\| \le T^{2+13\varrho}$. Since $J = J^{-1}$, $\|x\| = \|x^{-1}\|$. Every entry of $x - 1$ is therefore at most T^{99n} in modulus. Outside this ball every active point has $D_a(g, x) \ge T^2/\pi$ at some $a \ge j$. For $j = \varrho$ there are no active points there.

The uniform next-stage mass is bounded by

$$V_b = (\pi R_b^{24})^k R_b^{9999n}.$$

Rebase the starting point to the box by R_j, as in the first inverse-depth integral. On $\mathcal{U}_{j+1}$ the orbit vector has norm at most R_b^2, by projecting the column of a boxed matrix times a boxed inverse. By the two-coordinate sphere bound of Section 11.1, its box mass is at most $4\pi^2 R_b^4 (\pi R_b^4)^{r-2j-2} \le V_b$. The same estimate at the top uses two positive coordinates.

Let $H = 2^i$, with $i \ge 0$ least such that the entries of $x - 1$ have modulus at most H. One position ball with coordinate bound LH covers this shell. If $0 < \tau < 1$ and $\Lambda_L^{1200n^2} \le \tau^{400n^2} H$, take $T = L^{11n+1}/\tau$. Then $T^{-1} \le \tau/2$ and

$$2T^{99n} \le L^{2400n^2} \tau^{-400n^2} \le H.$$

Minimality of the shell index puts x outside the position ball of radius T^{99n}. Thus the deep-stage bound applies with $D_a \ge \tau^{-2}$.

11.10 Summation and real frequency decay

In this subsection, we sum the weighted real integrals on the dyadic sets of Section 11.9. The resulting Fourier bound controls the nonconstant terms in Section 12.2.

Let $\mathcal{M}_j(x)$ be the majorant obtained by integrating the two pivot moduli against M_j. Take any measurable $0 \le f \le \mathcal{M}_j$ and $t \ge 1$, with

$$f(x)^2 \le V_b^2 t^{-2s} \mathcal{Q}_j(x) \qquad (t > 1).$$

For $t = 1$ use the bound for $\mathcal{M}_j$. This also gives the estimate for the modulus of the form integrand. The energy estimate of Section 11.8 on the ball with coordinate bound LH shows that the one-shell integral times $\mathcal{D}_j(p, q)$ contributes at most

$$\Lambda_L^{2\cdot 10^6 n^3} H^{10^5 n^2} \pi^{2(k-5)\varrho} \mathcal{N}$$

times the square-root saving. Write $u = t^{-1}$.

For $H \le \Lambda_L^{2400n^2}$ use saving $V_b u^s$. On larger shells take $\tau = H^{-1/(800n^2)}$, which satisfies the condition of Section 11.9. The saving there is $V_b \min(u, \tau)^s$. Indeed the deep pivot bound with $\varepsilon = \tau^2$ gives $\mathcal{M}_j^2 \le V_b^2 \tau^{2s} \mathcal{Q}_j$. Taking the smaller of this bound and the given frequency bound for f^2, then taking the square root, gives the stated factor. For $g_* = u$ in the first case and $g_* = \min(u, \tau)$ in the second,

$$g_*^{9\cdot 10^7 n^4} H^{1+10^5 n^2} \le \Lambda_L^{29\cdot 10^7 n^4}.$$

For the smaller shells use $2400n^2(1 + 10^5 n^2) \le 29 \cdot 10^7 n^4$. On a larger shell $g_* \le H^{-1/(800n^2)}$, and hence

$$H^{10^5 n^2} g_*^{9\cdot 10^7 n^4} \le H^{-12500n^2}.$$

For the first index i_0 with $2^{i_0} > \Lambda_L^{2400n^2}$, the sum of these factors over $i \ge i_0$ is at most $2^{-12500n^2 i_0}/(1 - 2^{-12500n^2})$, uniformly in the two heads satisfying the bounds for L. The rank bounds give $s \ge 10^{10} n^4$. Since $g_* \le t^{-1}$ and $t \ge 1$, its remaining power satisfies

$$g_*^{s - 9\cdot 10^7 n^4} \le t^{9\cdot 10^7 n^4 - s} \le t^{10^8 n^4 - s}.$$

Summing $\sum_{i \ge 0} 2^{-i} = 2$ gives

$$\mathcal{D}_j(p, q) \int f \, d\mu \le V_b \, t^{10^8 n^4 - s} \Lambda_L^{4\cdot 10^8 n^4} \pi^{2(k-5)\varrho} \mathcal{N}.$$

This in particular proves integrability of the majorant at $t = 1$ for each pair of heads. By Section 11.8,

$$d_I C_F \mathcal{N} = \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)} \frac{\mathcal{N}}{\lambda^2 d_I} \le \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)}.$$

Inserting $V_b = (\pi R_b^{24})^k R_b^{9999n}$ gives

$$\begin{aligned} d_I C_F \, \mathcal{D}_j(p, q) \int f(x) \, d\mu(x) &\le \Theta L^{10^9 n^4} R_b^{24k} t^{10^8 n^4 - s}, \\ \Theta &= \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)} \pi^{(2\varrho+1)k}. \end{aligned} \tag{19}$$

Here $k + 2(k-5)\varrho \le (2\varrho + 1)k$ and $R_b^{9999n} L^{8\cdot 10^8 n^4} \le L^{10^9 n^4}$.

Lemma 11.1 (Real frequency decay)**.** *For a pivot tuple p, set $P_h(p)=\prod_{a<h}p_a$ and*

$$w_j(p)=\mathbf{1}_{\max_{1\le h\le j}|P_h(p)|\le b^{60n^2}}\prod_{h=1}^{j}\max(1,|P_h(p)|^{-1})^{n^6},\qquad K_*=k-n^7.$$

For $j=0$ use $w_0=1$. Fix $0\le j\le \varrho$ and $p,q\in(\mathbb{C}^\times)^j$. Put $Z_j(g)=h(f_j,gf_j)$ and write $\mathcal{I}_j(p,q;\omega)$ for the integral with weight ω in Section 12.2. With C_F as in Section 11.8 and Θ as in (19), one has

$$d_I|C_F\mathcal{I}_j(p,q;\omega)|\le\Theta b^{2000n^9}|\xi|^{-K_*}\min(1,|\xi|^{-1})^6w_j(p)w_j(q) \tag{20}$$

for either of the following choices of weight and frequency:

$$\begin{aligned}&\omega(g)=\psi(\xi Z_j(g)),\quad \xi\in\mathbb{C}^\times,\\ &\omega(g)=1,\qquad\qquad j=i+1,\quad \xi=p_i^{-1}-q_i^{-1}\neq0.\end{aligned}$$

Proof. Set $t=\max(1,R_b^{24}|\xi|)$ at frequency $\xi\neq0$. If $t>1$, then

$$(\pi/|\xi|)^{k-5}\le(\pi R_b^{24})^kt^{-s},$$

which bounds the frequency factor in either case. Let $f(x)$ be the integral against $dM_j(g)$ of the absolute value of the paired pivot integral at (x,g). For the character weight, the Fourier estimate on the two quadrics in Section 11.2 and (15) give

$$f(x)^2\le(\pi/|\xi|)^{2(k-5)}\mathcal{Q}_j(x).$$

For the constant weight the two pivots separate. At distinct last entries the section estimate at i gives the same inequality by the crossed integrations in Section 11.5. The triangle inequality gives $f\le\mathcal{M}_j$ for both weights. Where both pivot moduli contribute, their nonzero terminal contributions imply $g,w_p^{-1}g\in\mathcal{U}_j$, and likewise $gx,w_q^{-1}gx$.

The leading minors then bound the partial products by R_b^{2j}, so a head violating the threshold gives zero. For the heads satisfying the threshold, take L to be the largest of R_b and the inverse initial segment moduli for p,q. It meets the box hypothesis since $8r\le8nr\le b\le R_b$ by Section 12.10. Thus $L^{n^6}\le R_b^{n^6}w_j(p)w_j(q)$. The parameter inequalities $b\ge2$, $n\ge10^8$, $k\ge n^7$ give $10^9n^4\le n^6$ and $10^8n^4+K_*+6\le s$. Use $R_b^{24}/t\le|\xi|^{-1}$ and $t^{-1}\le\min(1,|\xi|^{-1})$ in (19). Since $\mathcal{I}_j$ includes the density $\mathcal{D}_j(p,q)$ and $R_b^{n^6+24n^7}\le b^{2000n^9}$, we obtain

$$\begin{aligned}d_I|C_F\mathcal{I}_j(p,q;\omega)|&\le d_IC_F\mathcal{D}_j(p,q)\int f\,d\mu\\ &\le\Theta b^{2000n^9}|\xi|^{-K_*}\min(1,|\xi|^{-1})^6w_j(p)w_j(q).\end{aligned}$$

□

12 Formal degrees and real Bessel functions

In this section, we pass from the weighted real bounds of Section 11 to the discrete-series normalization. The stationary point, coefficient estimates, and truncation bound apply to the Poincaré map of Section 8.

12.1 The local mass in the formal-degree normalization

In this subsection, we compare the initial orbital pairing for the truncated and full discrete-series vectors. The formal degree in Section 12.3 gives the full norm, and the truncation estimate of Section 12.4 controls the comparison.

For d the formal degree, let dg be the orbital Haar, $d\bar{g}$ its quotient Haar and $d\mu = \lambda\, dg$. For Bessel functions of the prescribed type define

$$P(A,B)(x) = \int_{U_{<n}\backslash G} \langle A(y), B(yx)\rangle\, d\bar{g}(y).$$

The original real vector satisfies

$$d > 0, \quad f \geq 4000n, \quad \|W\|_2^2 = d_I, \quad \|F\|_2 \leq \sqrt{d}_I, \quad \|F - W\|_2 \leq b^{-r^3 f}\sqrt{d}_I,$$
$$\|P(W,W)\|_{L^2(dg)}^2 \leq d_I^2/d, \qquad \lambda = \frac{\operatorname{vol} U(r)}{\operatorname{vol} U(k)}, \qquad d\Theta \leq b^{4r^2},$$
$$d\operatorname{vol}_{dg}\{x : \|x\| \leq Y\} \leq Y^{9r^2} b^{99r^2} \quad (Y \geq 1).$$

Section 12.6 proves the Haar comparison, and Section 12.10 gives $f \geq 4000n$ for the chosen datum. For empty heads, the two pivot correlations with weight 1 give $\lambda^2 |P(F,F)(x)|^2$, with head density one. By Cauchy–Schwarz and the isometry of right translation,

$$|P(F,F)(x) - P(W,W)(x)| \leq \|F - W\|_2(\|F\|_2 + \|W\|_2) \leq 2d_I b^{-r^3 f}.$$

Set $t_* = b^{6r}$, $T = t_* R_b^{11n}$, and take the position ball $\mathcal{B}_*$ of entry radius T^{99n}. On it $\|x\| \leq (1 + r^2)T^{99n} \leq R_b^2 T^{99n} \leq b^{700nr}$, since $R_b \leq b^{7n}$ and $14n + 7623n^3 + 594nr \leq 700nr$. The squared $L^2(dg)$ error on the ball is therefore at most

$$\frac{d_I^2}{d}\, 4b^{-2r^3 f + 6300nr^3 + 99r^2} \leq b^{-24}\frac{d_I^2}{d}.$$

As $C_F \lambda^3 = d_I^{-2}$, the L^2 triangle inequality bounds the normalized mass on the ball by $(1 + b^{-12})^2/d$.

On the complement, Section 11.9 gives, at each active point, some $0 \leq a < \varrho$ with $D_a \geq T^2/\pi \geq t_*^2$. The deep pivot estimate with $\varepsilon = t_*^{-2}$ applies to $f = \mathbf{1}_{\mathcal{B}_*^c}\mathcal{M}_0$ with $t = t_*$ and $L = R_b$. The empty heads have density one. The absolute contribution is at most $C_F \int f\, d\mu$, which is bounded by (19) since $d_I \geq 1$. It is therefore at most

$$\Theta R_b^{10^9 n^4 + 24k} t_*^{10^8 n^4 - s} \leq b^{-12}/d.$$

Indeed $k = s + 3n + 6$, $r = s + 5n + 5$, $s \geq 10^{10} n^4$ and $n \geq 10^8$ give $r \leq (11/10)s$, $k \leq r$ and

$$4r^2 + 7n(10^9 n^4 + 24k) + 6r \cdot 10^8 n^4 + 12 \leq 6rs.$$

Here $4r^2 \leq (22/5)rs$, both $7 \cdot 10^9 n^5$ and $168nk$ are at most $rs/10$, and the penultimate term is at most $(3/50)rs$. Thus the normalized form at the bottom satisfies

$$|J_0| \leq \frac{(1 + b^{-12})^2 + b^{-12}}{d} \leq \frac{1 + b^{-10}}{d}.$$

12.2 Descent through a common entry

In this subsection, we relate consecutive pairings when the two vectors at the current isotropic entry coincide. Section 11.10 bounds the other Fourier terms, while Section 11.3 supplies the terminal positive form.

For $a \le \varrho$, let p, q have length a, and use the kernels K_1, K_2 of Section 11.5 with twist $w_q^{-1} w_p$. Write $\mathcal{I}_a(p, q; \omega)$ for the integral against $d\mu(x) dM_a(g)$, with factor $\mathcal{D}_a(p, q)$, of

$$\iint \omega(\sigma\tau^{-1})\, C_{a+1}^{K_1}(\sigma g, \sigma g x)\, \overline{C_{a+1}^{K_2}(\tau w_p^{-1} g, \tau w_p^{-1} g x)}\, d\theta_a(\tau) d\theta_a(\sigma).$$

This is absolutely convergent for measurable $|\omega| \le 1$, by integrating the ordinary majorant at $t = 1$ in Section 11.10. For weight 1 the two pivot integrals separate into $C_a^{K_1}$ and $\overline{C_a^{K_2}}$ at their respective starting points.

Set $Z_a(s) = h(f_a, s f_a)$. Let $v(c) = w_a(c)$ for $a < \varrho$ and $v(c) = a(c)$ of Section 11.3 at $a = \varrho$. The law θ_a is the pushforward by $(c, \nu) \mapsto v(c)^{-1}\nu$ of $\rho_a(c)\, dc\, d\zeta_a(\nu)$, where

$$\begin{aligned} \rho_a(c) &= |c|^{2d_a}, \quad \zeta_a = \eta_a, && a < \varrho, \\ \rho_\varrho(c) &= \frac{2\pi^k}{(k-1)!}(|c|^2 - 1)_+^{k-1}, \quad \zeta_\varrho = \theta_n, && a = \varrho. \end{aligned}$$

These are the lower chart and top sphere densities in Section 11.8. Both ζ_a are carried by $R_{a+1} \cap G_a$, the stabilizer of f_a in G_a. The entry $Z_a(v(c)) = c$ wherever the density is nonzero, except possibly at $c = 0$, a null set.

Let $\omega = \phi \circ Z_a$ with $|\phi| \le 1$. For fixed x, g, σ, the weighted product in the double integral is unchanged under left multiplication of τ by the stabilizer of f_a. The entry is unchanged because that stabilizer fixes f_a, and the complex tail correlation is unchanged by the stabilizer invariance of Section 11.5. Thus the integral in τ depends only on $\tau^{-1} f_a$. Right multiplication of τ by σ sends the orbit point to $\sigma^{-1}\tau^{-1} f_a$, which preserves the norm-fiber measure.

This applies to the product and its absolute value. We may consequently substitute $\tau = v(c)^{-1}\nu\sigma$ with the above chart law. For $g' = \nu\sigma g$, the entry is

$$Z_a(\sigma\tau^{-1}) = h(f_a, \nu^{-1} v(c) f_a) = h(\nu f_a, v(c) f_a) = h(f_a, v(c) f_a) = c.$$

At the last step the z-coordinate of $a(c)z$ is $-c$ and $h(z, z) = -1$, giving the same sign. The first tail at σg equals that at g', by the stabilizer invariance. The other starting point becomes

$$\tau w_p^{-1} g = v(c)^{-1} w_p^{-1} g',$$

since w_p commutes with G_a. The same relation holds with $gx, g'x$ in the second variable. The variable g' now has measure $\zeta_a * \theta_a * M_a$. The absolute integrability just proved transfers through this substitution, so Fubini applies also to the new integral in c.

For $a < \varrho$, append the common entry $c \ne 0$ to both heads, writing p_c, q_c. Their new Weyl elements are $w_p v(c), w_q v(c)$, so the twist is unchanged. The next lower measure is $M_{a+1} = \eta_a * \theta_a * M_a$, by rearranging the lower Heisenberg and stage products using the same conjugation invariance. Also $\mathcal{D}_{a+1}(p_c, q_c) = \mathcal{D}_a(p, q)|c|^{2d_a}$. Thus the new inner integral is the next form with weight 1, by separation of its pivots. We obtain

$$\int_{\mathbb{C}} \phi(c) \mathcal{I}_{a+1}(p_c, q_c; 1)\, dc = \mathcal{I}_a(p, q; \phi \circ Z_a).$$

The values assigned to the appended head at $c = 0$ do not affect this statement.

At $a = \varrho$, $\theta_n * \theta_\varrho * M_\varrho = d\mu$ by the full stage decomposition, since the last two Heisenberg measures are trivial. The compact tail average is the probability average on H. The coefficient and integrand comparison in Section 11.8 now identifies the new integral, after multiplication by C_F, with $J_n(p_c, q_c)$. Hence

$$\int_{\mathbb{C}} \phi(c) J_n(p_c, q_c)\, dc = C_F \mathcal{I}_\varrho(p, q; \phi \circ Z_\varrho).$$

In particular its absolute convergence follows from the ordinary majorant, including at zero frequency.

Define $J_a(p, q) = C_F \mathcal{I}_a(p, q; 1)$ for $a \le \varrho$, with J_n the terminal Gram form. The preceding identities show that

$$\int_{\mathbb{C}} \psi(\xi c) J_{a+1}(p_c, q_c)\, dc = C_F \mathcal{I}_a(p, q; \psi(\xi Z_a)), \qquad a < n.$$

At $\xi = 0$ this integral equals $J_a(p, q)$. At $\xi \neq 0$, the character case of (20) bounds d_I times its modulus, including at $a = \varrho$. At unequal last entries with $a < \varrho$, the constant-weight case bounds $d_I |J_{a+1}|$ at $\xi = p_a^{-1} - q_a^{-1}$, with the weights of the extended heads. Since $d_I \ge 1$, the same right sides bound the respective moduli in Section 13.4.1. The terminal form is continuous and positive semidefinite on fixed last-entry lists, by Section 11.3. The remaining bottom norm estimate is supplied by Section 12.1 when its formal-degree and Haar bounds hold.

12.3 The reproducing coefficient and the norm-ball bound

In this subsection, we use Schur orthogonality to express the full coefficient norm in terms of the formal degree. The same normalization gives a bound for the volumes used in the truncation argument of Section 12.4.

Suppose the full vector W is reproduced on the right by $e(g) = d\,\overline{c(g)}$, where $c(g) = \langle v, \pi(g)v \rangle$ is the coefficient of a unit vector in the real discrete series, $c \in L^1(dg)$, and $d \int |c|^2\, dg = 1$. Let $U = U_{<n}$, let I be the compact value space, and write χ_U for the unitary character of U in the Bessel transformation law. Use the space of measurable functions $B : G \to I$ satisfying

$$B(uy) = \chi_U(u) B(y), \qquad \|B\|_2^2 = \int_{U\backslash G} \|B(y)\|^2\, d\bar{g}(y) < \infty,$$

modulo equality almost everywhere. Inner products of two such functions descend to $U\backslash G$, since $|\chi_U| = 1$. Right invariance of $d\bar{g}$ makes $R(g)B(y) = B(yg)$ unitary. Set $\mathcal{P} = \int_G e(g) R(g)\, dg$. Then $\mathcal{P}^* = \mathcal{P}$ by inversion and $\mathcal{P}W = W$ by reproduction. Schur orthogonality gives

$$\int e(a) e(x^{-1}a^{-1}t)\, da = d^2 \int \overline{c(a)}\, c(t^{-1}ax)\, da = c(x) e(t).$$

Indeed the second coefficient in that pairing has vectors $\pi(t)v, \pi(x)v$. The two L^1 kernels justify the operator multiplication, so $\mathcal{P}R(x)\mathcal{P} = c(x)\mathcal{P}$. Consequently $P(W, W)(x) = d_I c(x)$ in the notation of Section 12.1. Its squared norm is d_I^2/d.

For the norm-ball estimate it suffices to have $|c(g)| \ge 1/2$ on $\|g - 1\| \le \delta = (200\beta^5)^{-1}$, where $1 \le \beta \le b^4$ bounds r, and $b \ge 2000$. Here is a sufficient representation-theoretic condition for this near-identity bound. In trace-form normalization let the full and compact Casimirs act

on v by ω_G, ω_K, with $|\omega_G|, \omega_K \le r\beta^2$. For a real orthonormal basis X_i of the compact and noncompact summands in the Hilbert–Schmidt metric, the full Casimir is $\sum_{\mathfrak{p}} X_i^2 - \sum_{\mathfrak{k}} X_i^2$ and the compact one is $-\sum_{\mathfrak{k}} X_i^2$.

Skew-adjointness gives

$$\sum_i \|d\pi(X_i)v\|^2 = 2\omega_K - \omega_G \le 3r\beta^2.$$

For the lowest compact type use the Casimir eigenvalue formulas $\|\lambda\|^2 - \|\rho\|^2$ and $\|\mu + \rho_K\|^2 - \|\rho_K\|^2$ when each coordinate of $\lambda, \rho, \mu + \rho_K, \rho_K$ is bounded by β. Here $\mu = \lambda + \rho - 2\rho_K$ by the lowest-type formula [5, Theorem 8.5, p. 42]. For the unitary group one applies that theorem on the derived group, retaining the central character on the compact center in both Casimirs. By Cauchy–Schwarz and $\|X\|_{\mathrm{HS}} \le \sqrt{r}\|X\|$, $\|d\pi(X)v\| \le \sqrt{3}\, r\beta\|X\|$. For $\|g-1\| \le \delta$, the logarithm $X = \log g$ belongs to the Lie algebra, since the form adjoint sends g to g^{-1} and the logarithm to $-X$. The power series gives $\|X\| \le 2\|g-1\|$. Integrating the coefficient derivative along $\exp(tX)$ gives $|c(g) - 1| \le 2\sqrt{3}\, r\beta\delta \le 1/2$.

The unitary norm ball of radius $Y \ge 1$ can be covered by translates of the identity ball of radius δ using at most $(600\beta^5 Y^2)^{2r^2}$ centers. In fact choose a maximal δ/Y-separated set in the group ball. Disjoint ambient matrix balls of radius $\delta/(2Y)$ fit in the ambient ball of radius $Y + \delta/(2Y)$. Volume comparison in real dimension $2r^2$ gives the stated count. Each group point g is within δ/Y of some center h, so $\|gh^{-1} - 1\| \le \delta$, since $\|h^{-1}\| = \|h\| \le Y$. The identity ball has dg-mass at most $4/d$ by integrating $|c|^2$. Right Haar invariance now gives

$$d\,\mathrm{vol}_{dg}\{g : \|g\| \le Y\} \le 4(600\beta^5 Y^2)^{2r^2} \le b^{99r^2} Y^{9r^2}.$$

This uses the original dg throughout and gives the ball bound of Section 12.1.

12.4 Error from truncating the orbital integrals

In this subsection, we estimate the norm of the part removed from the full real vector. The decay of Section 12.7 and the volume estimate of Section 12.3 bound its contribution to the orbital pairing.

The same packing argument gives the stronger bound $d\,\mathrm{vol}_{dg}\{\|g\| \le Y\} \le b^{42r^2+1} Y^{4r^2}$. For $M_* = 9r^2$ define $\phi(g) = (2/\|g\|)^{M_*}$. Since $\|g\| \ge 1$, integration of the sublevel volumes against $M_* Y^{-M_*-1} dY$ yields

$$d\, 16^{M_*} \int_G \phi\, dg \le b^{99r^2}.$$

Indeed $M_*/(M_* - 4r^2) \le 2$, and the extra powers $2^{M_*} 16^{M_*}$ contribute at most b^{2M_*}.

Write $U = U_{<n}$ and $S = UH = R_n$. Use the Bessel graph of the projection $S \to H$, and write its elements as $s = hu$ in the first coordinate. Its Haar ds is the product of the original du and probability Haar on H. Conjugation by H preserves du, so this product is right invariant. The graph is closed, since S is a closed flag subgroup and the projection to the compact factor is continuous. We use the operator norm in the original unitary basis with prescribed signs, which is unchanged by transport to the chosen coordinates.

For a point x_0, let B_{x_0} be the graph set $\|x_0^{-1} s x_0 - 1\| \le 1$, with mass V_{x_0}. This is compact because the first coordinate ranges over a compact subset of G and the other coordinate is in the compact group H. It contains a neighborhood of the identity, so $0 < V_{x_0} < \infty$. Put

$\Phi(y) = \int_S \phi(x_0^{-1} sy)\, ds$. To bound this function, take a maximal $1/Y$-separated set of matrices $a_i = x_0^{-1} s_i y$ of norm at most Y.

Ambient packing gives at most $3^{2r^2} Y^{4r^2}$ centers for $Y \geq 1$. If $a = x_0^{-1} sy$ is within $1/Y$ of a_i, then

$$\|x_0^{-1} s s_i^{-1} x_0 - 1\| = \|(a - a_i) a_i^{-1}\| \leq 1.$$

Thus each piece is contained in $B_{x_0} s_i$, of mass V_{x_0} by right invariance. Integrating the sublevel estimate against $2^{M_*} M_* Y^{-M_*-1}\, dY$ gives $\Phi(y) \leq 16^{M_*} V_{x_0}$. For any integrable function γ on G, the original quotient disintegration gives

$$\int_{U\backslash G} \int_S \gamma(sy)\, ds\, d\bar{g}(y) = \int_G \gamma(g)\, dg.$$

Indeed write $s = hu$, unfold $du\, d\bar{g}$ for each h, and apply left Haar invariance. The same proof applies to nonnegative functions. With left translation by x_0^{-1} it gives $\int \Phi\, d\bar{g} = \int \phi\, dg$.

For $s = hu$, let $\sigma(u)$ be the simple-root sum in the real chart of Section 8.3, and put

$$\chi_U(u) = \exp(\sigma(u) - \overline{\sigma(u)}), \qquad \sigma_S(hu) = \chi_U(u) \pi_H(h).$$

The first superdiagonal entries of the unitriangular flag action add under multiplication. The compact group H fixes the ordered flag vectors, so conjugation by H preserves χ_U. Thus σ_S is a unitary representation of S. For the unit highest vector v_0 of the compact type, set

$$A(y) = \int_S c(x_0^{-1} sy) \sigma_S(s)^{-1} v_0\, ds.$$

The large-gap coefficient bounds used here are $|c(g)| \leq \phi(g)$ and $|c(g)| \leq (2/\|g\|)^{M_* + 3r^3 f}$, as proved in Section 12.7 for the prescribed lowest compact type and parameter gap. The volume estimate above uses square integrability and the coefficient bound at the identity. Since $|c| \leq \phi$ and $\int_G \phi\, dg < \infty$, we have $c \in L^1(G, dg)$ before using the reproducing convolution. The same majorant gives absolute convergence in the graph variable and $|A| \leq \Phi$. In particular A is bounded, and its norm belongs to $L^1 \cap L^2$ on $U\backslash G$. It is continuous on G: for $y = y_0 a$ with a in a compact neighborhood of 1, the norm bound is dominated by $\|a^{-1}\|^{M_*} \phi(x_0^{-1} s y_0)$. The substitution $s \mapsto st^{-1}$ gives $A(ty) = \sigma_S(t) A(y)$. Also

$$\int_G e(a) A(ya)\, da = A(y).$$

Indeed Schur orthogonality gives $\int e(a) c(ga)\, da = c(g)$. The graph integral commutes with this convolution by the bound $\|e\|_1\, 16^{M_*} V_{x_0}$ on the absolute double integral.

The period at x_0 is exactly $d\|A\|_2^2$. To check the constant, unfold the integrable function $d\, \overline{c(x_0^{-1} g)} \langle v_0, A(g) \rangle$. At $g = sy$, covariance and unitarity make it

$$d\, \langle c(x_0^{-1} sy) \sigma_S(s)^{-1} v_0, A(y) \rangle.$$

Its graph integral is $d\|A(y)\|^2$. On the full group, substitute $g = x_0 a$ and use the reproducing identity. The integral is $\langle v_0, A(x_0) \rangle$, which on expanding A is the stationary period on this same graph and Haar.

The quantitative stationary bound is supplied by Section 12.8 under its weight and gap conditions. It gives a point x_0 for which

$$\ell(x_0) \leq b^{5n}, \qquad V_{x_0} \leq b^{60(r+k)^2} |\langle v_0, A(x_0) \rangle|, \qquad \ell(y) = \inf_{s \in S} (\|sy\| + 1).$$

Then $A \neq 0$ by the positive mass of the graph ball. Take the positive scalar $c_0 = \sqrt{d_I}/\|A\|_2$ and put $W = c_0 A$, so $\|W\|_2^2 = d_I$. The length is left S-invariant and satisfies $\ell(x_0 g) \leq \ell(x_0)\|g\|$, since $\|g\| \geq 1$. For $g = x_0^{-1} s y$ this gives $\ell(y)/b^{6n} \leq \|g\|/2$.

If $H(y) = \max(1, \ell(y)/b^{6n}) > 1$, the second coefficient estimate absorbs $H(y)^{3r^3 f}$. When $H(y) = 1$, use the first estimate. In either case $H(y)^{3r^3 f}|W(y)| \leq |c_0|\Phi(y)$. With $w = |c_0|\Phi$, the period identity and the graph and group volume bounds give

$$\begin{aligned}(\sup w) \int w \, d\bar{g} &\leq d_I \frac{V_{x_0}}{\langle v_0, A(x_0)\rangle} \, d \, 16^{M_*} \int_G \phi \, dg \\ &\leq d_I b^{60(r+k)^2 + 99r^2} \leq d_I b^{400 r^2}.\end{aligned}$$

Here $\langle v_0, A(x_0)\rangle = d\|A\|_2^2 > 0$ uses the same quotient Haar and measured point as the normalization of W. ℓ is continuous by its right-translation norm inequality and the same inequality for the inverse. The scalar cutoff $\chi = \max(0, \min(1, 2 - 2\ell/R_b))$ is one on $\ell(y) \leq R_b/2$. If $\chi(y) > 0$ there is $s = hu$ with $\|sy\| < R_b$, hence $\|uy\| < R_b$ because H is Euclidean unitary.

Put $F = \chi W$. Since U is closed and $K_b = \{g \in G : \|g\| \leq R_b\}$ is compact, UK_b is closed. Thus $\operatorname{supp} F \subseteq UK_b$. For the right correlation of Section 12.1, a nonzero integrand requires $y = u_1 k_1$ and $yx = u_2 k_2$, with $u_i \in U$ and $k_i \in K_b$. Hence

$$\operatorname{supp} P(F, F) \subseteq K_b^{-1} U K_b, \qquad x = k_1^{-1} u_1^{-1} u_2 k_2.$$

This set is closed because the outside factors are compact and U is closed. For the left translate in the orbital tensor, $F(y) \otimes \overline{F(\gamma^{-1} y)} \neq 0$ gives instead $\gamma = u_1 k_1 k_2^{-1} u_2^{-1} \in U K_b K_b^{-1} U$.

On the support of $F - W$, one has $H(y) \geq b$ for $R_b = b^{6n+2}$ and $b \geq 2$. Thus

$$\|F - W\|_2^2 \leq b^{-6r^3 f} \int w^2 \, d\bar{g} \leq b^{400r^2 - 6r^3 f} d_I \leq b^{-2r^3 f} d_I$$

in the range $r \geq 100$, $f \geq 1$. The stationary-period and coefficient decay bounds in this argument concern the same x_0, A as the reproducing vector.

12.5 The formal-degree factor

In this subsection, we compare the formal degree in the orbital Haar measure with the compact normalization of the Weyl dimension formula. The resulting factor enters the initial norm calculation of Section 12.1.

In the Cayley normalization, the density at the identity of $U(J)$ in the skew-Hermitian coordinates $S = JX$ of its Lie algebra is $1/\operatorname{vol} U(r)$. The corresponding density on the compact dual is the same. In a unitary basis where J is diagonal, the Cartan decomposition keeps the two diagonal skew-Hermitian blocks and sends the off-diagonal noncompact part $X_{\mathfrak{p}}$ to $iX_{\mathfrak{p}}$. In S-coordinates this change has real determinant of absolute value one, since it multiplies the independent complex off-diagonal entries by a unit complex scalar.

For signature (p, q), assign probability Haar to $K = U(p) \times U(q)$. Probability Haar on $U(r)$ then induces mass one on the complex Grassmannian $U(r)/K$, the compact dual of $U(p, q)/K$. With the tangent measures identified above, these are the two measure normalizations in the Harish–Chandra degree formula [5, (3.15)–(3.16), p. 14, and Theorem 8.1, p. 41]. Indeed put $G_1 = SU(p, q)$ and $K_1 = S(U(p) \times U(q))$. The maps $(z, h) \mapsto zh$ from $U(1) \times G_1$ to $U(p, q)$ and from $U(1) \times SU(r)$ to $U(r)$ have the same kernel $\{(\xi, \xi^{-1} I) : \xi^r = 1\}$. On the compact

side the product of the two probability measures pushes forward to probability Haar. On the noncompact side take probability Haar dz on the circle and the degree-formula Haar dh on G_1. The quotient $G_1/K_1 = U(p,q)/K$ has the same tangent measure as its compact dual $SU(r)/K_1 = U(r)/K$, and probability Haar on $U(1)\times K_1$ pushes forward to probability Haar on K. Thus $dz\,dh$ pushes forward to dg. For a coefficient c of an irreducible unitary representation the circle acts by a scalar of modulus one, so its restriction to G_1 is irreducible and

$$\int_{U(p,q)} |c(g)|^2\,dg = \int_{U(1)\times G_1} |c(zh)|^2\,dz\,dh = \int_{G_1} |c(h)|^2\,dh.$$

For a regular parameter $\lambda_1,\dots,\lambda_r$, the type A_{r-1} root products give

$$d = \frac{\prod_{i<j}|\lambda_i-\lambda_j|}{\prod_{i=0}^{r-1} i!}.$$

In fact the product of the positive-root differences at ρ is $\prod_{j=2}^{r}(j-1)!$, by grouping the factors $j-i$ at fixed j. For the compact volume in these same coordinates, project $U(l)$ to its last column on the unit sphere in $\mathbb{C}^l$. At the identity the differential is $S\mapsto Se_l$. Its kernel has the skew coordinates of $U(l-1)$, and its remaining coordinates are the $l-1$ complex entries and the imaginary diagonal entry of that column. They give the Euclidean tangent density of the sphere. By invariance this identifies the quotient measure everywhere with surface measure. Polar integration of the Gaussian therefore gives

$$\frac{\operatorname{vol}U(l)}{\operatorname{vol}U(l-1)} = \operatorname{area}(S^{2l-1}) = \frac{2\pi^l}{(l-1)!}, \qquad \operatorname{vol}U(0) = 1.$$

Set $t = r-k = 2\varrho+1$ and $\Theta = \pi^{tk}\operatorname{vol}U(k)/\operatorname{vol}U(r)$ as in (19). Consequently

$$d\Theta = \frac{\prod_{i<j}|\lambda_i-\lambda_j|}{2^t\pi^{t(t+1)/2}\prod_{i=0}^{k-1} i!}.$$

The factorials with indices at least k cancel, and the exponent of π comes from $\sum_{i=k}^{r-1}(i+1)-tk = t(t+1)/2$. Thus $|2\lambda_i|\le b^4$ on the parameter implies $d\Theta \le b^{2r(r-1)} \le b^{4r^2}$, as used in Section 12.1.

12.6 Haar measure in the adapted real basis

In this subsection, we identify the real Haar measure obtained by successive isotropic quotients with the group measure used for discrete series. This gives the conversion needed in Section 12.1 and matches the finite convention of Section 10.2.

Use the basis ordered by the pairs, then z, then the k positive coordinates. Its Gram matrix J has the hyperbolic pair blocks and the diagonal blocks $-1, I_k$. Let $d\mu$ be the real stage Haar measure, with the terminal stabilizer $U(k)$ of mass one. Then

$$d\mu = \frac{\operatorname{vol}U(r)}{\operatorname{vol}U(k)}\,dg.$$

For a matrix Y of independent columns, put $\Gamma(Y) = Y^*JY$. Use Lebesgue measure on the complex entries of Y, and on the Gram entries use the real diagonals and the real and imaginary parts of the upper off-diagonals. The map Γ is a submersion since $Y^*J\dot{Y}$ can be prescribed.

With an initial segment Y_0 fixed, the differential in the new columns surjects onto the cross block and the new diagonal block by the same linear solve with Y^*J.

Thus local submersion coordinates disintegrate Lebesgue measure over these blocks in order. In fact the coordinate Jacobian is block triangular, with the old Gram coordinates depending only on Y_0. Consequently the full fiber measure is obtained by the successive conditional fiber integrations at the prescribed Gram entries. For an initial segment of the form gY_0 with $g \in U(J)$ and Y_0 the specified initial segment, the conditional measure is transported by g. Both ambient Lebesgue measure and all the Gram entries are preserved, since $|\det g| = 1$.

For the full square Gram fiber at J, its density at the identity is computed by

$$\dot{Y} = J(S + T/2), \qquad \dot{\Gamma} = T, \qquad S^* = -S, \quad T^* = T.$$

Left multiplication by J preserves Lebesgue measure. Each diagonal contributes $1/2$. For an off-diagonal pair the complex map from (S_{ij}, T_{ij}) to $(S_{ij} + T_{ij}/2, -S_{ij} + T_{ij}/2)$ has determinant one. Conjugating the second image preserves real volume. The fiber density is therefore 2^{-r} in the S-coordinates. The full fiber is left invariant, so the Cayley normalization gives its measure as $2^{-r} \operatorname{vol} U(r)\, dg$.

Inversion of the stage product gives, in order, the factors $\theta_i^{\mathrm{inv}}, \eta_i$ for each pair, then $\theta_\varrho^{\mathrm{inv}}$ for z, and finally probability Haar on $U(k)$. Here θ^{inv} denotes the image under inversion. Every new factor fixes the earlier columns. Fixing the cross products with the earlier pairs sets their opposite coordinates to zero, with Jacobian one. When adding e_i or z, the chart of Section 10.2 thus disintegrates in the remaining space.

In its hyperbolic pivot coordinates (t, tc, v), the ambient Jacobian is $|t|^2$ and

$$h((t, tc, v), (t, tc, v)) = 2|t|^2 \Re c + h(v, v).$$

The change from $\Re c$ to the real Gram entry gives $1/(2|t|^2)$. The top hyperbolic change of basis has determinant of modulus one, since both Gram determinants on the plane are -1. Thus half the stage quadric measure is the conditional fiber measure on its pivot chart. The discarded set $t = 0$ has measure zero for both measures. Indeed the nonzero isotropic vectors form one orbit, as do the vectors of the prescribed negative norm, by extension of isometries. Each orbit is connected because $U(p, q)$ is connected by Cartan decomposition. The real-analytic function $|t|^2$ is nonzero somewhere on each orbit by the pivot parametrization. Its zero set has measure zero for any smooth density there. The isotropic origin cannot occur in an independent frame.

For the dual column the additional cross product with e_i fixes the coefficient of e_i^* to one. Write the column as $e_i^* + v + (\alpha + i\beta)e_i$. Its norm is $2\alpha + h(v, v)$, so the fiber law at zero has density $dv\, d\beta/2$. This is half the Heisenberg measure on that column.

After the $2\varrho + 1$ columns have been inserted, their complement has Gram matrix I_k in the chosen coordinates. The cross products with the initial segment again give a coordinate map of determinant of modulus one. The calculation for the full Gram fiber, now with $J = I_k$, gives the terminal conditional measure $2^{-k} \operatorname{vol} U(k)$ times the compact probability Haar. All the pivot charts have full conditional measure. Multiplication of the $2\varrho + 1$ factors $1/2$ with the terminal factor 2^{-k} gives

$$2^{-r} \operatorname{vol} U(k)\, d\mu^{\mathrm{inv}} = 2^{-r} \operatorname{vol} U(r)\, dg.$$

The square-column map sends a group element to its matrix in the chosen basis. At each step the norm-fiber measure is invariant under the remaining unitary group, since the group preserves the norm map and ambient Lebesgue measure. Changing the orbit representative in right

translation changes the inner integral by a stabilizer translation. Induction from probability Haar on $U(k)$, as in Section 10.2, proves right invariance of $d\mu$. The displayed identity and the nonzero chart densities give $d\mu = \lambda\, dg$ with $0 < \lambda < \infty$, since G is unimodular as verified below.

To compare the quotient measures, order the Heisenberg factors by increasing a. The factor with parameters (v, β) fixes the earlier dual columns and sends e_a^* to

$$e_a^* + v + (i\beta - h(v,v)/2)e_a.$$

The product of earlier factors only adds earlier isotropic vectors to this column. The resulting column therefore has coordinates v in the remaining space and imaginary coordinate β along e_a. Moreover $u^{-1} = Ju^*J$ identifies these dual-column coordinates of u^{-1} with the free entries of u up to permutation, signs and conjugation. All have real Jacobian of absolute value one. Thus the product of the Heisenberg Haar measures is the original unipotent Haar, since inversion preserves Haar on a unipotent group. Dividing the group measures by this common Haar gives $dQ = \lambda\, d\bar{g}$ on $U_{<n}\backslash G$.

The adjoint map $S \mapsto g^{-*}Sg^{-1}$ has real determinant of absolute value $|\det g|^{-2r} = 1$. Thus G is unimodular, and inversion preserves both group measures. The Gram fiber identity gives $\lambda = \operatorname{vol} U(r)/\operatorname{vol} U(k)$. In particular the coefficient of the initial squared correlation is $C_F\lambda^3 = d_I^{-2}$ by Section 11.8.

12.7 Decay of the lowest compact type coefficient

In this subsection, we bound the lowest compact type coefficient using its Harish–Chandra parameter. The bound is used away from compact sets in Section 12.4 and in the tail of the stationary-phase estimate of Section 12.9.

Work in the unitary basis with prescribed signs $J = \operatorname{diag}(\varepsilon_i)$. Write $\lambda_1 > \cdots > \lambda_r$ for the Harish–Chandra parameter, $\rho_i = (r + 1 - 2i)/2$, and $a_i = \lambda_i + \rho_i$, which are integers. For $0 \le c_* \le h$ integral, where $|\lambda_i - \lambda_j| \ge h$ for opposite signs, we prove

$$|c(g)| \le \min\{1, (2/\|g\|)^{c_*}\}$$

for a coefficient of unit vectors in the lowest compact type.

For a matrix Z, let $D_I(Z)$ be the principal minor on I. For a weight $w \in \mathbb{Z}^r$ and $|w_i| \le M$ put

$$m_w(Z) = (\det Z)^{-M} \prod_{t=-M+1}^{M} D_{\{i:w_i \ge t\}}(Z).$$

If $Z \in U(J)$ is positive definite, then $Z^{-1} = JZJ$ and $\det Z = 1$. Its block diagonal part $D = (Z + Z^{-1})/2$ is positive definite. For each threshold t, take the partition

$$I = \{a_i \ge t\}, \qquad B = \{a_i < t \le a_i + c_*\}, \qquad C = \{a_i + c_* < t\}.$$

The signs on B agree since ρ_i is ordered as λ_i. Thus $D_B(D) = D_B(Z)$. Fischer's inequality (the positive Schur complement bound) and complementary minors give

$$D_{I\cup B}(D) \le D_I(D)D_B(Z) = D_I(D)D_{I\cup C}(Z) \le D_I(D)D_I(Z)D_C(Z).$$

Multiply using a common M for a and $a+c_*$. The product of $D_C(Z)$ is $m_{a+c_*}(Z)$ by complementary minors. Since $m_{a+c_*}(W) = m_a(W)(\det W)^{c_*}$, cancelling $m_a(D)$ gives

$$(\det D)^{c_*} \le m_a(Z)^2.$$

For the compact-Cartan Borel $\mathfrak{b}$ determined by λ, the discrete-series Harish–Chandra module is $A_{\mathfrak{b}}(\lambda-\rho)$ [2, §9, paragraph following Theorem 9.2 and Theorem 9.4], [39, §3.5, (3.4)]. Its inducing weight is in the good range, since $(\lambda-\rho)+\rho = \lambda$ pairs positively with every root of the nilradical in the chosen order. Let V be the lowest compact type with unit highest vector v and orthogonal projection E_V. On the positive factor $Z = e^X$ of the polar decomposition, Sun's bottom layer integral [71, Theorem 6.3 and (25)–(26), pp. 219–220] gives

$$E_V \pi(Z) E_V = \dim V \int_K m_a(k^{-1}Zk)^{-1} P_{\pi(k)v}\, dk,$$

where $K = U(J) \cap U(r)$, $dk(K) = 1$, and P_u is the orthogonal projector onto $\mathbb{C}u$. For the compact-Cartan Borel the inducing character has weight $\lambda - \rho$. For same-sign indices $i < j$, let s count the indices $i < h \le j$ of that sign. The proposed compact weight $\mu = a - 2\rho_K$ has $\mu_i - \mu_j = (\lambda_i - \lambda_j) + (j-i) - 2s \ge 0$ by integrality and regularity. Multiplication by the determinant character of the nilradical gives weight $(\lambda-\rho)+2\rho = a$. The lowest compact type has highest weight $(\lambda-\rho)+2(\rho-\rho_K) = a - 2\rho_K$. In Sun's formula take the compact argument to be 1, the positive argument to be $e^{X/2}$, and the compact integration variable to be k^{-1}.

In the upper triangular times unitary decomposition $k^{-1}e^{X/2} = udk'$, take u unitriangular and d positive diagonal. Sun's map $H_{\mathfrak{b}}$ in (7) reads d modulo the compact torus. The complexified Cartan involution is conjugation by J, and real-form conjugation sends g to $J(g^*)^{-1}J$. Hence the factor $d\,\overline{\theta(d^{-1})}$ in (25) is d^2. Also $k^{-1}Zk = ud^2u^*$. Trailing principal minors give $d_i^2 = D_{[1,i-1]}(k^{-1}Zk)/D_{[1,i]}(k^{-1}Zk)$, using $Z^{-1} = JZJ$ and complementary minors. The torus weight therefore gives $m_a(k^{-1}Zk)^{-1}$. The positive factor has determinant one, so this use of the derived group formula preserves the central character. Schur orthogonality gives $\dim V \int_K P_{\pi(k)v}\, dk = E_V$. The scalar $m_a(k^{-1}Zk)^{-1}$ is positive and at most $(\det D)^{-c_*/2}$, since K preserves the determinant of the block diagonal part.

Thus the same bound holds in operator norm. The compact factor of g acts unitarily on V, so it preserves that bound. Write $g = D_g + E_g$ for its equal-sign and opposite-sign blocks. The diagonal blocks of $g^*Jg = J$ give $D_g^*D_g = 1 + E_g^*E_g$, and $|\det D_g| = \det D$ by the polar decomposition. Therefore

$$|c(g)|^4 \le (1 + \|E_g\|_{\mathrm{HS}}^2)^{-c_*},$$

using $\det(1 + E_g^*E_g) \ge 1 + \operatorname{tr}(E_g^*E_g)$.

The eigenvalues u of Z contribute $(u+u^{-1})/2$ to $\det D$, each at least one. If $\|Z\| > 1$, then $\|Z\|$ and $\|Z\|^{-1}$ are distinct eigenvalues and their two factors give $\det D \ge (\|Z\|/2)^2$. The same inequality holds for $\|Z\| = 1$. Since $\|g\| = \|Z\|$, this proves the decay claim. In Section 12.4 use $c_* = 9r^2$ and $c_* = 9r^2 + 3r^3 f$, both at most the available gap whenever $3r^3 f \le A$ and $h \ge 500r^3 + A$.

12.8 The stationary point and its period

In this subsection, we construct the stationary point for the original pair of discrete series. The stationary-phase calculation of Section 12.9 gives the period lower bound used to normalize the full vector in Section 12.10.

Proposition 12.1 (Stationary point and period lower bound)**.** *Use the pair of discrete series with a common zero-weight positive block, integral weights μ_i on the remaining $2n$ coordinate lines of signatures $(-1)^i$, and remaining weight a in H. Put $N_0 = (r+k)^2$, $\beta = b^3$ and $T_0 = N_0 \log \beta$. Assume*

$$B_{\mathrm{wt}} \geq 1, \quad -3B_{\mathrm{wt}} \leq \mu_{2n-1} < \cdots < \mu_0 \leq -B_{\mathrm{wt}}, \quad a \geq 0, \quad 3B_{\mathrm{wt}} + a \leq b^2, \quad 8nr \leq b$$

and $\beta \geq 10^{12}$. Assume that all doubled Harish–Chandra parameters of both representations have absolute value at most L_0', with $r + k + L_0' \leq \beta$. For some integer $A_{\mathrm{gap}} \geq 10^{30} T_0^3$, assume $|\lambda_i - \lambda_j| \geq 500r^3 + A_{\mathrm{gap}}$ for every pair of ambient parameters on coordinate lines of opposite signs. Then there is a stationary point x_0 for which

$$\ell(x_0) \leq b^{5n}, \qquad V_{x_0} \leq b^{60N_0} |\langle v_0, A(x_0) \rangle|.$$

Proof. Let the lowest compact type weights of the original two representations be zero on a common positive block X. In the remaining $2n$ coordinates of G, take integral weights $-3B_{\mathrm{wt}} \leq \mu_i \leq -B_{\mathrm{wt}}$ in strictly decreasing order at signs $(-1)^i$, $0 \leq i < 2n$, with $B_{\mathrm{wt}} \geq 1$. The remaining coordinate in $H = U(k)$ has weight $a \geq 0$. Use $\beta = b^3$, and suppose $T = 3B_{\mathrm{wt}} + a \leq b^2$ and $8nr \leq b$. The following construction takes place in these two original weight bases.

Put $P(t) = \prod_i (t - \mu_i)$ and, on the $2n$ coordinates,

$$[F]_i = \frac{F(\mu_i)}{\sqrt{|P'(\mu_i)(\mu_i - a)|}}, \qquad V_j = [t^j (t-a)] \quad (0 \leq j \leq 2\varrho), \qquad I = \sqrt{P(a)}[1].$$

Extend these vectors by zero on X. For real polynomials the signed pairing of $[F]$ and $[G]$ is $-\sum_i F(\mu_i) G(\mu_i) / (P'(\mu_i)(\mu_i - a))$. By Lagrange interpolation, $\sum_i Q(\mu_i)/P'(\mu_i)$ is the top coefficient of Q for $\deg Q \leq 2n - 1$. It follows that

$$h(V_j, V_l) = 0 \quad (j + l < 2\varrho), \qquad h(V_j, V_l) = -1 \quad (j + l = 2\varrho), \qquad h(I, V_j) = 0.$$

Partial fractions of $1/(P(t)(t-a))$ have residues summing to zero, with residue $1/P(a)$ at a. This gives $h(I, I) = 1$. Let i_0 embed the compact space by the common coordinates on X and by I at its final coordinate. The embedding is isometric. The vectors $V_0, \ldots, V_\varrho$ are independent, since $t - a$ is nonzero on the $2n$ nodes and a polynomial of degree at most ϱ vanishing there is zero. Set

$$D_\mu = \mathrm{diag}(0_X, \mu_0, \ldots, \mu_{2n-1}), \qquad D_H = \mathrm{diag}(0_X, a).$$

Then

$$D_\mu V_j = V_{j+1} \quad (j < 2\varrho), \qquad D_\mu i_0 = i_0 D_H \bmod \langle V_0 \rangle.$$

Transport the basis with prescribed signs into the original stage space by its Euclidean unitary change of basis. The perpendicular complement of $i_0 \mathbb{C}^k \oplus \mathbb{C} V_\varrho$ is nondegenerate. In it let A have columns $V_0, \ldots, V_{\varrho-1}$ and choose B_0 with $A^* J B_0 = 1$. Replacing B_0 by $B_0 - A(B_0^* J B_0)/2$ gives isotropic duals. Together with i_0, V_ϱ the columns give an isometry x_0^{-1} from the original basis, sending e_j to V_j, z to V_ϱ, and the positive block to $i_0 \mathbb{C}^k$ with its chosen basis.

To bound the length of this same x_0, use Euclidean Gram–Schmidt on $(V_0, \ldots, V_\varrho)$, without rescaling. The resulting F_j differ from V_j by earlier columns, and retain the Gram and the pairings $h(F_j, V_{2\varrho - j}) = -1$. As $|P'(\mu_i)| \geq 1$, the coordinate formula gives $\|V_j\| \leq M_0 = \sqrt{r}\, T^{2n}$ through 2ϱ. Thus $M_0^{-1} \leq \|F_j\| \leq M_0$. For $j < \varrho$, the vectors $J F_j / \|F_j\|^2$ are isotropic duals,

with Euclidean norm at most M_0. The pairs form a J-invariant Euclidean orthogonal summand, and F_ϱ lies in its complement of signature $(k,1)$.

Write $F_\varrho = z_+ + z_-$ in the positive and negative eigenspaces, with $s = \|z_+\|$, $t = \|z_-\|$ and $t^2 - s^2 = 1$. For $s > 0$, set $v = tz_+/s + sz_-/t$. In the Hermitian form h of signature $(k,1)$,

$$h(v,v) = 1, \qquad h(v,F_\varrho) = 0, \qquad \|v\|^2 = s^2 + t^2 = \|F_\varrho\|^2.$$

For $s = 0$, take a Euclidean unit positive vector in that complement. Complete by the remaining orthonormal positive directions, orthogonal to these at most $2n$ exceptional columns. The new basis matrix Y is an isometry with $\|Y^{-1}\| = \|Y\| \le \sqrt{2n}M_0 + 1$. Put $s_0 = x_0 Y$, $A_j = \operatorname{span}(e_i : i < j)$, and $A = A_\varrho$ in the original basis. The relations $F_j - V_j \in \operatorname{span}(V_i : i < j)$ give

$$s_0 e_j - e_j \in A_j \quad (0 \le j < \varrho), \qquad s_0 z - z \in A.$$

Unitarity preserves $A^\perp$. The induced isometry on $A^\perp/A$ fixes $[z]$ and acts by $u \oplus 1$, with $u \in U(k)$. After removing the standard lift of u, the isometry is upper unitriangular on A and the identity on $A^\perp/A$. Unitarity forces the inverse-adjoint block on the opposite isotropic quotient. These are the defining conditions for $U_{<n}$. Thus $s_0 \in S$, and

$$s_0^{-1} x_0 = Y^{-1}, \qquad \ell(x_0) \le 1 + \|Y^{-1}\| \le 8nM_0 \le b^{5n}.$$

The representative $s_0^{-1}x_0$ is used in the length infimum, while the measured period is taken at the stationary point x_0.

Return to the basis with prescribed signs, and for a graph element s write $g = x_0^{-1} s_G x_0$ in that basis, with compact component $h = s_H$. Put $Y_1 = g - 1$, $Y_2 = h - 1$, $L = \langle V_0, \dots, V_{\varrho-1}\rangle$, and let $\sigma(s)$ be the sum of first predecessor coefficients on the original basis. Then $g - 1$ lowers the transported flag, and on $L^\perp/L$ the action of g is $h \oplus 1$ in the transported i_0, V_ϱ. Conversely these equations characterize the conjugated graph, since in the original Witt blocks they give exactly $U_{<n}H$ and its compact projection. In particular the diagonal $U(X)$ belongs to the transported graph, with $\sigma = 0$.

Choose a projection P_0 onto L killing $i_0\mathbb{C}^k$, put $D_1 = D_\mu P_0$, and let $D_2 = D_\mu - D_1 - D_1^\dagger$, with $\dagger$ the form adjoint. Since $D_\mu L \subset L^\perp$, D_2 kills L and has image in $L^\perp$. On $i_0\mathbb{C}^k$ it agrees with D_μ, since $D_\mu i_0 \mathbb{C}^k \subset L^\perp$. On $L^\perp/L = i_0\mathbb{C}^k \oplus \mathbb{C}[V_\varrho]$, Y_1 induces $Y_2 \oplus 0$. The compression of D_2 to the first summand is $i_0^\dagger D_\mu i_0 = D_H$, since $V_0 \perp i_0\mathbb{C}^k$. The traces of $D_2 Y_1$ on L and on the quotient by $L^\perp$ are zero. Thus $\operatorname{tr}(D_2 Y_1) = \operatorname{tr}(D_H Y_2)$. Since $D_\mu V_j = V_{j+1}$, the diagonal entry of $P_0 Y_1 D_\mu$ on L at V_j is the coefficient of V_j in $Y_1 V_{j+1}$ for $0 \le j < \varrho$. These entries sum to $\sigma(s)$, with the empty sum equal to zero if $\varrho = 0$. Applying this calculation to $Y_1^\dagger = g^{-1} - 1$ gives $\operatorname{tr}(D_1^\dagger Y_1) = -\bar\sigma(s)$, since inversion negates the first predecessor coefficients. The flag and its dual contribute zero to the unweighted trace. We obtain

$$\operatorname{tr} Y_1 = \operatorname{tr} Y_2, \qquad \operatorname{tr}(D_\mu Y_1) - \operatorname{tr}(D_H Y_2) = \sigma(s) - \bar\sigma(s).$$

With the unit highest vector v_0 of the compact type, put $c_H(h) = \langle v_0, \pi_H(h) v_0\rangle$. The period integrand for the average of Section 12.4 is $\theta(s) = c(g)\overline{c_H(h)} e^{\bar\sigma(s) - \sigma(s)}$. All following volumes and integrals use the original ds. Put $N_0 = (r+k)^2$ and $Q = (2/\|g \oplus h\|)^{9N_0}$. Since $\|h\| = 1$, the decay just proved gives $|\theta| \le Q$ when the gap is at least $9N_0$. If the doubled parameters are bounded by L_0' with $r + k + L_0' \le \beta$, the Casimir calculation of Section 12.3 on each factor gives $|\theta| \ge 1/4$ on the ball $\|g \oplus h - 1\| \le \delta = (200\beta^5)^{-1}$.

For the phase estimate put $T_0 = N_0 \log \beta$ and $\eta_0 = 10^{-13}/T_0$, and suppose the excess gap is at least $A_{\mathrm{gap}} \geq 10^{30} T_0^3$. Since $A_{\mathrm{gap}} \geq 18 N_0$, the decay bound also gives $|\theta| \leq Q^2$. If $\|E_g\|_{\mathrm{HS}} \geq \eta_0$, the preceding determinant estimate with $c_* = A_{\mathrm{gap}}$ gives

$$|\theta| \leq (1+\eta_0^2)^{-A_{\mathrm{gap}}/4} \leq \exp(-A_{\mathrm{gap}}\eta_0^2/8) \leq e^{-100T_0}.$$

Section 12.9 proves $|\theta| - 2\Re\theta \leq 12e^{-100T_0}$ when $\|E_g\|_{\mathrm{HS}} < \eta_0$. Away from that set the same bound follows from $|\theta| - 2\Re\theta \leq 3|\theta|$. Combining it with $|\theta| - 2\Re\theta \leq 3Q^2$ by the geometric mean gives

$$|\theta| \leq 2\Re\theta + 6\beta^{-50N_0} Q.$$

Let V_s be the mass of the ball of radius δ about 1, and V that of $\|g \oplus h\| \leq \beta + 1$.

Both are finite by closedness of the graph and compactness of the compact factor. Matrix packing as in Section 12.3 gives $V \leq \beta^{16N_0} V_s$, since $3(\beta+1)^2/\delta \leq \beta^8$ for $\beta \geq 10^{12}$. With radius one, the same packing bounds the mass of $\|g \oplus h\| \leq Y$ by $3^{2N_0} Y^{4N_0} V$ for $Y \geq 1$, since the ball of radius one about the identity lies in the set of mass V. Integrating in the original Haar measure gives

$$\int Q \, ds \leq 3^{2N_0} 2^{9N_0} V \int_1^\infty 9N_0 Y^{-5N_0-1} \, dY = \tfrac{9}{5}\, 3^{2N_0} 2^{9N_0} V \leq 16^{9N_0} V.$$

Hence

$$V \leq 8\beta^{16N_0} \left| \int \theta \, ds \right| + 24\beta^{-34N_0} 16^{9N_0} V.$$

The last coefficient is less than $1/2$. Since $V_{x_0} \leq V$ and $16\beta^{16N_0} \leq \beta^{20N_0}$, this yields $V_{x_0} \leq b^{60N_0} |\langle v_0, A(x_0)\rangle|$. This proves the quantitative period bound at the constructed point for these weights and gaps. □

12.9 The phase on the common compact block

In this subsection, we compare the phase-corrected coefficient near the stationary point of Section 12.8 with a positive Gaussian. The decay in Section 12.7 controls the complementary region in the original graph Haar measure.

Use the stationary graph and T_0, η_0 of Section 12.8. Write $x = g - 1$, $x' = h - 1$, $v_0' = \|E_g\|_{\mathrm{HS}}$ and $\phi = \sigma - \bar{\sigma}$. Suppose $v_0' \leq \eta_0$. For each row index i, the (i,i)-entry of $gJg^* = J$ gives

$$2\Re x_{ii} = 2\sum_j |(E_g)_{ij}|^2 - \sum_j |x_{ij}|^2.$$

The corresponding identity for h has zero opposite-sign term. Multiply by $-\mu_i$ on the first factor and by the nonnegative compact weight on the second. The weighted traces cancel in real part by the phase identity. Since $B_{\mathrm{wt}} \leq -\mu_i \leq 3B_{\mathrm{wt}}$ off X, and the weight vanishes on X, the sum of squared norms of the rows of x off X is at most $6(v_0')^2$. Apply the same calculation to the inverse pair, using $g^{-1} - 1 = Jx^*J$, to bound the columns off X.

For $a_0 = g_{XX}$ and $C = g_{X^cX}$, signed unitarity gives $a_0^* a_0 - 1 = -C^* J_{X^c} C$. Its Hilbert–Schmidt norm is at most $6(v_0')^2$. Thus $a_0 = up$ with u unitary and $p > 0$, and $\|p - 1\|_{\mathrm{HS}} \leq 6(v_0')^2 \leq \eta_0$ by the scalar inequality $|t - 1| \leq |t^2 - 1|$ for $t > 0$. Left multiply the pair by the diagonal copy of u^* in the graph. Both highest-weight vectors are fixed by this copy. Indeed the highest-weight coefficient on the compact group is m_μ, by the exterior-power realization.

Its threshold minors include all of X for $t \le 0$ and exclude it for $t > 0$, so the determinant factors on X cancel. A unit coefficient of value one forces the unit vector to be fixed. The translating element has phase zero, and ϕ is additive since the first predecessor coefficients add. Therefore θ has the same value after the move. The opposite-sign norm is unchanged and the X-block of the moved g is p. The row and column estimates now give $\|g-1\|_{\mathrm{HS}}^2 \le 13\eta_0^2$. Summing the unweighted row identity and using the trace equality gives $\|h-1\|_{\mathrm{HS}}^2 = \|g-1\|_{\mathrm{HS}}^2 - 2(v_0')^2$. Thus both distances are at most $4\eta_0$.

On this moved pair put $Z = \log g,\ Z' = \log h$. For the linear functional $\mathcal{L}(P,Q) = \operatorname{tr}(D_\mu P) - \operatorname{tr}(D_H Q)$, the phase identity applied to group powers gives $\mathcal{L}(g^j - 1, h^j - 1) = j\phi$ for nonnegative integers j. Taking finite differences at zero gives $\mathcal{L}(x^i, (x')^i) = 0$ for $i \ge 2$. The log series therefore gives $\mathcal{L}(Z, Z') = \phi$.

Triangular factors in the coefficient integral. We give the local calculation for either factor of dimension d_1. Use the weights $a = \lambda + \rho$ and $\mu = a - 2\rho_K$, and put $J_\mu = \int_K |m_\mu(k)|^2\, dk$. Write $c(w)$ for the coefficient of its unit highest vector. For fixed w in the sign group close to 1, consider

$$I(p, r_0) = \int_K \frac{m_\mu(pk) m_\mu(k^* r_0)}{m_a(k^* r_0 w^{-1} p k)}\, dk$$

for block diagonal p, r_0 near 1. The integral is holomorphic there, since all powers are integral and the denominator compressions are uniformly invertible on K. In the next computation the neighborhood can be taken with $\|p-1\|, \|r_0 - 1\| < 1/10$ and $\|w-1\| < 1/100$, for which $r_0 w^{-1} p$ is within distance one of 1.

12.9.1 Triangular coordinates on the sign blocks

In this subsection, we write the phase of Section 12.8 in triangular coordinates.

In each sign block take the upper triangular group for the parameter order. If b is upper triangular in these blocks, replacing k by kb in the integrand multiplies it by

$$\prod_i |b_{ii}|^{2(\mu_i - a_i)} = \prod_{i>j,\ \varepsilon_i = \varepsilon_j} |b_{ii}/b_{jj}|^2.$$

Indeed left lower and right upper factors scale each initial principal minor by its diagonal product. The thresholds for μ are initial within each block, and those for a are initial in the full order.

For block invertible t, write the block QR decomposition $tk = k'b$. This induces a diffeomorphism of the compact flag $K/\mathbb{T}$, where $\mathbb{T}$ is the diagonal torus. In left-translated tangent coordinates the strict lower complex entries of the derivative are the strict lower entries of bYb^{-1}, with Y skew-Hermitian in the sign blocks. The derivative of the upper factor has no strict lower part.

Each (i,j) depends on entries (i', j') with $i' \ge i,\ j' \le j$, so the complex diagonal factors are b_{ii}/b_{jj}. Thus the real Jacobian for invariant probability measure on the flag is the product just displayed. The integrand is right $\mathbb{T}$-invariant. Change of variables gives $I(pt, t^* r_0) = I(p, r_0)$ whenever the nearby arguments are defined. Differentiating at $t = 1$ in the directions Y and iY and using holomorphy gives separately $D_p I[pY] = 0$ and $D_{r_0} I[Y^* r_0] = 0$. Hence I is constant on the indicated neighborhood.

At $p = k_0,\ r_0 = 1$, for $w = k_0 Z_0$ its polar decomposition, the bottom-layer formula of Section 12.7 gives $I(k_0, 1) = J_\mu c(w)$. Indeed, $w^{-1} k_0 = Z_0^{-1}$, and $k^* Z_0^{-1} k = J(k^* Z_0 k) J$ since

k commutes with J. Conjugation by the diagonal sign matrix preserves every principal minor. The factor $m_\mu(k_0k)\overline{m_\mu(k)}$ follows by taking the highest-vector coefficient of $\pi(k_0)P_{\pi(k)v}$, and complementary minors give $m_a(D^{-1}) = m_{-a}(D)$ for invertible D. For $Z = \log w$, let U, V be its strict lower and upper equal-sign parts, respectively, and set $\widehat{w} = e^{-U}e^{Z}e^{-V}$. Evaluating the same integral at $p = e^U$, $r_0 = e^V$ uses $r_0w^{-1}p = \widehat{w}^{-1}$. Both triangular factors have unit diagonal, so the numerator is $|m_\mu(k)|^2$. For the moved matrices $\|w-1\|_{\mathrm{HS}} \le 4\eta_0$. The logarithm series and orthogonal truncation of matrix entries give $\max(\|U\|, \|V\|) \le \|\log w\|_{\mathrm{HS}} \le 8\eta_0$. Thus all these choices are in the indicated neighborhood. We obtain

$$c(w) = \frac{1}{J_\mu}\int_K \frac{|m_\mu(k)|^2}{m_{-a}(k^*\widehat{w}k)}\,dk.$$

Uniform quadratic remainder. Set $d_0 = \|Z\|_{\mathrm{HS}} \le 2\delta_0$, where $\delta_0 = 4\eta_0$, so $d_0 \le 1/20$. Let $A_1 = -U, A_2 = Z, A_3 = -V$. For an orthogonal projection P' let D' be the compression of $\prod_i e^{A_i}$ to its image and $D'_0 = \prod_i e^{P'A_iP'}$ there. Put $z_0 = \max_i \|[P', A_i]\|_{\mathrm{HS}} \le d_0$. In expanding $D' - D'_0$ into words in the diagonal and off-diagonal blocks relative to P', surviving words have at least two off-diagonal factors. The degree two trace is

$$S_{P'} = \sum_{i<j} \mathrm{tr}(P'A_i(1-P')A_j) + \tfrac{1}{2}\sum_i \mathrm{tr}(P'A_i(1-P')A_i).$$

The corresponding degree two matrix has trace norm at most $(9/2)z_0^2$. Bound two off-diagonal factors of each longer word in Hilbert–Schmidt norm and the rest in operator norm. The multinomial sum and the choices of blocks give the trace-norm bound for the tail

$$z_0^2\sum_{m\ge 3} 6^m d_0^{m-2}/m! \le 36e^{6d_0}d_0z_0^2 \le 49d_0z_0^2.$$

It follows that $\|D' - D'_0\|_1 \le 7z_0^2$, where $\|\cdot\|_1$ is the trace norm. Also $\|(D'_0)^{-1}\| \le e^{3d_0}$ and $\|(D'_0)^{-1} - 1\| \le 4d_0$. For $N = (D'_0)^{-1}(D' - D'_0)$, the log series gives

$$|\,\mathrm{tr}(\log(1+N) - N)| \le \frac{81d_0^2z_0^2}{1-9d_0^2} \le 5d_0z_0^2.$$

The linear trace of N differs from $\mathrm{tr}(D' - D'_0)$ by at most $28d_0z_0^2$. Hence

$$\det D' = \exp\{\mathrm{tr}(P'\textstyle\sum_i A_i) + S_{P'} + R_{P'}\}, \qquad |R_{P'}| \le 100d_0z_0^2.$$

12.9.2 The compact block estimate

In this subsection, we use the triangular coordinates of Section 12.9.1 to bound the compact phase.

Apply the preceding determinant expansion to $P' = kPk^*$, where P is a threshold projection for $-a$. Put $s_0 = \|[P, Z]\|_{\mathrm{HS}}$ and $h_0 = \|[P, k]\|_{\mathrm{HS}} = \|P' - P\|_{\mathrm{HS}}$. Then $z_0 \le s_0 + 2d_0h_0$. Both projections are block diagonal for the signs. The equal-sign off-diagonals of $\sum_i A_i$ vanish, and $\sum_i |(P' - P)_{ii}| = h_0^2$. Thus its linear trace differs from $\mathrm{tr}(PZ)$ by at most $d_0h_0^2$. Each factor $P'A_i(1-P')$ differs from its version at P by at most $2d_0h_0$ in Hilbert–Schmidt norm. Consequently $|S_{P'} - S_P| \le 18d_0h_0(s_0 + 2d_0h_0)$. For i inside the threshold and j outside, the parameter at i is smaller. For equal signs, the three cross terms and the A_2 square contribute

$(1-1+1-1/2)|Z_{ij}|^2$ to S_P. For opposite signs only the square contributes, giving $|Z_{ij}|^2/2$ since $Z^*J+JZ=0$. Thus $S_P=s_0^2/4$. The total error in the exponent for this threshold is at most $10^5 d_0(s_0^2+h_0^2)$.

Write $Q_b(F)=\sum_{i,j}|b_i-b_j||F_{ij}|^2$ for a weight b, and $q=Q_{-a}(Z)/4$. Summing squared commutators counts each pair once per threshold that separates it. Also $\det \widehat{w}=e^{\operatorname{tr} Z}$. Multiplying the threshold formulas gives

$$m_{-a}(k^*\widehat{w}k)=\exp\{q-\operatorname{tr}(\operatorname{diag}(\mu)Z)-R(k)\},\qquad |R(k)|\le 10^6\delta_0(q+Q_\mu(k)+d_1^2).$$

For the error use $|a_i-\mu_i|\le d_1$, $Q_{-a}(k)\le Q_\mu(k)+2d_1^2$, and $|\operatorname{tr}(\operatorname{diag}(a-\mu)Z)|\le d_1^{3/2}d_0$. The remainder is a sum of the local log series above. For unitary k, the squared modulus of each compressed determinant at P equals $\det(1-Pk^*(1-P)kP)$ on the compressed space. Since $1-t\le e^{-t}$, the product gives $|m_\mu(k)|^2\le e^{-Q_\mu(k)/2}$, using equality of the two off-diagonal squared masses of a unitary matrix. The compact Weyl dimension formula gives $J_\mu^{-1}=\dim V_\mu\le(1+6\beta)^{d_1^2}\le e^{5000T_0}$, since the weights have modulus at most 3β and $d_1^2\log\beta\le T_0$.

Put $W_0=q+Q_\mu(k)+d_1^2$. If $W_0\le 50000T_0$, then $\delta_0T_0\le 10^{-12}$ gives $|R|\le 1/20$ and $|e^R-1|\le 1/10$. Otherwise $W_0>50000T_0\ge 50000$ and the bound on R gives $|R|\le 10^{-6}W_0$. Summing the absolute values of the positive-degree terms in the exponential series gives $|e^R-1|\le e^{|R|}-1\le e^{W_0/4}$, so

$$|m_\mu(k)|^2e^{-q}|e^R-1|\le e^{-q-Q_\mu(k)/2+W_0/4}\le e^{-5100T_0}.$$

Indeed the exponent is at most $-W_0/4+d_1^2/2$. After integration and division by J_μ,

$$\left|c(w)e^{-\operatorname{tr}(\operatorname{diag}(\mu)\log w)}-e^{-q}\right|\le e^{-q}/10+e^{-100T_0}.$$

Apply the estimate to g and h. The sign-Lie identity makes both logarithmic traces imaginary. Conjugating the compact coefficient reverses its phase and preserves its real Gaussian. The phase in θ is therefore

$$\exp\bigl(\operatorname{tr}(D_\mu Z)-\operatorname{tr}(D_H Z')-\phi\bigr)=1.$$

The product of the two phase-corrected coefficients differs from the product a_1a_2 of the two numbers e^{-q} in $[0,1]$ by at most $0.21a_1a_2+4e^{-100T_0}$. Thus $|\theta|-2\Re\theta\le 12e^{-100T_0}$, as required at the original graph element.

12.10 The real Bessel functions for the construction

In this subsection, we apply the real estimates to the parameters of Section 4. This constructs the discrete-series vectors and verifies the normalization and truncation bounds required for Section 9.1.

Return to the datum of Section 4. The rank of the ambient form is r, with $k=r+1-m$ and $\varrho=m/2-1$. Thus $n=\varrho+1=m/2$ and $r=k+2\varrho+1$. Put $a=xb$, $\Delta=2r^8$, and $B_\tau=\kappa_\tau+\Delta/2$ at an infinite place τ. The character of lift number j has weight $B_\tau+j\Delta$. The initial parameter, in its numbered coordinate order, has $k-1$ positive entries followed by m alternating signs starting with $+$.

In this order the parameter entries are

$$\lambda_i=k/2-1-i\quad(0\le i<k-1),\qquad \lambda_{k-1+t}=-B_\tau-(t+1/2)\Delta\quad(0\le t<m).$$

Here the tail entry is the half gap minus the weight of lift $t+1$. The signs have signature $(k+\varrho, \varrho+1)$. To realize them in the original stage space one can take the unit eigenvectors $(e_j \pm e_j^*)/\sqrt{2}$ of the pair blocks, along with the positive basis and z, and permute the two sign sets into the parameter order. This gives a Euclidean unitary basis with prescribed signs for the original Gram matrix J. The local construction applies using any such chosen basis throughout.

The least weight satisfies $\min_\tau B_\tau \geq \Delta^2$ for large x, since $\kappa_\tau \geq xb/x_1$, $\log b - \log x_1 \geq x^\epsilon - x^{\epsilon/6}$, and $\Delta = 2r^8$ grows polynomially in x. The last initial entry is $1 - k/2$, and the first tail entry is $-B_\tau - \Delta/2$, so the entire parameter decreases strictly. Within the tail the smallest opposite-sign gap is Δ. For a mixed pair the initial coordinate is positive in signature, so the first possible opposite tail position is $t = 1$. Its distance from the last initial entry is $1 - k/2 + B_\tau + 3\Delta/2 \geq \Delta$. Thus the smallest opposite-sign gap equals Δ at every infinite place. The doubled parameter has the right parity because k is even and r is odd. Choose the discrete series with this Harish–Chandra parameter and a unit highest vector of its lowest compact type [5, Theorems 8.1 and 8.5, pp. 41–42].

On the derived group use the parameter with its mean removed. Its lowest compact type has the central restriction given by the original integral weight. The character of the scalar circle of weight $\sum_i \lambda_i$ agrees on the common scalar subgroup, since the root corrections have sum zero, so the representation descends from the product to the unitary group. The signed root count in this order gives

$$\mu_i = \lambda_i - \tfrac{1}{2}\left(\sum_{j>i} \varepsilon_i\varepsilon_j - \sum_{j<i} \varepsilon_i\varepsilon_j\right).$$

On the initial block the tail terms cancel in pairs and the count is $k-2-2i$, so $\mu_i = 0$. At tail position t, the initial block contributes $-(k-1)(-1)^t$, and the tail contributes $-(-1)^t$. Hence

$$\mu_{k-1+t} = -B_\tau - (t+1/2)\Delta + \tfrac{k}{2}(-1)^t.$$

Consecutive differences are $\Delta + k(-1)^t > 0$. Since $\Delta \geq k$ and $m\Delta + k/2 \leq 2B_\tau$, these entries lie in $[-3B_\tau, -B_\tau]$.

On X_0 the column counts at heights $0 < h < k$ are zero except at $h = 1$, where the count a_τ is a if $\tau \in \Phi_a$ and zero otherwise. The determinant weight on X_0 is zero. Thus the compact type is the symmetric power $\mathrm{Sym}^{a_\tau}(\mathbb{C}^k)$ of the defining representation of $U(k)$. Its highest weight is $(a_\tau, 0, \ldots, 0)$, and its doubled parameter entries are $2a_\tau \mathbf{1}_{j=0} + k - 1 - 2j$ for $0 \leq j < k$.

Match the other $k-1$ coordinates with the zero-weight positive block in the ambient parameter. The remaining compact coordinate is the highest-weight coordinate, also when $a_\tau = 0$. Use exactly the Schur matrices $S_\tau(h)$ on the value space. If $Q_\tau = B_0^* B_0$ is their invariant Weyl Gram matrix, set $\pi_H(h) = B_0 S_\tau(h) B_0^{-1}$. Since S_τ is defined over the reals and $h^{-1} = \bar{h}^{\mathsf{T}}$, invariance gives

$$Q_\tau S_\tau(h) = S_\tau(h^{\mathsf{T}})^{\mathsf{T}} Q_\tau, \qquad B_0^* \pi_H(h) = S_\tau(h^{\mathsf{T}})^{\mathsf{T}} B_0^*.$$

Let j_* be the chosen highest-weight coordinate, and let ξ be the highest monomial $e_{j_*}^{a_\tau}$ in the symmetric power ($\xi = 1$ if $a_\tau = 0$). The unit highest vector and its coefficient in the Bessel value space are

$$v_0 = \frac{B_0 \xi}{\|B_0 \xi\|}, \qquad c_H(h) = \frac{\xi^* Q_\tau S_\tau(h) \xi}{\xi^* Q_\tau \xi} = h_{j_* j_*}^{a_\tau}.$$

Indeed the diagonal torus acts unitarily for Q_τ, so distinct weights are orthogonal. The coefficient of $e_{j_*}^{a_\tau}$ in $(he_{j_*})^{a_\tau}$ gives the last equality. Thus the stationary integrand uses $\overline{c_H(h)}$ for this same Schur representation.

We check the required sizes uniformly in τ. For large x, $x^5 \le r < 3x^5$, $\sqrt{x}/2 \le n < \sqrt{x}$, $k \ge x^5/2$, and $4000n \le f \le x^{12}$, the last pair from $f = 3L_0 m^{22}\varphi(q_0)$. Thus $k \ge n^7$, $k - 3n - 6 \ge 10^{10}n^4$, $\varrho \ge 10^8$, and $8nr \le b$. Since $x_1 \ge x$ and $r^8 \le b$ eventually, $\kappa_\tau \in \{b, xb/x_1\}$ gives $B_\tau = \kappa_\tau + r^8 \le 2b$. Thus $3B_\tau + a_\tau \le (6+x)b \le b^2$ for all large x. The two doubled parameters have magnitude at most $L_* = 3a$, since $b + (m+1)\Delta \le a$ eventually and $2a + k \le 3a$. Put $\beta = b^3$ and $A_* = r^7$. Then $r + k + L_* \le \beta$, $\beta \ge 10^{12}$, and

$$500r^3 + A_* \le \Delta, \qquad 3r^3 f \le A_*,$$
$$10^{30}\big((r+k)^2 \log \beta\big)^3 \le 10^{30} 108^3 x^{30+3/100} \le r^7.$$

The second line uses $\log b \le x^{1/100}$ from the chosen scale bounds. The compact parameter has only plus signs. These size comparisons need one threshold in x, since the weights have the same upper and lower bounds at all places.

Apply Proposition 12.1 with $(B_{\mathrm{wt}}, a, A_{\mathrm{gap}}) = (B_\tau, a_\tau, r^7)$ at each place τ, including the places where $a_\tau = 0$. Write $x_\tau^\star$ for the stationary point in the original stage group, using the prescribed sign basis at τ. When $a_\tau = 0$, all interpolation nodes satisfy $\mu_i \le -B_\tau < 0$, hence $P(0) > 0$, and $c_H(h) = 1$. Let $\deg_\tau$ denote the formal degree in dg. With the induced quotient Haar, Section 12.4 and the proposition give at this point

$$\deg_\tau \|A_\tau\|_2^2 = \langle v_{0,\tau}, A_\tau(x_\tau^\star)\rangle \ge b^{-60(r+k)^2} V_{x_\tau^\star} > 0.$$

For $D_\tau = \dim \pi_H$, take

$$W = \frac{\sqrt{D_\tau}}{\|A\|_2} A, \qquad F = \chi W, \qquad R_b = b^{6n+2}.$$

For the point $g_{0,\infty}$ of Section 8.3 one has $T_\tau(g_{0,\tau}) = x_\tau^\star$ for every τ. Consequently

$$\max_\tau \ell_\tau(T_\tau(g_{0,\tau})) \le b^{5n}.$$

Tensoring the absolutely convergent graph integrals at these points gives

$$\prod_\tau \deg_\tau \|A_\tau\|_2^2 \ge b^{-60f(r+k)^2} \prod_\tau V_{x_\tau^\star} > 0.$$

In particular,

$$\langle w, W(g_{0,\infty})\rangle = \prod_\tau \frac{\sqrt{D_\tau}}{\|A_\tau\|_2} \langle v_{0,\tau}, A_\tau(x_\tau^\star)\rangle > 0.$$

Here $w = \bigotimes_\tau v_{0,\tau}$ and $W(g_{0,\infty})$ is in the product chart of the measured period. These choices precede the orbital tests.

Section 12.4 applies this period bound to the graph average and gives the decay majorant, the cut support, and $\|F - W\|_2^2 \le b^{-2r^3 f} D_\tau$. Conjugation by the compact group $H = U(k)$ preserves du, so its left action on $U\backslash G$ preserves the quotient $d\bar{g}_\tau$ of the orbital Haar dg_τ by du. Since the length used in χ is left UH-invariant, $W(hy) = \pi_H(h)W(y)$ and $F(hy) = \pi_H(h)F(y)$. Both Gram matrices therefore commute with π_H. In the Weyl isometry above this representation is $\mathrm{Sym}^{a_\tau}(\mathbb{C}^k)$. Its monomial weight spaces are one-dimensional, and the operators $X_i\partial_{X_j}$ of the complexified Lie algebra connect all monomials of degree a_τ. This proves irreducibility, also for $a_\tau = 0$. Schur's lemma and the traces now give

$$\int WW^*\, d\bar{g}_\tau = 1_{I_\tau}, \qquad \int FF^*\, d\bar{g}_\tau = s_\tau 1_{I_\tau}, \qquad (1 - b^{-r^3 f})^2 \le s_\tau = \frac{\|F\|_2^2}{D_\tau} \le 1.$$

The lower bound uses $\|F\|_2 \geq \|W\|_2 - \|F - W\|_2$, and the upper bound follows from $0 \leq \chi \leq 1$.

Section 12.3 gives the right reproduction, the correlation norm $D_\tau^2/\deg_\tau$, and the norm-ball bound. Sections 12.5 and 12.6 give the other degree and Haar bounds of Section 12.1. The shell estimates and Lemma 11.1 apply to F on this graph. The factor $(\pi/|\xi|)^{k-5}$ is the Plancherel factor from the $k-5$ free complex coordinates of Section 11.1. Section 12.1 gives the bottom bound $(1+b^{-10})/\deg_\tau$. Lemma 11.1 gives the character-weight and unequal-entry bounds, and Section 12.2 defines the real forms J_j and proves integration in a common entry. The derivative estimates for the full vector W in Section 12.11 enter the comparison of nonzero-discriminant orbits in Section 15.4. For a terminal element σ fixing the hyperbolic pairs, the top form at heads with the same last entry $h(z, \sigma z)$ is the inner product of the two actual normalized orbital tensors of F at σ, by Section 11.3. This uses the normalizing dimension D_τ for both tensors.

Lemma 12.2 (Comparison of the formal degrees)**.** *Put* $B = \log b$, $S_\kappa = \sum_\tau \log \kappa_\tau$, *and* $W_* = \log \mathcal{B} + f \sum_{l=k-1}^{r-1} \log(l!)$. *Then*

$$|W_* - kmS_\kappa| \leq m^2 fB + fmr/x.$$

Proof. The formal-degree product $\mathcal{B}$ used in Section 9.1 is

$$\log \mathcal{B} = \sum_\tau \sum_{l=k-1}^{r-1} \left(\sum_{i<l} \log(\lambda_i - \lambda_l) - \log(l!) \right).$$

In Section 12.5 the factors at $l < k-1$ cancel because $\lambda_i - \lambda_l = l - i$. At $l \geq k-1$, each initial entry contributes a difference at least κ_τ, and every tail-to-tail difference is at least one. All differences in this range are at most $\kappa_\tau + m\Delta + k$. Uniformly in τ,

$$\frac{x(m\Delta + k)}{\kappa_\tau} \leq \frac{(m\Delta + k)x_1}{b} = o(1),$$

since $m\Delta + k = O(x^{81/2})$ and $\log b - \log x_1 \geq x^\epsilon - x^{\epsilon/6}$. Using $\log(1+y) \leq y$ for $y \geq 0$ gives

$$\begin{aligned} m(k-1) \sum_\tau \log \kappa_\tau &\leq \log \mathcal{B} + f \sum_{l=k-1}^{r-1} \log(l!) \\ &\leq mr \sum_\tau (\log \kappa_\tau + 1/x). \end{aligned}$$

Subtract kmS_κ and use $r = k + m - 1$ and $0 \leq S_\kappa \leq fB$. This gives

$$-mfB \leq -mS_\kappa \leq W_* - kmS_\kappa \leq m(m-1)S_\kappa + fmr/x \leq m^2 fB + fmr/x. \qquad \square$$

12.11 The differentiated Schur bound

In this subsection, we transfer derivatives of the full Bessel vector to its discrete-series matrix coefficient by the reproducing identity of Section 12.3. This bounds the derivatives needed in Section 15.4.

Use the full vector W just constructed. Let π be the ambient discrete series and v its chosen unit highest vector, with $c(g) = \langle v, \pi(g)v \rangle$. Right translation R on the quotient satisfies $\langle W, R(g)W \rangle = D_\tau c(g)$, by Section 12.3. In particular the map $\pi(g)v \mapsto R(g)W$ extends linearly

to a bounded intertwiner T with $\|Tz\|_2 = \sqrt{D_\tau}\|z\|$: the two Gram matrices on each finite list of translates agree up to D_τ, and v is cyclic by irreducibility.

The right derivatives of W are bounded continuous Bessel functions. Indeed take a smooth compactly supported approximate identity and average it on the left and right by the compact matrix coefficient projector onto $\mathbb{C}v$. The latter projector comes from Schur orthogonality and multiplicity one of the lowest compact type. Choosing a bump sufficiently near the identity yields a smooth compactly supported kernel ξ with $\pi(\xi)v = \alpha v$, $\alpha \neq 0$, since the bump operator tends strongly to the identity.

Thus $R(\xi)W = \alpha W$, first in L^2 by the intertwiner and then everywhere by continuity. The change $u = \exp(tX)g$ in $\int \xi(g) W(y\exp(tX)g)\,dg$ puts each derivative on the smooth kernel by left translation. Compact support and the boundedness of W justify all differentiations, and left Bessel covariance is preserved. Complex linear extension supplies the derivatives in the matrix units. For a word ω of length j denote the resulting function by W_ω and the corresponding derivative of v by v_ω; the intertwiner gives $W_\omega = Tv_\omega$ in L^2.

For the derivative norm, let V be the lowest compact type. The span V_j of derivatives of order at most j of vectors of V is the image of $\bigoplus_{i=0}^{j} \mathfrak{g}_{\mathbb{C}}^{\otimes i} \otimes V$ as a compact module. Its highest compact weights have coordinates of modulus at most $L_* + r + j$. Indeed weights of V lie between the smallest and largest highest-weight coordinate in each unitary block, by the highest-weight order and Weyl permutation symmetry.

An adjoint weight changes any coordinate by at most one. The compact Casimir on V_j is therefore at most $r(L_* + 2r + j + 1)^2$ and the ambient Casimir has absolute value at most $r(L_* + 2r + j + 1)^2$ by the same eigenvalue formulas as in Section 12.3. For a real Hilbert–Schmidt orthonormal Lie basis X_i adapted to the compact and noncompact summands, the energy identity gives

$$\sum_i \|d\pi(X_i)u\|^2 \le 3r(L_* + 2r + j + 1)^2 \|u\|^2, \qquad u \in V_j.$$

For $X_i, X_l \in \mathfrak{u}(J)$, the relation $X_i^* = -JX_iJ$ and cyclicity give $\operatorname{tr}(X_i^* X_l) = \operatorname{tr}(X_l^* X_i)$. Thus (X_i) is also a complex Hilbert–Schmidt orthonormal basis. A matrix unit E_{ab} of the original basis becomes $M_{ab} = p_v^{-1} E_{ab} p_v = \sum_i c_i X_i$ in the signed basis of Section 8.3. Since p_v is Euclidean unitary, $\sum_i |c_i|^2 = 1$, and hence

$$\|d\pi(M_{ab})u\|^2 \le \sum_i \|d\pi(X_i)u\|^2 \le 3r(L_* + 2r + j + 1)^2 \|u\|^2.$$

As $\sqrt{3r}(L_* + 2r + 2n^2 + 1) \le b^5$ for large x, iteration gives $\|v_\omega\| \le b^{5j}$ for $j \le 2n^2$.

For $\phi \in L^1(dg) \cap L^2(dg)$, Schur orthogonality and Cauchy–Schwarz against an arbitrary unit vector give $\|\pi(\phi)v_\omega\|^2 \le \|\phi\|_2^2 \|v_\omega\|^2 / \deg_\tau$. Applying T gives the desired inequality in the quotient normalization,

$$\int_{U\backslash G} \left\| \int_G \phi(g) W_\omega(yg)\,dg \right\|^2 d\bar{g}(y) \le b^{10j} \frac{D_\tau}{\deg_\tau} \int_G |\phi(g)|^2\,dg, \qquad j \le 2n^2.$$

For integrable ϕ outside L^2 the right side is infinite. The estimate concerns the same W, chosen coordinates and Haar measure as the real sieve forms.

13 Adelic orbital comparison and definite masses

In this section, we combine the local estimates on the open rational cell and average at a fixed final vector. The definite lattice masses give the normalization for the moment comparison of Section 9.1.

13.1 The open rational cell and its finite restrictions

In this subsection, we identify the rational terms in the open range of the Petersson expansion. The fixed finite levels from Section 8 allow the local integrals to be compared on one set of places in Section 13.4.

Use the cut Bessel vector, the class representatives $c^\circ \in H(\mathbb{A}_{F_0,f})$ for $H = U(X_0)$, and the two class cutoffs of Section 8. In the rational basis $f = (e_0, \dots, e_{\varrho-1}, z)$ we have $h(z,z) = -N$. The rational-coordinate matrix $h_c^{\mathrm{rat}} = \iota_{\mathrm{rat}}(c^\circ)$ fixes all the hyperbolic pairs and z. In products with the glued matrix $h_c = B_{\pi_N}^{-1} h_c^{\mathrm{rat}} B_{\pi_N}$, write $g_L = B_{\pi_N}^{-1} g_f B_{\pi_N}$ for a rational isometry g, as in Section 8.1. For an original term γ above (c, s, c'), the matrix $h_c^{-1}\gamma_L h_{c'}$ is integral with reduction s. Thus the leading minors Δ_a of $(h(f_i, \gamma f_j))_{i,j}$ are integral. Their reductions are the minors of s, since the Gram entries are unchanged by the class translates. We take the order-zero minor to be 1. An open residue s has nonzero proper minors at one chosen place above p_0^{lev}, so $\Delta_a \neq 0$ for $1 \le a \le \varrho$.

Lemma 13.1 (Factorization on the open rational cell)**.** *If $\Delta_a \neq 0$ for $1 \le a \le \varrho$, with $\Delta_0 = 1$, the rational term γ has a unique factorization*

$$\gamma = u^{-1} w_p \sigma u', \qquad u, u' \in U(F_0), \quad \sigma \in G_\varrho(F_0), \qquad p_a = \Delta_{a+1}/\Delta_a \quad (a < \varrho). \tag{21}$$

Here $w_p e_a = p_a e_a^*$ and $w_p e_a^* = \bar{p}_a^{-1} e_a$.

Proof. At stage a, use the residual isometry γ_a fixing the earlier pairs. The pivot $c_a = h(e_a, \gamma_a e_a)$ is nonzero by the leading-minor elimination. More explicitly, clearing one pair makes the upper-left basis-product block (c_a) and its complementary block the next basis-product matrix. Since lower and upper unitriangular changes preserve each leading minor, the successive residual pivots are the stated ratios. Let $m = e_a^* + \omega + \mu e_a$ be isotropic. The matrix N_m sending e_a^* to m, fixing e_a, and sending a remaining vector v to $v - h(\omega, v) e_a$ is unitary, because $\mu + \bar{\mu} + h(\omega, \omega) = 0$.

Use $A = N_{c_a^{-1}\gamma_a e_a}$ and put $L = (A w_a(c_a))^{-1}\gamma_a$, which fixes e_a. Then $B = N_{L^{-1} e_a^*}$ is defined and LB fixes the pair. Iterate $\gamma_a = A w_a(c_a)(LB)B^{-1}$, using commutation of the earlier swaps with the later stage group. Each step is integral when its pivot is a unit and the residual matrix is integral: the column of N_m is integral, as is its inverse by the integral inverse Gram. This uses the column relation also above two. Uniqueness follows by writing two decompositions as $w_p \sigma v' = v w_q \tau$. Compare the e_a columns.

Their dual components have indices respectively at most and at least a, since v' and the lower dual block of v are unitriangular. The a-th coefficient gives $p_a = q_a$, and comparison of all components gives $v e_a^* = e_a^*$. In the order consisting of the isotropic block, the middle space and the dual block, the Bessel unipotent has the form

$$v = \begin{pmatrix} A & C & B \\ 0 & I & D \\ 0 & 0 & A^{-*} \end{pmatrix}.$$

Fixing the dual columns gives $B = D = 0$ and $A = I$. Pairing the middle columns with the fixed dual columns then gives $C = 0$. Hence $v = 1$, so $v' \in U \cap G_\varrho = \{1\}$ and $\sigma = \tau$. □

The link is the graph $x \mapsto \sigma^{-1}x$ in the middle quotient. Indeed a vector orthogonal to both isotropic bases of $w_p\sigma$ has zero hyperbolic coordinates, so the two quotient coordinates are x and $\sigma^{-1}x$. The left and right unipotents change the corresponding lifts by the original isotropic span, which preserves the quotient graph. Thus the link determines σ, independently of p. Put

$$t = h(z, \sigma z), \qquad v_0 = \mathrm{pr}_{X_0}(\sigma z), \qquad \sigma z = v_0 - (t/N)z.$$

If the discriminant is nonzero then $t\bar{t} - N^2 \neq 0$: the two last minors are $\Delta_\varrho = \prod p_a$ and $\Delta_{\varrho+1} = t\Delta_\varrho$, and the discriminant is $\Delta_\varrho\bar{\Delta}_\varrho(t\bar{t} - N^2)$. Moreover $Nh(v_0, v_0) = t\bar{t} - N^2$. By definiteness $|\Phi_v(t)| > N$ at every infinite place v. In particular the full minor is nonzero on this set.

All nonzero cut terms have bounded archimedean minors, with a bound depending on the chosen Bessel data. Indeed nonzero cut support at $y, \gamma^{-1}y$ implies $\gamma \in U_\infty \mathcal{K}\mathcal{K}^{-1}U_\infty$. Leading minors are unchanged by the two unipotents, and the original rational basis has fixed scales in this compact box. Thus each Δ_a lies in a finite set of bounded algebraic integers. Take $\mathcal{E}$ to consist of the quotients of the nonzero elements of this set with 1 included.

All p_a, t belong to $\mathcal{E}$. Choose the finite conjugation-stable set T containing the places above M, N, those needed for the lattice basis and its inverse, and the nonunit places of every element of $\mathcal{E}$. We may also include all places below the nonintegral entries of the finite list of class representatives and their inverses. This single T works for all terms of the chosen data and may depend on x. Its finite Haar factors cancel on dividing $\mathcal{I}_i^T$ by $\mathcal{R}_Z$ in Section 10.3. For $s > 1$, the Euler product for ideals supported at T and the complementary product are each bounded by $\zeta_E(s)$, independently of T. The global additive covolume factor in Fourier inversion is v_E.

In the lattice coordinates put $L_{\mathrm{lat}} = B_{\pi_N}$ and $g_L = L_{\mathrm{lat}}^{-1} g_f L_{\mathrm{lat}}$. The three factors in the original class lattice then give

$$h_c^{-1}\gamma_L h_{c'} = (h_c^{-1}u_L h_c)^{-1}\, w_p (h_c^{-1}\sigma_L h_{c'})(h_{c'}^{-1}u'_L h_{c'}).$$

Here $L_{\mathrm{lat}}, h_c, h_{c'}$ fix the hyperbolic pairs and commute with w_p. Off T, the left side is integral and every proper pivot is a unit in each local field factor. The integral elimination and uniqueness above therefore put both conjugated unipotents and the displayed middle factor in the integral group there.

For a cut block above (c, s, c') and link L, let $B^\flat_{c,c',s,L}$ be the sum of its terms on the nonzero-discriminant set. For each nonzero block the link determines its σ. The preceding factorization gives its expansion over $p \in \mathcal{E}^\varrho$ of the orbit sums $\mathrm{CosAmp}^\flat_{c,s,c'}(w_p\sigma)$. These are the sums of the cut amplitudes over the unipotent factors in (21) and the original class and residue conditions. The block parameters give distinct tuples (c, c', s, σ). The absolute unfolding for these orbit sums is given below.

13.2 Comparison at a fixed final vector

In this subsection, we express the terminal pairing at a fixed last vector as a finite class sum. The calculation matches the source level of Section 6 with the measures of Section 10.10.

Take Z for the prescribed fixer from Section 10.10, and use its $W_Z, \mathcal{R}_Z$ and $V_p(l)$ from Section 10.4. Let

$$A_{c,c',\sigma} = h_c^{-1}\sigma_L h_{c'}, \qquad l = (A_{c,c',\sigma})_T^{-1}, \qquad X = (w_p A_{c,c',\sigma})_T.$$

Write $\widetilde{s} \in Z$ for the finite projection of an integral lift of $s \in \Gamma_{\mathcal{A}}$. Here and below the products with class representatives are read in the original lattice.

The rational unipotents whose class translates are integral outside T, modulo the class unipotent subgroup U_c of Section 8, project bijectively to the coset set $U(A_T)/(U(A_T) \cap J)$, with representatives identified by right multiplication by $U(A_T) \cap J$. For surjectivity, prescribe the desired coset of $U(A_T) \cap J$ at T and the integral unipotent subgroup at each place outside T. Transport by $L_{\mathrm{lat}} h_c$ carries their product to a nonempty open set in the rational coordinates of $U(\mathbb{A}_{F_0,f})$. It meets $U(F_0)$ by the additive approximation in exponential coordinates of Section 8.2. Indeed the exponential and logarithm are mutually inverse polynomials over F_0 and preserve integral coordinates at almost all places. Thus the rational element has exactly the required class translate at T and is integral after translation outside it. For injectivity, two representatives with the same T-coset already have quotient in the principal level outside T and hence everywhere, since M is a unit off T. Their rational lifts therefore differ on the right by an element of U_c, and every such change preserves the local coset.

For a contributing (c, c', σ), the middle factor $A_{c,c',\sigma}$ is integral off T by the preceding elimination of one representative. For each head tuple $p \in \mathcal{E}^{\varrho}$, w_p and its inverse are integral there. Integrality off T of the relative term is therefore equivalent to integrality of both conjugated unipotents, by multiplication in one direction and integral elimination in the other. Thus the remaining residue condition on finite representatives a, b from these two unipotent quotients is $a^{-1} X b \in \widetilde{s} J$. Only finitely many pairs qualify: if $a^{-1} X b$ lies in this compact coset, the same fixed-pivot block elimination that proves compact support in Section 10.4 bounds a, b. Quotienting bounded sets by the open subgroup gives a finite set.

Within each such pair of cosets, sum the two cutoffs over the two rational class unipotent subgroups. If the central term is θ, the terms in those two sums have the form $v^{-1} \theta v'$. Change the real integration variable by v; the two corresponding cutoff sums are one by the partitions in Section 8. The class-unipotent character is trivial on these subgroups, since their finite translates are in the principal level.

This also follows from the adelic product of the additive characters at rational points. The remaining real tensor is $I^{\infty}(w_p \sigma) = \int F(y) \otimes F((w_p \sigma)^{-1} y)^{*} \, dy$, with its real embeddings understood. Its integral is absolute: with fixed nonzero pivots the two cut-support conditions put the unipotents in bounded sets by the same block calculation on the compact ratio box. Tonelli applied to

$$\int \|F(y)\| \, \|F((w_p \sigma)^{-1} y)\| \, dy < \infty$$

and the two nonnegative cutoff partitions justifies summing the translates before integrating the tensor.

13.2.1 Cancellation of the class phases

In this subsection, we combine the finite and real actions before applying the common-entry sum of Section 13.4.

For a rational unipotent u put $z_u = -\mathfrak{s}(u)/(M \delta_E)$. Our adelic character has archimedean component $z \mapsto \exp(-2\pi i \sum_v 2 \Re z_v)$, with one embedding from each complex pair. Since the real root entry is $\alpha_{\Psi,v} = -2\pi i \Phi_v(z_u)$, we have

$$\chi_{\infty}(u) = \prod_v \exp(\alpha_{\Psi,v} - \bar{\alpha}_{\Psi,v}) = \exp(-2\pi i \operatorname{Tr}_{E/\mathbb{Q}} z_u) = \psi_{E,f}(z_u)^{-1}.$$

This is the covariance character of F. The graph average uses its inverse. Conjugation by h_c preserves the root character because the definite Levi fixes the vectors e_i, z, e_i^* that give its simple-root entries. At places off T the root coordinate divided by M is integral and the finite character has conductor the integer ring. If u_1, u_2 lift a, b, respectively, the substitution $y = u_1^{-1}y'$ in the tensor at $u_1^{-1}w_p\sigma u_2$ therefore contributes $\chi_\infty(u_1)^{-1}\chi_\infty(u_2) = \chi(a)\overline{\chi(b)}$.

Let $\alpha_1, \alpha_2 \in Z/J$ with $\alpha_1\alpha_2^{-1} \in \widetilde{s}J$, and put $a_p^T = \mathcal{D}_\varrho^T(p, 1)$. The corresponding coefficient of $V_p(l)$ is $a_p^T \int \chi(y\alpha_1^{-1})\overline{\chi(X^{-1}y\alpha_2^{-1})}\, d\mu(y)$. Replace y by $y\alpha_2$. The integrand becomes $\chi(y\widetilde{s}^{-1})\overline{\chi(X^{-1}y)}$, using normality of J in K_T. A nonzero coset yJ has $yJ = a\widetilde{s}J = XbJ$ for a unique pair from the two unipotent quotients, and conversely. It contributes $\mu(J)$ times the same phase as in the real unfolding. Thus, with the real and finite Haar already fixed,

$$V_p(l)_{\alpha_1,\alpha_2}\, I^\infty(w_p\sigma) = a_p^T \mu(J)\, \mathrm{CosAmp}^\flat_{c,s,c'}(w_p\sigma). \tag{22}$$

Write I_v for the dimension index set of the local cut Bessel function at v, and c_G for the real open-cell constant of Section 11.3. In the real coordinates the squared basis scale is

$$s_{v,a} = \kappa_v^{2(\varrho-a)}/N,$$

where κ_v is the real scale for the additive character of level M. In fact the finite unipotent character on rational elements reads the root sum divided by M against the different-normalized character. Write δ_E for the skew generator of the different. At v, put $\kappa_v = M\Phi_v(\delta_E)/(2\pi i)$.

For the finite character at $-\mathfrak{s}(u)/(M\delta_E)$ the opposite archimedean phase is $\exp(c - \bar{c})$ with $c = \Phi_v(\mathfrak{s}(u))/\kappa_v$, where $\mathfrak{s}$ is the simple-root sum. Scaling e_a by $\kappa_v^{\varrho-a}/\sqrt{N}$, its dual by the reciprocal and z by $1/\sqrt{N}$ gives this same consecutive root coefficient: conjugation into the scaled basis multiplies each simple-root entry by κ_v^{-1}. Conjugating the Weyl element gives proper head entries $x_{v,a} = s_{v,a}\Phi_v(p_a)$; the last Gram entry is $x_{v,\varrho} = \Phi_v(t)/N$. With $d_a = k - 1 + 2(\varrho - a)$, set

$$R_0 = \prod_v \left(\frac{\sqrt{c_G}}{|I_v|} \prod_{a=0}^{\varrho} s_{v,a}^{d_a} \right), \qquad G(t) = \prod_v (|\Phi_v(t)|^2 - N^2)^{k-1}.$$

Let a_p^∞ be the product of the real tensor scales in Section 11.3.

For $a < \varrho$, the product formula gives

$$\mathrm{md}(p_a)^{1/2} \prod_v |\Phi_v(p_a)| = 1.$$

Consequently the proper-head factors cancel and

$$a_p^\infty a_p^T = R_0\sqrt{G(t)}. \tag{23}$$

The final factor comes from $(|x_{v,\varrho}|^2 - 1)^{(k-1)/2} = N^{-(k-1)}(|\Phi_v(t)|^2 - N^2)^{(k-1)/2}$. There is no finite t-factor in a_p^T. The Gram matrix $\mathcal{G}_t(p, q)$ of the $a_p^\infty I^\infty(w_p\sigma)$ is the product of the top forms of Section 11.3. For all blocks with the same t this is one matrix, as the real top forms depend on their heads and t.

13.2.2 The common-entry pairing

In this subsection, we group the terms with the same final vector using the phase identity of Section 13.2.1.

Define

$$\mathcal{H}_t(l) = \sum_{\alpha_1,\alpha_2 \in Z/J} \left\| \sum_p V_p(l)_{\alpha_1,\alpha_2} a_p^{\infty} I^{\infty}(w_p \sigma) \right\|^2 .$$

With the Hermitian products linear in the second argument, its expansion is

$$\mathcal{H}_t(l) = \sum_{p,q} \mathcal{G}_t(p,q) \langle V_p(l), V_q(l) \rangle .$$

For fixed l, only $\mathcal{G}_t$ involves the real representative σ. The two compact factors in Section 11.3 act by the same unitary operators for every head, so the expansion depends on that representative only through t. Fix c, c', σ, and sum the comparison over the blocks with these parameters. For each α_1 and such residue s, there is exactly one α_2 with $\alpha_1 \alpha_2^{-1} \in \widetilde{s} J$, since $\widetilde{s} \in Z$. The pairs for distinct s are disjoint. Equations (22) and (23) and nonnegativity give

$$\mu(J)^2 [Z : J] R_0^2 G(t) \sum_s \| B^{\flat}_{c,c',s,L} \|^2 \le \mathcal{H}_t(l). \tag{24}$$

Fix c, v_0, t and write $B = B_{\pi_T}$ for the basis of the glued middle lattice at T. In rational coordinates the class representative is c° on X_0 and fixes z. Since $h_c = \iota_L(c^{\circ})$ in the lattice basis and $\sigma z = v_0 - (t/N) z$, the common last column is

$$l^{-1} z = B^{-1} \left(((c^{\circ})^{-1} v_0)_T - (t/N) z \right) =: e.$$

The two factors from the finite point correction η cancel in the class residue condition of Section 8.5. If $l_1 l_2^{-1} \in J$, the quotient $h_{c_1'}^{-1} (\sigma_1^{-1} \sigma_2)_L h_{c_2'}$ is in the ambient principal level: it is integral outside T by the preceding elimination and in J at T. The rational element $\sigma_1^{-1} \sigma_2$ fixes the pairs and z, so belongs to $H(F_0)$. The pullback of the ambient principal level to H is the level J_H of Section 6.

For this lattice calculation use the glued generator $x_{k_0}/\bar{\pi}_N + z/N$ of the middle lattice. Write $x_{\mathrm{gl}} = x_{k_0}/\bar{\pi}_N + z/N$ for this lattice vector. Since $x_{k_0} = \bar{\pi}_N x_{\mathrm{gl}} - z/\pi_N$, the intersection with X_0 is the integer lattice with distinguished coordinate in $\pi_N \widehat{\mathcal{O}}_E$. If an isometry h fixing z preserves the middle lattice, its columns away from k_0 are in this intersection, as is $(h x_{k_0} - x_{k_0})/\bar{\pi}_N$.

Applying the same fact to h^{-1} shows that h is integral with integral inverse on X_0. It fixes the distinguished column modulo $\bar{\pi}_N$, equivalently the distinguished covector modulo π_N by adjunction. Conversely an integral unitary matrix with this condition on the covector preserves the intersection. It has the column condition as well. The distinguished diagonal difference is divisible by both coprime factors, so $(h x_{k_0} - x_{k_0})/\bar{\pi}_N$ is in the intersection.

This proves preservation of the glued lattice, for h and its inverse. At $p_0^{\mathrm{lev}}, p_1^{\mathrm{lev}} \mid M$, the integer N is a unit and the principal congruence is the original one modulo π_M. Hence the two source classes are equal, and the freeness in Section 6 then gives $\sigma_1 = \sigma_2$.

Thus these positions give disjoint cosets in the H-fiber over e. Indeed, fixing l_0 over e, use $(H \cap J) l l_0^{-1} \subset H$ for each position. On such a coset the integrand $\mathcal{H}_t(h l_0)$ equals $\mathcal{H}_t(l)$, since W_Z is invariant on the left under $H \cap J$. This invariance follows from its coordinate formula. The

group H fixes the hyperbolic pairs and z. Its middle-block action normalizes U and preserves the simple-root character.

An element of $H \cap J$ acts trivially after moving past U and using right J-invariance. Write $\bar{H}$ for the finite reduction of the integral definite group and m_U for the order of the reduced integral unipotent. In the chosen Haar, $\mu_H(H \cap J) = |\bar{H}|^{-1}$ and $\delta_U = m_U^{-1}$. Indeed the terminal measure gives $H \cap K_T$ mass one. At the pairs above N this is the covector stabilizer computed above; at the pairs above M its reduction has kernel $H \cap J$ and image $\bar{H}$. The integral unipotent has mass one in its own measure. The reductions are unchanged by projection to T, as representatives extend by the identity outside T. The integral Bessel subgroup is a product of its integral unipotent and definite factors, by reading the middle block. Its reduced image has order $|H_R| = m_U|\bar{H}|$. Thus

$$C_* = |H_R| m_U, \qquad \mu_H(H \cap J)\, \mu(J)^2 [Z : J] = \mathcal{R}_Z / C_*.$$

Let $\Psi_t(e) = \int_H \mathcal{H}_t(hl_0)\, dh$. This is nonnegative, since $\mathcal{G}_t$ is a Gram matrix. It is the sum of the top finite pairings $\Phi_{p,q}(e)$ of Section 10.4 weighted by $\mathcal{G}_t(p,q)$: the integrand is $\sum_{p,q} \mathcal{G}_t(p,q) \langle V_p(hl_0), V_q(hl_0) \rangle$ with both inner products linear in the second argument. It has a compactly supported, locally constant, nonnegative extension from the last-vector quadric to the ambient finite vector space by the compact step extension from that section, taking the positive part of the real sum. The extension can be chosen $H \cap J$-invariant by averaging.

On the quadric the fiber value is invariant since $V_p(lh) = V_p(l)$ for $h \in H \cap J$, by changing its integration variable using that same left invariance of W_Z. Summing (24) over the disjoint fiber cosets therefore proves

$$R_0^2 G(t)\, \frac{\mathcal{R}_Z}{C_*} \sum_{(c,c',s,L)\,:\,(c,v_0,t)\ \text{fixed}} \|B^{\flat}_{c,c',s,L}\|^2 \le \Psi_t(e). \tag{25}$$

This compares the original cut blocks with the same finite form at a fixed last vector. For the subsequent sum over (c, v_0), the rational norm to average is $h(v_0, v_0) = (t\bar{t} - N^2)/N$, and the vector $(c^\circ)^{-1} v_0$ is integral off T by the integral stage factor and the integral original column z.

13.3 Averaging at a fixed final vector

In this subsection, we average the finite class sum of Section 13.2 by interpreting the fixed last-vector value as a Hermitian representation problem. This supplies the terminal factor in Section 13.4.

Let $E^+ = F_0$, $d_C = \#\mathcal{C}$ for the source classes, and $v_E = 2^f / \sqrt{|D_E|}$, with D_E the field discriminant. Fix t as in Section 13.2 and put $\beta = (t\bar{t} - N^2)/N \in F_0$, totally positive. Write B_{π_T} for the finite lattice basis, with $\pi_T = \pi_N|_T$. On the finite adeles of X_0 take

$$F_t(y) = \mathbf{1}_{\{y \text{ integral off } T\}} \Psi_t(B_{\pi_T}^{-1}(y_T - (t/N)z)).$$

Here vectors in the finite model are read in the basis B_{π_T}; this is the same affine coordinate $e(t, y_T)$ as in Section 10.4. The image of this affine embedding is closed and its inverse is the X_0-projection after multiplication by B_{π_T}. The pullback of the compact step test is therefore locally constant and compactly supported. Off T the indicated lattice is compact open. The embedding intertwines the rational-coordinate action on X_0 with the lattice-coordinate action through B_{π_T}, since the source group fixes z. The extended function Ψ_t is invariant under $H \cap J$

on the ambient space. Thus F_t is invariant under the original source level, which also preserves the lattice off T. On the selected (c, v_0, t), $F_t((c^\circ)^{-1}v_0) = \Psi_t(e)$.

We use Weil's Siegel formula for the orbit of one vector [78, §52, Theorem 5, (41)]. In this case the group is $U(X_0)$ over F_0, for the quadratic algebra E/F_0, and the rank condition is $k > 2$. The group has compact adelic quotient by definiteness. An original split finite place meets the local orbit condition: there the norm is $x \cdot y$ on pairs of vectors over $F_{0,v}$, with the general linear group on a vector and its dual.

The full-rank locus has both vectors nonzero, and the group is transitive at each value of the pairing by extending a vector and covector to an adapted basis. In particular the full-rank zero-norm locus is nonempty there. On the orbit of the nonzero represented norm β, unfolding the Tamagawa integral and dividing by the volume of $U(X_0)(F_0)\backslash U(X_0)(\mathbb{A}_{F_0})$ gives the factor

$$\frac{\tau(U(v_0^{\perp}))}{\tau(U(X_0))} = \frac{2}{2} = 1.$$

Indeed both groups are unitary of positive rank over the quadratic field extension E/F_0 [30, §2.5, p. 755]. This recovers the stabilizer normalization in [78, §53, (43)] and applies as well to the consecutive ranks below. Probability Haar on the quotient assigns the same mass $1/d_C$ to each class, since each open class has Haar weight proportional to the reciprocal of its rational stabilizer order and these orders are one. At each real place take the Schwartz function $y \mapsto u_v(h_v(y, y))$, where $u_v \in C_c^\infty(\mathbb{R})$ and $u_v(\Phi_v(\beta)) = 1$. Definiteness gives compact support. The function is invariant under $H(F_{0,v})$ and equals one on the prescribed sphere. Weil's absolute convergence hypothesis concerns every adelic Schwartz function. Such a function is a finite sum of products of a real Schwartz function with a compactly supported locally constant finite function. For each product use a compact set of adelic quotient representatives. Translating its finite support by this set confines all rational vectors to one fractional lattice. The archimedean translates of the Schwartz factor have a uniform bound $C_a(1 + \|y\|)^{-a}$ for every $a > 0$. Choose $a > 2kf$ to get absolute and uniform convergence.

The measure on the norm fiber in (41) is the quotient of the additive vector measure by the additive norm-value measure, both with adelic covolume one [78, §44, Theorem 2]. We spell out its conversion. Use ordinary area on $\mathbb{C}$ at the complex places, so the product finite vector measure gives $\widehat{\mathcal{O}}_E$ mass v_E per coordinate, by the covolume of $\mathcal{O}_E$. At the real places of F_0 use ordinary Lebesgue measure in the norm value. In the finite norm-value Fourier variable $u \in \mathbb{A}_{F_0,f}$, take du_0 giving the finite inverse different of F_0 mass one.

The phase $\psi_{E,f}(u(h(y, y) - \beta)/2)$ is the standard F_0-phase, by the trace E/F_0. For this Fourier pairing the covolume-one measure is $\sqrt{|D_{F_0}|}\, du_0 = |D_E|^{1/4}\, du_0$: the archimedean covolume of the inverse different is $|D_{F_0}|^{-1/2}$, and E/F_0 is unramified at the finite places by the inertia calculation in Section 4.2. The real norm fiber in $\mathbb{C}^k$ has mass $\pi^k \Phi_v(\beta)^{k-1}/(k-1)!$, by differentiating the Euclidean ball volume in its squared radius. Accordingly Weil's formula gives

$$\sum_c \sum_{h(v_0,v_0)=\beta} F_t((c^\circ)^{-1}v_0) = d_C |D_E|^{1/4} \prod_v \frac{\pi^k \Phi_v(\beta)^{k-1}}{(k-1)!}\, \mathcal{D}_f(F_t, \beta), \tag{26}$$

where $\mathcal{D}_f$ is the finite norm density with the vector measure just specified and frequency measure du_0.

To bound the finite density, take out v_E^k by normalizing the integer vector box at every place to mass one. Put $b_u = -\delta_E u$, a skew frequency, where δ_E generates the different of E.

Because E/F_0 is unramified or split at every finite place, the different of E is the extension of the different of F_0. Thus du_0 becomes the measure on skew frequencies giving the integral skew box mass one.

With $\psi_m(a) = \psi_{E,f}(-a/\delta_E)$ and an integral trace-one scalar τ, the phase on the norm difference is

$$\psi_{E,f}\big(u(h(y,y)-\beta)/2\big) = \psi_m\big(\tau b_u(h(y,y)-\beta)\big).$$

Indeed $\mathrm{Tr}_{E/F_0}(\tau) = 1$. The character ψ_m is self-dual on the integer ring and is trivial on $a+\bar{a}$, since δ_E is skew. In particular $\psi_m(\tau b_u(a+\bar{a})) = \psi_m(b_u a)$. It follows that the annihilator of the fixed integral ring for the skew-frequency pairing is the integral skew ring: testing against fixed traces is equivalent to testing b_u against all integral a. The two measures giving these boxes mass one are therefore dual. These identities hold also at places above two.

For a finite place v of F_0 off T, let R_v be the integer ring of the local quadratic algebra and let q_v be the residue size of $F_{0,v}$. Let $G_v(b)$ be the integral of $\psi_m(\tau b(h(x,x)-\beta))$ on R_v^k, with its probability measure. Squaring its absolute value and writing the two variables as $x, x+d$ leaves the inner average of $\psi_m(bh(x,d))$ times a unit phase depending on d.

Hence that average vanishes unless $bd \in R_v^k$, and

$$|G_v(b)|^2 \le [R_v : R_v \cap b^{-1}R_v]^{-k}.$$

The skew lattice has a unit generator. At skew valuation $-j < 0$ the index is q_v^{2j}, in both the unramified and split cases. The skew shell has volume $(1-q_v^{-1})q_v^j$, so its contribution after integrating G_v in absolute value is at most $(1-q_v^{-1})q_v^{-j(k-1)}$. The integral over the skew integer box is at most one. Thus each exterior factor contributes at most $(1-q_v^{-(k-1)})^{-1}$. Their product is bounded by

$$\zeta_{F_0}(k-1) \le \zeta(3\varrho+4)^f \le 1 + 2^{-\varrho} \le 1 + b^{-6}.$$

Here $k-1 \ge 3\varrho+4$, $f \le 2^{2\varrho}$ and $\log\zeta(a) \le 2^{1-a}$ for $a \ge 2$; the last inequality uses the parameter bounds on b, n.

At T the Fourier integral is exactly the norm-fiber integral in Section 10.4, since its skew parameter is dual to the norm-value Haar with integer mass one. For absolute convergence of the Fourier inversion, decompose the compactly supported test into finitely many shifted boxes $x_0 + cR_v^k$ on each orbit of places, using products of such boxes across T. After the substitution $x = x_0 + cx'$, the same squaring argument bounds the integral by the conductor index for frequency $c\bar{c}b$, with the additional box volume factor.

In the squared modulus use the difference variable $d = x' - x''$ for the two scaled box coordinates. The shift multiplies each outer d-term by the unit phase of the cross term in x_0, cd. The inner x''-average has the linear frequency $c\bar{c}bd$, so it vanishes off the same conductor annihilator as in the centered box. Taking absolute values after this cancellation gives the stated bound. The skew integral is finite by $k-1 > 1$. Write $E_v = E \otimes_{F_0} F_{0,v}$ and $q(y) = h(y,y)$. At every finite v the image β_v of the nonzero rational norm is nonzero. Choose $\tau \in \mathcal{O}_{E_v}$ of trace one. At a split place take $(1,0)$; at an unramified nonsplit place lift a residue element of trace one and divide by its unit trace. For $q(y) = \beta_v$ the differential and an $F_{0,v}$-linear right inverse are

$$dq_y(u) = \mathrm{Tr}_{E_v/F_{0,v}}\, h(y,u), \qquad a \longmapsto a\beta_v^{-1}\tau y.$$

Thus the norm map is a submersion also at dyadic places. The pushforward of the compactly supported test has a continuous density near β_v in these norm charts. Fourier inversion with the dual Haar measures gives its value at β_v.

Outside T, β is integral because N, t are units there. The Euler-product calculation can therefore first be made with integral skew frequencies at all but finitely many places, each such integral box contributing one in the normalized measure, and then extended by the preceding bounds on the absolute integrals. In total,

$$|\mathcal{D}_f(F_t, \beta)| \le (1 + b^{-6}) v_E^k \int_{h(y,y)=\beta} \Psi_t(e(t, y))\, d\nu_{\beta,T}(y). \tag{27}$$

Let Q_t be the sum against $\mathcal{G}_t(p, q)$ of the unnormalized finite last-entry densities of the two heads. By the definition of Ψ_t and the terminal disintegration, it is nonnegative and

$$\int_{h(y,y)=\beta} \Psi_t(e(t, y))\, d\nu_{\beta,T}(y) = \mathrm{md}(\pi_T) Q_t = N^{-f} Q_t.$$

Indeed $\mathrm{md}(\pi_T)^2 = \mathrm{md}(N) = N^{-2f}$, since $\pi_T \bar{\pi}_T = N$, T is conjugation-stable, and N is a unit off T. Together with the $N^{-f(k-1)}$ from $\prod_v \Phi_v(\beta)^{k-1}$, this gives the constant

$$C_{\mathrm{S}} = |D_E|^{1/4} v_E^k \left(\frac{\pi^k}{(k-1)!\, N^k} \right)^f.$$

By positivity the sum of $\Psi_t(e)$ over the selected pairs (c, v_0) is bounded by (26). Consequently (25) and (27), with $G(t) > 0$, give the fixed-entry estimate

$$R_0^2 \sum_{(c,c',s,L):\, t_L = t} \|B^\flat_{c,c',s,L}\|^2 \le (1 + b^{-6}) C_* d_C C_{\mathrm{S}} \frac{Q_t}{\mathcal{R}_Z}. \tag{28}$$

This applies to the original cut blocks on any finite set at t, so the bound holds for their nonnegative sum.

13.4 Adelic summation of the common-entry forms

In this subsection, we multiply the finite and real Fourier estimates with the same quotient measures. The local bounds from Sections 10.9 and 11.10 then lead to the open-cell estimate of Section 13.8.

Since $n = \varrho + 1 = m/2$, Section 4 gives $N \le b^{25m} = b^{50n} \le b^{60n}$. At depth i, take the product of the real forms $J_{v,i}$ and of the finite form $\mathcal{I}_i^T / \mathcal{R}_Z$ at weight one. At depth $n = \varrho + 1$ use the finite last-entry density in place of $\mathcal{I}_n^T$. Write the resulting form as J_i^{ad}. For a rational entry $t \in \mathcal{E}$, use its scale $a_{v,i} \Phi_v(t)$ at v, with $a_{v,i} = s_{v,i}$ of Section 13.2, and t_T at the finite places. These coordinates also define the scaled heads of a rational tuple.

Write $W_i(p)$ for the product of the real head weights in Section 11.10 and the finite head weight in Section 10.9. Define $w_i = \sum_{p \in \mathcal{E}^i} W_i(p)$ and

$$D_i = \sum_{\substack{p,q \in \mathcal{E}^i \\ p_{i-1} = q_{i-1}}} \Re J_i^{\mathrm{ad}}(p, q) \quad (i \ge 1),$$

$$D_0 = \Re J_0^{\mathrm{ad}} \le M_0 = (1 + b^{-6}) b^{-6k|\mathcal{A}|} \prod_v \frac{1 + b^{-10}}{\deg_v}.$$

The finite bottom bound is for the original Z, by Section 10.10. At each real place let $D_v = |I_v|$. The stationary vector in Section 12.10 has squared norm D_v. Its full correlation is $D_v c_v$ for

the coefficient of a unit vector. The stage Haar factor is $\lambda = \operatorname{vol} U(r)/\operatorname{vol} U(k)$. Hence the normalized full bottom mass is exactly

$$C_F \lambda^3 \int |D_v c_v(g)|^2 \, dg = \frac{\lambda \operatorname{vol} U(k)}{\operatorname{vol} U(r)} \frac{1}{\deg_v} = \frac{1}{\deg_v}.$$

Section 12.1 bounds the cut mass by $(1 + b^{-10})/\deg_v$. Thus the product uses the same $\mathcal{B} = \prod_v \deg_v$ as the main moment. At each selected split pair, the finite bound uses the index $q^r - 1$ for either a vector or a covector stabilizer. Since the bottom masses factor over conjugate pairs, the same bound applies to every mixed choice of the two kinds of stabilizer. For $0 \le i < n$, let $F_i(x)$ be the real part of the sum of $J^{\mathrm{ad}}_{i+1}(p \parallel x, q \parallel x)$ over the two proper heads in $\mathcal{E}^i$. Here x is a common entry on the product of the real places and T, and $\parallel$ appends that entry in local coordinates.

We may replace a zero complex component or a zero component in a local field E_w, $w \in T$, by one in forming a head. Such replacements concern null sets for the integrals and do not affect the nonzero rational samples.

Put $Y = \prod_{v|\infty} \mathbb{C} \times A_T$ with the product of the stage additive measures, and let $Y^\times$ consist of the tuples nonzero in every field component. Its complement is null because there are finitely many components and each singleton has additive measure zero. For $x \in Y^\times$ and a common suffix $u \in (Y^\times)^{n-i-1}$, write

$$K_{pq}(x, u) = J^{\mathrm{ad}}_n(p \parallel x \parallel u, q \parallel x \parallel u), \qquad p, q \in \mathcal{E}^i.$$

The empty suffix has mass one. At the common last entry each local matrix is a Gram matrix by Sections 11.3 and 10.4. Their entrywise product is a Gram matrix, since one may tensor the representing vectors. In particular $K_{pp} \ge 0$ and $|K_{pq}|^2 \le K_{pp} K_{qq}$.

At each real place and on A_T, the unweighted stage integrals for fixed nonzero heads are finite by the absolute shell estimates of Sections 11.10 and 10.9. The terminal integrals are finite by the compact support bounds in Sections 11.3 and 10.4. Starting with the nonnegative local terminal diagonal, integrate the last entry first. The one-step identities of Sections 12.2 and 10.4 give the preceding stage diagonal. Iterating and using Tonelli for the finite product of local factors gives

$$\int K_{pp}(x, u) \, du = J^{\mathrm{ad}}_{i+1}(p \parallel x, p \parallel x), \qquad \iint K_{pp}(x, u) \, du \, dx = J^{\mathrm{ad}}_i(p, p) < \infty.$$

The right side of the first identity is finite at each $x \in Y^\times$. Cauchy–Schwarz on the suffix, and then on all common coordinates, gives absolute integrability of K_{pq} in both identities. Fubini and the same one-step identities now give

$$F_i(x) = \int \sum_{p,q} K_{pq}(x, u) \, du \ge 0, \qquad \int F_i = \Re \sum_{p,q \in \mathcal{E}^i} J^{\mathrm{ad}}_i(p, q).$$

The finite sum in the first integral is a squared norm. The local terminal entries are continuous on nonzero heads, so this integrand is measurable and continuous in $x \in Y^\times$ for almost every u. Every selected rational entry has nonzero components at all the indicated places, also after its real scalings. Fatou's lemma along any sequence tending to that entry gives lower semicontinuity there. For a last entry t with $|\Phi_v(t)| > N$ at every real place, the nonnegative summand of D_n at t is $Q_t/\mathcal{R}_Z$ of Section 13.3.

Extend F_i to the adeles by setting

$$G_i(z) = \mathbf{1}_{\{z_f \text{ integral off } T\}}\, F_i((a_{v,i}z_v)_v, z_T), \qquad P_i = \frac{v_E}{\prod_v a_{v,i}^2}.$$

The exterior integer box is compact and open in the restricted product off T, hence its indicator is continuous. The extension is measurable, is integrable by the displayed stage integral, and is lower semicontinuous at the selected rational entries. The additive Haar has E of covolume one, as in Section 13.3. The Fourier coefficient at rational y/δ_E vanishes unless y is integral off T, since the integral exterior translations preserve G_i. For such y a change of variables in the Fourier integral gives P_i times the transform of F_i at

$$\xi_v = \frac{2\pi i}{\Phi_v(\delta_E)a_{v,i}}\Phi_v(y), \qquad \xi_T = y_T.$$

Let S_i be the complex sum defining $F_i = \Re S_i$. Since $F_i = (S_i + \overline{S_i})/2$, its Fourier transform is

$$\widehat{F_i}(\xi) = \tfrac{1}{2}(\widehat{S_i}(\xi) + \overline{\widehat{S_i}(-\xi)}).$$

The character estimates for y and $-y$ have the same bound at each place. The factor P_i is the real area Jacobian times the finite integer-box mass, since the projected finite integer box has mass one in the local forms. Indeed $\psi(\xi_v a_{v,i} z_v) = \exp(4\pi i\, \Re(\Phi_v(y/\delta_E)z_v))$ at each complex place, while $\psi_{m,T}(y_T z_T) = \psi_{E,T}(-y_T z_T/\delta_E)$ at T. These are the local factors of the inverse adelic character in the Fourier coefficient at y/δ_E, with the character of Section 13.2. At zero frequency this is $P_i \int F_i$.

13.4.1 The sum over nonzero frequencies

In this subsection, we combine the finite bounds of Section 10.9 with Lemma 11.1.

Set $K_* = k - n^7$, $X_v = |\kappa_v|^{2\varrho+1}/(MN)$, $B_{\mathrm{fr}} = b^{23(2\varrho+1)}$, and

$$\mathcal{D}_{\mathrm{dec}} = \prod_v \Theta\, X_v^{K_*},$$

where Θ is the constant in (19). For a nonzero y integral off T, its finite denominator index $D(y_T)$ is the absolute norm of its denominator ideal $\mathfrak{b}_y$. In both applications of Lemma 11.1 the heads have length i. For the character weight in the Poisson coefficient take $h = i$ and $c_v = 2\pi i/\Phi_v(\delta_E)$, including $i = 0$ with empty heads. For unequal last entries in the constant-weight case, $1 \le i \le \varrho$, take $h = i - 1$, $c_v = M$ and $y = (t^{-1} - t'^{-1})/M$. Here h indexes the scale of the entry giving the frequency. In both cases $\xi_v = c_v\Phi_v(y)/a_{v,h}$. Since $1 \le |\kappa_v|$ and $a_{v,h} = |\kappa_v|^{2(\varrho-h)}/N$, one has $a_{v,h}/|c_v| \le X_v \le B_{\mathrm{fr}}$. The complex places contribute squared moduli to the product formula, so

$$\mathrm{md}(y_T)^{-K_*/2}\prod_v |\xi_v|^{-K_*} = \prod_{\substack{w\nmid\infty \\ w\notin T}} |y|_w^{K_*/2}\prod_v (a_{v,h}/|c_v|)^{K_*} \le \prod_v X_v^{K_*}.$$

For the residual real damping use

$$\min(1, |\xi_v|^{-1})^6 \le B_{\mathrm{fr}}^6 \max(1, |\Phi_v(y)|)^{-6}.$$

Using $D(y_T)^{-6} \le D(y_T)^{-3}$, multiply (9) and the real bounds (20). The resulting product before its real constant $\Theta^f b^{2000n^9 f}$ and its finite constant $b^{n^9 f}$ is bounded by

$$\left(\prod_v X_v^{K_*}\right) B_{\mathrm{fr}}^{6f} \frac{\prod_v \max(1, |\Phi_v(y)|)^{-6}}{(\mathrm{N}\mathfrak{b}_y)^3}.$$

For the sum of the last weights, fix a denominator ideal $\mathfrak{b}$. The elements with this denominator lie in the $\mathrm{N}\mathfrak{b}$ cosets of $\mathfrak{b}^{-1}/\mathcal{O}_E$. In each coset, a product of complex half-unit squares contains at most one element, since a nonzero integral difference cannot have modulus less than one at all complex places. If z lies in a square with corner s, $\max(1, |s|) \le 2\max(1, |z|)$.

There are at most $(4R+1)^2 \le 25R^2$ such corners of modulus at most $R \ge 1$, by packing disjoint discs of radius $1/4$. The sum of $\max(1, |s|)^{-6}$ over corners is then at most $6\int_1^\infty 25R^{-5}\, dR < 38$. Summing first over the cosets and then the ideals gives

$$\sum_{0 \ne y \in E} \frac{\prod_v \max(1, |\Phi_v(y)|)^{-6}}{(\mathrm{N}\mathfrak{b}_y)^3} \le (64 \cdot 38)^f \zeta_E(2) \le (64 \cdot 38e^4)^f.$$

Since $64 \cdot 38e^4 \le b^3$ and $3 + 138(2\varrho + 1) \le n^9$, the common frequency sum, including its real and finite constants, is bounded by

$$\varepsilon_0 = b^{2002n^9 f} \mathcal{D}_{\mathrm{dec}}.$$

It follows that the nonzero Fourier coefficients of the extension have absolute sum at most $P_i \varepsilon_0 w_i^2$.

For completeness, one-sided Poisson at the chosen entries follows directly from periodization. The nonnegative G_i is integrable, so its periodization over E is integrable on the compact additive quotient. Its Fourier coefficients are those just computed. Absolute summability makes their Fourier series continuous, equal almost everywhere to the periodization by uniqueness of Fourier coefficients. Near the identity in a quotient chart, the periodization dominates the sum over the finitely many selected rational translates of G_i.

Lower semicontinuity at those translates gives the same inequality at the identity against the continuous representative. Consequently

$$D_{i+1} \le P_i \left(\Re \sum_{p,q \in \mathcal{E}^i} J_i^{\mathrm{ad}}(p, q) + \varepsilon_0 w_i^2 \right).$$

In the unrestricted sum, equal last entries give D_i. For two different last entries t, t', take $y = (t^{-1} - t'^{-1})/M$. It is nonzero and integral off T, and supplies precisely the unequal-head frequency: y_T and $M\Phi_v(y)/a_{v,i-1}$ at the indicated places. The singleton version of the frequency sum therefore bounds that pair by $\varepsilon_0 W_i(p) W_i(q)$. There is no unequal contribution at $i = 0$. Thus for $0 \le i < n$,

$$D_{i+1} \le P_i(D_i + 2\varepsilon_0 w_i^2).$$

A head of nonzero weight has partial products δ_j integral at T. Each factor belongs to $\mathcal{E}$ and is a unit off T by its construction in Section 13.1. Thus $\delta_j \in \mathcal{O}_E$ and has finite modulus one off T. At each v, the initial segments of the scaled head have modulus at most b^{60n^2}. The inverse of the product of their scales is at most $N^i \le b^{60n^2}$. Hence $|\Phi_v(\delta_j)| \le b^{120n^2}$. For an initial segment with scale product c_v, the finite weight uses $\mathrm{md}(\delta_{j,T})^{-1/2} = \prod_v |\Phi_v(\delta_j)|$.

The real factors for this initial segment, before raising them to the power n^6, give the product

$$\prod_{v} \max\{|\Phi_v(\delta_j)|, c_v^{-1}\}.$$

It follows that $W_i(p) \le b^{120in^8 f}$. A head is recovered from its partial products. Packing these integer tuples using the maximum complex modulus gives at most $(1 + 2b^{120n^2})^{2if}$ heads of nonzero weight: the difference of distinct tuples has modulus at least one in some coordinate, so their radius-1/2 polydiscs have disjoint interiors. Therefore $w_i \le b^{130n^9 f}$. Also $a_{v,i} \le b^{46\varrho}$ and $v_E \ge b^{-f}$ by the root discriminant bound. Hence $\prod_{j<i} P_j \ge b^{-100n^2 f}$ and $w_i^2 \le b^{400n^9 f} \prod_{j<i} P_j$.

Put $A_i = \prod_{0 \le j < i} P_j$, so $A_0 = 1$ and $A_{i+1} = A_i P_i > 0$ by the formula for P_i. Dividing the recurrence by A_{i+1} and summing gives

$$\frac{D_n}{A_n} \le D_0 + 2\varepsilon_0 \sum_{i=0}^{n-1} \frac{w_i^2}{A_i} \le M_0 + 2n\, b^{(2002+400)n^9 f} \mathcal{D}_{\mathrm{dec}}.$$

Put

$$\mathcal{E}_+ = \{t \in \mathcal{E} : |\Phi_v(t)| > N \text{ for every } v \mid \infty\}.$$

Using $2n \le b^{n^9 f}$ gives

$$\sum_{t \in \mathcal{E}_0} Q_t / \mathcal{R}_Z \le D_n \le \left(\prod_{i<n} P_i\right)(M_0 + b^{2403n^9 f}\mathcal{D}_{\mathrm{dec}}) \qquad (\mathcal{E}_0 \subseteq \mathcal{E}_+). \tag{29}$$

The first inequality follows from nonnegativity at each common last entry. For Ω contained in the open residues of the prescribed stabilizer, fix any finite collection $\mathcal{F}$ of tuples (c, c', s, L) with $s \in \Omega$, retaining the full nonzero-discriminant amplitude at each tuple. Taking $\mathcal{E}_0$ to contain their last entries, (28) gives

$$R_0^2 \sum_{(c,c',s,L) \in \mathcal{F}} \|B^{\flat}_{c,c',s,L}\|_{\mathrm{HS}}^2 \le (1 + b^{-6}) C_* d_C C_{\mathrm{S}} \left(\prod_{i<n} P_i\right)(M_0 + b^{2403n^9 f}\mathcal{D}_{\mathrm{dec}}).$$

The supremum of the left side over these finite collections is $R_0^2 \mathcal{L}(\Omega)$ by the nonnegative sum defining $\mathcal{L}$ in Section 9.1.

13.5 Consecutive masses of definite unitary lattices

In this subsection, we compare the genus masses in two consecutive ranks, retaining the integral automorphism weights. This supplies the factors used in Sections 13.3 and 13.7.

Write μ_j for the mass of the genus of the standard positive definite Hermitian lattice $\mathcal{O}_E^j$. Let H_j be its unitary group, K_j the stabilizer of $\widehat{\mathcal{O}}_E^j$, and $L_c = E^j \cap c\widehat{\mathcal{O}}_E^j$ for $c \in H_j(\mathbb{A}_{F_0,f})$. These lattices give all classes in the genus. For a lattice described by its local completions, Landherr's theorem first identifies its rational Hermitian space with E^j in the standard form [43, pp. 246–247]. The lattice is recovered from its finite completions, and local isometries of the standard lattice give its representative c. Its stabilizer is $\Gamma_c = H_j(F_0) \cap cK_j c^{-1}$. Definiteness makes every rational unipotent in H_j trivial. The norm-one determinant torus has no nontrivial F_0-rational characters, and every rational character of H_j factors through it. The adelic quotient is therefore compact by [18, Theorem 5.6(ii), p. 21]. The compact real factor and the compact

open K_j make the class set a finite discrete quotient of the adelic space. The class of c has measure $\operatorname{vol}(H_j(F_{0,\infty}))\operatorname{vol}(K_j)/|\Gamma_c|$. Consequently

$$\mu_j = \sum_{[c]} |\Gamma_c|^{-1} > 0, \qquad \Pr([c]) = \frac{1}{\mu_j |\Gamma_c|}.$$

For each rank $j+1$, the weighted count of pairs consisting of a lattice in this genus and a norm-one vector equals μ_j. Indeed such a vector w splits off $\mathcal{O}_E w$ by the integral projection onto its line.

Over a split finite place the two components of a norm-one vector are a vector and a covector of pairing one, and an adapted basis sends them to the standard pair. At an unramified nonsplit place the complement is unimodular. In a unimodular Hermitian lattice one can split off a unit-norm line by reduction: if every residue norm vanishes, polarization using all scalars and the nondegenerate residue trace pairing forces the residue form to vanish.

This works also in residue characteristic two. Surjectivity of the norm on unramified local units allows the unit on each diagonal line to be made one. Thus the complement is standard locally. Rationally, Witt extension carries the norm-one line to the standard line in the standard Hermitian space. Conversely a rank-j genus lattice with an adjoined standard line belongs to the rank-$j+1$ genus. These operations give inverse isometries of the respective objects and identify their automorphism groups.

Fix $k \le j < r$, so $j+1 > 2$. Apply the one-vector formula of Section 13.3 at the full integral level, with the indicator of the finite integer box and norm value one. Probability Haar weights each class by its reciprocal automorphism order divided by the total mass. The split-place and rank conditions hold as before. Let x_j be the finite norm density divided by v_E^{j+1}, in the frequency Haar giving the codifferent of F_0 mass one. The formula gives

$$\frac{\mu_j}{\mu_{j+1}} = |D_E|^{1/4} v_E^{j+1} \left(\frac{\pi^{j+1}}{j!}\right)^f x_j.$$

Put $l = j+1$, and at a finite place v of F_0 let q_v be its residue size. Write $\chi_v = 1$ at a split place and $\chi_v = -1$ at an inert place. The number of solutions of $h(w,w) = 1$ in the residue quadratic algebra of rank l is

$$A_v = q_v^{2l-1} - \chi_v^l q_v^{l-1}.$$

At a split place a nonzero vector has q_v^{l-1} covectors taking value one on it. At an inert place choose a nontrivial additive character ψ of the fixed residue field. For $a \neq 0$, the single-coordinate sum is

$$\sum_{z \in \mathbf{F}_{q_v^2}} \psi(a z \bar{z}) = -q_v,$$

since the nonzero norms have $q_v + 1$ preimages. Orthogonality in the norm variable gives $A_v = q_v^{-1}(q_v^{2l} - (-q_v)^l)$. The derivative at a solution is $u \mapsto \operatorname{Tr}(h(w,u))$. Taking $u = tw$ gives the surjective residue trace $t \mapsto \operatorname{Tr}(t)$, including at residue characteristic two. Consequently every solution has q_v^{2l-1} lifts from one precision to the next. With the vector and norm-value integer boxes each of mass one, the norm fiber therefore has local density $q_v^{1-2l} A_v$.

The absolute shell estimate of Section 13.3 applies with l coordinates. Fourier inversion at every finite set of places, followed by absolute convergence of the tails, gives

$$x_j = \prod_{v \nmid \infty} (1 - \chi_v^l q_v^{-l}), \qquad |x_j - 1| \le \prod_{v \nmid \infty} (1 + q_v^{-l}) - 1 \le \zeta_{F_0}(l) - 1.$$

For $j \geq k$, using $f \leq 2^{2\varrho}$, $k \geq 4\varrho + 12$, and $\log \zeta_{F_0}(l) \leq f 2^{1-l}$, one has

$$|x_j - 1| \leq 2^{-\varrho}, \qquad |\log x_j| \leq 2^{1-\varrho}.$$

Define

$$\sigma_{v,l} = \frac{(2\pi)^l}{|\Phi_v(\delta_E)|^{l-1/2}(l-1)!}, \qquad \mathrm{fr}_v = N^k \prod_{j=k}^{r-1} \sigma_{v,j+1}.$$

Since $\prod_v |\Phi_v(\delta_E)| = \sqrt{|D_E|}$, the mass identity reads $\mu_j/\mu_{j+1} = x_j \prod_v \sigma_{v,j+1}$. The sizes $N \geq b^m$, $\log b = O(x^\epsilon)$, $m \geq \sqrt{x}$, $f \leq x^{12}$, and $r - k = m - 1$ give

$$(r-k)2^{1-\varrho} + 4f/N \leq \log(1 + b^{-9})$$

for all sufficiently large x, since each term on the left is $o(b^{-9})$. Taking the product over $k \leq j < r$ and using $\log(N^k/(N^k-1)) \leq 2/N$ yields

$$\left(\prod_v \mathrm{fr}_v\right) \mu_r \left(\frac{N^k}{N^k - 1}\right)^{2f} \leq (1 + b^{-9}) N^{kf} \mu_k. \tag{30}$$

13.6 The class count and the unipotent cutoff measure

In this subsection, we compute the change in counting measure from the integral stabilizer to the source level subgroup. Its finite covolume combines with the orbital measure of Section 8 in the moment comparison of Section 13.7.

Let K_X be the stabilizer of $\widehat{\mathcal{O}}_E^k$ in $H(\mathbb{A}_f)$, and let H_{int} be its subgroup preserving the ambient lattice. The glued-column calculation of Section 13.2 shows that H_{int} is the last-covector stabilizer modulo π_N. At each of the f chosen places above N, reduction from the split integral factor to $\mathrm{GL}_k(\mathbb{F}_N)$ is onto: lift the matrix over the valuation ring and put its inverse transpose at the conjugate place. The group of matrices over $\mathbb{F}_N$ acts transitively on its $N^k - 1$ nonzero covectors. The congruence at the two primes dividing M is independent of these conditions on the covectors since $N \nmid M$. Thus

$$[K_X : H_{\mathrm{int}}] = (N^k - 1)^f, \qquad [H_{\mathrm{int}} : J_H] = |\bar{H}|.$$

If a class at K_X has stabilizer Γ, its fiber at J_H is $\Gamma \backslash K_X / J_H$. The stabilizer acts freely on this fiber before quotienting, by Section 6. Summing the fiber sizes over the classes at K_X gives

$$d_C = |\bar{H}|(N^k - 1)^f \mu_k. \tag{31}$$

For the class cutoff put $B = B_{\pi_N}$, the original glued basis in the rational coordinates $e_a; x_j, z; e_a^*$. By Section 8.1, $h_c = B^{-1} \iota_{\mathrm{rat}}(c^\circ) B$. Thus the class lattice has basis

$$B_c = \iota_{\mathrm{rat}}(c^\circ) B = B h_c, \qquad B_c^{-1} u B_c = h_c^{-1} (B^{-1} u B) h_c.$$

For a rational unipotent u, membership in U_c is precisely $B_c^{-1} u B_c \in 1 + M M_r(\widehat{\mathcal{O}}_E)$. Here reduction modulo the rational integer M is equivalent to reduction at the half of its split places, by unitarity at the conjugate places. Use $a_{v,a} = s_{v,a}$ of Section 13.2. At a real place, write $b_{v,j}$ for the scale of a basis column. The scales on e_a, e_a^* are $\kappa_v^{\varrho - a}/\sqrt{N}$ and its inverse, those on x_j are one, and that on z is $N^{-1/2}$. The real image of a rational matrix is conjugated by this diagonal change of basis.

The free entries in the row indexed by e_a lie in the columns indexed by e_b, e_b^* for $b > a$ and in the middle block. Write $r_a = k + 1 + 2(\varrho - 1 - a)$ for their number. The other coordinate in this row is the imaginary part at e_a^*. To obtain the coordinate chart, take the product of the real Heisenberg factors at $0, \dots, \varrho - 1$ in that order. On $F_a^{\perp}/F_a$ for $F_a = \langle e_b : b < a \rangle$, the earlier factors act trivially. With the later factors fixed, the free covector from the factor at a is $-h(w, \cdot)$ on the remaining space, composed with the later isometry, and its imaginary coordinate is the imaginary part of the Heisenberg center in the half-trace chart.

The later factors fix e_a^*. The Gram matrix on the remaining space in the normalized basis is a signed permutation matrix. Thus the covector map and composition with the later isometry have real Jacobian of absolute value one, and the center also has Jacobian one. Descending in a gives the inverse chart and shows that the original product of Heisenberg Haar measures is ordinary product Lebesgue measure in these coordinates.

Let $B_{c,a}$ be the block of B_c on the free columns of row a. The earlier basis and dual columns of B_c are unchanged, so the free entries in row a of $(u - 1)/M$, regarded as a row vector, for $u \in U_c$ lies in

$$L_{c,a} = \{\ell \in E^{r_a} : \ell B_{c,a} \in \widehat{\mathcal{O}}_E^{r_a}\}.$$

Every such row vector is attained with the free and central coordinates of every later row equal to zero. Extend ℓ by zero and write it as $h(\omega, \cdot)$ in the original rational Gram form, conjugate-linear in the first argument.

The class lattice is self-dual because the glued vector and z span a plane with Gram determinant -1, and the other summands are unimodular. Thus ω lies in it. In particular $h(e_a, \omega) = 0$. Choose an integral trace-one scalar τ, which exists since E contains the sixth roots of unity. For an integral skew scalar s put $C = Ms - M^2 \tau h(\omega, \omega)$. The map

$$g(w) = w + Me_a h(\omega, w) - M\omega h(e_a, w) + Ce_a h(e_a, w)$$

is an isometry.

The change in the pairing on two vectors is $(C + \bar{C} + M^2 h(\omega, \omega))$ times the product of their two $h(e_a, \cdot)$ coordinates with conjugation on the first, and hence zero. It is integral and congruent to one modulo M in the class lattice. Its free entries in row a form $M\ell$. In the other top rows only column e_a^* can change, and that entry is neither free nor central in any row $b > a$.

Setting $\omega = 0$ realizes every central shift Ms. Conversely if the free entries in row a vanish, that center entry is skew: multiplication of the matrix with its adjoint gives $CA^* + AC^* + BJ_0^{-1}B^* = 0$ for the upper blocks A, B, C and the middle Gram J_0. Here row a of A equals row a of the identity matrix and that of B is zero. The center entry divided by M is integral since e_a^* is an unchanged class-basis column.

These lattices in the successive coordinates give a strict right fundamental domain. To see the translation rule, if all rows of $u \in U_c$ with index $b > a$ have their coordinates zero, its matrix on the tail quotient after a is the identity, by the real chart just described. Thus right multiplication by u adds its free coordinates in row a and preserves the free and central coordinates in all later rows.

If its free coordinates in row a also vanish, the tail block including row a has $A = I$ and $B = 0$; its C is skew-Hermitian. In this block the other entries in column e_a^* vanish, so right multiplication adds the central coordinate. Reduce in descending row order, free coordinates before center, to half-open fundamental parallelepipeds of the two additive lattices. The isometries above supply every needed translation. For uniqueness, compare two such reduced points in a right coset, starting at the last row. The change in each successive coordinate belongs to

its lattice, and the parallelepiped forces it to be zero. The coordinate chart then identifies the points.

The middle determinant of $B_{c,a}$ is $\det(c^\circ)/\bar{\pi}_N$, since $\iota_{\mathrm{rat}}(c^\circ)$ acts by c° on X_0, fixes z, and the glued column replaces x_{k_0} by $x_{k_0}/\bar{\pi}_N + z/N$. Its finite modulus is N^f: the unitary determinant has modulus one and $\pi_N \bar{\pi}_N = N$. Hence the rational lattice $L_{c,a}$ has Minkowski covolume $N^f v_E^{-r_a}$, using the finite box under $B_{c,a}^{-1}$ and the covolume-one additive adelic Haar. The real changes of variables in that row have complex scales $M b_{v,j}/b_{v,e_a}$. The scales at the two columns of every later hyperbolic pair cancel, while the z-column has squared scale N^{-1}. Thus the measures of the free and central coordinates in row a are

$$\left(v_E^{-1} \prod_v \frac{M^2}{a_{v,a}} \right)^{r_a}, \qquad |D_E|^{1/4} \prod_v \frac{M}{a_{v,a}},$$

respectively. For the second formula, the skew integers are $i\mathcal{O}_{F_0}$, with imaginary components equal up to signs to the embeddings of F_0. The unscaled center has covolume $\sqrt{|D_{F_0}|} = |D_E|^{1/4}$.

Put $d_U = \sum_a r_a = \varrho(k+\varrho)$ and $\Lambda_0 = \prod_{v,i<n} a_{v,i}^{k+2(\varrho-i)}$. At each split residue pair of characteristic $p \mid M$ the free entries and centers have $2d_U + \varrho$ scalar coordinates over $\mathbb{F}_p$. All choices lift to the local integer lattice: use the paired block chart, with the quadratic correction given by an integral trace-one scalar in the center. At the primes dividing M the middle lattice is unmodified since N is a unit. Thus $m_U = M^{f(2d_U+\varrho)}$. Multiplying the measures for the individual rows, using $r_a + 1 = k + 2(\varrho - a)$ and $a_{v,\varrho} = 1/N$, gives

$$\nu \Lambda_0 = m_U N^{-kf} v_E^{-d_U} |D_E|^{\varrho/4}. \tag{32}$$

The inverse of the right fundamental box is a left fundamental box of equal measure, by unimodularity. Since the translates of the nonnegative cutoff β_c sum to one as in Section 8, Tonelli gives $\int_U \beta_c(uy)\,du = \nu$ for every y, in the specified real Haar. The value is independent of the original class c.

13.7 The main and identity moments on the source

In this subsection, we express both normalization constants using the source class count and the dimensions of the real compact types. The mass comparison of Section 13.5 ensures that the ratio used in Section 9.1 is for the same measure.

Write $d_B = \prod_v |I_v|$. The source classes are free at J_H, and each class value is the tensor product of the compact types of Section 12.10. Thus $d_X = d_C d_B$. From $|H_R| = m_U |\bar{H}|$ and (31), the definitions of Section 9.1 give

$$R_* := \frac{D_{\mathrm{main}}}{D_{\mathrm{id}}} = \frac{d_B (N^k - 1)^f \mu_k}{\mathcal{B} \mu_r m_U}.$$

Set $B = \log b$, $Q_0 = M \operatorname{rd}(E)/(2\pi)$, and let S_κ be the sum of the logarithms of the weights of Section 4 over the f infinite places. The choice of N reads

$$\log N = m S_\kappa / f + (m-1) \log Q_0 - \tfrac{15}{2} B + \delta, \qquad 0 \le \delta \le \sqrt{B}.$$

Set

$$W_* = \log \mathcal{B} + f \sum_{l=k-1}^{r-1} \log(l!), \qquad \varepsilon_m = \log(\mu_k/\mu_r) - \sum_{j=k}^{r-1} \sum_v \log \sigma_{v,j+1}.$$

Section 13.5 gives $|\varepsilon_m| \le (m-1)2^{1-\varrho}$. Lemma 12.2 gives

$$|kmS_\kappa - W_*| \le m^2 fB + fmr/x.$$

Using $\sum_{j=k}^{r-1}(j+1) = (m-1)(k+\varrho+1)$ and $m_U = M^{f\varrho(2k+2\varrho+1)}$, the factorials in the moment ratio cancel except at $k-1$. More explicitly,

$$\begin{aligned}\log R_* - fk(\log M - \tfrac{15}{2}B) - \log\bigl(d_B((k-1)!)^f\bigr)\\ = fk\delta + f\log(1-N^{-k}) + \varepsilon_m + kmS_\kappa - W_*\\ - f(m-1)\left(\varrho\log Q_0 + \log\frac{\sqrt{\mathrm{rd}(E)}}{2\pi}\right).\end{aligned}$$

The right side is $o(fkB)$. Indeed the first term is at most $fk\sqrt{B}$, the second has absolute value at most $2f/N$, and the last term is $O(fm^2B)$ by the parameter bounds on Q_0 and the root discriminant. Here $m^2/k = O(x^{-4})$ and r/k is bounded.

In the compact type, the dimension of the symmetric-power representation of weight a_v is $\binom{a_v+k-1}{k-1}$. For $a_v \le a = xb$ it is at most $(a+k)^k$, also when $a_v = 0$. Since $(a+k)k \le b^{5/4}$ for large x,

$$1 \le d_B((k-1)!)^f \le b^{5kf/4}.$$

Using the logarithmic error bound $fkB/4$ and $M \ge b^{20}$ now gives

$$b^{11kf} D_{\mathrm{id}} \le D_{\mathrm{main}} \le (M/b^6)^{kf} D_{\mathrm{id}}. \tag{33}$$

13.8 The open-orbit bound at equal parameters

In this subsection, we insert the adelic estimate of Section 13.4 into the normalized open orbital sum. The mass identities in Sections 13.5 and 13.7 determine its constant.

Proposition 13.2 (Open-orbit bound for a prescribed stabilizer)**.** *Let Ω be a finite set of open residues in the stabilizer $\Gamma_{\mathcal{A}}$. On the nonzero-discriminant set, the cut blocks over these residues satisfy*

$$\nu^{-2}\mathcal{L}(\Omega) \le (1+b^{-4})b^{-6k|\mathcal{A}|} D_{\mathrm{main}}.$$

For each residue it includes all rational links, with the squared norm taken separately for each link of the cut kernel. Passage to the uncut kernel uses (36).

Proof. Set $\Delta_0 = R_0^{-2}\prod_{i<n} P_i$ and $C_0 = C_* d_C C_{\mathrm{S}}$. The definitions of R_0 and P_i give the exact ratio

$$\Delta_0 = \frac{v_E^n}{c_G^f}\left(\frac{d_B}{\Lambda_0}\right)^2.$$

The successive unitary compact volumes in the stage Haar satisfy $\mathrm{vol}\,U(l)/\mathrm{vol}\,U(l-1) = 2\pi^l/(l-1)!$, the unit sphere area. Hence from the formulas for c_G, fr_v and C_{S},

$$\left(\prod_v \mathrm{fr}_v\right) c_G^f |D_E|^{\varrho/2} = C_{\mathrm{S}} N^{2kf} v_E^{n+2d_U}.$$

For this substitution use $\prod_v |\Phi_v(\delta_E)| = |D_E|^{1/2}$ and $\sum_{l=k+1}^{r} l = (2\varrho+1)(k+\varrho+1)$. Using $C_* = m_U^2|\bar{H}|$ and (32) now gives

$$C_0\Delta_0 = \nu^2|\bar{H}|d_C d_B^2 \prod_v \mathrm{fr}_v \le (1+b^{-9})\nu^2 D_{\mathrm{main}}\mathcal{B}.$$

Indeed, substituting $d_X = d_C d_B$, $D_{\mathrm{main}}\mathcal{B} = d_X^2/\mu_r$ and (31) gives

$$\frac{C_0\Delta_0}{\nu^2 D_{\mathrm{main}}\mathcal{B}} = \frac{\mu_r \prod_v \mathrm{fr}_v}{(N^k-1)^f \mu_k} \le (1+b^{-9})(1-N^{-k})^f$$

by (30). In fact the exact ratio is

$$(1-N^{-k})^{-f} \prod_{j=k}^{r-1} x_j^{-1}, \qquad x_j = \prod_{v\nmid\infty}(1-\chi_v^{j+1}q_v^{-(j+1)}).$$

Each split place of F_0 over N therefore contributes $\prod_{l=k}^{r}(1-N^{-l})^{-1}$, including the rank-$k$ factor from the covector index. This factor can also be read from the original compact at that place. In the last stage of rank $l = k+1$, the split components of z are a primitive vector and a primitive covector of pairing $-N$, since $h(-x_{\mathrm{gl}}, z) = 1$ for the glued column $x_{\mathrm{gl}} = x_{k_0}/\bar{\pi}_N + z/N$. Send the vector to the first standard basis vector. The covector then has first coordinate $-N$ and a unit among the remaining coordinates, so a basis change on their span takes it to $(-N, 1, 0, \dots, 0)$. This proves transitivity. The residue count is $(N^l-1)(N^{l-1}-1)$, and the pairing has surjective differential there. Division by N^{2l-1} gives $(1-N^{-l})(1-N^{-(l-1)})$. The earlier isotropic stages supply the exponents $k+2, \dots, r$, so the compact mass is $\prod_{l=k}^{r}(1-N^{-l})$ in the stage Haar.

For the decay term, put $s = \sqrt{\mathrm{rd}(E)}$ and $h = r-k = 2\varrho+1$. Put $X_* = Q_0^h/(MN)$, so $\prod_v X_v = X_*^f$. The sphere-volume formula and the definition of fr_v give

$$\Theta = \frac{\prod_{l=k}^{r-1} l!}{2^h \pi^{h(\varrho+1)}}, \qquad \left(\prod_v \mathrm{fr}_v\right)^{1/f} = \frac{N^k (2\pi)^{h(k+\varrho+1)}}{s^{2h(k+\varrho+1/2)} \prod_{l=k}^{r-1} l!}.$$

Since $D_{\mathrm{id}} = m_U|\bar{H}|d_C d_B$ and $m_U = M^{f\varrho(2k+h)}$, the preceding equality for $C_0\Delta_0$ yields

$$\frac{C_0\Delta_0\mathcal{D}_{\mathrm{dec}}}{d_B D_{\mathrm{id}}\nu^2} = \frac{(\prod_v \mathrm{fr}_v)\Theta^f X_*^{f(k-n^7)}}{m_U} = \left[(2/M)^{\varrho h} s^{-h^2} X_*^{-n^7}\right]^f \le b^{n^9 f}.$$

Here $Q_0 = Ms^2/(2\pi) \ge 1$, $M \ge 2$, $s \ge 1$, and $X_*^{-1} \le MN \le b^{23+60n}$, with $(23+60n)n^7 \le n^9$ for large n. Also $d_B \le b^{2kf}$, $|\mathcal{A}| \le f$, and $k \ge 512000n^9$ by the rank bounds. Consequently

$$2404n^9 f + 2kf + 10 + 6k|\mathcal{A}| \le 11kf.$$

Applied to (33), this gives

$$C_0\Delta_0 b^{2403n^9 f}\mathcal{D}_{\mathrm{dec}} \le b^{-10}\nu^2 b^{-6k|\mathcal{A}|} D_{\mathrm{main}}.$$

For any finite set of open cut blocks on the nondegenerate orbital set, combine the two estimates with (29) and the summed fixed-entry bound immediately following it. The degrees in M_0 cancel $\mathcal{B}$, and $(1+b^{-10})^f \le 1+b^{-9}$ for large x since $f/b \to 0$. The remaining multiplier on $b^{-6k|\mathcal{A}|}D_{\mathrm{main}}$ is at most

$$(1+b^{-6})\big((1+b^{-9})^2(1+b^{-6})+b^{-10}\big) \le 1+b^{-4}.$$

For any choice of a vector or covector stabilizer at each place of $\mathcal{A}$, the nonnegative sum over all tuples above the given residues Ω is the supremum of the sums over $\mathcal{F}$. By the definition in Section 9.1 this sum is $\mathcal{L}(\Omega)$. Hence

$$\nu^{-2}\mathcal{L}(\Omega) \le (1+b^{-4})b^{-6k|\mathcal{A}|}D_{\mathrm{main}} \tag{34}$$

on the open part of $\Gamma_{\mathcal{A}}$ of Section 9.1, with Z formed from those same stabilizers as in Section 10.10. □

14 Truncation, finite projections, and lattice pair counts

In this section, we sum the truncation errors from Section 12.4, control the residue projections, and count pairs of lattices in the Petersson expansion. These estimates give the finite sums used in Section 15.

14.1 Reduced bases and the truncation error

In this subsection, we choose representatives for rational isometries in the Petersson expansion with controlled norms. This lets the real truncation estimate of Section 12.4 be summed in the full pairing of Section 9.1.

Let γ_f be the finite matrix in the rational Witt basis, $B = B_{\pi_N}$, and $\gamma_L = B^{-1}\gamma_f B$. By Section 13.6 the matrix in the class bases is

$$B_c^{-1}\gamma_f B_{c'} = h_c^{-1}\gamma_L h_{c'}.$$

Write $\mathcal{T}(c,s,c')$ for the rational isometries with this matrix integral and of residue s at the original principal level. We specify the class cutoff used in the orbital series. Set $A = \langle e_0,\dots,e_{\varrho-1}\rangle_E$. Write $F = (e_0,\dots,e_{\varrho-1},z)$ for the ordered basis of $A \oplus Ez$ and U_c^0 for its pointwise stabilizer in U_c. The action of U_c on F has image the upper unitriangular matrices whose strict upper entries belong to $M\mathcal{O}_E$.

The containment follows because the basis columns and the top coordinate functionals are unchanged in the integral class basis. For surjectivity, realize an elementary entry Ma in row i, column $j > i$, by the integral Eichler lift of Section 13.6. For $f_j = e_j$ the coordinate covector $e_j^\vee$ is integral on the lattice. For $f_j = z$ the coefficient of z in the finite class basis on the middle space is

$$\ell_c(x+tz) = t - \pi_N^{-1}\big((c^\circ)^{-1}x\big)_{k_0}.$$

Indeed it vanishes on $c^\circ x_{k_0}/\bar\pi_N + z/N$ and the other X_0-columns, and takes value one on z. To obtain a rational covector $x+tz \mapsto t+\xi(x)$ integral on the same lattice, the exact conditions are

$$(\xi c^\circ)_l \in \widehat{\mathcal{O}}_E \quad (l \ne k_0), \qquad (\xi c^\circ)_{k_0} + \pi_N^{-1} \in \bar\pi_N \widehat{\mathcal{O}}_E.$$

They define a nonempty open set of finite adelic rows containing the restriction of ℓ_c to X_0. Additive approximation gives a rational row ξ satisfying all the conditions, at the places of both π_N and $\bar\pi_N$ as well. Extend by zero on the hyperbolic pairs. In either case the covector is supported on the free columns in row i. Multiply it by a and apply the Eichler lift. Its restriction to F adds just the stated elementary entry.

Choose a measurable set P of representatives for archimedean tuples modulo $M\mathcal{O}_E$, bounded in each coordinate by b^{26}. The $2f$ elements $\zeta_{q_1}^j$, $\sqrt{-d}\zeta_{q_1}^j$ for $0 \le j < f$ span a full sublattice of $\mathcal{O}_E$ by linear disjointness. Its multiple by M has a half-open parallelepiped bounded by $2fM\max(1,\sqrt{d}) \le b^{26}$. On this parallelepiped the quotient of $M\mathcal{O}_E$ by the smaller lattice acts by translation and reduction. Choose the lexicographically first point in each finite orbit in real coordinates. This gives the measurable P.

At each infinite place write the normalized columns $y_v^{-1}f_a$ as their unnormalized orthogonal Gram–Schmidt columns times an upper unitriangular matrix. Read the matrix $T_{\mathrm{GS}}(y)$ back in the rational basis by multiplying entry a, j by $b_{v,f_a}/b_{v,f_j}$. Call y reduced if the strict upper entries, as tuples over the infinite places, lie in P. Replacing y by uy multiplies $T_{\mathrm{GS}}(y)$ on the right by the inverse basis matrix of u.

In fact such a column operation preserves the orthogonalized vectors. Right multiplication reduces each column in descending row order. The entry added in row i is a lattice shift plus terms determined by rows with larger index. The condition in P determines that shift uniquely. Thus the elements of U_c taking y to a reduced point form exactly one coset of U_c^0.

Multiply the reduced indicator by a nonnegative cutoff for U_c^0 whose translates sum to one. A measurable section exists by taking local sections of the discrete free left action on G_∞ over a countable cover of its quotient and partitioning by first chart. The product is β_c, with the orbit sum of Section 8 and the integral of Section 13.6.

Where β_c is nonzero, the normalized upper entries have modulus at most b^{26}. Indeed the basis scales have nonincreasing moduli in column order. The orthogonalized columns are invariant under the Bessel subgroup acting on the left. Indeed that subgroup fixes each successive line of the basis flag, with diagonal coefficients one. Each of those orthogonalized columns has norm at most the length $\ell_v(y_v)$ of Section 12.4. On any Bessel translate its norm is bounded by the original column norm, which is at most the operator norm of the translated matrix, since $\|g^{-1}\| = \|g\|$ in the normalized basis. Summing at most $n \le b$ contributions gives

$$\|y_v^{-1}f_j\| \le b^{27}\ell_v(y_v).$$

Put $H_v(y_v) = \max(1, \ell_v(y_v)/b^{6n})$ and $L(y) = \prod_v H_v(y_v)$. Fix c, s, y. For $R \ge 1$ and a term (c', γ) with $\beta_{c'}(\gamma^{-1}y) > 0$ and $L(\gamma^{-1}y) \le R$, the transported basis columns have norm at most $B_R = b^{37n}R$. For two such terms let $y' = \gamma_1^{-1}y$ and $g = \gamma_1^{-1}\gamma_2$, an integral transporter between their right classes reducing to one. If the maximum distance of the two tuples of columns is less than $(NB_R)^{-1}$, then $h(f_i, gf_j) = h(f_i, f_j)$ for every i, j. These pairings are integral on the class bases. At each v the normalized pairing scales the rational one by a number of modulus at least $1/N$. Pair $(y_v')^{-1}f_i$ with the difference of the two transported columns. The absolute value, after division by the scale, is bounded by NB_R times their distance. Every conjugate of the integral difference then has modulus less than one.

The equality of pairings gives zero dual and z coordinates of $gf_j - f_j$. Invariance of the norm and definiteness give zero X_0-component. Compare the remaining top coordinates in descending row order. Once the differences in the later rows vanish, a real unipotent moving y_v' to operator norm at most $b^{6n+2}R$ leaves the difference in the current row unchanged by unitriangularity. The modulus of that coefficient in the rational basis is therefore at most $b^{23\varrho}b^{6n+2}R$ times the column distance, since the reciprocal of the basis-scale ratio is at most $b^{23\varrho}$. The bound is less than one. The coefficient is integral because the class basis preserves the specified columns and the top coordinate functionals. Its field norm forces it to vanish. Thus $gF = F$ pointwise.

If $\gamma_1 F = \gamma_2 F$, then $g = \gamma_1^{-1}\gamma_2$ fixes F pointwise. Its action on $A^\perp/A = X_0 \oplus Ez$ fixes z, hence restricts to $a \in H(F_0)$ on X_0. Removing $\iota_{\mathrm{rat}}(a)$ leaves a basis-fixing unipotent. Write

d_1, d_2 for the right classes. The two class matrices $B_c^{-1}\gamma_{j,f}B_{d_j}$ have the same residue s, so their relative matrix $B_{d_1}^{-1}g_fB_{d_2}$ belongs to J_L. Its middle diagonal block is the glued matrix of $(d_1^\circ)^{-1}a_fd_2^\circ$. Hence

$$(d_1^\circ)^{-1}a_fd_2^\circ \in J_H$$

by Section 8.2. Thus $d_1 = d_2$. For the fixed representative of this class, Lemma 6.1 gives $a = 1$. Consequently $g \in U_{d_1}^0$. With γ_1 fixed, the terms with this basis image give distinct translations in $U_{d_1}^0$, so their cutoff weights sum to at most one by the orbit sum.

Distinct basis images give separated points in real dimension $2nrf$, inside a ball of radius $2B_R$ about one of the points. Disjoint balls of radius $(2NB_R)^{-1}$ bound their number by $(1 + 4NB_R^2)^{2nrf}$. Since $N \le b^{60n}$, this is at most $(b^{6n^2}R^2)^{2nrf}$ for large n. Writing $I = r^3f > 8nrf$ and integrating the layers against $IR^{-I-1}\,dR$ on $[1,\infty)$ gives

$$\sum_{c'}\sum_{\gamma\in\mathcal{T}(c,s,c')}\beta_{c'}(\gamma^{-1}y)L(\gamma^{-1}y)^{-I} \le 2b^{12n^3rf}. \tag{35}$$

Let $W = \bigotimes_v W_v$ and $W^\flat = \bigotimes_v F_v$ be the full and cut Bessel functions. Section 12.4 supplies majorants w_v with $A_v = \sup w_v$ and

$$H_v^{3I}\|W_v\| \le w_v, \qquad A_v\int_{U_v\setminus G_v} w_v \le b^{400r^2}|I_v|.$$

Put $w = \prod_v w_v$ and $A_* = \prod_v A_v$. The majorants are invariant under the corresponding real unipotent groups by the graph integral. For $W^\sharp = W, W^\flat$ write $K_\gamma^\sharp(y) = W^\sharp(y)\otimes\overline{W^\sharp(\gamma^{-1}y)}$. In orthonormal source coordinates this tensor is the matrix $W^\sharp(y)W^\sharp(\gamma^{-1}y)^*$, with

$$\|K_\gamma^\sharp(y)\|_{\mathrm{HS}} = \|W^\sharp(y)\|\,\|W^\sharp(\gamma^{-1}y)\|.$$

The kernels agree unless $L(y)L(\gamma^{-1}y) \ge b$. On the support of their difference, the majorants give

$$\|K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y)\|_{\mathrm{HS}} \le 2b^{-2I}w(y)A_*L(\gamma^{-1}y)^{-I}.$$

The same majorant with the factor $2b^{-2I}$ omitted bounds either kernel everywhere.

Fix a subset S_0 of the rational terms off the Bessel subgroup. Define a block by integrating $\beta_c(y)\beta_{c'}(\gamma^{-1}y)K_\gamma^\sharp(y)$ and summing over the terms of $\mathcal{T}(c,s,c')$ in S_0. Tonelli, (35), and the integral $\int_{U_\infty}\beta_c(uy)\,du = \nu$ show absolute convergence over all right classes and terms for fixed c, s. For the norm of the difference the same argument gives

$$4\nu b^{-2r^3f+12n^3rf+400r^2f}d_B \le \tfrac{1}{2}\nu b^{-k^3f}.$$

By Section 13.7, $d_B((k-1)!)^f \le b^{5kf/4}$, hence $d_B \le b^{2kf}$. From $\sqrt{x} \le m = 2^{\lceil\log_2\sqrt{x}\rceil} < 2\sqrt{x}$ we get $n = m/2 < \sqrt{x}$. Since $x^5 \le r$ and $k = r + 1 - m$, $12n^3 + 500 \le k < r$ for sufficiently large x. In particular

$$12n^3r + 400r^2 + 2k + 1 \le 402r^2 \le r^3, \qquad k^3 \le r^3.$$

The exponent after bounding d_B is therefore at most $-(k^3+1)f$. Since $4b^{-f} \le 1/2$ for $b \ge 8$ and $f \ge 1$, the stated bound holds for every subset of terms.

For fixed c, s, put $D_{c'} = A_{S_0}^{\mathrm{full}}(c,s,c') - A_{S_0}^{\mathrm{cut}}(c,s,c')$. The bound just obtained sums over all right classes, so

$$\sum_{c'}\|D_{c'}\|_{\mathrm{HS}}^2 \le \left(\sum_{c'}\|D_{c'}\|_{\mathrm{HS}}\right)^2 \le \tfrac{1}{4}\nu^2b^{-2k^3f}.$$

Freeness at J_H in Section 6 identifies X with one copy of the nonzero real coefficient space per source class. Thus $d_X = d_C d_B$. Since $d_B \geq 1$ and $|H_R| \geq 1$,

$$d_C \leq |H_R| d_C d_B = D_{\mathrm{id}}.$$

Since $|G| \leq M^{r^2 f} \leq b^{23 r^2 f}$, and $23r^2 f + 40 \leq 2k^3 f$, the following estimate holds.

Proposition 14.1 (Error from the cut integral)**.** *For every prescribed subset S_0 of the rational terms off the Bessel subgroup, the blocks formed from S_0 satisfy*

$$\nu^{-2} \sum_{c,c',s} \left\| A^{\mathrm{full}}_{S_0}(c,s,c') - A^{\mathrm{cut}}_{S_0}(c,s,c') \right\|^2_{\mathrm{HS}} \leq b^{-40} D_{\mathrm{id}}. \tag{36}$$

14.2 The residue estimates for the projection onto the selected types

In this subsection, we estimate the action of the finite idempotent of Section 7.3 on the level subgroups. The bounds supply the loss estimates for the diagonal and mixed moments in Section 9.

For each $q = p_i^{\mathrm{lev}}$, use the $\mathrm{GL}_r(\mathbb{F}_q)$-factors of G_i. On an algebraic closure of each residue field fix a faithful character $\jmath : \overline{\mathbb{F}}_q^{\times} \to \mathbb{C}^{\times}$. For $z \in \mathbb{F}_{q^a}^{\times}$ write $\jmath(z) = \exp(2\pi i b/(q^a - 1))$. Label z by the character $(\jmath|_{\mathbb{F}_{q^a}^{\times}})^b$. This is a bijection commuting with q-th powers, and extending the residue field pulls the character back by the norm. We use this choice again in Section 28.3.1. A local type parameter is a monic squarefree polynomial of degree r with nonzero constant term. It is allowed if no root in $\mathbb{F}_q$ is a q_2-th root of unity. A primitive type polynomial here is the minimal polynomial of a generator of $\mathbb{F}_{q^r}^{\times}$. It is allowed since $r > 1$. A tuple is selected if all its parameters are allowed and at least one is primitive. The original projector is the product of the two central idempotents h_i selecting these tuples, because the type polynomials are indexed by the same chosen places for both realizations.

For a squarefree type polynomial with distinct irreducible factors of degrees $d_1, \ldots, d_l$, the associated Green data have one box in each of the corresponding distinct cuspidal parameters. Green's character parametrization and degree formula [31, §8, Theorem 14 and Remark (1), pp. 443–444] give degree $\prod_{a=1}^{r}(q^a - 1)/\prod_{j=1}^{l}(q^{d_j} - 1)$. Distinct parameters give orthogonal central idempotents. The identity coefficient of each is its degree squared divided by the group order, namely

$$q^{-r} \frac{\prod_{a=1}^{r}(1 - q^{-a})}{\prod_j (1 - q^{-d_j})^2} \geq q^{-r}(1 - r/q) = u_q.$$

Use weight u_q for each monic degree-r polynomial to count the allowed parameters. Roots at zero or at one of the q_2-th roots exclude at most $(1 + q_2)q^{r-1}$ polynomials. A repeated monic factor of degree $j \geq 1$ excludes at most $q^j q^{r-2j} \leq q^{r-1}$ polynomials for each $j \leq r/2$.

Therefore the allowed weight is at least $1 - \epsilon_q$, with $\epsilon_q = (1 + q_2 + 2r)/q$. There are $\varphi(q^r - 1)/r$ primitive type polynomials, since all generators have degree r and Frobenius orbits of size r. If $L \geq 2$ has j distinct prime divisors, then

$$\frac{\varphi(L)}{L} \geq \prod_{a=1}^{j} \left(1 - \frac{1}{a+1}\right) = \frac{1}{1+j} \geq \frac{1}{1 + 2\log L}.$$

So the unweighted fraction of primitive polynomials is at least $\delta_q = 1/(2r(2r \log q + 1))$. Write A_q for the allowed parameters and $P_q \subset A_q$ for the primitive ones. Their comparison weights

satisfy

$$a_q = u_q|A_q| \geq 1 - \epsilon_q, \qquad c_q = u_q|A_q \setminus P_q| \leq q^{-r}(q^r - |P_q|) \leq 1 - \delta_q.$$

Each selected tuple has identity coefficient at least u_q^f. Thus $h_i(1) \geq a_q^f - c_q^f$, giving

$$(1 - h_i)(1) \leq f\epsilon_q + (1 - \delta_q)^f \leq f\epsilon_q + \exp(-f\delta_q).$$

For fixed q_0, d, ϵ, Section 4 gives

$$f = 3\varphi(q_0)L_0 m^{22} \asymp_{q_0} x^{11}\log x, \qquad q_2 = (q_0 d)^{s+6} \leq (q_0 d)^{28} x^{11\log_2(q_0 d)}.$$

Since $r < 3x^5$, $10B \leq \log q \leq 11B$ and $B \sim x^{\epsilon}$,

$$f\delta_q \gg_{q_0} x^{1-\epsilon}\log x \geq 9B, \qquad f\epsilon_q \leq f(1 + q_2 + 2r)b^{-10} = o(b^{-8})$$

for all sufficiently large x. It follows that

$$(1 - h_i)(1) \leq (2b^8)^{-1}. \tag{37}$$

For the Bessel trace on G_i, let ψ_U be the finite character of the integral unipotent subgroup on its reduction, extended by zero off that image. For two lifts of the same reduction the simple-root sums differ by an element of $M\widehat{\mathcal{O}}_E$, by the level character calculation of Section 8.3. After division by $M\delta_E$ the difference is in the inverse different, so this character is well-defined.

For a diagonal class term $\gamma = u\iota_{\mathrm{rat}}(\delta)$, put $y_H = (c^{\circ})^{-1}\delta_f c^{\circ}$. In the finite lattice basis

$$h_c^{-1}\gamma_L h_c = u_c \iota_L(y_H), \qquad u_c = h_c^{-1}L^{-1}u_f L h_c.$$

This transporter and its inverse are integral. Its middle block is that of $\iota_L(y_H)$, since the unipotent acts trivially on the middle quotient. The inverse gives the same assertion for y_H^{-1}. By the glued lattice calculation of Section 13.2.2, y_H, y_H^{-1} preserve the original lattice of X_0, also above N. For $s \in G_i \subset G_0 \times G_1$, the transporter has reduction s in the i-th factor and the identity in the other factor. Thus its middle block reduces to the identity at each chosen place over p_{1-i}^{lev}. At this prime the gluing basis change and its inverse are integral because N is a unit, so y_H also reduces to the identity there. Applied to the full integral definite level with this one principal congruence, Lemma 6.1 gives $\delta = 1$. Hence a diagonal term requires s in the reduced unipotent subgroup.

Conversely let k be an integral unipotent of reduction s. For each c, additive approximation in the rational unipotent group, using its Heisenberg vector and skew central coordinates, gives a rational u with $h_c^{-1}L^{-1}u_f L h_c \in kJ_L$. The class representative preserves the simple-root sum, so $\Psi_f(u) = \psi_U(s)$. By Section 13.2.1 the real covariance is $W(uy) = \Psi_\infty(u)W(y)$, with $\Psi_\infty(u) = \psi_U(s)^{-1}$. For the identity residue the diagonal Bessel amplitude is νI by the cutoff orbit sum and orthonormality of the real Bessel coordinates. For the lift of s its tensor is $W(y)W(u^{-1}y)^*$, so with A_{Bes} denoting the Bessel part of the full amplitude,

$$A_{\mathrm{Bes}}(c, s, c) = \nu\psi_U(s)^{-1}I, \qquad A_{\mathrm{Bes}}(c, s^{-1}, c) = \nu\psi_U(s)I.$$

The entrywise conjugation and division by ν in (1), summed over the d_C class blocks of size d_B, give $\operatorname{tr} r_\psi(s) = d_X\psi_U(s)$ for every $s \in G_i$.

Take T to consist of all places above M, and set $J_T = K_T(M)$, so $G = K_T/J_T$ and $\mu(K_T) = |G|\mu(J_T)$. For $k \in K_T$, writing $k = uj$ with $u \in U$, $j \in J_T$ forces $u \in K_T$. It extends

by the identity off T, and each global integral unipotent gives such a local lift. The definition on UJ_T and extension by zero therefore give $\chi(k) = \psi_U(\bar{k})$. If $\psi_U^\natural$ is the probability conjugacy average on G, then

$$\int_{K_T} \chi(yky^{-1})\,d\mu(y) = \mu(J_T) \sum_{t\in G} \psi_U(t\bar{k}t^{-1}) = \mu(K_T)\psi_U^\natural(\bar{k}).$$

Restrict the outer nonnegative integral in $\mathcal{M}(K_T)$ to K_T. Each reduction class has mass $\mu(J_T)$, so (10) with $Q_0 = 2$ and $t = n$ gives

$$\mu(J_T)\mu(K_T)^2 \sum_{s\in G} |\psi_U^\natural(s)|^2 \le \mathcal{M}(K_T) \le \mu(J_T)\mu(K_T)(\mu(K_T) + 2^{-n}).$$

Put $q_i = p_i^{\mathrm{lev}}$. The $2f$ places of E above each q_i form f conjugate pairs, each giving one split unitary factor. The calculation in the proof of Lemma 10.2 applies at either q_i, since N is a unit there. At each isotropic stage of rank l the primitive column and covector contribute $(1 - q_i^{-l})(1 - q_i^{-(l-1)})$. At the last stage, of rank $k+1$, the prescribed unit norm contributes $1 - q_i^{-(k+1)}$. Since $r = k + 2n - 1$, these factors use each exponent from $k+1$ through r once. The integral Heisenberg groups and the terminal compact each have mass one in Section 10.2. Thus in the Haar measure used for $\mathcal{M}(K_T)$,

$$\mu(K_T) = \prod_{i=0}^{1} \prod_{l=k+1}^{r} (1 - q_i^{-l})^f \ge 1 - f \sum_{i=0}^{1} \frac{q_i^{-(k+1)}}{1 - q_i^{-1}} \ge \tfrac{1}{2} \ge 2^{-n}$$

for all sufficiently large x, since $f \le x^{12}$ and $q_i \ge b^{10}$. Since G is the product of the two level factors, conjugacy averaging on G_i agrees with the restriction of averaging on G. Consequently the conjugacy average of $\operatorname{tr} r_\psi|_{G_i}$ has squared counting norm at most $2d_X^2$, as required in the diagonal estimate.

14.3 The weighted count at the additional ramified place

In this subsection, we bound the sublattices that occur in the fixed-vector contributions of Section 13.2. Their ramified level weights enter the count of pairs in Section 14.5.

Write $\mathcal{O}_f = \widehat{\mathcal{O}}_E$ and $\Lambda_c^0 = \Lambda_c \cap X_0(\mathbb{A}_f)$. A sublattice L of Λ_c^0 has rank $\dim_E V$ if it has finite index in $\Lambda_c^0 \cap V(\mathbb{A}_f)$ for an E-subspace $V \subset X_0$. For a finite idele a put $\mathrm{ind}(a) = |a|_f^{-1}$. The discriminant $d(L)$ is the square root of the index of the determinant of the Hermitian Gram matrix in a finite basis of L. It is independent of basis and at least one for an integral lattice. For $L \subseteq L'$ of index ι a basis change gives $d(L) = \iota d(L')$.

Put $\Delta_{\mathrm{lat}} = b^{n^3 f}$ and $\mathfrak{t} = b^{kf/4}$. For $0 \le l \le n^3$, the weighted sum satisfies

$$\sum_{c} \sum_{\substack{L\subseteq\Lambda_c^0 \\ \operatorname{rank} L = k-l,\ d(L)\le\Delta_{\mathrm{lat}}}} d(L)^{-k} \le \frac{d_C}{N^{kf}} \mathfrak{t}^{l+1}. \tag{38}$$

Use the group K_X of Section 13.6. Let K_N be its subgroup obtained by requiring the distinguished coordinate covector $e_{k_0}^\vee$ to be fixed modulo the idele π_N, where $[\widehat{\mathcal{O}}_E : \pi_N \widehat{\mathcal{O}}_E] = N^f$. The actual source level lies in K_N. The stabilizers for the source classes are trivial at the

principal level. Therefore each class has the same positive weight in quotient Haar measure. For any nonnegative function F_0 on the classes, a uniform bound

$$|K_X/K_N|^{-1} \sum_{\phi \in K_X/K_N} F_0([uy_\phi]) \le B_0 \quad (u \in U(X_0)(\mathbb{A}_f))$$

implies $\sum_c F_0(c) \le d_C B_0$. Indeed integrate the translate averages over the rational quotient, using right invariance of Haar. For unbounded F_0 the same argument first applies to its finite truncations.

For fixed u and ϕ, write the chosen representative as $\eta u y_\phi j_H$, with η rational unitary and $j_H \in J_H \subset K_N$. The class basis is $B_c = \iota_{\mathrm{rat}}(c^\circ)B$. By the glued intersection in Section 13.2.2,

$$\Lambda_c^0 = c^\circ\{v \in \mathcal{O}_f^k : v_{k_0} \in \pi_N \mathcal{O}_f\}, \qquad [c^\circ \mathcal{O}_f^k : \Lambda_c^0] = d(\Lambda_c^0) = N^f.$$

Indeed $c^\circ \mathcal{O}_f^k$ is unimodular and the distinguished coordinate quotient is $\mathcal{O}_f/\pi_N\mathcal{O}_f$. Consequently, with $U = u\mathcal{O}_f^k$,

$$\eta^{-1}\Lambda_c^0 = \{x \in U : e_{k_0}^\vee y_\phi^{-1} u^{-1} x \equiv 0 \pmod{\pi_N}\}.$$

Transport by η^{-1} preserves rational subspaces and discriminants. The map $yK_N \mapsto e_{k_0}^\vee y^{-1} \bmod \pi_N$ identifies K_X/K_N with the product of the nonzero covectors at the f places $w \mid \pi_N$, by split reduction and transitivity in Section 13.6. For a lattice $L \subset U$ of rank $h_0 = k - l$, put $r_w = \dim_{\mathbb{F}_N} \mathrm{im}(u^{-1}L_w \to \mathbb{F}_N^k)$. The covectors at w vanishing on L are the nonzero vectors of its annihilator. The fraction of admissible cosets is therefore

$$\frac{1}{[K_X : K_N]} \#\{\phi : L \subseteq \eta^{-1}\Lambda_c^0\} = \prod_{w \mid \pi_N} \frac{N^{k-r_w} - 1}{N^k - 1}.$$

It vanishes if $r_w = k$ at any chosen place. Unitarity fixes the action at the conjugate places by adjunction.

Let V be the rational subspace of L. Put $L^{\mathrm{sat}} = V(\mathbb{A}_f) \cap U$, $\iota = [L^{\mathrm{sat}} : L]$, and $d = d(L^{\mathrm{sat}})$. Thus $d(L) = d\iota$ and $d, \iota \ge 1$. At $w \mid \pi_N$, the saturated lattice is a direct summand of U_w. In Smith normal form for $L_w \subseteq L_w^{\mathrm{sat}}$, exactly $h_0 - r_w$ exponents are positive. Thus

$$\prod_{w \mid \pi_N} N^{h_0 - r_w} \le \iota, \qquad \prod_{w \mid \pi_N} (N^{k-r_w} - 1) \le N^{fl}\iota.$$

Put $\mu = 2\varrho + 4$. On $d\iota \le \Delta_{\mathrm{lat}}$, one has

$$\iota (d\iota)^{-k} \le \Delta_{\mathrm{lat}}^{\mu+2} d^{-(k+1)} \iota^{-(h_0+\mu)}.$$

Indeed the ratio to the right-hand power of d and ι is $d\iota^{h_0+\mu+1-k} \le (d\iota)^{\mu+2}$. The sum of $\iota^{-(h_0+\mu)}$ over finite-index sublattices in L^{sat} is bounded by $\zeta_E(\mu)^{h_0}$. Locally, triangular echelon bases with diagonal ideals $\mathfrak{p}^{a_1}, \dots, \mathfrak{p}^{a_{h_0}}$ have at most $q^{\sum_{j=1}^{h_0}(h_0-j)a_j}$ choices of off-diagonal entries, with q the residue cardinality. Multiplying the resulting geometric series gives factors $\zeta_E(h_0 + \mu - j)$ with $0 \le j \le h_0 - 1$, all bounded by $\zeta_E(\mu)$. The bound $n[E : \mathbf{Q}] \le 2^{2\varrho}$ and $\zeta_E(\mu) \le \zeta(\mu)^{[E:\mathbf{Q}]}$ yield $\zeta_E(\mu) \le 2$; for the latter use $\log \zeta(\mu) \le (1 + 2/(\mu - 1))2^{-\mu}$. Thus the index sum is at most 2^k.

Write $\operatorname{rd}(E) = |D_E|^{1/(2f)}$, where D_E is the field discriminant. The subspace sum is

$$\sum_{\operatorname{codim} V = l} d(V(\mathbb{A}_f) \cap U)^{-(k+1)} \le (200 f k \operatorname{rd}(E))^{f(k+1)l}. \tag{39}$$

For $l = 0$, the only subspace is X_0, its saturation in U is U, and $d(U) = 1$. Its perpendicular lattice is zero with empty Gram determinant one. For $l > 0$, the primitive lattices of V and $V^\perp$ in U have the same discriminant. In fact, local unimodularity identifies the quotient by their direct sum with either discriminant group. The pairing from U onto the dual of the first primitive lattice is surjective because a functional on a direct summand extends to U. Its kernel is the second primitive lattice. The image of the first lattice is its own pairing lattice in the dual. Interchange the two lattices to get the other identification. Their discriminants, defined as the square roots of the orders of these groups, are therefore equal.

Under one embedding from each complex pair, with the standard Euclidean norm at the infinite places, the real covolume of $V^\perp(E) \cap U$ in its span is

$$2^{-fl} |D_E|^{l/2} d(V(\mathbb{A}_f) \cap U).$$

Choose a rational basis matrix Y of $V^\perp$ and a finite coefficient matrix D so that YD is an adelic lattice basis. The rational coefficient lattice has real covolume $v_E^{-l} \operatorname{ind}(\det D)$, by comparison with a common integral sublattice.

At v, the real Jacobian of Y in the positive definite coordinates is $\det(\Phi_v(Y)^* \Phi_v(Y))$. The product of these Jacobians times $\operatorname{ind}(\det D)$ is $d(V(\mathbb{A}_f) \cap U)$, by the product formula for $\det(Y^*Y)$. Minkowski's second theorem [21, Ch. VIII, §4.3, Theorem V] in dimension $2fl$, using the unit Euclidean ball, gives independent E-vectors $x_1, \dots, x_l$ in this lattice such that

$$(\|x_1\| \cdots \|x_l\|)^{2f} \le (8 f l \operatorname{rd}(E))^{fl} d(V(\mathbb{A}_f) \cap U).$$

Let λ_h be the real Euclidean successive minima. After choosing $a - 1$ independent E-vectors, their E-span has real span of dimension $2f(a-1)$. Among $2f(a-1)+1$ real-independent lattice vectors of lengths at most $\lambda_{2f(a-1)+1}$, choose x_a outside that span. Then $\|x_a\|^{2f} \le \prod_{h=2f(a-1)+1}^{2fa} \lambda_h$.

Multiplying over a gives $(\prod_a \|x_a\|)^{2f} \le \prod_{h=1}^{2fl} \lambda_h$. The ball contains the cube of side $2/\sqrt{2fl}$, so Minkowski's upper bound with the stated covolume already gives the displayed inequality with room in its constant.

For distinct $x, y \in E^k \cap U$, the hermitian square $h(x-y, x-y)$ is a totally positive algebraic integer by unitary integrality at the finite places. Its norm is a positive integer, so at least one real conjugate is at least one. The diagonal archimedean images of x and y therefore have distance at least one. The packing bound in real dimension $d_0 = 2fk$ gives at most $(1 + 2R)^{d_0}$ lattice vectors of length at most R. Dyadic summation at exponent $2f(k+1) = d_0 + 2f$ then gives

$$\sum_{x \ne 0} \|x\|^{-2f(k+1)} \le 5^{d_0} \sum_{j \ge 0} 2^{-2fj} \le 5^{2f(k+1)}.$$

The chosen short tuple determines V as its orthogonal complement. Raising the short-basis bound to the $(k+1)$-st power and summing over tuples gives (39), since $8 \cdot 5^2 = 200$.

The unaveraged sum over ϕ and L is consequently at most

$$\Delta_{\text{lat}}^{\mu+2} 2^k \left(N^f (200 f k \operatorname{rd}(E))^{f(k+1)} \right)^l \le b^{-f} \mathfrak{t}^{l+1}.$$

For the last inequality use $k \le f$, $k/4 \ge 10^5 n^9$, $N \le b^{60n}$, $(200fk\,\mathrm{rd}(E))^{1000} \le b$, and $60n \cdot 1000 + (k+1) \le 250k$. In particular, $\mathrm{rd}(E) \le dq_1$ by Section 4. Together with $f, q_1 \le x^{12}$ and $k < 3x^5$, this gives $200fk\ \mathrm{rd}(E) = O_d(x^{29})$. By the class count in Section 13.6, $[K_X : K_N] = (N^k - 1)^f \ge (N^k/b)^f$. Dividing by this index and applying the Haar average gives (38).

14.4 Counting orbital representatives containing a fixed lattice

In this subsection, we count bases adapted to the isotropic flag and its final orthogonal vector inside a fixed lattice. Organizing them by their pairing matrices gives the multiplicity bound used in Section 14.5 with the representatives of Section 14.1.

In the rational Witt decomposition write A for the isotropic span of $e_0, \dots, e_{\varrho-1}$ and take $F = (e_0, \dots, e_{\varrho-1}, z)$ as a matrix of columns. Write $Z_\gamma = F^* h \gamma F$ for the matrix of their products and P_γ for the submatrix of its first ϱ rows. Here h is the matrix of the rational Hermitian form. Use the same reduced cutoffs as in Section 14.1. A boxed point is reduced and has $\ell_v \le b^{6n+2}$ at every place. A term γ is near if y and $\gamma^{-1}y$ are boxed for some y. For an orbital calculation one fixes a representative of each U_c^0-$U_{c'}^0$ double coset, choosing a near representative when there is one.

Let δ_a be the leading minor of order a in Z_γ, with $\delta_0 = 1$, and put

$$\mathfrak{g}_\gamma = \delta_n \overline{\delta_n} - N^2 \delta_{n-1} \overline{\delta_{n-1}}.$$

We say that γ has cell rank j, $0 \le j \le n$, if $\delta_j \ne 0$, if $\mathfrak{g}_\gamma \ne 0$ is equivalent to $j = n$, and if there exists $\zeta \in E$ with $\zeta\bar{\zeta} = 1$ such that the following holds for every $j \le a < n$. The spans of the first $a+1$ columns of F and γF contain a common vector whose two coefficients at index a are ζ and 1, respectively. Section 15.2 obtains these conditions for the terms used there.

For $\gamma \in \mathcal{T}(c, s, c')$, define its term lattice L_γ to consist of $x \in \Lambda_c^0$ for which some lift $x + Mw$, $w \in A(\mathcal{O}_f)$, is orthogonal to γF. In the rational coordinates this means $X_\gamma^* x + M P_\gamma^* w = 0$ for some integral w, with X_γ the definite block of γF. Choose representatives a_{sH_R} for the cosets sH_R in G, and define $t \in H_R$ by $s = a_{sH_R} t$. This is the parameter of the term.

Fix the left source class c, the Bessel parameter t, and the lattice L of rank $h_0 = k - l$ in Λ_c^0, with discriminant at most Δ_{lat}. A term counted here is a pair (c', γ), where γ is the fixed near representative, has a cell rank, and has L in its level-M term lattice. All moves in the proof are in U_c^0, the rational unipotents of level M in class c fixing the standard basis pointwise.

Suppose the moved rational representatives g_1, g_2 satisfy $g_1 F = g_2 F$. Write d_1, d_2 for their right classes and s_1, s_2 for their residues. The rational isometry $m = g_1^{-1} g_2$ fixes F pointwise. On $A^\perp/A$ it fixes z and restricts to an isometry of X_0; removing this isometry leaves a basis-fixing unipotent. Thus m is a rational Bessel term between the two right classes, with residue

$$\mathrm{red}(B_{d_1}^{-1} m_f B_{d_2}) = s_1^{-1} s_2 \in H_R$$

by the integral Bessel calculation following Proposition 8.1. Hence $s_1 H_R = s_2 H_R$. Writing $s_i = a_{s_i H_R} t$ with the common parameter gives $s_1 = s_2$. The relative transporter is now in the principal level. Its middle block identifies $d_1 = d_2$ at the original source level, and Lemma 6.1 makes its rational definite component the identity. Thus $m \in U_{d_1}^0$ as in Section 14.1. The original representatives belong to the same double coset and hence coincide.

We partition the terms by cell rank j and integral basis product matrix $Z = (\langle f_a, \gamma f_b \rangle)$, where $f_0, \dots, f_{n-1}$ is the standard basis. There are at most $n + 1 \le b^{2f}$ cell ranks. The entries

of Z are integral because the two bases are integral in the class basis, whose Gram form is integral. Their absolute values are bounded by $Nb^{74n} \le b^{134n}$ at all complex places.

Indeed both normalized basis columns have norm at most b^{37n} by the box bound, and the product of the two basis scales has modulus at least $1/N$. The lattice of integral elements of E in the product of the f complex embeddings has point separation at least one in the maximum norm, by the norm formula for a nonzero algebraic integer. Disjoint products of discs of radius $1/2$ show that there are at most $(1 + 2b^{134n})^{2f}$ values for each entry. Since $1 + 2b^{134n} \le b^{135n}$, the number of matrices is at most $b^{270n^3 f}$.

Fix j, Z, let Y be the matrix formed by the first j rows of Z and the remaining rows of the identity, and put $\delta = \det Y \neq 0$. Write P for the submatrix of the first ϱ rows of Z. For a term let X_γ be the X_0-block of its basis. In the splitting $A \oplus (X_0 \perp Ez) \oplus A^\vee$, the block along $A^\vee$ is P, and Z also determines the z-block by pairing with z of norm $-N$. These statements are in the rational basis. In the finite calculation both transported bases lie in the same left class lattice.

14.4.1 Changes of basis on the two blocks

In this subsection, we put the two blocks into the coordinates needed for the count of Section 14.4.

For a matrix η of rational columns in Λ_c^0, a move takes

$$X_\gamma \longmapsto X_\gamma - M\eta P. \tag{40}$$

To check it in the indicated splitting, let $T = -M\eta : A^\vee \to X_0 \perp Ez$, and let J_0 be the middle Gram matrix. Choose an integral scalar θ with $\theta + \bar{\theta} = 1$, for example a primitive sixth root in E. The block unipotent

$$\begin{pmatrix} 1 & -T^* J_0 & -\theta T^* J_0 T \\ 0 & 1 & T \\ 0 & 0 & 1 \end{pmatrix}$$

preserves the form, fixes A and z, and is congruent to the identity on Λ_c modulo M. For the congruence, use integrality of the column vectors of η in the middle lattice and of the pairings there. Its middle action on the basis gives (40). Also an integral skew-Hermitian matrix σ yields the move with upper-right block $M\sigma$ and other off-diagonal blocks zero. It changes the rows of the basis matrix indexed by A by adding $M\sigma P$ and leaves the other rows fixed. Both moves retain the original residue and the cell rank.

14.4.2 Lifts of the fixed lattice

In this subsection, we bound the number of lifts after the changes of basis in Section 14.4.1.

Write a finite basis of L as $m = ya$, with y a rational matrix of independent columns and a invertible over the finite adeles. From containment in the term lattice we have a rational matrix w with wa integral and

$$X_\gamma^* y + MP^* w = 0. \tag{41}$$

The defining containment solves this equation over the finite adeles with wa integral. A linear system over E with a finite-adelic solution has a rational solution. Its solutions form an affine space over the rational kernel, whose rational points are dense in the finite-adelic solutions by

additive approximation. The condition that wa be integral is an open neighbourhood of the given solution. This establishes (41) with its original lattice.

Sort the ϱ rows of wa by their classes modulo the lattice generated by the rows of m^*m. The number of classes is at most $d(L)^{2\varrho}$, since the Gram determinant has index $d(L)^2$. In one class, choose a base lift w_0. Each $w_0 - w$ is $\eta^* y$ for integral columns η in the lattice, given by

$$\eta = y(y^*y)^{-1}(w_0 - w)^*.$$

The row congruences mean $(w_0 - w)a = S^*m^*m$ with S integral. Cancelling a shows $\eta = mS$ over the adeles, so its columns indeed lie in L. Equation (40) makes (41) hold with the common w_0. All moved bases hence have the same matrix of products of their X_0-block with y.

14.4.3 Counting definite columns

In this subsection, we bound the definite columns for a lift chosen in Section 14.4.2.

For a fixed lift, the differences between two X_0-blocks lie columnwise in the lattice $L_c^\perp = \Lambda_c^0 \cap V(\mathbb{A}_f)^\perp$, of rank l, with V the rational span of L. For each column, write its rational middle coordinates as $x + \lambda z$ and put $v = (c^\circ)^{-1}x$. If the column is γf_b, then $\lambda = -Z_{n-1,b}/N$ by pairing with z. Thus the two rational coefficients of z agree before passing to class coordinates. The columns of F are unchanged in each right class basis, so $B_c^{-1}\gamma F$ is integral. Its middle coordinates are

$$\big((v_i)_{i \neq k_0},\ \bar{\pi}_N v_{k_0},\ \lambda - v_{k_0}/\pi_N\big).$$

Subtracting gives integral differences in the other coordinates and a difference in $\pi_N \mathcal{O}_f$ at k_0. Hence the difference of the two x's lies in Λ_c^0. Their difference pairs to zero with L by (41) with the common w_0. Taking residues modulo $M\delta L_c^\perp$ contributes at most $\operatorname{ind}(M\delta)^{ln}$.

In a residue class let $v = X_{\gamma'} - X_\gamma$ be the difference. The matrix vY^{-1}/M has columns integral in Λ_c^0, since v is divisible there by $M\delta$ and $\operatorname{adj} Y$ is integral. Its columns at positions $b \geq j$ vanish because the cell relations supply a shared vector at each such position, with translated leading coefficient one, and with higher coefficients zero. Their translated coefficients in the tail can be combined triangularly to give tail e_b. The first j products against the standard basis vanish, so the complete coefficient vector is $Y^{-1}e_b$. Its image lies in the span of the columns of F and hence has zero X_0-component. The same argument applies after a move, since the move fixes the shared vectors.

The remaining exceptional position is the last column in the top cell, where $Y = Z$. The vector $\gamma F Z^{-1} e_{n-1}$ is orthogonal to A and has z-coordinate $-1/N$. Since it is in the translated span of the basis vectors its Hermitian norm is nonpositive. The squared norm of its X_0-part is thus at most $1/N$ at each complex place. Their difference w is the last column of vY^{-1}, hence a rational vector integral in the class lattice. Its Hermitian square $h(w, w)$ belongs to $\mathcal{O}_{F_0}$ and is at most $4/N$ at every real embedding. If $w \neq 0$, definiteness gives

$$1 \leq N_{F_0/\mathbb{Q}}\big(h(w, w)\big) \leq (4/N)^f < 1.$$

Thus this column vanishes. The same proof works on the moved bases and uses only the cell-rank and basis identities.

Consequently vY^{-1}/M has support only in the first $\min(j, \varrho)$ columns. Multiplication by Y writes $v = M\eta P$ with integral η obtained from these columns. Equation (40) therefore moves all the bases in the residue class to a common X_0-block.

14.4.4 Counting isotropic columns

In this subsection, we complete the count using the bound for the definite columns in Section 14.4.3.

First sort by the matrix of shared vectors, meaning the A-coordinates of the shared columns of $\gamma F \operatorname{adj} Y$ at positions $b \geq j$ and zero at the other positions. These coordinates are unchanged by moves and integral because the coordinate functionals for the isotropic block in the class basis are integral and agree with those in the rational basis. There are at most $b^{366n^4 f}$ possible matrices. Here is the bound on their entries. At each complex place the transported chosen basis at a boxed point has column lengths at most b^{37n}. A Bessel unipotent makes the representative of that point of norm at most b^{7n}, by the definition of the Bessel length and the boxed bound b^{6n+2}.

Its action on the standard basis is unitriangular. The entries of that matrix are at most b^{44n}, since it is obtained by multiplying the transported basis by an operator of norm at most b^{7n}. Cramer's rule and the minor bound give, for any linear combination in the boxed basis, each coefficient bounded by b^{46n^2} times the length of the combination. The exponent is bounded by $1 + (2 + 44n)\varrho + 7n \leq 46n^2$. Apply this to the shared vector at the first point, whose expression in the other boxed basis uses a column of $\operatorname{adj} Y$.

Its coefficients have absolute values at most $b^{(2+134n)\varrho}$ before converting scales. The ratios of any two chosen basis scale factors are at most $b^{23\varrho}$, because $1 \leq |\kappa_v| \leq b^{23}$. Thus each shared coordinate has absolute value at most

$$b^{46n^2 + (2+134n)\varrho + 1 + 23\varrho + 37n} \leq b^{182n^2}.$$

Packing the integral coordinates in ϱn positions as for Z gives $(1 + 2b^{182n^2})^{2f\varrho n} \leq b^{366n^4 f}$.

For one shared matrix and one X_0-block, sort the remaining ϱn isotropic coordinates modulo $M\delta\bar{\delta}$, costing $\operatorname{ind}(M\delta\bar{\delta})^{\varrho n}$. Within this congruence class the two bases differ only in the isotropic block, by $\xi = M\delta\bar{\delta}\,\mu$ for integral μ. Since their Gram matrices agree and the difference lies in the isotropic space, one has $\mu^* P = -P^* \mu$. Put α equal to the first ϱ columns of $\operatorname{adj} Y$. The columns at $b \geq j$ of $\mu \operatorname{adj} Y$ vanish by equality of shared vectors. In the top cell its last column vanishes because $P \operatorname{adj} Y\, e_{n-1} = 0$, P^* is injective, and $P^*\mu = -\mu^* P$. Consequently $\mu\alpha P = \delta\mu$.

An integral skew-Hermitian solution is

$$\sigma = \bar{\delta}\mu\alpha - \delta(\mu\alpha)^* - \alpha^* P^* \mu\alpha.$$

The displayed skew relation gives $\sigma^* = -\sigma$; multiplying by P cancels the last two terms and gives $\sigma P = \delta\bar{\delta}\mu$. The central move with parameter $M\sigma$ therefore joins the two bases. Separation then identifies their terms. Neither this correction nor the previous two moves alter the level or the residue parameter.

All the moduli and Gram indices are positive and satisfy

$$d(L)^2, \quad \operatorname{ind}(M\delta), \quad \operatorname{ind}(M\delta\bar{\delta}) \leq b^{2n^3 f}.$$

The first inequality is the discriminant cutoff. The other two follow from $M \leq b^{23}$ and the squared minor bound b^{270n^2} at all complex places, using $n \geq 10^5$. The counts of ranks, product matrices and three successive rounds thus give the upper bound

$$b^{\{2+270n^3+366n^4+2n^3(\varrho+ln+\varrho n)\}f} \leq b^{2000n^6(l+1)f}.$$

Lemma 14.2 (Number of representatives containing a fixed lattice)**.** *Fix c, t and a lattice L of corank l with $d(L) \le \Delta_{\mathrm{lat}}$. Count the pairs (c', γ) over (c, t) for which γ has a cell rank. Then*

$$\#\{(c', \gamma) : \gamma \text{ is a chosen near representative, } L \subseteq L_\gamma\} \le b^{2000 n^6 (l+1) f}.$$

14.5 The weighted sum over nearby pairs

In this subsection, we reduce the weighted count of two orbital terms to a count of lattices in their common subspace. Section 14.3 supplies the local weights, and Section 14.7 bounds the relevant discriminants.

For a term γ, its link consists of pairs of middle classes of w and $\gamma^{-1} w$, where $w \in A^{\perp} \cap (\gamma A)^{\perp}$. For two terms γ, γ' over a common block, let $V_{12} \subseteq X_0$ consist of the first coordinates of the common link pairs with both coordinates in X_0. Intersect $V_{12}(\mathbb{A}_f) \cap \Lambda_c^0$ with the two term lattices to get L_{12}. Fix $0 \le l \le n^3$. Sum over ordered pairs of chosen near representatives, each having a cell rank, in the same block (c, s, c'). Retain those for which L_{12} has rank $k - l$ in V_{12} and $d(L_{12}) \le \Delta_{\mathrm{lat}}$. Section 14.7 shows that these coranks and discriminants cover all such pairs over the agreeing residue s.

Fix c and group such pairs by their common parameter t and lattice L_{12}. Both representatives have that parameter and contain the lattice in their respective term lattices. The assignment to the ordered pair of terms $((c', \gamma), (c', \gamma'))$ is injective, since for those classes the first term determines the residue. By Lemma 14.2 there are at most $b^{4000 n^6 (l+1) f} \le \mathfrak{t}$ ordered pairs for each fixed c, t, L_{12}. Indeed $l \le n^3$ and $k / n^9 \to \infty$. Every pair in that group has weight $d(L_{12})^{-k}$. Sum over the lattices, parameters, and left classes and use (38). With $n_p = |H_R| d_C / N^{kf}$, this gives

$$\sum_{\substack{\text{such ordered pairs} \\ \operatorname{corank} L_{12} = l}} d(L_{12})^{-k} \le n_p \mathfrak{t}^{l+2}. \tag{42}$$

The factor $|H_R|$ counts parameters. There is no additional multiplicity from the residues.

14.6 Divisibility of the pairing determinant

In this subsection, we deduce divisibility of a determinant of pairings with the two final vectors from agreement of the finite characters. The resulting product of residue norms enters the discriminant estimate of Section 14.7.

A residue s agrees if $\psi_U(u) = \psi_U(u')$ whenever u, u' are reduced integral unipotents with $us = su'$, using the character of Section 14.2. Consider one place above M, allowing either member of each conjugate pair. At the residue field at this place, write g for the reduction of the original transporter, and retain F for the reduced basis. Suppose $Fx = gFy$ for coefficient vectors supported through index $j < \varrho$.

Let m be any covector annihilating both basis initial segments through j. Then the two matrices

$$1 + t(Fx)m, \qquad 1 + t(Fy)mg$$

are reductions of integral unipotents, for every residue scalar t. To construct the lifts in the original lattice basis, lift the basis coefficients and the covectors m, mg supported at this one place. The covectors may be chosen to annihilate the corresponding standard initial segment, since it consists of basis vectors.

The integral Gram identifies the covectors with Hermitian products against integral vectors. Use the same Eichler formula as in Section 13.6, with the lifted isotropic vector replacing e_a and level factor one. It applies because the covector kills the initial segment. The map is upper unitriangular in the Witt order since the vector is supported through j and the covector vanishes there. At the chosen place the term using the conjugate of the lifted isotropic vector vanishes.

Outside this conjugate pair the matrix is the identity. The opposite reduction is forced by unitary adjunction. There the correction kills the basis, because the conjugate pairing of the isotropic vector with the basis is zero. Thus it contributes no simple-root entry. Thus the reduced unipotents are intertwined by s.

The simple-root sums of the two reductions at the test place are $tx_j m(f_{j+1})$ and $ty_j m(gf_{j+1})$, since the covector kills the earlier initial segment on each side. The local additive character on this root is nontrivial, by the conductor of the different-normalized character and the factor M^{-1}. Testing every t in the agreement condition gives

$$x_j m(f_{j+1}) = y_j m(gf_{j+1}).$$

Therefore $x_j f_{j+1} - y_j g f_{j+1}$ belongs to the sum of the two earlier spans, by the annihilator criterion. Moving that linear combination to the other side gives a common vector at index $j+1$ with last coefficient pair (x_j, y_j). Starting at the highest index of any nonzero common vector of the full bases and iterating this argument gives a relation with a nonzero pair of coefficients in the final positions.

Let T have two columns in the original integral left lattice orthogonal to both isotropic spans $A, \gamma A$, and apply the last two pairings with $z, \gamma z$ to these columns. Suppose the resulting two-by-two determinant is invertible over the finite adeles. At each meeting place, the preceding relation gives a nonzero kernel vector for this matrix after taking its adjoint and reducing, since all its other basis pairings vanish. If D denotes the two-column pairing matrix in the integral lattice coordinates, then $v_w(\det D^*) = v_{\bar{w}}(\det D) \geq 1$ at every meeting place w. Thus the ideal of $\det D$ is divisible by $\prod_{w \text{ meeting}} \bar{\mathfrak{p}}_w$. The places over M are split, and conjugation permutes them and preserves their residue norms. Multiplying the local contributions gives $\mathrm{ind}(\det D) \geq Q_{\cap}(s)$, with each meeting place counted once. The integral Gram matrix is included when taking these pairings in a lattice basis.

14.7 The discriminant bound for the pair lattice

In this subsection, we compare the discriminant of the lattice common to two orbital terms with the level and intersection determinants. The calculation of Section 14.6 supplies the local divisibility used in Section 14.5.

Use the rational spaces $A, \gamma A$ and the left lattice Λ_c throughout. Equivalently one may carry all three through the original left class basis. A term over (c, s, c') sends $\Lambda_{c'}$ to Λ_c. Set $I = A \cap \gamma A$. The form on $A^{\perp}/A$ has middle space $X_0 \perp Ez$. For the two middle coordinates of a link pair, exchange their z-coordinates to obtain two vectors in the positive space $X_0 \perp Ez^+$, where z^+ has norm N.

The difference of the two middle forms becomes the difference of the two positive forms. Both middle coordinates come from one lift, so this difference is zero on the link. Its dimension is $k+1$, as follows by taking the orthogonal complement of $A + \gamma A$ and quotienting by I. Definiteness makes both coordinate projections of the link injective. It is therefore the graph

of an isometry Υ_γ of the positive space. Its positive metric is the positive form of the first coordinate, or equivalently that of the second.

The calculations below use the cell rank j in the original basis. Put $j_0 = \min(j, \varrho)$ and denote the leading minor of order a of the basis product matrix by δ_a. Modulo I, the first j_0 isotropic basis columns are a basis of each isotropic basis space. Indeed their pairing is the leading j_0-minor below the top rank, and the common-vector flag relations supply all the other columns modulo I. At the top rank the full basis product matrix is invertible. Its first ϱ rows and first ϱ columns are each independent. A vector in I is orthogonal to both full bases, so $I = 0$. This argument also covers $\delta_\varrho = 0$. Thus a top cell of rank n can have a vanishing proper minor. The estimate for open residues in Section 13.8 uses the additional condition of Section 13.1. Set

$$\mu_\gamma = \frac{|N_{E/\mathbf{Q}}(\delta_j)|}{N^{2f(j-\varrho)_+}}.$$

Let t_γ be the index of $(\Lambda_c \cap A) + \gamma(\Lambda_{c'} \cap A)$ in $\Lambda_c \cap (A + \gamma A)$.

Let i_γ, i'_γ be the indices of the pairing lattices of the first j_0 isotropic columns inside the pairing lattices of all isotropic columns. Use respectively $\Lambda_c \cap A$ paired with γF, and $\gamma(\Lambda_{c'} \cap A)$ paired with F. Define the arithmetic height by

$$G_\gamma = \frac{M^{2fj_0}}{i_\gamma i'_\gamma} \max\{N^f \mu_\gamma, \mathfrak{g}_{\mathrm{norm}}(\gamma)\}, \qquad \mathfrak{g}_{\mathrm{norm}}(\gamma) = \prod_{v|\infty} |\mathfrak{g}(\gamma)|_v. \tag{43}$$

Here $|\cdot|_v$ is the complex modulus at one embedding in a conjugate pair. Below top rank the discriminant term vanishes, and at top rank both first indices are one. For a near term both transported real bases are bounded by b^{37n} in the box.

Removing the real basis scaling gives a basis product bound $Nb^{74n} \le b^{134n}$, since its other scaling factor is a nonnegative power of $|\kappa_v| \ge 1$. Expanding a minor and using $n \le b$ bounds each squared modulus by b^{270n^2}. For the discriminant, put $a = \mathrm{adj}(Z_\gamma)e_{n-1}$. The vector $\gamma F a$ has zero dual-isotropic coordinates and z-coordinate $-\delta_n/N$. Its norm equals that of Fa, namely $-N|\delta_{n-1}|^2$ at each complex place. Hence $\mathfrak{g} = \delta_n\bar{\delta}_n - N^2\delta_{n-1}\bar{\delta}_{n-1}$ is N times the squared norm of its X_0-component, and is at most the squared full minor. This holds also when Z_γ is singular. At top rank with $\delta_\varrho = 0$ and $\delta_n \ne 0$,

$$\mathfrak{g}_{\mathrm{norm}} = |N_{E/\mathbf{Q}}(\delta_n)| = N^{2f}\mu_\gamma > N^f\mu_\gamma.$$

This term therefore uses the dominant bound of Section 15.5. If $\mathfrak{g} \ne 0$, it is a totally positive integer of F_0. Hence on the near set

$$1 \le N_{F_0/\mathbf{Q}}(\mathfrak{g}), \qquad b^{-270n^2(f-1)} \le \Phi_v(\mathfrak{g}) \le b^{270n^2} \quad (v \mid \infty).$$

The lower bound follows by bounding the other $f-1$ conjugates. On the open rational cell $t = \delta_n/\delta_{n-1}$. Dividing by $|\Phi_v(\delta_{n-1})|^2 \le b^{270n^2}$ gives $|\Phi_v(t)|^2 - N^2 \ge b^{-270n^2 f}$. Equation (43), $M \le b^{23}$, and $N \le b^{60n}$ now give $0 < G_\gamma \le b^{n^3 f}$, using $n \ge 10^5$.

14.7.1 The lattice associated with one link

In this subsection, we describe the lattice for each link of Section 14.7.

Use the unimodular quotient lattice

$$Q = (\Lambda_c \cap I^\perp)/(\Lambda_c \cap I).$$

This is unimodular because $I \cap \Lambda_c$ is a primitive isotropic direct summand. Its pairing with Λ_c surjects onto its dual. In a basis consisting of it, a basis of its orthogonal quotient, and lifts dual to it, the determinant of the full form is a unit times the determinant on the quotient. All these bases may be taken place by place over the valuation rings. Let F_1 be the matrix of the two lists of the first j_0 isotropic columns in Q, and $p = (z, \gamma z)$ read in this quotient.

The classes of the integral lifts of the link form the primitive orthogonal lattice to F_1 in Q. In fact a lift is orthogonal to $A + \gamma A$, and the list F_1 spans the image of that space in the quotient, with the deleted kernel I. Under this lift map, the link metric is the quotient Gram form J augmented by $N^{-1} J p p^* J$, since the two additional squares are those of the two values of the z-form divided by N.

We give the determinant comparison with this augmentation. Write $F_0 = F_1 a_0$ for a basis of the saturation of F_1 in Q, $I_F = [\mathrm{sat}(F_1) : F_1]$, and v for a basis of the primitive orthogonal lattice. Choose an integral right inverse Q_0 to $F_0^* J$, so (Q_0, v) is integrally invertible. In the ambient space with Gram $\widetilde{J} = \mathrm{diag}(J, N I_2)$, let $W = \begin{pmatrix} F_1 & p \\ 0 & I_2 \end{pmatrix}$ and let $v_1 = \binom{v}{-N^{-1} p^* J v}$. Then $W^* \widetilde{J} v_1 = 0$, and $v_1^* \widetilde{J} v_1 = v^* (J + N^{-1} J p p^* J) v$. A right inverse to $W^* \widetilde{J}$ is

$$Q_1 = \begin{pmatrix} Q_0 a_0^* & 0 \\ -N^{-1} p^* J Q_0 a_0^* & N^{-1} I_2 \end{pmatrix}.$$

In the basis (Q_1, v_1) the first coordinate block of W is $W^* \widetilde{J} W$.

Put $C = (Q_1, v_1)$, $G_1 = W^* \widetilde{J} W$. Since $W^* \widetilde{J} C = (1, 0)$, the matrix $C^{-1}(W, v_1)$ is triangular by blocks with diagonal blocks $G_1, 1$. Orthogonality gives

$$\det(G_1) \det(v_1^* \widetilde{J} v_1) = \det(\widetilde{J}) \overline{\det C} \det C \, \det(G_1)^2.$$

The bordered determinant computed below is nonzero in each cell. Cancelling $\det G_1$, taking indices, and using unimodularity of J yields

$$\mathrm{ind}(\det(v_1^* \widetilde{J} v_1)) I_F^2 \, \mathrm{ind}(N)^2 = \mathrm{ind}(\det(W^* \widetilde{J} W)). \tag{44}$$

Indeed $\det(Q_1, v_1)$ up to sign is $\det(Q_0, v) \det a_0^* N^{-2}$, and the first determinant has index one. More precisely, put $s_w = \mathrm{length}_{\mathcal{O}_{E,w}}(\mathrm{sat}(F_1)_w / (F_1)_w)$, where $(F_1)_w$ denotes the column lattice, and $n_w = v_w(N)$. Then $v_w(\det a_0) = -s_w$ and $v_w(\det a_0^*) = -s_{\bar{w}}$, so

$$v_w(\det(v_1^* \widetilde{J} v_1)) + s_w + s_{\bar{w}} + 2 n_w = v_w(\det G_1).$$

Taking the product of residue norms accounts for the square I_F^2. The Gram matrix of W, reordered into the two isotropic lists with their bordered last vectors, has zero diagonal blocks and off-diagonal block equal to the subframe product matrix. On the upper cell this block is the full matrix.

Below the upper cell a common vector with last coefficients $(\zeta, 1)$, with $\zeta \bar{\zeta} = 1$, shows that its determinant is $-N \zeta \delta_j$. Reduce the isotropic coefficient vector on the right modulo I to its first j columns. Pairing with the opposite subframe gives a last column with entries $(0, \ldots, 0, -N\zeta)$ after adding the prescribed linear combination of the preceding columns. Thus the square root of the right side of (44) is $N^{2f} \mu_\gamma$ in both cases. Since $\mathrm{ind}(N) = N^{2f}$, the link lattice of integral lifts has discriminant μ_γ / I_F.

Write $A_L = \Lambda_c \cap A$ and $A_R = \gamma(\Lambda_{c'} \cap A) = \Lambda_c \cap \gamma A$. The second equality uses $\gamma\Lambda_{c'} = \Lambda_c$, also at the places dividing N. Let A_L^0, A_R^0 be generated by the first j_0 columns on the respective sides, and let $\pi : \Lambda_c \cap I^\perp \to Q$ be the quotient map. Put

$$B_0 = \pi(A_L^0 + A_R^0), \qquad B_1 = \pi(A_L + A_R^0), \qquad B_2 = \pi(A_L + A_R), \qquad B_3 = \mathrm{sat}_Q(B_2).$$

The first j_0 columns span each side modulo I, and the two images intersect trivially over E. Thus all four lattices have rank $2j_0$ in $(A+\gamma A)/I$, and the successive indices are finite. Pairing with γF kills A_R, including $\Lambda_c \cap I$. It sends B_1/B_0 onto the quotient of the pairing lattices of A_L and A_L^0, of order i_γ. Pairing with F kills A_L and likewise sends B_2/B_1 onto the quotient of order i'_γ. Since $I \subseteq A + \gamma A \subseteq I^\perp$, the inverse image of B_3 under π is $\Lambda_c \cap (A + \gamma A)$. The kernel $\Lambda_c \cap I$ lies in $A_L + A_R$, so

$$B_3/B_2 \simeq \frac{\Lambda_c \cap (A + \gamma A)}{A_L + A_R}, \qquad [B_3 : B_2] = t_\gamma.$$

Multiplying the three indices gives $i_\gamma i'_\gamma t_\gamma \mid [B_3 : B_0] = I_F$.

Impose the level condition on the lifts by demanding that their A-component be divisible by M, after adding an integral vector of I. This contributes index at most $[\widehat{\mathcal{O}}_E : M\widehat{\mathcal{O}}_E]^{j_0} = M^{2fj_0}$. The obstruction is the class of that component in

$$(\Lambda_c \cap A)/((\Lambda_c \cap I) + M(\Lambda_c \cap A)).$$

Its rank before reducing modulo M is j_0. Denote the resulting full link lattice by L_γ^+. We have proved

$$t_\gamma d(L_\gamma^+) \le \frac{M^{2fj_0}\mu_\gamma}{i_\gamma i'_\gamma}. \tag{45}$$

14.7.2 Decomposition of the two link lattices

In this subsection, we compare the two lattices constructed in Section 14.7.1.

Let γ, γ' be two terms of cell ranks j, j' over the same agreeing residue block (c, s, c'). Write V_1 for their link intersection and V_0 for the kernel in V_1 of the two z-forms. Let $i = k+1-\dim V_1$. Under the first middle projection V_0 is the space of common X_0-vectors. Split the first link orthogonally, first along V_1, then split V_1 along V_0. The discriminant of its full lattice L_γ^+ is the product of the three discriminants of its successive intersection and projection lattices.

Indeed the basis matrix in orthogonal coordinates is block triangular after lifting bases of the projection lattices; the shears along the earlier subspaces do not change its determinant. The first piece projects to a full sublattice of the part of the term lattice in V_0. The lift is orthogonal to both isotropic spaces, its two z-forms vanish, and its A-part is divisible by M. Hence it is orthogonal to the whole translated basis and satisfies the defining term lattice condition. Write d_γ for the discriminant of that part of the term lattice. It is at most the discriminant of the first piece. The latter metric is the original Hermitian form on X_0 because both z-forms vanish.

The second piece is detected injectively by the two z-forms. In the positive model they are pairings, up to signs, against z^+ and $\Upsilon_\gamma^{-1} z^+$. If u is the last coordinate of $\Upsilon_\gamma^{-1} z^+$, their Gram matrix has diagonal N, N and off-diagonal $Nu, N\bar{u}$. At top rank the adjugate last column of the basis product matrix sends, by Υ_γ, a vector with last coordinate δ_{n-1} to $-\delta_n z^+/N$. This follows by taking the corresponding combination of the translated basis vectors, orthogonal to

A, with zero second X_0-coordinate, and first z-coordinate $-\delta_n/N$. Therefore $u = -N\delta_{n-1}/\delta_n$, and the Gram determinant is $N^2\mathfrak{g}/|\delta_n|^2$. At top rank both δ_n and $\mathfrak{g}$ are nonzero by the cell condition. Since $\mathfrak{g}$ is N times a squared norm, the determinant is positive at every real place. When $\delta_{n-1} = 0$ it equals N^2. Below top rank the last common vector of the two flags has norm-one coefficient ratio and zero X_0-coordinates, so the two z-forms are proportional. Pairing with z^+ is nonzero, so their joint map on the full link has rank exactly one in these cells.

For the second piece, choose enough forms for a nonzero square pairing determinant. The values on a projected lattice vector can be computed on its integral lift because the shear lies in V_0. They are integral. If the links agree and the rank is top, the second piece has dimension two and Section 14.6 applies to its two integral lifts. Thus the determinant has index at least x if $1 \le x \le Q_\cap(s)$. In the other cases take $x = 1$, since the selected determinant is integral and nonzero.

The absolute norm of the Gram determinant of the selected form vectors is at most

$$\left(\frac{\max\{N^f\mu_\gamma, \mathfrak{g}_{\mathrm{norm}}(\gamma)\}}{\mu_\gamma}\right)^2.$$

In rank zero its value is one. In rank one choose either original form whose restriction is nonzero. Its ambient Gram entry is N, of absolute norm N^{2f}. In rank two the displayed Gram determinant gives $(\mathfrak{g}_{\mathrm{norm}}/\mu_\gamma)^2$. Cauchy–Schwarz on the exterior powers at every complex place, followed by multiplication of the finite basis indices, bounds the index of the pairing determinant by the product of the discriminant of the projected lattice and the square root of the absolute norm of the selected form Gram determinant. The second piece therefore has discriminant at least $x\mu_\gamma/\max\{N^f\mu_\gamma, \mathfrak{g}_{\mathrm{norm}}(\gamma)\}$. Writing c_γ for the discriminant of the third piece, the complement to V_1, equation (45) now gives

$$t_\gamma d_\gamma x c_\gamma \le G_\gamma. \tag{46}$$

14.7.3 The common residue and the intersection lattice

In this subsection, we use the decomposition in Section 14.7.2 to prove the discriminant bound.

For two terms γ, γ' with the same block, the factor $m_0 = \gamma\gamma'^{-1}$ is congruent to one on Λ_c modulo M, since unitarity determines the conjugate reductions from the chosen ones and M is a unit at all other places. On integral lifts y, y', the difference form between their two middle pairings is

$$\langle y, y'\rangle - \langle y, m_0 y'\rangle = \langle y, (1 - m_0)y'\rangle \in M\widehat{\mathcal{O}}_E.$$

This calculation is in the original lattice basis. In the positive model the same difference form between source vectors a, b is $\langle a, (1 - \Upsilon_\gamma^{-1}\Upsilon_{\gamma'})b\rangle$.

It vanishes along the link intersection. It pairs the two complementary spaces perfectly. A vector of the second link orthogonal to the whole first link for the difference form lies in the first link, whose graph is maximal isotropic. The integral lift congruence descends to the projected complementary lattices because the shears lie in the intersection. On each complex exterior power of degree i one has the determinant bound

$$\left|\det\left(\langle a, (1 - \Upsilon_\gamma^{-1}\Upsilon_{\gamma'})b\rangle/M\right)\right| \le (2/M)^i\sqrt{\det \mathrm{Gram}(a)\det \mathrm{Gram}(b)}.$$

It follows by Cauchy–Schwarz and $\|1-U\|_{\mathrm{op}} \le 2$ for a unitary U. Choose rational bases a, b of the two complements and write their finite lattice bases as aA_f, bB_f. Let C be the rational matrix of the difference pairing divided by M in the bases a, b. The nondegenerate integral pairing in the finite bases has matrix $A_f^{*}CB_f$. By the product formula and the definition of the finite index,

$$1 \le \operatorname{ind}\det(A_f^{*}CB_f) = \operatorname{ind}(\det A_f)\operatorname{ind}(\det B_f)\prod_{v\mid\infty}|\det C_v|^2.$$

The discriminant of the first complementary lattice is $c_\gamma = \operatorname{ind}(\det A_f)\prod_{v\mid\infty}\det \operatorname{Gram}_v(a)$, and likewise for $c_{\gamma'}$. Multiplying the squared exterior-power inequalities therefore gives

$$c_\gamma c_{\gamma'} \ge (M/2)^{2if}. \tag{47}$$

Lastly the common lattice has index at most $t_{\gamma'}$ in the part of the first term lattice in V_0. For v in this part choose the lift $y = v - Ma$ with integral $a \in A$ and y orthogonal to the basis of γ. The common middle pair lies in the image of the rational lift map for the second link. A linear section of this surjection over E extends to the finite adeles and gives a lift y', over the finite adeles without an integrality condition on y'. Thus $y - y' \in A$ and $\gamma^{-1}y - \gamma'^{-1}y' \in A$. Thus for $m_1 = \gamma'\gamma^{-1}$,

$$(m_1 - 1)y = (y' - y) + \gamma'(\gamma^{-1}y - \gamma'^{-1}y') \in A + \gamma' A.$$

The vector $w = (1 - m_1)y/M$ is integral on the left lattice by the same-residue congruence.

The products of v with the basis of γ' equal those of $M(a + w)$ because $m_1 y$ is orthogonal to that basis. Write

$$A_0 = \Lambda_c \cap A, \quad A_1 = \gamma'(\Lambda_{c'} \cap A), \qquad W_0 = \Lambda_c \cap (A + \gamma' A).$$

Let P' send a vector to its pairings against $\gamma' F$. Since P' vanishes on A_1, the finite module $W_0/(A_0 + A_1)$ surjects onto $P'(W_0)/P'(A_0)$. For v in the first term lattice restricted to V_0, the preceding calculation sends v to $P'(v)/M$ in this quotient. Indeed the class vanishes precisely when $P'(v) = MP'(a')$ for some $a' \in A_0$. Then $v - Ma'$ is an integral lift orthogonal to $\gamma' F$, which is exactly the second term-lattice condition. Membership in the first term lattice and in V_0 was already imposed. Thus the kernel is L_{12}, and its index divides $t_{\gamma'}$, by the definition of that torsion index. The same argument in the other order gives

$$d \le t_{\gamma'} d_\gamma, \qquad d \le t_\gamma d_{\gamma'}.$$

The first pieces of the two link lattices are full in V_0, so the finite indices just proved make L_{12} a full lattice there. It lies in the integral definite lattice Λ_c^0, hence $d \ge 1$. Put

$$l = k - \dim V_0, \qquad X = \begin{cases} x, & i = 0,\ j = n, \\ 1, & \text{otherwise}, \end{cases} \quad 1 \le x \le Q_{\cap}(s).$$

The equal-link case is $i = 0$, since both links have dimension $k + 1$. On the graph of Υ_γ, the space with both z-coordinates zero has dimension $k - 1$ at top rank, and k below top rank, by the two-form calculation above. This also shows that equal links have the same top-rank status, so the value of X is symmetric in the terms. If r_1 is the rank of the two forms restricted to V_1, then

$$\dim V_0 = k + 1 - i - r_1, \qquad l = i + r_1 - 1 \le i + (j - \varrho)_+.$$

In particular, equal links have $l = 1$ in the top cell and $l = 0$ in the lower cells.

Lemma 14.3 (Discriminant of the pair lattice)**.** *For the original terms over one agreeing residue s, with cell ranks as in Section 14.4, let d, X, i, l be as above. For $1 \le x \le Q_\cap(s)$,*

$$(dX)^2 \le (2/M)^{2if} G_\gamma G_{\gamma'}, \qquad l \le i + (j - \varrho)_+. \tag{48}$$

The first inequality follows by multiplying the two torsion bounds and applying (46) to each term and (47) to the complement. For near terms both G_γ and $G_{\gamma'}$ are at most Δ_{lat}. Taking $x = 1$ in (48) and using $M/2 \ge b^{18}$ gives

$$b^{18if} d \le \Delta_{\mathrm{lat}}.$$

In particular $d \le \Delta_{\mathrm{lat}}$ and $l \le i + 1 \le n^3$. Hence all such near pairs satisfy the rank and discriminant conditions of Section 14.5.

15 Character cancellation and Schur's test

In this section, we apply character orthogonality to the rational terms counted in Section 14. The resulting cell estimates and Schur bound give the off-diagonal sums used in Section 9.1.

15.1 Character cancellation on the blocks

In this subsection, we integrate the two actions of the Bessel unipotent subgroup on the terms from Section 14.1. Character orthogonality leaves the stabilizer agreement condition analyzed in Section 15.2.

Let T_0 be a rational term set invariant under U on both sides. On $(T_0 \cap \mathcal{T}(c, s, c')) \times G_\infty$ the group $U_c \times U_{c'}$ acts by

$$(u, u') \cdot (\gamma, y) = (u\gamma(u')^{-1}, uy).$$

This action preserves counting times Haar measure. It preserves the full and cut kernels $K^\sharp_\gamma(y)$ of Section 14.1, because the real characters on the two class lattices are trivial by the product formula and the original principal congruence. The cut uses a left unipotent-invariant length. One can replace the original weight $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$ by any nonnegative measurable weight with orbit sum one. Insert the new orbit sum, change variables in each summand, and sum the old weight. Tonelli and the absolute convergence proved in Section 14.1 justify this both for the norm of the kernel and for the kernel itself. The same argument applies to the norm of the full-minus-cut kernel.

For a double coset with representative γ, put $\Gamma_\gamma = U_c \cap \gamma U_{c'} \gamma^{-1}$. Write its terms uniquely as $u\gamma(u')^{-1}$, with u running over representatives of U_c/Γ_γ and $u' \in U_{c'}$. After the substitution $y \mapsto uy$, the sum of the right cutoffs over u' is one. The remaining weight $\sum_u \beta_c(uy)$ has orbit sum one for Γ_γ. The contribution is thus $\int_{\Gamma_\gamma \backslash G_\infty} K^\sharp_\gamma(y)\, dy$.

Say the rational characters agree at γ if $\chi_f(u) = \chi_f(u')$ whenever $u, u' \in U(F_0)$ and $u\gamma = \gamma u'$, where F_0 is the totally real subfield of E and χ_f denotes the finite Bessel character. On the real unipotent group $H_\gamma = U_\infty \cap \gamma U_\infty \gamma^{-1}$, the kernel transforms on the left by the quotient of the two real characters. This quotient is trivial on Γ_γ. If the rational characters disagree, their rational element gives a nontrivial value of the real quotient by the product formula. Integrate in stages over $H_\gamma \backslash G_\infty$ with fiber $\Gamma_\gamma \backslash H_\gamma$ by the quotient integral formula for the unimodular groups. Right translation by the real element preserves the fiber measure and multiplies the

integrand by the nontrivial value. The fiber integral is zero on almost every fiber by absolute integrability. Therefore every stable set of rationally disagreeing terms contributes zero.

There is also a finite test on a whole residue block. Let $\bar{U}$ be the reduction of the integral finite unipotent group, with its induced character ψ_U. For $\bar{u}, \bar{u}' \in \bar{U}$, lift them to the integral finite group. Approximate the lifts in the respective class bases by rational unipotents α, α' with the same reductions. The free coordinates and skew center give this approximation as in Section 14.2.

Each class representative fixes the hyperbolic pairs and z. Its action on X_0 normalizes the unipotent and preserves the simple-root sum. Hence $\Psi_\infty(\alpha) = \psi_U(\bar{u})^{-1}$ and $\Psi_\infty(\alpha') = \psi_U(\bar{u}')^{-1}$ by the product formula. The approximants normalize the class principal unipotent groups since their finite transports are integral and J is normal in the integral stabilizer. Under the change $(\gamma, y) \mapsto (\alpha\gamma(\alpha')^{-1}, \alpha y)$ the translated weight has orbit sum one in the new block. By the cutoff identity and the product formula,

$$A^{\sharp}_{T_0}(c, \bar{u}s(\bar{u}')^{-1}, c') = \psi_U(\bar{u})^{-1}\psi_U(\bar{u}')A^{\sharp}_{T_0}(c, s, c').$$

In particular an agreeing residue for a nonzero block must satisfy $\psi_U(\bar{u}) = \psi_U(\bar{u}')$ whenever $\bar{u}s = s\bar{u}'$. This identity cancels the complex block before taking its Hilbert–Schmidt norm. On the surviving blocks the ordered-pair bound (42) uses the nonnegative weights $d(L_{12})^{-k}$ with their ramified multiplicities from Section 14.3.

15.2 The common flag and the cell rank

In this subsection, we use agreement of the characters from Section 15.1 to compare the isotropic flags. Their intersections determine which ranks contribute to the open and closed terms in Section 15.3.

Suppose the rational characters agree at γ. Write $Fx = \gamma Fy$ for a common vector supported in both bases through index $a < \varrho$, with coefficient pair (x_a, y_a). Let λ be a linear functional vanishing on both initial segments through a, and write $\lambda = h(v, \cdot)$. For $A_1 = tFx$ consider

$$u(w) = w + h(v, w)A_1 - h(A_1, w)(v + \tfrac{1}{2}h(v, v)A_1).$$

Expansion using $h(A_1, A_1) = h(v, A_1) = 0$ gives $h(u(w), u(w')) = h(w, w')$. In the Witt order the dual coordinates of v through index a vanish, and $h(A_1, \cdot)$ reads only these dual columns.

Thus the corrections are strictly upper triangular in the Witt blocks, with identity middle block. In particular $u \in U(F_0)$. It acts on f_b by adding $\lambda(f_b)A_1$, so its simple-root sum is $tx_a\lambda(f_{a+1})$. The conjugate $\gamma^{-1}u\gamma$ has the same formula with $A_1 = t\gamma Fy$ written in the second basis, giving root sum $ty_a\lambda(\gamma f_{a+1})$. Put $c = \lambda(x_a f_{a+1} - y_a\gamma f_{a+1})$. Taking $t = -M\delta_E q$ for $q \in E$, agreement of the two characters gives $\psi_{E,f}(qc) = 1$ by Section 13.2. If $c \neq 0$, choose $q = (2[E : \mathbb{Q}]c)^{-1}$. Then $\mathrm{Tr}_{E/\mathbb{Q}}(qc) = 1/2$, so the additive product formula gives $\psi_{E,f}(qc) = -1$. Hence $c = 0$. The annihilator criterion puts this difference in the sum of the two initial segments. Rearranging supplies a common vector with coefficient pair (x_a, y_a) at $a + 1$.

Iteration to the last position and comparison of norms there give $x_a\bar{x}_a = y_a\bar{y}_a$, since the norm of a basis vector at last coefficient x_a is $-Nx_a\bar{x}_a$. Applied at the larger of the two highest indices of a nonzero common vector, this shows that both indices agree. A nonzero common vector of least top index j can be scaled to give a pair $(\zeta, 1)$, $\zeta\bar{\zeta} = 1$, which then occurs at every index at least j.

Write δ_j for the leading minor of Z_γ of order j, with $\delta_0 = 1$, and put $\mathfrak{g}_\gamma = \delta_n\bar{\delta}_n - N^2\delta_\varrho\bar{\delta}_\varrho$. For the last adjugate column q of Z_γ, the products of γFq with the basis vanish except at the last entry δ_n. Its last coefficient in the second basis is δ_ϱ. It lies in $A^\perp$ with z-coordinate $-\delta_n/N$. Comparing its norm in these two expressions gives $\mathfrak{g}_\gamma = Nh(w_0, w_0)$ for its X_0-part w_0. Thus a nonzero $\mathfrak{g}_\gamma$ implies $\delta_n \neq 0$, and proves rank n. If $\mathfrak{g}_\gamma = 0$ and $\delta_n \neq 0$, the same adjugate column gives a nonzero common vector. If $\delta_n = 0$, any nonzero $q \in \ker Z_\gamma$ gives $\gamma Fq \in A + X_0$ of nonpositive norm, hence in A. This also gives a nonzero common vector.

For zero discriminant with rational agreement, take the least top index j as above. The shared isotropic vectors of indices $j, \ldots, \varrho - 1$ span $A \cap \gamma A$ by descending elimination of their highest coefficients. If $j > 0$ and $\delta_j = 0$, a nonzero vector in the first j columns of γF pairs to zero with the first j columns of F. It is perpendicular to the shared isotropic vectors and hence to all of A. It also pairs to zero with the final shared basis vector, whose coefficient at z is nonzero, so it is perpendicular to z. It is isotropic in $A + X_0$ and thus belongs to A. Its top index is below j in the second basis and therefore in both bases, contradicting minimality. This gives $\delta_j \neq 0$ and the remaining cell-rank conditions of Section 14.4. This cell rank is unique. A candidate $j' < j$ would require a shared vector of top index j', contrary to minimality. If $j < j' < n$, the shared vector of index j is isotropic and lies in A. Its nonzero coefficient vector in the first j' translated columns therefore lies in the kernel of the leading j'-by-j' pairing matrix, so $\delta_{j'} = 0$. The top rank is excluded by $\mathfrak{g}_\gamma = 0$. The case $j = 0$ forces $\gamma A = A$ by the shared isotropic basis.

15.3 Scalar cosets and the surviving terms

In this subsection, we classify the terms satisfying the agreement and rank conditions of Sections 15.1 and 15.2. This identifies the scalar contributions and the terms estimated by the orbital bounds.

If the rational characters agree and $\gamma A = A$, the common vector f_0 has top index zero in both bases. Thus $\gamma f_0 = \zeta f_0$ for $\zeta\bar{\zeta} = 1$, and the common pairs above give $\gamma f_a - \zeta f_a \in \langle f_b : b < a\rangle_E$ by successive substitution. Hence $\zeta^{-1}\gamma$ is in the rational Bessel subgroup $R(F_0) = (U \rtimes H)(F_0)$. Indeed it acts trivially on the successive isotropic lines and fixes the class of z in $A^\perp/A$. In a term of $\mathcal{T}(c, s, c')$ the first diagonal entry ζ of the integral transporter in the class basis is integral at every finite place. So is $\zeta^{-1} = \bar{\zeta}$. It follows that $s \in \mathrm{red}(\zeta)H_R$.

Put

$$\Gamma_* = \{\gamma : \mathfrak{g}_\gamma \neq 0\} \cup \{\gamma : \mathfrak{g}_\gamma = 0,\ \gamma A \neq A, \text{rational characters agree}\}.$$

This is stable under both unipotent actions. On the basis they induce upper unitriangular column moves and lower unitriangular row operations on the product matrix, so the leading minors are preserved. They preserve A, and inner conjugations preserve the unipotent character. On $R(F_0)$ the first product column vanishes, since the first basis vector is fixed and orthogonal to the basis. Thus $R(F_0)$ and Γ_* are disjoint. Outside their union a rationally agreeing term must preserve A and hence belongs to a scalar coset as above. Rationally disagreeing terms outside this union contribute zero by Section 15.1. For residues outside the nontrivial integral norm-one scalar cosets we obtain

$$A^\sharp_{R(F_0)^c}(c, s, c') = A^\sharp_{\Gamma_*}(c, s, c'). \tag{49}$$

Every term on the right has a cell rank $1 \leq j \leq n$ by Section 15.2. The nonzero discriminant terms have $j = n$.

Each global scalar $\zeta \in \mathcal{O}_E^\times$ with $\zeta\bar{\zeta} = 1$ belongs to μ_{q_1} by Section 4.4. If $\zeta \neq 1$ reduces to one at a place above either level prime p, the identity $1+\zeta+\cdots+\zeta^{q_1-1} = 0$ gives $p \mid q_1$, whereas $p \equiv 1 \bmod q_1$. At each place the Bessel subgroup acts upper unitriangularly on the span of F. Thus an element of $\mathrm{red}(\zeta)H_R$ fixing a nonzero vector of this span forces ζ to reduce to one there. A fixed covector gives a fixed vector of the conjugate span by the inverse-transpose identification in Section 9.1. Consequently a fixer at even one selected pair excludes every nonidentity scalar coset. On $H_R G_i$, put $p = p_{1-i}^{\mathrm{lev}}$. At every place above p the reduction belongs to the Bessel subgroup. It fixes e_0, excluding every nontrivial scalar coset and giving $Q_\cap(s) \geq p^{2f}$ from the f split pairs. The divisibility used for equal top links can also be read directly on this factor. The class matrices $B_c, B_{c'}$ preserve the string columns, and the reduced Bessel action sends z to z modulo the span of A. Thus at each $w \mid p$, $\gamma z - z$ belongs to $(A \cap \Lambda_c)_w + p\Lambda_{c,w}$. For the integral columns orthogonal to A and γA in Section 14.6, the pairing rows with z and γz agree modulo p at the conjugate place. Their determinant is divisible by p at all $2f$ places, giving index at least $p^{2f} \geq (M/b)^f$, since $p_i^{\mathrm{lev}} \leq bp$ by Section 4. Every scalar coset has vanishing first leading minor at all chosen places, so the open set avoids them all. For a fixer at every first-prime place, any fixed nonzero vector in the span lies in the intersection of that span with its image. A fixed covector gives the same intersection at the conjugate place via the Hermitian pairing and inverse-transpose action. Thus the meeting norm is at least $(p_0^{\mathrm{lev}})^f$ on the full fixer.

15.4 The real estimate in the open Gram cell

In this subsection, we estimate a real orbital integral when the Gram matrix is in the open cell. The bounds of Section 12.11 control the derivatives in the passage from a Gram neighborhood to a fixed Gram matrix.

Work at one real embedding in the orthonormal basis $E = (e_0, \ldots, e_{\varrho-1}, z)$, with $n = \varrho + 1$ and $E^* J E = -Q_0$, where Q_0 projects onto the last column. Use the full Bessel vector W, its quotient mass $m_W = \|W\|^2_{U\backslash G}$, and its formal degree d_W in dg. Put

$$Z = E^* J \gamma E, \qquad \Delta(Z) = |\det Z|^2 - |\det Z_{<n-1,<n-1}|^2.$$

Assume $\Delta(Z) > 0$ and that some y_0 has both $y_0^{-1}E$ and $y_0^{-1}\gamma E$ bounded by $\rho = b^{37n}$ in each column.

15.4.1 Haar density in Gram coordinates

In this subsection, we compute the density for the real estimate of Section 15.4.

The pointwise basis stabilizer is $U_0 \rtimes H$ with $H = U(k)$. Its Haar is $du\, dh$, with dh a probability and du Lebesgue in the Eichler coordinates. This Haar is invariant under inversion since H preserves the coordinate volume and U_0 is unimodular.

The full Gram fibre has density $2^{-r} \operatorname{vol} U(r)\, dg$ in the Hermitian coordinates by Section 12.6. For a rectangular tuple Y, split $Y = JEZ + F_0 p$, where F_0 consists of the free isotropic and definite columns. The blocks JE, F_0 form a Euclidean orthonormal basis. For invertible Z, write $Y = Y'Z$. In $A \oplus X_0 \oplus \mathbb{C}z \oplus A^\vee$ the columns of Y' have blocks

$$Y' = \begin{pmatrix} A_0 & a \\ B & x \\ 0 & -1 \\ I & 0 \end{pmatrix}.$$

For $T' = (Y')^* J Y'$, its first Gram block is $A_0 + A_0^* + B^*B$, its cross block is $a + B^*x$, and its last entry is $\|x\|^2 - 1$. Solving $A_0 = S + T'_{00}/2 - B^*B/2$ gives $2^{-\varrho} du$ in the skew and free coordinates, with the remaining factor in dT'_{00}. The first columns are the Eichler translate of the columns with $B = S = 0$ at that first Gram block. On the remaining column, the Eichler action on the isotropic coordinates is a translation once its definite coordinates are fixed. Thus solving for a also gives Jacobian one. Polar integration in x gives $\frac{1}{2}\operatorname{area}(S^{2k-1})(1 + T'_{nn})^{k-1}\, dh$ times the last real Gram coordinate. The angular measure is the image of probability Haar on $H = U(k)$ on the sphere $U(k)/U(k-1)$.

Right multiplication by Z scales the Lebesgue measure in the free rows by $|\det Z|^{2(\varrho+k)}$. For $T = Z^*T'Z$, Hermitian volume changes by $|\det Z|^{2n}$. In fact unitary factors preserve it, and diagonal singular values s_i give s_i^2 on the i-th real diagonal and $s_i s_j$ on each of the two real upper coordinates for the pair $i < j$. The net Gram-fibre factor is $|\det Z|^{2(k-1)}$. At $T = -Q_0$, the radius satisfies $1 + T'_{nn} = \Delta(Z)/|\det Z|^2$.

The resulting basis fibre in the open cell has density

$$2^{-(\varrho+1)} \operatorname{area}(S^{2k-1}) \Delta(Z)^{k-1}\, dZ\, du\, dh.$$

Here dZ is ordinary complex entry measure. The comparison at the prescribed Gram value $T_0 = -Q_0$ follows on the open chart because $Z = E^*JY$ is invertible. Thus Y has independent columns and Y^*J has a right inverse C. For $D = D^*$,

$$d(Y^*JY)[CD/2] = D.$$

In submersion coordinates the fibre integral of a continuous, compactly supported test is continuous in the Gram value. Average over shrinking neighborhoods of T_0, then exhaust the open chart by compactly supported nonnegative tests to obtain domination by the basis fibre measure at T_0. Completing with dual isotropic and definite columns, as in Section 12.6, multiplies by $2^{-\varrho} du'$ and $2^{-k} \operatorname{vol} U(k)\, dh'$, respectively.

Comparison with the full Gram density gives

$$\int c_G \Delta(Y)^{k-1} v(s\gamma_Y s')\, dY\, ds\, ds' \le \int_G v\, dg, \qquad c_G = \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)} \frac{2\pi^k}{(k-1)!}.$$

The left variables range over the open cell and both basis stabilizers, with the section γ_Y given below. The inequality of measures extends to nonnegative measurable tests.

15.4.2 A neighborhood in the original group

In this subsection, we use the density of Section 15.4.1 to compare neighborhoods in the original measure.

In the original Eichler coordinates of U_0, write its parameters as (B, C) with $C + C^* = B^*B$. Its group law is

$$(B, C)(B', C') = (B + B', C + C' + B^*B').$$

Let V_R be the ball $\|B\| \le R$, $\|C\| \le R^2$. The group law shows $V_R V_{R'} \subset V_{R+R'}$. The Haar measure is Lebesgue measure in the real coordinates of B and the skew part of C. The dilation $(B, C) \mapsto (RB, R^2C)$ has Haar factor $R^{2\varrho(k+\varrho)}$. For $T \ge 8r$, take

$$\Omega = V_{r^2T^2}, \qquad A_0 = \Omega\{y : \|y\| \le T\}.$$

Both sets are compact and Ω has positive volume. The block formula gives $\|u\| \le (1+r^2T^2)^2$ on Ω, so $\|y\| \le T^{10}$ on A_0.

On a nonempty left U_0-slice there is a representative z of norm at most T. A point in the slice has $uz = sx$, $s \in \Omega$, $\|x\| \le T$. Then $s^{-1}u = xz^{-1}$ has norm at most T^2, since $\|z^{-1}\| = \|z\|$. Its two Eichler blocks have norm at most T^2, whence it lies in V_{T^2}. The slice has volume at most the volume of $V_{(r^2+1)T^2}$, at most $(1+r^{-2})^{2\varrho(k+\varrho)}\operatorname{vol}(\Omega) \le 3\operatorname{vol}(\Omega)$. The equality of volumes before and after changing the representative uses right invariance of du_0.

For the section γ_Z constructed below, put $M_{\mathrm{loc}} = \rho^{9n}/\Delta(Z)$. The boxed columns give $|Z_{ij}| \le \rho^2$, so

$$0 < \Delta(Z) \le |\det Z|^2 \le (n\rho^2)^{2n} \le \rho^{6n}.$$

Thus $M_{\mathrm{loc}} \ge \rho \ge 8r$ and $M_{\mathrm{loc}}^{-1} \le \Delta(Z)$. In this local calculation only, write $M = M_{\mathrm{loc}}$, set $R = M^{9n}$, and take $T = R^5$. The section is the identity outside the span of E and $E' = (e_0^*, \dots, e_{\varrho-1}^*, x_0)$, with x_0 the definite vector. Its defining basis equations give the following identity for every $a < \varrho$:

$$e_a^* = \gamma_Z^{-1} E (Z^*)^{-1} \mathbf{e}_a + \tfrac{1}{2} \sum_{i<\varrho} (Z^{-1})_{i,\mathrm{last}} ((Z^*)^{-1})_{\mathrm{last},a}\, e_i,$$

where $\mathbf{e}_a$ is the coordinate vector.

It results from the equation taking the section's Z^*-dual columns to $E(Z^*)^{-1}$; the last positive basis vector occurs only in the last column. For g in the support test, the columns of $g^{-1}E$ have norm at most M, and $\gamma_Z g$ can be moved by $u \in U$ to $z' = u\gamma_Z g$ of norm at most M, by the Bessel length bound. Indeed approximate the infimum of $\|s\gamma_Z g\| + 1$ for $s \in UH$ to within $1/2$, and remove the unitary compact factor.

Put $Z' = E^* J u \gamma_Z E$. Since the action of u on E is upper unitriangular, the leading minors are unchanged under $Z \mapsto Z'$, and $(Z')_{ij} = \langle (z')^{-1} e_i, g^{-1} e_j \rangle_J$ with the last index interpreted as z. Thus the entries of Z' have norm at most M^2; $\Delta(Z') = \Delta(Z)$. The adjugate estimate gives $\|(Z')^{-1}\| \le R$ and $\|Z^{-1}\| \le R$. If $uE = EU_E$, then $Z = U_E^* Z'$ and $U_E (Z^*)^{-1} = ((Z')^*)^{-1}$. Applying g^{-1} to the change of basis now replaces the first summand by $(z')^{-1} E ((Z')^*)^{-1} \mathbf{e}_a$.

It bounds each $g^{-1} e_a^*$ by $nMR + \varrho M R^2/2 \le R^4$. Finally, for $L = g^{-1}$ use $LL^* = J + L(1-J)L^*$. The columns of $1-J$ are zero or signed differences of the isotropic and dual vectors, and $2z$. Therefore $\|L(1-J)\| \le (4\varrho+2)R^4$, so $\|L\|^2 \le 1 + (4\varrho+2)R^4 \|L\|$. This gives $\|g\| + 1 \le 8nR^4 \le R^5$. Thus if $g^{-1}E$ is boxed and the Bessel length of $\gamma_Z g$ is at most ρ, then $\|g\| \le T$.

On the original double coset with Gram matrix Z, write $\gamma = s\gamma_Z s'^{-1}$ with s, s' fixing the basis. The ordered pairs $(E, \gamma E)$ and $(E, \gamma_Z E)$ have the common Gram matrix $B(1)$, where

$$B(t) = \begin{pmatrix} -tQ_0 & Z \\ Z^* & -tQ_0 \end{pmatrix}, \qquad \det B(t) = (-1)^n |\det Z|^2 (1 - t^2 |(Z^{-1})_{nn}|^2).$$

The determinant follows by swapping the column blocks and taking the Schur complement of Z. The cofactor formula and $\Delta(Z) > 0$ give $|(Z^{-1})_{nn}| < 1$, so $B(t)$ is nonsingular on $[0,1]$. Its signature is (n,n) at $t = 0$ and hence at $t = 1$. Both orthogonal complements are positive definite of dimension $k-1$. Matching orthonormal bases there extends the isometry between the ordered pairs. For $v \in \Omega$ and $h \in H$ choose $t = s'(vh)^{-1}$. The support test at the section applies to $g = (vh)^{-1} y$ because the first basis is unchanged and Bessel length is left invariant under s. If the columns of $y^{-1}E$ are bounded by ρ and $\gamma t y$ has Bessel length at most ρ, it gives $\|v^{-1}y\| = \|g\| \le T$, so $y \in A_0$. The support implication holds for each shift and all y.

15.4.3 The transverse integral

In this subsection, we apply the neighborhood estimate of Section 15.4.2 to the section of the Gram map.

Write $P_0 = \operatorname{diag}(1, \dots, 1, 0)$ and $Q_0 = 1 - P_0$, and put

$$\begin{aligned} H_Y &= Y^{-*} Q_0 Y^{-1}, & \eta_Y &= (1 - |(Y^{-1})_{\text{last,last}}|^2)^{1/2}, \\ A_Y &= -Q_0 - \tfrac{1}{2} P_0 H_Y (1 + Q_0), & L_Y &= E A_Y + E'(P_0 + \eta_Y Q_0). \end{aligned}$$

The block Gram matrix on (E, E') is $\left(\begin{smallmatrix} -Q_0 & P_0 \\ P_0 & Q_0 \end{smallmatrix}\right)$. In particular

$$E^* J L_Y = 1, \qquad L_Y^* J L_Y = -Q_0 + A_Y^* P_0 + P_0 A_Y + \eta_Y^2 Q_0 = -H_Y,$$

where $Q_0 H_Y Q_0 = (1 - \eta_Y^2) Q_0$. The isometry sending (E, L_{Y^*}) to $(L_Y Y, E Y^{-*})$ exists because the displayed identities give identical Gram matrices, and both ordered pairs are bases of the span of (E, E'). Extend it by the identity on the definite orthogonal complement to get γ_Y. In particular it has the original basis products Y. If $S_Y = L_Y Y$, its column formula is

$$\gamma_Y = 1 - E E^* - E' E'^* + S_Y E^* + (E Y^{-*} - S_Y A_{Y^*})(P_0 + \eta_Y^{-1} Q_0) E'^*.$$

These equations give the dual-column identity used above.

Suppose $P \geq 8r$ bounds $\|Y\|, \|Y^{-1}\|, \eta_Y^{-1}$. For a real tangent of unit norm,

$$\|\dot{Y}^{-1}\| \leq P^2, \quad |\dot{\eta}_Y| \leq P^3, \quad |(\eta_Y^{-1})^{\cdot}| \leq P^5.$$

The product rule gives $\|A_Y\| \leq 2P^2$, $\|\dot{A}_Y\| \leq 2P^3$, $\|S_Y\| \leq 3P^3$, and $\|\dot{S}_Y\| \leq 4P^4$. Thus $\|\gamma_Y\| \leq P^{18}$ and $\|\dot{\gamma}_Y\| \leq P^{25}$. The stable section neighbourhood can be taken as $\|Y - Z\| < P^{-6}$, with $M \geq 8r$, $M^{-1} \leq \Delta(Z)$, $|Z_{ij}| \leq M^2$, $R = M^{9n}$, and $P = (R^5)^n$. In fact the adjugate and determinant estimates give $\|Z\|, \|Z^{-1}\|, \eta_Z^{-1} \leq M^{8n} \leq P/2$. A perturbation of norm P^{-6} keeps the inverse below P and changes it by at most $P^{-4}/2$. The absolute change in $1 - |(Y^{-1})_{\text{last,last}}|^2$ is at most $2P^{-4}$, whereas its initial value is at least $4/P^2$. Thus $\eta_Y^{-1} \leq P$. The determinant ratio of $Z = (1 + (Z - Y)Y^{-1})Y$ is bounded above by $(1 + P^{-5})^n \leq 9/8$. The relative loss in η_Z^2 is at most $2P^{-4}/(4P^{-2}) \leq 1/4$. Since $\Delta(Y) = |\det Y|^2 \eta_Y^2$, these bounds give $\Delta(Y) \geq (3/4)(8/9)^2 \Delta(Z) > \Delta(Z)/2$.

Let $c = c_G(\Delta(Z)/2)^{k-1}$. For a nonnegative function v invariant under the full left UH, a measurable cutoff of integral one on each left U_0-orbit exists by local sections of $U_0 \backslash G$. In a section chart take a compactly supported continuous density on U_0 of mass one and multiply it by a partition of unity on the quotient. Restrict the cutoff to the inverse image of the Gram ball.

Its U_0-integral there remains one, since a basis stabilizer keeps the Gram matrix fixed. The lower Haar density above and the probability Haar of H then give

$$c \int_{\|Y - Z\| < P^{-6}} \int_{H \times U_0} v(\gamma_Y (uh)^{-1}) \, dh \, du \, dY \leq \int_G \chi(y) v(y) \, dg(y).$$

In the right integral unfold first along U_0, then along U/U_0, which acts on E by unitriangular matrices. If two points in a U-orbit lie over the Gram ball, the adjoint of the transition matrix on E is the product of the first Gram matrix with the inverse of the second. All its entries are

at most P^2 in modulus. Using the original Haar on the upper entries of the basis matrix, the last unfolding contributes at most $(2P^2)^{2n^2}$. This proves

$$c\int_{\|Y-Z\|<P^{-6}}\int_{H\times U_0} v(\gamma_Y(uh)^{-1})\,dh\,du\,dY \le (2P^2)^{2n^2}\int_{U\backslash G} v.$$

For matrix-entry words j of length at most $2n^2$, Section 12.11 gives

$$\int_{U\backslash G}\left\|\int_G \beta(y)W_j(gy)\,dg(y)\right\|^2 \le P^{2|j|}\frac{m_W}{d_W}\int_G |\beta|^2\,dg,$$

since $b^5\le P$. The derivatives have the same unitary left Bessel covariance as W.

For a test supported in A_0, the logarithmic tangent for differentiation at gy is $y^{-1}Xy$ when differentiating at g along gX. Since $\|y\|\le P^{20}$, each matrix-entry coefficient contributed by this conjugation, divided by P^{40}, has modulus at most one. More precisely attach the fixed weight $(y^{-1})_{ac}y_{db}/P^{40}$ to the differentiation generator E_{ab} and the external coefficient X_{cd}. After expansion each integral is a convolution with a weight bounded by $|\beta|$ and one of the W_j. Differentiation under the integral uses continuity and boundedness of these derivatives on the compact support, locally in the external variable. Each of the r^4 possible new generators contributes at most $P^{40}\|X\|$. On the cube of side P^{-8} based at Z in $2n^2$ entry coordinates, take $g=\gamma_Y(uh)^{-1}$ with $u\in\Omega$, so that $\|u^{\pm1}\|\le P^{20}$. For each real entry coordinate of Y,

$$X=g^{-1}\dot g=(uh)\gamma_Y^{-1}\dot\gamma_Y(uh)^{-1},\qquad \|X\|\le P^{20+18+25+20}\le P^{85},$$

since h is Euclidean unitary and $\|\gamma_Y^{-1}\|=\|\gamma_Y\|$. Therefore each first derivative has a sum of at most r^4 terms with coefficients at most P^{125}.

For a Hilbert-valued function on $[0,\epsilon]$,

$$\|a(0)\|^2\le 2\epsilon^{-1}\int_0^\epsilon \|a\|^2+2\epsilon\int_0^\epsilon\|a'\|^2.$$

Apply it successively in all $2n^2$ coordinates. To iterate, denote a convolution obtained so far by $H_\omega(Y)$, where its test β_ω is independent of Y and $|\beta_\omega|\le|\beta|$. In each remaining coordinate the derivative formula and Cauchy–Schwarz give

$$\|\partial H_\omega(Y)\|^2\le r^4P^{250}\sum_{\omega'}\|H_{\omega'}(Y)\|^2.$$

There are at most r^4 terms, each with one additional generator and a new fixed test bounded by $|\beta|$. Substitution into the one-dimensional inequality with $\epsilon=P^{-8}$ leaves only squared convolutions under the integral. The scalar coefficients are now independent of Y, so the next coordinate requires no derivative of the section coefficients already estimated. The total multiplier per coordinate is at most $2(1+P^{258})P^8$, since $r^8\le P^8$. Apply the Gram-ball inequality to each word, whose squared norm is left Bessel-invariant. The combined contribution per coordinate is at most P^{275}. The quotient Schur bound contributes at most another P^{4n^2}. This proves the transverse estimate

$$c\int_{H\times\Omega}\left\|\int_{A_0}\beta(y)W(\gamma_Z(uh)^{-1}y)\,dg(y)\right\|^2 dh\,du\le P^{2000n^2}\frac{m_W}{d_W}\int_{A_0}|\beta|^2\,dg.$$

The cube lies in the Gram ball since $2n^2P^{-8}<P^{-6}$.

For $a = (h, u)$, put $g_a = \gamma_Z(uh)^{-1}$ and use the shift t_a with $\gamma t_a = s g_a$ constructed above. The Bessel covariance is unitary on the value space, so

$$\int_{A_0} \phi(y) W(\gamma t_a y) W(t_a y)^* \, dg(y) = U(s) \left(\int_{A_0} \phi(y) W(g_a y) W(y)^* \, dg(y) \right) U(t_a)^*.$$

Its squared Hilbert–Schmidt norm is the sum of the squared norms of the inner columns. Writing W_j here for a coordinate of W, apply the scalar estimate to $\beta_j = 1_{A_0} \phi \overline{W_j}$ and sum, using $\sum_j |\beta_j|^2 = 1_{A_0} |\phi|^2 \|W\|^2$. Dividing by $c \operatorname{vol}(\Omega)$ gives the contraction with normalized measure $dh\, du / \operatorname{vol}(\Omega)$ on $H \times \Omega$. Thus the operator in the parameter a is a contraction from L^2 of the measure on the source to L^2 of the normalized measure with Hilbert–Schmidt values. The source measure has density

$$\frac{P^{2000 n^2} m_W}{d_W \, c \operatorname{vol}(\Omega)} 1_{A_0}(y) \|W(y)\|^2 \, dg(y).$$

The same slice volume cancels on unfolding $\int_{A_0} \|W\|^2 \, dg$, since the slice contribution is at most $3 \operatorname{vol}(\Omega)$, the upper-entry contribution is at most $(2P^{40})^{2n^2} \le (M P^{40})^{2n^2}$, since two points in the tube have norms at most P^{20} and the transition matrix on their basis has entries at most P^{40}, and $\|W\|^2_{U \backslash G} = m_W$. Thus the total contribution is

$$M^{10^5 n^4} \frac{m_W^2}{d_W \, c_G (\Delta(Z)/2)^{k-1}},$$

using $3 \le R^5$ and $P^{2000 n^2} R^5 (M P^{40})^{2n^2} \le M^{10^5 n^4}$. This bounds the mass of the measure on the source of the one-place contraction.

15.5 The contribution of large discriminants

In this subsection, we apply the real estimate of Section 15.4 when the discriminant factor dominates. Its decay can then be combined with the lattice comparison in Section 14.7.

For an agreeing block (c, s, c'), denote by $\mathcal{R}_{S_0}(c, s, c')$ the chosen near representatives in a stable set S_0 having a cell rank. For γ of rank j put

$$D_j = \begin{cases} D_{\mathrm{main}}, & j = n, \\ \mathfrak{t}^{-16} D_{\mathrm{id}}, & j < n, \end{cases} \qquad w_\gamma = \mathfrak{t}^2 \sqrt{\frac{D_j}{n_p G_\gamma^k}},$$

with $n_p = |H_R| d_C / N^{kf}$ as in (42) and G_γ as in (43).

For a stable subset of the surviving terms, take its part on which $\mathfrak{g}_{\mathrm{norm}}(\gamma) > N^f \mu_\gamma$. Both quantities are stable under the rational unipotent moves, by the leading-minor and cell-rank invariance. On this part the cell rank is n, both first indices are one, and (43) gives $G_\gamma = M^{2\varrho f} \mathfrak{g}_{\mathrm{norm}}(\gamma)$. For a near representative write $x_\gamma = \mathfrak{g}_{\mathrm{norm}}(\gamma)$. The basis products are integral after the transport, so $x_\gamma \ge 1$ by the norm formula. The box bound in Section 14.7 gives $x_\gamma \le b^{270 n^2 f}$. At v the normalized discriminant is

$$\Delta_v = \left(\prod_a a_{v,a} \right)^2 |\mathfrak{g}(\gamma)|_v.$$

Indeed the full minor scales by the product of all the squared basis scales, whereas the preceding minor omits the last one, $1/N$. Section 15.4 gives the bound for the local source measure

$$P_v = \left(\frac{\rho^{9n}}{\Delta_v}\right)^{10^5 n^4} \frac{d_v^2}{\deg_v \, c_G(\Delta_v/2)^{k-1}},$$

Here $\deg_v$ is the formal degree of the ambient discrete series in the chosen real Haar measure. The normalization of Section 12.10 gives

$$d_v = \|W_v\|^2_{U_v\backslash G_v} = \dim \mathrm{Sym}^{a_\tau}(\mathbb{C}^k) = \binom{k-1+a_\tau}{k-1}, \qquad d_B = \prod_v d_v, \qquad \mathcal{B} = \prod_v \deg_v .$$

The product contraction. Write $K_v^\epsilon(\lambda, z) = W_v^\epsilon(\lambda z) \otimes W_v^\epsilon(z)^*$, with $\epsilon \in \{\mathrm{full}, \mathrm{cut}\}$. For each v, use the compact set A_v, the probability m_v, the finite measure ν_v and the measurable basis-fixer shift $t_v(a_v)$ in the local restriction estimate. The linear map is

$$(T_v\phi)(a_v) = \int_{A_v} \phi(z_v) K_v^{\mathrm{full}}(\lambda_v, t_v(a_v) z_v)\, dg_v(z_v)$$

and its norm from $L^2(\nu_v)$ to $L^2(m_v; \mathrm{HS}_v)$ is at most one. For a bounded measurable ϕ on the product the tensor operator still has the displayed integral formula with $A = \prod_v A_v$, product Haar and the product kernel.

Indeed it has this formula for finite sums of rectangle indicators by Fubini. The contraction gives the extension on the Hilbert tensor product, where the squared norm is computed in the tensor Hilbert–Schmidt basis. To identify that extension with the integral for general bounded measurable ϕ, the bounded functions for which the formula holds form a vector space closed under uniformly bounded monotone limits of real functions. Convergence holds in the source norm by finite mass and pointwise in the ordinary integral by dominated convergence, since for a fixed shift every kernel is continuous on the compact domain. The contraction then identifies the two limits almost everywhere. The monotone class theorem applies to the product Borel sets, and the imaginary part is treated by linearity.

For $|\omega| \le 1$ invariant under the product of basis stabilizers, apply this contraction to ω. The mean squared Hilbert–Schmidt norm is at most $\|\omega\|^2_{L^2(\prod \nu_v)} \le \prod P_v$. The finitely many support tests hold on one full $\prod m_v$-measure set, independently of the point tested. There is a shift t in that set with squared norm at most $\prod P_v$. In fact a nonnegative measurable function strictly larger than that number throughout the set would have integral strictly larger than it. Fixer invariance identifies the value $\omega(z)$ with $\omega(tz)$.

Use $\lambda = \gamma^{-1}$, with the chosen element $\lambda^{-1} y_0' = y_0$, $y_0' = \gamma^{-1} y_0$, and conjugate the cutoff when taking the adjoint. For the actual weight $\omega = 1_{\mathrm{red}}$ the conjugation is trivial. Put $y = tz$. By the tensor formula for K_v^{cut}, the translated weighted integrand is nonzero exactly when

$$\omega(y) \ne 0, \qquad W_v^{\mathrm{cut}}(y_v) \ne 0, \qquad W_v^{\mathrm{cut}}(\gamma_v^{-1} y_v) \ne 0 \quad \text{at every real place } v.$$

The first two conditions put y in the reduced set and in the cut, so triangular reduction bounds the columns of $y_v^{-1} E_v$. These columns equal those of $z_v^{-1} E_v$ since t_v fixes E_v. The third condition bounds the Bessel length of $\gamma_v^{-1} t_v z_v$ by $b^{6n+2} \le \rho$. This is exactly the support test that puts z in A. The support of the actual weighted cut is thus contained in tA, although the

indicator alone need have no compact support. All kernels are bounded on tA, and left Haar gives

$$\Big\| \int \omega(y)\, K_\gamma^{\mathrm{cut}}(y)\, dg(y) \Big\|_{\mathrm{HS}} \le (\prod_v P_v)^{1/2} + \int |\omega(y)| \|K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y)\|_{\mathrm{HS}}\, dg(y).$$

The error is a nonnegative integral.

Its finiteness for the selected orbit representatives follows from the cutoff transfer below. The full kernel is integrated on tA. The local discriminant for γ^{-1} equals that for γ, because their basis product matrices are adjoints. This uses the product Haar and the same chosen bases.

Write $\Lambda_v = \prod_a a_{v,a}^{k+2(\varrho-a)}$, so $\Lambda_0 = \prod_v \Lambda_v$. Put $q = 10^5 n^4$. Then

$$\prod_v P_v(\gamma) = C_{\mathrm{pc}} / x_\gamma^{q+k-1}, \qquad C_{\mathrm{pc}} \le 2^{kf}\mathfrak{t}\, \frac{d_B^2}{\mathcal{B} c_G^f \Lambda_0^2}.$$

Indeed $b^{-60n^2} \le \prod_a a_{v,a} \le b^{46n^2}$, and the local factor $\prod_a a_{v,a}^{k+2(\varrho-a)}$ is at most $(\prod_a a_{v,a})^k b^{92n^3}$. Against $2^k d_v^2/(\deg_v c_G \Lambda_v^2)$, the ratio is at most $b^{453n^2 q + 92n^2 + 184n^3}$ at each place, whose product is at most $\mathfrak{t}$ by $k \ge 512000 n^9$. In fact putting $S_v = \prod_a a_{v,a}$, the per-place ratio, with the discriminant norm removed, is $\frac{1}{2}(\rho^{9n}/S_v^2)^q S_v^{-2(k-1)} \Lambda_v^2$.

Put $v_E = \mathrm{vol}(\widehat{\mathcal{O}}_E)$ in the additive Haar at finite places used for the Siegel calculation, and let C_0 denote the scaled Siegel constant of the prescribed theta datum. The two numerical comparisons are

$$2^{kf} n_p M^{2\varrho f k} \le \mathfrak{t} C_0 v_E^n, \qquad C_0 \le (1 + b^{-9}) D_{\mathrm{main}} \mathcal{B} c_G^f v_E^{-n} (\nu \Lambda_0 / d_B)^2.$$

For the first comparison set $X = 200 f k \,\mathrm{rd}(E)$. The scale bounds above give $X^{1000} \le b$. For the one-vector Siegel constant of Section 13.3, the bounds $v_E \ge X^{-f}$, $(k-1)! \le X^k$, and $n \le k$ give

$$C_{\mathrm{S}} v_E^n = |D_E|^{1/4} v_E^{k+n} \left(\frac{\pi^k}{(k-1)!\, N^k} \right)^f \ge N^{-kf} X^{-(2k+n)f} \ge N^{-kf} X^{-3kf}.$$

Use $\mathfrak{t} \ge X^{4kf}$, $X \ge 2$, and the unipotent order $m_U \ge M^{2\varrho f k}$ in $C_0 = |H_R| d_C m_U C_{\mathrm{S}}$, and use $n_p = |H_R| d_C / N^{kf}$. The second comparison is $C_0 \Delta_0 \le (1 + b^{-9}) \nu^2 D_{\mathrm{main}} \mathcal{B}$ from Section 13.8. Combining gives

$$C_{\mathrm{pc}} n_p M^{2\varrho f k} \le 2\mathfrak{t}^2 \nu^2 D_{\mathrm{main}} \le \tfrac{1}{4} \mathfrak{t}^4 \nu^2 D_{\mathrm{main}}.$$

Since $q + k - 1 \ge k$ and $x_\gamma \ge 1$, this bounds the local product square by one fourth of the claimed orbit square. The absolute cut error will use the other half of the orbit bound. Its admissibility follows from

$$b^{-k^3 f} \le \mathfrak{t}^2 \sqrt{D_{\mathrm{main}} / (n_p (M^{2\varrho f} x_\gamma)^k)},$$

using $n_p \le D_{\mathrm{id}} \le D_{\mathrm{main}}$, $x_\gamma \le b^{270 n^2 f}$, $M \le b^{23}$, and the rank inequality. This proves the normalization of the two halves.

Use the reduced-point indicator ω from the box partition. It is invariant under the real fixer of the basis, since the transported columns are unchanged, and on the real cut its support is in the box of radius ρ. For each orbit of $U_c^0 \backslash (S_0 \cap \mathcal{T}(c, s, c')) / U_{c'}$ choose one representative,

preferring a near representative of a smaller orbit with right group $U_{c'}^0$ from the fixed list of Section 14.4. This choice suffices because whenever the reduced indicator and cut kernel are simultaneously nonzero, one can bring the second cut point to its box by its original class unipotent lattice. The new term remains in the larger orbit. Within the smaller orbit boxing is invariant under both basis-fixing unipotent lattices, so its chosen representative is near.

For a nonzero-discriminant term the stabilizer of $\gamma U_{c'}$ in U_c^0 is trivial. Indeed if $a\gamma = \gamma v$, then $v \in U_{c'}$ fixes $\gamma^{-1}F$ pointwise. These columns pair perfectly with F since the full basis product matrix is invertible. Linear combinations therefore give ϱ fixed vectors whose e_a^*-coordinates form the identity. In Witt blocks such an element has diagonal blocks (B, I, B^{-*}). Fixing the independent dual coordinates gives $B = I$; the middle rows of the fixed-vector equations kill the middle-to-dual block. The unitary relation kills the isotropic-to-middle block, and the remaining fixed-vector equations kill the strictly upper corner block. Thus $v = a = 1$.

Give (γ, y) weight $\omega(y)$ when γ is the chosen representative of its larger orbit, and weight zero otherwise. This is an orbit cutoff on the set of terms times the real group. To check this, for each right coset of U_c^0 in U_c exactly one element of that coset, together with one element of $U_{c'}$, takes the term to the representative of the larger orbit of $u\gamma$, for u in that coset.

Existence is the orbit decomposition and uniqueness is the trivial stabilizer proved above. The sum of ω over the remaining cosets is one by the box partition. Apply the change of cutoff in Section 15.1. It transfers the full-minus-cut absolute bound as well, because the norm of the difference of kernels is invariant under the same action. In particular the absolute tail on each chosen term is at most $\nu b^{-k^3 f}/2$.

Any chosen term that contributes is near by the boxing argument. Its integral is bounded by the sum of the two halves calculated above. A larger orbit charges at most one smaller-orbit representative, so summing gives the bound

$$\|A_{S_0}^{\mathrm{cut}}(c, s, c')\|_{\mathrm{HS}} \le \nu \sum_{\gamma \in \mathcal{R}_{S_0}(c,s,c')} w_\gamma$$

for S_0 in the dominant-discriminant part. The sum uses the original fixed near representatives. On a nonagreeing residue the block is zero by Section 15.1.

15.6 The one-place Schur integral

In this subsection, we give the real integral bound for a contributing cell at a single place. The rank condition comes from Section 15.2, and the estimate will be multiplied across places in Section 15.7.

Use a surviving cell of rank $1 \le j \le n$, let $i = \min(j, \varrho)$, and put $I = A \cap \gamma A$. Put $\rho = b^{37n}$ and $\theta = b^{6n+2}$. Throughout, y is a real point such that the columns of $y^{-1}F$ have length at most ρ and the Bessel length of y is at most θ. At a real place use the orthonormal basis with last vector of norm -1. Let F_i consist of the first i isotropic columns and the last vector. In a lower cell the shared columns starting at j have leading pairs $(\zeta, 1)$ with $|\zeta| = 1$ in the orthonormal basis, by Section 15.2.

These pairs retain their coefficients because both bases have the same column scales. The shared isotropic columns span I. From the shared last vector remove the columns of I in the second basis, to obtain a vector in $\mathrm{span}(F) \cap \mathrm{span}(\gamma F_i)$ with coefficient one at γz and coefficient of modulus one at z. Let $B_1 = F_i^* h \gamma F_i$, $\delta = \det B_1$. Replacing its last column by the pairings of that vector preserves the determinant. Those pairings are zero except for the last, of modulus

one. Thus in a lower cell $|\delta|$ is the modulus of the leading j-minor in the orthonormal basis. In the top cell this is also true, directly since $F_i = F$.

Let V_- be the negative eigenspace of the Hermitian involution h in the orthonormal basis. The negative space $Y_- = yV_-$ pairs perfectly with the entire basis: a vector in Y_- orthogonal to the basis must be zero since the basis orthogonal complement is nonnegative. Pull the columns of $F^*h\gamma F_i B_1^{-1}$ back through this pairing to Y_-, and apply Gram–Schmidt for $-h$ in reverse column order. These columns pair to zero with I, which is orthogonal to γF_i. This gives a negative orthonormal basis ω of $Y_- \cap I^\perp$ and a lower triangular invertible L with

$$F^*h\omega = F^*h\gamma F_i B_1^{-1} L, \qquad F_i^* h\omega = L.$$

The norm of $y^{-1}\omega_a$ is one, so $|L_{aa}| \le \|y^{-1}(F_i)_a\|$, giving $|\det L| \le \rho^{i+1}$ on the box of radius ρ.

Let E_- have orthonormal columns spanning V_-. Under y^{-1}, the h-orthogonal projection onto $Y_- = yV_-$ becomes Euclidean projection onto V_-, with its $-h$ metric. Hence the ratio of the projected volumes is

$$\mathcal{V}(\gamma, y) = \frac{|\det(E_-^* y^{-1}\gamma F)|}{|\det(E_-^* y^{-1} F)|}, \qquad \mathcal{V}(\gamma^{-1}, \gamma^{-1}y) = \mathcal{V}(\gamma, y)^{-1}.$$

Both projected bases are independent since their spans are nonpositive and the kernel of the projection is positive definite. In a lower cell replace the omitted isotropic columns in each basis by their shared columns. The changes of basis have determinant of modulus one by the common leading pairs. The projected shared wedge volume cancels, leaving volumes in the orthogonal complement of the projected I in Y_-.

This complement is $Y_- \cap I^\perp$, since pairing against a vector in Y_- only sees the negative projection. Pairing the retained columns with its orthonormal basis ω yields

$$\mathcal{V}(\gamma, y) = \frac{|\det((\gamma F_i)^* h\omega)|}{|\det L|}.$$

Put $\omega' = \gamma F_i B_1^{-1} L$. The columns of $\omega' - \omega$ lie in $A + X_0$; let V_0 be the matrix of their X_0-coordinates, so their Gram is $V_0^* V_0$. If $C = \omega^* h\omega'$, expansion gives

$$-C - C^* = I + V_0^* V_0 - \omega'^* h\omega' \succeq I + V_0^* V_0.$$

For a matrix of positive Hermitian part Q, the factorization $Q^{1/2}(I + \mathrm{i}R)Q^{1/2}$, with R Hermitian, shows its determinant has modulus at least $\det Q$. Apply this to $-C$, and use the determinant monotonicity of positive matrices and $\det(I + V_0^* V_0) = \det(I + V_0 V_0^*)$. The formula for $\det C$ from its factors now gives

$$\mathcal{V}(\gamma, y) \ge \frac{|\delta| \det(I + V_0 V_0^*)}{2^{i+1} |\det L|^2}. \tag{50}$$

Changing the transported basis in X_0 by A_1^* times the first i rows of its product matrix changes V_0 by A_1^* times the first i rows of L, since right multiplication by $B_1^{-1}L$ takes those product rows to the corresponding rows of L. Let L_1 be the leading i-block and l the final diagonal entry. Lower triangularity leaves the final column v_z of V_0 fixed. Thus the determinant in (50) is $\det(G_1 + T^*T)$, with $T = L_1^* A_1 + C_1$, $G_1 = I + v_z v_z^*$, and C_1 fixed.

In a lower cell the last column of ω' is a multiple of the common vector with coefficient one at γz. Its norm is $-|l|^2$ by comparing its last pairing with L. It is orthogonal to the difference

$\omega_z' - \omega_z$, so the norm of ω_z gives $|l|^2 = 1 + \|v_z\|^2 = \det G_1$. In the top cell the isotropic pairings of ω_z vanish by triangularity. For u in the Bessel unipotent, $u^{-1}z - z \in A$, so the pairing of $u^{-1}z$ with ω_z equals that of z. The length bound gives such a u with $\|uy\| \le \theta$, because the infimum defining the length includes the additional one. Evaluating in the orthonormal basis after this move gives $|l| \le \theta$. Thus in both cases

$$|l|^2 / \det G_1 \le \theta^{2(j-i)}.$$

Use Lebesgue measure on complex matrix entries. The change $T' = (L_1^* A_1 + C_1) G_1^{-1/2}$ has inverse Jacobian $|\det L_1|^{-2k} \det(G_1)^i$. The integrand contributes $\det(G_1)^{-(k+i)}$. Thus the factor in the integral of the determinant part of the majorant is $|\det L_1|^{-2k} \det(G_1)^{-k}$ times

$$\int_{\mathbb{C}^{i\times k}} \det(I + T'^* T')^{-k-i}\, dT'.$$

Integrate one row at a time. If T_0 contains the remaining rows and $Q = I + T_0^* T_0$, appending a row t changes the determinant to $\det Q(1 + tQ^{-1}t^*)$. The substitution $t = uQ^{1/2}$ has Jacobian $\det Q$. The remaining exponent is $-(k + i - 1)$, and integration over this row contributes the radial integral $\pi^k \Gamma(i)/\Gamma(k+i)$. Thus the integral is $\pi^{ik} \prod_{h=1}^{i} \Gamma(h)/\Gamma(k+h)$. Each factor $\pi^k \Gamma(h)/\Gamma(k+h)$ is at most $\pi^k/k! \le 1$ for $k \ge 7$. Consequently the explicit determinant majorant for $\mathcal{V}(\gamma, y)^{-(k+i)}$ in the shift parameter has integral at most

$$\frac{(2\rho^2)^{(i+1)i} (2^{i+1} \theta^{2(j-i)})^k}{|\delta|^{k+i}}. \tag{51}$$

The majorant is a measurable function of the parameter alone, so this does not require a measurable choice of a basis at each parameter. All integrals use the original complex entry measure.

15.7 The contribution bounded by Schur's test

In this subsection, we combine the real Schur estimates of Section 15.6 with the finite orbital weights. This controls the part of the near-orbit sum complementary to the range treated in Section 15.5.

Let $U_c^0 \subset U_c$ fix the basis pointwise. On the orbit $U_c^0 \gamma U_{c'}^0$ the product matrix and the residue in the block (c, s, c') are constant. Use the index i_γ of (43). In the transported integral class coordinates it is the index of the lattice generated by the first i isotropic rows of the pairing matrix in the lattice generated by all its isotropic rows. At a finite place w, conjugate transposition identifies the column quotient for the adjoints with the conjugate of the row quotient at $\bar{w}$. Hence the lengths at a split pair are exchanged and the product of residue norms giving i_γ is preserved. Multiplication by M also identifies the row quotient with the quotient of the scaled lattices. For $u\gamma v$ in the orbit, v fixes the basis, and $(u-1)/M$ is integral in the class basis. Its middle component reads only the dual isotropic coordinates. These are exactly the isotropic rows of the pairing matrix.

Put $D = \Lambda_c^0$. Let $\Xi \in M_{\varrho,n}(E)$ consist of the isotropic rows of the product matrix and write $\Xi_i \in M_{i,n}(E)$ for its first $i = \min(j, \varrho)$ rows. Write Λ_{all} and $\Lambda_{<j}$ for their respective row lattices. The difference of two moved bases has zero z-component because their pairings with z agree. In the Witt blocks the middle part of $(u-1)/M$ on a free dual isotropic column belongs

to D. Indeed the result is integral, its z-pairing is zero since u fixes z, and the isotropic and dual isotropic lattices split off the middle lattice. The middle block on the other columns is zero. Consequently each middle difference divided by M, measured from one fixed term, is in $D \otimes \Lambda_{\mathrm{all}}$, with all tensor products over $\widehat{\mathcal{O}}_E$. Its residue modulo $D \otimes \Lambda_{<j}$ has at most

$$\#(D \otimes (\Lambda_{\mathrm{all}}/\Lambda_{<j})) = [\Lambda_{\mathrm{all}} : \Lambda_{<j}]^k = i_\gamma^k$$

possible values, since D is free of rank k at every finite place. Fix a base term p_0 in each residue class and put $X(p, p_0) = (pF - p_0F)_{X_0} \in M_{k,n}(E)$. By equality of residues, $X(p, p_0) = MQ_f\Xi_i$ for a finite adelic matrix Q_f with i columns in D. Below top rank, $i = j$ and the leading j-minor is nonzero. At top rank Ξ_i consists of the first ϱ rows of the invertible full matrix, also when $\delta_\varrho = 0$. Choose $R_i \in M_{n,i}(E)$ with $\Xi_i R_i = I_i$. Then

$$Q_\lambda = M^{-1}X(p, p_0)R_i \in M_{k,i}(E), \qquad (Q_\lambda)_f = Q_f, \qquad X(p, p_0) = MQ_\lambda\Xi_i.$$

Thus every column of Q_λ lies in $X_0(E) \cap D$. At fixed λ the basis differences lie in A. The packing estimate restricts to n copies of $y_v^{-1}A$ at each real place, of total real dimension at most $2n\varrho f$. The separation stays $(N\rho)^{-1}$ and the radius is 2ρ, with $\rho = b^{37n}$.

Distinct bases thus contribute at most $b^{6n^2 \cdot 2n\varrho f} \le \mathfrak{t} = b^{kf/4}$, as in the packing calculation in Section 14.1. The fixed-basis argument of Section 14.1 puts the relative element of two terms with the same basis in $U_{c'}^0$. Their right cutoff weights therefore sum to at most one. For $Q \in \mathbf{C}^{\varrho \times k}$ use the following real Eichler shift. It fixes z and on $A \oplus X_0 \oplus A^\vee$ is

$$s_Q = \begin{pmatrix} I & Q & -QQ^*/2 \\ 0 & I & -Q^* \\ 0 & 0 & I \end{pmatrix}, \qquad s_Q^{-1} = s_{-Q}.$$

In the coordinates of Section 15.4.2 this has $B = Q^*$ and $C = QQ^*/2$, so $C + C^* = B^*B$ and the skew-Hermitian part is zero. Thus s_Q is unitary and fixes the frame pointwise. Its inverse adds Q^* times the dual isotropic coordinates of a column to its X_0-coordinates. On the moved frame those dual coordinates are the isotropic rows of the pairing matrix. The diagonal rescaling $\mathcal{S}_v$ has entries $M/d_{v,a}$, $a < i$, with the signed complex basis scale $d_{v,a} = \kappa_v^{\varrho - a}/\sqrt{N}$. The change of variables in (51) multiplies the integral by $|\det \mathcal{S}_v|^{-2k}$.

Writing $w_v = |\kappa_v|^{\sum_{a<j}(\varrho - a)}$ and $P_v = |\Phi_v(\delta_j)|$, we have

$$|\det \mathcal{S}_v| = (M\sqrt{N})^i/w_v, \qquad |\delta| = w_v^2 P_v/N^j.$$

The powers of w_v cancel in the k-th power and $w_v \ge 1$ allows us to drop them in the other denominator. Since $\theta = b^{6n+2}$, $2^{i+1}\theta^{2(j-i)} \le b^{13i}$. In lower cells this follows from $j = i$ and $2^{i+1} \le b^i$. In the top cell use $i = \varrho$, $2^{i+1} \le b^{i-17}$ and $2(6n+2) = 12i+16$. The shifted majorant has integral at most

$$K_v = \left(\frac{(2\rho^2)^{i+1} N^j}{P_v} \right)^i \left(\left(\frac{b^{13}}{M^2} \right)^i \frac{N^{j-i}}{P_v} \right)^k.$$

Then $\mu_\gamma = \prod_v P_v^2/N^{2f(j-i)}$ is the quantity of Section 14.7, since $\prod_v P_v^2 = |N_{E/\mathbf{Q}}(\delta_j)|$ and $j - i = (j - \varrho)_+$. The rational minor is integral and nonzero, so $\prod_v P_v \ge 1$. From $N \le b^{60n}$ and $2 \le b$, the product of the first factors in K_v is at most $b^{135n^2 if} \le \mathfrak{t}$.

Consequently

$$\prod_v K_v \le \mathfrak{t}\left(\frac{b^{13}}{M^2}\right)^{ikf} \mu_\gamma^{-k/2}. \tag{52}$$

The lattice of column parameters has Lebesgue covolume at least one. Pairing two column tuples in the definite lattice gives an algebraic integer. Because E contains $\mathbf{Q}(\mathrm{i})$, the sum of its real parts over the chosen complex places is the real part of the trace to $\mathbf{Q}(\mathrm{i})$, hence an integer. The real Gram determinant of a lattice basis is therefore a positive integer. This gives the covolume bound in precisely the measure used in the matrix integral. Write $\beta_c^{\mathrm{box}} = \mathbf{1}_{\mathrm{box}}\beta_c$, where boxed means reduced with $\ell_v \le b^{6n+2}$ at every place, as in Section 14.4.

For a term p in the orbit define

$$J(p) = \int \beta_c^{\mathrm{box}}(y)\|W^\flat(y)\|^2 \beta_{c'}^{\mathrm{box}}(p^{-1}y) \prod_v \mathcal{V}(p_v, y_v)^{-(k+i)}\, dy.$$

For a fixed color with base p_0, write $p = p_\lambda$ for any term with lattice parameter $\lambda_v = \Phi_v(Q_\lambda)^*$ at each real place v, allowing several terms at each parameter. Let y_0 be boxed and let u belong to the real basis stabilizer U_∞^0. At every v, triangular reduction gives $\|(y_0)_v^{-1} f_a\| \le b^{27}\ell_v((y_0)_v) \le b^{6n+29} \le \rho$, and boxing gives $\ell_v((y_0)_v) \le \theta$. Use the Eichler shift $s(X)$ whose central parameter is zero and whose matrix parameter at v has first i rows $\mathcal{S}_v^* X_v$ and all subsequent rows zero. Write Z_i for the first i rows of the pairing matrix in the orthonormal basis.

For every term in the color, the product matrix of $(us(X))^{-1}p$ is independent of the term and of X. At a real place the X_0-parts satisfy

$$\big((us(X))^{-1}pF\big)_{X_0} = \big(u^{-1}p_0F\big)_{X_0} + \big(\mathcal{S}_v^*(\lambda_v + X_v)\big)^* Z_i.$$

Indeed an inverse basis-fixing unipotent with block Q adds Q^* times the rows of dual isotropic coordinates to the X_0-part. The u^{-1} additions at p, p_0 coincide because they have the same product matrix. The $s(X)^{-1}$ addition uses precisely Z_i; the rational lattice difference, read in the orthonormal basis, has coefficient $(\mathcal{S}_v^*\lambda_v)^*$. Indeed, with D_v the diagonal matrix of all frame scales and $D_{v,i}$ its first i entries as a diagonal matrix, one has

$$Z_i = D_{v,i}^*\Phi_v(\Xi_i)D_v, \qquad M\Phi_v(Q_\lambda\Xi_i)D_v = M\Phi_v(Q_\lambda)(D_{v,i}^*)^{-1}Z_i = (\mathcal{S}_v^*\lambda_v)^* Z_i.$$

Here $D_{v,i}^* = D_{v,i}$ since $\kappa_v = M\Phi_v(\delta_E)/(2\pi i)$ is real, and $\lambda_v = \Phi_v(Q_\lambda)^*$. The identity follows. This formula treats u as fixed before X, which is the order of right translation on the real basis-fixer fiber.

Since $us(X)$ fixes the basis, $\mathcal{V}(p, us(X)y_0)$ equals $\mathcal{V}((us(X))^{-1}p, y_0)$. Apply the one-place majorant with base $u^{-1}p_0$ and parameter $\lambda + X$. It has product integral at most $\prod_v K_v$, independently of u, y_0. The basis ω and the matrix L use the common product matrix and y_0. The unipotent shifts fix the basis pointwise, so the common leading pairs used in the same cell are preserved.

At each lattice point the sum of the second boxed cutoffs is at most $\mathfrak{t}$ by basis packing and the right cutoff partition at a fixed basis. Hence the color sum at $us(X)y_0$ is bounded by $\mathfrak{t}\sum_\lambda \varphi(\lambda + X)$ for that product majorant φ. If $\mathcal{D}$ is the measurable fundamental domain of the i copies of the definite lattice in the conjugate-transpose coordinates, write Λ for that full lattice. Conjugate transpose preserves its real covolume. The parameters attained in the color form a

subset of Λ, since they are measured relative to its chosen base term. Enlarge the nonnegative sum to all of Λ. Tonelli gives

$$\frac{1}{\operatorname{vol}(\mathcal{D})}\int_{\mathcal{D}}\sum_{\lambda\in\Lambda}\varphi(\lambda+X)\,dX=\frac{\int\varphi}{\operatorname{vol}(\mathcal{D})}\le\prod_{v}K_v.$$

Use the same domain for each of the at most i_γ^k colors.

15.7.1 Summation over the smaller orbits

In this subsection, we use the preceding Schur bounds to sum the smaller orbits required in Section 15.8.

For the integration in stages, let $\mathcal{F}$ be the sum over the whole smaller orbit of the second boxed cutoff times the inverse volume power. This sum is left invariant under U_c^0, which permutes the terms. On the fiber through y_0, the first boxed indicator and the squared Bessel norm are constant. Write its remaining cutoff as $a(u)$, so $\sum_{\eta\in U_c^0}a(\eta u)=1$.

For each X, right Haar on the nilpotent group gives

$$\begin{aligned}\int a(u)\mathcal{F}(uy_0)\,du&=\int a(us(X))\mathcal{F}(us(X)y_0)\,du\\&=\int a(u)\mathcal{F}(us(X)y_0)\,du.\end{aligned}$$

For the second equality insert $\sum_\eta a(\eta^{-1}u)=1$ in the first integral, substitute $u\mapsto\eta u$ in each term, and sum $\sum_\eta a(\eta us(X))=1$. Tonelli applies as all weights are nonnegative. Averaging in X and using the preceding bound now yields the inequality on each real fiber. Unipotent disintegration of the first boxed cutoff times $\|W^\flat\|^2$ contributes at most νd_B. Indeed $\int_{U_\infty}\beta_c(uy)\,du=\nu$ by Section 13.6, and each cut vector has quotient mass at most the mass d_v of the full vector since its multiplier has modulus at most one.

Combining the colors with (52) gives

$$\sum_{p\in U_c^0\gamma U_{c'}^0}J(p)\le\nu d_B\mathfrak{t}^2\left(\frac{b^{13}}{M^2}\right)^{ikf}(i_\gamma/\sqrt{\mu_\gamma})^k.$$

Let $d\mu_p(y)=\beta_c^{\mathrm{box}}(y)\beta_{c'}^{\mathrm{box}}(p^{-1}y)\,dy$ and $V_p(y)=\prod_v\mathcal{V}(p_v,y_v)>0$. The pushforward of μ_p by $y\mapsto p^{-1}y$ is $\mu_{p^{-1}}$ with the classes switched, and $V_{p^{-1}}(p^{-1}y)=V_p(y)^{-1}$. Define the absolute amplitude

$$a(p)=\int\|W^\flat(y)\|\,\|W^\flat(p^{-1}y)\|\,d\mu_p(y).$$

The integrand contains the Hilbert–Schmidt norm of a rank-one operator, the product of the two vector norms. On its support the boxed cutoffs equal the original cutoffs. Insert the factors $V_p(y)^{-(k+i)/2}$ and $V_p(y)^{(k+i)/2}$ in the two respective factors of the integrand. Cauchy–Schwarz gives $a(p)^2\le J(p)J(p^{-1})$ with the original Haar measure. Write $\mathcal{O}=U_c^0\gamma U_{c'}^0$. Inversion bijects $\mathcal{O}$ over (c,s,c') with $\mathcal{O}^{-1}=U_{c'}^0\gamma^{-1}U_c^0$ over (c',s^{-1},c). Cauchy–Schwarz on finite subsets and monotone convergence therefore give

$$\left(\sum_{p\in\mathcal{O}}a(p)\right)^2\le\left(\sum_{p\in\mathcal{O}}J(p)\right)\left(\sum_{q\in\mathcal{O}^{-1}}J(q)\right),$$

where the second J uses the interchanged classes. In the definition of the link, $w' = p^{-1}w$ exchanges the two quotient coordinates. The identity $Z_{\gamma^{-1}} = Z_\gamma^*$ preserves the absolute norms of the leading minors and hence μ_γ. Exchanging the two bases in each common vector and rescaling its coefficient pair $(1, \zeta)$ to $(\zeta^{-1}, 1)$ gives the same cell rank for γ^{-1}.

For $a \in \Lambda_{c'} \cap A$ and a basis vector f of F, the identity $h(a, \gamma^{-1}f) = h(\gamma a, f)$ identifies the full pairing lattices and their sublattices generated by the first $\min(j, \varrho)$ columns. Thus the inverse orbit has first index i'_γ and reverse index i_γ. The pairing matrices are constant on each smaller orbit, so these identities apply to the fixed representatives on both sides of the inversion. Define $g_{\min} = N^f \mu_\gamma / (i_\gamma i'_\gamma)$. We obtain

$$\left(\sum_p a(p) \right)^2 \le \left(\nu d_B \mathfrak{t}^2 (b^{13}/M^2)^{ikf} \right)^2 (N^f / g_{\min})^k.$$

With n_p, D_j as in Section 15.5, $d_B^2 N^{kf} = d_B D_{\mathrm{id}}/n_p$ and $b^{-8ikf} = \mathfrak{t}^{-32i}$. The bounds $b^{13}/M^2 \le (b^4 M)^{-1}$, $d_B \le \mathfrak{t}^8$ and $D_j \ge \mathfrak{t}^{8-32i} D_{\mathrm{id}}$ give

$$\sum_p a(p) \le \nu \mathfrak{t}^2 \left(\frac{D_j}{n_p (M^{2if} g_{\min})^k} \right)^{1/2}. \tag{53}$$

The last bound on D_j holds in lower cells because $i \ge 1$, and in the top cell because $D_{\mathrm{main}} \ge \mathfrak{t}^{44} D_{\mathrm{id}}$. For $\mathfrak{g}_{\mathrm{norm}}(\gamma) \le N^f \mu_\gamma$ the denominator uses $G_\gamma = M^{2if} g_{\min}$, so (53) is exactly νw_γ. A nonzero cut summand with the original two cutoffs puts both arguments in their boxes. Nearness is invariant throughout the smaller orbit, because both basis stabilizers preserve reduction and length. Hence a non-near smaller orbit contributes zero. Sum the bound over the original fixed near representatives of the complementary part.

15.8 Summing the pair weights

In this subsection, we sum the pair multiplicities with the bounds from Sections 15.5 and 15.7. This completes the estimate for the corresponding terms of the full Petersson matrices in Section 9.1.

Use the weights w_γ of Section 15.5. In each agreeing block (c, s, c'), take a family of ordered pairs from $\mathcal{R}_{\Gamma_*}(c, s, c')^2$, using the representatives fixed in Section 14.4 for both terms. Suppose for each such pair of ranks j, j' that $(d\eta)^2 \le G_\gamma G_{\gamma'}$ and $D_j, D_{j'} \le D_0$. Here $d = d(L_{12})$ and l is its corank. If $\mathfrak{t}^{l+6} D_0 \eta^{-k} \le B_0$ for every pair of the family, then

$$w_\gamma w_{\gamma'} \le \frac{\mathfrak{t}^4 D_0}{n_p d^k \eta^k} \le \frac{B_0}{n_p \mathfrak{t}^{l+2}} d^{-k}.$$

By Sections 14.5 and 14.7 the sum at any l is at most B_0. There are at most $n^3 + 1 \le \mathfrak{t}$ coranks, so the total is at most $\mathfrak{t} B_0$.

For distinct links, where the codimension $i = k + 1 - \dim V_1$ of Lemma 14.3 satisfies $i \ge 1$, take $\eta = (M/2)^{if}$ and $D_0 = D_{\mathrm{main}}$. Using $l \le i + 1$ and $\xi = \mathfrak{t}(2/M)^{kf} \le 1$ gives

$$\mathfrak{t}^{l+6} D_0 \eta^{-k} \le \mathfrak{t}^7 \xi^i D_{\mathrm{main}} \le \mathfrak{t}^8 (2/b^6)^{kf} D_{\mathrm{id}} \le \mathfrak{t}^{-12} D_{\mathrm{id}},$$

where the middle step uses (33). The rank of the two z-coordinate forms restricted to a rational link depends only on the link. It is two at top rank and one below top rank by Section 14.7.2.

Thus equal links have the same top-rank status. For equal lower links $i = l = 0$. Use $\eta = 1$ and $D_0 = \mathfrak{t}^{-16} D_{\mathrm{id}}$, giving $\mathfrak{t}^{l+6} D_0 \eta^{-k} = \mathfrak{t}^{-10} D_{\mathrm{id}}$. For each ordered pair in either family these bounds give

$$w_\gamma w_{\gamma'} \le \frac{\mathfrak{t}^{-10} D_{\mathrm{id}}}{n_p \mathfrak{t}^{l+2}} \, d(L_{12})^{-k}.$$

Apply (42) once to the union at each corank l, then sum over the at most $\mathfrak{t}$ coranks. The combined family has total pair weight at most

$$\mathfrak{t}^{-9} D_{\mathrm{id}} \le b^{-kf} D_{\mathrm{id}}. \tag{54}$$

For equal top links with meeting norm at least $z \ge 1$, equation (48) gives $\eta = z$ and $l = 1$. Take $D_0 = D_{\mathrm{main}}$.

Lemma 15.1 (Weight of equal top links)**.** *Fix $z \ge 1$. In each agreeing block (c, s, c') with $Q_\cap(s) \ge z$, let $\mathcal{E}_z(c, s, c')$ be the ordered pairs in $\mathcal{R}_{\Gamma_*}(c, s, c')^2$ whose terms both have cell rank $j = n$ and equal rational links in $(A^\perp/A)^2$. In every other block put $\mathcal{E}_z(c, s, c') = \varnothing$. Then*

$$\sum_{c,s,c'} \sum_{(\gamma,\gamma') \in \mathcal{E}_z(c,s,c')} w_\gamma w_{\gamma'} \le \mathfrak{t}^8 z^{-k} D_{\mathrm{main}} = (b^{2f}/z)^k D_{\mathrm{main}}. \tag{55}$$

Proof. Every pair has $l = 1$ and individual weight at most

$$\frac{\mathfrak{t}^4 D_{\mathrm{main}} z^{-k}}{n_p d(L_{12})^k}.$$

Thus (42) gives

$$\sum_{c,s,c'} \sum_{(\gamma,\gamma') \in \mathcal{E}_z(c,s,c')} w_\gamma w_{\gamma'} \le \mathfrak{t}^7 D_{\mathrm{main}} z^{-k} \le \mathfrak{t}^8 D_{\mathrm{main}} z^{-k}. \qquad \square$$

For a stable subset $S_0 \subseteq \Gamma_*$ the cut blocks satisfy

$$\| A^{\mathrm{cut}}_{S_0}(c, s, c') \|_{\mathrm{HS}} \le \nu \sum_{\gamma \in \mathcal{R}_{S_0}(c,s,c')} w_\gamma. \tag{56}$$

Here $A^{\mathrm{cut}}_{S_0}$ is the cut block of Section 14.1, with the rational terms restricted to S_0, and $\mathcal{R}_{S_0}$ takes the chosen near representatives in S_0 with cell ranks over the given agreeing block. Stability means $U(F_0) S_0 U(F_0) = S_0$. In particular the rational lifts of the finite unipotents in Section 15.1 preserve S_0. Fix the same near representatives for every piece. At a nonagreeing residue take $\mathcal{R}_{S_0} = \varnothing$. Split each cell at $\mathfrak{g}_{\mathrm{norm}} = N^f \mu_\gamma$, assigning equality to the Schur part.

For a unipotent move the product matrix is multiplied on either side by triangular matrices with diagonal one, upper triangular on the right and lower triangular on the left. Hence its leading minors, including the determinant, are preserved. The ranks are preserved because the changes of basis preserve the flags and the common leading pairs. Sections 15.5 and 15.7 give (56) on the two stable parts with the same near representatives. The nonagreeing block vanishes by the finite cancellation in Section 15.1.

Now partition the cut sum by links at top rank and group all lower ranks in one piece. To see stability of a link, write its two quotient coordinates as the images of w and $\gamma^{-1} w$ in $A^\perp/A$. For a move $\gamma \mapsto u^{-1} \gamma u'$, use $u^{-1} w$ and $(u')^{-1} \gamma^{-1} w$. Both unipotents fix A as a subspace and act trivially on $A^\perp/A$, so the two quotient coordinates are unchanged.

In a fixed block (c, s, c'), let A_{low} be the sum of the cut amplitudes of ranks $j < n$, and let A_Λ be the cut amplitude of rank n with link Λ. By absolute convergence in Section 14.1, for $A = A_{\mathrm{low}} + \sum_\Lambda A_\Lambda$ we can expand

$$\begin{aligned}\|A\|_{\mathrm{HS}}^2 - \sum_\Lambda \|A_\Lambda\|_{\mathrm{HS}}^2 &= \|A_{\mathrm{low}}\|_{\mathrm{HS}}^2 + 2\Re \sum_\Lambda \langle A_{\mathrm{low}}, A_\Lambda\rangle_{\mathrm{HS}} \\ &\quad + \sum_{\Lambda \neq \Lambda'} \langle A_\Lambda, A_{\Lambda'}\rangle_{\mathrm{HS}} \\ &\le \|A_{\mathrm{low}}\|_{\mathrm{HS}}^2 + 2\|A_{\mathrm{low}}\|_{\mathrm{HS}} \sum_\Lambda \|A_\Lambda\|_{\mathrm{HS}} \\ &\quad + \sum_{\Lambda \neq \Lambda'} \|A_\Lambda\|_{\mathrm{HS}} \|A_{\Lambda'}\|_{\mathrm{HS}}.\end{aligned}$$

Summing the squares of the top pieces gives the equal-link moment after normalization. The last sum uses ordered pairs, and the factor 2 accounts for both orders of a lower and a top term. Applying (56) to each piece leaves on the right only pairs of distinct links or equal lower links. For fixed c, c', the first term determines $s = \mathrm{red}(B_c^{-1}\gamma_f B_{c'})$, and the second term must have the same residue. Each term belongs to its unique top link piece or to the single lower piece. Thus every ordered pair on the right occurs in just one product over its two classes. Within a top square the first term also determines the common rational link, so summing the top squares introduces no further multiplicity.

Combine the block estimates first and then restrict to any $\Omega \subseteq G$. By (54),

$$\mathcal{L}(\Omega) = \sum_{s \in \Omega} \sum_{c,c'} \sum_\Lambda \|A_\Lambda(c, s, c')\|_{\mathrm{HS}}^2,$$

$$\nu^{-2} \sum_{s \in \Omega} \sum_{c,c'} \|A^{\mathrm{cut}}_{\Gamma_*}(c, s, c')\|_{\mathrm{HS}}^2 \le \nu^{-2}\mathcal{L}(\Omega) + b^{-kf} D_{\mathrm{id}}.$$

Applying the same size estimate twice within each top piece and using (55) bounds the normalized equal-link moment by $(b^{2f}/z)^k D_{\mathrm{main}}$ on residues of meeting norm at least z. The partition of the blocks is justified by the absolute convergence in Section 14.1. Thus the pair summation gives the bounds needed in Section 9.1. The reduction to the cell ranks used in the sum is (49).

16 The common automorphic family and its translated span

In this section, we assemble the data of Section 8 and Sections 10–15 on one family. The moment inequalities of Section 9 then give the dimension bound in Proposition 5.1.

16.1 Construction of the common automorphic family

In this subsection, we assemble the real and finite data and verify that the resulting Poincaré map has the specified levels at every sufficiently large scale. The normalization of Section 8 and the dimension in Section 9.4 apply to this same construction.

Fix q_0, d, ϵ. For each sufficiently large integer x, take the primes, weights and skew different of Section 4. The grading, finite point and levels in Section 8 use these choices and the specified u. The source level is exactly the preimage of the original principal level under the definite embedding, by the glued-lattice calculation in Section 13.2.2. At the chosen split places reduction

of the full integral group is surjective: lift each prescribed general linear matrix at the chosen place, use the inverse adjoint at its conjugate and use the identity at the remaining places. The change of finite isometry has identity reduction at every chosen place.

Use the real signs and weights of Section 12.10. The different and scales in Section 4 give

$$|\Phi_v(\delta_E)| \le b, \qquad 1 \le |\kappa_v| \le b^{23}, \qquad N \le b^{25m} \le b^{60n}.$$

Use the adelic point x^{pt} and the pair (g_0, t) of Section 8 in the Poincaré map. These choices precede the selection of forms and residue classes. At each real place the value space is a determinant twist of the symmetric power of degree $a_\tau \le a$ of the standard rank-k representation, so its dimension is at most $(a+k)^k \le b^{2k}$.

In particular $0 < d_B \le b^{2kf}$ and the source class set is nonempty. Equation (33) gives both bounds for $D_{\mathrm{main}}/D_{\mathrm{id}}$. Before imposing the principal level at M, the inverse image of the glued lattice stabilizer fixes the specified nonzero covector at each chosen place above N. The last-row criterion of Section 8.1 and transitivity on nonzero covectors give the index $(N^k-1)^f \ge (N^k/b)^f$. Finally the mass comparison in Section 13.8 gives

$$C_0\Delta_0 \le (1+b^{-9})\nu^2 D_{\mathrm{main}}\mathcal{B}, \qquad \Delta_0 = \frac{v_E^n}{c_G^f}\left(\frac{d_B}{\Lambda_0}\right)^2.$$

This is the same constant bound used in the dominant orbit and in the open equal-link estimates. All these conditions hold on the constructed Bessel data, with the terminal definite comparison of Section 9.4. Their numerical thresholds depend only on the fixed q_0, d, ϵ. The arithmetic comparisons below use the measured map already specified by g_0, t and this level.

16.2 Selecting the translates

In this subsection, we convert the two moment inequalities of Section 9 into a lower bound for the dimension of a sum of translated subspaces. We then pass from the full source to the ramified subspace of Section 6.

Lemma 16.1 (A dimension bound for translated subspaces)**.** *Let a finite group G act unitarily on T by ρ. Let p be the orthogonal projection onto an invariant subspace W, and let $B : S \to T$ be linear between Hilbert spaces with $1 \le d_S = \dim S < \infty$ and $1 \le D = \dim W < \infty$. Write P_U for orthogonal projection onto $U \subseteq S$. Suppose $0 < \eta_1 < 1/16$ and*

$$\|pBP_U\|_{\mathrm{HS}}^2 \ge \dim U - \eta_1 d_S \quad (U \subseteq S), \qquad D\,\mathbb{E}_g\|B^*\rho(g)B\|_{\mathrm{HS}}^2 \le (1+\eta_1)d_S^2.$$

Then there is $t \subseteq G$ such that

$$|t|d_S \le \tfrac{5}{2}\left(1+\log\frac{1}{16\eta_1}\right)D, \qquad \dim\sum_{g\in t} p\rho(g)B(S) \ge (1-16\eta_1)D.$$

Proof. Set $A_g = \rho(g)BB^*\rho(g)^*$ and $C_{\mathrm{av}} = \mathbb{E}_g A_g$. Each A_g has finite rank. Cyclicity of the trace gives $\mathrm{tr}(A_gA_h) = \|B^*\rho(g^{-1}h)B\|_{\mathrm{HS}}^2$. Since $h \mapsto g^{-1}h$ preserves the uniform measure,

$$\|C_{\mathrm{av}}\|_{\mathrm{HS}}^2 = \mathbb{E}_{g,h}\,\mathrm{tr}(A_gA_h) = \mathbb{E}_g\|B^*\rho(g)B\|_{\mathrm{HS}}^2.$$

For $F \subseteq W$ of dimension αD, some g satisfies

$$\|P_F\rho(g)B\|_{\mathrm{HS}}^2 \le \mathrm{tr}(P_FC_{\mathrm{av}}) \le \sqrt{\alpha(1+\eta_1)}\,d_S \le (1+\alpha+\eta_1)d_S/2.$$

On the preimage $U = (p\rho(g)B)^{-1}F$, $P_F\rho(g)BP_U = p\rho(g)BP_U = \rho(g)pBP_U$. Restrict to U in the near-isometry inequality and compare with the full Hilbert–Schmidt norm. This gives $\dim U \le (1+\alpha+3\eta_1)d_S/2$. By rank-nullity in W/F, adding the new image adds at least $(1-\alpha-3\eta_1)d_S/2$ dimensions.

Let F_j be the span after j choices, with $F_0 = 0$. Put $m_j = D - \dim F_j$ and $a_1 = 2d_S/(5D)$. Apply the first moment inequality to $U = \ker(pB)$. Its left side vanishes, so

$$(1-\eta_1)d_S \le \operatorname{rank}(pB) \le D, \qquad 0 < a_1 \le \frac{2}{5(1-\eta_1)} < 1.$$

The dimension increment just proved gives

$$m_j > 16\eta_1 D \quad \Longrightarrow \quad m_{j+1} \le (1-a_1)m_j \le e^{-a_1}m_j.$$

Indeed the increment is at least $\frac{1}{2}(m_j/D - 3\eta_1)d_S \ge (13/32)(m_j/D)d_S \ge a_1 m_j$. It is positive at every step before termination, so the selected group element is new. Iteration up to the first $m_j \le 16\eta_1 D$, with $m_0 = D$, gives

$$j \le \left\lceil a_1^{-1} \log \frac{1}{16\eta_1} \right\rceil, \qquad jd_S \le \tfrac{5}{2}\left(1 + \log \frac{1}{16\eta_1}\right) D. \qquad \square$$

Proposition 16.2 (Span of the translates)**.** *For fixed q_0, d, ϵ and sufficiently large x, there is $t \subseteq G$ such that*

$$|t| \dim S \le 5D \log(1+2b^2), \qquad \dim \sum_{g\in t} P_g(S) \ge (1-4/b^2)D.$$

Here $P_g = c_B|G|c_{\mathrm{ss}}^{-1}p\rho(g)\Pi|_S$, where c_B is the positive rational scalar of Proposition 7.10 and $c_{\mathrm{ss}} \ne 0$ is the scalar in (73).

Proof. Use the Euclidean realization $C(X)$ of the full source, where $C = C_X R_H(\eta^{-1})$, with η as in Section 8, is the change of coordinates in Section 8. The geometric map is $B : C(X) \to T_{\mathrm{full}}$ and the projection onto the selected types is p. Section 9 gives, for $U \subseteq C(X)$,

$$\|pBP_U\|_{\mathrm{HS}}^2 \ge \dim U - 5b^{-4}d_X, \qquad D\,\mathbb{E}_g\|B^*\rho(g)B\|_{\mathrm{HS}}^2 \le (1+9b^{-4})d_X^2.$$

The free class coordinates have $d_X > 0$. Applying the first inequality to $U = \ker(pB)$ gives

$$\dim\ker(pB) \le 5b^{-4}d_X, \qquad D \ge \operatorname{rank}(pB) \ge (1-5b^{-4})d_X > 0.$$

The same inequality with $U = C(S)$ and the codimension estimate of Section 6 give

$$\|pBP_{C(S)}\|_{\mathrm{HS}}^2 \ge (1 - b^{-8} - 5b^{-4})d_X > 0.$$

Write $p = \sum_c p_c$, where c runs over the selected tuples at both old rational primes. Orthogonality gives $p_cB(C(S)) \ne 0$ for some c. This component has an allowed type at every chosen place and a primitive type at one place above each of the two primes. The commuting normal split Hecke operators away from the old level preserve the c-isotypic space, so some simultaneous eigenspace has nonzero projection of this component. Conversely, for any nonzero eigenvector $z \in pT_{\mathrm{full}}$, a summand $p_cz \ne 0$ has the same outside-level Hecke eigenvalues as z, as used in Section 28.1. Both error constants are at most $\eta_1 = (8b^2)^{-1}$ for large b.

The selected translates span all but $2D/b^2$ dimensions, and

$$|t|d_X \le \tfrac{5}{2}(1+\log(b^2/2))D \le 5D\log(1+2b^2).$$

Section 6 gives $\dim(X/S) \le fd_X/(N-1) \le b^{-8}d_X$. The operator $R_H(\eta^{-1})$ commutes with the central scalar averages defining S, so the restricted source in the Euclidean coordinates is $C(S)$. The maps $p\rho(g)B$ induce the surjection

$$\bigoplus_{g\in t} C(X)/C(S) \twoheadrightarrow \frac{\sum_{g\in t} p\rho(g)B(C(X))}{\sum_{g\in t} p\rho(g)B(C(S))}.$$

The dimension of the quotient is therefore at most

$$|t|\dim(X/S) \le |t|b^{-8}d_X \le \tfrac{5}{2}b^{-8}(1+\log(b^2/2))D \le 2D/b^2.$$

With $v_G = \mathrm{vol}(J)$ and ν the unipotent covolume, the Euclidean coordinates of Section 8.5 give

$$B = (v_G\nu)^{-1/2}\Pi C^{-1},$$
$$P_g = c_B|G|c_{\mathrm{ss}}^{-1}p\rho(g)\Pi|_S = c_B|G|c_{\mathrm{ss}}^{-1}(v_G\nu)^{1/2}p\rho(g)BC|_S.$$

The scalar is nonzero. Hence $\operatorname{im} P_g = p\rho(g)B(C(S))$, giving the span estimate on the datum of Section 16.1. □

For this set t, lift each residue translation integrally at the specified split places, with the identity at all other finite places. Its action depends only on its residue by principal-level invariance. The vectors $P_g\phi$, $g \in t$, with ϕ among the everywhere integral generators of Lemma 7.5, span the displayed sum. A maximal independent subset therefore gives a complex basis of it before any coefficient place or twist is chosen.

Part III Theta identities and the upper covolume bound

17 The Weil action and the measured seesaw

In this section, we verify rational invariance and relate the doubled Weil action to the normalized theta integral. This fixes the scalar and measures in the comparison with the Poincaré map of Section 8.

17.1 Rational invariance for the Weil action

In this subsection, we verify rational invariance of the Weil action used in Section 17.2. We compute the Levi scalar in a split polarization and pass to a general Hermitian space through $B \perp B^-$.

Lemma 17.1 (Composition of the normalized local Weil operators)**.** *Let g, h belong to the local unitary group of the given Hermitian space. In the signed real and normalized finite coordinates write $T(x)$ for the symmetric Weyl displacement and v_0 for the unit vacuum. There is a unitary operator U_g implementing $T(x) \mapsto T(gx)$ with*

$$\langle v_0, U_g v_0\rangle = 1/\overline{\Delta_g}.$$

Here $\Delta_g = \overline{\det d_g}$ at a real place, where d_g is the negative diagonal block of g. At a finite place v of F_0, put $E_v = E \otimes_{F_0} F_{0,v}$ and $L = \mathcal{O}_{E_v}^n$ in the normalized coordinates. The norm and character on $E_v^\times$ are products when v splits. Every nonsplit finite place is unramified by Section 4.2. The finite multiplier is $\Delta_g = (v_v| \cdot |_{E_v}^{-1/2})(\det H_g)$, where $H_g L = L \cap g^{-1}L$. Any other choice is $H_g k$ with $k \in \mathrm{GL}(L)$. Both v_v and $|\cdot|_{E_v}$ are trivial on $\det k$, since v_v is unramified. Then $U_g U_h = U_{gh}$, also on the restricted tensor product.

Proof. Write $e(x) = \langle v_0, T(x)v_0\rangle$, which is real and even in both models. For the self-dual displacement measure,

$$P_0 = |v_0\rangle\langle v_0| = \int \overline{e(x)} T(x)\, dx.$$

Indeed, in Schrödinger coordinates $x = (a, b)$, integration over the modulation b is Fourier inversion, and integration over the translation a gives $f \mapsto v_0\langle v_0, f\rangle$. By the Heisenberg intertwining theorem [77, no. 10, Theorem 1, p. 157], an isometry g has a unitary implementer unique up to a scalar. Its image of P_0 by conjugation is $Q_g = \int e(g^{-1}x)T(x)\, dx$. It satisfies

$$\langle v_0, Q_g v_0\rangle = \int e(x)e(g^{-1}x)\, dx = |\Delta_g|^{-2}.$$

At a finite place the integral is $\mathrm{vol}(L \cap gL) = [L : L \cap g^{-1}L]^{-1}$, since g preserves volume and the lattice phases agree. At a real place $e(x) = e^{-x^*x/2}$. With P the positive coordinate projection, set $D(a) = P + (I - P)a$. Pseudo-unitarity gives $(I + a^*a)/2 = D(a)^*D(a)$, so the normalized complex Gaussian integral is $|\det d_g|^{-2}$ after the substitution $x = gy$. Consequently $U_g v_0 = \Delta_g Q_g v_0$ fixes the unique required phase. For $k = gh$, covariance gives $U_g Q_h U_g^{-1} = Q_k$, whence

$$U_g U_h = c(g, h)U_k, \qquad c(g, h) = \Delta_g \Delta_h \overline{\Delta_k}\langle v_0, Q_k Q_g v_0\rangle, \qquad |c(g, h)| = 1.$$

Writing $T(x)T(y) = \beta(x, y)T(x + y)$ and substituting $u = -k(x + y),\ v = kx$ in this last double integral gives

$$\langle v_0, Q_k Q_g v_0\rangle = \iint e(hx)e(ky)\beta(x, y)e(x + y)\, dx\, dy. \tag{57}$$

Indeed $k^{-1}u = -(x + y),\ g^{-1}v = hx,\ u + v = -ky$ and $\beta(u, v) = \beta(x, y)$ by alternation and unitary invariance.

At a finite place write $c_0 = (1 - j_{\mathrm{W}}^{-1})/2$, $\psi_0(z) = \psi_{E,v}(j_{\mathrm{W}}\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}z)$, and $e_0(x) = \psi_0(c_0 x^*x/2)$. $c_0 = (1 + \sqrt{-d})/2$ is integral and satisfies $c_0 + \bar{c}_0 = 1$. The different identity of Section 8.1 makes ψ_0 self-dual on $\mathcal{O}_{E_v}$, and the skew sign of j_{W} gives $\psi_0(\bar{z}) = \psi_0(z)^{-1}$. In these coordinates $\beta(x, y) = \psi_0(y^*x/2)$ and $e = \mathbf{1}_L e_0$. The finite translation fixing the vacuum for integral x is $T_\sharp(x) = e_0(x)T(x)$, as in the finite vacuum formula of Section 7.4.3. Its multiplier is $e_0(x)e_0(y)e_0(x + y)^{-1}\beta(x, y)$. Finite unitary isometries preserve x^*x, so both translations have the same unitary covariance, including at places above two. The character ψ_0 is trivial on $\mathcal{O}_{E_v}$, so $e_0^2 = 1$ on L. Put

$$D = \{(x, y) : hx \in L,\ ky \in L,\ x + y \in L\}.$$

On this compact open lattice $e_0(hx) = e_0(x)$, $e_0(ky) = e_0(y)$ and $e_0(x + y) = e_0(x + y)^{-1}$. Expanding the three quadratics in (57) gives the integrand $\psi_0(\bar{c}_0 y^*x)$ on D and zero elsewhere.

At a split place $g = (A, A^{-\mathsf{t}})$. If the elementary divisor exponents of A are a_i, the intersection exponents at the two places are $\max(0, -a_i)$ and $\max(0, a_i)$. Thus $\Delta_g/|\Delta_g| = v_w(\det A)^{-1}$ at

the chosen factor w, so $\Delta_g\Delta_h\overline{\Delta_k} > 0$. Writing $D = D_+ \times D_-$ at the two factors, the phase $\psi_0(\bar{c}_0 y^* x)$ is a bicharacter between D_+ and D_-. Integration over D_- gives its volume times the indicator of its annihilator in D_+, which contains an open subgroup. Hence $c(g,h) > 0$, and therefore $c(g,h) = 1$. This argument applies at the dyadic split places.

At a nonsplit place let q be the cardinality of the residue field of $F_{0,v}$. The extension is unramified quadratic, and $\Delta_g = (-q)^{a_g}$, where $a_g = \operatorname{length}_{\mathcal{O}_{E_v}} L/(L \cap g^{-1}L)$. Here we used $\upsilon_v(\varpi) = -1$ on a base uniformizer. Choose $m \geq 0$ so that $B = (\varpi^m L)^2 \subset D$ and the phase is constant on cosets of B in D. The coordinates of x, y on D have valuations bounded below. Taking m sufficiently large makes the change in $\bar{c}_0 y^* x$ integral after adding a vector of B. With $\operatorname{vol}(L) = 1$ and residue cardinality q^2 over E_v, (57) gives

$$q^{4nm} c(g,h) = (-q)^{a_g + a_h + a_k} \mathcal{S}, \qquad \mathcal{S} = \sum_{(x,y) \in D/B} \psi_0(\bar{c}_0 y^* x).$$

Simultaneous multiplication by an integral unit of norm -1 conjugates every summand, so $c(g,h)$ is real and equals ± 1. Such a unit exists by norm surjectivity on unramified units. The Teichmüller group of order $q+1$ and norm one fixes the summands and acts freely on $(D/B) \setminus \{0\}$, since $\xi - 1$ is a unit for every nonidentity member ξ. Consequently $\mathcal{S} \equiv 1 \pmod{(q+1)\mathcal{O}'}$, where $\mathcal{O}'$ is the ring of integers of the cyclotomic field generated by the finitely many phase values. Reducing the preceding equality gives $c(g,h) \equiv 1 \pmod{(q+1)\mathcal{O}'}$. Since $(q+1)\mathcal{O}' \cap \mathbb{Z} = (q+1)\mathbb{Z}$ and $q + 1 > 2$, this forces $c(g,h) = 1$.

At a real place put $J = 2P - I$, $Q = I - P$, and $C_a = (I + a^* a)/2 = D(a)^* D(a)$. In these coordinates $\beta(x,y) = \exp((y^* J x - x^* J y)/2)$. The integral in (57), with measure $\pi^{-2n} dx\, dy$ in real Lebesgue coordinates, has matrix

$$A = \begin{pmatrix} C_h & P \\ Q & C_k \end{pmatrix}, \qquad A + A^* = \operatorname{diag}(h^* h, k^* k) + \begin{pmatrix} I & I \\ I & I \end{pmatrix} > 0.$$

Thus its value is $(\det A)^{-1}$ by the complex Gaussian integral (first for positive Hermitian matrices and then by holomorphic continuation on matrices with positive Hermitian part). Write $a = \left(\begin{smallmatrix} a_{11} & a_{12} \\ c_a & d_a \end{smallmatrix}\right)$ in signed coordinates. Since $D(a) = \left(\begin{smallmatrix} I & 0 \\ c_a & d_a \end{smallmatrix}\right)$, the off-diagonal blocks of C_a^{-1} are $-c_a^* d_a^{-*}$ and $-d_a^{-1} c_a$. The Schur complement and $g = k h^{-1}$ give

$$\begin{aligned} \det A &= |\det d_h \det d_k|^2 \det(I - d_h^{-1} c_h c_k^* d_k^{-*}) \\ &= |\det d_h \det d_k|^2 \frac{\det(d_h d_k^* - c_h c_k^*)}{\det d_h \det d_k^*} = \overline{\det d_h \det d_g} \det d_k. \end{aligned}$$

This is precisely $\Delta_g \Delta_h \overline{\Delta_k}$, also when either signed summand is zero. Hence $c(g,h) = 1$. The map $g \mapsto e(g^{-1}\cdot)$ is continuous in L^1. Since Δ_g is continuous, $U_g v_0 = \Delta_g Q_g v_0$ is norm continuous. Covariance gives continuity on the span of the Weyl translates of v_0, which is dense by irreducibility. Unitarity extends this to every vector. The operators preserve Schwartz space by [77, nos. 11–12, Proposition 2, pp. 157–159]. The finite integral stabilizers fix v_0 since $\Delta_g = 1$ there, giving the restricted tensor product representation. □

Let B be the given Hermitian space and ω_B its Weil action before inserting determinant characters. Write J_B for its rational Gram matrix, X_B for its point matrix, and $\Gamma_B = X_B^{-1} E^{\dim B}$ in adelic coordinates before the local rescalings. A rational unitary element γ acts in these

coordinates by $X_B^{-1}\gamma X_B$. For $x, y \in E^{\dim B}$ the symmetric Weyl displacements have adelic multiplier

$$\beta(X_B^{-1}x, X_B^{-1}y) = \psi_E(j_{\mathrm{W}} y^* J_B x/2) = 1.$$

Indeed the finite factors of $\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}$ cancel the two normalized coordinate factors, and the signed real coordinates give the infinite factors of the same rational trace character. Moreover $\beta(z, X_B^{-1}y)^2 = \psi_E(j_{\mathrm{W}} y^* J_B X_B z)$. As y varies, the covectors $j_{\mathrm{W}} y^* J_B$ fill $E^{1\times \dim B}$. The annihilator of E in $\mathbb{A}_E$ under ψ_E is E, so triviality of all these characters is equivalent to $z \in \Gamma_B$. In a rational Schrödinger polarization the functional θ_B is a nonzero multiple of rational summation. Indeed rational modulations force its support to be on the rational lattice by adelic self-duality. At each support point the modulation differentials span the real cotangent space and eliminate derivatives of point masses. Rational translations make all the point masses equal. Heisenberg covariance implies that $\theta_B \omega_B(\gamma)$ has the same rational Heisenberg invariance for $\gamma \in U(B)(F_0)$. Thus

$$\theta_B \omega_B(\gamma) = c_B(\gamma)\theta_B,$$

where c_B is a character by Lemma 17.1. We evaluate it using the multipliers of the action.

First let B be split, with rational dual isotropic bases k, ℓ of rank p. In the mixed model $f[x]$ is the theta value after displacement by ℓx and averaging the k-modulations on the additive quotient with probability Haar. In the Schrödinger realization write $\langle \xi, u\rangle$ for the character of the modulation parameter u at ξ. With $[E^p] = E^p \backslash \mathbb{A}_E^p$, rational summation gives

$$f[x] = C \sum_{\xi \in E^p} f_{\mathrm{Sch}}(x+\xi) \int_{[E^p]} \langle \xi, u\rangle \, du = C f_{\mathrm{Sch}}(x).$$

For fixed x, compact finite support restricts the sum to a translate of a fractional lattice, on which the archimedean Schwartz bound is summable. Character orthogonality then gives the second equality. In particular

$$\theta_B(f) = \sum_{x \in E^p} f[x], \qquad \int \overline{f[x]} g[x] \, dx = |C|^2 \langle f, g\rangle.$$

The original theta functional has mean square one on the Heisenberg translates of a unit vacuum. Orthogonality of the rational modulation characters and unfolding the translation variable give $|C|^2 = 1$, as computed in Section 41.5.1.

For the Levi $m(A)$ acting by A on k and A^{-*} on ℓ, covariance gives $(\omega_B(m(A))f)[x] = c(A) f[A^* x]$. Composing with the inverse substitution commutes with all Heisenberg operators, so its effect is scalar. The Jacobian and unitarity give $|c(A)|^2 = |\det A|_{\mathbb{A}_E}$.

For the phase, let h carry the signed orthonormal frame of the standard hyperbolic model to the adapted frame in B, and test on $\omega_B(h)v_0$. By unitarity its matrix coefficient at $m(A)$ is

$$\langle \omega_B(h)v_0, \omega_B(m(A))\omega_B(h)v_0\rangle = \langle v_0, \omega_B(h^{-1}m(A)h)v_0\rangle.$$

This identity holds for an adelic frame change, including one that moves the original finite lattice. At infinity the change from the signed orthonormal basis to the hyperbolic coordinates, up to the ordering of signs, uses normalized sums and differences. The negative block of the Levi in the adapted signed basis is $(A^{-*} + A)/2$. Its conjugate determinant times $\det A$ is the

positive determinant of $(I + A^*A)/2$. At finite places an integral change from the hyperbolic Gram to the identity Gram is given by the matrix

$$\begin{pmatrix} I & -cI \\ \varepsilon I & c\bar{\varepsilon} I \end{pmatrix}, \qquad c\bar{c} = -1, \qquad \varepsilon + \bar{\varepsilon} = 1.$$

The norm and trace conditions have integral solutions with c a unit. This holds even at the places over two: at an unramified quadratic place use the residue norm and trace followed by successive lifts, and at a split place use the product and sum. In our family 2 splits in K by Section 4. Since $E = F_0K$, for every $v \mid 2$ one has $E \otimes_{F_0} F_{0,v} = F_{0,v} \times F_{0,v}$, with the two factors exchanged by conjugation. The choices $c = (1, -1)$, $\varepsilon = (1, 0)$ give $c\bar{c} = -1$ and $\varepsilon + \bar{\varepsilon} = 1$. Multiplying by the hyperbolic Gram and its adjoint gives the identity Gram, and the determinant of the change is c^p. It preserves the elementary divisors of the lattice intersection. In these coordinates, with $L = \widehat{\mathcal{O}}_E^p$ and $\chi_f = v_f| \cdot |_f^{-1/2}$, the finite multiplier of A is

$$\Delta_f(A) = \chi_f(\det H), \qquad HL = L \cap A^{-1}L.$$

Since v_f is unramified, changing H by an integral basis change leaves $\Delta_f(A)$ unchanged. Inversion replaces this intersection by AHL, transpose preserves its elementary divisors and conjugation conjugates the multiplier. Hence the finite multiplier of the pair is

$$\Delta_f(A^{-*})\Delta_f(A) = \overline{\chi_f(\det A)}|\Delta_f(A)|^2.$$

Together the real and finite calculations give $\Delta v(\det A) > 0$, where Δ is the multiplier of the Levi written in these adapted coordinates.

The adapted vacuum has slice a nonzero scalar times a positive real Gaussian and a finite integral-box indicator. At infinity this follows by computing the Heisenberg coefficient of the standard Gaussian using the orthonormal hyperbolic basis: the cross terms cancel in the product of the paired coordinates. At finite places use normalized displacement coordinates, dividing the original coordinates by α_{W}. The self-dual lattice is L in these coordinates by the different identity of Section 8.1. Indeed let $w \mid p$, $\mathfrak{D}_w = \mathfrak{D}_{E_w/\mathbf{Q}_p}$, and $a_w = (j_{\mathrm{W}}\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}})_w$. In the scalar trace pairing for the usual character of $\mathbf{Q}_p$ of conductor $\mathbf{Z}_p$, the annihilator of $\mathcal{O}_{E_w}$ is $\mathfrak{D}_w^{-1}$. Fourier inversion gives this ring volume $|\mathfrak{D}_w|_w^{1/2}$ in the self-dual measure. Scaling the pairing by a_w changes the volume to $|a_w\mathfrak{D}_w|_w^{1/2} = 1$. Conjugation identifies the dual coordinate with the opposite component at a split pair. At $p = d$ one has $\alpha_{\mathrm{W}} = 1$ locally and $\mathfrak{D}_w = (\sqrt{-d})$. At $p \mid q_1$ the conjugate halves supply the inverse relative different through $\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}$, also at $p = 2$. The k- and ℓ-displacements with normalized parameters in L fix the vacuum, by isotropy and the integral change of basis just computed.

In the mixed realization the modulations force support in the self-dual box and translations force constancy there. The product of the vacuum slice and its Levi substitute therefore has positive integral. By the slice inner-product formula, the phase of $c(A)$ is the phase of the vacuum coefficient $1/\bar{\Delta}$ of the Levi in these coordinates. Thus

$$(\omega_B(m(A))f)[x] = v(\det A)^{-1}|\det A|_{\mathbb{A}_E}^{1/2} f[A^*x]. \tag{58}$$

For rational A, the scalar is one by the product formula and the triviality of v on $E^\times$. The substitution permutes the rational points of summation. Each element of either opposite unipotent radical is a commutator with a rational scalar Levi: conjugation by the Levi with

scalar 2 multiplies the additive radical parameter by 4 or 1/4. The character c_B is therefore one on both radicals.

To see that these elements and the Levi generate, write the image of k under a rational isometry as $ka + \ell b$, so $a^*b + b^*a = 0$. In a complex embedding $(a+b)^*(a+b) = a^*a + b^*b > 0$ since the image has rank p. The polynomial $\det(a + Tb)$ is nonzero. Choose a skew $t \in E$ where it is nonzero. One radical changes the upper block to $A = a + tb$. The other, with parameter $-bA^{-1}$, makes the lower block zero. This parameter is skew by the isotropy equation after the first operation. The remaining element stabilizes k and belongs to its Siegel parabolic, generated by the Levi and its radical. This proves $c_B = 1$ for split B.

For general B, the rational diagonal and antidiagonal split $B \perp B^-$, where B^- has the opposite form. In direct-sum coordinates $(\gamma, 1)$ acts on product vectors as $\omega_B(\gamma) \otimes 1$. To verify the scalar, the real negative minors multiply, the finite lattice intersections are direct sums, and the vacuum is a product. The nonzero vacuum coefficient is therefore the product coefficient. Covariance and irreducibility identify the two intertwiners. Rational summation on the product lattice factors. Since each theta functional is a nonzero multiple of rational summation, there is a fixed $C_\oplus \neq 0$ with

$$\theta_{B \perp B^-}(f \otimes f^-) = C_\oplus \, \theta_B(f) \theta_{B^-}(f^-).$$

Choose f^- with $\theta_{B^-}(f^-) \neq 0$. Split invariance for $(\gamma, 1)$ cancels the same factor $C_\oplus \theta_{B^-}(f^-)$ on the two sides. This proves

$$\theta_B \omega_B(\gamma) = \theta_B \quad (\gamma \in U(B)(F_0)).$$

For the pair S, T use $B = \mathrm{Hom}_E(S, T)$. The determinant characters in the pair action are one at rational arguments. In particular rational invariance holds for the normalized theta kernel after all the automorphic twists used below.

17.2 The doubled action in the Siegel–Weil formula

In this subsection, we compute the Levi scalar in the doubled Weil action using the rational normalization of Section 17.1. This scalar determines the normalization of the paired lift in Section 34.2.

Write $n = \mathrm{rk}_E S$, $s = \mathrm{rk}_E T$. Take the mixed model for a rational Lagrangian of $S \perp S^-$. For a source Siegel Levi $m(A)$ and a target isometry h, the argument of the mixed-model function becomes $h^{-1} X A$, with determinant $(\det h)^{-n} (\det A)^s$ on the space of s-by-n matrices. On the isotropic frame in (58), the action in target coordinates is $Y \mapsto hYA^*$. Indeed, h acts on the output and precomposition by $m(A)^{-1}$ acts by A^* on this frame of source covectors. Thus the frame block has determinant $d = (\det h)^n \overline{\det A}^{\,s}$. Since $\det h$ has norm one and $\upsilon(\bar{a}) = \upsilon(a)^{-1}$, its scalar is

$$\upsilon(d)^{-1} |d|_{\mathbb{A}_E}^{1/2} = \upsilon(\det h)^{-n} \upsilon(\det A)^s |\det A|_{\mathbb{A}_E}^{s/2}.$$

The target character $\upsilon(\det h)^n$ of the doubled pair cancels the first factor.

The Siegel parabolic preserves the line of evaluation at zero. Indeed it preserves the modulation subgroup, whose invariant distributions in this model are the multiples of evaluation at zero. Its action on this line is a character. Conjugation by the scalar Levi $2I$ scales the upper radical parameter by 4 or 1/4, so the radical consists of commutators and acts trivially there. Write $u(Z)$ for the radical element $\ell \mapsto \ell - k j_{\mathrm{W}}^{-1} Z$ with $Z = Z^*$. Moving the displacement by

X through $u(Z)$ adds a k-displacement absorbed by the average and gives

$$(\omega(u(Z))f)[X] = \psi_E\Big(\tfrac{1}{2}\operatorname{tr}(ZX^*J_TX)\Big)\, f[X].$$

Here $\psi_E = \psi_{F_0} \circ \operatorname{Tr}_{E/F_0}$ for the rational trace characters. The matrix trace inside the phase is fixed by conjugation. To identify the unipotent coordinate, convert Yamana's matrix convention by $g \mapsto (g^*)^{-1}$. The ordered basis $(2j_{\mathrm{W}}\ell, k)$ has Hermitian Gram matrix $-2j_{\mathrm{W}}\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Removing the scalar gives the skew-Hermitian matrix of the classical model with the same isometry group. The upper unipotent $n(b)$, for $b = b^*$, sends $2j_{\mathrm{W}}\ell$ to $2j_{\mathrm{W}}\ell - kb$ and fixes k. Thus $Z = b/2$. The mixed variable on k is unchanged and

$$\psi_E\Big(\tfrac{1}{2}\operatorname{tr}(ZX^*J_TX)\Big) = \psi_{F_0}(\operatorname{tr}(bX^*J_TX/2))\,.$$

This is the unipotent law in [79, §1.3]. The rational change of basis preserves probability Haar on the rational additive quotient. The dual Fourier measures transport with the pairing. In the opposite polarization the law follows by self-dual Fourier transform, with its scalar canceled by conjugation.

These are the Schrödinger laws of the splitting character υ^s with the additive character. The two opposite parabolics generate over each local field. For a nonsplit place the block proof of Section 17.1 works after a Siegel Levi move and a change of frame. Choose invertible P, Q with $b' = PbQ = \operatorname{diag}(I_j, 0)$, and put $a' = P^{-*}aQ$. Isotropy gives $a'_{12} = 0$, and joint injectivity gives invertible a'_{22}. Thus

$$\det(a' + Tb') = \det(a'_{11} + TI_j)\det(a'_{22}) \neq 0 \quad \text{in } E_v[T].$$

Choose a skew scalar t outside its finite set of roots. In the original frame the required upper radical parameter is $N_t = tP^*P$, which is skew, and $a' + tb' = P^{-*}(a + N_tb)Q$. Put $A = a + N_tb$. Isotropy gives $A^*b + b^*A = 0$, so $-bA^{-1}$ is skew. The lower radical with this parameter sends $\binom{A}{b}$ to $\binom{A}{0}$, placing the resulting isometry in the upper parabolic. At a split place the two block radicals and their common Levi contain the elementary transvections and invertible diagonal matrices of $\mathrm{GL}_{2n}(F_{0,v})$. The actions agree on the integral vacua outside a finite set.

At infinity the Fock polynomial space in [80, pp. 236–237] is relative to the chosen positive and negative orthogonal subspaces. The corresponding signed decompositions of the doubled source and of the target can each be carried to the original ones by an isometry: map signed orthonormal bases to signed orthonormal bases. Write g_0, h_0 for these real isometries. On the Hom space their action carries the complex coordinates, the Gaussian and its polynomial multiples to those of the transported Fock model. Thus the original tensor is $\omega(g_0, h_0)\Phi_0$ for a polynomial vector Φ_0 in the cited model. The cancellation of $\upsilon(\det h_0)^{-n}$ by the doubled target character gives $(\omega(1, h_0)\Phi)(X) = \Phi(h_0^{-1}X)$. This fixes evaluation at zero and is absorbed by right translation in the target quotient integral. By real Iwasawa decomposition one may take $g_0 \in P_\infty$, the Siegel parabolic, absorbing its right compact factor into Φ_0. Let K be the maximal compact of the doubled source in Yamana's model, with the given finite factors, and put $K' = g_0Kg_0^{-1}$. Writing $u = pk'$ gives $|a_{K'}(u)| = |a_K(ug_0)|/|a_K(g_0)|$. With $s_0 = (s-n)/2$, the sections satisfy

$$f^{(z),K'}_{\omega(g_0,h_0)\Phi_0}(u) = |a_K(g_0)|^{s_0-z} f^{(z),K}_{\Phi_0}(ug_0).$$

The factor is one at $z = s_0$. This transports the polynomial identity also at a holomorphic value obtained by continuation.

In [80, Theorem 2] the division algebra is E of index one. The splitting character restricts to χ_{E/F_0}^s. At $s_0 = (s-n)/2$ the normalized Siegel section is $g \mapsto (\omega(g,1)f)[0]$, and the theta integrand is

$$\sum_{X \in \mathrm{Hom}_E(E^n, T)} (\omega(g,h)f)[X].$$

The zero value and the rational sum use the same mixed realization, so a fixed scalar in this realization multiplies both sides of the Siegel–Weil identity. In every strict-range doubling below $s > 2n$, so

$$s - \mathrm{Witt}(T) \geq s/2 > n.$$

The ordinary theta integral converges absolutely for every Schwartz–Bruhat test by [78, §51, Proposition 8, p. 75]. The spherical Eisenstein majorant converges since $s_0 > n/2$. In Weil's case I_2, the parameter is $1/2$, so his dimension condition is $s > 2n$ [78, §§26–27].

At an old split place v, put $k = F_{0,v}$. A full-rank tuple consists of an injection $X : k^n \to k^s$ and a surjection $Y : k^s \to k^n$, with Gram matrix $B = YX$ in one split factor. In $k^s = k^n \oplus k^n \oplus k^{s-2n}$, take

$$X(a) = (a, 0, 0), \qquad Y(a, b, c) = Ba + b.$$

These maps realize every B, including $B = 0$, with X injective and Y surjective. For two pairs (X_i, Y_i) with the same B, match $X_1 a$ with $X_2 a$. Since

$$\operatorname{im} X_i \cap \ker Y_i = X_i(\ker B),$$

extend this matching on the intersections to an isomorphism $\ker Y_1 \to \ker Y_2$, hence to the sums $\operatorname{im} X_i + \ker Y_i$. Complete it by matching lifts of the same basis of a complement of $\operatorname{im} B$ in k^n under Y_1, Y_2. The resulting $g \in \mathrm{GL}_s(k)$ satisfies $gX_1 = X_2$ and $Y_2 g = Y_1$. Thus $U(T)(k)$ acts transitively on the full-rank part of every Gram fibre and preserves the probability Haar measure on the adelic quotient.

For polynomial Fock vectors use [80, Theorem 2]. For arbitrary real Schwartz vectors use [78, §52, Theorem 5, pp. 76–77]. Both formulas give coefficient one with probability quotient measure. In Section 30.2.1, the doubled polynomial Fock datum has definite Hermitian target and the target quotient is compact. For $m = n = 2$ the holomorphic value is $E(g, 0) = 2I(g)$ with the ordinary theta integral and probability target measure [80, Theorem 2, p. 237] [79, §1.3, p. 1007, and Proposition 5.8(i), p. 1015].

17.3 Translation and scalar normalization in the seesaw

In this subsection, we fix the normalization of the theta operators used in the seesaw. The source index is the one in Section 18.3, and equivariance carries the finite group action through the entire composition.

The lower chain is a composition of the theta operators $L_{\xi,j}$ from Y_j to Y_{j+1} with the tame character at the j-th arrow, followed by the terminal algebraic coefficient map. Each $L_{\xi,j}$ is the original theta integral multiplied by the source index $A_{\xi,j} = [K_j^\circ : J_{\xi,j}]$, where $J_{\xi,j}$ is the principal subgroup at the old and twist places. In particular, split reduction gives

$$A_{\xi,j} = \left|\mathrm{GL}_{r+jt}(\mathcal{O}_E/(\pi_M Q(\xi)))\right|.$$

Section 18.3 computes the transpose with this very index. Here Y_m is the lower-chain endpoint, of a different rank from the dimension-comparison space W_m. Use the lattice inclusion ι_j between the standard finite coordinates of Y_j, Y_{j+1}. At the original level places and the additional

twist places, the lattice test has the inclusion reduction $\bar{\iota}_j$. The finite twist factor of the Weil action on an integral pair is the quotient of the tame character of its two determinants.

Lemma 17.2 (Equivariance of the weighted lower lifts)**.** *For each $s \in G$ there are integral lifts $k_j(s)$ with first-block residue* $\mathrm{diag}(s, I)$ *at the old places and identity at the twist places. They satisfy*

$$L_{\xi,j}\rho_j(k_j(s)) = \rho_{j+1}(k_{j+1}(s))L_{\xi,j}.$$

The successive intertwinings give the action of the original finite projector on the terminal values.

Proof. For $s \in G$, choose integral lifts $k_j(s)$ on each lower space with reduction $\mathrm{diag}(s, I)$ in the first-block coordinates at the original places and identity at the twist places. Such lifts exist because these places are split. At each step

$$\overline{k_{j+1}(s)}\,\bar{\iota}_j = \bar{\iota}_j\,\overline{k_j(s)}.$$

The simultaneous translate of the lattice test therefore preserves its support: integral matrices are carried to integral matrices, and the inclusion reduction is fixed.

The two reduced determinants agree, since the extra diagonal block is the identity. Their quotient therefore has trivial tame character. Away from these places take the pair to be the identity. The accompanying real test is unchanged. In the theta kernel, apply the finite source translation by changing the source integration variable. The Haar measure is right invariant and the simultaneous action on the datum has just been shown to fix it.

Thus

$$L_{\xi,j}\rho_j(k_j(s)) = \rho_{j+1}(k_{j+1}(s))L_{\xi,j}.$$

The principal kernel $J_{\xi,j+1}$ in the target variable fixes the inclusion test and its tame determinant character, as in Section 18.3. Thus the image of $L_{\xi,j}$ is fixed by $J_{\xi,j+1}$. Moreover, for $s, t \in G$,

$$k_j(ts)^{-1}k_j(t)k_j(s) \in J_{\xi,j}.$$

Together with principal-kernel invariance this gives the action of G on the translated coefficients independently of the integral lifts. The conversion to the algebraic coefficient at the definite endpoint commutes with this finite action. Write L_ξ for the resulting composed lower chain. Expanding the finite-type projector $p = \rho(e_0)$ in the original group algebra gives the identity of terminal values

$$(L_\xi(|G|p\rho(s)\Pi\phi))(h) = \sum_{t\in G} |G|e_0(t)(L_\xi\Pi\phi)(h\,k_m(ts)).$$

Indeed $k_m(t)k_m(s)$ and $k_m(ts)$ have the same reduction, so they act identically on the right $J_{\xi,m}$-invariant endpoint. This uses the same measured Π for every translate. □

Fix c_0, C_0 independently of ξ. Suppose the unfolded pairing gives, on the original source, $L_\xi\Pi\phi = c_0\,\mathcal{U}_\xi(T_\xi)\phi$, where $\mathcal{U}_\xi$ is the upper chain with its terminal restriction and is linear in its tensor. Suppose the tensor is a finite combination of translated vacua $T_\xi = C_0 \sum_z b_{\xi,z} v_z$, and put $B_* = M^{fn_m^2}$ for terminal upper rank n_m. If $B_*^{2m} b_{\xi,z}$ is integral off the twist, and the untwisted sum of coefficient moduli in each complex embedding is at most B_*^m, then the coefficients $B_*^{2m} b_{\xi,z}$ give the integral tensor with untwisted bound B_*^{3m}.

The common scalar multiplying that tensor is c_0C_0/B_*^{2m} when this is nonzero. If $c_0C_0 = 0$, use the zero integral tensor and scalar one. In either case the scalar is nonzero and independent

of the twist. Support and invariance under the upper levels are inherited from the same evaluated tensor, provided its translated vacua have the prescribed support. The unfolded pairing is (72). Section 19.3 supplies the tensor expansion and its coefficient bounds.

17.4 Rational orbits for consecutive lifts

In this subsection, we describe the rational maps that contribute to the Bessel integral for two consecutive Hermitian spaces. The character of Section 16.1 determines these maps up to the Bessel actions, which are used in Section 18.1.

Use the rational graded coordinates of the two consecutive spaces Y, Y', with Hermitian spaces V, V' and string lengths $s, \max(0, s-1)$, respectively. For $s \geq 1$ write $a_i = s-1-2i$ and $a'_b = s-2-2b$ for the string grades, starting at index zero. The orthogonal definite blocks have grade zero. The Hermitian forms are conjugate-linear in the first argument. On a string, $i^\vee$ denotes the reversed index and the Gram matrix has entries $g_i = h(e_i, e_{i^\vee}) = 1$, except at a middle index, where $g_i = -N$. The definite orthonormal coordinates are their own dual indices. A matrix has degree at least d if its entry is zero when the target grade minus the source grade is less than d.

Let $c \in F_0^\times$ be the string constant and $T : V \to V'$ the degree--1 map which kills the definite block and the last string vector. Its other string entries are $Te_i = t_i e'_i$, with

$$t_i = \begin{cases} 1, -Nc, c & s \text{ odd, in grades } > 0, = 0, < 0, \\ c, 1 & s \text{ even, in grades } > 0, < 0. \end{cases}$$

Taking the adjoint in the antidiagonal Gram matrices gives $T^\dagger T e_i = t_i t_{s-2-i}(g'_i/g_{i+1}) e_{i+1}$ for $i < s-1$. This coefficient is c, except at an odd string's grade zero, where it is $-Nc$, and at an even string's grade one, where it is $-Nc^2$. In particular all the shift coefficients are nonzero. Write R, R' for the Bessel subgroups over F_0.

Every element of R fixes each nonnegative-grade string basis vector modulo vectors of strictly higher grade. The same property holds for R' on its string. The matrices of these elements and their inverses have degree at least zero. Indeed the stated column property preserves the positive flag and, by orthogonality, the corresponding negative flag. On the grade-zero quotient the orthogonal complement of the central string vector, when present, can carry the definite unitary block. Write $U_R \subset R$ for the unipotent group of matrices whose difference from the identity has degree at least one.

Say that w agrees with T on the positive columns if $w_{bi} = T_{bi}$ for $a_i > 0$ and $a'_b < a_i$. If w agrees there and $\gamma \in R(\mathbb{A}_{F_0})$, then $w\gamma$ also agrees: the positive columns are stable under γ, and $T(\gamma - 1)$ has degree at least zero on them. For $\delta \in U(V')(\mathbb{A}_{F_0})$,

$$\delta w \gamma \text{ agrees with } T \quad \Longleftrightarrow \quad \delta \in R'. \tag{59}$$

For the forward implication put $\widetilde{w} = w\gamma$. In a nonnegative string column k of V', use the positive source pivot i with $a_i = a'_k + 1$ and $Te_i = t_i e'_k$. For every index b with $a'_b \leq a'_k$, the entry $((\delta - 1)\widetilde{w})_{bi}$ is zero by agreement.

Proceeding in descending order of a'_k gives $0 = t_i(\delta - 1)_{bk}$: in column i, all entries in rows of grade at most a'_k are already those of $t_i e'_k$, and entries of $\delta - 1$ in higher columns vanish in row b by induction. These are precisely the defining conditions for R'. Conversely, if $\delta \in R'$, then $(\delta - 1)T$ has degree at least zero on the positive columns, as does $\delta(\widetilde{w} - T)$. This proves (59).

Use the additive character ψ_E of $\mathbb{A}_E/E$ from the absolute trace and the skew element j_{W} of Section 16.1. Set

$$\beta(x,y) = \psi_E\big((j_{\mathrm{W}}/2)\,\mathrm{tr}(y^\dagger x)\big).$$

For target grades $D = 0$ or $D = a'$, the phase test at w requires

$$\beta(Tu,T) = \beta(wu,w) \quad \text{if} \quad u \in U_R(\mathbb{A}_{F_0}), \quad \deg(u-1) \geq 2, \quad \deg_D(w(u-1)) \geq 1. \tag{60}$$

Here $D = 0$ assigns grade zero to every target coordinate. The degree-two condition is automatic for a strictly grade-raising matrix when s is odd.

Suppose w passes (60). Fix a source index k of grade d and an adelic vector v supported in grades at least $\max(1, d+2)$. Suppose wv is supported in target grades $D > \max(-1, d)$ and $we_{k^\vee}$ is supported in $D > -a_i$ for every i in the support of v.

Lemma 17.3 (Eichler moment on the supported grades)**.** *Under the stated grade hypotheses and the phase test* (60)*,*

$$[(T^\dagger T)v]_k = [(w^\dagger w)v]_k. \tag{61}$$

Proof. Put $x = \lambda e_{k^\vee}$ and $L = vh(x,\cdot) - xh(v,\cdot)$ for arbitrary $\lambda \in \mathbb{A}_E$. Since $h(x,v) = h(v,v) = 0$,

$$L^\dagger = -L, \qquad L^2 = -h(x,x)vh(v,\cdot), \qquad L^3 = 0.$$

Thus $u = 1 + L + L^2/2$ is unitary. The three terms of $u-1$ raise the source grade by at least two, and their images under w have D-degree at least one by the support conditions. For $A = y^\dagger y$,

$$\mathrm{tr}(Au) = \mathrm{tr}\,A + h(x,Av) - h(v,Ax) - \tfrac{1}{2}h(x,x)h(v,Av).$$

The first and last terms are invariant under conjugation, and $h(v,Ax) = \overline{h(x,Av)}$. Since $\overline{j_{\mathrm{W}}} = -j_{\mathrm{W}}$ and ψ_E is invariant under conjugation, the phase is $\psi_E(j_{\mathrm{W}}h(x,Av))$. Apply (60) for $y = T, w$. Since $h(x,Av) = \bar{\lambda} g_{k^\vee}(Av)_k$ and $j_{\mathrm{W}} g_{k^\vee} \neq 0$, nondegeneracy of the adelic trace pairing as λ varies gives (61). □

Take first $D = 0$ and w rational. For positive string indices i, l, use $v = e_i, k = l^\vee$ in (61). The resulting identity is $h'(Te_l, Te_i) = h'(we_l, we_i)$. On the span of these columns w is injective. In fact if $wv = 0$ for a nonzero vector supported there, let i be its least-grade occupied string index and use $k = i+1$. The support conditions hold even if $a_{i+1} \geq 0$, because $wv = 0$. The k-th coordinate of $T^\dagger Tv$ is a nonzero multiple of v_i, contradicting (61). The same positive-column space injects under T. Apply Witt extension [65, Ch. 7, Theorem 9.1, p. 265] to the regular quadratic space $(V', [h'/2])$ over E with its CM involution. Symmetrization gives h', and division by two identifies Hermitian forms with their quadratic classes in characteristic zero. Thus the E-linear isometry $we_i \mapsto Te_i$ on the positive-column span extends to an element of $U(V')(F_0)$, including when that span is degenerate. In particular the rational orbit contains a matrix agreeing with T on the positive columns. By (59), the rational target elements giving agreement have the form $R'(F_0)\delta_0$, also after a source change by $R(F_0)$.

For the second test assume this agreement and take $D = a'$. We show that $w - T$ has degree at least zero, by induction in increasing target space grade. A possibly different entry (b,i) has $a'_b < a_i \leq 0$. Use the positive source pivot $v = e_j$ with $Te_j = t_j e'_{b^\vee}$ and $a_j = 1 - a'_b$, and put $k = i^\vee$, of grade $-a_i$. Then $a_j \geq 2 - a_i$. The vector we_j is Te_j plus terms of grade at least $1 - a'_b$, by agreement.

The entries of we_i of grade at most $-a_j = a'_b - 1$ vanish by induction and by the degree of T. These are the conditions for (61), which now gives

$$h'(we_i, we_j) = h'(Te_i, Te_j).$$

The extra terms of we_j pair only with the already matched coordinates of we_i of grade at most $a'_b - 1$. Their pairing there is zero by the same degree bound on T. Hence the remaining equation is $\overline{(w-T)_{bi}}\, t_j g'_b = 0$, and the induction closes. This includes the definite grade-zero source columns. In particular all surviving rational terms of the second test are in the single class of T modulo the rational matrices of degree at least zero.

18 Consecutive theta lifts and their coefficients

In this section, we unfold consecutive theta integrals using the rational orbits of Section 17.4. The even-step coefficient formula and the finite support determine the sum evaluated in Section 19.

18.1 Unfolding the integral for consecutive lifts

In this subsection, we unfold the adjacent theta integral by the orbit computation of Section 17.4. The Haar measures on the original quotients determine the constant in the resulting expression.

Use the datum of Section 17.4. For $A = \mathbb{A}_{F_0}$, give the two quotients

$$[R] = R(F_0)\backslash R(A), \qquad [G'] = U(V')(F_0)\backslash U(V')(A)$$

their inherited Haar measures. Write $\rho(w)$ for the Heisenberg translate on $\mathrm{Hom}_E(V, V')$, with $\rho(w)\rho(z) = \beta(w,z)\rho(w+z)$. The theta functional θ is invariant under rational Heisenberg translations by reindexing and triviality of the additive character on rational arguments. For the canonical rational lift, Weil gives invariance of the theta sum in [77, no. 41, (39) and Theorem 6, p. 193]. Section 17.1 verifies the compatibility of the splitting used here with that rational action. Use the chosen adelic splitting characters for the pair $U(V), U(V')$. For a rational unitary pair its two determinants are rational norm-one elements, so the full automorphic characters take value one on them. On each unipotent group the determinant is one at every place.

For $D = 0, a'$ let I_D be the space of strictly positive-degree maps and $\mathcal{C}_D$ the rational maps modulo the rational maps of nonnegative degree. Reversal of both indices exchanges the degrees $d, -d$, so I_D is isotropic and has as rational dual basis the opposite coordinates with the inverse Gram entries. The arrow test is Schwartz–Bruhat in rational Schrödinger coordinates. At infinity the creation polynomial of the joint harmonic acts on the Gaussian vacuum.

At the finite places the vacuum is fixed by a small open lattice of Heisenberg translations and modulations. Translation invariance makes it locally constant, and modulation invariance cuts the support to the compact annihilator lattice. The inclusion test is a finite sum of translates of that vacuum. A fixed outer Weil translate preserves this class. Use the same theta normalization in the two gradings.

For a Schwartz test f in this realization put

$$C_D(f; w) = \int_{I_D(F_0)\backslash I_D(A)} \theta(\rho(z)\rho(w)f)\, dz, \qquad \int dz = 1.$$

For rational w, commutation and invariance give the integrand $\beta(z,w)^2\theta(\rho(z)f)$. Thus $C_D(f;w)$ depends only on the class of w in $\mathcal{C}_D$. These are the Fourier coefficients at the rational dual frequencies of $z \mapsto \theta(\rho(z)f)$, with the frequency sign specified by the displayed integrand.

In particular

$$\theta(f) = \sum_{w \in \mathcal{C}_D} C_D(f;w).$$

The sum is absolute. In rational Schrödinger coordinates extending the positive block and its dual basis, the I_D variable acts by a character on the first Schrödinger coordinate. Orthogonality cuts the theta sum to one such rational coordinate ξ, leaving the sum of the test over all rational coordinates in the remaining Lagrangian. The sum of the coefficient moduli is bounded by the sum of the modulus of the test over the full rational Schrödinger lattice, times the fixed modulus of the theta normalization.

The finite test has compact support, so the relevant rational coordinates lie in a fixed fractional ideal to a finite power. Multiply an archimedean Schwartz supremum by $\sum_\lambda (1 + \|\lambda\|)^{-a}$, with a greater than the real dimension. This gives a bound uniform on any bounded set of archimedean Schwartz tests and bounded finite tests in a fixed finite-dimensional space of compactly supported tests.

For $z_0 \in I_D(A)$, translation by z_0 fixes the averaged theta functional. Hence

$$C_D(f;w) = \beta(w + z_0, w) C_D(f; w + z_0). \tag{62}$$

For a source rational Bessel element γ and a rational target element δ preserving the D-filtration, covariance gives

$$C_D(\omega(\gamma r, \delta h)\Phi; w) = C_D(\omega(r,h)\Phi; \delta^{-1} w \gamma). \tag{63}$$

This is rational theta invariance and change of variable on the compact quotient of I_D. The change preserves Haar probability. The same covariance holds for the adelic source $u \in U_R(A)$ with $u - 1$ strictly raising the source grade, and for a strictly D-grade-raising target unipotent. Write $W = \bigoplus_j W_j$ for the graded Hom space. The form pairs W_j only with W_{-j}, so $I_D = W_{>0}$ and $I_D^\perp / I_D = W_0$.

Put $E_D(f) = C_D(f;0)$. In partial Schrödinger coordinates this is the residual theta distribution on the slice at zero. An $I_D(A)$-modulation multiplies a test by an additive character of the transverse coordinate. At infinity, differentiating this invariance makes the distribution vanish on multiples of each real transverse coordinate, so it factors through evaluation at zero. At the finite places, a test vanishing on the zero slice is covered by finitely many compact open pieces on each of which a modulation is a constant different from one. These pieces have zero evaluation. Invariance under rational Heisenberg translations in W_0 then makes the residual distribution a multiple of E_D by the rational uniqueness in Section 17.1. Use the Weil representation of $U(V)(A) \times U(V')(A)$ from that section. On the subgroup preserving I_D and acting as the identity on $I_D^\perp / I_D$, covariance gives these two invariances for $E_D \circ \omega(g)$. In the residual translation the added component in $I_D(A)$ pairs trivially with W_0. Since $E_D \neq 0$, the scalar in $E_D \circ \omega(g) = c(g) E_D$ is a character of this subgroup.

Let D_b be the target grade of coordinate b for the chosen D. Take the rational pair

$$d = (\mathrm{diag}(2^{a_i}), \mathrm{diag}(2^{D_b})).$$

Opposite grades pair, so the two diagonal matrices are unitary. On the Hom entry (b,i) the pair acts by $2^{D_b - a_i}$, and hence fixes W_0. In each unitary factor let N_k consist of the elements

whose difference from the identity raises its own grades by at least k. Both factors' N_1 preserve I_D and act trivially on the quotient. For $g \in N_k$, $k \geq 1$, put

$$h = \exp((2^k - 1)^{-1} \log g).$$

The series are finite and $(\log g)^\dagger = -\log g$, so h lies in N_k of the same factor. Conjugation by d multiplies the degree-k part by 2^k. It follows that $[d, h] \equiv g \pmod{N_{k+1}}$. The character is one on the commutator. Since the filtration terminates, descending induction gives $c|_{N_1} = 1$ in each factor. This applies to the two unipotent actions, whose determinant characters are trivial. Moving $\rho(w)$ through their Weil actions gives the stated covariance.

18.1.1 Phase cancellation and rational orbits

In this subsection, we use Lemma 17.3 to identify the terms that survive the phase average.

Let W be continuous on $[R]$ with $W(ur) = \beta(Tu, T)W(r)$ for $u - 1$ of degree at least two. By (62) the product $F_w(r, h) = W(r)C_D(\omega(r, h)\Phi; w)$ satisfies

$$F_w(ur, h) = \frac{\beta(Tu, T)}{\beta(wu, w)} F_w(r, h) \qquad (\deg(u - 1) \geq 2, \quad \deg_D(w(u - 1)) \geq 1).$$

For a class that fails (60), unfold its rational source orbit. If Γ_w is its stabilizer, the orbit sum over $[R]$ is the single integral over $\Gamma_w \backslash R(A)$, with quotient measures from counting measure. The sum is absolutely integrable for each h. Indeed R is the semidirect product of its definite unitary Levi and unipotent radical, so $[R]$ is compact. For each r choose a rational left translate into a fixed compact cover. Equation (63) permutes the classes, and the Schwartz bound applies to the compact family of translated tests.

Put $J_D = I_D^\perp$, the space of maps of nonnegative D-degree. Thus $\Gamma_w = \{\gamma \in R(F_0) : w\gamma - w \in J_D(F_0)\}$. Inside the source group let

$$H_w = \{u \in U_R : \deg(u - 1) \geq 2, \quad w(u - 1) \in I_D\}.$$

It is a closed unipotent F_0-subgroup. Indeed $w(uv - 1) = w(u - 1)v + w(v - 1)$, and right composition by $v^{\pm 1}$ preserves I_D. For $\gamma \in \Gamma_w$,

$$w(\gamma u \gamma^{-1} - 1) = \{w(u - 1) + (w\gamma - w)(u - 1)\}\gamma^{-1} \in I_D.$$

The middle product has D-degree at least two. Conjugation also preserves the source degree bound, so Γ_w normalizes H_w, and $H_w(A) \cap \Gamma_w = H_w(F_0)$.

Let $L = \Gamma_w H_w(A)$. In the algebraic normalizer $N_R(H_w)(A)$, this is the inverse image of the image of Γ_w in $(N_R(H_w)/H_w)(A)$. That image is closed and discrete, since it consists of rational points in an affine algebraic group. Thus L is closed. Conjugation by $\gamma \in \Gamma_w$ preserves adelic Haar on $H_w(A)$: its modulus is the product of the local absolute values of $\det(\operatorname{Ad} \gamma|_{\operatorname{Lie} H_w}) \in F_0^\times$, which is one. Hence L is unimodular, and the invariant measure on $\Gamma_w \backslash L$ is the measure on the compact quotient $H_w(F_0) \backslash H_w(A)$. The modular function of $R(A)$ is trivial on L: for $H_w(A)$ the adjoint action is unipotent, and for Γ_w the product formula applies to the rational adjoint determinant on $\operatorname{Lie} R$. Thus the Haar quotient formula applies both to $\Gamma_w \backslash L$ and to $L \backslash R(A)$.

For $v \in H_w(A)$, the multiplier of $F_w(vr, h)$ is

$$\chi(v) = \frac{\beta(Tv, T)}{\beta(wv, w)}.$$

If the fibre value is nonzero, two successive translations show that χ is a character, trivial on $H_w(F_0)$. A failed phase test gives $u \in H_w(A)$ with $\chi(u) \neq 1$. Right translation by u preserves the fibre measure and multiplies its integral by $\chi(u)$, so that integral vanishes. Quotient integration along $\Gamma_w\backslash L$ and $L\backslash R(A)$ now gives zero for the original orbit, using the absolute integrability proved above.

For $D = 0$, the first phase test and the rational orbit calculation show that only the translates of the classes agreeing with T on positive columns contribute. Agreement is a condition on entries with $a_i > \max(0, a'_b)$, so it is well defined in both quotients. For a matrix $v : V' \to V$, put $y_{ib} = v_{ib}\mathbf{1}_{a_i > \max(0,a'_b)}$. The sum of the agreeing terms, for either grading, is

$$\sum_{w \text{ agreeing}} C_D(f; w) = \int_{\operatorname{Hom}(V',V)(F_0)\backslash \operatorname{Hom}(V',V)(A)} \frac{\theta(\rho(y^\dagger)f)}{\psi_E(j_{\mathrm{W}} \operatorname{tr}(yT))}\, dv, \qquad \int dv = 1. \tag{64}$$

Adjunction reverses the two indices, so $z_0 = y^\dagger \in I_D(A)$. Translate the variable of the probability integral defining C_D. Commuting $\rho(w)$ past $\rho(z_0)$ gives $\beta(w, z_0)^2 = \beta(z_0, w)^{-2}$, so

$$C_D(\rho(z_0)f; w) = \beta(w, z_0)^2 C_D(f; w) = \psi_E(j_{\mathrm{W}} \operatorname{tr}(yw)) C_D(f; w).$$

Indeed $\operatorname{tr}(w^\dagger z_0) = \overline{\operatorname{tr}(yw)}$, $\overline{j_{\mathrm{W}}} = -j_{\mathrm{W}}$, and $\psi_E(\bar{x}) = \psi_E(x)$. After division by the character at T the frequency is $\psi_E(j_{\mathrm{W}} \operatorname{tr}(y(w - T)))$. The coordinates y_{ib} with $a_i > \max(0, a'_b)$ vary independently in $E\backslash\mathbb{A}_E$. Their product mean is one precisely when every corresponding $(w - T)_{bi} = 0$, and zero otherwise. This proves (64) for both $D = 0$ and $D = a'$. Indeed, the character identity and the absolute convergence of the I_D-Fourier expansion give

$$\int \sum_{w \in \mathcal{C}_D} |C_D(\rho(y^\dagger)f; w)|\, dv = \sum_{w \in \mathcal{C}_D} |C_D(f; w)| < \infty$$

because $\int dv = 1$. Division by the unit character at T preserves this majorant. Fubini thus justifies the termwise projection after the I_D-average defining each coefficient.

Evaluate this sum again at $D = a'$. The rational Bessel group preserves agreement and fixes the class of T in $\mathcal{C}_{a'}$. On nonzero graded pieces it fixes the string coordinates by the positive flag and its dual, and on the grade-zero piece it fixes the central string coordinate. The definite grade-zero part is killed by T. Thus $T(\gamma - 1)$ has degree at least zero. Likewise $(\delta - 1)T$ has degree at least zero for $\delta \in R'(F_0)$, since T has no component in the definite target block. The second phase test and orbit cancellation leave only the class of T. By (59), the translated agreeing sets for $D = 0$ are indexed by the cosets $\delta R'(F_0)$ in $U(V')(F_0)$. Their integrals are obtained from the original agreeing set by replacing h with $\delta^{-1}h$ in (63). We obtain

$$\int_{[R]} W(r)\theta(\omega(r, h)\Phi)\, dr = \sum_{\delta \in R'(F_0)\backslash U(V')(F_0)} I(\delta h), \qquad I(h) = \int_{[R]} W(r) C_{a'}(\omega(r, h)\Phi; T)\, dr. \tag{65}$$

The invariance of the class of T gives left $R'(F_0)$-invariance of I.

18.1.2 Absolute convergence and unfolding

In this subsection, we bound the sum over the orbits of Section 18.1.1 before unfolding.

For integration over $[G']$, choose signed orthonormal coordinates at each real place, so the Hermitian matrix satisfies $J_\tau^* = J_\tau = J_\tau^{-1}$. Measure a rational vector of the moved finite

lattice by $\max_\tau \|v_\tau\|_2$. If $\lambda_1(h)$ is the least such length of a nonzero vector, put $H(h) = \max(1, \lambda_1(h)^{-1})$. Choose an integer $d_H \geq 1$ that clears the Hermitian pairings on the fixed finite lattice, and set $\eta = d_H^{-1/2}$. The moved lattice has fixed covolume, since the determinant of a unitary isometry has adelic absolute norm one. If $H(h) \leq 2/\eta$, the first minimum is at least $\eta/2$. Minkowski's second theorem [21, Ch. VIII, §4.3, Theorem V, p. 218] bounds the product of the successive minima by a constant times the covolume, so the last minimum is bounded as well. This gives a bounded rational basis in the adelic coordinates.

Use a full set of bounded independent $\mathbb{Z}$-lattice vectors and take an E-basis among them. Its finite coordinates are integral in the chosen basis. The rational Gram matrices of these bases lie in a compact adelic set and therefore take finitely many values. Fix a rational basis for each Gram matrix. The map from the moving basis to that fixed one is a rational isometry. The change of basis puts the inverses of the resulting translates in a compact set of adelic matrices, and thus the translates themselves in a compact set of the unitary group. Here inversion on the latter uses the fixed Hermitian form.

If $H(h) > 2/\eta$, take the lattice vectors of length below η. For any two such vectors, their rational Hermitian pairing multiplied by d_H is integral at all finite places, since h is unitary and both vectors lie in the fixed lattice in the moved coordinates. At each real place the absolute value is at most $d_H\|v_\tau\|_2\|w_\tau\|_2 < d_H\eta^2 = 1$. The norm of a nonzero algebraic integer has absolute value at least one. Thus every such pairing is zero and the short vectors span a totally isotropic space. If U_τ denotes the resulting isotropic space, $J_\tau U_\tau$ is Euclidean-orthogonal to it and pairs perfectly with it for the Hermitian form. Dilate the isotropic space by 2 and its opposite by $1/2$, and fix the joint orthogonal complement. This is unitary and both it and its inverse have operator norm at most 2. The short lengths double. Every other length stays at least $\eta/2$. The resulting height is at most $\max(2/\eta, H(h)/2)$. Repetition gives $\delta h = zx$, with δ rational, z in a fixed compact set and x purely real, with $\|x_\tau\|, \|x_\tau^{-1}\| \leq H(h)$. Indeed there are at most $\lfloor \log_2 H(h) \rfloor$ dilations, and each dilation and its inverse has norm at most two. The compact set and the finite pairing denominator depend only on the fixed group and lattice of this unfolding.

In the real Schrödinger model use the L^2-norms of monomials in coordinate multiplication and differentiation operators. By Heisenberg covariance, conjugation by the Weil operator sends the linear operators to linear combinations with coefficients of the symplectic matrix or its inverse. By L^2-unitarity, these norms of the translate of the test are bounded by the corresponding norms of the original test times a polynomial in the matrix and inverse norms.

Sobolev embedding bounds the weighted suprema above by finitely many such L^2-norms. The symplectic action is $w \mapsto hwr^{-1}$. Its matrix and inverse are polynomially bounded on the reduced representatives with r in a compact set. At the finite places the translates of the test over a compact set lie in a finite set by smoothness of the Weil action. Therefore

$$\sum_{w \in \mathcal{C}_0} |F_w(r,h)| \leq C H(h)^B.$$

Here B depends on the fixed spaces and test Φ. The constant C also includes $\sup_{[R]} |W|$, finite because $[R]$ is compact and W is continuous. At $D = 0$ every rational target isometry preserves the filtration. Equation (63) therefore permutes all frequencies under either rational reduction used in this estimate.

For any continuous automorphic y with $\sup_h H(h)^B |y(h)| < \infty$ for all B, the product majorant is bounded on $[R] \times [G']$. Each $I(\delta h)$ is the integral of the sum over its own agreeing

set at $D = 0$, by (64) and the second orbit cancellation. The distinct sets are disjoint, whence

$$\sum_{\delta} |I(\delta h)y(h)| \le |y(h)| \int_{[R]} \sum_{w \in \mathcal{C}_0} |F_w(r,h)|\, dr \le C \operatorname{vol}[R] H(h)^B |y(h)|.$$

Both quotients have finite volume. Fubini and quotient integration give

Proposition 18.1 (Unfolding consecutive lifts)**.** *For the data let W be continuous on $[R]$, with $W(ur) = \beta(Tu,T)W(r)$ for $\deg(u-1) \ge 2$. Let y be continuous on $[G']$ and satisfy $\sup_h H(h)^B|y(h)| < \infty$ for every $B > 0$. The two integrals below converge absolutely and unfolding with the inherited target measure gives*

$$\int_{[R]} W(r) \int_{[G']} \theta(\omega(r,h)\Phi) y(h)\, dh\, dr = \int_{R'(F_0)\backslash U(V')(A)} I(h) y(h)\, dh. \tag{66}$$

The quotient measure on the right uses counting measure on the rational subgroup, and the unfolding has scalar one.

At the final arrow $Y_{m-1} \to Y_m$ both strings are empty. All grades are zero, $I_D = 0, \mathcal{C}_D = \{0\}$, and $T = 0$. The Bessel groups are $R = U(V)$ and $R' = U(V')$, both definite. Hence

$$C_D(f;0) = \theta(f), \qquad I(h) = \int_{[R]} W(r)\theta(\omega(r,h)\Phi)\, dr.$$

Both quotients are compact. Fubini gives (66) with the original Haar measures.

18.1.3 Rapid decay of the projected tests

In this subsection, we verify the decay hypothesis of Proposition 18.1 for the projected test.

For the target test coming from an integrable discrete series projector, rapid decay follows as follows. Let y be any bounded finite-level form reproduced by the coefficient projector of the prescribed integrable real discrete series. For the target tests in the lower chain, Section 18.3 verifies these hypotheses for the conjugate projector. Write the reproduction identity as $y = R(e)y$, where $e = d_\pi \overline{\langle v_0, \pi(\cdot)v_0\rangle}$ is the convolution projector for a unit K_∞-finite vector v_0 of the specified integrable discrete series π.

Take the inner product linear in the second variable. For v in the Harish-Chandra module of π, the map

$$v \longmapsto d_\pi \int y(hx)\langle \pi(x)v_0, v\rangle\, dx$$

takes v_0 to y. Denote it by T, and put $b_v(x) = \langle \pi(x)v_0, v\rangle$. Write $G_\infty = Z_c G_\infty^1$, with Z_c the compact scalar torus and G_∞^1 the product of the special unitary groups. The product map has finite kernel, and restriction of π to G_∞^1 is irreducible since the centre acts by scalars. If ω_π is its central character, then

$$b_v(zs) = \overline{\omega_\pi(z)} b_v(s) \qquad (z \in Z_c,\ s \in G_\infty^1).$$

Hence the restriction of the nonzero coefficient b_{v_0} belongs to $L^1 \cap L^2$ on G_∞^1. Apply [58, arXiv:1610.05483v2, Lemmas 5-3 and 5-5(i), (iii), (v), (vii)] with $p = 1$ to this connected semisimple group with finite centre. After inversion of the coefficient argument, these lemmas give smoothness in L^1 for the left translates of b_v; the central factor differentiates by the

displayed character. Since $b_v(k^{-1}x) = b_{\pi(k)v}(x)$, the change of integration variable and differentiation give $R(k)Tv = T\pi(k)v$ for $k \in K_\infty$ and $dR(X)Tv = T\,d\pi(X)v$. Each Tv is bounded, as y is bounded and the coefficient is integrable. The quotient has finite volume, so $Tv \in L^2$.

If $y \neq 0$, the pullback $\langle Tv, Tw\rangle_{L^2}$ is a nonzero invariant Hermitian form on the irreducible Harish-Chandra module of π. On each finite-dimensional K_∞-isotypic subspace it is represented, in the Hilbert inner product of π, by a positive semidefinite operator. Their direct sum intertwines the Lie algebra by infinitesimal invariance and intertwines K_∞. Schur's lemma therefore gives $\|Tv\|_2^2 = c\|v\|_\pi^2$ for one $c > 0$. Thus T extends to the Hilbert completion. For a real Lie algebra element X and a K_∞-finite vector v,

$$dR(X)^n Tv = T\,d\pi(X)^n v \qquad (n \geq 0).$$

The analytic-vector series for v converges in norm for small $|t|$, and the norm identity gives the same convergence for Tv. Their sums give $R(\exp tX)Tv = T\pi(\exp tX)v$. By density and generation by the identity component and K_∞, the extension is equivariant. Its closed image is a discrete-series summand.

Borel's class-number theorem gives finitely many adelic double classes at the standard integral compact K_f^0 [18, Theorem 5.1, p. 19]. For the specified compact open level J, each such class splits into at most $[K_f^0 : K_f^0 \cap J]$ classes at the intersection; hence there are finitely many arithmetic real quotient components at J. The determinant of each element of its arithmetic subgroup Γ is an integral unit of absolute value one at every embedding, hence a root of unity. Thus $\Gamma^1 = \Gamma \cap G_\infty^1$ has finite index in Γ. Lift to $\Gamma^1\backslash G_\infty$. The centre acts by a unitary character, so integration over Z_c shows that restriction to $\Gamma^1\backslash G_\infty^1$ preserves the squared norm up to a fixed positive scalar. Here the product map has finite central kernel. The restriction of π is irreducible because Z_c acts by scalars, and remains square integrable by the same central integration.

On each component with nonzero image let G^1 be the restriction of the special unitary group from F_0 to $\mathbf{Q}$. At the finite representative h_f, the compact open $J^1 = h_f J h_f^{-1} \cap G^1(\mathbb{A}_f)$ preserves a lattice in the underlying rational representation, obtained locally by taking the sum of the translates of an integral lattice under the compact group and retaining the standard lattice almost everywhere. Thus J^1 has finite index in the adelic stabilizer of this lattice in $G^1(\mathbb{A}_f)$, and intersection with $G^1(\mathbf{Q})$ puts Γ^1 at finite index in the integral arithmetic group. For each right differential operator D, the intertwining identity gives $D(Tv) = T(d\pi(D)v)$, which is bounded by the integrability of its coefficient. Together with compact finiteness and the infinitesimal character this gives Wallach's automorphic-function hypotheses. His theorem places this tempered summand in the cuspidal spectrum [76, Theorem 4.3, p. 233]. Each rational parabolic of the unitary group has the same unipotent radical N as its intersection with the derived group, and $\Gamma \cap N = \Gamma^1 \cap N$. At the finite representative h_f of a component, put $J_N = N(\mathbb{A}_f) \cap h_f J h_f^{-1}$. Exponential coordinates and additive approximation give $N(\mathbb{A}_f) = N(F_0)J_N$. The resulting quotient is

$$N(F_0)\backslash N(\mathbb{A})/J_N \simeq (N(F_0) \cap J_N)\backslash N(F_{0,\infty}),$$

where the intersection uses the finite embedding of $N(F_0)$. The quotient integration formula with probability measures identifies the adelic constant term on this component with the real one, which vanishes. The restriction of $T(v_0)$ on each arithmetic component is a bounded smooth cuspidal form, hence of moderate growth. The rapid decay theorem [47, arXiv:1106.2149v1,

Theorems A and 2.23–2.24] gives $O_A(\|g\|^{-A})$ on each of the finitely many Siegel sets for every $A > 0$. Central translates have the same absolute value.

The finite lattice is fixed on each real component, so $H(h) \ll 1 + \|h_\infty\|$ there. Indeed the original nonzero lattice vectors have a fixed positive minimum and the inverse real action decreases lengths by at most $\|h_\infty\|$. This proves the required decay in $H(h)$ for the projected tests.

18.2 Two successive Bessel integrals

In this subsection, we combine two copies of the unfolding in Section 18.1. The middle Hermitian space then appears in a single theta integral, and its constant term gives the paired operator needed in Section 18.3.

Take three lower spaces Y, Y', Y'' with the same c, N, odd initial string length s, and remaining lengths $s-1, \max(0, s-2)$. Write T, T' for the two string maps and let

$$S = (T^\dagger, T')^{\mathsf{T}} : V' \longrightarrow V^- \oplus V'',$$

where V^- has form $-h$. For an interior pair, adjunction in the antidiagonal forms gives

$$TT^\dagger e'_i = t_{s-2-i} t_{i+1} (g'_i / g_{i+1}) e'_{i+1}$$

before the last coordinate. This is the middle shift, with coefficient c except at grade one, where it is $-Nc^2$. Thus $TT^\dagger = T'^\dagger T'$, and $S^\dagger S = -TT^\dagger + T'^\dagger T' = 0$. The same adjoint computation for even s gives the odd target shift, with $-Nc$ at grade zero. At $s = 1$ the maps are zero.

Let $u \in U_{R'}(\mathbb{A}_{F_0})$ satisfy $\deg(u-1) \geq 2$. Target covariance replaces T by $u^{-1}T$, and $(u^{-1}-1)T$ has degree at least one. Equation (62) and $u^\dagger = u^{-1}$ give

$$I(uh) = \beta(T, u^{-1}T) I(h),$$
$$\beta(T, u^{-1}T) = \psi_E\big((j_{\mathrm{W}}/2)\operatorname{tr}(TT^\dagger u)\big) = \beta(T'u, T'),$$

since $TT^\dagger = T'^\dagger T'$. This gives the middle Bessel character on the subgroup of degree at least two. Section 18.2.1 computes the contribution of the degree-one entries. Continuity follows from the compact $[R]$ integral and the local Schwartz bounds. We may therefore use the resulting I on the next Bessel group.

The adjoint identifies the symplectic spaces $\operatorname{Hom}(V, V')$ and $\operatorname{Hom}(V', V^-)$. The Hermitian trace on the latter is the negative conjugate of the trace pairing on the former, which preserves the alternating form with j_{W}. Use the corresponding linear oscillator transport and tensor it with the second datum. Write $\mathcal{B}(f, f')$ for the joined datum. This intertwines the Heisenberg translations by w and $w^\dagger$.

For the opposite finite basis, multiply the original basis by the conjugate of $\eta\alpha_{\mathrm{W}}/\bar{\alpha}_{\mathrm{W}}$, where $\eta \in \widehat{\mathcal{O}}_E^\times$ has norm -1. Such a unit exists at every finite place since E/F_0 is unramified or split there. Put $\lambda = \eta\alpha_{\mathrm{W}}/\bar{\alpha}_{\mathrm{W}}$ and $A^* = \bar{A}^{\mathsf{T}}$. The pullback on finite oscillator coordinates is $A \mapsto \lambda A^*$, with inverse $A \mapsto -\lambda A^*$ since $\lambda\bar{\lambda} = -1$. In the split Schrödinger coordinates on the conjugate places, $\lambda(\alpha_{\mathrm{W}} Y)^* = \alpha_{\mathrm{W}} \eta Y^*$; hence the pulled-back test is $Y \mapsto \Phi(\eta Y^*)$.

For a finite adelic matrix X, use the quadratic phase and translation of Section 7.4.3,

$$e_0(X) = \psi_0(c_0 \operatorname{tr}(X^*X)/2)\,, \qquad T_\sharp(X) = e_0(X)\rho_f(\alpha_{\mathrm{W}} X),$$

with the product of the local characters ψ_0 and $c_0 = (1 - j_{\mathrm{W}}^{-1})/2$. Since $\operatorname{tr}((\eta A^*)^*(\eta A^*)) = -\operatorname{tr}(A^*A)$, one has $e_0(\eta A^*) = e_0(A)^{-1}$. The quadratic phase of a column of two blocks is the product of their phases. Heisenberg covariance therefore gives

$$T_\sharp \begin{pmatrix} A \\ B \end{pmatrix} \mathcal{B}(f, f') = e_0(A)^2\, \mathcal{B}(T_\sharp(\eta A^*) f, T_\sharp(B) f').$$

For integral A, the argument $c_0 \operatorname{tr}(A^*A)$ is integral, so $e_0(A)^2 = 1$ also above 2. For a fractional displacement the displayed phase is retained. The constant terms below use the unnormalized adelic translations ρ of Section 18.1.

In rational coordinates the forward map is ordinary adjunction. Transporting a rational Schrödinger polarization by this symplectic map gives the same sum over the rational Lagrangian. The two theta normalizations therefore satisfy

$$\theta(\mathcal{B}(f, f')) = z_\theta\, \theta(f)\theta(f'), \qquad z_\theta \neq 0.$$

The scalar is fixed by the models and is independent of the two tests.

18.2.1 The constant terms in the joined block

In this subsection, we group the constant terms before the lower-lift comparison of Section 17.3.

Let Z_1 be the zero term along the joined positive-degree block. The zero term along an isotropic space K, at a vector in the residual space, is the theta functional after translating by that vector and averaging over $K(F_0)\backslash K(A)$ with probability measure. The map $(w, w') \mapsto (w^\dagger, w')$ takes the two positive blocks to the joined block, since adjunction reverses both grades. It is a rational bijection and pushes product probability to probability. Consequently

$$C_{a'}(f; w) C_{a''}(f'; w') = z_\theta^{-1} [Z_1 \rho((w^\dagger, w')) \mathcal{B}(f, f')](0). \tag{67}$$

For an element n of the middle unipotent radical, the two frequencies in (67) become $n^{-1}T, T'n$ by covariance. Since $n^\dagger = n^{-1}$, their joined frequency is $((n^{-1}T)^\dagger, T'n)^{\mathsf{T}} = Sn$. The cross term is trivial because $S^\dagger S(n-1) = 0$, so $\rho(Sn) = \rho(S(n-1))\rho(S)$. The map $S(n-1)$ has nonnegative degree. If $s > 1$, the outer strings have even grades and the middle string has odd grades. The grade-zero part of Hom is thus the maps from X' to the outer grade-zero space. The two outer middle vectors z, z' have norms $N, -N$ in the joined target. They form the hyperbolic plane with basis $(z - z')/(2N), z + z'$. The grade-one middle vector e_+ has $Se_+ = c(z + z')$: the first coefficient is the adjoint coefficient $(-Nc)/(-N)$ and the second is c. It follows that the degree-zero part of $S(n-1)$ is $(z + z')b(n)$, with $b(n)_i = c(n-1)_{+,i}$ for $i \in X'$. Its positive part is orthogonal to the degree-zero part and is absorbed by Z_1. The product is z_θ^{-1} times the value of $Z_1\rho(S)\mathcal{B}(f, f')$ at $(z + z')b(n)$.

The map b is additive on the middle unipotent, since the product of two positive-degree differences has degree at least two. For $s > 1$ it is onto over F_0 and over A. To prescribe b_i, take $x = e_+$, $v = \sum_i c^{-1}\bar{b}_i e_i$ in X', and $L = xh'(v, \cdot) - vh'(x, \cdot)$. Here $L^\dagger = -L$, $L^3 = 0$, and $Le_i = c^{-1}b_i e_+$, $L^2 e_i = 0$ for each definite basis vector e_i. Hence $u(t) = 1 + L + L^2/2$ is unitary, strictly grade-raising off the identity, and $b(u(t)) = t = (b_i)_i$. Since b sends rational elements to rational covectors, it descends to the left rational quotients. For invariant probability dn on the unipotent quotient, right translation by $u(t)$ preserves dn and $b(nu(t)) = b(n) + t$. Thus $b_* dn$ is the translation-invariant probability on the adelic covectors modulo rational covectors.

Its average gives a second zero term Z_2 along $z+z'$. The two slices compose to $\mathcal{Z} = Z_2 Z_1$, with Haar probability on the direct sum of the isotropic directions.

For $s = 1$ the two string maps vanish, all three gradings are zero and there is no plane to remove. The vector z/π_N has norm $N/(\pi_N \bar{\pi}_N) = 1$ in V^-. It is the last rational coordinate of X'', whose other coordinates are the definite part of Y''. Thus $\widehat{k} = \operatorname{diag}(k, 1)$, and both zero-term operators are the identity in these coordinates. Section 19.2 computes the original lattice in this basis at the conjugate places above N. In the interior $\widehat{k} = k$.

For the determinant factors, write υ_1 for the splitting character on norm-one ideles. On the isotropic block of a parabolic Weil action let A_0 be its matrix. The scalar of the zero slice, relative to the action on the residual space, is $\upsilon(\det A_0)^{-1} |\det A_0|^{1/2}$. The modulus is the square root of the Jacobian in the dual coordinates $A_0^{\dagger} x$. For the phase, use an adapted Gaussian and finite lattice vacuum.

In the real basis with prescribed signs, the negative block of the hyperbolic Levi has matrix $(A_0^{-\dagger} + A_0)/2$. Its conjugate determinant times $\det A_0$ is the positive determinant of $(1 + A_0^{\dagger} A_0)/2$. At a finite place put $L = \mathcal{O}_E^{\dim A_0}$ in the corresponding local coordinates. The multiplier for an invertible g is $\chi_f(\det B)$, where $BL = L \cap g^{-1} L$ and $\chi_f = \upsilon_f |\cdot|^{-1/2}$. For the hyperbolic pair $A_0^{-\dagger}, A_0$ this has multiplier $\overline{\chi_f(\det A_0)} |\Delta_f(A_0)|^2$.

Indeed A_0 carries $L \cap A_0^{-1} L$ to $L \cap A_0 L$, so inversion multiplies Δ_f by $\chi_f(\det A_0)$. Transposition leaves the intersection determinant ideal unchanged, by elementary divisors, and conjugation conjugates Δ_f. Thus the hyperbolic multiplier has phase $\upsilon(\det A_0)^{-1}$. The Gaussian overlap and the overlap of the finite zero slices are positive in the adapted coordinates. The phase of the scalar is therefore the phase of this multiplier.

The parabolic radical has trivial scalar: it acts on the zero slice by a scalar by Heisenberg covariance and every element of that radical is a commutator with an isotropic dilation on its two successive positive-grade quotients. Equivalently, in the canonical Schrödinger splitting the terms of its phase vanish at the zero coordinate. This also gives the unipotent covariance used in Section 18.1.

The arrow of parameter a multiplies the raw action of (g, h) by

$$\frac{a(\det h)}{a(\det g)} \upsilon_1(\det h)^{\lfloor n/2 \rfloor} \upsilon_1(\det g)^{-\lfloor n'/2 \rfloor},$$

where n, n' are the ambient ranks. The adjoint transport on the first arrow contributes $\upsilon_1(\det g)^{n'} \upsilon_1(\det h)^{-n}$. Indeed, under the transpose of the inverse matrix of the action on Hom, the opposite sign interchanges the real positive and negative blocks.

Put $D(w) = hwg^{-1}$ on the first Hom space. At a real place v write $\Delta_{\epsilon_v}(D_v) = \overline{\det((D_v)_{--})}$. Jacobi's complementary minor identity and $|\det D_v| = 1$ give

$$\Delta_{-\epsilon_v}(D_v^{-1}) = \overline{\det((D_v^{-1})_{++})} = (\det D_v) \Delta_{\epsilon_v}(D_v).$$

At each finite place $D_v(L \cap D_v^{-1} L) = L \cap D_v L$. The determinant has norm one, so $|\det D_v|_{E_v} = 1$, with both factors included at a split place. The intersection formula gives

$$\Delta_f(D_f^{-1}) = \upsilon_f(\det D_f) \Delta_f(D_f).$$

Transposition preserves the principal minors and the finite elementary divisors. Reordering the matrix indices permutes the corresponding coordinates in both bases. The original multiplier

divided by the opposed multiplier is therefore $\upsilon_1(\det D)^{-1}$, with $\det D = (\det h)^n(\det g)^{-n'}$. The exponents on the three determinants in the joined action are consequently

$$(\lceil n'/2\rceil,\ -\lceil n/2\rceil - \lfloor n''/2\rfloor,\ \lfloor n'/2\rfloor).$$

The factors $a(\det h)/a(\det g)$ and $b(\det k)/b(\det h)$ are retained.

Let $d_0 = (s-1)/2$, let p_0 count the positive outer grades, and let q_0 be one in the interior and zero at $s = 1$. On the positive Hom block the outer determinant occurs once per negative middle coordinate and the middle inverse determinant once per positive outer coordinate. Both determinants have norm one. The first slice changes the exponents by $(-d_0, p_0, -d_0)$. The plane slice changes them by $(0, q_0, 0)$: the plane is fixed and the column action is h^{-1}. If x, x', x'' are the final ranks, then

$$n' = x' + 2d_0, \qquad n + n'' = x + x'' + 2p_0 + 2q_0.$$

Here $n'' = n + 2t$, so each of the pairs (n, n''), (x, x'') has equal parity. In particular

$$-\lceil n/2\rceil - \lfloor n''/2\rfloor + p_0 + q_0 = -(x + x'')/2 = -\lceil x/2\rceil - \lfloor x''/2\rfloor.$$

Subtracting d_0 from the two outer exponents gives $\lceil x'/2\rceil$, $\lfloor x'/2\rfloor$, respectively, and $\det \widehat{k} = \det k$. The joined motions of the definite groups commute with $\rho(S)$, since they fix all the string maps. Thus $\mathcal{Z}$ carries the ambient action to the joined definite action.

18.2.2 The two quotient volumes

In this subsection, we compare the quotient measures after the preceding unfolding with the normalization of Section 8.2.

Write $\kappa_R = \operatorname{vol}[R]/\operatorname{vol}[U(X)]$ and similarly $\kappa_{R'}$. The Levi acts on the unipotent radical with modulus one, since a determinant of its action is a rational character of a unitary group. Disintegrating the original quotient Haar therefore gives the Levi quotient Haar times this constant and Haar probability on each unipotent fibre. For odd s, the source unipotent acts with degree at least two. Its phase in the first constant term cancels the character of W by (62). In the interior the middle mean gives Z_2 by the surjectivity of b. At $s = 1$ the two unipotent radicals are trivial and their probability integrals are evaluations.

Let $\omega_{a,b}^{\mathrm{def}}$ be the joined action on the two definite arrows, and $\iota_X, \iota_{X_0''}$ the original definite group embeddings into the two outer unitary groups. For the two arrows write

$$D_f(r, r') = C_{a'}(\omega_a(r, r')f; T), \qquad D_{f'}(r', h) = C_{a''}(\omega_b(r', h)f'; T').$$

The actions include the determinant characters of their respective arrows. For every continuous W with the prescribed source unipotent character, the two integrations just computed give

$$\begin{aligned}&\int_{[R']}\int_{[R]} W(r)D_f(r, r')D_{f'}(r', \iota_{X_0''}(k))\,dr\,dr' \\ &\quad = \frac{\kappa_R\kappa_{R'}}{z_\theta}\int_{[U(X)]}\int_{[U(X')]} W(\iota_X(g))\,\theta_{\mathrm{def}}\Big(\omega_{a,b}^{\mathrm{def}}(g, h, \widehat{k})\mathcal{Z}\rho(S)\mathcal{B}(f, f')\Big)\,dh\,dg.\end{aligned} \tag{68}$$

The integrands descend continuously to the compact Bessel quotients. This justifies the order of the means. At the endpoint the Bessel subgroup is the full definite unitary group. The coefficient on it is evaluated at $\operatorname{diag}(k, 1)$ with the branching normalization of Section 7.4. The two quotient-mass constants in (68) are for the original Haar measures.

18.3 The coefficients after an even number of steps

In this subsection, we iterate the paired identity of Section 18.2. We keep the source indices and identify the real coefficients after each pair of theta operators.

Write R_j for the Bessel group of Y_j, $J_{\xi,j}$ for the principal kernel at the chosen level and twist places, and $A_{\xi,j} = [K_j^\circ : J_{\xi,j}]$ for its index in the standard compact. Put $G_j = U(Y_j)$ and write $K_{\xi,j}(g,h) = \theta(\omega_{\xi,j}(g,h)\Phi_{\xi,j})$ for the kernel with the given level test, rank twists and splitting character. For a source form f and a test z of the conjugate target type, the two operators for the bilinear pairing are

$$(L_{\xi,j}f)(h) = A_{\xi,j}\int_{[G_j]} K_{\xi,j}(g,h)f(g)\,dg,$$
$$(L^{\mathrm{t}}_{\xi,j}z)(g) = A_{\xi,j}\int_{[G_{j+1}]} K_{\xi,j}(g,h)z(h)\,dh.$$

Thus $\int_{[G_{j+1}]} zL_{\xi,j}f = \int_{[G_j]}(L^{\mathrm{t}}_{\xi,j}z)f$. Indeed the compact finite translates give finitely many Schwartz tests. On reduced representatives the real symplectic matrix for $w \mapsto hwg^{-1}$ and its inverse are bounded by powers of the product of the two heights. The Schwartz seminorm estimate of Section 18.1.2 gives $|K_{\xi,j}(g,h)| \ll H_j(g)^C H_{j+1}(h)^C$ for the fixed test. The rapid decay of f, z and finite quotient volumes make the double integral absolute, so Fubini applies. In either variable the principal kernel fixes the lattice-inclusion test and the determinant character. In the terminal group, an integral element with block-upper-triangular reduction at the twist and identity at the chosen level places has leading blocks that lift to the preceding unitary groups. The inclusions intertwine those blocks in the residues. The successive determinant ratios are the tame characters on the added diagonal blocks. The initial block fixes the source vector, which has maximal level at the twist places. This proves the terminal finite character.

In the lower grid the added positive blocks of length $t = 5r$ are centered at $-w_i$ and have entries $-w_i + (t-1)/2 - b$ for $0 \le b < t$. The common half shift at odd stages leaves all gaps unchanged. The base tails lie midway between consecutive centers. Only base entries carry negative sign. Thus opposite signs have gaps at least $\Delta/2 - t > 500(r+mt)^3$ for large x. Let e_j denote the normalized discrete-series coefficient on $U(Y_j)_\infty$, with $R(e)f(g) = \int e(k)f(gk)\,dk$. Write $e^\vee(k) = e(k^{-1}) = \overline{e(k)}$. The pair (Y_j, Y_{j+1}), $0 \le j < m$, is the horizontal step in grid $z = 0$ with zero offset, no up blocks and j previous down blocks. The ranks are $r+jt, r+(j+1)t$. Writing c for the new centre, its positive block has doubled entries $2c+t-1-2b$, $0 \le b < t$. The common odd-stage shift cancels after the opposite determinant exponent is subtracted. Thus the signed parameter and joint-harmonic calculation of Section 34.3.2 applies to this step and, with conjugate types interchanged, to its transpose.

For clarity, take either orientation, with incoming and outgoing coefficients $(e, e') = (e_j, e_{j+1})$ or $(\bar{e}_{j+1}, \bar{e}_j)$. Write P for its real harmonic and L_P for the integral in the incoming variable with the given finite test. In the real oscillator put

$$p_{\mathrm{in}} = \int e^\vee(k)\,\omega(k,1)\,dk, \qquad p_{\mathrm{out}} = \int e'(k)\,\omega(1,k)\,dk.$$

The local line identity of Section 34.3.2 gives $p_{\mathrm{out}}p_{\mathrm{in}}P = p_{\mathrm{in}}P$ in this orientation. For a rapid incoming form $y = R(e)y$, replace $y(g)$ by $\int e(k)y(gk)\,dk$ in the bilinear lift and set $g' = gk$. The kernel at $g'k^{-1}$ contributes $\omega(k^{-1},1)P$. Inversion in k gives p_{in}. Right convolution in the target gives

$$L_P y = L_{p_{\mathrm{in}}P}y, \qquad R(e')L_P y = L_{p_{\mathrm{out}}p_{\mathrm{in}}P}y = L_P y.$$

To justify both projections before applying the seesaw, denote the incoming and outgoing unitary groups by $G_{\mathrm{in}}, G_{\mathrm{out}}$ and their ranks by a, b'. Put $X(k) = \prod_{\tau} \max(1, \|k_{\tau}\|)$. For fixed target point h, source reduction and the coherent Gaussian estimate in Section 34.3.1 give, for the original finite test,

$$|K_P(g'k^{-1}, hk')| \le C_h H(g')^B X(k)^{ab'} X(k')^{ab'}.$$

Indeed the source and target real moves act on the same Hom space, whose operator norm is bounded by the product of their norms, and finite rational reduction leaves a compact set of finite translates. After $g' = gk$, an absolute majorant for the integral over g', k, k' therefore has integral at most

$$C_h \int_{[G_{\mathrm{in}}]} |y(g')| H(g')^B \, dg' \int_{G_{\mathrm{in},\infty}} |e(k)| X(k)^{ab'} \, dk \int_{G_{\mathrm{out},\infty}} |e'(k')| X(k')^{ab'} \, dk'.$$

The first factor is finite by rapid decay, and the coefficient moments in Section 34.3.2 give the other two in both orientations. Thus Fubini applies with the original quotient measure. The finite character and test, together with the factor $A_{\xi,j}$, are the same on both sides of each identity. The truncation argument of Section 34.3.2 now places $L_P y$ in the cuspidal coefficient space at its fixed finite level. Section 18.1.3 gives its rapid decay. Starting at the definite endpoint and applying this argument successively in the transposed orientation supplies the rapid test required at every unfolding.

Fix outer points $g_j = (u_j, y_j)$ and let W_0 be the original Bessel source coefficient of ϕ with the functional t. For $j \ge 0$ define the raw functions

$$I_{j+1}(h) = \int_{[R_j]} W_j(r) C_{a_{j+1}}(\omega_{\xi,j}(rg_j, h)\Phi_{\xi,j}; T_j) \, dr, \qquad W_{j+1}(r) = I_{j+1}(rg_{j+1}).$$

The successor covariance of Section 18.2 gives the required character of W_j on $\deg(u-1) \ge 2$ at every step. At an even stage this is the entire unipotent radical. At an odd stage its degree-one entries contribute the second isotropic mean of Section 18.2.1. The function I_j is independent of the outer j-th point before its evaluation. For a rapid automorphic z of the conjugate target type, Proposition 18.1 gives

$$\int_{[R_j]} W_j(r)(L^{\mathrm{t}}_{\xi,j} z)(rg_j) \, dr = A_{\xi,j} \int_{R_{j+1}(F_0)\backslash U(Y_{j+1})(A)} I_{j+1}(h) z(h) \, dh.$$

This is the unnormalized unfolding for one theta lift with the source-level factor of the transpose. Put $G' = U(Y_{j+1})$ and $R' = R_{j+1}$. The quotient integral on its right disintegrates as

$$\int_{R'(A)\backslash G'(A)} \int_{[R']} I_{j+1}(rh) z(rh) \, dr \, d\dot{h}.$$

Here $d\dot{h}$ is the quotient of the given group Haar measures, since both groups are unimodular. Absolute convergence in Proposition 18.1 applies also to this disintegration. For $z = L^{\mathrm{t}}_{\xi,j+1} z'$, fix a representative $h = g_{j+1}$ in the outer integral. The inner source is precisely $W_{j+1}(r)$, so the next unfolding contributes $A_{\xi,j+1}$.

For $j = 2i$, apply (68) to

$$f = \omega_{\xi,j}(g_j, g_{j+1})\Phi_{\xi,j}, \qquad f' = \omega_{\xi,j+1}(g_{j+1}, g_{j+2})\Phi_{\xi,j+1}.$$

Indeed the two original factors have arguments $(rg_j, r'g_{j+1})$ and $(r'g_{j+1}, \iota_{X_0''}(k)g_{j+2})$. The representation law puts the same middle point $r'g_{j+1}$ in both. For the raw function $I_{j+2}(\iota_{X_0''}(k))$ take $g_{j+2} = 1$. Join the translated tests and apply the S-translation and both slices of Section 18.2. The result is a pure real factor times a joined finite factor. Indeed in an adapted adelic Schrödinger polarization the slices are restrictions at zero in the dual coordinates of the two isotropic directions.

On a pure tensor this is the product of real and finite restrictions. Heisenberg translations, Weil moves and the adjoint transport also factor by place. Put the fixed theta normalization in the real factor. Thus the finite factor is the joined finite zero slice of $\rho_f(S_f)$ on the two translated finite vectors, with the opposite-basis scale. Denote its inverse image under the finite tensor identification by $F_i(\xi, y)$.

This inverse exists since compactly supported locally constant tests on a product are finite sums of product tests, by a finite rectangular partition. The real factor depends on the u_j and fixed weights. Every finite tame character has trivial real part, so the real factor is independent of ξ, y.

Write V_S, V_T for the complex column spaces of the definite source and target. For the pullback $P(Z) \mapsto P(h^{-1}Zg)$ with $Z : V_S \to V_T$, the degree-D polynomials have the Cauchy decomposition

$$\operatorname{Sym}^D(V_T^\vee \otimes V_S) = \bigoplus_{\substack{\lambda \vdash D \\ \ell(\lambda) \le \min(\dim V_S, \dim V_T)}} \mathcal{S}_\lambda(V_T^\vee) \otimes \mathcal{S}_\lambda(V_S).$$

Each summand occurs once [37, §4(d), p. 560]. The two determinant weights of the arrow twist the indicated factors. Source integration selects the summand of the source column partition, read in the next rank by padding with zero rows. For the highest coefficient the polynomial is the product of the minors.

In normalized Bargmann coordinates the Gaussian pairing satisfies

$$\int_{\mathbb{C}^d} \bar{z}^\alpha z^\nu e^{-\|z\|^2} \frac{dz}{\pi^d} = \mathbf{1}_{\alpha=\nu}\, \nu!,$$

so pairing with a Fock monomial gives its unscaled differential at the origin. Let d_X be the source Weyl dimension, D the degree of the coupled polynomial, and $c_{\mathrm{jet}} = \pi_M s_{\mathrm{jet}} \neq 0$ the scale used in this algebraic lift. Its homogeneous jet includes c_{jet}^D, so converting that jet back to the unscaled theta coefficient gives c_{jet}^{-D}. More generally, write the component paired with the source as a sum of rectangular Schur coefficients $Z \mapsto A_i \mathcal{S}(Z) v_i$, where v_i belongs to the source Weyl module and A_i to the target dual. These coefficients span the matched compact summand by the Cauchy decomposition. For real points u, u', denote the source and target compact Weyl actions by σ, σ'. Substitution in the minors, including the determinant twists, carries the pair (A_i, v_i) to $(A_i \sigma'(u'^{-1}), \sigma(u) v_i)$. At finite points g, h, with finite test f and finite Weil action ω_f, write $J_c F$ for the Taylor jet at scale c from Section 7.4.3. The rectangular matrix is $\mathcal{J}_c(F) = (\mathcal{S}(\partial_Z)(J_c F))(0)$, with the differential operators applied entrywise. The resulting theta coefficient is

$$c_{\mathrm{jet}}^{-D} A_i \sigma'(u'^{-1}) \mathcal{J}_{c_{\mathrm{jet}}}(\omega_f(g,h) f) \sigma(u) v_i.$$

The incoming source functional at (u, g) is $t\sigma(u^{-1})\phi(g)$, where ϕ is the source vector. Compact integration uses the mass-one identity

$$\int \sigma(u)(v_i \otimes t)\sigma(u^{-1})\, du = \frac{t(v_i)}{d_X} \operatorname{Id}.$$

The average commutes with the irreducible source action, and its trace is $t(v_i)$. Write K for the fixed level on the definite source X_j, with the prescribed last-row condition modulo π_N, and K° for its integral stabilizer. For a finite test choose a compact open subgroup $J \subseteq K$ fixing the scalar integrand. Its quotient integral for $\mathrm{vol}(K^\circ) = 1$ is the class sum at J with inverse stabilizer orders, multiplied by $[K^\circ : J]^{-1} = [K^\circ : K]^{-1}[K : J]^{-1}$. The algebraic lift, normalized by $\mathrm{vol}(K) = 1$, has the factor $[K : J]^{-1}$ in the same sum. For this upper source of rank n_j, split reduction gives

$$[K^\circ : K] = |\mathrm{GL}_{n_j}(\mathcal{O}_E/(\pi_M))|\,(N^{n_j} - 1)^f.$$

At each of the f chosen places over N the residue group acts transitively on the nonzero row covectors, and K fixes the last covector itself. The opposite component is determined by the unitary condition. Thus the scalar from the highest coefficient of the algebraic lift to the theta integral is

$$\frac{c_{\mathrm{harm}}}{d_X\, c_{\mathrm{jet}}^D\, [K^\circ : K]},$$

where c_{harm} compares the chosen real harmonic with the polynomial of minors. All four factors are fixed at this definite step as ξ varies.

In particular both maps have the same finite-datum dependence. For the preceding Schur expansion the outgoing functional is

$$\frac{\sum_i t(v_i) A_i}{d_X c_{\mathrm{jet}}^D [K^\circ : K]}.$$

This contracts the rectangular Schur coefficient with t and is independent of the expansion, the finite test and its fixing subgroup. It includes the cases $t = 0$ and $t(v_i) = 0$.

For the joined real polynomial, the first block of variables has the transposed indices of the first arrow and the second has those of the second arrow. The sign change of the first form makes the flipped Fock variables holomorphic in the new basis. Products of monomials of the two blocks span all polynomials. Write z_{def} for the nonzero scalar in the theta-product identity on the definite triple (X, X', X''). It is fixed by its rational Schrödinger models, by the argument preceding (67). On a product polynomial and two finite tests the residual theta kernel equals z_{def} times the product of the definite kernels. Integrating the first and then the second compact coefficient therefore composes their two contractions, multiplied by $z_{\mathrm{def}}\kappa_R\kappa_{R'}/z_\theta$ from (68). Extending linearly in the finite tensor gives the outgoing functional at $F_i(\xi, y)$.

For a general Schwartz test after the slices, approximate by Gaussian times polynomials in the Schwartz topology, by the Hermite expansion. All the integral evaluations are continuous in that topology by the compact definite quotients and absolute theta bounds. For each fixed source functional the possible families of target coefficients, with the twists, tensors, forms and target points as indices, form the image of the finite-dimensional target Weyl dual in a product of copies of $\mathbb{C}$. Choose a complement of the common kernel of these evaluations in the target dual. Finitely many coordinates give an injective map on the complement, with a linear inverse on its image. Convergence in these coordinates therefore gives a limiting functional in the complement, and convergence in every coordinate gives the identity for every finite datum. Its dependence on the real points and the incoming functional is continuous by the same finite inverse and the compact integral formulas. This also covers a zero-dimensional image.

At the first pair the incoming functional is the original t. For the measured period the initial point is $(g_{0,\infty}, c)$ in the finite level trace of Section 19.1; the intermediate point is the common

argument of both kernels. At the outgoing place the real argument can vary with the outgoing finite component held at y_2. Inductively the even-stage function on $U(X_{2i})$ is a coefficient of the first i definite pairs with tensors $F_0, \dots, F_{i-1}$. The coefficient functional depends only on the outer real points and the fixed datum. Its extension to R_{2i} is determined by the unipotent character. The terminal coefficient is read at $\mathrm{diag}(h, 1)$, by the last pair calculation. This statement is pointwise before any outer integration. It also applies to the support test for the finite cuts.

At the terminal space let B be the branching map from the Weyl space of X_m to that of Y_m. For a terminal functional ℓ_u and normalized highest vector v_Y, average

$$\int_{U(Y_m)_\infty} \sigma_Y(k)(v_Y \otimes \ell_u)\sigma_X(\mathrm{diag}(k^{-1}, 1))\, dk = c_{\ell_u} B.$$

The average intertwines by Haar invariance. The terminal weights interlace by Lemma 7.9. After a common determinant twist, removing the boxes occupied by the last letter leaves the smaller diagram, and the removed boxes form a horizontal strip. Thus the compact branching multiplicity is one. Let z belong to the original terminal type and use $\overline{z}$ as the test of the conjugate type in the preceding bilinear transpositions. With the Hermitian product linear in its second argument,

$$\overline{z(k, h)} = \langle z_{\mathrm{alg}}(h), \sigma_Y(k) v_Y \rangle.$$

If ψ is the upper-chain output, its coefficient at $(\widehat{k}, \widehat{h})$ is $\ell_u(\sigma_X(\widehat{k}^{-1})\psi(\widehat{h}))$. Multiplication by $\overline{z(k, h)}$ and integration in k therefore insert precisely the displayed averaged map. Let J_B be the finite class integral of

$$\langle z_{\mathrm{alg}}(h), B\psi(\mathrm{diag}(h, 1)) \rangle$$

with maximal-compact mass one. At a common fixing level J this is its weighted class sum divided by $[K_Y^\circ : J]$, where K_Y° is the terminal maximal compact. Here $R_m(F_0) = U(Y_m)(F_0)$, so the last quotient $R_m(F_0)\backslash U(Y_m)(\mathbb{A}_{F_0})$ has the ambient Petersson measure. With probability Haar at infinity the terminal integral against $\overline{z}$ is $c_{\ell_u} J_B$. The restriction of Section 7.4 is B/b_N, where $b_N = \prod_\tau \tau(\bar{\pi}_N)^{B_\tau}$ and $B_\tau = a_\tau - m h_0$ is the exponent denoted by d_τ in that section. With d_Y the full terminal Weyl dimension, Schur orthogonality for the unit vector v_Y gives

$$\langle z, \mathrm{Res}\, \psi \rangle = (d_Y b_N)^{-1} J_B, \qquad c_{\ell_u} J_B = d_Y b_N c_{\ell_u} \langle z, \mathrm{Res}\, \psi \rangle.$$

Thus the terminal coefficient uses the same quotient measure and the same finite representative on both sides.

It separates into a factor of the real outer points and the indicated linear expression in the finite tensors. The two quotient-mass factors of each pair are included in ℓ_u.

18.4 Finite support on the original Bessel orbits

In this subsection, we show that the period expression of Section 18.3 is supported on a compact union modulo the original Bessel actions. The lattice inclusions below give the compact sets used in the class sum of Section 19.1.

For the space Y_j, let $L_j(x)$ be the lattice in the chosen basis at the finite point x, and $\Lambda_j = L_j(1)$. On its string subspace S_j, write $L_j^- = L_j \cap S_j$ and $L_j^+ = \mathrm{pr}_{S_j} L_j$. The full lattices

L_j and Λ_j are Hermitian self-dual. Write $\#$ for the dual on the nondegenerate string space. A string vector pairs integrally against L_j precisely when it pairs integrally against L_j^+, so

$$(L_j^+)^\# = L_j^-, \qquad (L_j^-)^\# = L_j^+.$$

Use $q_{\mathrm{id}} = b_{\mathrm{id}}/c_M$, where b_{id} is the full modulus idele of the lower arrow and c_M is the original level generator. Then $q_{\mathrm{id}}, \bar{q}_{\mathrm{id}}$ are integral and are units above N. Put $a_E = \alpha_{\mathrm{W}}/\bar{c}_M$.

We first make a graded representative in a Bessel orbit. At each local place take a basis B of the moved lattice adapted to the decreasing grade filtration, by extending successive bases of the saturated intersections. Its degree-zero matrix part D_0 is invertible and $BD_0^{-1} - 1$ strictly raises grades. Let $G = B^* J B$ and $G_0 = D_0^* J D_0$, where J is the rational Gram matrix. The form pairs opposite grades.

Thus $G - G_0$ has entries with sum of the two grades negative. After reversing the grading on one axis, G is block triangular, with its diagonal blocks those of the reversed G_0. Since G, G^{-1} are integral, the diagonal blocks and their inverses are integral. Hence G_0 is unimodular. Write $G_0^{-1} G = 1 + P$; P is self-adjoint for G_0 and strictly raises grades. At every finite place E/F_0 is split or unramified, so choose a local integral λ with $\lambda + \bar{\lambda} = 1$.

For $D = 1 - \lambda P$, self-adjointness gives

$$G_0^{-1} D^* G D = (1 - \bar{\lambda} P)(1 + P)(1 - \lambda P) \equiv 1 \pmod{P^2}.$$

Thus a correction doubles the minimum grade increase of the error. Each correction and its inverse are integral, since P is integral and nilpotent. Iterating gives a unitriangular H with $H^* G H = G_0$. For $d_j = \dim_E Y_j$, the isometry BHD_0^{-1} takes the graded lattice $D_0 \widehat{\mathcal{O}}_E^{d_j}$ to L_j and acts trivially on the associated graded space. Its inverse is the required finite Bessel move. Almost everywhere one starts with the original graded basis and the move is the identity.

For $0 \le j \le m - 1$ the string of Y_j has length $s = m - 1 - j = 2\varrho + 1 - j$. In the basis of normalized string powers the line ideals for $0 \le i < s$ are

$$\Lambda_i^- = p_{s,i} N^{-\mathbf{1}_{2i \ge s}} \widehat{\mathcal{O}}_E, \qquad \Lambda_i^+ = p_{s,i} N^{-\mathbf{1}_{2i+1 \ge s}} \widehat{\mathcal{O}}_E, \qquad p_{s,i} = a_E^{s-i} \bar{a}_E^{-1-i}.$$

For $2i < s$ the coefficient of the i-th power in the original string basis is c^d, with $d = i - \lfloor s/2 \rfloor$. For $2i \ge s$ it is $-N c^d$, with $d = i - \lfloor (s-1)/2 \rfloor$. These signs make $T_{j,f}$ send each surviving power to the power of the same index, by the three odd-string and two even-string coefficients of Section 17.4. Put $u = (a_E/\bar{a}_E)^{\lfloor (s+2)/2 \rfloor}$. In the original finite frame the string multiplier is u when s is odd. For even s it is u/a_E on the first half and $u\bar{a}_E$ on the second half. Substitution of the two displayed exponents d gives $p_{s,i}(a_E \bar{a}_E)^d$ in all cases.

The valuations of c and $a_E \bar{a}_E$ agree, so the intersection ideals are as displayed. In odd length the central glued vector has nonzero component in the last definite line, so its coefficient must vanish in the string intersection. With $u = (a_E/\bar{a}_E)^{\lfloor (s+2)/2 \rfloor}$, its projection adds uz/N on the middle line, giving the extra factor N^{-1} for the projection exactly there. For $s \ge 1$ the shift kills the last power. For $i < s - 1$, $2i + 1 \ge s$ is equivalent to $2i \ge s - 1$, and $p_{s,i} = a_E p_{s-1,i}$. Extending a shorter coefficient tuple by zero proves surjectivity, so

$$T_{j,f} \Lambda_j^+ = a_E \Lambda_{j+1}^-.$$

The quotient Λ^+/Λ^- is therefore zero outside N and for even length, and one residue line at each place above N in odd length.

18.4.1 The lattice conditions on a finite slice

In this subsection, we express the surviving slices of Section 18.1.1 as inclusions of lattices.

Consider a nonzero finite slice in the step from Y_j to Y_{j+1}. In the finite Hom model write L_H for the self-dual vacuum lattice, K_I for the basis of the positive-degree space and z_T for the finite coordinate of $T_{j,f}$. In these coordinates use $\psi_H(t) = \psi_f(j_{\mathrm{W}}\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}t)$. The signed Heisenberg operators obey

$$\rho_f(x)\rho_f(y) = \psi_H(y^*x)\rho_f(y)\rho_f(x), \qquad \rho_f(l)v_{\mathrm{vac}} = v_{\mathrm{vac}} \quad (l \in L_H).$$

Let $\mathfrak{m}$ be the ideal generated by b_{id}, $h = \dim_E \mathrm{Hom}_E(Y_j, Y_{j+1})$, and $R = (\mathcal{O}_E/\mathfrak{m})^h$. For the prescribed inclusion $a_0 \in R$ the test is

$$F = \frac{1}{|R|}\sum_{v\in R} \psi_{\mathfrak{m}}(-a_0 \cdot v)\, \rho_f(t_v)v_{\mathrm{vac}}, \qquad t_v = \overline{1_{\mathfrak{m}} b_{\mathrm{id}}^{-1}}\widetilde{v}.$$

Here $1_{\mathfrak{m}}$ selects the modulus places, $\widetilde{v}$ is an integral lift, and $\psi_{\mathfrak{m}}$ is the primitive residue character in the chosen split coordinates. Indeed the sum of the characters $\psi_{\mathfrak{m}}((a-a_0)\cdot v)$, divided by $|R|$, is the indicator of $a = a_0$. Since $\bar{b}_{\mathrm{id}}t_v \in L_H$, the commutator of $\rho_f(b_{\mathrm{id}}l)$ with $\rho_f(t_v)$ is one for every $l \in L_H$. Therefore $\rho_f(b_{\mathrm{id}}l)F = F$.

Let g be the unitary Hom action of the two finite points, set $f_1 = \omega_f(g)F$, $M_1 = b_{\mathrm{id}}g$ and $M_2 = \bar{b}_{\mathrm{id}}^{-1}g$. Then $M_1M_2^* = 1$ and $\rho_f(M_1l)f_1 = f_1$ by covariance. In the mixed model let $\mathcal{Z}$ denote restriction to zero in the coordinate dual to the positive-degree space I. With probability Haar on $I(F_0)\backslash I(\mathbb{A}_{F_0})$, character orthogonality selects exactly this coordinate. On the real and finite tensor the restrictions factor, so nonvanishing of the adelic constant term implies $\mathcal{Z}(\rho_f(z_T)f_1) \neq 0$. For $u \in I$ the signed translation coincides with the Heisenberg translation used to define the zero slice. Indeed its quadratic normalizing phase is evaluated on $\mathrm{tr}(u^\dagger u)$, as in Section 7.4.3. Adjunction preserves the degree, since the Hermitian forms pair opposite graded pieces. Thus $u^\dagger u$ strictly raises grades and has trace zero, including at the places over two. In the mixed model $\rho_f(u)$ is modulation by a character that is one at the zero coordinate. Hence $\mathcal{Z}\rho_f(u) = \mathcal{Z}$. If also $u \in M_1L_H$, then $\rho_f(u)f_1 = f_1$ by the lattice covariance above. Applying $\mathcal{Z}$ to

$$\rho_f(z_T)\rho_f(u)f_1 = \psi_H(u^*z_T)\rho_f(u)\rho_f(z_T)f_1$$

gives $\psi_H(u^*z_T) = 1$. Choose columns C so that $M_2^*K_IC = B_I$ is an integral basis of the saturation of its image in L_H.

This is possible over every local DVR, with the original basis almost everywhere. Let $J_IB_I = 1$ be an integral left inverse. For every integral column x, the vector $K_ICx = M_1B_Ix$ is in $I \cap M_1L_H$. Thus $\psi_H(x^*C^*K_I^*z_T) = 1$ for all such x, and self-duality gives $v = C^*K_I^*z_T$ integral. Taking $\ell_I = J_I^*v$ gives $C^*K_I^*M_2\ell_I = B_I^*J_I^*v = v$. The adjoint uses the transposed matrix at the conjugate place of a split pair. Cancelling C^* yields

$$K_I^*(z_T - M_2\ell_I) = 0.$$

In the original rational bases the two terms are $\alpha_{\mathrm{W}}^{-1}T_{j,f}$ and $\bar{b}_{\mathrm{id}}^{-1}W$, where W is the integral Hom matrix ℓ_I transported to the actual lattices L_j, L_{j+1}.

Orthogonality to I means all negative-degree coordinates of the difference vanish. Thus $W - (\bar{b}_{\mathrm{id}}/\alpha_{\mathrm{W}})T_{j,f}$ has degree at least zero. Move the two lattices by Bessel elements r, r' to graded representatives. The degree-zero actions are the identity on each string and preserve the

definite blocks. Since $T_{j,f}$ kills the definite block and has degree -1 on the string, $r' T_{j,f} r^{-1} - T_{j,f}$ has degree at least zero. Write P_d, P'_d for the source and target grade projections. The map $r' W r^{-1}$ is integral between the moved lattices. Its degree--1 part

$$\sum_d P'_{d-1}\, r' W r^{-1} P_d = (\bar{b}_{\mathrm{id}}/\alpha_{\mathrm{W}}) T_{j,f}$$

is still integral since these projections preserve the graded lattices. Factoring through the source string projection and taking the target string intersection gives

$$(\bar{q}_{\mathrm{id}}/a_E) T_{j,f} L_j^+ \subset L_{j+1}^-.$$

This computation concerns the original finite factors at these points.

18.4.2 Exclusion by the ramified character

In this subsection, we use the source condition of Section 6 in the lattice calculation of Section 18.4.1.

Let the initial finite point y_0 lie in the chosen level, so $L_0(y_0) = \Lambda_0$. If $I_{j+1}(a, x) \neq 0$ at an archimedean and finite pair, the integral in Section 18.3 has a point r at which its two factors are both nonzero. The source finite point is $r_f y_j$. For $j = 0$, applying r_f^{-1} recovers the original graded lattice. For $j > 0$ choose a graded representative of the source orbit and use the inductive bound. Keep an arbitrary graded representative of the target orbit. Duality takes the source bound $q_{\mathrm{id}}^j L_j^- \subseteq \Lambda_j^-$ to $\bar{q}_{\mathrm{id}}^j \Lambda_j^+ \subseteq L_j^+$. Apply the finite slice inclusion and the standard shift formula, then dualize on the target, to obtain

$$\bar{q}_{\mathrm{id}}^{j+1} \Lambda_{j+1}^- \subseteq L_{j+1}^-, \qquad q_{\mathrm{id}}^{j+1} L_{j+1}^+ \subseteq \Lambda_{j+1}^+.$$

A failure of the desired upper bound for L_{j+1}^- can occur only at a place above N on an odd string. There q_{id} is a unit, and the image of a vector witnessing failure generates the residue line of Λ^+/Λ^-. Both inclusions then force $L^- = L^+ = \Lambda^+$ at that place. At the conjugate place duality also gives $L^- = L^+$. In particular the moved lattice splits orthogonally into its definite and string parts at the pair of places.

The split option occurs only at an even stage $l = j + 1 < m$. Let $w, \bar{w}$ be its pair above N, choosing $w \mid \pi_N$. Let z_w be the original norm-one scalar unit supported at that pair, with primitive reduction at w. Its embedding Z acts by that scalar on the definite block of Y_l and by one on the string. The splitting of $L_l(x_*)$ for the moved representative x_* implies $x_*^{-1} Z x_* \in J_{\xi,l}$: it is integral at the split pair and the identity elsewhere, and the pair is disjoint from the chosen level and twist places.

Use x_* at the l-th outer point in the even-stage coefficient formula of Section 18.3. The last lower finite test of that stage, after the outer move, is fixed by $\omega(1, Z)$ by the conjugate level condition. The exact slice covariance passes this action to multiplication by z_w on the last definite group. Each earlier definite arrow intertwines scalar translation by z_w on its source and target. Indeed their simultaneous matrix action on Hom is trivial and the finite determinant characters are unramified, hence trivial on these units.

In the last arrow the translation can be moved to the finite test, where it fixes the evaluated tensor. By linearity the resulting partial chain L therefore satisfies $L \rho_X(z_w) = L$ on the original

source forms. For ϕ in the ramified source, the sum of its translates over the residue-unit action at w is zero. Thus

$$(N-1)L\phi = L\sum_{b=0}^{N-2}\rho_X(z_w)^b\phi = 0.$$

Thus $L\phi = 0$ in the characteristic-zero space of definite forms. The same conclusion holds with x_* replaced by any finite Levi translate hx_*, since the scalar Z on the definite block commutes with that Levi. Every finite Bessel translate is uhx_*, with u unipotent. At this even stage the string has odd length, so $u-1$ has degree at least two. Successor covariance multiplies the value at hx_* by its unipotent character. Thus the even-stage coefficient vanishes on the entire finite Bessel orbit at every real point, before outer integration. If $x_* = r_*x$ was the graded representative, take the finite Bessel translate r_*^{-1}. This gives zero at the original point (a,x), contradicting $I_l(a,x) \neq 0$.

Lemma 18.2 (The support on the ramified source)**.** *Suppose the source form is ramified and the initial finite point lies in the chosen level. For $1 \leq j \leq m-1$, if $I_j(a,x) \neq 0$, then every finite Bessel translate of x with graded lattice L_j satisfies*

$$\bar{q}_{\mathrm{id}}^{j}\Lambda_j^- \subseteq L_j^- \subseteq q_{\mathrm{id}}^{-j}\Lambda_j^-. \tag{69}$$

18.4.3 Compact representatives of the surviving terms

In this subsection, we use Lemma 18.2 to choose representatives for the measured sum.

To pass from the string lattices to compact representatives, suppose $L(x)$ and $L(y)$ have the same string intersection. It is saturated in both lattices. Its local bases therefore give two primitive bases in the standard coordinates, with equal Gram matrices. At a split pair write the bases as (A, A'), (B, B'). The equations for one matrix are $gA = B$ and $B'^{\mathsf{T}}g = A'^{\mathsf{T}}$. Over the residue field, first identify the basis spans with their given bases and extend the identification on their intersections with the two kernels to an isomorphism of the kernels.

The two maps A'^{T}, B'^{T} are surjective and agree on these identified spans by Gram equality. Complete with lifts of the remaining quotient basis. This gives an invertible residue solution. Over the DVR choose $DA = 1$, $B'^{\mathsf{T}}R = 1$ and put

$$g_0 = BD + RA'^{\mathsf{T}}(1 - AD).$$

Lifting the difference of the residue solution and g_0 to H, replace g_0 by $g_0 + (1 - RB'^{\mathsf{T}})H(1 - AD)$. The correction is homogeneous for both equations and gives the chosen invertible reduction. The conjugate matrix is $g^{-\mathsf{T}}$.

At a nonsplit unramified place choose a uniformizer π of the fixed field; it is also a uniformizer of the quadratic extension and $\bar{\pi} = \pi$. Finite-field Hermitian Witt extension matches the two primitive residue bases, allowing degenerate spans. Lift the isometry by successive Gram corrections. At each residue step the equation $C + C^* = H$ for Hermitian H is solvable using residue trace on the diagonal and one triangle off it. The trace is surjective also in residue characteristic two. If the unitary basis images agree modulo π^a, their discrepancy D at the next order satisfies $A^*D + D^*A = 0$.

The prescription $CA = D$ extends to a skew-adjoint matrix: prescribe the resulting skew-Hermitian form on the basis and fill the complementary block with zero. Then $1 + \pi^a C$ can be corrected to a unitary lift starting at order $a+1$, so the bases agree one order further. The

limit agrees exactly. The resulting finite integral isometry k has ykx^{-1} fixing the entire string, since it fixes its full lattice basis. It therefore lies in the Bessel Levi. Thus a common string intersection determines the Bessel orbit modulo the standard compact.

The bounds (69) give only finitely many string intersections, all equal to the standard ones off $Q\cup\bar{Q}$, where Q is the set of twist places: the quotient of the two bounding lattices is finite and supported on this union. Start with a graded representative for each occurring intersection. At a place outside $Q\cup\bar{Q}$, its string intersection agrees with the one at the identity. The local frame comparison just proved gives a Bessel Levi element on the left and a standard compact element on the right that take this component to the identity. At almost every place the component already belongs to the standard compact, so one may use the identity Levi element there. At the twist places keep the given components. The local choices assemble to finite adelic points, and the resulting representative remains graded everywhere. Its right standard compact coset meets the same finite Bessel orbit. This use of $R_j(\mathbb{A}_{F_0,f})$ is inside the integration in stages over the adelic Bessel quotient in Section 18.3. This gives a compact union at each intermediate stage in the original basis. At stage zero use the classes of the original chosen level modulo $J_{\xi,0}$.

19 The measured theta tensor and its integral lattice

In this section, we evaluate the finite sum from Section 18 with the prescribed Haar measures. The resulting tensor and its comparison at the additional ramified prime give the local identities used to bound derivatives in Section 20.

19.1 The measured finite sum

In this subsection, we integrate the paired formula of Section 18.3 over the finite classes furnished by Section 18.4. The computation retains the stabilizer factors and shows how the compact indices cancel.

For each intermediate $U(Y_j)$, use all string intersections satisfying (69) that occur for a graded moved lattice. The bounds depend only on the level and the twist. By Section 18.4 this gives a finite union of standard compact cosets. Its image modulo $J_{\xi,j}$ is finite since $J_{\xi,j}$ has finite index in K_j°. Choose one class $xJ_{\xi,j}$ in this image from each $R_j(A_f)$-orbit meeting it. These choices work for every source form, terminal test and set of outer points. Write $B_j = R_j(A_f)$ and $B_j^0 = B_j \cap K_j^\circ$. Let Ω_j be the union of the B_j^0-orbits of the selected classes. The unitary Levi acts on the Lie algebra of the unipotent radical with determinant one: that determinant is a rational character, and a unitary group has no nontrivial character over F_0. The Levi and the unipotent radical are unimodular, so R_j is unimodular at each place. Take finite Haar with $\mathrm{vol}(B_j^0) = 1$, rescaling the real Haar reciprocally to keep the original Bessel Haar. At an anchor $xJ_{\xi,j}$ put

$$d_j(xJ_{\xi,j}) = [R_j(A_f) \cap xJ_{\xi,j}x^{-1} : R_j(A_f) \cap xJ_{\xi,j}x^{-1} \cap K_j^\circ].$$

The Haar measure of the r with $rxJ_{\xi,j} \in \Omega_j$ is this integer on the represented orbit.

Indeed, with $Z_c = B_j \cap xJ_{\xi,j}x^{-1}$ at the anchor $c = xJ_{\xi,j}$, this set is $B_j^0 Z_c$. Its disjoint right translates of B_j^0 are indexed by $(B_j^0 \cap Z_c)\backslash Z_c$. For a starting point bc in the same B_j-orbit the set is $B_j^0 Z_c b^{-1}$, of the same measure. Use this anchor's integer on the whole orbit. If the twist is empty all compact representatives are integral and $d_j = 1$. In general d_j divides a power of the product of the residue characteristics of the twist places.

Indeed, at each twist place w, let $K_w(h)$ denote the principal kernel of depth h in the integral matrix group. Choose h_w with $x_w K_w(h_w) x_w^{-1} \subseteq K_{j,w}^{\circ}$. The level $J_{\xi,j,w}$ is contained in $K_w(1)$. The kernel of

$$Z_c \longrightarrow \prod_{w \mid Q(\xi)} K_w(1)/K_w(h_w), \qquad z \longmapsto (x_w^{-1} z_w x_w)_w,$$

is contained in $B_j^0 \cap Z_c$, since x is integral away from these places. Each successive quotient $K_w(h)/K_w(h+1)$ embeds in the additive matrix group over the residue field by $1 + \varpi_w^h D \mapsto \bar{D}$. Thus the index d_j is supported on the residue characteristics of the twist places.

Choose a nonnegative real cutoff α_j with $\int_{R_j(A_\infty)} \alpha_j(ru)\, dr = 1$ on each real orbit. Such a cutoff comes from the proper translation action of the closed Bessel subgroup. The cutoff depends only on the fixed inclusion $R_j(A_\infty) \subset U(Y_j)(A_\infty)$. Then

$$w_j(u,x) = \alpha_j(u) \frac{\mathbf{1}_{\Omega_j}(x J_{\xi,j})}{d_j(x J_{\xi,j})}$$

has full Bessel orbit integral one on the support of the unfolded integrand. Outside Ω_j the fraction is zero. Quotient integration in stages can therefore be performed with w_j, using counting measure for the rational subgroup and the inherited Haar for the adelic groups. Indeed, for a nonnegative function F on $R_j(F_0)\backslash G_j(\mathbb{A}_{F_0})$, integration first over $[R_j]$ and then over $G_j(\mathbb{A}_{F_0})$ gives

$$\int w_j(g) \int_{[R_j]} F(rg)\, dr\, dg = \int_{R_j(F_0)\backslash G_j(\mathbb{A}_{F_0})} F(h) \left(\int_{R_j(\mathbb{A}_{F_0})} w_j(b^{-1}h)\, db \right) dh.$$

To see this, write $g = b\dot{h}$ using the quotient integral formula over $R_j(\mathbb{A}_{F_0})\backslash G_j(\mathbb{A}_{F_0})$. Right invariance on $[R_j]$ replaces $F(rb\dot{h})$ by $F(r\dot{h})$ under its integral. The cutoff integral over b is constant on the remaining orbit. Tonelli applies to the nonnegative integrand. Unimodularity and the two cutoff integrals identify the parenthesis with the indicator of the represented finite orbits. Those contain the support of I_j. For $1 \le j < m$, with the earlier outer points fixed and $z = L_{\xi,j}^{\mathrm{t}} z'$ the rapid transposed test, apply this identity to $F = |I_j z|$. Its right side is $\int_{R_j(F_0)\backslash U(Y_j)(A)} |I_j z|$, finite by Proposition 18.1 at the preceding arrow. Fubini gives the same identity for $I_j z$.

Write $\Xi_j(z)$ for the integral of $I_j(h) z(h)$ on $R_j(F_0)\backslash U(Y_j)(A)$. Its remaining outer points are understood as parameters. The product is right $J_{\xi,j}$-invariant when $z = L_{\xi,j}^{\mathrm{t}} z'$, by the original level invariance of the last datum in I_j and the transpose. Factor the given adelic Haar on $U(Y_j)$ as $du\, dg_f$, with $dg_f(K_j^{\circ}) = 1$. Any rescaling of dg_f is compensated by the reciprocal rescaling of du. In the integral with w_j a coset of $J_{\xi,j}$ has finite measure $1/A_{\xi,j}$. The one-step identity of Section 18.3 has multiplier $A_{\xi,j}$. These two factors cancel, leaving exactly

$$\Xi_j(L_{\xi,j}^{\mathrm{t}} z') = \int_{U(Y_j)(A_\infty)} \alpha_j(u) \sum_{c \in \Omega_j} \frac{\Xi_{j+1}(z';\, g_j = (u,c))}{d_j(c)}\, du. \tag{70}$$

For $1 \le j < m$ set $z = L_{\xi,j}^{\mathrm{t}} z'$. For fixed earlier outer points the one-step identity at g_j gives

$$A_{\xi,j} |\Xi_{j+1}(z'; g_j)| \le \int_{[R_j]} |I_j(r g_j) z(r g_j)|\, dr.$$

Integrate this inequality against $w_j(g_j)\,dg_j$. On the left, each finite coset contributes $A_{\xi,j}^{-1}$. On the right, quotient integration and the orbit integral of w_j give

$$\int_{U(Y_j)(A_\infty)} \alpha_j(u) \sum_{c\in\Omega_j} \frac{|\Xi_{j+1}(z';\, g_j=(u,c))|}{d_j(c)}\,du \le \int_{R_j(F_0)\backslash U(Y_j)(A)} |I_j(h)z(h)|\,dh < \infty.$$

The last integral is absolute by Proposition 18.1 at the preceding step, since the transposed test z is rapid. For $j=0$, apply the measured level formula of Section 8.4 with $J_{\xi,0}\subseteq J_M$ to the first transpose. It gives the sum over $c\in J_M/J_{\xi,0}$, divided by $[J_M : J_{\xi,0}]$, of the original periods at the right translate $g_0(1,c)=(g_{0,\infty},c)$, where $g_{0,\infty}$ is the real point of Section 8.3. After unfolding, the factor $A_{\xi,0}$ leaves $[K_0^\circ : J_M]$ because indices multiply. It is independent of ξ.

By the terminal calculation of Section 18.3, the last period is a product of a function $a_m(u)$ of the outer real points and the terminal pairing on the finite tensor $\bigotimes_i F_i(\xi,y)$. At each step of (70), update the finite factor by the sum over $c\in\Omega_j$ with weight $d_j(c)^{-1}$. With the earlier real variables fixed, write $a(u)b_c$ for the factored integrand before multiplication by $\alpha_j(u)d_j(c)^{-1}$. Set the new real factor to $\int \alpha_j(u)a(u)\,du$ when this integral is absolute, and set it to zero otherwise. This defines a measurable function of the earlier real points: the integral of $\alpha_j|a|$ is measurable by Tonelli, and on its finite locus the signed real and imaginary parts give the parameter integral. Absolute integrability in (70) at any coset with $b_c\neq 0$ implies $\int|\alpha_j a|<\infty$. In the divergent case every b_c therefore vanishes and the same factorization holds. The rule uses only the fixed real data and works for all twists and finite tests. Applying it successively proves the identity successively at the intermediate groups.

Define the evaluated tensor by the weighted sum

$$T_\xi = \sum_{c_0,\dots,c_{m-1}} \left(\prod_{j=1}^{m-1} d_j(c_j)^{-1}\right) \bigotimes_{i=0}^{m/2-1} F_i(\xi,c), \tag{71}$$

where c_0 runs over $J_M/J_{\xi,0}$, $c_j\in\Omega_j$ for $j>0$, and the terminal finite point is one. Each finite tensor uses the same actual representative as its outer period. The real cutoffs, integrals, and point $g_{0,\infty}$ are fixed before ξ. Define c_{meas} as the iterated real factor times the fixed index $[K_0^\circ : J_M]$. The terminal test $\overline{z}$ may have any finite level. The finite invariance used in (70) is that of $L_{\xi,j}^{\mathrm{t}} z'$ at $J_{\xi,j}$ for $j<m$, which follows from the inclusion test in its source variable. Both terminal outputs have the prescribed real type, by the coefficient identity and the terminal branching map. They are smooth at the finite places, so their difference is fixed by a common compact open subgroup. Take z to be that difference. The terminal quotient is compact, hence this is an admissible test, and the equality of pairings gives zero for its squared Petersson norm. This proves equality of the terminal forms.

Proposition 19.1 (Measured seesaw identity)**.** *Let $\phi\in S$ and let ξ be an allowed twist. With the fixed real point and original finite levels, the measured lifts satisfy*

$$L_\xi \Pi\phi = c_{\mathrm{meas}}\,\mathcal{U}_\xi(T_\xi)\phi. \tag{72}$$

The scalar c_{meas} is independent of ξ.

For the trivial twist the number of choices of each c_j is at most $M^{fn_m^2}=B_*$, since the classes lie in $K_j^\circ/J_{\xi,j}$ and the principal index there is a product of $|\mathrm{GL}_{\dim Y_j}(k_w)|$. The sum of the cut weights is at most B_*^m in that case. The expansion of the $F_i(\xi,c)$ in translated vacua is given in Section 19.3.

19.2 The lattice of an evaluated pair

In this subsection, we calculate the lattice obtained when a summand of Section 19.1 is restricted to the paired oscillator variables. At the places over N, its two components satisfy a common residue condition that determines the sum in Section 19.3.

Use the finite bases in a pair from Y_{2i} through Y_{2i+2}. Put $R_f = \widehat{\mathcal{O}}_E$ in this calculation, let $\pi = \pi_N$ as a principal idele, and write J_N for the idempotent of the places dividing π. Thus $J_N \bar{J}_N = 0$. Use $\eta_- = \eta$, the norm--1 unit from the opposite finite basis of Section 18.2. Its basis scalar is s_- below. Put

$$\delta = \frac{c_M}{\bar{c}_M}, \qquad s_- = \bar{\eta}_- \frac{\bar{\alpha}_{\mathrm{W}}}{\alpha_{\mathrm{W}}}, \qquad u_j^{\mathrm{str}} = (a_E/\bar{a}_E)^{\lceil (m-j)/2 \rceil}.$$

With $\beta = \pi_M$ the integral level generator in valuation, the identities are

$$s_- u_{2i}^{\mathrm{str}} = \bar{\eta}_- \delta \, u_{2i+2}^{\mathrm{str}}, \qquad v_w(\bar{\beta}\delta) = v_w(\beta), \qquad \delta = 1 \text{ above } N.$$

They follow from $a_E = \alpha_{\mathrm{W}}/\bar{c}_M$ and the successive two-step exponents. In an odd string the lattice contains the glued vector $e/\bar{\pi} + (u_j^{\mathrm{str}}/N)z$ and $u_j^{\mathrm{str}} z$, where e, z are the last definite and central string coordinates. The coefficients of a vector with ordinary coordinates x and string coordinate $u_j^{\mathrm{str}}(\pi^{-1}x_e + d)$ in this pair of lattice vectors are $\bar{\pi}x_e, d$. Thus membership is tested by those two integral values and the other ordinary integral coordinates. The two-by-two Gram block is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$. The even string has unglued definite coordinates. The lattices in this calculation are over $R_f = \prod_w \mathcal{O}_{E_w}$. At each conjugation orbit a finite frame identifies the original lattice with a free module over its integer rings. The outer points of (71) act on the two tests before taking their zero slice. The matrices below are in the fixed original frames. All lattice inclusions can therefore be checked on the completions, also when the intersection of a class lattice with its rational space is a nonfree $\mathcal{O}_E$-module.

Let e, e'' be the last indices of the two definite blocks in the target of the joined space. Define

$$\begin{aligned} v_0 &= e + \bar{\eta}_- \bar{J}_N e'', \\ D_N &= J_N(\pi - 1) + \bar{J}_N(\bar{\pi}^{-1} - 1), \\ \Gamma &= 1 + D_N v_0 v_0^{\dagger}, \\ b_e &= D_N \pi^{-1} \bar{J}_N, \qquad b_{e''} = D_N \pi^{-1} \eta_- J_N. \end{aligned}$$

Set $b_j = 0$ for the other indices of this block. The vector v_0 has norm one, and Γ acts on its line by the norm-one idele $1 + D_N$. In order e, e'' its two blocks are

$$\begin{pmatrix} \pi & (\pi - 1)\eta_- \\ 0 & 1 \end{pmatrix} \quad (w \mid \pi), \qquad \begin{pmatrix} \bar{\pi}^{-1} & 0 \\ (\bar{\pi}^{-1} - 1)\bar{\eta}_- & 1 \end{pmatrix} \quad (w \mid \bar{\pi}),$$

and it is the identity elsewhere. For the column $x = \Gamma e_j$ with $b = b_j$, direct subtraction gives

$$\bar{\pi}x \in R_f^{\dim x}, \qquad b - \pi^{-1}x_e \in R_f, \qquad \bar{\eta}_- b - \pi^{-1}x_{e''} \in R_f, \qquad \delta b = b.$$

The first and second differences are $-[J_N + (1 - J_N)\pi^{-1}]$ in the column e or e'', respectively, and zero in the other columns. The bracket is integral.

For an interior pair put $u = u_{2i}^{\mathrm{str}}$. Lift the column (x_-, x_+) to ordinary coordinates $(s_- x_-, x_+)$ and an additional component $s_- u b(z + z'')$ on the removed plane. Dividing the

first block by its opposite-basis scale s_- leaves string coordinate ub. In the other block it is $u^{\mathrm{str}}_{2i+2}\bar{\eta}_- b$. Both are integral in the required glued lattice by the two difference congruences. In the terminal pair the extra target index e'' is represented by z/π on the negative source side. Use the diagonal scaling $\kappa_{e''} = \delta$, with all other κ's one. The corresponding original source string coordinate is then $\pi^{-1}\delta x_{e''}$. Since $s_-^{-1}\delta = -\eta_- u$, after dividing the first block by s_- its integrality test is

$$-\pi^{-1}x_e - \eta_-\pi^{-1}x_{e''} = (b - \pi^{-1}x_e) + \eta_-(\bar{\eta}_- b - \pi^{-1}x_{e''}) \in R_f.$$

The remaining ordinary target entries are integral. Thus the same lifted procedure works in both pairs.

Let C have these lifted columns in the original finite coordinates. Let x_f be the finite basis matrix of Y_{2i+1} in its graded coordinates, and j_0 the inclusion of its definite summand. Its definite-coordinate projection is

$$R' = j_0^\dagger x_f, \qquad (R')^\dagger = x_f^{-1} j_0.$$

Here $j_0^\dagger$ extracts the definite coordinates, and the second identity uses the isometry equation for x_f. The even middle string has unglued definite coordinates, so R' is integral. On matrices the lift $B : L_{\mathrm{def}} \to L_{\mathrm{big}}$ is $X \mapsto CXR'$, hence integral.

If i_f is the orthogonal residual inclusion and k_f the isotropic basis in the two slices, its columns give

$$B = i_f g_{\mathrm{nb}} + k_f T_{\mathrm{nb}}, \qquad g_{\mathrm{nb}} = (\operatorname{diag}\kappa\ \Gamma) \otimes 1.$$

Here T_{nb} is supported above N, and g_{nb} is unitary since the diagonal scaling is one above N and has norm one. Write $I = \operatorname{im} k_f$, and let L_{big}, L_{def} denote the standard finite vacuum lattices.

The quotient image of $L_{\mathrm{big}} \cap I^\perp$ in $I^\perp/I$ is exactly $g_{\mathrm{nb}} L_{\mathrm{def}}$. One inclusion follows from the integral columns of B. Conversely, for $x \in L_{\mathrm{def}}$ and $y \in L_{\mathrm{big}}$, the pairing of x with $B^\dagger y$ equals the integral pairing of Bx with y. Self-duality of L_{def} gives $B^\dagger L_{\mathrm{big}} \subseteq L_{\mathrm{def}}$. Since $i_f^\dagger i_f = 1$ and $k_f^\dagger i_f = i_f^\dagger k_f = k_f^\dagger k_f = 0$,

$$B^\dagger(i_f z + k_f v) = g_{\mathrm{nb}}^\dagger z = g_{\mathrm{nb}}^{-1} z.$$

Thus every integral lift has $z \in g_{\mathrm{nb}} L_{\mathrm{def}}$.

Write S_f for the joined string shift in the finite vacuum coordinates, namely α_{W}^{-1} times the stacked rational shift in the chosen basis. It satisfies $B^\dagger S_f = 0$, because the two string maps vanish on the definite summand of the middle space, so $S_f(R')^\dagger = 0$. For y vanishing above N one therefore has

$$g_{\mathrm{nb}} B^\dagger(S_f + y) = i_f^\dagger y.$$

On the ambient source and target spaces the denominators are $\beta, \bar{\beta}$, respectively. At the finite places away from N, $\Gamma = 1$ and the lifted columns lie in the corresponding block of the original inclusion. The additional diagonal scaling at the terminal index converts a source bound by β into the target bound by $\bar{\beta}$, by $v_w(\bar{\beta}\delta) = v_w(\beta)$. More explicitly, let d_i be β or $\bar{\beta}$ according to the ambient block of coordinate i, and let d'_b be the corresponding denominator at residual coordinate b. At $w \nmid N q_\xi$, each nonzero entry $(B^\dagger)_{bi,w}$ has $v_w(d'_b \kappa_b) = v_w(d_i)$ by the block calculation above. In the split case the entry uses the transpose at $\bar{w}$. Since g_{nb} acts by κ_b in this coordinate,

$$d'_b (g_{\mathrm{nb}} B^\dagger y)_{b,w} = \sum_{i:\,(B^\dagger)_{bi,w} \neq 0} (d'_b \kappa_b / d_i)_w \, (B^\dagger)_{bi,w} \, (d_i y_i)_w \in \mathcal{O}_{E_w}$$

whenever $(d_i y_i)_w \in \mathcal{O}_{E_w}$ for all i. At $w \mid N$ both generators are units and the retained shift vanishes.

19.3 Integral coefficients of the measured tensor

In this subsection, we expand the tensor of Section 19.1 using the lattice of Section 19.2. The resulting translates have controlled denominators at coefficient places and an absolute coefficient bound at complex embeddings.

Put $d_j^{\mathrm{H}} = \dim(Y_j)\dim(Y_{j+1})$, and let $\mathfrak{m}_\xi = (\beta)\prod_{w\in Q_\xi}\mathfrak{p}_w$ be the ideal of the lower residue test. The Fourier expansion of its one prescribed reduction in $(\mathcal{O}_E/\mathfrak{m}_\xi)^{d_j^{\mathrm{H}}}$ gives one character per element of the dual group, with coefficient $\zeta/|\mathcal{O}_E/\mathfrak{m}_\xi|^{d_j^{\mathrm{H}}}$ for a root of unity ζ. Multiplying the finite vacuum by such a character is its Heisenberg translate in the conjugate split half. Thus the translation has denominator $\bar{\beta}$ away from Q_ξ and its conjugates. At a coefficient prime away from the twist its coefficient denominator divides $|\mathcal{O}_E/(\beta)|^{d_j^{\mathrm{H}}}$. When Q_ξ is empty the sum of the absolute coefficients is one under every complex embedding.

Here is the effect of moving that expansion by an outer cut point. In the original self-dual finite lattice L write v_L for the vacuum and ρ for the signed Heisenberg translations. If g is the action on the oscillator and $H = L \cap g^{-1}L$, then

$$\omega(g)v_L = \frac{\Delta(g)}{[L:H]}\sum_{z\in L/H}\rho(gz)v_L, \qquad |\Delta(g)|^2 = [L:H], \quad \Delta(g^{-1}) = \overline{\Delta(g)}.$$

Indeed the fixed line for gL is the transport of the vacuum line.

Projection to that line by averaging the translations over gL gives the indicated sum. The translates by distinct cosets are orthogonal, so unitarity gives the displayed modulus. The coefficient at the zero coset is the finite Weil lattice multiplier divided by $[L:H]$. The signed translations are the vacuum-normalized ones, so the other phase factors in multiplying translations are values of the finite additive character, hence roots of unity.

For a translated vector $\rho(x)v_L$, the shifts become $g(x+z)$. Outside the twist g is an integral isometry, so the denominator of the shift is unchanged there and the local vacuum multiplier is one. For the move supported at the twist places, the lattice multiplier identity of Section 7.4.3 gives

$$\Delta(g)^2 = [L:H]\,\upsilon_f(\det g)^{-1}.$$

Every prime divisor of $[L:H]$ divides q_ξ. To check the character at $t = \det g$, use its ideal I, which satisfies $I\bar{I} = (1)$ and is supported above q_ξ. If $I^h = (z)$, then $y = z/\bar{z}$ has norm one, $(y) = I^{2h}$, and t^{2h}/y is an integral unit idele. The character on this idele has finite order. Its finite value on y is the inverse of the infinity-weight monomial on the conjugates of y, which are units at every coefficient place away from q_ξ. Thus $\upsilon_f(t)$ and $\Delta(g)/[L:H]$ are units at every such place. The same argument applies to the norm-one determinant arguments of the additional pair characters.

The extra pair characters on the integral components have finite order on the compact level. At empty twist all actions are integral and their coefficients have modulus one in every complex embedding. The Hilbert norm normalization of the multiplier above contributes the factor

$$\frac{\Delta(g)}{[L:H]} = \frac{\Delta(g)}{|\Delta(g)|^2} = \Delta(g^{-1})^{-1}.$$

The arrow product joins the first shift, by its adjoint into the negative block, to the second shift in its original order. In the finite vacuum coordinates its two blocks are

$$z_- = -\eta_-\,{}^t\bar{x}_1, \qquad z_+ = x_2.$$

Since $\eta_- z_-^* = x_1$, the joined product of the two normalized translations equals $e_0(z_-)^{-2}T_\sharp(z) = e_0(x_1)^2 T_\sharp(z)$ on the joined datum, by Section 18.2. This phase is a root of unity. The joined translation z therefore has denominator β on the first block and $\bar{\beta}$ on the second. As both are units above N, the N-component can be dropped on the ambient vacuum. The retained and removed parts have zero Heisenberg pairing since they have disjoint conjugation-stable supports.

Let Z_I denote the finite zero slice for the two isotropic bases, and let L_{big} be the ambient vacuum lattice, and let S_f be the string shift. We explain the finite support test and its coefficient. If $Z_I\rho(S_f + z)v_{L_{\mathrm{big}}} \neq 0$, its commutator against translations by $L_{\mathrm{big}} \cap I$ must be trivial. For $\lambda \in L_{\mathrm{big}} \cap I$, the signed translation fixes the vacuum. Its phase on the zero slice is

$$e_0(\lambda) = \psi_0\big(c_0 h(\lambda, \lambda)/2\big) = 1$$

because I is totally isotropic, including at the places above 2.

Put $A = L_{\mathrm{big}} \cap I$ and $x_0 = S_f + z$. The character condition also holds on all R_f-multiples of each $u \in A$. The annihilator of R_f under the normalized trace character is R_f, so $u^\dagger x_0 \in R_f$. At each conjugation orbit A is saturated in the free ambient lattice. Extend a basis in each field factor and extend the integral pairing functional by zero on the added basis vectors. Hermitian self-duality represents it by λ in the ambient lattice, with the two field factors paired at a split orbit. Almost everywhere one may take $\lambda = x_0$, which is integral there. The local choices yield $\lambda \in L_{\mathrm{big}}$ with $x_0 - \lambda \in I^\perp$, so $S_f + z - \lambda = i_f w + k_f v$. The translation by λ fixes the vacuum, and the translation by $k_f v$ acts trivially on the zero slice.

The result is therefore a phase times $\rho(w)Z_I v_{L_{\mathrm{big}}}$. The same argument applies to any translate used to test whether the zero slice itself vanishes. The zero slice is nonzero in an adapted split model, and translates of the finite vacuum span its Schwartz space. Thus $Z_I v_{L_{\mathrm{big}}} \neq 0$.

For $a = i_f w + k_f v \in I^\perp$, orthogonality and isotropy give $\langle a, a\rangle_{\mathrm{big}} = \langle w, w\rangle_{\mathrm{def}}$ in the Hermitian trace pairings on displacements. Hence the quadratic factors in the vacuum-normalized translations of a and w agree, including at places above two. The two summands of a also have trivial Heisenberg commutator and half-pairing. Restriction to zero in the coordinates dual to I thus gives

$$Z_I \rho_{\mathrm{big}}(i_f w + k_f v) = \rho_{\mathrm{def}}(w) Z_I.$$

For $w \in g_{\mathrm{nb}} L_{\mathrm{def}}$, Section 19.2 gives an integral lift a fixing the ambient vacuum. Thus the signed translation by w fixes its zero slice. In the Schrödinger model the original lattice invariants are one-dimensional: translation invariance makes a function constant on additive lattice cosets and invariance under the dual characters restricts its support to the integral coset. Weil transport gives the invariant line for $g_{\mathrm{nb}} L_{\mathrm{def}}$, so the zero slice of the vacuum is a fixed nonzero multiple of $\omega(g_{\mathrm{nb}}) v_{L_{\mathrm{def}}}$. The residual shift $w = g_{\mathrm{nb}} B^\dagger(S_f + z - \lambda)$ can be replaced by $g_{\mathrm{nb}} B^\dagger z$ up to a phase: $B^\dagger S_f = 0$, and $g_{\mathrm{nb}} B^\dagger \lambda \in g_{\mathrm{nb}} L_{\mathrm{def}}$ fixes the slice. Section 19.2 gives the same two denominators for the new shift and shows that it vanishes above N.

19.3.1 Translation at the additional level

In this subsection, we express the neighbor sum of Section 19.2 in translated vacua at N.

For each $w \mid \pi$, choose a covector $y_w \in \mathbb{F}_N^{1\times n_{2i+1}}$. Put $r_{y,w} = \widetilde{y}_w/N$ for any integral lift of this row and $r_{y,\bar{w}} = 0$. The linked representative n_y has last coordinates $(\eta_- \bar{r}_y, -\bar{r}_y)$ in the two blocks. At $w \mid \pi$ the matrix Γ has integral entries, so that factor of the additive quotient is zero. At $\bar{w}$, let a, b be the two rows in the distinguished coordinates. Their image modulo the integral lattice is

$$(a/\bar{\pi}, \bar{\eta}_- a/\bar{\pi}).$$

It equals n_y modulo the lattice precisely when $\overline{y_w} \equiv -\pi\bar{\eta}_- a \pmod{\bar{\pi}}$ in every middle coordinate.

The chosen split places have residue field $\mathbb{F}_N$, and $\pi\bar{\eta}_-$ is a unit in the conjugate half. Thus every class occurs once. In fact $a = \bar{\pi}\eta_- \bar{r}_y$ and $b = \bar{\eta}_- a$ give n_y exactly. Applying the inverse arrow product to n_y gives the shifts $(-{}^t r_y, -\bar{r}_y)$. As y ranges over all residues this is exactly the original linked gluing, denoted by v_x^{gl} when applied to an upper pair with additional translate x.

The two factors are both supported in the prescribed last coordinates. The signs may simultaneously be reversed by $y \mapsto -y$, which permutes the sum. All pairings between n_y and the shift away from N vanish. In signed vacuum coordinates the adjoint transport of a first block u has phase $e_0(u)^2$ by Section 18.2. For n_y the first block is supported on only one split half, so its Hermitian square is zero and the phase is one. The entire sum at N is retained in v_x^{gl}.

Away from N, let b_i be the number of classes of $g_{\mathrm{nb}} L_{\mathrm{def}}$ modulo its intersection with L_{def}. On an interior pair $\kappa = 1$ in every coordinate, so $b_i = 1$. On the terminal pair only the coordinate e'' is scaled, by δ in each of the n_{2i+1} middle coordinates. The ideals of c_M and β agree, and $(\beta), (\bar{\beta})$ are coprime. Consequently $\delta R_f/(\delta R_f \cap R_f)$ has order $|\mathcal{O}_E/(\bar{\beta})| = M^f$, so $b_i = M^{f n_{2i+1}}$ on this pair. Above N the quotient has the independently computed $N^{f n_{2i+1}}$ classes in every pair. These are already summed in the glued vector. Put the factor b_i into the common pair scalar and assign the b_i terms coefficients of size b_i^{-1}, including their phases. To check divisibility away from the twist, set $D_M = |\mathcal{O}_E/(\beta)| = M^f$, using $\beta\bar{\beta} = M$ and $[E : \mathbf{Q}] = 2f$. The two residue tests have denominators dividing $D_M^{d_{2i}^{\mathrm{H}}}$ and $D_M^{d_{2i+1}^{\mathrm{H}}}$ there. The quotient just computed has order $b_i \mid D_M^{r_i n_{2i+1}}$, with $r_i = n_{2i} + n_{2i+2}$. For $j \le m$ the ranks are $\dim X_j = n_j = k + j(t+1) \le n_m$ and $\dim Y_j = r + jt = k + m - 1 + jt \le n_m$. Hence

$$d_{2i}^{\mathrm{H}} + d_{2i+1}^{\mathrm{H}} + r_i n_{2i+1} \le n_m^2 + n_m^2 + 2n_m^2.$$

Consequently the product of the three denominators divides $D_M^{4n_m^2} = B_*^4$ away from the twist.

At the empty twist, after factoring out the common pair scalar, the sum of the absolute coefficients of the glued vectors is at most one at every complex embedding. Each residue test has absolute coefficient sum one, and the b_i weights have absolute sum one. Opening the linked sum inside a glued vector contributes the separate factor $N^{f n_{2i+1}}$ used in Section 7.4.5. The vacuum line scalar, the normalization of its neighbor sum, and the nonzero normalizing factors of the two upper vacua depend only on the original pair. If s_{vac} is the one-coordinate vacuum normalization, the respective factors are $\kappa_j = s_{\mathrm{vac}}^{n_j n_{j+1}}$. More precisely, write $\mathcal{C}_i$ for the chosen representatives of $g_{\mathrm{nb}} L_{\mathrm{def}}/(g_{\mathrm{nb}} L_{\mathrm{def}} \cap L_{\mathrm{def}})$. The unshifted slice coefficient $c_i \neq 0$ is determined by

$$Z_I v_{L_{\mathrm{big}}} = c_i \sum_{\nu \in \mathcal{C}_i} \rho(\nu) v_{L_{\mathrm{def}}}.$$

Write $Z_I v_{L_{\mathrm{big}}} = a_i \omega(g_{\mathrm{nb}}) v_{L_{\mathrm{def}}}$. Multiplication by g_{nb} identifies $L_{\mathrm{def}}/(L_{\mathrm{def}} \cap g_{\mathrm{nb}}^{-1} L_{\mathrm{def}})$ with the quotient represented by $\mathcal{C}_i$; hence its order is $N^{f n_{2i+1}} b_i$. The finite vacuum expansion gives

$$c_i = \frac{a_i \Delta(g_{\mathrm{nb}})}{N^{f n_{2i+1}} b_i}.$$

The linked sum at N has coefficient one on every residue, and the b_i classes away from N carry weights b_i^{-1}. The common pair scalar for the two integral vacua is therefore

$$\frac{c_i b_i}{\kappa_{2i} \kappa_{2i+1}} = \frac{a_i \Delta(g_{\mathrm{nb}})}{N^{f n_{2i+1}} \kappa_{2i} \kappa_{2i+1}}.$$

Applying this argument to the tensor in (71) gives a fixed nonzero C_0 and an expansion

$$T_\xi = C_0 \sum_x b_{\xi,x} v_x^{\mathrm{gl}}.$$

Here $B_*^{2m} b_{\xi,x}$ is integral at every coefficient embedding off the twist. Indeed the cut weights have denominators supported on q_ξ and each of the $m/2$ pairs contributes at most B_*^4. At the trivial twist the cut sum gives

$$\sum_x |\sigma(b_{\varnothing,x})| \le B_*^m \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Terms with coincident translates can be combined by the triangle inequality.

The expanded tensor is fixed by the original product of upper levels with the terminal action one. For $j < m$ let $X = X_j$ be the definite block of the Bessel Levi and let Λ be the original lower lattice. Write L_X for the standard definite lattice and e for its last basis vector. The Levi element attached to g in the upper level fixes the string, so its difference from the identity acts on Λ by $A \,\mathrm{pr}_X$ with $A = g - I$. For an even string, $\mathrm{pr}_X \Lambda = \Lambda \cap X = L_X$. For an odd string the two columns $e/\bar{\pi}_N + uz/N, uz$ of Section 19.2 give

$$\mathrm{pr}_X \Lambda = D_{\bar{\pi}_N}^{-1} L_X, \qquad \Lambda \cap X = D_{\pi_N} L_X, \qquad D_v = \mathrm{diag}(1, \dots, 1, v).$$

Indeed a linear combination of the two columns with zero string coordinate has first coefficient divisible by N. The upper last-row condition gives $e^t A \equiv 0 \bmod \pi_N$. The same condition on $g^{-1} = g^*$ gives $Ae \equiv 0 \bmod \bar{\pi}_N$. The last diagonal entry is divisible by N, since $(\pi_N), (\bar{\pi}_N)$ are coprime. Thus $D_{\pi_N}^{-1} A D_{\bar{\pi}_N}^{-1}$ is integral in the odd case, and A is integral in the even case. In both cases $A \,\mathrm{pr}_X \Lambda \subseteq \Lambda \cap X$, also for g^{-1}, so the Levi action preserves Λ. At the chosen places above M both diagonal scalings are units and $g \equiv I \bmod \pi_M$, giving identity reduction on Λ.

At each intermediate stage the embedded element fixes the string, preserves Λ, and belongs to $B_j^0 = R_j(A_f) \cap K_j^\circ$. The set Ω_j is a union of left B_j^0-orbits and d_j is constant on each orbit by Section 19.1. Thus the element permutes the cut classes with their weights. At the first stage it permutes the chosen level classes. Equivariance of the two-slice evaluation gives this upper action on each summand, on both occurrences of every shared index. Thus it preserves the measured sum. Before applying the linked sum at N, write each additional shift as $x = x^{\mathrm{lev}} + x^{\mathrm{away}}$, supported above Mq_ξ and away from those places, respectively. Both supports are stable under conjugation and are disjoint. The signed Heisenberg multiplication law therefore gives

$$\rho(x) = \rho(x^{\mathrm{lev}}) \rho(x^{\mathrm{away}}), \qquad \rho(x^{\mathrm{away}}) v_L = v_L,$$

where the second identity uses integrality of x^{away} and the vacuum normalization of the signed translations. Thus $\rho(x)v_L = \rho(x^{\mathrm{lev}})v_L$ on each arrow. Applying the linked sum at N to these equal tensors preserves the evaluated tensor.

Apply Section 17.3 to (72) with this tensor. Put $a_{\mathrm{ss}} = c_{\mathrm{meas}}C_0/B_*^{2m}$ and define

$$(c_{\mathrm{ss}}, \Psi_\xi(x)) = \begin{cases} (a_{\mathrm{ss}}, B_*^{2m} b_{\xi,x}) & a_{\mathrm{ss}} \neq 0, \\ (1, 0) & a_{\mathrm{ss}} = 0. \end{cases}$$

In the second case $c_{\mathrm{meas}} = 0$, since the two-arrow vacuum calculation gives $C_0 \neq 0$. In both cases $c_{\mathrm{ss}} \neq 0$. Linearity and (72) give

$$L_\xi(c_{\mathrm{ss}}^{-1}\Pi\phi) = \mathcal{U}_\xi\left(\sum_x \Psi_\xi(x) v_x^{\mathrm{gl}}\right)\phi \qquad (\phi \in S). \tag{73}$$

The scalar $c_{\mathrm{ss}} \neq 0$ is fixed before the twist. For each ξ, one tensor gives the identity for every $\phi \in S$. For the empty twist

$$\sum_x |\sigma\Psi_\varnothing(x)| \leq B_*^{3m} = M^{3fmn_m^2}$$

for every complex embedding σ. The resulting upper lift evaluates the same source and uses the same terminal branching as $L_\xi\Pi$. The equality on the original ramified source is now in integral tensor form.

19.4 The comparison at the additional ramified place

In this subsection, we compare the two integral realizations at N for the tensor of Section 19.3. The comparison gives an integral submodule of the ramified source on which the local theta estimate holds uniformly.

Use the spaces X_j, their original finite levels, and the norm equation $\pi\bar{\pi} = N$. These are the upper definite groups and original glued vacua of Section 19.3. Fix an N-adic embedding $\jmath$. Put $P = \Sigma_N$, the set of conjugates of the places induced by $\jmath$ and the infinite embeddings. Put $S = \{w : w \mid \pi\}$. Each of P, S contains one place from every conjugate pair over N, and at each such place the completion and the residue field are $\mathbf{Q}_N, \mathbf{F}_N$.

For a tuple a of matrices at P, let $\widehat{a}$ be its unitary embedding by inverse conjugate transpose at $\bar{P}$ and identity away from N. Put $\delta_j(P) = \widehat{A}_j$, where $(A_j)_w = \mathrm{diag}(I_{n_j-1}, N^{\mathbf{1}_{w\in S}})$ at $w \in P$. Write $\delta_j = \delta_j(P)$ at this embedding. Thus at the chosen half P the point takes the standard lattice to L_j, the lattice of integral vectors whose last coordinate is divisible by π. For right translation R_j on forms, its expression in these bases is $\rho_j(a) = R_j(\delta_j\widehat{a}\delta_j^{-1})$. The original local level becomes the stabilizer of $e = e_{\mathrm{last}}$ modulo N. The conjugate coefficient embedding uses $\delta_j(\bar{P})$ with the same S and the same original level.

Let

$$B = \prod_{w\in P} \mathbf{F}_N, \quad D_j = \mathrm{diag}(1, \ldots, 1, N^{-1}), \quad \psi_N(y) = \psi_0(-y/N).$$

Here D_j acts at all of P in the appropriate rank. More explicitly, let e_P be the finite idempotent of the places in P and let $\widetilde{y}$ lift the residue coordinates there. Then

$$\psi_{N,P}(y) = \psi_f(-j_{\mathrm{W}}\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}e_P\widetilde{y}/N) = \psi_N(y).$$

The inverse different identity of Section 8.1 makes this independent of the lift and nontrivial on each $\mathcal{O}_{E_w}/N = \mathbb{F}_N$. Since $\bar{j}_{\mathrm{W}} = -j_{\mathrm{W}}$ and the trace is invariant under conjugation, $\psi_{N,\bar{P}}(\bar{y}) = \psi_{N,P}(y)^{-1}$. All functions of a residue matrix below are extended by zero off the integral matrices.

19.4.1 Translation laws and the ramified sum

In this subsection, we apply the local oscillator action to the summands of Proposition 19.1.

Let v_j be a finite Weil vector. In the split Schrödinger model at P, let $\mathcal{B}_j$ be the integral matrix box, and normalize its additive volume to one. The projection onto its vacuum line at N is

$$v_P \otimes v^{(N)} \longmapsto \mathbf{1}_{\mathcal{B}_j} \otimes \left(\int_{\mathcal{B}_j} v_P(T)\, dT \right) v^{(N)}$$

on pure tensors. Write the projection of v_j as $\mathbf{1}_{\mathcal{B}_j} \otimes u_j^{(N)}$, extending the formula linearly. Inserting Φ gives $\Phi \otimes u_j^{(N)}$. Write $L_j(\Phi)$ for the lift of this datum in the L_j, L_{j+1} coordinates. A group element supported at N therefore acts on the inserted function through its split Weil action. For a, b invertible over the local integers, the laws are

$$\rho_{j+1}(b) L_j(\Phi(bTa)) \rho_j(a) = L_j(\Phi(T)), \tag{74}$$

$$\rho_{j+1}(D_{j+1}) L_j(\mathrm{cov}_e) = \gamma_j L_j(\mathrm{col}_e) \rho_j(D_j). \tag{75}$$

Here $\mathrm{cov}_e(T) = \mathbf{1}_{\mathrm{row}_{\mathrm{last}}(\bar{T})=e^t}$ and $\mathrm{col}_e(T) = \mathbf{1}_{\bar{T}e=e}$ on integral matrices. Put $n = n_j$, $n' = n_{j+1}$, and write χ_j for the base change of the $(j+1)$-st lift character. The substitution $T \mapsto b^{-1}Ta$ has additive Jacobian $J = |\det a|_P^{n'} |\det b|_P^{-n}$. The unitary oscillator acts on functions by this substitution times $\chi_j(\det b / \det a) J^{1/2}$. Changing the source variable in the lift by right translation gives

$$L_j(\Phi(b^{-1}Ta)) = \chi_j(\det a / \det b) J^{-1/2}\, \rho_{j+1}(b) L_j(\Phi) \rho_j(a^{-1}).$$

It gives (74) since the characters are unramified there. The equality of functions $\mathrm{cov}_e(T) = \mathrm{col}_e(D_{j+1} T D_j^{-1})$ includes their supports. In blocks the change sends $\left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} A & Nb \\ c/N & d \end{smallmatrix}\right)$. Testing the last column and integrality therefore tests $c \equiv 0$, $d \equiv 1 \bmod N$ and integrality of T. The determinant character factor in (75) uses $a = D_j^{-1}$ and $b = D_{j+1}^{-1}$ in $T \mapsto b^{-1}Ta$. Both determinants have component N at each place of P, so their character values cancel and

$$\gamma_j = (N^{-|P|})^{-n_{j+1}/2 + n_j/2} = N^{|P|(n_{j+1}-n_j)/2}, \qquad |\gamma_j|_N \le 1.$$

Conjugation by δ_j in these calculations uses the same change on the datum and the source action. The source and target lattices in the two laws are transported by δ_j, δ_{j+1}, respectively.

A ramified source vector u on the original level has zero mean at each scalar group over N. For each $w \in P$ write $u = (1 - \rho_j(c_w)) u_w$, using a generator of that local cyclic residue unit group in the scalar action, supported at w and its conjugate. Write $C = \rho_j(c_w)$ and $L = N - 1$. Since $C^L = 1$ on the level invariants,

$$(1 - C) \frac{1}{L} \sum_{r=0}^{L-1} (L - 1 - r) C^r = 1 - \frac{1}{L} \sum_{r=0}^{L-1} C^r.$$

The right side fixes u. The scalar translations preserve the transported integral lattices and $N \nmid L$, so this formula also preserves integrality at N. At $j = 0$ the source condition uses S, and at $w \in P$ it reads through the member of S in the same conjugate pair.

Conjugation by δ_j does not change a scalar. Every $L_j(\Phi)$ for a residue function carries the difference form of ramification to the target by (74), taking opposite scalar changes of variables. Moreover, if the datum is invariant under scaling T at one place w, then it annihilates such a difference at w. This applies to the indicator that a given matrix column or row reduces to a vector that vanishes at w. It also applies to the datum that a specified matrix entry vanishes there.

For a residue column y in the target let $C_y(T) = \mathbf{1}_{\bar{T}e=y}$, and for a column in the source let $H_y(T) = \psi_N(e^t\bar{T}y)$, with the integral support in both cases. Write $L_j^{\mathrm{d}} = L_j(C_e)$. For the double lift the middle vector y must be primitive, meaning nonzero in each residue field. The other terms in $\sum_y L_{j+1}(H_y)L_j(C_y)u$ vanish by the ramified difference property. For primitive y choose k integral invertible with $\bar{k}e = y$ by extending the column to a basis in every residue field and lifting. The two laws for its insertion in the middle are

$$L_j(C_y)u = \rho_{j+1}(k)L_j^{\mathrm{d}}u, \qquad L_{j+1}(H_y)\rho_{j+1}(k) = L_{j+1}(H_e).$$

These follow from (74), since $\overline{kT}e = \bar{k}(\bar{T}e)$ and $e^t\overline{Tk}e = e^t\bar{T}y$. Summing gives

$$\sum_y L_{j+1}(H_y)L_j(C_y)u = c_{\mathrm{prim}}\, L_{j+1}(H_e)L_j^{\mathrm{d}}u, \qquad c_{\mathrm{prim}} = (N^{n_{j+1}} - 1)^{|P|}.$$

Indeed each residue column has $N^{n_{j+1}} - 1$ nonzero choices at each $w \in P$, independently. In particular $|c_{\mathrm{prim}}|_N = 1$.

Define the same-level operator

$$U_j = \sum_{y:\, y_{\mathrm{last}} \in B^\times} \psi_N(y_{\mathrm{last}})\rho_j(D_j k_y),$$

where the integral invertible k_y has both last column y and last row y^t modulo N. For $y = (v, t)$ with t a unit the residue matrix $\left(\begin{smallmatrix} 1+vv^t/t & v \\ v^t & t \end{smallmatrix}\right)$ has determinant t by the Schur complement and is such a lift modulo N. The actual U_j uses the choice of the lifts. For the actions on mirabolic fixed vectors only their last columns matter. Conjugating a mirabolic h by D_j^{-1} makes an integral matrix $D_j^{-1}hD_j$ whose last row reduces to e^t. Left multiplication by that matrix permutes the residue y with unit last entry, preserving their last entries. The error after replacing k_y by the chosen $k_{\bar{D_j^{-1}hD_j y}}$ fixes the last column. This proves that U_j preserves the original mirabolic fixed vectors. It commutes with the original level away from N by disjoint support. It preserves ramification because it commutes with the unit scalar actions.

At a ramified source vector one may omit matrices with nonunit corner from the function H_e, by decomposing that function according to the corner and using the scalar difference property at a place where the corner is zero. Sort the remaining function first by its last column y and then, in a separate expression, by its last row y^t. Using (74) in both sorts and (75) for the last-row indicator gives

$$\rho_{j+1}(D_{j+1})L_j(H_e)u = U_{j+1}L_j^{\mathrm{d}}u = \gamma_j L_j^{\mathrm{d}}U_j u.$$

These are identities of the same theta lift, so no change of integral structures occurs when they are used in succession.

19.4.2 The direct lift at the ramified place

In this subsection, we identify the gluing of the summed tensors in Section 19.4.1.

At N the normalized vacuum projects to the integral box. In the model at P, the original gluing from S is the sum of the tensors

$$C_y(TA_j) \otimes H_y(B_{j+2}R), \qquad y \in B^{n_{j+1}},$$

where $A_j = \mathrm{diag}(1, \dots, 1, N)$ on $P \cap S$ and $B_{j+2} = \mathrm{diag}(1, \dots, 1, N)$ on $P \setminus S$, both with the identity elsewhere. Write e_i for the last basis vector at rank n_i, and set $t_w = T_w e_j$, $r_w = e_{j+2}^t R_w$. Let $b = (b_v)_{v \in S}$ be the residue column indexing the original pair of translates in Section 19.3.1, and let red_w denote reduction modulo N at w. Write $\widetilde{b}_v$ for any integral lift. At $w \in P$, all entries outside the indicated last column and last row must be integral. For the remaining entries the two translated box indicators impose

w	integral entries	unique original class
$P \cap S$	$t_w - \widetilde{b}_w/N,\ r_w$	$b_w = \mathrm{red}_w(Nt_w)$
$P \setminus S$	$t_w,\ r_w - \widetilde{\overline{b}}_{\bar{w}}^{\,t}/N$	$b_{\bar{w}} = \overline{\mathrm{red}_w(Nr_w)}^{\,t}$.

Here conjugation identifies the residue fields at w and $\bar{w}$. Write $\psi_{0,w}$ for the restriction of ψ_0 to the w-factor. In split coordinates $(a, b^\vee)$, with a in the position space at w and $b^\vee$ the covector from the conjugate component, the displacement acts by

$$(\mathsf{T}(a, b^\vee)F)(z) = \psi_{0,w}(b^\vee(z + a/2))\, F(z + a).$$

Indeed its multiplier on a second displacement $(a', b'^\vee)$ is $\psi_{0,w}((b'^\vee a - b^\vee a')/2)$, the split expression for $\beta(x, y) = \psi_0(y^* x/2)$. Each original displacement has only one nonzero component, so its quadratic phase is one. In the first row of the table the conjugate translate on R_w contributes $\psi_{0,w}(-r_w \widetilde{b}_w/N)$. In the second row the first translate on T_w contributes $\psi_{0,w}(-\widetilde{\overline{b}}_{\bar{w}}^{\,t} t_w/N)$. Both arguments are congruent to $-r_w t_w$ modulo the local integers. Since $\psi_{N,w}(\mathrm{red}_w\, c) = \psi_{0,w}(-c/N)$ for integral c, the factor in either row is $\psi_{N,w}(\mathrm{red}_w(Nr_w t_w))$. With $T' = TA_j$ and $R' = B_{j+2}R$, the sum is zero unless T', R' are integral. On that support its value is

$$\psi_N(e_{j+2}^t \overline{R'}\, \overline{T'} e_j).$$

Sorting by $y = \overline{T'} e_j$ recovers the displayed tensor sum with coefficient one. The semicharacter of each translation is one by its support on a single isotropic half, and Fourier interchange preserves the integral box with its self-dual measure of volume one.

For this pair $j = 2i$, let $\widetilde{L}_l(\Phi)$ denote the lift with Φ in the standard bases, so

$$L_l(\Phi) = R_{l+1}(\delta_{l+1})\, \widetilde{L}_l(\Phi)\, R_l(\delta_l^{-1}).$$

Write $\chi_{l,h}(a) = \chi_l(\det a)|\det a|_P^{-h/2}$ with the character of arrow l in Section 19.4.1. Its split change of variables gives

$$\widetilde{L}_j(C_y(TA_j)) = \chi_{j,n_{j+1}}(A_j)\, R_{j+1}(\delta_{j+1}^{-1}) L_j(C_y),$$
$$\widetilde{L}_{j+1}(H_y(B_{j+2}R)) = \chi_{j+1,n_{j+1}}(B_{j+2})\, \rho_{j+2}(D_{j+2}) L_{j+1}(H_y) R_{j+1}(\delta_{j+1}).$$

In the first row take $(a, b) = (A_j, 1)$ in $b^{-1}Ta$; in the second take $(1, B_{j+2}^{-1})$. Here $A_{j+2}B_{j+2} = D_{j+2}^{-1}$, whence $\rho_{j+2}(D_{j+2})R_{j+2}(\delta_{j+2}) = R_{j+2}(\widehat{B}_{j+2}^{-1})$. On composition the two middle translations cancel. For the current twist and coefficient embedding the scalar is

$$g_{j,\xi,\jmath} = \chi_j(\det A_j)\chi_{j+1}(\det B_{j+2})|\det A_j \det B_{j+2}|_P^{-n_{j+1}/2}.$$

Each $w \in P$ occurs in exactly one determinant. Through the conjugate embedding a factor N on P reads as N^{-1} in the norm-one determinant.

The algebraic character formula therefore gives an N-adic factor at most N^{2b} there, because the lift weight at that embedding is $\le 2b$ and the modulus power has N-adic norm ≤ 1. Thus $|\jmath(g_{j,\xi,\jmath})|_N \le N^{2fb}$. The tame twist has weight zero and is unramified at N. Its uniformizer values are roots of unity and hence N-adic units. For the embedding $\jmath \circ \mathrm{conj}$, use $\bar{P}$ with its partition by the same S. The argument gives the same bound at that embedding. Apply the ramified summation and commute U through the direct lifts one at a time.

All intermediate vectors remain ramified. For pair i, the commutations contribute

$$\prod_{j=0}^{2i+1} \gamma_j = N^{|P|(n_{2i+2}-n_0)/2}.$$

The product for the fixed twist and embedding is therefore

$$c_{\xi,\jmath} = \prod_{i=0}^{m/2-1} \left(g_{2i,\xi,\jmath}\,(N^{n_{2i+1}} - 1)^{|P|} N^{|P|(n_{2i+2}-n_0)/2} \right).$$

The ranks increase, so $|\jmath(c_{\xi,\jmath})|_N \le (N^{2fb})^{m/2} = N^{mfb}$. The scalar is independent of the input vector and the finite translates. Thus on the ramified source

$$\mathcal{U}_\xi(\Psi)(u) = \mathrm{res}_{Y_m}\left(c_{\xi,\jmath} \sum_x \Psi_\xi(x)\, L_{m-1,x}^{\mathrm{d}} \cdots L_{0,x}^{\mathrm{d}} U_0^{m/2} u \right).$$

Here each $L_{j,x}^{\mathrm{d}}$ uses the same translate of the normalized vacuum as the x-term of the original chain. The permitted translates are vacuum-projected above N before gluing, as given by admissibility of the chain.

To bound a direct lift, use the standard lattices at the identity. Write e_- and e_+ for the last basis vectors in the source and target, respectively. On integral matrices, finite Fourier inversion on the last column gives

$$C_e(T) = N^{-|P|n_{j+1}} \sum_{a \in B^{n_{j+1}}} \psi_N\Big(a^t(\bar{T}e_- - e_+)\Big).$$

Write $v = \bar{T}e_- - e_+$. The sum factors over the places and the row coordinates. Each one-coordinate sum is

$$\sum_{a_w \in \mathbb{F}_N} \psi_{N,w}(a_w v_{w,i}) = N\mathbf{1}_{v_{w,i}=0}.$$

Thus any single nonzero coordinate kills the sum, including when its entry in B is a zero divisor. The inverse character gives the same identity in the opposite polarization. The denominator therefore divides the common clearing power $N^{[E:\mathbf{Q}]n_j n_{j+1}}$.

Multiplication by a character of these residue variables is a Heisenberg translate by a vector with at most simple poles at the conjugate places $\bar{P}$, by the self-dual pairing. For the projected vacuum the original translate is either integral above N, or the projection is zero. All induced places $\bar{P}$ restrict to the same quadratic prime of K. Choose the quadratic integer $\delta = \omega_K - c_0$, where $\omega_K = (1 + \sqrt{-d})/2$ and $c_0 \in \mathbb{Z}$ lifts the residue of ω_K at that quadratic prime. The Fourier translations have at most simple poles above N, all in $\bar{P}$. Thus for every surviving term the total translation z satisfies $\delta z_w \in \mathcal{O}_{E_w}^{n_j n_{j+1}}$ for every $w \mid N$. Write $\mathfrak{n}$ for the chosen prime of K. Then $v_{\mathfrak{n}}(\delta) \geq 1$, whereas $\bar{\delta} \equiv -\sqrt{-d} \pmod{\mathfrak{n}}$ gives $v_{\bar{\mathfrak{n}}}(\delta) = 0$. Thus $\jmath(\bar{\delta} d)$ is a unit. For the conjugate coefficient embedding the pole set is P above $\bar{\mathfrak{n}}$; using $\bar{\delta}$ there clears the poles and gives the same unit bound. Apply Section 20.1 to these translates of the normalized vacuum. At its original scale the required factor in the lattice norm is $(|\jmath(\pi_M)|_N / |\jmath(\bar{\delta} d)|_N)^{\#\mathrm{boxes}} \leq 1$. After dividing by the Fourier clearing factor, the direct lattice norm contribution is at most $N^{2 f n_j n_{j+1}} \leq N^{f b}$.

The estimate in the standard lattice bounds the kernel at every pair of finite points. Indeed choose rational translates of the two points independently so that their finite coordinates above N are integral. This is possible by weak approximation on the two unitary groups, using their rational Cayley charts. At the integral representatives the action at $w \in P$ has the form $T \mapsto b_w^{-1} T a_w$ with a_w, b_w integral and invertible. It preserves the integral matrix lattice. Writing $a_{w,0}, b_{w,0}$ for the two reductions, the transported column condition is

$$\bar{T} a_{w,0} e_- = b_{w,0} e_+.$$

The same character expansion applies to this equation. Its linear forms are integral in the residue coordinates and its constant phases are roots of unity. Thus the Fourier denominator is still $N^{|P| n_{j+1}}$, and the translating vectors have at most simple poles at $\bar{P}$. The unramified character factors above N are units. Let A be the forward action on the displacement space away from N, extended by the identity above N. In the finite Weil expansion of Section 19.3, write $H = L \cap A^{-1} L$ for the intersection with the vacuum lattice L. The index $[L : H]$ is prime to N, and the representatives $y \in L/H$ can be chosen zero above N. The splitting multiplier and the pair characters are N-adic units by the class-number power calculation in Section 22.2 with coefficient prime N. Each shift $A(z + y)$ retains the component z_w at $w \mid N$. Thus $N^{|P| n_{j+1}}$ times the transformed datum is in the coefficient valuation ring span of vacua with $\delta(A(z + y))_w \in L_w$ for every $w \mid N$, with the same δ as above. The ordinary jet bound of Section 20.1 applies at the integral representatives, and rational covariance transports the bound to the Weyl lattice norms at the original points. Translating the datum by (δ_j, δ_{j+1}) changes the two transported lattices to those at $g\delta_j$ and $h\delta_{j+1}$. In particular, writing $\|v\|_a$ for the lattice norm at a, the direct kernel satisfies

$$\|K_{j,x}^{\mathrm{d}}(g, h) v\|_{h\delta_{j+1}} \leq N^{f b} \|v\|_{g\delta_j} \qquad \text{for every } g, h.$$

To sum the products of these kernels, project the glued tensor onto the original integral vacuum above N in each factor. This is the product of the mass-one additive averages defining the original vacuum projections. For a linked nonzero residue choose a place $w \in P$ where $y_w \neq 0$, identifying residues on conjugate halves as in the gluing formula. If $w \in P \cap S$, the first factor is a translate of the integral box by a last column $-\widetilde{y}_w / N$. Its support is disjoint from the box on which the vacuum projection integrates. If $w \in P \setminus S$, the first factor on that box is the additive character paired with the nonzero residue. Write O_w for the integer ring, Q_w for

its residue cardinality and n for the number of column entries. The residue pairing is perfect and the box has additive mass one. Thus

$$\int_{O_w^n} \psi_{N,w}(\langle z \bmod N, y_w\rangle)\, dz = Q_w^{-n} \sum_{z\in (O_w/NO_w)^n} \psi_{N,w}(\langle z, y_w\rangle) = \mathbf{1}_{y_w=0}.$$

For $y_w \neq 0$, translation by a residue vector with nontrivial character value multiplies the sum by that value. This equality in characteristic zero precedes the coefficient valuation at N. Only $y = 0$ survives. Both factors have value one on their original boxes of additive volume one, so the projected tensor is

$$\sum_x \Psi_\xi(x) \bigotimes_{j=0}^{m-1} V_j(x),$$

where $V_j(x)$ denotes the original translated vacuum. The original levels preserve the projection lattices and their probability Haar, so covariance makes the projection commute with their simultaneous action. In particular the displayed tensor is jointly invariant. The direct insertions also commute with this action. In their conjugated bases the original levels preserve the integral box, and the test $\bar{T}e = e$ is equivalent to $b^{-1}\bar{T}ae = e$ for the two residue actions. Hence the tensor with the direct insertions is jointly invariant under the same levels.

Let $u = U_0^{m/2}\varphi$ have lattice norm at most one at $g\delta_0$ for every g, as required by the definition of the preimage lattice. For a fixed terminal point g_m the product integral of the m kernels, summed with $\Psi_\xi(x)$, is

$$\sum_{(g_j)_{j<m}} \sum_x \Psi_\xi(x) K^{\mathrm{d}}_{m-1,x}(g_{m-1}, g_m) \cdots K^{\mathrm{d}}_{0,x}(g_0, g_1) u(g_0).$$

Indeed simultaneous level invariance makes the contracted integrand constant on each product of the original level classes. Their stabilizers are trivial at the old odd principal level and the class masses in the mean integrals are one. Since N is prime to the twist modulus, $|\jmath(\Psi_\xi(x))|_N \leq 1$. The m bounds multiply to N^{mfb}, and the ultrametric inequality gives that same bound for the entire sum in the lattice norm at $g_m\delta_m$. Multiplication by $c_{\xi,\jmath}$ gives the bound N^{2mfb} before restriction.

For $h \in U(Y_m)(\mathbb{A}_{F_0,f})$ put $\widetilde{h} = \mathrm{diag}(h, 1) \in U(X_m)(\mathbb{A}_{F_0,f})$. Let w_τ be the place induced by $\jmath \circ \tau$, and set $\chi_\tau = \mathbf{1}_{\bar{w}_\tau|\pi}$. The weight component is read at $\bar{w}_\tau \in P$, where $(\widetilde{h}\delta_m)^{-1}$ has blocks h^{-1} and $s_\tau = N^{-\chi_\tau}$. The last-coordinate exponent of the branching matrix B_Y is

$$B_\tau = |\mu(X_m)_\tau| - |\mu(Y_m)_\tau| = a_\tau - mh_0$$

by Lemma 7.9. Since $\pi\bar{\pi} = N$, the restriction scalar satisfies

$$\left| \jmath\left(\prod_\tau \tau(\bar{\pi})^{-B_\tau} \right) \right|_N = \left| \prod_\tau s_\tau^{B_\tau} \right|_N .$$

Indeed $v_N(\jmath(\tau(\bar{\pi}))) = v_{\bar{w}_\tau}(\pi) = \chi_\tau$, since N splits completely. Both sides are $N^{\sum_\tau \chi_\tau B_\tau}$. For the conjugate embedding $\jmath \circ \mathrm{conj}$, form its own half P and the corresponding lattice coordinate change $\delta_m = \widehat{A}_m$ as above. Then χ_τ is replaced by $1 - \chi_\tau$ on both sides. For each signed exponent B_τ, the corresponding scalar quotient is a unit, so

$$u_\jmath = \jmath\left(\prod_\tau \tau(\bar{\pi})^{-B_\tau} \right) \prod_\tau s_\tau^{-B_\tau}, \qquad |u_\jmath|_N = 1.$$

Write ρ_X, ρ_Y for the Weyl actions in the coefficient coordinates. The integral branching matrix satisfies

$$B_Y \rho_X((\widetilde{h}\delta_m)^{-1}) = \left(\prod_{\tau} s_{\tau}^{B_\tau}\right) \rho_Y(h^{-1}) B_Y.$$

Thus the restriction in transported coordinates is $u_J B_Y$. Since B_Y has integral entries, the upper lift Φ just bounded satisfies

$$\|(\operatorname{res}\Phi)(h)\|_h \le \|\Phi(\widetilde{h})\|_{\widetilde{h\delta_m}} \le N^{2mfb} \qquad (h \in U(Y_m)(\mathbb{A}_{F_0,f})).$$

Its contribution to the logarithmic denominator bound is

$$e_N \log N = 2mfb \log N = O(fm^2 bB) = O(faB) = o(faB^3)$$

by $\log N = O(mB)$, $m^2 \le 4x$, and $a = xb$, as in Lemma 7.1.

19.4.3 Integral clearing on the original source

In this subsection, we bound the colength required for the direct lift of Section 19.4.2.

Keep the coefficient embedding $\jmath$ above N. Set $\Delta_N = D_0^{-1}$ on the P half and use its inverse conjugate transpose on the other half. Thus the dual point is $\delta_0^\vee = \delta_0 \Delta_N^{-1}$. At a place in $P \cap S$ the original δ_0 has last entry N; on $P \setminus S$ it is the identity. Every embedding in the coefficient lattice norm reads the point through P. Therefore $(\delta_0^\vee)^{-1}$ acts in the Weyl coordinates by integral diagonal matrices. The source representation is polynomial, with determinant exponent zero, so integrality at 1 implies integrality at $\delta_0^\vee$.

For a class representative c of the original source let K_N be the product of the integral groups above N. Choose a rational unitary point γ_c such that

$$\gamma_c c \delta_0^\vee \in K_N \cap \Delta_N K_N \Delta_N^{-1}.$$

Weak approximation here follows from the Cayley transform $X \mapsto (1-X)(1+X)^{-1}$ on skew-Hermitian matrices. The chart is dense in each local unitary group and the coefficients of X can be approximated at the finitely many places.

The required condition is open and nonempty as a condition on γ_c over the local fields. Put $r_c = \gamma_c c$, so $r_c \delta_0^\vee \in K_N$ and $r_c \delta_0 \in \Delta_N K_N$. For k in the original source level, $\delta_0^{-1} k \delta_0$ is integral above N and fixes the last column on reduction in P. At $P \cap S$ this follows by conjugating the original condition on the last row, which divides the lower left block by N and multiplies the upper right by N.

At $P \setminus S$ it follows by applying the condition on the last row to k^{-1} on the other half and taking the inverse conjugate transpose in the unitary coordinates. Both coordinates have unit determinant. Thus the unit lattices at the two points are tested in the same class bases r_c, with the indicated diagonals. Rational equivariance transports these tests to arbitrary representatives.

For a Weyl basis index I let w_I be the total degree in the last variable over all embeddings. Integrality at $\delta_0^\vee$ gives integral coordinates. Integrality at δ_0 requires the I-th coordinate to belong to $N^{w_I}\mathcal{O}$. Here $\mathcal{O}$ is the valuation ring pulled back along $\jmath$ to the coefficient field, with $v_N(N) = 1$. The ratios used in integral column operations therefore lie in that same field. For a finite tuple of vectors impose these conditions one coordinate row at a time. In a nonzero row

choose a coefficient of least valuation, clear the other coefficients by integral column operations, and multiply the remaining column by N^{w_I}. A zero row already satisfies the condition.

The operations preserve the divisibility already imposed on the preceding rows. The determinant contribution is at most

$$h_{\deg} = \sum_{c,I} w_I \le d_X f a / k,$$

where d_X is the dimension of the full original source. Indeed the sum of the degrees in each variable is the same, by permuting the diagonal in the Weyl representation. Summing the degrees over all k variables gives at most fa for each (c, I). The source level is free: a stabilizer at an old odd distinguished prime lies in the torsion-free principal subgroup proved in Section 30.1.

The right translation in U_0 at y is by $u_y = \delta_0^\vee k_y \delta_0^{-1}$. Thus $u_y \delta_0 = \delta_0^\vee k_y$ differs from $\delta_0^\vee$ by an integral invertible right factor above N. For an arbitrary point g the lattice norm of the corresponding term $u(gu_y)$ at $g\delta_0^\vee$ is precisely its lattice norm at $gu_y\delta_0$. The coefficient $\psi_N(y_{\mathrm{last}})$ is a unit. It follows that U_0 takes the submodule of integral vectors at δ_0 into the one at $\delta_0^\vee$. The previous clearing step, applied first to the original tuple and then to the tuple at each U_0-iterate, gives contribution $(m/2+1)h_{\deg}$ to obtain integrality at δ_0 after $m/2$ iterations. Each clearing acts on the original tuple by linearity. Since $a = xb$ and $k \ge x$ for sufficiently large x, the contribution is at most $mfbd_X$. Intersecting this preimage with the original ramified source preserves this clearing bound, since every operation takes integral linear combinations of the original ramified columns. Explicitly, for a finite adelic element d put

$$\mathcal{A}(d) = \{u : \|u(g)\|_{gd} \le 1 \text{ for every finite source point } g\}.$$

Here the argument of the form is g and its Weyl lattice is transported by gd. Put

$$\mathcal{A}'_N = \mathcal{A}(1) \cap S \cap (U_0^{m/2})^{-1} \mathcal{A}(\delta_0).$$

For every tuple $(v_1, \dots, v_s)$ in $\mathcal{A}(1) \cap S$ the operations give a matrix $H \in M_s(\mathcal{O})$ such that

$$(v_1, \dots, v_s)H \in (\mathcal{A}'_N)^s, \qquad \det H \mid N^{(m/2+1)h_{\deg}}, \qquad v_N(\det H) \le mfbd_X.$$

On a finite coefficient DVR this bounds the length of the quotient by the intersection with $\mathcal{A}'_N$, divided by the ramification index, for the lattice spanned by any independent tuple. Indeed the sublattice generated by its transformed tuple has that normalized index $v_N(\det H)$.

The preimage is fixed by the level and $\jmath$ before choosing the twist or translated arrow data. The loss in the lattice norm $2mfb$ has zero saving at N since $N \ge x_1$. The separate branching factor at infinity occurs in the norm calculation, with the same branching exponent $B_\tau = a_\tau - mh_0$.

20 Integral theta sections and local derivatives

In this section, we bound the derivatives of the integral theta sections occurring in Section 19. The exceptional primes require the additional Hecke calculation in Section 21.

20.1 Integral sections and ordinary jets

In this subsection, we construct an integral CM model and control the derivatives of translated theta sections in that model. The derivative estimate applies to the translates in Section 19.3 and retains their actual translation denominators.

Let $K = \mathbb{Q}(\sqrt{-d})$, d a prime congruent to 7 modulo 8, and $\beta = (1+\sqrt{-d})/2$. Write $\Omega_0 = \omega_0$ for the period of Section 7.4.3 and $\Lambda = \Omega_0\mathcal{O}_K$ for its CM lattice, A for its area, and t for its marked half-period. Modulo Λ, the three nonzero points of order two are $t, \beta t, \bar{\beta} t$. The scale is fixed by $\wp''(t) = 2$. Put

$$x(z) = \wp(z+t) - \wp(t), \quad a_1^2 = 12\wp(t), \quad y(z) = \tfrac{1}{2}(\wp'(z+t) - a_1 x(z)).$$

The equation and differential relations are

$$y^2 + a_1 xy = x^3 + x, \qquad x' = 2y + a_1 x, \qquad y' = 1 + 3x^2 - a_1 y.$$

For every ideal class and every normalization at a half-period u_i, the equation in completed-square coordinates is $Y^2 = X(X^2 + 3e_i X + 1)$.

Its invariant is

$$j = 256\,\frac{((3e_i)^2 - 3)^3}{(3e_i)^2 - 4}.$$

Complex multiplication makes j algebraic [51, Theorem 12.8], so each normalized e_i and a_1 is algebraic. The coordinates of the torsion points are algebraic as well, since the multiplication maps are defined over $\overline{\mathbb{Q}}$. Here $(x(z), y(z))$ are the Weierstrass coordinates of $z+t$ on the group curve. This shifted parametrization sends $z = 0$ to $(0,0)$ and $z = t$ to the point at infinity. Thus $y(az)$ below evaluates the Weierstrass coordinate at $az+t$. The Taylor parameter at $(0,0)$ is y, with $y(0) = 0$ and $y'(0) = 1$. At the fixed coefficient place above p, use $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$ of Section 2.7 and set $R = \mathcal{O}_{\mathbb{C}_p}$. Apply this embedding coefficientwise to each Taylor series.

The discriminant at all primes. Let e_i range over the three half-period values on a CM lattice of any ideal class. If u_i is the corresponding half-period, set

$$s(u_i) = 16\frac{(e_j - e_k)^2}{(e_i - e_j)(e_i - e_k)} \qquad (\{i,j,k\} = \{1,2,3\}).$$

This expression is invariant under homothety. The identity $\wp''(u_i) = 2(e_i - e_j)(e_i - e_k)$ shows that $s(u_i)$ is the discriminant after normalizing $\wp''(u_i) = 2$. Taking a common denominator in the sum and multiplying the three expressions gives

$$\sum_i s(u_i) = -48, \qquad \prod_i s(u_i) = -2^{12}.$$

For the quotient by the point u_i, translate e_i to zero. In the equation $Y^2 = X(X^2 + aX + b)$ one has $a = (e_i - e_j) + (e_i - e_k)$, $b = (e_i - e_j)(e_i - e_k)$. The quotient by $(0,0)$ has equation $Y^2 = X(X^2 - 2aX + a^2 - 4b)$, with dual kernel $(0,0)$. Indeed, the isogeny is given off the kernel by $(X,Y) \mapsto (X + a + b/X, Y(1 - b/X^2))$, as substitution in the equations verifies. The other two old half-periods map to $(0,0)$. Hence the two values of s at the original and dual kernels are $16(a^2 - 4b)/b$ and $2^8 b/(a^2 - 4b)$, respectively.

Index a family of CM lattices by the ideal classes c, including the original lattice. On an invertible ideal $\mathfrak{a}$, $(\frac{1}{2}\mathfrak{a})/\mathfrak{a}$ is free of rank one over $\mathcal{O}_K/(2) \simeq \mathbb{F}_2 \times \mathbb{F}_2$. At each class use its unique

generator t, which is nonzero on both summands. The idempotents $\beta, \bar\beta$ give the two other nonzero points. Homotheties preserve these choices. The quotient by βt replaces an ideal $\mathfrak{a}$ by $(1, \beta/2)\mathfrak{a}$ and carries $\bar\beta t$ to the nonzero $\bar\beta$-half-period of the quotient. In fact β is the projection to the line generated by βt on $\frac{1}{2}\mathfrak{a}/\mathfrak{a}$, so this is the indicated quotient lattice.

Use g for the class of $(1, \beta/2)$. With $s_c, s'_c, s_{0,c}$ the values at $\beta t, \bar\beta t, t$, the product calculation gives

$$s_c s'_{gc} = 2^{12}, \qquad s_{gc} s_{0,gc} = -s_c.$$

If $|s_c|_p > 1$, then $|s'_{gc}|_p \le 1$, $|s_{gc} + s_{0,gc}|_p \le 1$, and $|s_{gc} s_{0,gc}|_p = |s_c|_p > 1$. The ultrametric inequality gives $|s_{gc}|_p = |s_{0,gc}|_p = |s_c|_p^{1/2} > 1$. Iterating around the finite cycle contradicts $|s_c|_p > 1$. Interchanging $\beta, \bar\beta$ proves $|s'_c|_p \le 1$. The sum shows $|s_{0,c}|_p \le 1$, and multiplying the second identity through all classes yields $\prod_c s_{0,c} = (-1)^{h_K}$. Each $s_{0,c}$ is a unit. In the chosen Weierstrass model this gives $\Delta = a_1^4 - 64 \in R^\times$ and $a_1 \in R$, including at $p = 2$.

Theta sections and exact multiplication. Set $H(u, z) = \pi \bar u z / A$. Use the odd sigma section σ with $\sigma'(0) = 1$ and period law

$$\sigma(z + l) = \varepsilon(l) e^{H(l,z) + H(l,l)/2} \sigma(z), \qquad \varepsilon(l) = \begin{cases} 1, & l \in 2\Lambda, \\ -1, & l \in \Lambda \setminus 2\Lambda. \end{cases}$$

These Appell–Humbert data specify a degree-one line bundle. Its space of sections has dimension one. With this multiplier, σ is the reduced theta function of the divisor $[0]$; the area parameter in [9, Definition 1.1 and (1.2)–(1.3), p. 190] is A/π in our notation. For a degree-m section set

$$(T_v^{(m)} G)(z) = e^{-m(H(v,z) + H(v,v)/2)} G(z + v), \qquad G[u]_m = (T_u^{(m)} G)(0).$$

Put $\theta = T_t^{(1)} \sigma / (T_t^{(1)} \sigma)(0)$. Oddness of σ and the sign $\varepsilon(2t) = -1$ in the period law give $\theta(-z) = \theta(z)$. Its degree-one semicharacter is

$$\chi(l) = \exp(i \operatorname{Im} H(\bar\beta l, l)) = (-1)^{|l|^2 \sqrt{d}/(2A)} \qquad (l \in \Lambda).$$

For the sign calculation, the sigma sign at a period v is $+1$ if $v/2 \in \Lambda$ and -1 otherwise. Twisting sigma by t gives the sign at al, for $a = \beta, \bar\beta$, as the negative of the sigma sign at $2t + al$. For $z = t + al/2$ one has $(1 - a)z = (1 - a)t + |a|^2 l/2 \notin \Lambda$ since $|a|^2 = (d + 1)/4$ is even. Thus $\chi(al) = 1$. The semicharacter rule at $l = \beta l + \bar\beta l$ gives the displayed formula. Write L_θ for the line bundle of θ. A section G of $L_\theta^{\otimes m}$ satisfies $T_l^{(m)} G = \chi(l)^m G$. Both ε and χ take values in $\{1, -1\}$, so σ^2 is a section of $L_\theta^{\otimes 2}$.

If multiplication by a is an isogeny into another such lattice of area A' and $|a|^2 A = N A'$, then $H'(au, az) = N H(u, z)$ and $\chi'(al) = \chi(l)^N$. Consequently pullback takes degree m sections to degree mN, with

$$(G \circ [a])[u]_{mN} = G[au]_m.$$

Also $(T_v^{(m)} G)[u]_m = e^{\pi i m E(u,v)} G[u + v]_m$ where $E(u, v) = \operatorname{Im} H(u, v)/\pi$. The phase is a root of unity at torsion points since E is integral on Λ. In particular if $\delta u \in \Lambda$, $\delta \in \mathcal{O}_K \setminus \{0\}$, then

$$z \longmapsto (T_u^{(1)} \theta)(\bar\delta z) \quad \text{has degree } N = \delta\bar\delta.$$

Indeed $T_{\bar\delta l}^{(1)}$ commutes with $T_u^{(1)}$ because $E(\bar\delta l, u) = E(l, \delta u) \in \mathbb{Z}$. Its remaining multiplier is $\chi(\bar\delta l) = \chi(l)^N$.

Interpolation on the curve. The classical Weierstrass addition and duplication formulas give

$$\sigma^2 = x\theta^2, \qquad \theta(2z) = \theta(z)^4 - \sigma(z)^4, \qquad \sigma(2z) = x'(z)\theta(z)^4.$$

For the normalization in these formulas, the addition formula at $(z+t, t)$ proves the first identity after comparing the constant at $z = 0$ of $\sigma(z)^2/(x(z)\theta(z)^2)$. Indeed this quotient is elliptic with no zero or pole and tends to one there. Twisting twice by t gives the sigma sign -1 at $2t$. If $c = (T_t^{(1)}\sigma)(0)$, translating $\sigma^2 = x\theta^2$ by t gives $c^4 = x(z)x(z+t) = 1$, the last identity being the addition formula at a half-period with $\wp''(t) = 2$. The sigma duplication formula at $z + t$ now gives the last displayed identity, with the sign at $2t$ cancelling its minus sign. For the middle identity, apply the sigma addition formula first to $(z + t, v)$ and divide out the common exponentials. This gives $\theta(z + v)\theta(z - v) = \theta(z)^2(\theta(v)^2 - x(z)\sigma(v)^2)$. Set $v = z$.

For $w(u) = (\theta[u]_1^2, \sigma[u]_1^2)$, the identities imply $w(2u) = D(w(u))$ with

$$D(U, V) = ((U^2 - V^2)^2, UV(4U^2 + a_1^2UV + 4V^2)).$$

First, the two coordinates of $w(u)$ are algebraic when u is torsion. The identity $\sigma^2 = x\theta^2$ gives its projective coordinate as $[1 : x(u)]$, or $[0 : 1]$ at the unique zero of θ. Write $w(u) = \lambda v$ for a nonzero $v \in \overline{\mathbb{Q}}^2$. Choose $a < b$ with $2^a u \equiv 2^b u \bmod \Lambda$. The section law at torsion gives $D^b(w(u)) = \zeta D^a(w(u))$ for a root of unity ζ. In characteristic zero the two forms of D have no common projective zero, since the curve discriminant is nonzero. For a nonzero coordinate of $D^a(v)$ the corresponding coordinate of $D^b(v)$ is nonzero as well. The homogeneity of D then gives

$$\lambda^{4^b - 4^a} = \zeta \, \frac{D^a(v)_i}{D^b(v)_i} \in \overline{\mathbb{Q}}^\times.$$

This justifies the coefficient-place absolute values on $w(u)$.

The reduction of these forms has no common projective zero. A zero of the first must have $U = \pm V \neq 0$; the second then vanishes only if $a_1^2 = \pm 8$ in the residue field, contrary to $\Delta \in R^\times$. This also applies in characteristic two. Therefore $\|D(w)\|_p = \|w\|_p^4$ in the maximum norm, by reducing after dividing by a coordinate of largest norm. The norm is unchanged modulo Λ because sigma and theta have sign multipliers at periods and the remaining torsion phases have norm one. A torsion orbit under doubling takes only finitely many norms, all positive. The identity forces every one to equal one.

The section $G_3 = y\theta^3$ is entire and its twisted torsion values are in R, by the monic quadratic obtained from the curve equation,

$$G_3[u]_3^2 + a_1\theta[u]_1\sigma[u]_1^2 G_3[u]_3 = \sigma[u]_1^6 + \theta[u]_1^4\sigma[u]_1^2.$$

For $u_i = t/2^i$, $i \geq 1$, the reductions of both coordinates of $w(u_i)$ are nonzero. Indeed both coordinate points map under D to $(1 : 0)$, which is fixed, whereas $D^i(w(u_i))$ reduces to $(0 : 1)$. Thus $x(u_i), y(\pm u_i) \in R$. The differences $x(u_i) - x(u_j)$ are units for $i < j$, by applying D^i to the projective reductions and comparing $(0 : 1)$ with the interior point from u_{j-i}. The normalized duplication identity is

$$\sigma[2u_i]_1 = x'(u_i)\theta[u_i]_1^4.$$

The denominator $\theta[u_i]_1^4$ and the numerator are units, including at $2u_1 = t$. Hence $y(u_i) - y(-u_i) = x'(u_i)$ is a unit also at $p = 2$.

Let G be a section of $L_\theta^{\otimes m}$ whose twisted torsion values are in R. The elliptic function G/θ^m has its only possible pole at t, of order at most m. Since x, y have pole orders two and three there, successive elimination of leading terms gives

$$G/\theta^m = P_0(x) + yP_1(x), \qquad 2\deg P_0 \le m, \qquad 3 + 2\deg P_1 \le m$$

with missing terms omitted. Elimination uses the fact that a function with a sole pole of order one is impossible on an elliptic curve, by the residue theorem.

Evaluating at $u_i, -u_i$ gives integral values because $\theta[\pm u_i]_1$ is a unit. Their difference is $x'(u_i)P_1(x(u_i))$. Unit Vandermonde interpolation implies $P_1 \in R[X]$, then $P_0 \in R[X]$. The curve equation solves for x in $R[[y]]$ at the origin, since its partial derivative in x at $(0,0)$ is a unit. In particular

$$\frac{(T_u^{(1)}\theta)(\bar\delta z)}{\theta(z)^N} \in R[[y(z)]] \qquad (\delta u \in \Lambda).$$

Here the hypothesis on the numerator's torsion values follows from the pullback identity, the torsion translation phase and the norm computation.

For multiplication by any nonzero $a \in \mathcal{O}_K$, apply the same interpolation to the pullbacks of θ and G_3. It gives

$$A_a = \frac{\theta(az)}{\theta(z)^{a\bar a}} \in R[[y(z)]], \qquad B_a = \frac{G_3(az)}{\theta(z)^{3a\bar a}} \in R[[y(z)]].$$

Since $A_a(0) = 1$, division gives $y(az) = B_a/A_a^3 \in R[[y(z)]]$. The finite Vandermonde systems above also show that whenever the twisted torsion values of the numerator are algebraic, the resulting coefficients of P_0, P_1 are algebraic. This applies to these pullbacks and to the translated numerator above. The differential equations for x, y, with algebraic initial values, give algebraic Taylor coefficients by recursion. For $g = -\log\theta$, the quadratic coefficient of A_β is $\beta(\bar\beta - \beta)g''(0)/2$, so $g''(0)$ is algebraic. To compute g'', write the reduced sigma section as $\exp(-e_2^* z^2/2)\sigma_{\mathrm{W}}(z)$, with σ_{W} the Weierstrass sigma function [9, (1.2), p. 190]. Since $-(\log\sigma_{\mathrm{W}})'' = \wp$ and the translation contributes a linear term to the logarithm, $g''(z) = \wp(z+t) + e_2^*$. Hence $g''(z) - g''(0) = x(z)$, and $g(0) = g'(0) = 0$ gives algebraic coefficients for g and θ. The translated series follows from its quotient and rescaling by the nonzero $\bar\delta$.

For the limits, let $Z(Y)$ be the compositional inverse of y. For every j the coefficients of Y^j in

$$y(aZ(Y)), \qquad \theta(aZ(Y))\exp(sg(Z(Y)))$$

are polynomials in a, s over $\mathbb{C}_p$. Indeed $Z(Y) \in Y\mathbb{C}_p[[Y]]$ and $g(Z(Y)) \in Y^2\mathbb{C}_p[[Y]]$. Writing $\theta(z) = \sum_m \theta_m z^m$ gives

$$[Y^j]\theta(aZ(Y))e^{sg(Z(Y))} = \sum_{m+2n\le j} \frac{\theta_m a^m s^n}{n!}[Y^j]Z(Y)^m g(Z(Y))^n.$$

Likewise $[Y^j]y(aZ(Y))$ has degree at most j in a. These polynomials are continuous at every coefficient prime. Since their values lie in the closed valuation ring R at $(a,s) = (a, a\bar a)$ for $0 \ne a \in \mathcal{O}_K$, they lie in R at limits of those pairs. At an inert prime take CM integers tending to $\zeta \in \mu_{p^2-1}$ in $\mathcal{O}_{K,p}$. Conjugation gives the limit pair (ζ, ζ^{p+1}). At a ramified prime take rational integers tending to $\zeta \in \mu_{p-1} \subset \mathbf{Z}_p^\times$. The limit pair is (ζ, ζ^2). Both specializations use

the same normalized even theta section. For every $0 \neq a \in \mathcal{O}_K$ the quotient A_a is even. Since x is even and $y(z) - y(-z) = x'(z) \neq 0$, its interpolating polynomial has $P_1 = 0$. Thus $A_a \in R[x]$ and $2 \deg_x A_a \leq a\bar{a}$. For odd $a\bar{a}$ this gives $2 \deg_x A_a \leq a\bar{a} - 1$, both for $a = p$ at an inert prime and for $a = \sqrt{-d}$ at the ramified prime $p = d$.

In the same way, the even section $\sigma(az)^2$ has integral quotient $S_a(x) = \sigma(az)^2/\theta(z)^{2a\bar{a}}$ with $\deg S_a \leq a\bar{a}$. At least one coefficient of S_a is a unit. At $u = t/a$, its expression before division by $\theta^{2a\bar{a}}$ is a sum of monomial sections $\theta^{2a\bar{a}-2i}\sigma^{2i}$, each with integral torsion evaluation, while the pullback identity gives the unit value $\sigma[t]_1^2$ for their linear combination. The quotient $x(az) = S_a(x)/A_a^2$ is therefore nonzero after reduction and has order at most $2a\bar{a}$ in y. Since $x = y^2 + O(y^3)$ over R, the order of the reduction of $y(az)$ is at most $a\bar{a}$. This proves the needed degree bound on the original formal parameter from the torsion interpolation itself. For the inert multiplication-by-p calculation at the Teichmüller-averaged parameter, the reduced series has zero linear term and every allowed term has degree 1 (mod $p^2 - 1$); thus its first nonzero term has degree exactly p^2.

A basis with denominator prime to the residue characteristic. Use the cyclotomic field $E = K(\zeta)$, with ζ of order $q = q_1$, and its lattice $\mathfrak{a}_0 = E \cap \alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E$. In this model $\mathfrak{a}_0$ is the product of the inverse relative different at the primes over β, and the unit ideal elsewhere. It is stable under $\mathrm{Gal}(E/K)$, it contains $\mathcal{O}_E$, and it is self-dual for $\mathrm{Tr}_{E/K}(x\bar{y})$: conjugation interchanges the two sets of primes of the relative different above the split primes dividing q. Write Φ for the embeddings over the fixed embedding of K. The powers of ζ give Gram sum qI under these embeddings by the finite geometric sum. For $p \nmid q$ this already has suitable length $b = q$.

For $p \mid q$, let $j = \lceil \log_2 q \rceil$ and take $e_i = \bar{\beta}^j d_i$, where d_i is trace-dual to ζ^i, for $0 \leq i < [E : K]$. Then $e_i \in \mathfrak{a}_0$. Indeed d_i is in the relative codifferent. The codifferent denominators are supported over q and are cleared by q, because $\Phi_q'(\zeta)$ divides q by differentiating the factorization of $X^q - 1$. For every rational $\ell \mid q$, $v_\ell(q) \leq j$ and $\beta\bar{\beta}$ is divisible by ℓ.

Since $\beta + \bar{\beta} = 1$, the factor $\bar{\beta}^j$ clears the codifferent at primes over $\bar{\beta}$. Take ζ^{-k} for $0 \leq k < q$ and all the vectors $\zeta^k\sigma(e_i)$ with $0 \leq k < q$, $\sigma \in \mathrm{Gal}(E/K)$. Their Gram sum is $q(1 + \sum_i \nu_i)I$, where $\nu_i = \mathrm{Tr}_{E/K}(e_i\bar{e}_i)$ is a nonnegative integer by self-duality and reality. Replace each pair ζ^{-i}, e_i with $\zeta^{-i} + e_i$. This adds $(\beta^j + \bar{\beta}^j)I$ by trace duality, since the products of the two square embedding matrices in either order are identity.

The new scalar $b = q(1 + \sum_i \nu_i) + \beta^j + \bar{\beta}^j$ is positive and prime to q. At any prime over $\ell \mid q$, one of $\beta, \bar{\beta}$ is zero and the other is one, so the last two terms sum to one modulo ℓ. In both cases repeat each vector b times to obtain $w_i \in \mathfrak{a}_0$ with Gram sum $b^2 I$.

The finite vacuum expansion. Let $L = \mathfrak{a}_0^n$, embedded in the real Fock coordinates by $r(\gamma)_{\tau,A} = a_d\tau(\gamma_A)$, where $a_d = \sqrt{2\pi/\sqrt{d}}$. This is the self-dual lattice for the Riemann form. Put

$$A_{i,A}(z) = \frac{\Omega_0}{b\,a_d} \sum_{\tau \in \Phi} \overline{\tau(w_i)} z_{\tau,A}, \qquad S(z) = \prod_{i,A} \theta(A_{i,A} z).$$

The area $A = |\Omega_0|^2\sqrt{d}/2$ and the basis identity give $\sum A_{i,A}^* H_C = H_{\mathrm{Fock}}$. Also $A_{i,A}(r(bL)) \subset \Lambda$ by trace self-duality. For $l = b\gamma \in bL$ each period is $A_{i,A}(r(l)) = \Omega_0 \,\mathrm{Tr}_{E/K}(\gamma_A \overline{w_i})$. By the frame

identity its product of theta signs is

$$\prod_{i,A} \chi(A_{i,A}(r(l))) = (-1)^{\sum_{i,A} |\operatorname{Tr}_{E/K}(\gamma_A \overline{w_i})|^2}$$
$$= (-1)^{b^2 \sum_A \operatorname{Tr}_{E/K}(\gamma_A \bar{\gamma}_A)} = (-1)^{\sum_A \operatorname{Tr}_{E/K}(l_A \bar{l}_A)}.$$

This is the finite vacuum sign of Section 7.4.3, also when b is odd. We have $S(0) = 1$.

The same finite Fourier calculation constructs the product vector and resolves the vacuum. Write U_l for signed periods and T_t for ordinary twisted real translations. The groups L/bL and $b^{-1}L/L$ are perfectly paired by the translation commutator and both have order $M_L = b^{2|\Phi|n}$. Let $\mathcal{V}_{\mathrm{int}}$ be the entire theta image of $s_{\mathrm{vac}}^n v_0^{\otimes n}$, with v_0 the rank-one unit vacuum of (AV). On the bL-periodic space,

$$P_t S = \sum_{l \in L/bL} U_l T_t S = d_t \mathcal{V}_{\mathrm{int}} \quad (t \in b^{-1}L/L).$$

Indeed the sum is invariant under L, and the line of theta sections on the principally polarized torus is one-dimensional.

The commutator orthogonality gives

$$M_L S = \sum_t T_{-t} P_t S = \sum_t d_t T_{-t} \mathcal{V}_{\mathrm{int}}.$$

This identifies S as the entire theta image of a finite-part vector F. In the finite Weil model a rational real translation by $r(\gamma)$ is induced by the operator at the opposite finite component, by rational invariance of the theta functional. To prove injectivity, let f have zero entire theta image. The Gaussian factor is nonzero, so $\theta(R(z_\infty)(v_\infty \otimes f)) = 0$ for every real displacement z_∞.

For any finite displacement z_f, choose a rational γ with $z_f - \gamma_f$ in an open translation stabilizer of f. Rational Heisenberg invariance identifies the evaluation at $z_\infty + z_f$, up to a unit phase, with that at $z_\infty - \gamma_\infty$. Hence all adelic displacement evaluations vanish. The polarized norm identity of Section 41.5.1 gives $\|v_\infty \otimes f\|^2 = 0$. Since $v_\infty \neq 0$, $f = 0$, and the identities hold in the finite Weil model.

Evaluating at zero, $M_L = \sum_t d_t (T_{-t}\mathcal{V}_{\mathrm{int}})(0)$, and substituting $d_t \mathcal{V}_{\mathrm{int}} = P_t S$ gives

$$\mathcal{V}_{\mathrm{int}} = M_L^{-1} \sum_t (T_{-t}\mathcal{V}_{\mathrm{int}})(0) \sum_{l \in L/bL} U_l T_t S.$$

Each $(T_{-t}\mathcal{V}_{\mathrm{int}})(0)$ is integral. Indeed (AV) expresses the scaled rank-one vacuum as a sum of finite translates of the cyclotomic theta product. Tensoring this identity in the n coordinates and then translating gives a sum of products of the integral twisted torsion values, with roots of unity as coefficients. Since $p \nmid M_L$, the displayed equality expresses the original vacuum as an integral combination of translates of S. Each translating point becomes integral after multiplication by b.

In rank one the integral vacuum is $s_{\mathrm{vac}} v_0$ from (AV); in rank n it is $(s_{\mathrm{vac}} v_0)^{\otimes n}$. The theta evaluations of its signed finite translates are algebraic integers by the torsion calculation and the signed translation formula. The expansion of this fixed vector may use a different b at each p.

Ordinary integral jets. Let f lie in the integral span of the signed finite translates of this integral vacuum by x with δx integral above p, where $\delta \in \mathcal{O}_K \setminus \{0\}$. Using the preceding expansion, it suffices to treat finitely many translates of F with the same condition. In standard finite coordinates a rational vector γ acts by $\alpha_{\mathrm{W}}^{-1}\gamma$. For each fixed x, choose an open neighborhood of zero on which $T(h)F = F$, h is integral above p, and the phases of Lemma 17.1 satisfy $\beta(x,h) = 1$ and $e_0(x+h) = e_0(x)$. The two phase conditions are open because the finite additive character is locally constant. By density choose γ so that $h = \alpha_{\mathrm{W}}^{-1}\gamma - x$ lies in this neighborhood. Then $T(x+h)F = T(x)F$, also with the signed normalization $T_\sharp(x) = e_0(x)T(x)$. Above p we have $\delta\alpha_{\mathrm{W}}^{-1}\gamma$ integral, so $\delta\gamma \in L \otimes \mathbb{Z}_p$. Only finitely many other places contribute denominators, so one integer m prime to p makes $m\delta\gamma \in L$ in every term.

For the translate by γ, the factor (i, A) has shift $t_{i,A} = -(\Omega_0/b)\operatorname{Tr}_{E/K}(\gamma_A\overline{w_i})$. Choose $k = bm$, so $k\delta t_{i,A} \in \Lambda$ by trace self-duality. For each such shift $s = t_{i,A}$, interpolation with $k\delta$ gives

$$(T_s^{(1)}\theta)(k\bar{\delta}Z) = \theta(Z)^{k^2N}\,P_s(y(Z)), \qquad P_s \in R[[Y]], \quad N = \delta\bar{\delta}.$$

In degrees zero and one this reads

$$P_s(0) = (T_s^{(1)}\theta)(0), \qquad P_s'(0) = k\bar{\delta}(T_s^{(1)}\theta)'(0)$$

because $\theta(0) = y'(0) = 1$ and $\theta'(0) = 0$. This includes shifts where the translated section vanishes at the origin. At the argument $\bar{\delta}(a_d/\Omega_0)z$, substitute $Z = Z_{i,A} = (w_i/(bk))^\dagger z_{\cdot,A}$. The product $\prod_{i,A}\theta(Z_{i,A})^{k^2N}$ is independent of the translating vector because k was chosen for the entire finite sum. The remaining factor is an integral sum of the products $\prod_{i,A}P_{t_{i,A}}(y(Z_{i,A}))$.

Write $s_{\mathrm{jet}} = p_F a_d/\omega_0$. In the original normalization the absolute value of p_F at p is that of α_{W}^{-1} at each weight embedding $a \mapsto \iota_p(\overline{\tau(a)})$. Indeed Section 7.4.3 gives $p_F = s_0/u$, with u a unit at every coefficient embedding by the class-cycle argument, and $s_0^f = \lambda_f(\alpha_{\mathrm{W}})^{-1}$ for $\lambda = v|\cdot|_E^{-1/2}$. Let $h > 0$ and $y \in E^\times$ satisfy $(y) = \mathfrak{a}_0^h$, and put $w_\tau^- = w(\iota_p \circ \bar{\tau})$, where $\bar{\tau}(a) = \overline{\tau(a)}$. Normalize the valuations v_w by $v_w(p) = 1$. The identity $s_0^{-fh} = \prod_\tau \overline{\tau(y)}$ in Section 7.4.3 gives

$$v_p(\iota_p(p_F)) = -\frac{1}{f}\sum_\tau v_{w_\tau^-}(\alpha_{\mathrm{W}}) = -v_{w_\tau^-}(\alpha_{\mathrm{W}}) \qquad (\tau \in \Phi).$$

All the w_τ^- restrict to the same place of K, and $\mathfrak{a}_0$ is invariant over K. For $p \mid q_1$ write $q_1 = p^a m'$, $p \nmid m'$. At either place of K above p the completion of K is $\mathbb{Q}_p$; adjoining $\zeta_{m'}$ is unramified. Over this unramified field $\Phi_{p^a}(1+X)$ is Eisenstein, and $\zeta_{p^a} - 1$ generates the ring of integers as an algebra. The different therefore has normalized valuation

$$v_p(\Phi_{p^a}'(\zeta_{p^a})) = v_p\left(\frac{p^a\zeta_{p^a}^{-1}}{\zeta_{p^a}^{p^{a-1}} - 1}\right) = a - \frac{1}{p-1}.$$

Here the last equality follows from $\Phi_p(1) = p$ and the equal valuations of $1 - \zeta_p^j$ for $1 \le j < p$. Thus

$$v_p(\iota_p(p_F)) = \begin{cases} a - \dfrac{1}{p-1}, & w_\tau^- \mid (\beta), \\ 0, & w_\tau^- \mid (\bar{\beta}). \end{cases}$$

Conjugating the coefficient embedding interchanges the cases. Since $w_i \in \mathfrak{a}_0$, in both cases $v_p(\iota_p(p_F\overline{\tau(w_i)})) \ge 0$. This includes $p = 2$. Every choice of the prescribed root differs by a root

of unity. Before applying the coefficient embedding the fixed jet argument is

$$A_{i,A}(s_{\mathrm{jet}}z) = \frac{p_F}{b}\sum_{\tau\in\Phi}\overline{\tau(w_i)}z_{\tau,A}.$$

The factors a_d and $\Omega_0 = \omega_0$ cancel in this identity. Since $w_i/(bk) \in \mathfrak{a}_0 \otimes \mathbb{Z}_p$ and $p \nmid bk$, every coefficient of $\nu = p_F(w_i/(bk))^\dagger$ lies in R.

Let $g = -\log\theta$, so $g'' = g''(0) + x$ and $g'(0) = 0$. For $a = \beta, \bar\beta$, the even interpolation above gives $A_a(z) = \theta(az)/\theta(z)^{a\bar a} = P_a(x(z))$, $P_a \in R[X]$. Since $x(z) = z^2 + O(z^4)$ by $\wp''(t) = 2$, the quadratic coefficient is

$$\frac{a(\bar a - a)}{2}g''(0) = P_a'(0) \in R.$$

Subtract the value at β from that at $\bar\beta$, using $\beta+\bar\beta = 1$, to get $\sqrt{-d}\,g''(0)/2 \in R$. Multiplication by $-2\sqrt{-d}$ gives $dg''(0) \in R$, so $dg'' \in R[[y]]$. For $\mu_i = d\nu_i$, take the ring generated over R by integral series in the $y(\mu_i z)$, together with $\theta(\mu_i z)$ and $g'(\mu_i z)$.

Write $Y_i = y(\mu_i z)$, $X_i = x(\mu_i z) \in R[[Y_i]]$. For $B \in R[[(Y_i)_i]]$,

$$\partial_j B((Y_i)_i) = \sum_i (\partial_i B)((Y_i)_i)\,\mu_{i,j}(1 + 3X_i^2 - a_1 Y_i).$$

The other generators satisfy

$$\partial_j\theta(\mu_i z) = -\mu_{i,j}\theta(\mu_i z)g'(\mu_i z), \qquad \partial_j g'(\mu_i z) = \nu_{i,j}(dg'')(\mu_i z).$$

Thus the ring is preserved by all partial derivatives, and every element has integral constant term. Applying any polynomial $Q(\partial)$ with coefficients in R and evaluating at zero thus gives an integral value for the actual theta vector at scale $\bar\delta d\, s_{\mathrm{jet}}$.

If Q is homogeneous of degree l, rescaling gives the ordinary bound at $c\, s_{\mathrm{jet}}$,

$$\Big|Q(\partial)\Theta_f(c\, s_{\mathrm{jet}}\, z)\big|_{z=0}\Big|_p \le \left|\frac{c}{\bar\delta d}\right|_p^l.$$

For a multi-index $\mathbf{m}$, with $\mathbf{m}! = \prod_i m_i!$, this bounds the ordinary derivative by

$$\mathbf{m}!\,[z^{\mathbf{m}}]\Theta_f(c\, s_{\mathrm{jet}}z) \in (c/(\bar\delta d))^{|\mathbf{m}|}R.$$

For an integral source basis vector v and a coordinate covector A of the integral target basis, put $Q_{A,v}(Z) = A\mathcal{S}_j(Z)v = \sum_{|\alpha|=l} q_\alpha Z^\alpha$. Let F_s, F_t embed the source and target Weyl coordinates in their exterior column tensors. The row of each matrix at a column filling gives the straightening coefficients of its minor product. Reading the rows indexed by the standard tableaux gives a left inverse $R_tF_t = I$. At τ, the two diagrams have the same columns below n_j, and the target has $c_\tau = m_{\tau,n_j}$ additional columns of height n_j. Form $\widehat{F}_s$ from F_s by inserting $e_1 \wedge \cdots \wedge e_{n_j}$ in each of those slots. Let $C(Z)$ be the tensor product of the exterior powers of Z over the target columns. For a source frame X, let X_τ be the square block of its first n_j columns and write v_s, v_t for the standard minor evaluation vectors. Cauchy–Binet and the minor expansion defining $\mathcal{S}_j$ give

$$\begin{aligned} F_tv_t(ZX) &= \Big(\prod_\tau \det(X_\tau)^{c_\tau}\Big)C(Z)\widehat{F}_sv_s(X) \\ &= \Big(\prod_\tau \det(X_\tau)^{c_\tau}\Big)F_t\mathcal{S}_j(Z)v_s(X). \end{aligned}$$

The vectors $v_s(X)$ with every X_τ invertible span the rational source, since a linear combination of the independent standard minor polynomials vanishing on this dense open set is zero. Thus

$$C(Z)\widehat{F}_s = F_t\mathcal{S}_j(Z), \qquad \mathcal{S}_j(Z) = R_tC(Z)\widehat{F}_s.$$

The latter matrix has integral polynomial entries, so $q_\alpha \in \mathbf{Z}$. By the monomial pairing of Section 18.3, the corresponding entry of the jet matrix is

$$j_{Q_{A,v}}(J_c\, s_{\mathrm{jet}} f) = \sum_{|\alpha|=l} q_\alpha\, \alpha!\, [z^\alpha]\Theta_f(c\, s_{\mathrm{jet}} z) \in (c/(\bar{\delta}d))^l R.$$

This coordinate computation also applies at $p = 2$ and at every $p \mid q_1$ on the same trace lattice.

For a fixed finite sum of the signed translates with algebraic coefficients, take a positive integer D_f clearing those coefficients and $0 \neq \delta \in \mathcal{O}_K$ with every δx integral at all finite places. In the finite expansion above, torsion interpolation gives algebraic coefficients for the translated sections. Their linear arguments have coefficients in $\overline{\mathbf{Q}}$ after multiplication by s_{jet}. Thus, for c an algebraic integer, each multi-index $\mathbf{m}$ satisfies

$$D_f(d\,\mathrm{N}_{K/\mathbf{Q}}\delta)^{|\mathbf{m}|}\, \mathbf{m}!\, [z^{\mathbf{m}}]\Theta_f(c\, s_{\mathrm{jet}} z) \in \overline{\mathbf{Z}}.$$

Indeed the local bound for $D_f f$ applies at every coefficient embedding, and $d\,\mathrm{N}_{K/\mathbf{Q}}\delta/(\bar{\delta}d) = \delta$. The preceding local bound retains the sharper denominator at any single prime. In particular, in the fixed integral source basis and target coordinate dual, the polynomial $Q_{A,v}$ has integral coefficients. The same integer $D_f(d\,\mathrm{N}_{K/\mathbb{Q}}\delta)^l$ clears $j_{Q_{A,v}}(J_c\, s_{\mathrm{jet}} f)$ for each entry of degree l, simultaneously at all coefficient embeddings. Applying an embedding to these cleared coordinates preserves integrality.

To pass from these derivatives to the kernel at finite arguments, let a, b be integral invertible above p and let G be the induced isometry $X \mapsto bXa^{-1}$. In normalized finite coordinates write L_f for the standard lattice and $H = L_f \cap G^{-1}L_f$. The lattice vacuum formula of Section 7.4.3 expresses the image of the translate by x as a sum of translates by $G(x + z)$, $z \in L_f/H$. The coefficients are $\Delta_{G^{-1}}^{-1}$ times translation phases and the pair characters. Above p, $H = L_f$ locally and $\delta G(x+z) = G(\delta x+\delta z)$ is integral. The determinant ideal defining $\Delta_{G^{-1}}$ is a unit at every place over p, as are $\det a, \det b$. To evaluate an algebraic finite character on any of these determinants, take a power making the ideal principal. Its generator is a unit at every place over p, so the algebraic weight monomial is a p-unit. The residual finite-order factor and all translation phases are roots of unity. For a norm-one determinant one may take twice the class-group exponent and a norm-one generator, as in Section 7.4.3. Thus every coefficient is a p-unit. The entire span with δx integral above p is preserved even when a, b have denominators away from p. For arbitrary finite arguments g, h, choose rational isometries γ, η with $\gamma g, \eta h$ integral invertible above p. Indeed every prescribed local open set meets the Cayley chart $\det(1+u) \neq 0$. Multiply a point of the set by a norm-one scalar arbitrarily near one, avoiding the finitely many roots of $\det(1 + tu)$ in each field component. Such scalars are given by $(1 + s\zeta)/(1 - s\zeta)$ for a nonzero skew scalar ζ and s in the local fixed field near zero. Weak approximation in the skew-adjoint matrix space and the rational Cayley map give the required isometry, also at dyadic places.

Write T_g for transport of the source coordinates by g^{-1} at the coefficient embedding, and similarly on the target. Rational covariance of the kernel K gives

$$T_{\eta h}K(\gamma g, \eta h)T_{\gamma g}^{-1} = T_hK(g, h)T_g^{-1}.$$

Thus the bound is on the values at the original points. Left rational translation carries the entire right coset gJ to $(\gamma g)J$ at the same source level J, and likewise at the target level.

20.2 Valuations of the derivative coefficients

In this subsection, we use the highest weights to count the columns entering the derivative estimates of Section 20.1. This expresses the local valuation in terms of the prime powers in the arithmetic sum of Section 5.

Fix a coefficient prime p and a twist ξ with $p \nmid q_\xi$. For $p \neq N$, each displacement in an individual arrow of the expanded tensor is made integral above p by $\bar{\pi}_M$. Indeed the two adjoint blocks are returned to their original order in the individual arrows, the additional translations are supported above Mq_ξ, and the linked residues in Section 19.3.1 are supported above N. At $p = N$ these residues can have denominator N. Section 19.4.2 treats the resulting sum.

Use $n_j = k + j(t+1)$, $t = 5r$, $\Delta = 2r^8$, and put $w_j(\tau) = \kappa_\tau + (j + \frac{1}{2})\Delta$. The weight on X_j has determinant exponent zero at $j = 0$ and last entry $\lfloor n_{j-1}/2 \rfloor - w_j$ for $j > 0$. Its columns of height $h < n_j$ have counts

h	number
1	$a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$
k	$w_1(\tau) - \lfloor (k+t+1)/2 \rfloor$
$k + i(t+1),\ i \geq 1$	$\Delta - (t+1)$.

All other counts vanish. The successive blocks of the parameter have size $t+1$, and have centers $-w_i(\tau)$. In each such block the parameter and the highest-weight shift both decrease by one, so the highest weight is constant there. At adjacent block boundaries the parameter centers differ by Δ, while the shifts differ by $t+1$.

For $j \geq 1$, put $l = (t+1)/2$. Subtracting the rank-n_j half-sum of positive roots from the parameter gives the entries on the first two blocks, in zero-based indices,

$$\mu_j(b) = -jl + a_\tau \mathbf{1}_{b=0} \quad (0 \leq b < k), \qquad \mu_j(k+b) = -jl + \lfloor n_1/2 \rfloor - w_1(\tau) \quad (0 \leq b < t+1).$$

The common shift $-jl$ cancels in the difference at height k, giving the table above. It remains in the determinant exponent. For the diagram below n_{j+1}, its first k rows each contain $\kappa_\tau + O(m\Delta)$ boxes from the low columns other than height 1. The remaining rows have $O(m\Delta)$ such boxes. In particular $2(m+1)\Delta < r^{16}$ gives the additional term for these rows used in the pointwise size calculation. For part with index zero its additional term $b + a/x_1$ absorbs $\kappa_\tau \leq b$.

For $j < m$ let c_τ, b_τ count the columns and the boxes at heights $1 < h < n_{j+1}$, respectively. The height-k count is between κ_τ and $\kappa_\tau + 2\Delta$, and each subsequent height has at most Δ columns. Thus

$$k\kappa_\tau \leq b_\tau \leq k\kappa_\tau + x^{60}, \qquad c_\tau \leq \kappa_\tau + x^{60}, \tag{CG1}$$

using $2k\Delta + n_m^2\Delta \leq x^{60}$.

Let $p < x_1$ be nonsplit in K, I_p the positive odd exponents if p is inert and all positive exponents if $p = d$. Put $v = 1$ in the inert case and $v = 1/2$ when $p = d$. Write $\tau_e = \langle -p^e \rangle \tau$ and $\gamma_p = v \sum_{e \in I_p} p^{-e}$. For $\mathcal{E} \subset \Phi$ erase the height-one columns at $\mathcal{E}$, setting $u_\tau = a\mathbf{1}_{\Phi_a \setminus \mathcal{E}}(\tau)$. Define $C_p(\mathcal{E})$ as the sum of a/p^e over $\tau, \tau_e \in \Phi_a \setminus \mathcal{E}$, $e \in I_p$. The positive-exponent contribution after erasing is

$$D_{\tau,e} = (u_\tau/p^e - u_{\tau_e} - c_{\tau_e})_+ + (b_\tau/p^e - b_{\tau_e})_+ - c_\tau/p^e.$$

For the full number B_j of boxes, put

$$v \sum_{\tau,\, e\in I_p} D_{\tau,e} + \gamma_p a|\Phi_a \cap \mathcal{E}| - f n_{j+1}^3(1+\log_p \max(B_j,1)).$$

Denote this value by $d_{pj}(\mathcal{E})$ at inert primes. At $p=d$ subtract from the same expression the exponent-zero term

$$v\sum_{\tau}(\min(u_\tau, u_{\tau_0}+c_{\tau_0}) + c_\tau + \min(b_\tau, b_{\tau_0})), \qquad \tau_0 = \langle -1\rangle\tau.$$

This specifies the uncharged jet exponent in the ramified case. The inert exponent is derived by the null-direction argument in Section 20.3.

Set $U = a/(pm)$. The saving contribution $(a/p^e - \kappa_{\tau_e})_+$ for $\tau \in \Phi_a$ is at most

$$(u_\tau/p^e - u_{\tau_e} - c_{\tau_e})_+ + x^{60} + ap^{-e}\mathbf{1}_{\mathrm{coll}} + ap^{-e}\mathbf{1}_{\tau\in\mathcal{E}},$$

where coll means that both large parts survive. When $\tau \in \mathcal{E}$ and $e=1$, the bound still holds with a/x_1 added to the left, since $(a/p-\kappa_{\tau_e})_+ + a/x_1 \le a/p$. When the large part at τ survives and its partner does not, use (CG1) and the Lipschitz bound for the positive part. When both survive the collision term bounds the left side. At $\tau \notin \Phi_a$ there is no saving from a long row. Also (CG1) yields

$$k(\kappa_\tau/p^e - \kappa_{\tau_e})_+ \le (b_\tau/p^e - b_{\tau_e})_+ + x^{60}.$$

For $p^e \ge x_1$ the saving terms vanish because $\kappa_{\tau_e} \ge a/x_1$, so no error is needed in that range.

Summing contributes at most $2x^{60}(1+\log_2 x_1) \le U$ per place. The negative column term contributes $\gamma_p c_\tau \le 2(b+x^{60})/p \le U$. At the ramified prime τ, τ_0 cannot both be a-places or both be b-places. Thus each minimum in the exponent-zero subtraction is at most $a/(dm)$, using $b_\tau \le ka/x_1 + x^{60} \le a/(dm)$ at a non-b-place. The same bound applies to c_τ, so the subtraction is at most $2U$ per place. Finally $n_{j+1}^3(1+\log_p\max(B_j,1)) \le U$. To see that one cutoff suffices, use $f, q_1 \le x^{12}$, $n_{j+1} \le n_m \le x^6$ and $B_j \le f(a+kb+x^{60})$. Thus $\log\max(B_j,1) \ll B + \log x$, with $B = \log b$. Uniformly in $0 \le j < m$, $p \le x_1$ and the erased set,

$$\frac{x^{60}(1+\log_2 x_1) + n_{j+1}^3(1+\log_p\max(B_j,1))}{U} \ll \frac{mx_1}{xb}\left[x^{60}(1+\log x_1) + x^{18}(1+B+\log x)\right] = o(1).$$

Indeed $U = xb/(pm)$, $x_1 \le b^{1/10^6}$, $m \le 2\sqrt{x}$, and $B = x^\epsilon + O(1)$.

Lemma 20.1 (Local derivative valuation)**.** *Fix $p < x_1$ nonsplit in K, with $p \nmid q_\xi$, and $0 \le j < m$. Let $\mathcal{E} \subseteq \Phi$ index the erased height-one columns of the j-th map. For fixed q_0, d, ϵ and sufficiently large x,*

$$d_{pj}(\mathcal{E}) \ge g_p/m - C_p(\mathcal{E}) + v|\Phi_a \cap \mathcal{E}|a/x_1 - faB^2/(mp). \tag{76}$$

Proof. Multiply the positive-exponent comparisons above by v and sum over τ, e. The erased terms va/p^e sum to $\gamma_p a|\Phi_a\cap\mathcal{E}|$; at $e=1$ the comparison retains $v|\Phi_a\cap\mathcal{E}|a/x_1$. Equation (CG1) and the four error bounds above therefore give

$$\frac{g_p}{m} + v|\Phi_a\cap\mathcal{E}|\frac{a}{x_1} \le d_{pj}(\mathcal{E}) + vC_p(\mathcal{E}) + f(U+U+2U+U).$$

Now $v \le 1$, $U = a/(mp)$, and $B^2 \ge 5$ for sufficiently large x. □

Call an inert prime $p < x_1$ exceptional if $p \not\equiv \pm 1 \bmod m$ and $o_0 = \operatorname{ord}_{q_0}(-p)$ is odd with $o_0 \le r_p = \lceil \log_p m \rceil$. For such a prime and a coefficient embedding ι, put

$$\operatorname{ind}_\iota(\tau) = \{z \in \mathcal{O}_E : v_p((\iota \circ \Phi_\tau)(z)) > 0\}.$$

Choose one member of each conjugate pair of primes induced by Φ_a, and denote this set of representatives by Λ.

At $p = d$ there are no collisions. A partner belongs to the same fibre modulo q_0 only if $d^e \equiv -1 \bmod q_0$, which is excluded by Lemma 4.1. At an inert prime $p \nmid q_0$, so o_0 is defined and a collision requires an odd e divisible by o_0. For $p \not\equiv \pm 1 \bmod m$ and a nonexceptional prime, such an e is at least $r_p + 1$, where $r_p = \lceil \log_p m \rceil$. For $p \equiv \pm 1 \bmod m$ use $e \ge 1$. The series starting at either lower bound e_0 is at most $2fa/p^{e_0}$. Thus every nonexceptional nonsplit $p < x_1$ satisfies $mC_p(\varnothing) \le L_p = faB^2 c_p/p$, where c_p is the multiplier in the local error bound.

Now suppose that p is exceptional. To construct Λ, let o be the order of $-p \bmod q_1$. It is even because $-p \not\equiv 1 \bmod m$ and the reduction modulo $2^s = m^{22}$ has power-of-two order. The exceptional condition $p \not\equiv \pm 1 \bmod m$ gives $v_2(p \pm 1) < \log_2 m$. Since $p^o = (-p)^o \equiv 1 \bmod 2^s$,

$$s \le v_2(p^o - 1) = v_2(p - 1) + v_2(p + 1) + v_2(o) - 1.$$

The equality follows by factoring differences of squares after removing the odd part of o. Exactly one of $v_2(p-1), v_2(p+1)$ is one, so $v_2(o) > \log_2 m$ and $m \mid o$. In $\mathrm{Gal}(E/\mathbb{Q}) = \mathrm{Gal}(K/\mathbb{Q}) \times (\mathbb{Z}/q_1)^\times$, write c_K for conjugation on K. The decomposition group is generated by the arithmetic Frobenius (c_K, p). Complex conjugation on E is $c = (c_K, -1)$. If the inducing prime were fixed by c, then $c = (c_K, p)^j$ with odd j. This would give $(-p)^j = 1 \bmod q_1$, although its order o is even. The two primes are therefore distinct. The identity

$$(1, (-p)^j) = (c_K, p)^j c^j$$

shows that even steps preserve the inducing prime and odd steps exchange the two primes. Conversely, put $w_\tau = \operatorname{ind}_\iota(\tau)$. Then w_σ belongs to $\{w_\tau, \bar{w}_\tau\}$ precisely when $(1, \sigma\tau^{-1})$ lies in $\langle (c_K, p), (c_K, -1) \rangle$. In this subgroup the elements fixing K have exponents of the same parity on its two generators. Their second coordinates are exactly the powers of $-p$. Thus

$$w_\sigma \in \{w_\tau, \bar{w}_\tau\} \quad \Longleftrightarrow \quad \sigma\tau^{-1} \in \langle -p \rangle \subseteq (\mathbb{Z}/q_1)^\times.$$

Multiplication by $(-p)^{o_0}$ preserves Φ_a, so every conjugate pair induced by that set occurs on both sides. Take one member of each pair to get Λ, with $\mathcal{E}$ its inducing set. Then $C_p(\mathcal{E}) = 0$. No b-place induces a prime in either Λ or $\bar{\Lambda}$, because -1 is outside the odd-order subgroup generated by $-p \bmod q_0$. On the full set of embeddings inducing λ or $\bar{\lambda}$, the odd step $\tau \mapsto \tau_1$ exchanges the two halves and all the weights κ_τ are a/x_1. In the column table, only the height-k multiplicity $w_1(\tau) - \lfloor (k + t + 1)/2 \rfloor$ depends on τ among the heights $1 < h < n_{j+1}$. Thus both the column count c_τ and the box count b_τ at these heights agree on the two halves under $\tau \mapsto \tau_1$. On the subsets in Φ_a, the step $\tau \mapsto \tau_{o_0}$ is the bijection exchanging the halves. Thus the sums of the lift weights $w_j(\tau)$, the determinant exponents, and the height-one counts balance separately at each pair.

For $\tau \in \mathcal{E}$ use $c_\tau \le a/x_1 + x^{60}$. If $p \le m$, then $r_p(a/x_1 + x^{60}) \le a/(pm)$ for large x. If $p > m$, then $r_p = 1 = o_0$ and $-p \equiv 1 \bmod q_0$. The orbit identity gives $\mathcal{E} \subseteq \Phi_a$, since Φ_a is the fibre of $1 \bmod q_0$. Hence $c_\tau \le a/x_1 + a/(pm)$ for $\tau \in \mathcal{E}$. Thus $r_p \sum_{\tau \in \mathcal{E}} c_\tau$ is at most

$|\mathcal{E}|a/(pm) + |\Phi_a \cap \mathcal{E}|a/x_1$. Also $|\Lambda|\nu_p \le L_p - fa/(pm)$ for $\nu_p = fr_p(n_m + r_p)$, with the sharper bound $|\Lambda|\nu_p \le L_p/m - fa/(pm)$ since $a/(pm)$ dominates every fixed power of x. This proves both column comparisons required on the chosen set of primes Λ.

At a nonexceptional small nonsplit prime, use the uncharged jet bound $p^{-d_{pj}(\varnothing)}$ from Section 20.3 in the inert case and Section 22.1 in the ramified case. Every translate in the evaluated tensor is standard at that prime, since $p \nmid MNq_\xi$, and the gluing is supported at N. For an inert prime take the erased set empty and all extra exponents zero in (80). Inequality (76) gives the full-chain lattice norm $p^{mC_p(\varnothing)+L_p-g_p} \le p^{2L_p-g_p}$. The terminal branching matrix is integral and its scalar has unit norm at $p \ne N$. At a split prime or $p \ge x_1$, except N, the saving is zero. Apply the ordinary jet estimate with $\delta = \bar{\pi}_M$, $c = \pi_M$ and $|c/(\bar{\delta}d)|_p = 1$, since $p \ne d$. In each case we use the entire integral ramified source, with denominator contribution zero. In particular every $p \mid q_1$ splits in K by Lemma 4.1, including 2 since $-d \equiv 1 \pmod 8$. At every such p, the auxiliary Gram scalar b in Section 20.1 satisfies $b \equiv \beta^j + \bar{\beta}^j \equiv 1 \pmod p$. Thus its finite vacuum average has unit denominator. The two valuations of the period factor computed in Section 20.1 give $p_F\overline{\tau(w_i)}$ integral at every coefficient embedding. At 2, put $n_2 = v_2(q_1)$. Over the unramified part of the completion, the polynomial $(1+X)^{2^{n_2-1}} + 1$ is Eisenstein and its derivative at $\zeta_{2^{n_2}} - 1$ has valuation $n_2 - 1$. Thus the two local Weil ideals have valuations $-(n_2-1)$ and 0. The corresponding period valuations $n_2 - 1$ and 0 cancel these possible denominators. At every such embedding, $c/(\bar{\delta}d) = d^{-1} \in \mathbb{Z}_p^\times$; hence every ordinary derivative $\mathbf{m}![z^{\mathbf{m}}]\Theta_f(\pi_M s_{\mathrm{jet}} z)$ is integral in every degree. The identity matrix therefore clears the entire upper tuple at $p \mid q_1$ for every allowed twist prime to p, with $v_p(\det I) = e_p = g_p = 0$.

20.3 Twisted derivatives at inert primes

In this subsection, we prove the derivative estimate at each small inert prime. At the exceptional primes Section 20.3.3 applies it to Hecke operators.

Fix an inert prime $p < x_1$, a twist with $p \nmid q_\xi$, and the coefficient embedding ι into $\mathbf{C}_p$. The prime divisors of MN are at least x_1, and every prime divisor of q_1 splits in K by Section 4. Thus $p \nmid MNq_1q_\xi$. Normalize $v(p) = 1$ and put $\mathcal{O} = \mathcal{O}_{\mathbf{C}_p}$. For positive odd e write $\tau_e = \langle -p^e \rangle \tau$ in the chosen CM type Φ, and put $n_j = k + j(t+1)$. Take any $\mathcal{E} \subseteq \Phi$ and nonnegative extra exponents satisfying the pairing condition of Section 20.3.2. For $\mathcal{E} = \varnothing$ take every extra exponent to be zero.

The ordinary-coordinate exponential. The one-variable calculation through (77) holds at every inert prime p. Such a prime is odd since 2 splits in K. We use $p < x_1$ and $p \nmid q_\xi$ when applying the calculation to the translated tensor. Use the integral section identities and vacuum resolution of Section 20.1. In their notation the quotient for $\theta(pz)$ is $R(x)$, $R(0) = 1$, with integral coefficients and $2 \deg R < p^2$, and $g = -\log\theta$ satisfies $g'' = g''(0) + x$, $g(0) = g'(0) = 0$. The order bound on the reduced multiplication-by-p series there also applies after the parameter change below.

Set $q = p^2$. Average $y(\zeta z)$ with weight ζ^{-1} and $g(\zeta z)$ with weight ζ^{-p-1} over $\zeta^{q-1} = 1$, obtaining T_0 and $g^\flat$, respectively. The CM pullback formulas extend coefficientwise to the Teichmüller units, for which $\bar{\zeta} = \zeta^p$. Applying the formula at ζ^{-1} and substituting ζz gives

$$\exp(\zeta^{-p-1}g(\zeta z) - g(z)) \in \mathcal{O}[[y]].$$

Multiplying over ζ and taking the unique constant-one $(q-1)$-st root gives $\exp(g^\flat - g) \in \mathcal{O}[[y]]$.

Also $T_0 = y+O(y^2)$, so the two parameters give the same integral series ring. The covariance of T_0 gives

$$T_0(pz) = pT_0(z) + b_1T_0(z)^q + \sum_{j\geq 2} b_jT_0(z)^{1+j(q-1)}, \qquad b_1 \in \mathcal{O}^\times.$$

The order of the reduced parameter under $[p]$ is at most the degree p^2 of the isogeny. Thus b_1 is a unit. The permitted degrees above the vanishing linear term start at $q = p^2$. Choose $\varpi^{q-1} = -p/b_1$ in $\mathbb{C}_p$ and set $Y = T_0(\varpi z)/\varpi$. With $v(p) = 1$ one has $v(\varpi) = 1/(q-1)$. Multiplying ϖ by $\zeta \in \mu_{q-1}$ leaves Y and $g^\flat(\varpi z)/\varpi^{p+1}$ unchanged by their character weights $1, p+1$. The resulting equation is $Y(pz)/p = Y - Y^q + pE(Y)$, with $E \in U^2\mathcal{O}[[U]]$. In each degree $d \geq 2$, division by $1-p^{d-1}$ determines an integral coefficient from earlier coefficients. Modulo p, the equation becomes $Y - Y^q = z$, whence

$$Y \equiv \sum_{u\geq 0} z^{q^u} \pmod{p}.$$

For clarity, the logarithm calculation giving the nonintegral exponential is short. Since $T_0 = z + O(z^q)$, integral series in y have integral coefficients in degrees below q. For $3 \leq n < q = p^2$, the identity $n(n-1)g_n = [z^{n-2}]x$ and $v_p(n(n-1)) \leq 1$ give $pg_n \in \mathcal{O}$. In degree two the CM pullback gives $a(\bar{a}-a)g_2 \in \mathcal{O}$. For $a = \beta = (1+\sqrt{-d})/2$ and $a = \bar{\beta}$, the difference of the two multipliers is $(\beta+\bar{\beta})(\bar{\beta}-\beta) = -\sqrt{-d}$, a unit at the inert prime p. Hence $g_2 \in \mathcal{O}$. Thus $qg-g(pz)$ lies in $p\mathcal{O}[[z]]$ through degree $q-1$. Its exponential does also after subtracting the constant, since $v(n!) \leq (n-1)/(p-1)$ for $n \geq 1$. Comparing coefficients with $R(x(z))$, using $x = z^2 + \cdots$ and $2\deg R < q$, gives $R-1 \in p\mathcal{O}[X]$. Its formal logarithm shows $qg-g(pz) \in p\mathcal{O}[[T_0]]$. Project this equation to degrees congruent to $p+1$ modulo $q-1$. The same covariance in T_0 gives

$$qg^\flat(z) - g^\flat(pz) = p(a_0T_0^{p+1} + T_0^{p+q}H(T_0)), \qquad a_0 \in \mathcal{O}, \quad H \in \mathcal{O}[[U]].$$

20.3.1 Integral rescaling of the local exponential

In this subsection, we normalize the exponential in the local coordinates of Section 20.3.

Set $c_\varpi = \varpi^{p+1}/p$ and $\Xi = -g^\flat(\varpi z)/c_\varpi$. After rescaling, the leading scalar $-p\varpi^{p+1}/(qc_\varpi)$ is a unit, and the higher terms have the additional factor $\varpi^{q-1} \in p\mathcal{O}$. The coefficient of Ξ in degree d is multiplied by $1-p^d/q$, a unit congruent to one for $d \geq p+1$, while all lower degrees are zero. Thus Ξ is integral and

$$\theta(\varpi z) = W(z)\exp(c_\varpi \Xi(z)), \qquad W \in \mathcal{O}[[z]],$$
$$\Xi \equiv \lambda Y^{p+1} \equiv \sum_{e \text{ odd}, u\geq 0} \lambda (z^{p^e} z)^{p^u} \pmod{p}.$$

In fact the rescaled equation has the form

$$(1-p^{n-2})[z^n]\Xi = -a_0[z^n]Y^{p+1} + pR_n, \qquad R_n \in \mathcal{O} \quad (n \geq p+1).$$

The degrees below $p+1$ vanish by the Teichmüller projection. Modulo $p\mathcal{O}$, multiplying Y by Y^p pairs one even Frobenius index with one odd index. Their minimum u and positive odd difference e give degree $p^u(p^e+1)$. Conversely $u = v_p(n)$ and $p^e = n/p^u - 1$ recover the two indices. Thus $[z^n]\Xi \equiv \lambda \bmod p\mathcal{O}$ in exactly these degrees once λ is shown to be a unit below; in every other degree the coefficient belongs to $p\mathcal{O}$.

We determine the leading coefficient for use in Section 37.1.1. Put $h_j = [z^{p-1}]x(z)^j$. Since the projection defining $g^\flat$ retains degree $p+1$ and $g'' = g''(0) + x$,

$$[z^{p+1}]\Xi = -p[z^{p+1}]g = -\frac{h_1}{p+1}. \tag{77}$$

The identities $(x')^2 = 4x^3 + a_1^2x^2 + 4x$ and $x'' = 6x^2 + a_1^2x + 2$ follow from the normalized Weierstrass equation. For $j \geq 0$ they give

$$(4j+6)h_{j+2} + (j+1)a_1^2h_{j+1} + (4j+2)h_j = [z^{p-1}](x^jx')' = p[z^p](x^jx') \in p\mathcal{O}.$$

Indeed x and $x' = 2y + a_1x$ have integral coefficients in degrees below p^2 by the preceding parameter calculation. Put $k_p = (p-1)/2$. Then $h_0 = 0$ and $h_{k_p} = 1$ since $x = z^2 + O(z^3)$. For $0 \leq j \leq k_p - 2$, $4j + 6 = 2(2j+3)$ is a unit. If h_1 vanished in the residue field, the recurrence would force h_{k_p} to vanish. For $p = 3$ we already have $h_1 = h_{k_p} = 1$. Thus $[z^{p+1}]\Xi$ is a unit for every inert p, and comparison with Y^{p+1} shows that λ is a unit.

For the theta series of standard level-one data on the matrix space, the standard-vacuum resolution writes the series at normalized scale a_d/Ω_0 as a product of $\theta(u_i^*z)^{m_i}$ times an integral series in $y(u_i^*z)$, with u_i integral above p and $\sum_i m_iu_iu_i^* = 1$. The resolution uses the powers of ζ_{q_1}, whose Fourier orthogonality gives the scalar basis matrix q_1I. Repeat each vector q_1 times and use the basis scale q_1^{-1}. Since $p \nmid q_1$, the signed finite Fourier inversion from the basis product to the integral vacuum has unit denominator at p. Write v_i for the repeated vectors, so that their Gram sum is q_1^2I. Take the rational representatives of the finite translations as in Section 20.1. They are integral above p, so one may take $\delta = 1$ there and clear their torsion shifts by a common integer k_0 prime to p, divisible by q_1. The integral section formula then gives

$$u_i = v_i/(q_1k_0), \qquad m_i = k_0^2, \qquad \sum_i m_iu_iu_i^* = I$$

in the embedding coordinates, separately for each matrix entry. All terms have this same theta prefactor.

For each odd e, the Frobenius congruence in the field gives $\overline{\tau(u)}^{p^e} \equiv \tau_e(u) \pmod p$ in the coefficient embedding for u integral above p. On K the odd Frobenius cancels conjugation, and on the cyclotomic part their composite acts by $-p^e$. Since $p \nmid dq_1$, each completion of E above p is unramified, with maximal ideal generated by p. Thus the residue-field identity gives the displayed congruence in $\mathcal{O}/p\mathcal{O}$. Reduction of the basis identity gives

$$\sum_i m_i(u_i^*z)^{p^e}(u_i^*z) \equiv \Gamma_e(z) := \sum_{\tau,b,a} z_{\tau,b,a}^{p^e} z_{\tau_e,b,a} \pmod p.$$

The integers m_i satisfy $m_i^{p^u} \equiv m_i \pmod p$. Raising the preceding congruence to p^u therefore gives the coupling $\Gamma_e^{p^u}$ with coefficient λ. The difference of the two integral exponents is $pR(z)$, with $R(0) = 0$. For $a \geq 1$, Legendre's formula gives

$$v((pc_\varpi)^a/a!) = s_p(a)/(p-1),$$

where $s_p(a)$ is the sum of the base-p digits. Thus $\exp(pc_\varpi R)$ is an integral unit as a formal series: only finitely many powers contribute in each degree. In the linear forms of Section 20.1, the factors a_d and Ω_0 cancel at the analytic scale $\pi_M s_{\mathrm{jet}}$. The remaining scalar is $\alpha = \iota(cp_F)$,

with $c = \pi_M$. All primes of q_1 split in K, so here $p \nmid q_1$ and p_F is a unit at ι. Also $p \nmid M$ and $c\bar{c} = M$, so c and α are units. Homogeneity permits the choice $l_{eu} = \lambda\alpha^{(p^e+1)p^u}$ for every e, u. Thus the actual Taylor series G, in the ordinary matrix coordinates of the arrow, satisfies

$$G(\varpi z) = W_0(z) \exp\left(c_\varpi \sum_{e \text{ odd} \geq 1,\, u \geq 0} l_{eu} \Gamma_e(z)^{p^u} \right), \qquad W_0 \in \mathcal{O}[[z]], \quad l_{eu} \in \mathcal{O}. \tag{78}$$

For each tuple x in (73), let $t(y)$ be the tuple of linked shifts indexed by the residues y of Section 19.3.1. Write $V_j(u)$ for the original integral vacuum of the j-th arrow translated by u. That section gives the tensor identity

$$\sum_x \Psi_\xi(x) v_x^{\mathrm{gl}} = \sum_{x,y} \Psi_\xi(x) \bigotimes_{j=0}^{m-1} V_j(x_j + t_j(y)). \tag{79}$$

The supports of x and $t(y)$ are disjoint, so their mutual Heisenberg phase is one. Equal tuples on the right have coefficients equal to sums of the original $\Psi_\xi(x)$, integral above $p \nmid q_\xi$. Since $p \nmid MNq_\xi$, the local factor of every summand above p is the same vacuum. In particular (78) applies to each arrow. Applying the linear chain operator and terminal restriction to this tensor identity gives equality on the full source space. Thus it also applies to the individual summands of $Y_\Lambda^m u + p^{j_p} v$, $u, v \in M_\iota$, in Section 20.3.3. The identification with $L_\xi c_{\mathrm{ss}}^{-1}\Pi$ on the ramified subspace then follows from (73).

20.3.2 Valuations of the jets

In this subsection, we bound the Schur coefficients of the expansion in Section 20.3.1.

Let $n = n_{j+1}$, $\kappa_{\tau,h}$ be the number of columns of height $h < n$, and B_j the total number of their boxes. Erase height-one columns at $\mathcal{E}$, writing B_j' for the remaining number of boxes. Let t_τ, c_τ, b_τ count the remaining height-one columns, the other columns, and the boxes of those other columns, respectively. For each relevant τ, e define

$$\ell_{\tau,e} = \min(t_\tau/p^e, t_{\tau_e} + c_{\tau_e}) + c_\tau/p^e + \min(b_\tau/p^e, b_{\tau_e}).$$

Consider directions $y_{\tau,a,k} \in \mathcal{O}$ and a fixed target basis I'. Give the row index b of a matrix entry in a column of height h at τ an extra exponent $w_{\tau,h,b} \in \mathbf{Z}_{\geq 0}$. Require that one factor has positive extra exponent when the paired entries have the same row index and their columns have heights $(1, h)$ at (τ, τ_e) if $\tau \in \mathcal{E}$ or if $h = 1$ and $\tau_e \in \mathcal{E}$. Let $C(I')$ sum the extra exponents over its boxes.

Use only heights with $\kappa_{\tau,h} > 0$ in the original diagram, including its erased height-one columns. Variables at absent heights have no derivatives in the minor product and may be set to zero. For each occurring (τ, h) change the first h directions by a matrix $U_{\tau,h}$ over $\mathcal{O}$. For the positive odd e with $p^e \leq B_j$, the changes can make all Frobenius pairings $\sum_a \beta_a^{p^e} \beta_a'$ divisible by p except the pairs with a positive extra exponent. The contribution in determinant valuation, with each height weighted by its column count, is at most $\sum_{\tau,e:p^e \leq B_j} \ell_{\tau,e}$. For fixed (τ, e) use the following constraints.

1. For an unerased height one at the source, either multiply that direction by a p^e-th root of p, or impose at every nonexempt height on the partner side one linear congruence against the original source height-one direction. Choose the cheaper option.

2. For source height above one against partner height one, impose the linear congruence $\sum_a \sqrt[p^e]{y_{\tau_e,a,0}}\,\beta_a = 0 \pmod{p^{1/p^e}}$ on the source. Taking p^e-th powers is additive modulo p. Use the original partner direction even if its place is erased, since this pair has source height above one.

3. For heights above one on both sides, make all directions on the cheaper side divisible by p^{1/p^e} on the source or by p on the partner.

Each linear constraint contributes at most the valuation of its modulus. Choose a coefficient of least valuation and clear the other components by integral subtraction before multiplying the pivot direction by the modulus. Collect the forms in the original directions and impose them successively on the changed directions. All earlier constraints persist under integral changes of the directions. A changed height-one direction is a scalar multiple of its original direction, so the constraints give the required pairings after all changes. The three contributions are the three summands of $\ell_{\tau,e}$.

In (78) substitute

$$z_{\tau,b,a} = \sum_{\substack{h<n,\,\kappa_{\tau,h}>0\\ k<h}} p^{w_{\tau,h,b}}\beta_{\tau,h,k,a}s_{\tau,h,k,b}.$$

For fixed row b, heights h,h' and directions k,k', the corresponding pairing modulo p has coefficient

$$p^{p^e w_{\tau,h,b}+w_{\tau_e,h',b}}\sum_a \beta^{p^e}_{\tau,h,k,a}\beta_{\tau_e,h',k',a}.$$

A positive extra exponent makes this zero; the change of directions imposes the congruence for every other pair. Each term is divisible by p even if its two variables coincide. For $p^e > B_j$ the coupling has degree above B_j. In the quotient by monomials of total degree greater than B_j, the substituted exponent before multiplication by c_ϖ therefore belongs to $p\mathcal{O}[[s]]$. The same factorial estimate for $(pc_\varpi)^a/a!$ proves integrality of every coefficient through degree B_j. There are at most fn^3 variables s. For a multi-index α of total degree B_j, the factorial estimate is

$$v(\alpha!) \ge \frac{B_j}{p-1} - fn^3\bigl(1+\log_p \max(B_j,1)\bigr).$$

Apply the product of column determinants in the differential variables ∂_s indexed by I'. On a single determinant the chain rule replaces its derivative matrix by the original matrix times $U^{\mathsf{T}}_{\tau,h}$, with row factors $\varpi p^{w_{\tau,h,b}}$. Thus the derivative is the original minor-product jet J_y times $\varpi^{B_j}p^{C(I')}\prod_{\tau,h}\det(U_{\tau,h})^{\kappa_{\tau,h}}$.

Using the factorial estimate and the determinant contribution gives

$$v(J_y)+C(I') \ge \gamma_p B_j - \sum_{\tau,e:p^e\le B_j}\ell_{\tau,e} - fn^3\bigl(1+\log_p\max(B_j,1)\bigr),$$

$$\gamma_p = \frac{1}{p-1}-\frac{1}{p^2-1}.$$

If $B_j = 0$ use the integral constant term. Otherwise enlarge the nonnegative deduction sum to all positive odd e. Since $\gamma_p = \sum_{e\ \mathrm{odd}\ge 1} p^{-e}$, the resulting bound is

$$v(J_y)+C(I') \ge d_{pj}(\mathcal{E}),$$

Indeed $B'_j = \sum_\tau (t_\tau + b_\tau)$ and $B_j - B'_j = a|\Phi_a \cap \mathcal{E}|$. For the contribution $D_{\tau,e}$ of Section 20.2,

$$\frac{t_\tau + b_\tau}{p^e} - \ell_{\tau,e} = (t_\tau/p^e - t_{\tau_e} - c_{\tau_e})_+ + (b_\tau/p^e - b_{\tau_e})_+ - c_\tau/p^e = D_{\tau,e}.$$

Summation over odd $e \geq 1$ gives $\gamma_p B_j - \sum_{\tau,e} \ell_{\tau,e} = \gamma_p a|\Phi_a \cap \mathcal{E}| + \sum_{\tau,e} D_{\tau,e}$. The series converge absolutely.

The same bound applies to each Schur jet entry $J_{I',I}$. Indeed the product of the flag minors of $z_\tau y_\tau$ expands as $\sum_I A_I(y) S_{I',I}(z)$, where S is the matrix of the Weyl map and the A_I are the source flag-minor basis multiplied by $\prod_\tau \det(Y_\tau)^{c_{\tau,j}}$. Here Y_τ is the square block of the first n_j direction columns of the n_j-row matrix y_τ, and $c_{\tau,j}$ is the number of target columns of height n_j. Each such target column contributes one source top exterior power by the minor identity of Section 20.1. For the diagram the only possible heights other than one are $k + h(t+1)$, so there are no heights strictly between n_j and n_{j+1}. The A_I are independent after reduction.

Order the variables by the direction column and then by increasing basis index, putting the earlier variable first in lexicographic order. Every flag minor has leading coefficient one on its diagonal monomial. In their product the variables of direction d give the multiset of entries in tableau row d; its weak order recovers that row. Distinct tableaux therefore give distinct leading monomials in every residue characteristic. The common determinant factor is nonzero after reduction, so multiplication by it preserves independence. An integral change of source basis preserves this independence. The target index I' is kept in the standard tableau basis.

If all $J_{I',I}$ vanish there is nothing to bound. Otherwise choose I_0 maximizing $|J_{I',I_0}|_p$. The polynomial

$$J_{I',I_0}^{-1} \sum_I A_I(y) J_{I',I}$$

has integral coefficients and nonzero reduction modulo the maximal ideal by this independence. The residue field of $\mathbf{C}_p$ is infinite, so some integral y gives it a unit value. Then $|J_y|_p = \max_I |J_{I',I}|_p$. The bound for every J_y gives

$$v(J_{I',I}) + C(I') \geq d_{pj}(\mathcal{E}). \tag{80}$$

20.3.3 The resulting Hecke operators

In this subsection, we insert the valuations of Section 20.3.2 into the integral Hecke action.

Fix an exceptional p as in Section 20.2 and its coefficient embedding ι. The conjugate pairs above p are split over F_0, and p is a uniformizer on both sides. For the construction below fix any allowed twist ξ with $p \nmid q_\xi$. Among the primes induced by Φ_a, choose one member of each conjugate pair that occurs. The choice depends on (p, ι) and the original datum. Write Λ for this set and put $\mathcal{E} = \{\tau : \mathrm{ind}_\iota(\tau) \in \Lambda\}$. Odd partners induce conjugate primes, so $C_p(\mathcal{E}) = 0$.

At $\lambda \in \Lambda$ each allowed target neighbor has the form $k_1 \,\mathrm{diag}(pI_i, I_{n_{j+1}-i}) k_2$, with $k_1, k_2 \in \mathrm{GL}_{n_{j+1}}(\mathcal{O}_\lambda)$ and $0 \leq i \leq r_p$. Its lattice index is q_λ^i. Write J_λ for the set of i positions with elementary divisor p. Form the product of the chosen neighbors, written $k_1 D k_2$ with D diagonal at all places above p, using the inverse conjugate transpose on the opposite halves. Let K_- and K_+ be the products of the integral source and target groups above p.

For arbitrary finite arguments g, h, weak approximation on the rational Cayley charts gives rational source and target isometries γ, η with

$$(\gamma g)_p \in K_-, \qquad k' = (\eta h k_1 D)_p \in K_+ \cap D_p^{-1} K_+ D_p.$$

Both conditions specify nonempty open sets. At the transformed arguments $(\gamma g, \eta h k_1 D k_2)$ both acting points are integral above p. The translated vacuum then has integral displacements there. Away from p the vacuum expansion of Section 19.3 has p-unit indices and character values. Thus the datum is still an integral combination of the allowed vacuum translates, and the source lattice is standard. For the target,

$$(\eta h)_p = D_p^{-1}(D_p k' D_p^{-1})k_{1,p}^{-1} \in D_p^{-1} K_+.$$

Thus its inverse acts on the transported value by D_p followed by an integral invertible matrix. By rational covariance, the charged row estimate for the transformed jet applies at the original arguments g, h. The weight embedding $\iota \circ \bar{\Phi}_\tau$ induces the place conjugate to $\mathrm{ind}_\iota(\tau)$. The diagonal entries above $\lambda, \bar{\lambda}$ at a position in J_λ are p, p^{-1}, respectively. A box read at the conjugate place therefore contributes $|p^{-1}|_p = p$ when $\mathrm{ind}_\iota(\tau) = \lambda$, and $|p|_p = p^{-1}$ when $\mathrm{ind}_\iota(\tau) = \bar{\lambda}$. Thus its contribution to the base-p logarithm of the lattice norm multiplier is

$$\mathbf{1}_{b \in J_\lambda} \left(\mathbf{1}_{\mathrm{ind}_\iota(\tau) = \lambda} - \mathbf{1}_{\mathrm{ind}_\iota(\tau) = \bar{\lambda}} \right).$$

At each pair take

$$w_{\tau,h,b} = \begin{cases} \mathbf{1}_{h=1} \mathbf{1}_{b \notin J_\lambda}, & \mathrm{ind}_\iota(\tau) = \lambda, \\ \mathbf{1}_{b \in J_\lambda}, & \mathrm{ind}_\iota(\tau) = \bar{\lambda}. \end{cases}$$

Set $w_{\tau,h,b} = 0$ at every other embedding. An odd partner has the same J_λ on the other side. For an exempt pair the assigned exponent is one on the inducing side if the entry is outside J_λ, and one on the conjugate side if it is in J_λ. This verifies the condition on the extra exponents for every neighbor. The determinant weights cancel, since their sums on the two sets of embeddings inducing λ and $\bar{\lambda}$ agree. The sum of the box exponent and the extra exponent is zero on the conjugate side, is one in each height-one box on the inducing side, and contributes at most r_p per other column there by strict increase down each column. If

$$A_\lambda = a \, |\{\tau \in \Phi_a : \mathrm{ind}_\iota \, \tau = \lambda\}|, \qquad L_j(\mathcal{E}) = \sum_{\tau \in \mathcal{E}} \sum_{h \neq 1,\, h < n_{j+1}} \kappa_{\tau,h},$$

then (80) bounds the neighbor arrow by the factor in the lattice norm

$$p^{\sum_{\lambda \in \Lambda} A_\lambda + r_p L_j(\mathcal{E}) - d_{pj}(\mathcal{E})}.$$

Fix a uniformizer ϖ_λ. Normalize Haar measure on $\mathrm{GL}_d(E_\lambda)$ by $\mathrm{vol}(\mathrm{GL}_d(\mathcal{O}_\lambda)) = 1$. For $d = n_j$, let $T_{j,\lambda,i}$ denote the double-coset operator of $\mathrm{diag}(\varpi_\lambda I_i, I_{d-i})$, $0 \le i \le d$. Write $\chi_{j,\lambda}$ for the local base change of the norm-one character of the lift $X_j \to X_{j+1}$, obtained by composition with $u \mapsto u/\bar{u}$. With $q_\lambda = z_\lambda^2 = p^{2h_\lambda}$ and $2h_\lambda \le f$, put

$$T_{j,\lambda}(U) = \sum_i (-U)^i q_\lambda^{\binom{i}{2}} T_{j,\lambda,i}.$$

On source forms spherical at λ, the identity is

$$T_{j+1,\lambda}(U) \Theta_{x,j} = R_{j,\lambda}(U) \Theta_{x,j} T_{j,\lambda}(z_\lambda^{t+1} U),$$
$$R_{j,\lambda}(U) = \prod_{k=0}^{t} (1 - q_\lambda^k z_\lambda^{n_j} \chi_{j,\lambda}(\varpi_\lambda) U).$$

In the split type-II correspondence for irreducible representations, the retained Langlands factors are the contragredients of the smaller factors [52, Corollary 6.3(2)]. We verify the operator identity on the local Schwartz function, using the matrix calculation of Section 28.1. The pair $\lambda, \bar{\lambda}$ avoids the original level and the twist. The translating coordinates of the weak-chain tensor are integral there by Section 19.3, and the rank twists are unramified by Section 4.3. Thus the local factor in self-dual coordinates is the indicator of $M_{n',n}(\mathcal{O}_\lambda)$, where $n = n_j$ and $n' = n + t + 1$. Put $q = q_\lambda$, $z = z_\lambda$, and $c = \chi_{j,\lambda}(\varpi_\lambda)$. For the additive matrix measure the map $Z \mapsto h^{-1} Z g$ has modulus $J = |\det h|_\lambda^{-n} |\det g|_\lambda^{n'}$, by its action on the n columns and n' rows. The unitary Schrödinger action on functions uses $J^{1/2}$ because the substitution in the integral of the squared norm contributes J^{-1}. Thus the target coset contributes $q^{ni/2}$, and the inverse source coset contributes $q^{n'i/2}$. Both ranks are even, since $k = r + 1 - m$ and $t + 1 = 5r + 1$ are even. Write ν for the character of this arrow, so $\nu^{\mathrm{bc}}(a) = \nu(a/\bar{a})$. In either rank $d = n, n'$,

$$(\nu \upsilon_1^{d/2})^{\mathrm{bc}} \upsilon^{-d} = \nu^{\mathrm{bc}}, \qquad \upsilon_1^{\mathrm{bc}}(a) = \upsilon(a/\bar{a}) = \upsilon(a)^2$$

by Section 4.3. The two character factors are consequently both c^i. Indeed the local action, with source matrix g and target matrix h, is

$$(\omega(g,h)\phi)(Z) = \chi_{j,\lambda}(\det h / \det g) |\det h|_\lambda^{-n/2} |\det g|_\lambda^{n'/2} \, \phi(h^{-1} Z g).$$

For the positive target coset and the inverse source coset this gives $\alpha^i = (z^n c)^i$ and $\beta^i = (z^{n'} c)^i$, respectively. In λ-coordinates the positive source coset at $\bar{\lambda}$ is the inverse coset at λ, by inverse transpose.

At an integral matrix Z, let s be its rank modulo ϖ_λ. The target cosets contributing to the indicator are indexed by the i-planes in $\ker \bar{Z}^{\mathsf{t}}$, and the inverse source cosets by the i-planes in $\ker \bar{Z}$. Put $K_d = \mathrm{GL}_d(\mathcal{O}_\lambda)$, $a_i = \mathrm{diag}(\varpi_\lambda I_i, I_{d-i})$ and $D_i^{(d)} = K_d a_i K_d$. Choose $\gamma_V \in D_i^{(d)}$ with $\ker \bar{\gamma}_V = V$ for each i-plane V of the residue space. The coset $K_d \gamma_V$ is determined by $\gamma_V^{-1} \mathcal{O}_\lambda^d$, whose image under multiplication by ϖ_λ and reduction is V. Thus $D_i^{(d)} = \bigsqcup_V \gamma_V^{\mathsf{t}} K_d$ and $(D_i^{(d)})^{-1} = \bigsqcup_V \gamma_V^{-1} K_d$, with each coset of measure one. If $Z \gamma_V^{-1}$ is integral, so is Z, since γ_V is integral. At integral Z, the further condition is $V \subseteq \ker \bar{Z}$. Transposition gives the target criterion for $\gamma_V^{-\mathsf{t}} Z$, including vanishing outside the integral matrix box. The finite q-binomial theorem thus gives the polynomials $P_{n'-s}(\alpha U)$ and $P_{n-s}(\beta U)$, where $P_b(V) = \prod_{k=0}^{b-1} (1 - q^k V)$. Since $q^{t+1} \alpha / \beta = z^{t+1}$,

$$P_{n'-s}(\alpha U) = P_{t+1}(\alpha U) \, P_{n-s}(\beta z^{t+1} U).$$

For the theta integral write $B(\phi, v)$ for the lift of a source form v with datum ϕ, and ρ_d for right translation. Translation in the two variables gives

$$B(\omega(g,1)\phi, v) = B(\phi, \rho_n(g^{-1}) v), \qquad B(\omega(1,h)\phi, v) = \rho_{n'}(h) B(\phi, v).$$

The target is spherical since $K_{n'}$ fixes ϕ. Inversion carries Haar measure on $(D_i^{(n)})^{-1}$ to Haar measure on $D_i^{(n)}$, since $\mathrm{GL}_n(E_\lambda)$ is unimodular. Integrating the first identity therefore gives the positive source operator at λ on v. Bilinearity preserves the scalar coefficients in the kernel identity and gives the asserted formula for $\Theta_{x,j}$. The unramified character value c is a unit at ι. In fact, let a_λ be the idele supported at λ with component ϖ_λ. Evaluation of the weight character on $a_\lambda / \bar{a}_\lambda$ gives

$$|\iota(c)|_p = p^{\sum_{\mathrm{ind}_\iota(\tau) = \bar{\lambda}} w(\tau) - \sum_{\mathrm{ind}_\iota(\tau) = \lambda} w(\tau)} = 1,$$

where w is the weight of this arrow character. To obtain the first equality, raise the idele ideal to a principal power and use the infinity character on a generator as in Section 4.3. Finite unit characters have absolute value one. The local uniformizer has $v_p = 1$ since $E_\lambda/\mathbf{Q}_p$ is unramified. The twist has weight zero, and the two weight sums agree by Section 20.2. For the fixed twist, $R_{j,\lambda}$ is the same for every translate in the support of the chain datum. Its coefficients are ι-integral and its constant term is one. For the original initial segment $C = C_{x,j+1} = \Theta_{x,j} \cdots \Theta_{x,0}$, iteration gives

$$\left(\prod_{a=0}^{j} R_{a,\lambda}(z_\lambda^{(j-a)(t+1)} U) \right)^{-1} T_{j+1,\lambda}(U) C = C T_{0,\lambda}(z_\lambda^{(j+1)(t+1)} U).$$

The substitution in each factor is by an integer multiple of U, so the product still has integral coefficients and constant term one. Its formal reciprocal is integral by the coefficient recurrence. Since $r_p \le m \le k = n_0$, the coefficient of U^{r_p} on the right is $(-1)^{r_p} p^{u_j} C T_{0,\lambda,r_p}$, with

$$u_j = h_\lambda r_p (r_p - 1 + (j+1)(t+1)) \le f r_p (n_m + r_p) = \nu_p.$$

Here $2h_\lambda \le f$ and $j < m$. Multiplying the coefficient identity by $(-1)^{r_p} p^{\nu_p - u_j}$ gives integral coefficients $c_{j,\lambda,i}$ satisfying

$$\sum_{i \le r_p} c_{j,\lambda,i} T_{j+1,\lambda,i} C_{x,j+1} = p^{\nu_p} C_{x,j+1} T_{0,\lambda,r_p}, \qquad \nu_p = f r_p (n_m + r_p).$$

Put $H_\lambda = p^{A_\lambda} T_{0,\lambda,r_p}$ and $Y_\Lambda = \prod_{\lambda \in \Lambda} H_\lambda$. Move the outgoing Hecke sums into the data by Weil equivariance, using the operators $p^{A_\lambda - \nu_p} \sum_i c_{j,\lambda,i} T_{j+1,\lambda,i}$ on the target side of each datum. At each distinguished pair every supported datum has the local vacuum factor. A double-coset operator preserves its $K_{n_{j+1}}$-invariance and commutes with the source action and with the level actions at the other places. The resulting sums preserve the original level equivariance. The operations at different pairs commute. The coefficients $c_{j,\lambda,i}$ may depend on ξ but are common to all the supported translates for that twist. The source operator Y_Λ uses only the fixed arithmetic datum and (p, ι, Λ). The resulting arrow applied to $C_{x,j}$ is $C_{x,j+1} Y_\Lambda$, so the modified full chain gives exactly the original chain on Y_Λ^m. The factors $p^{A_\lambda - \nu_p}$ cancel the preceding $\sum A_\lambda$ in the exponent in the lattice norm. Taking the maximum over the integral Hecke coefficients gives the per-arrow exponent

$$|\Lambda| \nu_p + r_p L_j(\mathcal{E}) - d_{pj}(\mathcal{E}).$$

The column comparison in Section 20.2 gives the following two bounds with $L_p = faB^2/p$ and $e_p = 2L_p$. At these primes $p \not\equiv \pm 1 \bmod m$, so $c_p = 1$ in the definition of L_p.

$$d_{pj}(\mathcal{E}) \ge g_p/m + |\Phi_a \cap \mathcal{E}| a/x_1 - L_p/m,$$
$$|\Lambda| \nu_p + r_p L_j(\mathcal{E}) \le L_p/m + |\Phi_a \cap \mathcal{E}| a/x_1.$$

The first uses zero collisions on the unerased large parts. Together they give an exponent at most $(2L_p - g_p)/m$. The composite therefore has lattice norm at most $p^{2L_p - g_p}$ on $Y_\Lambda^m M_\iota$, where M_ι is the original integral source. With no erased columns, the same comparison gives $p^{mC_p + L_p - g_p}$ on M_ι, where $C_p \le 2fa/p$ by the sum of a/p^e over the colliding large parts. Thus on $p^{j_p} M_\iota$, with $j_p = \lceil 2mfa/p \rceil$, the exponent is at most $L_p - g_p$. Ultrametricity gives $p^{2L_p - g_p}$ on the sum. The terminal restriction uses the original integral Weyl matrix and the original scalar, which is a unit here because $p \ne N$.

Proposition 21.1 gives $H_\lambda M_\iota \subseteq M_\iota$ for the scaled operator $H_\lambda = p^{A_\lambda} T_{0,\lambda,r_p}$ and

$$\operatorname{length}_p\Big(M_\iota/(H_\lambda M_\iota + p^{j_p} M_\iota)\Big) \le d_X \left(\frac{12A_\lambda}{mp} + \frac{j_p}{m^2 f}\right).$$

Compute normalized lengths over the integers $\mathcal{O}_F$ of a finite extension $F/\mathbb{Q}_p$ containing the entries of all H_λ and of the operators $R_H(z_w)$, $w \mid \pi_N$, defining the ramified source, in the original class and Weyl bases. These entries use finite sums of algebraic characters and matrix entries in the local coefficient fields. Divide each length by $e(F/\mathbb{Q}_p)$. Smith normal form identifies it with the sum of the invariant-factor valuations for $v(p) = 1$, which is unchanged after scalar extension. On the resulting finite quotient $R = M_{\iota,F}/p^{j_p} M_{\iota,F}$ the cokernel lengths of two endomorphisms bound the length for their composition. For endomorphisms A, B, the exact sequence

$$0 \longrightarrow AR/ABR \longrightarrow R/ABR \longrightarrow R/AR \longrightarrow 0$$

has first term a quotient of R/BR, via A. After normalization the length inequality is unchanged. Apply it successively to all factors in the m copies of the product over Λ. The normalized length of $M_\iota/(Y_\Lambda^m M_\iota + p^{j_p} M_\iota)$ is at most

$$d_X m \sum_{\lambda\in\Lambda} \left(\frac{12A_\lambda}{mp} + \frac{j_p}{m^2 f}\right) \le d_X(14fa/p + 1/m) \le d_X L_p.$$

Here $\sum A_\lambda \le af$, $|\Lambda| \le f$, and the last inequality is the parameter inequality $14fa + p/m \le faB^2$.

Put $J = Y_\Lambda^m M_\iota + p^{j_p} M_\iota$. Over this same field F, by Section 6, the characteristic-zero source S_F is the common kernel of

$$\sum_{j=0}^{N-2} R_H(z_w)^j = (N-1)A_w \qquad (w \mid \pi_N).$$

Each z_w has identity components above p, since $p \neq N$. Thus $R_H(z_w)$ and its inverse preserve the transported integral source lattice, and the displayed sums have integral matrices. Their common kernel on $M_{\iota,F}$ is $M_{\iota,F} \cap S_F$, even when $p \mid N-1$, since the kernel is taken before reduction. Put $J_F = Y_\Lambda^m M_{\iota,F} + p^{j_p} M_{\iota,F}$. The inclusion

$$(M_{\iota,F} \cap S_F)/(J_F \cap S_F) \hookrightarrow M_{\iota,F}/J_F$$

bounds the length of its source. The valuation ring R_ι is flat over $\mathcal{O}_F$, since it is torsion-free over a DVR. Scalar extension therefore preserves these kernels and intersections. The normalized colength on the original integral ramified source is bounded by the same $d_X L_p$, also when $p \mid N-1$. The tuple clearing is given in Section 22.3. The resulting module is fixed after the chosen arithmetic datum and (p, ι) and before any twist or translate. For every allowed ξ with $p \nmid q_\xi$, the local datum at the pair is spherical and the preceding estimates apply on this same module to all evaluated translates. The twist can change the unramified character values in the transfer identity; their weight-zero factors are roots of unity, hence p-adic units even if their orders are divisible by p. The operators H_λ defining the module act on the original source independently of these values. The set of representatives and the module may depend on the coefficient embedding. The loss and denominator bounds depend on p alone.

21 Hecke operators and colength at exceptional primes

In this section, we construct the source Hecke operators that supply the divisibilities in Section 20.3. The local choices of Section 20.2 ensure that the colength bounds hold on the same source lattice for all permitted twists.

Keep the exceptional prime p, its coefficient embedding and the set of representatives Λ of Section 20.3. The two members above each chosen place of the real subfield are distinct by the decomposition-group calculation in Section 20.2. The exceptional prime p is inert in K, whereas d ramifies and every divisor of q_1 splits in K. Thus $p \nmid dq_1$, and $E = K(\zeta_{q_1})$ is unramified above p. Since $p < x_1$ and every prime divisor of MN is at least x_1, the local factor of J_H at the pair of each $\lambda \in \Lambda$ is the hyperspecial group $\mathrm{GL}_k(\mathcal{O}_{E_\lambda})$. Let X_0 be the rank-k Hermitian space and X the full complex space of algebraic automorphic forms at its original level J_H of Section 6. Set $R_\iota = \{z \in \overline{\mathbf{Q}}_p : v_p(z) \geq 0\}$ with $v_p(p) = 1$. In the given coefficient embedding let $M = M_\iota$ be the R_ι-lattice of integral forms. We may calculate the matrices over $\mathcal{O} = \mathcal{O}_{\mathbf{C}_p}$. Their maximal minors below determine the colength over R_ι. Equivalently, fix a finite extension $F/\mathbf{Q}_p$ containing the images of E at the coefficient embeddings in the source, the entries of the finitely many original class transitions, and $w = \sqrt{p}$. The matrices of all Levi elements on the active factors have entries in $\mathcal{O}_F$, since the active embeddings read the same unramified completion. For every integer $l \geq 1$, the tests by these matrices modulo w^l depend only on the Levi elements modulo $p^{\lceil l/2 \rceil}$. Hence each intersected kernel at a fixed depth is an intersection of finitely many kernels. The occurrence-matrix operators in Section 21.5 and their images of the specified tensor basis vectors define the trace functional over this same DVR. Flat scalar extension preserves the kernels. Smith form with $v(p) = 1$ preserves the valuations of the elementary divisors. Put $d_X = \dim_{\mathbf{C}} X$, $r = r_p = \lceil \log_p m \rceil$, and $A_\lambda = a\,\#\{\tau \in \Phi_a : \mathrm{ind}_\iota(\tau) = \lambda\}$. In the unnormalized double-coset convention the operator of Section 20.3.3 is $Y_\lambda = H_\lambda = p^{A_\lambda} T_{0,\lambda,r}$.

The Frobenius orbit of embeddings. Fix $\lambda \in \Lambda$ and choose $\tau \in \Phi_a$ inducing it. Put $u = \langle -p \rangle$, $e = \mathrm{ord}_{q_0}(-p)$, and write $N_{\mathrm{orb}} = \mathrm{ord}(u^e)$ for the orbit length in this calculation. The infinite places inducing λ or $\bar{\lambda}$ are the u-orbit of τ, by decomposition at the inert prime of the imaginary quadratic field. The defining q_0-condition of Φ_a is preserved by u^j exactly when $e \mid j$. Thus

$$t(s) = u^{es}\tau, \qquad s \in \mathbb{Z}/N_{\mathrm{orb}}\mathbb{Z}$$

lists precisely the original Φ_a-places above $\lambda, \bar{\lambda}$, each once. The integer e is odd, and the step $u^e = \langle -p^e \rangle$ exchanges the two primes. With $t(0) = \tau$, the even places induce λ. Hence N_{orb} is even and $A_\lambda = aN_{\mathrm{orb}}/2$.

Write $W = (\mathcal{O}_E)_\lambda = E \cap \mathcal{O}_{E_\lambda}$ for the localization and $\widehat{W} = \mathcal{O}_{E_\lambda}$ for its completion. As p is unramified, $pW = \lambda W$; the common residue field is $W/pW = \widehat{W}/p\widehat{W}$. Put $P = p^e$. At an even embedding use the embedding of $t(s)$, and at an odd embedding compose that embedding with complex conjugation on the field. These embeddings ι_s all induce λ. On the field the step $\langle -p^e \rangle$ followed by conjugation is the e-th power of arithmetic Frobenius at p. Indeed in $E = K\mathbb{Q}(\zeta_{q_1})$ the step fixes K and takes ζ_{q_1} to $\zeta_{q_1}^{-p^e}$.

Conjugation changes the cyclotomic sign and acts nontrivially on K, where p is inert and e is odd. More explicitly, with $\sigma_p = (c_K, p)$ and $c = (c_K, -1)$, the alternating embeddings are $\iota_s = \iota_0 \circ u^{es} c^s = \iota_0 \circ \sigma_p^{es}$. Arithmetic Frobenius satisfies the residue congruence modulo $\lambda W = pW$. Elements of $\mathcal{O}_E \setminus \lambda$ have unit images under every ι_s, so the congruence extends to

W and holds modulo $p\mathcal{O}$. Consequently

$$\iota_{s+1}(x) \equiv \iota_s(x)^P \pmod{p\mathcal{O}}, \qquad q = |W/\lambda W| = p^{eN_{\mathrm{orb}}} = P^{N_{\mathrm{orb}}}.$$

Indeed arithmetic Frobenius is u times conjugation, and conjugation has order two and is independent of $\mathrm{Gal}(E/K)$. The residue degree is $\mathrm{lcm}(\mathrm{ord}(u), 2) = eN_{\mathrm{orb}}$, since N_{orb} is even.

To check the exceptional order condition, write $m = 2^H$. The order eN_{orb} satisfies $u^{eN_{\mathrm{orb}}} \equiv 1 \bmod 2^{22H+v_2(q_0)}$. It is even, because an odd exponent would give $v_2(u^{eN_{\mathrm{orb}}} - 1) = v_2(u-1) < H$. For an even exponent the factorization of the difference of powers gives

$$v_2(u^{eN_{\mathrm{orb}}} - 1) = v_2(u-1) + v_2(u+1) + v_2(eN_{\mathrm{orb}}) - 1.$$

One of the first two terms is 1, and both are less than H since $p \not\equiv \pm 1 \bmod m$. It follows that $v_2(eN_{\mathrm{orb}}) \geq H$, so $m \mid eN_{\mathrm{orb}}$. This construction keeps λ and its parameters fixed.

Coordinates for the exact operator. Represent each class x at the original level by a point a_x integral above p. For any representative c_x prescribe $\gamma_p \in K_{X_0,p}(c_x)_p^{-1}$, with $K_{X_0,p}$ the product of the integral groups above p. Since $p \nmid M$, one may simultaneously prescribe the old principal subgroups at the primes dividing M. Weak approximation on the rational Cayley chart of Section 19.4.3 gives γ in these open sets. Then $a_x = \gamma c_x$ represents the same class. The original level has trivial class stabilizers. Evaluation at these representatives therefore identifies M with the module of all integral coordinate vectors at the classes. Indeed left rational translation in the lattice norm cancels the rational transformation of the value, and right level translation preserves the integral lattice norm.

Since p generates the maximal ideal of $\widehat{W}$, the uniformizers ϖ_λ and p give the same hyperspecial double coset and the same unramified character value. Put $d = \mathrm{diag}(pI_r, I_{k-r})$ at λ. Put $K_\lambda = \mathrm{GL}_k(\widehat{W})$, $\widehat{\mathcal{P}} = K_\lambda \cap dK_\lambda d^{-1}$, and

$$\mathcal{P} = \widehat{\mathcal{P}} \cap \mathrm{GL}_k(W) = \{g \in \mathrm{GL}_k(W) : d^{-1}gd \in \mathrm{GL}_k(W)\}.$$

The incoming edges at x are indexed by $K_\lambda/\widehat{\mathcal{P}}$. Reduction identifies this quotient with the $(k-r)$-planes over the residue field. It gives the same flag space for $\mathrm{GL}_k(W)/\mathcal{P}$: the upper right block vanishes modulo p, and every invertible residue matrix lifts to a matrix over W with unit determinant. Choose representatives $\beta_b \in \mathrm{GL}_k(W)$ in this way. With $\mathrm{vol}(K_\lambda) = 1$,

$$K_\lambda d K_\lambda = \bigsqcup_b \beta_b d K_\lambda, \qquad \mathrm{vol}(\beta_b d K_\lambda) = 1, \qquad \mathrm{vol}(\widehat{\mathcal{P}}) = [K_\lambda : \widehat{\mathcal{P}}]^{-1}.$$

Thus every edge has coefficient one in the unnormalized Hecke sum. At λ replace the component of a_x by $\beta_b d$ and make the dual change at $\bar{\lambda}$. This gives the same Hecke coset as right multiplication by $a_{x,\lambda}^{-1}\beta_b d$. Write the resulting point as $\gamma_e a_{y(e)} v_e$, with γ_e rational unitary and v_e of original level. At λ this identity gives

$$\gamma_e^{-1} = g_e d^{-1} \beta_b^{-1}, \qquad g_e \in \mathrm{GL}_k(W).$$

At λ, $g_e = \gamma_e^{-1}\beta_b d$ has entries in E, and equals $a_{y(e),\lambda} v_{e,\lambda}$ in $\mathrm{GL}_k(\widehat{W})$. Thus $g_e \in \mathrm{GL}_k(W)$. At each $w \mid p$ outside $\{\lambda, \bar{\lambda}\}$, the modified representative still has component $a_{x,w}$, so

$$\gamma_{e,w}^{-1} = a_{y(e),w}\, v_{e,w}\, a_{x,w}^{-1} \in \mathrm{GL}_k(\mathcal{O}_{E_w}).$$

Here both class representatives are integral above p and the original level is hyperspecial there.

For use on the dual edges, the map $e \mapsto (y(e), (g_e^{\mathsf{T}})^{-1}\mathcal{P})$ is a bijection. If two edges have the same image, $g_e^{\mathsf{T}}(g_{e'}^{\mathsf{T}})^{-1} \in \mathcal{P}$. Taking inverse transpose in the conjugation test defining $\mathcal{P}$ shows that $dg_e^{-1}g_{e'}d^{-1}$ is integral with integral inverse. Let x, x' be the two termini and put $\delta = \gamma_e\gamma_{e'}^{-1}$. At λ, $\delta = \beta_b(dg_e^{-1}g_{e'}d^{-1})\beta_{b'}^{-1}$ is integral with integral inverse. The dual component has the same property. Write J for the original level. Away from the pair, the two factorizations at their common origin give $a_x^{-1}\delta a_{x'} = v_e^{-1}v_{e'}$ locally. At the pair the representatives are integral and J is hyperspecial. Hence $a_x^{-1}\delta a_{x'} \in J$, so $x = x'$. Now δ is in the trivial rational stabilizer of a_xJ, so $\gamma_e = \gamma_{e'}$. The displayed equation for γ_e^{-1} then gives $\beta_b^{-1}\beta_{b'} \in \mathcal{P}$. This proves injectivity, and equal finite cardinalities give bijectivity.

At an embedding τ the one-row weight has degree $a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ and determinant exponent zero. Every column has height one, so its evaluation vector in the integral tableau coordinates is $e(z) = (z^\alpha)_{|\alpha|=a_\tau}$. For the symmetric-power matrix $C(A)_{\beta,\alpha} = [z^\beta](A^{\mathsf{T}}z)^\alpha$, coefficient comparison gives

$$e(Az) = C(A^{\mathsf{T}})^{\mathsf{T}}e(z).$$

This is an identity over $\mathbb{Z}[A]$, also for $a_\tau = 0$. Use $\rho_s(g)$ for the symmetric power of $\iota_s(g)$ at even s and of $\iota_s(g^{-\mathsf{T}})$ at odd s, and let $\rho_{\mathrm{let}}(g) = \bigotimes_s \rho_s(g)$. On even embeddings, p times the transpose of the conjugated rational matrix is $p\iota_s(\gamma_e^{-1})$, with middle factor pd^{-1}. On odd embeddings the transpose instead is $\iota_s(\gamma_e^{\mathsf{T}})$, with middle factor d. Outside this orbit, every positive-degree factor lies away from $\lambda, \bar{\lambda}$, where γ_e and its inverse are integral. A factor of degree zero has trivial action even above the pair. Therefore the p^{A_λ}-scaled source matrix for this edge is the tensor product of an integral invertible transition on the other Weyl factors and the transpose of

$$\rho_{\mathrm{let}}(g_e)\ \mathrm{diag}_\mu(p^{\mathrm{dep}(\mu)})\ \rho_{\mathrm{let}}(\beta_b^{-1}).$$

Number the coordinates from 0 to $k-1$. Then $\mathrm{dep}(\mu)$ counts exponents in coordinates $r, \dots, k-1$ on even embeddings and $0, \dots, r-1$ on odd embeddings. This follows by taking symmetric powers of the two diagonal matrices. In particular Y_λ preserves M. The formula is the edge sum on the original class lattice.

Size conditions for the local calculation. Set $\theta = (m^2 f)^{-1}$. The polynomial estimates below use

$$0 < 2r \le k, \quad 0 < \theta \le 1, \quad a^4 \le q, \quad (N_{\mathrm{orb}}k\log q/\theta)^5 P^{3r} \le a.$$

These hold uniformly for the primes in Λ for all sufficiently large x. Indeed $m \asymp \sqrt{x}$, $k \asymp x^5$, $f = O(x^{11}\log x)$, $N_{\mathrm{orb}} \le f$, $r = r_p = O(\log x)$, and $e \le r$. In particular $2r \le k$ and $r^2 m \le k$. Since $a = xb$ and $\log b = x^\epsilon + O(1)$,

$$a^4 \le b^5 \le \exp(m) \le p^{eN_{\mathrm{orb}}} = q.$$

The exceptional prime p is inert in K, whereas all primes below 13 divide q_0 and split in K by Lemma 4.1. Thus $p \ge 13$. Since $p < x_1$, we have $\log q \le eN_{\mathrm{orb}}\log x_1$. Also $P^{3r} \le x_1^{3r^2} \le \sqrt{b}$, because $(\log x)^2 \log x_1 = o(\log b)$. Finally $(eN_{\mathrm{orb}}^2 k\log x_1/\theta)^5$ is bounded by a fixed power of x, uniformly in p, λ. Its product with $\sqrt{b}$ is at most a for large x. This gives the second large-degree inequality.

Put

$$C_{\mathrm{div}} = N_{\mathrm{orb}}a\left(P^{-2r} + \frac{r^2}{kP}\right).$$

Let $w^2 = p$ on extending scalars if necessary. Let b_μ be the diagonal symmetric-power pairing, with integral complementary weights b'_μ satisfying $b_\mu b'_\mu = p^{h_0}$. For the exact common exponent

put $h_0 = N_{\mathrm{orb}} k^2 e N_{\mathrm{orb}}$. If $M_s = a!/\prod_i \mu_{s,i}!$, take $b_\mu = p^{h_0}/\prod_s M_s$ and $b'_\mu = \prod_s M_s$. Since $a < q = p^{eN_{\mathrm{orb}}}$, the factorial valuation formula bounds $v_p(M_s)$ by keN_{orb}. This proves integrality. The multinomial theorem proves that substitution by a matrix and by its transpose are adjoint for the weights $1/M_s$, and hence for their scaled tensor product. Finally $k^2 e N_{\mathrm{orb}} P^{2r} \le a$ follows from the displayed large-degree bound and $eN_{\mathrm{orb}} \le \log q$, so $h_0 \le N_{\mathrm{orb}} a/P^{2r} \le C_{\mathrm{div}}$. Write $s_{\mathcal{P}} = [\mathrm{GL}_k(W) : \mathcal{P}]$ and d_{let} for the number of monomials of degree a in each of the N_{orb} embedding factors. Thus

$$d_{\mathrm{let}} = \binom{a+k-1}{k-1}^{N_{\mathrm{orb}}}.$$

Write $W_0 = \mathrm{diag}_\mu(w^{\mathrm{dep}(\mu)})$. A saturated halfway restriction is a vector $W_0 F$ with F over the coefficient fraction field such that $W_0 \rho_{\mathrm{let}}(g) F$ is integral in every parahoric direction. To see that representatives suffice, let $u \in \mathcal{P}$. In the linear substitution at each embedding, an entry with shallow row and deep column is divisible by $p = w^2$. For even embeddings this is the upper-right block of u, and for odd embeddings it is the lower-left block of $u^{-\mathsf{T}}$. Conjugating by the diagonal scaling of deep coordinates by w divides those entries by w and preserves integrality. Applying this also to u^{-1} and taking symmetric and tensor powers gives

$$W_0 \rho_{\mathrm{let}}(u) W_0^{-1} \in \mathrm{GL}_{d_{\mathrm{let}}}(\mathcal{O}).$$

Choose $h_{\mathrm{clr}} \ge 0$ such that $F_{\mathrm{int}} = w^{h_{\mathrm{clr}}} F$ is integral. The condition on a saturated restriction is precisely

$$W_0 \rho_{\mathrm{let}}(g) F_{\mathrm{int}} \in w^{h_{\mathrm{clr}}} \mathcal{O}^{d_{\mathrm{let}}} \qquad (g \in \mathrm{GL}_k(W)).$$

21.1 The depth count for the endpoint restrictions

In this subsection, we bound the valuations of the elementary divisors of the two restrictions of Section 21. The proof counts coefficients satisfying the divisibility conditions in coordinates adapted to the parahoric.

Work over the residue coefficient field K_0 of characteristic p for the cycle of embeddings above. Write $k = 2r + k_1$. A variable $X_{b,s}$, $0 \le b < k$, is deep when s is even and $b \ge r$, or s is odd and $b < r$. Let D_j be the space of degree-a forms in each embedding whose translates in all parahoric directions have no coefficient of deep degree less than j. The condition then holds after every $\mathrm{GL}_k(W)$ translate. In fact the parahoric preserves the deep-degree filtration modulo p in both parities. The action on the reduced coefficients factors through $\mathrm{GL}_k(W/pW)$.

Treat the complementary variables as coefficients in the field

$$L_0 = K_0(X_{b,s} : (s \text{ even}, b < r) \text{ or } (s \text{ odd}, b \ge r)).$$

For each deep variable $v = (b, s)$ put

$$\kappa_v = \begin{cases} P, & b < 2r, \\ P^{2r}, & b \ge 2r. \end{cases}$$

Give a monomial in the deep variables weight $\sum_v \mu_v/\kappa_v$. We show that any nonzero form in D_j has a nonzero coefficient of weight at least j.

Let B run through matrices over $\mathbb{F}_q$ supported on entries $(b, b+r)$, $b < r$, and (b, t), $b < r$, $2r \le t < k$. There are $r(k_1+1)$ independent entries. Such matrices satisfy $B^2 = 0$.

The translation by $(1+B)^{-1}$ fixes every complementary variable. It takes a deep variable X_v to $X_v - z_v(B)$, where $z_{t,s}$ at even s is $\sum_{b<r} X_{b,s}\iota_s(B_{bt})$, for $t \geq r$, and $z_{b,s}$ at odd s is $-\sum_{t\geq r} X_{t,s}\iota_s(B_{bt})$, for $b < r$. For $F \in D_j$ the deep degree of every term in the translate is at least j, so $F(X - z(B)) \in (X_v : v \text{ deep})^j$. Using $-B$ gives $F \in (X_v - z_v(B) : v \text{ deep})^j$ at each point $z(B)$.

We claim that all the coordinate functions $\iota_i(B_{bt})$, for any fixed embedding i, are L_0-linear combinations of the functions $z_v(B)^{P^u}$ with $s + u = i$ and $P^u < \kappa_v$. For a column $t \geq 2r$, choose the r distinct integers $0 \leq u < 2r$ for which $s = i - u$ is even. The r values $z_{t,i-u}(B)^{P^u}$ have coefficient matrix

$$\left(X_{b,i-u}^{P^u}\right)_{u,b}.$$

This uses $\iota_s(y)^{P^u} = \iota_{s+u}(y)$ on the residue field. The determinant is nonzero. Distinct permutations use different monomials, because within any one column the variables are distinguished either by their embedding or by the distinct exponent P^u.

Thus these values solve for every entry of the column. For $b < r$ the remaining entries satisfy

$$\begin{aligned} z_{b+r,i}(B) &= X_{b,i}\iota_i(B_{b,b+r}) && (i \text{ even}),\\ -z_{b,i}(B) &= X_{b+r,i}\iota_i(B_{b,b+r}) + \sum_{2r\leq t<k} X_{t,i}\iota_i(B_{bt}) && (i \text{ odd}). \end{aligned}$$

The displayed coefficients of $\iota_i(B_{b,b+r})$ are nonzero in L_0. Subtracting the known sum in the odd case proves the claim, also when $k = 2r$ and the sum is empty. When $r = 1$, the interpolation matrix for each $t \geq 2r$ has a single nonzero entry $X_{0,i-u}^{P^u}$.

It follows first that the points $z(B)$ are distinct. It also follows, by taking the P-power or the P^{2r}-power of the coordinate expression at its original embedding, that for every deep variable v

$$z_v(B)^{\kappa_v} = g_v(z(B)),$$

where g_v is a polynomial of weight strictly less than one. Indeed the power moves the coefficient functions to the corresponding later embedding, where the claim expresses them using the allowed lower powers. There are $N_{\mathrm{orb}}r$ deep coordinates with $b < 2r$ and $N_{\mathrm{orb}}k_1/2$ others, the latter all in even embeddings. Since $q = P^{N_{\mathrm{orb}}}$, the number of points equals

$$q^{r(k_1+1)} = P^{N_{\mathrm{orb}}r}(P^{2r})^{N_{\mathrm{orb}}k_1/2} = \prod_v \kappa_v.$$

For completeness, the weighted interpolation argument for these points is as follows. Let $\psi_v = X_v^{\kappa_v} - g_v$, of weight one. The ideals of the points are pairwise comaximal, as are their j-th powers. The map to truncated polynomials of degree less than j at all these points is therefore surjective. Division using $X_v^{\kappa_v} = \psi_v + g_v$ expresses each polynomial in the span of $X^\alpha\psi^\gamma$, $0 \leq \alpha_v < \kappa_v$. In each replacement the term with g_v strictly lowers weight and the other term removes $X_v^{\kappa_v}$ from the monomial preceding ψ^γ.

At each step the pair $(\mathrm{wt}(\alpha)+|\gamma|, \mathrm{wt}(\alpha))$ decreases in lexicographic order on pairs of elements of $P^{-2r}\mathbb{Z}_{\geq 0}$. Thus division terminates without increasing total weight. For $j \geq 1$, all products with $|\gamma| \geq j$ vanish in the jets, since each ψ_v vanishes at every point. Let d be the number of deep coordinates. Translation to each point identifies its jet algebra with $L_0[X_v]/(X_v)^j$, where the ideal is generated by all monomials of total degree j. The remaining products span the target and number

$$\Big(\prod_v \kappa_v\Big)\binom{d+j-1}{d} = \dim_{L_0} \bigoplus_B L_0[X_v]/(X_v - z_v(B) : v)^j.$$

Their images are therefore independent. If F had weight less than j, division would use only products with $|\gamma| < j$. Its zero image would force $F = 0$. The change of complementary variables to coefficients is injective, so it follows that a nonzero original coefficient of F has weight at least j.

Projection onto the monomial coordinates of weight at least j is consequently injective on D_j. A monomial of weight u is counted at no more than u positive integer thresholds. In the uniform average over monomials of degree a in each embedding, each variable has mean exponent a/k. There are $N_{\mathrm{orb}} r$ deep variables with $b < 2r$, and at most $N_{\mathrm{orb}} k$ other deep variables. Thus the mean weight is at most

$$\frac{a}{k}\left(\frac{N_{\mathrm{orb}} r}{P} + \frac{N_{\mathrm{orb}} k}{P^{2r}}\right) \le N_{\mathrm{orb}} a \left(P^{-2r} + \frac{r^2}{kP}\right) = C_{\mathrm{div}}.$$

Summing the injective dimension bounds gives $\sum_{j \ge 1} \dim D_j \le C_{\mathrm{div}} d_{\mathrm{let}}$, as required for both endpoint restrictions.

21.2 The rank-two defect estimate

In this subsection, we make the root calculation needed for the source Hecke operator of Section 21. The two-dimensional calculation bounds the coefficients on the saturated restrictions of Section 21.1.

Fix two distinct indices b, b' and an embedding i. Set $(X_s, Y_s) = (X_{b,s}, X_{b',s})$ for even s and $(X_s, Y_s) = (X_{b',s}, X_{b,s})$ for odd s. Thus Y_s is the positive variable in the defect. Put $P_s = P^{(s-i) \bmod N_{\mathrm{orb}}}$. For an exponent α define

$$D = q \sum_s P_s \alpha(X_s), \qquad E = q\alpha(Y_i) + \sum_{s \ne i} P_s \alpha(Y_s).$$

The defect of α at i is $(E - D/q)/(q-1)$.

First work over the integral coefficient ring with $w^2 = p$. Let c give weights on the variables with $1/2 \le c(Y_s) - c(X_s) \le 1$. For a coefficient a_α, give the term $a_\alpha X^\alpha$ total weight $\mathrm{ord}_w(a_\alpha) + c \cdot \alpha$, where for $a \ne 0$

$$\mathrm{ord}_w(a) = \max\{j \in \mathbf{Z}_{\ge 0} : a \in w^j \mathcal{O}\} = \lfloor 2v(a) \rfloor, \qquad \mathrm{ord}_w(0) = +\infty.$$

Writing $o = \mathrm{ord}_w$, we have

$$o(ab) \ge o(a) + o(b), \qquad o(a+b) \ge \min\{o(a), o(b)\}.$$

Thus lower bounds on total weight pass to products and sums. Over the residue field every nonzero coefficient has order zero. Suppose all terms of all translates of F by the root $\mathrm{SL}_2(W)$ have weight at least $l + c \cdot \alpha$, where the coefficient a_α of X^α is nonzero and $l = \mathrm{ord}_w(a_\alpha)$. Thus $a_\alpha \ne 0 \bmod w^{l+1}$. Over the residue field take $l = 0$ and omit reduction. We prove $Eq \le D$.

Put $\epsilon_s = -1$ if s has the parity of i, and $\epsilon_s = 1$ otherwise. In a new variable T make the substitution $\mathcal{A}_T$ given by

$$\begin{aligned} X_i &\longmapsto T^q X_i, & Y_i &\longmapsto X_i, \\ X_s &\longmapsto T^{qP_s} X_s - \epsilon_s T^{P_s} Y_s, & Y_s &\longmapsto Y_s - \epsilon_s X_s \quad (s \ne i). \end{aligned}$$

All other variables are fixed. Let $\bar{\alpha}$ be the exponent obtained by merging Y_i into X_i, and let $\Phi(T)$ be the coefficient at $\bar{\alpha}$ of $\mathcal{A}_T F$, reduced modulo w^{l+1}.

For the expansion at infinity put $\rho = (c(Y_i) - c(X_i))/q$. Give T weight $-\rho$, X_s weight $c(X_s) + qP_s\rho$, and every other variable its original weight. The main term of each substituted variable has its original weight. At i this uses $c(X_i) + q\rho = c(Y_i)$. At $s \neq i$ the cross terms have strictly greater weight, since

$$P_s\rho \leq 1/P < 1/2, \qquad qP_s\rho \geq P/2 > 1.$$

The variable weight of $T^D X^{\bar{\alpha}}$ is $c \cdot \alpha$. Any contribution using a cross term has strictly greater total weight than $l + c \cdot \alpha$, so its integral coefficient order is at least $l + 1$. For a contribution using only main terms the exponents away from X_i, Y_i are unchanged, and

$$\alpha(X_i) = D/q - \sum_{s \neq i} P_s \bar{\alpha}(X_s), \qquad \alpha(Y_i) = \bar{\alpha}(X_i) - \alpha(X_i).$$

Hence $[T^D]\Phi = a_\alpha \bmod w^{l+1}$ is nonzero. For degree $n > D$ the variable weight drops by $(n - D)\rho > 0$; the coefficient order is again at least $l + 1$. Hence $\deg \Phi = D$.

For each finite point take $r_0 \in W$, put $t_0 = \iota_i(r_0)$, and shift $T \mapsto T + t_0$. The root matrix is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -r_0 \end{smallmatrix}\right)$ if i is even, with $+r_0$ in the last entry if i is odd. Denote it by g_{r_0}, embedded by the identity on the other coordinates. It has determinant one. As r_0 runs through residue representatives, its second column $(1, \pm r_0)^{\mathsf{T}}$ represents the q affine directions in $\mathbf{P}^1(W/pW)$. The remaining direction is the second column of the identity. Put $\delta_s = -1$ at even s and $\delta_s = 1$ at odd s. Reading its two columns at even embeddings and those of its inverse transpose in the reversed order at odd embeddings gives

$$\rho_{\mathrm{let}}(g_{r_0}) X_s = \delta_s Y_s, \qquad \rho_{\mathrm{let}}(g_{r_0}) Y_s = -\delta_s (X_s + \epsilon_s \iota_s(r_0) Y_s).$$

Compose that root substitution with $\mathcal{A}_{T+t_0}$. The result on each of X_s, Y_s is the signed image under $\mathcal{A}_T$ of the swapped variable, up to errors on the image of Y_s with coefficients in $p\mathcal{O}[T]$. Indeed for $s \neq i$ the change in $\mathcal{A}_T(X_s)$ modulo p is

$$t_0^{P_s}(X_s - \epsilon_s Y_s) = -\epsilon_s \iota_s(r_0) \mathcal{A}_T(Y_s) \pmod{p}.$$

Here the shifts of T^{qP_s} and T^{P_s} have the same constant modulo p, by the Frobenius cycle. At i the change is $t_0^q X_i = t_0 X_i$ modulo p, and $-\epsilon_i = 1$. The errors consist only of p-multiples of polynomials times X_s, Y_s, or of X_i after merging. This follows by expanding the two shifted p-powers. Over the residue field the errors vanish.

For this finite expansion give T weight $+\rho$, X_s weight $c(X_s)$, Y_s weight $c(Y_s) - P_s\rho$ for $s \neq i$, and Y_i weight $c(Y_i) - q\rho$. The signed image of the swapped variable has weight at least the weight of the source vector. For $s \neq i$ the required two inequalities are $c(Y_s) - P_s\rho \geq c(X_s)$ and $c(X_s) + qP_s\rho \geq c(Y_s)$; at i they hold with the stated ρ. Each error still has weight at least the source weight, because a factor $p = w^2$ contributes two, while the difference lost in either variable is at most $1 + 1/P$. Applying this to the inverse root translate of F shows that all terms of $\mathcal{A}_{T+t_0} F$ have weight at least $l + c \cdot \alpha$ in this finite expansion. Now $T^E X^{\bar{\alpha}}$ has weight $c \cdot \alpha$. Every lower T-degree at that exponent has zero coefficient modulo w^{l+1}. Thus Φ vanishes at t_0 to order at least E.

Choose one $r_0 \in W$ lifting each of the q classes of W/pW. Distinct lifts differ by a unit of W, whose image under ι_i is a unit even modulo w^{l+1}. Therefore the ideals generated by

the distinct $T - t_0$ are pairwise comaximal in the quotient polynomial ring, as are their E-th powers. Their product divides Φ. Since this product is monic of degree Eq, multiplication by it raises the degree of every nonzero polynomial by Eq, even over $\mathcal{O}/(w^{l+1})$. Thus $Eq \le D$. This proves that the defect of the extreme term is nonpositive. Taking $l = 0$ over the residue field with the height weights proves the least-height fact used in the flag count.

For the ordinary depth estimate take an ordered root $b < r \le b'$, in particular the boundary root $(r-1, r)$. In both parities X_s is shallow and Y_s is deep, so $\operatorname{dep}(Y_s) - \operatorname{dep}(X_s) = 1$. Fix i and change these weights by subtracting one third of the defect functional at i. The increments of the changed weights are still in $[1/2, 1]$: the increments of the defect are $1 + 2/(q-1)$ at i and $2P_s/(q-1)$ elsewhere, all in $[0, 3/2]$.

Suppose every root translate initially has order plus ordinary depth at least v_0. If the changed bound failed in even one translate, choose an extreme term of minimal order plus changed weight over all translates. Such a minimum exists, because the exponents lie in a finite set and nonnegative integral orders with weight below v_0 are bounded. The estimate just proved says that the extreme term has nonpositive defect. Its ordinary depth plus order would therefore be below v_0, a contradiction. Thus

$$\operatorname{ord}_w(a_\beta) + \operatorname{dep}(\beta) \;\ge\; v_0 + \operatorname{defect}_i(\beta)/3.$$

In particular let $v = w^{-h_{\mathrm{clr}}} W_0 F$ be a saturated halfway restriction with F integral and $w^{-h_{\mathrm{clr}}} W_0 \rho_{\mathrm{let}}(g) F$ integral in every direction g. For a block-diagonal Levi element g_L and any root element g_2, the product $g_2 g_L$ is an allowed direction. The depth hypothesis therefore holds for every root translate of $F_L = \rho_{\mathrm{let}}(g_L) F$ with $v_0 = h_{\mathrm{clr}}$. Since g_L preserves the block degrees, its action commutes with W_0. For every nonzero coefficient this gives

$$\operatorname{ord}_w\big((\rho_{\mathrm{let}}(g_L) v)_\beta\big) = \operatorname{ord}_w\big((F_L)_\beta\big) + \operatorname{dep}(\beta) - h_{\mathrm{clr}} \ge \operatorname{defect}_i(\beta)/3.$$

Here the equality uses $\lfloor 2v_p(z) + d \rfloor = \lfloor 2v_p(z) \rfloor + d$ for $d \in \mathbf{Z}$. If $\operatorname{defect}_i(\beta) \ge 3u$ for $u \in \mathbf{Z}_{\ge 0}$, this coefficient belongs to $w^u \mathcal{O}$. Writing e_β for the monomial vector and using the integral pairing weights b_β, we also have

$$\langle \rho_{\mathrm{let}}(g_L^{-1}) e_\beta, v \rangle = b_\beta (\rho_{\mathrm{let}}(g_L^{-\mathsf{T}}) v)_\beta \in w^u \mathcal{O},$$

since $g_L^{-\mathsf{T}}$ belongs to the same Levi.

21.3 Counting forms with bounded local defect

In this subsection, we use the two-variable estimate of Section 21.2 to bound a dimension in the residue representation. We count the permitted monomials after imposing the same conditions under all elements of the Levi.

Write $b_0 = r - 1$ and take the defect bound $3t$ at the root (b_0, r), where t is an integer with $t \ge 2(k+1)$. Work in the residue coefficient field of characteristic p. Let S be the degree-a forms in the embedding factors for which every translate by the reduction of the two-block Levi has all defects along this root at most $3t$. The reduction is $\mathrm{GL}_r(\mathbb{F}_q) \times \mathrm{GL}_{k-r}(\mathbb{F}_q)$ acting through the prescribed Frobenius embeddings into K_0. The other simple roots lie inside the Levi.

The rank-two estimate of Section 21.2 gives the following fact at the roots inside one Levi block. Among the nonzero coefficients of all Levi translates of a nonzero polynomial, choose one of least height, with height of $X_{b,s}$ given by b on even embeddings and $-b$ on odd embeddings.

The defect at that coefficient is nonpositive at each embedding for every simple root inside the block. The required root calculation applies without a homogeneity requirement on the polynomial, since the root substitutions are linear and preserve the degree at each embedding.

A least-height coefficient occurs at a representative of a Levi flag. Write $g = ux$, with u lower triangular modulo p and x chosen for the quotient by those matrices acting on the left. On even embeddings an off-diagonal term of u strictly increases the variable index, and on odd embeddings inverse transpose strictly decreases it. Each increases the height. Thus at the least-height coefficient u contributes only its diagonal action, with a nonzero factor. The coefficient at x is therefore already nonzero.

For a positive integer $M = q^j$, take a family $(F_u)_u$ in S indexed by the vectors u with $0 \le u_{b,s} < M$. In each embedding put

$$\gamma_{u,b,s} = u_{b,s} + \mathbf{1}_{b=b_0}\, M \left(k - \left\lfloor \frac{\sum_c u_{c,s}}{M} \right\rfloor \right).$$

Then X^{γ_u} keeps the original residue u modulo M, while the degree of γ_u in each embedding belongs to $[Mk, M(k+1))$. Pack the family as

$$F_{\mathrm{pack}}(X) = \sum_u X^{\gamma_u} F_u(X^M).$$

This linear map is injective by distinct exponent residues. The residue entries of every translated matrix lie in the images of $\mathbb{F}_q$, so the substitutions commute with $X \mapsto X^M$. Consequently any nonzero coefficient at α in a Levi translate of the packed polynomial comes from a term with exponent $\alpha = M\beta + \eta$, where β occurs in a translate of some F_u and η occurs in the translate of its monomial X^{γ_u}. The degrees of η/M are in $[k, k+1)$, because linear substitutions preserve them.

The defects along (b_0, r) of β are at most $3t$. In the defect of η/M, the term with factor q is bounded by $(k+1)q/(q-1)$; the singled-out term with factor -1 is nonpositive. Including the other weighted differences gives

$$\frac{k+1}{q-1}\left(q + \sum_{l=1}^{N_{\mathrm{orb}}-1} P^l \right) = (k+1)\left(1 + \frac{1}{P-1}\right) \le 2(k+1).$$

Thus each of these defects of α/M is at most $3t + 2(k+1) \le 4t$. Its degree at each embedding is in $[a+k, a+k+1]$.

We move the boundary-root bound into the zero-defect cone by a common integral shift. Put $\mu_b = P^{2(b_0-b)}$ for $b \le b_0$ and $\mu_b = 0$ otherwise. Put $\nu_{b,s} = \mu_b$ on even embeddings and $P\mu_b$ on odd embeddings. Put $\varepsilon_s = (-1)^s$. The cyclic steps $\sigma_{b,s} = \varepsilon_s(P\nu_{b,s+1} - \nu_{b,s})$ are $(P^2-1)\mu_b$ on even embeddings and zero on odd embeddings. It is integral and nonnegative, with degree at most $P^{2r} - 1$ at every embedding. Along a simple root, its defects are $P^2\mu_{b+1} - \mu_b$ on even embeddings and $P(\mu_{b+1} - \mu_b)$ on odd embeddings. Thus they are at most -1 at the boundary root and at most zero at all other simple roots.

Write $\mathcal{C}_0$ for the cone in which all simple-root defects are nonpositive, and $\mathcal{D}$ for the nonnegative degree slab with the bounds below. For a nonzero packed family, choose a least-height coefficient α at a flag representative x. The fact about roots inside the Levi and the last estimates show

$$\alpha/M + 4t\sigma \in \mathcal{C}_0 \cap \mathcal{D}, \qquad D_s = a + k + 1 + 4t \sum_b \sigma_{b,s} \le a + 4tP^{2r}.$$

Since α and $4t\sigma$ are integral arrays, the point lies in $M^{-1}\mathbb{Z}^{kN_{\mathrm{orb}}}$.

Therefore the linear map that lists the coefficients at $M(y-4t\sigma)$ in the flag translates, for $y \in \mathcal{C}_0 \cap \mathcal{D} \cap M^{-1}\mathbb{Z}^{kN_{\mathrm{orb}}}$, is injective on the packed families. Points giving negative coordinates can be skipped. It follows that $M^{kN_{\mathrm{orb}}} \dim S$ is bounded by the number of Levi flags times the number of these lattice points. Divide by $M^{kN_{\mathrm{orb}}}$ and let $M = q^j \to \infty$. The bounded polytope has boundary of measure zero, and its normalized lattice counts converge to its volume by the ordinary Riemann sums. This is the required lattice-point bound for the volume calculation.

21.4 The volume factor in the residue-defect estimate

In this subsection, we compute the volume ratio in the dimension estimate of Section 21.3. Ordering the exponents gives interlacing inequalities. Their volume normalization follows by determinant expansion.

Use the same parity sign $\varepsilon_s = (-1)^s$. For permutations $\sigma_s \in S_k$, set

$$(T_\sigma \nu)_{b,s} = \varepsilon_s (P\nu_{b,s+1} - \nu_{\sigma_s(b),s}).$$

Write $q = P^{N_{\mathrm{orb}}}$. The sum of the entries at each embedding is independent of the permutations. Let $\mathcal{D}$ be any slab of nonnegative vectors whose degree in the embedding s is in $[D_s - 1, D_s]$. In coordinates $T_1\nu$, the zero-defect cone $\mathcal{C}_0$ is given by the additional inequalities

$$\begin{cases} P\nu_{b+1,s+1} \le \nu_{b,s}, & \varepsilon_s = 1, \\ \nu_{b+1,s} \le P\nu_{b,s+1}, & \varepsilon_s = -1, \end{cases} \qquad (b < k-1).$$

Indeed put $\beta = T_1\nu$ and $d_s = \nu_{b,s} - \nu_{b+1,s}$. The signed weighted difference of the two relevant beta coordinates is $d_s - Pd_{s+1}$. Summing with weights $P^{(s-i) \bmod N_{\mathrm{orb}}}/(q-1)$ gives $-d_i$, because the cycle closes with $P^{N_{\mathrm{orb}}} = q$. The remaining term of the defect is $\beta_{b+1,i}$ if i is even and $\beta_{b,i}$ otherwise. Thus zero defect is the displayed pair of inequalities. We prove

$$\mathrm{vol}(\mathcal{C}_0 \cap \mathcal{D}) = \frac{\mathrm{vol}(\mathcal{D})}{\prod_{b=1}^{k}(1 + q + \cdots + q^{b-1})}.$$

First take ν with distinct entries in each embedding, avoiding also equality in adjacent comparisons. These restrictions discard only finitely many hyperplanes. For each s let M_s be the k-square matrix of indicators of the inequalities $\varepsilon_s(P\nu_{b,s+1} - \nu_{c,s}) > 0$. The product $\prod_s \det M_s$ is unchanged if the entries in any one embedding are independently permuted, since this permutes the rows of one determinant and the columns of another with the same sign.

Sort the entries of ν decreasingly in each embedding. For two decreasing sequences x, y, the determinant of $(1_{x_b > y_c})_{b,c}$ is the indicator of interlacing $x_0 > y_0 \ge x_1 > y_1 \ge \cdots$. Indeed each row consists of zeros followed by ones, and the first positions of the ones are nondecreasing. The determinant vanishes if two rows coincide or there is a zero row. In the remaining case the first positions are $0, \ldots, k-1$ and the matrix is triangular with diagonal ones. For the reversed inequality transpose and interchange the two sequences. Thus the product of determinants indicates the cone inequalities and positivity on the sorted arrays.

Conversely the cone inequalities and positivity force the entries of ν to be decreasing in each embedding, by the same interlacing inequalities. Therefore the number of tuples of permutations of the embeddings sending ν to a point satisfying these conditions equals that product of determinants. Expanding the determinants gives the signed sum over the tuples σ of the

indicators of $T_\sigma \nu > 0$, with sign $\prod_s \operatorname{sgn} \sigma_s$. Multiply by the degree-slab condition, common to all terms, and integrate.

To compute the resulting volumes, eliminate the embedding vectors cyclically in $P\nu_{s+1} = \Pi_{\sigma_s}\nu_s + \varepsilon_s\beta_s$, where Π_{σ_s} is the corresponding permutation matrix. Eliminating $N_{\mathrm{orb}} - 1$ vectors gives diagonal factors P on each eliminated coordinate. The final block is $P(1 - q^{-1}\Pi_\sigma)$, with Π_σ the product round the cycle. Thus

$$|\det T_\sigma| = P^{kN_{\mathrm{orb}}} |\det(1 - q^{-1}\Pi_\sigma)|.$$

Each determinant is nonzero since permutation eigenvalues have absolute value one and $q > 1$. For a prescribed product π there are $(k!)^{N_{\mathrm{orb}}-1}$ tuples yielding it.

Each permutation of the embedding coordinates has Jacobian of absolute value one. Integrating the indicator identity gives

$$\frac{(k!)^{N_{\mathrm{orb}}}}{|\det T_1|} \operatorname{vol}(\mathcal{C}_0 \cap \mathcal{D}) = \operatorname{vol}(\mathcal{D}) \sum_{\sigma \in (S_k)^{N_{\mathrm{orb}}}} \frac{\prod_s \operatorname{sgn}(\sigma_s)}{|\det T_\sigma|}.$$

Grouping by the product permutation yields

$$\operatorname{vol}(\mathcal{C}_0 \cap \mathcal{D}) = \operatorname{vol}(\mathcal{D}) \frac{(1 - q^{-1})^k}{k!} \sum_{\pi \in S_k} \frac{\operatorname{sgn} \pi}{|\det(1 - q^{-1}\Pi_\pi)|}.$$

The same determinant calculation holds for each choice of the embedding signs. We can therefore evaluate the factor, which depends only on q, k, with one positive embedding, $P = q$, and the simplex $\sum_b \beta_b \le 1$, $\beta_b \ge 0$, of positive volume. In this case the step map is scalar multiplication by $q - 1$, and the cone is the cone of $\beta_b \ge q\beta_{b+1}$, $\beta_{k-1} \ge 0$. Put $d_b = 1 + q + \cdots + q^b$ and $\beta_k = 0$. The triangular map $\beta_b \mapsto d_b(\beta_b - q\beta_{b+1})$ maps this cone to the nonnegative orthant, preserves the sum of coordinates and has determinant $\prod_b d_b$. This proves the claimed factor for every N_{orb}.

Here is the consequence of the corresponding lattice-point bound. Section 21.3 shows that the bounded-defect space S has dimension at most the number F_L of flags of the two-block Levi times $\operatorname{vol}(\mathcal{C}_0 \cap \mathcal{D})$, with $1 \le D_s \le a + E$ and $E = 4tP^{2r}$. The parahoric index times F_L is the number of all flags of $\mathbb{F}_q^k$, namely $\prod_{b=1}^{k}(1 + q + \cdots + q^{b-1})$. A complete flag is uniquely its $(k - r)$-plane with the flags on that plane and on the quotient. Thus the volume factor cancels this index and

$$[\mathrm{GL}_k(W) : \mathcal{P}] \dim S \le \operatorname{vol}(\mathcal{D}).$$

The nonnegative vectors in one embedding with degree in $[D_s - 1, D_s]$ have volume $(D_s^k - (D_s - 1)^k)/k! \le D_s^{k-1}/(k-1)!$. If $8N_{\mathrm{orb}}ktP^{2r} \le \theta a$, then $N_{\mathrm{orb}}(k-1)E/a \le \theta/2$. Therefore

$$\operatorname{vol}(\mathcal{D}) \le \exp\!\left(\frac{N_{\mathrm{orb}}(k-1)E}{a}\right)\left(\frac{a^{k-1}}{(k-1)!}\right)^{N_{\mathrm{orb}}} \le (1+\theta)\binom{a+k-1}{k-1}^{N_{\mathrm{orb}}}.$$

The last step uses $0 < \theta \le 1$ and $\exp(\theta/2) \le 1 + \theta$.

21.5 An integral average in one matrix variable

In this subsection, we construct one integral functional on polynomials in a generic matrix. The invariance and vanishing properties are chosen for the Levi in Section 21, while the congruence to evaluation at the identity will control the pairing in Section 21.6.

Let W be the local ring of the distinguished prime, with residue field of order $q = p^{eN_{\mathrm{orb}}}$, and fix one embedding into the coefficient ring $\mathcal{O}$. Let $L_W \subseteq \mathrm{GL}_k(W)$ be the two-block Levi. For integral weights ω_i on the indices, call a monomial X^α in the k^2 matrix entries bad if its row weight $\sum_{i,j} \omega_i \alpha_{ij}$ exceeds a given threshold t_0. Suppose $D + k \le q$ and $D < q$. There is an integral functional I on polynomials in X which is invariant under conjugation by L_W, kills bad monomials, and satisfies

$$I(F) \equiv p^{k^2 eN_{\mathrm{orb}}} F(1) \pmod{\mathfrak{b}}$$

whenever $\deg F \le D$ and every Levi conjugate of F has its bad coefficients in the ideal $\mathfrak{b}$.

Here is the construction. Work in degree $d \le D$, on $U_d = (\mathcal{O}^k)^{\otimes d}$. Let $\mathcal{S}_d$ be the Schur algebra with basis the occurrence matrices of pairs of tensor indices with the same sequence of block parameters. For such an occurrence matrix α, the corresponding basis element is the coefficient of X^α in $(X_{\mathrm{block}}^{\mathsf{T}})^{\otimes d}$, where off-block entries of X_{block} are set to zero. The basis elements have disjoint supports in the matrix entries. For independent block matrices X, Y, put $A(X) = (X^{\mathsf{T}})^{\otimes d}$. Comparing coefficients of $X^\alpha Y^\beta$ in $A(X)A(Y) = A(YX)$ expresses each product of two occurrence operators as an integral combination of these operators. For an exponent involving an off-block variable the occurrence operator is zero.

The tensors of matrices in L_W span $\mathcal{S}_d$ over $\mathcal{O}$. To verify this, first work over W. If a homogeneous polynomial of degree d over $\mathbb{F}_q$ in the block entries vanishes on all invertible block matrices, its product with the block determinant vanishes at every point of the affine space. This product is homogeneous of degree $d+k \le q$. A homogeneous polynomial of degree at most q that vanishes on all of this affine space is zero.

Indeed reduction by the relations $X_i^q = X_i$ gives a polynomial of degree less than q in each variable, and such polynomials are tested injectively by evaluation on the grid. In degree q only the pure powers X_i^q have to be changed, and their reductions of degree one cannot cancel the other homogeneous terms. The block determinant is a nonzero polynomial. Thus the evaluations at invertible block matrices span the dual of the space of degree-d polynomials over $\mathbb{F}_q$. Lift the matrices to L_W, apply Nakayama to the finite free coefficient space over W, and extend along the embedding.

For $|\alpha| = d$, put $u_\alpha = p^{k^2 eN_{\mathrm{orb}}} / \binom{d}{\alpha}$. These weights are integral. By the factorial valuation formula each of the eN_{orb} relevant floors contributes less than k^2 to $v_p\binom{d}{\alpha}$. Use u_α to pair the coefficients of polynomials in the generic matrix variables. Let $F^\sharp \in \mathcal{S}_d$ be the operator whose coefficient at the occurrence matrix α is $u_\alpha [X^\alpha] F$. For a matrix A set $\ell_A(X) = \sum_{i,j} A_{ij} X_{ij}$, and let A_{block} be its block-diagonal part. Writing E_α for the occurrence operators and using $u_\alpha \binom{d}{\alpha} = p^{k^2 eN_{\mathrm{orb}}}$ gives

$$(\ell_A^d)^\sharp = p^{k^2 eN_{\mathrm{orb}}} \sum_\alpha A^\alpha E_\alpha = p^{k^2 eN_{\mathrm{orb}}} (A_{\mathrm{block}}^{\mathsf{T}})^{\otimes d}.$$

Since $g \in L_W$ is block diagonal, $\ell_A(g^{-1} X g) = \ell_{g^{-\mathsf{T}} A g^{\mathsf{T}}}(X)$, and taking the block-diagonal part commutes with this change of A. Thus the identity

$$\big(F(g^{-1} X g)\big)^\sharp = g^{\otimes d} F^\sharp (g^{-1})^{\otimes d}$$

holds on the d-th powers of linear forms. These span in characteristic zero by polarization. Both sides are integral on integral polynomials, so equality over the fraction field proves the stated integral identity. For a tensor index $j = (j_1, \dots, j_d)$ let its weight be $\sum_l \omega_{j_l}$. Let U_+ be

the submodule generated under $\mathcal{S}_d$ by all tensors of weight $> t_0$. It is finitely generated and torsion-free, hence free over the valuation ring. It is invariant under both actions in the last conjugation identity.

Define $I_d(F)$ to be the trace of $F^\sharp$ on U_d minus its trace on U_+. This is integral and conjugation invariant. If $F = X^\alpha$ is bad, its operator is zero on tensor indices of weight at most t_0, because the first indices of X^α are the column indices of $F^\sharp$. On the higher-weight basis tensors its image lies in U_+, since $F^\sharp \in \mathcal{S}_d$ preserves U_+. Thus the two traces agree by the cyclic trace identity for a map factoring through U_+. Thus I_d kills bad monomials.

Suppose now all the bad coefficients of every conjugate of F are in $\mathfrak{b}$. For a high-weight basis tensor v, write E_α for the occurrence operator of α. Then

$$F^\sharp v = \sum_{\alpha:\, E_\alpha v \neq 0} [X^\alpha]F \, u_\alpha E_\alpha v \in \mathfrak{b}U_+.$$

Indeed the input indices in each nonzero term have the weight of v, hence α is bad. Also $E_\alpha \in \mathcal{S}_d$, so $E_\alpha v \in U_+$ before reduction. For a translate $g^{\otimes d}v$ the same assertion follows by the conjugation identity. Such translates generate U_+ by the integral Schur spanning just proved. Hence $F^\sharp U_+ \subseteq \mathfrak{b}U_+$, so its trace there belongs to $\mathfrak{b}$. Finally the trace on U_d of $(X^\alpha)^\sharp$ is zero unless α is diagonal. For a diagonal α there are $\binom{d}{\alpha}$ identical diagonal coefficients, so that trace is $p^{k^2 e N_{\mathrm{orb}}}$. Thus the trace on U_d is $p^{k^2 e N_{\mathrm{orb}}}$ times evaluation at 1 in degree d. Sum I_d on the homogeneous components for $0 \le d \le D$ and set the functional equal to zero in higher degrees. Conjugation preserves degree.

Conjugation invariance also kills every monomial with different row and column weights. Indeed conjugation by the diagonal matrix with entries $(1+p)^{\omega_i}$ scales that monomial by a nontrivial integer power of $1+p$ when the weights differ. The coefficient ring has characteristic zero and no zero divisors.

21.6 The simultaneous average on the coefficient space

In this subsection, we tensor the functionals of Section 21.5 across the embeddings in the local representation. The estimates of Sections 21.2 and 21.3 yield both a pairing congruence and a dimension bound for its range after reduction.

Keep $N_{\mathrm{orb}} = 2h$, $P = p^e$, $q = P^{N_{\mathrm{orb}}}$, and r as above. Put $c_0 = k^2 e N_{\mathrm{orb}}$, $h_0 = N_{\mathrm{orb}} c_0$, $c_1 = 5h_0$, and $t = 3kc_1$. Write $\rho_b = 1_{b=r} - 1_{b=r-1}$. For a fixed copy i, give a variable in row b and column c of the generic matrix in copy j weight

$$\omega_i(j,b,c) = P^{-h} \begin{cases} \rho_b, & j = i, \\ q^{-1} P^{(j-i) \bmod N_{\mathrm{orb}}} (\rho_b - \rho_c), & j \neq i. \end{cases}$$

Denote the weight of a monomial in all copies by $\omega_i(\nu)$. Write def_i for the defect at an embedding at the boundary root, in the convention of the rank-two calculation.

Take one generic k-square matrix X_j for each copy. At embedding s use the product of the entrywise P^d-th powers of X_{s-d}, in decreasing order of d, with $0 \le d < h$ for a shallow row and $0 \le d \le h$ for a deep row. In odd embeddings take the cofactor matrix $\mathrm{adj}(M)^{\mathsf{T}}$ of the corresponding product M before extracting the row. Assemble these rows into a matrix at each embedding. Let $\mathcal{B}$ be the tensor product of the degree-a symmetric powers of these matrices.

Its evaluation at $X_j = 1$ for all j is 1. For a change of basis g, replace the factor in embedding s at position d by

$$g_s\big(g_{s-d}^{-1} X_{s-d} g_{s-d}\big)^{[P^d]} g_s^{-1}, \qquad g_s = \iota_s(g).$$

Brackets denote entrywise power. Call the resulting substitution $\mathcal{B}_g$. Modulo p these conjugations cancel, because the P^d-power map is a ring map there and takes g_{s-d} to g_s. The difference from the original factor has all coefficients divisible by $p = w^2$.

We give the weight estimate on $\mathcal{B}_g$. Fix $|\epsilon| \le 1/2$ and give the generic variables weights $\epsilon\omega_i$, adding the w-order to the weight of each monomial coefficient. Every monomial of any single entry of the factor at d has degree P^d and weight between -1 and 1, since $|\epsilon\omega_i(j,b,c)| \le P^{-h}$ and $d \le h$. Adding w-order two therefore makes every error term weigh at least as much as the original entry monomial. In particular each entry has weight at least $\epsilon P^d \omega_i(s-d,b,c)$.

For the matrix product put

$$A_{s,d}(b) = \frac{\epsilon\rho_b}{P^h} \frac{P^{d+(s-d-i) \bmod N_{\mathrm{orb}}}}{q}, \qquad u = \frac{1}{2P^h}.$$

At $d = 0$, $|A_{s,0}(b)| \le u$. For each $d \le h$ one has

$$\Big|A_{s,d+1}(b) - A_{s,d}(c) - \epsilon P^d \omega_i(s-d,b,c)\Big| \le u.$$

Write $[v]$ for the residue of v in $\{0,\dots,N_{\mathrm{orb}}-1\}$ and put $j = [s-d-i]$. For $j > 0$ one has $[j-1] = j-1$, whereas for $j = 0$ one has $P^{1+[j-1]} = q$. Substitution gives

$$A_{s,d+1}(b) - A_{s,d}(c) - \epsilon P^d \omega_i(s-d,b,c) = -\mathbf{1}_{j=0}\, \epsilon\rho_c P^{d-h}/q.$$

Its absolute value is at most u, including at $d = h$. Along a path in a product of $m \le h+1$ factors, the potentials telescope, so all entries in row b weigh at least $A_{s,m}(b) - (h+2)u$. In an odd embedding a cofactor omits its own row and takes one entry in each other row. The potentials sum to zero over the rows. Thus in both parities the row bound is $\varepsilon_s A_{s,m}(b) - k(h+2)u$, with $\varepsilon_s = 1$ for even embeddings and -1 otherwise.

For $m = h+1$ on deep rows and h on shallow rows, the term $\varepsilon_s A_{s,m}(b)$ is exactly $\epsilon(1-1/q)$ times the coefficient of the variable (b,s) in the defect at $i+h$. For shallow rows the power simplifies to $P^{(s-i-h) \bmod N_{\mathrm{orb}}}$ after removing P^h. For deep rows it is the same power except at $s = i+h$, when it is q, giving the additional deep term of the defect. The signs are those of the two root variables, and the other potentials vanish. Taking the symmetric powers and then the tensor product adds these weights according to the row exponents β. Hence, for every monomial ν in an entry of row β of $\mathcal{B}_g$, that coefficient is divisible by w to the nonnegative ceiling of

$$\epsilon\big((1-1/q)\,\mathrm{def}_{i+h}(\beta) - \omega_i(\nu)\big) - N_{\mathrm{orb}} ak(h+2)u.$$

The row estimate uses exactly a factors in each embedding, one row factor for each variable counted with its exponent.

The size hypotheses give $N_{\mathrm{orb}} ak(h+2) + a + k < P^h$. Indeed $P^h \ge a^2$, $h+2 \le N_{\mathrm{orb}} k$, and $a \ge (N_{\mathrm{orb}}k)^5 \ge (N_{\mathrm{orb}}k)^2 + 2$, with $k < a$. Thus the loss $N_{\mathrm{orb}} ak(h+2)u$ is at most $1/2$. The sum of the cyclic geometric weights of the defect is $1/(P-1)$, and each embedding has degree a, so $|\,\mathrm{def}_{i+h}(\beta)|/q \le 2a/q \le 1$. Taking $\epsilon = 1/2, -1/2$ shows that if the coefficient survives modulo w^{c_1}, then

$$|\,\mathrm{def}_{i+h}(\beta) - \omega_i(\nu)| \le 2c_1 + 2 \le t.$$

The degree in each generic copy is at most $D = N_{\mathrm{orb}} ak(h+2)P^h$. This follows by adding the degrees of the factors in each product, using at most k factors of product entries for a cofactor and a row factors in each embedding. The same bound holds in any basis. Our strict size inequality gives $D + k < q$, so the one-copy average applies. At each copy j use its embedding and row weights ρ_b, with bad threshold $2tP^h$. Let I_j be the resulting invariant functional, and let I_j^0 be p^{c_0} times evaluation at the identity. Both are invariant under the Levi.

Apply some such invariant functionals in all copies but j, leaving a polynomial in X_j. If a coefficient of it survives modulo w^{c_1}, some original term survives and its other-copy monomials are not killed by the chosen functionals. Each has equal row and column weights by invariance under diagonal conjugation. Consequently for the surviving term ω_j is just its row weight in copy j, divided by P^h. The last weight bound has two consequences. A defect above $3t$ at $j+h$ forces the row weight in this copy above $2tP^h$, and row weight above $2tP^h$ forces defect above t.

For g in the Levi, write C_g for the substitution $X_s \mapsto g_s^{-1} X_s g_s$ in every copy, and put $H_s = g_s$ for even s, $H_s = g_s^{-\mathsf{T}}$ for odd s. In each product defining $\mathcal{B}_g$ the adjacent factors $g_s^{-1} g_s$ cancel. At an odd embedding we use the polynomial identity

$$\operatorname{adj}(g_s A g_s^{-1})^{\mathsf{T}} = g_s^{-\mathsf{T}} \operatorname{adj}(A)^{\mathsf{T}} g_s^{\mathsf{T}}.$$

The two products of lengths $h, h+1$ contribute to different depth blocks. Projection onto either block commutes with H_s. Taking the symmetric powers and their tensor product therefore gives the identity over the integral polynomial ring

$$C_g \mathcal{B} = \rho_{\mathrm{let}}(g)^{-1} \mathcal{B}_g \rho_{\mathrm{let}}(g).$$

Fix j and let l_s be any of the invariant functionals above for $s \neq j$. For an integral coefficient vector v, put

$$F_g(X_j) = \left(\bigotimes_{s \neq j} l_s \right) (\mathcal{B}_g)\, \rho_{\mathrm{let}}(g) v.$$

Applying these functionals to the polynomial identity leaves

$$F_1(g_j^{-1} X_j g_j) = \rho_{\mathrm{let}}(g)^{-1} F_g(X_j).$$

Write $\mathcal{I} = \bigotimes_s I_s$ and $V = \mathcal{I}(\mathcal{B})$, applying the functional entrywise. Invariance in every copy and the integral basis identity give, for every $g \in L_W$,

$$\rho_{\mathrm{let}}(g) V v = \mathcal{I}(\mathcal{B}_g) \rho_{\mathrm{let}}(g) v.$$

For a row of defect $> 3t$ at i, contract first all copies except $j = i - h$. Modulo w^{c_1} every remaining monomial in that row is bad in copy j by the first weight implication. The functional I_j kills those monomials, and its integrality keeps all other coefficients in $w^{c_1}\mathcal{O}$. Thus the row of every Levi translate of Vv vanishes modulo w^{c_1}.

For the pairing, replace I_j^0 by I_j one copy at a time. Fix the other copies and a saturated halfway vector v'. Pair the uncontracted polynomial with v' using the diagonal weights. Its degree is at most D. Write this polynomial as $\Psi(X_j) = \sum_\beta b_\beta v'_\beta F_{1,\beta}(X_j)$. The adjunction of $\rho_{\mathrm{let}}(g^{-1})$ with $\rho_{\mathrm{let}}(g^{-\mathsf{T}})$ for the diagonal pairing gives

$$[X_j^\alpha] \Psi(g_j^{-1} X_j g_j) = \sum_\beta [X_j^\alpha] F_{g,\beta}(X_j)\, b_\beta \big(\rho_{\mathrm{let}}(g^{-\mathsf{T}}) v'\big)_\beta.$$

For a bad α, any first factor surviving modulo w^{c_1} forces the defect of β to exceed $t = 3kc_1$. Since $g^{-\mathsf{T}}$ belongs to the same Levi, the rank-two estimate of Section 21.2 makes the second factor divisible by w^{c_1}. Thus all the bad coefficients of every conjugate of the pairing polynomial are in (w^{c_1}). The congruence property of I_j shows that replacing I_j^0 does not change the pairing modulo w^{c_1}. With all identity evaluations the contraction is $p^{N_{\mathrm{orb}}c_0}1 = p^{h_0}1$. This proves the desired pairing congruence for V.

Finally let C consist of the integral coefficient vectors whose L_W-translates have coefficients of defect $> 3t$ divisible by w^{c_1}. The rank-two estimate puts the saturated halfway restrictions in C, and the full contraction takes values there. Suppose an integral v satisfies $w^{c_1-1}v \in C$. Lift any residue Levi element $\bar{g}$ to $g \in L_W$ by lifting its two invertible blocks with unit determinants. For each coordinate α of defect $> 3t$ at some embedding,

$$w^{c_1-1}(\rho_{\mathrm{let}}(g)v)_\alpha \in w^{c_1}\mathcal{O} \quad \Longrightarrow \quad (\rho_{\mathrm{let}}(g)v)_\alpha \in w\mathcal{O}.$$

Reduction commutes with the symmetric powers and inverse transposes of the Levi blocks. Since $v_p(w) > 0$, all these coordinates of $\bar{g}\bar{v}$ vanish in the residue field, so $\bar{v} \in S$. The flag count uses the inequalities: $t \geq 2(k+1)$ and

$$8N_{\mathrm{orb}}ktP^{2r} = 120N_{\mathrm{orb}}^2k^4eN_{\mathrm{orb}}\,P^{2r} \leq (N_{\mathrm{orb}}k\log q)^5P^{3r} \leq \theta a.$$

For the first inequality use $eN_{\mathrm{orb}} \leq \log q$, $N_{\mathrm{orb}}, k \geq 2$, and $\log q \geq 40\log 2 > 27$, which follows from $q \geq a^4$ and $a \geq (N_{\mathrm{orb}}k)^5$. The final bound follows from the large-degree hypothesis and $0 < \theta \leq 1$. The flag estimate therefore gives $s_{\mathcal{P}}\dim S \leq (1+\theta)d_{\mathrm{let}}$. This supplies both the congruence and the residue dimension used in the edge Smith argument.

21.7 The colength of the source Hecke operator

In this subsection, we combine the two restriction estimates of Section 21.1 with the congruence of Section 21.6. Smith normal form gives an explicit bound on the length of the quotient by the image of the Hecke operator and the prescribed power of p.

The depth count of Section 21.1 bounds the sum of the dimensions of the residue spaces above all positive depths by $C_{\mathrm{div}}d_{\mathrm{let}}$. Use C, V, S from Section 21.6. In particular $s_{\mathcal{P}}\dim S \leq (1+\theta)d_{\mathrm{let}}$, C contains the saturated halfway restrictions, and $w^{c_1-1}u \in C$ implies $\bar{u} \in S$. The pairing of V with a saturated halfway restriction agrees with that of $p^{h_0}1$ modulo w^{c_1}.

Here is the linear algebra that uses those estimates. On an edge the two restrictions are the origin value acted on by $W_0\rho_{\mathrm{let}}(g_e^{\mathsf{T}})$ and the terminus value acted on by $W_0\rho_{\mathrm{let}}(\beta_b^{-1})$, respectively. At either endpoint every class is incident to an edge, by the forward coset description and the dual-edge bijection. On a fixed edge the restriction at that class is invertible over the coefficient field, since W_0 and each change of coordinates are invertible there. Both restrictions therefore have column rank d_X. Factor them as $R_1 = Q_1D_1$, $R_2 = Q_2D_2$, with D_i square and the saturated bases Q_i admitting integral left inverses L_i. Write $\mathcal{E}$ for the integral module of edge values. Since $L_iQ_i = 1$, an element Q_ic of the fraction-field image is in $\mathcal{E}$ exactly when $c = L_iQ_ic$ is integral. Thus $Q_i\mathcal{O}^{d_X}$ is the saturation of $R_i\mathcal{O}^{d_X}$ in $\mathcal{E}$. Put $g_{1,e} = g_e^{\mathsf{T}}$, $g_{2,e} = \beta_b^{-1}$. For either endpoint, a vector $Q_ic = R_iD_i^{-1}c$ of the saturation has, at the edge e, active component $W_0\rho_{\mathrm{let}}(g_{i,e})F$, where F is the corresponding component of $D_i^{-1}c$ at that class. Fix the same coordinate in the remaining tensor factors. Integrality of the saturated column in $\mathcal{E}$ gives $W_0\rho_{\mathrm{let}}(g_{i,e'})F \in \mathcal{O}^{d_{\mathrm{let}}}$ for every edge e' incident to this class, with this same F. As e' runs

through those edges, the $g_{i,e'}$ represent $\mathcal{P}\backslash \mathrm{GL}_k(W)$, since the cosets $g_{i,e'}^{-1}\mathcal{P}$ exhaust $\mathrm{GL}_k(W)/\mathcal{P}$. For $g \in \mathrm{GL}_k(W)$, write $gg_{i,e} = ug_{i,e'}$ with $u \in \mathcal{P}$. Then

$$W_0\rho_{\mathrm{let}}(gg_{i,e})F = \big(W_0\rho_{\mathrm{let}}(u)W_0^{-1}\big)W_0\rho_{\mathrm{let}}(g_{i,e'})F \in \mathcal{O}^{d_{\mathrm{let}}}.$$

At the fixed edge e, multiply $\rho_{\mathrm{let}}(g_{i,e})F$ by a sufficiently large power of w to clear its finitely many denominators. The displayed identity already gives the integral tested values of the unscaled vector in every direction g. Thus the integral edge values of each Q_i are saturated halfway restrictions for the common local test in Section 21.6.

Let $B = \mathrm{diag}(b_\mu)$ at each edge and B_X be the same diagonal matrix repeated at the classes. Write $U = \bigoplus_e U_e$ for the transitions on the remaining tensor factors, acting as the identity on the active factors, and set $A_0 = Q_2^{\mathsf{T}}UB$. On the tensor product of the active factors the identity

$$(W_0\rho_{\mathrm{let}}(\beta_b^{-1}))^{\mathsf{T}}BW_0\rho_{\mathrm{let}}(g_e^{\mathsf{T}}) = \big(\rho_{\mathrm{let}}(g_e)W_0^2\rho_{\mathrm{let}}(\beta_b^{-1})\big)^{\mathsf{T}}B$$

follows from $B\rho_{\mathrm{let}}(g_e^{\mathsf{T}}) = \rho_{\mathrm{let}}(g_e)^{\mathsf{T}}B$. Here $W_0^2 = \mathrm{diag}(p^{\mathrm{dep}(\mu)})$. Tensoring with U_e and summing incoming edges gives, in the original class coordinates,

$$D_2^{\mathsf{T}}A_0Q_1D_1 = R_2^{\mathsf{T}}UBR_1 = Y_\lambda B_X.$$

Its image is therefore contained in $Y_\lambda M$ after scalar extension. To bound $v(\det D_i)$, write t_j for the Smith factors of one restriction, and v_j for the corresponding input basis. All parahoric cosets occur at that endpoint by the two edge descriptions above. Thus t_j divides every coefficient of $W_0\rho_{\mathrm{let}}(g)v_j$ in each active factor at every class, for every $g \in \mathrm{GL}_k(W)$, by the parahoric invariance of the weighted lattice proved above. The remaining tensor factors change by integral invertible matrices.

If $2v(t_j) > \delta$ for an integer $\delta \geq 0$, every coefficient of depth at most δ in the unweighted translates of v_j vanishes modulo the maximal ideal. Thus $\bar{v}_j$ belongs to the direct sum of d_X/d_{let} copies of $D_{\delta+1}$. These reduced basis columns are independent. Since $v(p) = 1$ and $w^2 = p$, we obtain

$$\begin{aligned} 2v(\det D_i) &= \sum_j 2v(t_j) \leq \sum_{\delta\geq 0} \#\{j : 2v(t_j) > \delta\} \\ &\leq \frac{d_X}{d_{\mathrm{let}}} \sum_{\delta\geq 0} \dim D_{\delta+1} \leq C_{\mathrm{div}}d_X. \end{aligned}$$

Applying this at both endpoints gives $v(\det D_1) + v(\det D_2) \leq C_{\mathrm{div}}d_X$.

Let $B' = \mathrm{diag}(b'_\mu)$ and let $\mathcal{V}$ act by V in every active edge factor. Since $BB' = p^{h_0}1$ and $L_2Q_2 = 1$, we have $A_0B'U^{-1}L_2^{\mathsf{T}} = p^{h_0}1$. The matrix $B_0 = \mathcal{V}B'U^{-1}L_2^{\mathsf{T}}$ has edge values in C. The matrix U acts on the remaining tensor factors and commutes with $B, \mathcal{V}, B'$. The pairing congruence, tested against the active edge values of the columns of Q_2, gives

$$\begin{aligned} A_0B_0 &= Q_2^{\mathsf{T}}B\mathcal{V}B'L_2^{\mathsf{T}} \\ &\equiv p^{h_0}Q_2^{\mathsf{T}}BB'L_2^{\mathsf{T}} = p^{2h_0}1 \pmod{w^{c_1}}. \end{aligned}$$

The values of Q_1 also belong to C.

Take Smith form for the image of B_0 in the complement of Q_1 given by L_1. The projection $1 - Q_1L_1$ preserves C, since integral linear combinations of the columns of Q_1 belong to C. Keep those nonzero divisors t_j dividing w^{c_1-1}, and write u_j for their primitive columns. Then

$w^{c_1-1}u_j$ lies in C on each edge, so each u_j reduces into S at every edge. The columns of Q_1 also reduce into S. These two lists together are independent modulo the maximal ideal, since the primitive columns are basis vectors in the complement. There are $s_{\mathcal{P}}d_X/d_{\mathrm{let}}$ copies of the residue space at the edges, including the multiplicities from the remaining tensor factors. Thus

$$d_X + \#\{u_j\} \le \frac{s_{\mathcal{P}}d_X}{d_{\mathrm{let}}} \dim S \le (1+\theta)d_X.$$

The other Smith terms are all divisible by w^{c_1-1}.

Let G be the sum of the image of A_0Q_1 and the span of the kept A_0u_j. The near-inverse congruence gives $p^{2h_0}\mathcal{O}^{d_X} \subset G + w^{c_1-1}\mathcal{O}^{d_X}$. Since $c_1 = 5h_0$ and $h_0 \ge 2$, $w^{c_1-1} = p^{2h_0}w^{h_0-1}$. The image of $p^{2h_0}\mathcal{O}^{d_X}$ in $\mathcal{O}^{d_X}/G$ is finitely generated and equals its multiple by w^{h_0-1}. Nakayama's lemma therefore gives $p^{2h_0}\mathcal{O}^{d_X} \subseteq G$. For finitely generated lattices $L' \subseteq L''$ of the same rank over the coefficient valuation ring, successive elimination at an entry of least valuation diagonalizes the inclusion. Thus $\mathrm{length}_p(L''/L') = \sum_i v(t_i)$, where t_i are the elementary divisors and $v(p) = 1$. This normalized length is additive along inclusions by multiplicativity of the determinants of the basis changes. Write $L = \mathcal{O}^{d_X}$, $z = p^{j_p}$, and $\Phi = A_0Q_1$. The truncated sums of images used here are finitely generated and contain zL.

The containment $p^{2h_0}L \subset G$ bounds the normalized length of $L/(G+zL)$ by $2h_0d_X$. The quotient between $\Phi L + zL$ and $G + zL$ is generated by the kept columns, each killed by z, so its normalized length is at most $j_p\#\{u_j\}$. Replacing ΦL by $\Phi D_1 L$ adds at most $v(\det D_1)$, since the intervening quotient is a quotient of L/D_1L.

Put $T = \Phi D_1 L + zL$. Since $D_2^{\mathsf{T}}(zL) \subseteq zL$, $D_2^{\mathsf{T}}T \subseteq D_2^{\mathsf{T}}\Phi D_1 L + zL$. The injection D_2^{T} identifies L/T with $D_2^{\mathsf{T}}L/D_2^{\mathsf{T}}T$. The inclusion $zD_2^{\mathsf{T}}L \subseteq D_2^{\mathsf{T}}T$ gives the full rank needed for the normalized lengths. Additivity along $D_2^{\mathsf{T}}T \subseteq D_2^{\mathsf{T}}L \subseteq L$ gives

$$\mathrm{length}_p\big(L/(D_2^{\mathsf{T}}\Phi D_1 L + zL)\big) \le v(\det D_2) + \mathrm{length}_p(L/T).$$

The sum of these endpoint valuations is at most $C_{\mathrm{div}}d_X$ and $h_0 \le C_{\mathrm{div}}$. This proves

$$\mathrm{length}_p\big(M/(Y_\lambda M + p^{j_p}M)\big) \le (3C_{\mathrm{div}} + \theta j_p)d_X.$$

In a class basis of M, let $H = [Y_\lambda \mid p^{j_p}I_{d_X}]$. Its maximal minors generate the zeroth Fitting ideal of the quotient. Smith form therefore gives its normalized length, with $v_p(0) = +\infty$, as

$$\min_{\#J = d_X} v_p(\det H_J) = \sum_{i=1}^{d_X} \min\{v_p(s_i(Y_\lambda)), j_p\},$$

where the s_i include the zero Smith factors. Indeed, if $PY_\lambda Q = \mathrm{diag}(s_i)$ is a Smith form, left multiplication by P and right multiplication by $\mathrm{diag}(Q, P^{-1})$ preserve the ideal of maximal minors and keep the second block equal to $p^{j_p}I$. In these coordinates each nonzero maximal minor selects one entry from each row. Selecting an entry of lesser valuation in every row realizes the sum above, including rows with $s_i = 0$. In the original class basis the minimum ranges over all column sets and can depend on the coefficient place. The valuations of its minors are unchanged after any extension of the coefficient field with $v_p(p) = 1$. Hence the bound applies to the original local module.

Finally $p^r \ge m$, $r \ge 1$, and $P \ge p$ give $P^{2r} \ge mp$. The size bound $r^2 m \le k$ gives $r^2/(kP) \le 1/(mp)$. Hence

$$3C_{\mathrm{div}} \le \frac{6N_{\mathrm{orb}}a}{mp} = \frac{12A_\lambda}{mp}.$$

Proposition 21.1 (Colength of the source Hecke operator)**.** *For each $\lambda \in \Lambda$, Y_λ preserves the original source lattice M. With $j_p = \lceil 2mfa/p \rceil$, its normalized colength modulo p^{j_p} satisfies*

$$\operatorname{length}_p(M/(Y_\lambda M + p^{j_p} M)) \le d_X \left(\frac{12 A_\lambda}{mp} + \frac{j_p}{m^2 f} \right).$$

22 Norm and index estimates for the translated images

In this section, we combine the derivative estimates of Sections 20 and 21 with the calculations at the level primes. The common scalar of Proposition 7.10 gives the upper estimates on the family of Section 16.1.

22.1 Derivatives at the ramified prime

In this subsection, we specialize the integral CM calculation of Section 20.1 to $p = d$. The ramified uniformizer changes the exponent in the estimate of Section 20.2, including the contribution at exponent zero.

Put $p = d$, $\pi = \sqrt{-d}$, and normalize $v(p) = 1$. Here $v(\pi) = 1/2$, $p < x_1$, and $p \nmid MNq_1$. Fix any allowed twist ξ with $p \nmid q_\xi$. By Section 19.3, the translations in the evaluated tensor can be supported on the primes dividing MNq_ξ. The signed action of an integral displacement fixes the local vacuum. Thus each arrow has the standard level-one datum at p. At the actual scale $\pi_M s_{\mathrm{jet}}$ the algebraic multiplier in the normalized Taylor variables is $\iota_p(\pi_M p_F)$. Section 20.1 gives $v_p(\iota_p(p_F)) = 0$, since $p \nmid q_1$ and α_{W} is integral invertible above p. Also $\pi_M \bar{\pi}_M = M$ with $p \nmid M$, so this multiplier is a unit at either coefficient embedding.

On the CM curve with its chosen half-period, put $g = -\log \theta$. Average $y(az)$ and $g(az)$ over $a^{p-1} = 1$, with weights a^{-1} and a^{-2}, respectively, and denote the results by T and $g^\flat$. At the ramified place these roots lie in $\mathbb{Q}_p$, so their CM conjugates are themselves. The integral CM pullbacks give $y(az) \in y\mathcal{O}[[y]]$ for each such a, with linear term ay. Thus $T = (p-1)^{-1} \sum_a a^{-1} y(az) = y + O(y^2)$ over $\mathcal{O}$. Since $a\bar{a} = a^2$, the corresponding pullback quotient is

$$Q_a(z) = \exp(a^2 g(z) - g(az)) \in 1 + y\mathcal{O}[[y]].$$

Substituting $y(az)$ for y in the quotient at a^{-1} gives

$$\prod_{a^{p-1}=1} Q_{a^{-1}}(az) = \exp((p-1)(g^\flat(z) - g(z))) \in 1 + y\mathcal{O}[[y]].$$

In each positive degree the equation for its constant-one $(p-1)$-st root has leading coefficient $p - 1 \in \mathcal{O}^\times$. It follows recursively that $\exp(g^\flat - g) \in \mathcal{O}[[y]] = \mathcal{O}[[T]]$. The torsion calculation in Section 20.1 bounds the order of the reduced parameter under $[\pi]$ by $\pi\bar{\pi} = p$. The linear coefficient vanishes on reduction and all allowed degrees are $1 \pmod{p-1}$. The reduced order is therefore p, giving

$$T(\pi z) = \pi T(z) + b_1 T(z)^p + \sum_{j \ge 2} b_j T(z)^{1 + j(p-1)}, \qquad b_1 \in \mathcal{O}^\times.$$

Choose $\varpi^{p-1} = -\pi/b_1$ in $\mathbb{C}_p$ and write $Y = T(\varpi z)/\varpi$. Thus $v(\varpi) = 1/(2(p-1))$. If the chosen root is multiplied by $\zeta \in \mu_{p-1}$, then $T(\zeta z) = \zeta T(z)$ and $g^\flat(\zeta z) = \zeta^2 g^\flat(z) = \zeta^{p+1} g^\flat(z)$.

Both Y and $g^\flat(\varpi z)/\varpi^{p+1}$ are therefore independent of the root. The equation $Y(\pi z)/\pi = Y - Y^p + \pi E(Y)$, with $E \in U^2\mathcal{O}[[U]]$, gives

$$Y \in \mathcal{O}[[z]], \qquad Y \equiv \sum_{j\geq 0} z^{p^j} \pmod{\pi}.$$

Indeed the recursion divides in degree $n > 1$ only by the unit $1 - \pi^{n-1}$, and the reduction is the unique zero-constant solution of $Y - Y^p = z$.

The pullback $\theta(\pi z)$ has degree $\pi\bar{\pi} = p$. The elliptic quotient by $\theta(z)^p$ is even and has its only possible pole at the theta zero, of order at most p. The integral interpolation of Section 20.1 expresses it as $P_0(x) + yP_1(x)$. Comparing $z, -z$ and using $y(z) - y(-z) = x'(z) \neq 0$ gives $P_1 = 0$. Since the pole of x has order two and p is odd, we obtain

$$\exp(pg - g(\pi z)) = R(x), \qquad R \in \mathcal{O}[X], \qquad R(0) = 1, \qquad 2\deg R \leq p - 1.$$

For $2 \leq n < p$, $\pi[z^n]g$ is integral. In degree two use the two CM pullbacks at $\beta, \bar{\beta}$ and subtract their quadratic coefficients. For $n > 2$ use $g'' = g''(0) + x$, since the factors $n(n-1)$ are units and $x \in \mathcal{O}[[T]]$ with $T = z + O(z^p)$. It follows that $pg - g(\pi z)$ belongs to $\pi\mathcal{O}[[z]]$ through degree $p - 1$, since $p = -\pi^2$.

Exponentiation preserves the congruence through those degrees because $k/2 - v(k!) \geq 1/2$ for $k \geq 1$. Triangular comparison using $x = z^2 + \cdots$ gives $R - 1 \in \pi\mathcal{O}[X]$. Taking the formal logarithm gives $pg - g(\pi z) \in \pi\mathcal{O}[[T]]$. The component of degree 2 modulo $p - 1$ thus satisfies

$$pg^\flat(z) - g^\flat(\pi z) = \pi(aT^2 + T^{p+1}H), \qquad a \in \mathcal{O}, \quad H \in \mathcal{O}[[T]].$$

With $c_\varpi = \varpi^{p+1}/p$ and $\Xi = -g^\flat(\varpi z)/c_\varpi$, the scalar $-\pi\varpi^2/(pc_\varpi)$ is a unit. The preceding identity gives, coefficientwise, $(1 - \pi^n/p)\Xi_n \equiv \lambda[z^n]Y^2 \pmod{\pi}$ for some $\lambda \in \mathcal{O}$.

Indeed $-\pi\varpi^2/(pc_\varpi) = -\pi/\varpi^{p-1} = b_1$; the higher terms contain an additional factor $\varpi^{p-1} \in \pi\mathcal{O}$. Since $p = -\pi^2$, the multiplier is 2 for $n = 2$ and $1 + \pi^{n-2} \equiv 1 \pmod{\pi}$ for $n > 2$. The constant and linear terms vanish. Hence

$$\Xi \in \mathcal{O}[[z]], \qquad \Xi \equiv \lambda(Y^2 - z^2/2) \pmod{\pi}.$$

In particular the coupling indices now include $e = 0$. The coefficients of $(z^{p^e}z)^{p^j}$ are 2λ for $e > 0$, λ for $e = 0, j > 0$, and $\lambda/2$ for $e = j = 0$.

Use the same resolution of the integral vacuum as for the exceptional jet. The basis entries are integral above p, since $p \nmid q_1$. On K, conjugation is identity modulo π; on the cyclotomic generator, the p^e-power is the corresponding Frobenius. Therefore

$$\overline{\tau(w)}^{p^e} \equiv \tau_e(w) \pmod{\pi}, \qquad \tau_e = \langle -p^e \rangle \tau.$$

In fact $p = d \nmid q_1$, so the completion of the cyclotomic factor at p is unramified, and adjoining $\pi^2 = -p$ gives ramification index two and uniformizer π. Thus equality of residues for these elements of E is congruence modulo $\iota_p(\pi)\mathcal{O}_{\mathbf{C}_p}$ after embedding, including at $e = 0$. Denominators prime to p preserve it, and either conjugate of π generates the same ideal. The basis identity turns the single-curve terms into $\Gamma_e = \sum_{\tau,b,a} z_{\tau,b,a}^{p^e} z_{\tau_e,b,a}$. The congruence error has zero constant term and coefficients divisible by πc_ϖ. Since $v(\pi c_\varpi) = 1/(p-1)$, Legendre's formula gives

$$v\big((\pi c_\varpi)^n/n!\big) = s_p(n)/(p-1) \geq 0,$$

where $s_p(n)$ is the sum of the base-p digits. Thus its formal exponential is integral. Scaling by the actual integral factor $\pi_M p_F$ preserves this form. We obtain the same matrix normal form as in the exceptional case, now for all $e \geq 0$ and with $v(\varpi) = 1/(2(p-1))$.

Since $\lambda \in \mathcal{O}$, a direction that kills the coupling modulo π also kills its integral multiple λ, including $\lambda = 0$. Use the null-direction proof with no extra exponents or discarded columns and with each congruence modulus p there replaced by π. The coefficient ring modulo π has characteristic p, so the same linear congruences, using p^e-th roots of π where needed, work also for $e = 0$. Each determinant contribution is multiplied by $v(\pi) = 1/2$. In detail, with $u_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ and c_τ, b_τ as in the column counts, write

$$E_{\tau,e} = \min(u_\tau/p^e, u_{\tau_e} + c_{\tau_e}) + c_\tau/p^e + \min(b_\tau/p^e, b_{\tau_e}).$$

The factorial bound gives the lower bound for every minor jet, and therefore every Schur entry,

$$\frac{B_j}{2(p-1)} - \frac{1}{2}\sum_{\tau,e\geq 0} E_{\tau,e} - f n_{j+1}^3 (1 + \log_p \max(B_j, 1)).$$

If $B_j = 0$, the constant term is integral.

Here $B_j = \sum_\tau (u_\tau + b_\tau)$ since no columns were erased. Subtracting $v(\varpi^{B_j}) = B_j/(2(p-1))$ from the factorial contribution $B_j/(p-1)$ leaves $\frac{1}{2}\sum_{e\geq 1} B_j/p^e$. For each $e \geq 1$,

$$(u_\tau + b_\tau)/p^e - E_{\tau,e} = (u_\tau/p^e - u_{\tau_e} - c_{\tau_e})_+ + (b_\tau/p^e - b_{\tau_e})_+ - c_\tau/p^e = D_{\tau,e}.$$

The $e = 0$ coupling gives solely $-\frac{1}{2}\sum_\tau E_{\tau,0}$, with partner $\langle -1\rangle\tau$. This is the required $d_{pj}(\varnothing)$, including the exponent-zero subtraction. The argument applies to every arrow with the same original tensor. Transport of bases and composition of the finite class sums preserve the lattice norm estimates. Since $p \neq N$, the final branching matrix is integral and its extra scalar is a unit. Thus the restricted chain has valuation at least $g_p - L_p$ at $p = d$, since (76) gives $\sum_{j<m} d_{pj}(\varnothing) \geq g_p - L_p$ when $C_p(\varnothing) = 0$, as proved for $p = d$ in Section 20.2.

22.2 Integrality at the two original level primes

In this subsection, we treat the two primes dividing M. The tensor in Section 19.3 can have denominators there, so we use the actual translation parameter in Section 20.1 to check the unit bound.

Let $\ell \mid M$, so $\ell = p_0$ or p_1. These primes split in K and are distinct from d, N. For a twist prime to ℓ, the admissible translations of the actual tensor satisfy $\bar{\pi}_M x$ integral above ℓ. This is the datum of standard level $\delta = \bar{\pi}_M$; the argument allows translations with denominators at those places. The genuine arrow scale in the notation of the ordinary jet bound is $c = \pi_M$, so for each homogeneous Schur entry of degree B_j the ordinary multiplier is

$$\left|\frac{c}{\bar{\delta} d}\right|_\ell^{B_j} = |d^{-1}|_\ell^{B_j} = 1.$$

For the source and target arguments separately, weak approximation in the Cayley chart provides rational isometries whose left translates of those arguments are integral at every place above ℓ.

Write g_1, h_1 for these integral translates. The action on displacements is $A(Z) = h_1 Z g_1^{-1}$, as in Section 20.1. The pullback on functions uses A^{-1}. Let L be the vacuum lattice and

$H = L \cap A^{-1}L$. At every split place w above ℓ, $AL_w = L_w$. Thus $H_w = L_w$ and $\ell \nmid [L : H]$. The vacuum expansion of Section 19.3 expresses the translate of $T(x)v_L$ as a sum of translates $T(A(x + y))v_L$, $y \in L/H$. Their coefficients are $\Delta_A/[L : H]$ times the character of the pair and a root of unity. Write $a_H = \det H_A$ with $H_A L = H$. Put $\vartheta = \upsilon_E|\cdot|_{\mathbb{A}_E}^{-1/2}$, where υ_E is the full splitting character. Lemma 17.1 gives $\Delta_A = \vartheta_f(a_H)$. The character ϑ is unramified at the finite places. At infinity, $\vartheta_\infty((z_v)_v) = \prod_v \overline{z_v}^{-1}$. Put $h_E = |\operatorname{Cl}(E)|$ and write $a_H^{h_E} = \alpha u$ with $\alpha \in E^\times$ and $u \in \widehat{\mathcal{O}}_E^\times$. Since ϑ is trivial on $E^\times$,

$$\Delta_A^{h_E} = \vartheta_f(a_H)^{h_E} = \prod_v \overline{\Phi_v(\alpha)}.$$

The generator α is a unit at every place above ℓ since a_H is. Thus $v_\kappa(\Delta_A) = 0$ at every $\kappa \mid \ell$. For the determinant t of either finite argument, Section 4.4 gives $t^{h_E} = \alpha_1 u_1$ in the norm-one torus by the surjection from $\operatorname{Cl}(E)$ onto its finite quotient. Here $\alpha_1 \in T_1(F_0)$ and $u_1 \in T_{1,f}^0$. The finite character on u_1 is a root of unity, and its value on α_1 is the inverse infinite weight monomial on a unit at every place above ℓ. Thus the full pair character is a ℓ-unit, even at a place of nontrivial tame character. At each $w \mid \ell$, E-linearity gives $\delta A(x+y) = A(\delta x+\delta y) \in L_w$. Thus the span of the allowed translated vacua over the coefficient valuation ring is preserved, including under the actions at the other finite places.

The ordinary jet estimate applies at the two integral translates. Rational equivariance transports it to the two original arguments in their respective Weyl lattices. The original principal levels give free class sums. In the normalized tensor, the zero coefficients are integral. In the case $a_{\mathrm{ss}} \neq 0$ the coefficients $\Psi_\xi(x) = B_*^{2m} b_{\xi,x}$ include the point multiplicities. Section 19.3 clears the two residue-test denominators and the neighbor denominator at each pair. Since $\ell \nmid q_\xi$, these coefficients are integral also at the two old primes.

The final branching has integral entries and unit scalar at $\ell \neq N$, since $\pi_N \bar{\pi}_N = N$. Consequently the full integral source has lattice norm at most one after restriction. The saving at $\ell \mid M$ is zero because its positive partner set is empty for a split prime. The distinct prime N uses the original preimage clearing and has saving zero from $N \geq x_1$.

22.3 Norm and index bounds for the Poincaré images

In this subsection, we combine the integral estimates with the complex norm bound of Section 7.4. The normalized tensor of Section 19.3 gives the upper realization of $L_\xi(c_{\mathrm{ss}}^{-1}\Pi)$ in (73). Here the finite tensor in (71) is the argument of $\mathcal{U}_\xi$. In Section 44.1 the symbol T_ξ denotes the weighted map L_ξ, with $\Theta_\xi = L_\xi|_V$. We apply the scalar of Proposition 7.10.

Lemma 22.1 (Class sum of the upper tensor)**.** *The contracted upper integral with the jointly invariant tensor of Section 19.3 is the sum over the product of the original upper class sets, with every class given mass one. The tensor can then be expanded at those representatives.*

Proof. The tensor is jointly invariant under the upper levels with the last action one. At an intermediate class variable the two adjacent kernels transform by inverse coefficient actions, and the contracted product with the invariant tensor is unchanged by the level. The source factor is already invariant in its original level. Thus the contracted integrand can be summed over the Cartesian product of the original upper classes.

Each quotient measure at a definite stage is multiplied by that stage's level index in the definition of the algebraic lift. Its mass on a class is the reciprocal of the class stabilizer order.

All the intermediate levels are free by reduction at an old odd distinguished place, so every class has mass one in this calculation. One can therefore expand the finite tensor after choosing these class representatives. Individual translated vacuum terms need not have the separate level invariance of the combined tensor. Their terms are evaluations at the same representatives in this one finite sum.

Apply (79) at these representatives. Each linked shift $t_j(y)$ is supported above N and $Nt_j(y)$ is integral there. The shifts x_j vanish above N, so their mutual Heisenberg phases are one. Collecting equal tuples preserves coefficient integrality off the twist; the translation denominators and supports away from N are unchanged. Linearity of the upper integral and terminal restriction gives this expansion of the upper operator on the full original source. Equation (73) identifies its value with $L_\xi(c_{\mathrm{ss}}^{-1}\Pi\phi)$ for $\phi \in S$. At $p = N$ the estimate on S uses Section 19.4.2. For the exceptional Hecke operators, the local Hecke operators act on the spherical place and commute with the original level actions on these data, so the modified tensor has the same joint invariance and the same class-sum formula. A one-arrow lattice norm bound in the transported coordinates therefore composes inside each summand in the class sum. The ultrametric inequality then preserves the resulting bound under summing the classes and the integral tensor coefficients. □

Proposition 22.2 (Norm and index on the original translates)**.** *Fix q_0, d, ϵ as in Section 4. For every sufficiently large integer x there is a spanning set $\mathcal{S}$ of everywhere integral forms in S such that, for every complex automorphism σ and $g \in G$,*

$$\|\sigma\Theta_1 P_g\phi\|^2 \le e^{A-m\mathcal{R}+L} \qquad (\phi \in \mathcal{S}).$$

For each $t \subseteq G$ and each independent tuple y of images $P_g\phi$ with $g \in t$ and $\phi \in S$ everywhere integral,

$$\mathrm{Ind}(y) \le 2|t| \dim(S)\, faB^3.$$

At each coefficient place over p, its clearing matrix works for every allowed twist of level prime to p.

Proof. Fix p and a coefficient place $\kappa = (p, \iota)$. Write S_ι for the ramified subspace in these coefficient coordinates and M_ι for the full integral source lattice. When $p \neq N$ and either p is split or $p \ge x_1$, the ordinary derivative estimate gives lattice norm at most one on $M_\iota \cap S_\iota$. At a nonexceptional inert prime $p < x_1$ or at $p = d$ use the same module, with the bounds in Sections 20.2 and 22.1, respectively. In the exceptional case use

$$J_\iota \cap S_\iota, \qquad J_\iota = Y_\Lambda^m M_\iota + p^{j_p} M_\iota,$$

as in Sections 20.3 and 21. For $\varphi = Y_\Lambda^m u + p^{j_p} v$ in the intersection, apply the upper tensor operator of Lemma 22.1 on the full integral source M_ι. Its values on $Y_\Lambda^m u$ and $p^{j_p} v$ have lattice norms at most $p^{2L_p - g_p}$ and $p^{L_p - g_p}$, respectively. Linearity and the ultrametric inequality bound its value on φ by $p^{2L_p - g_p}$, also when $p \mid N - 1$. It is this sum $\varphi \in S_\iota$ to which the measured seesaw identity (73) applies. At N, use the preimage under $U_0^{m/2}$ of the integral lattice at δ_0, intersected with the original integral ramified source. These modules are fixed after the embedding and before choosing any twist or translate. At an exceptional prime the natural map gives

$$(M_\iota \cap S_\iota)/(J_\iota \cap S_\iota) \hookrightarrow M_\iota/J_\iota.$$

For a tuple $\phi_1, \dots, \phi_s$ of integral preimages, work over the ring of integers O of a finite extension of $\mathbf{Q}_p$ over which the lattice, the subspace, the Hecke matrices and the tuple are defined. At an exceptional prime choose a Smith basis for the inclusion in Section 20.3.3. After extension to $R_\iota = \iota(\mathcal{O}_\kappa)$ this writes $M_\iota = \bigoplus_i R_\iota e_i$ and $J_\iota = \bigoplus_i d_i R_\iota e_i$, with all $d_i \neq 0$ because J_ι contains $p^{j_p} M_\iota$. In a nonzero coordinate of the tuple choose a pivot of least valuation, subtract its multiples from the other columns, and multiply the pivot column by d_i. The ratios used to clear the other columns belong to R_ι. In a zero coordinate no operation is needed. These operations preserve previous divisibilities and membership in S_ι. Writing $\Phi : R_\iota^s \to M_\iota \cap S_\iota$ for the map with the given tuple as columns, the resulting matrix C therefore satisfies

$$\Phi C(R_\iota^s) \subseteq J_\iota \cap S_\iota, \qquad v_\iota(\det C) \le \sum_i v_\iota(d_i) = \frac{\operatorname{length}_O(M_{\iota,O}/J_{\iota,O})}{v_O(p)} \le d_X L_p.$$

Here $M_{\iota,O}, J_{\iota,O}$ are the two lattices over O and v_O assigns value one to a uniformizer of O. The bound is unchanged on finite extension, since the length and $v_O(p)$ are multiplied by the same ramification degree. All operations pull back to $\mathcal{O}_\kappa$. At $p = N$, Section 19.4.3 provides the clearing directly, with determinant valuation at most $(m/2+1)d_X f a/k \le m f b d_X$. In the other cases the whole integral module suffices. The determinant valuation in each case is at most $d_X e_p$. For every allowed twist prime to p the chain lattice norm after restriction is at most $p^{e_p - g_p}$ on this module.

Let $z = c_{\mathrm{ss}}^{-1} \Pi \varphi$, using the common scalar of Section 19.3. Then (72) and the integral tensor expansion identify $L_\xi z$ with the restricted upper chain at φ. Set

$$U_g \varphi = |G|\, \rho(e_0)\rho(g) z.$$

By the lower-chain equivariance of Section 17.3, $\Theta_\xi U_g \varphi$ is the sum of the values of $L_\xi z$ at the finite translates. More precisely,

$$(\Theta_\xi U_g \varphi)(h) = \sum_{s \in G} |G| e_0(s) (L_\xi z)(h k_m(sg)),$$

because $U_g \varphi \in V$ and $\Theta_\xi = L_\xi|_V$. The coefficients are those of $|G| e_0$. They are algebraic integers: for each irreducible character the coefficient of the central idempotent multiplied by $|G|$ is its degree times the character on the inverse class.

The indicated terminal translations are integral at the places over p. The transported lattice norm for a point and its right translate by an integral invertible matrix is the same on a fixed value. Thus the same bound $p^{e_p - g_p}$ holds after the projected translate at every terminal point, with no change of the local source module. This uses the original Θ_ξ and its prescribed level, so its unit lattices are the ones tested in the index bound.

Lemma 7.5, Proposition 7.8 and Lemma 7.3 give

$$\|\sigma \Theta_1 U_g \varphi\|^2 \le \exp\left(A - m\mathcal{R} + 2 \sum_{p \in F} g_p \log p + 2E_0 + x^{99} \right).$$

Here $K_{\mathrm{in}} = E_0$ for the integral generators, the terminal restriction contributes another E_0, and the passage from pointwise to class norm costs at most $\log h_{\mathrm{term}} + 4 \log |G| \le x^{99}$. The exponents in the lattice norm are $2L_p - g_p = e_p - g_p$ at nonsplit $p < x_1$ other than d, at most $L_p - g_p \le e_p - g_p$ at d, and $2mfb = e_N$ at N, where $g_N = 0$. They are zero elsewhere. Using

$\sum_p e_p \log p \le E_0$, the rational rescaling in Proposition 7.10 now gives the short length and index bounds for $P_g = c_B U_g$ on the original family. For the given ordered tuple y, let P be a permutation matrix grouping its columns by their chosen translates. If C_g clears the group for g, take

$$U_\kappa = P\,\mathrm{diag}(C_g)P^{-1}, \qquad a_p = |t| d_X e_p.$$

In the original column order this matrix applies each clearing within its own group. The determinant bound $v_\iota(\det U_\kappa) \le a_p$ holds with the same a_p at every embedding above p. The scalar c_B leaves the cleared images integral since $e_p - g_p - v_p(c_B) \le 0$, and

$$|t| d_X \sum_p e_p \log p \le 2|t| \dim(S) E_0.$$

At the infinite places the same scalar gives logarithmic squared norm at most $A - m\mathcal{R} + 4E_0 + 2b + x^{99} \le A - m\mathcal{R} + L$ by Proposition 7.10.

For the set t of Proposition 16.2, choose a basis $y_i = P_{g_i}\phi_i$ of the sum of the images from $g_i \in t$ and $\phi_i \in \mathcal{S}$. Such a basis exists because $\mathcal{S}$ spans S and $c_B \neq 0$. Make this choice before the coefficient place or twist. For each $\sigma \in \mathrm{Aut}(\mathbb{C})$, use the same n selected columns. Hadamard's inequality in the positive untwisted terminal class pairing gives

$$\det \mathrm{Gram}(\sigma\Theta_1 y) \le \prod_{i=1}^{n} \|\sigma\Theta_1 P_{g_i}\phi_i\|^2 \le \exp\bigl(n(A - m\mathcal{R} + L)\bigr).$$

Combining this inequality with the index bound gives

$$\mathrm{Cov}(y) \le n(A - m\mathcal{R} + L), \qquad \mathrm{Ind}(y) \le 10D \log(1 + 2b^2) E_0 = D I_0. \qquad \square$$

23 The mixed Gram determinant and the product formula

In this section, we use the product formula to pass from valuations of mixed pairings to a global determinant inequality. We use the upper covolume bounds of Section 22, the critical-value bounds of Section 38, and the lattice completion of Section 44.

Use the normalized absolute value $|\ell|_\kappa = \ell^{-1}$ at a coefficient place $\kappa = (\ell, \iota)$, where $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ is a field isomorphism. Write $\bar{\kappa} = (\ell, \iota \circ \mathrm{conj})$. Say that a family f_κ has height at most j if there is a single $H \neq 0$ and bounds $\alpha \in \mathbb{R}$, $e_\ell \ge 0$ such that

$$|\sigma H| \le e^{\alpha}, \qquad |H f_\kappa|_\kappa \le \ell^{e_\ell}, \qquad \alpha + \sum_{\ell \in U} e_\ell \log \ell \le j$$

for every complex automorphism, every coefficient place and every finite set U of primes.

Height upper bounds add under multiplication by multiplying the multipliers and adding the bounds. Suppose that $z \neq 0$ is a constant family of height at most j and $|\sigma z| \le e^{u}$ for every σ. Choose its multiplier H and put $h = Hz$. Since $|\sigma h| \le e^{\alpha+u}$ for every σ, h is algebraic. Indeed, a transcendental number can be carried to a transcendental number of arbitrarily large modulus by extending a bijection of transcendence bases to the algebraic closures. Likewise a fixed cleared coordinate that is bounded at every coefficient embedding above a single ℓ is algebraic. If it were transcendental, its images could include $\ell^{-m}\theta$ for every $m \ge 0$, with $\theta \in \overline{\mathbb{Q}}_\ell$ transcendental over $\mathbb{Q}$. Both algebraically closed fields have cardinality continuum; the algebraic closure of $\mathbb{Q}(\mathcal{B})$ has cardinality $\max(\aleph_0, |\mathcal{B}|)$ for a transcendence basis $\mathcal{B}$. Thus both

transcendence degrees are continuum. Thus any embedding of a number field in either field extends to a map of the required type. A finite list of such fixed algebraic coordinates lies in a single number field.

Write $K = \mathbb{Q}(h)$, $d_h = [K : \mathbb{Q}]$ and $q_h = N_{K/\mathbb{Q}}(h)$. With $|\ell|_\lambda = \ell^{-1}$ at every $\lambda \mid \ell$, the bounds give

$$\log|q_h| = \sum_{\tau : K \hookrightarrow \mathbb{C}} \log|\tau h| \le d_h(\alpha + u),$$
$$\log|q_h|_\ell = \sum_{\lambda \mid \ell} [K_\lambda : \mathbb{Q}_\ell] \log|h|_\lambda \le d_h e_\ell \log \ell.$$

The first sum counts both members of each pair of complex embeddings. For every ℓ, the local degrees in the second sum satisfy $\sum_{\lambda \mid \ell} [K_\lambda : \mathbb{Q}_\ell] = d_h$. Take the finite set

$$U = \{\ell : \ |h|_\lambda \neq 1 \text{ for some finite place } \lambda \mid \ell \text{ of } K\}.$$

The rational product formula [48, Theorem 7.13] applied to q_h gives

$$0 = \log|q_h| + \sum_{\ell \in U} \log|q_h|_\ell \le d_h\Big(u + \alpha + \sum_{\ell \in U} e_\ell \log \ell\Big) \le d_h(u + j).$$

Let $w = (y_1, \dots, y_n, z_1, \dots, z_s)$ be a basis of the target space V of the Poincaré map. The comparison lift Θ_ξ has domain V and is injective. Use the fixed Hermitian metric on V and the class sum on the target of each lift. For a twist ξ, including the trivial twist 1, put

$$G_\xi = \det \operatorname{Gram}(\Theta_\xi w), \qquad Z_\xi = \det_V(\Theta_\xi^* \Theta_\xi).$$

Choose an orthonormal basis e of V, write $w = eP$ and set $M_\xi = \operatorname{Gram}(\Theta_\xi e)$. Then

$$G_\xi = \det(P^* M_\xi P) = |\det P|^2 Z_\xi = \det \operatorname{Gram}(w)\, Z_\xi.$$

Proposition 23.1 (Mixed Gram determinant inequality)**.** *For the basis $w = (y, z)$ and the injective comparison lifts above, suppose that the following bounds hold.*

- *For every $\sigma \in \operatorname{Aut}(\mathbb{C})$, the Gram determinant of the coordinatewise conjugates $\sigma\Theta_1 y_i$ is at most e^c, and each $\sigma\Theta_1 z_i$ has squared norm at most $e^{b_{\mathrm{short}}}$. The positive definite Hermitian matrix of the trivial lift's coordinate metric is rational.*
- *Fix $a_\ell \ge 0$, independent of the place above ℓ, with $\sum_{\ell \in U} a_\ell \log \ell \le \delta$ for every finite set of primes U. At each κ above ℓ take a clearing matrix U_κ for the y_i, with κ-integral entries and $|\det U_\kappa|_\kappa \ge \ell^{-a_\ell}$. Its cleared vectors lift integrally at κ for every allowed twist at ℓ.*
- *For each κ choose an allowed twist ξ_κ and write $\gamma = \gamma(\xi_\kappa) \neq 0$. The vectors $\gamma\Theta_{\xi_\kappa} z_i$ are integral at both κ and $\bar\kappa$. The pairings of the $\bar\kappa$-integral lattice against the κ-integral lattice have absolute value at most 1 at κ.*
- *The families $(\gamma\bar\gamma)^{-1}$ and Z_1/Z_{ξ_κ} have heights at most j_γ, j_Z.*

Then

$$0 \le c + 2\delta + s(b_{\mathrm{short}} + j_\gamma) + j_Z.$$

Proof. The terminal mixed Gram form. The pairing in the determinant is at the original lift level, principal at the old chosen places and parabolic at the twist places. The dual delta used for the period at the end of the path has the smaller principal level at the twist places. The pointwise Weyl lattice norm checked by those deltas is the lattice norm of the same lifted vector. Thus the check applies to its values on the class representatives at the lift level.

The terminal archimedean group is compact, so each class stabilizer at the lift level is finite by discreteness of the rational points in the adelic group. At any one old odd completely split place it embeds in $1 + p_i M_{n_{\mathrm{term}}}(\mathbf{Z}_{p_i})$ after conjugation by the class representative. A finite-order element of this group is one. Indeed, for $g = I + p_i^a B$ with $a \geq 1$ and some entry of B a unit, the minimum valuation of the entries of $g^m - I$ is $a + v_{p_i}(m)$ for $m \geq 1$, by the binomial expansion first for $p_i \nmid m$ and then for successive p_i-th powers. Thus every class stabilizer is trivial, and the lift-level Gram pairing is the plain class sum. Write K_{par} for this level and $K_{\mathrm{pr}} = K(\pi_M Q(\xi))$ for the level of (113). Over the class of g, the principal-level representatives are gk for $k \in K_{\mathrm{par}}/K_{\mathrm{pr}}$. Both vectors acquire the same unitary level character at k, so the principal-level class sum multiplies the parabolic one by $[K_{\mathrm{par}} : K_{\mathrm{pr}}] = I_\xi/I_{\mathrm{par}}(\xi)$, as computed in Section 44.1. We check the lattice at each representative g of the parabolic level, using its pointwise dual-delta value. Write J_{c} for the signed complement in Section 44.3.4, and $G_\vee$ for the Gram matrix of the dual tensor basis. Its isometry on the two Weyl spaces gives the primal Gram matrix

$$J = J_{\mathrm{c}}^{\mathsf{T}} G_\vee J_{\mathrm{c}}.$$

Indeed both sides are the restriction of the exterior tensor inner product in the primal integral basis. This is the metric in the upper estimate of Section 22.3. The principal-level delta reads the bilinear form with matrix $G_\vee J_{\mathrm{c}}$. The integral coordinate extraction in Section 44.3.4 therefore recovers the components in this same primal basis. At a representative g, choose both transported Weyl coordinates at their own coefficient embeddings.

Write T for the transport by g^{-1} at the embedding κ, and $T^{\#}$ for the equivalent transport on the conjugate coefficient vector. With a the local matrices of g read at κ, unitarity reads $T = \rho(a^{-1})$, $T^{\#} = \rho(a^{\mathsf{t}})$. The integral rational Weyl Gram matrix J satisfies $\rho(a^{\mathsf{t}})^{\mathsf{t}} J = J\rho(a)$, by the transpose rule on the exterior tensor construction. Hence $(T^{\#})^{\mathsf{t}} J T = J$. Consequently the mixed term at κ is

$$\iota\big(\langle v_-, v_+\rangle_J\big) = \big(T^{\#}\iota(\overline{v_-})\big)^{\mathsf{t}} J \big(T\iota(v_+)\big).$$

The two transported vectors have integral coordinates when v_- has lattice norm at most one at $\bar{\kappa}$ and v_+ has lattice norm at most one at κ. Since J is integral, their scalar pairing is integral as well. Summing over the free classes is integral. For the trivial twist the coordinate metric is a direct sum of the same rational Gram matrices, which is the metric used in the archimedean exterior-product inequality.

Use the convention that the first variable of the Hermitian form is conjugate-linear. Write the basis vectors as columns. At a coefficient place $\kappa = (\ell, \iota)$, a clearing matrix U_κ acts by column combinations. The entries of $\iota(U_\kappa)$ lie in a common finite extension of $\mathbb{Q}_\ell$. Regard the matrix in its ring of integers $\mathcal{O}$, with integer-valued valuation $v_{\mathcal{O}}$. Smith normal form gives

$$\mathrm{Fitt}_0(\mathcal{O}^n/U_\kappa\mathcal{O}^n) = (\det U_\kappa), \qquad -\log_\ell |\det U_\kappa|_\kappa = \frac{\mathrm{length}_{\mathcal{O}}(\mathcal{O}^n/U_\kappa\mathcal{O}^n)}{v_{\mathcal{O}}(\ell)}.$$

This normalized length is unchanged by finite extension of the coefficient field. Under scalar extension both the module length and the valuation of ℓ acquire the same relative ramification

factor. For a fixed κ, use the same allowed twist $\xi = \xi_\kappa$ at κ and $\bar{\kappa}$; both lie over ℓ. Put $M = \mathrm{Gram}(\Theta_\xi w)$ and

$$C_+ = \mathrm{diag}(U_\kappa, \gamma I_s), \qquad C_- = \mathrm{diag}(U_{\bar{\kappa}}, \gamma I_s).$$

By the clearing hypotheses for this tuple and $\ell \nmid q_\xi$, the columns of $(\Theta_\xi w)C_+$ are integral at κ, and those of $(\Theta_\xi w)C_-$ are integral at $\bar{\kappa}$. Their mixed pairing matrix $C_-^* M C_+$ therefore has integral entries at κ. Its determinant is $\overline{\det U_{\bar{\kappa}}}\,(\gamma\bar{\gamma})^s G_\xi \det U_\kappa$. Since $|\bar{t}|_\kappa = |t|_{\bar{\kappa}}$,

$$|(\gamma\bar{\gamma})^s G_\xi|_\kappa \le |\det U_{\bar{\kappa}}|_{\bar{\kappa}}^{-1} |\det U_\kappa|_\kappa^{-1} \le \ell^{2a_\ell}.$$

Thus $((\gamma\bar{\gamma})^s G_{\xi_\kappa})_\kappa$ has height at most 2δ, with multiplier 1. Now

$$G_1 = ((\gamma\bar{\gamma})^s G_{\xi_\kappa})\,(\gamma\bar{\gamma})^{-s}\,\frac{Z_1}{Z_{\xi_\kappa}},$$

so the constant G_1 has height at most $2\delta + s j_\gamma + j_Z$.

Let B be the coordinate matrix with columns $\Theta_1 w$ and Q the rational matrix of the coordinate metric. Write $\bar{\sigma} = \mathrm{conj} \circ \sigma \circ \mathrm{conj}$. Entrywise application of σ gives

$$\sigma G_1 = \det\big((\bar{\sigma}B)^* Q (\sigma B)\big).$$

Write $H_\tau(v) = \det \mathrm{Gram}(\tau\Theta_1 v)$. Gram–Schmidt and the norm bounds give

$$H_\tau(w) \le H_\tau(y) \prod_{i=1}^{s} \|\tau\Theta_1 z_i\|^2 \le e^{c + s b_{\mathrm{short}}}.$$

Cauchy–Schwarz in the $(n+s)$-th exterior power gives

$$|\sigma G_1|^2 \le H_\sigma(w) H_{\bar{\sigma}}(w) \le e^{2(c + s b_{\mathrm{short}})}.$$

Since $G_1 \neq 0$, the preceding product-formula argument yields

$$0 \le c + 2\delta + s(b_{\mathrm{short}} + j_\gamma) + j_Z.$$

The same twist is needed on the two sides of each mixed pairing. The individual clearing matrices may depend on the two coefficient places. The volume and denominator heights are separate arithmetic estimates on the allowed twists. □

Proposition 44.3 gives the spanning set and the clearing-scalar height. Proposition 44.2 gives the determinant-ratio height. Section 45 substitutes them into this inequality to prove Proposition 5.2.

Part IV Selmer bounds and Galois congruences

24 Length bounds after removing auxiliary Selmer conditions

In this section, we estimate the change in the length of the cocycle module when auxiliary local relations are discarded. The local bounds below are used in the polynomial estimate of Section 25.

Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_\ell$. Normalize $v(\ell) = 1$ and put $\mathcal{B} = \mathcal{O}/\ell^M\mathcal{O}$ for $M \ge 1$. In this section lengths are measured by $\lambda(X) = \mathrm{length}_{\mathcal{O}}(X)/v_{\mathrm{ord}}(\ell)$, where v_{ord} gives a uniformizer valuation one.

Lemma 24.1 (Length lost by imposing local relations)**.** *Let ρ be a continuous integral Galois representation of rank n. At each place in a finite list let ϕ_j be an arithmetic Frobenius lift or its inverse, let γ_j belong to inertia, and suppose $\rho \bmod \ell^M$ is unramified there. All these places have residue characteristic different from ℓ and residue cardinality $1 \bmod \ell^N$. Let $s_j, y \in \mathcal{O}$, and let ψ be a continuous scalar character into $\mathcal{B}^\times$. Suppose*

$$\psi(\phi_j) = 1 + y, \quad \psi(\gamma_j) = 1 + s_j y, \qquad s_j \neq 0 \implies \rho(\phi_j) \equiv I \bmod \ell^M.$$

For every element of inertia at the listed places, suppose its scalar α under ψ satisfies

$$\sum_{a=0}^{\ell^N - 1} \alpha^a = 0 \qquad \text{in } \mathcal{B}.$$

Put $R = \psi(\rho \bmod \ell^M)$. Let C be an $\mathcal{O}$-submodule of the continuous R-cocycles with values in $\mathcal{B}^n$. Write C_0 for its submodule defined by $z(\gamma_j) = s_j z(\phi_j)$ for all j. Then

$$\lambda(C/C_0) \leq v\left(\prod_j \det((1+y)\rho(\phi_j) - I)\right).$$

Proof. For $z \in C$, write $A_g(u) = z(g) + R(g)u$ for the affine action on $\mathcal{B}^n$. If g belongs to a listed inertia group, $R(g) = \alpha I$, so

$$A_g^{\ell^N}(u) = \alpha^{\ell^N} u + \left(\sum_{a=0}^{\ell^N - 1} \alpha^a\right) z(g) = u.$$

Indeed $\alpha^{\ell^N} - 1 = (\alpha - 1)\sum_{a<\ell^N} \alpha^a$. The finite continuous affine image of wild inertia is trivial: it is a finite group of order a power of the residue characteristic and all its elements have order dividing ℓ^N. Let q_j be the residue cardinality and f_j an arithmetic Frobenius lift. The tame inertia relation gives $A_{f_j} A_{\gamma_j} A_{f_j}^{-1} = A_{\gamma_j}^{q_j}$ [17, §5.1, Lemma 5.1]. Since $q_j \equiv 1 \pmod{\ell^N}$ and $A_{\gamma_j}^{\ell^N} = 1$, both A_{f_j} and its inverse commute with A_{γ_j}.

Put $F_j = (1+y)\rho(\phi_j) - I \in M_n(\mathcal{O})$ and let $\bar{F}_j$ be its reduction in $M_n(\mathcal{B})$. Evaluating the commuting affine maps at zero and applying the condition on $\rho(\phi_j)$ gives, on $\mathcal{B}^n$,

$$\bar{F}_j z(\gamma_j) = s_j y z(\phi_j), \qquad s_j \bar{F}_j = s_j y I.$$

These identities give the injection

$$C/C_0 \hookrightarrow \bigoplus_j \ker(\bar{F}_j : \mathcal{B}^n \to \mathcal{B}^n), \qquad [z] \longmapsto (z(\gamma_j) - s_j z(\phi_j))_j.$$

Let $(u_{j,i})_i$ be the Smith diagonal of F_j over $\mathcal{O}$, with $v(0) = +\infty$. The kernel of multiplication by u on $\mathcal{B}$ has ordinary length $\min\{M v_{\mathrm{ord}}(\ell), v_{\mathrm{ord}}(u)\}$. Thus

$$\lambda(C/C_0) \leq \sum_j \lambda(\ker \bar{F}_j) = \sum_{j,i} \min\{M, v(u_{j,i})\} \leq v\left(\prod_j \det F_j\right).$$

The middle sum is finite also for singular F_j. □

For the auxiliary twist constructed from homomorphisms χ_w into $\mathbb{Z}/\ell^N\mathbb{Z}$, use variables $x_w \in y\mathcal{O}$ and put $\psi(g) = \prod_w (1+x_w)^{\chi_w(g)}$ in $\mathcal{B}$. The scalar condition in Lemma 24.1 follows if $\sum_{a<\ell^N} \alpha^a$ is divisible by ℓ^M for every $\alpha \in 1+y\mathcal{O}$. In particular each $(1+x_w)^{\ell^N} = 1$ in $\mathcal{B}$. This makes the character well defined. With a first coordinate $x_0 = y$, the other coordinates are $x_w = s_w y$. At each subsequent place, the character values needed for the lemma are the additive values of the χ_w in $\mathbf{Z}/\ell^N\mathbf{Z}$, namely 1 in coordinate 0 and 0 elsewhere at ϕ_j, and 1 in coordinate j and 0 elsewhere at γ_j. These statements concern the chosen pairs at both conjugate places, with inverse lifts where used. They give the two evaluations of ψ in the lemma. The polynomial whose valuation bounds the length loss is consequently $\prod_j \det((1+Y)\rho(\phi_j) - I)$.

For a common lower bound $v(y) \geq a_0 > 0$, the choice of N ensuring the scalar condition can be made uniformly. For $z \in y\mathcal{O}$ the polynomial identity

$$\sum_{a=0}^{\ell^N-1} (1+z)^a = \sum_{j=1}^{\ell^N} \binom{\ell^N}{j} z^{j-1}$$

shows that the right side has valuation at least $N - \sup_{j\geq 1}\{\log_\ell j - (j-1)a_0\}$. Here $j\binom{\ell^N}{j} = \ell^N \binom{\ell^N-1}{j-1}$, and the supremum is finite. At the first place and its conjugate use the untwisted specialization $y = 0$, with slope zero. The scalar condition then reduces to $N \geq M$. In this specialization the kernel matrices only depend on the reductions of the first Frobenius matrices. Thus their length contribution is bounded by $\sum_j v(\det(\rho(g_j) - I))$ for any chosen g_j with $\rho(g_j) \equiv \rho(\phi_j) \bmod \ell^M$, with the sum over the two first-place parameters. This uses the kernel bound before replacing the lifts in the determinant.

24.1 The denominator bound for odd cocycles

In this subsection, we bound the odd cocycles in the paired Tate-twist estimate of Section 29.5. We reduce their estimation to the annihilator of Section 24.2 and the detection argument of Section 24.3.

Set $N_c = dq_1$, $t = 5r$, and $y = N_c + t + 3$. The field E of Section 4 embeds in $\mathbb{Q}(\zeta_{N_c})$. For a coefficient DVR $\mathcal{O}$ as above, use ordinary length in this subsection. Write $Z^-_{e,\nu}$ for the module of continuous cocycles $G_E \to (\mathcal{O}/\ell^\nu)(-e)$ that vanish on inertia outside ℓ. For $e > 0$ require

$$z(cgc^{-1}) = -z(g), \qquad z|_{D_v \cap \ker \chi_{\mathrm{cyc}}} = 0$$

for every decomposition group D_v at a prolongation of a place above ℓ, using the full ℓ-adic cyclotomic character. For $e = 0$ require vanishing on inertia above ℓ and impose no sign. Lengths are taken on these modules of cocycles themselves.

For $e = 0$ the action is trivial. Since E has no real places, global reciprocity [50, Ch. V, Theorems 5.3 and 5.5] gives

$$Z^-_{0,\nu} = \mathrm{Hom}(\mathrm{Cl}(E), \mathcal{O}/\ell^\nu).$$

Write $h_E = |\mathrm{Cl}(E)|$. If the class group has cyclic factors of orders d_i, then

$$\mathrm{length}_{\mathcal{O}} Z^-_{0,\nu} = \sum_i \min\{\nu v_{\mathrm{ord}}(\ell), v_{\mathrm{ord}}(d_i)\} \leq v_{\mathrm{ord}}(h_E).$$

Detection at small coefficient primes. Suppose that $\ell \leq y^2$. There is one list of at most y^{20} elements of G_E whose evaluations detect every cocycle unramified outside ℓ with values

in $(\mathcal{O}/\ell^{\nu})(-e)$, for all $\nu, e, \mathcal{O}$. Put $m_0 = N_c\ell^2$, $K' = \mathbb{Q}(\zeta_{m_0})$ and $s' = [K' : \mathbb{Q}] \le m_0$. Use representatives of $G_E/G_{K'}$, together with lifts of generators of the maximal abelian quotient of $G_{K'}$ of exponent dividing m_0 unramified outside ℓ. Denote this quotient by D_{m_0} and put $U_w = \mathcal{O}_{K'_w}^{\times}$. The ray class sequence and global reciprocity give

$$\prod_{w|\ell} U_w/U_w^{m_0} \longrightarrow D_{m_0} \longrightarrow \mathrm{Cl}(K')/\mathrm{Cl}(K')^{m_0} \longrightarrow 1.$$

Indeed, take the inverse limit over moduli supported above ℓ and then the quotient by m_0-th powers. The kernel before taking powers is the image of the local units, and K' has no real places [50, Ch. V, Theorems 1.7, 5.3 and 5.5]. Thus

$$|D_{m_0}| \le h_{K'} \prod_{w|\ell} |{K'_w}^{\times}/({K'_w}^{\times})^{m_0}|.$$

The valuation factor, the cyclic group of roots of unity, and the free part of the principal units give the bound $m_0^{2+[K'_w:\mathbb{Q}_\ell]}$ at w. The free rank is $[K'_w : \mathbb{Q}_\ell]$, since the local logarithm identifies sufficiently deep principal units with a full additive lattice, also for $\ell = 2$.

Their product is at most $m_0^{3m_0}$. In a number field L' of degree s', each class has an integral ideal of norm at most $\sqrt{|D_{L'}|}$ by Minkowski's bound [48, Theorem 4.3]. There are at most $h^{s'}$ ideals of norm h, since over each rational prime p there are at most s' prime ideals and the number of exponent assignments is at most $(1 + v_p(h))^{s'}$. Hence $h_{L'} \le (2|D_{L'}|)^{s'+1}$. Using $|D_{K'}| \le m_0^{s'}$ bounds the size of the class-field quotient by $2^{m_0^3+5m_0^2+m_0}$. Thus it has a generating set of size at most $m_0^3+5m_0^2+m_0$. With the coset representatives the total is at most $m_0^4 \le y^{20}$, since $m_0 \ge 4$ and $m_0 \le y^5$.

If a cocycle z vanishes at those elements, induct on the power of ℓ annihilating its values. The induction hypothesis applied to $m_0 z$ gives $m_0 z = 0$. On these values $G_{K'}$ acts trivially, since it fixes the m_0-th roots of unity. The restriction of z is now a homomorphism through the indicated class-field quotient. It vanishes on the generators, hence on $G_{K'}$, and the coset representatives give vanishing on all of G_E by the cocycle identity. This includes $\ell = 2$.

Assembly of the bound. For the positive twists, take $\mathcal{O}$ to be the integers of a finite extension of $\mathbb{Q}_\ell$ containing a primitive root of unity of order $\exp((\mathbb{Z}/N_c)^{\times})$. For $0 < e < t$, Section 24.2 provides an annihilator $Y_e \le 2^{4y^6}$ independent of $\ell, \nu, \mathcal{O}$ under this condition. For $\ell \le y^2$ the same set of at most y^{20} evaluations works for all exponents and depths. For $\ell > y^2$, Section 24.3 gives at most y^{20} evaluations for each fixed $e, \nu, \mathcal{O}$. In either case the bound on the number of evaluations is uniform. Evaluation therefore embeds $Z^-_{e,\nu}$ into at most y^{20} copies of $(\mathcal{O}/\ell^{\nu})[Y_e]$, of length at most $v_{\mathrm{ord}}(Y_e)$ each. Together with the zero twist this gives one integer

$$X_{\mathrm{odd}} = h_E \left(\prod_{0<e<t} Y_e \right)^{y^{20}}, \qquad \mathrm{length}_{\mathcal{O}} Z^-_{e,\nu} \le v_{\mathrm{ord}}(X_{\mathrm{odd}}), \qquad \log X_{\mathrm{odd}} \le \log h_E + 4y^{27} \log 2.$$

For an arbitrary coefficient ring $\mathcal{O}$, adjoin the specified roots of unity and let $\mathcal{O}'$ be the resulting ring of integers. In an $\mathcal{O}$-basis of $\mathcal{O}'$, the cocycle, inertia, cyclotomic-kernel and parity conditions hold coordinatewise because the coefficient action is scalar. Thus

$$Z^-_{e,\nu}(\mathcal{O}') = Z^-_{e,\nu}(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'.$$

Both the length and the valuation of X_{odd} multiply by the ramification index. Hence the normalized bound $v_\ell(X_{\mathrm{odd}})$ per exponent holds over $\mathcal{O}$. The ideal-class estimate above, applied to E, and $\mathrm{rd}(E) \le N_c$ give

$$\log h_E = O(f^2 \log N_c), \qquad N_c = O_d(x^{11} \log x), \quad t = O(x^5).$$

Consequently, for every finite set of primes U,

$$\sum_{\ell \in U} v_\ell(X_{\mathrm{odd}}) \log \ell \le \log X_{\mathrm{odd}} \le x^{2000}$$

for all sufficiently large x. Summing the odd contributions of at most t ordered exponents therefore contributes at most tx^{2000} to the logarithmic denominator bound.

Application to a paired cocycle test. For $e > 0$ let S be a module of cocycles unramified outside ℓ satisfying the local vanishing above ℓ. Suppose evaluations at g_0 and a finite set of points detect all unramified-outside-ℓ cocycles of this type. Put $d_e = \chi_{\mathrm{cyc}}(g_0)^{-e} - 1$. Assume $d_e z(g_0) = 0$ for $z \in S$, and that at each test point a the paired Frobenius conditions give $2(z + z \circ c)(a) = 0$, where $(z \circ c)(g) = z(cgc^{-1})$. Under these hypotheses $z \mapsto z \circ c - z$ sends S to $Z^-_{e,\nu}$. The local conditions are invariant under conjugation by c.

In the kernel $z = z \circ c$, hence $4z(a) = 0$ at all test points. The cocycle $4d_e z$ therefore vanishes at every test point and at g_0, so it is zero. If $4d_e \mid C$, in particular for the prescribed multiple of $4d_e^2$, the kernel is contained in $S[C]$, and consequently

$$\mathrm{length}_{\mathcal{O}}(S/S[C]) \le \mathrm{length}_{\mathcal{O}} Z^-_{e,\nu}.$$

The paired evaluations and separation scalar act on the same cocycle module over $\mathcal{O}$. The coefficient descent above gives the bound for this ring, also at $\ell = 2$.

24.2 A uniform annihilator for the positive twists

In this subsection, we construct a common integer that controls the positive exponents appearing in Section 24.1. The distribution identity for the Bernoulli polynomial also supplies the auxiliary units needed in Section 24.3.2.

Fix $e > 0$, abbreviate N_c to N, and put $u = N + e + 3$. We construct an integer $0 < X_e \le 2^{u^6}$. For each prime ℓ it supplies a positive integer $a \equiv 1 \pmod N$, prime to ℓ, with $(a^e - 1)(a^{e+1} - 1) \mid X_e$. For a character ψ of $(\mathbf{Z}/N)^\times$ with $\psi(-1) = -(-1)^e$ and values in $\mathcal{O}$, set

$$S_m(a, \psi) = \sum_{b \in (\mathbf{Z}/Nm)^\times} b^e \psi(b) \left(\left\lfloor a \left\langle \frac{b}{Nm} \right\rangle \right\rfloor - \left\lfloor a \left\langle \frac{-b}{Nm} \right\rangle \right\rfloor \right) \pmod m,$$

where angle brackets denote fractional part. For $m = \ell^\nu$, this scalar in $\mathcal{O}/m\mathcal{O}$ divides $(a^{e+1} - 1)X_e$ for every positive $a \equiv 1 \pmod N$ prime to ℓ.

Construction of the integer. For the Bernoulli polynomials put

$$B(b) = (e+2)! N^{e+1} B_{e+1}(b/N), \qquad P(X) = (e+1)! N^{e+2} (B_{e+2}(X/N) - B_{e+2}(0)).$$

Both have integer coefficients. In fact $(j+1)!B_j$ is integral by the Bernoulli recursion. In the coefficient of X^k in P, $j = e + 2 - k$. For $k \ge 2$, $(j+1)! \mid (e+1)!$, and for $k = 1$ the binomial

coefficient contributes $e+2$. The identities $P'=B$ and $P(b+N)-P(b)=N(e+2)!b^{e+1}$ now give

$$(e+2)!\sum_{i=0}^{m-1}(b+Ni)^{e+1}\equiv mB(b)\pmod{m^2}.$$

Indeed, summing the difference gives

$$N(e+2)!\sum_{i=0}^{m-1}(b+Ni)^{e+1}=P(b+Nm)-P(b)=NmB(b)+(Nm)^2R,\qquad R\in\mathbf{Z}.$$

Division in $\mathbf{Z}$ by N gives the congruence also when $\ell\mid N$.

Extend ψ by zero on the nonunits modulo N. In particular T_m includes multiples of ℓ when $\ell\nmid N$. Use the sums

$$T_m=\sum_{0\le b<Nm}\left\lfloor\frac{ab}{Nm}\right\rfloor b^e\psi(b),\qquad U_\psi=\sum_{0\le b<N}B(b)\psi(b).$$

Multiplication by a permutes residues modulo Nm and preserves ψ. Modulo $(Nm)^2$, expand

$$\big(ab-Nm\lfloor ab/(Nm)\rfloor\big)^{e+1}.$$

Multiply by $(e+2)!$, sum, group by the residues modulo N, and cancel m in the congruence modulo m^2. This gives

$$(a^{e+1}-1)U_\psi\equiv(e+2)!(e+1)Na^eT_m\pmod{m}.\tag{81}$$

On $G=(\mathbf{Z}/N)^\times$ define the integer matrix

$$A_{s,t}=2B(t/s)+(-1)^e\mathbf{1}_{t=s}+\mathbf{1}_{t=-s},$$

with arguments read in $[0,N)$. Work first in the cyclotomic number field of character values, using a complex embedding to test nonvanishing. The vector $(\psi(s))_{s\in G}$ has eigenvalue $2U_\psi+(-1)^e+\psi(-1)$. Grouping the Dirichlet series by residue and applying the Hurwitz special value gives $U_\psi=-(e+2)!(e+1)NL(-e,\psi)$. For $N=1$ the sole character has $L(s,\psi)=\zeta(s,1)$; here $B_{e+1}(0)=B_{e+1}(1)$ since $e+1\ge2$. At primitive conductor one the zeta reflection formula [60, (25.4.2)] gives vanishing at $-e$ exactly when e is even. At primitive conductor greater than one the parity factor in the Dirichlet functional equation gives vanishing at $-e$ exactly when $\psi(-1)=(-1)^e$. In both cases the Euler product at $1+e$ converges to a nonzero value. For a character induced by ψ^*, every omitted factor $1-\psi^*(p)p^e$ is nonzero. We have used [60, (25.11.14) and (25.15.3)–(25.15.5)]. The eigenvalues are therefore $2(-1)^e$ on the vanishing parity and $2U_\psi\neq0$ on the other parity, so $\det A\neq0$.

Take

$$X_e=|\det A|\prod_{a\in\{1+N,1+2N\}}(a^e-1)(a^{e+1}-1).$$

The two choices for a are relatively prime. The Bernoulli Fourier series [60, (24.8.3)] bounds $|B_{e+1}(x)|$ by $17(e+1)!$ on $[0,1]$. With $d'=\varphi(N)$ and $M'=(e+2)!N^{e+1}(e+1)!$, this gives $|\det A|\le d'!(64M')^{d'}$. Use $d'\le u$, $M'\le u^{3u}$ and bound the four factors by $(2u)^u$ each. Then $\log_2 X_e\le3u^3+5u^2+10u\le u^6$ for $u\ge5$.

For the odd parity the eigenvalue is $2U_\psi$. In the identity coordinate the character vector has entry 1. Apply $\mathrm{adj}(A)A = \det(A)I$ there and embed its cyclotomic integer coefficients in $\mathcal{O}$. This shows $X_e = 2U_\psi q_\psi$ with $q_\psi \in \mathcal{O}$. Multiply (81) by $2q_\psi$ to obtain

$$(a^{e+1}-1)X_e \equiv 2wT_m \pmod{m}, \qquad w = q_\psi(e+2)!(e+1)Na^e.$$

The same w works at every power of ℓ. Write $T_{\ell^j} = V_j + \ell^e\psi(\ell)T_{\ell^{j-1}}$, where V_j is the sum on units modulo $N\ell^j$. Subtract the scaled congruence at level $j-1$, using $e \geq 1$, to get

$$(1-\ell^e\psi(\ell))(a^{e+1}-1)X_e \equiv 2wV_j \pmod{\ell^j}.$$

For $j = 1$ the level-zero congruence is automatic. Replacing the representative b by that of $-b$ changes $b^e\psi(b)$ to its negative modulo ℓ^j, so $S_{\ell^j}(a,\psi) = 2V_j$ in the quotient. The scalar $1-\ell^e\psi(\ell)$ is a unit there. This proves the asserted divisibility, also for $\ell = 2$.

24.2.1 The Jacobi quotient

In this subsection, we compute the split-prime Jacobi quotient for the odd cocycles of Section 24.1.

Let $m = \ell^\nu$ with $\nu > 0$, let $p \equiv 1 \pmod{Nm}$ be prime, and fix a place B above p. Write θ_j for the multiplicative characters of $\mathbf{F}_p^\times$, indexed by $j \in \mathbf{Z}/(p-1)$. Fix a nontrivial additive character ξ and put $G_j = \sum_{b\neq 0}\theta_j(b)\xi(b)$. Normalize $v_B(p) = 1$. Inertia fixes the multiplicative characters and acts on ξ by scaling its argument. The resulting scalars for G_j multiply in j. Thus $G_{j+k}/(G_jG_k)$ belongs locally to the unramified subfield and has integral order with $v_B(p) = 1$. It follows that $j \mapsto v_B(G_j) \bmod \mathbf{Z}$ is additive. For $j \neq 0$, reduction of the additive character to one gives positive order, and $G_j\sum_{b\neq 0}\theta_{-j}(b)\xi(-b) = p$ gives order less than one. The product identity follows by summing first over the ratio of the two nonzero arguments: the inner additive sum is $p-1$ at ratio one and -1 otherwise. For $j = 0$, $G_0 = -1$. Hence $j \mapsto v_B(G_j) \bmod \mathbf{Z}$ identifies the character group with the $(p-1)$-torsion of $\mathbf{Q}/\mathbf{Z}$.

If x corresponds to j, put

$$J(x) = \frac{G_j^a/G_{aj}}{G_{-j}^a/G_{-aj}}.$$

A Galois change of additive character cancels in each ratio G_j^a/G_{aj}. A change of multiplicative character sends j to its cyclotomic multiple. Thus if x has denominator dividing Nm, then $J(x) \in \mathbf{Q}(\zeta_{Nm})^\times$, and its conjugate under $b \in (\mathbf{Z}/Nm)^\times$ is $J(bx)$. The Gauss product shows that these numbers are units off p. By the least nonnegative orders above,

$$v_B(J(x)) = \lfloor a\langle x\rangle\rfloor - \lfloor a\langle -x\rangle\rfloor.$$

24.2.2 Annihilation on character components

In this subsection, we apply reciprocity to the Jacobi quotient of Section 24.2.1.

Write $C = \mathcal{O}/\ell^\nu\mathcal{O}$, $K' = \mathbf{Q}(\zeta_{Nm})$ and $\Theta_\psi = \chi_{\mathrm{cyc}}^e\psi$. For this annihilation argument take $a \in \{1+N, 1+2N\}$ prime to ℓ. Such a choice exists since $2(1+N)-(1+2N) = 1$; at $\ell = 2$ one may take $1+2N$. The product defining X_e contains $(a^e-1)(a^{e+1}-1)$ as an integer factor. Let $z_\psi : G_{\mathbf{Q}} \to C$ be a cocycle for Θ_ψ^{-1}, with $z_\psi(c) = 0$ at complex conjugation c. Assume it vanishes on inertia off ℓ lying in $G_{\mathbf{Q}(\zeta_N)}$, and on the elements of decomposition groups over ℓ

that fix ζ_N and have $\chi_{\mathrm{cyc}} = 1$. Suppose $\Theta_\psi(c) = -1$. On $G_{K'}$ the character Θ_ψ^{-1} is trivial in $C^\times$, so $z_\psi|_{G_{K'}}$ is an additive homomorphism. At any place over ℓ, choose inertia δ with $\chi_{\mathrm{cyc}}(\delta) = a$. This is possible by cyclotomic total ramification over $\mathbf{Q}_\ell$. It fixes ζ_N, since inertia fixes the prime-to-ℓ part and $a \equiv 1 \pmod N$ handles the ℓ-part. For $g \in G_{K'}$ in that decomposition group, $[\delta^{-1}, g]$ fixes ζ_N and has trivial cyclotomic character. Its cocycle value is $(a^e - 1)z_\psi(g)$ since g acts trivially on C. It follows that $2(a^e-1)z_\psi|_{G_{K'}}$ vanishes on the decomposition groups above ℓ, the inertia groups off ℓ, and any real decomposition groups.

Apply global reciprocity over K' to this homomorphism and the principal idele $J(1/(Nm))$ [50, Ch. V, Proposition 5.2 and Theorem 5.3(a)]. At the chosen place over p write ϕ for an arithmetic Frobenius and choose lifts s_b of $b \in (\mathbf{Z}/Nm)^\times$. The other places above p are $s_b^{-1}B$. Their Frobenius values are

$$z_\psi(s_b^{-1}\phi s_b) = b^e\psi(b)z_\psi(\phi)$$

because ϕ acts trivially on C. Their orders on the principal idele are $v_B(J(b/(Nm)))$. The sum of local Artin images is zero, giving

$$S_m(a,\psi)\,2(a^e - 1)z_\psi(\phi) = 0.$$

Write $(a^{e+1} - 1)X_e = S_m k$ in C and $X_e = (a^e - 1)(a^{e+1} - 1)d_0$ in $\mathbf{Z}$. Multiplication by kd_0 gives $2X_e^2 z_\psi(\phi) = 0$.

The cocycle $2X_e^2 z_\psi$ is zero on $G_{K'}$ by Chebotarev's density theorem [48, Theorem 8.31, p. 147]. Indeed take a finite normal quotient carrying its values and the cyclotomic quotient modulo Nm. On the kernel of the latter quotient, conjugacy changes a cocycle value by a unit, and each conjugacy class in the kernel is represented by a Frobenius above a rational prime split in K'. So $2X_e^2 z_\psi$ factors through the abelian group $\mathrm{Gal}(K'/\mathbf{Q})$. For a cocycle on that group with value zero at c, commuting an element with c, which acts by -1, shows that twice every value is zero. We obtain $4X_e^2 z_\psi = 0$.

24.2.3 Integral recombination of character components

In this subsection, we pass from Section 24.2.2 to the integral lattice.

For $z \in Z^-_{e,\nu}$, let s_v be lifts of $v \in (\mathbf{Z}/N)^\times$, with $s_1 = 1$. Form the character components

$$z_\psi(g) = \sum_v \Theta_\psi(s_v)z(s_v[c,g]s_v^{-1}).$$

Here $[c,g] = cgc^{-1}g^{-1}$. Each conjugated commutator lies in $G_{\mathbf{Q}(\zeta_N)} \subseteq G_E$ and acts trivially on C. For $h \in G_E$ acting trivially, the cocycle rule gives $z(khk^{-1}) = \chi_{\mathrm{cyc}}(k)^{-e}z(h)$ for $k \in G_E$. It follows that each summand as a function of s_v is unchanged by left multiplication by $G_{\mathbf{Q}(\zeta_N)}$. Using $[c,gg'] = [c,g]g[c,g']g^{-1}$ and reindexing $s_v g$ proves the cocycle rule for Θ_ψ^{-1}. The value at c is zero. At an element g in either prescribed local subgroup, the two conjugates $s_v cgc^{-1}s_v^{-1}$ and $s_v g s_v^{-1}$ are in the corresponding prescribed local subgroups of G_E. Their values under z vanish, and their difference is the commutator value. Thus the local conditions above hold for every component.

Assume $\mathcal{O}$ contains the roots of unity of order the exponent of $(\mathbf{Z}/N)^\times$. Character orthogonality holds as an identity in $\mathcal{O}$ before reduction. For $g \in G_E$ it gives

$$\sum_\psi z_\psi(g) = \varphi(N)(z(cgc^{-1}) - z(g)) = -2\varphi(N)z(g).$$

If the odd components vanish, the even components are invariant under conjugation by c by their cocycle rule. Comparing the sum at g and cgc^{-1} gives $4\varphi(N)z = 0$. Apply this to $4X_e^2 z$, whose odd components vanish by the previous argument. Hence

$$Y_e Z_{e,\nu}^- = 0, \qquad Y_e = 16\varphi(N)X_e^2 \le 2^{4u^6}. \tag{82}$$

The bound is uniform in ℓ, ν, and the coefficient ring. Its integral character argument includes the primes dividing $2\varphi(N)$.

24.3 Detection at large coefficient primes

In this subsection, we use the integer of Section 24.2 to choose a torsion order at a large coefficient prime. Reciprocity and Jacobi derivatives will then detect the nonzero odd component from Section 24.1.

Fix $0 < e < t$, a character ψ with $\psi(-1) = (-1)^{e+1}$ as in Section 24.2, a coefficient ring $O = \mathcal{O}$, and the component depth ν. Let $C_{\psi,\nu}$ consist of the continuous cocycles $G_{\mathbf{Q}} \to (O/\ell^\nu O)((\chi_{\mathrm{cyc}}^e\psi)^{-1})$ vanishing at c. On restriction to $G_{\mathbf{Q}(\mu_{N_c})}$, they vanish on inertia away from ℓ and on the cyclotomic kernel in each decomposition group above ℓ, as in Section 24.2.2. Put $h = v_\ell(X_e) + 1$. We prove

$$\exists g_1, \dots, g_h \in G_{\mathbf{Q}}, \qquad \{z \in C_{\psi,\nu} : z(g_i) = 0 \ (1 \le i \le h)\} = 0.$$

It suffices to detect $C_{\psi,\nu}[\ell]$: in a nonzero evaluation kernel the last nonzero ℓ-multiple of any nonzero element is killed by ℓ. The inequality $\ell > y^2$ gives $e + 1 < \ell - 1$ and $\ell \nmid N_c$. For the detection argument choose a positive $a \equiv 1 \bmod N_c$ with primitive residue modulo ℓ, by the Chinese remainder theorem. For this a, Section 24.2 gives

$$(a^{e+1} - 1)X_e \in S_{\ell^h}(a, \psi)\,(O/\ell^h O).$$

Indeed the Voronoi congruence there gives this membership after multiplication by the unit $1 - \ell^e\psi(\ell)$. Then $2, \varphi(N_c\ell), a^e - 1, a^{e+1} - 1$ are units in O. At a place above ℓ choose γ in inertia, fixing μ_{N_c}, with $\chi_{\mathrm{cyc}}(\gamma) = a$. The existence follows by restricting to the compositum of the unramified prime-to-ℓ cyclotomic extension and the totally ramified ℓ-power cyclotomic extension. The value of the dual character $\Upsilon = \chi_{\mathrm{cyc}}\Theta_\psi = \chi_{\mathrm{cyc}}^{e+1}\psi$ at γ is a^{e+1}, with $\Upsilon(\gamma) - 1$ a unit.

Here and below, a cocycle with finite coefficient depth has open zero set, which is a subgroup. If $w(g_0) = 0$ and $(\chi_w(g_0) - 1)w \neq 0$, the zero set cannot contain the subgroup fixing any cyclotomic modulus. Indeed, vanishing on that subgroup would give $w(g_0 g) = w(gg_0)$, since $(g_0 g)^{-1} g g_0$ belongs to it. The cocycle identity would then give $(\chi_w(g_0) - 1)w(g) = (\chi_w(g) - 1)w(g_0) = 0$ for every g. For a nonzero odd component take $g_0 = c$, so its character difference is -2. For a nonzero normalized dual cocycle take $g_0 = \gamma$. These differences are units in the large-prime range.

Suppose no h evaluations detect the odd component. For any set T of at most h primes $r \equiv 1 \bmod N_c\ell$, use their Frobenius elements as tests and take the last nonzero ℓ-multiple of a missed cocycle. This gives a nonzero component z, killed by ℓ, trivial at all test Frobenii. At these split primes the action on the values of z is trivial. It vanishes on inertia, so it vanishes on each tested decomposition group. Conjugating a tested Frobenius multiplies its value by the character at the conjugating element, since the Frobenius action on the coefficients is trivial. The same applies to inertia, so z kills every decomposition group above each $r \in T$.

For sets T of primes split modulo $N_c\ell$, call T tight at depth h when every dual cocycle in O/ℓ^h killed by ℓ and unramified over $\mathbb{Q}(\mu_{N_c})$ outside ℓ, T is unramified there outside ℓ. The empty set is tight. If T is tight and z is nonzero on a decomposition group at a new split r and trivial above T, the modulo-ℓ reciprocity argument in Section 24.3.1 implies that $T \cup \{r\}$ is tight.

Choose a finite set Q' of primes $r \equiv 1 \bmod aN_c\ell^h$ detecting every pair (u, z) in which z is a nonzero ℓ-killed odd component and u is a dual cocycle at depth h, unramified outside ℓ over $\mathbb{Q}(\mu_{N_c})$, and nonzero on the subgroup fixing $\mu_{aN_c\ell^h}$. Both cocycle spaces are finite. Over the cyclotomic field killing the coefficient action, each restriction is a homomorphism to a finite additive coefficient module. Its kernel fixes an extension of degree at most the order of that module, unramified outside ℓ. There are finitely many such extensions [48, Theorem 8.42, p. 151]. Each restriction and the values on finitely many coset representatives then have only finitely many possibilities.

On the subgroup fixing $\mu_{aN_c\ell^h}$ the two zeros are proper subgroups. Their union is proper. If neither contains the other, a product of elements exclusive to the two subgroups lies in neither. Choose such an element and pass to a finite Galois quotient through which the two cocycles, their coefficient actions and the cyclotomic quotient factor. Chebotarev gives infinitely many unramified rational primes in its conjugacy class. Every such prime splits completely in $\mathbf{Q}(\mu_{aN_c\ell^h})$. Conjugating a prime of the finite extension above it, and then prolonging that prime to $\overline{\mathbf{Q}}$, realizes the prescribed element as the image of arithmetic Frobenius. Both cocycle values are nonzero at this same prolongation. Choosing a distinct prime for each pair gives Q'. The preceding abelian calculation ensures that every nonzero normalized dual cocycle here qualifies for this pair test.

Choose a nonzero odd component killed by ℓ and apply its cyclotomic nonvanishing to the modulus $N_c\ell^h \prod_{r\in Q'} r$. Chebotarev gives a prime p split at this modulus, a distinguished place $B \mid p$, and a nonzero component value at its Frobenius. This uses the same finite-quotient argument as above with one cocycle. In particular $p \notin Q'$, and reciprocity with $T = \varnothing$ makes $\{p\}$ tight. Section 24.3.2 gives κ_S for $S \subseteq Q'$ at depth h, with this p, B, γ. It uses $r \equiv 1 \bmod aN_c\ell^h$ for $r \in Q'$ and $p \equiv 1 \bmod N_c\ell^h \prod_{r\in Q'} r$.

Prove by descending induction for $i = h, \dots, 0$ that $\ell^i\kappa_S$ is unramified outside ℓ over $\mathbb{Q}(\mu_{N_c})$ whenever $|S| \le i$ and $\{p\} \cup S$ is tight. At $i = h$ it is zero. For $i < h$, if $u = \ell^{i+1}\kappa_S \neq 0$, then it is unramified by induction and normalized at γ. Since $|\{p\} \cup S| \le h$, choose $z \neq 0$, killed by ℓ, trivial above $\{p\} \cup S$. Use the pair test to choose $r \in Q'$, a prolongation $A \mid r$ and a lift ϕ of arithmetic Frobenius at A with $u(\phi), z(\phi) \neq 0$. It lies outside S and z makes $\{p\} \cup S \cup \{r\}$ tight. Since $|S \cup \{r\}| \le i + 1$, the induction hypothesis at $(i + 1, S \cup \{r\})$ gives vanishing on inertia above r. Apply the Frobenius–inertia identity of Section 24.3.3 at the same prolongation to obtain

$$u(\phi) = \ell^{i+1}\kappa_S(\phi) = \ell^{i+1}\kappa_{S\cup\{r\}}(\tau_r) = 0, \qquad \tau_r \in I_A,$$

a contradiction. Thus $u = 0$ as a cocycle and $\ell^i\kappa_S$ is killed by ℓ. By construction it is unramified outside $\ell, \{p\} \cup S$, so tightness removes the additional primes. This proves the induction.

At $i = 0$ the cocycle $\kappa_\varnothing$ kills inertia at B. The base derivative identity gives $(\Upsilon(\gamma) - 1)S_{\ell^h}(a, \psi) = 0$ in O/ℓ^h. Cancel the unit and use the Voronoi divisibility of Section 24.2 to get $\ell^h \mid (a^{e+1} - 1)X_e$ in O. The factor $a^{e+1} - 1$ is a unit. This contradicts $h = v_\ell(X_e) + 1$, in the normalized valuation $v_\ell(\ell) = 1$. This proves existence of the h component tests.

For each ψ with $\psi(-1) = (-1)^{e+1}$, take its h tests g. Evaluate the original cocycle at all $s_u[c, g]s_u^{-1}$ in the character sums of Section 24.2. At most $\varphi(N_c)^2(v_\ell(X_e) + 1) \le y^{20}$ points are

used, since $v_\ell(X_e) \le y^6$. If z vanishes there, every component of total character $\chi_{\mathrm{cyc}}^e\psi$ of sign -1 vanishes. Then $4\varphi(N_c)z = 0$ and $4\varphi(N_c) < \ell$, so $z = 0$.

Thus these evaluations detect the original module for $\ell > y^2$.

24.3.1 Local reciprocity at an auxiliary prime

In this subsection, we apply local reciprocity at a split auxiliary prime as part of the detection argument of Section 24.3. The local calculation prescribes how an unramified cocycle is evaluated against the auxiliary units.

Let $K = \mathbf{Q}(\mu_{N_c\ell})$ and suppose a split prime $r \equiv 1 \bmod N_c\ell$ lies outside T. Let x be an ℓ-killed odd component, trivial on the decomposition groups above T and nonzero at one Frobenius over r. On the decomposition groups over r the action on its values is trivial, and x is unramified there. Conjugation therefore shows $2(a^e-1)x(\phi) \neq 0$ for an arithmetic Frobenius ϕ at any $B \mid r$.

Take an inertia element τ at B. Let κ be a dual cocycle for $\Upsilon = \chi_{\mathrm{cyc}}^{e+1}\psi$, killed by ℓ and unramified over $\mathbf{Q}(\mu_{N_c})$ off $\ell, T \cup \{r\}$. Write $z = \kappa(\tau)$. Take an $\mathbf{F}_\ell$-basis b_i of Oz inside the ℓ-torsion of the coefficient module and extend its coordinate maps L_i to that whole $\mathbf{F}_\ell$-space. On G_K the cocycle action on these coefficients is trivial.

With ζ_ℓ fixed, Hilbert's Theorem 90 [50, Ch. II, Proposition 1.22] applies over the finite Galois extension of K cut out by $L_i \circ \kappa|_{G_K}$. It gives r_i in that extension with

$$g(r_i)/r_i = \zeta_\ell^{L_i(\kappa(g))} \qquad (g \in G_K).$$

Hence $y_i = r_i^\ell \in K^\times$. Apply Artin reciprocity over K to y_i and $2(a^e-1)x|_{G_K}$. This unramified homomorphism is zero on the decomposition groups above ℓ by the cyclotomic commutator calculation in Section 24.2, and on those above T by hypothesis. Elsewhere off r the root r_i is inertia-fixed, so y_i has normalized order divisible by ℓ.

Such orders contribute zero since the homomorphism is killed by ℓ. Thus only the places above r contribute. For a cyclotomic lift s_u, with $u \in (\mathbf{Z}/N_c\ell)^\times$, put $m_i(u) = v_B(s_u y_i)$ with $v_B(r) = 1$. Fix $\pi^\ell = r$ and write $\tau(\pi)/\pi = \zeta_\ell^j$. Since r splits in K, $m_i(u) \in \mathbb{Z}$. The relations $r_i^\ell = y_i$ and $\pi^\ell = r$ give

$$v_B(\pi^{-m_i(u)} s_u(r_i)) = -m_i(u)/\ell + v_B(s_u y_i)/\ell = 0.$$

The ℓ-th power of this local unit lies in K, which τ fixes. Thus its quotient under τ belongs to μ_ℓ and reduces to 1 at B by inertia. Reduction is injective on μ_ℓ since $r \neq \ell$. The equality of exponents is therefore

$$jm_i(u) = uL_i(\kappa(s_u^{-1}\tau s_u)) \quad \text{in } \mathbf{F}_\ell.$$

Since τ fixes $\mu_{N_c\ell}$, its dual coefficient action is trivial.

The cocycle identity gives $\kappa(s_u^{-1}\tau s_u) = \Upsilon(s_u)^{-1}z$. Together these identities read in the chosen basis as

$$(u^e\psi(u)) \sum_i jm_i(u)b_i = z.$$

The weighted Artin identity gives

$$\left(\sum_u m_i(u)u^e\psi(u)\right) 2(a^e-1)x(\phi) = 0.$$

Each scalar in parentheses thus lies in the maximal ideal $\mathfrak{m}_O$. Summing the basis identity over u yields $\varphi(N_c\ell)z \in \mathfrak{m}_O Oz$. Since $\varphi(N_c\ell)$ is a unit in this range, the cyclic module Oz vanishes by Nakayama. This holds at all places over r, so κ is unramified off ℓ, T. If T is tight, it follows that $T \cup \{r\}$ is tight.

24.3.2 The Jacobi derivative cocycles

In this subsection, we differentiate the Jacobi family built from the distribution identity of Section 24.2. The resulting cocycles satisfy the relations needed for the detection calculation of Section 24.3.

We use Kolyvagin's derivatives of Gauss sums in the form discussed by Rubin [64]. Use $N = N_c$, $m = \ell^h$, $L_1 = Nm$, and the primes $r \in Q'$, p and place B from the descent. In the construction of J in Section 24.2 use the chosen multiplier a. All the arguments used below have denominators dividing $p-1$. Since $(a, r) = 1$, the distribution quotient $\prod_{b \bmod r} J(x+b/r)/J(rx)$ is a root of unity. Its valuations disappear by the floor distribution identity at B, its conjugate versions at all primes above p, and the units away from p.

The equality of absolute values $|G_j| = |G_{-j}|$ at each embedding makes every absolute value of the quotient one, so Kronecker's theorem applies. The floor distribution follows by writing each floor as $a\langle x\rangle - \langle ax\rangle$ and using $\sum_{b \bmod r}\langle x + b/r\rangle = (r-1)/2 + \langle rx\rangle$ twice for x and $-x$. If $r \mid p-1$ is prime and $rb = 0$ in $\mathbf{Q}/\mathbf{Z}$, the Gauss sums at the shifted indices for $x + b$ and x are congruent at each place above r.

The extra character takes r-th root values, which reduce to 1, and the Gauss sums are units there. Therefore $J(x + b)/J(x)$ is a principal unit there. For the derivative system, let $m = \ell^h$, choose $y(x)^m = J(x)$, and define $F_g(x)$ by

$$\zeta_m^{F_g(x)} = \frac{\gamma(y(x))}{(g\gamma g^{-1})(y(x))}.$$

On the $M_S = L_1 \prod_{r\in S} r$-torsion this quotient has order dividing m, since the action of the commutator on $J(x)$ is trivial. Use the sum

$$\mathcal{D}_S F = \sum_{b\in(\mathbf{Z}/L_1)^\times} b^e \psi(b) \left[\prod_{r\in S}\sum_{u\in(\mathbf{Z}/r)^\times} \log_r(u)\, T_{u/r}\right] F\left(\frac{b}{L_1}\right), \qquad (T_b F)(x) = F(x+b).$$

For each r, $\log_r$ is a fixed surjective homomorphism to $\mathbf{Z}/m$, available since $m \mid r-1$.

The exponent $F_g(x)$ is independent of the choice of m-th root $y(x)$: the two conjugate Galois operators act alike on μ_m. Write $b_{g,S} \in (\mathbf{Z}/M_S)^\times$ for the exponent of g on μ_{M_S}. For $b = b_{g,S}$ and $M_S x = 0$, $g(y(x))$ and $y(bx)$ differ by an m-th root of unity. Cancelling that root gives, for $g' \in G_{\mathbf{Q}}$,

$$F_{gg'}(bx) - F_g(bx) = bF_{g'}(x).$$

The quotient on the left is the g-translate of $\gamma y(x)/(g'\gamma g'^{-1})y(x)$.

Let T be a set of the allowed auxiliary primes and write $\mathcal{D}_T^{\mathbf{d}}$ for the same sum with weights $\log_s(u) + d_s$, $\mathbf{d} \in (\mathbf{Z}/m)^T$. For an auxiliary prime $q \notin T$, set $\boldsymbol{\lambda}_T(q) = (\log_s(q))_{s\in T}$. Since $q \equiv 1 \bmod L_1$, the base indices are fixed and substituting $u' = qu \bmod s$ gives

$$\mathcal{D}_T^{\mathbf{d}}(F(q\,\cdot)) = \mathcal{D}_T^{\mathbf{d}-\boldsymbol{\lambda}_T(q)} F.$$

The quotient $\prod_{b \bmod q} y(x+b/q)/y(qx)$ is a root of unity by the Jacobi distribution identity. The two Galois actions defining F_g agree on roots of unity, so $\sum_{b \bmod q} F_g(x+b/q) = F_g(qx)$. For $S = T \sqcup \{q\}$, changing only d_q by $h_0 \in \mathbf{Z}/m$ therefore gives

$$\mathcal{D}_S^{\mathbf{d}+h_0\mathbf{e}_q} F_g - \mathcal{D}_S^{\mathbf{d}} F_g = h_0 (\mathcal{D}_T^{\mathbf{d}|_T - \boldsymbol{\lambda}_T(q)} F_g - \mathcal{D}_T^{\mathbf{d}|_T} F_g),$$

where $\mathbf{e}_q$ is the coordinate vector at q. Starting with $T = \varnothing$, induction removes all shifts. For any b prime to M_S, also substitute $a' = ba \bmod L_1$ in the base sum. The original base weight is $(b^e\psi(b))^{-1}(a')^e\psi(a')$, and the same shift calculation at the auxiliary primes gives

$$\mathcal{D}_S(F_g(b\,\cdot)) = (b^e\psi(b))^{-1}\mathcal{D}_S F_g.$$

With $b = b_{g,S}$, applying this identity to the equation for $F_{gg'} - F_g$ proves

$$\kappa_S(gg') - \kappa_S(g) = b^{e+1}\psi(b)\kappa_S(g'), \qquad \kappa_S(g) = \mathcal{D}_S F_g.$$

Thus κ_S has coefficients in O/mO with character $\chi_{\mathrm{cyc}}^{e+1}\psi$. It is continuous because each sum uses finitely many roots. The cocycle at γ is zero termwise. Away from ℓ, p, S, inertia fixing μ_N fixes μ_{M_S}. For g in this inertia, $gy(x)/y(x)$ is a principal unit in μ_m, since $y(x)$ is a unit and its m-th power is fixed. The same holds for $g\gamma y(x)/\gamma y(x)$, since $(\gamma y(x))^m = J(b_{\gamma,S}x)$. Here $b_{\gamma,S} \equiv a \bmod L_1$: γ has ℓ-adic cyclotomic value a, fixes μ_N, and $a \equiv 1 \bmod N$. For each $r \in S$ one has $b_{\gamma,S} \equiv 1 \bmod r$, since γ belongs to inertia above $\ell \neq r$. Injectivity of reduction on μ_m gives $F_g(x) = 0$.

For the base derivative $S = \varnothing$, take $x \in L_1^{-1}\mathbf{Z}/\mathbf{Z}$. Then $J(x) \in \mathbf{Q}(\mu_{L_1})$ and $b_{\gamma,\varnothing}x = ax$. At the chosen prolongation $B \mid p$, take a tame inertia element τ that multiplies the m-th root ϖ_m of p by ζ_m. It exists by total ramification of the uniformizer extension over the unramified field containing μ_m. It fixes μ_{L_1}. Write

$$s_x = v_B(J(x)) = \lfloor a\langle x\rangle \rfloor - \lfloor a\langle -x\rangle \rfloor.$$

Then $y(x)/\varpi_m^{s_x}$ is a unit whose m-th power lies in $\mathbf{Q}(\mu_{L_1})$. The quotient of this unit under τ is in μ_m and reduces to one. Therefore $\tau y(x)/y(x) = \zeta_m^{s_x}$, and likewise $\tau\gamma y(x)/\gamma y(x) = \zeta_m^{s_{ax}}$, since $\gamma(y(x))^m = J(ax)$. It follows that $F_\tau(x) = as_x - s_{ax}$. Summing over $x = b/L_1$ and reindexing b by ab, one obtains

$$a^e\psi(a)\,\mathcal{D}_\varnothing F_\tau = (a^{e+1}\psi(a) - 1)S_m(a,\psi).$$

This proves the base Stickelberger identity on the same B.

24.3.3 Comparison of Frobenius and inertia

In this subsection, we compute the Frobenius evaluation of the Jacobi cocycles of Section 24.3.2 in terms of inertia. This gives the precise local relation used with the reciprocity calculation of Section 24.3.1.

Retain the Jacobi family and $m = \ell^h$. Fix $r \in Q' \setminus S$, any prolongation $A \mid r$ and any lift ϕ of arithmetic Frobenius at A. Let x be an M_S-torsion point. Choose a unit $v \bmod r$ with $\log_r(v) = 1$ and choose τ in the inertia group at A acting by v^{-1} on the r-torsion roots of unity. It acts trivially on μ_{M_S} because $r \nmid M_S$. Use additive notation for $\overline{\mathbf{Q}}^\times$ in the following calculation. Write $Y_b = y(x+b/r)$, $Y_* = y(rx)$, and $\lambda(u) \in \mathbf{Z}$ for any integral lift of $\log_r(u)$.

For $\delta \in G_{\mathbf{Q}}$ set

$$W_\delta = \sum_{b \bmod r} Y_b - (\delta\phi\delta^{-1})Y_0 + \sum_{u\in(\mathbf{Z}/r)^\times} \lambda(u)\big((\delta\tau\delta^{-1})Y_{vu} - Y_u\big).$$

At the conjugate place δA, all Y_b are units. Its inertia acts trivially on their reductions and its Frobenius acts on the residue field by the r-th power. With $Z_b = Y_b - Y_0$, reduction of W_δ modulo principal units there gives

$$\sum_{u\in(\mathbf{Z}/r)^\times} \big(1 + \lambda(v^{-1}u) - \lambda(u)\big)Z_u.$$

Each coefficient is divisible by m; the Jacobi congruence says mZ_u is a principal unit at every place above r.

Thus W_δ is a principal unit at δA. This uses the Gauss-sum congruence at the conjugate place as well as at A. Set $R_u = \tau Y_{vu} - Y_u$ and $R_* = \phi Y_0 - Y_*$. The action of ϕ on μ_{M_S} is the r-th power, as reduction is injective there. Thus R_u, R_* are m-torsion by Jacobi equivariance. The distribution quotient gives that $m(\sum_b Y_b - Y_*)$ is torsion, hence $\sum_b Y_b - Y_*$ is torsion.

It follows that W_1 is a root of unity, so γW_1 is a power of W_1 and is still a principal unit at A. Also $\gamma W_{\gamma^{-1}}$ is a principal unit at A. The actions of γ and any of its conjugates agree on all roots of unity. Expand the difference of the two terms, using $\phi Y_0 = Y_* + R_*$ and $\tau Y_{vu} = Y_u + R_u$, to obtain

$$\gamma W_1 - \gamma W_{\gamma^{-1}} = \sum_u \lambda(u)\big(\gamma Y_u - (\tau\gamma\tau^{-1})Y_u\big) - \big(\gamma Y_* - (\phi\gamma\phi^{-1})Y_*\big).$$

The R_u and R_* terms cancel by the agreement on roots of unity.

The right side is m-torsion and a principal unit. Since $r \neq \ell$, reduction on μ_m is injective. The exponent therefore gives

$$\sum_{u\in(\mathbf{Z}/r)^\times} \log_r(u) F_\tau(x + u/r) = F_\phi(rx).$$

Here $r \notin S$ is prime to M_S and $r \equiv 1 \bmod L_1$, so $r^e\psi(r) = 1$. The scaling identity just proved integrates the local equation as

$$\mathcal{D}_{S\cup\{r\}}F_\tau = \mathcal{D}_S(F_\phi(r\,\cdot)) = \mathcal{D}_S F_\phi.$$

This gives $\kappa_{S\cup\{r\}}(\tau) = \kappa_S(\phi)$ for the given ϕ and $\tau \in I_A$. In particular the identity holds for the prolongation and Frobenius lift selected by simultaneous detection in Section 24.3.

25 Polynomial bounds for the Selmer lengths

In this section, we represent the finite cocycle conditions by a matrix over a polynomial ring in the twist variables. Its determinantal ideals control the lengths from Section 24 after specializing the twist.

Retain the normalized valuation and length from Section 24. Fix a number field E, a finite set Σ of finite places, an integral continuous representation ρ of G_E, and finitely many continuous additive characters $\chi_w : G_E \to \mathbb{Z}/\ell^N\mathbb{Z}$, where $N \geq 1$. For a finite extension of

coefficient rings $\mathcal{O} \subset \mathcal{O}'$ we use $\lambda(C) = \operatorname{length}_{\mathcal{O}'}(C)/\operatorname{ord}_{\mathcal{O}'}(\ell)$, so $v(\ell) = 1$ in every extension. For each $g \in G_E$ set

$$R_X(g) = \rho(g) \prod_w (1 + X_w)^{[\chi_w(g)]},$$

where the brackets denote the representative in $[0, \ell^N - 1]$. Take a fixed collection of homogeneous linear relations with coefficients in $\mathcal{O}$, each involving only finitely many evaluations. Over the ring of integers $\mathcal{O}'$ of a finite extension of $\operatorname{Frac}(\mathcal{O})$, denote by $C(x)$ the continuous functions into $(\mathcal{O}'/\ell^M)^n$ satisfying the cocycle equations for R_x, the chosen linear relations, and vanishing on every inertia group at the finite places outside Σ. At every integral x the equation is $z(gh) = z(g) + R_x(g)z(h)$. It is a homogeneous linear equation on the values of z even when exponent reduction modulo ℓ^N introduces a carry. At a character specialization these are the usual cocycle equations.

Lemma 25.1 (Polynomial minors for Selmer lengths)**.** *There is a finite set of integral polynomials P_j such that*

$$\lambda(C(x)) = \min_j v(P_j(x))$$

over every such coefficient ring and for each integral x. If j_0 attains the minimum at x_0, then

$$\lambda(C(x_0)) = v(P_{j_0}(x_0)), \qquad \lambda(C(x)) \le v(P_{j_0}(x)) \quad \textit{for every integral } x.$$

Proof. Let H be the open normal subgroup in the kernels of all χ_w and of $\rho \bmod \ell^M$, and let E_H be its fixed field. Every $z \in C(x)$ is additive on H, since $R_x(h) = I \bmod \ell^M$ there. The continuous homomorphisms from H to $\mathbb{Z}/\ell^M$ unramified outside the places above Σ form a finite set. Indeed their kernels cut out cyclic extensions of E_H of degree at most ℓ^M with this ramification bound, and those form a finite set by Hermite's finiteness theorem [48, Theorem 8.42]. Each such field admits finitely many homomorphisms. Let H_1 be the core in G_E of the intersection of their kernels in H. The dual into $\mathbf{Z}/\ell^M\mathbf{Z}$ separates points of the finite abelian group $(\mathcal{O}'/\ell^M)^n$. Applying its characters to each $z|_H$ gives

$$z|_{H_1} = 0, \qquad z(gh) = z(g) + R_x(g)z(h) = z(g) \quad (h \in H_1).$$

Thus z factors through G_E/H_1.

The finite quotient G_E/H_1 depends only on the original data and M, N, and works for every $\mathcal{O}'$ and x. In the original cocycle, inertia and linear equations replace each value $z(g)$ by the coordinate indexed by gH_1. Conversely, any solution on the quotient pulls back to a continuous solution on G_E. The rows of these equations generate a submodule of a finite free $\mathcal{O}[X]$-module. Since $\mathcal{O}[X]$ is Noetherian, finitely many of the rows span it. Let $A(X)$ be their matrix, with q columns. For every specialization the kernel of $A(x)$ acting on $(\mathcal{O}'/\ell^M)^q$ is exactly $C(x)$, since all the equations are equivalent to this finite selection after evaluation. Put

$$A^+(X) = \begin{pmatrix} A(X) \\ \ell^M I_q \end{pmatrix}, \qquad C(x) = \ker\Big(A^+(x) \bmod \ell^M\Big).$$

The bottom q rows give the constant maximal minor ℓ^{Mq}, so $A^+(x)$ has full column rank at every specialization. For $q = 0$ the empty minor is 1, and the length is zero. For $q > 0$ its Smith invariants $u_1, \dots, u_q$ divide ℓ^M, since its row module contains $\ell^M(\mathcal{O}')^q$. Multiplication

by u_i on $\mathcal{O}'/\ell^M$ has kernel $(\ell^M/u_i)\mathcal{O}'/\ell^M\mathcal{O}'$ of normalized length $v(u_i)$. The maximal minors $P_j(x)$ generate the ideal $(\prod_i u_i)$ by Smith reduction. Hence

$$\lambda(C(x)) = \sum_{i=1}^{q} v(u_i) = \min_j v(P_j(x)). \qquad \square$$

25.1 The line through the untwisted point

In this subsection, we restrict the polynomial matrix of Section 25 to a line through the untwisted specialization. At zero the distinguished pair has already been released. The determinant product bounds the cost of releasing the relations at the other auxiliary pairs along this line.

Let w_0 be the distinguished place, and fix integral slopes $s_w \in \mathcal{O}$ for $w \neq w_0$. Define the line by

$$x_{w_0}(Y) = Y, \qquad x_w(Y) = s_w Y \quad (w \neq w_0).$$

The first coordinate has coefficient one, while the untwisted local relation at w_0 used slope zero. At all other pairs assume the Frobenius and inertia values, norm congruences, and the conditions on $\rho \bmod \ell^M$ of Lemma 24.1. Write $C_0(x)$ for the module obtained by imposing the auxiliary relations at these pairs, so the first place is excluded from this set of relations. Suppose $\lambda(C_0(0)) \leq \beta$. This is the untwisted length bound after accounting for the first place and its conjugate.

Let J_{aux} index the auxiliary pairs where the relations are imposed. For each $w \in J_{\mathrm{aux}}$, put $\phi_{w,1} = \phi_w$, $\phi_{w,2} = (\phi_w^c)^{-1}$. Both evaluations of Lemma 24.1 enter the product

$$D_{\mathrm{aux}}(Y) = \prod_{w \in J_{\mathrm{aux}}} \prod_{i=1}^{2} \det((1+Y)\rho(\phi_{w,i}) - I).$$

Choose by Lemma 25.1 a polynomial G for C_0 that is sharp at 0. For every y satisfying the scalar condition of Lemma 24.1, the exact sequence for $C_0(x) \subset C(x)$ gives

$$\lambda(C(x)) \leq v(G(x)) + v(D_{\mathrm{aux}}(y)).$$

The degree of D_{aux} is $D_0 = 2n|J_{\mathrm{aux}}|$ and its leading coefficient $\prod_{w,i} \det \rho(\phi_{w,i})$ is a unit. The bound on $|J_{\mathrm{aux}}|$ from Section 27.3 is independent of N. Fix a positive rational radius a_0 and require $N \min(1, a_0) \geq M$ before selecting the auxiliary places at level ℓ^N. For $1 \leq j \leq \ell^N$ put $k = v_\ell(j)$. As $k \leq N$ and $j - 1 \geq k$, for $v(z) \geq a_0$ we have

$$v\left(\binom{\ell^N}{j} z^{j-1}\right) \geq N - k + (j-1)a_0 \geq N \min(1, a_0).$$

This gives the scalar condition of Section 24 on the whole line in every coefficient extension.

For $q(Y) = \sum_k q_k Y^k$ put $v_{a_0}(q) = \min_k(v(q_k) + ka_0)$. With $g(Y) = G(x(Y))$, the constant coefficient of g and the leading coefficient of D_{aux} give

$$v_{a_0}(g) \leq v(G(0)) \leq \beta, \qquad v_{a_0}(D_{\mathrm{aux}}) \leq D_0 a_0.$$

Choose y_0 of valuation a_0. For $q = g, D_{\mathrm{aux}}$, divide $q(y_0 U)$ by a coefficient of least valuation to obtain a nonzero reduction. Lift a nonzero residue avoiding the zeros of both reductions to a unit u in a finite coefficient extension. At $y = y_0 u$ both evaluations attain their Gauss valuations. Choose a minor P for C attaining its minimum at $x = x(y)$. It satisfies

$$v(x_w) \geq a_0, \qquad v(P(x)) \leq \beta + D_0 a_0, \qquad \lambda(C(x')) \leq v(P(x')) \quad \text{for every } x'.$$

25.2 Transfer to divisible coefficients

In this subsection, we apply the finite-length estimate of Section 25 to divisible coefficient modules at torsion specializations. We check that the same inertia and linear conditions identify the finite modules with their torsion subgroups.

At a tuple of ℓ^N-power torsion points $x'_w = \zeta_w - 1$ with $\zeta_w^{\ell^N} = 1$, the specialized twist is a scalar character. Let C_∞ be the module of cocycles for this twist with coefficients in $(\mathrm{Frac}(\mathcal{O}')/\mathcal{O}')^n$, with the same inertia and linear conditions. Write ϖ' for a uniformizer and $e' = \mathrm{ord}_{\mathcal{O}'}(\ell)$. The equivariant coefficient isomorphism

$$\mathcal{O}'/\ell^M\mathcal{O}' \longrightarrow (\mathrm{Frac}(\mathcal{O}')/\mathcal{O}')[\ell^M], \qquad b \longmapsto b/\ell^M \pmod{\mathcal{O}'}$$

preserves all homogeneous relations, hence identifies $C(x')$ with $C_\infty[\ell^M]$. Every continuous cocycle has finite image since G_E is compact and its target is discrete. If a cocycle has annihilator (ϖ'^b) with $b > e'M$, multiplication by $\varpi'^{b-e'M}$ gives a cyclic submodule of $C(x')$ of normalized length $e'M/e' = M$. Hence, for the full set of maximal minors in Lemma 25.1,

$$\min_j v(P_j(x')) < M \quad\Longrightarrow\quad C_\infty = C_\infty[\ell^M], \qquad \lambda(C_\infty) = \lambda(C(x')) = \min_j v(P_j(x')).$$

In particular, the minor P chosen on the test line gives $\lambda(C_\infty) \le v(P(x'))$ whenever $v(P(x')) < M$. This holds over every finite extension containing the coefficients and the torsion tuple.

26 Simultaneous evaluation on torsion

In this section, we choose torsion specializations at which finitely many polynomials satisfy their valuation bounds simultaneously. The choice will apply to the Selmer minors of Section 25 and the group-ring evaluations of Section 27.

Let v be the additive valuation on $\overline{\mathbb{Q}}_\ell$, normalized by $v(\ell) = 1$, and write $\mathcal{O}$ for its valuation ring. The following evaluation lemma gives a torsion exponent before the auxiliary primes or their polynomials are chosen.

Lemma 26.1 (Simultaneous torsion specialization)**.** *Fix a variable bound $s_0 \in \mathbb{Z}_{\ge 0}$, an integer $m \ge 1$, a positive rational radius a, a tolerance $\epsilon > 0$, and a finite list of real bounds $B_i \ge 0$. Use the convention $v(0) = +\infty$. Set*

$$\rho = a/m, \qquad B = m\sum_i B_i, \qquad \delta = \frac{\min(1,\epsilon)}{s_0+2}, \qquad \rho_h = \varphi(\ell^h)^{-1} \quad (h \ge 1).$$

Let H_1 be the least positive integer with $\rho_{H_1} \le \rho$. Any $H \ge H_1$ satisfying

$$B + 1 < \delta \sum_{h=H_1}^{H} \frac{1}{h+1}$$

works. For example one can take

$$H = \lceil (H_1+1)\exp((B+1)/\delta) \rceil$$

since

$$\sum_{h=H_1}^{H} \frac{1}{h+1} \ge \log\frac{H+2}{H_1+1} > \frac{B+1}{\delta}$$

for this choice. For every family of integral polynomials P_i in the same $s \le s_0$ variables and points x_i such that

$$v(x_{i,w}) \ge a, \qquad v(P_i(x_i)) \le B_i,$$

there are roots of unity $\zeta_w^{\ell^H} = 1$ with $v(\zeta_w - 1) \le \rho$ and

$$v\big(P_i((\zeta_w^j - 1)_w)\big) \le v(P_i(x_i)) + \epsilon \qquad (1 \le j \le m,\ \textit{all } i).$$

Proof. One coordinate. For a nonzero integral polynomial g, write

$$v_t(g) = \min_b \{v(g_b) + bt\}.$$

This valuation is multiplicative on products in additive notation. To see this for rational t, scale the variable by an element of valuation t, then divide each scaled polynomial by a coefficient of minimum valuation. The reductions are nonzero polynomials over a field and have nonzero product. By factorization, with leading coefficient g_{top},

$$v_t(g) = v(g_{\mathrm{top}}) + \sum_\alpha \min(t, v(\alpha)),$$

where roots are counted with multiplicity. In particular $v_t(g) \le v(g(y))$ for $v(y) \ge t$, and $v_t(g)$ is increasing in t.

Suppose $v_\rho(g) \le B + 1$. The polynomial is integral, so $v_0(g) \ge 0$. Subtracting the root formula at radius zero from that at radius ρ gives a sum of the nonnegative terms $\min(\rho, v(\alpha)) - \min(0, v(\alpha))$. Since the radii $\rho_h \le \rho$ are distinct, the multisets R_h of roots with $v(\alpha) = \rho_h$ obey

$$\sum_{h=H_1}^{H} |R_h| \rho_h \le v_\rho(g).$$

Thus there is an h with $|R_h|\rho_h(h+1) \le \delta$, by the harmonic-sum inequality. For each $\alpha \in R_h$, the polynomial

$$F_\alpha(Y) = (1 + \alpha + Y)^{\ell^h} - 1$$

is monic and has linear coefficient of valuation h, because $v(\alpha) > 0$.

Its roots include $Y = \zeta - 1 - \alpha$ for the primitive ℓ^h-th roots ζ. All roots of F_α are of positive valuation. Applying the Gauss valuation at radius 1,

$$\sum_{\zeta \text{ primitive}} \min\{1, v(\zeta - 1 - \alpha)\} \le v_1(F_\alpha) \le h + 1.$$

There are $1/\rho_h$ such roots of unity. Sum over $\alpha \in R_h$ and average to find one ζ with the sum of the truncated valuations at most $|R_h|\rho_h(h+1) \le \delta$.

Since $\delta < 1$, each truncation is inactive. All primitive roots of this order have $v(\zeta - 1) = \rho_h$, by taking the norm of $1 - \zeta$ or using the cyclotomic polynomial at 1. For $y = \zeta - 1$, the roots of g with valuation different from ρ_h have $v(y - \alpha) = \min\{\rho_h, v(\alpha)\}$. The other roots give

$$v(g(y)) - v_{\rho_h}(g) = \sum_{\alpha \in R_h} \big(v(y - \alpha) - \rho_h\big) \le \delta.$$

Now let there be finitely many polynomials g_i, with their old arguments u_i of valuation at least ρ and $\sum_i v(g_i(u_i)) \le B+1$. Apply this argument to $g = \prod_i g_i$. Since $v_{\rho_h}(g_i) \le v_\rho(g_i) \le v(g_i(u_i))$ and $v(g_i(y)) \ge v_{\rho_h}(g_i)$, multiplicativity gives

$$\sum_i \max\{0, v(g_i(y)) - v(g_i(u_i))\} \le \sum_i (v(g_i(y)) - v_{\rho_h}(g_i)) = v(g(y)) - v_{\rho_h}(g) \le \delta.$$

All coordinates and powers. For each pair (i,j), $1 \le j \le m$, replace $P_i(X)$ by the integral polynomial

$$Q_{i,j}(Y) = P_i(((1+Y_w)^j - 1)_w).$$

For each old coordinate $x = x_{i,w}$ the monic polynomial $(1+Y)^j - (1+x)$ has j integral roots whose product up to sign is x. For $x \neq 0$, the sum of their valuations is $v(x)$, so one root has valuation at least $a/j \ge \rho$. For $x = 0$ choose the root 0. Use these roots as the test point of $Q_{i,j}$. Its evaluation is $P_i(x_i)$; the sum of the initial valuations over (i,j) is at most B.

Process the coordinates one at a time and freeze the others at their current values for each test point. On the coordinate currently being replaced the old valuations are at least ρ. The sum of the evaluation valuations after k replacements is at most $B + k\delta \le B+1$. Hence the one-coordinate argument applies with the same H at every step, including when the previous replacements have valuations below ρ. They remain integral, so the resulting one-variable polynomials have integral coefficients. Each is nonzero since its evaluation at the current test point has finite valuation. The common final points are $(\zeta_w - 1)_w$. Summing the positive changes bounds the increase at each (i,j) by $s\delta \le \epsilon$. This proves the assertion. □

27 A common twist for the Selmer and Euler-factor conditions

In this section, we choose one twist for the cocycle conditions of Section 25 and the Euler-factor relations of the norm systems. The auxiliary primes and torsion specialization are arranged so that both estimates apply to this same character.

Fix a coefficient place $\kappa = (\ell, \iota)$, normalize $v(\ell) = 1$, and take a finite family of the Selmer data of Section 25. Use a common finite coefficient DVR for their lattices. In this section N denotes the exponent of the auxiliary level ℓ^N. Fix also finitely many norm systems, each consisting of a family $(\Theta_Q)_Q$ of integral group-ring elements over $\overline{\mathbf{Q}}_\ell$, indexed by the allowed auxiliary sets. For each system let ϕ_w denote a lift of geometric Frobenius at the new place w, in the reciprocity convention of Section 27.2. For a character ψ factoring through level Q, require that its evaluation at $\Theta_{Q\cup\{w\}}$ is its evaluation at Θ_Q multiplied by

$$E_w(\vartheta(\phi_w)\psi(\phi_w)), \qquad E_w(A) = \prod_{e\in\mathcal{E}} (1 - Aq_w^{-e})(1 - A^{-1}q_w^{-e}).$$

Here q_w is the residue cardinality, $\vartheta : G_E \to \mathcal{O}^\times$ is the continuous character of that system, unramified at the allowed auxiliary places, and $\mathcal{E}$ is its fixed finite list of integer exponents. Thus $\vartheta(\phi_w)$ is independent of the lift of Frobenius at each such place. In the application to the signed systems of Section 38.1, ψ is the inverse ℓ-adic avatar of the auxiliary norm-one character. Both signs are included among the systems before the choice of ψ.

27.1 A specialization of a norm system

In this subsection, we express each group-ring evaluation from Section 27 as a polynomial in the chosen character coordinates. The norm relation determines the divisibility along the line used for the simultaneous specialization.

Use one list J of auxiliary primes with distinguished prime w_0, equipped with characters χ_w of order dividing ℓ^N as in Section 24. Write $\chi_w : G_E \to \mathbf{Z}/\ell^N\mathbf{Z}$ additively. It comes from a character of the norm-one classes at level w, is zero on the decomposition groups above ℓ, and takes the value 1 at the chosen tame generator at w. The other characters are zero there. At the Frobenius and its conjugate inverse for every subsequent prime, the first character has value 1 and the other characters are zero. All q_w are 1 mod ℓ^N. For a norm-one class represented by $u/\bar{u}$, read $\chi_w(\mathrm{Art}\, u)$ using the induced class character. Put $h = \ell^N$ and take $[c]_h \in \{0, \dots, h-1\}$ for $c \in \mathbf{Z}/h\mathbf{Z}$. The assignment on class elements

$$[u/\bar{u}] \longmapsto \prod_{w \in J} (1 + X_w)^{[\chi_w(\mathrm{Art}\, u)]_h}$$

is multiplicative in $\mathcal{O}[X_w : w \in J]/((1+X_w)^h - 1 : w \in J)$. Define P_J in the polynomial ring by summing these displayed representatives with the coefficients of Θ_J. Each variable has degree less than h. For $X_w = \zeta_w - 1$ with $\zeta_w^h = 1$ the value is the evaluation at $\prod_w \zeta_w^{\chi_w}$.

Restrict P_J to the line with $X_{w_0} = Y$ and all other variables zero, and call the resulting polynomial $P_J^0(Y)$. Let $R_0(Y)$ denote the polynomial for the same one-variable evaluation of $\Theta_{\{w_0\}}$. Put $j_0 = |\mathcal{E}|(|J| - 1)$. With $A_w = \vartheta(\phi_w)$, the product

$$R_1(Y) = \prod_{w \neq w_0} \prod_{e \in \mathcal{E}} (1 - A_w q_w^{-e}(1 + Y))(1 + Y - A_w^{-1} q_w^{-e})$$

has degree $2j_0$ and unit leading coefficient. The norm relation iterated away from w_0 gives

$$(1 + Y)^{j_0} P_J^0(Y) \equiv R_0(Y) R_1(Y) \pmod{(1 + Y)^{\ell^N} - 1}.$$

Indeed at $Y = \zeta - 1$ the character is $\zeta^{\chi_{w_0}}$. Its value on the Frobenius at $w \neq w_0$ is ζ, so each pair of factors of R_1 equals ζ times the Euler-factor pair. This gives equality at every root of the modulus, which is monic and has simple roots. Polynomial division gives the congruence over the valuation ring.

For the first-prime Euler factor, fix g_0 with the following finite bound

$$\beta = v(1(\Theta_\varnothing)) + 2|\mathcal{E}| v(\vartheta(g_0) - 1).$$

The second term is zero for an empty exponent list. Choose $M' > \beta + 1$, and require $N \geq M'$ and $v(\vartheta(\phi_{w_0}) - \vartheta(g_0)) \geq M'$. If $\mathcal{E}$ is nonempty, each of the two factors for each e at the first prime and trivial character has valuation $v(\vartheta(g_0) - 1)$. Indeed $q_{w_0}^{-e}$ is 1 mod $\ell^{M'}$, and $\vartheta(\phi_{w_0})$ and its inverse agree with $\vartheta(g_0)$ and its inverse, respectively, to that precision. For a nonempty list the integrality of the augmentation and $M' > \beta$ imply $v(\vartheta(g_0) - 1) < M'$. For an empty list the Euler product is one. In both cases the norm relation gives $v(R_0(0)) = \beta$.

Let $0 < a_0 \leq 1$ be rational. The Gauss valuation of $R_0 R_1$ at radius a_0 is at most $\beta + 2j_0 a_0$, by the constant coefficient of R_0 and the leading coefficient of R_1. For $v(y) = a_0$, the binomial expansion shows

$$v((1 + y)^{\ell^N} - 1) \geq N a_0.$$

Indeed, for $1 \le j \le \ell^N$, put $k = v_\ell(j) \le N$. The identity $j\binom{\ell^N}{j} = \ell^N \binom{\ell^N - 1}{j-1}$ gives

$$v\binom{\ell^N}{j} + ja_0 \ge N - k + ja_0 \ge Na_0, \qquad (N-k)(1-a_0) + (j-k)a_0 \ge 0.$$

The coefficient bound just proved gives $v_{a_0}((1+Y)^{\ell^N} - 1) \ge Na_0$. The quotient in the cyclic polynomial congruence is integral, so the difference of its two sides has Gauss valuation at least Na_0. Since $v_{a_0}(1+Y) = 0$, whenever $\beta + 2j_0a_0 < Na_0$ this gives

$$v_{a_0}(P_J^0) = v_{a_0}(R_0R_1) \le \beta + 2j_0a_0.$$

For finitely many such systems i on J, choose c of valuation a_0 and divide each $P_{i,J}^0(cZ)$ by a coefficient of minimum valuation. All these reductions are nonzero. Their product has finitely many roots in $\overline{\mathbf{F}}_\ell$. Lift a nonzero residue outside this set to a unit u, passing to a finite unramified extension of the coefficient field containing c and the polynomials if needed. At $y = cu$ we obtain simultaneously

$$v(P_{i,J}^0(y)) = v_{a_0}(P_{i,J}^0) \le \beta_i + 2|\mathcal{E}_i|(|J| - 1)a_0.$$

The other coordinates are zero on this line. The degree of the product of reductions need only be known after J is chosen.

27.2 The distinguished auxiliary prime

In this subsection, we use the class-field conditions of Section 27 to choose the first auxiliary prime. Its Frobenius lies in the specified neighborhood, and its local character has the prescribed restrictions at the other places.

Let E be CM and let $g_0 \in G_E$ lie in the normal subgroup generated by inertia at a finite set of places R_{in} away from ℓ. Fix a neighborhood $\mathcal{W}$ of g_0, a finite set of forbidden primes, and a finite set S_{sp} of places at which the new character must be trivial on decomposition groups. Require the conjugate closure of S_{sp} to be disjoint from R_{in}. For each $N \ge 1$ there is a prime of prime residue cardinality 1 mod ℓ^N, outside the forbidden set with its conjugate and distinct from its conjugate, equipped with a character of the kind used in Section 27. Its geometric Frobenius lift can be chosen in $\mathcal{W}$ with character value zero.

To prove this, choose a finite set T, stable under conjugation, containing S_{sp} and the places above ℓ, and avoiding R_{in} and its conjugate. Require the primes in T to generate the ideal class group. Such a set exists by Chebotarev in the Hilbert class field, which gives a prime in every ideal class outside any finite set [50, Ch. V, Theorems 3.23 and 5.5]. Let $\mathbb{A}_{E,f}^{\times}(T)$ denote the finite ideles that are units outside T. Since T generates the class group, the ideal of any finite idele differs from an ideal supported on T by a principal ideal. Thus

$$\mathbb{A}_{E,f}^{\times} = E^{\times}\mathbb{A}_{E,f}^{\times}(T), \qquad E^{\times} \cap \mathbb{A}_{E,f}^{\times}(T) = \mathcal{O}_{E,T}^{\times}.$$

Put $h = \ell^N$ and take the field obtained by adjoining μ_h and the h-th roots of a finite generating set of the T-units. The generating set exists by the unit theorem, since valuation at T has kernel the ordinary unit group and image in $\mathbb{Z}^T$. This Kummer extension is Galois over E and unramified at each place of R_{in}: the T-units are units there and h is prime to the residue characteristic.

Their h-th roots and μ_h can be obtained in an unramified extension by lifting the distinct roots over a residue field extension. Choose a finite Galois extension of E such that the coset of the kernel containing g_0 lies in $\mathcal{W}$, and enlarge it by the Kummer extension. Prescribe the conjugacy class of g_0^{-1} for arithmetic Frobenius in this compositum. Chebotarev gives w_0 and a place above it with this arithmetic element in the quotient [50, Ch. V, Theorem 3.23]. Its inverse lifts to geometric Frobenius in $\mathcal{W}$. We may avoid T, the forbidden primes and their conjugates, and the primes ramified over $\mathbb{Q}$. We may also take w_0 of residue degree one over $\mathbb{Q}$, since prime ideals of higher absolute residue degree have density zero. Then $w_0 \neq \bar{w}_0$, as an unramified prime fixed by the nontrivial conjugation of E/E^+ has relative residue degree two.

The Frobenius of w_0 fixes the Kummer extension, so $q_{w_0} = 1 + hk$ for an integer k. For each generating T-unit u, its root $u^{1/h}$ reduces to a nonzero element fixed by the residue Frobenius. Hence the reduction of u is an h-th power in the residue field of w_0. The discrete logarithm of residue units modulo h kills these h-th powers and therefore all T-units.

The residue logarithm on the w_0-component of a T-idele kills $\mathcal{O}_{E,T}^{\times}$, so the displayed quotient identifies it with a character θ of the finite idele classes. Its kernel contains an open subgroup of finite ideles, giving continuity. Extend it trivially at the complex places. By reciprocity [50, Ch. I, Theorem 1.1 and Ch. V, Theorems 5.3 and 5.5], $\theta(u/\bar{u})$ defines an anticyclotomic Galois character, trivial on the decomposition groups at S_{sp} and above ℓ, and unramified away from $w_0, \bar{w}_0$. Indeed a local idele supported at a place of T and its conjugate have unit component 1 at w_0, even for a local uniformizer. Thus the character vanishes on the full decomposition groups there. If u is a unit idele supported at w_0, the w_0-component of $u/\bar{u}$ is u_{w_0}, since $w_0 \neq \bar{w}_0$. Its residue logarithm maps onto $\mathbb{Z}/h$, including for $h = 2^N$. Normalize the logarithm to read 1 on the chosen tame generator γ_{w_0}. It then reads 1 on $(c\gamma_{w_0}c^{-1})^{-1}$ as well.

Multiplying the Frobenius lift by a power of this tame generator makes the character value of Frobenius zero. This correction does not alter the chosen finite Frobenius, as the compositum used for Chebotarev is unramified at w_0. The corrected lift still lies in $\mathcal{W}$.

27.3 Selmer length descent and the uniform bound

In this subsection, we iterate the local relations of Section 24 until the remaining module can be treated by Section 27.4. The argument bounds the number of added primes independently of the larger torsion order.

Fix the coefficient ring $\mathcal{O}$, its uniformizer ϖ, the finite data indexed by i of ranks n_i, and M. Set $e = \mathrm{ord}_{\mathcal{O}}(\ell^M)$. If necessary, extend $\mathcal{O}$ by a finite unramified extension so that its residue field has more than $2en_i + 1$ elements for every i. This preserves e and the ranks. Use this ring throughout the descent, before computing the initial length bound or choosing the auxiliary exponent N. In this subsection lengths are unnormalized. For an auxiliary list with prescribed congruences J and $1 \leq a \leq e$, let $C_{i,a}(J)$ be the untwisted cocycles modulo ϖ^a with the original linear conditions, allowed ramification at J and its conjugates, and the given congruences at each auxiliary pair. Let $C_{i,a}(\varnothing)$ have just the original ramification and linear conditions. The lengths are finite by the finite-presentation argument.

Apply Section 27.2 at exponent $N > M$ to the inertial reference g_0 in the prescribed common neighborhood $\mathcal{W}$, with $S_{\mathrm{sp}} = \varnothing$. The neighborhood and its precisions are specified below before increasing N. Forbid every place above the rational primes dividing the norms of the original bad places. Start with slope zero at both members of the distinguished pair. For each datum i, depth a, and eligible new pair, consider the following condition.

The two Frobenius values of every member of $C_{i,a}(J)$, at that prime and its conjugate inverse, can be realized as the two tame values of a cocycle with the old congruences and ramification also allowed at the new prime pair. If this condition holds for every eligible pair for i at depth a, and some cocycle in $C_{i,a}(J)$ has a unit coordinate at some group element, Section 27.4 bounds the length by $G_i + L_i$, where G_i, L_i are the datum's global and original-place terms, in unnormalized length.

An eligible prime P has prime norm, is distinct from its conjugate, and both P and $\bar{P}$ are outside the forbidden places and the previously chosen pairs. The prime P does not lie over the rational norm of w_0. Its character is trivial on the decomposition groups above ℓ and at both members of each old pair other than $\{w_0, \bar{w}_0\}$, and vanishes on its own Frobenius lift.

On this lift, the old character at w_0 reads 1 and the other old characters read 0. The new prime has the same tame-generator and norm conditions as the original list. For the selected datum i, the representation modulo ℓ^M is trivial on both Frobenius parameters. This makes every nonzero slope admissible in that datum.

Here is the local calculation used to choose a slope. At an eligible new prime pair let C be the enlarged group with the old congruences, and for this datum and depth write $A, B : C \to (\mathcal{O}/\varpi^a)^{2n_i}$ for the two tame and the two Frobenius evaluations respectively. Put

$$C' = \sum_{u \in \mathcal{O}} \ker_C(uA - B).$$

For $c \in \ker(uA - B)$ one has $Bc = A(uc)$, so $B(C') \subseteq A(C')$. Let L' be the inverse image of $A(C')$ in $\mathcal{O}^{2n_i}$, with reduction $\pi : L' \twoheadrightarrow A(C')$.

It is free of rank $2n_i$ since it contains $\varpi^a \mathcal{O}^{2n_i}$. Lift π through $A : C' \twoheadrightarrow A(C')$ to $\sigma : L' \to C'$, then lift $B\sigma$ through π to an endomorphism U' of L'. This gives $(uA - B)\sigma = \pi(uI - U')$. Exclude the roots of the characteristic polynomial of U' modulo ϖ. There are at most $2n_i$ excluded residues. Outside these residues $uI - U'$ is invertible, so the image of $uA - B$ on C' equals $A(C')$.

Since its kernel on C is contained in C', the length identity for the two surjections onto $A(C')$ gives

$$\operatorname{length} \ker_C(uA - B) = \operatorname{length}(C' \cap \ker A) \le \operatorname{length}\big(\ker_C A \cap B^{-1}(A(C))\big).$$

The representations are unramified at the new prime pair, and the additive inertia restrictions with these coefficients factor through the pro-ℓ tame quotient. Thus $\ker_C A = C_{i,a}(J)$, since the chosen generators detect vanishing on that quotient. For a unit u the new congruence has slope u^{-1}, and the inequality bounds its length by that of the old cocycles satisfying the Frobenius-value property at the new pair.

If the Frobenius-value property fails at any datum and depth, choose an eligible prime where the condition fails. For this datum exclude the union of the bad residues over $1 \le a \le e$ and zero. The residue-field bound supplies one unit u for the chosen datum at all depths. Give it slope u^{-1}, and give the other data slope zero. For the selected datum the upper bound is the length of the old submodule $C_{i,a}(J) \cap B^{-1}(A(C))$. At the failing depth this is a proper submodule, hence the length drops by at least one. For the other data, slope zero recovers unramifiedness at the new pair, so their lengths do not increase. The sum of lengths over all i, a decreases strictly. For the initial singleton with slope zero this sum is at most

$$B_0 = \sum_i \sum_{a=1}^{e} \operatorname{length} C_{i,a}(\varnothing).$$

The same inertia argument justifies this bound at the first prime. It follows that the process terminates with at most $1 + B_0$ primes, independently of N.

At termination the Frobenius-value property holds at every depth for each eligible prime. To apply the primitive bound to the group at depth e, consider any depth a whose group has no member with a unit value coordinate. Every value then lies in $\varpi\mathcal{O}/\varpi^a$. The isomorphism $T/\varpi^{a-1}T \xrightarrow{\varpi} \varpi T/\varpi^a T$ therefore identifies the group at depth $a-1$ with that at depth a. Indeed the same integral homogeneous linear equations define the cocycle, local, and auxiliary conditions at both depths, and $\varpi t = 0 \bmod \varpi^a$ exactly when $t = 0 \bmod \varpi^{a-1}$. Descend until there is a unit value coordinate or the depth is zero. Thus the terminal bound gives length at most $G_i + L_i$ for $C_{i,e}(J)$. The first-place bound of Section 24 then bounds releasing the distinguished pair at zero. After dividing by $\operatorname{ord}_{\mathcal{O}}(\ell)$ this gives the bound β_i for the untwisted length used in Section 25. The argument uses the hypotheses on the representation of Section 27.4.

27.4 Duality for the final local representation

In this subsection, we analyze the residual classes at the end of the descent in Section 27.3. The Cartier duality calculation gives the necessary relation between the two representations at the chosen torsion depth.

Consider an untwisted specified space at a torsion depth a with $V = T/\varpi^a T$, $T = \mathcal{O}^n$, and $a \le \operatorname{ord}_{\mathcal{O}}(\ell^M)$. Put $\Gamma = G_E$ and write $g^c = cgc^{-1}$ for the specified complex conjugation. For the Cartier dual put $V^* = \operatorname{Hom}_{\mathbf{Z}}(V, \mu_{\ell^N})$, with scalar action $(tb)(v) = b(tv)$ and the contragredient Galois action twisted by the cyclotomic character. Here $\ell^M V = 0$ and $N > M$. Suppose that on the original lattice T there is an element $s \in \Gamma$ satisfying

$$\rho(s) = \rho((s^c)^{-1}) = \xi I, \qquad \chi_{\mathrm{cyc}}(s) = 1, \qquad \xi^2 - 1 \in \mathcal{O}^\times.$$

Suppose s lies in inertia at an original place prime to ℓ and the matrices of ρ span the full matrix algebra over $\mathcal{O}$. The cocycles z satisfy $z(s) = 0$. At termination assume the Frobenius-value property of Section 27.3 at this depth. We prove its length bound for a group containing a cocycle z with a unit coordinate at some group element.

Let S' contain the original ramification set of the representation, including the places above ℓ, and the later auxiliary primes and their conjugates. The first auxiliary pair is outside S'. For $v \in S'$, let A_v be the restrictions to the local group of cocycles satisfying the original local relations and the old congruences, with neither a ramification constraint nor the normalization at s.

Assume the original local conditions contain the coboundaries. The congruence conditions also contain them, since inertia acts trivially at the auxiliary primes and Frobenius acts trivially when the slope is nonzero. Let $\mathcal{L}_v$ be the image of A_v in local cohomology. At each finite $v \notin S'$, impose the unramified condition. Here V is unramified and its order is prime to the residue characteristic. The unramified subgroups for V and V^* are exact annihilators under local Tate duality [49, Ch. I, Theorem 2.6]. In particular, the dual condition is unramified at the first auxiliary pair and at each eligible new pair outside S'. The local length estimate needed here is

$$\sum_{v \in S'} \operatorname{length} A_v \le (|S'| + f) \operatorname{length} V + L_i.$$

The old auxiliary places contribute at most $\operatorname{length} V$ each because the Frobenius value determines the tame value by the congruence and determines the rest of the cocycle with that inertia

value. Thus the remaining local bounds concern the excess L_i at the original places away from ℓ and the crystalline rank bound at the places above ℓ.

The cocycle module C injects into $H^1_{\mathcal{L}}(E,V)$. At the first pair the zero slope forces unramifiedness, and if a coboundary vanishes at s its defining vector is killed by the unit $\xi-1$. Write $D^\vee_{\mathcal{L}} = H^1_{\mathcal{L}^\perp}(E,V^*)$ for the ordinary Selmer group using the local Tate orthogonal conditions. Since all infinite places of E are complex, the archimedean factors below apply also at $\ell = 2$. The Greenberg–Wiles formula [27, Theorem 2.19], in unnormalized length, gives

$$\operatorname{length} H^1_{\mathcal{L}}(E,V) = \operatorname{length} D^\vee_{\mathcal{L}} + \sum_{v\in S'} (\operatorname{length} A_v - \operatorname{length} V) - f \operatorname{length} V.$$

Indeed both global invariants vanish by the action of s. Since A_v contains the coboundaries, the local ratio $|\mathcal{L}_v|/|H^0(E_v,V)|$ is $|A_v|/|V|$. Unramified places give ratio one and complex places give $1/|V|$. Taking logarithms to the residue field cardinality gives the displayed identity. In particular $\operatorname{length} C \le \operatorname{length} D^\vee_{\mathcal{L}} + L_i$.

Put $U = \ker(\rho \bmod \ell^M) \cap \ker(\chi_{\mathrm{cyc}} \bmod \ell^N)$. For a dual class $[b]$, write $y = b|_U$. This is additive and equivariant under conjugation, and is independent of the representative since U acts trivially. Restriction is injective on dual classes. Indeed its kernel inflates from $H^1(\Gamma/U, V^*)$, where the image of s is central and acts by ξ^{-1}. Comparing a cocycle at sg and gs shows that such a cocycle is a coboundary because $s-1$ acts invertibly on V^*.

Let H be generated by $[g,s]$ and $[g,s^c]$, $g\in\Gamma$, with $[g,s] = gsg^{-1}s^{-1}$. The identity $[ag,s] = a[g,s]a^{-1}[a,s]$, also for s^c, gives normality, and conjugation by c interchanges the two sets of generators. Both scalar actions commute with the whole image on $T/\ell^M T$; the cyclotomic character kills commutators. Thus $H \subset U \cap U^c$. The identity $z([g,s]) = (1-\xi)z(g)$ and equivariance on U show that the $\mathcal{O}$-span of $z(H)$ is V: its reduction contains a nonzero vector, is stable under all matrices, and Nakayama's lemma applies.

Identify μ_{ℓ^N} additively using one fixed primitive root of unity. For $t\in\mathcal{O}$ and $u,u'\in U\cap U^c$ put

$$F_t(u,u') = y(u')(tz(u)) - y(u^c)(tz(u'^c)).$$

For $h\in H$, apply the prime construction of Section 27.2 to the target $\gamma_{w_0}h$. The commutators defining H lie in the normal subgroup generated by inertia at the original place of s and its conjugate. Together with the first prime these places form R_{in}.

Let J index the old auxiliary pairs, with distinguished representative w_0. Take

$$S_{\mathrm{sp}} = \bigcup_{w\in J\setminus\{w_0\}} \{w,\bar w\}.$$

The distinguished pair $\{w_0,\bar w_0\}$ belongs to R_{in}, so the splitting conditions apply only to the other pairs. Exclude the old prime pairs, the original forbidden primes and all places above the rational norm of w_0. The target and its conjugate are in U. Every old character kills h, since H is generated by commutators. At γ_{w_0} only the first character is nonzero and reads 1. Prescribe these finitely many residual and character evaluations, as well as the locally constant values of y,z and their conjugates, in the Frobenius neighborhood.

The new character vanishes on its own Frobenius, on every G_{E_v} for $v\mid\ell$, and on G_{E_w} and $G_{E_{\bar w}}$ for $w\in J\setminus\{w_0\}$. It is unramified at the original level places away from ℓ. Anticyclotomicity preserves the character values on the inverse conjugate parameter. Hence this produces an eligible pair with $\phi,\phi^c\in U$. The cocycles y,z vanish at the first tame pair because both local classes there are unramified and inertia acts trivially. Thus $F_t(h,h) = F_t(\phi,\phi)$.

Choose z' by the terminal property, with $z'(\gamma_P) = z(\phi)$ and $z'((\gamma_P^c)^{-1}) = z((\phi^c)^{-1})$. Apply global reciprocity to $b \cup (tz')$. Its local invariants vanish on S' by orthogonality, and outside S' and the new pair by unramifiedness. At the new pair the coefficient actions are trivial and b is unramified. The local invariant reads the Frobenius value of b on the tame value of tz', with respect to the generator determined by the fixed root of unity.

For this formula, subtract from z' the Kummer cocycle of a uniformizer times the normalized tame value of z'. The difference is unramified and its cup product with b has zero invariant. To check the sign, put $d_0 = \ell^N$, choose a local uniformizer π and $r^{d_0} = \pi$, and write $\kappa_\pi(g) = g(r)/r$. For the resulting unramified character χ with values in $\mathbf{Z}/d_0\mathbf{Z}$, take integer lifts $\widetilde{\chi}(g)$. The coboundary of $g \mapsto r^{\widetilde{\chi}(g)}$ at (g, h) is

$$\pi^{(\widetilde{\chi}(g)+\widetilde{\chi}(h)-\widetilde{\chi}(gh))/d_0}\kappa_\pi(g)^{\widetilde{\chi}(h)}.$$

Thus graded commutativity identifies $\chi \cup \kappa_\pi$ with $\pi \cup \delta(\chi/d_0)$ in the Brauer group, where χ/d_0 takes values in $\mathbf{Q}/\mathbf{Z}$. The local invariant is $\chi(\phi)/d_0$ for arithmetic Frobenius ϕ [50, Ch. III, Proposition 3.6 and Lemma 3.7]. For the chosen geometric Frobenius and its geometric conjugate the invariant has the opposite sign at both places. The inverse conjugate of a normalized tame generator is normalized by the same root of unity at the conjugate place. The two chosen generators therefore differ from the normalized generators by the same unit modulo ℓ^N. Multiplying the reciprocity sum by this common unit gives

$$0 = y(\phi)(tz(\phi)) + y(\phi^c)(tz((\phi^c)^{-1})) = F_t(\phi, \phi).$$

Here $z((\phi^c)^{-1}) = -z(\phi^c)$. Hence $F_t(h, h) = 0$ on H.

Additivity makes F_t antisymmetric on $H \times H$. Conjugating the first argument by s replaces t by ξt, and conjugating the second replaces t by $\xi^{-1}t$, by the two scalar actions at s, s^c. Indeed

$$F_{\xi t}(h, k) = F_t(shs^{-1}, k) = -F_t(k, shs^{-1}) = -F_{\xi^{-1}t}(k, h) = F_{\xi^{-1}t}(h, k).$$

Replacing t by ξt gives $F_{(\xi^2-1)t} = 0$. Because $\xi^2 - 1$ is a unit, F_t vanishes for every t. It follows that $z(h) \mapsto y(h^c)$ extends to a unique $\mathcal{O}$-linear map $V \to (V^*)^c$.

In fact, if $\sum_h c_h z(h) = 0$, then for every $k \in H$ and $t \in \mathcal{O}$, the identities $F_{tc_h}(h, k) = 0$ give

$$\Big(\sum_h c_h y(h^c)\Big)(tz(k^c)) = y(k)\Big(t\sum_h c_h z(h)\Big) = 0.$$

The elements $tz(k^c)$ generate V additively since $H^c = H$ and $z(H)$ spans over $\mathcal{O}$. Equivariance follows from the conjugation formulas, and the map depends $\mathcal{O}$-linearly on $[b]$ by uniqueness. If the map vanishes, $y|_H = 0$; for $u \in U$ the commutator $[s, u]$ then gives $y(u) = y(sus^{-1}) = \xi^{-1}y(u)$. Thus $y = 0$. We have proved the injection

$$D_{\mathcal{L}}^\vee \hookrightarrow \mathrm{Hom}_{\mathcal{O}[\Gamma]}(V, (V^*)^c).$$

The bound G_i for the length on the right gives the terminal bound $G_i + L_i$ for C. This proof uses the stated local length and hypotheses on the representation. The first-place determinant losses are paid separately at the untwisted specialization.

28 Galois representations, finite types, and local lengths

In this section, we attach Galois representations to the original Hecke eigenspaces and compute the normalizations at the finite places. The resulting representations supply the data for Section 29.

28.1 The original Hecke system and the normalized trace

In this subsection, we isolate the admissible finite type in the original Hecke eigenspace using Section 9.4. We then identify the normalization of the Frobenius trace needed for the attachment in Section 28.2.

Let $z \neq 0$ have Hecke system $b = (b_{w,i})$ in the final projected space. On the principal level the finite-type idempotent is a sum of the central idempotents attached to admissible tuples of finite parameters. Since the sum fixes z, some term acts nontrivially on z. The corresponding component keeps b: the unramified Hecke operators act at places away from the residue group and commute with its action. The joint trace comparison of Section 9.4 gives a nonzero component on the definite group with the same prescribed finite type and outside-level Hecke system.

The terminal rank-r space W_m is definite with standard form and finite basis. Its level is principal at the primes dividing π_M and the prescribed maximal integral compact elsewhere, as in Section 9.4. The last-covector condition at π_N is imposed on the rank-k source X of Section 6. Let σ be the compact coefficient representation of the weight, v its unit highest vector and $d_\sigma = \dim \sigma$. On a form reproduced by the real projector set

$$Lz(g_f) = d_\sigma \int_{U(W_m)_\infty} z(k, g_f)\sigma(k)v \, dk, \qquad \int dk = 1.$$

The compact projector kernel in these coordinates is $d_\sigma \langle v, \sigma(k)v\rangle$, with the inner product linear in its second variable. Indeed $\langle v, \sigma(k^{-1})Lz(g_f)\rangle = z(k, g_f)$ by the projector equation and change of variable in Haar. Thus L is injective. For a rational unitary element γ, the change $k \mapsto \gamma_\infty k$ uses left rational invariance of z and gives $Lz(\gamma_f g_f) = \sigma(\gamma_\infty)Lz(g_f)$. This is the coefficient transformation of the algebraic form. The integral commutes with every finite right translation, so it preserves the original level, finite type projector and outside-level Hecke system.

The split polynomial. Let $w, \bar{w}$ be a split pair off the level and twist of a descent with source rank r_0 and target rank $n = r_0 + t_0$. The local Schrödinger test there is the matrix-lattice vacuum $\Phi_w = \mathbf{1}_{\mathrm{Mat}_{n\times r_0}(\mathcal{O}_w)}$. The other local factors are unchanged. For an i-space U in a residue vector space of dimension k, choose an integral γ_U with elementary divisors ϖ_w in i positions and 1 elsewhere, whose reduction has kernel U. These kernels parameterize the Hecke representatives. The target representatives are read by transpose and those at the opposite source place by inverse. The split Schrödinger rules give the two terms

$$\nu_w^i \Phi_w(\gamma_U^{-\mathsf{T}} X), \qquad (\nu'_w)^i \Phi_w(X\gamma_U^{-1}).$$

The target determinant modulus factor is $q^{r_0 i/2}$, the source one is $q^{ni/2}$, and the splitting characters at the uniformizer give the other factors of $\nu_w^i, (\nu'_w)^i$ specified below.

The two terms require X integral. For integral X the tests are $U \subseteq \ker \bar{X}^{\mathsf{T}}$ and $U \subseteq \ker \bar{X}$, respectively, where here $\bar{X}$ denotes reduction. Indeed γ_U^{-1} has only a simple pole and the image of $\varpi_w \gamma_U^{-1}$ modulo ϖ_w is U. Let s be the residue rank of X. For $P_d(Z) = \prod_{j=0}^{d-1}(1 - q^j Z)$, the subspace identity

$$P_d(Z) = \sum_{i=0}^{d} (-Z)^i q^{\binom{i}{2}} \begin{bmatrix} d \\ i \end{bmatrix}_q$$

gives target and source series $P_{n-s}(\nu_w Y)$ and $P_{r_0-s}(\nu'_w Y)$.

28.1.1 Hecke multiplicities and the trace

In this subsection, we compute the multiplicities used to normalize the original trace in Section 28.1.

The identity follows by counting i-spaces contained in and transverse to a fixed hyperplane, which gives the recurrence for multiplying by $1 - q^{d-1}Z$. Because $n = r_0 + t_0$,

$$P_{n-s}(\nu_w Y) = P_{t_0}(\nu_w Y) P_{r_0-s}(q^{t_0}\nu_w Y).$$

Thus on the actual test the target Hecke polynomial action is the source one at $q^{t_0}\nu_w Y/\nu'_w$ times $P_{t_0}(\nu_w Y)$. Moving the source action through the bilinear theta pairing inverts the coset under the inherited Haar, changing its component at $\bar{w}$ to w. Both actions preserve the original data at the other local factors. This proves the split Hecke comparison when the lift is nonzero.

Apply the split polynomial identity to a nonzero lifted Hecke eigenform. Fix a descent pair with ranks r_0 and $n = r_0 + t_0$, with n even, and let μ be its unitary character before the tame twist ϑ. At an unramified split place w of norm q, use the reciprocal Hecke polynomial

$$P_{b,w}(Y) = \sum_{i=0}^{r_0} (-Y)^i q^{\binom{i}{2}} b_{w,i}.$$

Here $b_{w,0} = 1$. For the lift let $H_w(X)$ be the monic degree-n polynomial in the Galois normalization.

The split Hecke comparison needed here is

$$Y^n H_w(Y^{-1}) = \prod_{a=0}^{t_0-1} (1 - \nu_w q^a Y)\, P_{b,w}(q^{t_0}\nu_w Y/\nu'_w).$$

The nonzero scalars ν_w, ν'_w are the rank-r_0 and rank-n Frobenius values of the splitting characters. For a norm-one character χ and a rank n', this value is

$$q^{n'/2}\chi(\varpi_w/\bar{\varpi}_w)\upsilon(\varpi_w)^{-n'},$$

where υ is the splitting character of the Weil data. In the even rank the character for the lift is $\chi = \mu\vartheta(\upsilon|_{U(1)})^{n/2}$. By Section 4.3, $\upsilon(\varpi_w/\bar{\varpi}_w) = \upsilon(\varpi_w)^2$, so $\nu'_w = q^{n/2}(\mu\vartheta)(\varpi_w/\bar{\varpi}_w)$, where ϖ_w denotes the uniformizer idele.

Comparing linear terms gives the exact formula

$$-\nu_w^{-1}[X^{n-1}]H_w = \frac{q^{t_0}}{\nu'_w} b_{w,1} + \sum_{a=0}^{t_0-1} q^a.$$

In the geometric Frobenius convention, $\chi_{\mathrm{cyc}}(\mathrm{Fr}_w)^{-1} = q$. The scalar q^{t_0}/ν'_w is the product of the characters at Fr_w obtained from ϑ^{-1} and from the algebraic character

$$u \longmapsto \mu^{-1}(u/\bar{u})|u|^{(r_0-t_0)/2}$$

by reciprocity in the same convention. This follows from $|\varpi_w| = q^{-1}$ and $n = r_0 + t_0$. Write θ for the avatar of the latter character and ϑ_ℓ^{-1} for the former avatar.

Let a be the source splitting character of the original pair. The normalizing avatar α_{nor} is that of the algebraic Hecke character

$$x \longmapsto a(x/\bar{x})^{-1}\upsilon(x)^{r_0}|x|^{r_0/2}.$$

On the chosen CM type, the last two factors have infinity type z^{r_0}, by the infinity type of υ. The first factor has integral powers of $z, \bar{z}$ since a is a character of the norm-one torus. Its finite unit action is trivial at the original and auxiliary level places, since the source character is a prescribed power of the norm-one restriction of υ. At the selected geometric Frobenius the avatar is ν_w^{-1}. Multiplying the Hecke character by its conjugate gives $|x|^{r_0}$, hence $\alpha_{\mathrm{nor}}\alpha_{\mathrm{nor}}^{c_E} = \chi_{\mathrm{cyc}}^{r_0}$.

If R is the representation attached to b and R_{lift} to H_w, put $\widetilde{R}_{\mathrm{lift}} = \alpha_{\mathrm{nor}} R_{\mathrm{lift}}$. After transporting coefficients by ι, the normalized trace equality is

$$\operatorname{tr} \widetilde{R}_{\mathrm{lift}}(g) = \vartheta_\ell^{-1}(g) \operatorname{tr}((\theta R)(g)) + \sum_{a=0}^{t_0-1} \chi_{\mathrm{cyc}}(g)^{-a}.$$

Use the cofinite Frobenius comparison of Section 28.2 for both systems. In every finite quotient of G_E, the conjugacy class of the image of g has positive Chebotarev density [50, Ch. V, Theorem 3.23]. The primes of E of absolute residue degree greater than one have density zero, and every remaining unramified degree-one prime is split over F_0. Using both members of split pairs and omitting the two systems' finite excluded sets, we obtain the linear coefficient identity on dense conjugacy classes. Both sides are continuous class functions on G_E, giving the displayed identity for every g. By semisimplicity and the independence of irreducible trace characters, the main summand of $\widetilde{R}_{\mathrm{lift}}$ at twist ϑ is $\vartheta_\ell^{-1} \otimes \theta R$. The representation θR and its stable lattice can be fixed before choosing ϑ; at $\vartheta = \xi_\kappa^j$ the scalar twist of this lattice is exactly ψ^j, where ψ is the inverse avatar of ξ_κ.

Removing auxiliary character factors. Suppose two continuous semisimple representations R_i in the same coefficient field have characteristic polynomial at the prescribed Frobenius elements equal, in the limit, to a common monic polynomial times the linear factors of a list of characters U_i. Assume the two character lists are disjoint as Galois characters. On taking first coefficients and limits, the virtual characters $\operatorname{tr} R_i - \operatorname{tr} U_i$ agree on G_E. Characters of pairwise nonisomorphic irreducible representations over an algebraically closed field of characteristic zero are linearly independent.

Indeed the algebra generated by the group on their direct sum is finite-dimensional and its faithful module is semisimple, so its simple factors are the full matrix algebras on the distinct simple modules. Their trace functions are independent on these factors. For any constituent in U_i, comparison with the other list shows that its multiplicity in R_i is at least its multiplicity in U_i. Thus both R_i contain U_i and have the same semisimple remainder R. The summands are continuous as restrictions to invariant subspaces.

This proves the cancellation at the level of representations. The resulting characteristic polynomials satisfy $\operatorname{charpol}(R_i(g)) = \operatorname{charpol}(R(g)) \operatorname{charpol}(U_i(g))$. At the specified Frobenius elements, division by the monic character factor recovers the common attachment polynomial in the limit. Indeed coefficients of the monic quotient are obtained by successively subtracting products of higher coefficients, so this division is continuous.

If both larger attachments are unramified, crystalline, or potentially crystalline at a place, these properties pass to the remainder as a direct summand. The jumps at a coefficient embedding are obtained by subtracting the multiset of jumps of the corresponding character list. At an unramified place, if the two larger annihilating polynomials are $P(X)Q_i(X)$ and Q_0, Q_1 are coprime there, P annihilates the remainder by the Bézout identity applied to its Frobenius matrix. These reductions require the two larger attachments and the local polynomial comparisons on the form.

28.2 Galois attachment and spherical compatibility

In this subsection, we attach a Galois representation to the spherical eigensystem of Section 28.1. The contragredient compact type fixes the normalization, which is used again for the Frobenius tests of Section 29.2.

Use an eigensystem on a definite group of rank n, with the chosen compact coefficient representation σ_{alg}. Its scalar coefficient is $\langle \sigma_{\mathrm{alg}}(k)v, Lz(g_f)\rangle$. At the chosen CM embedding the coefficient action on Weyl coordinates is $k \mapsto S_\mu(\bar{k})$, with the determinant power included in S_μ; here $\bar{k} = (k^{-1})^{\mathsf{T}}$ on the definite unitary group. Right translation acts in the conjugate-linear first variable, so the compact automorphic constituent has type $\sigma_{\mathrm{alg}}^\vee \simeq S_\mu$. Its Harish–Chandra entries are $\mu_q + (n-1-2q)/2$, $0 \le q < n$; at the conjugate embedding their multiset is negated. This fixes the sign of the raw jumps in Section 29.1.1. One may choose an irreducible automorphic constituent containing a component of the eigenvector: the space is compact modulo rational points, and the outside-level eigenvalues and the specified finite type are preserved by taking components. By the compact coefficient realization in Section 28.1 this is the same finite-level Hecke system.

Apply [42, Corollaire 5.3, p. 43 of the author's version]. The corollary specifies local compatibility at each split place and at each unramified place. In the paired split coordinates the transfer of π_v is $\pi_v \boxtimes \pi_v^\vee$, including at a ramified split component. At a spherical inert place the transfer uses the unramified Satake parameter [33, §2.4, Theorem 2.2, pp. 1040–1041]. Section 4 gives $[F_0 : \mathbb{Q}] = \varphi(q_0)3L_0Q$, a multiple of four, so the degree condition for the definite group also holds in even rank. The extension E/F_0 is unramified at finite places. The definite groups used here have the standard positive Hermitian matrix. At a nonsplit finite place choose a unit u of norm -1, using the unit norm surjectivity of Section 4.3. The vectors $e_{2j-1} + ue_{2j}$, $1 \le j \le \lfloor n/2 \rfloor$, span a totally isotropic subspace of maximal dimension. Thus the group is quasi-split there. At a split place it is GL_n. The real groups are compact, so their representations are discrete series and Labesse's condition $(*)$ holds.

The rank-n transfer is an isobaric sum of discrete conjugate-self-dual representations over E. It agrees with the local base change at every split finite place and every inert place where the representation is spherical. Its infinitesimal character is the base change of the compact weight. By Mœglin–Waldspurger [53, theorem of the introduction, pp. 605–606], each block of rank ab is a length-b Speh representation from a unitary cuspidal σ of rank a, with exponents $u_j = (b-1)/2 - j$, $0 \le j < b$. Conjugate self-duality and uniqueness of the cuspidal support give $\sigma^c \simeq \sigma^\vee$. In both complex coordinates at infinity, the infinitesimal weights of the full base change lie in $(n-1)/2 + \mathbb{Z}$ and are pairwise distinct. Subtracting u_j shows that the weights of σ are regular and lie in $(n-b)/2 + \mathbb{Z}$.

Use the unitary character υ of Section 4.3. It is everywhere unramified at finite places, satisfies $\upsilon^c = \upsilon^{-1}$, and has infinity type $z/|z|$ at the chosen CM embeddings. Set

$$\psi = \begin{cases} 1, & n-a-b+1 \text{ even}, \\ \upsilon, & n-a-b+1 \text{ odd}. \end{cases}$$

Then $\sigma \otimes \psi$ is regular algebraic and conjugate self-dual, since its infinitesimal exponents lie in $(a-1)/2 + \mathbb{Z}$.

For its polarizing character on F_0 use 1 if a is even and the quadratic character of E/F_0 if a is odd. Its pullback by the norm is trivial, and its sign at each real place is $(-1)^a$. The sign in the Galois multiplier is then $(-1)^{1-a}(-1)^a = -1$. Hence the hypotheses of Barnet-

Lamb–Gee–Geraghty–Taylor [12, Theorem 2.1.1] apply for every coefficient prime ℓ, including 2.

For each j the character

$$\alpha_j = \psi^{-1}|\cdot|^{(a-n)/2+u_j}$$

is algebraic. Put $d_j = a-n+b-1-2j$ and let e_ψ be 0 for $\psi = 1$ and 1 for $\psi = \upsilon$. Since $d_j \equiv n-a-b+1 \equiv e_\psi \pmod 2$, its infinity character on the chosen CM type is

$$z^{(d_j-e_\psi)/2}\bar{z}^{(d_j+e_\psi)/2}.$$

The two integral exponents exchange at the conjugate embedding. Let $\rho_{\ell,\iota}$ be the direct sum over all blocks and j of

$$r_{\ell,\iota}(\sigma\otimes\psi)\otimes r_{\ell,\iota}(\alpha_j).$$

Here $r_{\ell,\iota}(\alpha_j)$ is the class field avatar in the same geometric Frobenius convention. The twists total $(1-a)/2+(a-n)/2+u_j = (1-n)/2+u_j$. Thus this direct sum, on the chosen split coordinate, has the local Langlands normalization of the base change twisted by $|\det|^{(1-n)/2}$. This is the direct sum computation in the proof of [33, proof of Theorem 2.3, pp. 1041–1042]. The summand representations here come from the cuspidal attachments of [12, Theorem 2.1.1] for every ℓ, including when $\ell = 2$ or a place above ℓ is nonsplit. At a nonsplit spherical place $w \nmid \ell$, Labesse gives the unramified base change. The factors ψ_w and ψ_w^{-1} cancel in each shifted block, so the direct sum has trivial Weil–Deligne inertia and zero monodromy. Thus its Galois representation is unramified at w.

At an unramified split place w let λ_i be the normalized Satake parameters. With Haar volume one for the maximal compact and right translation by the positive diagonal coset with j uniformizers, the eigenvalue is $t_{w,j} = q^{j(n-j)/2}e_j(\lambda_1,\dots,\lambda_n)$. The Hall–Littlewood polynomial for (1^j) is e_j by [46, Ch. III, (2.8)]. With $q^{-s_i} = \lambda_i$, the Hecke algebra map and its spherical specialization in [46, Ch. V, (2.7) and (3.3)] give

$$q^{-\binom{j}{2}}e_j(q^{(n-1)/2}\lambda_1,\dots,q^{(n-1)/2}\lambda_n) = q^{j(n-j)/2}e_j(\lambda_1,\dots,\lambda_n).$$

Proposition 28.1 (Spherical Frobenius polynomial)**.** *For an unramified split place away from ℓ and the finite excluded set of the attached system, transport the Hecke and Satake coefficients to $\overline{\mathbb{Q}}_\ell$ by ι. The characteristic polynomial of geometric Frobenius in the chosen split coordinate is*

$$\begin{aligned}\det(XI_n-\rho_{\ell,\iota}(\mathrm{Frob}_w^{\mathrm{geom}})) &= \sum_{j=0}^{n}(-1)^j q^{\binom{j}{2}}t_{w,j}X^{n-j}\\ &= \prod_{i=1}^{n}(X-q^{(n-1)/2}\lambda_i).\end{aligned} \tag{83}$$

Proof. On the unramified quotient geometric reciprocity sends the class of ϖ_w to geometric Frobenius. The local comparison above twists the unitary parameter by $|\det|^{(1-n)/2}$, whose value on one ϖ_w is $q^{(n-1)/2}$. The Frobenius eigenvalues are therefore $\alpha_i = \iota(q^{(n-1)/2}\lambda_i)$, counted with multiplicity. For $0 \le j \le n$, the minuscule coset formula gives

$$e_j(\alpha_1,\dots,\alpha_n) = \iota\Big(q^{j(n-1)/2}e_j(\lambda_1,\dots,\lambda_n)\Big) = q^{\binom{j}{2}}\iota(t_{w,j}).$$

Expansion of $\prod_i(X-\alpha_i)$ gives (83) with the indicated coefficient transport. □

All coefficients on the left are computed at w; the inverse Satake tuple belongs to $\bar{w}$. Each parity-corrected cuspidal attachment lies in a regular compatible system by [12, Theorem 2.1.1(3) and §5.1, final paragraph]. After a finite coefficient extension its integral models at the coefficient places are given by [12, Lemma 5.3.1(3)]. The compatible systems of the polarized characters are regular because their rank is one. For each fixed ℓ, the direct sum with the specified character twists is a continuous representation into $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$.

Lemma 28.2. *A continuous representation of a profinite group in $\mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ is defined over a finite extension $K_\ell/\mathbf{Q}_\ell$ and preserves an O_{K_ℓ}-lattice of rank n.*

Proof. Exhaust $\overline{\mathbf{Q}}_\ell$ by an increasing sequence of finite extensions, using finiteness of the extensions of each fixed degree. The inverse images of their matrix groups are closed subgroups covering the given group. By Baire one has interior and hence finite index. Adjoin the matrix entries of finitely many coset representatives to its coefficient field to obtain K_ℓ. For an initial O_{K_ℓ}-lattice L_0, compactness gives $a \geq 0$ with

$$L_0 \subseteq \sum_g \rho(g) L_0 \subseteq \ell^{-a} L_0.$$

The middle module is invariant and finite free of rank n. □

Apply the lemma to each attachment. For finitely many systems take the compositum of their coefficient fields and extend the lattices by scalars. Thus the coefficients in the ℓ-adic embedding are integral and the polynomial annihilates the actual Frobenius operator. These claims hold at every such split place outside a finite set including the coefficient and level places.

Fix ℓ and the original level. For the family in Section 29.2, fix the representations and the union of their finite excluded sets before choosing the depth and the auxiliary primes. At the resulting raised level, Section 30.3 uses the finite union for its own systems before applying Chebotarev at that level. An existing semisimple attachment specified by convergence to the same Frobenius polynomials has the same trace by continuity on the dense selected Frobenius classes. Independence of irreducible characters as in Section 28.1 identifies the two semisimple representations. In particular (83) applies to the same systems used in the residual comparison. It supplies the spherical part of Section 29.2.

28.3 Inertial parameters of the finite types

In this subsection, we read the inertia eigenvalues of the attachment in Section 28.2 from the original finite types. These local data verify the spectral conditions for the length estimates of Section 28.4.

For a split place of residue characteristic different from ℓ, the attachment of Section 28.2 has the full Frobenius-semisimplified Weil–Deligne comparison in the chosen coordinate. Labesse supplies local base change at split places, and [12, Theorem 2.1.1(2)] gives the comparison for each parity-corrected cuspidal block. In rank at least two the monodromy assertion is [20, Theorem 1.1, p. 2312]. The regular algebraic cuspidal representation $\sigma \otimes \psi$ is cohomological at the complex infinite places by [22, Lemme 3.14, p. 114]. The auxiliary quadratic character in that lemma is trivial at every complex place since $\mathbf{C}^\times$ is connected. Conjugate self-duality therefore gives temperedness at all finite places by [20, Theorem 1.2, p. 2312]. Rank-one blocks are unitary characters. Removing the unitary parity twist preserves temperedness. Thus the

factors of a length-b Speh block, ordered by $u_0 > \cdots > u_{b-1}$, form a standard module on each finite general linear group. By the Langlands quotient description, its parameter is the sum of the shifted tempered parameters. Each discrete automorphic block is unitary, so its irreducible local component is unitarizable. Bernstein's theorem [72, Theorem 2.5(i), p. 346] gives irreducibility of the normalized induction of these local components across the discrete blocks. By [72, Corollary A.7(i), p. 375], its Langlands datum is the union of the block data with multiplicities. Its Weil–Deligne parameter is therefore the direct sum of the block parameters, including their monodromy operators. The algebraic twists in the direct sum cancel the parity corrections at this very place. This proves the required comparison on the same attachment. Frobenius semisimplification preserves the Weil–Deligne inertia action and the monodromy operator. When the latter vanishes, the inertia action on the original ℓ-adic representation at $w \nmid \ell$ equals its finite Weil–Deligne inertia action.

We describe the level-zero local calculation for the finite type of that attachment. Let $k = \mathbb{F}_q$ be the residue field, $K_w = \mathrm{GL}_n(\mathcal{O}_w)$, and $K_{w,1}$ its principal kernel. For an irreducible factor of the squarefree type polynomial of degree a, the finite type is the genuine cuspidal character $\tau_\theta = (-1)^{a-1} R_T^\theta$ of $\mathrm{GL}_a(k)$, where $T(k) = \mathbb{F}_{q^a}^\times$ and θ is indexed by a root as in Section 28.3.1. Its Frobenius orbit has size a, since the root has degree a and the character parametrization commutes with Frobenius. This is the primitivity hypothesis in [28, §2.4]. Full multiplicative order is required only at the separately chosen primitive type polynomials. The corresponding depth-zero supercuspidal has on inertia the a Frobenius conjugates of this character, with zero monodromy [28, Proposition 2.4.1(i), p. 10 of the authors' version]. This is a characteristic-zero statement.

For the full parameter, the $K_{w,1}$-invariants of a parabolic induction are the finite Harish–Chandra induction of the inducing $K_{w,1}$-invariants [69, Proposition 6.18]. Indeed the maximal compact and the parabolic give Iwasawa decomposition, and reducing the intersection with the parabolic gives the finite one. Invariants are exact in characteristic zero by compact averaging. Use the finite irreducible constituent $\sigma_{\mathcal{P}}$ of [69, Definitions 6.6–6.7 and Proposition 6.9]. A tame Frobenius orbit of multiplicity one contributes its finite cuspidal representation. On a determinant block of size d, untwist its determinant character. The trivial representation occurs in the finite induction from the Borel; by Frobenius reciprocity it is absent from any induction having a Steinberg block of size greater than one. Thus it is the constituent labelled by (1^d). Finite parabolic induction combines the constituents on the distinct cuspidal lines.

For an irreducible local representation containing this type, take its Langlands standard module. A supercuspidal factor with nonzero principal-kernel invariants contributes precisely the finite cuspidal representation of its depth-zero type, by the Cartan decomposition in Section 31.1. The finite cuspidal support then identifies their multiset [69, §6.3]. If $\mathcal{P}'$ is the family of segment lengths in this module, its principal-kernel invariants are the finite induction $\pi_{\mathcal{P}'}$ of the corresponding generalized Steinberg blocks. Exactness implies $\mathrm{Hom}(\sigma_{\mathcal{P}}, \pi_{\mathcal{P}'}) \neq 0$. The finite multiplicity formula gives $\mathcal{P}' \preceq \mathcal{P}$ on every line [69, Corollaries 6.10 and 6.19, Theorem 6.20]. All the partitions in $\mathcal{P}$ have only singleton parts, so the monodromy is zero. This is the forward implication of [69, Theorem 3.7(1) and Corollary 6.21(1)] for the specified type. For the smooth complex calculation one may use an auxiliary coefficient prime different from the residue characteristic and transport its Weil–Deligne parameter to $\mathbf{C}$. The local–global comparison at the prescribed coefficient prime is applied to this same complex parameter. In particular the squarefree type gives one depth-zero factor per irreducible polynomial, and a determinant block of size d gives d rank-one factors on its tame line. There is no nontrivial

segment on a line occurring once.

On a repeated line, twist out the common tame character. A segment of length $a > 1$ on the remaining rank-one unramified line is an unramified twist of Steinberg. Put $H_a = \mathrm{GL}_a(k)$ and let B_a denote its upper Borel. Reducing the flag functions at the principal kernel and taking invariants by compact averaging gives

$$\mathrm{St}_{H_a} = \mathbf{C}[H_a/B_a] \Big/ \sum_{B_a \subsetneq P \subseteq H_a} \mathbf{C}[H_a/P].$$

Each standard parabolic summand is pulled back along its flag projection. For $a > 1$, the summand $P = H_a$ contains the constants, which are the only H_a-invariants in the numerator. Exactness gives $(\mathrm{St}_{H_a})^{H_a} = 0$. By Frobenius reciprocity the finite Harish–Chandra induction from such segments cannot contain the trivial representation of the repeated block.

Fix an old place $w \in S_M$. Let $P \subset \mathrm{GL}_{r_0+t_0}(k_w)$ stabilize the image of the original rank-r_0 block, and let τ_w be its selected lower residue type. The pointwise stabilizer of the original columns contains the unipotent radical and the whole complementary $\mathrm{GL}_{t_0}(k_w)$; its action there is trivial by the original level. On the first block the projector uses right translation multiplied by $\chi_{M,w}^{j} \circ \det$, as in Section 32.3. Thus ordinary right translation there has type $\tau_w \otimes (\chi_{M,w}^{-j} \circ \det)$. The raised level has the same local conditions at w. On every upper constituent at either level, finite Frobenius reciprocity gives

$$\mathrm{Hom}_{\mathrm{GL}_{r_0+t_0}(k_w)}\Big(\mathrm{Ind}_{P}^{\mathrm{GL}_{r_0+t_0}(k_w)}((\tau_w \otimes (\chi_{M,w}^{-j} \circ \det)) \boxtimes \mathbf{1}), \pi_w^{K_{w,1}}\Big) \neq 0.$$

For $r_0 = r$, the lower type excludes the roots of order dividing q_2, so its rescaled roots exclude 1. For $r_0 = 1$, the unscaled polynomial is $X - 1$ and $\chi_{M,w}^{-j}$ has order $q_2/\gcd(q_2, j) > 1$, since $1 \le j \le m < q_2$. In both cases the two blocks have disjoint cuspidal supports. At a twist place of exact ramification the two determinant characters are distinct. In finite parabolic Mackey theory, a double coset moving a positive-rank subblock between distinct cuspidal lines would require the corresponding two Jacquet modules to share a cuspidal factor. Their supports are disjoint by the parabolic product formula [31, §2, Theorem 2, and §8, Theorem 14]. Thus only the identity double coset can contribute when the Levi blocks are grouped by cuspidal line in the same order on both sides. The multiplicity of the prescribed type in the finite induction is the product of its multiplicities on the distinct cuspidal lines [69, Proposition 6.9 and Corollary 6.10]. The zero multiplicity on the repeated line therefore remains zero after adjoining the other lines. Thus each segment in the determinant block has length one and its monodromy is zero. The same proof applies to a pair of distinct tame determinant characters on two blocks. In particular at a split auxiliary prime with ramification the normalized characters are ϑ_ℓ^{-1} on the original rank block and 1 on the added block. For each inertia element g the polynomial $(X - \vartheta_\ell^{-1}(g))(X - 1)$ annihilates this type. Away from ℓ this is actual Galois inertia, since monodromy is zero. At the coefficient place the polynomial calculation concerns the linear inertia on potentially crystalline periods, as proved below.

28.3.1 The inertia characters

In this subsection, we compute the inertia characters of the finite type on the generator used in the trace comparison of Section 28.1.

To specify the action of the generator, fix the faithful multiplicative character $\jmath : \bar{k}^\times \to \mathbb{C}^\times$ from the parameter convention. If $\jmath(\xi) = \exp(2\pi i b/(q^a - 1))$, the Green character of the degree-a

field is $\jmath|^{b}_{\mathbb{F}^{\times}_{q^a}}$. For each a, let F_a/E_w be unramified of degree a. On inertia set

$$\omega_a(s) = \overline{\mathrm{Art}_{F_a}^{-1}(s^{\mathrm{ab}})} \in \mathbb{F}_{q^a}^{\times},$$

using geometric reciprocity and abelianization in W_{F_a}. For $a \mid D$ the norm relation gives $\omega_a = \omega_D^{(q^D-1)/(q^a-1)}$. Take D divisible by all degrees in the finitely many systems at this place. The tame quotient is procyclic, and ω_D maps it onto $\mathbb{F}^{\times}_{q^D}$. Choose one tame generator s with $\jmath(\omega_D(s)) = \exp(2\pi i/(q^D-1))$. For every such system the Green character on s then equals $\jmath(\omega_a(s))^b = \jmath(\xi)$.

Its Frobenius conjugates are the q-th powers. Fix any $w \in S_M$ above $p_\nu \neq \ell$, and put $q = p_\nu$. Let $c_w \in k_w[X]$ be the original monic type polynomial of a rank-n main summand. Write $u_{j,w} \in k_w^{\times}$ for the parameter corresponding under the degree-one Green convention above to $(\chi_{M,w}^{j})^{-1}$, where $\chi_{M,w}$ is the original character of order q_2 and $q_2^2 \mid q-1$. At w the character θ on the lower summand θR of Section 28.1 has tame character $\chi_{M,w}^{-j}$, by the choice of μ. The character α_{nor} multiplying the upper attachment has trivial smooth finite-unit character there. On that attachment the first parahoric block already carries $\chi_{M,w}^{-j} \circ \det$, so its inertia roots receive the same single multiplier $u_{j,w}$ as those of θR on the lower side. The absolute norm factor and the auxiliary twist are trivial on inertia there. Pullback from $k_w^{\times}$ to the degree-a extension uses the norm exponent $(q^a-1)/(q-1)$. Thus the same multiplier $u_{j,w}$ acts on each root, and the normalized polynomial is

$$C_w(X) = u_{j,w}^{n} c_w(X/u_{j,w}).$$

The specified tame generator has eigenvalues $\imath\jmath(\xi)$ for the roots ξ of C_w. In rank r the raw parameter is squarefree with nonzero roots and excludes all q_2-th roots of unity in k_w. Since $u_{j,w}^{-1}$ is one of these excluded values, none of the scaled roots equals one. In rank one $c_w = X-1$, so $C_w = X - u_{j,w}$. Its root has order $q_2/\gcd(q_2, j) > 1$, since $1 \le j \le m < q_2$.

A root of C_w has degree $h \le n$ over the residue field and order dividing q^h-1. It follows that the inertia eigenvalues at the tame generator have orders dividing $q^{n!}-1$. The local comparison gives zero monodromy and a sum of tame characters. Hence wild inertia acts trivially, and the image of tame inertia is finite cyclic, generated by the image of the chosen generator. Its order divides $q^{n!}-1$, which is prime to the residue characteristic.

For any eigenvalue $\alpha \neq 1$ of order dividing q^h-1, the identity $\sum_{a=0}^{q^h-2} \alpha^a = 0$ gives

$$v_\ell(\alpha - 1) \le v_\ell(q^h - 1) :$$

subtract the geometric sum at 1 and divide by $\alpha - 1$, with integral quotient. Since $q^h - 1$ divides the order of $\mathrm{GL}_{rt}(\mathbb{F}_q)$, multiplying the eigenvalue differences proves the determinant bound $nv_\ell(A_{\mathrm{aut}})$. At the conjugate place use the inverse conjugate of the chosen element. In the polarization trace relation the cyclotomic factor is trivial on this inertia.

Applying it to every power of the chosen element gives equal power traces at the two parameters. Newton's identities give equal characteristic polynomials and hence the same estimate. In this way the determinant bounds concern the specified tame generator at each chosen place and its inverse conjugate. Choosing one distinguished place on a rational prime different from ℓ supplies the common reference element in the bound.

28.3.2 The matrix span at the coefficient place

In this subsection, we use the inertia characters of Section 28.3.1 to obtain the matrix span for the trace comparison.

For the full matrix span, choose $p_i^{\mathrm{lev}} \neq \ell$. The selected tuple at p_i^{lev}, as defined in Section 14.2, has a primitive type polynomial at a distinguished place over p_i^{lev}. Its original root generates $\mathbb{F}_{q^r}^{\times}$. Multiplication by any element of order dividing q_2 preserves this property: the exponent of the multiplier in a generator of this group is a multiple of $(q^r-1)/q_2$. Every prime dividing q^r-1 divides this quotient, since $q_2^2 \mid q-1$. Thus the new exponent remains prime to q^r-1. The normalized roots form the q-Frobenius orbit of a generator α_0.

This orbit stays distinct after every ℓ-power. Indeed r is a positive power of 3 with $r \geq 3$. Put $Q' = q^{r/3}$. The factor $(q^r-1)/(q^{r/3}-1) = Q'^2+Q'+1$ is congruent to 3 mod 9, since $9 \mid q-1$, and exceeds 3. It has at least two distinct prime divisors. It cannot divide a power of ℓ, so no $\alpha_0^{\ell^a}$ belongs to $\mathbb{F}_{q^{r/3}}$. Every proper subfield over $\mathbb{F}_q$ lies in that field. Under the faithful character map into the coefficient field the orbit consequently has pairwise unit differences: a root of unity reduces to 1 only when its order is an ℓ-power, by the separability of roots of order prime to ℓ. The same argument applies if the tame generator raises all eigenvalues to an exponent prime to their orders.

At such a place, the orbit is the full spectrum of the inertia generator. To see this from the annihilating polynomial, one eigenvalue occurs, all its q-powers occur by tame Frobenius conjugation, and there are r eigenvalues counted with multiplicity. Raising the generator to the exponent $(q^r-1)/(q-1)$ gives a scalar whose root-of-unity order is $q-1$, since every normalized root is a Frobenius conjugate of a generator.

For the rank-one main pair, its own normalized root already gives a scalar with order $q_2/\gcd(q_2, j)$. Let ξ be the resulting scalar and n_{up} the upper rank. For $a \geq 0$ and $1 \leq h \leq 4n_{\mathrm{up}}+2$, the difference $\xi^{h\ell^a}-1$ is a unit. Otherwise the order of ξ would divide $h\ell^{a+b}$ for some b, giving $q_2 \mid jh\ell^{a+b}$ in either case. Take $p_* \in \{2,5\}$ different from ℓ. Then $p_*^s \mid q_2 = (q_0 d)^{s+6}$. The main rank is r or 1, and $n_{\mathrm{up}} = r+t$ or $2z \leq 2r$, respectively. Thus

$$jh \leq (4n_{\mathrm{up}}+2)m \leq (24r+2)m = O(x^{11/2}) < x^{11} \leq m^{22} = 2^s \leq p_*^s.$$

This contradicts $p_*^s \mid jh$. In particular ξ^2-1 is a unit. The scalar comes from the original tame inertia.

For lift j and real place τ, the two conjugate jumps at index u are $(t-1)/2 \pm (\lambda_u(\tau)+w_j(\tau))$ at rank r, and $z-1\pm c_\tau$ at rank one, by (H1) and (H2). The two sums are $t-1$ and $2z-2$, respectively, both nonnegative. Each pair thus contains at most one negative jump.

At the coefficient place. Apply [11, Theorems 1.1 and 3.1, pp. 170 and 172] to each parity-corrected regular algebraic conjugate-self-dual cuspidal block of rank at least two. Theorem 3.1 allows every coefficient prime ℓ, including 2. At $w \mid \ell$ its conclusion identifies the semisimplified Weil representations and bounds the Galois monodromy partition by the automorphic partition on each unramified twist class. For the fixed compact type the automorphic monodromy on the whole parameter is zero by [69, Theorem 3.7(1) and Corollary 6.21(1)]. In the standard-module description above that operator is the direct sum of the monodromy operators of the shifted cuspidal blocks. Every one of these operators is therefore zero, with the same conclusion for repeated summands. On a class of multiplicity d, the resulting partition bound on the Galois side is $\lambda \preceq (1^d)$. Its largest part is at most one, so its monodromy also vanishes. The Frobenius-semisimplified parameters consequently agree. For a rank-one block let δ be its local

avatar at $F = E_w$, $w \mid \ell$, and fix a uniformizer ϖ. At the chosen coefficient embedding extend the coefficient field finitely to L containing all images of F and the values of the smooth units character ε_w. In geometric reciprocity the defining formula for the avatar on units is

$$\delta(\mathrm{Art}_F(u)) = \varepsilon_w(u) \prod_{\tau : F \hookrightarrow L} \tau(u)^{b_\tau}, \qquad u \in \mathcal{O}_F^\times.$$

Here b_τ are the exponents of the algebraic factor. Extend ε_w by $\varepsilon_w(\varpi) = 1$ and view it as a finite Galois character via the same Artin map. The Lubin–Tate character χ_ϖ satisfies $\chi_\varpi(\mathrm{Art}_F(u)) = u$ on units and $\chi_\varpi(\mathrm{Art}_F(\varpi)) = 1$. Hence

$$\delta = \varepsilon_w \, \delta_{\mathrm{ur}} \prod_\tau (\tau \circ \chi_\varpi)^{b_\tau}, \qquad \delta_{\mathrm{ur}}|_{I_F} = 1.$$

The unramified factor is crystalline of weight zero, and $\tau \circ \chi_\varpi$ is crystalline with filtration jump -1 at τ and 0 at the other embeddings [40, Hypothesis 6.0.1, Definition 6.2.5 and Example 6.2.6(1),(3), pp. 1091–1092, 1095–1096]. In particular the jump at τ for δ is $-b_\tau$; for $F = \mathbf{Q}_\ell$ and $\varpi = \ell$, the Lubin–Tate character is χ_{cyc}. Over the field cut out by ε_w, a basis e of the finite-character line gives the period $1 \otimes e$. Its descent action at $s \in I_F$ is multiplication by $\varepsilon_w(s)$. The remaining factor is crystalline over F and has trivial linear inertia on its periods. Thus the inertial character on the potentially crystalline periods of δ is exactly its smooth units character under reciprocity. With $f_w = [k_w : \mathbf{F}_\ell]$, the eigenvalue of φ^{f_w} on the crystalline factor $\delta\varepsilon_w^{-1}$ is

$$\delta(\mathrm{Art}_F(\varpi)) \prod_{\tau : F \hookrightarrow L} \tau(\varpi)^{-b_\tau}$$

by [40, Example 6.2.6(1),(3)]. Since $\varepsilon_w(\varpi) = 1$, this is the value of the smooth Weil–Deligne character at the geometric Artin image of ϖ. Period spaces commute with the finite extension of coefficients, since tensoring with a finite-dimensional coefficient space commutes with the defining Galois invariants. Thus these attachments are potentially crystalline at the coefficient places, with the specified tame inertial types. When the finite unitary representation is spherical, Labesse's local base change Π_w is spherical. Put $K_s = \mathrm{GL}_s(\mathcal{O}_{E_w})$. By the Speh description in Section 28.2, Π_w is a subquotient of the normalized parabolic induction of the $\sigma_w|\det|^{u_j}$ over all blocks and shifts. Taking K_n-invariants is exact by compact averaging. By Iwasawa decomposition, evaluation at 1 identifies the invariants of the induction with $\bigotimes_{\mathrm{blocks}} \bigotimes_{j=0}^{b-1} \sigma_w^{K_a}$, since the modulus and norm twists are trivial on the integral Levi. As $\Pi_w^{K_n} \neq 0$, each $\sigma_w^{K_a} \neq 0$. The unramified parity character preserves these fixed vectors. By [12, Theorem 2.1.1(4)] each parity-corrected attachment is crystalline there. The algebraic characters α_j in the direct sum have trivial smooth units character. Taking $\varepsilon_w = 1$ in the preceding factorization gives crystallinity of their avatars over E_w. With arithmetic reciprocity, the Lubin–Tate character on units is inversion, as in the proof of [25, Proposition B.4(i), pp. 35–36]. When the smooth units character is trivial, the avatar therefore restricts in this convention to the inverse algebraic character described above. The same criterion applies to the normalizing characters off the level. Tensor products and direct sums preserve crystallinity. In particular every place above 2 is spherical when $\ell = 2$, since the original level and the auxiliary twist places are odd and the finite level there is hyperspecial.

At $w \mid \ell$ the Frobenius comparison concerns the Frobenius-semisimplified Weil–Deligne parameter. On each shifted block the two parity characters cancel, giving the local parameter of $\sigma_w|\det|^{(1-n)/2+u_j}$ after transport of coefficients. For the inverse auxiliary avatar $\psi_{\mathrm{aux}} =$

$\prod_{u\in Q}\zeta_u^{\chi_u}$ in Section 27, each additive character χ_u is zero on the entire decomposition group G_{E_w} by Sections 24 and 27.2. Thus $\psi_{\mathrm{aux}}^h|_{G_{E_w}}=1$ for every integer h. Every power preserves the local representation, its period inertia and its labeled Hodge–Tate numbers, also when $\ell\mid M$.

If $\ell=p_i$, complete splitting of the original level prime gives $E_w=\mathbb{Q}_\ell$ at every $w\mid\ell$. Let $B_w(s)$ denote the linear inertia action on the potentially crystalline periods. Each eigenvalue of $B_w(s)$ from a degree-a tame parameter has order dividing ℓ^a-1, with $a\le r$. Polarization gives

$$B_{\bar w}(c_E s c_E^{-1})\sim (B_w(s)^{-1})^{\mathsf T},$$

since the Tate power in the polarization has trivial linear period inertia. Thus $c_E s^{-1}c_E^{-1}$ at $\bar w$ has the same eigenvalue multiset as s at w. The common exponent

$$e_r=\operatorname*{lcm}_{1\le a\le r}(\ell^a-1)$$

kills inertia on these periods. Put $m_r=\mathrm{lcm}(1,\dots,r)$. Since $\ell=p_i>r$, one has $\ell\nmid m_re_r$. The unramified extension of degree m_r contains μ_{e_r}, since $e_r\mid\ell^{m_r}-1$. Adjoining an e_r-th root of a uniformizer then gives a totally ramified extension of degree e_r. The composite is Galois of degree m_re_r, prime to ℓ. Since $m_r\mid r!$, it is contained in the chart field of Section C.2.2, obtained by taking the unramified part of degree $r!$ and the same root.

Here the Swan conductor is that of the Weil–Deligne representation. Its wild inertia acts trivially on the potentially crystalline periods, since the finite inertia exponent e_r is prime to $\ell=p_i$. The tame inertia of the extension maps into the e_r-th powers of the original tame quotient. For a de Rham representation the Weil–Deligne representation is obtained from its potentially semistable periods. Its restriction here has trivial inertia and zero monodromy, so the representation is crystalline over this extension. The same extension works for the finitely many types under consideration, including the degree-one normalizing characters.

Fix the individual tuple of old types for the descent pair before varying the upper system. For each inertia element the upper spectrum, including its determinant block, consists of at most $r+t$ eigenvalues. An eigenvalue of order dividing ℓ^a-1 lies in the unramified field of degree $a\le r$. The product of the distinct monic minimal polynomials over $\mathbf{Q}_\ell$ of these eigenvalues is in $\mathbf{Z}_\ell[X]$ and has degree at most $(r+t)r$. Finite inertia acts semisimply in characteristic zero, so this polynomial annihilates the linear period action throughout the fixed type, and hence on subrepresentations and direct sums of arbitrary multiplicity. Inversion of the eigenvalues gives the same degree bound on Tate duals since the Tate character has trivial linear period inertia. At the fixed comparison $\kappa=(\ell,\iota)$, every $\mathbf{Q}_\ell$-embedding of the local coefficient field preserves this polynomial, since its coefficients lie in $\mathbf{Z}_\ell$. For another comparison above ℓ take the minimal polynomials of the resulting eigenvalues anew. Their common degree bound $(r+t)r$ depends only on the ranks. This gives the inertial polynomials of Section 29.1 for each coefficient place with the same bound.

The paired jumps. The raw rank-n attachment satisfies $R^c\simeq R^\vee\chi_{\mathrm{cyc}}^{1-n}$: in each Speh string j pairs with $b-1-j$, and $\alpha_j^c\alpha_{b-1-j}=|\cdot|^{a-n}$. Combine this with the polarization of the rank-a cuspidal attachment. At the given embedding let v be the infinitesimal exponent of σ in the contragredient compact type, and let b_ψ be the exponent of the parity character there. It is 0 for $\psi=1$ and $\pm\frac12$ for $\psi=\upsilon$. By [12, Theorem 2.1.1(3)] and the rank-one computation above, the algebraic factors $r_{\ell,\iota}(\sigma\otimes\psi)$ and $r_{\ell,\iota}(\alpha_j)$ have integral jumps

$$\frac{a-1}{2}-(v+b_\psi),\qquad -\left(\frac{a-n}{2}+u_j-b_\psi\right).$$

Their sum is $(n-1)/2-(v+u_j)$. The parentheses run through the infinitesimal exponents of the rank-n base change.

In the two conjugate embeddings the raw jumps match u with $n-1-u$. The normalizing character and its conjugate have product $\chi_{\mathrm{cyc}}^{n-t_0}$ in the rank-n attachment of a descent with rank difference t_0. Thus the normalized jumps pair as u, t_0-1-u. The main summand keeps this pairing after subtracting the extra character string, which is invariant under the same involution. Hence at most one jump per pair is negative. The rank-one chart has the same property, with jumps $z-1\pm c_w$.

28.4 Local length bounds for a main summand

In this subsection, we use the inertial eigenvalues of Section 28.3 and the stable coefficient lattices to bound the lengths needed in Section 27.4. The estimates are normalized by the chosen valuation at ℓ.

Fix the coefficient place $\kappa=(\ell,\iota)$ and the original main summand of rank $n\le r$. We describe the precise data on the summand that suffice for the previous count. Work over a finite ℓ-adic coefficient field, enlarged to contain the eigenvalues and the images of the designated embeddings of the local fields. The representation is unramified outside the places above $\ell p_0 p_1$, is de Rham at the places above ℓ, and has trace relation

$$\operatorname{tr}\rho(g^c)=\chi_{\mathrm{cyc}}(g)^{-d_{\mathrm{pol}}}\operatorname{tr}\rho(g^{-1}),\qquad 1\le d_{\mathrm{pol}}+1\le t,\quad t=5r.$$

Assume the prescribed tame type gives a cyclic Frobenius orbit of n distinct inertia eigenvalues whose pairwise differences are units at a distinguished place prime to ℓ.

There the norm is p_ν for $\nu\in\{0,1\}$. Assume also a scalar inertia element s away from ℓ whose scalar ξ has ξ^2-1 a unit. At the original places above p_0p_1 away from ℓ there is a bound on the exponent of the original inertia image prime to the residue characteristic. At each such place a specified inertial element has determinant of $\rho(g)-I$ of valuation at most nu_ℓ, where

$$A_{\mathrm{aut}}=\prod_{w\in S_M}|\mathrm{GL}_{rt}(k_w)|,\qquad u_\ell=v_\ell(A_{\mathrm{aut}}).$$

There are $f=[F_0:\mathbb{Q}]$ chosen places over each p_i, so $|S_M|=2f$. Local ramification can occur on both members of a pair, giving at most $4f$ original places. At the common reference g_0 and its inverse conjugate require the same determinant bounds. These are local hypotheses on the original summands.

Take a stable lattice, for example the span of the orbit of a coefficient lattice, which is finitely generated because the image is compact and the orbit is bounded. The integral spectral idempotents of the tame generator split it into rank-one summands, by Lagrange interpolation with unit denominators. Frobenius permutes those lines with unit multipliers in integral bases. The idempotents and the powers of Frobenius give all the matrix units over the valuation ring.

For g_1 the n-th power of this arithmetic Frobenius, $\rho(g_1)=\alpha I$, since on each line its multiplier is the product of the same n units around the cycle, and $\chi_{\mathrm{cyc}}(g_1)=p_\nu^n$. The trace relation and the full matrix span show that

$$\rho(g)=\alpha_g I\quad\Longrightarrow\quad\rho((g^c)^{-1})=\chi_{\mathrm{cyc}}(g)^{d_{\mathrm{pol}}}\alpha_g I.$$

Indeed for every x, apply $\operatorname{tr}\rho((u^c)^{-1})=\chi_{\mathrm{cyc}}(u)^{d_{\mathrm{pol}}}\operatorname{tr}\rho(u)$ to $u=gx^{-1}$, then use the formula at x^{-1}. Pairing the difference of the claimed matrices with all $\rho(x^c)$ gives trace zero. The

trace pairing on the full matrix algebra is nondegenerate. At s, the cyclotomic value is one by inertiality away from ℓ, giving the two evaluations in Section 27.4.

For $V = T/\varpi^a T$, an equivariant map $V \to (V^*)^c$ is determined by its value on the first coordinate vector, whose orbit spans V. The scalars of g_1 on source and target are α and $\alpha p_\nu^{n(d_{\mathrm{pol}}+1)}$, respectively. Hence the evaluation lies in the torsion killed by $1 - p_\nu^{n(d_{\mathrm{pol}}+1)}$. For $0 \neq u \in \mathcal{O}$, the scalar action on the Cartier dual gives

$$V^*[u] = \mathrm{Hom}_{\mathbb{Z}}(V/uV, \mu_{\ell^N}), \qquad \mathrm{length}_{\mathcal{O}} V^*[u] = \mathrm{length}_{\mathcal{O}}(V/uV) = n \min\{a, \mathrm{ord}_{\mathcal{O}}(u)\}.$$

The middle equality follows from equality of cardinalities: V/uV has exponent dividing ℓ^N, as in Section 27.4. Dividing by $\mathrm{ord}_{\mathcal{O}}(\ell)$ gives a bound of $nv_\ell(u)$. For $1 \le n(d_{\mathrm{pol}} + 1) \le rt$, $p_\nu^{n(d_{\mathrm{pol}}+1)} - 1$ divides $|\mathrm{GL}_{rt}(\mathbb{F}_{p_\nu})|$. This bounds the normalized global term by nu_ℓ.

At an original place away from ℓ, write b_v for the given exponent of the inertia image. On cocycles modulo ϖ^a, the affine action on V shows that every inertia element raised to b_v acts by translation. Thus its order divides $b_v \ell^j$ for some common j, prime to the residue characteristic. The affine action kills wild inertia. Evaluation at a tame generator has image of length at most $\mathrm{length}\, V$. On its kernel the cocycle vanishes on inertia, and its Frobenius value is inertia-invariant and determines the cocycle. It therefore lies in the kernel of $\rho(g) - I$ on V for the chosen inertia element. By Smith normal form the excess normalized length is at most $v_\ell \det(\rho(g) - I) \le nu_\ell$. Summing over $(S_M \cup \bar{S}_M) \setminus \{v : v \mid \ell\}$ therefore costs at most $4fnu_\ell$ in excess of the baseline local lengths.

At $v \mid \ell$, impose the full annihilator of the integral cocycles whose rational classes lie in H^1_f. For the de Rham representation W over the coefficient field K, put $D = D_{\mathrm{dR}}(W)$. Since K contains the images of all embeddings $\tau : E_v \hookrightarrow K$, the decomposition $E_v \otimes_{\mathbf{Q}_\ell} K = \prod_\tau K$ gives

$$d_v = \dim_K(D/\mathrm{Fil}^0 D) = \sum_\tau \dim_K(D_\tau/\mathrm{Fil}^0 D_\tau).$$

Divide the $\mathbf{Q}_\ell$-dimension formula of Bloch–Kato [16, Corollary 3.8.4, p. 355] by $[K : \mathbf{Q}_\ell]$. This gives

$$\dim_K H^1_f(E_v, W) = d_v + \dim_K H^0(E_v, W), \qquad \dim_K B^1(E_v, W) = n - \dim_K H^0(E_v, W).$$

Thus the preimage of H^1_f in the rational cocycles has dimension $n + d_v$. To see the lattice assertion, finitely many coordinate evaluations embed the rational cocycle space in a finite product of coefficient fields. The integral submodule is then finite free.

Each rational cocycle can be cleared into it by compactness of its image. The quotient of the finite-support evaluation module by the annihilator is a submodule of the dual of this lattice, free of rank at most the same rank. Allowed values modulo ϖ^a define homomorphisms from that quotient to $\mathcal{O}/\varpi^a$, so the local length is at most a times the rank. Thus the sum over the places above ℓ satisfies the required crystalline bound if the total number of negative de Rham jumps is at most fn, as follows from a bound of n at each complex conjugate pair of embeddings.

Adding the global term, the original-place excess and the two first-prime determinant contributions give a normalized bound of at most

$$(4f + 3)nu_\ell \le (4f + 3)ru_\ell.$$

This bound uses the original lattice and all its original local data. For a finite family of such summands a common torsion cutoff is $M = (4f + 3)ru_\ell + 2$. After putting the covectors into

one finite coefficient field, an unramified enlargement increases the residue cardinality without increasing the ramification index. It can therefore meet the residue-field requirement in the length descent for all the covectors at once.

The extended local equations still annihilate lattice-valued crystalline cocycles after twisting by a character constructed in Section 27. At each $v \mid \ell$ its local restriction is trivial: every auxiliary character is trivial on the full decomposition group there by Section 27.2, and so is every product of their powers. To verify the claim after a finite coefficient extension $\mathcal{O} \subseteq \mathcal{O}'$, apply an $\mathcal{O}$-linear functional $\mathcal{O}' \to \mathcal{O}$ to the values of a cocycle. Applying its rational extension to the crystalline period-ring equation shows that the result satisfies the original local condition. Evaluating a finite-support relation commutes with the functional, and these functionals separate the elements of the finite free module $\mathcal{O}'$. This proves the required inclusion of crystalline cocycles in the module tested by the extended equations. Framing at the scalar inertia element is preserved because the twist is unramified at its place.

28.5 The order of the simultaneous choices

In this subsection, we specify which choices in Sections 27.3 and 24 are made before the torsion order grows. This allows the finite Selmer family and the norm systems of Section 27 to use one specialization.

Proposition 28.3 (One auxiliary twist for the Selmer and norm tests)**.** *Fix a coefficient place $\kappa = (\ell, \iota)$, the finite family of original integral Selmer data satisfying the hypotheses of Sections 27.4 and 28.4, the finite family of norm systems of Section 27, m, and $0 < \eta < 1$. Use the same element g_0 of inertia at an original place away from ℓ for the first-prime tests in every system. Specifically, require the two first-prime determinant bounds of Section 28.4 at $g_0, (g_0^c)^{-1}$, and $\vartheta_i(g_0) \neq 1$ for every norm system whose exponent list is nonempty. Assume all norm-system augmentations are nonzero. One can choose a common bound ℓ^H for the torsion orders before the auxiliary level ℓ^N and the data $J, (\chi_w), (P_i)$ at that level. There is then one tuple $(\zeta_w)_{w \in J}$ with*

$$\zeta_w^{\ell^H} = 1, \qquad \psi = \prod_{w \in J} \zeta_w^{\chi_w}, \qquad v(P_i((\zeta_w^j - 1)_w)) \leq \beta_i + \eta \quad (\textit{all } i,\ 1 \leq j \leq m).$$

The bounds β_i are independent of N. Each ψ^j is ramified on $J \cup \bar{J}$. For a Selmer test of rank n_i, let O' be the ring of integers in any finite extension of the original coefficient field containing the tuple $(\zeta_w)_{w \in J}$. Put $T_{i,j} = (L_i \otimes_{O_0} O') \otimes \psi^j$, where O_0 is the original coefficient DVR, and put $K' = \mathrm{Frac}(O')$. Let $\mathcal{S}_{i,j}$ consist of the continuous cocycles with coefficients in $T_{i,j} \otimes_{O'} (K'/O')$ vanishing at the scalar inertia element of this summand and on inertia outside the places above $\ell p_0 p_1$ and $J \cup \bar{J}$. At each $v \mid \ell$ impose all O'-linear relations of finite support in the cocycle evaluations that annihilate the integral cocycles of $T_{i,j}$ whose rational class lies in $H^1_f(E_v, T_{i,j} \otimes_{O'} K')$. For the uniformizer valuation $v_{O'}$, the valuation $v_\ell(\ell) = 1$, and a tame generator γ_w, one has

$$\frac{\mathrm{length}_{O'} \mathcal{S}_{i,j}}{v_{O'}(\ell)} \leq (4f+3) n_i u_\ell + \eta, \qquad 2 \sum_{w \in J} v_\ell(\psi^j(\gamma_w) - 1) \leq \eta \qquad (1 \leq j \leq m).$$

Proof. Write $\beta_i^{\mathrm{Sel}} = (4f+3) n_i u_\ell$ for the bounds after releasing the distinguished pair. Adjoin zero to each finite list of bounds before taking its maximum. Fix an integer $M > 1 + \max(\{0\} \cup$

$\{\beta_i^{\mathrm{Sel}}\}_i$) for the Selmer rows. Make the finite unramified coefficient extension of Section 27.3 at this M. Over the resulting fixed DVR the descent bounds $|J|$ by

$$1+\sum_{i}\sum_{a=1}^{Mv_O(\ell)}\operatorname{length}_O C_{i,a}(\varnothing)$$

over the fixed coefficient DVR O of Section 27.3. This integer is independent of N. For these J the congruence Selmer lengths at zero satisfy the bounds after the first place has been released as in Section 24. Independently, for each norm system with character ϑ_j choose an integer $M_j' > \beta_j^{\mathrm{norm}} + 1$, using its augmentation and first Euler-factor bound.

In the descent require the first Frobenius in $\mathcal{W}$, the intersection at g_0 of

$$\begin{aligned}\rho_i(g) &\equiv \rho_i(g_0) \pmod{\ell^M},\\ \rho_i((g^c)^{-1}) &\equiv \rho_i((g_0^c)^{-1}) \pmod{\ell^M},\\ \vartheta_j(g) &\equiv \vartheta_j(g_0) \pmod{\ell^{M_j'}},\end{aligned}$$

over all original main data and signed norm systems. This is an open neighborhood by continuity. At $q_{w_0} \equiv 1 \bmod \ell^N$ and $N \geq \max(\{0\} \cup \{M_j'\}_j)$, the last conditions preserve each first norm bound. The M_j' are independent of the Selmer depth M. All these conditions refer to the original main summands. The later trace systems at the raised level retain their prescribed type and contain the twisted main systems. Write β_i for the combined list of the Selmer and norm bounds.

Take $s_0 \geq 1$ bounding $|J|$, the degrees of the shifted Frobenius products in the Selmer tests, and $2|\mathcal{E}|(|J|-1)$ for each norm system. For a prescribed number m of powers and $0 < \eta < 1$, choose a positive rational a_0 with $s_0 a_0 \leq \eta/2$. Apply Lemma 26.1 with s_0, m, a_0, tolerance $\eta/2$, and the list $\beta_i + 1$. This fixes H before the primes or their polynomials. Choose an integer $N > \max\{H, M\}$ with $Na_0 \geq M$ and, for each norm system j,

$$N \geq M_j', \qquad Na_0 > \beta_j^{\mathrm{norm}} + s_0 a_0.$$

Since $a_0 < 1$, the first scalar inequality gives $N\min(1, a_0) \geq M$ on every Selmer test line in Section 25. All these lower bounds are fixed before the primes are chosen. Then choose J, the characters, and all the polynomials. Each test has a point of depth at least a_0 where its valuation is at most $\beta_i + s_0 a_0$. One tuple $(\zeta_w)_w$ now suffices for all tests and all powers $1 \leq j \leq m$.

In particular

$$\psi = \prod_{w \in J} \zeta_w^{\chi_w}, \qquad v(P_i((\zeta_w^j - 1)_w)) \leq \beta_i + \eta \quad (1 \leq j \leq m).$$

Let O' be any coefficient DVR in the statement and let R be the ring of integers of the compositum of $\mathrm{Frac}(O')$ with the field used to choose the slopes. Over R, Lemma 25.1 bounds the normalized length by the valuation of the same minor. Since $\beta_i^{\mathrm{Sel}} + \eta < M$, Section 25 gives this bound for divisible coefficients.

At $v \mid \ell$ the twist is trivial. Write T for the local O'-lattice and $T_R = T \otimes_{O'} R$. Expand an integral crystalline cocycle in an O'-basis of R. The coordinate projections applied to its period primitive give primitives for its coordinates, and scalar extension preserves each primitive. Thus

$$Z_f(T_R) = Z_f(T) \otimes_{O'} R.$$

On any finite set of evaluations the same basis expansion gives the corresponding equality for the modules of relations annihilating these cocycles. Indeed a relation over R vanishes on the extended cocycles from T exactly when each coefficient relation vanishes on $Z_f(T)$. These cocycles span $Z_f(T_R)$ over R. The same argument modulo ℓ^a, for every $a \geq 1$, applies to the images of the cocycles in the finite coefficient modules, using the reduced basis of $R/\ell^a R$ over $O'/\ell^a O'$.

The twist values and original matrices lie in O', so the global cocycle equations, inertia conditions and normalization at the scalar element also hold coordinatewise. Since $(\mathrm{Frac}(O')/O') \otimes_{O'} R = \mathrm{Frac}(R)/R$, this gives $\mathcal{S}_{i,j}(R) = \mathcal{S}_{i,j}(O') \otimes_{O'} R$. The coordinate argument applies to every finite extension of coefficient DVRs $O_0 \subseteq O$ over which the selected twisted lattice is obtained by scalar extension from O_0. It gives

$$\frac{\mathrm{length}_O \mathcal{S}_{i,j}(O)}{v_O(\ell)} = \frac{\mathrm{length}_{O_0} \mathcal{S}_{i,j}(O_0)}{v_{O_0}(\ell)}, \tag{84}$$

since the ramification index multiplies both the length and the uniformizer valuation of ℓ. Taking $O_0 = O'$, $O = R$ gives the same normalized bound over O'. The norm tests are the values of ψ^j on their systems. Each $v(\zeta_w - 1) \leq a_0/m$. If ζ_w has order ℓ^h and $j = \ell^u j'$ with $\ell \nmid j'$, then

$$\frac{j}{\varphi(\ell^h)} \leq a_0 < 1 \quad \Longrightarrow \quad u < h, \qquad v(\zeta_w^j - 1) = \frac{\ell^u}{\varphi(\ell^h)} \leq a_0.$$

Thus ψ^j is ramified at each $w \in J$. At $\bar{w}$ the values on the chosen generators have the same valuation, by anticyclotomicity and invariance under prime-to-ℓ powers. Hence

$$\sum_{w \in J \cup \bar{J}} v(\psi^j(\gamma_w) - 1) \leq 2|J|a_0 \leq \eta. \qquad \square$$

29 Crystalline bounds and auxiliary prime selection

In this section, we bound the relative cocycle lengths for the lattices of Section 28 and select auxiliary primes through paired Kummer evaluations. These choices enter the trace determinant in Section 32.

29.1 Crystalline colength for the relative cocycles

In this subsection, we bound the local indices in the relative cocycle modules from the original potentially crystalline lattices and their Tate duals. Appendices C and E give the local bounds used in Section 28.4.

Let $O = \mathcal{O}$ be the common coefficient DVR. For a main or character lattice L, write $L^\dagger = \mathrm{Hom}_O(L, O)(1)$. Fix the lift number j, its descent pair, the upper weight and the individual tuple of old finite types selected by Q in Section 32.3 before varying the upper Hecke system x. For a rank-one pair this fixes z and the character of weights $-c_\tau$ in (H2). The auxiliary pairs at positive depth satisfy $\ell^\nu \mid \mathrm{N}w - 1$, so no such place lies over ℓ. Raising the level at these pairs preserves each old type and preserves the hyperspecial level at $v \mid \ell \nmid M$. Indeed at an old split place let P be the stabilizer in the residue group of the image of the original block inclusion. The subgroup fixing its columns pointwise contains the unipotent radical of P and its complementary general linear block. Its prescribed character is trivial on the unipotent radical.

On the remaining block the projector Q selects the chosen finite type with the determinant twist in Section 32.3. Thus each upper constituent in this subspace contains the indicated Levi type on its unipotent invariants. By finite Frobenius reciprocity it contains the corresponding finite parabolic induction: the cuspidal lines of the two blocks are disjoint, so this induction is irreducible [69, Proposition 6.9]. This verifies the compact-type hypothesis of Section 28.3 on every upper constituent at the raised level. At $v \mid \ell$ with $\ell \nmid M$, the original level contains the integral stabilizer of the standard unimodular Hermitian lattice. Raising the level away from ℓ preserves this stabilizer. The quadratic algebra of local integers is étale. After this étale base change the unitary group of the lattice is GL_n, since the inverse adjoint determines the second factor from the first. Smoothness and connected reductive geometric fibres descend, so the stabilizer is hyperspecial also at residue characteristic two. The spherical comparison of Section 28.3 therefore gives crystallinity over E_v. For $\ell \mid M$ the annihilating polynomials for inertia on the potentially crystalline periods at each old place are common to the whole family for this tuple. For each $v \mid \ell$, define $\mathcal{Q}_v$ by the jump support of the full upper list in Section 29.1.1 and the local conditions in Section 28.3. Off M the local class requires crystallinity over E_v. For $v \mid \ell$ with $\ell \mid M$ it requires potential crystallinity and, for every $s \in I_v$,

$$B_Y(s)^{e_r} = 1, \qquad q_{v,s}(B_Y(s)) = 0$$

on the linear period action. Here $e_r = \operatorname{lcm}_{1 \le a \le r}(\ell^a - 1)$ and $q_{v,s} \in \mathbf{Z}_\ell[X]$ is the common monic polynomial for the upper type of degree at most $(r+t)r$ from Section 28.3.2. Since e_r is prime to ℓ, the first equation forces trivial wild inertia. The jumps at each embedding must lie in the specified support, with arbitrary multiplicities. As checked below, this class contains the restrictions of L and of every U_x for this pair and is closed under equivariant inclusions into finite direct sums of its members. The local class, including the jumps and inertial polynomials, is fixed before the cocycles are chosen. At depth $\nu = 0$ the coefficient quotient is zero. Take $\nu \ge 1$ for the finite-level argument below.

Write $Z_f(A)$ for the integral cocycles of a local lattice A that become coboundaries over $B_{\max}$. This agrees with the usual crystalline condition: $\varphi(B_{\max}) \subseteq B_{\mathrm{cris}} \subseteq B_{\max}$ [13, §13, before Theorem 13.1]. Applying $\varphi \otimes 1$ to a $B_{\max}$-primitive gives a B_{cris}-primitive for the same cocycle, and the opposite inclusion uses the ring inclusion.

Define the cup annihilator of $\mathcal{Q}_v$ in $Z_f(L^\dagger)$ using all members $Y \in \mathcal{Q}_v$, all depths ν, and all O-linear forms $Y \otimes L^\dagger \to O(1)$ equivariant modulo ℓ^ν. For each form compose with any $\mathbf{Z}_\ell$-linear functional $O \to \mathbf{Z}_\ell$. The test is that the cup with each $Z_f(Y)$, reduced at $a = \ell^\nu$, vanish in $H^2(E_v, \mu_a)$. By Kummer and Hilbert 90 this is the a-torsion subgroup $\mathrm{Br}(E_v)[a]$. Let $\Delta_v(L)$ be the O-length of the quotient by this annihilator. Put $J_{\mathrm{pair}} = \mathrm{Hom}_O(Y \otimes_O L^\dagger, O(1))$. Since $Y \otimes_O L^\dagger$ is finite free, every equivariant pairing modulo a lifts to an O-linear form $q \in J_{\mathrm{pair}}$. Equivariance gives the integral cocycle $z_q(g) = (gq - q)/a$ with $B_{\max}$-primitive q/a. Replacing q by $q + ab$ adds $(g-1)b$ to the cocycle. The map $(j, c) \mapsto (j + cb, c)$ identifies the extension for $q + ab$ with that for q and satisfies

$$a(j + cb) + cq = aj + c(q + ab).$$

Thus the integral evaluation is independent of the lift under this identification.

The common integral presentation. A relative cocycle $z : G_E \to L/aL$ obeys all the O-linear relations among $d(g) = (u_x(g) - 1)_x$. Let W be the finite product of the matrix lattices of the u_x, acted on by left multiplication, and let $U \subset W$ be generated by the $d(g)$. The map

$d(g) \mapsto z(g)$ defines $U \to L/aL$. It is equivariant since

$$h\, d(g) = d(hg) - d(h).$$

Choose a Smith basis of W in which U is generated by $\varpi^{s_i} e_i$ for a subset of its indices, with ϖ a uniformizer of O. Choose $b_1 \geq 1$ with $(b_1 - 1)v_{\mathrm{ord}}(a) \geq \max_i s_i$ if $U \neq 0$. Then $a^{b_1} W \cap U \subseteq aU$. The map kills this intersection and extends by zero on $a^{b_1} W$ to $Y = U + a^{b_1} W$. Both summands are stable, so the extended map is equivariant. The columns embed Y into copies of the upper lattices, and $a^{b_1} W \subseteq Y \subseteq W$ gives $Y[1/\ell] = W[1/\ell]$ at each completion. Thus Y has the required jumps and local period conditions, including when $\ell = 2$. The identity tuple $I = (1)_x \in W \subset Y[1/\ell]$ satisfies $d(g) = gI - I$, so the integral cocycle d lies in $Z_f(Y)$. Therefore the original relative cocycle at v is the image of a crystalline cocycle in this same class under an equivariant map $t : Y \to L/aL$.

Finite Cartier duality. Write $A_\nu, A_\nu^\dagger$ for the reductions of $Z_f(L), Z_f(L^\dagger)$ at v, and $C_\nu \subseteq A_\nu^\dagger$ for the reduction of the cup annihilator. The trace against a generator of the inverse different of $\mathrm{Frac}(O)/\mathbf{Q}_\ell$ gives a perfect $\mathbf{Z}_\ell$-pairing between $L, L^\dagger$ with values in $\mathbf{Z}_\ell(1)$. Its reduction gives Cartier duality between L/aL and $L^\dagger/aL^\dagger$. Write bars for the images of cocycles in cohomology.

The image of any relative cocycle as above annihilates $\bar{C}_\nu$. Lift t to an O-linear map $q : Y \to L$ by lifting the images of an O-basis of Y. Write $b_q(y, \lambda) = \lambda(qy)$. For the usual action on $\mathrm{Hom}_O(Y, L)$ one has

$$(gq - q)(y) = gq(g^{-1}y) - q(y) \in aL, \qquad (gb_q - b_q)(y, \lambda) = \lambda((gq - q)y).$$

Thus $b_q \bmod a$ is equivariant, and the divided integral cocycle has rational primitive b_q/a. Replacing q by $q + au$ changes that cocycle by the integral coboundary of b_u. Use the inverse-different trace functional of the Cartier pairing among the allowed functionals. Cup functoriality gives the claimed orthogonality in the Brauer group, hence in $H^2(E_v, \mu_a)$ by the Kummer sequence and Hilbert 90. The same reasoning with the reduction of L itself applies to A_ν.

The two finite conditions $\bar{A}_\nu, \bar{A}_\nu^\dagger$ are exact annihilators. For the cardinality calculation let $V = L[1/\ell]$, $h_0 = \dim_{\mathbf{Q}_\ell} H^0(E_v, V)$, $d_f = \dim_{\mathbf{Q}_\ell} H^1_f(E_v, V)$, and let Q_L be the torsion of $H^1(E_v, L)$. The integral finite condition is the preimage of the rational one [16, (3.7.3)], hence is saturated and contains Q_L. Its intersection with $aH^1(E_v, L)$ is $aH^1_f(E_v, L)$, so $\bar{A}_\nu \simeq H^1_f(E_v, L)/aH^1_f(E_v, L)$. The cohomology sequence for multiplication by $a = \ell^\nu$ gives

$$0 \longrightarrow H^0(E_v, L)/aH^0(E_v, L) \longrightarrow H^0(E_v, L/aL) \longrightarrow Q_L[a] \longrightarrow 0.$$

The free $\mathbf{Z}_\ell$-ranks of $H^1_f(E_v, L)$ and $H^0(E_v, L)$ are d_f and h_0, respectively. Since $|Q_L/aQ_L| = |Q_L[a]|$, it follows that

$$|\bar{A}_\nu| = \ell^{\nu d_f} |Q_L[a]|, \qquad |H^0(E_v, L/aL)| = \ell^{\nu h_0} |Q_L[a]|.$$

Use the same identities on the dual. The dimension formula for de Rham representations [16, Corollary 3.8.4, p. 355] and the complementary filtrations give

$$d_f + d_f^\dagger - h_0 - h_0^\dagger = [E_v : \mathbf{Q}_\ell] \operatorname{rank}_{\mathbf{Z}_\ell} L.$$

Rational orthogonality [16, Proposition 3.8] gives integral orthogonality by the injection

$$H^2(E_v, \mathbf{Z}_\ell(1)) \longrightarrow H^2(E_v, \mathbf{Q}_\ell(1)).$$

The local invariant identifies this map with $\mathbf{Z}_\ell \hookrightarrow \mathbf{Q}_\ell$ by passing to the inverse limit of the invariants on $H^2(E_v, \mu_{\ell^j})$ and inverting ℓ. The finite $H^1(E_v, \mu_{\ell^j})$ satisfy the Mittag–Leffler condition, so this limit computes the integral group. For $M = L/aL$, finite local duality [49, Ch. I, Corollary 2.3] gives

$$|H^2(E_v, M)| = |H^0(E_v, L^\dagger/aL^\dagger)|.$$

The local Euler characteristic [49, Ch. I, Theorem 2.8] gives

$$|H^1(E_v, M)| = |H^0(E_v, M)|\,|H^2(E_v, M)|\,|M|^{[E_v:\mathbf{Q}_\ell]}.$$

Substituting the cardinalities above yields

$$\bar{A}_\nu \subseteq (\bar{A}_\nu^\dagger)^\perp, \qquad |\bar{A}_\nu|\,|\bar{A}_\nu^\dagger| = |H^1(E_v, L/aL)|.$$

The finite Tate pairing is perfect, so the inclusion is an equality.

If S_ν denotes the local image of the relative cocycles, coboundaries modulo a lift to integral coboundaries. The kernel of passage to cohomology on S_ν is therefore contained in A_ν, and

$$\operatorname{length}_O\big(S_\nu/(S_\nu \cap A_\nu)\big) \le \operatorname{length}_O(\bar{C}_\nu^\perp/\bar{A}_\nu) = \operatorname{length}_O(\bar{A}_\nu^\dagger/\bar{C}_\nu) \le \Delta_v(L).$$

The two middle quotients have equal cardinality by the perfect Tate pairing. They are O-modules because the inverse-different trace pairing satisfies $\langle bx, y\rangle = \langle x, by\rangle$ for $b \in O$. For any finite O-module D, writing $f_O = [O/\mathfrak{m}_O : \mathbf{F}_\ell]$, one has

$$\log_\ell |D| = \operatorname{length}_{\mathbf{Z}_\ell} D = f_O \operatorname{length}_O D.$$

This gives the equality of O-lengths. The last quotient is an image of the defining defect quotient.

Every member of A_ν obeys all the integral linear relations of $Z_f(L)$. Thus imposing those linear relations on S_ν contributes at most $\Delta_v(L)$. At each place v this argument uses the restrictions of the actual integral lattices $Y, L, L^\dagger$ to G_{E_v}. If $v \neq \bar{v}$, repeat it on $G_{E_{\bar{v}}}$ with the same integral coefficient maps. The polarization compares the rational spectra there. The local cup pairings at the two places are formed separately.

Lemma 29.1 (Crystalline colength over the coefficient prime)**.** *If $c_{\mathrm{crys}}(L, \nu)$ denotes the global colength for the crystalline relations at the places above ℓ, then*

$$c_{\mathrm{crys}}(L, \nu) \le \sum_{v|\ell} \Delta_v(L). \tag{85}$$

Proof. Write Z for the relative cocycle module at depth ν and Z_{lin} for its submodule satisfying the crystalline relations at all $v \mid \ell$. With $S_{\nu,v}$ the local restriction of Z and $A_{\nu,v}$ the reduction of the integral crystalline cocycles, put $Z_{\mathrm{red}} = \bigcap_{v|\ell} \mathrm{res}_v^{-1}(A_{\nu,v})$. For finitely supported $F_g \in \mathrm{End}_O(L)$, write $R_F(z) = \sum_{g \in G_{E_v}} F_g z(g)$. The relations are those with $R_F(Z_f(L)) = 0$, reduced modulo $a = \ell^\nu$. They all vanish on $A_{\nu,v}$, so $Z_{\mathrm{red}} \subseteq Z_{\mathrm{lin}}$. Restriction induces an injection

$$Z/Z_{\mathrm{red}} \hookrightarrow \bigoplus_{v|\ell} S_{\nu,v}/(S_{\nu,v} \cap A_{\nu,v}).$$

The preceding finite-duality estimate bounds each summand's O-length by $\Delta_v(L)$, giving the assertion for $c_{\mathrm{crys}}(L, \nu) = \operatorname{length}_O(Z/Z_{\mathrm{lin}})$. □

Summing the defects. On the descent pair the upper lattice for its own parameter is $T \oplus \bigoplus_{0 \le e < t'} O(-e)$, with $t' \le t$. Here $r' = \operatorname{rank}_O T$ is r in (H1) and 1 in (H2). At a coefficient embedding the jumps of this direct sum are the multiset sum of the original main jumps and $0, \dots, t'-1$. In the Tate dual the jumps u are replaced by $-u-1$ and the inertia operator is the transpose of the inverse period operator. For this ℓ, take the coefficient fraction field to be a finite Galois extension of $\mathbb{Q}_\ell$ containing the matrix entries, local type eigenvalues, all embedded completions E_v and their residue-field Teichmüller roots. Take the normal closure of their finite compositum. This gives one field for all places above ℓ. For a coefficient embedding extended by $g \in G_{\mathbf{Q}_\ell}$, read the jumps at the embedding $g^{-1}e_v$ of E, where e_v is the chosen local embedding, transported to the complex side by the fixed comparison. This choice depends only on the coefficient embedding, because $g^{-1}e_v(E)$ lies in the coefficient field. The upper class uses the full original upper list at this embedding.

Use the local class $\mathcal{Q}_v$ defined above. An equivariant inclusion into a sum of these lattices at each embedding gives a jump multiset contained, with multiplicity, in that of the ambient direct sum. Its jump support therefore lies in the prescribed union, and the inertia operators satisfy the same polynomials. Indeed one may read the crystalline periods after a common finite crystallizing extension, using the exact crystalline period functor there. In particular the direct summands of the upper lattice for its own parameter belong to that same class.

29.1.1 Hodge–Tate intervals at the coefficient places

In this subsection, we determine the intervals for the lattices used in Appendix E.

Write $h = r^8$, $G = xb/x_1$, $Q_* = kt$, $W_* = 4t$, and $\eta = h^2 + 1$. For the fixed lift $1 \le j \le m$ put

$$w_j(\tau) = \kappa_\tau + j\Delta + h, \qquad \delta_j = j\Delta + h - 2t, \qquad S_j(\tau) = \kappa_\tau + \delta_j.$$

Let ε_σ be 1 if σ restricts to the chosen embedding of K, and -1 otherwise, and let τ be its infinite place. Write J_σ for the multiset of main jumps. For the rank-r pair, $t' = t$ and

$$J_\sigma = \left\{ \frac{t-1}{2} - \varepsilon_\sigma(\lambda_q(\tau) + w_j(\tau)) : 0 \le q < r \right\}. \tag{H1}$$

Indeed, if $n = r + t$, the upper highest-weight entry μ_q gives the doubled parameter $P_q = 2\mu_q + n - 1 - 2q$.

The raw jump is $(n - 1 - \varepsilon_\sigma P_q)/2$. The normalizing character subtracts $r - r_-$ on the chosen CM type and r_- on its conjugate, where r_- counts the negative lower signs. Thus the normalized formula replaces $n-1, P_q$ by $t-1, P_q - 2r_- + r$. By the parameter correspondence of the descent pair, the latter shifted entries consist of $2(\lambda_q + w_j)$ and $t - 1 - 2a$ for $0 \le a < t$. In the order of this parameter list the second block reads a on the chosen CM type and $t-1-a$ on its conjugate. Reindexing the latter by $a \mapsto t - 1 - a$ gives the same Tate string at both embeddings. The fixed global summand $O(-a)$ has jump a at every embedding, and its Tate dual $O(a+1)$ has jump $-a-1$. Equivalently one gets (H1) directly by lowering the original rank-r jumps by $(r-t)/2 + \varepsilon_\sigma w_j$.

For a rank-one pair the parameter z and its character of weight $-c_\tau$ satisfy

$$m < z \le r, \qquad w_j + r - t \le c_\tau \le w_j, \qquad t' = 2z - 1, \qquad J_\sigma = \{z - 1 - \varepsilon_\sigma c_\tau\}. \tag{H2}$$

The last equality is the jump of the character obtained by normalizing the trivial rank-one system. In both cases $\varepsilon_{\bar\sigma} = -\varepsilon_\sigma$, with τ unchanged. Corresponding main jumps sum to $t' - 1$, so their total negative count over the two embeddings is at most r'.

For $0 \le q < k-1$ we have $\lambda_q = k/2 - 1 - q$ by Section 12.10. The corresponding jump in (H1) therefore satisfies

$$|\varepsilon_\sigma u + w_j| = \left|\varepsilon_\sigma \frac{t-1}{2} - \lambda_q\right| \le \frac{t+k-3}{2} < t.$$

The rank-one formula (H2) gives $|\varepsilon_\sigma u + w_j| = |\varepsilon_\sigma(z-1) + w_j - c_\tau| \le (r-1) + (t-r) < t$. For the remaining m indices of (H1),

$$\lambda_{k-2+i} + w_j = (j-i)\Delta + h = (2(j-i)+1)h \qquad (1 \le i \le m).$$

Since $1 \le |2(j-i)+1| \le 2m-1$, either CM sign gives

$$\frac{h}{2} \le h - \frac{t-1}{2} \le |u| \le (2m-1)h + \frac{t-1}{2} < 2mh \le h^2.$$

For every jump with $|\varepsilon_\sigma u + w_j| < t$, the reverse triangle inequality gives $|u| > w_j - t > 2h^2$, since $w_j \ge G \ge 99h^3$.

At a coefficient embedding γ, use σ as above and put $\mathcal{U}_\gamma = \operatorname{supp}(J_\sigma) \cup \{0, \dots, t'-1\}$. For the Tate dual of an individual summand L, let $\mathcal{U}'_{\gamma,L}$ be its actual jump set. These sets lie in opposite intervals:

$$\begin{aligned} \mathcal{U}_\gamma &\subset [-\eta, \eta] \ \cup\ -\varepsilon_\sigma[S_j(\tau), S_j(\tau) + W_*], \\ \{-u-1 : u \in \mathcal{U}_\gamma\} &\subset [-\eta, \eta] \ \cup\ \varepsilon_\sigma[S_j(\tau), S_j(\tau) + W_*]. \end{aligned}$$

The 1 in η absorbs the dual shift for low jumps, and the far-interval half-width $2t$ absorbs it for high jumps. For the main summand, every negative member of $\mathcal{U}_\gamma \cup \mathcal{U}'_{\gamma,L}$ is at most $-h/2$. Its Tate dual has at most $\lfloor r'm/k \rfloor$ jumps in $[-\eta, \eta]$, counted with multiplicities: at full rank there are at most m tail indices, and at rank one there are none. For an individual Tate character of jump a, the dual set is $\{-a-1\} \subset [-\eta, 0]$.

Residue-field parameters and sparsity. Write E_v for the embedded local field at $v \mid \ell$, with residue field k_v, degree d_v, ramification index e_v^{ram} and residue degree f_v. Then $e_v^{\mathrm{ram}} f_v = d_v = [G_{\mathbf{Q}_\ell} : G_{E_v}]$ and $d_v \le [E : \mathbf{Q}] \le h$, since $[E : \mathbf{Q}] \le 2q_1 = O_{q_0}(x^{11} \log x)$ and $h = r^8 \ge x^{40}$. Choose a primitive $(\ell^{f_v} - 1)$-st root ζ_v in the maximal unramified subfield of E_v. A coefficient embedding γ has parameter $\gamma(\zeta_v)$. The chart is indexed by all $i \in k_v^\times$, identified with their Teichmüller lifts. The parameters occurring at embeddings are precisely $\{\zeta_v^{\ell^h} : 0 \le h < f_v\}$: the restrictions to $\mathbf{Q}_\ell(\zeta_v)$ are its Frobenius powers, each of which extends to an embedding of E_v. Witt Frobenius sends i to i^ℓ, so the level-a test at i reads $\varphi^{-a} i$. Call a parameter thick if it occurs at an embedding whose read infinite place lies in T, where $\kappa_\tau = b$.

If $\ell \nmid q_1 d$, the completion is unramified and ζ_v generates it. Two embeddings with the same parameter agree on the completion, hence read the same σ. On the orbit of ζ_v the parameter therefore determines both thickness and sign. If $\ell \nmid q_1 d$, then $\ell > 7$ since $2 \cdot 3 \cdot 5 \cdot 7 \mid 18480 \mid q_0 \mid q_1$. Every prime dividing $q_1 d$ divides $q_0 d$, so is at most Q_* for large x. If $\ell \mid q_1$, then $\ell \mid q_0$. By Lemma 4.1, K splits at ℓ, so all embeddings at a fixed completion have the same restriction to K and therefore the same sign. At $\ell = d$, the completion of the cyclotomic field is unramified, and adjoining $\sqrt{-d}$ is totally ramified quadratic. Each embedding of the maximal unramified subfield consequently extends to two embeddings of E_v with opposite CM signs. If two thick embeddings at this completion had opposite signs, the ratio of their cyclotomic coordinates on the chosen CM type would be $-d^a \bmod q_0$ for some a, since Frobenius acts by d and conjugation

negates the coordinate. Both thick coordinates are $-1 \bmod q_0$, so $d^a \equiv -1 \bmod q_0$. This is impossible by $-1 \notin \langle d \bmod q_0 \rangle$ in Lemma 4.1. At $\ell \mid q_1$, set $a = v_\ell(q_1)$ and $q' = q_1/\ell^a$. At either place of K the completion is $\mathbf{Q}_\ell$. The roots of order q' generate the unramified extension of degree $\operatorname{ord}_{q'}(\ell)$, and $\Phi_{\ell^a}(1+X)$ remains Eisenstein over it. At $\ell = d$, the roots of order q_1 generate an unramified extension, over which $X^2 + d$ is Eisenstein. Thus

$$(e_v^{\mathrm{ram}}, f_v) = \begin{cases} (\varphi(\ell^a), \operatorname{ord}_{q'}(\ell)), & \ell \mid q_1, \\ (2, \operatorname{ord}_{q_1}(d)), & \ell = d. \end{cases}$$

These degrees are independent of the coefficient embedding. In the ramified cases assign all parameters the common sign of the thick embeddings, with either choice if there are none.

Suppose $\ell \le Q_*$ and three thick parameters are $\omega, \omega^{\ell^a}, \omega^{\ell^{a+b}}$, where $1 \le a, b \le \lfloor \log_\ell h^3 \rfloor$. Represent them by coefficient embeddings extended to $g_0, g_1, g_2 \in G_{\mathbf{Q}_\ell}$. Put $D = \ell^{f_v} - 1$ and $q' = (q_1)_{\ell'}$. The image under e_v of the prime-to-ℓ part of ζ_{q_1} has order q' and is a power of ζ_v, so $q' \mid D$. Write $g_i(\zeta_v) = \zeta_v^{u_i}$ with $u_i \in (\mathbf{Z}/D\mathbf{Z})^\times$. The three parameters give $u_1 = \ell^a u_0$, $u_2 = \ell^b u_1$ modulo D. The corresponding embedding is $g_i^{-1} e_v$, so its exponent on the root of order q' is multiplied by u_i^{-1}. If τ_i denotes its infinite place on the chosen CM type and ε_i its sign on K, complex conjugation multiplies the exponent by -1. Consequently, modulo q',

$$\tau_1/\tau_2 = \varepsilon_1\varepsilon_2\, u_2/u_1 = \varepsilon_1\varepsilon_2 \ell^b, \qquad \tau_0/\tau_2 = \varepsilon_0\varepsilon_2\, u_2/u_0 = \varepsilon_0\varepsilon_2 \ell^{a+b}.$$

Since

$$0 < b < a + b \le 2\lfloor \log_\ell h^3 \rfloor \le 24s < L_0,$$

where $2\log_\ell h^3 \le 48 \log_2 r \le 24s$ for large x, by $r < 3x^5$ and $s \ge 11 \log_2 x$. Also $L_0 = 27 \cdot 3^{\lfloor \log_3 s \rfloor + 1} > 27s$. Thus the sparsity proved in Section 4 excludes this triple. At $\ell = 2$, two thick parameters at any distance $0 < i < L_0$ would already give $4^i \equiv 1 \pmod{3L_0}$: their signed ratio is $\pm 2^i$, while both places are $-1 \bmod 3L_0$ and $3L_0 \mid q'$. Since $v_3(4^i - 1) = 1 + v_3(i)$, this would force $L_0 \mid i$.

For a parameter take the union of the jump sets over its coefficient embeddings and adjoin 0. At an index outside the orbit of ζ_v, the two original sides consequently have jump sets $\{0\}$ after adjoining zero. This index is thin. Assign it the thin center $G + \delta_j$, with either sign in the unramified case and the common sign chosen above in the ramified case. When $\ell > Q_*$ the resulting intervals are the displayed ones with the sign and center read on the parameter. When $\ell \le Q_*$ the low radius can be enlarged to $2G$, since for a thin embedding $\eta \le G$ and $G + \delta_j + W_* \le 2G$ put both of its intervals in $[-2G, 2G]$. In particular at $\ell = d$ a thick residue parameter also has an embedding of E_v with the opposite CM sign. That embedding is thin and all its jumps belong to $[-2G, 2G]$. The other intervals still have the sign and center read on the thick parameters. For dual sets the signs of the separated intervals are reversed.

The Kummer coordinate and the scale inequalities. Choose a uniformizer π_v of E_v and compatible roots $\underline{\pi_v}$. In Fontaine's ring $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbf{C}_\ell^\flat})$ use $Z = [\underline{\pi_v}]$, and embed $W(k_v)[[Z]]$ by evaluation, which is injective modulo ℓ by the valuation of the first nonzero term. Write θ for Fontaine's map [14, Lemma 3.10 and Remark 3.11]. Lift the unit $-\ell/\pi_v^{e_v^{\mathrm{ram}}}$ to a unit A_v of $W(k_v)[[Z]]$ by the uniformizer expansion. Then $Z^{e_v^{\mathrm{ram}}} A_v + \ell$ generates $\ker\theta$: it lies in that ideal, and its reduction modulo ℓ has the same valuation as the reduction of the generator $[\underline{\ell}] - \ell$.

Their quotient is therefore a unit. If $e_v^{\mathrm{ram}} = 1$, choose $\pi_v = -\ell$ and $A_v = 1$. The group fixing $W(k_v)$ pointwise and fixing π_v is G_{E_v}, since $E_v = W(k_v)[1/\ell](\pi_v)$. If $\underline{\epsilon}$ is a compatible system of primitive ℓ^n-th roots of unity, for every integer $a \ge 0$ some element of G_{E_v} sends

Z to $[\epsilon]^{a}Z$. Indeed $X^{\ell^{n}}-\pi_{v}$ is Eisenstein over E_{v}, so the closed sets imposing this action at level n are nonempty and nested. Compactness produces the result. The residue action of the arithmetic Frobenius is $y\mapsto y^{\ell^{f_{v}}}$.

The size conditions for these intervals are

$$Q_{*}>0,\quad 99(Q_{*}^{2}+(W_{*}+1)Q_{*}+Q_{*}+W_{*}+m)\le h,$$
$$99h^{3}\le G,\qquad 99(Gh^{3}+G)\le b,\qquad 2\log(4b)<m,$$
$$0\le\delta_{j}<\eta,\qquad 99((\eta+W_{*})Q_{*}+\eta+h)\le h^{3}.$$

Indeed $Q_{*}\le 5r^{2}$, $W_{*}=20r$, $m<r$, $h=r^{8}$, and $\delta_{j}\le(2m+1)h$. For the comparisons with G,b use $b\ge x_{1}^{10^{6}}$, $x_{1}\ge x^{10^{6}}$, $G/b=x/x_{1}$, and $(\log b)^{100}\le x$. At a prescribed level prime $\ell\mid M$, the congruence $\ell\equiv 1 \bmod q_{1}d$ makes $E_{v}=\mathbf{Q}_{\ell}$, so all coefficient embeddings read the same σ. The tame interval threshold holds because $\ell\ge b^{10}$:

$$S_{j}(\tau)+6(W_{*}+\eta)+r^{2}((r+t)r+1)^{2}+2<\ell.$$

For a local cup exponent q_{v}, meaning that $\ell^{q_{v}}Z_{f}(L^{\dagger})$ lies in the cup annihilator, the normalized defect is at most

$$\frac{\Delta_{v}(L)}{v_{\mathrm{ord}}(\ell)}\le q_{v}\,\mathrm{rank}_{O}\,Z_{f}(L^{\dagger})\le 2q_{v}[E_{v}:\mathbf{Q}_{\ell}]\,\mathrm{rank}_{O}\,L.$$

Put $K_{O}=\mathrm{Frac}(O)$ and $V=L^{\dagger}\otimes_{O}K_{O}$. The rational cocycles $Z_{f}(V)$ are the inverse image of $H^{1}_{f}(E_{v},V)$ under passage to cohomology. Their subspace of coboundaries has dimension $\dim_{K_{O}}V-\dim_{K_{O}}H^{0}(E_{v},V)$. The representation V is de Rham since it is potentially crystalline. The Bloch–Kato formula [16, Corollary 3.8.4] on the underlying $\mathbf{Q}_{\ell}$-representation, with dimensions divided by $[K_{O}:\mathbf{Q}_{\ell}]$, gives

$$\dim_{K_{O}}Z_{f}(V)=\dim_{K_{O}}V+\dim_{K_{O}}(D_{\mathrm{dR}}(V)/\operatorname{Fil}^{0})\le(1+[E_{v}:\mathbf{Q}_{\ell}])\,\mathrm{rank}_{O}\,L.$$

Here $D_{\mathrm{dR}}(V)$ has $\mathbf{Q}_{\ell}$-dimension $[E_{v}:\mathbf{Q}_{\ell}]\dim_{\mathbf{Q}_{\ell}}V$. The common kernel of all evaluations on $G_{E_{v}}$ is zero. While a finite collection has nonzero joint kernel, evaluation at a point where a cocycle in that kernel is nonzero strictly lowers its dimension. The resulting finite evaluations embed $Z_{f}(L^{\dagger})$ into a finite direct sum of copies of $L^{\dagger}$, so it is finite free over O. Every rational continuous cocycle has compact image, and a power of ℓ makes it integral. Hence $Z_{f}(L^{\dagger})$ has rank $\dim_{K_{O}}Z_{f}(V)$.

For the fixed $\mathcal{Q}_{v}$, apply the same definition of Δ_{v} to the sum of the character lattices. Then

$$\Delta_{v}\left(\bigoplus_{a=0}^{t'-1}O(-a)\right)=\sum_{a=0}^{t'-1}\Delta_{v}(O(-a)).\tag{86}$$

Indeed each allowed pairing restricts to one on each dual summand at the same depth, and the cup of a cocycle tuple is the sum of the component cups with the same partner and coefficient functional. Conversely, every test on one summand extends by zero to the others. Thus the cup annihilator is the direct sum of the individual annihilators over the original coefficient DVR.

The estimates needed at v are normalized defects at most $[E_{v}:\mathbf{Q}_{\ell}]r'F_{\ell}$ for T and $[E_{v}:\mathbf{Q}_{\ell}]F_{\ell}/t$ for each character, where

$$F_{\ell}=10^{6}\frac{m}{k}\left(bv_{\ell}(M)+v_{\ell}\left((2b)!\prod_{\substack{b-30h_{2}^{3}<j\le b+30h_{2}^{3}\\ j>0}}j^{h_{2}^{3}}\right)\right),\qquad h_{2}=r^{8}.$$

29.1.2 Uniform bounds for the local defects

At the primes dividing the original level we use Appendix C. At the other coefficient primes we use Appendix E, which applies the lifts of Appendix D to the intervals of Section 29.1.1.

Put

$$\lambda_\ell = v_\ell\left((2b)!\prod_{\substack{b-30h^3<n\le b+30h^3\\ n>0}} n^{h^3}\right), \qquad K_\ell = \left\lfloor \frac{2\lambda_\ell}{Q_*}\right\rfloor .$$

Then $\ell(\lambda_\ell+1) > 2b$, since each multiple of ℓ up to $2b$ contributes at least one to the factorial valuation. If $\lambda_\ell < h^3$, no integer in $(b-30h^3, b+30h^3]$ is divisible by ℓ. In this case set $N_\ell = 6(2(\lambda_\ell+1)+\eta+W_*)$. The scale inequalities give $\delta_j + N_\ell < 30h^3$ and

$$\ell > \frac{2b}{h^3+1} > G + \delta_j + N_\ell.$$

For either center $S = \kappa_\tau + \delta_j$, no nonzero multiple of ℓ is within N_ℓ of 0 or of either signed center $\pm S$. For $S = G+\delta_j$ all such integers have absolute value smaller than ℓ. For $S = b+\delta_j$ take the absolute value in the separated intervals to reduce to the preceding exclusion near b.

For fixed q_0, d, the prime divisors of $q_1 d$ are those of $q_0 d$, since $q_1/q_0 = 3L_0\,2^s$ and $6 \mid q_0$. As $Q_* \to \infty$, every $\ell > Q_*$ is therefore unramified in E for sufficiently large x. Moreover

$$\ell \le Q_* \quad\Longrightarrow\quad \lambda_\ell \ge \lfloor 2b/\ell\rfloor \ge \lfloor 2b/Q_*\rfloor \ge h^3,$$

since $b/(Q_* h^3) \to \infty$. For these small primes the factorial bound also implies

$$Q_*(K_\ell+3) > 2\lambda_\ell + 2 > 4b/Q_*.$$

Since b/Q_*^2 grows faster than any power of r, this gives $K_\ell \ge 2b/Q_*^2$ for large x. In particular $2(b+\delta_j) < h^2 K_\ell$. For the normalization of the local defect, write ρ for the rank of the summand being tested and $c_* = 1$ on the main summand, $c_* = t$ on each Tate character. Put $\mu = \lfloor \rho m/(kc_*)\rfloor$, and write $d_v = [E_v : \mathbf{Q}_\ell]$ and $e_O = v_{\mathrm{ord}}(\ell)$. For $\ell \nmid M$, Proposition E.1 applies to the actual dual of L and the whole upper jump set at the fixed lift. It gives

$$\frac{\Delta_v(L)}{e_O d_v} \le 10^5(\rho K_\ell m + \mu\lambda_\ell) \le 3\cdot 10^5 \frac{\rho m \lambda_\ell}{kc_*} \le \rho F_\ell/c_*.$$

The last two comparisons use $Q_* = kt$, $c_* \le t$, whence $0 \le K_\ell \le 2\lambda_\ell/(kt) \le 2\lambda_\ell/(kc_*)$, and $\mu \le \rho m/(kc_*)$. For $\ell \mid M$, Proposition C.1 with $\eta = h^2+1$, $W_* = 4t$ and $k_0 = r^2((r+t)r+1)^2$ gives the sufficient exponent

$$q_{\mathrm{tame}} = 3r^2((r+t)r+1)^2 + 150(h^2+4t+2).$$

For $i = 0, 1$, Proposition 4.2 gives $p_i \ge b^{10} > \max(2b, b+30h^3)$ for sufficiently large x. Since $M = p_0 p_1$ with distinct primes,

$$v_{p_i}(M) = 1, \qquad \lambda_{p_i} = 0, \qquad F_{p_i} = 10^6\frac{m}{k}b, \qquad k = r+1-m.$$

At each $v \mid p_i$, complete splitting gives $d_v = 1$ and $\mathrm{rank}_O\, Z_f(L^\dagger) \le 2\rho$. The inclusion of $p_i^{q_{\mathrm{tame}}} Z_f(L^\dagger)$ in the annihilator gives

$$\frac{\Delta_v(L)}{e_O d_v} \le 2\rho q_{\mathrm{tame}} \le \rho F_{p_i}/c_*.$$

The last inequality follows from $c_* \le t$, $m \ge 1$, and $ktq_{\mathrm{tame}} = o(b)$.

With these estimates, (85) and $\sum_{v|\ell}[E_v : \mathbf{Q}_\ell] = 2f$ give the following normalized bound at every depth ν:

$$\frac{c_{\mathrm{crys}}(T,\nu) + \sum_{0\le j<t'} c_{\mathrm{crys}}(O(-j),\nu)}{r'e_O} \le 2f\Big(1 + \frac{t'}{r't}\Big)F_\ell \le 4fF_\ell.$$

This is the crystalline contribution per unit rank in (RB1). A uniform cup exponent bounds the index by the rank of the cocycle lattice. In the small-loss range for the main summand, Appendix E bounds the colength of a common sublattice directly. Both bounds hold at every finite pairing depth ν.

For sufficiently large x, $b + 30h^3 < 2b$. Thus $F_\ell = 0$ for $\ell > 2b$ with $\ell \nmid M$. The same primes have zero defect by the local bound. For a finite set I of coefficient primes, put $B = \log b$ and $P_h = \prod_{b-30h_2^3<j\le b+30h_2^3,\ j>0} j$. By nonnegativity and factorization over the finite support we have

$$\sum_{\ell\in I} 4fF_\ell \log\ell \le \sum_{\ell \text{ prime}} 4fF_\ell \log\ell = 4\cdot 10^6 f\frac{m}{k}(b\log M + \log((2b)!) + h_2^3 \log P_h).$$

The half-open interval in P_h contains at most $60h_2^3$ positive integers. Since $h_2^6 = r^{48} < (3x^5)^{48}$ and $b \ge x_1^{10^6} \ge x^{10^{12}}$, for fixed q_0, d, ϵ and all sufficiently large x we have $120h_2^6 \le b$. Hence

$$h_2^3 \log P_h \le 60h_2^6 \log(b + 30h_2^3) \le 120h_2^6 B \le bB.$$

Both primes dividing M satisfy $p_i \ge b^{10} > 2b$, so $\lambda_{p_i} = 0$. Appendix E treats the primes $\ell \nmid M$ in three disjoint cases, $\ell \le Q_*$, $\ell > Q_*$ with $\lambda_\ell \ge h_2^3$, and $\ell > Q_*$ with $\lambda_\ell < h_2^3$. The triangle in Section 29.2 uses $Y_{\mathrm{tr}}^2 = h_2^2 > Q_*$. Throughout $Q_* < \ell \le h_2^2$, $\lambda_\ell \ge \lfloor 2b/\ell \rfloor \ge h_2^3$, since b dominates h_2^5. Thus these primes use the two-start colength estimate and the small-prime triangle estimate. The contribution of λ_ℓ is counted once per rational coefficient prime in the displayed sum. Also $M \le b^{22}$ and $\log((2b)!) \le 4bB$. Dividing the right side by $fxbB^3/(mk)$ gives at most $108\cdot 10^6 m^2/(xB^2)$, which tends to zero because $m^2 \le 4x$ and $B \to \infty$.

29.2 Cyclotomic-kernel evaluations for the auxiliary primes

In this subsection, we refine the finite detection argument of Section 24.1 to choose elements in the cyclotomic kernel. The normalization of Section 28.1 lets these elements test the representations in the prime selection of Section 29.4.

The finite set of evaluation elements in Section 24.1 constructed by adjoining $\mu_{N_c\ell^2}$ works for every ℓ. The bound y^{20} on its size was used only for small ℓ. To put its points in the cyclotomic kernel, let P be that finite set and take a cyclotomic generator g_{gen}, meaning that its cyclotomic value generates the image of G_E topologically. For $p \in P$ choose q_p in the closure of $\langle g_{\mathrm{gen}}\rangle$ with $\chi_{\mathrm{cyc}}(q_p) = \chi_{\mathrm{cyc}}(p)$. Such a choice exists by continuity and compactness. Set $s_p = pq_p^{-1}$, so $s_p \in \ker\chi_{\mathrm{cyc}}$. The zero set of a continuous cocycle is a closed subgroup. If it contains g_{gen} and every s_p, it contains all $p = s_pq_p$, so the cocycle is zero. This procedure detects all unramified-outside-ℓ cocycles in the stated cyclotomic characters, without a parity condition. The chosen points are independent of $e, \nu, \mathcal{O}$.

Put $Y_{\mathrm{tr}} = r^8 = \Delta/2$, so $N_c + t + 3 \le Y_{\mathrm{tr}}$ for large x. The image of χ_{cyc} on G_E is procyclic, since it is a closed subgroup of $\mathbb{Z}_\ell^\times$ for odd ℓ, and of $1 + 4\mathbb{Z}_2$ at $\ell = 2$ since $i \in E$. Choose

$\gamma_\ell \in G_E$ whose cyclotomic value ξ_ℓ generates that image topologically. If w is any value in the image, $\xi_\ell^a - 1$ divides $w^a - 1$, by taking limits of powers of ξ_ℓ in the closed ideal generated by $\xi_\ell^a - 1$. For $\ell \le Y_{\mathrm{tr}}^2$ take $w = p_i^{\mathrm{lev}}$ from arithmetic Frobenius at a distinguished place not above ℓ. For $0 < a < t$, the integer $w^a - 1$ divides A_{aut} since it is a factor of the order of the general linear group of rank rt. For $\ell > Y_{\mathrm{tr}}^2$, inertia at ℓ fixes $E \subseteq \mathbb{Q}(\zeta_{N_c})$ and its cyclotomic reduction fills $\mathbb{F}_\ell^\times$. Choose w reducing to a generator. Then $w^a - 1$ is a unit for $0 < a < t$. Thus, with

$$h_\ell = \begin{cases} v_\ell(A_{\mathrm{aut}}), & \ell \le Y_{\mathrm{tr}}^2, \\ 0, & \ell > Y_{\mathrm{tr}}^2, \end{cases}$$

every $\xi_\ell^a - 1$, $0 < a < t$, divides ℓ^{h_ℓ}.

For the triangle take $g_{\mathrm{gen}} = g_{\mathrm{tr}} = \gamma_\ell^{-1}$, whose cyclotomic value also generates the image, and use the characters $\chi_e = \chi_{\mathrm{cyc}}^{-e}$. Then $\chi_e(g_{\mathrm{tr}}) = \xi_\ell^e$. For the actual rank increment $t' \le t = 5r$ of the fixed pair, define

$$\Delta_e = \prod_{\substack{0 \le j < t' \\ j \ne e}} (\xi_\ell^e - \xi_\ell^j), \qquad C = \ell^{Y_{\mathrm{tr}} h_\ell}.$$

The scalar $4\Delta_e$ divides C for $0 \le e < t'$, and $4(\xi_\ell^e - 1)^2$ divides C for $0 < e < t'$. Indeed, for $j \ne e$ the difference $\xi_\ell^e - \xi_\ell^j$ is a unit times $\xi_\ell^{|j-e|} - 1$, with valuation at most h_ℓ. At $\ell = 2$, $4 \mid \xi_\ell - 1$ implies $v_2(4) \le h_\ell$; at odd primes 4 is a unit. The inequalities $t - 1 + 1 \le Y_{\mathrm{tr}}$ and $2 + 1 \le Y_{\mathrm{tr}}$ prove the claims.

Let R be the raw lower attachment of rank n and $T = \vartheta_\ell^{-1} \otimes \theta R$ the main summand in the normalized trace of Section 28.1. At rank one the raw attachment is trivial and θ supplies the inverse old tame character. Sections 28.3.1 and 28.3.2 supply the off-ℓ inertia element γ such that $T(\gamma) = \xi I$, $\chi_{\mathrm{cyc}}(\gamma) = 1$, and $\xi^{a\ell^\nu} - 1$ is a unit for $1 \le a \le n$. Here the auxiliary character is unramified at the original inertia used for γ. Fix any $g \in G_E$ and depth ν. The eigenvalues of $T(g)$ are integral, since the image is compact and preserves a lattice over its coefficient field. For λ in this multiset, at most one $i \in \{0, \dots, n\}$ satisfies $\overline{\xi^{i\ell^\nu}\lambda} = 1$, where bars denote residue classes. Indeed the $n + 1$ powers of $\overline{\xi^{\ell^\nu}}$ are distinct. Choose i outside all the forbidden indices, and set $\delta = \gamma^i$. Then

$$\chi_{\mathrm{cyc}}(\delta) = 1, \qquad \det(1 - T(\delta^{\ell^\nu} g)) = \prod_\lambda (1 - \xi^{i\ell^\nu}\lambda) \in \mathcal{O}^\times.$$

The unit condition persists on an open neighborhood of $\delta^{\ell^\nu} g$ by continuity. This is the determinant condition when selecting new Frobenius points for the triangle.

An original-level system x with normalized representation N_x is a weak lift when $\det(1 - N_x(g)) = 0$ for every $g \in G_E$. At an off-level split spherical place with residue norm q, let $N_H = r_0 + t'$ be the rank of a weak lift and $t' > 0$ the chain length for this descent pair. Write H for its monic Hecke polynomial. Suppose the normalized weak-lift representation has eigenvalue 1 at geometric Frobenius ϕ, and the unnormalized representation at ϕ is killed by H in the coefficient embedding. If $R^{\mathrm{nor}} = \chi R$, the unit eigenvalue of $R^{\mathrm{nor}}(\phi)$ gives the root

$$X_0 = \chi(\phi)^{-1}, \qquad |X_0| = q^{r_0/2}$$

in the complex realization, by the normalization of Section 28.1. At this split place the original local unitary-group representation is spherical and unitarizable. Apply the classification before the common arithmetic shift.

Tadić's spherical unitary classification [73, Theorem 3.2, p. 54] expresses the Satake roots in strings. In arithmetic normalization a string of length h has terms

$$aq^{(N_H-1)/2+\theta-(h-1)/2+j}, \qquad 0 \le j < h, \quad |a| = 1, \quad -\tfrac{1}{2} < \theta < \tfrac{1}{2}.$$

The generators in the classification are unitary determinant characters and complementary pairs of their norm twists. The norm of a uniformizer is q^{-1}; reversing each string writes its powers of q in increasing order. The shift $(N_H - 1)/2$ is the arithmetic normalization of Section 28.2. If X_0 is at position u in this string, with the positions starting at zero, subtracting the common shift gives

$$\theta - (h-1)/2 + u = (1-t')/2, qquad 2\theta = h - t' - 2u \in \mathbb{Z}.$$

The strict bound $|\theta| < 1/2$ forces $\theta = 0$ and $h = t' + 2u$. The $h - 1 - u = t' - 1 + u$ steps above X_0 include the distinct roots $X_0 q^j$ for $0 \le j < t'$.

Put $B_{t'}(X) = \prod_{0 \le j < t'}(X - q^j)$. In the coefficient embedding the exact chain gives

$$H(\chi^{-1}X) = B_{t'}(X)U(X), \qquad \chi \in \mathcal{O}^\times, \qquad \deg H = N_H,$$

where χ is the normalizing character at the chosen ϕ. Let $C(X)$ be its normalized Galois characteristic polynomial. Assume the coefficientwise comparison $C \equiv \chi^{N_H} H(\chi^{-1}X)$ in the residue field, with the rescaled polynomial integral, and a residual factorization $C \equiv SB_{t'}$ with $S(1)$ a unit. At a depth-ν triangle prime, $q \equiv 1 \bmod \ell^\nu$. For $\ell = 2$ one also has $q \equiv 1 \bmod 4$, since the rational prime splits completely in $E \supset \mathbb{Q}(i)$. Thus for $0 \le i < j < t'$,

$$v_\ell(q^j - q^i) = v_\ell(q-1) + v_\ell(j-i) \ge \nu.$$

Indeed an exponent prime to ℓ leaves the valuation of $q - 1$ unchanged, and raising to the ℓ-th power increases it by one by the binomial formula, using $4 \mid q - 1$ at 2. Thus the roots are distinct in characteristic zero and all reduce to 1 modulo ℓ^ν; in particular $\overline{B}_{t'} = (X-1)^{t'}$ in the residue field. Monic division in $\mathcal{O}[X]$, followed by cancellation of $\overline{B}_{t'}$ in the residue polynomial ring, gives

$$\chi^{N_H} U \in \mathcal{O}[X], \qquad \chi^{N_H} U \equiv S.$$

Thus $U(1)$ is a unit and $U/U(1)$ is an integral polynomial of value one at 1. For finitely many residual-near weak lifts at the same prime, multiply these polynomials to obtain a single such polynomial divisible by their normalized quotients.

For an original system x put $C_x(g; X) = \det(X - N_x(g))$. The product uses the weak lifts satisfying

$$C_x(g; 1) = 0, \qquad C_x(g; X) \equiv C_{x_0}(g; X) \pmod{\mathfrak{m}_\mathcal{O}} \qquad (g \in G_E),$$

where x_0 is the distinguished normalized lift and the congruence is coefficientwise. Before choosing the auxiliary primes, take a basis of the original-level space for each of the finitely many cofinite spherical systems. By definition each basis vector has the specified spherical eigenvalues away from finitely many split places. Exclude the union of these exceptions and the exceptions in the Galois comparisons of Section 28.2. At each remaining split place every spherical operator acts by its specified scalar on the full original space for each system. In particular, at each subsequently chosen auxiliary prime, the root chain used in Section 32.3 applies to every vector of each original system in the product. The trace identity of Section 28.1, applied to every power of g, and Newton's identities in the characteristic-zero fraction field give

the distinguished polynomial as the product of that of $T(g)$ and $\prod_{0\le j<t'}(X-\chi_{\mathrm{cyc}}(g)^{-j})$. In the determinant-unit neighborhood the first factor has value a unit at 1. Thus the residual quotient condition holds for all the near systems there. The weak-lift condition gives eigenvalue 1 in each normalized representation itself. The local argument applies simultaneously to all these systems.

Eligible Frobenius elements of prime norm are dense in this neighborhood, also after any further finite exclusion. For any point h of the given open set, choose an open normal subgroup $V \triangleleft G_E$ with hV in that set. Let $L/\mathbb{Q}$ be the normal closure of the fixed field of V and put $h_L = h|_L \in \mathrm{Gal}(L/E)$. Chebotarev over $\mathbb{Q}$ gives a positive density of unramified rational primes with arithmetic Frobenius conjugate to h_L^{-1} in $\mathrm{Gal}(L/\mathbb{Q})$ [48, Theorem 8.31]. Choose a prime of L over such a prime whose arithmetic Frobenius is exactly h_L^{-1}, and prolong it to $\overline{\mathbb{Q}}$. Since $E/\mathbb{Q}$ is Galois and $h_L|_E = 1$, the rational prime splits completely in E. Its geometric Frobenius at the chosen prolongation belongs to G_E and restricts to h_L, hence belongs to hV. Excluding the rational primes below the earlier places and the finite exceptional sets gives distinct primes of F_0. This proves abundance at $\delta^{\ell^\nu} g$ for Section 29.4, on the stated scalar inertia and distinguished lift data. The finite family and its off-level comparisons are fixed before choosing the depth ν.

29.3 The fresh Kummer maps

In this subsection, we construct the Kummer maps for adding a prime to the places allowed to ramify. Their local restrictions are the ones prescribed in the inductive selection of Section 29.4.

Fix $\nu > 0$ and $e > 0$. A ν-split place is a degree-one prime $w \nmid \ell$ of E with a chosen prolongation and geometric Frobenius there fixing μ_{ℓ^ν}. Take k_0 earlier such places of type e with distinct restrictions to $F_0 = E^+$. Let S_0 be the set of their primes in F_0 together with the primes over ℓ. Put $F_\nu = E(\mu_{\ell^\nu})$, $U = G_{F_\nu}$, and $\kappa = \chi_{\mathrm{cyc}}\chi_{E/F_0}$ on G_{F_0}, where the second factor has values in $\{\pm 1\}$. Then $\kappa|_{G_E} = \chi_{\mathrm{cyc}}$ and $\kappa(c) = 1$ at every real conjugation. It is unramified off ℓ since E/F_0 is finite unramified. Let

$$M_\nu = (\mathbb{Z}/\ell^\nu\mathbb{Z})(\kappa^{-e}), \qquad A = M_\nu[\ell], \qquad M_\nu^\vee = \mathrm{Hom}(M_\nu, \mu_{\ell^\nu}).$$

The earlier primes of F_0 split in F_ν, so both coefficient modules are trivial there.

Use primal local conditions of zero at real places and at the earlier primes, vanishing on $D_v \cap \ker\kappa$ at $v \mid \ell$, and unramified elsewhere. These give conditions on cohomology classes as coboundaries obey the required vanishings. Let P be a finite set of auxiliary primes split in F_ν, outside S_0, where the primal condition is relaxed to the full local group. Write $Z_\nu(P)$ for the resulting group of actual cocycles. The same conditions for A give $Z_A(P) = Z_\nu(P)[\ell]$.

To choose P, take all residual dual classes with coefficients in $A^\vee = \mathrm{Hom}(A, \mu_\ell)$ unramified outside S_0 and infinity. These classes form a finite group. Their restrictions to U are homomorphisms. For each nonzero restriction, take an element of U detecting it. For a cocycle representative b, the map $g \mapsto (b(g), g|_{F_\nu})$ is a homomorphism to $A^\vee \rtimes \mathrm{Gal}(F_\nu/F_0)$, since the action on $A^\vee$ factors through this finite Galois group. Intersect the kernels for the finitely many representatives with U. In the resulting finite quotient of G_{F_0}, each detecting element has trivial image in $\mathrm{Gal}(F_\nu/F_0)$. Chebotarev applied to its inverse conjugacy class gives a prime splitting completely in F_ν. Choose its prolongation so that geometric Frobenius has the detecting image. The restriction of b to U is independent of the representative, since U acts trivially. Exclude S_0 and the earlier choices at each step.

At finite $v \notin S_0$, the coefficients are unramified of order prime to the residue characteristic. For either A or M_ν, the unramified conditions for the module and its Cartier dual are exact

annihilators [49, Ch. I, Theorem 2.6]. At P the relaxed primal condition is the full group, with zero annihilator. Thus the residual dual classes are unramified outside S_0 and infinity and vanish on the chosen Frobenius elements at P. By the choice of P their restrictions to U vanish. Inflation–restriction gives the inclusion in $H^1(\mathrm{Gal}(F_\nu/F_0), A^\vee)$. The Galois group is generated by at most two elements. For odd ℓ it is a subgroup of the product of the order-two group of E/F_0 and the cyclic units modulo ℓ^ν. For $\ell = 2$, use $E = F_0(i)$: at $\nu = 1$ the group has order two and for $\nu \geq 2$ it is a subgroup of the units modulo 2^ν. Cocycles are determined by their values on generators, so the residual dual group has dimension at most two.

Apply Greenberg–Wiles to A with the relaxed conditions [27, Theorem 2.19], which applies to finite modules of arbitrary order. At P the local factor is ℓ by the unramified and tame directions in local H^1. Each earlier prime contributes ℓ^{-1}. At a real place v, the local condition is zero and $\kappa(c) = 1$, so $|\mathcal{L}_v|/|H^0(F_{0,v}, A)| = \ell^{-1}$, also for $\ell = 2$. At $v \mid \ell$, an allowed cocycle factors through the procyclic $\kappa(D_v)$ and has at most ℓ possible values on a lift of a generator. After division by $\ell/|H^0(F_{0,v}, A)|$ coboundaries the local factor is at most one. Elsewhere the unramified factor is one. At $\ell = 2$, the character of $E = F_0(i)$ is the sign modulo 4 of χ_{cyc}, so κ takes values in $1 + 4\mathbb{Z}_2$ and its local images are procyclic. The global coboundary factor $\ell/|H^0(F_0, A)|$ cancels the numerator of the global invariant ratio. With the residual dual bound this gives

$$|Z_\nu(P)[\ell]| \leq \ell^{|P|-k_0-f+3} \leq \ell^{|P|-k_0-1},$$

since $f \geq 4$.

At full depth let $B \subset M_\nu^P$ be the image of tame evaluation at P on $Z_\nu(P)$. Let $B^\vee \subset (M_\nu^\vee)^P$ be the Frobenius evaluations of the global dual classes with the orthogonal conditions for the unrelaxed primal group. Put $d_\nu = \ell^\nu$. At every $v \in P$ both coefficient modules are trivial. Use a single primitive d_ν-th root of unity ζ_{d_ν} for the Kummer coordinates, transported to each completion by its chosen prolongation. For a uniformizer ϖ_v of $F_{0,v}$, choose τ_v in tame inertia with

$$\tau_v(\varpi_v^{1/d_\nu})/\varpi_v^{1/d_\nu} = \zeta_{d_\nu}.$$

A primal class with tame value t differs from $t\kappa_{\varpi_v}$ by an unramified class. Two unramified classes have zero cup product, since the unramified quotient has ℓ-cohomological dimension one. For the unramified dual coordinate χ, the calculation of Section 27.4 gives

$$\mathrm{inv}_v(\chi \cup t\kappa_{\varpi_v}) = -t\,\chi(\phi_v)/d_\nu \quad \text{in } \mathbf{Q}/\mathbf{Z},$$

where ϕ_v is geometric Frobenius. Indeed the local Artin map in [50, Ch. III, Proposition 3.6 and Lemma 3.7] sends ϖ_v to arithmetic Frobenius on the unramified quotient. Reversing the cup factors changes the sign at every v. In the chosen root-of-unity coordinates the product of the local pairings is the perfect pairing on the two copies of the coefficient modules indexed by P. Global cup reciprocity gives $B^\vee \subset B^\perp$. The kernels of tame and Frobenius evaluation are the unrelaxed primal and the strict dual groups, respectively. The ratio of the two Greenberg–Wiles identities for M_ν with and without relaxation at P yields

$$|B|\,|B^\vee| = \ell^{\nu|P|}.$$

Indeed each relaxed prime changes the local numerator of the Greenberg–Wiles formula from the order of the unramified line to the order of the full local group, a factor ℓ^ν. The global invariant terms cancel. Since $|B^\perp| = \ell^{\nu|P|}/|B|$, this proves $B^\vee = B^\perp$.

Every character of $A^P/(B \cap A^P)$ extends to M_ν^P/B by divisibility of $\mathbb{Q}/\mathbb{Z}$, and therefore arises from one of the dual classes. The coefficient map here is restriction

$$M_\nu^\vee \longrightarrow A^\vee, \qquad b \longmapsto b|_{\ell^{\nu-1}M_\nu}.$$

It is Galois-equivariant. In coordinates from the chosen roots, $b(\ell^{\nu-1}x) = b(x)^{\ell^{\nu-1}} \in \mu_\ell$, so it reduces the exponent of b modulo ℓ. On U coboundaries in $M_\nu^\vee$ vanish. The resulting homomorphism on U determines these values on A^P since the Frobenius elements of P belong to U. Moreover $|B[\ell]| \le |Z_\nu(P)[\ell]|$, by surjectivity on the quotients by multiplication by ℓ and the equality of the kernel and cokernel sizes of multiplication on a finite abelian group. Hence there are at least ℓ^{k_0+1} different reductions of the restricted dual classes. Some restriction is outside the span of the k_0 earlier maps modulo ℓ.

The restricted cocycle on U is a homomorphism ψ to $\mathbb{Z}/\ell^\nu\mathbb{Z}$ in the chosen root-of-unity coordinates. Conjugation acts through $\chi_{\mathrm{cyc}}\kappa^e$. This gives oddness under c and exponent $e+1$ under conjugation by γ_ℓ. The global dual class is unramified at every finite place outside S_0. At a real place its local condition is unrestricted, the orthogonal complement of the zero primal condition. The exploratory set P can be discarded.

For the reciprocity law, let z be an even cocycle over E allowed to ramify at the later split pairs and above ℓ, vanishing on $D_v \cap \ker \chi_{\mathrm{cyc}}$ for every $v \mid \ell$, with earlier tame and type-e Frobenius tests zero. Extend it to G_{F_0} by $\widetilde{z}(gc) = z(g)$ for $g \in G_E$. This is a cocycle by evenness and $\kappa(c) = 1$. Use $2\widetilde{z}$ as the primal cocycle. At every real conjugation $2\widetilde{z}$ vanishes by squaring.

For $g \in D_v \cap \ker \kappa$ at $v \mid \ell$, $2\widetilde{z}(g) = \widetilde{z}(g^2) = 0$, since $g^2 \in D_v \cap G_E$ has cyclotomic value one. At an earlier prime of type e, unramifiedness and the zero Frobenius value give zero on its decomposition group. Thus the terms at S_0 and infinity vanish by the orthogonal local conditions. At earlier primes of other types and all remaining finite primes outside the new pairs, both classes are unramified. Their cup inflates from the procyclic residue Galois group of ℓ-cohomological dimension one, so its local term is zero. At each new prime of F_0 split in E, take τ_i^+ with $\tau_i^+(\varpi_i^{1/d_\nu})/\varpi_i^{1/d_\nu} = \zeta_{d_\nu}$ for a uniformizer of F_{0,v_i}, using the same primitive root. Represent the local class of z by a Kummer element $b_i \in F_{0,v_i}^\times$ in these coordinates. The d_ν-th roots of units lie in the maximal unramified extension, since $v_i \nmid \ell$. Thus $z(\tau_i^+) = v_i(b_i) \bmod d_\nu$. Local reciprocity pairs the class with the unramified dual value as $\pm\psi(\phi_i)v_i(b_i)$, with the same sign at every prime from the geometric Frobenius and cup-product conventions. This gives the term $\pm 2\psi(\phi_i)z(\tau_i^+)$ for $2\widetilde{z}$, and proves the reciprocity relation. Inertia acts trivially on the coefficients, so each arbitrarily prescribed tame generator has value an A_ν-multiple of $z(\tau_i^+)$, with $A_\nu = \mathbf{Z}/\ell^\nu\mathbf{Z}$. For O/ℓ^ν-valued cocycles apply the relation after all linear functionals $O/\ell^\nu \to \mathbb{Z}/\ell^\nu\mathbb{Z}$, which separate values since O/ℓ^ν is free over $\mathbb{Z}/\ell^\nu\mathbb{Z}$. This proves the fresh-map assertion at every positive depth.

29.4 The ordered prime selection from the Kummer relations

In this subsection, we combine the cyclotomic-kernel evaluations of Section 29.2 with the Kummer maps of Section 29.3. The order of the choices ensures that each newly selected prime satisfies the conditions imposed by the earlier ones.

Keep g_{tr}, $C = \ell^{Y_{\mathrm{tr}}h_\ell}$, and a finite set of evaluation elements in the cyclotomic kernel P_0 as in Section 29.2. For $\ell \le Y_{\mathrm{tr}}^2$, the finite detection bound of Section 24.1 gives

$$m_0 = N_c\ell^2 \le Y_{\mathrm{tr}}^5, \qquad |P_0| \le m_0^4 \le Y_{\mathrm{tr}}^{20}.$$

For the actual rank increment $t' \le t$, order the pairs (e, s), $0 < e < t'$, $s \in P_0$, lexicographically with the exponent first. The uniform bound uses t. Fix this list before choosing the depth. For each $\nu > 0$, put $A_\nu = \mathbb{Z}/\ell^\nu\mathbb{Z}$ and

$$G_\nu = G_{E(\mu_{\ell^\nu})} = \{g \in G_E : \chi_{\mathrm{cyc}}(g) \equiv 1 \pmod{\ell^\nu}\}.$$

At each step Section 29.3 gives a Kummer map ψ_j on G_ν with values in A_ν for the earlier split pairs of type e, and a finite excluded set of primes of F_0. The map is a continuous homomorphism, is odd under complex conjugation, and conjugation by γ_ℓ multiplies its values by ξ_ℓ^{e+1}. Its reduction modulo ℓ is outside the span of the previous maps of type e.

The required reciprocity law is as follows. Let an even cocycle z of type e vanish on the full cyclotomic kernel in the decomposition groups above ℓ. Suppose it is unramified at the previous pairs and outside ℓ and a finite family $\{w_i, \bar{w}_i\}$ of further ν-split pairs with pairwise distinct restrictions to F_0, all outside the excluded set. If it vanishes on the earlier Frobenius elements of type e, then

$$\sum_i 2\psi_j(\phi_i) z(\tau_i^+) = 0.$$

Here ϕ_i is geometric Frobenius and τ_i^+ is the generator normalized over F_0 in the chosen decomposition group as in Section 29.3.

The primes over ℓ and the previously specified local tests belong to the excluded set of the new map. The set of usable Frobenius points must be abundant near a translate of every prescribed point. At each depth ν and $g \in G_E$ there must be $\delta \in \ker \chi_{\mathrm{cyc}}$ such that every open neighborhood of $\delta^{\ell^\nu} g$ contains usable Frobenius elements of new places of prime norm outside any given finite excluded set. At those places use the determinant-unit and Hecke conditions of Section 29.2.

Under this Frobenius condition, consider the index $j = (e, s)$. The earlier maps of type e and the new ψ_j are independent modulo ℓ. Their joint evaluation on G_ν has image all of the appropriate power of A_ν: modulo ℓ this follows by the absence of an annihilating linear combination, and Nakayama's lemma gives the assertion over A_ν. Choose $a \in G_\nu$ on which the new map reads one and the earlier maps of type e read zero. For each $k \in \{0, \dots, t'-1\} \setminus \{e\}$, replace a successively by

$$(\gamma_\ell a \gamma_\ell^{-1})^{\xi_\ell^{-1}} a^{-\xi_\ell^k}.$$

The exponents are integer representatives modulo ℓ^ν. Every map of type d is multiplied by $\xi_\ell^d - \xi_\ell^k$. Thus the resulting $a_* \in G_\nu$ has every earlier value zero and $\psi_j(a_*) = \Delta_e \bmod \ell^\nu$.

Apply the auxiliary-prime theorem at $g = scsc^{-1}a_*(ca_*c^{-1})^{-1}$. All factors and δ^{ℓ^ν} lie in G_ν. An odd Kummer map reads $2\psi(a_*)$ at $\delta^{\ell^\nu} g$, since the first two factors cancel and ℓ^ν kills its value on δ. An even cocycle of type e, restricted to G_ν, is additive and reads $2z(s)$ there. The finitely many Kummer evaluations persist on an open neighborhood.

For the even evaluations take the finitely many A_ν-valued cocycles unramified outside ℓ using the finite evaluation set. Their values persist on one open neighborhood. Applying the additive A_ν-linear duals of O/ℓ^ν, which separate its elements, gives the same equalities for the O/ℓ^ν-valued cocycles, on that neighborhood for every coefficient ring. Set $h = \delta^{\ell^\nu} g$. Intersect the open kernels in G_ν of the earlier and current Kummer maps and the restrictions of these finitely many cocycles. Each restriction is additive on G_ν. Take an open normal subgroup of G_E inside this intersection, shrinking it so that its coset at h also preserves the residual determinant of $1 - T(h)$. The Frobenius construction of Section 29.2 now gives one prime and prolongation

satisfying all these conditions. Exclude ℓ, the rational primes under the original level, the twist support, the exceptional places of the finitely many original systems, the excluded sets of the Kummer maps chosen so far, and every earlier pair. These are finitely many rational primes at this step. Even when the same Frobenius class is prescribed again, it still has positive density after these exclusions. Use the conjugate prolongation at $\bar{w}_j$. Writing $\phi'_j = (c\phi_j c^{-1})^{-1}$, oddness and evenness on G_ν give

$$\psi_i(\phi'_j) = \psi_i(\phi_j), \qquad z(\phi'_j) = -z(\phi_j)$$

for the Kummer maps and even cocycles just tested. The geometric Frobenius at the conjugate prolongation is $c\phi_j c^{-1}$. The paired test uses its inverse. Its prime norm implies it is split over F_0, and $\ell^\nu \mid \mathrm{N}w_j - 1$. It has

$$\psi_i(\phi_j) = 0 \ (i < j), \qquad \psi_j(\phi_j) = 2\Delta_e, \qquad z(\phi_j) = 2z(s)$$

for the stated unramified even cocycles.

For an even cocycle ramified only on the new pairs and above ℓ, first impose vanishing at all earlier tame generators. Evenness gives vanishing at their conjugates, so the cocycle is unramified at those earlier pairs. Impose the earlier Frobenius tests of type e. The reciprocity law for ψ_j now applies to the current and later pairs. The later coefficients vanish by construction. The remaining term is $2\psi_j(\phi_j)z(\tau_j^+) = 4\Delta_e z(\tau_j^+)$. Write $C = u_e\, 4\Delta_e$ in $\mathbb{Z}_\ell$. Multiplication by u_e gives $Cz(\tau_j^+) = 0$ at every depth, including those for which $\Delta_e \equiv 0 \pmod{\ell^\nu}$. The value at the prescribed τ_j is a multiple of the value at τ_j^+ in the cyclic pro-ℓ tame quotient modulo ℓ^ν, so $Cz(\tau_j) = 0$ as well. This gives the ordered tame test and the reads of the original evaluation map at the new Frobenius elements. The total loss exponent counts $(t'-1)|P_0|$ ordered indices, $|P_0|$ evaluation points and one reference point. Thus

$$(t'|P_0| + 1)Y_{\mathrm{tr}}h_\ell \le (t|P_0| + 1)Y_{\mathrm{tr}}h_\ell \le Y_{\mathrm{tr}}^{22}h_\ell.$$

When $\ell > Y_{\mathrm{tr}}^2$ this holds because $h_\ell = 0$. There is no bound on the norms of the new primes in this argument.

29.5 The paired tame and Frobenius tests

In this subsection, the successive primes of Section 29.4 give two local evaluations for each relative cocycle. We combine the tame and Frobenius evaluations with the original local conditions to bound the length of their common kernel.

At depth ν, let Z be an O-submodule of the relative character cocycles for $(O/\ell^\nu)(-e)$ of Section 29.1. Put $\chi_e = \chi_{\mathrm{cyc}}^{-e}$. With γ_π the prescribed scalar inertia element and g_{tr} the reference element of Section 29.2, assume

$$z \in Z, \quad z(\gamma_\pi) = 0 \quad \Longrightarrow \quad (\chi_e(g) - 1)z(g_{\mathrm{tr}}) = 0 \quad (g \in G_E).$$

The congruence modules used below satisfy this implication by the normalizer calculation. Suppose the old prescribed tame tests and their conjugate inverses vanish and cover all original and auxiliary ramification away from ℓ except the new pairs $w_j, \bar{w}_j$. Outside these places the matrices $u_x(g) - 1$ vanish for inertia g, so evaluation at g is a relative linear relation. At an old place away from ℓ, the character cocycle on inertia is a homomorphism to an ℓ-primary finite

group. It factors through pro-ℓ tame inertia. The topological tame generator and its conjugate inverse therefore give vanishing throughout those groups. Thus Z is unramified outside ℓ and the new pairs.

Let τ_j generate pro-ℓ tame inertia at w_j, and let the chosen ϕ_j fix μ_{ℓ^ν}. Set $\tau_j' = (c\tau_j c^{-1})^{-1}$ and $\phi_j' = (c\phi_j c^{-1})^{-1}$. Thus ϕ_j is geometric Frobenius and ϕ_j' is arithmetic Frobenius at the conjugate prolongation. For $q_j = \mathrm{N}w_j$,

$$\chi_{\mathrm{cyc}}^{-1}(\phi_j) = q_j, \qquad \chi_{\mathrm{cyc}}^{-1}(\phi_j') = q_j^{-1}, \qquad q_j \equiv q_j^{-1} \equiv 1 \pmod{\ell^\nu}.$$

Let $Z_0 \subseteq Z$ require vanishing on the local sets above ℓ used for $Z_{e,\nu}^-$ in Section 24.1. This concerns inertia for $e = 0$, and the full cyclotomic kernel in each decomposition group for $e > 0$. The crystalline linear relations imply this vanishing. For positive e, the cyclotomic image is infinite and the period $t_{\mathrm{cyc}}^e \otimes v_{-e}$ generates the de Rham module in Fil^0, where v_{-e} is a basis of the Tate character and t_{cyc} is the cyclotomic period. Bloch–Kato's formula gives $H_f^1(E_v, \mathrm{Frac}(O)(-e)) = 0$. Thus an integral crystalline cocycle is a rational coboundary and vanishes on that kernel. For $e = 0$ the formula gives dimension one over $\mathrm{Frac}(O)$ [16, Corollary 3.8.4, p. 355]. Let f_v be the residue degree of $E_v/\mathbf{Q}_\ell$. The equation $\varphi^{f_v} u - u = 1$ has a solution in $W(\overline{\mathbf{F}}_\ell) \subset B_{\mathrm{cris}}$. Solve successively modulo the powers of ℓ using surjectivity of $x \mapsto x^{\ell^{f_v}} - x$ over $\overline{\mathbf{F}}_\ell$. Since arithmetic Frobenius acts on this Witt ring by φ^{f_v}, the unramified cocycle sending Frobenius to 1 is the coboundary of u in B_{cris}. Hence the finite condition is the unramified line. Each asserted evaluation is therefore an integral crystalline relation. It remains a relation modulo ℓ^ν, including for cocycles that only obey the relations. Conjugation transfers vanishing from the designated decomposition group to its conjugates since the character at an element of the local set is one. Consequently

$$\mathrm{length}_O(Z/Z_0) \le c_{\mathrm{crys}}(O(-e), \nu).$$

For $z \in Z_0$, paired agreement at either (τ_j, τ_j') or (ϕ_j, ϕ_j') is equivalent to vanishing of $z + z \circ c$ at the first member. In fact the character acts trivially on both pairs and hence inversion negates the value. The sum $z + z \circ c$ is an even cocycle with the same ramification and local vanishing. Suppose the new places have the ordered positive types and satisfy the conditions of Section 29.4, with the separation scalar $C = \ell^{Y_{\mathrm{tr}} h_\ell}$. Its even-cocycle test at an index of type e says that vanishing on earlier tame generators and earlier Frobenius elements of type e forces the value at the current tame generator to be killed by C. For the paired tests this gives agreement up to C at τ_j, τ_j', using all earlier tame pairs and only earlier Frobenius pairs of type e.

Now let $S \subseteq Z_0$ require the two tame values at every new pair to vanish individually and the Frobenius values to agree for type e. Killing the tame generators removes all new ramification as above. For the reference test use the normalizing element for the covector congruences. Write $d(g)$ for simultaneous augmentation coordinates in a finite generating list g_i for the tuples $(u_x(g) - 1)_x$. Thus

$$u_x(g) - 1 = \sum_i d(g)_i (u_x(g_i) - 1) \qquad \text{for every } x.$$

For a group-algebra covector v, its value is $\sum_i v_i(g_i - 1)$. The congruence ideal contains the common action kernel of the u_x and the left multiples of the normalizer N_0 by the augmentation ideal.

Here the normalizer is

$$E_\pi = c_\pi(\gamma_\pi - 1), \qquad N_0 = E_\pi + (1 - E_\pi)(g_{\mathrm{tr}} - 1),$$

where $c_\pi = (y_\pi - 1)^{-1}$ for the scalar y_π of γ_π. Its inverse is integral since the reduction of the scalar differs from one. The covector $v_0 = c_\pi d(\gamma_\pi) + (1 - E_\pi)d(g_{\mathrm{tr}})$ has value N_0 modulo the common action kernel. Hence $(g-1)v_0$ satisfies the congruences for every g. Evaluation at χ_e kills E_π, since γ_π is in inertia off ℓ. The character cocycle given by a functional annihilating the vectors satisfying the congruences therefore satisfies

$$(\chi_e(g) - 1)(c_\pi z(\gamma_\pi) + z(g_{\mathrm{tr}})) = 0.$$

This gives the required implication for the coordinate quotients in the Fitting estimate, as verified there.

For $z \in S$ the value on γ_π is zero by unramifiedness, giving $(\chi_e(g_{\mathrm{tr}})-1)z(g_{\mathrm{tr}}) = 0$. For $e > 0$, use the triangle's detecting points besides g_{tr}, its prescription that twice the even value at each point is an appropriate Frobenius value of type e, and its divisibility $4(\chi_{\mathrm{cyc}}(g_{\mathrm{tr}})^{-e} - 1)^2 \mid C$. The paired-cocycle argument in Section 24.1 gives $\mathrm{length}_O(S/S[C]) \le \mathrm{length}_O Z^-_{e,\nu}$. For $e = 0$ there is no index of type zero, and S is already everywhere unramified, hence a submodule of $Z^-_{0,\nu}$. Thus in both cases

$$\mathrm{length}_O(Z/Z_0) + \mathrm{length}_O(S/S[C]) \le c_{\mathrm{crys}}(O(-e), \nu) + v_{\mathrm{ord}}(X_{\mathrm{odd}}). \tag{87}$$

Together with the tests for unramified cocycles, evaluations at both tame generators of each new pair detect all of Z: their vanishing removes the new ramification, and the remaining tests give zero. The reference test uses the original congruence ideal. The prime triangle uses the scalar inertia, distinguished lift and residual comparison of the same representation.

30 Measured norm tests and the integral trace algebra

In this section, we normalize the two theta norm tests for the critical values and construct the integral trace algebra on the raised levels. The norm tests are used in Section 38, and the algebra is used in Section 31.

30.1 Norm tests from a definite line

In this subsection, for $z \ge 2$, the theta lift from the definite line gives a measured norm involving the critical value at z. We compute the finite factors and the archimedean normalization here, and establish integrality for the same test in Section 44.4.

Let $p = (\nu, z)$ be a critical pair on the prescribed old level with $z \ge 2$. Its tame residue characters at the old places may be trivial. The infinity weights of ν are $-c_v$ with $c_v \ge z$. Use the standard definite line and the standard definite group of rank $a_0 = 2z$ over the same E/F_0. In this section S_M denotes the chosen halves of the old split level places. Let Q be a set of the auxiliary split places allowed in the tests, disjoint from the old places and their conjugates. Write

$$q_Q = \prod_{w \in Q} q_w, \qquad S_Q = (S_M \cup Q) \cup (\bar{S}_M \cup \bar{Q}), \qquad C_G = \prod_{w \in S_M} |\mathrm{GL}_{rt}(k_w)|.$$

Let τ be a finite-order character of the adelic norm-one quotient, trivial at infinity, unramified away from Q and tame at Q. Its ramification set may be smaller than the chosen modulus. For a norm-one character χ write $\chi^{\mathrm{bc}}(a) = \chi(a/\bar{a})$. We use the finite L-function of this base change and omit the places in S_Q.

Use the vacuum multiplied by $s_{\mathrm{vac}}^{a_0}$. At each old place restrict the column to the reduction of the specified basis vector. At a place of Q prescribe the distinguished coordinate to be one. Use the same arrow scale $\pi_M = \bar{\alpha}$ and $s_{\mathrm{jet}} = \gamma = p_F a_d/\omega_0$ from (AJ). Use the jet of weight $\bigotimes_v \mathrm{Sym}^{c_v - z}(\mathbf{C}^{a_0})$ at scale $\pi_M s_{\mathrm{jet}}$, with the fixed character ν.

Denote the resulting kernel with trivial auxiliary target character by $K_Q(g,h)$. Here g is on the definite line and h on the rank-a_0 space, both with the original measures. For line classes g_i take the old principal-level classes times integral lifts of inverse unit residues at Q. Set $u_i = \det(1, g_i)$. The values of

$$\Phi_\tau(h) = \sum_i \tau(u_i)^{-1} K_Q(g_i, h)$$

are the class lift of the line character, since summing over the inverse unit residues gives the test on which the distinguished coordinate is nonzero and weighted by the tame character. The sum on the line has no stabilizer denominators at the old principal level. Its quotient stabilizer is trivial by reduction at an old odd split prime in rank one.

On the target the level J_Q is the old vector fixer and the inverse image of the parabolic preserving the complementary hyperplane at each new place. For $k \in J_Q$, write $t_w(k) \in k_w^\times$ for its action on this one-dimensional quotient and $\psi_{w,\tau}$ for the residue character of $\nu\tau$. In the integral column, the inverse pullback satisfies $(k^{-1}X)_1 = t_w(k)^{-1} X_1$ modulo the maximal ideal. Therefore

$$\Phi_\tau(hk) = \prod_{w \in Q} \psi_{w,\tau}(t_w(k))^{-1} \Phi_\tau(h).$$

The old vector fixers preserve the prescribed column. If the auxiliary target character a is used, the lift is multiplied by $a(\det h)$, so the displayed transformation law acquires the factor $a(\det k)$. All factors are unitary, and the squared Weyl norm is invariant under J_Q for either choice. The class pairing is the sum of the transported Weyl pairings divided by the finite class stabilizer orders. Every such order divides C_G. Indeed the stabilizer is finite, since the real group is definite and its rational integral points are discrete. After conjugation into an old compact group at an odd split place with $E_w = \mathbf{Q}_{p_i}$, the kernel of reduction is torsion-free. In that kernel a nonidentity $1 + p_i^s A$, for $s \geq 1$ and A nonzero modulo p_i, has the same depth s after raising to a power prime to p_i and depth $s+1$ after raising to p_i, by the binomial expansion. Reduction therefore injects the finite stabilizer into the finite linear group. The rank is at most rt.

Proposition 30.1 (Measured norm of the line lift)**.** *For the line lift Φ_τ with the original class pairing at J_Q,*

$$\|\Phi_\tau\|_{J_Q}^2 = c_p \, q_Q^{a_0 - 1} L^{S_Q}(z, ((\nu\tau)^{\mathrm{bc}})^{-1}), \qquad c_p > 0. \tag{88}$$

The scalar c_p is independent of Q and τ.

Proof. Put $u = v|_{U(1)}$, $b = (\nu\tau)u^z$. Use the column test f after the sum over inverse unit residues at Q, and put $f_g = \omega_b(g,1)f$, where the source action includes $b(\det g)^{-1}$. Let $\eta = (1, \eta_f)$,

$\eta_f \bar{\eta}_f = -1$, be the opposite-form coordinate of Section 40.2.1. In the two original blocks the doubled test is

$$f_g(X)\,\overline{f_{g'}(\eta^{-1}Y)}.$$

Here the phase of the opposite theta functional is chosen to give the conjugate evaluation. Its modulus is one by Section 17.1.

If the auxiliary character on the common target in the first factor is a, its opposed character in the rank-one negative factor is $a^{-1}u$, by the intrinsic determinant multiplier of that factor. Their product is u. Thus this is the target action of Section 17.2 with Siegel rank one, with both source translations already in the test. The partial Fourier transform into the split polarization uses the original dual additive measures. The doubled line has a one-dimensional Lagrangian. In [80, Theorem 2] the Siegel index is therefore $n = 1$, the Hermitian dimension is $a_0 = 2z$, and the Witt index is zero. Thus $a_0 > n$, $s_0 = (2z-1)/2$, and evaluation at the identity of the double has coefficient one for target quotient probability Haar.

Every rational isotropic line of the doubled source is a graph of an isometry δ, since both projections are injective by definiteness. Evaluation at zero in the transverse polarization evaluates the two blocks at $(M\widehat{\delta}, \eta M)$ in the original point coordinates of Section 40.2.1. Thus the second argument returns to M. Rational invariance and unitarity give

$$\langle f_{g'}, \omega_b(\delta, 1) f_g \rangle = \langle f, \omega_b(g'^{-1}\delta g, 1) f \rangle.$$

The source character here is $b(\det(g'^{-1}\delta g))^{-1}$, since $b(\det \delta) = 1$. Use the original target volume to undo the probability normalization. The quotient formula unfolds the rational graph sum to the ambient line group, leaving one original source quotient volume. The absolute local sum at a split place outside S_Q is bounded by $\sum_{j\in\mathbb{Z}} q_w^{-z|j|}$. The product converges for $z > 1$, and the remaining local factors are finite. Together with the compact real groups this justifies the absolute unfolding and factorization.

Here are the local quantities, with each standard finite compact of mass one. At a split place outside S_Q, a line element with valuation pair $(j, -j)$ has unitary Schrödinger coefficient of absolute value

$$q_w^{-a_0 j/2} q_w^{-a_0 \max(0,-j)} = q_w^{-a_0|j|/2}.$$

The two factors are the square root of the Jacobian and the volume of the overlap of the integral column boxes. The split Weil character factor, after multiplication by the source character, is A_w^{-j} for $A_w = (\nu\tau)^{\mathrm{bc}}(\varpi_w)$: indeed the determinant factor from the splitting is $v(\varpi_w)^{2zj}$ and the source factor contains its inverse.

The sum of the overlaps is

$$\sum_{j\in\mathbb{Z}} A_w^{-j} q_w^{-z|j|} = \frac{1 - q_w^{-2z}}{(1 - A_w q_w^{-z})(1 - A_w^{-1} q_w^{-z})}.$$

At a nonsplit finite place v of F_0, write w for the place of E above it and $q_v = |k_v|$. The norm-one line group is its integral compact, the test is fixed, and its local integral is one. Since $E_w/F_{0,v}$ is unramified quadratic, $q_w = q_v^2$ and a uniformizer of $F_{0,v}$ is also one of E_w. The base change of $\nu\tau$ is trivial on this uniformizer. Hence its inverse has Euler factor $(1 - q_w^{-z})^{-1}$, whose product with $1 - q_v^{-2z}$ is one. Let S_Q^+ be the finite places of F_0 below S_Q, and let I_v denote the

local integral. Since $a_0 = 2z$ and $S_M \cup Q$ contains one place above each $v \in S_Q^+$,

$$\prod_{\substack{v\nmid\infty\\ v\notin S_Q^+}} I_v = \frac{L^{S_Q}(z,((\nu\tau)^{\mathrm{bc}})^{-1})}{\zeta_{F_0}^{S_Q^+}(2z)} = \frac{L^{S_Q}(z,((\nu\tau)^{\mathrm{bc}})^{-1})}{\zeta_{F_0}(2z)} \prod_{w\in S_M\cup Q}(1-q_w^{-a_0})^{-1}.$$

At a distinguished place a nonunit moves the primitive column support off itself. The old column has squared mass $q_w^{-a_0}$, and among source units its support is fixed exactly by reduction one. On this subgroup every allowed tame character is trivial, so the old matrix coefficient integral is $q_w^{-a_0}/(q_w-1)$ for each old residue character, including the trivial one. At a new place the sum over the inverse unit residues replaces the point condition by the character-weighted condition $X_1 \neq 0$ on the distinguished coordinate X_1.

Every source unit preserves this support. The residue multiplier cancels the source character, so the coefficient integral is the mass $(q_w-1)/q_w$. The same statement holds with a trivial new-place character: the residue sum still gives the same support with coefficient one. Thus in that case the new quotient level and volume factor are the same as for an exact-support place. The passage to the target class pairing multiplies by $q_w^{a_0}-1$ at an old place and $(q_w^{a_0}-1)/(q_w-1)$ at a new place. These are the coset counts for the two target stabilizers. Turning the line Haar lift into the old principal-level class sum multiplies its norm square by the square of the old source index, whose old local factor is q_w-1.

At infinity the line character cancels the rotation of the selected highest monomial $X_1^{c_v-z}$ and its squared norm is $(c_v-z)!$ in the Fock pairing. For a highest covector of norm one, Schur orthogonality on the compact group gives the squared scalar coefficient integral as the squared Weyl norm divided by $\dim\bigl(\bigotimes_{v|\infty}\mathrm{Sym}^{c_v-z}(\mathbb{C}^{a_0})\bigr)$. Thus passing to the Weyl norm multiplies by this dimension. At an old place the two class-sum source indices, the source coefficient integral, the target index and the removed Euler numerator give

$$(q_w-1)^2\frac{q_w^{-a_0}}{q_w-1}\frac{q_w^{a_0}-1}{1-q_w^{-a_0}} = q_w-1.$$

At a new place the corresponding factor is

$$\frac{q_w-1}{q_w}\frac{q_w^{a_0}-1}{q_w-1}(1-q_w^{-a_0})^{-1} = q_w^{a_0-1}.$$

In particular, with V_j the volume of the standard definite rank-j quotient for these Haar measures, put

$$t_v = c_v - z, \qquad d_\infty = \prod_{v|\infty}\binom{t_v+a_0-1}{a_0-1}.$$

For the vacuum and jet scales specified above the constant in (88) is

$$c_p = \frac{V_1V_{a_0}}{\zeta_{F_0}(a_0)}\,d_\infty\prod_{v|\infty}t_v!\prod_{w\in S_M}(q_w-1)\left|s_{\mathrm{vac}}^{a_0}(\pi_M s_{\mathrm{jet}})^{\sum_{v|\infty}t_v}\right|^2.$$

All factors are positive and independent of Q,τ. □

For the integral coefficient test put

$$d_p = M^{2fa_0}(Md_K)^{\sum_v(c_v-z)}, \qquad F_{p,Q}(\tau) = C_G d_p^2 q_Q^{1-a_0}\|\Phi_\tau\|_{J_Q}^2.$$

Section 44.4 proves the unit bound on $d_p K_Q$ in the transported coordinates at every embedding away from q_Q, including the old level primes and 2. To apply it in the pairing, let A be the tuple of original coefficient embeddings of h, G_{W} the integral Weyl Gram matrix, and S the weight representation. The conjugate value of h^{-1} is A^t, since the standard definite form is unitary. The pairing identity

$$S(A^t)^t G_{\mathrm{W}} S(A^{-1}) = G_{\mathrm{W}}$$

follows on tensor powers of the coordinate pairing and hence on the Weyl space.

For the Gram form linear in its second entry write

$$F_{p,Q}(\tau) = \sum_{i,j} C_{ij}(Q)\tau(u_j u_i^{-1}),$$
$$C_{ij}(Q) = \frac{C_G d_p^2}{q_Q^{a_0-1}} \sum_{[h]} \frac{\mathrm{Gram}(K_Q(g_j,h), K_Q(g_i,h))}{|\operatorname{Aut}(h, J_Q)|}.$$

Indeed $\overline{\tau(u_j)^{-1}}\tau(u_i)^{-1} = \tau(u_j u_i^{-1})$. Use the given embedding on the second vector and its composite with complex conjugation on the first. The displayed Gram identity then evaluates both cleared vectors in integral coordinates at the same class point. The integer q_Q is a unit at each prime being tested. Thus the stabilizer bound and the unit bound give integral $C_{ij}(Q)$ at every embedding away from q_Q. Use the same representatives for all τ, since the unitary level character cancels in the squared norm of the complete lift. Equation (88) proves

$$\frac{F_{p,Q}(\tau)}{F_{p,\varnothing}(1)} = \frac{L^{S_Q}(z, ((\nu\tau)^{\mathrm{bc}})^{-1})}{L^{S_\varnothing}(z, (\nu^{\mathrm{bc}})^{-1})}.$$

The denominator is positive by absolute convergence and pairing the two conjugate Euler factors at each split place. Using τ and τ^{-1} gives the two sign tests with the same clearing constant.

30.2 The rank-two determinant norm

In this subsection, when $z = 1$, we instead evaluate a determinant period on the rank-two group. Its ratio to the untwisted period will enter the same integral comparison as the norm of Section 30.1.

Use the same finite bases and the sets S_M, Q, S_Q, and q_Q, C_G as in the preceding subsection. Let $p = (\nu, 1)$ be a ramified critical pair, with infinity weights $-c_v$, where $c_v \geq 1$. Put $k_v = c_v - 1$ and

$$\alpha = \nu\tau\, \upsilon|_{U(1)}, \qquad P(X) = \prod_{v|\infty} \det(X_v)^{k_v}.$$

Use the original oscillator on the pair of standard definite rank-two spaces. Write $H = U(I_2)$, and K_2 for the stabilizer of its standard finite lattice. Give each finite compact factor of K_2 and the real compact group mass one. Write v_2 for the volume of $[H] = H(F_0)\backslash H(\mathbb{A}_{F_0})$ for this Haar normalization.

The finite test is $\chi_{\mathfrak{m}}(\det X)$ in the original split Schrödinger coordinates at the modulus $\mathfrak{m} = (\pi_M)^2 \prod_{w \in Q} \mathfrak{p}_w$. The residue character $\chi_{\mathfrak{m}}$ is the finite unit character of α in these coordinates, extended by zero on nonunits. At every other place the finite test is the original vacuum.

Denote by f_Q this finite test with the real Fock vector P, and by $\theta_{f_Q}(g,h)$ its kernel with the two characters α,α. The determinant period is

$$\Lambda_Q(\tau)=\sum_{h\in H(F_0)\backslash H(\mathbb{A}_f)/K_2}\frac{\theta_{f_Q}(1,h)}{|\operatorname{Aut}(h,K_2)|}. \tag{89}$$

The local actions and the quotient integral. At a split place let X be the 2-by-2 Schrödinger variable. Identify both groups there with $\mathrm{GL}_2(F_{0,v})$ by the same chosen member of the split pair. The action is

$$(\omega(g,h)f)(X)=B(\det h)B(\det g)^{-1}|\det g|_v|\det h|_v^{-1}f(h^{-1}Xg),\qquad B=\alpha^{\mathrm{bc}}\upsilon^{-2}=(\nu\tau)^{\mathrm{bc}}.$$

Here B and υ are read on the chosen split factor. The powers of the absolute values are the square roots of the two matrix Jacobians. The original split self-dual coordinate measure gives the integral matrix box mass one. The splitting factor on each side is υ^{-2}, and $\upsilon(t/\bar{t})=\upsilon(t)^2$ on this split coordinate.

On integral units B is precisely the finite character. In particular the test is fixed by both unit compacts: multiplying the argument of $\chi_{\mathfrak{m}}\circ\det$ on the left by h^{-1} and on the right by g cancels the two unit character factors. This holds also on the noninvertible residue matrices, on which both sides vanish. Away from the chosen places, the original unit compacts fix the vacuum and the characters are unramified. At infinity α has weight $-c_v+1=-k_v$. The bare Fock substitution on $P(X_v)=\det(X_v)^{k_v}$ contributes $(\det h_v)^{k_v}(\det g_v)^{-k_v}$. The factors $\alpha_v(\det h_v)\alpha_v(\det g_v)^{-1}$ cancel these powers. The oscillator vector and the two theta variables are therefore invariant under their respective real compact groups and under K_2 at finite places.

The kernel is left rationally invariant in each adelic variable. In the finite evaluation in (89), this includes the real factor of a rational move: substitution in P multiplies it by the inverse power of the rational determinant, which cancels the real multiplier of that move. The definite quotients have compact sets of representatives. The finite translates of the fixed test over each such set are a finite family, by its open stabilizer. On the real compact groups its Schwartz seminorms are uniformly bounded. Accordingly the rational theta sum converges absolutely and uniformly on these sets of representatives, and the kernel is continuous and bounded. Haar disintegration of a class of representative h gives the mass $|\operatorname{Aut}(h,K_2)|^{-1}$, since the real compact and K_2 have mass one. Thus $\Lambda_Q(\tau)=F_Q(1)$, where

$$F_Q(g)=\int_{[H]}\theta_{f_Q}(g,h)\,dh.$$

Constancy on the original group. Fix a chosen split place and regard the integral over h as a linear functional in the local test, with all other factors and the source point fixed. On $C_c^\infty(\mathrm{GL}_2(F_{0,v}))$ this functional satisfies

$$T(f(l^{-1}X))=\lambda(\det l)\,T(f),\qquad \lambda(t)=B(t)^{-1}|t|_v.$$

Indeed right translation by l of the target point preserves the quotient Haar, and its oscillator action has scalar $B(\det l)|\det l|_v^{-1}$. For a small compact open L on which $\lambda\circ\det$ is trivial, replace each test by its finite expansion in indicators of left cosets lL.

This shows that on every such test the functional is a constant times $\int f(X)\lambda(\det X)\,d^\times X$. More explicitly, for any finite set of tests one may choose one L so that all their compact

supports are disjoint finite unions of its cosets. The constant for these tests is $T(\mathbf{1}_L)/\operatorname{vol}(L)$, by the displayed covariance. Both the test and its conjugate by any fixed matrix satisfy this condition since their supports are compact subsets of the open invertible locus. Simultaneous translation of source and target by the same matrix sends the test to its conjugate and has scalar one.

Conjugation preserves the weighted integral. Absorb the target translation into its Haar integral to obtain right invariance of F_Q by the full source split group at this place. The argument applies at every place of $S_M \cup Q$, including a place of a larger modulus at which τ is unramified.

The group $\mathrm{SU}(I_2)$ is simply connected and absolutely almost simple. At an old split place it is SL_2, which is noncompact. Strong approximation [62, Theorem 7.12] away from this place implies $F_Q(sg) = F_Q(g)$ for every $s \in \mathrm{SU}(I_2)(\mathbb{A}_{F_0})$. Indeed, approximate s by rational s_0 away from the place so that $g^{-1}s_0^{-1}sg$ is integral at all other finite places and tends to the identity at the real places. Apply rational left invariance, right invariance by the full group at the omitted place, and continuity. The determinant map is onto the adelic norm-one group, by $u \mapsto \operatorname{diag}(u, 1)$. The lift therefore factors through the determinant.

At a split place outside S_Q both tests are the same vacuum. The minuscule source and target actions on it agree when the target acts at w and the source acts at $\bar{w}$. Namely choose integral elementary matrices γ_U with determinant valuation one and reduction kernel U, for the residue lines $U \subset k_w^2$. Use γ_U^{T} on the target coordinate at w, and γ_U^{-1} on the source coordinate at w, the latter being the positive elementary class at $\bar{w}$.

Take the corresponding right coset representatives. The target and source translates are respectively

$$qA\,\mathbf{1}_{\mathrm{int}}(\gamma_U^{-\mathsf{T}}X), \qquad qA\,\mathbf{1}_{\mathrm{int}}(X\gamma_U^{-1}), \qquad A = B(\varpi_w).$$

They require integral X, and then require U to lie respectively in the kernel of $\bar{X}^{\mathsf{T}}$ and $\bar{X}$. Indeed the image of $\varpi_w\gamma_U^{-1}$ on reduction is U. The two sums are therefore equal because these kernels have the same dimension.

The group on the integrated variable can be moved through the integral, including the inverse transpose if that coordinate is used. Thus the corresponding source Hecke operator acts on F_Q by $q+1$. On a determinant function it acts by $q+1$ times translation by the determinant of the diagonal with one uniformizer (or its inverse). This proves invariance by that translation and hence by the whole split group. At a nonsplit finite place every norm-one element is integral and $\operatorname{diag}(u, 1)$ belongs to the standard compact. At infinity the real compact invariance was verified above. This proves that F_Q is constant, so

$$\|F_Q\|_{[H]}^2 = v_2|\Lambda_Q(\tau)|^2.$$

30.2.1 Doubling and the local integrals

In this subsection, we apply the doubled action of Section 17.2 to the determinant norm.

For this equal-rank calculation put $u = v|_{U(1)}$ and write θ_c for the kernel at the same vector f_Q with pair characters (u, u). With $\chi = (\nu\tau) \circ \det$ one has

$$\theta_{f_Q}(a, h) = \chi(a)^{-1}\chi(h)\theta_c(a, h).$$

Here h is originally in the target slot. To double that slot as a source, take the rational adjoint $X \mapsto X^*$ from the original Hom space to the Hom space with reversed factors and opposite

second form. The opposite sign preserves the Heisenberg pairing. In the opposite basis of Section 36.1.6 its unitary intertwiner J takes the two characters (u,u) to $(u^{-1}u^2,u^{-1}u^2)=(u,u)$. The rational theta functional changes by a unit scalar, since the adjoint identifies the rational lattices and the normalized vacuum translates have mean square one. The same J acts on both factors. Thus its theta phase cancels, and for $v_h=\omega_c(1,h)f_Q$ the identity-graph pairing is $\langle Jv_h,Jv_{h'}\rangle=\langle v_h,v_{h'}\rangle$. The passage to the opposite form leaves the rational unitary group, its integral compact and its quotient Haar unchanged.

On negating a rank-two factor, conjugation inverts its character b, and the intrinsic Weil multiplier on the conjugate Hom action contributes u^2 on each determinant in the respective numerator or denominator. Hence the opposed character is $b^{-1}u^2$. Both centered characters u are unchanged. On doubling the integrated group, the pair characters are u for the doubled source and u^2 for the common rank-two target, in the convention that the action has the target determinant in the numerator.

Since $\upsilon(\bar y)=\upsilon(y)^{-1}$ for every idele y, we have $u^{\mathrm{bc}}(y)=\upsilon(y/\bar y)=\upsilon(y)^2$. Thus both splitting characters are trivial by the identities

$$u^{\mathrm{bc}}\upsilon^{-2}=1,\qquad (u^2)^{\mathrm{bc}}\upsilon^{-4}=1.$$

In the doubled model the source-character factor contributes $(u^{\mathrm{bc}})^{-1}=\upsilon^{-2}$ to the untwisted Siegel Levi factor of Section 17.2, so the centered section has the trivial splitting character. The doubled theta kernel on the pullback is the product of the two centered theta kernels with the first factor conjugated. Both inputs use the above f_Q before conjugation. On the pullback the two real factors are $\overline{P}$ and P in the corresponding Fock models. In common graph coordinates the finite tensor is $\overline{f_{Q,f}}\otimes f_{Q,f}$, where $f_{Q,f}$ is the finite determinant test at $\mathfrak{m}=(\pi_M)^2\prod_{w\in Q}\mathfrak{p}_w$. At the diagonal this gives $|f_{Q,f}(X)|^2$, including when the determinant character is nonquadratic. In taking the partial Fourier transform, the passage from midpoint and difference to the two arguments is

$$(Z,U)\longmapsto(Z+U/2,Z-U/2).$$

Its determinant in each scalar coordinate is -1. At $v\mid 2$ the place splits in E/F_0 by Section 17.1. For the two vacuum indicators on $L=M_2(\mathcal{O}_v)$, the conditions in these coordinates are

$$U\in L,\qquad Z\in U/2+L.$$

Thus Fourier transformation in Z with negative sign gives $\mathbf{1}_L(U)\mathbf{1}_{L^\perp}(\Xi)\psi_v(-\langle U/2,\Xi\rangle)$, since L has mass one. In particular the phase is one at the graph evaluation at zero. More generally, after the original translations, evaluation at $U=\Xi=0$ integrates $\overline{v_h(Z)}v_{h'}(Z)$ in the original measure. This explains the graph pairing also when 2 is not a local unit.

At a real place v, use the oscillator complex structure given by the two definite summands of the doubled source and the definite target. The corresponding maximal compact of $U(2,2)$ is $U(2)\times U(2)$. In the opposite and original Fock factors, respectively, $\overline{P_v}\otimes P_v$ is a polynomial of total degree $4k_v$ in creation operators applied to the tensor vacuum. The partial Fourier transform $\mathcal{F}$ to the mixed realization intertwines the Weyl displacements. Differentiating gives $\mathcal{F}a^*(u)\mathcal{F}^{-1}$ as the creation operator for the transported complex structure, and carries the tensor vacuum to its vacuum line. Choose a real isometry taking the ordered positive and negative subspaces in Yamana's doubled model to these two subspaces. Write it as $p_\infty k$ by Iwasawa decomposition, with p_∞ in the Siegel parabolic and k in the standard maximal compact K_∞. Since k preserves each subspace, p_∞ effects the same change. The target is definite. Thus

$\Phi_0 = \omega_c(p_\infty^{-1})\Phi$ is polynomial in Yamana's Fock model, where Φ is the transformed tensor and ω_c denotes the centered doubled action. With $K'_\infty = p_\infty K_\infty p_\infty^{-1}$ and the same finite compacts, Section 17.2 gives

$$f^{(s)}_{\Phi,K'}(g) = |a_K(p_\infty)|^{-s} f^{(s)}_{\Phi_0,K}(gp_\infty), \qquad \Theta(g,h;\Phi) = \Theta(gp_\infty,h;\Phi_0).$$

Both rank parameters in Yamana are 2, so the height factor is one at $s_0 = 0$. Both evaluations use the same centered Weil action and its trivial Siegel splitting character. The graph variables evaluate the section of the original transformed tensor [80, pp. 236–237].

Write E_c for the Eisenstein series of this section and $\iota(h,h')$ for the graph embedding. The norm integrand carries $\chi(h)^{-1}\chi(h')$ outside the Eisenstein series, because $\overline{\theta_{f_Q}(a,h)}\theta_{f_Q}(a,h')$ has that factor. The Hermitian and Siegel ranks are both two and the Hermitian space is definite. Thus [80, Theorem 2, p. 237] at $s = 0$, with the probability target measure of [79, §1.3, p. 1007, and Proposition 5.8(i), p. 1015], gives

$$\|F_Q\|^2_{[H]} = \frac{v_2}{2}\int_{[H]\times[H]} \chi(h)^{-1}\chi(h')E_c(\iota(h,h'),0)\,dh\,dh'.$$

Indeed the squared norm integrates the product of the two theta kernels in the common variable a. Yamana's identity reads $E_c = 2\int_{[H]} \theta_c^{\mathrm{dbl}}\,da/v_2$. Thus integration in a contributes $v_2/2$. The two remaining quotient integrations use the original measures.

All rational maximal isotropic subspaces are graphs of rational isometries by definiteness of the two factors. The graph zero term is the centered coefficient $\langle f_Q, \omega_c(1,g)f_Q\rangle$ at $s = 0$: the self-dual partial Fourier transform at zero gives the inner product on the common polarization. For $\Re s > 1$, extend the coefficient by the parabolic height to the power s, using the compact stabilizing the original doubled self-dual lattice.

Its Iwasawa decomposition is the integral Witt completion of an isotropic direct summand. The change to the classical skew coordinates in Section 17.2 can be nonintegral, so the compact is transported with the lattice. The choice of extension leaves the value at zero unchanged: two holomorphic sections agreeing at zero differ by s times a holomorphic section, and in rank two with trivial inducing character [80, Theorem 1(2), p. 236] gives no pole at zero for the series of the latter section. The diagonal copy of H stabilizes the identity graph. Its action on this Lagrangian has determinant of absolute value one over E, since $\det h\,\overline{\det h} = 1$. The centered inducing character is trivial, so left diagonal translation preserves the flat section for every s. Thus on $\Re s > 1$ the rational graph sum unfolds with relative variable $g = h^{-1}\gamma h'$, $\gamma \in H(F_0)$.

The external factor equals $\chi(g)$ since $\chi(\gamma) = 1$. The relative variable has ambient Haar, and the remaining quotient variable has volume v_2. Writing $Z(s)$ for the unfolded integral including $\chi(g)$, continuation and the constancy already proved give

$$v_2|\Lambda_Q(\tau)|^2 = \|F_Q\|^2_{[H]} = \frac{v_2^2}{2}Z(0).$$

For evaluation under the integrals, choose finite open compacts in the two factors fixing the pullback of the flat section and the external characters. The definite finite class sets at these levels are finite. At a fixed pair of representatives the real restriction is in a fixed finite sum of compact matrix coefficient spaces. Indeed the embedded real groups lie in the doubled maximal compact and the flat height is right invariant under that compact.

In the Fock model the polynomial tensor lies in the sum of its finitely many homogeneous degrees. The doubled real maximal compact preserves each of these finite-dimensional spaces;

the flat extension has the same compact restriction. Thus the section is finite under its right action. Finitely many point evaluations recover the coefficients in this fixed space, and every such evaluation is holomorphic at zero by [80, Theorem 1(2), p. 236]. Integration against the external characters on the compact quotients is a fixed linear functional on the finite sum of these spaces. It therefore commutes with evaluation at zero.

Here are the local integrals of this section. Invert the ambient variable, preserving its unimodular Haar. The height is unchanged by inversion since the unitary matrix and its inverse have equal absolute lattice intersection index. At the split places the centered operator is $\omega_c(1, g^{-1})$ and acts by $X \mapsto gX$ with its intrinsic half-density. The external character becomes $\chi(g)^{-1}$. In particular on elementary divisors j, k it is A^{-j-k}, where $A = (\nu\tau)^{\mathrm{bc}}(\varpi_w)$. In fact $\chi(g)^{-1}\omega_c(1, g^{-1}) = \omega_{\alpha,\alpha}(1, g^{-1})$. In particular, if $g \in K_{2,v}$ at a place of the determinant test, this coefficient is the squared norm of that test. At a finite place the doubled compact preserves the doubled self-dual lattice. A graph intersects it in

$$\{(gx, x) : x \in \Lambda \cap g^{-1}\Lambda\}, \qquad \Lambda = \mathcal{O}_{E,v}^2.$$

The inverse index of $\Lambda \cap g^{-1}\Lambda$ in Λ is the absolute determinant of the graph coordinate in the Levi, hence the parabolic height.

This follows by taking an integral basis of the graph intersection and completing it to a hyperbolic basis of the doubled lattice. A primitive isotropic summand in the split or unramified self-dual lattice admits such a completion: choose vectors paired dually with its basis and subtract multiples of the isotropic basis to cancel their Gram matrix. In the unramified case the diagonal equations use surjectivity of the integral trace, also at residue characteristic two.

At a split pair outside S_Q with norm q and $A = B(\varpi_w)$, the Cartan representative with elementary divisors $j \geq k$ has height $q^{-|j|-|k|}$, using both members of the pair. The overlap of the two integral matrix boxes, with the square-root Jacobian in the local action, gives the coefficient

$$A^{-j-k} q^{-|j|-|k|}.$$

For example in a coordinate with valuation j the matrix Jacobian factor is q^{-j}, and the overlap condition in the two entries of that row has probability $q^{-2\max(0,-j)}$. The double coset has Haar mass 1 if $j = k$, and mass $q^{j-k}(1 + q^{-1})$ otherwise, by counting free lines over $\mathcal{O}_v/\varpi_v^{j-k}$. With $x = q^{-1-s}$, $t = A^{-1}x$, $u = Ax$, summing the weighted Cartan double cosets gives

$$\begin{aligned} &\frac{1}{(1-t)(1-qt)} + \frac{1}{(1-u)(1-qu)} - 1 + \frac{(1+q^{-1})q^2tu}{(1-qt)(1-qu)} \\ &\qquad = \frac{(1-x^2)(1-qx^2)}{(1-Ax)(1-A^{-1}x)(1-qAx)(1-qA^{-1}x)}. \end{aligned}$$

For $j \geq k \geq 0$, set $j = k + d$. The local sum on this cone is

$$\sum_{k\geq 0} t^{2k}\Big(1 + (1+q^{-1})\sum_{d\geq 1}(qt)^d\Big) = \frac{1}{(1-t)(1-qt)}.$$

The cone $0 \geq j \geq k$ gives the same expression in u, with the origin counted a second time. The sector $j > 0 > k$ gives the last term.

At an unramified nonsplit place let q be the norm in F_0, so the norm in E is q^2. An integral hyperbolic basis of the standard local plane exists because the unit norm is onto. It may be

constructed by first taking a primitive isotropic vector and then making its integral dual vector isotropic using the trace. The special unitary group in the hyperbolic basis is $\mathrm{SL}_2(F_{0,v})$ under conjugation by a suitable diagonal matrix. Indeed determinant one forces the diagonal entries to be fixed by conjugation and the off-diagonal ones to be skew. Choose a skew integral unit δ, for example $z-\bar{z}$ with the residue of z outside the fixed residue field. Conjugating by $\mathrm{diag}(\delta,1)$ gives the asserted isomorphism and identifies the integral compact with $\mathrm{SL}_2(\mathcal{O}_v)$, including above two.

Let K_v be the unitary integral compact, let $K_v^1 = K_v \cap \mathrm{SU}(2)$, and put $a_j = \mathrm{diag}(\varpi^j, \varpi^{-j})$, $j \geq 0$. The determinant of $K_v \cap a_j K_v a_j^{-1}$ is all norm-one units. In fact the integral isometries $\mathrm{diag}(z, \bar{z}^{-1})$ with $z \in \mathcal{O}_{E_w}^\times$ commute with a_j and have determinant $z/\bar{z}$. A Hilbert 90 preimage of a norm-one unit can be made a unit by dividing by a power of the fixed-field uniformizer. In particular every unitary double coset has special unitary Cartan representative a_j and

$$[K_v : K_v \cap a_j K_v a_j^{-1}] = [K_v^1 : K_v^1 \cap a_j K_v^1 a_j^{-1}] = q^{2j}(1+q^{-1}) \quad (j>0).$$

The last equality counts free lines modulo ϖ^{2j}. The identity coset has mass one.

The graph height at a_j is q^{-2j} by the intersection lattice formula. In the split-plane polarization the centered matrix coefficient is the overlap of the integral box in E_w^2 with its dilation by the rank-one Levi. In the convention $X \mapsto \varpi^j X$, the old box is the smaller box. The intrinsic half-density is $|\varpi^j|_{E_w}^{2/2} = q^{-2j}$ and the centered splitting character is trivial. The external character is one on a_j. Thus the vacuum coefficient is q^{-2j} in this same model. The local sum is therefore

$$1 + (1+q^{-1}) \sum_{j \geq 1} q^{-2sj} = \frac{1+q^{-1-2s}}{1-q^{-2s}}.$$

At a chosen split place, overlap of the two determinant tests requires $X, gX \in \mathrm{GL}_2(\mathcal{O}_v)$ and thus $g \in \mathrm{GL}_2(\mathcal{O}_v)$. Conversely for every such g the finite determinant phase cancels by the local action checked above. The height is one, the ambient compact has mass one, and the integral of the squared test is exactly

$$\frac{|\mathrm{GL}_2(k_w)|}{q_w^4} = (1-q_w^{-1})(1-q_w^{-2}).$$

In the original point tests, put $e = 1$ or 2, $R_e = \mathcal{O}_v/\varpi_v^e$ and $\delta_y(X) = \mathbf{1}_{y+\varpi_v^e M_2(\mathcal{O}_v)}(X)$. For $g \in K_{2,v}$ and $y, y' \in \mathrm{GL}_2(R_e)$,

$$\int \delta_y(X)\delta_{y'}(gX)\, dX = q^{-4e}\, \mathbf{1}_{\{y' = \bar{g}y\}}.$$

Here $q = q_w$ and $\bar{g}$ denotes reduction modulo ϖ_v^e. Weighting these two tests by the conjugate and ordinary determinant characters gives $B(\det g)$ on every surviving pair. The external factor is its inverse. Thus the mass is exactly $q^{-4e}|\mathrm{GL}_2(R_e)| = q^{-4}|\mathrm{GL}_2(k_w)|$, also at an inserted split place with trivial unit character. At infinity the parabolic height is one on the two compact groups, and the local integral is $\|\det(X_v)^{k_v}\|^2$. In the normalization of Bargmann's monomial basis [10, (1.6)], polar coordinates give

$$\frac{1}{\pi}\int_{\mathbb{C}} z^a \bar{z}^b e^{-|z|^2}\, dA(z) = \delta_{ab}\, a! \qquad (a,b \in \mathbb{Z}_{\geq 0}).$$

Applying this identity in the four matrix entries gives

$$\|\det(X)^k\|^2 = \sum_{j=0}^{k} \binom{k}{j}^2 (j!)^2((k-j)!)^2 = k!(k+1)!.$$

Let $\beta = (\nu\tau)^{\mathrm{bc}}$ and let ξ be the quadratic character of E/F_0. The local formulas give the absolute Euler product for $\Re s > 1$. For example the nonidentity contributions at places away from S_Q are $O(q^{-\Re s})$ uniformly on closed half-planes $\Re s \geq 1+\delta$. The remaining finite local factors are constant in s. The product away from S_Q is

$$\frac{L^{S_Q}(s,\beta^{-1})L^{S_Q}(s+1,\beta^{-1})}{L_{F_0}^{S_Q}(2s+1,\xi)\,\zeta_{F_0}^{S_Q}(2s+2)}.$$

In a superscript on an F_0-function, omit the places below S_Q. At a spherical split place the parameter in [30, §11.6, pp. 823–824] is our normalized induction parameter s. The standard factor there is evaluated at $s+\frac{1}{2}$: the two Satake exponents of a determinant character of GL_2 are $\pm\frac{1}{2}$, giving the shifts $s, s+1$ in the numerator. The two places above v supply the factors with A and A^{-1}. The split denominator is $\zeta_v(2s+1)\zeta_v(2s+2)$, as in the Cartan sum above.

At an inert v, put $q = \mathrm{N}v$. The character β is unramified and trivial on the fixed-field uniformizer, the residue norm upstairs is q^2, and $\xi_v(\varpi_v) = -1$. Hence the local Euler quotient is

$$\frac{(1+q^{-2s-1})(1-q^{-2s-2})}{(1-q^{-2s})(1-q^{-2s-2})} = \frac{1+q^{-2s-1}}{1-q^{-2s}}, \qquad \Re s > 1.$$

The individual nonsplit integrals $(1+q^{-1-2s})/(1-q^{-2s})$ have poles at zero. The Euler product was taken on $\Re s > 1$. Meromorphic continuation of the Eisenstein series and of the Hecke functions extends the equality of their integrals from $\Re s > 1$.

The continued Eisenstein pullback may be evaluated under the compact quotient integrals at the holomorphic Siegel–Weil point. At zero the local masses at the omitted places insert exactly the omitted reciprocal denominator factors. The positive residues at one in $\zeta_E(s) = \zeta_{F_0}(s)L_{F_0}(s,\xi)$ show that $L_{F_0}(1,\xi) > 0$. Thus the denominator $L_{F_0}(1,\xi)\zeta_{F_0}(2)$ is positive, giving

$$|\Lambda_Q(\tau)|^2 = D_p L^{S_Q}(0,\beta^{-1})L^{S_Q}(1,\beta^{-1}), \qquad D_p > 0, \tag{90}$$

where D_p depends only on the original field, Haar and real test at p. With the stated probability real measures and unscaled Bargmann test it is $v_2 \prod_{v|\infty} k_v!(k_v+1)!/(2L_{F_0}(1,\xi)\zeta_{F_0}(2))$. Using the fixed normalization of the theta vector throughout multiplies this constant by a fixed positive factor. The same normalization is used with and without Q.

For a twist of exact support Q, the conductor of β^{-1} contains both members of every pair with exponent one. The old factors are fixed, since p is ramified at each old place and the auxiliary support is disjoint. The archimedean gamma factors are $\Gamma_{\mathbb{C}}(s+c_v)$ and are finite and nonzero at zero and one. With $\mathfrak{f}_\beta$ the conductor, set $D_\beta = |\operatorname{disc}(E)|\,\mathrm{N}_{E/\mathbb{Q}}\mathfrak{f}_\beta$. The completed function is

$$D_\beta^{s/2} \prod_{v|\infty} \Gamma_{\mathbb{C}}(s+c_v) L(s,\beta^{-1}).$$

Tate's global functional equation [74, §2.5, Theorem 4.4.1 and §4.5] exchanges this function with the completion for β at $1-s$, up to a factor of modulus one. On the real axis the two finite L-values are complex conjugates. Exact support gives $D_\beta/D_{\nu^{\mathrm{bc}}} = \prod_{w\in Q} q_w^2 = q_Q^2$; the

archimedean factors are the same for every twist. Since $\Gamma_{\mathbf{C}}(1+c_v)/\Gamma_{\mathbf{C}}(c_v) = c_v/(2\pi)$, the absolute values at the two Mellin points satisfy

$$|L(0,\beta^{-1})| = D'_p q_Q |L(1,\beta^{-1})|, \qquad D'_p = D_{\nu^{\mathrm{bc}}}^{1/2} \prod_{v|\infty} \frac{c_v}{2\pi} > 0.$$

In this calculation the omitted factors of the full Hecke function at the exact ramified support are all one.

For real $s > 1$, applying $3 + 4\cos\theta + \cos(2\theta) \geq 0$ to prime powers outside the conductors gives

$$\zeta_E^S(s)^3 |L^S(s,\beta^{-1})|^4 |L^S(s,\beta^{-2})| \geq 1,$$

where S contains all ramified primes. The omitted local factors are finite and nonzero at one. The first factor has pole order three, and $L^S(s,\beta^{-2})$ is holomorphic there since its infinity weights are nonzero. A zero of $L^S(s,\beta^{-1})$ at one would force the left-hand side to tend to zero. Hence $L(1,\beta^{-1}) \neq 0$. These facts give $\Lambda_\varnothing(1) \neq 0$ and

$$\frac{|\Lambda_Q(\tau)|^2}{|\Lambda_\varnothing(1)|^2} = q_Q \left| \frac{L^{S_Q}(1,((\nu\tau)^{\mathrm{bc}})^{-1})}{L^{S_\varnothing}(1,(\nu^{\mathrm{bc}})^{-1})} \right|^2 . \tag{91}$$

Keep the character τ fixed and insert a split place w at which β is unramified. Put $q = q_w$, $A = \beta(\varpi_w)$ and $f = \mathbf{1}_{M_2(\mathcal{O}_w)}$. The new local test is $\mathbf{1}_{\mathrm{GL}_2(\mathcal{O}_w)}$. For each line $U \subset k_w^2$ choose h_U of determinant ϖ_w with $h_U \mathcal{O}_w^2$ the inverse image of U. Counting the lines containing the image of a matrix of rank zero, one or two gives, on $M_2(E_w)$,

$$\mathbf{1}_{\mathrm{GL}_2(\mathcal{O}_w)}(X) = f(X) - \sum_U f(h_U^{-1}X) + q f(\varpi_w^{-1}X).$$

All other local tests and both characters are fixed. Let T be the quotient integral defining the period with variable test at w. Right invariance of target Haar and the split oscillator action give

$$T(f(h^{-1}\,\cdot)) = |\det h|_w B(\det h)^{-1} T(f).$$

The $q+1$ line terms each contribute $q^{-1}A^{-1}T(f)$; the last test, with $h = \varpi_w I_2$, contributes $q^{-2}A^{-2}T(f)$ before its coefficient q. Consequently

$$\Lambda_{Q\cup\{w\}}(\tau) = (1-A^{-1})(1-q^{-1}A^{-1})\Lambda_Q(\tau).$$

Since $|A| = 1$, the squared period is multiplied by

$$(1-A)(1-A^{-1})(1-q_w^{-1}A)(1-q_w^{-1}A^{-1}). \tag{92}$$

This identity is a multiplication identity and remains valid when the multiplier or the period is zero.

30.2.2 Integral coefficients of the rank-two norm

In this subsection, we express the norm of Section 30.2.1 in the original finite coordinates.

Let h_c be representatives in (89). For $y \in \mathrm{GL}_2(\mathcal{O}_E/\mathfrak{m})$, let $\widetilde{y}$ be the split integral lift in the original coordinates at the chosen places, with identity components elsewhere, and set

$h_{c,y} = h_c\widetilde{y}$, $u_{c,y} = \det h_{c,y}$. Let $K^{\mathrm{pt}}_{Q,1}(h)$ be the kernel at the same modulus with the point indicator of the identity and with $\tau = 1$. Expand the determinant character as the sum of the point indicators at invertible matrices y, weighted by $\chi_{\mathfrak{m}}(\det y)$. Write δ_y for the indicator of $X \equiv y$ at this modulus, and $\alpha_1 = \nu\upsilon|_{U(1)}$. The split action gives

$$\omega_{\alpha,\alpha}(1,\widetilde{y})\delta_1 = \chi_{\mathfrak{m}}(\det y)\delta_y, \qquad \theta^{\alpha,\alpha}_{\delta_1}(1,h_{c,y}) = \tau(u_{c,y})\,\theta^{\alpha_1,\alpha_1}_{\delta_1}(1,h_{c,y}).$$

In the second equality the modulus of δ_1 still includes every prime in Q, even where the character under evaluation is unramified. Thus

$$\Lambda_Q(\tau) = \sum_{c,y} \frac{K^{\mathrm{pt}}_{Q,1}(h_{c,y})\tau(u_{c,y})}{|\operatorname{Aut}(h_c,K_2)|}.$$

The stabilizer order here divides C_G, by reduction at an old odd completely split place as in the preceding subsection.

Use the fixed nonzero integral-jet scale

$$s_p = (M^2)^{8f}\, s_{\mathrm{vac}}^4 (M^2 d_K s_{\mathrm{jet}})^{2\sum_v k_v}$$

for the original rank-two point test at modulus $\mathfrak{m}$. Section 44.4 gives

$$\iota(s_p K^{\mathrm{pt}}_{Q,1}(h_{c,y})) \in \mathcal{O}_{\overline{\mathbb{Q}}_\ell} \qquad (\ell \nmid q_Q)$$

at every coefficient embedding, with the same s_p for all Q, c, y.

Set

$$a_{c,y}(Q) = \frac{C_G s_p K^{\mathrm{pt}}_{Q,1}(h_{c,y})}{|\operatorname{Aut}(h_c,K_2)|}.$$

Its coefficients are integral away from q_Q by this jet bound. With i, j ranging over the pairs (c, y), the squared norm system is

$$F_{p,Q}(\tau) = |C_G s_p \Lambda_Q(\tau)|^2 = \sum_{i,j} a_i(Q)\overline{a_j(Q)}\,\tau(u_i u_j^{-1}).$$

The conjugate coefficients are integral by applying the bound at the conjugate embedding. Formula (91) gives its exact-support ratio, with the same scale s_p on numerator and denominator.

30.3 The integral trace algebra on the auxiliary levels

In this subsection, the attachment of Section 28.2 identifies traces with Hecke data at a cofinite set of primes. We use the integral Hecke action there to place traces at all Galois elements in one coefficient algebra.

Fix a coefficient embedding above ℓ and any of the finitely raised levels under consideration. The old type and twist characters are fixed. Raising changes only the invariance at the auxiliary split places away from ℓ. Let $V_{\mathrm{orig}} \subseteq V_{\mathrm{raised}}$ be these two spaces. Their Weyl lattices are defined by the same inequalities on the values in transported coordinates. Consequently

$$L_{\mathrm{orig}} = V_{\mathrm{orig}} \cap L_{\mathrm{raised}}, \qquad V_{\mathrm{orig}} \cap aL_{\mathrm{raised}} = aL_{\mathrm{orig}} \quad (a \neq 0),$$

by dividing the vector by a in V_{orig}. On the ambient integral space all local translations supported away from ℓ preserve this lattice norm. In fact the lattice norm of a value at a point is

read in the transported coordinates of that point at the places above ℓ. Right translating by a point with identity components above ℓ changes neither those coordinates nor the list of points where the lattice norm is bounded. The finite local sums that preserve the raised invariants therefore act on the raised integral lattice.

For a system x of spherical eigenvalues at almost all split places, identify systems that agree off a finite set. Write W_x for the space of forms with these eigenvalues at almost all split places. For each level the corresponding space is $V \cap W_x$. Thus

$$V_{\mathrm{orig}} \cap W_x \subseteq V_{\mathrm{raised}} \cap W_x.$$

The set $Y = \{x : V_{\mathrm{raised}} \cap W_x \neq 0\}$ is finite by finite dimensionality and orthogonality of the distinct systems. Auxiliary translations preserve W_x, since they commute with spherical Hecke operators away from their support. Define $\mathbb{T}$ to be the ring of functions $Y \to \overline{\mathbb{Q}}_\ell$ whose diagonal operators preserve the raised integral lattice. It contains the scalar valuation ring. It is closed in the product of the coefficient topologies: for each vector in the integral lattice and each test in the lattice coordinates, the result of the diagonal operator is a continuous linear function of the finitely many scalars. The condition that this result be in the valuation ring is closed, and the intersection of all these conditions is closed. This argument uses the original lattice norm at every class point, so it is valid with stabilizers as well.

For this fixed raised level apply Section 28.2 to the finite set Y. Choose a basis of each $V_{\mathrm{raised}} \cap W_x$. Remove the selected triangle primes, the exceptional places of these finitely many basis vectors, and the excluded places of the Galois comparisons for Y from the split places away from ℓ. On the remaining set, an original eigenvector and its image in the raised space have the same spherical Hecke polynomials, since the inclusion commutes with the local translations there. Density and semisimplicity therefore identify their Galois attachments. For newly occurring systems, the comparison at a selected triangle prime uses the split local parameter of Section 31.1. On the common good set the minuscule spherical sums act by the stipulated scalar on each outside-level isotypic space. As finite sums of the integral translations above, their diagonal scalar functions belong to $\mathbb{T}$. The coefficients of the normalized Hecke polynomial are integral multiples of these operators, with powers of q_w as in Section 28.2. Fix $g \in G_E$. The cofinite Frobenius comparison for each member of Y, and Chebotarev, approximate the coefficients of $\det(X - R_x(g))$ simultaneously, for all $x \in Y$, by these Hecke coefficients.

Indeed for any required precision, continuity gives an open normal subgroup on whose coset at g the coefficients agree to that precision. The functions are conjugacy invariant. Chebotarev's theorem [48, Theorem 8.31] gives positive density for the conjugacy class in the finite quotient. The prime ideals of E of residue degree at least two over $\mathbb{Q}$ have Dirichlet density zero, since their sum of $(\mathrm{N}w)^{-s}$ for $s > 1$ is bounded by $[E : \mathbb{Q}] \sum_p p^{-2}$. A degree-one prime outside the finite ramified set must split over F_0, by multiplicativity of residue degrees. Thus the required class occurs in the common good set. Closedness now gives each coefficient function in $\mathbb{T}$.

For an integral character a, putting $N_x = aR_x$ multiplies the coefficient of X^{n-i} by $a(g)^i$; hence these twisted characteristic coefficients also belong to $\mathbb{T}$. In particular $x \mapsto \mathrm{tr}(N_x(g))$ belongs to $\mathbb{T}$ for every g, including products involving the selected tame generators and Frobenius lifts. The trace functions used below are integral linear combinations of these. For instance expansion of p in monomials writes

$$\mathrm{tr}\big(p(N_x(\phi))(N_x(\tau) - 1)N_x(g)\big) = \sum_i p_i\big(\mathrm{tr}(N_x(\phi^i \tau g)) - \mathrm{tr}(N_x(\phi^i g))\big).$$

All of these diagonal operators commute with the local translations preserving the decomposition into outside-level eigensystems. Thus the raised local comparison uses the same integral endomorphism ring and original invariant lattice throughout.

31 Hecke operators and the common trace coefficient

In this section, we identify the tame and Frobenius operators in the algebra of Section 30.3. Spherical compatibility gives a common Hecke coefficient to which the Fitting and trace arguments of Section 32 apply.

31.1 Tame and Frobenius operators at a raised prime

In this subsection, we represent tame inertia and Frobenius by operators on the K_1-invariants at the same split place. The resulting trace identities refine the integral trace comparison of Section 30.3 by retaining the local finite type.

Fix a coefficient prime ℓ, a chosen split place $w \nmid \ell$ off the original level, and a congruence depth ν. Write $k = k_w$, $q = |k|$, $H = \mathrm{GL}_n(k)$, $K = \mathrm{GL}_n(\mathcal{O}_w)$, and $K_1 = \ker(K \to H)$. At w the original level is K and the raised level is K_1. The other local factors stay fixed. We assume $\ell^\nu \mid q - 1$ and fix the prescribed geometric Frobenius lift ϕ and tame generator τ in the local Weil group. In its abelianization choose $\varpi \in E_w^\times$ and $u \in \mathcal{O}_w^\times$ corresponding to their images under geometric local reciprocity. The Frobenius degree of ϕ gives $v_w(\varpi) = 1$. Let $\lambda : k^\times \to A = \mathbb{Z}/\ell^\nu\mathbb{Z}$ be the chosen homomorphism with $\lambda(\bar{u}) = 1$. Put $d = \mathrm{diag}(\bar{u}, 1, \dots, 1)$.

Write ρ for the H-action on the raised space and $\mathcal{A}$ for the integral endomorphisms of its lattice. The raised lattice is the relevant invariant space in the ambient integral lattice on which all translates at w act. Section 30.3 gives the integral diagonal trace action on this same lattice. The outside-level eigensystems decompose the raised space over characteristic zero, and the local operators at w preserve each summand.

On every raised system x use its semisimple attachment R_x^{up} of rank n and the full split-place comparison from Section 28.3. At a split level place this comparison follows from the same split base change on each constituent of the eigenvector. The cofinite comparison of Section 28.2 identifies the semisimple attachments of all constituents with system x. Choose the normalization $N_x = aR_x^{\mathrm{up}}$, where $a(\tau) = 1$. For each Galois element g, write

$$F_x = N_x(\phi), \qquad T_{x,\tau} = N_x(\tau), \qquad M_x = N_x(g),$$

and let $C_p, D_p \in \mathcal{A}$ be the diagonal operators whose entries for an integral polynomial p are

$$C_p(x) = \mathrm{tr}(p(F_x)M_x), \qquad D_p(x) = \mathrm{tr}(p(F_x)(T_{x,\tau} - 1)M_x).$$

All scalars are evaluated at the same embedding.

Let $\bar{P} \subset H$ be the stabilizer of ke_1, $J \subset K$ its inverse image, and $I_1 \subset K$ the inverse image of the upper unitriangular group. For a character ψ of A, put $\chi = \psi \circ \lambda$. On the H-type $\chi \circ \det$, the inverse determinant twist has a K-fixed vector. Its parameter is spherical. Twisting back gives zero monodromy, and τ acts on the entire Galois representation by $\psi(1)$, the eigenvalue of $\rho(d)$ on the same type.

Consider now the J-type $j \mapsto \chi(\bar{j}_{11})$ with $\chi \neq 1$. In an irreducible local constituent π, finite Frobenius reciprocity gives a common constituent of π^{K_1} and $\mathrm{Ind}_{\bar{P}}^{H}(\chi \boxtimes 1)$. The latter

embeds into the Borel induction from $(\chi, 1, \dots, 1)$ by transitivity of induction. Write π as a subquotient of an induction from its supercuspidal support. Iwasawa decomposition and exactness of compact invariants give the K_1-invariants of that induction as finite induction from the corresponding principal-kernel invariants of its factors.

For a supercuspidal factor of rank b, set $K_b = \mathrm{GL}_b(\mathcal{O}_w)$. If its $K_{b,1}$-invariants are nonzero, it has depth zero at the vertex of the standard lattice. By [57, Propositions 6.6 and 6.8, pp. 114–115, 117–118], it is compactly induced from a representation of a maximal parahoric normalizer containing an inflated finite cuspidal representation. In $\mathrm{GL}_b(E_w)$ every such vertex is a lattice class; choose a basis for the lattice to take its parahoric to K_b. The normalizer $E_w^\times K_b$ acts on the same finite cuspidal space, since the centre acts by scalars. To calculate the invariants of this compact induction, take Cartan representatives $s = \mathrm{diag}(\varpi^{v_1}, \dots, \varpi^{v_b})$, with $v_1 \le \cdots \le v_b$, modulo $E_w^\times K_b$ on the left and K_b on the right. For any strict cut $v_i < v_{i+1}$, the reduction of $K_b \cap sK_{b,1}s^{-1}$ contains the upper rectangular unipotent radical at that cut, since conjugation by s^{-1} multiplies the entry in row $a \le i$, column $b' > i$, by $\varpi^{v_{b'} - v_a}$. The intersection is the same for sk_0, $k_0 \in K_b$, because $K_{b,1}$ is normal in K_b. Cuspidality forces the value of every $K_{b,1}$-invariant vector at those cosets to vanish. The central Cartan class gives the finite cuspidal factor.

31.1.1 The restriction to the tame subgroup

In this subsection, we calculate the tame subspace using the local Hecke data of Section 28.2.

Now use permutation representatives in finite Mackey theory between the Borel and the block parabolic. For each representative, the intersection in each conjugated Levi block contains the unipotent radical of a Borel of that block. A finite cuspidal of rank greater than one has no such invariants, because that radical contains the radical of a proper parabolic. Thus every block has rank one. Comparing the characters on the diagonal torus gives the tuple $(\chi, 1, \dots, 1)$, up to permutation. Unramified twists preserve restriction to $\mathcal{O}_w^\times$. The entry with character $\chi \neq 1$ therefore forms a segment of length one on its own cuspidal line. The classification by segments and the product over distinct lines give

$$\pi = \mathrm{Ind}_P^{\mathrm{GL}_n(E_w)}(\chi_0 \boxtimes \pi_1) \quad \text{with normalized induction,}$$

where P is the stabilizer of the first coordinate line, $\chi_0|_{\mathcal{O}_w^\times} = \chi$ via reduction, and π_1 has all-unramified rank-one support [81, Theorem 6.1 and Proposition 8.6, pp. 188, 197].

Apply finite Mackey theory to $\bar{P}\backslash H/\bar{P}$. For $n \ge 2$ the two representatives are 1 and $s = (12)$. Choose $b \in k^\times$ with $\chi(b) \neq 1$. The element $h = \mathrm{diag}(1, b, 1, \dots, 1)$ belongs to $\bar{P} \cap s\bar{P}s^{-1}$ and acts trivially by the prescribed character $j \mapsto \chi(j_{11})$. On the conjugated inducing representation it acts by $\chi(b)$, since $s^{-1}hs = \mathrm{diag}(b, 1, \dots, 1)$ acts on the first block alone. The intertwining space for this coset is therefore zero. The identity coset gives $\pi_1^{\mathrm{GL}_{n-1}(\mathcal{O}_w)}$. In particular π_1 is spherical and evaluation at 1 is injective on the required J-type. For $n = 1$ there is only χ_0 and the evaluation is immediate.

Set $t = \mathrm{diag}(\varpi, 1, \dots, 1)$ and let U be the unnormalized operator of $I_1 t I_1$. For $b \in k^{n-1}$, put $u_b = \left(\begin{smallmatrix} 1 & \widetilde{b} \\ 0 & I_{n-1} \end{smallmatrix}\right)$. Both $I_1 \cap tI_1t^{-1}$ and $J \cap tJt^{-1}$ impose precisely the extra divisibility by ϖ on the off-diagonal entries of the first row. Thus the cosets are $u_b t I_1$ and $u_b t J$, respectively. The J-character agrees with its t-conjugate on the intersection because the $(1,1)$ entry is unchanged. Left multiplication by $j \in J$ permutes the cosets and contributes its character to each term, so U preserves the type. In the induced model every term has value

$$f(u_b t) = q^{-(n-1)/2}\chi_0(\varpi) f(1).$$

There are q^{n-1} terms. Evaluation at 1 therefore shows that the U-scalar is $q^{(n-1)/2}\chi_0(\varpi)$.

Both blocks have zero monodromy, and the complementary block has trivial inertia. On the unique $\psi(1)$-inertia line the Frobenius scalar after the twist $|\det|^{(1-n)/2}$ is $q^{(n-1)/2}\chi_0(\varpi)$. Frobenius semisimplification preserves this scalar on the one-dimensional line. Hence $T_{x,\tau}-1$ acts only on this line, where F_x has the scalar of $\widetilde{U}=a(\phi)U$. Write $\alpha_\phi=a(\phi)q^{(n-1)/2}\chi_0(\varpi)$. Then $p(F_x)(T_{x,\tau}-1)=p(\alpha_\phi)(T_{x,\tau}-1)$, since the image of $T_{x,\tau}-1$ lies on that line. Right multiplication by M_x and taking the trace yield $D_p=p(\widetilde{U})D_1$ on the given J-type for every g. This and the determinant-type calculation apply on the corresponding types of every raised eigensystem, by taking irreducible components.

For an integer-valued weight f on A, define

$$L_J(f)=\sum_{k\in\bar{P}} f(\lambda(k_{11}))\rho(k),\quad L_H(f)=\sum_{h\in H} f(\lambda(\det h))\rho(h),$$
$$B(f)=p(\widetilde{U})L_J(f).$$

Left multiplication by I_1 preserves the line weight, so the image of $L_J(f)$ is I_1-fixed. The translates in the coset sum U act on the ambient integral lattice, and their sum on I_1-invariants preserves those invariants. As $K_1\subset I_1$, $B(f)$ is an integral operator on the raised lattice made from translates at w. Let $\mathcal{E}(f)=B(f)D_1-L_J(f)D_p$. Fourier expansion in A and the evaluations on the two types give

$$L_H(f)D_p=(\rho(d)-1)L_H(f)C_p,\qquad \mathcal{E}(f(\cdot-n!)-f)=0.$$

The weight ψ^{-1} has image in the ψ-type, and its coefficient in $f(\cdot-n!)-f$ is multiplied by $\psi(n!)-1$. The equality is tested in characteristic zero between the integral operators.

Let $\omega:A\to\mathbb{Z}$ give the signed representatives. Taking $f(s)=\omega(s)(\omega(s)+n!)$ gives $f(s-n!)-f(s)\equiv -2n!\,\omega(s)\pmod{\ell^\nu}$, so $2n!\mathcal{E}(\omega)\in\ell^\nu\mathcal{A}$. Let σ be a reduction homomorphism on $\mathcal{A}$ which kills every X with $2n!X\in\ell^\nu\mathcal{A}$. For example, use reduction modulo $\ell^\nu/(2n!)$ when that quotient belongs to the coefficient valuation ring. Write $\mathcal{N}_w=\sum_{h\in H}\rho(h)$. The determinant identity with the constant weight gives $\mathcal{N}_wD_1=0$ exactly.

Choose representatives r_s with $r_s\bar{P}$ running over the lines, and set

$$C=\sum_s \rho(r_s),\quad C^\vee=\sum_s\rho(r_s^{-1}),\quad \partial=-CB(\omega)C^\vee.$$

Write $hr_s=r_{hs}k_{h,s}$ with $k_{h,s}\in\bar{P}$. In the group algebra the coefficient of h in $CL_J(\omega)C^\vee$ is $\sum_s\omega(\lambda((k_{h,s})_{11}))$. The product $\prod_s(k_{h,s})_{11}$ is a homomorphism in h by the coset cocycle identity and is independent of the representatives because their changes telescope. On $\mathrm{diag}(b,1,\dots,1)$ the product is b.

For each line other than ke_1 choose a representative vector with its first nonzero coordinate among coordinates 2 through n equal to 1. The diagonal element preserves this normalization, so the only contribution is that of ke_1. A transvection has residue-characteristic order, prime to $|k^\times|$, and therefore contributes 1. These elements generate H, so the product is the determinant. It follows that

$$CL_J(\omega)C^\vee\equiv L_H(\omega),\qquad (\rho(d)-1)L_H(\omega)\equiv-\mathcal{N}_w\pmod{\ell^\nu\mathcal{A}}.$$

The second congruence uses $\omega(s-1)-\omega(s)\equiv-1\pmod{\ell^\nu}$.

Since the diagonal trace operators commute with the local action, the two congruences and $\sigma(\mathcal{E}(\omega)) = 0$ give

$$\begin{aligned}\sigma(\partial D_1) &= -\sigma\big(CL_J(\omega)C^{\vee}D_p\big) = -\sigma\big(L_H(\omega)D_p\big)\\ &= -\sigma\big((\rho(d)-1)L_H(\omega)C_p\big) = \sigma(\mathcal{N}_w C_p).\end{aligned}$$

If $p(X) = (X - q^b)H_0(X)$, with $b \geq 0$ and H_0 integral, then $q^b \equiv 1 \pmod{\ell^\nu}$ and $\sigma(\partial D_1) = \sigma(\mathcal{N}_w C_{(X-1)H_0})$. The operator ∂ is independent of g and uses the original unnormalized norm on the full raised lattice. At every selected place the argument uses the same depth and reduction, with the hypotheses stated at the start of the subsection.

31.1.2 Integral clearing from the tame action

In this subsection, we use the tame restriction of Section 31.1.1 to pass to integral vectors.

For use with clearing, suppose an integral endomorphism Ψ and scalar c satisfy

$$\sum_{k\in\bar{P}} \rho(k^{-1})\Psi\rho(k) = c\,1.$$

Let $\mathcal{N}_{\mathrm{all}}$ be the product of the selected full norms, including $\mathcal{N}_w$, and suppose the spherical annihilation on the supported original systems gives $\delta\mathcal{N}_{\mathrm{all}}B(1) = 0$, with 1 the constant weight. For $h \in H$ set $A_h = \sum_s \omega(\lambda((k_{h,s})_{11}))\rho(r_s^{-1})$. The equations

$$B(f)\rho(k) = B(f(\cdot - \lambda(k_{11}))), \qquad r_{hs}^{-1}h = k_{h,s}r_s^{-1}$$

and the signed weight congruence give

$$\partial(\rho(h) - 1) \equiv CB(1)A_h \pmod{\ell^\nu\mathcal{A}}.$$

Here the difference of the weights at $k_{h,s}$ is congruent to the negative of its logarithm.

The double-coset calculation for the trivial J-character gives $\rho(k)B(1) = B(1)$. Applying the identity for Ψ to $B(1)$ yields $cCB(1) = \mathcal{N}_w\Psi B(1)$. Set $Z = \Psi B(1)A_h$. Then

$$\sigma\big(c\partial(\rho(h) - 1)\big) = \sigma(\mathcal{N}_w Z).$$

Before applying C to that identity for Ψ, multiply on the left by $\delta\mathcal{N}_{\mathrm{all}}$. Norm invariance gives

$$c\delta\mathcal{N}_{\mathrm{all}}B(1) = |\bar{P}|\delta\mathcal{N}_{\mathrm{all}}\Psi B(1).$$

The left side vanishes. Integral endomorphisms have no integer torsion, so $\delta\mathcal{N}_{\mathrm{all}}Z = 0$.

31.2 Spherical root chains on the original level

In this subsection, we factor the normalized spherical polynomial into successive roots in the pro-unipotent Iwahori space. The construction preserves the original spherical vector and keeps the local normalization used in Section 31.1.

In this subsection r is the local rank and t is the length of the specified chain of roots. Let $K_0 = \mathrm{GL}_r(\mathcal{O}_w)$ with residue field of size q, and use the fixed ordered coordinates and uniformizer ϖ at the auxiliary place. Let $I_1 \subset K_0$ be the pro-unipotent Iwahori. Work in the given smooth unitary representation with positive invariant pairing B, conjugate linear in

the first argument. Let $x \neq 0$ be the original K_0-fixed eigenvector, with eigenvalues t_j for the unnormalized spherical coset sums of $\mathrm{diag}(\varpi I_j, I_{r-j})$.

The polynomial is

$$P(X) = \sum_{j=0}^{r} a_j X^{r-j}, \qquad a_j = (-1)^j q^{j(j-1)/2} t_j, \quad t_0 = 1.$$

Put $d = \mathrm{diag}(\varpi, 1, \ldots, 1)$. The operator U in the derivative congruence is the unnormalized I_1-double-coset sum of d. On I_1-fixed vectors, it is $q^{r-1} M_{I_1} \rho(d) M_{I_1}$, where M_{I_1} is probability Haar averaging. Indeed $I_1 \cap d I_1 d^{-1}$ imposes the vanishing modulo ϖ of the off-diagonal entries in the first row, giving index q^{r-1}. The adjoint is $U^\dagger = q^{r-1} M_{I_1} \rho(d^{-1}) M_{I_1}$.

For $m, j \geq 0$, let $T_{m,j}$ be the sum of $\rho(g)x$ over the lattices $L = g\mathcal{O}_w^r$ between $\varpi\mathcal{O}_w^r$ and $\mathcal{O}_w^r$, of colength j, containing the first m coordinate vectors. A representative g gives a well-defined vector because x is K_0-fixed. In particular $T_{0,j} = t_j x$. Put $T_j = T_{1,j}$. These vectors are fixed by I_1, which permutes the relevant lattice sets.

The interval of lattice sums gives a forward recursion. For each residue covector ξ with first entry 1, take $n_\xi \in I_1$ the identity with first row lifting ξ. For a lattice in $T_{0,j+1}$, its occurrence in $n_\xi^{-1} d T_j$ means that its reduction lies in the hyperplane $\xi = 0$. To check the condition, $n_\xi L$ must have first residue coordinate zero; then $d^{-1} n_\xi L$ is integral, contains $\varpi\mathcal{O}_w^r$ and contains e_1 since $n_\xi L \supset \varpi\mathcal{O}_w^r$. The colength has dropped by one. There are no such covectors if L contains e_1. Otherwise the requirement that ξ vanish on a subspace of dimension $r-j-1$ and take value 1 on e_1 leaves j free coordinates, hence q^j covectors. Applying M_{I_1}, which kills the action of each n_ξ^{-1} on the left, gives

$$U T_j = q^j (t_{j+1} x - T_{j+1}). \tag{SC1}$$

In this notation $T_0 = x$ and $T_r = 0$.

For the inverse count use the representation $\rho'(g) = \rho((g^{\mathsf{T}})^{-1})$. Under duality the lattice L becomes $L^\vee = (g^{\mathsf{T}})^{-1}\mathcal{O}_w^r$ for the standard integral bilinear pairing. For L containing the first m vectors, the first m dual coordinates are integral. With $C_m = \mathrm{diag}(I_m, \varpi^{-1} I_{r-m})$, the lattice $C_m^{-1} L^\vee$ is between $\varpi\mathcal{O}_w^r$ and $\mathcal{O}_w^r$, contains those same coordinate vectors, and its colength is $r-m-j$. This is a bijection, by applying duality and the inverse diagonal scaling. It proves

$$T'_{m,j} = \rho(C_m) T_{m,r-m-j},$$

where primes refer to ρ'. In the forward count for ρ' test by the linear map $\Psi = M_{I_1}\rho(C_1^{-1})$.

For every first-row unipotent n_ξ of that count, $C_1^{-1} n_\xi^{\mathsf{T}} C_1$ is in I_1: its off-diagonal first-column entries are multiplied by ϖ. Hence $\Psi\rho'(n_\xi^{-1}) = \Psi$, as required in the same hyperplane count. Use $d = C_0^{-1} C_1$. For $1 \leq i < r$, take the dual colength $j = r-1-i$. The left side of the dual count, after applying Ψ, is $q^{r-1}\Psi\rho'(d) T'_{1,j} = U^\dagger T_i$, since d commutes with C_1 and T_i is I_1-fixed. The two lattice sums on the right give

$$\Psi T'_{0,j+1} = t_i M_{I_1} \rho(d^{-1}) x, \qquad \Psi T'_{1,j+1} = T_{i-1}.$$

The multiplicity remains $q^j = q^{r-1-i}$, so

$$U^\dagger T_i = q^{r-1-i}(q^{-(r-1)} t_i U^\dagger x - T_{i-1}). \tag{SC2}$$

At the end, T_{r-1} is the single lattice of C_1^{-1}, and $t_r x = \rho(\varpi I_r)x$. Thus $t_r \neq 0$,

$$U^\dagger x = q^{r-1} t_r^{-1} T_{r-1}, \qquad U T_{r-1} = q^{r-1} t_r x, \qquad U U^\dagger x = q^{2r-2} x.$$

Write $P_i(X) = \sum_{j=0}^{i} a_j X^{i-j}$. Then $P_r = P$ and $P_{i+1} = X P_i + a_{i+1}$.

Lemma 31.1 (The spherical root relation)**.** *For the spherical vector x, the Hecke operator U, and its polynomials P_i, one has*

$$P_i(U)x = (-1)^i q^{\binom{i}{2}} T_i \quad (0 \le i \le r), \qquad P(U)x = 0.$$

Proof. For $i = 0$ both sides are x. Using (SC1) and the definition of a_{i+1}, the induction step is

$$\begin{aligned} P_{i+1}(U)x &= (-1)^i q^{\binom{i}{2}+i}(t_{i+1}x - T_{i+1}) + a_{i+1}x \\ &= (-1)^{i+1} q^{\binom{i+1}{2}} T_{i+1}. \end{aligned}$$

At $i+1 = r$, the only lattice of colength r between $\varpi\mathcal{O}_w^r$ and $\mathcal{O}_w^r$ is $\varpi\mathcal{O}_w^r$, which omits the first basis vector. Thus $T_r = 0$ and $P(U)x = 0$. □

For $1 \le i \le r$, (SC2) and the endpoint identity give

$$U^\dagger P_i(U)x = a_i q^{-i} U^\dagger x + q^{r-2} P_{i-1}(U)x. \tag{SC3}$$

For $i = r$ both terms on the right cancel, using the formula for $U^\dagger x$. For any scalar c, set $P_{r+1} = XP_r$ and $D_i(X) = (P_i(X) - P_i(c))/(X - c)$ for $0 \le i \le r+1$. Then $D_{i+1} = P_i + cD_i$ for $0 \le i \le r$. By (SC3), induction gives

$$U^\dagger D_{i+1}(U)x = q^{-i} P_i(qc) U^\dagger x + q^{r-2} D_i(U)x, \qquad 0 \le i \le r.$$

The case $i = 0$ is $D_1 = 1$, $D_0 = 0$. In the step use $q^{-i}P_i(qc) = a_i q^{-i} + cq^{-(i-1)}P_{i-1}(qc)$.

If $P(c) = 0$, put $z = D_r(U)x$. Since the constant term of P is nonzero, $c \neq 0$. The divided difference identity and $P(U)x = 0$ give $Uz = cz$ and

$$cU^\dagger z = q^{-r} P(qc) U^\dagger x + q^{r-2} z.$$

Pair this last identity first with $U^\dagger x$ in the first variable. From $UU^\dagger x = q^{2r-2}x$ one obtains

$$c(q^r - 1)B(x, z) = P(qc)B(x, x).$$

Pairing with z in the second variable instead and substituting this equality gives

$$(|c|^2 - q^{r-2})B(z, z) = \frac{|P(qc)|^2}{q^r(q^r - 1)} B(x, x).$$

Since $x \neq 0$, positive definiteness gives $B(x, x) > 0$. If $P(qc) \neq 0$, the first identity gives $z \neq 0$, and the second gives $|c|^2 > q^{r-2}$ since $B(z, z) > 0$.

If $P(qc) = 0$, the first identity gives $B(x, z) = 0$, and the operator identity gives $U^\dagger z = q^{r-2}c^{-1}z$. Therefore $B(U^j x, z) = 0$ for every j, by moving the adjoint. Since z is a polynomial in U applied to x, it is orthogonal to itself, and $z = 0$. Thus when c, qc are both roots, $(P/(X - c))(U)x = 0$. Suppose now $P(X/a)$ is $\prod_{i=0}^{t-1}(X - q^i)\, R(X)$, with $a \neq 0$. Let H be any polynomial divisible by R. At the auxiliary prime indexed by j we use the normalized annihilating polynomial H_j constructed from the complementary factors in Section 29.2. If $t = 1$, $(aU - 1)R(aU) = P(U)$ already annihilates x and hence its cyclic space. For $t > 1$, the minimal polynomial of U on the cyclic space $\mathcal{C} = \mathbb{C}[U]x$ divides $P/(X - q^i/a)$ for every $i < t-1$, by the calculation for adjacent roots.

For $t > 1$ these roots are distinct, so taking the greatest common divisor cancels one copy of each indicated factor from P. In either case $(aU - q^{t-1})R(aU)$ kills $\mathcal{C}$, and so does $p(aU)$ for $p(X) = (X - q^{t-1})H(X)$. The space $\mathcal{C}$ is spanned by $T_0, \dots, T_{r-1}$ and is stable under $U^{\dagger}$ by (SC2). The adjoint of $p(aU)$ applied to x belongs to $\mathcal{C}$ and is orthogonal to it, hence is zero. For every I_1-fixed y, therefore, $B(x, p(aU)y) = 0$. Apply this to $y = L_J(1)v$, with $a = a(\phi_w)$, on a system in the support of the diagonal operator δ for which the displayed factorization holds. Since $L_J(1)v$ is I_1-fixed, $B(1)v = p(aU)L_J(1)v$ is orthogonal to every original spherical vector in this system. Over $\mathbf{C}$ the product of full norms $\mathcal{N}_{\mathrm{all}}$ is $\prod_w |H_w|$ times the orthogonal projection onto the original spherical subspace at all selected places. It follows, system by system, that $\delta \mathcal{N}_{\mathrm{all}} B(1) = 0$ on the raised lattice, as required in Section 31.1.2.

The same calculation also supplies the full root chain from its initial modulus. The adjoint of the j-th spherical double-coset sum is the $(r-j)$-th such sum times $\rho(\varpi^{-1} I_r)$, by inverting the diagonal representative and permuting coordinates. Thus $\overline{t_j} = t_r^{-1} t_{r-j}$, since the original vector is a joint eigenvector and t_r is the unitary central eigenvalue. Substitution into P gives the reciprocal identity, for $z \neq 0$,

$$q^{r(r-1)/2} \overline{P(z)} = (-\bar{z})^r \bar{t}_r \, P(q^{r-1}/\bar{z}).$$

For example, reindex the right-side term of order $r - j$ by j and use $\binom{r}{2} + \binom{j}{2} = \binom{r-j}{2} + (r-1)j$. The roots are therefore stable under $z \mapsto q^{r-1}/\bar{z}$. If $P(c) = 0$ and $|c|^2 q^k = q^r$, this reflection interchanges cq^i and cq^{k-1-i}. For $0 < i \leq (k-1)/2$, the previous root cq^{i-1} has squared modulus $< q^{r-2}$. The endpoint bound forces cq^i to be a root. Induction covers the lower half and reflection covers the upper half.

31.3 The common coefficient of the Hecke polynomial

In this subsection, we construct one polynomial over the integral Hecke algebra of Section 30.3 on the entire finite set of systems. The coefficient of interest is isolated by specializing the common relations on the selected lifts.

Fix the descent pair and a nonzero original lift generator with system π. Identify systems by their spherical eigenvalues at all but finitely many split places, as in Section 28.2. Let $\mathcal{X}_0$ be the systems of the original nonzero lift generators for the chosen twist. At a chosen auxiliary level let Y be the full set of systems occurring there, as in Section 30.3, and put $\mathcal{X} = Y \cap \mathcal{X}_0$. The form defining π remains in its system after raising level, since both realizations agree at all split places outside the old and auxiliary levels. Thus $\pi \in \mathcal{X}$. On this intersection use the block decomposition and twisted lattice of the original lift. Write r' for the main rank and t' for the rank difference. Use the ℓ-adic embedding in the congruence construction and put

$$G = C_G, \qquad v = v_\ell(G), \qquad H = Y_{\mathrm{tr}}, \qquad h = h_\ell, \qquad C = \ell^{Hh}, \qquad 0 \leq h \leq v.$$

Let J be the ordered set of new primes and $\mathrm{ty}\colon J \to \{1, \dots, t' - 1\}$ the monotone type.

The coefficient of $X_J = \prod_{j \in J} X_j$ in the common Ribet polynomial at π is d. The bound for the actual passage to the Hecke algebra is

$$v_\ell(G^2 d) \leq \mathrm{sel} + 2t' \mathrm{ram} + r' \mathrm{loc}_\ell + \mathrm{err}_\ell, \qquad \mathrm{loc}_\ell = 4 f F_\ell, \tag{RB1}$$

where f is the number of infinite places of E, and

$$\mathrm{err}_\ell = t \, v_\ell(X_{\mathrm{odd}}) + 8 f t \, v + 2v + \big((4t+1)H^{22} + H\big)v.$$

For the selected twist η, write ψ_η for the ℓ-adic Galois avatar of η^{-1}. If $\eta = \xi_\kappa^j$ and ψ is the inverse avatar of ξ_κ, then $\psi_\eta = \psi^j$. Here $\mathrm{ram} = \sum_{w\in Q} v_\ell(\psi_\eta(\sigma_w) - 1)$, where Q contains one place of each pair in the twist support and σ_w is the tame generator there. For the main lattice L_0 below, sel bounds the normalized length of the divisible-coefficient Selmer module with $z(\gamma_\pi) = 0$, that is, $\mathrm{length}_{O_0}(\mathcal{S}_0) \le \mathrm{sel}\, v_{O_0}(\ell)$. Fix Q, η and the lattice carrying this Selmer bound before choosing the congruence depth ν and the distinct auxiliary set J for the trace comparison. In the application, Proposition 28.3 gives the bound for each power of the same twist, independently of ν and J.

For the relative traces, c may be any common multiple of the ambient compact stabilizer orders. For the original type projector and the congruence depth of Section 32.3, fix $c = G = C_G$ before selecting the raised level. Reduction at one old split place and block inclusion show that every ambient stabilizer order divides C_G. The product of the block inclusions at $w \in S_M$ also gives $\prod_{w\in S_M} |\mathrm{GL}_{r'}(k_w)| \mid C_G$, so cQ preserves the original lattice. With $v_\ell(\ell) = 1$, (RB1) bounds $v_\ell(c^2 d) = 2v + v_\ell(d)$.

For this raised family take the finite compositum of the fields of definition of the representations N_y, $y \in Y$, the prescribed lattices, characters, congruence polynomials and block identifications. Adjoin the torsion values and all conjugates of the completions of Section 29.1. Use the valuation ring O of its normal closure over $\mathbb{Q}_\ell$ for the entire family. On the distinguished lift π take $L_\pi = L_0 \otimes_{O_0} O$, where L_0 is the main-summand lattice over O_0 carrying the selected Selmer bound. Let $\mathcal{S}_0$ be the divisible-coefficient Selmer cocycle module for L_0, normalized by $z(\gamma_\pi) = 0$. In (RB1) we use

$$\mathrm{sel} = \frac{\mathrm{length}_{O_0} \mathcal{S}_0}{v_{O_0}(\ell)}, \qquad \mathrm{ram} = \sum_{w\in Q} v_\ell(\psi_\eta(\sigma_w) - 1),$$

with the same representatives and tame generators. Replacing the tame generator raises its ℓ-power root of unity to an ℓ-adic unit power and leaves the valuation unchanged. Equation (84) bounds the normalized length for L_π by sel.

For $x \in \mathcal{X}$, let R_x^{main} be the rank-r' summand before the auxiliary twist. For the source attachment R of Section 28.1.1 this summand is θR. On each other lift take its specified twisted lattice. In the semisimple block decomposition add the standard Tate lattices in the prescribed order. Thus the integral model is

$$u_x = T_x \oplus \bigoplus_{e=0}^{t'-1} \xi_e, \qquad T_x = \psi_\eta R_x^{\mathrm{main}}, \quad \xi_e = \chi_{\mathrm{cyc}}^{-e}.$$

For each $y \in Y \setminus \mathcal{X}$ choose any G_E-stable O-lattice in N_y and let u_y be its representation. Such a lattice is the O-span of the translates of an initial lattice. Compactness of the continuous image bounds all translates in one scalar multiple of the initial lattice, so this span is a full finite free O-module and is stable. All these models have the specified rank $n = r' + t'$. If $O_* \subseteq O$ is the valuation ring used for the original systems in the prime selection, restriction of the normalized valuation gives

$$\ell^\nu O \cap O_* = \ell^\nu O_*, \qquad \mathfrak{m}_O \cap O_* = \mathfrak{m}_{O_*}.$$

Hence a ramified enlargement of coefficients after raising the level preserves the original depth-ν congruences and the residual determinant-unit test. The newly occurring systems enter the trace comparison at the chosen level.

The element γ_x acts on T_x by y_x with $y_x - 1 \in O^\times$, and trivially on all ξ_e. The $T_\pi(g)$ span $\mathrm{Mat}_{r'}(O)$. The original local conditions in Section 29.1 hold on every u_y by inclusion in N_y. The auxiliary primes are away from ℓ. At places over ℓ the original type projector and the original determinant conditions are kept on the raised space, so all u_y belong to the same class $\mathcal{Q}_v$, by the inertial and de Rham comparisons there. Since ψ_η is trivial on the full decomposition group at each $v \mid \ell$ by Section 27.2, the integral matrices on the chosen main lattice satisfy $T_\pi(g) = R_\pi^{\mathrm{main}}(g)$ for every $g \in G_{E_v}$. Globally $T_\pi = \psi_\eta R_\pi^{\mathrm{main}}$ on L_π carries the selected Selmer bound.

Use the doubled pairs (τ_j, ϕ_j) and $(\overline{\tau_j}^{-1}, \overline{\phi_j}^{-1})$, with the same variable X_j for the two pairs. The raw rank-n attachments have polarization exponent $1 - n$. The normalizing avatar of Section 28.1 has product with its conjugate $\chi_{\mathrm{cyc}}^{r'}$. Consequently

$$N_y^{c_E} \simeq N_y^\vee \chi_{\mathrm{cyc}}^{1-t'} \qquad (y \in Y).$$

Put $\chi = \chi_{\mathrm{cyc}}^{-1}$ and $g^\sharp = (c_E g c_E^{-1})^{-1}$. On each $y \in Y$, polarization and cyclicity of trace give

$$\operatorname{tr} N_y((b^\sharp)^i g) = \chi(g)^{t'-1} \chi(b^\sharp)^{i(t'-1)} \operatorname{tr} N_y(b^i g^\sharp),$$

since the conjugate inverse of $(b^\sharp)^i g$ is $g^\sharp b^i$. At j, let H_j be the product of the normalized quotients $U/U(1)$ constructed in Section 29.2. Thus $H_j \in O[X]$ and $H_j(1) = 1$, also for the empty product. Write $P = (X-1)H_j = \sum_i P_i X^i$. With $u = (u_y)_{y \in Y}$, use $R_j(g) = \operatorname{tr}(u((\tau_j - 1 - X_j P(\phi_j))g))$ for the tuple of trace relations on Y, and put

$$R_j^\sharp(g) = \operatorname{tr} u((\tau_j^\sharp - 1 - X_j P(\phi_j^\sharp))g).$$

Thus the tame and Frobenius elements are replaced by their conjugate inverses, with the trace tested at the same g. Tame inertia gives $\chi(\tau_j^\sharp) = 1$. The auxiliary Frobenius condition gives $\chi(\phi_j^\sharp) \equiv 1 \bmod \ell^\nu$. Therefore $b_i = (\chi(\phi_j^\sharp)^{i(t'-1)} - 1)/\ell^\nu$ belongs to O, and

$$R_j^\sharp(g)_y = \chi(g)^{t'-1} \left(R_j(g^\sharp)_y - X_j \sum_i P_i b_i \ell^\nu \operatorname{tr} N_y(\phi_j^i g^\sharp) \right).$$

The terms in parentheses belong to the original trace ideal by its ordinary congruences and its ℓ^ν-multiples of traces. The coefficients are common to Y. The same trace identities hold after changing to the integral realizations u_y.

Since the auxiliary Frobenius fixes the ℓ^ν-th roots of unity, its value under each cyclotomic power ξ_e is 1 mod $\ell^\nu O$, also at the conjugate inverse. By integrality and $H_j(1) = 1$, this gives $H_j(\xi_e(\phi)) - 1 \in \ell^\nu O$ on both sides. Take $a = g_{\mathrm{tr}} = \gamma_\ell^{-1}$ from Section 29.2. Its values are $\xi_e(a) = \xi_\ell^e$. The cyclotomic image has finite index in $\mathbb{Z}_\ell^\times$ since $E/\mathbb{Q}$ is finite, so its topological generator has infinite order. The values are pairwise distinct. By Section 29.2, each $\xi_\ell^b - 1$ for $0 < b < t$ divides ℓ^h. Each difference below is a unit multiple of such a factor. Since $t' - 1 \le H$, its product satisfies

$$\prod_{e' \neq e} (\xi_e(a) - \xi_{e'}(a)) \mid C \qquad (0 \le e < t').$$

Use the tame elements δ_w at the places of S_M away from ℓ and at the twist support Q, together with their conjugate inverses, as in Section 31.3.2. The cyclotomic character is unramified at these places, so $\xi_e(\delta_w) = 1$ for every e. The finite types give $(u_y(\delta_w) - 1)o_w(u_y(\delta_w)) = 0$ for

every $y \in Y$. At an old place over ℓ, Section 29.1 uses the inertial polynomials on the potentially crystalline periods. Let s' consist of the test set P_0 of Section 29.4, the reference a, and the two tame generators at each new prime pair. The triangle gives, with n_{test} counting the tests and a,

$$(|J| + n_{\mathrm{test}})Hh \le H^{22}h, \qquad n_{\mathrm{det}} := |s'| \le 2(|J| + n_{\mathrm{test}}).$$

31.3.1 The main-block rows

In this subsection, we use the trace relations above and the congruence of Section 29.2 to select the main-block rows for Section 32.

Put $\Lambda = O[X_j : j \in J]$ and $A = \Lambda[\mathrm{Gal}(\overline{E}/E)]$. Use one finite list g_i, $i \in I$, generating the augmentation module

$$\mathcal{M} = \mathrm{span}_O\{(u_y(g) - 1)_{y \in Y} : g \in G_E\} \subseteq \prod_{y \in Y} \mathrm{Mat}_n(O).$$

This module is finitely generated over the DVR O. Include the old tame elements and s' in the list. Let $z(g) \in O^I$ express $(u_y(g) - 1)_{y \in Y}$ in these generators. Extend each u_y to A by Λ-linearity, and put $I_{\mathrm{aug}} = \ker(A \to \Lambda)$ for the augmentation ideal. Write $(\tau_{j,+}, \phi_{j,+}) = (\tau_j, \phi_j)$ and $(\tau_{j,-}, \phi_{j,-}) = (\tau_j^\sharp, \phi_j^\sharp)$. For $\epsilon \in \{+, -\}$, put

$$b_{j,\epsilon} = \tau_{j,\epsilon} - 1 - X_j H_j(\phi_{j,\epsilon})(\phi_{j,\epsilon} - 1),$$
$$\varepsilon_\pi = (y_\pi - 1)^{-1}(\gamma_\pi - 1), \qquad \rho_0 = \varepsilon_\pi + (1 - \varepsilon_\pi)(a - 1).$$

Define the left ideal $\mathfrak{b} \subset A$ and its module of rows by

$$\mathfrak{b} = \Big(\bigcap_{y \in Y} \ker u_y\Big) + \ell^\nu I_{\mathrm{aug}} + \sum_{j,\epsilon} A b_{j,\epsilon} + I_{\mathrm{aug}} \rho_0,$$
$$\mathcal{N} = \Big\{ r \in A^I : \sum_i r_i (g_i - 1) \in \mathfrak{b} \Big\}.$$

Let ψ_0 be evaluation at $X_j = 0$ followed by the T_π action.

The spanning hypothesis makes ψ_0 onto $\mathrm{Mat}_{r'}(O)$. The zero-variable image of the row $z(\tau) - X_j H_j(\phi) z(\phi)$ is $z(\tau)$ acting by scalar entries. The row

$$(\gamma_\pi - 1)((y_\pi - 1)^{-1} z(\gamma_\pi) + (1 - \varepsilon_\pi) z(a))$$

satisfies the congruences and has image $z(\gamma_\pi)$ with scalar entries, since $\psi_0(\varepsilon_\pi) = 1$. These statements about values are read modulo the common kernel. The same kernel contains the values of the cocycle vectors $z(gg') - z(g) - gz(g')$, the coordinate rows $z(g_i) - \mathbf{e}_i$, and the rows of all linear relations among $(u_y(g) - 1)_{y \in Y}$.

Let $L \subset O^{I \times r'}$ be the span of the individual scalar rows of $\psi_0(\mathcal{N})$. For $w \in (K/O)^{I \times r'}$ annihilating L, $K = \mathrm{Frac}(O)$, put $c_w(g) = \sum_i z_i(g) w_i$. The displayed relations give the cocycle rule for T_π, the simultaneous linear relations, and $c_w(\tau) = c_w(\gamma_\pi) = 0$. Also $c_w(g_i) = w_i$, so this map is injective. Put $a_\nu = \ell^\nu$. The rows $a_\nu \mathbf{e}_i$ belong to $\mathcal{N}$, so $a_\nu w_i = 0$ for every i. Over the fixed DVR O, multiplication by a_ν induces

$$(K/O)[a_\nu] \xrightarrow{\sim} O/a_\nu O, \qquad x + O \longmapsto a_\nu x + a_\nu O.$$

Write $\widetilde{c}_w$ for the corresponding cocycle in $L_\pi/a_\nu L_\pi$. By the simultaneous linear relations its evaluation on the augmentation tuple factors through $\mathcal{M}/a_\nu\mathcal{M}$. In a Smith basis of $W_{\mathrm{mat}} = \prod_{y\in Y}\mathrm{Mat}_n(O)$, write $\mathcal{M} = \bigoplus_i \varpi^{s_i} O e_i$. For $n_0 \geq v_O(a_\nu) + \max(\{0\}\cup\{s_i\})$ one has $\mathcal{M}\cap \varpi^{n_0} W_{\mathrm{mat}} \subseteq a_\nu\mathcal{M}$. Continuity of the tuple of original representations therefore gives continuity of $\widetilde{c}_w$ and c_w.

Sections 29.4 and 31.1 choose the triangle places $J\cup\bar{J}$ off ℓ, the original level, and $Q\cup\bar{Q}$ for the chosen twist. Thus T_π is unramified at both members of every pair. Its cocycle on each inertia group is a continuous additive homomorphism into the finite ℓ-primary torsion module. If $p\neq\ell$ is the residue characteristic, wild inertia is pro-p and maps to zero, as do the prime-to-ℓ factors of tame inertia. The two equations $c_w(\tau_{j,\epsilon}) = 0$ test generators of the remaining pro-ℓ quotient at the respective places. Hence the inertia restrictions at these places vanish. Off the prescribed support all the tuple differences vanish on inertia. Let D_ν be the image of $w\mapsto\widetilde{c}_w$ in the relative cocycles for this tuple. With $A_{\nu,v}$ the reduction of the integral crystalline cocycles, put $D_{\nu,\mathrm{red}} = D_\nu \cap \bigcap_{v|\ell} \mathrm{res}_v^{-1}(A_{\nu,v})$. Restriction induces

$$D_\nu/D_{\nu,\mathrm{red}} \hookrightarrow \bigoplus_{v|\ell} S_{\nu,v}/(S_{\nu,v}\cap A_{\nu,v}).$$

The local bounds in the proof of (85) give $\mathrm{length}_O(D_\nu/D_{\nu,\mathrm{red}}) \leq \sum_{v|\ell}\Delta_v(T_\pi)$. For $z\in D_{\nu,\mathrm{red}}$, choose at each $v\mid\ell$ an integral crystalline cocycle c_v lifting $\mathrm{res}_v z$. Under the inverse coefficient identification

$$L_\pi/a_\nu L_\pi \hookrightarrow L_\pi[1/\ell]/L_\pi, \qquad y + a_\nu L_\pi \longmapsto a_\nu^{-1}y + L_\pi,$$

its local restriction is $a_\nu^{-1}c_v \bmod L_\pi$ and satisfies the homogeneous crystalline relations. The inertia tests and $z(\gamma_\pi) = 0$ are preserved. This gives an injection of $D_{\nu,\mathrm{red}}$ into $\mathcal{S}_O$ on actual cocycles. Since evaluation at the g_i recovers every w_i, the annihilator is isomorphic to D_ν.

At the fixed depth ν and over the coefficient ring O of this raised family, the ordinary O-length of the annihilator is at most

$$b_T = \mathrm{length}_O \mathcal{S}_O + \sum_{v|\ell}\Delta_v(T_\pi),$$

where $\mathcal{S}_O$ is the Selmer module with divisible coefficients and $z(\gamma_\pi) = 0$. For each depth and corresponding raised family, the coefficient extension bound and the main-block estimate of Section 29.1.2 give

$$\frac{b_T}{v_O(\ell)} \leq \mathrm{sel} + 2fr'F_\ell,$$

using $\sum_{v|\ell}[E_v:\mathbf{Q}_\ell] = 2f$. The injection is applied at each such depth. The common twist and its Selmer bound were chosen at the earlier cutoff M. Smith form identifies this annihilator with $\mathrm{Hom}_O(O^{I\times r'}/L, K/O)$. A free summand in the quotient would give a K/O summand of infinite length in the annihilator. For a summand $O/(\varpi^a)$ the two lengths are both a. It follows that L has full rank and colength at most b_T.

Choose an O-basis of L. Each basis vector is an O-linear combination of scalar rows of the matrices $\psi_0(r)$, $r\in\mathcal{N}$. To put it in row c of a block row, multiply each contributing r on the left by a lift of the corresponding matrix unit $E_{c,d}$, where d indexes the chosen scalar row. Sum with the given coefficients. The lifts exist by surjectivity of ψ_0, and $\mathcal{N}$ is a left A-submodule. Group the basis vectors into $|I|$ blocks of r' rows. This gives $B_0\in M_I(A)$ with rows in $\mathcal{N}$ and

$$v_O(\det\psi_0(B_0)) = \mathrm{length}_O(O^{I\times r'}/L) \leq b_T. \tag{RB2}$$

Thus the whole matrix, rather than a multiple of its determinant by a rank-dependent denominator, realizes the cocycle length bound.

The character matrices and the top coefficient. Order the generator coordinates as the disjoint blocks $I_{\mathrm{old}} \sqcup I_{\mathrm{aux}} \sqcup s'$. The first block consists of the old tame tests and the last of the test set. Adjoin a finite list I_{aux} of generators as needed to span the simultaneous augmentation tuples. The coordinate relation $z(g_i)-\mathbf{e}_i$ remains valid if a generator is repeated. Let $\eta_e : A \to \Lambda$ evaluate each group element by ξ_e, with the polynomial variables fixed. We need rows satisfying the congruences $U_i \in \mathcal{N}$, $i \in I_{\mathrm{aux}}$, whose I_{aux}-parts are the respective unit vectors under every η_e.

First fix e. Let P_e be the image of $\mathcal{N}$ under η_e restricted to the I_{aux} coordinates. It contains $\ell^\nu \Lambda^{I_{\mathrm{aux}}}$. Put $B_e = O^{I_{\mathrm{aux}}}/(P_e \cap O^{I_{\mathrm{aux}}})$. For a functional $b : B_e \to O/\ell^\nu O$, extend it by zero on the two other constant coordinate blocks. The cocycle, simultaneous linear, coordinate and normalizer rows have η_e-values in O^I. Their auxiliary parts therefore belong to $P_e \cap O^{I_{\mathrm{aux}}}$, so b kills these rows. In particular $c(g) = b(z(g))$ satisfies the cocycle and simultaneous linear relations. The Smith form argument for the simultaneous tuple module gives continuity. The coordinate relations give zero at all old tame and test points, including both tame generators of each new pair. The vectors $(g-1)((y_\pi-1)^{-1}z(\gamma_\pi)+(1-\varepsilon_\pi)z(a))$ give

$$(\xi_e(g)-1)((y_\pi-1)^{-1}c(\gamma_\pi)+c(a))=0.$$

These are precisely the conditions for the injectivity criterion in Section 29.5, so $c=0$. Evaluating at the coordinate generators gives $b=0$. Since B_e is a finite-length O-module killed by ℓ^ν, finite DVR duality gives $B_e=0$. Thus each unit vector of I_{aux} has a lift $U_{e,i} \in \mathcal{N}$ under this evaluation.

For simultaneous lifting, start with $U_i = 0$. At the next character e replace U_i by

$$U_i + \sum_{j \in I_{\mathrm{aux}}} (\mathbf{1}_{j=i} - (U_i)_j)\, U_{e,j}.$$

The coefficients multiply on the left in A, so the result still satisfies the congruences. Its evaluation on I_{aux} is the unit vector at e. At every previously processed character all coefficients of the correction evaluate to zero. Induction gives U_i as required. This construction is integral.

For each e eliminate the auxiliary block of a row w satisfying the congruences by subtracting $\sum_{i \in I_{\mathrm{aux}}} w_i U_i$ before evaluating. This operation makes the auxiliary part zero at e. It defines a test-coordinate image in $\Lambda^{s'}$, the same image used in the next subsection. For $J_e = \{j : \mathrm{ty}(j) = e\}$, its Fitting calculation gives determinant sums $d_e(X)$ and polynomials Γ_e, ζ_e such that

$$d_e + \zeta_e = \Gamma_e \prod_{j \in J_e} X_j, \qquad \zeta_e \prod_{\mathrm{ty}(j)<e} X_j \in (X_j^2 : j \in J), \qquad v_O(\Gamma_e(0)) \le b_e + (|J_e| + n_{\det}) v_O(C). \tag{RB3}$$

Write each d_e as a weighted sum of test-block determinants. Each is formed from rows that lift to the congruence module after elimination at e. Choose one such matrix B_e^{test} for each e and denote the eliminated lifts by $w_{e,l}$, $l \in s'$. In the common test block use $\sum_e q_e^{\mathrm{sep}} w_{e,l}$, with the separation element

$$q_e^{\mathrm{sep}} = \frac{C}{\prod_{e' \neq e}(\xi_e(a) - \xi_{e'}(a))} \prod_{e' \neq e} (a - \xi_{e'}(a)).$$

The values $\xi_e(a)$ are distinct in $\mathrm{Frac}(O)$. Their difference product divides C in O by Section 29.2. Form the displayed quotient in O, before reducing at depth ν. Thus $q_e^{\mathrm{sep}} \in A$ and

$\eta_{e'}(q_e^{\mathrm{sep}}) = C\mathbf{1}_{e=e'}$, also when two characters have the same reduction. Use the common U_i in the auxiliary block. For $w \in I_{\mathrm{old}}$, use the row supported in that coordinate with entry $o_w(\delta_w)$. The congruences follow since $(\delta_w - 1)o_w(\delta_w)$ annihilates every system. In the order $I_{\mathrm{old}}, I_{\mathrm{aux}}, s'$, evaluation at η_e gives

$$\begin{pmatrix} \mathrm{diag}_{w\in I_{\mathrm{old}}}(o_w(1)) & 0 & 0 \\ * & I_{I_{\mathrm{aux}}} & * \\ * & 0 & CB_e^{\mathrm{test}} \end{pmatrix}.$$

Elimination with the common U_i gives the zero in the lower middle block. Expansion along the old coordinates gives determinant $c_0 \det B_e^{\mathrm{test}}$, where $c_0 = (\prod_w o_w(1))C^{n_{\mathrm{det}}}$. In particular the separation factor is used only on the test block. This establishes the simultaneous specialization for each tuple of determinant summands.

For a fixed tuple of test-block determinants let M be the corresponding simultaneous character matrix. In the integral group algebra A put

$$k_{\mathrm{char}} = \prod_{x\in\mathcal{X}}^{\mathrm{ord}} \frac{y_x - \gamma_x}{y_x - 1}, \qquad B = (1 - k_{\mathrm{char}})B_0 + k_{\mathrm{char}}M.$$

Each denominator is a unit. The factor indexed by x acts by zero on T_x, and every factor acts by one on each ξ_e. The multiplications are on the left of the entries, so the rows of B remain in the left A-module $\mathcal{N}$. The main and character blocks of its evaluations are $T_x(B_0)$ and $\eta_e(M)$, respectively. Use this same B for the trace determinant on the full family Y. Sum over the tuples of chosen test-block determinants with the product of their weights from the individual character sums. The common trace determinant construction applied to these congruence matrices gives a single multiplier f for all $\det(1 - u_x(g))$, with specialization on $\mathcal{X}$

$$f_x(X) = C\, c_0^{t'} D_x(X) \prod_e d_e(X),$$

where D_x is the block determinant. For the fixed M the trace calculation of Section 32.2 uses $q_0^{\mathrm{sep}} k_{\mathrm{char}}$. On each lift this element acts by C on ξ_0 and by zero on the main block and all other characters. Since $\xi_0(\rho_0) = 0$, the trace of this element times the Ribet polynomial evaluated at ρ_0 is C times its constant term. This gives the displayed block product.

Replace the factors d_e in increasing order by $d_e + \zeta_e$. At each replacement the difference is a multiple of $\zeta_e \prod_{\mathrm{ty}(j)<e} X_j$. It is zero modulo $(X_j^2 : j \in J)$. Hence on $\mathcal{X}$ all the coefficients of proper divisors of X_J vanish and

$$d = Cc_0^{t'} D_\pi(0) \prod_e \Gamma_e(0).$$

Write $L_{0,e}$ and $b_{A,e}$ for L_0 and b_A in the Fitting calculation for ξ_e below, and put

$$m_T = \mathrm{length}_O(O^{I\times r'}/L), \qquad m_e = \mathrm{length}_O(O^{s'}/L_{0,e}).$$

Since $\Gamma_e(0) = \varpi^{b_{A,e}} C^{|J_e|} d_T(0)$ with $v_O(d_T(0)) = m_e$ in that calculation, the product gives

$$\begin{aligned} v_O(d) = m_T + \sum_e (b_{A,e} + m_e) + t' \sum_w v_O(o_w(1)) \\ + (1 + t' n_{\mathrm{det}} + |J|) v_O(C). \end{aligned}$$

Here $\sum_e |J_e| = |J|$, including when some J_e is empty. The estimates $m_T \le b_T$ and $b_{A,e} + m_e \le b_e + n_{\mathrm{det}} v_O(C)$ now give

$$v_O(d) \le b_T + \sum_e b_e + t' \left(\sum_w v_O(o_w(1)) + 2n_{\mathrm{det}} v_O(C) \right) + (|J| + 1) v_O(C). \tag{RB4}$$

31.3.2 The old tame relations in the coefficient comparison

In this subsection, we use the tame operators of Section 31.1 to bound the remaining coefficient contributions.

Fix the coefficient prime ℓ, the chosen embedding, and the prescribed twist at this embedding. Recall $C_G = A_{\mathrm{aut}}$ and $v = v_\ell(C_G)$. Use all the places above M prime to ℓ, together with the twist support Q. At an old place above p_i, the normalized upper representations have finite inertia and every inertia eigenvalue has order dividing $p_i^a - 1$ for some $1 \le a \le r$. The conjugate-inverse element has the same bound, by the local types on the conjugate place. The integer $p_i^a - 1$ divides the order of the corresponding general linear group in C_G. Finite order in characteristic zero implies semisimplicity. Thus, on every upper representation and for either choice of inertia element, $X^{C_G} - 1$ annihilates its action. Take $o_w(X) = \sum_{j=0}^{C_G - 1} X^j$, so $o_w(1) = C_G$.

The old local projector is imposed on the whole raised space. At a twist place the prescribed two determinant types give the quadratic annihilator of Section 28.3 on every raised system. Choose the tame generator σ_w used in the ramification bound. For $g = \sigma_w$ or its conjugate inverse the quadratic relation permits $o_g(X) = \psi_\eta(g) - X$. The anticyclotomic identity gives $\psi_\eta((c_E g c_E^{-1})^{-1}) = \psi_\eta(g)$. Hence the two copies give the same valuation $v_\ell(\psi_\eta(\sigma_w) - 1)$. These annihilations pass to all the fixed integral models by their injections into the representations over the coefficient field. There are at most $4f$ old places above the two rational level primes and at most twice that many old inertia elements in the list. Thus on this same family

$$\sum_g v_\ell(o_g(1)) \le 2\mathrm{ram} + 8fv.$$

Places above ℓ omitted here are handled by the crystalline colength bound. The auxiliary triangle uses separate places outside both the old and the twist level.

Divide (RB4) by $v_O(\ell)$. In normalized units,

$$\frac{b_T}{v_O(\ell)} = \mathrm{loss}_T + \mathrm{sel}_T, \qquad \frac{b_e}{v_O(\ell)} = \mathrm{loss}_e + \mathrm{odd}_e.$$

The local defect estimate bounds the normalized sum of all losses by

$$\sum_{v|\ell} [E_v : \mathbf{Q}_\ell](r' + t'/t)F_\ell = 2f(r' + t'/t)F_\ell \le 4fr'F_\ell = r'\mathrm{loc}_\ell.$$

Here $r' \ge 1$ and $t' \le t$. The normalized sel_T is at most sel; the normalized odd_e at most $v_\ell(X_{\mathrm{odd}})$; and the normalized old tame sum at most $2\mathrm{ram} + 8fv$. Finally the two triangle inequalities give

$$(2t' n_{\mathrm{det}} + |J| + 1)Hh \le (4t+1)(|J| + n_{\mathrm{test}})Hh + Hh \le \big((4t+1)H^{22} + H\big)v.$$

Adding $2v$ to (RB4) proves (RB1). The next two subsections establish the character Fitting estimate and construct f on Y. The annihilation of f by the detecting determinants modulo the trace ideal is proved on Y, and the coefficient bound is computed by the block evaluation at π. The quantitative bound uses the local defect bounds of Section 29.1 and the Selmer bound on the prescribed representation.

32 Fitting ideals and trace determinant specialization

In this section, we specialize the common coefficient of Section 31 on the chosen characters. The resulting Fitting inclusion and trace determinant give the finite coefficient used with the norm tests in Section 38.

32.1 The Fitting element for the ordered characters

In this subsection, the specialization on a character summand of Section 31.3 reduces to an inclusion between Fitting ideals. We derive it by eliminating the unit relations and applying the bounds on the paired evaluations of Section 29.5.

Fix the character ξ_e in the common Ribet construction and write $I = J_e$, $s = s'$, $R = O[X_j : j \in J]$, $p = \ell^\nu$ and $F = R^s$. Write $\mathfrak{q} = (X_j^2 : j \in J)$ and $m_{<e} = \prod_{\mathrm{ty}(j)<e} X_j$, with empty product 1. Let $M_e \subset F$ be the image obtained by evaluating the congruence covectors at ξ_e, eliminating the auxiliary coordinates with the unit covectors, and retaining the test coordinates. Put

$$N = M_e + (\mathfrak{q} : m_{<e})F.$$

Write $\Phi : O^{I_{\mathrm{coord}}} \to F$ for the inclusion of constant coordinate covectors followed by this evaluated elimination and projection to the test block. Here I_{coord} is the full list of the common augmentation generators.

Write $y(g) = \Phi(z(g))$. The cocycle vectors show that the forms on $O^{I_{\mathrm{coord}}}$ modulo p which annihilate $\Phi^{-1}N$ give the character cocycles by z. The coordinate relations $z(g_i) - \mathbf{e}_i$ show that z recovers each such form from its cocycle. On the test positions $g \in s$ one has $y(g) \equiv \mathbf{e}_g$ (mod N). The two congruences are $y(\tau_j^{\pm}) \equiv X_j y(\phi_j^{\pm})$ (mod N), because $H_j(\xi_e(\phi_j^{\pm})) \equiv 1$ (mod p) and $pF \subset N$. Indeed the p-multiples of the pure test rows are congruence covectors and have zero auxiliary coordinates. For an earlier $i < j$ with $j \in I$, either $i \in I$ or $\mathrm{ty}(i) < e$; in the latter case $X_i F \subset N$ by the colon ideal.

For each R-submodule L containing N, put

$$A_L = O^{I_{\mathrm{coord}}}/\Phi^{-1}L, \qquad Z_L = \mathrm{Hom}_O(A_L, O/(p)).$$

The finite O-module A_L is killed by p. A functional $\lambda \in Z_L$ gives the cocycle $c_\lambda(g) = \lambda([z(g)])$. The relations at g_i give $c_\lambda(g_i) = \lambda([\mathbf{e}_i])$, so this identifies Z_L with its image in the character cocycles. We use the following determinant operation throughout. For $w \in F$, if $aw \in L$ for $a \in R$, then $a\,\mathrm{Fitt}_0(F/(L+Rw)) \subseteq \mathrm{Fitt}_0(F/L)$. Expand a determinant of covectors in $L + Rw$. Terms with two or more covectors proportional to w are zero, and in each remaining term the factor a can be placed in the w-covector if it is present. The Fitting ideal here is generated by all determinants of $|s|$ covectors, so this proves the inclusion.

For each old tame generator $\delta = g_i$, $\Phi(z(\delta) - \mathbf{e}_i) \in N$ and $\Phi(\mathbf{e}_i) = 0$. Thus $c_\lambda(\delta) = 0$ for every $\lambda \in Z_L$. The simultaneous linear relations make c_λ factor through $\mathcal{M}/p\mathcal{M}$. Continuity follows from the Smith basis of Section 29.1. The row $(g-1)v_0$ lies in $\mathcal{N}$. With $a = g_{\mathrm{tr}}$, its evaluation at ξ_e is the constant row

$$m_g = (\xi_e(g) - 1)\big(c_\pi z(\gamma_\pi) + z(a)\big), \qquad \xi_e(\gamma_\pi) = 1.$$

Eliminating the auxiliary coordinates and retaining the test coordinates gives $\Phi(m_g) \in N \subseteq L$. Applying λ gives $(\xi_e(g)-1)(c_\pi c_\lambda(\gamma_\pi) + c_\lambda(a)) = 0$. In particular the required reference relation

holds for all $\lambda \in Z_L$ and we may take $Z = Z_N$ in Section 29.5. Let $A \subset G$ be the union of the inertia groups at $v \mid \ell$ if $e = 0$, and of $D_v \cap \ker \chi_{\mathrm{cyc}}$ at $v \mid \ell$ if $e > 0$. Write $Z^0 = Z_0$ for the submodule of Z_N that vanishes on A. The O-submodule of A_N spanned by the $z(a)$, $a \in A$, has length $b_A = \mathrm{length}(Z_N/Z^0)$, by finite DVR duality. In fact $O/(p)$ identifies with the p-torsion of K/O, and duality with K/O is exact on finite O-modules. Choose cyclic direct summands of that submodule, and lift their generators to constant vectors w_i in the preimage of the submodule in $O^{I_{\mathrm{coord}}}$. If the summand has length a_i, $\varpi^{a_i}\Phi(w_i) \in N$, where ϖ is a uniformizer. Set $K_1 = N + \sum_i R\Phi(w_i)$. Applying the determinant operation to the successive augmentations gives

$$\varpi^{b_A}\,\mathrm{Fitt}_0(F/K_1) \subseteq \mathrm{Fitt}_0(F/N).$$

The generators $y(a)$, $a \in A$, belong to K_1. Consequently $Z_L \subseteq Z^0$ whenever $L \supset K_1$.

For $j \in I$ set $v_j = y(\phi_j^+) - y(\phi_j^-)$. Starting at K_1, add Rv_j in increasing order of j. Immediately before adding v_j, call the current module L. Every $c \in Z_L$ vanishes on the differences of the earlier tame generators: for $i \in I$ this follows from $v_i \in L$ and the two congruences, and for $i \notin I$ from $X_iF \subset N$. It also vanishes on the earlier Frobenius differences of type e, by $v_i \in L$. The triangle condition therefore gives $C(c(\tau_j^+) - c(\tau_j^-)) = 0$. The finite dual Z_L separates classes of A_L. Applied to the constant vector $C(z(\tau_j^+) - z(\tau_j^-))$, this gives

$$C(y(\tau_j^+) - y(\tau_j^-)) \in L, \qquad CX_jv_j \in L.$$

After all augmentations, let T be the resulting module. The determinant operation gives

$$C^{|I|}\left(\prod_{j\in I} X_j\right)\mathrm{Fitt}_0(F/T) \subseteq \mathrm{Fitt}_0(F/K_1).$$

Let $L_0 \subset O^s$ be the image of T at $X = 0$. It contains pO^s. The evaluation map $v \mapsto \Phi(v)(0)$ surjects onto O^s/L_0, since the images of $z(g)$, $g \in s$, are the test basis modulo L_0. Its kernel modulo L_0 is $\Phi^{-1}(T + (X_j : j \in J)F)$, because L_0 is the image of T at zero. Thus $Z_1 = Z_{T+(X_j:j\in J)F}$ is the dual of O^s/L_0. Every member of Z_1 is in Z^0, vanishes at both tame generators for each j by the two congruences specialized at zero, and vanishes on the Frobenius differences for $j \in I$ by $v_j \in T$.

Thus $Z_1 \subseteq S$ for the module S of Section 29.5, and $Z_1/Z_1[C]$ injects into $S/S[C]$. Our choice $Z = Z_N$ gives $b_A = \mathrm{length}_O(Z/Z_0)$. Equation (87) therefore gives

$$b_A + \mathrm{length}_O(Z_1/Z_1[C]) \le c_{\mathrm{crys}}(O(-e), \nu) + v_{\mathrm{ord}}(X_{\mathrm{odd}}) = b_e.$$

Put $e_O = v_O(\ell)$. The modulus $p = \ell^\nu$ generates $(\varpi^{e_O\nu})$ in O. Since $Z_1 \subset (O/(p))^s$ via the test basis, the contribution of its C-torsion is

$$\mathrm{length}_O\, Z_1[C] \le |s|\min\{v_O(C), e_O\nu\} \le |s|v_O(C).$$

Choose a basis of L_0 and lift its elements to T. Their determinant $d_T \in \mathrm{Fitt}_0(F/T)$ satisfies

$$v_O(d_T(0)) = \mathrm{length}(O^s/L_0) = \mathrm{length}(Z_1) \le b_e - b_A + |s|v_O(C).$$

Take $\Gamma_e = \varpi^{b_A}C^{|I|}d_T$. The two determinant inclusions prove

$$\Gamma_e \prod_{j\in I} X_j \in \mathrm{Fitt}_0(F/N), \qquad v_O(\Gamma_e(0)) \le b_e + (|I| + |s|)v_O(C).$$

In a determinant with rows in $N = M_e + (\mathfrak{q} : m_{<e})F$, every expansion term containing a row in $(\mathfrak{q} : m_{<e})F$ belongs to $(\mathfrak{q} : m_{<e})$. Hence

$$\mathrm{Fitt}_0(F/N) \subseteq \mathrm{Fitt}_0(F/M_e) + (\mathfrak{q} : m_{<e}).$$

Writing $\Gamma_e \prod_{j\in I} X_j = d_e + \zeta_e$ accordingly gives a polynomial combination d_e of determinants from M_e and $m_{<e}\zeta_e \in \mathfrak{q}$, as in (RB3).

32.2 The trace determinant under the Frobenius relations

In this subsection, we use the trace pairing on the algebra of Section 31.3 to obtain a determinant over its coefficient ring. Its annihilation property holds on the full family of representations and permits specialization at the chosen system.

Use the coefficient DVR O of Section 31.3 and the full raised family Y of Section 30.3. The diagonal copy of O preserves the raised lattice. Put

$$\mathbb{T}_O = \mathbb{T} \cap O^Y, \qquad S = \mathbb{T}_O[X_j : j \in J] \hookrightarrow L = \prod_{y\in Y} O[X_j : j \in J].$$

The map records each coefficient at each system. For every $g \in G_E$ the characteristic coefficients of $(u_y(g))_{y\in Y}$ lie in $\mathbb{T}$ by the Hecke comparison and in O^Y by the stable integral models. The local polynomials H_j have coefficients in O. Projection to any $y \in Y$ fixes the diagonal $O[X_j : j \in J]$. In this trace calculation write $G = G_E$, and extend the tuple of representations to $u : S[G] \to \mathrm{Mat}_n(L)$. Define the scalar ideal and its trace annihilator by

$$\mathcal{J} = \big(\ell^\nu \operatorname{tr} u(g) : g \in G\big)_S + \big(R_j(g) : j \in J,\ g \in G\big)_S,$$

$$\mathfrak{a} = \{w \in S[G] : \operatorname{tr}(u(wv)) \in \mathcal{J} \text{ for every } v \in S[G]\}.$$

Explicitly the local relations are

$$R_j(g) = \operatorname{tr} u\big((\tau_j - 1 - X_j P_j(\phi_j))g\big), \qquad P_j(U) = (U-1)H_j(U) \quad (g \in G,\ j \in J).$$

The relations $R_j^\sharp(g)$ also lie in $\mathcal{J}$ by the computation in Section 31.3. The trace condition defining $\mathfrak{a} \subset S[G_E]$ is two-sided by cyclicity of the trace, and it contains $\ker u$. Write $y_i = g_i - 1$, and let $\mathcal{I} = \sum_i S[G]y_i$ be the left ideal they generate.

Let $m = |I_{\mathrm{coord}}|$, and let B be an m-by-m matrix over $S[G]$ with rows given by the congruence covectors. View $y = (g_i - 1)_i$ as a column. The trace relations give

$$By \equiv B_1 y\, \rho_0 \pmod{\mathfrak{a}}$$

for a matrix B_1 over $S[G]$. Use the coefficient inclusion $O[X_j : j \in J] \hookrightarrow S$ to view the rows in $S[G]^m$. To test membership in $\mathfrak{a}$ it suffices by S-linearity to multiply on the right by each $h \in G$. For $p = \ell^\nu$ we have

$$\operatorname{tr} u\big(p(g-1)h\big) = p \operatorname{tr} u(gh) - p \operatorname{tr} u(h) \in \mathcal{J}.$$

The tests on the paired generators are $R_j(h), R_j^\sharp(h)$; the common kernel maps to zero. For any $v = \sum_g c_g(g-1)$ in the augmentation ideal, with finite support, the simultaneous coordinates give

$$v\rho_0 \equiv \sum_i \Big(\sum_g c_g z_i(g)\Big) y_i \rho_0 \pmod{\ker u}.$$

Applied row by row, this constructs the required B_1.

The characteristic coefficients of $u(g)$ lie in S. The same holds for $u(w)$, $w \in S[G]$, and for all flattened block matrices over this image, also with polynomial coefficients. For square matrices $A_1, \dots, A_h$ of the same size over a commutative ring, the primitive-word form of Amitsur's identity [4][63, p. 301, (6)] gives, in a central formal variable t,

$$\det(1 - t\sum_i A_i) = \prod_{[v]\ \text{primitive}} \det(1 - t^{|v|}A_v),$$

where the product is over primitive cyclic words and A_v is the ordered product of the matrices of a representative word.

Only words of length at most k contribute to the coefficient of t^k. This coefficient is an integral polynomial in the characteristic coefficients of the word matrices. Applying it to scalar multiples of $u(g)$ proves the assertion for $u(w)$. For block matrices, split into individual blocks $E_{ij} \otimes u(w_{ij})$. Every product is zero or $E_{ab} \otimes u(w)$. If $a \neq b$, the matrix $1 - tE_{ab} \otimes u(w)$ is block upper unitriangular when a precedes b. Its determinant is one over the coefficient ring. If $a = b$, its only contributing diagonal block is $u(w)$. This proves coefficient integrality in S and in $S[T]$.

Put

$$B_T = B - TB_1, \qquad P(T) = \det(u(B_T)) \in S[T].$$

Flatten $u(B_T)$ to an mn-by-mn matrix $M(T)$; the variable T is central. With $N_B = mn \geq 1$, write $\det(zI - M(T)) = z^{N_B} + \sum_{k=0}^{N_B-1} c_k(T)z^k$ and $c_{N_B}(T) = 1$. Then

$$P(T) = (-1)^{N_B}c_0(T), \qquad \operatorname{adj} M(T) = (-1)^{N_B-1}\sum_{k=1}^{N_B} c_k(T)M(T)^{k-1}.$$

Lifting the right-hand polynomial in the block entries gives a matrix B'_T over $S[G][T]$ such that

$$B'_T B_T \equiv P(T)\mathbf{1} \pmod{\ker u}$$

coefficientwise. View $S[G]/\mathfrak{a}$ as an $S[G][T]$-module by left multiplication from $S[G]$ and right multiplication by ρ_0 for T. The two actions commute by associativity. The covector relation says that B_T annihilates y in this module, so the adjugate relation gives

$$y_i P(\rho_0) \in \mathfrak{a}, \qquad \mathcal{I}P(\rho_0) \subset \mathfrak{a}.$$

Taking $T = 0$ in the definition yields $P(0) = \det(u(B))$.

Use the ordered product k_{char} of Section 31.3. On each lift it acts by zero on the main summand and by one on the characters. The separation element q_0^{sep} acts on ξ_e by $C\mathbf{1}_{e=0}$. Thus $\rho_1 = q_0^{\mathrm{sep}}k_{\mathrm{char}}$ acts on ξ_0 by C and vanishes on all other summands of each lift. Now put $f_B = \operatorname{tr}(u(\rho_1 P(\rho_0)))$. The trace belongs to S, by the characteristic coefficient result applied to the group-algebra element inside the trace. For $w \in \mathcal{I}$ put $b = \det(u(w)) \in S$. The adjugate of $u(w)$ is the image of an element $w' \in S[G]$, again by Cayley–Hamilton with coefficients in S.

Thus $u(w'w) = b\mathbf{1}$, and

$$bf_B = \operatorname{tr}(u(\rho_1\, w'wP(\rho_0))) \in \mathcal{J},$$

since $w'wP(\rho_0) \in \mathfrak{a}$. At a lift system x, one has $\xi_0(\rho_0) = 0$, and $u_x(\rho_1)$ is zero off that character. Write $\mu_0 = \xi_0(\rho_1)$, the value of the character projector multiplied by the first factor of the common multiplier. Hence

$$f_B(x) = \mu_0 P(0)(x) = \mu_0 \det(T_x(B)) \prod_{e=0}^{t'-1} \det(\xi_e(B)).$$

For each $g \in G$ take $w_g = -\sum_i z(g)_i(g_i - 1) \in \mathcal{I}$. Then $u(w_g) = 1 - u(g)$ on the full family, so that $\det(1 - u(g))f_B \in \mathcal{J}$ with the indicated sign for every g. Combining the matrices satisfying the congruences and summing their integral weights gives the common f used above. The argument does not introduce factorial denominators or require changing the coefficient embedding or the system set.

32.3 The highest squarefree coefficient and the original test space

In this subsection, the polynomial of Section 31.3 must be evaluated on the space at the original level. We use the spherical root construction of Section 31.2 to track the top squarefree coefficient under this restriction.

The type projection and the order of the congruence. Write r' for the source rank of this descent pair and $G_c = \prod_{w \in S_M} \mathrm{GL}_{r'}(k_w)$. For the irreducible finite types λ with the prescribed parameters the coefficient of $g \in G_c$ in the projector is

$$|G_c|^{-1} \sum_{\lambda} \dim(\lambda)\ \mathrm{tr}\,\lambda(g^{-1}).$$

Each trace is a sum of roots of unity. The residue action of g is right translation by an integral lift of the first block of g, multiplied by $\prod_{w \in S_M} \psi_w(\det g_w)$. The scalar is a root of unity. Let k be the finite lift and ρ_{W} the Weyl representation in the transported coordinates. For each finite point x the value of its right translate satisfies

$$\rho_{\mathrm{W}}(x_\ell^{-1})y(xk) = \rho_{\mathrm{W}}(k_\ell)\rho_{\mathrm{W}}((xk)_\ell^{-1})y(xk).$$

The first matrix on the right preserves the integral Weyl lattice: k_ℓ and its inverse are integral at every place over ℓ, including the selected split places when $\ell \mid M$. Thus every summand preserves the lattice before the division by $|G_c|$ in the characteristic-zero idempotent.

The lift is supported at the specified old places and their conjugates. The block embeddings in the groups defining C_G show $|G_c| \mid C_G$. Hence $C_G Q y$ is integral for integral y, also when ℓ is an old prime. This conclusion holds in the original level with the type condition imposed: one takes the given level and character space at the old and twist places, its Q-fixed subspace, and the spherical invariants outside those places. Raising auxiliary level changes only the last invariance condition.

Fix the chosen original system x_0. There are finitely many systems at the original level in the fixed type space. For each original system x_k outside the weak lifts, the definition in Section 29.2 gives $g_k \in G_E$ with $\lambda_k = \det(1 - N_{x_k}(g_k)) \neq 0$. These values are integral: the powers of the continuous Galois matrix are bounded by compactness, so its eigenvalues have absolute value at most one. Put $n_0 = \prod_k \lambda_k$, with the empty product equal to one.

The triangle construction first gives its index set I, with a choice of primes and root chains at every positive depth for the weak original systems congruent to x_0. Put

$$n_1 = C_G^2 n_0, \qquad n_2 = 2n! n_1^{|I|+1}.$$

At the fixed embedding, n_2 is a nonzero element of the original coefficient valuation ring. Choose an integer $\nu \geq \max\{1, \lceil v_\ell(n_2) \rceil\}$, where $v_\ell(\ell) = 1$. This choice remains valid on extending that valuation to the raised coefficient field. Only now select the actual depth-ν primes and the

common raised trace algebra. At each raised system use $\beta_k(x) = \det(1 - N_x(g_k))$ with the same g_k. The scalar $\eta = \ell^\nu/(2n!)$ satisfies

$$v_\ell(\eta) = \nu - v_\ell(2n!) \geq (|I| + 1)v_\ell(n_1).$$

In the coefficient DVR O of any resulting raised family this is

$$v_O(\eta/n_1^{|I|+1}) = v_O(\ell)(\nu - v_\ell(2n!) - (|I| + 1)v_\ell(n_1)) \geq 0.$$

Hence $\eta \in n_1^{|I|+1}O$ at the depth fixed from the original systems and their index set.

Use the full raised trace algebra $\mathbb{T}$ acting on its integral lattice L. The raised places are indexed by I. Write $\mathcal{N}_j, \partial_j$ for the local norm and derivative from the local comparison at j, and $\mathcal{N} = \prod_{j\in I} \mathcal{N}_j$. The groups at different places commute, as do their local operator algebras. The original-level space is the fixed space for the product $H = \prod_{j\in I} G_j$, where G_j is the full finite residue group at j. Its integral lattice is L^H: at either level the integral test requires $\rho_{\mathrm{W}}(x_\ell^{-1})y(x)$ to lie in the same Weyl lattice for every finite point x. Passing to the original level adds precisely the H-invariance condition to this pointwise test. Use $c = C_G$ for the common multiple of the ambient compact stabilizer orders.

For the local polynomial H_j of Section 31.3, put $P_j(X) = (X-1)H_j(X)$. At a raised system x and $g \in G_E$,

$$t_{j,g}(x) = \mathrm{tr}((N_x(\tau_j) - 1)N_x(g)),$$
$$s_{j,g}(x) = \mathrm{tr}(P_j(N_x(\phi_j))N_x(g)).$$

Under $J = I$, $X_j = \alpha_j$, extend the scalar ideal from S to $\mathcal{J}_{\mathbb{T}} = \mathcal{J}\,\mathbb{T}[\alpha_j : j \in I]$ by the coefficient inclusion $\mathbb{T}_O \hookrightarrow \mathbb{T}$. Its generators are the traces $\ell^\nu \operatorname{tr} N(g)$ and $t_{j,g} - \alpha_j s_{j,g}$. In this calculation reductions are modulo $\eta = \ell^\nu/(2n!)$ in the coefficient valuation ring. On the full raised lattice, Section 31.1.1 gives

$$\mathcal{N}_j t_{j,g} = 0, \qquad \partial_j t_{j,g} \equiv \mathcal{N}_j s_{j,g} \pmod{\eta}.$$

The local operator in the second relation uses $(X - q_j^{t'-1})H_j(X)$. Since $q_j \equiv 1 \bmod \ell^\nu$, its trace reduces to the one with P_j modulo η. Let $A \in \mathbb{T}$ be the indicator of the residual congruence class of the chosen lift. It belongs to the integral diagonal algebra. To construct it, for each system outside this class choose an integral characteristic-polynomial coefficient separating it modulo the maximal ideal from the chosen lift. Subtract the value at the outside system and divide by the unit value at the chosen lift.

The product T of these factors is zero outside the class and is a unit on the class. The polynomial

$$1 - \prod_{x \text{ in the class}} (1 - X/T(x))$$

evaluated at T gives the indicator. All scalars here are in the coefficient valuation ring. Put

$$\delta_0 = \prod_k (\lambda_k - \beta_k), \qquad n_0 = \prod_k \lambda_k, \qquad n_1 = c^2 n_0, \qquad \delta' = A\delta_0.$$

On every lift system the trivial character ξ_0 gives $\beta_k = 0$ for all k and $\delta_0 = n_0$. At the nonweak original system x_k, the k-th factor in δ_0 is zero. Thus an original system in the support of δ' is weak and residually congruent to x_0. For each $j \in I$ the complementary spherical factor for that system divides H_j by the product construction in Section 29.2. The depth-ν prime choice gives the spherical root chain on every such system.

We construct the relative trace operators and verify the spherical annihilation on these systems. Section 31.1.2 then yields $Z_{j,h}$ for $h \in G_j$ with

$$c\partial_j(\rho(h)-1) \equiv \mathcal{N}_j Z_{j,h} \pmod{\eta}, \qquad \delta'\mathcal{N}Z_{j,h} = 0.$$

For the relative trace on endomorphisms, see [36, §13, pp. 507–508]. To construct the local and full relative traces, take K to be the ambient standard maximal finite compact. Reduction at the original auxiliary primes gives $\pi : K \to H$, where $H = \prod_{j\in I} G_j$ uses the full rank n at each place. Put $K^1 = \ker \pi$. On $\Gamma\backslash G_f/K^1$, with Γ the rational left group, let D be H or a line stabilizer in one factor. The stabilizer at $x = \Gamma g K^1$ is

$$D_x = D \cap \pi(K \cap g^{-1}\Gamma g).$$

Indeed a lift in K of a residue element fixing x differs from a rational stabilizer by an element of K^1. Its order therefore divides the order of the ambient compact rational stabilizer. At an old split place over the odd prime p_i its integral action injects under reduction into $\mathrm{GL}_n(\mathbb{F}_{p_i})$: the principal congruence kernel in $\mathrm{GL}_n(\mathbb{Z}_{p_i})$ is torsion-free. Block inclusion for $n = r' + t' \le rt$ and the definition of C_G in Section 30.1 give the asserted divisibility.

On each D-orbit define an integer-valued weight m_D equal to $c/|D_x|$ at one point and zero elsewhere. Each orbit point occurs $|D_x|$ times under translation by D, so $\sum_{d\in D} m_D(xd^{-1}) = c$. Multiplication by the pullback of this weight commutes with every right K^1-translation. The old and twist compact factors, including conjugate values, lie in K^1 since their places are off the auxiliary support. The multiplier therefore respects the original projection Q, both character equations and the auxiliary principal levels. It preserves L because it is an integer scalar at each point with specified lattice coordinates. For this endomorphism Ψ_D the orbit identity gives

$$\sum_{d\in D} \rho(d^{-1})\Psi_D\rho(d) = c\,1_L.$$

In particular $\mathcal{N}\Psi_H v = cv$ on the original-level invariants. Use D equal to the line stabilizer in the local derivative calculation.

For the spherical annihilation at j, let q_j be the residue cardinality and $a = a(\phi_j)$. Here $t' = n - r' > 0$. The actual operator is $B_j(1) = p_j(aU)L_J(1)$, where $p_j(X) = (X - q_j^{t'-1})H_j(X)$. Its input $L_J(1)y$ is I_1-fixed. The congruence $q_j \equiv 1 \bmod \ell^\nu$ gives $p_j(X) \equiv (X-1)H_j(X) \bmod \ell^\nu$, the polynomial appearing in $s_{j,g}$. Over the characteristic-zero space, if W_x is a cofinite Hecke eigenspace in the raised space, the action of H preserves W_x and

$$\mathcal{N}|_{W_x} = |H|\,\mathrm{pr}_{W_x^H}.$$

Here W_x^H is the original-level space of that system. In particular, the norm kills the systems which have no original-level vectors. The integral operator is the sum of the H-translations. For an original spherical vector x on a system supported by δ', the norm is self-adjoint and acts on x by $|H|$. Therefore

$$\langle x, \mathcal{N}B_j(1)y\rangle = |H|\langle x, B_j(1)y\rangle = 0$$

by the adjoint orthogonality in Section 31.2. The image of $S_j = \mathcal{N}B_j(1)$ lies in the full original-level invariants. For every original-level w, the vector $\delta' w$ is a sum of supported vectors, and hence $\langle \delta' w, S_j y\rangle = 0$. The local operators defining S_j commute with the diagonal trace operator δ'. Applying the orthogonality to $w = S_j y$ and to $S_j(\delta' y)$ gives $\langle \delta' S_j y, S_j \delta' y\rangle = \|\delta' S_j y\|^2 = 0$.

This proves $\delta' \mathcal{N} B_j(1) = 0$ on the characteristic-zero space and on the integral lattice. Together with the local Higman operator it gives the two stated properties of $Z_{j,h}$.

Put $\Delta = c^2 \delta'$ and

$$\widetilde{\mathcal{N}} = \prod_{j \in I} (\mathcal{N}_j + \alpha_j \partial_j), \qquad \widetilde{\mathcal{N}}_S = [\alpha_S] \widetilde{\mathcal{N}}, \qquad \alpha_S = \prod_{j \in S} \alpha_j.$$

In the polynomial operator ring modulo η and $\alpha_j^2 = 0$, the two local trace equations give $\widetilde{\mathcal{N}}(t_{j,g} - \alpha_j s_{j,g}) = 0$. Coefficients in $\mathbb{T}$ commute with the local operators. The remaining generators $\ell^\nu \operatorname{tr} N(g)$ are zero modulo η since $\ell^\nu = (2n!)\eta$. Hence $[\alpha_I](\widetilde{\mathcal{N}} B) \equiv 0 \pmod{\eta}$ for every $B \in \mathcal{J}_{\mathbb{T}}$. The quotient by α_j^2 does not change any squarefree coefficient.

For $i \geq |S|$ there is an integral $Y_{S,i}$ such that

$$\Delta^i \widetilde{\mathcal{N}}_S \equiv Y_{S,i} \mathcal{N} \pmod{\eta}, \qquad Y_{\varnothing,i} = \Delta^i.$$

Indeed the empty case is $\widetilde{\mathcal{N}}_\varnothing = \mathcal{N}$. For $S \neq \varnothing$ put $X = c \delta' \Delta^{i-1} \widetilde{\mathcal{N}}_S$. For a place $j \notin S$, $X(\rho(h) - 1) = 0$ for $h \in G_j$, since $\mathcal{N}_j$ is a factor. For $j \in S$, apply the local derivative congruence at j and commute δ' to the right of $\widetilde{\mathcal{N}}_{S \setminus \{j\}}$. By induction, $X(\rho(h) - 1)$ reduces to $Y_{S \setminus \{j\}, i-1} \mathcal{N} \delta' Z_{j,h} = 0$. Thus $X \rho(h) \equiv X$ for all $h \in H$. Multiplying the full Higman identity on the left by X gives $cX \equiv X \Psi_H \mathcal{N}$. Since $cX = \Delta^i \widetilde{\mathcal{N}}_S$, take $Y_{S,i} = X \Psi_H$ with this order of composition. The Hecke operators ΔD_S in the calculation below commute with $\mathcal{N}$ and act before this multiplier. When $I = \varnothing$ only $Y_{\varnothing,i}$ occurs.

Let $D(\alpha) = f((\alpha_j)_{j \in I})$, where f is the common trace multiplier of Section 32.2 on the same raised space, with distinguished lift $\pi = x_0$. Under $J = I$ and $X_j = \alpha_j$, the relations $R_j(g)$ become $t_{j,g} - \alpha_j s_{j,g}$. For the augmentation element $w_{g_k} \in \mathcal{I}$ constructed there, $u(w_{g_k}) = 1 - u(g_k)$, so $\det u(w_{g_k}) = \beta_k$. The trace determinant gives $\beta_k D \in \mathcal{J}_{\mathbb{T}}$. On a lift system x, the successive replacements $d_e \mapsto d_e + \zeta_e$ give

$$f_x \equiv C c_0^{t'} D_x(0) \prod_e \Gamma_e(0) X_J \pmod{(X_j^2 : j \in J)}.$$

Indeed the e-th difference contains $\zeta_e \prod_{\mathrm{ty}(j) < e} X_j$, which vanishes in the squarefree quotient by (RB3). Thus $D(x) \equiv [\alpha_I] D(x)\, \alpha_I$ in that quotient and $[\alpha_I] D(x_0) = d$. Write $D_S = [\alpha_S] D$, and put $i = |I|$, $b = i + 1$. The trace relation $\beta_k D \in \mathcal{J}_{\mathbb{T}}$ gives $(\delta_0 - n_0) D \in \mathcal{J}_{\mathbb{T}}$ by expanding the product. The first-order lower coefficient vanishing means $D_S = 0$ on the lift systems when $S \neq I$. The original-level choice above gives $2n!\, n_1^b \mid \ell^\nu$ in the coefficient valuation ring before the raised prime selection.

Let l, v be original-level vectors with $d_0 l \in L$ for an integral scalar d_0. Assume $\delta_0 v = n_0 v$, $D_S v = 0$ for $S \neq I$, and $A \delta_0 l = n_0 A v$. In the raised lattice L of the prescribed old type, take

$$G = [\alpha_I](\widetilde{\mathcal{N}} D) = \sum_{S \subseteq I} \widetilde{\mathcal{N}}_{I \setminus S} D_S \in \mathrm{End}_O(L).$$

Indeed only complementary subsets contribute to the squarefree monomial α_I. Each $D_S \in \mathbb{T}$ acts by a scalar on every cofinite Hecke eigenspace, which the auxiliary local operators preserve by Section 30.3. Thus D_S commutes with those operators. Reduction on $L/\eta L$ preserves the coefficient identity. Since $\Delta^b - n_1^b A^b$ is divisible by $\delta_0 - n_0$, the vanishing modulo the congruence ideal and the factorization of $\widetilde{\mathcal{N}}_S$ give the congruence below. In each summand Δ commutes with the local norm coefficients, so put

$$\Delta^b \widetilde{\mathcal{N}}_{I \setminus S} D_S = (\Delta^i \widetilde{\mathcal{N}}_{I \setminus S}) \Delta D_S$$

before substituting the factorization of the parenthesized operator. Thus

$$n_1^b A^b G \equiv \Delta^b G \equiv \sum_{S \subseteq I} Y_{I\setminus S,i} \mathcal{N}\Delta D_S \pmod{\eta}.$$

Apply this congruence to $u = \Psi_H(d_0 l) \in L$. Since $d_0 l$ is fixed by H, $\mathcal{N}u = c d_0 l$. The term for S on the right is

$$n_1 c d_0\, Y_{I\setminus S,i} A D_S v.$$

Only $S = I$ survives. Since $Y_{\varnothing,i} = \Delta^i$, the sum is $n_1^b c d_0 A^b D_I v$. There is therefore a vector $z \in L$ with

$$n_1^b (A^b G u - c d_0 A^b D_I v) = \eta z.$$

Since $n_1 \neq 0$, $\eta/n_1^b \in O$ and $A^b G u \in L$, division in the ambient vector space gives $c d_0 A^b D_I v = A^b G u - (\eta/n_1^b) z \in L_{\mathrm{raised}}$. The functions A, D_I act by scalars on each original summand $V_{\mathrm{orig}} \cap W_x$. Hence this vector belongs to $V_{\mathrm{orig}} \cap L_{\mathrm{raised}} = L_{\mathrm{orig}}$ by Section 30.3. Thus the exact clearing function before the type denominator is

$$s = A^b D_I = A D_I.$$

At the chosen lift system x_0 it has value $D_I(x_0)$.

For the fixed descent pair and twist, let Z be the span at the original level of a collection of good lift eigenvectors. Each generator belongs to one old finite type and one outside-level Hecke system. Several generators may belong to the same joint summand. The distinguished system x_0 is the system of one of these generators. Taking all the generators gives the full good lift space. The old central idempotents split the projected source into its type summands. The commuting spherical operators give the isotypic decomposition there, and the finite-type and spherical intertwining relations of Sections 7.3 and 28 give the stated supports on the lifts. Choose a good ambient finite-type idempotent e_0 whose image contains the types of the selected generators. Write $K_w(X_w)$ for the original spherical Hecke polynomial at the lift root, for each split w outside the descent level and its conjugate. The false-lift subspaces are

$$F_w = \operatorname{ran} e_0 \cap Z^{\perp} \cap \ker K_w(X_w).$$

Let Q be the self-adjoint projection to the fixed lift type used for x_0, with $\operatorname{ran} Q \subseteq \operatorname{ran} e_0$.

On a lift generator of finite type tuple λ, the central idempotent Q acts by one or zero according as λ occurs in its defining sum or does not. Likewise each orthogonal isotypic projection E_x acts on the generator by one or zero. Thus both preserve Z, and self-adjointness gives preservation of $Z^{\perp}$. Moreover Q commutes with each E_x, since the old residue actions commute with the spherical Hecke actions away from the level. For an original integral test $y \perp F_w$ for all w, take $l = Qy$ and $v = Q\pi_Z y$, where π_Z is orthogonal projection. Choose one split place $w \nmid \ell$ off the level, the exceptional places of bases of the original summands, and the excluded places of their Galois comparisons, as in Section 30.3. Write $q = \mathrm{N}w$, let P_x be the monic degree-n spherical polynomial of system x there, and put $\nu_w = \alpha_{\mathrm{nor}}(\mathrm{Fr}_w)^{-1}$ as in Section 28.1.1. On a weak system, $\det(1 - N_x(\mathrm{Fr}_w)) = 0$ and hence $P_x(\nu_w) = 0$. Since $|\nu_w|^2 = q^{r'}$, the spherical reciprocity identity of Section 31.2 gives

$$P_x(q^{n-1}/\overline{\nu_w}) = P_x(q^{t'-1}\nu_w) = 0.$$

Consequently the reciprocal Hecke operator $K_w(X_w)$ acts by $X_w^n P_x(X_w^{-1}) = 0$ on that system at the original spherical level, with $X_w = (q^{t'-1}\nu_w)^{-1}$. The projections E_x, Q, π_Z therefore

commute. For a weak original system x, the vector $u_x = E_x Q(1 - \pi_Z)y$ belongs to F_w at such a place w, since $\operatorname{ran} Q \subseteq \operatorname{ran} e_0$. As $E_x Q(1 - \pi_Z)$ is an orthogonal projection, we obtain $\|u_x\|^2 = \langle y, u_x \rangle = 0$ by the test hypothesis. Hence $l = v$ on the weak original systems. Moreover v is supported on lift systems since Z is spanned by the lift generators and Q preserves each of their isotypic components. There $\delta_0 = n_0$ and $D_S = 0$ for $S \neq I$; on the nonweak original systems $\delta_0 = 0$. This proves all three required relations for l, v.

The integral realization of Q at its base level uses $cQy \in L$ for integral y. Taking $d_0 = c$ gives the following bound.

Proposition 32.1 (Clearing on the original test space)**.** *The operator $c^2 s Q \pi_Z$ is integral on the original integral tests perpendicular to F_w for all w. Its eigenvalue on $Z \cap W_{x_0} \cap \operatorname{ran} Q$, the joint summand containing the selected lift generator, satisfies*

$$v_\ell(c^2 s(x_0)) \leq \mathrm{sel} + 2t'\mathrm{ram} + r'\mathrm{loc}_\ell + \mathrm{err}_\ell.$$

Proof. Take $l = Qy$, $v = Q\pi_Z y$ and $d_0 = c$ in the original-level congruence above. The type denominator gives $d_0 l \in L$; the orthogonality to every F_w gives $A\delta_0 l = n_0 A v$. The lift supports give $\delta_0 v = n_0 v$ and $D_S v = 0$ for $S \neq I$. Hence $c d_0 A^b D_I v = c^2 s Q \pi_Z y$ belongs to L_{orig}. On the indicated joint summand Q and π_Z act as the identity and $A(x_0) = 1$. Its eigenvalue is $c^2 D_I(x_0) = c^2 d$; (RB1) gives the valuation bound with r' the source rank and $t' = n - r'$. □

For $Z = 0$ the determinant is the empty product. Otherwise take a basis $b_1, \ldots, b_d$ from the lift generators, each in a single finite-type summand and a single outside-level Hecke eigenspace. For each b_i use its clearing operator $D_i = c^2 s_i Q_i|_Z$ from the proposition. Write sel_i for the normalized Selmer length in this application. The ranks r', t' and the terms $\mathrm{ram}, \mathrm{loc}_\ell, \mathrm{err}_\ell$ are fixed for this lift image and twist. Put

$$\mathrm{sel}_Z = \max_{1 \leq i \leq d} \mathrm{sel}_i, \qquad B_{\mathrm{tr}} = \mathrm{sel}_Z + 2t'\mathrm{ram} + r'\mathrm{loc}_\ell + \mathrm{err}_\ell.$$

Write $D_i b_k = d_{ik} b_k$. Indeed the type projectors act on a fixed type by zero or one, and the trace coefficients act by scalars on each Hecke system. The selected diagonal coefficient satisfies $v_\ell(d_{kk}) \leq B_{\mathrm{tr}}$.

For each k, choose a nonzero entry a_k of least valuation in the column $(d_{ik})_i$, using the fixed coefficient embedding. The reduction of $\sum_i (d_{ik}/a_k) Y_i$ is a nonzero linear form. Over $\overline{\mathbb{F}}_\ell$ the product of these forms is nonzero at some tuple. Its entries lie in a finite field $\mathbb{F}_{\ell^a}$. Lift zero entries to zero and nonzero entries to their Teichmüller lifts, roots of unity of order dividing $\ell^a - 1$. Transport these coefficients u_i back by the chosen embedding. The individual clearing operators and the column minima are taken at that same embedding. The operator $D_Z = \sum_i u_i D_i$ still clears $\pi_Z y$ for every original integral y orthogonal to all F_w. Its eigenvalue on b_k has valuation $v_\ell(a_k) \leq v_\ell(d_{kk})$, so

$$v_\ell(\det{}_Z D_Z) \leq d B_{\mathrm{tr}}.$$

The application in Section 42.1.3 uses this combination on the common original test space.

Section 38.4 multiplies D_Z by the stabilizer multiple C_G when clearing the class pairing.

Part V Period comparison and the lower determinant bound

33 Period vanishing and the integral comparison lattice

In this section, we use the finite support of Section 18.1 to determine the terms that survive in the comparison periods. The integral first factors and their common scalar give the lattice used in Section 34.

33.1 Vanishing of the comparison periods

In this subsection, the finite support of the data in Section 18.1 forces certain kernels to vanish after integration. We establish the vanishing with the original local characters and measures before using it in Section 34.1.

Fix $1 \le j \le m$ and an allowed twist ϑ. Use the pair $W = W_j, V = I_n$, the coordinate inclusion $\iota_0 : E^r \hookrightarrow E^n$ into the first r rows, and the split sets S_M, Q of chosen and twist places. The places in S_M are the divisors of π_M, and every residue character on the exact support Q is nontrivial. These sets and their conjugates are disjoint in the construction. Let $S = S_M \cup Q$. At $w \in S$ write k_w for its residue field and χ_w for the exact residue character in the descent pair. Fix the false lift Ξ: it lies in the image of the finite-type idempotent, is perpendicular to the image of the projected lift at this level, and is killed by the Hecke polynomial at a split place off $S \cup \bar{S}$.

At a chosen place and a nonzero residue covector λ of length $r = \dim W$, put

$$H_\lambda = \{1 + u\lambda : \det(1 + u\lambda) \neq 0\}, \qquad \mathcal{A}_\lambda = \sum_{s \in H_\lambda} \chi_w(\det s)^{-1} s.$$

In a basis with $\lambda = (1, 0, \dots, 0)$, H_λ is the product of the lower unipotent radical for the $(1, r-1)$ parabolic and the first $\mathrm{GL}_1(k_w)$. The operator $|H_\lambda|^{-1}\mathcal{A}_\lambda$ first projects to unipotent invariants and then to type χ_w on the first GL_1.

For a squarefree Green parameter c_w, the finite Harish–Chandra support is the collection of cuspidal types of the irreducible factors of c_w. Any GL_1 of type χ_w in those unipotent invariants would give, by finite parabolic induction and Frobenius reciprocity, the factor of degree one named by χ_w. This contradicts the condition on the selected parameters: $\chi_w^{q_2} = 1$ and no q_2-torsion root of $k_w^\times$ occurs in the polynomial. The q_2-power condition is for the original χ_w, since the auxiliary twist is unramified at w and hence is trivial on $\mathcal{O}_{E_w}^\times$. Thus the weighted sum kills every selected finite type at w and hence kills Ξ. This is the parabolic product in Green's classification [31, §2, pp. 410–411, and §8, Theorem 14].

Let $\mathcal{D}\Xi$ be the normalized descent with the original block test. First consider it in the Petersson space at the old principal level. Choose $w \mid \pi_M$. In the split Schrödinger coordinates its test is supported on $X \equiv \binom{I_r}{0} \pmod{w}$. The top minor is a unit, so $X : E_w^r \to E_w^n$ is injective throughout the support. Changing the source or target point at w multiplies X on the right or left by an invertible matrix and preserves this injectivity, including at every representative of the definite target classes.

Fix such a representative c, its real polynomial P, and write $H = \mathrm{ht}_W(g)$. The source-variable estimate of Section 34.3.1 starts from $|L_{g,c}(P)| \ll_c H^{A_0} s(P)$ for the theta evaluation

and a Schwartz seminorm s. For $H > 2$ take a shortest isotropic source vector v. In the moved finite coordinates the modulation detected by a definite target basis is nonzero, since $Xv_w \neq 0$. Thus the isotropic average annihilates the evaluation for every real test. Its real displacements are $O_c(H^{-1})$. Iterating the difference from the average $A_0 + B$ times as in that section gives, for every integer $B \geq 0$,

$$|L_{g,c}(P)| \ll_{c,B} H^{-B} s_B(P). \tag{93}$$

On bounded heights the initial estimate suffices. The class sum is finite, so the descent is rapid and square integrable. Its real type is the one computed in Section 33.1.1, using the transposed coefficient projection of Section 34.3.2. At the old places the source principal level fixes the inclusion test. At a twist place the character of the invertible top block cancels the inverse determinant character on the integral source group, and away from these places the vacuum is fixed.

The pairing of $\mathcal{D}\Xi$ against any source form y, up to the fixed nonzero normalizing scalar, is the pairing of Θy with Ξ. Write e_{src} and $e_{\mathrm{tar}} = e_0$ for the actions of the same central idempotent of $\prod_{w \in S_M} \mathrm{GL}_r(k_w)$ on the source and target. For a tuple s lift s_w on the source and $\mathrm{diag}(s_w, I_{n-r})$ on the target. Their simultaneous action fixes the reduction of the block test, since $\mathrm{diag}(s_w, I_{n-r})\iota_0 = \iota_0 s_w$. The simultaneous Weil multiplier on this test is $\prod_{w \in S_M} \chi_w(\det s_w)^{-1}$, as in the split calculation of Section 38.2. Write $\rho_{\mathrm{tar}}(s)$ for the target translation multiplied by $\prod_{w \in S_M} \chi_w(\det s_w)$. The twist is trivial on the unit groups at the old places. At each $w \in Q$, the character of the invertible block cancels the inverse character of the integral source translation. Thus the simultaneous change of variables gives $\rho_{\mathrm{tar}}(s)\Theta = \Theta \rho_{\mathrm{src}}(s)$. For an integral source lift k of s, apply the same change of variables against compactly supported smooth source tests and take adjoints. The two residue actions are unitary, hence

$$(\mathcal{D}\Xi)(gk) = \mathcal{D}\big(\rho_{\mathrm{tar}}(s)\Xi\big)(g).$$

The equality of these continuous functions follows from the equality of the pairings against the smooth tests. Summing with the central idempotent coefficients gives $e_{\mathrm{tar}}\Theta = \Theta e_{\mathrm{src}}$. Adjointness and self-adjointness of the two projectors give

$$e_{\mathrm{src}}\mathcal{D}\Xi = \mathcal{D}e_{\mathrm{tar}}\Xi = \mathcal{D}\Xi \qquad \text{in } L^2([U(W)]).$$

Each summand retained by e_{src} has a primitive residue type at a split old place, by Section 14.2. To check cuspidality, take a proper rational parabolic with unipotent radical N. At this place use Iwasawa decomposition in computing the constant term at any translate. Right invariance of the integral on $[N]$ allows insertion of the probability average over a compact conjugate of $N(F_{0,w}) \cap K_w$ on the local vector, with the parabolic in standard position. Its reduction is the unipotent radical of a proper residue parabolic and its average on the cuspidal residue type is zero. Hence the constant term vanishes for every selected summand. Together with the real-line and level conditions above this puts $\mathcal{D}\Xi$ in the original projected source. Taking $y = \mathcal{D}\Xi$ in the adjunction, the pairing $\langle \Theta y, \Xi \rangle$ vanishes by $\Xi \perp Z$. Thus $\|\mathcal{D}\Xi\|_2^2 = 0$. The descent is continuous at every source point. Indeed, over a compact set of source points and compact representatives of the definite target quotient, the finite translates of its test form a finite set. Their rational support lies in a fixed fractional lattice. The real Schwartz seminorms of the translated test are uniformly bounded there, so a summable inverse power on that lattice dominates the kernel. Dominated convergence gives continuity after pairing with Ξ. The quotient Haar measure has full support, hence $\mathcal{D}\Xi(g) = 0$ for every g. At the original block finite datum the kernel at the harmonic highest vector is this normalized descent up to conjugation and a nonzero scalar.

33.1.1 The real type in the bottom descent

In this subsection, we determine the type for the descent used in Section 34.1.

At a place τ, order the entries of the source Harish–Chandra parameter as $\varsigma_0 > \cdots > \varsigma_{r-1}$. Write $a = k + j - 1$, $b_- = m - j$, $n = r + t$, and $w = w_j(\tau)$. Then $a + b_- = r$. The matrices for the positive source sector use the first a source coordinates in decreasing order. Those for the negative sector use the last b_- in increasing order. The two families of column counts are

$$k^+(h) = \big(\varsigma_{h-1} - \max(\varsigma_h, (t-1)/2 - w) - 1\big)_+,$$
$$k^-(h) = \big(\min(\varsigma_{r-h-1}, -w - (t-1)/2) - \varsigma_{r-h} - 1\big)_+.$$

Here one uses the first formula only through height a and the second only through height b_-. In these formulas use $\varsigma_r = -\infty$ and $\varsigma_{-1} = +\infty$ at a missing endpoint. Write $\mu_\tau^+(h) = k^+(h)$ and $\mu_\tau^-(h) = k^-(h)$, extended by zero at height zero and beyond the corresponding sign block. The inequality $\varsigma_i + w \geq \Delta/2$ for $i < a$, and $\varsigma_i + w \leq -\Delta/2$ for $i \geq a$, makes all entries beyond these heights zero.

In the compact target the column multiplicity at height h is $k^+(h) + k^-(n-h)$, and the determinant exponent is $-\sum_h k^-(h)$. The differences of consecutive twice Harish–Chandra parameters are twice these column multiplicities plus two. At position a, the determinant exponent cancels the whole negative contribution. It follows that after subtracting $2u - 1$, where $u = b_- + w - \lfloor r/2 \rfloor$, the compact twice parameter is the doubled source list with the inserted block

$$-2w + t - 1,\ -2w + t - 3,\ \ldots,\ -2w - t + 1$$

between the two sectors. This is the regular discrete-series theta correspondence with one inserted compact block, after the known determinant shifts of the Weil action. In Harish–Chandra units the retained entries are $\eta_i = \varsigma_i + w$ and the inserted entries are $\gamma_b = (t-1)/2 - b$, $0 \leq b < t = n - r$. Thus

$$\min_i |\eta_i| \geq \Delta/2 \geq (t+1)/2, \qquad \min_{i,b} |\eta_i - \gamma_b| \geq (\Delta - t + 1)/2 > 500(r+n)^3$$

for sufficiently large x. The positive source sector has a positive entries and the negative sector has b_- negative entries. They all give positive target signature, and the inserted string contributes t further positive directions. Hence the target is $U(a + b_- + t, 0) = U(n, 0)$. Use Li's regular discrete-series correspondence [44] in the formulation of [61, Theorem 2.7(b), p. 641]. Here r is odd and $n = r + t = 6r$ is even, so the genuine Harish–Chandra lattice in this normalization is $\mathbb{Z} + (r + n + 1)/2 = \mathbb{Z}$. The entries η_i are integers since both ς_i and w are integral, and the inserted entries γ_b are integral because t is odd. The four counts are $(a, 0, 0, b_-)$, the rank difference is t, and the first signature case is $(a + b_- + t, 0)$. It gives a nonzero theta lift with compact parameter

$$(\eta_0, \ldots, \eta_{a-1}, \gamma_0, \ldots, \gamma_{t-1}, \eta_a, \ldots, \eta_{r-1}).$$

The last string is empty if $b_- = 0$. Restoring the determinant shifts gives the compact representation computed above. For this regular range see also the proof of [39, Lemma 8.2].

The polynomial is the product of powers of leading minors in the two sectors. For the positive sector use the first target rows in the prescribed order and the positive source columns just specified. For the negative sector use the last target rows in reverse order and the negative source columns in their prescribed order. The two sets of target rows are disjoint since $a + b_- = r < n$.

Each leading minor is a nonzero polynomial. A raising operator annihilates it or produces a determinant with two equal rows or columns. A mixed Laplacian differentiates one positive and one negative variable with the same target row. Their row sets are disjoint, so all these contractions vanish. The product is therefore jointly harmonic and highest for the two compact orders. Its target weight is the list of positive row lengths followed, after the zero coordinates, by the negatives of the reversed negative row lengths. The splitting character cancels the vacuum determinant shift.

It is exactly the compact representation with column multiplicities and determinant exponent described above. The polynomial is the image of its highest vector by the Cartan product, since evaluating both highest vectors on the ordered coordinate arrays gives these same leading minors. Thus projection to that compact target highest line is nonzero. Write τ for this compact type and M_τ for its multiplicity space. On $\tau \widehat{\otimes} M_\tau$ the compact group generates $\mathcal{B}(\tau) \otimes 1$. Its real dual partner generates the commutant $1 \otimes \mathcal{B}(M_\tau)$ by [38, Theorem 6.1, p. 550]. Projection onto the chosen highest line in τ therefore gives one irreducible source representation. This proves the occurrence used in the compact bump argument.

The local oscillator correspondence expresses the lift with any polynomial Fock vector at that same finite datum as a real convolution of the harmonic lift. The normalized descent therefore vanishes there for every polynomial Fock vector. For this assertion let $K = U(n)$ at one real place, with probability Haar measure, and let v be the unit highest vector in the compact type τ of the harmonic. Set $c(k) = \langle v, \tau(k) v \rangle$, $d_\tau = \dim \tau$. Schur orthogonality gives the matrix unit

$$p = \int_K d_\tau \overline{c(k)}\, \omega(1,k)\, dk$$

on the oscillator. The conjugate scalar coefficient y of Ξ satisfies $R(d_\tau c) y = y$. In the bilinear descent, changing h to hk^{-1} replaces $c(k)$ by $c(k^{-1}) = \overline{c(k)}$; hence the descent of Q equals the descent of pQ. Take the product over the real places.

The projected joint harmonic is the original highest vector. Its orbit under the smaller real group has dense linear span in the matched irreducible theta representation. A polynomial vector projected to it is compact-type finite. For its finitely many compact types the image of compactly supported smooth bump averages of the projected harmonic is dense and has finite dimension after the compact-type projection. It is therefore the entire finite-type subspace.

The compact-type projection can be absorbed in the smooth bump by averaging over the real maximal compact. This expresses the projected polynomial exactly as one compact bump average. Moving that average to the source variable gives the claimed convolution. The finite vector stays fixed and the compact support justifies the interchange on the definite target quotient. Trivializing the second character of the kernel multiplies the descent by a character of $\det g$, so leaves its vanishing intact.

Here is the extension from the block datum to every finite test off the original vacuum places. Fix a real polynomial and a source point g. Call the corresponding descent pairing $B(\Psi, \Xi)$, linear in Ψ and conjugate-linear in Ξ. At $w \in S$, write a matrix as $z = (z', z'')^{\mathsf{t}}$ in the blocks, with z' an r-by-r block. Say that the block ratio is allowed at w if z' is invertible over the local field and $z'' z'^{-1}$ is integral, with zero reduction at $w \in S_M$. The block function Ψ_{bl} on integral matrices is the indicator of the inclusion modulo $w \in S_M$, multiplied at $w \in Q$ by $\chi_w(\det z')$, extended by zero at singular reductions. It has the original vacuum at the other places. The conjugate place is prescribed by the split unitary identification.

Suppose first that z has allowed block ratio at every place of S. Take the finite source element A whose component at each $w \in S$ is z'^{-1} and which is the identity off $S \cup \bar{S}$. Then

$\Psi_0(x) = \Psi_{\mathrm{bl}}(xA)$ takes the value one at z. A source translation shows $B(\Psi_0, \Xi)$ is the original block descent at gA times an invertible Weil scalar. Thus it is zero.

The nonzero support of Ψ_0 is one orbit of the target level, with its character action, in the S-coordinates. Indeed after right multiplication by A, a matrix in the support extends to an integral invertible block matrix with that matrix as its first columns. At S_M these columns are the coordinate columns modulo the residue place, and at Q the completed matrix is block lower triangular with the indicated first block. These are exactly the two level conditions. On this orbit all equivariant functions are determined by their value at the original inclusion multiplied by A^{-1}.

Let $C = \prod_{w \in S}(z_w + \varpi_w^{a_w} M_{n,r}(\mathcal{O}_w))$. First suppose that the ratio is allowed at every point of C. Then every possible first block $z' + \varpi_w^{a_w} B$ is invertible. In particular every entry of $\varpi_w^{a_w} z'^{-1}$ lies in the maximal ideal. Otherwise, for an entry u of modulus at least one, take B to have the single transposed entry $-u^{-1}$. The rank-one determinant formula then gives determinant zero. It follows that $x'z'^{-1} \equiv I \bmod \varpi_w$ and $x''z'^{-1} \equiv z''z'^{-1} \bmod \varpi_w$ for every $x \in C$. Thus C is contained in the orbit supporting Ψ_0. Average $\mathbf{1}_C$ over this same compact target level, including the conjugate of its level character. The translates have finite span and factor through a finite quotient. Equivariance of the Weil action and covariance of Ξ give the same value of $B(\cdot, \Xi)$ before and after averaging. The averaged function is supported on this orbit. The average is therefore a scalar multiple of Ψ_0. It follows that $B(\mathbf{1}_C, \Xi) = 0$.

Otherwise choose a point $z \in C$ where the ratio fails at a place w, and use that point as the center of the same ball. There exists an integral covector $l = (\lambda, \lambda')$ with $lz = 0$, $\bar{\lambda} \neq 0$, and $\bar{\lambda}' = 0$ when $w \in Q$. If z' is singular take an integral multiple of a nonzero covector of its left kernel, normalized to have a unit entry. Otherwise use a covector of $(z''z'^{-1}, -I)$ whose first part has a coefficient of modulus at least one at S_M, or strictly greater than one at Q.

Divide by an entry of largest modulus in that first part. This gives exactly the asserted integrality and residue conditions. For each $s = 1 + \bar{u}\bar{\lambda}$ in $H_{\bar{\lambda}}$, lift $\bar{u}$ to an integral column u and take the target shear

$$h_s = 1 + \iota_0(u) l.$$

It is integral invertible since its determinant reduces to $\det s$, and fixes z. Extend it by the identity at the other split places with the corresponding conjugate components.

Since h_s fixes z and preserves $M_{n,r}(\mathcal{O}_w)$, it fixes each such additive ball. Its level decomposition is upper block triangular at a chosen place, and at a twist place the reduction of the off-diagonal top-right block is zero. By that decomposition, its action on Ξ is $\chi_w(\det s)^{-1}$ times the action of s at a chosen place, and just $\chi_w(\det s)^{-1}$ at a twist place.

The split Weil action of h_s on the target variable preserves $\mathbf{1}_C$. Its determinant is a unit, and the rank-twisted source splitting character is unramified on this unit group. Thus the determinant multiplier as well as the modulus factor equals one. Changing the target integration variable gives $B(\mathbf{1}_C, h_s \Xi) = B(\mathbf{1}_C, \Xi)$. At a chosen place the sum of these actions kills Ξ by $\mathcal{A}_{\bar{\lambda}}$. At a twist place it also kills Ξ, since $\det : H_{\bar{\lambda}} \to k_w^{\times}$ is onto with equal fibers, and $\chi_w \neq 1$ on its exact support. Adding the pairings therefore gives $|H_{\bar{\lambda}}| B(\mathbf{1}_C, \Xi) = 0$.

Every compactly supported locally constant test on the finite split coordinates is a finite linear combination of these cosets, each of which is covered by one of the two cases. This proves the bottom vanishing for all finite Schwartz–Bruhat factors with the prescribed vacuum off $S \cup \bar{S}$. Fix one of these finite tests f_f, let u be the real Schrödinger vector, and let C be a

compact set of source points. The same lattice majorant justifies

$$\sup_{g\in C}|B_g(u\otimes f_f,\Xi)|\le C_{C,f_f}\ \mathrm{vol}([U(V)])\|\Xi\|_\infty p_j(u)$$

for a fixed real Schwartz seminorm p_j. In fact the translates on the two compact sets are bounded by a higher seminorm of u; summing the resulting inverse power costs a constant independent of u. For any real Schwartz datum, the Taylor polynomials times the Gaussian in Section 34.3.2 converge to its Fock vector in every weighted absolute moment. The coherent-state bound of Section 34.3.1 bounds p_j of the inverse Bargmann transform by such a moment. Passing to this limit proves vanishing for that finite test. Every Schwartz–Bruhat datum with the prescribed off-level vacuum is a finite sum of such tensors: on the remaining finite support it is constant on the cosets of a common compact open additive subgroup. This establishes the all-datum bottom vanishing used in the tower.

33.1.2 Extension through the hyperbolic planes

In this subsection, we extend the type of Section 33.1.1 through the original tower.

Keep the original pair, level, support S and false lift Ξ of the bottom-descent argument above. Write $V=I_n=I_{d_1}\oplus I_{d_2}$ for the prescribed gluing blocks, with $2r+2d_1+3<n$, and let $\mathbb{H}^l$ be the standard hyperbolic summand of rank $2l$. In the rank-r chain $n=r+t$, $t=5r$, and $d_1=r+9m+1$. The displayed inequality is then $4r+18m+5<6r$, which follows from the original bound $100m+9<k$ and $r+1=k+m$.

Let Y denote the conjugate of the matrix coefficient of Ξ at its original highest vector. The chosen split place of the false-lift test is w with norm q, off $S\cup\bar{S}$. Conjugation of this scalar coefficient commutes with right translation and conjugates the spectral argument of the Hecke polynomial. Thus Y is spherical at w and $\mathcal{P}_{n,w}(X)Y=0$, where

$$|X|=q^{1-n+r/2},\qquad \mathcal{P}_{a,w}(X)=\sum_i(-X)^i q^{i(i-1)/2}T_{w,i}. \tag{F1}$$

The $T_{w,i}$ are the original right-coset sums, using compact mass one.

With $t'=n-r$, the Frobenius value of the original source character in Section 28.1.1 is

$$\nu_w=\alpha_{\mathrm{nor}}(\mathrm{Fr}_w)^{-1}=q^{r/2}\chi_W(\varpi_w/\bar{\varpi}_w)\upsilon(\varpi_w)^{-r}.$$

The auxiliary twist enters ν'_w. The last factor of the complementary string gives $X_w=(q^{t'-1}\nu_w)^{-1}$, the parameter in Proposition 32.1. The right-coset sums have real coefficients. On the conjugate matrix coefficient their polynomial in (F1) therefore vanishes at $X=\overline{X_w}$, of the stated modulus.

First suppose that the descents to $W\oplus\mathbb{H}^{l-1}$ vanish for all data vacuum off $S\cup\bar{S}$, for the source character $a\upsilon^{l-1}$, where υ denotes the norm-one restriction and $a=\chi_W=\upsilon^{r_-}$ for the negative index r_- of W. Set $W_l=W\oplus\mathbb{H}^l$, $r_l=r+2l$. Split off the last plane, of Gram matrix $J_{\mathbb{H}}=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. For its finite adelic coordinates take the fixed integral trace-one element $e=c_{\mathrm{tr}}$ and a unit c of norm -1 by Section 4.3. The matrix

$$B_f=\begin{pmatrix}1&-c\\ e&c\bar{e}\end{pmatrix},\qquad B_f^*J_{\mathbb{H}}B_f=I_2,\qquad \det B_f=c$$

identifies the hyperbolic lattice with the original finite unitary lattice, including at dyadic places. At infinity use $2^{-1/2}\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$, which is orthogonal and carries $J_{\mathbb{H}}$ to $\mathrm{diag}(1,-1)$. Removing successive planes only reorders these blocks and leaves the summand W fixed.

Write $g = n_0(u, g_0)k$ in Iwasawa form. At a split place the integral compact moves an isotropic line to the specified line by extending its primitive vector inside the kernel of its primitive dual covector to a basis. At a nonsplit finite place the extension is unramified. For a primitive isotropic vector take an integral dual vector with pairing one, then subtract a multiple of the isotropic vector to make the dual vector isotropic, using the surjective integral trace. These two vectors split off a unimodular plane. Its orthogonal complement admits an integral orthonormal basis: lift a vector of nonzero norm over the quadratic residue field, normalize by surjectivity of the unit norm, and repeat. A nondegenerate residue Hermitian form has such a vector even in characteristic two, by nondegeneracy of the residue trace pairing. These local choices are integral at almost all places and give the adelic decomposition. Let θ_0 denote the isotropic zero slice of the theta functional. The compact factor carries the vacuum subspace at every place off $S \cup \bar{S}$ to itself, and the radical n_0 acts as the identity on the zero slice.

The slice of the Levi action is the Weil action of the smaller pair at g_0, times

$$\big((1)_n(\det u)\big)^{-1} |\det u|_{\mathbb{A}_E}^{n/2}. \tag{F2}$$

Here $(1)_n$ is the rank-n space character associated to the trivial target automorphic character. Write $b_d(z) = b(z/\bar{z})\upsilon_E(z)^{-d}$ for the splitting character of a norm-one character b at rank d. Since $\upsilon_E(\bar{z}) = \upsilon_E(z)^{-1}$, its restriction υ satisfies $\upsilon(z/\bar{z}) = \upsilon_E(z)^2$. Hence

$$(a\upsilon^l)_{r+2l}(z) = (a\upsilon^{l-1})_{r+2l-2}(z) = a_r(z).$$

Let s_0 denote evaluation at zero in the mixed coordinate of the removed plane and let ω_j denote the Weil action for $(W \oplus \mathbb{H}^j, V)$ with characters $(a\upsilon^j, 1)$. For $g = n_0(u, g_0)k$, the radical has zero Heisenberg displacement and trivial quadratic phase at this coordinate. The Levi substitutes $x \mapsto h^{-1}xu$ and contributes (F2). Consequently

$$s_0\big(\omega_l(g,h)\phi\big) = (1)_n(\det u)^{-1} |\det u|_{\mathbb{A}_E}^{n/2}\, \omega_{l-1}(g_0,h)\, s_0\big(\omega_l(k,1)\phi\big).$$

At each finite place off $S \cup \bar{S}$, the integral compact preserves the vacuum line. In the integral plane frame the mixed vacuum is the integral-box indicator times the smaller vacuum, up to the fixed nonzero normalization. Evaluation at zero therefore preserves the prescribed vacuum projection. The scalar is independent of h, so the integral of this term against Y vanishes by the induction hypothesis on the same datum.

The remainder gives a bounded descent on $U(W_l)$. Here are the analytic details. Write $\Lambda = \theta - \theta_0$ and denote by k_0 the rational isotropic column of the plane. The zero slice is the probability mean of the theta functional along the adelic isotropic translations

$$Z = \{zk_0^* J : z \in E^{n\times 1}\} \subset \mathrm{Hom}_E(W_l, V).$$

Each mixed-model character indexed by a rational vector is trivial on $Z(E)$, since the global additive character is trivial on rational arguments. On $Z(\mathbb{A})/Z(E)$ the zero index gives the constant character, and every nonzero index gives a nontrivial character by the nondegenerate trace pairing. Their probability means are therefore one and zero, respectively. Thus Λ is invariant under the rational translations $Z(E)$ and has probability mean zero on every datum.

33.1.3 The mixed-model height bound

In this subsection, we bound the mixed theta sum used in the vanishing argument after Section 33.1.2.

Let δ be the first Euclidean minimum of the moved integral source lattice, with its original positive Euclidean archimedean metric.

Put $H = \max(1, \delta^{-1})$. If $\delta < 1/4$, the E-span of the vectors of length $< 4\delta$ is totally isotropic. Indeed, the pairings of such vectors are integral at every finite place and have absolute value < 1 at every embedding, so vanish by the product formula. At each infinite place τ, with sign matrix J_τ, the completion I_τ of this isotropic subspace and $J_\tau I_\tau$ are Euclidean orthogonal and dual for the signed pairing. At every τ simultaneously, in the Euclidean splitting into $I_\tau, J_\tau I_\tau$, and their common orthogonal complement, multiply the first factor by two, the second by $1/2$, and the third by one.

This is a signed unitary map of norm at most two. It doubles the lengths of the short vectors and decreases the others by at most a factor of two. The new first minimum is therefore at least 2δ. Iteration of the inverse moves on the adelic point puts its minimum above $1/4$, using a purely real product of norm $O(H)$. For the resulting point, the lattices in the Euclidean coordinates have fixed covolume, since g is unitary, and first minimum at least $1/4$. Minkowski's second theorem bounds their last minimum. Choose a full set of independent lattice vectors of bounded length. They span a sublattice of bounded index by Hadamard's inequality. Its possible enlargements, in the coordinates of these vectors, lie between $\mathbb{Z}^N$ and $\frac{1}{D}\mathbb{Z}^N$, where N is the real dimension and D is a common multiple of the possible indices. There are finitely many such enlargements. Choosing a basis in each gives a uniformly bounded $\mathbb{Z}$-basis of the moved lattice, with uniformly bounded inverse basis matrix since its covolume is fixed.

In the corresponding rational basis, the matrices for multiplication by a fixed integral basis of $\mathcal{O}_E$ have bounded integer entries. The signed Hermitian pairings of the basis vectors lie in the fixed fractional ideal of pairing values and are bounded at every embedding, so they also have only finitely many possible values. Two bases with the same multiplication matrices and pairings are carried into one another by an E-linear unitary map defined over F_0. Fix one rational basis and one finite lattice representative for each realized set of these values. After the corresponding rational left translation, the finite point lies in that representative times the compact stabilizer of the original finite lattice. The real point and its inverse are bounded because they carry the fixed real basis to the bounded moved basis. This gives a compact set of representatives for all points with first minimum at least $1/4$. The definite target has a positive uniform lower bound on its first minimum, by integrality of the signed pairing and the product formula. The same argument gives a compact set of representatives there.

The theta distribution on a rational Schrödinger polarization is a rational lattice sum. On a fixed finite Schwartz space it is bounded by a finite-order Schwartz seminorm at infinity, by summing a decaying power on the rational lattice. Compact sets of finite translates of such a space are contained in another fixed finite Schwartz space. At infinity compact sets of Weil translates act boundedly on each fixed seminorm, allowing a higher fixed seminorm on the source vector.

For a general real move the conjugation formula on the position and differentiation operators is its linear symplectic action. Together with unitarity on L^2, this bounds a fixed Hermite Sobolev norm of the image by a polynomial in its matrix norm times a fixed Hermite Sobolev norm of the source vector. The real symplectic action of the pair is the linear Hom action, whose norm is polynomial in the two matrix norms. Rational invariance of theta and the preceding

compact reduction consequently give

$$|\theta(\omega(g,h)\phi)| \le CH^A p(\phi)$$

with one fixed Schwartz seminorm p on the finite Schwartz spaces in use, uniformly in the definite target h.

In the short-translation argument each difference is conjugated past $\omega(g,h)$, after choosing the rational representative with the fixed short isotropic vector. The transformed finite shifts fix the test because its finite stabilizer contains the fixed ideal used for the coefficient representatives. The real shifts have size $O(H^{-1})$ and are in one isotropic family, so commute. The fundamental theorem of calculus along a real Heisenberg one-parameter group bounds a difference by its length times the seminorm of the corresponding position or derivative operator. Repeating for A or more differences gives a factor H^{-A} in a higher fixed seminorm of the original polynomial Fock test. The following averaging identity applies this estimate directly to the full theta functional.

If $H > 2$, choose a shortest rational integral source vector of length at most $2/H$. Its norm pairing is integral at finite places and of absolute value less than one at the archimedean places, so it is isotropic by the product formula. Witt's theorem moves it to k_0 by a rational translate γ of the source. Rational invariance of the full kernel allows the descent at γg in place of g; the zero-slice integral at that point is again zero. At the definite target, a rational basis in its moved lattice has uniformly bounded lengths: Euclidean covolume is fixed and the first minimum is bounded below, so Minkowski's second theorem bounds the last minimum. With this basis d_j, the maps $d_j k_0^* J$ span Z, are integral in the moved Hom lattice, and have real lengths $O(H^{-1})$.

Choose a fixed compact open additive ideal in the finite coefficient coordinates which makes the transformed finite shifts act trivially on the fixed Schwartz vector. Such an ideal works uniformly because the moved finite rank-one maps are integral. Additive approximation gives representatives for the adelic quotient with finite coefficients in this ideal, and the archimedean coefficients can then be taken in a fixed compact fundamental domain for the corresponding rational coefficient lattice.

Give $[Z] = Z(E)\backslash Z(\mathbb{A})$ probability Haar measure. For $j \ge 1$ and the moved datum f, rational periodicity and isotropy give

$$(\theta - \theta_0)(f) = \int_{[Z]^j} \theta\left(\prod_{i=1}^{j}(1 - \varrho(z_i))f\right)\prod_{i=1}^{j} dz_i.$$

Indeed each nonempty term of the expanded product averages to $\theta_0(f)$, since translation preserves Haar measure on $[Z]$; their signed coefficients sum to -1. The rational basis $d_j k_0^* J$ identifies this probability measure with that on the coefficient quotient, independently of the chosen basis. After conjugation each finite shift fixes the test and each real shift has length $O(H^{-1})$. Integrating its Heisenberg generator gives one factor H^{-1} per difference in the Schwartz seminorm bound.

For $H > 2$, at the rational representative just chosen the difference estimate bounds the remainder by $C_j H^{A-j} p'_j(\phi)$, uniformly in the definite target point, for a higher real Schwartz seminorm and the fixed finite datum. The zero slice contributes zero after pairing with Y, by the induction hypothesis. At $H \le 2$ bound the full theta kernel by its initial estimate. Thus for every integer $B \ge 0$, choosing $j \ge A + B$ gives the descent bound

$$|F_l(g)| \le C_{l,B,f_f} \operatorname{vol}([U(V)])\|Y\|_\infty H(g)^{-B} p_{l,B}(\phi_\infty).$$

Here the seminorm is on the original real test and the finite factor is fixed. In particular the boundedness required below holds for each polynomial datum at this level.

The local Hecke transfer now kills this bounded function. For a polynomial Fock datum with the required off-level vacua write F_l for its descent to W_l, $k = n - r_l$, $\alpha = \mathrm{frob}_{r_l}(av^l, \bar{w})$, and $\beta = \mathrm{frob}_n(1, \bar{w})$. The vacuum at $w, \bar{w}$ makes F_l spherical and gives the exact adjoint transfer

$$0 = \prod_{i=0}^{k-1} (1 - q^i \alpha X) \mathcal{P}_{r_l, w}(X') F_l, \qquad X' = q^k \alpha \beta^{-1} X. \tag{F3}$$

For a target double coset $D = KdK$, write $T_D Y(h) = \int_D Y(hs)\, ds$, with $\operatorname{vol} K = 1$. For the kernel $H_g(h) = \theta(\omega_l(g, h)\phi)$, right invariance of the target quotient measure gives the bilinear identity

$$\int_{[U(V)]} H_g(h)(T_D Y)(h)\, dh = \int_{[U(V)]} \left(\int_{D^{-1}} H_g(hs)\, ds \right) Y(h)\, dh.$$

The double coset and the target quotient are compact. Inversion preserves Haar measure on the unimodular local group. The inverse coset may therefore be decomposed afresh into right K-cosets on the spherical kernel. In the $\bar{w}$-coordinate the split inverse-transpose relation gives the positive target coset; on the source it reads the inverse coset there as the positive one at w. The determinant factors on the vacuum have moduli $|\alpha| = q^{r_l/2}$, $|\beta| = q^{n/2}$. There is no conjugation of X at this step, since the kernel integration with Y is linear. To check the local rectangular vacuum identity in (F3), use the chosen coordinate at $v = \bar{w}$, with residue cardinality q. Write $L = \mathcal{O}_v^a$. For each $U \subseteq L/\varpi_v L$ take an integral matrix γ_U with elementary divisors $1, \varpi_v$ and reduction kernel U. Its transpose and inverse determine the image lattices

$$\gamma_U^{\mathsf{t}} L/\varpi_v L = U^{\perp}, \qquad \varpi_v \gamma_U^{-1} L/\varpi_v L = U.$$

The first equality is the row-space identity over the residue field, and the second follows by reducing $\gamma_U(\varpi_v \gamma_U^{-1} L) = \varpi_v L$ and comparing dimensions. Right cosets are determined by their image lattices. Use γ_U^{t} on the large group. On the small group the inverse double coset uses γ_U^{-1}. Their actions on the vacuum matrix $Z \in M_{n, r_l}(E_v)$ are the substitutions $\gamma_U^{-\mathsf{t}} Z$ and $Z \gamma_U^{-1}$, respectively. Taking $Y' = Z^{\mathsf{t}}$, $a = n$ on the large side and $Y' = Z$, $a = r_l$ on the small side, both tests have the form $Y' \gamma_U^{-1}$. In a basis adapted to U,

$$Y' \gamma_U^{-1} \text{ integral} \quad \Longleftrightarrow \quad Y' \text{ integral and } U \subseteq \ker \overline{Y'}.$$

Indeed the columns in the directions of U must be divisible by ϖ_w; the remaining columns must be integral. Thus both operators vanish off the integral matrices. On an integral matrix of residue rank a_0, the contributing subspaces lie in kernels of dimensions $n - a_0$ and $r_l - a_0$, respectively. Both base change characters at $\bar{w}$ are unramified, since the original source character and υ are unramified there, and the target character is trivial. A subspace of dimension i therefore contributes determinant factor α^i or β^i. For a residue space of dimension d, the finite binomial identity is

$$\sum_{U \subset k_w^d} (-s)^{\dim U} q^{\binom{\dim U}{2}} = \prod_{j=0}^{d-1} (1 - q^j s).$$

With $\beta X' = q^k \alpha X$, the two sums are consequently related by

$$\prod_{j=0}^{n-a_0-1} (1 - q^j \alpha X) = \prod_{j=0}^{k-1} (1 - q^j \alpha X) \prod_{j=0}^{r_l - a_0 - 1} (1 - q^j \beta X').$$

This gives (F3) for all X in the original Haar normalization.

The absolute values in (F1) and (F3) give

$$q^{k}|\alpha X| = q^{1-l} \le 1, \qquad q^{r_l+1}|X'| = q^{(r_l+r+4-n)/2} \le 1.$$

Since $i < k$, we have $|q^{i}\alpha X| \le q^{-1} < 1$, so each factor in that prefactor is nonzero. The second inequality follows from $2r + 2d_1 + 3 < n$ and $l \le d_1$. For a bounded spherical function F on the source, the absolute sum of the coefficients of the Hecke right-coset operator $\mathcal{P}_{r_l,w}(X')$ is

$$\sum_{U \subset k_w^{r_l}} q^{\binom{\dim U}{2}} |X'|^{\dim U} = \prod_{i<r_l} (1 + q^{i}|X'|) \le \exp(q^{r_l}|X'|) \le e^{1/2} < 2.$$

The term from $U = 0$ has weight one and acts as the identity.

Isolating it proves $\|F_l\|_\infty \le (e^{1/2} - 1)\|F_l\|_\infty$, so $F_l = 0$. This uses no spectral decomposition of the noncompact source. With each finite factor fixed, the compact-set Schwartz seminorm bound and polynomial approximation proved at the bottom apply also at rank r_l. They extend this pointwise vanishing to every Schwartz datum with the prescribed off-level vacuum. Induction starting with the bottom-descent argument above gives vanishing at $l = d_1$. Every conjugate matrix coefficient of Ξ has the same vanishing: translates of the highest vector span the irreducible Weyl space, and a real translation of the coefficient corresponds to the inverse real translation of the datum, leaving its finite projection fixed.

Next we pass from these descents to the prescribed block period. Write T for the complement of $S \cup \bar{S}$. For a continuous function Y_0 on $[U(V)]$ fixed by the integral group at T, suppose first that its descents to the split space $\mathbb{H}^{d_1}$ with character υ^{d_1} vanish for all data vacuum at T. Fix $h_1 \in U(I_{d_1})(\mathbb{A})$ and set

$$g_0 = \operatorname{diag}(h_1, 1), \qquad I(h) = \int_{[U(I_{d_2})]} Y_0(\operatorname{diag}(h_1, h_2)h)\, dh_2.$$

Let o_1 be the original orthonormal first-block basis. For $u \in \operatorname{Mat}_{n\times d_1}(\mathbb{A}_E)$ use the original absolute-trace character ψ_E to define

$$x(g) = g^{-1}o_1h_1, \qquad W(x) = |\operatorname{vac}(\alpha_0 x)|^2, \qquad \psi_u(x) = \psi_E\left(\delta \sum_{i,j} u^{\circ}_{ij} x_{ij}\right).$$

Take $j_{\mathrm{W}} = -1/\sqrt{-d}$ and the finite idele α_{W} of Section 8.1. Put $\alpha_f = (\alpha_0)_f = \alpha_{\mathrm{W}}$ and $(\alpha_0)_\tau = (2\pi \operatorname{Im}\tau(j_{\mathrm{W}}))^{-1/2}$. The vacuum support in each finite coordinate is $\alpha_f\widehat{\mathcal{O}}_E$, so $W(x) > 0$ precisely when x_f is integral. The idele $\delta = j_{\mathrm{W}}\alpha_0\bar{\alpha}_0$ generates the inverse different at finite places, and u° is obtained by setting the finite components of u at T to zero.

Parametrize the Siegel center by the skew block $C - C^{*}$, where $C \in \operatorname{Mat}_{d_1}(\mathbb{A}_E)$. The additive matrix quotient by rational matrices maps onto the central quotient, carrying probability Haar to probability Haar. Write $G_z = z^{*}Jz$ for the Gram matrix of a rational term. The central multiplier in the mixed model is

$$\psi_E\left(-\tfrac{1}{2}j_{\mathrm{W}} \operatorname{tr}(G_z(C - C^{*}))\right) = \psi_E(-j_{\mathrm{W}} \operatorname{tr}(G_z C)).$$

Here ψ_E is the character through the absolute trace, $\overline{j_{\mathrm{W}}} = -j_{\mathrm{W}}$, and $\operatorname{tr}(G_z C^{*}) = \overline{\operatorname{tr}(G_z C)}$. Multiplying the zero descent by $\psi_E(j_{\mathrm{W}} \operatorname{tr}(I_{d_1} C))$ and averaging therefore retains exactly the bases with $G_z = I_{d_1}$:

$$\int \psi_E(j_{\mathrm{W}} \operatorname{tr}((I_{d_1} - G_z)C))\, dC = \mathbf{1}_{G_z = I_{d_1}}.$$

For $d_1 > 0$, these matrices are injective. For $d_1 = 0$ there is just the empty matrix. They are one rational orbit by Witt's theorem, with stabilizer the original rational $U(I_{d_2})$. Fix the Siegel Levi point and take compact sets of representatives for the definite quotient and for the additive unipotent quotient. The finite supports of the translated mixed slices impose $z \in \mathfrak{a}^{nd_1}$ for a fixed fractional ideal $\mathfrak{a}$ of E. Their real Schwartz seminorms are uniformly bounded on these compact sets, and the unipotent character has modulus one. Thus the sum of absolute values is at most

$$C_N \sum_{z \in \mathfrak{a}^{nd_1}} (1 + \|z\|)^{-N} < \infty, \qquad N > nd_1[E : \mathbb{Q}].$$

The last inequality follows by counting lattice points in dyadic annuli. The bounded factor Y_0 preserves this majorant. We may therefore take the coefficient term by term and unfold the surviving orbit to the quotient by its rational stabilizer with the original ambient Haar measure.

Read the coefficient at the Siegel Levi element with matrix $\alpha_0 h_1$, and use the Heisenberg translate of the original vacuum by the isotropic vector whose adapted coordinates are $\alpha_0 \overline{u^\circ}$. It is vacuum at T. In the integral hyperbolic frame used above, the finite vacuum on the position coordinates is the indicator of $\alpha_{\mathrm{W}} \widehat{\mathcal{O}}_E^{nd_1}$ times its fixed nonzero normalization. Its opposite lattice is dual by the trace identity of Section 8.1, including above 2. At an infinite place the orthogonal plane rotation gives a positive radial Gaussian. On the frame orbit $x^* x = I_{d_1}$ its value and the local factor of $W(x)$ are fixed positive constants, since $\|x\|^2 = d_1$ there.

For the pure momentum translation the Schrödinger multiplier is the full character of the displacement paired against the position. Here its argument is

$$j_{\mathrm{W}} \operatorname{tr}\big((\alpha_0 \overline{u^\circ})^* (\alpha_0 x)\big) = \delta \sum_{i,j} u^\circ_{ij} x_{ij}.$$

Thus the translated slice on this orbit is a common nonzero scalar times $W(x)\psi_u(x)$. The target action on the position frame is $x \mapsto g^{-1}x$ because the pure hyperbolic source character satisfies $(\upsilon^{d_1})_{2d_1} = 1$. The remaining Levi scalar is the nonzero factor (F2) for $\alpha_0 h_1$, independent of the frame. For $h_2 \in U(I_{d_2})(\mathbb{A})$, $\operatorname{diag}(h_1, h_2) g_0^{-1} g = \operatorname{diag}(1, h_2) g$. The original Haar disintegration therefore gives the inner period $I(g_0^{-1} g)$. Canceling the common scalars, the vanishing coefficient gives

$$\int_{U(I_{d_2})(\mathbb{A}) \backslash U(V)(\mathbb{A})} W(x(g)) \psi_u(x(g)) I(g_0^{-1} g)\, d\dot{g} = 0 \qquad \text{for every } u. \tag{F4}$$

The quotient measure is the quotient of the original group Haar measures, and the inner integral defining I uses the original adelic quotient Haar measure.

The positive weight in (F4) detects the period at the chosen basis. On its support the finite basis is integral and orthonormal. Integral Witt extension at each finite place extends this basis to an integral unitary matrix. At a split place this is completion of a direct summand. At an inert place the basis splits off a unimodular orthogonal summand. The integral orthonormalization in Section 8.1 identifies its complement with the standard one, including in residue characteristic two.

The finite nonsplit extensions are unramified by Section 4.2. The residue and integral trace maps are surjective also in residue characteristic two, as in Section 4.3. At infinity orthonormal frames extend by compact unitary transformations.

More precisely, put $G = U(V)$, $H_2 = U(I_{d_2})$, and $K = G_\infty \prod_{v < \infty} G(\mathcal{O}_v)$ for the standard lattice. When $W(x(g)) > 0$, choose $k \in K$ with $x(g) = k^{-1} o_1$. Then $g k^{-1} o_1 = o_1 h_1$, whence

$$g \in H_2(\mathbb{A}) g_0 K.$$

The support of the integral in (F4) is therefore contained in the compact image of g_0K in $H_2(\mathbb{A})\backslash G(\mathbb{A})$. The inner period is unchanged by left multiplication of g by $H_2(\mathbb{A})$, by right invariance of its quotient measure. Two integral representatives with the same characters ψ_u have identical basis coordinates off T, by nondegeneracy of the additive pairing. Multiplying on the right by the T-component of their relative integral matrix then identifies their full bases and leaves Y_0 fixed. Thus $\overline{I}$ is a continuous function of the array of these characters on the compact character image.

On that compact image the span of the characters separates points, contains $1 = \psi_0$, and satisfies $\psi_u\psi_v = \psi_{u+v}$ and $\overline{\psi_u} = \psi_{-u}$. By Stone–Weierstrass there are finite character sums p_j converging uniformly to $\overline{I}$. Equation (F4) gives $\int W(x)p_jI = 0$, and

$$\left|\int W(x)(p_j - \overline{I})I\right| \le \|p_j - \overline{I}\|_\infty \int W(x)|I| \longrightarrow 0$$

by the unfolded absolute convergence. Thus $\int W(x)|I|^2 = 0$. Continuity and full support of quotient Haar measure give $W(x)|I|^2 = 0$ everywhere. At $g = g_0$ the basis is o_1, which has strictly positive vacuum weight. Hence $I(1) = 0$, the desired block period. This argument uses the chosen basis and complement throughout.

33.1.4 Vanishing for the gluing adjoint

In this subsection, we apply the vanishing calculation of Section 33.1.3 to the original false-lift coefficient.

Allow a finite right translate γ of Ξ supported on S, as in the prescribed divided lattice. The proved vanishing for $W \oplus \mathbb{H}^{d_1}$ remains valid after this translate because the transported datum is unchanged at T. Let ϕ_2 be the actual W, I_{d_2} datum of the slice, with target character b, and fix its source point g.

Choose a W, I_{d_1} datum ϕ_1 vacuum at T whose kernel, with target character b^{-1}, is nonzero at (g, h_1). Choose a rational Schrödinger polarization with position space D over F_0, and take the original finite vacuum. Its finite Weil translate at $(g_f, h_{1,f})$ is a nonzero locally constant Schwartz function f_f on $D(\mathbb{A}_{F_0,f})$. Additive strong approximation gives $x_0 \in D(F_0)$ with $f_f(x_0) \neq 0$. In rational coordinates the compact support of f_f lies in the profinite completion of a fractional lattice. The rational points in that support therefore form a discrete set in $D(F_{0,\infty})$. Take a smooth real bump supported in a neighborhood meeting this set only in x_0, with value one there, and pull it back by the inverse real Weil action at $(g_\infty, h_{1,\infty})$. The rational theta sum then has exactly one nonzero term, with value $f_f(x_0)$ up to the nonzero normalization of the distribution. This choice keeps the finite vacuum at T.

For a conjugate scalar coefficient Y of Ξ, set

$$Y_0(h) = Y(h\gamma)\,\theta^{a,1}_{W,V}(\phi_1 \boxtimes \phi_2; g, h), \qquad a = \upsilon^{m-j}, \quad 1 = b^{-1}b.$$

This is continuous and left rationally invariant. If k belongs to the original integral group supported on T, its support is disjoint from that of γ. Hence $k\gamma = \gamma k$ and $Y(hk\gamma) = Y(h\gamma)$. Both finite tests are vacuum at T and a is unramified there. Thus $\omega(1, k)$ fixes the tensor at T, also after the commuting source action at any g. Consequently $Y_0(hk) = Y_0(h)$, as required in (F4). The split-space descents of Y_0 vanish, since the product with the $\mathbb{H}^{d_1}, V$ kernel is the $W \oplus \mathbb{H}^{d_1}, V$ kernel with the box datum and source character $a\upsilon^{d_1}$. On the block $I_{d_1} \oplus I_{d_2}$, the W, V kernel factors as the product of the two chosen kernels. Its first factor is the fixed

nonzero value at (g, h_1). Cancel it from the block period and conjugate to obtain zero for the pairing of the original translated matrix coefficient with the conjugate W, I_{d_2} kernel. To apply this to the vector appearing in the gluing map, take the first-block class representative with real component one. Let b_t be the negative tableau polynomials, and write $b(UX) = S_-(U)b(X)$ for the column of these polynomials. For a real second-block point u and finite point h_f, put

$$\theta_t(u) = \theta_{b_t}(g; (u, h_f)), \qquad v(u) = (\overline{\theta_t(u)})_t.$$

In the target action the character of weight $m-j$ cancels the vacuum determinant of the $m-j$ negative source directions. The negative matrix variables undergo the substitution $X \mapsto u^{-1}X$. The entries of the tableau substitution matrix are polynomials with real coefficients in its matrix argument. Thus with the conjugate Weyl action $\rho_-(u) = S_-(\bar{u})$ this gives

$$v(u) = \rho_-(u^{-1})v(1).$$

These identities hold place by place, also for the constant polynomial when $j = m$.

For the fixed first factor let $C(v) = \mathrm{Cartan}(y_1, \bar{v})$. In the coefficient action of Section 28.2, each complemented minor of height $n-h$ contributes a determinant and an inverse-transpose minor of height h. At τ the exponent $-\sum_h \mu_\tau^-(h)$ cancels the determinants. The first block of $\mathrm{diag}(1, \bar{u})$ is the identity, and the second inverse-transpose block is $(\bar{u})^{-\mathsf{T}} = u$. Thus

$$\rho_V(\mathrm{diag}(1, u))C(v) = C(\rho_-(u)v).$$

Write $\Xi_f = \Xi(\mathrm{diag}(h_{1,f}, h_f)\gamma)$ and let e_t be the tableau coordinate vectors. The two conjugate-linear maps, C and the first variable of the pairing, give

$$\langle C(v(1)), \Xi_f \rangle = \sum_t \overline{\theta_t(u)}\, \langle \rho_V(\mathrm{diag}(1, u))C(e_t), \Xi_f \rangle.$$

Indeed $\rho_V(\mathrm{diag}(1, u))C(v(u)) = C(v(1))$. Each term on the right is the conjugate theta kernel times an original scalar coefficient at the prescribed finite translate. Its measured block period vanishes by the preceding calculation.

Write $\varpi = \pi_M$. The finite function is fixed by $J(\varpi^2)$. On the kernel vector, a right integral action precomposes the residue function by the inverse of the matrix reduction at that level. This reduction is the identity, and the determinant factor in the kernel transformation is one for these integral translates, since the source splitting character is unramified there. Conjugation by a divided lattice point with $\varpi\mathcal{O}^n \subset L_i\mathcal{O}^n \subset \mathcal{O}^n$ loses at most one factor of ϖ in the conjugated difference from the identity.

Hence it is the identity modulo ϖ at the chosen places. At a twist place the divided lattice point is the identity and the second block fixes the original first-block covectors, with level character one. This verifies the exact right invariance in the original finite level. Normalize the maximal finite compact and the compact real group to have mass one, or use the same common positive constant on both sides.

Each class at $J(\varpi^2)$ then has measure $[K_{\mathrm{std}} : J(\varpi^2)]^{-1}/|\Gamma_c|$. The stabilizers Γ_c are trivial by the original principal level trace argument in Section 6, which applies in rank at most $r+t$. Thus the vanishing measured integral is the plain sum over the second-block classes of the given gluing map times one positive constant. For a second-block class c, let h_c be its finite representative, and put $\kappa(g, h_c)$ equal to the vector of *conjugates* of the second-block theta kernels in the negative tableau basis. Here the two characters are $\upsilon^{m-j}, \upsilon^{m/2-j}$ and the residue test is the inclusion modulo ϖ^2. Write $\eta_{i,g}$ for the tuple supported on choice i with value $\overline{\kappa(g, h_c)}$ at (i, c). We use the inner product linear in the second variable.

Lemma 33.1 (Vanishing against the gluing adjoint)**.** *For the gluing map G_ϑ of Section 34.1, each false lift Ξ at this twist satisfies, for every i, g,*

$$\overline{K^\circ_{(G^*_\vartheta \Xi)_i}(g)} = \langle \eta_{i,g}, G^*_\vartheta \Xi \rangle = 0.$$

Here K° is the component pairing of Section 39.4.3.

For a fixed class c, let b_a be the standard tableau polynomials in the negative coordinates and put $l_{c,a} = \theta_{b_a}(g, (1, h_c))$ with the given finite test. At each infinite place write $s_1, \dots, s_{m-j}$ for the negative source indices in the prescribed reverse order. A tableau column $1 \le i_1 < \cdots < i_h \le d_2$ then gives the minor

$$\det\bigl(Z_{i_\alpha, s_\nu}\bigr)_{1 \le \alpha, \nu \le h}.$$

Here $h \le m - j$ by Section 33.1.1. The substitution uses distinct source indices and retains all d_2 target indices, so it is injective on the polynomial ring in the original matrix entries. Thus the standard tableau products form a basis of the negative test polynomials. When $j = m$, the diagram is empty and its basis is 1. The real positive definite Gram matrix G_- is invertible. By the tableau expansion in Section 34.1, $\sum_a (G_- \overline{\zeta_c})_a b_a$ is the Weyl polynomial of $\overline{\zeta_c}$, and these polynomials span exactly the specified tests as ζ_c varies. Thus $\kappa(g, h_c) = \overline{l_c}$, and

$$\overline{K^\circ_\zeta(g)} = \sum_c \overline{l_c}^{\mathsf{t}} G_- \zeta_c = \sum_c \langle l_c, \zeta_c \rangle_{G_-}.$$

This is the asserted first equality at $\zeta = (G^*_\vartheta \Xi)_i$. Write $y_i(h) = \overline{c'' \Upsilon_{\vartheta,i}(h)}$ for the first factor constructed below. By adjunction the inner product is

$$M^{l_1} \sum_{[h] \bmod J_\Lambda} \sum_c \Bigl\langle \mathrm{Cartan}\bigl(y_i(h), \overline{\kappa(g, h_c)}\bigr), \Xi\bigl(\mathrm{diag}(h, h_c) L_i\bigr) \Bigr\rangle_{\mathrm{W}}.$$

The stabilizer weights are one at these free levels. For each fixed h, the inner sum is the vanishing block period just proved.

33.2 The lattice for the comparison operator

In this subsection, we decompose the target lattice into the two Hermitian summands used in Section 34.1. The calculation identifies the subgroup and quotient of the original level needed for the tensor contraction.

On the j-th pair W_j, V_j the target splits in its prescribed coordinate order as $V'' \oplus V^\circ$, of ranks

$$d_+ = r + 9m + 1, \qquad d_- = t - 9m - 1.$$

The first r coordinate indices of V'' label the underlying basis of W_j. The same index set is used on the source of the exchange to form

$$W' = W_j \perp I_j^- \perp I_{3m-j} \perp W^\flat, \qquad W^\flat = I_{6m+1}.$$

The signs in this decomposition belong to W'. All coordinate directions in the first target factor $V'' = I_{d_+}$ are positive.

The second factor uses V° with its original level $J(\pi_M^2)$. Let I_e, I_z denote the inclusions of r columns in the fixed coordinates of V'', V°, respectively. Let $a_0 = 3m$ be the number of rows

of Λ, the rank of the auxiliary Hermitian space of signature $(2m, m)$. The residue choices are matrices $\Lambda \in M_{a_0,d_+}(\mathcal{O}_E/(\pi_M^2))$ and a level-π_M^2 class $h^\flat$ of $U(W^\flat)$ with its chosen short rows. For each choice use the level J_Λ on V'' that is 1 modulo π_M, fixes Λ modulo π_M^2, and has the identity columns off W_j modulo $\pi_M^2 Q$. Thus J_Λ contains the indicated principal level and is contained in the original free chosen level.

Here is the matrix of the lattice point in this description. It suffices to construct it at each place dividing π_M, in its split matrix realization, and extend by inverse conjugate transpose. Put $\pi = \pi_M$ locally and lift Λ to an integral matrix. Let D be a basis matrix for the row lattice generated by the rows of $\binom{\Lambda}{\pi I_{d_+}}$. This row lattice contains $\pi\mathcal{O}^{d_+}$ and lies in $\mathcal{O}^{d_+}$, so both D and $B_+ = \pi D^{-1}$ are integral. The equality of the row lattices gives, for all matrices X, Y of compatible size, the equivalence modulo π^2

$$DX = DY \quad \Longleftrightarrow \quad \binom{\Lambda}{\pi I_{d_+}} X = \binom{\Lambda}{\pi I_{d_+}} Y.$$

Write $J_z = I_z I_z^{\mathsf{t}}$.

Set

$$A = \begin{pmatrix} D & -DI_e I_z^{\mathsf{t}} \\ 0 & \pi J_z + 1 - J_z \end{pmatrix}, \qquad C = \begin{pmatrix} B_+ & I_e I_z^{\mathsf{t}} \\ 0 & J_z + \pi(1 - J_z) \end{pmatrix}.$$

The product is $CA = \pi I$. Permute the rows of A and the source columns of C inversely, so that the bottom z-rows of A are the prescribed inclusion rows of the original pair. Take $L_i = C$ at this place, and use the identity away from the chosen pairs. In particular L_i is chosen using the old modulus only and is independent of the twist.

To verify the exact block congruence, write x_+, x_- for the blocks of x and put $x_z = I_z^{\mathsf{t}} x_-$. The condition $(\pi L_i^{-1})x = \pi I_{\mathrm{incl}}$ modulo π^2 says

$$x_z \equiv I \pmod{\pi}, \qquad (1 - J_z)x_- = 0, \qquad D(x_+ - I_e x_z) = 0 \pmod{\pi^2}.$$

The first two conditions determine the unique invertible matrix $Y = x_z^{-1}$ modulo π^2 with $Y \equiv I$ modulo π for which $x_- Y = I_z$. Then the last condition is $D(x_+ Y - I_e) = 0$. By the row-lattice equivalence it is exactly

$$x_+ Y \equiv I_e \pmod{\pi}, \qquad \Lambda x_+ Y = \Lambda I_e \pmod{\pi^2}.$$

The implications reverse by multiplication by Y^{-1}.

Thus the original inclusion indicator equals the sum over $Y \equiv I \bmod \pi$ of the two block indicators. At every residue point this sum has zero or one term. The character of a member of this congruence group is trivial, since the original determinant twist factors through reduction modulo π. When the class periods are assembled, the right changes by these matrices give equal integrals on the original projected source, which is fixed by this old congruence group. Consequently the unnormalized sum in that period contributes its actual order, once.

For the denominator estimate, a change of columns in V'' puts the kernel of $\Lambda \bmod \pi$ on all but at most a_0 coordinates. In these coordinates the corresponding columns of D are zero modulo π, by the same equality of row lattices. Thus $L_i^{-1} = \pi^{-1} A$ is integral on every first-block basis vector except at most a_0; on the second block all d_- vectors may be divided once.

Cofactor matrices at the conjugate places are given by the unitary inverse transpose convention, and the integral inverse there follows from the displayed integral C. For a height-s column

of the first or second Weyl diagram, the exterior minor contains at most a_0 or d_- divided variables, respectively. The maximal heights in the two diagrams are at most $a = k + j - 1 < d_+$ and $b_- = m - j < d_-$, so no full-height columns have been omitted. This gives the common clearing exponent

$$l_1 = \sum_{\tau} \left(a_0 \sum_{s<d_+} \mu_\tau^+(s) + d_- \sum_{s<d_-} \mu_\tau^-(s) \right).$$

Since $\pi_M \bar{\pi}_M = M$, the rational integer M^{l_1} clears these denominators at every coefficient embedding. To check the Cartan map on Weyl vectors, take the positive exterior columns in V'' and the dual negative columns in $(V^\circ)^*$, included in the dual of the whole space by extension by zero on V''. Order the positive basis first and the negative basis last, in the orders of Section 33.1.1. For each negative column use the integral exterior isomorphism

$$\bigwedge^{s} V^* \simeq \bigwedge^{n-s} V \otimes (\det V)^{-1}, \qquad n = d_+ + d_-.$$

With the volume signs, the highest tensor of the two blocks now has the leading wedge in every column of the target diagram. The entire tensor product of the two block Weyl spaces lies in the target Weyl span: it is spanned in characteristic zero by the orbit of this tensor under the two independent block linear groups. Hence the normalized Cartan projection is the identity on these included tensors. The signed complement maps preserve the integral exterior coordinates, and the target tableau embedding has an integral left inverse by Section 7.5. After transport by L_i^{-1} the tensor remains in the same target Weyl span by equivariance. On a positive column the transport is exterior power of L_i^{-1}, and on a dual negative column it is exterior power of L_i^{t}. This last matrix is integral at the chosen split place, and at the conjugate place its restriction to an s-fold wedge has denominator at most $\bar{\pi}^s$, with $s \le d_-$. Together with the positive-column bound this proves the asserted M^{l_1}-clearing in the actual target Weyl coordinates. For a unitary matrix h in the definite standard bases, $h^{-\mathsf{t}} = \bar{h}$. Thus in the lattice norm at h the second Cartan factor acts by $(h^{-1})^{-\mathsf{t}} = h^{\mathsf{t}} = \bar{h}^{-1}$. Its input norm is therefore computed on the conjugate vector at $\iota \circ \mathrm{conj}$ when the first and output norms are computed at ι. This also applies at $\ell \mid M$.

33.3 A common integral scale for the first factors

In this subsection, from the lattice decomposition of Section 33.2, we bound denominators and archimedean norms of the first factors on one integral scale. These bounds are uniform over all the finite classes in the subsequent comparison.

Write d_K for the positive quadratic parameter d of the datum. In this subsection the letter d denotes a polynomial degree. Fix the original j and the gluing choices of Section 33.2. The space V'' has rank $d_+ = r + 9m + 1$. Use the same modulus $\mathfrak{m}_\vartheta = \varpi^2 Q$ and the original split residue coordinates. The first factor is a vector $\overline{c'' \Upsilon_{\vartheta,i}(h)}$ of weight μ^+ at $h \in U(V'')(\mathbb{A}_f)$, where $\Upsilon_{\vartheta,i}$ is the left class series on the original levels. The scalar c'' is independent of ϑ, i, h.

We use the following normalized definite kernel. Let $K(P, \Psi; g, h)$ be the kernel on the standard definite pair (I_l, I_n) of positive ranks l, n, at finite group points g, h with trivial archimedean components, the prescribed algebraic infinity types and tame characters, and a polynomial P homogeneous of degree d. Put

$$s_{l,n,d} = (M^2)^{2fln}\, s_{\mathrm{vac}}^{ln}(M^2 d_K s_{\mathrm{jet}})^d. \tag{I1}$$

Here s_{vac} is the rank-one vacuum scale, s_{jet} the jet scale, and d_K the positive integer of the original quadratic field. All factors are nonzero. Let $q_Q = \operatorname{Nm} Q$.

Let $\mathfrak{m}$ be the split modulus of the individual kernel. If $q_Q M^2 \in \mathfrak{m}$, the residue values of Ψ are integral away from q_Q, and $\ell \nmid q_Q$, then at integral points above ℓ one has

$$|s_{l,n,d} K(P, \Psi; g, h)|_\iota \le 1 \tag{I2}$$

whenever P has integral coefficients at ι. Indeed the residue datum applied to the integral vacuum is a finite Fourier expansion in translates of that vacuum, with translation coordinates integral after multiplication by $q_Q M^2$.

The denominators of Fourier inversion divide $|\mathcal{O}_E/\mathfrak{m}|^{ln}$, which divides $\operatorname{Nm}((q_Q M^2))^{ln}$ since $(q_Q M^2) \subseteq \mathfrak{m}$. As $\ell \nmid q_Q$, their ℓ-parts are cleared by $(M^2)^{2fln} = \operatorname{Nm}((M^2))^{ln}$. At ι the same unit q_Q can be removed from the translation denominator. The resulting scaled datum belongs to the local module of standard data at level M^2. The two integral group points preserve it by the finite Weil action and the prescribed norm-one character weights. The integral finite transport of Section 19.3 and the ordinary derivatives of Section 38.4 give integrality at scale $M^2 d_K s_{\mathrm{jet}}$ on that module. Applying the homogeneous polynomial gives precisely the scale (I1), by the homogeneous jet formula in Section 20.1.

We also use the conjugate size part of the same jet estimate. For the prescribed characters suppose that $l, n \le \Delta$, $d \le M$, and that the split modulus $\mathfrak{m}$ contains (M^2). A bound A_0 for all conjugates of the residue values and a bound B_0 for the conjugate polynomial on the polydisc of radius M^4 give

$$|\sigma(s_{l,n,d} K(P, \Psi; g, h))| \le e^{x_1} A_0 B_0. \tag{I3}$$

Each normalized finite Fourier coefficient is an average of the residue values times roots of unity. Its conjugate absolute value is at most A_0. The total is therefore at most

$$|\mathcal{O}_E/\mathfrak{m}|^{ln} A_0 \le (M^2)^{2fln} A_0,$$

since $(M^2) \subseteq \mathfrak{m}$ and $[E : \mathbb{Q}] = 2f$. At the unit ideal the finite group has one element and the constant datum gives the vacuum itself.

33.3.1 Bounds for the normalized derivatives

In this subsection, we apply the derivative estimates of Part III to the first factors of the original lift.

The normalized derivative estimate applies to each translated vacuum and to every finite point of the definite groups. The usual homogeneous rescaling uses

$$\frac{(M^2 d_K)^2}{\Omega}(2d + 2) \le (M^4)^2$$

in its coefficient ball, where Ω is the original positive period parameter. At radius R in the squared norm of that ball, homogeneity bounds the polynomial by $B_0 e^{dR}$, because $\sqrt{R} \le e^R$. The jet exponential factor is

$$\exp(C_0(q_1 ln + 1)^2 + fln \log(2d + 2))$$

in the original numerical notation. Multiplying by the Fourier cardinality and the denominator-clearing factor gives at most e^{x_1} by the cutoff inequality.

Here is the polynomial to which to apply (I2)–(I3). Write $w_{\mathrm{off}} = a/x_1$ for the weight at the non-b-places. The parameter bounds give $w_{\mathrm{off}} \le b$. The total determinant count in the specified space is bounded by q_*, where

$$q_* = f(b + w_{\mathrm{off}} + \Delta + m\Delta).$$

Let Y be the square matrix of short covectors used for the $W^\flat = I_{6m+1}$ class of $h^\flat$ in choice i. The definite lattice $L = E^{6m+1} \cap h^\flat \widehat{\mathcal{O}}_E^{6m+1}$ is Hermitian self-dual. Its Euclidean first minimum is at least one, since the squared norm $\mathrm{Tr}_{F_0/\mathbf{Q}}\langle v, v\rangle$ of a nonzero $v \in L$ is a positive integer. Apply the transference bound in Section 41.4 and take the columns of Y^{t} to be the resulting E-basis in L. Then

$$(h^\flat)^{-1} Y^{\mathsf{t}} \in M_{6m+1}(\widehat{\mathcal{O}}_E), \qquad \max_{\tau,a,b} |\tau(Y_{ab})| \le 6 d_K q_1 \big(f(6m+1)\big)^2 \le M^4.$$

This one choice gives the finite integrality and the bound at every complex embedding used below. The product of stacked determinants consists of constant covectors, some given by Y, over variable covectors.

Separate the W_j positive variables from the ones on

$$W^{\mathrm{add}} = I_{3m-j} \oplus W^\flat \oplus I_{3m}.$$

Let Z denote the positive variables of W_j, let $b_s(Z)$ be the standard tableau products of minors in those variables, and let G_{W} be the Gram matrix of the corresponding integral Weyl basis. For the stacked determinant product P, write

$$P = \sum_s d_s(Y) b_s(Z), \qquad d(Y) = G_{\mathrm{W}} \zeta(Y),$$

where $\zeta(Y)$ gives its Weyl coordinates over the coefficient field. Both the coefficients of this expansion and the entries of $\zeta(Y)$ are polynomials in the added variables. The number of variable columns remaining in a determinant attached to height h is $d_+ - h$. For $1 \le h < k-1$ the initial parameter entries satisfy $\varsigma_{h-1} - \varsigma_h = 1$, hence the formula for $k^+(h)$ in Section 33.1.1 gives $\mu_\tau^+(h) = 0$. Also $\mu_\tau^+(0) = 0$ by its definition. Thus $d_+ - h \le d_+ - (k-1) = 10m + 1$ at every contributing height, using $r = k + m - 1$. Their common homogeneous degree is

$$d = \sum_{\tau,h} (d_+ - h)\mu_\tau^+(h) \le (10m+1) q_* \le M.$$

The constant rows have entries $0, 1$, or $\tau(Y_{ab})$ for one of the chosen complex embeddings $\tau : E \hookrightarrow \mathbb{C}$.

Write $R_Y = \mathbb{Z}[\tau(Y_{ab}) : \tau, a, b]$ and denote the added variables by X_{add}. Expanding first along the fixed unit rows and then along the remaining Z-columns expresses P as a sum of products of column minors in Z with coefficients in $R_Y[X_{\mathrm{add}}]$. In the transposed diagram convention of [1, Theorem II.2.16, p. 232], the exterior factors are our columns. The Schur map alternates their entries and multiplies along our rows. Its standard tableaux give a basis over $\mathbb{Z}$, so straightening these products gives $d_s(Y) \in R_Y[X_{\mathrm{add}}]$. We control the denominator of $\zeta(Y) = G_{\mathrm{W}}^{-1} d(Y)$ by the complementary Gram form below.

33.3.2 Inverse Gram matrices in the chosen coordinates

In this subsection, we pass from the derivative bounds of Section 33.3.1 to the integral Gram scale.

For the complementary shape let $P^\vee$ be the product over τ, h of the leading minors of size $d_+ - h$, each to the power $\mu_\tau^+(h)$. Put

$$H = P^\vee(\partial)P^\vee(0) = \sum_\alpha \alpha!\,\big([X^\alpha]P^\vee(X)\big)^2.$$

Complementation in exterior powers, with reversed order of the covectors and application of the determinant tensor, is an integral isometry between the Gram spaces for the complementary shapes. On one exterior column of rank d_+, put $J_h e_I = \operatorname{sgn}(I, I^c) e_{I^c}$. The cofactor identity is

$$J_h \bigwedge^h g = (\det g) \bigwedge^{d_+-h} (g^{-\mathsf{t}}) J_h.$$

Reorder the complementary columns by decreasing height. The leading column wedges go to the final coordinate wedges, which are the images of the leading complementary wedges under the reversing permutation, up to sign. Thus the signed tensor isometry carries the rational span of the highest orbit onto its complementary span. Integral tableau straightening identifies both Weyl coordinate lattices with the intersections of these rational subspaces and their integral tensor lattices. Indeed the rows of the integral filling matrix indexed by the semistandard tableaux form an identity submatrix, giving an integral left inverse for each embedding. Write U for its unimodular coordinate matrix and $G^\vee$ for the complementary Gram matrix. The Fock Gram identity in Section 34.1 gives $H(G^\vee)^{-1}$ as an integral differential pairing. Thus

$$G_{\mathrm{W}} = U^{\mathsf{t}} G^\vee U, \qquad H G_{\mathrm{W}}^{-1} = U^{-1}\big(H(G^\vee)^{-1}\big)U^{-\mathsf{t}} \ \in \mathrm{Mat}(\mathbb{Z}).$$

Thus $H\zeta(Y) = (HG_{\mathrm{W}}^{-1})d(Y)$ is integral in the original Weyl basis at every embedding where the entries of Y are integral. The complementary shape has d boxes. For $d > 0$ the product of factorials of the minor sizes is at most $d!$, so $\|P^\vee\|_1 \le d!$ and $0 < H \le (d!)^3 \le d^{3d}$. For $d = 0$, $H = 1$. For fixed j, all choices of the short rows have the same d, H.

For the norm estimate, specialize the added variables to a point of the M^4-polydisc at the chosen conjugate embedding. The short-covector entries of Y also have absolute values at most M^4 there. Set $a = k + j - 1$. For $h < a$, the contributing heights satisfy $a - h \le j \le m$. The unit rows specified in Section 39.4.1 thus include all the positive W_j columns after the first h. Expand along those unit rows. For $h = a$, the short rows are supported on $W^\flat$, so the a positive W_j columns are already variable columns with zero short-row entries. In the resulting determinant let C be the matrix of the specialized columns remaining beside the h columns in Z. Laplace expansion in these h columns gives coefficients $\pm \det C_{I^c}$, where the h-element row sets I are restricted to the variable rows. Cauchy–Binet on all row sets and Hadamard's inequality give

$$\sum_I |\det C_{I^c}|^2 \le \det(C^* C) \le \prod_s \|C_s\|_2^2.$$

The number of specialized columns is at most $\dim W^{\mathrm{add}} + (k + j - 1) - h \le 12m + 1$. Each squared column length is at most $(r + 11m + 1)M^8 \le M^9$. Let c be the tensor product of these exterior coefficient vectors after specialization, and let F be the integral filling matrix above. Straightening gives $d(Y) = F^{\mathsf{t}} c$. Since $G_{\mathrm{W}} = F^{\mathsf{t}} F$, the exterior tensor

$$F\zeta(Y) = F(F^{\mathsf{t}}F)^{-1}F^{\mathsf{t}}c$$

is the orthogonal projection of c. Its norm is at most the product of the exterior coefficient norms at every complex embedding.

Put $V = H\zeta(Y)$ and apply σ to the coefficients to form V^{σ}. For $\|z\|_{\infty} \le M^{4}$,

$$\left\|V^{\sigma}(z)\right\|_{\mathrm{Gram}} \le B := HM^{9(12m+1)q_{*}/2}. \tag{I4}$$

Extend σ to $\overline{\mathbf{Q}}$ and take a dual row u of Gram norm one with algebraic entries. Put $Q = (\sigma^{-1}u)V$. With the other arguments fixed, let $K_{\mathrm{alg}}(P) = s_{l,n,d}K(P,\Psi;g,h)$ be the algebraic normalized kernel of (I3). Then

$$Q^{\sigma}(z) = uV^{\sigma}(z), \qquad \sigma(K_{\mathrm{alg}}(Q)) = u\big(\sigma(K_{\mathrm{alg}}(V_{s}))\big)_{s}.$$

Apply (I3) with $B_{0} = B$ to Q. Algebraic rows are dense in the dual Gram unit sphere, so taking their supremum gives the same norm bound for the vector of normalized kernel values.

To expand the first factor, let

$$c_{0} = s_{m,3m,0}, \qquad c_{1} = s_{j,d_{+},0}, \qquad c_{2} = s_{\dim W^{\mathrm{add}},d_{+},d}.$$

The scalar theta kernel K_{0} on (I_{m}, I_{3m}) has the unit ideal as its finite modulus and 1 as both polynomial and finite datum. Its characters are $\chi_{M}^{j}\vartheta$ and 1. The second kernel K_{1} is on (I_{j}, V'') with polynomial one, trivial first character, and the prescribed inclusion indicator evaluated at the norm-minus-one residue unit times its argument. Its second character is the inverse of the first-factor character times $\upsilon^{d_{+}}$. The negative-source identity in the Weil model gives $\overline{K_{1}} = \epsilon K_{I_{j}^{-}}$, where $|\epsilon| = 1$ and $K_{I_{j}^{-}}$ is the negative-source kernel in $\Upsilon_{\vartheta,i}$.

Indeed the conjugating map of Section 39.3 takes a residue datum $D(X)$ to $\overline{D(\eta^{-1}X)}$. For $D(X) = \mathbf{1}_{\mathrm{incl}}(\eta X)$ this is the original inclusion indicator. On the rectangular Hom space the determinant of $X \mapsto hXg^{-1}$ is $(\det h)^{j}(\det g)^{-d_{+}}$. Thus the characters $(1, b_{1})$ become $(\upsilon^{j}, b_{1}^{-1}\upsilon^{d_{+}})$, with b_{1} the second character of K_{1}. The phase ϵ comparing the two normalized theta functionals depends only on this fixed pair. The third kernel K_{2} is the linear map on polynomials for (W^{add}, V'') with the first-factor character on both sides. Write its residue columns as $(z_{0}, z_{\flat}, z_{3})$ on $I_{3m-j} \oplus W^{\flat} \oplus I_{3m}$, and put $R_{0} = \mathcal{O}_{E}/(\varpi^{2})$. Let $\phi : \mathcal{O}_{E}/\mathfrak{m}_{\vartheta} \to R_{0}$ be reduction. Writing $J_{0,\flat}$ for the coordinate inclusion of the first two blocks in V'', the residue function is

$$\mathbf{1}_{(z_{0},z_{\flat})=J_{0,\flat} \bmod \mathfrak{m}_{\vartheta}} \sum_{A \in \mathrm{GL}_{3m}(R_{0})} \psi\big(\mathrm{tr}(\Lambda\, \phi(z_{3})A)\big).$$

This is the factor on the added columns in (*). Its values are integral and every conjugate absolute value is at most

$$\mathcal{N} = |\mathrm{GL}_{3m}(\mathcal{O}_{E}/(\varpi^{2}))|.$$

The double class sum is over I_{3m} and I_{m} at their integral levels. Both class weights are the reciprocals of the actual stabilizer orders. Put $n_{A} = C_{G}$: reduction at an old odd split prime embeds each stabilizer into a finite general linear group of rank at most rt, so its order divides n_{A}. Expanding that sum and the coordinate minor product gives the identity

$$\begin{aligned} c'' &= n_{A}^{2}c_{2}Hc_{0}\overline{c_{1}}\epsilon \neq 0, \\ \overline{c''\Upsilon_{\vartheta,i}(h)} &= n_{A}^{2}\sum_{w,v}^{\mathrm{cl}} \overline{(c_{0}K_{0}(v,w))\,\overline{c_{1}K_{1}(h)}\,c_{2}K_{2}(H\zeta(Y);w,h)}. \end{aligned} \tag{I5}$$

In (I5) the arguments of K_2 on the first group are $\operatorname{diag}(1, h^\flat, w)$.

For (I2), integral representatives can be chosen by rational translation at all places over ℓ. When moving the representative of the short covectors by γ_1, transport the matrix Y to $Y\gamma_1^{\mathsf{t}}$. This matrix is integral at the conjugate embedding because

$$\gamma_1 Y^{\mathsf{t}} = (\gamma_1 h^\flat)((h^\flat)^{-1} Y^{\mathsf{t}})$$

is a product of two integral matrices there. The last factor is integral by the original choice of short covectors from that moved lattice. Rational covariance of the kernel brings the transformed value back to the lattice norm at the original h. For source and target rational elements Γ, γ' and character exponents e_τ, e'_τ, the exact complex identity is

$$\begin{aligned} K_P(\Gamma g, \gamma' h) &= \left(\prod_\tau \tau(\det \gamma')^{-e_\tau} \tau(\det \Gamma)^{e'_\tau} \right) K_{P^{A,B_1}}(g, h), \\ P^{A,B_1}(X) &= P\big((A_\tau^{\mathsf{t}} X_\tau B_{1,\tau}^{\mathsf{t}})_\tau\big), \\ A_\tau &= \tau(\gamma'^{-1}), \qquad B_{1,\tau} = \tau(\Gamma). \end{aligned}$$

Rational invariance of theta and the compact Fock action at the inverse real pair give the identity before any coefficient embedding. In the stacked case use $B' = \operatorname{diag}(1, B_1)$ on the full column set. Each stacked determinant changes by $\det(A)\det(B')$. Put $e_j^{(1)} = j + \lfloor (t+k-1)/2 \rfloor$. There are $m_\tau = w_j(\tau) - e_j^{(1)}$ stacked determinants at τ, the first row length of μ_τ^+. The first-factor character has weight $-m_\tau$ at τ on both sides. Thus the determinant multiplier $\tau(\det \gamma')^{-m_\tau} \tau(\det \Gamma)^{m_\tau}$ cancels the character multiplier in the displayed transformation. The remaining Schur action on conjugate coordinates is exactly the lattice norm transformation at $\gamma' h$. Thus no polynomial coefficient integrality is asserted before moving the rational representatives.

In the conjugated product in (I5), use the embedding $\iota \circ \mathrm{conj}$ for K_0, K_2 and ι for K_1. The scalar kernels have the indicated integral values after moving their representatives, with constant polynomials. Both exponents of K_0 are zero. In K_1 the source point stays at one, and its trivial first character makes the multiplier on the determinant of the rational target change equal to one. Thus the rational moves preserve each scalar value. The vector kernel is integral in the original lattice norm by the covariance just proved. The factors in the two class denominators divide n_A^2. Therefore (I5) has lattice norm at most one for every ι above every $\ell \nmid q_Q$.

For fixed ϑ, i, h, choose one such ℓ where h is integral at every place above ℓ. The change to the transported lattice coordinates and its inverse are then integral, so each fixed coordinate of $\overline{c'' \Upsilon_{\vartheta,i}(h)}$ is bounded by one for all field isomorphisms $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. A transcendental coordinate could be sent to an element of negative valuation by extension of transcendence bases. Thus these cleared coordinates are algebraic. In applying (I3) to (I5), the two barred factors are read at $\sigma \circ \mathrm{conj}$ and the middle factor at σ. Both are among the automorphisms in the conjugate estimate.

Suppose the auxiliary conductor is empty, $Q = (1)$, as for the trivial twist. The three moduli in (I5) are $(1), (\pi_M^2), (\pi_M^2)$, each containing $(M^2) = (\pi_M^2 \bar{\pi}_M^2)$. Since $1 \le j \le m \le r$, the ranks $m, 3m, j, d_+$ and $\dim W^{\mathrm{add}} = 12m - j + 1$ are positive and at most $\Delta = 2r^8$. The degrees are $0, 0, d$ with $d \le M$. Both scalar kernels have residue and polynomial bounds one. Their conjugate norms are at most e^{x_1} by (I3). For the vector kernel the residue bound is $\mathcal{N}$ and the

polynomial bound of (I4) is B, giving $e^{x_1}\mathcal{N}B$. For every choice of short rows, trace matrix and finite group points, the triangle inequality and the bound $|\Gamma|^{-1} \le 1$ on each stabilizer weight therefore bound the squared conjugate norm of (I5) by

$$(n_A^2 h_{3m} h_m e^{3x_1}\mathcal{N}B)^2,$$

where h_a denotes the number of classes of I_a at its integral level. The integral-level counting argument in the proof of Lemma 7.2, at $n = m, 3m$, gives $\log h_n = O((2fn)^3 \log x) \le x^{53}$ for sufficiently large x. Also $\mathcal{N} \le (M^2)^{2f(3m)^2}$ and $\log B \le 3d \log M + \frac{9}{2}(12m+1)q_* \log M$. The logarithm of the displayed square is thus at most

$$4\log n_A + 4x^{53} + 6x_1 + 8f(3m)^2 \log M + (6d + 9(12m+1)q_*) \log M \le fmb(\log b)^2.$$

Indeed $q_* = O(fb)$, $d = O(fmb)$, and $\log M \le 22 \log b$. The definition of n_A gives $\log n_A \le f(rt)^2 \log M$. The ranks and f grow at most polynomially in x, while $x_1 \le b^{1/10^6}$ by Section 4.1. The displayed left side is therefore $o(fmb(\log b)^2)$ for fixed q_0, d_K, ϵ as x tends to infinity. Thus the scale in (I5) gives both first-factor bounds for the gluing. Its nonvanishing precedes both the three-exchange identity and the proof of injectivity of the common period used to establish $c'_\vartheta \ne 0$.

34 The comparison operator and the first theta lifts

In this section, we define the contraction on the lattice of Section 33 and compute the Gram forms and paired periods that enter the exchange of Section 36.

34.1 The comparison operator on the original lattice

In this subsection, we contract the target tensor against the first factors of Section 33.3. The operator takes values in the measured direct sum over the original classes, and the vanishing of Section 33.1 identifies the period terms it retains.

Keep the actual twist ϑ, original descent level and quotient measures fixed. The tuple space $\mathcal{H}$ is the direct sum over the gluing choices and the $J(\pi_M^2)$-classes in V° of the second-factor Weyl spaces with their Gram forms. This direct sum has its plain class metric, since the second-factor level is free. Put $n_A = C_G$. For the gluing classes the bottom rows $\pi_M I$ of the stabilizer matrix force the reduction of every element of J_Λ to be the identity at each old place. A definite class stabilizer is finite.

At one old odd split place it injects into a conjugate of the torsion-free principal congruence subgroup, so it is trivial. The same argument applies to the second-factor $J(\pi_M^2)$-classes. Consequently the reciprocal stabilizer weights in the original gluing operator G_ϑ are all one, and the operator is

$$G_\vartheta f = M^{l_1} \sum_{i,c} \sum_{[h] \bmod J_\Lambda} \delta_{(h,c_f)L_i}(\phi''_{\vartheta,i}(h) \cdot f(i,c)).$$

The dot is the integral Cartan product in the original block order.

The notation $\delta_z(v)$ means the class form obtained by K-equivariant averaging of the specified value v at z; its values sum over the class stabilizer without division. In the target pairing against an equivariant value this multiplicity cancels the reciprocal target stabilizer order, so

$\langle \delta_z(v), a\rangle = \langle v, a(z)\rangle$. The first-factor coefficient is $\overline{c''\Upsilon_{\vartheta,i}(h)}$ from Section 33.3. Its lattice norm is at most one. For the second factor use the finite family of integral vectors constructed below. Section 33.2 shows that the power M^{l_1} clears the divided lattice in the Cartan product; the point-form evaluation sums unitary quotient transports with unit level characters, so preserves lattice norm. This verifies the integral gluing for these definite coordinates.

To construct this family, work first with the fixed column shape μ^{-}. Let $b_i(X)$ denote the standard minor products and F the integral matrix of the embedding of their dual coordinates into the tensor product of the exterior powers for the columns. Explicitly, the row of F at a tableau T is the list of coefficients of its product of minors in the b_i.

The matrices here use the original space enumerations. Writing G for the Gram matrix in the coordinate space, Cauchy–Binet in every column gives

$$K(X,Y) = \sum_{T} b_T(X) b_T(Y) = \sum_{i,k} G_{ik} b_i(X) b_k(Y), \qquad G = F^{\mathsf{t}} F.$$

Here $K(X,Y)$ is the product of the prescribed leading minors of $Y^{\mathsf{t}}X$. In the sum over T, choose an increasing list of indices independently in each exterior column, including repeated columns. The indices i,k enumerate semistandard tableaux. Straightening each exterior filling in this basis gives the integral rows of F. For a standard filling T_i, $F_{T_i,k} = \delta_{ik}$. The monomial coefficients $D_{i\alpha} = [X^{\alpha}] b_i(X)$ instead give the differential pairing

$$H_{ik} = b_i(\partial) b_k(X)\big|_{X=0} = \sum_{\alpha} D_{i\alpha}\, \alpha!\, D_{k\alpha}, \qquad h_{-} = b_{\mathrm{high}}(\partial) b_{\mathrm{high}}(X)\big|_{X=0}.$$

The last integer is positive because the highest minor product takes the value one on the standard array, and the Fock pairing of a polynomial with real coefficients with itself is a positive weighted sum of its squared coefficients.

The Gram identity holds for any of these column shapes. Write b for the column of basis polynomials and $b(gX) = S(g) b(X)$ for the change of basis under an invertible matrix g at each embedding. The identity $K(gX, Y) = K(X, g^{\mathsf{t}} Y)$ gives

$$S(g)^{\mathsf{t}} G = G S(g^{\mathsf{t}}).$$

The differential pairing moves g from one argument to g^{t} on the other, so $S(g) H = H S(g^{\mathsf{t}})^{\mathsf{t}}$. It follows that HG commutes with the Schur action and is scalar by irreducibility. At the standard array I, put $v = b(I)$. Then $v^{\mathsf{t}} G v = K(I,I) = 1$ and $b(X)^{\mathsf{t}} G v = K(X,I)$ is the highest minor polynomial. If $HG = cI$, its differential square is $v^{\mathsf{t}} G H G v = c$. Thus $HG = h_{-} I$ for the displayed shape, with the same identity and its own highest minor square for any complementary shape.

34.1.1 The comparison in the Fock model

In this subsection, we evaluate the comparison operator on the monomials in the metric of Section 33.3.2.

For the Hermitian Fock pairing the adjoint of substitution by a complex matrix a is substitution by a^{*}. For the bilinear differential pairing used to define H it is substitution by a^{t}. Since the basis polynomials have real integral coefficients, the two pairings have the same Gram matrix in this basis. The preceding computation gives $HG = GH = h_{-} I$. Thus for a vector e_T

in the orthonormal basis of the full exterior tensor product the vector $\eta_T = HF^{\mathsf{t}}e_T$ has integral coordinates and

$$F\eta_T = h_- \operatorname{pr}_{\operatorname{im} F}(e_T).$$

It follows that

$$\sum_T \langle \eta_T, \xi\rangle_G \eta_T = h_-^2 \xi, \qquad \|\eta_T\|_G^2 \le h_-^2. \tag{A}$$

For a given prime $\ell \nmid \operatorname{Nm} Q$, move each original second-factor representative h_c by a rational unitary element δ_c, so that $\delta_c h_c$ is integral above ℓ.

For an adelic point g and the integral lattice stabilizer K_v, require $\delta_v \in K_v g_v^{-1}$ at the places v of F_0 above ℓ. At any other finite set of places one can also require $\delta_v \in g_v J_v g_v^{-1}$ for prescribed open level subgroups J_v, which keeps $\delta_v g_v$ in $g_v J_v$. Each of these open sets contains a unitary point u_v with $\det(1+u_v) \neq 0$: multiplying a point by a norm-one scalar arbitrarily close to one avoids the finitely many excluded eigenvalues. The Cayley parameters $A_v = (1-u_v)(1+u_v)^{-1}$ are skew-Hermitian. Approximate them in their F_0-vector space by one rational parameter A and set $\delta = (1-A)(1+A)^{-1}$. Continuity gives all the prescribed open conditions. The inverses are in the local fields, including at the places above two.

Only finitely many places above ℓ are involved. Transport (A) at coordinate (i,c) by

$$\eta_{c,T} = \overline{\rho(\delta_c^{-1})\overline{\eta_T}}, \qquad \rho(\delta_c)\overline{\eta_{c,T}} = \eta_T.$$

Here ρ is the Weyl action of weight μ^- and determinant exponent zero. The transport is Gram-unitary, since the rational point is unitary at every embedding and the Gram matrix is real. For $x \in E$, the CM involution gives

$$\sigma(\overline{\tau(x)}) = (\sigma\tau)(\bar{x}) = \overline{(\sigma\tau)(x)}.$$

Thus the rational unitary equation for δ_c holds at $\sigma\tau$ also when σ acts nontrivially on K. The rational Weyl Gram matrix and its exterior tensor realization then make the transported action Gram-unitary at every σ.

Therefore each tuple f_a, supported at the single coordinate (i,c) with value $\eta_{c,T}$, has squared norm at most h_-^2 at every σ, and these tuples have sum of rank-one operators $h_-^2 I$ in $\mathcal{H}$. Rational covariance of the lattice norm and the displayed integral coordinate identity show that the value of f_a at h_c has lattice norm at most one at every embedding above ℓ.

In this calculation the embedding applied to the unbarred coordinate is the conjugate of the embedding applied to $\rho(\delta_c)\overline{\eta_{c,T}}$. The same family therefore works at both coefficient embeddings. Write $v_a = G_\vartheta f_a$. All v_a are integral at every embedding above ℓ when ℓ is prime to the twist modulus. Define the comparison operator by

$$U_\vartheta = n_A h_-^2 G_\vartheta G_\vartheta^* = n_A \sum_a v_a \langle v_a, \cdot\,\rangle.$$

Indeed the class coordinates of $\mathcal{H}$ have the unweighted direct-sum Gram metric. Thus for every target vector z,

$$\sum_a v_a \langle v_a, z\rangle = G_\vartheta \left(\sum_a f_a \langle f_a, G_\vartheta^* z\rangle_{\mathcal{H}} \right) = h_-^2 G_\vartheta G_\vartheta^* z$$

by (A) on each coordinate.

At the chosen embedding use the conjugate coefficient embedding in the first Gram factor. The two transported coordinates cancel by the transpose identity for the Weyl Gram matrix.

The rational class weights have denominators dividing n_A. For integral z, the coefficients $n_A\langle v_a, z\rangle$ are therefore integral. This proves preservation of the original integral target lattice by U_ϑ.

For the fixed twist let y belong to the original projected automorphic space on W_j, and set $Z = \operatorname{im} u_{j,\vartheta}$ on that space. For $\xi \in F_w$, Lemma 33.1 applies at that same w and twist. In its notation the tuple $\eta_{i,g}$ has entries $l_c = \overline{\kappa(g, h_c)}$ in the i-th summand and zero entries elsewhere. With G_- the negative tableau Gram matrix, the lemma gives

$$\overline{K^\circ_{(G^*_\vartheta \xi)_i}(g)} = \sum_c \overline{l_c}^{\mathsf{t}} G_-(G^*_\vartheta \xi)_{i,c} = \langle \eta_{i,g}, G^*_\vartheta \xi\rangle = 0.$$

For $z = u_{j,\vartheta} y$ and $\xi \in F_w$, insert $\zeta = G^*_\vartheta \xi \in \mathcal{H}$ in the unscaled gluing formula (110). Each factor $K^\circ_{\zeta_i}(g)$ vanishes by the individual pairing above and its complex conjugate. Hence $\langle G^*_\vartheta \xi, G^*_\vartheta u_{j,\vartheta} y\rangle_{\mathcal{H}} = 0$. Multiplying by $n_A h_-^2$ and taking the adjoint gives

$$\langle \xi, U_\vartheta z\rangle = 0 \qquad (\xi \in F_w,\ z = u_{j,\vartheta} y,\ \text{every } w). \tag{94}$$

Thus U_ϑ sends integral vectors in Z to integral tests for the same false-lift subspaces. In particular a clearing operator Λ with the bound above makes $\Lambda \pi_Z U|_Z$ preserve the original integral lift lattice. This proves the required compatibility of the clearing and comparison operators on that lattice.

Nonvanishing of the real first-arrow constant. At one real place let $P \neq 0$ be the prescribed joint harmonic polynomial. In the source action the variables are grouped by target index. In each such group, the source signs give the two Fock signs up to a common reversal. The polynomial is termwise harmonic: $\partial_\alpha \partial_\beta P = 0$ when the two variables have opposite signs in a single such group. Use the source discrete-series parameter λ so that its lowest compact highest weight is the weight of P for the oscillator action, including the determinant twist. In the real factor of the doubled integral $\overline{e_j(g)}/e_j(1)$ is then the conjugate of the normalized coefficient $c_\lambda(g)$ of the unit highest vector of this discrete series. This convention is the contragredient on the scalar form side.

If z is a positive definite element of the source unitary group, its action on the matrix space is positive definite, since on each group of variables with a common target index it is a copy of z, its transpose, or its inverse transpose in the indicated sign conventions. Its blocks on the positive and negative Fock variables are positive definite. Its determinant twist is one, since $\det z = 1$.

In the Gaussian Bargmann kernel for this action the two pure quadratic terms involve only opposite-sign pairs with the same target index. The mixed bilinear term has positive definite coefficient given, up to transposition with the variables, by the inverse of the complex-linear part of the action. Indeed that linear part is the direct sum of the positive sign blocks, with complex conjugation on the negative variables. The scalar of the kernel is the positive vacuum coefficient. It is the reciprocal determinant of the negative principal block in the unitary normalization.

34.1.2 The Gaussian pairing

In this subsection, we evaluate the corresponding Gaussian integral in the Fock model of Section 34.1.1.

Pair the Gaussian kernel against P in both Bargmann variables by differentiation at zero. Every contribution from a pure quadratic term vanishes by termwise harmonicity. The remaining mixed kernel pairs P with its linear substitution by the positive definite coefficient matrix

T. For $(S_aP)(X) = P(aX)$ one has $S_aS_b = S_{ba}$ and $S_a^* = S_{a^*}$ in the Hermitian Fock metric. With the positive definite square root $D = T^{1/2} = D^*$ this gives

$$\langle P, S_T P\rangle_F = \|S_D P\|_F^2 > 0,$$

since D is invertible and $P \neq 0$. Consequently

$$\langle P, \omega(z)P\rangle > 0.$$

In particular, one can compute this coefficient with both oscillator signs present. Write the positive element as $z = kak^{-1}$, with k compact and a a product of real two-dimensional boosts. Write $P_k = \omega(k)^{-1}P = \sum_\alpha p_\alpha Z^\alpha$. Compact substitution preserves the mixed derivative conditions within each group of variables with a common target index. For a boost parameter t_l, the two participating coordinates have opposite signs. Its Bargmann kernel, with the Gaussian in the measure, is

$$\frac{1}{\cosh t_l}\exp\left(\varepsilon_l \tanh t_l(Z_+Z_- - \bar{Y}_+\bar{Y}_-) + \frac{Z_+\bar{Y}_+ + Z_-\bar{Y}_-}{\cosh t_l}\right).$$

Differentiation in t_l gives the action of $\varepsilon_l(Z_+Z_- - \partial_+\partial_-)$, and at zero the kernel is the identity. Passing to the inverse transpose on a dual factor may change $\varepsilon_l \in \{1, -1\}$. In either case the two pure quadratic terms vanish in the pairing with P_k. Index all boost pairs in the Hom space, with repetitions over target coordinates. Orthogonality of the Bargmann monomials gives

$$\langle P, \omega(z)P\rangle = \sum_\alpha \alpha!\,|p_\alpha|^2 \prod_l (\cosh t_l)^{-1-\alpha_{l,+}-\alpha_{l,-}} > 0.$$

Coordinates fixed by a retain their usual monomial norm. The determinant twist is one at a, and the compact scalar at k cancels that at k^{-1}.

To deduce the assertion about the projector, let K_∞ be the maximal compact of this source group, with probability Haar, and d_K the dimension of its lowest compact type. Insert Sun's positive-factor formula for this coefficient, as in the real coefficient decay calculation, and average the integrand over $K_\infty \times K_\infty$. At a positive factor z the result is

$$\begin{aligned}\int_{K_\infty^2} &\overline{c_\lambda(k_1^{-1}zk_2)}\langle P, \omega(k_1^{-1}zk_2)P\rangle\, dk_1\, dk_2 \\ &= \frac{1}{d_K}\int_{K_\infty} m_{-\lambda-\rho}(u^{-1}zu)^{-1}\langle P, \omega(u^{-1}zu)P\rangle\, du.\end{aligned}$$

In fact Sun's formula is an integral of rank-one compact projectors times the displayed weight of the positive factor. For a fixed u, substitute $k_1 \mapsto uk_1$, $k_2 \mapsto uk_2$ and apply compact Schur orthogonality on each side of the oscillator matrix coefficient.

The two factors d_K^{-1} cancel one of the factors from the compact integral formula, leaving the displayed d_K^{-1}. A preceding compact factor in the polar decomposition is absorbed by Haar in k_1. For $Z = u^{-1}zu$ we have $\det Z = 1$ and $Z^{-1} = JZJ$. Jacobi's complementary-minor identity gives $D_I(Z) = D_{I^c}(Z)$. Since $\{i : -a_i \geq t\} = \{i : a_i \geq 1 - t\}^c$ for integral a, the definition in Section 12.7 gives $m_{-a}(Z) = m_a(Z) > 0$. Thus either sign of $a = \lambda + \rho$ gives the displayed weight, with ρ indexed in decreasing λ order. The right side is strictly positive. Absolute integrability follows from integrability of the original sufficiently separated discrete-series coefficient and the unitary norm bound for ω. Integrating over the real group proves nonvanishing. The tensor product over the real places gives the nonzero constant in the original doubled first-arrow formula.

34.2 Gram forms for the first lifts

In this subsection, we compute the Gram form of the initial theta lift with the finite and archimedean data fixed. Along with the paired period of Section 34.3, this identity determines the normalization for composing the lifts.

Use the split doubled source and the diagonal target action of [30, §11.3, pp. 820–821]. The tensor of a Weil datum with the conjugate datum on the opposite source gives the product of the two theta kernels on the diagonal target. The strict-range formula of Section 17.2 applies to each pair of separately translated Schwartz–Bruhat tests at the ranks below. Unfolding the graph terms gives the Gram form with the specified vectors and quotient measures.

Put $j = i+1$, $\mathfrak{m} = \pi_M Q(\xi)$ and let S be the union of its support and the conjugate support. Write r for the original source rank and $d = r + t$ for the rank at the first arrow, so $d > 2r$. Let K_0 be the original standard compact in the finite source group. The finite bases identify this group and K_0 for every space in the calculation. Give every local factor of K_0 mass one. Denote the source principal level at the old places by J_M and at the whole modulus by $J_{\mathfrak{m}}$.

The first arrow $a_{i,0}$ uses the character of index $j = i + 1$. At target rank d its rank twist is

$$b_j = \mu_j^0 \xi^j \upsilon_1^{\lfloor d/2 \rfloor}, \qquad \mu_j^0 = \chi_0(\upsilon_1^{-\Delta}\chi_M)^j, \qquad \upsilon_1 = \upsilon|_{U(1)}.$$

For an allowed m-fold twist the local component of ξ^j is nontrivial at each new place for every $1 \le j \le m$. Thus $Q(\xi^j) = Q(\xi)$, including the empty support when $\xi = 1$. For the single lift on W_j at ξ^j, the rank twist of its defining character is the same expression. This is the character in the source factor of each Weil matrix coefficient. In column zero the doubled entries are the stem $2\lambda_a$ at every offset. Passing to the next row negates all signature signs, so J changes to $-J$ while the ordered parameter stays fixed. The equations defining $U(J)$ and $U(-J)$ coincide. Compact roots are determined by equality of two signature signs, which is preserved by simultaneous negation. Thus the compact Cartan, positive roots, lowest compact highest line and its normalized coefficient agree under this identification. The group equation $g^* J g = J$ is the same for the two forms over F_0, so their diagonal rational subgroups also agree. The finite change of frame is scalar and acts trivially by conjugation on the group and principal level. The identity on the resulting adelic quotient preserves its normalized Haar measure and the right translations, giving the isometry j_i of the triangle base copies. The finite target character is unramified off S on both source spaces.

34.2.1 The masses of the first lifts

In this subsection, we compute the local measures required for Proposition 34.1.

At a chosen member of a chosen split pair use the original Schrödinger matrix $X \in M_{d,r}(E_w)$. For g in the source integral compact, the action on the residue test is

$$F(X) \longmapsto b_{j,w}(\det g)^{-1} F(Xg).$$

Here the finite character is read on the chosen split coordinate. The absolute-value factor on such g is one and the unramified Weil splitting has trivial unit factor. The opposite coordinate is already accounted for by the dual basis in this model. On the triangle the test at every distinguished place is the point $\binom{I_r}{0}$. On the single lift it is this point at an old place, and at a new place it is

$$F_w\left(\begin{pmatrix} U \\ V \end{pmatrix}\right) = \chi_w(\det U), \qquad U \in M_r(k_w), \quad V \in M_{t,r}(k_w).$$

The value is zero for noninvertible U. Here χ_w is the unit character of $b_{j,w}$. At the other places both data are the original vacuum.

Every matrix in either prescribed support has an integral left inverse. If X and Xg belong to their respective supported integral boxes, applying these left inverses shows g, g^{-1} are integral. For the point support the overlap then occurs exactly when $g \equiv I_r$. On the new block support the transformation $\chi_w(\det(Ug)) = \chi_w(\det U)\chi_w(\det g)$ cancels the phase. The stabilizer of the coefficient of the block test is therefore the entire integral source group at that place.

At an old place the stabilizer is the principal subgroup. Normalizing the residue measure by the original integral matrix box of mass one, put

$$B_\xi = |M_{d,r}(\mathcal{O}_E/\mathfrak{m})|, \qquad w_0 = [K_0 : J_\mathfrak{m}] = |\mathrm{GL}_r(\mathcal{O}_E/\mathfrak{m})|.$$

The matrix-coefficient integrals on the two products are exactly

$$c_\triangle = \frac{B_\xi^{-1}}{w_0}, \qquad c_{\mathrm{single}} = B_\xi^{-1} \frac{\prod_{w|Q(\xi)} |\mathrm{GL}_r(k_w)| q_w^{rt}}{\prod_{w|\pi_M} |\mathrm{GL}_r(k_w)|}.$$

Indeed the coefficient is the squared norm of the local test on its stabilizer and zero elsewhere. The squared norm of a block test is its support proportion, and that support has rt unconstrained entries and an invertible r-block. There is no extra Haar factor at a conjugate distinguished place.

For the single lift let $J_T \subset K_T^0$ be the target level in the standard finite compact, and let D_∞ be the dimension of the target Weyl representation. Give K_T^0 and the compact real target group mass one. For two lifted Weyl-valued forms F_1, F_2 let f_1, f_2 be their scalar coefficients at the prescribed unit highest vector. With c ranging over the finite target classes at J_T, quotient disintegration and compact Schur orthogonality give

$$\langle f_1, f_2 \rangle_{\mathrm{Pet}} = \frac{1}{[K_T^0 : J_T] D_\infty} \sum_c \frac{\langle F_1(c), F_2(c) \rangle_{\mathrm{W}}}{|\operatorname{Aut}(c)|}.$$

Indeed each class has measure $\mathrm{vol}(J_T)/|\operatorname{Aut}(c)|$, the unitary level character cancels in the product, and the real integral contributes $1/D_\infty$. Hence the class pairing multiplies the Petersson pairing by $[K_T^0 : J_T] D_\infty$. Only the finite index in this multiplier varies with the twist. Write

$$F_w = \prod_{h=t+1}^{t+r} (1 - q_w^{-h}).$$

At an old place the index of the target vector-basis stabilizer is the number of full-rank d-by-r matrices, namely $q_w^{dr} F_w$.

At a new place write the upper blocks of a residue matrix as $(A\ B)$, in block sizes (r, t). If it preserves the support of the test, then A is invertible by testing $(U, V) = (I_r, 0)$. If $Bb \neq 0$ for some $b \in k_w^t$, put $a = -A^{-1}Bb$ and choose $V \in M_{t,r}(k_w)$ with $Va = b$. Then $(A + BV)a = 0$, contradicting support preservation at (I_r, V). Thus $B = 0$. Conversely, an invertible lower block triangular matrix sends U to AU, so it acts on the test through the determinant character of A, with inversion for the inverse action. The resulting parabolic preserves the span of the first r coordinate covectors under the dual action and the kernel of the top-block projection under the column action. There are $q_w^{dr} F_w$ ordered independent covector frames and $|\mathrm{GL}_r(k_w)|$ ordered bases of their span. The target index is therefore $q_w^{dr} F_w / |\mathrm{GL}_r(k_w)|$. The target index times

c_{single} consequently contributes $F_w/|\mathrm{GL}_r(k_w)|$ at each old place and $q_w^{rt}F_w$ at each new one. The first group of factors is fixed and the second is precisely

$$c_Q(\xi) = \prod_{w \mid Q(\xi)} q_w^{rt} F_w.$$

For the triangle Petersson pairing there is no such target conversion at the first arrow.

34.2.2 Doubling and the common series

In this subsection, we use the masses of Section 34.2.1 to compare the two Gram forms.

Let A be the complement of S and put $G_A = U_r(\mathbb{A}_{F_0,A})$, with the finite restricted product interpreted in the original finite bases. Let K_A be its standard integral compact. Represent K_A itself by 1 in G_A/K_A. Extend the chosen representatives γ by the identity at S and write

$$u_j(\gamma) = \big(b_j(\det\gamma)\Delta(\gamma)^d\big)^{-1}, \qquad R_j = \sum_{\gamma \in G_A/K_A} u_j(\gamma) R(\gamma).$$

The series acts on K_A-fixed vectors. Its coefficient is constant on every double coset. Indeed, for the standard lattice Λ and $k, k' \in K_A$,

$$\Lambda \cap (k\gamma k')^{-1}\Lambda = (k')^{-1}(\Lambda \cap \gamma^{-1}\Lambda).$$

The determinant ideals of the two intersection lattices coincide. Thus their multipliers Δ coincide by the unramified formula of Section 40.2.2. Also b_j is trivial on determinants of K_A. This applies at every conjugation orbit in A. Both source levels contain K_A, so $R(\gamma k)y = R(\gamma)y$ for $k \in K_A$. On each of the d target covectors the source acts by $\gamma^{-\mathsf{t}}$. In the finite unitary bases this is $\bar\gamma$. Conjugation preserves the absolute value and conjugates the unitary character in the intersection multiplier, so $\Delta(\gamma^{-\mathsf{t}}) = \overline{\Delta(\gamma)}$ on each conjugation orbit. The vacuum coefficient $1/\overline{\Delta(\gamma^{-\mathsf{t}})}$ thus contributes $\Delta(\gamma)^{-1}$ per covector. In particular its absolute value is

$$|u_j(\gamma)| = \left(\prod_{w \in A} [\gamma\Lambda_w : \gamma\Lambda_w \cap \Lambda_w] \right)^{-d/2},$$

where the product includes both members of split pairs. This follows by multiplying the integral lattice overlap by its unitary square-root factor. The complex multiplier also agrees under the finite lattice isometry $\jmath_f : \Lambda \longrightarrow \Lambda'$. Indeed, with $\gamma' = \jmath_f \gamma \jmath_f^{-1}$ and $B\Lambda = \Lambda \cap \gamma^{-1}\Lambda$, one has

$$(\jmath_f B \jmath_f^{-1})\Lambda' = \Lambda' \cap (\gamma')^{-1}\Lambda', \qquad \det(\jmath_f B \jmath_f^{-1}) = \det B, \qquad \det\gamma' = \det\gamma.$$

Thus both the splitting character in Δ and the factor $b_j(\det\gamma)$ have the same values on the two groups.

Absolute convergence and injectivity can both be seen by a lattice count in the present ranks $t = 5r$, $d = 6r$. At a split pair a lattice at the chosen place determines its dual at the other. If it is not contained in Λ_w, choose x in it with $\varpi_w x$ primitive in Λ_w. Enlarge Λ_w by x and take its dual at the opposite place. The new intersection with the target lattice maps onto the quotient of the enlarged lattice by Λ_w, generated by the class of x and of order q_w. Its kernel is the old intersection at the chosen place. The dual lattice shrinks at the opposite place, but contains the old intersection there by integral pairing with x. Thus the intersection there

is unchanged. The chosen-place index is unchanged because both lattices grew by the same factor. The opposite-place index, and hence the product of the two indices, is divided by q_w. If containment is strict in the other direction, start on the opposite member. Each enlargement is specified by the line of $\varpi_w x$ in the residue space of dimension r. Thus there are at most $2(1 + q_w + \cdots + q_w^{r-1})$ moves at one split pair.

At a nonsplit unramified place the two lattices are self-dual. Choose $\varpi_w = \varpi_v$ in the fixed field. If the lattices differ, take x as above in Λ'_w and put $y = \varpi_w x$. The lattice $\Lambda_0 = \{z \in \Lambda_w : h(y, z) \in \varpi_w \mathcal{O}_w\}$ has index q_w. The lattice $\Lambda_0 + \mathcal{O}_w x$ is integral because $h(x, \Lambda_0) \subseteq \mathcal{O}_w$ by definition and $h(x, x) \in \mathcal{O}_w$ since $x \in \Lambda'_w$. As $\varpi_w x = y \in \Lambda_0$ and $x \notin \Lambda_w$, it contains Λ_0 with index q_w. Its covolume equals that of Λ_w, so integrality and self-duality of Λ_w imply that it too is self-dual. Put $P = \Lambda_w \cap \Lambda'_w$. Then $P \subseteq \Lambda_0$ since $h(x, P) \subseteq \mathcal{O}_w$, and $y \in P$. The new intersection maps onto $(\Lambda_0 + \mathcal{O}_w x)/\Lambda_0$, since it contains x, and the kernel is P. Hence it equals $P + \mathcal{O}_w x$ and the intersection index is divided by q_w. For a fixed residue line of y, two choices of x differ by $z \in \Lambda_w$ after fixing that line's representative. Modulo ϖ_v one has

$$\varpi_v\big(h(x + z, x + z) - h(x, x)\big) \equiv \operatorname{Tr}_{k_w/k_v} h(\bar{y}, \bar{z}).$$

The residue pairing with $\bar{y} \neq 0$ identifies Λ_w/Λ_0 additively with k_w. Since k_w/k_v is finite separable of degree two, its trace is surjective, also in characteristic two. Thus at most $|k_v| = \sqrt{q_w}$ classes give an integral square for this line. Each new self-dual lattice has an integral orthonormal basis. Its reduction is a nondegenerate Hermitian space over k_w/k_v. If all residue norms were zero, expanding the norm of $x + ay$ would give $\operatorname{Tr}_{k_w/k_v}(ah(x, y)) = 0$ for all a, x, y, contrary to nondegeneracy of the residue trace pairing. A lift of a vector with nonzero norm has unit norm and can be scaled to norm one by surjectivity of the norm on units (Section 4.3). Its orthogonal complement in the lattice is self-dual. Induction gives the basis, including in residue characteristic two.

For each such fixed move choose an isometry taking the new standard lattice back to the original one. Repeat on the moved Λ'. The product of indices in $|u_j|$ decreases by a factor at least q_w at each step. Reversing the resulting word recovers the original lattice coset. Its coefficient is bounded by the product of the step weights $q_w^{-d/2}$. The sum of those step weights over all choices of a single step is

$$\eta \ll [E : \mathbb{Q}] \sum_{n \geq 2} n^{-2r} \ll [E : \mathbb{Q}]\, 2^{-2r}.$$

Indeed either count above is $O(q_w^r)$, and there are at most $[E : \mathbb{Q}]$ places of a given prime power norm.

Here $[E : \mathbb{Q}] \ll x^{12}$ and $r \geq x^5$, so $\eta < 1/3$ for all sufficiently large x. Choose one word for each nonidentity coset. The geometric series gives absolute convergence in operator norm and

$$\|R_j - I\| \leq \sum_{\gamma \neq 1} |u_j(\gamma)| \leq \frac{\eta}{1 - \eta} < \frac{1}{2}.$$

The estimate holds on the whole K_A-fixed space, including the full triangle base and the two projected spaces. On each of these spaces the identity coset acts by the identity, so the corresponding operator is injective. More precisely, let η_0 be the same sum of one-step weights allowing every finite place. It has the same bound $O([E : \mathbb{Q}]2^{-2r})$. Each b_j is unitary, so its value on a coset contributes a factor of absolute value one. Omitting the old and new places

only decreases the sum. Thus $\|R_j - I\|, \|R_j^W - I\| \le \eta_0/(1-\eta_0)$ with one bound for all the allowed supports and all $1 \le j \le m$. Let $\ell(\gamma)$ be the least number of the preceding neighboring lattice changes joining Λ to $\gamma\Lambda$. The standard compact preserves adjacency and fixes Λ, so ℓ is constant on double cosets. Write $R_{j,<L}$ for the absolutely convergent subsum with $\ell(\gamma) < L$. Every omitted coset has chosen reducing word of length at least L; hence on either realization

$$\|R_j - R_{j,<L}\| \le \sum_{h \ge L} \eta_0^h = \frac{\eta_0^L}{1-\eta_0} \qquad (L \ge 1).$$

Equivalently, index the finite conjugation orbits in A by places v of F_0. Let b_v be the sum of the absolute values of the nonidentity local coset coefficients, and let η_v be the sum of the one-step weights there. The same word count gives $b_v \le \eta_v/(1-\eta_v)$ and $\sum_v \eta_v \le \eta_0 < 1/3$. Multiplicativity of the coefficient and monotone convergence give

$$\sum_{\gamma K_A} |u_j(\gamma)| = \prod_{v \in A} (1 + b_v) < \infty.$$

At a split place the two dual lattices enter one local factor. In particular the places over N occur in this product: $(q_\xi, MN) = 1$, the glued rank-r lattice is unimodular there, and the last-covector condition is on the rank-k source of Section 6.

The doubled action is identified in Section 17.2. The doubled source has dimension $2r$ and a Lagrangian of dimension r. Thus the Siegel index in [80, Theorem 2] is $n = r$. For the target T of dimension $d = 6r$ and Witt index $\iota(T)$,

$$d - \iota(T) \ge d/2 = 3r > r, \qquad s_0 = (d-r)/2.$$

Weil's convergent identity applies to the full Schwartz–Bruhat space with coefficient one and probability target measure [78, §52, Theorem 5, pp. 76–77], by the verification in Section 17.2. The doubled datum is the original tensor times the conjugate tensor on the opposite source. In a rational Schrödinger polarization the partial Fourier transform in the difference variable gives the product of the theta kernels on the diagonal group, and its graph zero term gives the original unitary matrix coefficient. The finite opposite-basis scale cancels against its inverse in the conjugate tensor. The graph change of variables preserves the probability measure on the additive quotient, by the calculation in the principal-level doubling calculation.

With source character b_j, the opposite source carries $b_j^- = b_j^{-1} v_1^d$. Indeed its conjugate character is b_j^{-1}, and the intrinsic multiplier on the opposite Hom space supplies v_1^d, as in Section 40.2.1. Write $f_g = \omega_{b_j}(g,1)f$, and take the antiunitary C of that calculation, normalized by $\theta_-(Cw) = \overline{\theta_+(w)}$. The doubled test is $f_g \otimes C f_{g'}$. The source multipliers are $b_j(\det g)^{-1}$ and $b_j^-(\det g')^{-1}$. The graph parameter $Z \mapsto X_T^{-1} J_T^{-1} Z J_S X_S$ preserves probability Haar between the rational additive quotients. The graph indexed by $\delta \in U_r(F_0)$ therefore gives $\langle f, \omega_{b_j}(g'^{-1}\delta g, 1) f \rangle$.

For clarity, the other rational Lagrangians give no term on either of the present tests. If a Lagrangian of the doubled source is not a graph, its intersection with the first source has a nonzero vector. Indeed if its second projection is injective it is an isomorphism, and the isotropy equation then makes the first projection an isometry. A vector in that intersection yields a Heisenberg translation in the first tensor factor alone under which the doubled zero-term functional is invariant.

In the split Schrödinger coordinates at an old place this translation is multiplication by $\psi_w(\mathrm{tr}(ZXc))$, for arbitrary Z, where $c \neq 0$ is the transported first-source vector at w. Invariance forces the functional to be supported on $Xc = 0$. To see this directly, partition a compact open support in X into finitely many compact open pieces on each of which some one multiplier is a constant different from one. Test invariance on each piece. The support has injective X, so all pieces are of this kind. This is the usual open-graph unfolding with a full-rank Schrödinger test.

In the convergent range the absolute sum of the Siegel–Weil sections is bounded by a Schwartz seminorm of the transformed datum times the convergent spherical Eisenstein series at the identity. This follows by writing each section in Iwasawa coordinates and bounding its restriction to the maximal compact by that seminorm. At reduced representatives of the two source quotients the seminorm of the tensor of the Weil translates grows polynomially in the reduction height. At fixed source levels the finite class representatives range over finite sets. Their remaining compact factors have finite orbits on the locally constant finite data. Thus the seminorm estimates apply to both original residue tests for all such representatives.

The source forms in the prescribed real coefficient types are rapidly decreasing by the coefficient-projector argument in Section 18.1. That argument applies to every bounded finite-level form of these types. For a fixed-level L^2 vector reproduced by the projector, truncate its values and project again. The projected truncations converge in L^2 and lie in the same finite-dimensional cusp space of the given compact type and infinitesimal character, as in Section 42.2. This gives a rapid representative also for the full projected space. For a finite right translate t, choose an integer $c \geq 1$ such that the fixed finite lattice satisfies $c\widehat{\Lambda} \subset t\widehat{\Lambda} \subset c^{-1}\widehat{\Lambda}$. Multiplication by g_f and intersection with the rational vectors give the corresponding inclusions for the moved lattices. Hence $c^{-1}H(g) \leq H(gt) \leq cH(g)$. Such a translate preserves rapid decay, commutes with the real projector, and changes its finite level J to tJt^{-1}. In particular both source forms may be used even when only the second is fixed by $J_{\mathfrak{m}}$.

Before unfolding, the absolute product of the original theta kernels is integrable in all three quotient variables. Indeed, with each source test paired with its own conjugate, the pointwise convergent Siegel–Weil formula above gives

$$\int_{[U_d]} |\theta_f(g,h)|^2 \, dh \ll_f H(g)^{2B}.$$

Here the diagonal doubled section has the preceding polynomial bound and the target quotient volume is fixed. Cauchy–Schwarz in h bounds the integral of the absolute mixed product at g, g' by $CH(g)^B H(g')^B$, increasing B for the two fixed data. Integration against the rapid source forms is finite. The absolute graph sum satisfies the same product majorant by the Iwasawa bound. Thus Fubini applies first to the theta product and then to the graph sum, as in Section 40.2.1. Unfolding the rational graph sum by the quotient Haar formula leaves the integral over the original adelic source of the Weil coefficient times $\langle x, R(g)y\rangle$, multiplied by the original target quotient volume.

At infinity a rank-one discrete-series coefficient projector gives

$$\langle x, R(g_\infty)z\rangle = \frac{\overline{e_j(g_\infty)}}{e_j(1)}\langle x, z\rangle.$$

This is the compression of the real right regular action to the unit highest coefficient line, by Schur orthogonality. The source form in the second variable is fixed by $J_{\mathfrak{m}}$ for the triangle and

by J_M for the single lift. Fubini therefore gives the local masses computed above and the same series over the remaining places. The real factor is

$$\operatorname{vol}([U_d]) \int_{U_r(F_0\otimes\mathbb{R})} \langle v_j, \omega(g_\infty,1)v_j\rangle \frac{\overline{e_j(g_\infty)}}{e_j(1)}\, dg_\infty.$$

For the two pairs use their respective joint harmonic and quotient volume. These constants are independent of ξ, since the real parameters and vectors do not include the tame twist. At a real place let d_λ be the formal degree for the given Haar measure. With c_λ in the oscillator convention above, put $p_\lambda = \omega(d_\lambda \overline{c_\lambda})$. Schur orthogonality makes this an orthogonal projection, and the compact average in Section 34.1.2 gives strict positivity for the indicated harmonic:

$$\int \langle P, \omega(h)P\rangle \overline{c_\lambda(h)}\, dh = d_\lambda^{-1}\|p_\lambda P\|^2 > 0.$$

The same calculation applies to the first true return: the rank $r+1$ source and rank $r+t$ target have the common entries and inserted block computed in the signed parameter comparison. Their joint harmonic has only one Fock sign in each group of variables with the same target index, so both pure quadratic terms vanish in the Bargmann pairing as above.

For x, y in the original full triangle base, y is fixed by $J_{\mathtt{m}}$, as required in the unfolding. This gives the following two identities in the original Petersson metrics.

Proposition 34.1 (Gram forms of the first lifts)**.** *On the full triangle base, with $\mathcal{V}_i$ the convergent spherical series,*

$$w_0\langle a_{i,0}j_i x, a_{i,0}j_i y\rangle = C_i B_\xi^{-1}\langle x, \mathcal{V}_i y\rangle.$$

For $j = i+1$, the restriction of $\mathcal{V}_i$ to the projected source is R_j. On the projected source of the single lift,

$$\langle u_{j,\xi^j}x, u_{j,\xi^j}y\rangle_{\text{class}} = A_j c_Q(\xi)\langle x, R_j^W y\rangle,$$

where R_j^W uses the same coefficients and cosets as R_j, acting on W_j. On each right-hand side use the Petersson pairing on the corresponding source. The two restrictions satisfy $\det R_j = \det R_j^W$.

For either lift U, write K for its spherical operator and c for the scalar multiplying its quadratic form. The Haar, Gaussian and finite factors give $c > 0$. The preceding series estimate gives

$$\|Uy\|^2 = c\,\mathrm{Re}\langle y, Ky\rangle \geq c(1 - \|K - I\|)\|y\|^2 \geq \tfrac{c}{2}\|y\|^2$$

in the respective source and target metrics.

34.2.3 The two spherical actions

In this subsection, we compare the actions of the coefficient series of Proposition 34.1 on the projected spaces.

For a double coset $K_A\gamma K_A$, its right cosets are represented by $k\gamma$ with k ranging over $K_A/(K_A \cap \gamma K_A \gamma^{-1})$. Because u_j is bi-invariant, the portion of the series on this double coset is exactly $u_j(\gamma)$ times its unnormalized Hecke operator. These operators commute with the finite residue action on the projected subspace.

Take finite subsums h_n over complete double cosets exhausting the series. They belong to the common finite spherical Hecke algebra, including its factors at nonsplit places. Their actions

converge in operator norm on the two projected spaces by absolute convergence and unitarity. For each positive integer k, (4) applied to h_n^k gives, on passing to the limit,

$$\operatorname{tr}(R_j^k) = \operatorname{tr}((R_j^W)^k).$$

The same trace identity at the unit gives equality of dimensions. Newton's identities now give $\det R_j = \det R_j^W$ with the original Petersson metrics on both spaces.

For the first arrow $a = a_{i,0}$, take $u \in K_0$ and write $u_{\mathfrak{m}}$ for its reduction at the chosen split members of $\mathfrak{m} = \pi_M Q(\xi)$. Lift $\operatorname{diag}(u_{\mathfrak{m}}, I_t)$ to an integral target element h, using inverse transpose at the conjugate members and the identity away from these pairs. Writing $h_{\mathfrak{m}}$ for its chosen reduction and $X_0 = \binom{I_r}{0}$, one has

$$h_{\mathfrak{m}}^{-1} X_0 u_{\mathfrak{m}} = X_0.$$

Thus the simultaneous action preserves the point indicator. The tame determinant factors cancel on the matched reductions, and the unramified factors fix the vacuum on integral elements off the modulus. More generally, writing λ for the unit scalar of the simultaneous action, covariance and the change of source variable $g = g'u$ give $R(h)a = \lambda a R(u)$. The integral elements normalize the respective principal levels. With their original Petersson pairings,

$$\langle aR(u)x, aR(u)y\rangle = \langle R(h)ax, R(h)ay\rangle = \langle ax, ay\rangle.$$

Pairing with every vector of the full triangle base gives $R(u)^{-1}a^*aR(u) = a^*a$. In particular the return preserves the J_M-fixed subspace of the $J_{\mathfrak{m}}$-fixed base. Identify the base copy by the isometry j_i. For every y in this entire original base, Proposition 34.1 and the uniform estimate $\|\mathcal{V}_i - I\| < 1/2$ give

$$\langle y, a^*ay\rangle = \|ay\|^2 = c\ \operatorname{Re}\langle y, \mathcal{V}_i y\rangle \geq \tfrac{c}{2}\|y\|^2, \qquad c = \frac{C_i}{w_0 B_\xi} > 0.$$

Thus the return is injective on the original base level. The first adjoint descent gives this same return on the next isometric base copy.

34.3 The period pairing for the four unitary groups

In this subsection, we write the composition of the two theta lifts as a paired period on the direct sum of the source and target spaces. This places both choices of intermediate space in the normalization fixed in Section 34.2.

Let U_1 and U_2 be the source and target Hermitian spaces and A and A' the two middle spaces. All quotient integrals below use the measures of the theta operators. Write $[B] = U(B)(E^+)\backslash U(B)(\mathbb{A}_{E^+})$. Set $W = U_1 \perp U_2$, with the source block first. Write $X = (X_1, X_2)$ and $Y = (Y_1^{\mathrm{t}}, Y_2^{\mathrm{t}})^{\mathrm{t}}$ for the two specified point arrays. Write

$$T_X = D_{2,X_2} L_{1,X_1}, \qquad T'_Y = L'_{2,Y_2} D'_{1,Y_1}.$$

Each arrow uses the indicator of the indicated block at the modulus and the original vacuum away from it. The two lower real polynomials are P'_1, P'_2, and the two upper real polynomials are P_1, P_2.

Write (a_i, b_i) for the upper splitting pairs and (b'_i, a'_i) for the lower splitting pairs. Put $N = \dim W = \dim A + \dim A'$ and $\gamma = \nu^{-1}\upsilon^{N/2}$, where υ is restricted to the norm-one ideles.

Each pair of opposite rank corrections has exponents $\lfloor p/2 \rfloor$ and $\lceil q/2 \rceil$, with $p+q=N$ even, in one order or the other. Their sum is $N/2$, and the product of the two factors before rank correction is ν^{-1}. Applying this calculation at the two kept ranks and at the two middle ranks gives

$$a_1 a_2 = a_1' a_2' = b_1 b_1' = b_2 b_2' = \gamma.$$

In particular $(\alpha, \alpha') = (b_1, b_1')$ satisfies the character hypothesis of Proposition 36.1. Apply that proposition on W with $V = A$ and $U = A'$. Its rank, signature and weight hypotheses for the chosen squares are checked in Section 34.3.3. For the fixed lower real tensor $v_- = P_1' P_2'$ (with its standard Fock Gaussian), the comparison supplies a real Fock vector v_+ of finite moments, before choosing the admissible modulus or character. For matched finite arrays (Φ, Ψ) it gives

$$\int_{[A]} K_{W,A}^{\gamma,b_1}(\Phi, v_+; g, h)\, dh = c_\xi \int_{[A']} K_{A',W}^{b_1',\gamma}(\Psi, v_-; h', g)\, dh' \qquad (g \in [W]). \tag{95}$$

Here $c_\xi \neq 0$. If $\dim A - \dim A' = 2t$, the comparison asserts

$$|c_\xi|^2 = C \prod_{w \mid \mathfrak{m}} q_w^{t^2} \prod_{k=1}^{t} |L^{S_\mathfrak{m}}(k, \nu)|^2, \qquad C > 0.$$

The constant C and v_+ may depend on the ordered real spaces, the fixed lower datum and archimedean weight. The finite characters and arrays are varied after this choice. Equation (95) is a period comparison for two kernels, with the individual original quotient measures of A and A'.

Set $\nu_0 = b_1/b_2 = b_2'/b_1'$. Our convention is $K^{a,b}(g,h) = a(\det h) b(\det g)^{-1} K^{1,1}(g,h)$. Multiplying the upper kernel with a pure real tensor by $\nu_0(\det g_2)$ changes the factor $b_1(\det g_2)^{-1}$ to $b_2(\det g_2)^{-1}$. Multiplying the lower kernel by the same factor changes $b_1'(\det g_2)$ to $b_2'(\det g_2)$. Use the product normalization of the rational theta functional on the two Hom blocks, as in Section 39.4.1. In a product rational polarization the rational points form the product of the two rational lattices. The two Heisenberg actions commute and their tensor product is the block action. Thus the normalized theta functional on a pure tensor is the product of its evaluations on the two factors. The Gaussian, the real pure tensor and the finite point indicator are products in those blocks. Thus the two corrections have the same sign.

Take the source vector f and the target vector z in their original real types, and set $f_1 = f$, $f_2 = \overline{z}$. Pairings of automorphic forms are linear in the second argument. For a real Fock vector v set

$$H_{\xi,X,f,z}(v) = \int_{[U_1]\times[U_2]\times[A]} f_1(g_1) f_2(g_2) \nu_0(\det g_2) K_{W,A}^{\gamma,b_1}(\mathbf{1}_X, v; g_1 \oplus g_2, h)\, dg_1\, dg_2\, dh.$$

Only primitive X are needed. Indeed, the support of a matched Φ is formed by the last rows of k^{-1}, and the support of its partner Ψ by the first columns of k, for invertible k. Thus X has a right inverse and Y has a left inverse at every factor of the nonempty modulus.

We prove a bound uniform in the integrated middle variable. Write $\rho = \mathrm{ht}_W(g)$ and $H = \mathrm{ht}_A(h)$ for the reciprocal minima as in Section 18.1. In the real Schrödinger model write u for the image of v. With the finite test fixed, reduction in both variables and Weil covariance of position and differentiation give the bound

$$|L_{g,h}(u)| \leq C(\rho H)^A s_j(u)$$

for the evaluation $L_{g,h}$ of the kernel and a fixed Schwartz seminorm s_j. Indeed the reduced finite translates range over a finite set and the real changes of variable have matrix and inverse norms bounded by powers of ρH. Take A to be a positive integer.

34.3.1 Uniform bounds for the two kernels

In this subsection, we bound the kernels in (95) by averaging over a rational isotropic subspace.

When $H > 2$, take a shortest vector d of the moved middle lattice. Its integral Hermitian norm has absolute value below one at every infinite place, so it is isotropic. In the kept lattice take an E-basis of vectors c_j with lengths $O(\rho^a)$, for a fixed $a \geq 1$. Indeed the real lattice has fixed covolume and first minimum at least ρ^{-1}, so Minkowski's second theorem bounds its last minimum by $O(\rho^{D-1})$ in real dimension D. Among D independent $\mathbb{Z}$-lattice vectors one can take an E-basis.

For the upper kernel average the theta functional over the adelic quotient of the rational isotropic space spanned by $dc_j^* J_W$, with probability Haar. In the moved finite coordinates these displacements are integral. As in Section 36.1.1, the modulations at a split place read $d_h^* \widetilde{X} C_g$, where C_g is the moved basis matrix and $\widetilde{X}$ is a local matrix in the support with residue X. This covector is nonzero over the local field: right invertibility of the residue implies right invertibility of $\widetilde{X}$.

For the lower kernel take the short isotropic vector c in the middle source and a basis d_j in the kept target. The modulations read $D_g^* \widetilde{Y} c_{h'}$, nonzero by left invertibility of $\widetilde{Y}$ over the local field. In either case, partition the compact finite support into pieces on each of which one modulation is a constant different from one. Invariance of the averaged functional under this modulation makes it vanish on the piece for every real vector u.

Rational Heisenberg invariance and additive strong approximation allow computing the average with bounded real coefficients of the rational generators and with finite coefficients in a fixed ideal of $\widehat{\mathcal{O}}_E$. The ideal can be chosen so that every such finite displacement fixes the finite test, by integrality of the generators in the moved coordinates. This yields an average A_h of real translations and modulations of size $O(\eta)$, where $\eta = \rho^a / H$. The vanishing gives $L_{g,h}(A_h u) = 0$ for every u. For $\eta \leq 1$, differentiating the real Heisenberg law along the segment from zero to each displacement gives $s_j((1 - A_h)u) \leq C_j \eta s_{j'}(u)$ with a higher fixed seminorm $s_{j'}$. Now

$$|L_{g,h}(u)| = |L_{g,h}((1 - A_h)^A u)| \leq C(\rho H)^A \eta^A s_{j''}(u) = C \rho^{A(a+1)} s_{j''}(u).$$

For $\eta > 1$ use $H < \rho^a$ in the initial bound. For $H \leq 2$ the initial bound suffices.

In the Fock coordinates the coherent-state formula is

$$v(w) = \pi^{-d} \int_{\mathbb{C}^d} v(y) e^{-(|w|^2 + |y|^2)/2 + \bar{y} \cdot w} \, dy.$$

The inverse Bargmann transform of the exponential is a translated and modulated normalized Gaussian, up to a unit scalar, by the Gaussian integral in the Bargmann transform. By Leibniz's rule and translation of the polynomial weights, its $s_{j''}$-seminorm is $O((1 + |y|)^N)$ for a fixed N.

Integration in that seminorm gives $s_{j''}(u) \leq C p_N(v)$. Thus

$$\left| K_{W,A}^{\gamma, b_1}(\mathbf{1}_X, v; g, h) \right| \leq C_X p_N(v) \, \mathrm{ht}_W(g)^B \qquad \text{uniformly in } h.$$

More generally, for every integer $D \ge 0$, the target height can be given the factor H^{-D}. For $H > 2$ and $\eta = \rho^a/H \le 1$, use $A + D$ differences. The preceding Schwartz estimate becomes

$$C_D(\rho H)^A \eta^{A+D} s_{j_D}(u) = C_D \rho^{A+a(A+D)} H^{-D} s_{j_D}(u).$$

If $H \le 2$ or $\eta > 1$, then $H \le 2\rho^a$, and the initial bound gives the same inequality after increasing C_D. The lower kernel uses the integral left inverse of Y in the transposed model, in place of the integral right inverse of X. The proof applies termwise to each finite sum of such residue tests. In particular the estimate is uniform in h' for $K_{A',W}^{b'_1,\gamma}(\mathbf{1}_Y, v_-; h', g)$. Here p_N is the weighted absolute Fock L^1 moment in the prescribed normalization. In the original block coordinates the moved integral lattice at $g_1 \oplus g_2$ is the direct sum of the two moved integral lattices. The length of a vector in the orthogonal sum dominates the length of each component, and each summand embeds isometrically. Thus its first minimum is the lesser of the two first minima and

$$\mathrm{ht}_W(g_1 \oplus g_2) = \max(\mathrm{ht}_{U_1}(g_1), \mathrm{ht}_{U_2}(g_2)).$$

The source tests are rapidly decreasing, and the quotient measures are finite. These inequalities give absolute integrability and, for each fixed $j = (\xi, X, f, z)$, the continuity bound

$$|H_j(v) - H_j(v')| \le C_j p_{N_j}(v - v').$$

The constants C_j, N_j can vary with j. In particular these bounds do not change the normalization of the quotient measures.

With lower real datum v_-, the block product formula and Fubini give the integral of the two transposed lower kernels against f_1, f_2, paired bilinearly on $[A']$. By transposition of the second lower arrow this is exactly $\langle z, T'_Y f\rangle_{[U_2]}$. The identical upper calculation with datum P_1P_2 gives $\langle z, T_X f\rangle_{[U_2]}$. These transpositions use the product quotient measure and introduce no indices.

For an arbitrary upper pure polynomial tensor Q_1Q_2, the real edge convolution identity proved below expresses the two lifts on $[A]$ as $C_{\alpha_1}F_1$, $C_{\alpha_2}F_2$, where F_j are the lifts with P_j, and

$$C_\alpha F(h) = \int_{U(A)_\infty} \alpha(k) F(hk)\, dk.$$

The two compactly supported real smoothing functions depend on the real edge and Q_1, Q_2, and precede the finite arrays and forms. The distinguished real types of F_1 and F_2 are conjugate. Primitivity concerns the combined matrix $X = (X_1, X_2)$ or $Y = (Y_1^{\mathsf{t}}, Y_2^{\mathsf{t}})^{\mathsf{t}}$. The Gaussian estimate below justifies the individual coefficient projections and proves square integrability of the two lifts.

Consider one edge of source and target ranks a, b', with the joint harmonic P and a rapidly decreasing source test. At each infinite place use the lowest compact type coefficient projectors e_S, e_T for the source and target lines, with the inverse in the source convolution as prescribed by the invariant pairing. Both kernels are integrable with weight $\|k\|^{ab'}$. Indeed the opposite-sign gaps in the types exceed $500d_*^3$, where d_* bounds the individual ranks, by the half-gap comparison of the base and inserted blocks. For the whole grid take $d_* = r + 1 + 2mt$, $t = 5r$. Indeed the node (z, a_1, a_2) has rank $r + z + (t + z)a_1 + (t - z)a_2$, with $0 \le a_1, a_2 \le m$ and $z = 0, 1$. This includes the two nodes at $(a_1, a_2) = (m, m)$. The gap before the inserted blocks is at least $\Delta/2$, and the subtractions are bounded by $5r + 2 + 2m$. Since $m = O(\sqrt{x})$, $r \ge x^5$, and $\Delta = 2r^8$,

$$500(2d_*)^3 + 5r + 2 + 2m < \Delta/2$$

for sufficiently large x. Reversing or conjugating the parameters preserves the gaps. Section 12.7 gives at least the exponent $ab' + 8d_*^2 + 1$.

The Cartan density of $U(p,q)$, $p+q \le d_*$, in coordinates $t_1 \ge \cdots \ge t_{\min(p,q)} \ge 0$ is a product of powers of $\sinh \alpha(t)$. The restricted roots have at most $2d_*^2$ total multiplicity and $|\alpha(t)| \le 2t_1$, so the density is $O(e^{4d_*^2 t_1})$. Since $\|k\| = e^{t_1}$ on the Cartan part, the stated moment is integrable. The compact cases are immediate.

Fix the finite test f_f in a rational Schrödinger polarization. Its compact support at the finite places restricts the rational positions to a single fractional lattice Λ in the real position space. Let G be the fixed Schrödinger Gaussian and let u be the inverse Bargmann transform of the weighted Fock vector v. A coherent vector at z gives, up to a modulation, the translate $G(\,\cdot - t_z)$. Consequently

$$\sum_{q \in \Lambda} |f_f(q) u(q)| \le \|f_f\|_\infty p_0(v) \sup_t \sum_{q \in \Lambda} |G(q-t)|.$$

The Gaussian series converges locally uniformly in t and is Λ-periodic, so its supremum is finite. Here $u(q)$ is the pointwise coherent integral. For $p_0(v) < \infty$ it gives a continuous position representative by dominated convergence against $|v(z)|$. For a Hilbert Fock vector it agrees with the inverse Bargmann transform by the coherent resolution of the identity. Tonelli applied to the Gaussian majorant proves the displayed inequality also for the projected vectors used below.

Write $d = ab'$ at one real place, let v_0 be the Fock vacuum, and set $p_0(v) = \pi^{-d} \int |v(z)|\, dz$, including the Gaussian in v. For the real linear symplectic transformation σ of the Hom space, write its positive singular values in pairs e^{t_l}, e^{-t_l} with $t_l \ge 0$, $1 \le l \le d$. The two orthogonal symplectic factors in its singular-value decomposition preserve p_0 and preserve the vacuum line. In the resulting coordinates the Gaussian kernel of Section 34.1.2 gives the transformed vacuum, up to phase, as

$$\prod_{l=1}^{d} (\cosh t_l)^{-1/2} \exp\left(-\frac{|z_l|^2}{2} + \frac{\tanh t_l}{2} z_l^2 \right).$$

Integration in $\Re z_l, \Im z_l$ contributes a factor $(1 - \tanh^2 t_l)^{-1/2} = \cosh t_l$ relative to the unsqueezed Gaussian. Thus

$$\frac{p_0(\omega(\sigma) v_0)}{p_0(v_0)} = \prod_l (\cosh t_l)^{1/2} \le \|\sigma\|^{d/2}.$$

For a coherent vector $\rho(y) v_0$, covariance gives $\omega(\sigma)\rho(y)v_0 = \rho(\sigma y)\omega(\sigma)v_0$. Heisenberg displacements preserve p_0. Applying the coherent integral first to the input and then the fixed-lattice theta bound to its image therefore gives, for the theta functional L,

$$|L(\omega(\sigma)v)| \le C_L p_0(\omega(\sigma)v) \le C_L' \|\sigma\|^{d/2} p_0(v).$$

The phases of the unitary splitting characters have modulus one. Reduce the source point as in Section 18.1.

For a fixed target h the resulting kernel at (g, hk) is bounded by

$$C_h \operatorname{ht}_S(g)^B \prod_{v \mid \infty} \|k_v\|^{ab'}.$$

This follows from $\|\sigma_v\| \le \|(hk)_v\| \|g_v^{-1}\|$ and the finitely many finite translates in the reduction. Integrating the rapid source test gives the same bound in k for the lift. For either edge write

$\mathcal{L}_f(v;h) = \int_{[S]} f(g)K(v;g,h)\,dg$ for its bilinear source pairing after the determinant correction above. For $k \in U(S)_\infty$ and $k' \in U(T)_\infty$, use $R(k)f(g) = f(gk)$. Covariance and the source substitution $g' = gk$ give the two translation laws

$$\mathcal{L}_f(\omega(k,1)v;h) = \mathcal{L}_{R(k^{-1})f}(v;h), \qquad \mathcal{L}_f(\omega(1,k')v;h) = \mathcal{L}_f(v;hk').$$

Put $e_S^\vee(k) = e_S(k^{-1})$. In the scalar realization of the source forms the unit highest line is cut out by $R(e_S^\vee)$, with the contragredient coefficient for the bilinear pairing, as in Section 34.2.2. Thus $R(e_S^\vee)f = f$ on every copy of that line. The analogous weighted bound in the source real translation gives Fubini against e_S; the target bound gives Fubini against e_T. Integrating the two laws yields

$$\mathcal{L}_f(p_S v;h) = \mathcal{L}_f(v;h), \qquad \mathcal{L}_f(p_T v;h) = \big(R(e_T)\mathcal{L}_f(v;\cdot)\big)(h).$$

The calculation holds for every multiplicity vector at the original level, including the conjugate test on the second edge. All the translations used in this projection have finite component one.

34.3.2 The real line projections

In this subsection, we derive the real convolution identity, using the uniform bounds of Section 34.3.1 to justify integration.

On the real oscillator, let p_S, p_T be the line projections given by these coefficients, with the dual source representation used in the pairing. We first verify the parameter rule and $p_S P \neq 0$ for the following block insertions, including the column edges of Section 34.3.3. At fixed offset o in grid $z \in \{0,1\}$, a node with coordinates (u,v) has rank $N = r + z + u(t+z) + v(t-z)$. The triangle position (i,n) uses offset $o = i$ and grid coordinates $(0,n)$. Its unlowered doubled entries are the stem $2\lambda_a + z$ and the block entries $2c + d - 1 - 2b$, $0 \le b < d$, with $d = t + z$ for an up block and $d = t - z$ for a down block. The lowering ϵ_N is 1 at odd $u+v$ in grid 0, and 0 otherwise. For either kind of edge the opposite rank N' has parity ϵ_N.

For a down insertion at c, the opposite determinant exponent is $c + \lfloor N'/2 \rfloor$. Its shifted entry is therefore $(u - \epsilon_N) - 2(c + \lfloor N'/2 \rfloor) + N' = u - 2c$ for an unlowered entry u. For an up insertion at c, the conjugate parameter and opposite exponent $-c + \lceil N'/2 \rceil$ give $(-u + \epsilon_N) - 2(-c + \lceil N'/2 \rceil) + N' = -u + 2c$. Thus the matched shifted entries coincide. The inserted entries form the length d compact block with a single signature sign. The signature of a retained entry contains one factor -1 per absent down centre above it and per absent up centre below it. Inserting c changes this factor precisely for a negative shifted entry. With coordinates (o,u,v) for the offset and the two block counts, the insertions are

step	centre	length
$(0,u,v) \longrightarrow (0,u+1,v)$	w_{u+1}	$t+z$
$(o,u,v) \longrightarrow (o,u,v+1)$	$-w_{o+v+1}$	$t-z$
$(o+1,0,v) \longrightarrow (o,0,v+1)$	$-w_{o+1}$	$t-z$.

In a rectangular grid $o = 0$; in a triangle $u = 0$. For the last insertion give the smaller space the opposite form to put both sign rules over $(-1)^o$. At stem index p, let $a = (p+1-k)_+$. The first a down centres are above this entry, and $\min((a-o)_+, v)$ of them are present. Its sign is $(-1)^{o+a+\min((a-o)_+,v)}$. A present up block has no missing centre contributing to its sign. For a present down block the o missing down centres above it cancel the offset sign. Thus the

inserted blocks are positive in the arithmetic signatures. Opposite signs at a node require a missing centre strictly between the two entries. In doubled coordinates each entry is at least $\Delta - t - 2$ from that centre, by the half-gap and the block widths in Section 42.2. This gives the undoubled gap used in Section 34.3.1.

For the prescribed opposite characters and parameters use the substitution

$$(l, l', s, s'; \gamma, \gamma') \longmapsto (-l, -l', -s, s'; N - \gamma, N' - \gamma'),$$

where s, s' are the two sign vectors for this insertion. The shifted entries change sign and the smaller signature returns to its original one. The retained rule is again equal signs at positive entries and opposite signs at negative entries. On reversing an arrow the two commuting actions still act on the same harmonic polynomial. Interchange of the two factors in the bilinear theta pairing takes both coefficient representations to their contragredients. The two kernels of the reversed arrow are consequently $\overline{e_T}, \overline{e_S}$ in this order.

To apply Li's regular discrete-series correspondence [44] in the form of [61, Theorem 2.7(b), p. 641], remove the determinant twists and use the tensor realization $\mathrm{Hom}(S, T) = S^\vee \otimes T$: on the source dual the compact weights change sign and their order in each compact factor reverses. More explicitly, if s_a, s'_b are the two signatures, the vacuum weights after removing the scalar twists are $-s_a \sum_b s'_b/2$ and $s'_b \sum_a s_a/2$ in the Hom model. Dualizing the first factor gives exactly the two vacuum weights of [61, Lemma 2.5]. The doubled stem entries have parity z, since k is even and the λ_a are integral. Every block has the same parity, since t is odd and its length is $t \pm z$. Shifting by $2c$ and negating preserve this parity. Thus for rank difference $d = t \pm z$ the genuine Harish–Chandra entries lie in $\mathbf{Z} + (d-1)/2$, as required on both groups. Let $n_{\epsilon,\eta}$ count the retained entries with signature ϵ and shifted sign η on the smaller side. They contribute

$$(n_{+,+} + n_{-,-},\ n_{-,+} + n_{+,-})$$

to the larger signature. The remaining entries are $(d-1)/2, (d-3)/2, \dots, (1-d)/2$, all in one compact factor. In fact $|\xi_a| \geq \Delta - 1$ at every retained index. For a tail entry of excess h, its difference from the doubled down centre $-2w_j$ is $(2j - 2h - 1)\Delta + z$, by the stem formula in Section 42.2. An up centre is positive and has distance greater than Δ from the tail. The doubled head has modulus at most $k+1$, whereas $w_j \geq \Delta^2$. For an earlier block the distinct centres differ by at least Δ, and its doubled half-width is at most t. This gives $|\xi_a| \geq 2\Delta - t$ on all retained blocks. In particular $|\xi_a|/2 \geq (d+1)/2$. Use the whole-grid rank bound $N, N' \leq d_* = r+1+2mt$ of Section 34.3.1. Its inequality $500(2d_*)^3 + 5r + 2 + 2m < \Delta/2$ implies the required undoubled separation $(\Delta - t - 2)/2 > 500(N + N')^3$. The bound $|\xi_a|/2 \geq (d+1)/2$ and the two choices of the signature of the inserted block are the hypotheses of [61, Theorem 2.7(b)]. The stronger separation is used for the coefficient moment in Section 34.3.1. Simultaneous reversal of the two Hermitian forms preserves the correspondence. Reversing just one form conjugates the oscillator and takes the contragredient on both representation factors [61, Proposition 2.9 and Corollary 2.10].

Write ξ_a for the doubled shifted entry at a retained index and ξ_b also for the doubled entries of the inserted string. Let C_a and R_b^{wt} be the relative lowest compact weights of the dual smaller factor and the larger factor. The lowest-type formula $\lambda + \rho - 2\rho_K$ and the vacuum weights above

give

$$2C_a = -\xi_a + s_a \left(\sum_{a'} s_{a'} \operatorname{sgn}(\xi_a - \xi_{a'}) + \sum_{b'} s'_{b'} \right),$$

$$2R_b^{\mathrm{wt}} = \xi_b - s'_b \left(\sum_{b'} s'_{b'} \operatorname{sgn}(\xi_b - \xi_{b'}) + \sum_{a'} s_{a'} \right).$$

Here a' ranges over the smaller parameter and b' over the larger parameter. Set $\eta_a = \operatorname{sgn} \xi_a$, so $s'_a = s_a \eta_a$ at a retained index. For another retained index a', its contribution to $2(R_a^{\mathrm{wt}} + C_a)$, divided by $s_a s_{a'}$, is

$$(1 - \eta_a \eta_{a'}) \operatorname{sgn}(\xi_a - \xi_{a'}) + \eta_{a'} - \eta_a = 0.$$

For an inserted index b, the contribution divided by $s_a s'_b$ is $1 - \eta_a \operatorname{sgn}(\xi_a - \xi_b) = 0$. At an inserted index b, the retained terms in $2R_b^{\mathrm{wt}}$ cancel because $\operatorname{sgn}(\xi_b - \xi_a) = -\eta_a$. The inserted entries have one compact sign and spacing two, so

$$s'_b \sum_{b' \text{ inserted}} s'_{b'} \operatorname{sgn}(\xi_b - \xi_{b'}) = \xi_b.$$

Thus $R_a^{\mathrm{wt}} = -C_a$ at retained indices and $R_b^{\mathrm{wt}} = 0$ on the inserted string. Also

$$|2R_a^{\mathrm{wt}} - \xi_a| \le N + N' - 1 < |\xi_a|, \qquad \operatorname{sgn} R_a^{\mathrm{wt}} = \eta_a.$$

In either compact order let L_i^* be the decreasing doubled parameter entries and μ_i the relative weights. For $i < j$ of the same signature, let c be the number of indices of that signature in $(i, j]$. Their common vacuum correction cancels. The successive entry gaps are at least two, so

$$\mu_i - \mu_j = (L_i^* - L_j^*)/2 + (j - i) - 2c \ge 2(j - i - c) \ge 0.$$

In each sign quadrant take the nested products of minors on the matched positions of degree $|R_a^{\mathrm{wt}}| \ge h$, $h \ge 1$. These are initial segments in the compact order for positive weights and in the dual order for negative weights. In the signed Fock coordinates every compact raising operator kills a minor or replaces an index by one already present in it. At the matrix with ones on the matched pairs and zeros elsewhere the minors have determinants ± 1. Each source or target index occurs with one oscillator sign, so the mixed derivatives with opposite signs and a common source or target index kill P. The two lowest compact types occur together with multiplicity one in the joint harmonics by [61, Theorem 2.2(1) and Lemma 2.5].

The positive Bargmann pairing and compact Schur averages of Section 34.1.2 now apply to either group as source. On $\operatorname{Hom}(S, T)$, a positive source matrix g acts by $X \mapsto X g^{-1}$ and a positive target matrix h acts by $X \mapsto hX$. Their matrices on the tensor factors are $g^{-\mathsf{T}}$ on the dual source and h on the target, both Hermitian positive. At either positive factor the determinant twist is one, the pure quadratic terms vanish by the mixed derivative calculation, and the mixed term gives the squared Fock norm after an invertible substitution, times the positive vacuum coefficient. The two compact Schur averages against the source matrix coefficient consequently have strictly positive integral at each positive factor. Integrate in the polar decomposition and multiply by the positive formal degree. The coefficient moment verified above justifies this step and gives $\langle P, p_S P \rangle > 0$.

Write $c_S(g) = \langle v_S, \pi_S(g) v_S \rangle$ for the coefficient in the source oscillator convention. The convolution calculation of Section 12.3 gives $p_S \omega(g) p_S = c_S(g) p_S$, and similarly for the target.

For any unit $u \in \operatorname{im} p_S$, its source translates therefore have the same Gram matrix as the translates of v_S. Their closed span is a copy of V_S under $\pi_S(g)v_S \mapsto \omega(g)u$. In particular $p_S P$ belongs to the full source discrete-series isotypic component.

Let Q_S project onto this component of the oscillator Hilbert space $\mathcal{H}_\omega$. This is a central projection in the source von Neumann algebra. On $Q_S\mathcal{H}_\omega = V_S \widehat{\otimes} M$, the source algebra is $\mathcal{B}(V_S)\overline{\otimes}1$. Howe's commutant theorem [38, Theorem 6.1, p. 550], restricted by this central projection, identifies the target algebra with $1\overline{\otimes}\mathcal{B}(M)$. Thus M is irreducible and admissible. Scalar twists leave these algebras unchanged.

Here admissibility follows from Harish–Chandra's admissibility theorem for irreducible unitary representations of real reductive groups, applied to the target and its metaplectic cover. The map Q_S intertwines both derived actions on Fock polynomials and has dense polynomial image in $V_S \widehat{\otimes} M$. Projection onto any fixed $K_S \times K_T$-type preserves the polynomials, since the compact action preserves degree. The image fills this finite-dimensional isotypic space by density and admissibility. Consequently $(V_S)_{K_S} \boxtimes M_{K_T}$ is a quotient of the polynomial oscillator module. Its target factor is the one identified by the preceding discrete-series correspondence, using the smaller group first and reciprocity for the reversed orientation.

On this component $p_S = P_{v_S} \otimes 1$, where v_S spans the source compact highest line. Since p_S commutes with the target action, $0 \neq p_S P$ has the target compact highest weight computed above. For a noncompact real factor the corresponding lowest compact type occurs once by [5, Theorem 8.5, p. 42] on the derived group. Adjoining the compact central torus with its fixed character preserves this multiplicity. For a definite factor the highest line is one-dimensional by highest-weight theory. Thus the highest line of the target type is one-dimensional and $p_T p_S P = p_S P$.

The nonzero vector $p_S P$ is cyclic in the target representation. Smearing its translates by smooth compactly supported functions has dense linear span. Project this span onto the finite sum of target compact types of $p_S Q$, for a polynomial Q. This sum is finite dimensional by admissibility. The projected span is therefore the whole sum, and its projection is still obtained by smooth compactly supported convolution, by averaging the convolution functions over the target compact group with the compact-type character projectors. Thus one such function sends $p_S P$ to $p_S Q$.

Use the weighted Fock estimate of Section 34.3.1 to evaluate this identity by theta summation. On a compact neighborhood of any real group element the Gaussian quadratic forms of the moved vacuum have uniformly positive real part in the decaying exponent. Consequently the moved vacuum is continuous in p_0 by dominated convergence. For each fixed displacement the same holds for its translate, by covariance and continuity of Heisenberg translations and modulations in L^1. Their p_0-norms are bounded on this neighborhood independently of the displacement. Integrating against the coherent coefficient of v gives strong p_0-continuity, with $p_0(\omega(g)v) \le C\|g\|^{ab'} p_0(v)$ as above. In the Hilbert norm the action is strongly continuous and unitary. The coefficient moment proved in Section 34.3.1 and the compact support of the smoothing function therefore give convergence of the source projection and smoothing integrals in $L^1 \cap L^2$ with norm $p_0 + \|\cdot\|_2$. Thus $p_S Q = \omega(\alpha) p_S P$, for the chosen smoothing function α, holds in this space: the L^2 identity identifies the two L^1 representatives.

At fixed finite datum, the coherent Gaussian lattice bound in Section 34.3.1 makes theta evaluation continuous for p_0. At the translated source points its bound is multiplied by at most the fixed power of the source height displayed there, which is integrable against the rapidly decreasing source test. Hence we may apply the pointwise theta lift to the smoothing identity

and interchange the convergent integrals. Taking the product at the infinite places gives the asserted replacement of the real polynomial lifts. The same weighted coefficient bound applies to the target projector p_T. Evaluating $p_T p_S P = p_S P$ therefore gives $R(e_T)F = F$ as an absolutely convergent pointwise integral for the lift F of P, at the same finite datum and with the same quotient Haar measures.

Fix a compact open subgroup J of the finite target group that fixes the chosen finite Weil vector with its splitting character. Smoothness gives such a J. Then F and all $F_j^{\mathrm{tr}} = F\mathbf{1}_{\{|F|\le j\}}$ are right J-invariant. Put $F_j = R(e_T)F_j^{\mathrm{tr}}$. The quotient has finite volume, so $F_j^{\mathrm{tr}} \in L^2$, and convolution gives

$$\|F_j\|_\infty \le j\|e_T\|_1, \qquad R(e_T)F_j = F_j.$$

The last identity follows from $e_T * e_T = e_T$ by Fubini. The convolution gives a continuous representative on each real component, since translations of e_T are continuous in L^1. Real convolution commutes with the finite right action of J, so the same J fixes every F_j.

Apply Section 18.1.3 with $y = F_j$. On each arithmetic component its coefficient integral intertwines the Harish–Chandra module of the target discrete series with right translation and sends the distinguished vector to F_j. The matrix coefficients and their derivatives are integrable, so the image consists of smooth, compact-finite automorphic forms of the given infinitesimal character. Each is bounded by $\|F_j\|_\infty$ times the L^1-norm of its coefficient, including the formal-degree factor, and hence is square-integrable. Restricting to the arithmetic quotients of the derived group as in Section 18.1.3, Wallach's theorem gives cuspidality of this tempered module [76, Theorem 4.3, p. 233]. The adelic constant terms are the corresponding real integrals by the same unipotent approximation. All F_j have the same finite level, archimedean compact type and infinitesimal character. As in Section 18.1.3, there are finitely many arithmetic quotient components. On each, the compact centre acts by the fixed character, so restriction to the derived group determines the form. The restricted compact types form a finite set, and the kernel of the derived infinitesimal character has codimension one in the centre of its enveloping algebra. Intersecting the arithmetic subgroup of each component with the special unitary group gives an arithmetic subgroup of the derived group. Inversion identifies its left quotient with the right quotient in [34, Ch. I, §2, Theorem 1], intertwining right translations with left translations. Boundedness gives the required moderate growth. Finiteness on these components therefore places all F_j in a single finite-dimensional cusp space.

For each fixed h, all truncations have the integrable majorant

$$C_h|e_T(k)| \prod_{v|\infty} \|k_v\|^{ab'}$$

by the preceding coefficient moment bound. Hence $R(e_T)F_j^{\mathrm{tr}}(h) \to R(e_T)F(h) = F(h)$. Take finitely many evaluations that are injective on the finite-dimensional space. Convergence at these points gives convergence of the coefficients in a basis, so F lies in that same space. In particular the two individual upper lifts and the two transposed lower lifts are square-integrable. They are rapidly decreasing by Section 18.1.3.

For fixed real elements s, t of the source and target groups, write $\omega_S(s) = \omega(s, 1)$, $\omega_T(t) = \omega(1, t)$. The central source projection Q_S commutes with the source compact group. Hence $Q_S P$ has the compact highest weight of P, that of the lowest compact type of V_S. The same multiplicity result gives one occurrence in V_S, with highest line $\mathbb{C}v_S$. Thus $Q_S P \in \mathbb{C}v_S \widehat{\otimes} M$ and $Q_S P = p_S P$, for the full multiplicity space M. Since $p_S = p_S Q_S$ and Q_S commutes with

$\omega_S(s)$, the source compression and the two translation laws give

$$p_S\omega_S(s)P = c_S(s)p_SP, \qquad \mathcal{L}_f(\omega_T(t)\omega_S(s)P) = c_S(s)R(t)F.$$

With $e_T^t(k) = e_T(t^{-1}kt)$, substitution $k = tlt^{-1}$ in the convolution gives

$$R(e_T^t)R(t) = R(t)R(e_T).$$

Right translation by t preserves the finite level and commutes with the left unipotent integrations defining the constant terms. The lattice norms at h and ht differ by bounded factors depending on t, t^{-1}, so $H(ht) \asymp_t H(h)$ and $R(t)F$ is again rapidly decreasing.

Write e_A for the normalized matrix coefficient of right translation on the real line of F_1. With the projector kernel e of this type, its argument convention is $e_A(g) = e(g^{-1})/\overline{e(1)}$. Then

$$\int_{[A]} F_2(u)F_1(ug)\,du = e_A(g)\int_{[A]} F_2(u)F_1(u)\,du \qquad (g \in U(A)_\infty).$$

Indeed $R(e)F_1 = F_1$, and the conjugate distinguished type of F_2 gives $R(e)\overline{F_2} = \overline{F_2}$. The integral on the left is $\langle \overline{F_2}, R(g)F_1\rangle$. Inserting the self-adjoint projector on both sides and using $R(e)R(g)R(e) = e_A(g)R(e)$ from Section 12.3 proves the identity on every multiplicity space.

The change $u = hk_2$ preserves the Haar measure of $[A]$. Consequently, for real edge data Q_1, Q_2,

$$H_j(Q_1Q_2) = r(Q_1, Q_2)\langle z, T_X f\rangle_{[U_2]},$$
$$r(Q_1, Q_2) = \int_{U(A)^2_\infty} \alpha_1(k_1)\alpha_2(k_2)e_A(k_2^{-1}k_1)\,dk_1\,dk_2.$$

Fubini here also follows directly from Cauchy–Schwarz, unitarity of the real translations on $[A]$, and compact support of the smoothing functions. The integrals are bilinear, so neither smoothing function is conjugated. Every polynomial in both blocks is a sum of pure tensors, and the same conclusion holds with a scalar $r(Q)$.

To approximate v_+, use the signed Fock coordinates $\zeta(w)$ and Gaussian $G(w) = e^{-|w|^2/2}$. For each m, the degree-m Taylor kernel

$$G(w)G(y)\sum_{k=0}^{m} \frac{(\overline{\zeta(y)}\cdot\zeta(w))^k}{k!}$$

integrated against $v_+(y)\,dy/\pi^d$ gives a polynomial Fock vector $v_m(w)$, where d is the complex dimension. The full exponential is the Fock reproducing kernel, by the Gaussian integral for orthogonal monomials. The absolute value of each partial kernel and of its limit is at most $e^{-(|w|-|y|)^2/2}$.

For every N there is a constant $C_{d,N}$ such that

$$\int_{\mathbb{C}^d} (1+|w|)^N e^{-(|w|-|y|)^2/2}\,dw \le C_{d,N}(1+|y|)^{N+2d-1}.$$

Indeed, use polar coordinates and bound the polynomial in the radial variable r by a constant times $(1+|y|)^{N+2d-1}(1+|r-|y||)^{N+2d-1}$. For $d = 0$ the Fock space consists of constants. Thus dominated convergence in (y, w) proves $p_N(v_m - v_+) \to 0$ for every N, using the finite moments of v_+. Set $B_j = \langle z, T_X f\rangle_{[U_2]}$.

If some $B_{j_0} \neq 0$, the identities for polynomials give

$$H_j(Q)B_{j_0} = H_{j_0}(Q)B_j.$$

Continuity for each j gives the same formula at v_+; put $\kappa = H_{j_0}(v_+)/B_{j_0} \in \mathbf{C}$. If all the B_j vanish, all the $H_j(v_+)$ vanish, and we may take $\kappa = 0$. In both cases

$$H_j(v_+) = \kappa\langle z, T_X f\rangle_{[U_2]} \qquad \text{for every } j.$$

This gives one κ for the entire family.

Integrate (95) at $g = g_1 \oplus g_2$ against $f_1(g_1)f_2(g_2)\nu_0(\det g_2)$. Expanding the two finite point arrays and applying the preceding identities yields

$$\kappa \sum_X \Phi(X)\langle z, T_X f\rangle_{[U_2]} = c_\xi \sum_Y \Psi(Y)\langle z, T'_Y f\rangle_{[U_2]}.$$

The target pairing is nondegenerate. Thus the two composites on the original four Hermitian spaces satisfy the matched operator identity

$$\sum_Y \Psi(Y)T'_Y = \frac{\kappa}{c_\xi}\sum_X \Phi(X)T_X.$$

This argument permits $\kappa = 0$. Its needed nonvanishing for the forward triangle follows from injective first returns and the weighted chain argument.

34.3.3 Application to the unitary groups

In this subsection, we apply the two pairings of Section 34.3.2 with the original levels and measures.

For the forward square the nodes U_1, U_2, A, A' are

$$(i, n+1), \qquad (i+1, n+1), \qquad (i, n+2), \qquad (i+1, n),$$

respectively. These are triangle positions, so each has first grid coordinate zero. For grid parameter $z_0 \in \{0, 1\}$, the rank in column j is $d_j = r + z_0 + j(t_0 - z_0)$. Hence $\dim W = 2d_{n+1} = d_n + d_{n+2}$ and the exchange defect is $t = t_0 - z_0$. The signs of the added blocks are positive. In the stem, the head stays on the same side of every inserted centre at each infinite place. The relative positions of the tail entries and centres are independent of τ, since the common κ_τ-shift cancels. Thus all negative counts here are constant across the infinite places. On a fixed base coordinate with excess a, the sign in column j at offset i is

$$(-1)^{i+a+\min((a-i)_+, j)}.$$

Moving $(i, n+1)$ to $(i, n+2)$ and $(i+1, n+1)$ to $(i+1, n)$ flips both signs exactly when $a - i \geq n + 2$. In that case the two old signs are opposite. Thus the number of negative signs agrees between the two diagonals, as the period comparison requires. That number is at most $2(r + z_0) < d_{n+1}$, using $t_0 \geq r + 6$. The inverse quotient character in the comparison has weight $(n+1)$ times the original gap at each real place. The inequality $d_{n+1} \leq (n+1)\Delta$ gives $\dim W \leq 2(n+1)\Delta + 2q$, with q the number of negative signs in W.

For the reversed square in the $z_0 = 0$ triangle, the source and target are $(i+1, n)$ and $(i, n+2)$. The upper and lower middle nodes in the ordered comparison are then $(i+1, n+1)$

and $(i, n+1)$. Their ranks are equal, so $t = 0$ in the measured period comparison. Interchanging the two diagonals preserves their common negative index q and their common total rank $2d_{n+1}$. The inverse quotient character has weight $(n+1)\Delta$, so

$$2q + 2 \le 2d_{n+1} \le 2(n+1)\Delta + 2q.$$

Apply the same paired calculation with the two reversed Fourier edge data, with the two row blocks in their prescribed order. The modulus of the period scalar is constant on this family.

To identify the adjoint consequence, let A_0, A_1 be the two reversed lifts with the Fourier data of the original test and its extended test, B_0, B_1 the corresponding transposed descents, and L_+, L_- the upper and lower ordinary lifts. Let S_n, S_{n+1} be the two original central reversal translations. Write the two reflection identities, the reversed square and the central intertwining as

$$\begin{aligned} \langle B_0 x', y\rangle &= d_0 \langle x', A_0 S_n^{-1} y\rangle, \\ L_+ A_0 &= g_0 A_1 L_-, \\ L_- S_n^{-1} &= \eta S_{n+1}^{-1} L_-, \qquad |\eta| = 1, \\ \langle x, A_1 S_{n+1}^{-1} u\rangle &= d_1 \langle B_1 x, u\rangle. \end{aligned}$$

For $x' = L_+^* x$, the second-linear convention gives $\langle x', v\rangle = \langle x, L_+ v\rangle$. It follows directly that

$$\langle B_0 x', y\rangle = d_0 g_0 \eta d_1 \langle B_1 x, L_- y\rangle.$$

In particular none of the four scalars is conjugated in this passage. The factors in d_0, d_1 come from the two matrix reversals (MR). For matched arrays the scalar of the reversed square is κ/c_ν, with κ the real projection scalar and c_ν the equal-rank period scalar. For the Fourier data in A_0, A_1, Section 36.2.2 gives

$$|g_0| = \frac{|\kappa|}{|c_\nu|\,|\mathcal{G}|\,|R|^{d_n t_0}}, \qquad R = \mathcal{O}_E/(\pi_M Q(\xi)), \qquad |\mathcal{G}|^2 = |R|^{t_0^2}.$$

Here the triangle has $z_0 = 0$, so its column increment is t_0.

35 Real restrictions and self-dual Fourier tests

In this section, we compute the local restrictions of the paired periods of Section 34.3. The real slice and the finite self-dual transform give the normalization for the exchange in Section 36.

35.1 The real restriction of the larger period

In this subsection, the real integrals in the paired periods of Section 34.3 involve the original Fock vectors. We evaluate the larger and smaller real slices with their quotient measures to determine the factor in the exchange.

Let $W_0 = H \perp W$ have rank $2j$, let $B_0 = U^-$ have rank b, and let $X = H^b \perp W_0$. Use the original two isotropic bases of X and the original mixed Schrödinger–Fock models for its pairs with $B_0 \perp W_0$ and B_0. Fix the lower real Schwartz vectors v for (W_0, B_0) and v_1 for (W_0, W_0). Fix the common infinity weight $\mu = (\mu_\tau)_\tau$. It is positive on the admissible family, so

the exponents $2\mu_\tau$ below are nonnegative integers. All matrices in the real formula are families at the infinite places. Let $J = \operatorname{diag}(J_1, J_2)$ be the matrix of $B_0 \perp W_0$, and set $x_0 = (1; 0)$.

The two real vectors. Near x_0, choose a smooth unitary $k(y)$ carrying $\operatorname{im} x_0$ onto $\operatorname{im} y$. For the graph of $T : \operatorname{im} x_0 \to W_0$, the derivative at zero of the graph chart on the action of

$$\exp \begin{pmatrix} 0 & -J_1^{-1} T^* J_2 \\ T & 0 \end{pmatrix}$$

is $T \mapsto T$. The real inverse function theorem produces the section on a neighborhood. Multiplying its Lie-algebra coordinates by a cutoff and exponentiating extends k smoothly while keeping it unitary. If $b > 0$, take at each real place a ball about $x_{0,\tau}$ with closure in this chart such that on the closure

$$\|(x_{0,\tau}^\dagger x_{0,\tau})^{-1}(x_\tau^\dagger x_\tau - x_{0,\tau}^\dagger x_{0,\tau})\| < 1/2.$$

Such balls exist by continuity and nondegeneracy of the fixed first block. Take a smooth nonnegative σ, supported in their product, with $\sigma(x_0) = 1$. For an empty first block use its unit function. These choices use only the prescribed real bases.

For $x^\dagger = x^* J$ choose a smooth matrix function ι equal to inversion on a neighborhood of $x_0^\dagger x_0$ and of all the Gram matrices from the support of σ. This is obtained by multiplying inversion by a smooth cutoff supported where the determinant is nonzero. Set

$$P(x) = x\, \iota(x^\dagger x) x^\dagger x_0, \qquad x^\flat = k(P(x))^{-1} x.$$

The large real vector Φ and small real vector Λ have the following slices in the initial polarization:

$$\Phi[x] = \kappa\, \sigma(P(x)) \sigma(x^\flat) \prod_\tau \overline{\det(x_\tau^\flat)_1}^{2\mu_\tau}\ \omega_{j-\mu}(1, k(P(x)))(v \boxtimes v_1), \tag{96}$$

$$\Lambda[B] = \kappa\, \sigma(x_0 B) \prod_\tau |\det B_\tau|^{2(j+\mu_\tau)}\, v. \tag{97}$$

In the first formula $(x_\tau^\flat)_1$ is the first b-block. The second-group determinant exponent in $\omega_{j-\mu}$ is $j - \mu_\tau$ at each place.

These sections are Schwartz in the mixed model. In the first formula $x = k(P(x))x^\flat$, so the support is contained in the compact image of the two bump supports. All scalar factors are smooth, including at rank-deficient matrices, since no inverse is taken outside the invertible Gram neighborhood. On that compact support the Weil operators form a smooth compact family on Schwartz functions. For every derivative and Schwartz seminorm this follows locally from the linear-change, quadratic-multiplier and Fourier formulas for the metaplectic action.

The small section is supported on a compact set because $B \mapsto x_0 B$ is a closed linear embedding. The mixed-model isomorphism therefore gives Schwartz vectors with these slices. Equivalently, in signed Fock coordinates the condition on the real source can be stated as finiteness of all weighted absolute moments. To see that this condition gives the Schwartz estimates, use the coherent-state reproducing integral. The modulus of its kernel in real coordinates x, y is $e^{-|x-y|^2/2}$, and differentiation contributes a polynomial in x, y. For any prescribed weighted derivative, bound the resulting Gaussian times the weight by a fixed polynomial in $|y|$, uniformly in x, and use finiteness of the corresponding moment.

In the opposite source polarization the zero slice is the integral of the slices in the initial polarization, with the Fourier measure of the chosen bases. To compute its normalization for

Λ, write $t_\Lambda(B) = \sigma(x_0 B) \prod_\tau |\det B_\tau|^{2(j+\mu_\tau)}$. Choose κ so that the real integral of κt_Λ is the real integral of the scalar vacuum slice. The integral of t_Λ is positive since it is continuous, nonnegative, and equal to one at the identity.

The Gaussian vacuum integral is nonzero. With any finite complementary test, the adelic opposite zero slice factors into this real integral and the same finite slice integral as for the vacuum. The linear change of slice variables carries the self-dual Haar to the same positive multiple of the product real and finite Haar in these two evaluations. Equality of the real integrals therefore gives equality for every finite test. The vacuum has the normalized opposite zero slice equal to the smaller vacuum. Consequently the opposite zero slice of Λ is v. This normalization uses the actual mixed model and places no condition on the finite data later used in the period.

A single translate for every test. Fix a real point $h \in U(B_0 \perp W_0)_\infty$ and put $x_h = h^{-1} x_0$, $y = P(x_h)$, and $\rho(h) = \sigma(y)$. Here P uses the true inverse, since the Gram matrix of x_h equals that of x_0. The function ρ is nonnegative and equals one at the identity. It depends only on the plane $\mathrm{im}(h^{-1} x_0)$.

If $\rho(h) \neq 0$, y is a basis spanning the plane of x_h. Since $k(y) x_0$ and x_h have the same Gram matrix and span, there is $u_1 \in U(B_0)_\infty$ such that $k(y) x_0 u_1 = x_h$. The isometry $hk(y)$ preserves the first summand and its orthogonal complement. Write

$$hk(y) = \mathrm{diag}(u_1^{-1}, u_2), \qquad p = \mathrm{diag}(u_1, u_2^{-1}).$$

Regard p as an adelic point with finite component 1. For every invertible B in the chosen adelic basis, the relevant real basis is

$$x = (ph)^{-1} x_0 B = x_h u_1^{-1} B, \qquad k(y)^{-1} x = x_0 B, \qquad phk(y) = 1.$$

If $\sigma(x_0 B) \neq 0$, the Gram matrix of x is that of $x_0 B$ and the inversion in $P(x)$ is genuine.

The orthogonal projection is independent of the choice of basis of the same plane, so $P(x) = y$. If $\sigma(x_0 B) = 0$, a nonzero value in (96) would force $k(P(x))^{-1} x$ to lie in the bump. Its Gram matrix is the Gram matrix of x, so again the projection is genuine and $P(x) = y$. This makes the second bump $\sigma(x_0 B) = 0$, a contradiction. If $\rho(h) = 0$, choose $p = 1$. The same argument shows that the large slice at $(ph)^{-1} x_0 B$ vanishes, since otherwise the first bump would equal $\rho(h)$.

Lemma 35.1 (Real slice at a common translate)**.** *For each $h \in U(B_0 \perp W_0)_\infty$, there is $p = p(h) \in (U(B_0) \times U(W_0))_\infty$, with finite component 1, such that for every invertible B in the chosen adelic basis,*

$$\omega_{j-\mu}(1, ph) \Phi[(ph)^{-1} x_0 B] = \rho(h) \kappa \, \sigma(x_0 B) \prod_\tau \overline{\det B_\tau}^{2\mu_\tau} \; (v \boxtimes v_1). \tag{98}$$

The product and denominator. Let ν be an admissible split tame character of weight μ, with residue character χ. Its tame support consists of the split pairs represented by a nonempty set S; $\chi_w \neq 1$ for every $w \in S$. Put $\gamma_0 = \nu^{-1} \upsilon^j$. For (W_0, W_0) at the finite places use the original determinant datum $\phi_\chi = \mathbf{1}_{\mathrm{GL}_{2j}(O_S)} \chi^{-1}(\det)$ at S, with the original vacuum off the support. Let F be any finite Schwartz vector for (W_0, B_0). The real vectors are evaluated against the theta distribution on the finite vector $F \boxtimes \phi_\chi$ and its small member F, respectively. In the

large functional the Heisenberg translate is $(c, 0)$, the group point is $(g, y_0 \oplus l)$, and the splitting characters are $(\gamma_0, \gamma_0\beta_0)$.

On the small functional these data read $c, (g, y_0)$ and (γ_0, β_0), for any remaining splitting character β_0. The block product of the rational lattice sums, together with the direct-sum Heisenberg action, factors the theta functional of $v \boxtimes v_1$ into that small functional times the equal-rank kernel of v_1, ϕ_χ at (g, l) with characters (γ_0, γ_0). This formula is for the original coordinates. The second Heisenberg move is zero and the determinant twist is the product of the twists of the two blocks.

The equal-rank kernel has absolutely convergent period on $[W_0]$ by the primitive support of ϕ_χ and the target-bounded theta estimate. Its period is independent of g, as follows. For a source element s supported at the chosen split places, simultaneously translate the target by s. The split Weil action on the square finite matrix reads $\phi(X) \mapsto \phi(s^{-1}Xs)$ with scalar one, since the two ranks and characters are equal.

Set $\lambda(a) = \nu^{\mathrm{bc}}(a)|a|^j$ on the split local factors. If $I_g(\phi)$ is the integrated kernel, the split target action and right invariance give

$$I_g(\phi(l^{-1}\,\cdot)) = \lambda(\det l) I_g(\phi).$$

Thus I_g on compactly supported locally constant functions on the invertible matrices depends through $\int \lambda(\det X)\phi(X)\, d^\times X$. Indeed, choose a common compact open subgroup K on whose right cosets the tests are constant and on which $\lambda \circ \det = 1$.

The equality on each coset lK follows from the target-action formula, giving the assertion by finite linearity. This weighted integral is invariant under conjugation by s. The determinant period is therefore right invariant at the chosen split places, and off them it is invariant under the original standard compact level away from both selected split halves.

Strong approximation [62, Theorem 7.12], applied to $\mathrm{SU}(W_0)$ now gives invariance under left translation by $\mathrm{SU}(W_0)(\mathbb{A}_{E^+})$. Since $W_0 = H \perp W$, its rank $2j$ is at least two. The special unitary group is simply connected and absolutely almost simple, and at every selected split place it is the noncompact group SL_{2j}. Explicitly, for $s' \in \mathrm{SU}(W_0)(\mathbb{A}_{E^+})$, approximate s' outside the selected places by a rational point s_0 so that $g^{-1}s_0^{-1}s'g$ lies in the original level at the other finite places and tends to the identity at infinity. At the selected places the full right factor is allowed. Automorphy and continuity give the claimed left invariance.

The period therefore equals its normalized radical mean on the one-plane parabolic of W_0. For this equal-rank datum the one-plane unfolding has lower source and target ranks equal, with empty deleted block. At its source Levi point (a_0, g_0), the unfolded integrand uses the idele variable a solely in the combination aa_0: the source and target Levi powers combine to $|aa_0|^{h+1}$, multiplied by the mixed slice factor $|aa_0|^{-h/2}$ in lower rank h.

The character and the slice variable are likewise read at aa_0. Right translation in the absolutely convergent idele-class integral proves that the mean is independent of a_0. Hence the period at $\mathrm{levi}(a_0, 1)$ equals its value I_0 at 1. The determinant of the Levi point is $a_0/\bar{a}_0$. Local Hilbert 90, with unit representatives away from finitely many places, gives every norm-one idele this way. As two points of equal determinant differ by the special unitary group, the equal-rank period is the constant I_0. This uses the equal-rank one-plane unfolding on the same finite determinant datum.

Finally let z_B be the idele with component $\det B_\tau$ at infinity and 1 at every finite place. With the normalized complex absolute value,

$$|z_B|^j = \prod_\tau |\det B_\tau|^{2j}, \qquad \nu^{\mathrm{bc}}(z_B) = \prod_\tau (\det B_\tau/|\det B_\tau|)^{2\mu_\tau}.$$

Consequently

$$\prod_{\tau} \overline{\det B_\tau}^{2\mu_\tau} = \frac{\prod_\tau |\det B_\tau|^{2(j+\mu_\tau)}}{\nu^{\mathrm{bc}}(z_B)|z_B|^j}.$$

Use (97) and (98), and integrate the equal-rank factor over its original $[W_0]$-measure. The result is the small functional of $\Lambda[B]$ multiplied by

$$\frac{\rho(h)I_0}{\nu^{\mathrm{bc}}(z_B)|z_B|^j}.$$

This holds simultaneously for every B, β_0, F, c, g, y_0, with the fixed choice of $p(h)$. The real vectors, bump, normalizing constant κ, and correction $p(h)$ are chosen before the character and modulus.

35.2 Extension of the self-dual Fourier tests

In this subsection, to pass from the real computation of Section 35.1 to an adelic identity, the finite data must match under self-dual Fourier transform. We establish the required Fourier identity on the full-rank locus and then extend the period formula.

Let D be the b-plane basis of the large source X, and A the additional plane inside its complement. The D-basis embeds from the complement of A. All conjugate-transpose and Fourier coordinates are those of these bases. Let S be the nonempty set of selected split places of E, and let P be the finite places outside $S \cup \bar{S}$. Use the integral compact level at P, and give every unipotent rational quotient its probability Haar measure.

In this subsection put $2j = \dim W_0$, $N = \dim X = 2b + 2j$ and $a = \dim(B_0 \perp W_0) = b + 2j$, so $N = a + b$. Write (K_D, I_D, L_D) for the ordered Witt basis with K_D spanning the given isotropic subspace. In these coordinates the Gram pairing with K_D is the row block $s = L_D^\vee$ of the inverse basis matrix. Thus $sK_D = 0$, $sI_D = 0$ and $sL_D = I_b$. At the selected places put

$$\Psi_0(Y) = \chi^{-1}(\det(sY)) \quad (Y \text{ integral, } \det(sY) \text{ a unit}),$$

with zero value elsewhere.

The common-section space and the equivariant difference. Let $F_S = \prod_{w \in S} E_w$ and let O_S be its integral subring. Put $\lambda(u) = \nu^{\mathrm{bc}}(u)|u|^{N/2}$ on $F_S^\times$, embedding the selected components as ideles with value one elsewhere. Indeed the split target change $X \mapsto l^{-1}X$ acts on each of the N columns and contributes $|\det l|^{-N/2}$ to the Weil action. For a full-column-rank matrix Y, define a section as follows. In each rank write $G = \mathrm{GL}(F_S)$ and $K = \mathrm{GL}(O_S)$. Choose a fixed set of representatives simultaneously for G/K and $K\backslash G$. Indeed each compact double coset has equally many left and right cosets by unimodularity, with each coset of measure one when K has measure one.

Each left and right coset in KgK meet, since $k_1 g k_2 \in k_1 gK \cap Kgk_2$. Pair the two finite sets and select an element in each intersection. Define

$$S_Y(\Psi) = \sum_r \lambda(\det r)\Psi(Yr).$$

On X^{t} use the same construction with GL_a. This defines

$$S_X^{\mathrm{cov}}(\Phi) = \sum_r \lambda(\det r)\Phi(r^{\mathsf{t}}X).$$

Here the tests are compactly supported in the open full-rank locus. For fixed full-rank Y, choose a left inverse A. If Yr is in that compact support, then $r = A(Yr)$ is bounded, and every limit of these r's is invertible since the corresponding limit of Yr has full rank. Thus r ranges over a compact subset of $\mathrm{GL}_b(F_S)$, which meets only finitely many cosets $r\mathrm{GL}_b(O_S)$. The covector sum follows by transpose. In the column space $W \subset E_w^N$ of a full-rank matrix, the intersection $W \cap O_w^N$ is a saturated lattice. Its basis gives the integral reference for comparing the fractional lattice generated by the matrix columns.

If $Y = Y_0 t$ with Y_0 a primitive integral basis of this saturated module at every place, the coset Kt is independent of the primitive basis. The set on which $Kt = Kr$ is given by primitivity of Yr^{-1}. This is a clopen condition since its entries are integral and its reduction has full column rank at each place. The disjoint open sets for $Kr \in K\backslash G$ cover the full-rank locus. A compact support therefore meets only finitely many of them. The same argument applies to X^{t}. The integral determinant covariance is $\Psi(Yu) = \lambda(\det u)^{-1}\Psi(Y)$ for $u \in \mathrm{GL}_b(O_S)$, and $\Phi(vX) = \lambda(\det v)^{-1}\Phi(X)$ for $v \in \mathrm{GL}_a(O_S)$. Integral matched functions have these covariances by changing the complementary block of an integral completion. Both are preserved by all split source translations. Each weighted summand is therefore independent of the choice of representative in rK.

For $k \in \mathrm{GL}_N(F_S)$, take Y to be its first b columns and X the last a covectors of k^{-1}. Take pairs of Schwartz–Bruhat functions on the full-rank loci with those integral covariances and satisfying

$$S_X^{\mathrm{cov}}(\Phi) = \lambda(\det k) S_Y(\Psi) \quad \text{for every } k.$$

For an integral matched pair, choose bases a_0X, Yb_0 of the saturated covector and column modules. The corresponding covector and column sections each have only one potentially nonzero term, up to the integral change of bases. Its weight cancels the determinant covariance on the integral group. The primitive map $a_0X : O_S^N \to O_S^a$ is onto at every factor, hence has an integral right inverse Z. Its kernel has basis Yb_0, so $k_0 = (Yb_0, Z) \in \mathrm{GL}_N(O_S)$. In the fixed block order,

$$k^{-1}k_0 = \begin{pmatrix} b_0 & * \\ 0 & a_0^{-1} \end{pmatrix}, \qquad \det k_0 = \det k \, \det b_0 / \det a_0.$$

The matching law on k_0 gives the displayed section identity. The section space is stable under

$$(\Phi, \Psi) \longmapsto (\lambda(\det g)\Phi(\,\cdot\, g), \Psi(g^{-1}\,\cdot\,)).$$

In fact its identity at k is the old one at $g^{-1}k$, multiplied by $\lambda(\det g)$.

35.2.1 Factorization through the Fourier test

In this subsection, we use the real restriction of Lemma 35.1 to compare the large period and its Fourier test.

The large period $\mathcal{L}(\Phi)$ factors linearly through Ψ on this space. To prove this, split the compact support of Φ into the finitely many clopen coset pieces Φ_r for which $X^{\mathsf{t}}r^{-1}$ is primitive integral. If $X^{\mathsf{t}} = Y_0 t$ for a primitive basis Y_0, this is exactly $t \in Kr$. Thus $K\backslash G$ indexes the partition. At a primitive covector X, the term $\Phi_r(r^{\mathsf{t}}X)$ equals $\Phi(r^{\mathsf{t}}X)$, and the same representatives enumerate G/K in the section sum.

Translate each piece to primitive integral covectors, with datum $\Phi_r(r^{\mathsf{t}}X)$. By changing the target variable at the selected split places and using the determinant multiplier of the Weil

action,

$$\mathcal{L}(\Phi_r) = \lambda(\det r)\mathcal{L}\big(\Phi_r(r^{\mathsf{t}}\,\cdot\,)\big).$$

Consequently $\mathcal{L}(\Phi)$ is the large period of the function $S_X^{\mathrm{cov}}(\Phi)$ on primitive integral covectors, extended by zero. At each such X, choose an integral completion k_X with that last block.

The section identity reads this function directly from Ψ as $\lambda(\det k_X)S_{Y(k_X)}(\Psi)$. Thus it depends linearly on Ψ. For convergence, apply the ordinary theta estimate at the end of Section 36.1.1 with the full rational source basis. A compactly supported covector test of full row rank satisfies its surjectivity hypothesis at a selected split place. The transpose of a full-column-rank test satisfies it at the conjugate place for the small integral. Cutting into clopen pieces and multiplying the matrix argument by a fixed invertible matrix on either side preserve compact support and full rank, also for a nonintegral matrix. For each resulting finite test and compact set of source points, the absolute value of the whole theta kernel has a common constant majorant on the finite-volume target quotient. Hence the changes of target variable use absolutely convergent integrals and the original Haar measure. Fix ν and take $c_{\mathrm{top},\nu}$ to be the explicit scalar in (1). For the small period $\mathcal{L}_{\mathrm{sm}}$ on the original column test define

$$\mathcal{D}(\Psi) = \mathcal{L}(\Phi) - c_{\mathrm{top},\nu}\mathcal{L}_{\mathrm{sm}}(\Psi).$$

The factorization makes this independent of the choice of matching covector test Φ. It is linear on the image of the section space in the column functions, including when $c_{\mathrm{top},\nu} = 0$.

Let d be the finite idele equal to $\det g_S$ at the selected places and to one elsewhere. On the large side the covectors transform by $X \mapsto Xg_S^{-1}$ with section multiplier $\lambda(d)^{-1}$. Since $a + b = N$, the combined factor is

$$\big((\nu^{\mathrm{bc}})^{-1}\upsilon_E^b\big)(d)^{-1}|d|^{a/2}\lambda(d)^{-1} = \big(\upsilon_E(d)|d|^{1/2}\big)^{-b}.$$

On the small side the transposed columns at the conjugate places read $\bar{d}^{-1}$. Their rank-b character is υ_E^b. As $\upsilon_E(d\bar{d}) = 1$, the factor is

$$\upsilon_E(\bar{d}^{-1})^{-b}|\bar{d}^{-1}|^{b/2} = \big(\upsilon_E(d)|d|^{1/2}\big)^{-b}.$$

Both calculations use the original quotient measures in $\mathcal{L}$ and $\mathcal{L}_{\mathrm{sm}}$. This gives the stated covariance of $\mathcal{D}$.

The self-dual tests first force only those central coefficients of $f = \mathcal{D}(\Psi_0)$ whose index is primitive at the selected split places to vanish. The finite Fourier combination B will then satisfy $\mathcal{D}B = 0$, and the translates of its dilate span the locally constant matched column data. For rank $b = 0$ there is one column datum up to scalar and one central coefficient with empty determinant equal to one. The self-dual test already gives its vanishing on the full parabolic, which in this case is the whole group. The line shifts and the Gauss sum are unnecessary in that case.

In these coordinates the covariance is

$$\mathcal{D}(\Psi \circ g_S)(xg_S) = \big(\upsilon_E(\det g_S)|\det g_S|^{1/2}\big)^{-b}\mathcal{D}(\Psi)(x).$$

For every such pair the compact-set bound above gives continuity of both periods by dominated convergence. Rational automorphy follows from that of the theta kernels. At P the fixed vacua and the unramified determinant characters give the indicated right compact invariance. These properties consequently hold for $\mathcal{D}$.

Interchange of the two line averages. For $T \in M_b(E)$, set $n_D(M) = 1 + K_D(M - M^*)s$. With probability Haar measure on the additive quotient the coefficient is

$$F_T(g) = \int_{M_b(E)\backslash M_b(\mathbb{A}_E)} \psi_E(\operatorname{tr}(TM))\, f(n_D(M)g)\, dM.$$

Since $sK_D = 0$, the map $M \mapsto n_D(M)$ is additive. Translation in the integral gives

$$F_T(n_D(Z)g) = \psi_E(-\operatorname{tr}(TZ))F_T(g).$$

For Hermitian Z we have $n_D(Z) = 1$, so a nonzero coefficient requires $T^* = -T$ by nondegeneracy of the trace pairing. Let $n_+(a)$ and $n_-(b')$ be the shifts using the two opposite isotropic lines of A, where a, b' are covectors of length b. Define

$$\phi_g^+(a) = \int_{E^b\backslash \mathbb{A}_E^b} F_T(n_-(b')n_+(a)g)\, db',$$

and use the opposite order for ϕ_g^-. Take the dual isotropic basis of A with cross pairing one. In the two orders of multiplication the upper-right blocks of the products are $-a^*b'$ and $-(b')^*a$, respectively. Hence

$$n_+(a)n_-(b') = n_D(-a^*b')\, n_-(b')n_+(a).$$

Moving $n_+(a)$ to the left in the opposite mean introduces $n_D(a^*b')$. Its central multiplier therefore gives

$$\phi^+_{n_+(a)g}(u) = \phi_g^+(u + a), \qquad \phi^-_{n_+(a)g}(b') = \psi_E(-\operatorname{tr}(Ta^*b'))\phi_g^-(b').$$

For a rational covector a_0, move $n_+(a_0)$ across the opposite shift and use rational automorphy. The value $\phi_g^+(a+a_0)$ is then the Fourier coefficient at $-Ta_0^*$ of $b' \mapsto F_T(n_-(b')n_+(a)g)$, with the displayed Fourier sign. If $\det T \neq 0$, these are all the characters of the covector quotient. For fixed a its continuous function $b' \mapsto F_T(n_-(b')n_+(a)g)$ is therefore determined everywhere by $\phi_g^+(a + a_0)$, by Fourier uniqueness. Commuting the two shifts with the stated central character and integrating over the other compact covector quotient recovers ϕ_g^-. Reversing the lines recovers ϕ_g^+, and these operations respect scalar multiplication. The line and central averages integrate continuous functions against probability Haar on compact sets of representatives.

Shears on the actual standard datum. At a point fixing the D-basis away from P, the partial mean at a shifted point, as a linear evaluation on the equivariant data, is invariant under

$$Y \longmapsto Y + uwsY \qquad (w \in M_{1,b}(E_S)),$$

where u is the moved extra-line vector at S. Indeed these are the split components of the opposite root shifts used in the mean. They have determinant one in the split covariance, and right translation of the covector quotient preserves its Haar measure. The covector here is still s: it is the adjoint covector of the fixed basis, and the added extra-line vector is orthogonal to it. Consequently $su = 0$. For a general allowed rank-one split shear the change in the local datum is $Y \mapsto Y + urY$, with $ru = 0$. The other split half has the conjugate unitary coordinates.

Such a shear also fixes the value of this linear evaluation on Ψ_0. Average the two functions over the restricted maps $Y \mapsto Y + urtsY$ with t in the compact additive group $M_{\dim X,b}(O_S)$.

If for some covector ρ the point $Y + u\rho$ lies in $\operatorname{supp}\Psi_0$, then sY is integral invertible and $t_0 = (Y + u\rho)(sY)^{-1}$ is integral. Since $ru = su = 0$,

$$Y + urY + urtsY = Y + ur(t + t_0)sY.$$

Translation by t_0 preserves Haar on the compact group, so the two averages agree. If no such ρ exists the averages vanish. The averages are finite linear combinations of translates because the datum is smooth for the compact group. The restricted invariance therefore proves the needed invariance under the full split shear with the exact datum.

For either extra line this gives partial-mean invariance under the adelic line radical, between the points under consideration. First, its centre fixes the partial mean for rational parameters by automorphy and for parameters supported over the selected split pairs by the shear argument. A central parameter at these pairs is specified by its coordinates at S, with the conjugate coordinates determined by the skew-Hermitian condition. The rational and selected parameters together are dense in the full adelic centre by additive strong approximation. Averaging over this centre now writes the partial mean as the coefficient of the combined isotropic basis with index $\operatorname{diag}(0, T)$.

The mixed blocks of its centre give the opposite line shift already integrated in the mean. A rational parameter in the line radical acts on the combined basis by an upper block unipotent and preserves this bordered index by congruence. For the noncentral column parameter c, take support on $\bar{S}$. Write i for the middle-block inclusion, $i^\vee$ for its coordinate projection, and J_0 for the middle Hermitian matrix in the Witt basis. At $w \in S$, $c_w = 0$, so the linear term with image in the middle block and the quadratic central term vanish. The action on this half is

$$1 + u_w (c^*)_w J_{0,w} i_w^\vee, \qquad i_w^\vee u_w = 0.$$

It is the shear just treated on the selected datum. The opposite component is determined by unitarity. Let N_1 be the extra-line radical, and let H_S be its subgroup with column parameter supported on $\bar{S}$ and centre supported over the selected pairs. For $h \in H_S$, multiplying by a selected central element still gives a single shear on the selected datum at the original point. Rational central translations preserve the partial mean by automorphy. Density in the centre and continuity then show that the bordered coefficient at hg equals the partial mean at g, for every original point g under consideration.

The bordered coefficient is invariant under $N_1(F_0)$ by the rational congruence above. The product $N_1(F_0)H_S$ is dense in $N_1(\mathbb{A}_{F_0})$. For the column parameter use additive strong approximation away from $\bar{S} \neq \varnothing$. The product of a rational column lift and a column lift supported on $\bar{S}$ differs from the lift of their sum by a central term supported over the selected pairs. The rational and selected central parameters then approximate the centre as above. Continuity gives equality of the bordered coefficient at ng with the partial mean at g for every $n \in N_1(\mathbb{A}_{F_0})$. At a point ng under consideration the partial mean equals its bordered coefficient, which gives the stated invariance.

35.2.2 Integral extension of the Fourier test

In this subsection, we extend the test of Section 35.2.1 in the local self-dual coordinates.

Suppose $F_T(g) \neq 0$ at a point fixing the D-basis away from P, with $\det T \neq 0$. The line interchange gives a nonzero shifted mean. At the places in P, act on the right by the integral

group so that the action on the combined isotropic basis is upper block triangular for the flag given by the extra line and the D-plane. This is the integral flag decomposition: intersect the two isotropic spaces of the transported flag with the standard lattice, take primitive bases in succession, and complete to an integral Witt basis.

For a primitive isotropic basis matrix X, unimodularity and primitivity give integral dual columns Y with $X^{*}JY = 1$. Choose integral H with $H + H^{*} = Y^{*}JY$, using the upper triangular entries off the diagonal and surjectivity of the integral trace on the diagonal. Every nonsplit finite place of E/F_0 is unramified by Section 4.2. For an unramified extension this surjectivity follows from the surjective residue trace: a lift with unit trace can be divided by that trace. The split trace is the sum of the two components. Now $Y - XH$ is isotropic and still dual to X, so the hyperbolic span splits off with unimodular complement. This construction includes residue characteristic two. The compact action is trivial at the other places. The complementary factor of the upper-triangular action is in the extra-line radical, which may be discarded by the invariance just proved. This leaves a point which at P agrees with a block action on the chosen bases with scalars t, t' on the two lines and matrix γ on the D-basis, where $(t')^{*}t = 1$. Call it a straight point. It still fixes the D-basis off P.

Put $\delta = j_{\mathrm{W}}\alpha_0\bar{\alpha}_0$, the additive scale, and set $C = \delta^{-1}\gamma^{*}T\gamma$ at P. At every straight point where the shifted mean is nonzero, C is integral at P. Indeed some shifted value of the coefficient is nonzero at a point with the same D-basis action. The compact level allows all integral central translations at P there. At $w \in P$, take the integral matrix parameter $M = aE_{ji}$, $a \in \mathcal{O}_w$, supported at that place. Write $z(M)$ for the central translation with skew block $M - M^{*}$. In the defining coefficient the matrix parameter U has weight $\psi_E(\mathrm{tr}(TU))$. This weight factors through $U - U^{*}$: for $H = H^{*}$, $\mathrm{Tr}_{E/E^{+}}\,\mathrm{tr}(TH) = 0$. At the coefficient point g, we have $gK_D = K_D\gamma$ on P; unitarity gives $sg^{-1} = \gamma^{*}s$. Thus $gn_D(M)g^{-1} = n_D(\gamma M\gamma^{*})$ for M supported there. Changing the integration variable to $U + \gamma M\gamma^{*}$ gives

$$F_T(gn_D(M)) = \psi_{E,w}(-\delta_w aC_{ij,w})F_T(g).$$

Here $\mathrm{tr}(MC) = aC_{ij,w}$, so the multiplier is $\psi_{\mathrm{sd}}(-aC_{ij,w})$, where $\psi_{\mathrm{sd}}(u) = \psi_E(\delta u)$ on these finite coordinates. For the absolute trace character the trace dual of $\mathcal{O}_w$ is $\mathfrak{D}^{-1}_{E_w/\mathbb{Q}_p}$, and $\delta_w\mathcal{O}_w = \mathfrak{D}^{-1}_{E_w/\mathbb{Q}_p}$ by Section 8.1. Thus the annihilator of $\mathcal{O}_w$ for this scaled pairing is $\mathcal{O}_w$. Since the compact translation fixes the nonzero coefficient, $C_{ij,w}$ is integral. The central parameter introduces no factor of two: for any integral skew block Z one can take $M = c_{\mathrm{tr}}Z$, since $c_{\mathrm{tr}} + \bar{c}_{\mathrm{tr}} = 1$ and $M - M^{*} = Z$. This works in residue characteristic two as well. Choose a straight point with nonzero shifted mean minimizing the positive integer

$$I(C) = \prod_{w \in P} q_w^{v_w(\det C)}.$$

The factors are one almost everywhere. For a fixed straight g, the two functions ϕ_g^{+} and ϕ_g^{-} are either both zero or both nonzero, by Fourier uniqueness in the line interchange. Straight points preserve the two opposite lines at P, so exchanging them preserves this set of points and the D-block γ. Thus a single minimizing point works for both functions.

For a covector a with $\phi_g^{+}(a) \neq 0$, one has

$$\gamma^{-1}a^{*}t \in O_P^{b}. \tag{99}$$

At $w \in P$ the bottom block row of the shifted basis matrix is $(-a^{*}t, \gamma)_w$. Its image on $\mathcal{O}_w^{b+1}$ is

$$\Lambda'_w = \gamma_w\mathcal{O}_w^{b} + (a^{*}t)_w\mathcal{O}_w.$$

Choose a basis γ'_w of Λ'_w. The surjection onto this free lattice splits. A basis of its rank-one kernel, followed by lifts of the columns of γ'_w, gives an integral invertible column change that turns the bottom row into $(0, \gamma'_w)$. At almost every place $a, t, \gamma, \gamma^{-1}$ are integral, so $\Lambda'_w = \mathcal{O}_w^b$. All the column changes are integral and together give an element of the product compact. Extend the resulting integral change on the combined isotropic plane by its inverse adjoint on the opposite plane and by the identity on the complement. This is a right compact move. The remaining upper off-diagonal block lies in the extra-line radical, under which the shifted mean is invariant. For the inverse column change M we have $-a^*t = \gamma' M_{21}$ and $\gamma = \gamma' M_{22}$. Thus $\gamma_w \mathcal{O}_w^b \subseteq \Lambda'_w$, and the inclusion is strict at some w if (99) fails. The new Gram is integral since the new mean is nonzero, and

$$C = M_{22}^* \, (\delta^{-1}(\gamma')^* T \gamma') \, M_{22}.$$

Write $C' = \delta^{-1}(\gamma')^* T \gamma'$. Since P is stable under conjugation and $q_w = q_{\bar{w}}$,

$$I(C) = I(C') \prod_{w \in P} q_w^{2v_w(\det M_{22})} > I(C').$$

All the valuations on the right are nonnegative and at least one is positive, contradicting minimality. The opposite argument gives $\gamma^{-1}(b')^* t' \in O_P^b$ whenever $\phi_g^-(b') \neq 0$.

Take any column x with $Cx \in O_P^b$, and set $a = (\gamma x (t')^*)^*$ on P and zero at the other places. On the support of ϕ_g^- the multiplier of the shift $n_+(a)$ is one. Indeed the scaled argument of its additive character is $-(\gamma^{-1}(b')^* t')^* Cx$, which is integral. Off P the argument is zero. Thus a is a period of ϕ_g^+ by the interchange. Apply (99) to a nonzero point of ϕ_g^+ and its translate by a, and subtract. Since $(t')^* t = 1$, this gives $x \in O_P^b$. Use the columns of C^{-1} as x. The minimizing Gram is therefore unimodular at every place in P.

A nonzero shifted mean gives a nonzero coefficient at a point with this same D-basis action on P and identity on that basis elsewhere, because both line shifts fix D. Suppose now that $\delta^{-1}T$ is primitive at S. Since $T^* = -T$ and $\bar{\delta} = -\delta$, the matrix $D_0 = \delta^{-1}T$ and its inverse are Hermitian. At a split pair, conjugate transposition therefore transports integrality of both matrices from $w \in S$ to $\bar{w}$. Write $T = j_{\mathrm{W}} \beta$ and $q = \alpha_0 q_{\mathrm{Lev}}$ for the parabolic Levi matrix of the point. Its finite action on the D-basis is γ at P and the identity on the selected split pairs. Hence

$$Q = (q_{\mathrm{Lev}}^* \beta q_{\mathrm{Lev}})_f = (\delta^{-1} q^* T q)_f, \qquad Q|_P = C, \quad Q|_{S \cup \bar{S}} = D_0|_{S \cup \bar{S}}.$$

This Hermitian matrix and its inverse are integral everywhere finite. The integral unitary basis construction in Section 36.1.1 gives $k \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ with $k^* Q k = I_b$. If β is outside the real congruence class of $B_0 = U^-$, its coefficient vanishes. For the chosen real slice this follows from its support and the empty small orbit in Section 36.1.1. For the arbitrary real vectors in Section 36.1.6, the compact averaging argument there gives this vanishing.

Otherwise Landherr's theorem [43, pp. 246–247] gives $d \in \mathrm{GL}_b(E)$ with $\beta = d^* J_{B_0} d$: the forms have the same real signatures and are isometric at each finite place. Let $m_D(d)$ be the rational Levi element acting by d on K_D. In the coefficient integral put $M = dM'd^*$. Rational automorphy and cyclicity of trace give

$$n_D(M) m_D(d) = m_D(d) n_D(M'), \qquad F_{j_{\mathrm{W}} J_{B_0}}(m_D(d) g) = F_T(g).$$

The change is an automorphism of the rational additive quotient, so it preserves probability Haar. On $M_b(E)$ its E-linear determinant is $(\det d \, \overline{\det d})^b$, whose absolute values over the places

of E have product one. The moved Levi matrix is $dq = \alpha_0 X_{B_0} Y$ with $Y = X_{B_0}^{-1} dq_{\mathrm{Lev}}$. From $(X_{B_0}^{*} J_{B_0} X_{B_0})_f = I_b$ we get

$$(Y_f k)^{*}(Y_f k) = k^{*} Q k = I_b.$$

This is precisely the finite basis condition of the self-dual test at index $j_{\mathrm{W}} J_{B_0}$, so the coefficient vanishes. The contradiction proves that $F_T(1, g_f) = 0$ when $\delta^{-1} T$ is primitive at S and g_f is trivial there and at the conjugate split places.

Finite detection of the other coefficients. At S use the opposite ordering $(K, L) = (L_D, K_D)$ in this calculation. For the corresponding coordinate rows of the inverse basis matrix we have $s = K^{\vee} = L_D^{\vee}$, $K^{\vee} L = L^{\vee} K = 0$ and $K^{\vee} K = L^{\vee} L = I_b$. In particular $L z K^{\vee} = K_D z L_D^{\vee}$ is the original central block on the chosen split half. For $z \in M_b(E_S)$ put

$$\Psi_z(Y) = \Psi_0(Y + L z K^{\vee} Y).$$

Extend z by zero outside S. The component of $n_D(z)$ on the chosen half is $1 + L z K^{\vee}$, and its determinant is one. Split covariance gives $\mathcal{D}(\Psi_z)(x) = \mathcal{D}(\Psi_0)(x n_D(-z))$. At the indicated points $g = (1, g_f)$ with $g_f = 1$ on $S \cup \bar{S}$, this inverse shear commutes with g. Apply the central translation formula with $Z = -z$, and put $\psi_S(u) = \prod_{w \in S} \psi_{E,w}(\delta_w u_w)$. Then

$$\Lambda_{T,g}(\Psi_z) = \psi_S(\mathrm{tr}((\delta^{-1} T)_S z)) \Lambda_{T,g}(\Psi_0),$$

where $\Lambda_{T,g}$ is the central coefficient of $\mathcal{D}$ at g as a linear form of the datum. Since $\delta_w \mathcal{O}_w = \mathfrak{D}_{E,w}^{-1}$, the conductor of $u \mapsto \psi_{E,w}(\delta_w u)$ is $\mathcal{O}_w$. Thus $r \mapsto \psi_{E,w}(\delta_w \varpi_w^{-1} \widetilde{r})$ is a well-defined nontrivial additive character of each k_w. Let k_S be the product of residue fields, and let $\widetilde{r}$ be integral lifts of their matrices with $\widetilde{1} = 1$. Put

$$B(Y) = \sum_{r \in M_b(k_S)} \chi^{-1}(\det r) \Psi_{\varpi^{-1} \widetilde{r}}(Y).$$

The character is extended by zero on nonunits. This detector uses the residue field even if the level at w is $\mathfrak{p}_w^2$. Indeed, on the deeper ring a determinant character inflated from k_w is constant on $r_0 + \varpi_w M_b(k_w)$. Summation over this fiber against a primitive additive character of the deeper ring gives zero unless the Fourier index lies in $\varpi_w M_b(\mathcal{O}_w / \mathfrak{p}_w^2)$, by additive orthogonality. The shifts $\varpi^{-1} \widetilde{r}$ in B pair directly with indices reduced modulo ϖ. The full depth of a matched datum enters the subsequent compact average through c.

If $\Lambda_{T,g}(\Psi_0) \neq 0$, invariance under integral shears forces $(\delta^{-1} T)_S$ integral. Write $A = (\delta^{-1} T)_S$. Since ψ_S is trivial on O_S, $\psi_S(\mathrm{tr}(A \varpi^{-1} \widetilde{r}))$ depends only on r. By the primitive test, $\bar{A}_w$ is singular for some $w \in S$. The tame residue character χ_w is nontrivial at each selected place. Choose a unit a with $\chi_w(a) \neq 1$, a nonzero $v \in \ker \bar{A}_w$ and a residue-field linear form ℓ with $\ell(v) = 1$. Take $u_w = 1 + (a - 1) v \ell$ and $u_{w'} = 1$ for $w' \neq w$. Then $\bar{A} u = \bar{A}$ and $\det u_w = a$. Reindex the Gauss sum by $r \mapsto u r$. The additive term is unchanged and the determinant character changes by a nontrivial scalar. Thus the sum is zero. This also handles a singular rational T, and all other terms with $\Lambda_{T,g}(\Psi_0) = 0$ are already zero.

The skew coefficients exhaust the characters of the central quotient in the original convention. Fourier uniqueness gives $\mathcal{D}B(g) = 0$ at these points. Their rational translates are dense by weak approximation for the unitary group at the specified real and split finite places, with all other finite components unrestricted. Weak approximation here follows from its rational Cayley chart and weak approximation for the vector space of skew matrices. Continuity and automorphy give $\mathcal{D}B = 0$.

Translate by $Y \mapsto -\varpi^{-1}Y$ on the split source. Put $p = L^\vee$ and $U = -\varpi^{-1}K^\vee Y$. The argument of Ψ_0 in the r-summand is

$$X_r = \varpi^{-1}(L\tilde{r}U - Y), \qquad K^\vee X_r = U.$$

A nonzero term requires $U \in \mathrm{GL}_b(O_S)$, $r \in \mathrm{GL}_b(k_S)$, and X_r integral. It follows that Y is integral and $\overline{pY} = r\bar{U}$, so at most one r contributes. Conversely, write $Y = LV + \varpi Z$ with $pZ = 0$. If $V = pY$ and $K^\vee Z$ are integral invertible matrices and Z is integral, then $U = -K^\vee Z$ is invertible, $r = \bar{V}\bar{U}^{-1}$ gives X_r integral, and its weighted value is $\chi^{-1}(\det V)$. Thus, writing Ω for this set of Y, we have the exact formula

$$G(Y) := B(-\varpi^{-1}Y) = \mathbf{1}_\Omega(Y)\chi^{-1}(\det(pY)).$$

On integral matrices membership in Ω depends only on $Y \bmod \varpi^2$: first test $(1 - Lp)Y \equiv 0 \bmod \varpi$, then test invertibility of pY and of $K^\vee(1-Lp)Y/\varpi$ after reduction. At $x_0 = L - \varpi K$ the term $r = 1$ has argument K, so $G(x_0) = 1$. Apply the compact averaging identity (102) with F multiplied by the indicator of Ω, and with this x_0, p. Its positive scalar gives all the integral matched data from G.

36 The normalized period exchange

In this section, we compare the periods using the local factors of Section 35. The real identity and its extension to the finite tests supply the comparison used in Section 39.

36.1 The normalized comparison of the two periods

In this subsection, we compare the paired periods of Section 34.3 by removing hyperbolic planes. The real factor from Section 35.1 and the finite Fourier transform of Section 35.2 fix the final scalar.

Let W, V, U have ranks N, a, b, their original standard unimodular models at the finite places, and respective negative indices $q(W), q(V), q(U)$ independent of the infinite place. Assume $N = a + b$, $a - b = 2t$ for $t \in \mathbb{Z}$, $q(W) = q(V) + q(U)$, and

$$2q(W) + 2 \le N \le 2\mu_\tau + 2q(W) \qquad (\tau \mid \infty).$$

Fix the ordered splitting into b, a coordinates and a real Schwartz vector y for (U, W), equivalently a Fock vector with all weighted absolute moments by Section 35.1. For the admissible split tame character ν, put

$$\gamma = \nu^{-1}\upsilon^{N/2}, \qquad M_\nu(j) = \prod_{w\in S} q_w^{j^2} \prod_{k=1}^{j} |L^S(k,\nu)|^2 \quad (j \ge 0), \qquad M_\nu(-j) = M_\nu(j)^{-1}.$$

The positive-index factors are nonzero by the argument preceding (4), so the reciprocal defines the negative-index values. Here υ_E is the fixed splitting Hecke character and υ is its restriction to the norm-one ideles. Let $S \ne \varnothing$ consist of one chosen member of each split pair in the modulus. Write $\nu^{\mathrm{bc}}(z) = \nu(z/\bar{z})$. Admissibility requires ν^{bc} to be unramified at every finite place outside $S \cup \bar{S}$. At each $w \in S$ its restriction to $\mathcal{O}_w^\times$ is the inflation of a nontrivial character of $k_w^\times$. Its restriction at $\bar{w}$ is determined by conjugation and inversion. The product L^S omits

exactly these pairs from the base-change L-series. The characters have weight μ at infinity, which is fixed. The Haar measures and vacua throughout are the original ones for the models.

We give the assembly from the Fourier and constant-term identities, specifying the versions needed to normalize the comparison. For the top calculation use

$$W_0 = H \perp W, \qquad X = H^b \perp W_0, \qquad V_{\mathrm{top}} = U^- \perp W_0.$$

The small side is (X, U^-), with U^- the opposite space. Write $j_0 = \dim W_0/2$. We use the real slice of Section 35.1 with $j = j_0$. Given the two fixed real vectors, one for the small side and one z_0 for (W_0, W_0), the slice identity on the original self-dual finite lattices has coefficient

$$\frac{\rho(h) I_{W_0,\nu}(z_0)}{\nu^{\mathrm{bc}}(z_B)|z_B|^{j_0}}.$$

Here B is the invertible small slice matrix, z_B is $\det B$ at infinity and 1 at finite places, and $I_{W_0,\nu}(z_0)$ is the equal-rank $[W_0]$-period for the finite determinant datum $\chi^{-1}(\det)$.

The real factor and the real coordinate corrections are chosen before ν, B and the finite data. The self-dual Fourier unfolding and the primitive-coefficient vanishing and spanning on its test space give the identity on standard partners with scalar

$$c_{\mathrm{top},\nu} = R_0 \kappa_0 \varepsilon\, \nu^{\mathrm{bc}}(d_0) I_{W_0,\nu}(z_0), \qquad \kappa_0 = (v_{\mathrm{big}}/v_{\mathrm{sm}})|d_0|^{j_0}. \tag{1}$$

The nonzero vacuum constants $v_{\mathrm{big}}, v_{\mathrm{sm}}$ and the coordinate-change idele $d_0 = \det(\alpha_0 X_{U^-})$ belong to the original splittings, with X_{U^-} the small adelic basis.

At the selected places let $B_D = (L_D, K_D, I_D)$ be the integral basis matrix in this order, with $s = L_D^\vee$. For the stacked covectors $C_s(Z) = \binom{s}{Z}$,

$$C_s(Z) B_D = \begin{pmatrix} I_b & 0 \\ ZL_D & Z(K_D, I_D) \end{pmatrix}.$$

Consequently, on the integral unit support,

$$\chi^{-1}(\det C_s(Z)) = \varepsilon\, \chi^{-1}(\det[Z(K_D, I_D)]), \qquad \varepsilon = \chi(\det B_D), \qquad |\varepsilon| = 1.$$

The column partner is $\chi^{-1}(\det(sY))$ on the support specified in Section 35.2. The positive integral R_0 is the integral of $\rho(h)\mathbf{1}_{H_f K_f^{\mathrm{std}}}(u)$ on the quotient by the block stabilizer. In particular the compact set in this integral is the full fixed standard level. The exponent here is j_0 because the complement in the mixed model is W_0.

36.1.1 Absolute summability of the mixed terms

In this subsection, we justify the unfolding needed for Proposition 36.1.

In this mixed model write K for the isotropic columns spanning D, denoted by L in Section 35.2.2, and K' for the opposite columns. With $n_0 = \dim W_0$ and I the n_0 middle columns, $A = (K, I)$ is a rational basis of $K^\perp$. Let A_f^{rat} give the finite coordinates of these rational columns in the standard model. The adapted frame $A_f = (K_f, I_f)$ uses the chosen adelic basis of W_0 in its middle block, so $A_f = A_f^{\mathrm{rat}} \operatorname{diag}(1_b, x_{W_0,f})$ with $x_{W_0,f}$ invertible. For a source point g in the parabolic fixing the span of K, one has

$$A_g = g_f^{-1} A_f^{\mathrm{rat}}, \qquad A_g C_g = A_f.$$

Indeed the columns of $g_f A_f$ and A_f^{rat} are bases of the same space with the same initial isotropic subspace. Their change of basis C_g is invertible and upper triangular by blocks.

For a target of rank d, the condition used below is that $Y A_g$ have a right inverse over the local field for every matrix Y in the support at one fixed split place. For the large test, $d = b + n_0$, and the integral basis matrix at a chosen $w \in S$ is $P = (K'_f, K_f, I_f)_w$. The covector s_w from Section 35.2 is the first row block of P^{-1}. Thus

$$\begin{pmatrix} s_w \\ Y \end{pmatrix} P = \begin{pmatrix} 1_b & 0 \\ Y K'_{f,w} & Y A_{f,w} \end{pmatrix}.$$

On the support of the determinant test both square basis changes are integrally invertible. Hence $Y A_{f,w}$ is invertible, and $C_{g,w}(Y A_{f,w})^{-1}$ is a right inverse to $Y A_{g,w}$.

For the small test, $d = b$. Read the transpose of its column variable at $\bar{w}$, identifying $E_{\bar{w}}$ with E_w by conjugation. In the standard finite Hermitian form, s_w^{t} is then $K_{f,\bar{w}}$, since s is the Hermitian dual of K. Thus the character is supported where $\det(s_w Y^{\mathsf{t}}) = \det(Y K_{f,\bar{w}})$ is a unit. With $J_K = \binom{1_b}{0}$ the required right inverse is

$$C_{g,\bar{w}} J_K (Y K_{f,\bar{w}})^{-1}.$$

If the target has rank zero its theta sum has one term. These calculations use a single selected split pair. The vacuum at all other places enters through its integral lattice and open stabilizer. For a change of finite coordinates both the support and the frame are transported in the product $Y A_g$. Full row rank of Y alone suffices for the ordinary period with a full source basis. In a mixed slice it is the restriction of Y to the moved $K^{\perp}$ that must be surjective.

For a fixed source point let q_x denote the theta functional of the mixed slice at the rational matrix x. Let h be the target point and $H = H(h)$ its reciprocal lattice minimum cut off below at one. There are A_0, M and C_g such that, for an arbitrary real Schwartz vector u and the fixed finite datum f_f,

$$\sum_x |q_x(\omega(g,h)(u \otimes f_f))| \le C_g H^{A_0} p_M(u).$$

Here p_M is a fixed finite-order Schwartz seminorm.

Indeed the absolute sum is invariant under rational target translation, which permutes the rational matrices x. Write Λ_h for the rational vectors integral in the moved finite target lattice and $\|v\|_h = \max_\tau \|v\|_{h,\tau}$ in its signed orthonormal real bases. The finite Hermitian form is integral on Λ_h, since the original finite lattice is integral and h_f is unitary. Thus vectors of length below one pair to zero: every conjugate of their integral pairing has modulus below one. At each infinite place double their complex span, halve its image under the sign matrix, and fix the joint Euclidean orthogonal complement, as in Section 18.1.2. This simultaneous unitary move sends the first minimum λ to at least $\min(2\lambda, 1/2)$. After at most $\lfloor \log_2 H \rfloor$ moves the height is at most two. The compact reduction of Section 33.1.3 therefore gives $\gamma h = z a$ with γ rational, z in a fixed adelic compact set and a purely real, with $\|a_\tau^{\pm 1}\| \le H$ at every τ. In any rational polarization of the smaller mixed block, the absolute sum of the slice theta evaluations is bounded by the sum of the absolute Schwartz values over the rational coordinates of both blocks.

The finite coordinates in the support range over a compact set. In a fixed rational basis it is contained in $m^{-1}\widehat{\mathcal{O}}_E^r$ for some positive integer m, so the rational coordinates in the sum lie in $m^{-1}\mathcal{O}_E^r$. Summing $(1 + \|x\| + \|y\|)^{-M}$ for sufficiently large M gives the claimed bound before

the real move. The position and differentiation operators transform by the real symplectic matrix under Weil conjugation. Their finite products, Sobolev's inequality and unitarity give the stated polynomial factor for the real move, increasing the fixed order M if needed. The finite translates in the compact reduction are taken from a finite set by smoothness.

For $H > 2$ take a shortest nonzero $d \in \Lambda_h$. The rational lattice is discrete in the product of its archimedean completions, so the minimum is attained and $\|d\|_h = H^{-1}$. Its Hermitian norm $\langle d, d\rangle$ belongs to $\mathcal{O}_{E^+}$ and $|\tau\langle d, d\rangle| \le \|d\|_{h,\tau}^2 \le H^{-2} < 1$ for every τ. Hence d is isotropic. Multiply the columns c_j of the fixed rational basis of $K^\perp$ by a single positive integer to put them in the moved source lattice.

Their lengths are bounded in terms of g. Let Z be the rational isotropic span of the maps

$$d\, c_j^* J_X.$$

These maps have integral finite coordinates and real lengths $O_g(H^{-1})$ in the moved Hom space. Use the opposite frame K', and write J_{tar} for the rational target form and Θ_0 for the theta functional of the remaining block. For $y \in Z(E)$ one has $yK = 0$. In rational coordinates the mixed Heisenberg formula gives

$$q_x(\rho(y)f) = \psi_E\big(j_{\mathrm{W}} \operatorname{tr}((yK')^* J_{\mathrm{tar}} x)\big)\, \Theta_0(\rho_0(yI) f[x]) = q_x(f).$$

Indeed the argument of the additive character lies in E, and the rational translation yI preserves Θ_0. As $d^* J_{\mathrm{tar}} d = 0$, the Heisenberg cocycle on $Z(\mathbb{A}_E)$ is trivial. Consequently, for any finite sum $L = \sum_x \epsilon_x q_x$, $|\epsilon_x| = 1$, the formula

$$L_0(f) = \int_{Z(E)\backslash Z(\mathbb{A}_E)} L(\rho(z) f)\, dz$$

defines a $Z(\mathbb{A}_E)$-invariant functional using the probability Haar measure.

The mean annihilates the translated datum. In the moved finite coordinates, the covector $d_h^* Y A_g$ is nonzero at a selected split place whenever the datum at Y is nonzero. For a right inverse R of $Y A_g$ we have

$$(d_h^* Y A_g) R = d_h^* \neq 0.$$

In the opposite split component the corresponding Heisenberg shift multiplies the Schrödinger datum by the additive character paired with this covector. For every support point one can choose a parameter for which this character is different from one. It is constant on a small open neighborhood. A finite partition of the compact support into such neighborhoods, and invariance of L_0 under the corresponding shifts, show that L_0 vanishes on each piece, with any real part.

For clarity the estimate for the difference from this mean is as follows. Choose a positive integer m_0 such that all finite Heisenberg shifts in $m_0 \widehat{\mathcal{O}}_E$ in the chosen integral coordinates fix the fixed finite datum. For each coefficient of the displayed generators, reduce modulo E to an adele with finite part in $m_0 \widehat{\mathcal{O}}_E$ and bounded real part. This uses additive strong approximation followed by a fundamental parallelepiped of $m_0 \mathcal{O}_E$.

These measurable representatives fix the finite test after conjugation and give real displacements s of size $O_g(H^{-1})$. The finite rank condition gives the exact vanishing of the mean. The displacement bound uses the shortest vector at the real places and the bounded real representatives for the scalar coefficients. Let A_h be their probability average on the real test and set $L_{g,h}(u) = L(\omega(g,h)(u \otimes f_f))$. Weil covariance gives

$$L_{g,h}(A_h u) = L_0(\omega(g,h)(u \otimes f_f)) = 0.$$

This holds for every real Schwartz vector, so $L_{g,h}(u) = L_{g,h}((1 - A_h)^n u)$ for every $n \geq 0$.

Use real self-dual coordinates and the seminorms

$$p_j(u) = \max_{|\alpha|+|\beta|\leq j} \sup_t |t^\alpha \partial^\beta u(t)|.$$

For $s = (a, b)$, the generator of the Weyl displacement $\rho(\lambda s)$ is $D_s = a \cdot \nabla + 2\pi i\, b \cdot t$ in coordinates with translation by $+a$ and modulation by $+b$. Leibniz's rule gives $p_j(D_s u) \leq C_j \|s\| p_{j+1}(u)$. Translation of the polynomial weights and differentiation of the modulation bound $p_j(\rho(\lambda s)u) \leq C_{g,j} p_j(u)$ for $0 \leq \lambda \leq 1$ on this bounded set of displacements. Integrating $\partial_\lambda \rho(\lambda s)u = \rho(\lambda s) D_s u$, then averaging and iterating, gives

$$p_j((1 - A_h)u) \leq C_{g,j} H^{-1} p_{j+1}(u), \qquad p_M((1 - A_h)^n u) \leq C_{g,M,n} H^{-n} p_{M+n}(u).$$

The initial estimate for $L_{g,h}$ is independent of the finite set of x's and their phases.

Taking phases so that the finite sum is the sum of the absolute values, then increasing the finite set, gives

$$\sum_x |q_x(\omega(g, h)(u \otimes f_f))| \leq C_{g,n} H^{A_0 - n} p_{M+n}(u).$$

At $H \leq 2$ use the first bound. In particular this is bounded on the target quotient, which has finite volume. The radical quotient is compact. Over a compact set of representatives its source points stay in the parabolic, so the same rank proof applies.

The constants can be taken uniformly there. The bounded representative set has compact finite projection, so a single integer clears the coordinates of the fixed rational basis throughout, and its real basis coordinates are uniformly bounded. The initial Schwartz bound is uniform on that compact set as well. The central parameter itself only multiplies each mixed term by a unit phase. Thus the absolute sum is integrable also in the radical variables.

Tonelli and Fubini now justify both the extraction of the central coefficient term by term and the subsequent unfolding of the surviving rational orbit. For the ordinary period, take a full rational source basis instead of the basis of the isotropic perpendicular and use the whole theta functional in place of q_x. Suppose that, at one split place and in the orientation for the integrated group, every matrix in the support of f_f has full row rank over the local field. Multiplication by the matrix of the moved full source basis preserves this rank at every source point. The mean then vanishes by the same finite modulation calculation. The initial bound, the denominator clearing for the fixed source basis, and the real bounds for its columns are uniform for g in a compact subset C of the source group. Taking sufficiently many differences gives a Schwartz seminorm p_{N_B} with

$$\sup_{g \in C} |\theta(\omega(g, h)(u \otimes f_f))| \leq C_{C, f_f, B}\, H(h)^{-B} p_{N_B}(u) \qquad (B \geq 0). \tag{100}$$

For a rank-zero integrated group the quotient consists of one point and continuity follows from the ordinary lattice sum. For positive rank the quotient has finite volume by [18, Theorem 5.6(i), p. 21]. A rational character of the unitary group factors through its determinant torus, and inversion by the nontrivial element of $\mathrm{Gal}(E/E^+)$ on the character lattice of that norm-one torus leaves only the zero character invariant. Thus $B = 0$ gives an integrable bound uniform on C, and dominated convergence gives continuity of the ordinary period. By the coherent-state formula of Section 34.3.1, the same bound can be written with a sufficiently high absolute Fock moment in place of p_{N_B}. These constants are for the fixed finite datum.

Here is the scalar computation on the original self-dual tests. Put $B_0 = U^-$ and $F_0 = E^+$. For its rational Hermitian matrix $\beta = J_{B_0}$ the skew central Fourier index in Section 35.2.2 is $T = j_{\mathrm{W}}\beta$. Let the parabolic Levi matrix be $q = \alpha_0 X_{B_0} Y_0$, where $Y_{0,f}k$ is unitary for an integral finite invertible matrix k. For a rational frame x with target Gram matrix $G_x = x^* J x$, use the central variable $C - C^*$ of Section 33.1.3. Its action contributes $\psi_E(-j_{\mathrm{W}} \operatorname{tr}(G_x C))$. The Fourier test contributes $\psi_E(\operatorname{tr}(TC))$. With probability Haar on the additive matrix quotient,

$$\int_{\mathrm{Mat}_b(E)\setminus \mathrm{Mat}_b(\mathbb{A}_E)} \psi_E(\operatorname{tr}((T - j_{\mathrm{W}} G_x)C))\, dC = \mathbf{1}_{G_x=\beta}.$$

Over $F_0 = E^+$, Witt extension gives one orbit of ordered isometric embeddings on each side, with representatives $(1; 0)$ and 1, and pointwise stabilizers $U(W_0)(F_0)$ and 1. Indeed fixing the ordered frame fixes its span pointwise, while its orthogonal complement can move by an arbitrary isometry.

Let $c \in \mathrm{Mat}_{2j_0\times b}(\mathbb{A}_E)$ be the noncentral radical coordinate and J_{std} the Gram matrix of W_0 in its adelic basis. In the adapted isotropic frame, the radical sends a middle vector v to v plus the first isotropic frame applied to $c^* J_{\mathrm{std}} v$. On a mixed slice with frame value x' in target coordinates, the induced Heisenberg translation is therefore $x' c^* J_{\mathrm{std}}$. For the two representatives the frame values are $\binom{X_{B_0}^{-1}}{0}$ and $X_{B_0}^{-1}$. Thus with $d_c = X_{B_0}^{-1} c^* J_{\mathrm{std}}$ the translations are $\binom{d_c}{0}$ and d_c, respectively. Write z for the unreduced central parameter. In the ordered Witt basis the upper-right block of the radical element is

$$-\frac{z + z^*}{2j_{\mathrm{W}}} - \tfrac{1}{2} c^* J_{\mathrm{std}} c.$$

The mixed slice at a frame of Gram matrix G_x has multiplier $\psi_E(\frac{1}{2}\operatorname{tr}(zG_x))$: in the Heisenberg phase the quadratic c-term has zero absolute trace since $\operatorname{tr}(c^* J_{\mathrm{std}} c G_x)$ is fixed by conjugation and j_{W} changes sign. Replacing z by $z - 2j_{\mathrm{W}}C$ adds $C - C^*$ to the upper-right block and multiplies the slice by $\psi_E(-j_{\mathrm{W}} \operatorname{tr}(G_x C))$. Both frames have $G_x = \beta$, so the multipliers agree before integration. The equal-rank block in the large integrand has zero translation. This gives the large unfolded quotient by $U(W_0)(F_0)$ and the small unfolded integral over $U(B_0)(\mathbb{A}_{F_0})$, with the original Haar normalizations.

For the slices of Lemma 35.1, a nonzero large slice in (96) requires $x^\flat \in \operatorname{supp}\sigma$. Unitarity of $k(P(x))$ gives $(x^\flat)^* J x^\flat = x^* J x$. For $B = x_0^* J x_0$ and $G = x^* J x$, the Gram bound $\|B^{-1}(G - B)\| < 1/2$ keeps $B + u(G - B)$ invertible for $0 \le u \le 1$. Hence G has the signature of J_{B_0} at every real place. The source frame changes act by invertible congruence and preserve rank and signature. Thus an index of different signature or deficient rank contributes zero. A nondegenerate index with a different signature has no small orbit. At a singular index every small square frame is singular since $\det(B^* J_{B_0} B) = \det(J_{B_0}) |\det B|^2$; its bump $\sigma(x_0 B)$ vanishes.

At a selected split pair write the source columns in the integral order (K'_f, K_f, I_f) of the mixed model. Put $A = YK'_f$, $B_1 = YK_f$, and $C = YI_f$ on the chosen half. The stacked determinant of the standard test is, up to the fixed basis determinant, $\det[\, B_1\ C\,]$. Its dependence on A is therefore exactly the indicator of the integral matrix box. The partial Fourier transform in A reads the frame on the conjugate half, since K'_f is dual to K_f. In the normalized coordinates its entrywise calculation is

$$\int_{O_w} \psi_{E,w}(\delta_w a t)\, da = \mathbf{1}_{O_w}(t), \qquad \delta = j_{\mathrm{W}} \alpha_0 \bar{\alpha}_0, \quad \mathrm{vol}(O_w) = 1.$$

It holds also with the inverse additive character. Thus both halves of the frame must be integral. For the small transposed test at $\bar{w}$ the determinant depends on $YK_{f,\bar{w}}$, since $s_w^{\mathsf{t}} = K_{f,\bar{w}}$ in the paired coordinates. The other line block has the same integral-indicator transform. The calculation uses the full integral box also at a place of exponent two; only the multiplicative character is read in the residue field.

A nonzero finite slice forces the large group point to lie in $H_f K_f^{\mathrm{std}}$, where H_f is the block group for $B_0 \perp W_0$. Indeed multiply the integral basis by k to obtain an integral orthonormal basis. It extends to a unitary integral basis by splitting off its unimodular orthogonal complement. At split places extend the chosen basis and its dual by a complementary basis. At an unramified nonsplit place, reduction of the complement is a nondegenerate Hermitian space over the quadratic residue extension. A vector of nonzero norm exists by the residue trace pairing: if all norms vanished, polarization after multiplying one argument by every residue scalar would make every pairing zero. Lift this vector and scale its unit norm to one. The residue-field norm is onto, and for $j \geq 1$ the norm of $1 + \varpi^j z$ is congruent to $1 + \varpi^j \operatorname{Tr}(z) \bmod \varpi^{j+1}$ for a fixed-field uniformizer. The residue trace is onto also in characteristic two. Successive lifting and completeness give the required unit norm. Splitting off each resulting norm-one line diagonalizes the unimodular complement. This proves the claim in all the finite models, which are split or unramified.

On a supported finite coset choose its representative in K_f^{std}. Right translation by this compact fixes each large mixed term. Indeed, at a chosen split place let A be the integral target matrix. In the determinant test

$$\begin{pmatrix} s_w \\ A^{-1}Y \end{pmatrix} = \operatorname{diag}(1_b, A^{-1}) \begin{pmatrix} s_w \\ Y \end{pmatrix}.$$

The matrix preserves the integral box. The determinant character χ_w^{-1} produces the factor $\chi_w(\det A)$ on the prime-power test, using the inflated residue character. The target character ν^{-1} supplies the inverse factor, and the unramified splitting and modulus factors are one on units. The opposite split component is determined by unitarity; away from these pairs the integral vacuum is fixed. Thus the finite Weil datum itself is fixed before the mixed functional is applied, so on this coset its formula is the one at the identity. Let $u \in U(B_0)(\mathbb{A}_{F_0})$ be its small group variable, $B = u^{-1}Y_0$ and $X_f = B_f$. Both data vanish unless X_f is integral. On this support $X_f k$ is integral unitary, so its adjoint is its integral inverse. Since k is integral invertible, $x = \det X_f$ is a unit at every finite place. At $w \in S$, the large residual determinant in the adapted row order is

$$\det \begin{pmatrix} X_{f,w} & Z_1 \\ 0 & Z_2 \end{pmatrix} = \det X_{f,w} \det Z_2.$$

The large slice has the integral vacuum on Z_1 and the original determinant datum on the W_0-block Z_2, with scalar $\varepsilon_w \chi_w^{-1}(x_w)$, where $\varepsilon_w = \chi_w(\det B_{D,w})$. On the small side the residual variable has the integral vacuum with scalar $\chi_w^{-1}(\overline{x_{\bar{w}}})$. Here conjugation identifies $E_{\bar{w}}$ with E_w. The quotient of the two factors depending on the frame is

$$\frac{\chi_w^{-1}(x_w)}{\chi_w^{-1}(\overline{x_{\bar{w}}})} = \chi_w(x_w/\overline{x_{\bar{w}}})^{-1}.$$

Its product over S is $\nu^{\mathrm{bc}}(x)^{-1}$, since away from these split pairs ν^{bc} is trivial on units. The product of the fixed phases ε_w is ε in (1). Put $n = \dim X$, $d = \dim V_{\mathrm{top}}$, so that $n = d + b$ and

$d-b=2j_0$, and put $\gamma_n=\nu^{-1}\upsilon^{n/2}$. The standard large pair is (γ_n,γ_n). The signed adjoint of the pair (γ_n,γ_n) in the reverse order has characters $(\gamma_n^{-1}\upsilon^n,\gamma_n^{-1}\upsilon^b)$ by the adjoint calculation below. Dividing its kernel by $\gamma_n(\det g)$ in the source variable gives the standard small pair $(\nu\upsilon^{n/2},\upsilon^b)$. Since $K^{a_1,a_2}(g,h)$ has character multiplier $a_1(\det h)a_2(\det g)^{-1}$, the ratio of the large and small target twists at u is $\nu(\det u)^{-2}$. For the source Levi use $\beta_k(z)=\beta(z/\bar z)\upsilon_E(z)^{-k}$ as in (F2). Then

$$(\gamma_n)_d=(\nu^{\mathrm{bc}})^{-1}\upsilon_E^b,\qquad (\upsilon^b)_b=\upsilon_E^b,$$

because $\upsilon(z/\bar z)=\upsilon_E(z)^2$. At $\det q$ the quotient of the two Levi factors is therefore $(\gamma_n)_d^{-1}(\upsilon^b)_b|\cdot|^{(d-b)/2}=\nu^{\mathrm{bc}}|\cdot|^{j_0}$. Since $z_B=\det(Y_0)(\det u)^{-1}(\det X_f)^{-1}$ as ideles, and $\nu^{\mathrm{bc}}(\det u)=\nu(\det u)^2$, the resulting phase and positive factor are exactly

$$\frac{\nu^{\mathrm{bc}}(\det q)}{\nu^{\mathrm{bc}}(z_B)\nu^{\mathrm{bc}}(\det X_f)\nu(\det u)^2}=\nu^{\mathrm{bc}}(d_0),\qquad \frac{|\det q|^{j_0}}{|z_B|^{j_0}}=|d_0|^{j_0}.$$

The remaining block stabilizer quotient integral of the bump gives the factor R_0 of (1).

36.1.2 The measure on the open orbit and the signed comparison

In this subsection, the first stage of the comparison in Section 36.1 unfolds over the full-rank orbit. We keep track of the quotient measure and the source involution to identify its signed scalar.

Let $G=U(B_0\perp W_0)(\mathbb{A}_{E^+})$, $L=U(B_0)(\mathbb{A}_{E^+})$, $H_1=U(W_0)(\mathbb{A}_{E^+})$, and $H=L\times H_1$ inside G. This is the stabilizer of the nondegenerate subspace B_0. Let $\Gamma=\{1\}\times U(W_0)(F_0)$ be the stabilizer of its ordered rational frame in the rational group. The groups G and H are unimodular, and H is closed. Give the homogeneous space $H\backslash G$ the invariant quotient measure determined by dG and $dL\,dH_1$ via [29, Theorem 2.51 and (2.52)]. The same Haar disintegration, first by Γ and then by H, gives for an integrable unfolded term F

$$\int_{\Gamma\backslash G}F=\int_{H\backslash G}\int_L\int_{\Gamma\backslash H_1}F((l,h_1)g)\,dh_1\,dl\,d\bar g.$$

This follows from the quotient integral formula first for compactly supported lifts to G, then for integrable functions by applying the positive formula to their absolute values. Right translation by each component of $p(h)$ in the two inner integrals preserves the original Haar, including the measured $[W_0]$. For $B_0=0$ the frame is empty, $H_1=G$, and L is the trivial group. If dL has mass v_0 on that point, the quotient integral formula gives $H\backslash G$ mass $1/v_0$, since $dH_1=dG$. Fix the source point and the matched self-dual test. Denote by A_{sm} its small Fourier coefficient, unfolded as the integral over L of the term at the identity basis. Applying (98) inside the two inner integrals gives

$$\int_L\int_{\Gamma\backslash H_1}F((l,h_1)g)\,dh_1\,dl=\kappa_0\varepsilon\nu^{\mathrm{bc}}(d_0)I_{W_0,\nu}(z_0)A_{\mathrm{sm}}\,\rho(g_\infty)\mathbf{1}_{H_fK_f^{\mathrm{std}}}(g_f).$$

The H_1-integral contributes the equal-rank period, and the L-integral is precisely A_{sm} with its original Haar measure. The only dependence on the outer coset is in the last two factors.

This weight descends to $H\backslash G$. The finite indicator is left H_f-invariant. At infinity $\rho(h)$ depends only on the span of $h^{-1}x_0$. On its support $hk(P(h^{-1}x_0))\in H_\infty$, so the corresponding cosets lie in the compact image of $k(\operatorname{supp}\sigma)^{-1}$. At finite places the supported cosets are the

image of K_f^{std}, also compact. The weight is positive on a neighborhood of the identity coset. The translate $p(h)$ of Lemma 35.1 is smooth where $\rho(h) > 0$ by its block formula and is taken to be the identity on the complementary closed set. It is therefore measurable. The quotient integral of this continuous compactly supported weight is the positive finite R_0 in (1). When $b = 0$, the basis is empty, the determinant of its empty block is one, $H = G$, and the outer quotient is a single point. The same formula integrates its unit weight with the quotient measure induced by the fixed Haar measures. Consequently integrating the last display gives the large coefficient $c_{\mathrm{top},\nu} A_{\mathrm{sm}}$, with the same R_0 even if $I_0 = 0$.

For completeness, at each plane removal the old target need not initially exhibit the next hyperbolic summand. If its rank and negative count are those of the new target plus one plane, form $H \perp V$ with its standard chosen basis and permute the basis so that its signs agree entry by entry with the old target. At infinity the forms are thus identified.

At finite places the forms have the same standard unimodular model, and Landherr's theorem [43, pp. 246–247] gives a rational isometry C. If X_{old} and X_{ad} are their adelic bases, write $X_{\mathrm{old}} = C^{-1} X_{\mathrm{ad}} u$ with u adelic unitary. Their rational groups in chosen coordinates are conjugate by u. Their theta distributions have the same rational vectors after the corresponding unitary change of point.

The real representation and the finite vacuum and test, in chosen coordinates, are unchanged because they depend on the common signs and on the given covector ordering. Transport the theta functional along the isometry. Its rational summation agrees up to a scalar with that in the adapted polarization. The unit-vacuum mean-square normalization gives this scalar modulus one, as in the calculation following (MR). With transported functionals the target argument changes by $h \mapsto uh$ and the character contributes $\alpha(\det u)^{-1}$, where α is the character of the kept source. Conjugation preserves the counting measure on rational points and left translation preserves ambient Haar, so this map preserves the induced quotient measure.

Thus the same finite covector datum occurs, and only a unit phase changes in the scalar. In the block-reindexed adapted model the plane basis and the new target inclusion have their prescribed integral and orthonormal forms. We compute the two constant terms in these coordinates. Through the entire source sequence use the fixed plane blocks and fixed complement in the original adapted basis. Successive zero slices are evaluations in the product mixed model, so they give its combined zero slice. Thus the same coordinates supply the adapted vectors at each removal. At the selected finite places let π_i be the middle projection at the i-th removal and j_i its integral basis inclusion, so $\pi_i j_i = 1$. Given a lower covector parameter p, use successively $p\pi_i$ on the preceding spaces. For two consecutive removals the upper parameter is $p\pi_{i+1}\pi_i$, and restriction along $j_i j_{i+1}$ recovers p. Each projection and inclusion is defined before reduction to any chosen residue depth. Thus the induction applies to every lower standard parameter, with the same real vectors and scalar. If p has deficient rank in a selected residue field, both standard determinant tests vanish there. For the empty parameter the same identities use the unique zero-row matrix.

For a single plane removal write e for the new defect $h - N_1/2$, using the new target rank h and source rank N_1. The old defect is $e + 1$. Under the quotient radical mean the standard partner covector p is read on the old spaces as $p\,\mathrm{pr}_{W_1}$, where pr_{W_1} is the projection in the dual basis. The two constant-term formulas on this datum have the form

$$\mathrm{CT}\,\Theta^{+}_{\mathrm{old}} = \gamma_{\mathrm{pl},\nu}\Theta^{+}_{\mathrm{new}}, \qquad \mathrm{CT}\,\Theta^{-}_{\mathrm{old}} = \kappa_A \Theta^{-}_{\mathrm{new}}. \tag{2}$$

The lower small vector is the zero slice of the old one. The large vector is the cone slice at $s(e) = e + 1$ for $e \geq 0$, or its dual cone slice at $s(e) = -e$ for $e < 0$. The small vacuum slice

scalar $\kappa_A \neq 0$ depends on the splitting and the fixed measures. These operations choose the new real vectors before the finite data and twist.

36.1.3 The two radical terms after removing one hyperbolic plane

We compute the two unipotent integrals on the standard finite data after removing one hyperbolic plane. The large constant term gives the equal-rank Levi invariance used in Section 35.1. The two constant terms together give (2).

Use the adapted decompositions $\widetilde{W} = H \perp W_1$ and $\widetilde{V} = H \perp V$ on the original tower. Put $N_1 = \dim W_1$, $h = \dim V$, and $e = h - N_1/2$. Retain the given middle and isotropic bases. Let p be the integral covector parameter in the lower basis. The two upper data use $p\,\mathrm{pr}_{W_1}$, where the projection is the middle dual covector of the source decomposition. The upper determinant test on covectors and its column partner are the functions $\chi^{-1}(\det[p\,\mathrm{pr}_{W_1}; Y])$ and $\chi^{-1}(\det(p\,\mathrm{pr}_{W_1} Z))$, respectively, restricted to integral matrices with unit determinant in the indicated formula.

Finite restriction of the upper space. Write (K, I, L) and $(K^\vee; I^\vee; L^\vee)$ for the column and dual covector bases of the source, and use the same notation with subscript V for the target. At the selected split places these are integral inverse bases. Extend the target basis to the stacked test by the identity on the fixed covectors of p. The resulting square change of basis G takes the first and last line coordinates to K_V, L_V on the moving block and acts on the middle by the identity on the fixed covectors and by I_V on the moving covectors.

Let $R = (K^\vee; I^\vee; L^\vee)$; both G and R have integral inverses. In the two partial Fourier models, the matrix inside the determinant is $GM_\pm R$, with

$$M_+ = \begin{pmatrix} a & z_1 & z_3 \\ 0 & [p;Y] & z_2 \\ 0 & 0 & u \end{pmatrix}, \qquad M_- = \begin{pmatrix} u & z_1 & z_3 \\ 0 & [p;Y] & z_2 \\ 0 & 0 & a \end{pmatrix}.$$

The column z_2 is zero on the fixed covectors. The entry z_3 is the coordinate of $K_V L^\vee$. These block expressions are in the chosen order of the covectors: multiplication by R reads the three dual source coordinates, and multiplication by G restores the target coordinates on the three moving blocks. On the fixed covectors the diagonal line blocks vanish and the middle p is read by precisely $\mathrm{pr}_{W_1} = I^\vee$.

Thus restriction to the line leaves the old fixed covectors on the same middle coordinates. Integrality of the determinant test is equivalent to integrality of $M_\pm$, and $\det(GM_\pm R) = \det(RG)\, a\, u\, \det[p;Y]$. The basis phase is $\varepsilon = \chi^{-1}(\det(RG))$, of absolute value one. The two diagonal line entries merely switch places.

The slice at $\alpha_0 a$ in the self-dual measure Fourier transforms z_1, z_2, z_3, u. The frequency is $\bar{a}$ at u in the scaled coordinates. Each of z_1, z_2, z_3 has zero frequency and enters the test only through its integral lattice indicator. The self-dual Haar gives each of these lattices mass one at the chosen split places. Let $\varpi = (\varpi_w)_{w \in S}$ on the chosen split side. The u-integral at $w \in S$ is

$$\int_{O_w^\times} \chi_w(u)^{-1} \psi_w(\bar{a}_w u)\, du = \begin{cases} g_w\, \chi_w(\varpi_w \bar{a}_w), & \varpi_w \bar{a}_w \in O_w^\times, \\ 0, & \text{otherwise}, \end{cases} \qquad |g_w| = q_w^{-1/2}.$$

The additive character has conductor O_w in this coordinate. To see the conductor in the formula, write the units as the fibers $u_0 + \varpi_w O_w$ of reduction to $k_w^\times$. The character χ_w^{-1} is

constant on each fiber. If $v_w(\bar{a}_w) < -1$, its additive integral there is zero. If $v_w(\bar{a}_w) \geq 0$, the integral is $q_w^{-1} \sum_{u_0 \in k_w^\times} \chi_w(u_0)^{-1} = 0$. At $v_w(\bar{a}_w) = -1$ the field Gauss sum has absolute value $q_w^{1/2}$, and each fiber has mass q_w^{-1}. More explicitly, if $\psi_{0,w}(t) = \psi_w(\widetilde{t}/\varpi_w)$ on the residue field, then

$$g_w = q_w^{-1} \sum_{t \in k_w^\times} \chi_w(t)^{-1} \psi_{0,w}(t).$$

For $v = \varpi_w \bar{a}_w \in O_w^\times$, replacing $\psi_{0,w}(t)$ by $\psi_{0,w}(vt)$ multiplies the sum by $\chi_w(v)$, by the substitution $t \mapsto v^{-1}t$. The calculation applies at every modulus exponent $d_w \geq 1$, since the determinant character always factors through $k_w^\times$. Together with the diagonal a in $M_\pm$ it gives the standard lower covector datum on the same $[p; Y]$, with scalar $l_\pm \mathfrak{g}\, F_\varpi(a_f)$. Here $l_\pm$ is the nonzero vacuum slice constant in the indicated Fourier model,

$$\mathfrak{g} = \varepsilon \prod_{w \in S} g_w, \qquad F_\varpi(a_f) = \chi^{-1}(a_S)\chi((\bar{a}\,\varpi)_S)$$

on $a_S, (\bar{a}\,\varpi)_S \in O_S^\times$ and a integral outside $S \cup \bar{S}$, and zero otherwise. At a selected split pair, write x_1, x_2 for the two coordinates, identifying the second with E_w by conjugation. The scaled alternating trace pairing is $\psi_w(x_1 y_2 - x_2 y_1)$, since the scale $j_{\mathrm{W}} \bar{\alpha}_0 \alpha_0$ changes sign under conjugation. Both additive measures give O_w mass one. Put $g(\theta) = q_w^{-1} \sum_{t \in k_w^\times} \theta(t) \psi_{0,w}(t)$. The transform of the local factor $\chi_w^{-1}(x_1)\chi_w(\varpi_w x_2)$, with its two unit conditions, has the same support and the same values: the x_1-integral contributes $g(\chi_w^{-1})\chi_w(\varpi_w y_2)$, and substitution $u = \varpi_w x_2$ in the x_2-integral contributes $q_w g(\chi_w)\chi_w^{-1}(-y_1)$. Indeed $q_w g(\chi_w) g(\chi_w^{-1}) = \chi_w(-1)$, by complex conjugation and $|g(\chi_w)|^2 = q_w^{-1}$. This also works with the inverse Fourier character, since the two minus signs cancel in the original test. Away from the chosen pairs the integral is the original vacuum slice. Thus the two line tests have the same F_ϖ with their fixed vacuum constants $l_\pm$. In particular both line data have zero value at the zero line coordinate.

The large radical mean. At a source Levi point (u, g), take $u = 1$, or allow arbitrary u if $h = N_1$. Both original kernels can be integrated in either order over the compact source radical quotient and the finite-volume target quotient. Indeed the large determinant test has right-invertible moving rows and the small column test gives a left-invertible Z at a conjugate split place, since pZ is invertible there. The ordinary whole-kernel estimate at the end of Section 36.1.1, or the primitive row and column estimates of Section 34.3.1, is uniform as the radical point ranges over compact representatives. A target of rank zero contributes one value.

Now fix the target point. On the compact radical representatives the translated archimedean tests form a bounded set in Schwartz space, and their finite parts range over a finite set. In rational coordinates for the mixed slice and its residual polarization, the rational indices belong to a fixed fractional lattice. The sum of $(1 + \|x\| + \|y\|)^{-M}$ on this lattice is finite for sufficiently large M. Thus the mixed expansion at this target point can be integrated term by term over the radical. The resulting vector terms are absolutely summable at the fixed point and can be grouped by their rational lines. The central average of the source line retains exactly the rational vectors of zero Hermitian norm in $\widetilde{V}$. Its other radical coordinates produce the line zero-term functional for each retained nonzero vector. For an isotropic vector $x = \xi k_V$, $\xi \in E^\times$, write c for the noncentral radical coordinate in the adapted Witt basis. The central action on this slice is trivial. The action on the residual theta test is the Heisenberg translation by $xc^* J_{W_1}$, where J_{W_1} is the Gram matrix of the retained source space in a rational basis. The map $c \mapsto xc^* J_{W_1}$ is a rational additive isomorphism onto $Z = \mathrm{Hom}_E(W_1, Ek_V)$ and sends the

normalized Haar of the compact rational quotient to normalized Haar. Take the self-dual mixed polarization in which Z acts by modulation. Its opposite block is $Z^\vee$, paired with Z by the additive character of the Weil representation. For each rational index λ in that block the mean contributes

$$\int_{Z(F_0)\backslash Z(\mathbb{A}_{F_0})} \psi\big(\langle z,\lambda\rangle\big)\,dz = \mathbf{1}_{\lambda=0}. \tag{101}$$

The absolute bound above permits this termwise integration. It leaves the residual theta on the target-line zero slice, with scalar one. The conjugate polarization replaces the character by its inverse and gives the same mean. For the zero vector the source slice at a chosen $w \in S$ restricts the moving covector matrix Y to $YK = 0$, with K the first isotropic basis column. The fixed rows also kill K, since $I^\vee K = 0$. Thus $[pI^\vee; Y]K = 0$, contrary to invertibility on the support and $K^\vee K = 1$. This source slice vanishes before taking a target-line term. Moving the target point preserves its vanishing by equivariance of the zero slice.

36.1.4 Unfolding the isotropic line orbit

In this subsection, we group the nonzero radical terms before the equal-rank vanishing in Section 36.1.6.

Witt extension gives one orbit of rational isotropic lines. Group the nonzero vectors by this orbit, then by their coordinate $\xi \in E^\times$ on the line. Let $P \subset U(\widetilde{V})$ stabilize Ek_V, and let N be its unipotent radical. In the ordered Witt decomposition,

$$m(b,h_0) = \operatorname{diag}(b,h_0,\bar{b}^{-1}), \qquad P = N \rtimes \big(\operatorname{Res}_{E/F_0}\mathrm{GL}_1 \times U(V)\big).$$

The stabilizer of k_V is $N \rtimes U(V)$; the rational scalars moving k_V on its line form $E^\times$. In coordinates for N with a column c in V and a central coordinate t on an F_0-line, conjugation by $m(b,h_0)$ acts by $(c,t) \mapsto (h_0 c\bar{b}, b\bar{b}t)$. Its absolute Jacobian is $|b|_{\mathbb{A}_E}^{h+1}$, since $|\det h_0|_{\mathbb{A}_E} = 1$. Hence for $\tilde{h} = nm(b,h_0)k$ the Iwasawa density is $|b|_{\mathbb{A}_E}^{-(h+1)}\,dn\,d^\times b\,dh_0\,dk$ up to its fixed positive Haar scalar. Here k ranges over the original maximal compact of $U(\widetilde{V})(\mathbb{A}_{F_0})$; the finite Iwasawa decomposition in these lattices is proved in Section 33.1.2. Since $m(b,h_0)^{-1}k_V = b^{-1}k_V$, the slice coordinate is $a = b^{-1}$, and its density is $|a|_{\mathbb{A}_E}^{h+1}$.

For the positive line term at ξk_V, write $S_{\xi k_V}$ for the source slice and T_0 for the subsequent target zero slice. For $n \in N(\mathbb{A}_{F_0})$ the mixed model gives

$$S_{\xi k_V}\omega(1,n)f = \omega_{\mathrm{low}}(1,n)S_{\xi k_V}f, \qquad T_0\omega_{\mathrm{low}}(1,n) = T_0.$$

In the first equality $n^{-1}k_V = k_V$ and $\det n = 1$. For the second, in the target line model the displacement from the off-diagonal column is linear in the slice coordinate, and the central phase is quadratic in it. At coordinate zero the displacement and phase exponent vanish, and n induces the identity on $k_V^\perp/Ek_V$. Any remaining scalar on this zero slice is a character of N invariant under $m(2,1)$. Conjugation multiplies the centre by 4, so the character is trivial on the centre. On the vector quotient conjugation multiplies by 2, so the character there is trivial as well. Thus the full target radical fixes the term before summing over ξ. Its quotient by $N(F_0)$ has a fixed positive finite volume: the column and central coordinates have compact additive adelic quotients, with product Haar since the group law acts by successive translations in these coordinates. When $h = 0$, this central line remains and the column is absent.

Let $\ell_\pm(z;h_0,k)$ be the lower kernel at the line slice at the actual parameter z, target point h_0 and old target compact point k; the retained source point is g. Put $x = au$, where $u = 1$

unless $h = N_1$. In the positive model, first slicing on the source gives $|u|^{(h+2)/2}$ from the old target rank. Slicing on the target then gives $|a|^{-N_1/2}$ from the lower source rank. In the opposite order these factors are $|a|^{-(N_1+2)/2}$ and $|u|^{h/2}$. The line arguments for a rational index $\xi \in E^\times$ are ξx and $\xi \bar{x}^{-1}$, respectively. To compute the unitary factors, write $\gamma_j = \nu^{-1}\upsilon^{j/2}$ and $\eta_k(z) = \eta(z/\bar{z})\upsilon_E(z)^{-k}$. A source-line Levi of parameter u contributes $\eta_{\dim(\text{target})}(u)^{-1}$, and a target-line Levi of parameter $b = a^{-1}$ contributes $\eta_{\dim(\text{source})}(a^{-1})$, with the character of the opposite space in each case. Removing a plane lowers its space character by one power of υ. For the source first these two factors are $(\gamma_{N_1+2})_{h+2}(u)^{-1}$ and $(\gamma_{N_1})_{N_1}(a^{-1})$; for the target first they are $(\gamma_{N_1+2})_{N_1+2}(a^{-1})$ and $(\gamma_{N_1})_h(u)^{-1}$. In both orders their product is

$$\nu^{\mathrm{bc}}(au)\upsilon_E(u)^{h-N_1} = \nu^{\mathrm{bc}}(x),$$

since $u = 1$ or $h = N_1$. Including the target orbit density gives the two scalar factors

$$\nu^{\mathrm{bc}}(x)|a|^{h+1}|u|^{(h+2)/2}|a|^{-N_1/2} = \nu^{\mathrm{bc}}(x)|x|^{e+1},$$
$$\nu^{\mathrm{bc}}(x)|a|^{h+1}|a|^{-(N_1+2)/2}|u|^{h/2} = \nu^{\mathrm{bc}}(x)|x|^{e}.$$

In the second expression $x' = \bar{x}^{-1}$ satisfies $\nu^{\mathrm{bc}}(x') = \nu^{\mathrm{bc}}(x)$ and $|x'| = |x|^{-1}$, so the exponent in the actual negative line variable is $-e$. These coordinates give the following integral for the contribution of one line modulo its rational stabilizer, with the exponent $s = e + 1$ supplied by the densities. We verify its absolute convergence before unfolding the sum over the orbit. Put

$$Z(s) = \int \nu^{\mathrm{bc}}(au)|au|^s \sum_{\xi \in E^\times} \ell_+(\xi au; h_0, k)\, d^\times a\, dh_0\, dk,$$

where $a \in E^\times\backslash\mathbb{A}_E^\times$, $h_0 \in U(V)(F_0)\backslash U(V)(\mathbb{A}_{F_0})$, and dk is Haar on the original maximal compact. The scalar $c_{\mathrm{unf}} > 0$ is the Iwasawa Haar scalar times the volume of $N(F_0)\backslash N(\mathbb{A}_{F_0})$. The quotients use counting measure on their rational subgroups. In particular the full $E^\times$-sum unfolds with this quotient measure, including all integral units and roots of unity. For $V = 0$ the h_0-integral has the prescribed mass of the trivial group. The Fourier formula in the two opposite line coordinates gives

$$|a| \sum_{\xi \neq 0} \ell_+(\xi a; h_0, k) = \sum_{\xi \neq 0} \ell_-(\xi \bar{a}^{-1}; h_0, k).$$

To identify the two polarizations, write $A = EK$, $A' = EL$, $B = EK_V$, $B' = EL_V$ in the ordered rational Witt bases. The common zero slice removes the isotropic subspace

$$D = \mathrm{Hom}(W_1 \oplus A', B) \oplus \mathrm{Hom}(A', V) \ \subset \mathrm{Hom}(\widetilde{W}, \widetilde{V}).$$

Extend each map by zero on the other displayed summands. The trace pairing of the matrix units pairs opposite Witt indices. Thus

$$D^\perp/D = \mathrm{Hom}(W_1, V) \ \perp \ (EK_VK^\vee \oplus EL_VL^\vee).$$

The last two maps have zero norms and pair to one for the Hermitian trace form. In the first order of slicing the translating map is $aK_VK^\vee$, and in the reverse order it is $aL_VL^\vee$. The remaining Weyl change is therefore Fourier transformation in this single adelic line with its original self-dual measure. It fixes the $\mathrm{Hom}(W_1, V)$ coordinates. This description also applies

when that summand is zero. For fixed h_0, k, g, the remaining theta sum commutes with this transform: its rational finite coordinates range in a fractional lattice, and Schwartz decay in the rational and line variables bounds the absolute sum of the line integrals. Write $P_+ = K_V K^\vee$, $P_- = L_V L^\vee$, and abbreviate the fixed lower and compact arguments of $\ell_\pm$. Let F be the common complementary zero slice and Θ its residual theta functional. The commutator of $\mathcal{R}(P_+x)$ and $\mathcal{R}(P_-y)$ is $\langle x, y\rangle = \psi_E(j_{\mathrm{W}}\bar{y}x)$, since P_+, P_- pair to one. In the first polarization these operators translate by x and multiply by $\langle\,\cdot\,, y\rangle$, respectively. In the second their roles are reversed and the commutator is inverted. Thus the restrictions of the displaced theta value to the two isotropic lines have the expansions

$$\Theta(\mathcal{R}(P_-y)F) = \sum_{\xi\in E} \ell_+(\xi)\langle\xi, y\rangle,$$
$$\Theta(\mathcal{R}(P_+x)F) = \sum_{\eta\in E} \ell_-(\eta)\langle x, \eta\rangle^{-1}.$$

Both sums at the identity equal $\Theta(F)$. More generally, put $[E] = E\backslash\mathbb{A}_E$ with probability Haar. The opposite slice is the mean after modulation and translation in the first polarization. For every $y \in \mathbb{A}_E$ it gives

$$\begin{aligned}\ell_-(y) &= \int_{[E]} \Theta(\mathcal{R}(P_+x)\mathcal{R}(P_-y)F)\,dx \\ &= \int_{[E]} \sum_{\xi\in E} \ell_+(\xi + x)\langle\xi + x, y\rangle\,dx = \int_{\mathbb{A}_E} \ell_+(z)\langle z, y\rangle\,dz.\end{aligned}$$

The last Haar measure has rational covolume one. In this Fourier convention the transform of $z \mapsto f(az)$ is $y \mapsto |a|^{-1}\widehat{f}(\bar{a}^{-1}y)$. The line function and its transform are Schwartz–Bruhat: derivatives and polynomial weights in that line are bounded after summing the remaining rational lattice by taking sufficiently many further powers of Schwartz decay in the lattice variables. Their periodizations converge uniformly on compact representatives. Adelic Poisson summation [74, Lemma 4.2.4] therefore gives the displayed equality before the substitution $z = \alpha_0 a$ in the finite line test. Both zero-frequency contributions are zero by the selected finite unit support of the respective slices. Put $a' = \overline{au}^{-1}$. Then $\nu^{\mathrm{bc}}(a') = \nu^{\mathrm{bc}}(au)$ and $|a'| = |au|^{-1}$. Multiplicative Haar on the idele class group is invariant under translation, conjugation and inversion. Substituting the Poisson formula shows that $Z(s)$ uses the negative line sum with exponent $1 - s$. In the equal-rank case, $a \mapsto au$ shows directly that the constant term is independent of u.

For $s > 1$ unfold the sum over $E^\times$ in the positive-sign formula for $Z(s)$ or in the negative-sign formula for $Z(1 - s)$. From the finite factor just obtained, the idele translation is $K = \alpha_0 c_0^{-1}$, where c_0 is $\bar{\varpi}_w$ at $\bar{w}$ and one elsewhere. Thus $|K| = |\alpha_0|Q_S$, with $Q_S = \prod_{w\in S} q_w$. In the original line slices substitute $z = Ka$. Since $(c_0)_\infty = 1$, this gives

$$(z/\alpha_0)_\infty = a_\infty, \qquad F_\varpi((z/\alpha_0)_f) = F_1(a_f),$$

where F_1 uses the same formula with $\varpi = 1$. Multiplicative Haar is invariant, and the character and modulus supply the factor $\nu^{\mathrm{bc}}(K)|\alpha_0|^s Q_S^s$.

Normalize the local multiplicative measures by $\mathrm{vol}_{d^\times a_w}(\mathcal{O}_{E_w}^\times) = 1$ and take their restricted product on the finite ideles. The original finite measure is $v_\pm$ times this product, where $v_\pm > 0$ is its mass on $\widehat{\mathcal{O}}_E^\times$, fixed before S. At $w \in S \cup \bar{S}$ the translated function is supported on

units and $\nu^{\mathrm{bc}}F_1 = 1$ there, so the normalized local integral is one. Elsewhere the integral is $\sum_{n\ge 0}\nu_w^{\mathrm{bc}}(\varpi_w)^n q_w^{-ns}$, since each multiplicative shell has mass one. The full finite factor is therefore $v_\pm L^S(s,\nu^{\mathrm{bc}})$, abbreviated to $v_\pm L^S(s,\nu)$ below. In absolute value the sum is bounded by the partial Dedekind zeta series at $s>1$. The real integral is the cone vector $C_\pm(s)$ formed by integrating the real line slice against $\nu^{\mathrm{bc}}(a_\infty)|a_\infty|^s$ and the original target compact Haar. For a finite point k of the original target compact, let k_S be its component on the chosen split side. In the upper covector model set $\phi_p(Y)=\chi^{-1}(\det[p\,\mathrm{pr}_{W_1};Y])$ on the integral unit support and zero elsewhere. Multiplication by $\mathrm{diag}(I,k_S^{-1})$ preserves that support and gives

$$\phi_p(k_S^{-1}Y)=\chi(\det k_S)\phi_p(Y),\qquad (\nu^{-1}\upsilon^{\dim(\widetilde{W})/2})(\det k)=\chi(\det k_S)^{-1}.$$

The second equality uses the unramified splitting character and the prescribed tame character on units. The two factors cancel. The unselected compact factors preserve the integral vacuum. Thus the finite compact fixes the standard vector in the twisted Weil action, and its images in the two slice formulas have the same invariance.

For either polarization let $v_\pm(k_\infty,a_\infty)$ be the real line slice before insertion of the Mellin weight. The common zero slice is restriction in a fixed mixed model. After the possible one-line Fourier transform it is still Schwartz jointly in the line and the remaining coordinates. As k_∞ varies in the maximal compact these functions form a bounded Schwartz family. Thus for every remaining Schwartz seminorm p_j and integer M,

$$\sup_{k_\infty} p_j(v_\pm(k_\infty,a_\infty)) \ll_{j,M} \prod_{\tau|\infty}(1+|a_\tau|)^{-M},$$

where the absolute values are ordinary complex moduli. For $0<\sigma_0\le \Re s\le\sigma_1$, the weighted majorant at each place is $r^{2\sigma_0}$ for $r\le 1$ and $r^{2\sigma_1-M}$ for $r>1$, with radial measure dr/r. Choosing $M>2\sigma_1$ gives absolute convergence in every seminorm, also after any fixed power of $\log|a_\infty|$. Schwartz completeness gives the cone vector. For either order of slicing, the retained covector test has full row rank on its support at a selected split place, by the invertibility of $[p;Y]$. The fixed lower source point preserves this rank by invertible right multiplication. Use the full lower source basis in the kernel estimate of Section 34.3.1. Equation (100) thus bounds the absolute value of the lower theta kernel uniformly on the target quotient by a fixed Schwartz seminorm. For rank zero use evaluation on the one-point quotient. At $z=\alpha_0 a$ the finite slice is $l_\pm\mathfrak{g}F_\varpi(a_f)$ times the retained covector test. Together with the real seminorm estimate this gives, for each $M>0$,

$$\sup_{h_0,k}|\ell_\pm(\alpha_0 a;h_0,k)| \ll_{g,M} |F_\varpi(a_f)|\prod_{\tau|\infty}(1+|a_\tau|)^{-M}.$$

Choose an integral ideal $\mathfrak{d}$ clearing the finite line-coordinate denominators in both slices. Such an ideal exists because their finite tests are compactly supported and have only finitely many translates under the compact variable k_f. In the normalized finite multiplicative measure, the integral of $|z_f|^\sigma$ on $\mathfrak{d}^{-1}\widehat{\mathcal{O}}_E\cap\mathbb{A}_{E,f}^\times$ is $\mathrm{N}(\mathfrak{d})^\sigma\zeta_E(\sigma)$, by summing the valuation shells. For $\sigma>1$ and $M>2\sigma$ this gives

$$\int_{\mathbb{A}_E^\times}\int_{[U(V)]\times K_{\widetilde{V}}}|z|^\sigma|\ell_\pm(z;h_0,k)|\,dh_0\,dk\,d^\times z \ll \mathrm{N}(\mathfrak{d})^\sigma\zeta_E(\sigma)\prod_{\tau|\infty}\int_0^\infty r^{2\sigma}(1+r)^{-M}\,\frac{dr}{r}<\infty.$$

Here $K_{\widetilde{V}}$ denotes the original maximal compact of the target, and the constant includes $\mathrm{vol}([U(V)])$ and the fixed measure factors. The bound justifies termwise unfolding for each sign in its own half-plane $\operatorname{Re} s > 1$. For the Mellin integral on the original quotient, write

$$D(a, h_0, k) = \nu^{\mathrm{bc}}(au) \sum_{\xi \in E^\times} \ell_+(\xi au; h_0, k), \qquad r = |au|.$$

The positive unfolding bounds the integral of $|D(a, h_0, k)| r^{\sigma_+}$ for $\sigma_+ > 1$. For $\sigma_- < 0$, apply Poisson summation to the sum before taking its absolute value. The zero terms vanish as above. With $a_- = \overline{au}^{-1}$, the Poisson identity and unitarity of ν^{bc} give

$$\begin{aligned} |D(a, h_0, k)| r^{\sigma_-} &= |a_-|^{1-\sigma_-} \left| \sum_{\xi \in E^\times} \ell_-(\xi a_-; h_0, k) \right| \\ &\le |a_-|^{1-\sigma_-} \sum_{\xi \in E^\times} |\ell_-(\xi a_-; h_0, k)|. \end{aligned}$$

The negative unfolding bounds the integral because $1 - \sigma_- > 1$ and $d^\times a_- = d^\times a$.

Consequently $r^\sigma \le r^{\sigma_-} + r^{\sigma_+}$ for $\sigma_- \le \sigma \le \sigma_+$ gives an integrable majorant for the grouped integral. Widening the interval absorbs every power of $\log r$, giving holomorphy of the Mellin integral.

To finish the unfolding, put $G = U(\widetilde{V})$ and $[G] = G(E^+)\backslash G(\mathbb{A}_{E^+})$. Let P_L stabilize the chosen rational isotropic line L, and write $B_L(\tilde{h})$ for the sum of the radical terms indexed by the nonzero rational vectors on L. Rational covariance permutes these vectors, so B_L is left $P_L(E^+)$-invariant. Each line term is invariant under the adelic target radical by the zero-slice calculation above. In the Iwasawa coordinates above, $|B_L|$ times the orbit density is $|D(a, h_0, k)|\, r^{e+1}$, with the fixed measure scalar. The radical quotient is compact. The preceding strip bound therefore gives

$$\int_{P_L(E^+)\backslash G(\mathbb{A}_{E^+})} |B_L| = \int_{[G]} \sum_{\delta \in P_L(E^+)\backslash G(E^+)} |B_L(\delta \tilde{h})|\, d\tilde{h} < \infty.$$

The equality is the quotient integration formula for nonnegative functions. The zero-vector term vanishes, and Witt extension gives one orbit on the remaining isotropic lines. We can now integrate the grouped line expansion and unfold it to $c_{\mathrm{unf}} Z(e+1)$, including at $e + 1 = 0, 1$.

In both cases the value required is $Z(e+1)$. If $e \ge 0$, the positive cone has exponent $s(e) = e + 1$. If $e < 0$, the negative cone has exponent $s(e) = 1 - (e+1) = -e$. The boundary values at $e = 0, -1$ are

$$Z(1) = \lim_{t \downarrow 1} Z(t), \qquad Z(0) = \lim_{t \downarrow 1} Z(1 - t),$$

respectively. The real cone factors are continuous at one by their Schwartz bounds, and the nontrivial Hecke L-function is regular there. The exact scalar multiplying the lower measured period in the respective model is

$$\gamma_{\pm,\nu} = c_{\mathrm{unf}}\, l_\pm\, v_\pm\, (\mathfrak{g}\sqrt{Q_S})\, \nu^{\mathrm{bc}}(K)\, |\alpha_0|^{s(e)} Q_S^{s(e)-1/2} L^S(s(e), \nu).$$

In particular $|\mathfrak{g}\sqrt{Q_S}| = 1$. This gives the first identity of (2) and its stated constant with all measure-dependent quantities $c_{\mathrm{unf}}, l_\pm, v_\pm$ fixed before the twist.

36.1.5 The small radical mean and its zero term

In this subsection, we compute the second radical integral of Section 36.1.3 and show that only its zero term remains.

For the small side read the transposed column datum at $\bar{S}$. In the original paired bases the middle projection read at S is the transpose of the middle inclusion read at $\bar{S}$. Consequently the test in the old basis only reads the lower complementary block of the transpose inside $\det(pZ)$. All other factors are the integrality conditions of the original lattice. Its source-line slice at $\alpha_0 x$ is zero unless x_f is integral, and otherwise equals the standard lower column datum, times the real slice at x_∞ and the fixed vacuum slice constant $\kappa_A \neq 0$. Indeed the extra sliced coordinates have exactly the constant integral-indicator test of the vacuum in this projection basis. In particular the slice at zero uses exactly the same lower finite datum at $\bar{S}$.

Take the source radical mean of this small kernel at $\mathrm{levi}(1, g)$. The central average again retains the isotropic rational vectors of the fixed small space U. At a nonzero isotropic vector, apply (101) with the transported line in the small target. This first takes the source slice and then the target zero term. The surviving rational indices have zero component in the opposite block. Hence the resulting theta functional is invariant under all Heisenberg translations along the transported target line.

The transported line vector at any selected place is nonzero. In the finite Schrödinger polarization at $\bar{S}$ this invariance is invariance under all multiplications by the additive characters paired against $x^* Z^{\mathrm{t}}$, where x is that line coordinate and Z the column variable. On the support, pZ is invertible, so Z is left-invertible; hence $x^* Z^{\mathrm{t}} \neq 0$. Nondegeneracy of the additive pairing makes one of these multipliers different from one at each support point.

It is constant on a sufficiently small compact open neighborhood. Compactness gives a finite disjoint refinement of this cover, and on each part the invariant functional must be zero. This proves vanishing of the whole nonzero isotropic term, with no size estimate and no change of finite datum. Anisotropic terms have already vanished in the central mean.

Only the zero source slice remains. At that slice the retained group action is the old one restricted to W_1 and U. The source splitting character drops by the prescribed one power of υ when the rank drops by two, which is the lower rank twist. The finite part is exactly the same standard column datum and the small real vector is the zero slice. Thus the small radical mean is κ_A times the lower measured period, as asserted in (2). For $U = 0$ this argument simply retains the unique zero vector on the zero-group quotient.

In particular,

$$|\gamma_{\mathrm{pl},\nu}| = C_{\mathrm{pl}} L_\nu^\sharp(s(e)), \qquad L_\nu^\sharp(s) = |L^S(s, \nu)| \prod_{w \in S} q_w^{s-1/2}, \tag{3}$$

where $C_{\mathrm{pl}} > 0$ includes the original line-orbit, slice and unit-volume factors, all fixed before the twist.

For the real section needed below, let x_B be the target isotropic vector. Give the lower vector v a target-line mixed-model extension v_B by tensoring with a Schwartz function of value one at zero in the removed coordinates. At each real place $x_{B,\tau} \neq 0$, and Witt extension gives the product orbit

$$\prod_{\tau \in \Phi} \{x_\tau \in \widetilde{V}_\tau \setminus \{0\} : h_\tau(x_\tau, x_\tau) = 0\}.$$

Its orbit maps are submersions. Take smooth local sections $u_x x_B = x$ with $u_{x_B} = 1$ on this product near x_B. At every real place the nonzero isotropic cone is a smooth real hypersurface

near $x_{B,\tau}$: the differential of $x \mapsto \langle x, x\rangle$ there is the nonzero real form $v \mapsto 2\Re\langle x_{B,\tau}, v\rangle$. A submanifold chart gives a smooth retraction onto the cone, so the local section extends to an ambient neighborhood by composition with this retraction. Choose a smooth nonnegative β with $\beta(x_B) = 1$ supported in a product of balls whose closures lie in that neighborhood and whose radii are less than $\|x_{B,\tau}\|/4$. Multiplication by β and extension by zero gives a global Schwartz slice. Prescribe on the source-line mixed model the slice

$$\beta(x)\omega_{N_1/2-\mu}(1, u_x)v_B.$$

On the support of the cone integral at (κ, a_∞) we have $x = \kappa_\infty^{-1} x_B a_\infty$. Then $\kappa_\infty u_x$ sends x_B to $x_B a_\infty$.

Let $h \in U(V)_\infty$ be its induced isometry on $(E_\infty x_B)^\perp/(E_\infty x_B)$. The diagonal blocks in the adapted Witt basis are $a_\infty, h, \bar{a}_\infty^{-1}$. The earlier target radical calculation applies to the real zero slice: the displacement and central phase exponent vanish there. Thus only the three Levi blocks act on that slice. At τ put $u_\tau = a_\tau/|a_\tau|$, where $|a_\tau|_{\mathbb{C}} = |a_\tau|^2$ is the normalized local absolute value. The isotropic block on $\mathrm{Hom}(W_{1,\tau}, \mathbb{C}x_{B,\tau})$ is $a_\tau I_{N_1}$. Equation (58) gives its intrinsic multiplier $u_\tau^{-N_1}|a_\tau|_{\mathbb{C}}^{N_1/2}$. The determinant of the three Levi blocks divided by $\det h_\tau$ is $a_\tau/\bar{a}_\tau = u_\tau^2$. The source slice reduces the target exponent from $N_1/2 + 1 - \mu_\tau$ to $N_1/2 - \mu_\tau$. Relative to the action of h_τ, the zero-slice multiplier is therefore

$$u_\tau^{-N_1}|a_\tau|_{\mathbb{C}}^{N_1/2}(u_\tau^2)^{N_1/2-\mu_\tau} = |a_\tau|_{\mathbb{C}}^{N_1/2} u_\tau^{-2\mu_\tau}.$$

The phase over all τ cancels $\nu^{\mathrm{bc}}(a_\infty)$, and $|a_\infty| = \prod_\tau |a_\tau|_{\mathbb{C}}$. The translation by h disappears from the lower period by right invariance on its original quotient.

The cone period of the constructed real vector is therefore the prescribed lower period times

$$B(s) = \int_{\kappa, a_\infty} \beta(\kappa_\infty^{-1} x_B a_\infty)|a_\infty|^{s+N_1/2}\, d\kappa\, d^\times a_\infty.$$

On the support, Euclidean unitarity of κ_∞ gives $3/4 < \|\kappa_\tau^{-1} x_{B,\tau} a_\tau\|/\|x_{B,\tau}\| = |a_\tau| < 5/4$ at every real place, where $|a_\tau|$ is the ordinary complex modulus. At $(\kappa, a_\infty) = (1, 1)$ the bump is one, so the integrand is positive on an open set of positive measure for every real s. Differentiation in s on the fixed compact annuli proves that B is entire. The same bump and normalization apply at every modulus and finite character. At a fixed positive exponent we may divide the real vector by $B(s)$ to have exactly the desired period. For the continued plus identity at a negative exponent the same section works without altering its finite test.

The Euler product proves nonvanishing for real $s > 1$. Taking logarithms of Euler products and using $3 + 4\cos\theta + \cos 2\theta \geq 0$ prime power by prime power gives

$$1 \leq \zeta_E^S(s)^3 |L^S(s, \nu^{\mathrm{bc}})|^4 |L^S(s, (\nu^{\mathrm{bc}})^2)|.$$

All three products omit $S \cup \bar{S}$. The pole of ζ_E^S at 1 is simple. A zero of $L^S(s, \nu^{\mathrm{bc}})$ at 1 would therefore make the right side tend to zero as $s \downarrow 1$. The square character is nontrivial because the weights are positive, so its L-function is regular at 1 by the Hecke continuation theorem.

Since $M_\nu(j+1)/M_\nu(j) = L_\nu^\sharp(j+1)^2$ for $j \geq 0$, and $M_\nu(-j) = M_\nu(j)^{-1}$, (3) gives

$$|\gamma_{\mathrm{pl},\nu}|^2 M_\nu(e) = C_{\mathrm{pl}}^2 M_\nu(e+1).$$

Let $\epsilon_{\mathrm{ad},\nu}$ be the unit multiplier of the preceding target transport, with the convention that the upper identity in the adapted coordinates reads $\Theta_{\mathrm{old}}^+ = \epsilon_{\mathrm{ad},\nu} c_{\mathrm{old},\nu} \Theta_{\mathrm{old}}^-$. It is the product of the

functional and character multipliers of the chosen isometry and is defined even when $c_{\mathrm{old},\nu}=0$. For an already adapted model take $\epsilon_{\mathrm{ad},\nu}=1$. Taking the two means in (2) gives

$$c_{\mathrm{new},\nu}=\epsilon_{\mathrm{ad},\nu}c_{\mathrm{old},\nu}\kappa_A/\gamma_{\mathrm{pl},\nu}.$$

The positive normalizing factor changes from D_{old} to $D_{\mathrm{old}}C_{\mathrm{pl}}^2/|\kappa_A|^2$, so

$$\frac{D_{\mathrm{new}}|c_{\mathrm{new},\nu}|^2}{M_\nu(e)}=\frac{D_{\mathrm{old}}|c_{\mathrm{old},\nu}|^2}{M_\nu(e+1)}. \tag{4}$$

The recurrence is valid for every integral defect allowed by the rank and signature conditions. If $e=-j<0$, then

$$\frac{M_\nu(e+1)}{M_\nu(e)}=\frac{M_\nu(j)}{M_\nu(j-1)}=L_\nu^\sharp(j)^2.$$

Thus the negative case uses the positive Mellin exponent $j=-e$, including the later application with $e=-k$ at $(J,0,J)$. Both positive Mellin exponents are at least one, and ν is nontrivial. Thus the factor $\gamma_{\mathrm{pl},\nu}$ of (3) is nonzero. The two scalars $c_{\mathrm{old},\nu}$ and $c_{\mathrm{new},\nu}$ can vanish until the choice of the equal-rank vector below.

At the top the factor is $D_{\mathrm{top}}=(R_0|\kappa_0|)^{-2}$ and the value of this invariant is

$$R_{W_0,\nu}(z_0)=|I_{W_0,\nu}(z_0)|^2/M_\nu(j_0). \tag{5}$$

The small target U^- stays fixed. After i removals, $0\le i\le b+1$, the source and large target ranks are $N_i=N+2(b+1-i)$ and $a_i=a+2(b+1-i)$. The large target has $q(V)+b+1-i$ negative signs, since $q(U^-)=b-q(U)$. Thus its defect is $e_i=a_i-N_i/2=j_0-i$, from $e_0=j_0$ to $e_{b+1}=t$. Writing $C_i,\kappa_{A,i}$ for the constants of step i, the telescoped identity is

$$|c_{\mathrm{final},\nu}|^2=|c_{\mathrm{top},\nu}|^2\frac{M_\nu(t)}{M_\nu(j_0)}\prod_{i=1}^{b+1}\frac{|\kappa_{A,i}|^2}{C_i^2}.$$

It applies also when $b=0$, with one removal, and when $a=0$, with final defect $-N/2$. Successive source zero slices compose, since they are evaluations of separate blocks in the product mixed model. Let y^{ad} be the image of the prescribed vector y for (U,W) under the signed real adjoint, so its pair is (W,U^-). This coordinate change preserves each coordinate modulus up to permutation and preserves the Gaussian and Lebesgue measure. The weighted Fock moments of y^{ad} are finite, with the coordinatewise moment indices permuted. In the one-plane mixed model on (W_0,U^-), extend y^{ad} by a Schwartz tensor of value one at zero to obtain v. Use this v in (97), whose opposite zero slice after the other b planes is v by its integral normalization. The combined zero slice is consequently y^{ad}. On the large side take the successive cone slices of (96) at their prescribed exponents $s(e)$. Their real integrals use only the fixed infinity type μ, the splittings and the two initial real vectors. Thus, for a fixed z_0, they give one large vector before the modulus and the finite matched arrays are chosen. After $b+1$ removals (4) holds at (W,V,U^-) with defect t and small vector y^{ad}; adjoint transport returns it to the prescribed y below.

From standard data to the actual finite data. In the product $F_S=\prod_{w\in S}E_w$, put $O_S=\prod_{w\in S}\mathcal{O}_w$. Let p be an integral block of size b and use the two standard data $\chi^{-1}(\det[p;X])$ and

$\chi^{-1}(\det(pY))$, each supported on integral arguments with unit determinant in the indicated formula. For X the last covectors of k^{-1} and Y the first columns of k, one has

$$\det[p; X] \det k = \det(pY).$$

Thus these are matched, with factor $\chi(\det k)$.

For use with the equivariant difference of the two theta periods, we prove finite generation. A locally constant matched column function Ψ is supported on primitive integral columns. It satisfies $\Psi(Yu) = \chi(\det u)^{-1}\Psi(Y)$ for $u \in \mathrm{GL}_b(O_S)$: replace a completion k by $k\,\mathrm{diag}(u, 1)$. Choose x_0 primitive and an integral p with $px_0 = 1$. Let $F(Y) = \chi^{-1}(\det(pY))$ on its standard support. Take a common depth $c \geq 1$ of local constancy of Ψ, and the product uniformizer $\varpi = (\varpi_w)_{w \in S}$.

Set

$$d = x_0 p + \varpi^{-c}(1 - x_0 p), \qquad K = \mathrm{GL}_N(O_S).$$

If $F(dkY) \neq 0$, put $Z = dkY$ and $u = pZ$. Then u is invertible and

$$Y = k^{-1}x_0 u + \varpi^c k^{-1}(Z - x_0 u), \qquad \Psi(Y) = \Psi(k^{-1}x_0)F(dkY).$$

Conversely any nonzero value of $F(dkY)$ forces Y primitive integral. Thus

$$\int_K \Psi(k^{-1}x_0)F(dkY)\,dk = c_F \Psi(Y) \tag{102}$$

everywhere, where $c_F = \int_K |F(dkx_0)|\,dk > 0$. For a datum $m(Z)F(Z)$ with nonnegative locally constant multiplicity m, the same calculation replaces c_F by $\int_K m(dkx_0)|F(dkx_0)|\,dk$, which is positive when $m(x_0) > 0$. To see constancy of c_F on primitive Y, use the transitivity of K on primitive columns. Positivity follows on a neighborhood of $k = 1$.

All functions of k in the average are locally constant. The translated F's themselves range over finitely many functions on K, by smoothness of the translation action on locally constant compactly supported functions. Consequently this is a finite linear combination of translates of F, or of the function with multiplicity in the preceding formula. For the detector $G = B(-\varpi^{-1}\,\cdot)$ the average in fact factors through $\mathrm{GL}_N(O_S/\varpi^{c+2}O_S)$. If $k_1 \equiv k_2 \bmod \varpi^{c+2}$ and Y is integral, then $dk_1 Y - dk_2 Y \in \varpi^2 M_{N,b}(O_S)$. Both evaluations vanish on nonintegral Y, since d^{-1} is integral. The factor $\Psi(k^{-1}x_0)$ is constant on the same cosets. If this finite quotient has order h, the scalar multiplying $\Psi(Y)$ is $h^{-1}\sum_{\bar{k}} |G(dkx_0)| \geq h^{-1}$: the identity coset contributes $|G(x_0)| = 1$. The split covariance of the difference $\mathcal{D}$ takes the form

$$\mathcal{D}(\Psi \circ g_\Sigma)(xg_\Sigma) = \lambda_b(\det g_\Sigma)^{-1}\mathcal{D}(\Psi)(x), \qquad \lambda_b(u) = (\upsilon_E(u)|u|^{1/2})^b.$$

Its kernel is therefore stable under the translates. The common-section calculation in Section 35.2 identifies the domain with the original integral matching and its split translates. This proves the comparison for every locally constant matched pair.

For a modulus at S, the reduction $O_S \to \mathcal{O}_E/\mathfrak{m}$ is surjective and detects units by the Chinese remainder theorem. Each invertible residue matrix lifts to an invertible integral matrix by lifting its entries and using the determinant test. A primitive integral covector or column block extends to such a matrix by completion over each local ring. The residue matching law and its support thus lift exactly, by pullback and extension by zero, to the same law and support on O_S. The unit character is the same product of nontrivial residue characters on the selected

support S. More explicitly, write $\mathfrak{m} = \prod_{w\in S} \mathfrak{p}_w^{d_w}$ with arbitrary $d_w \geq 1$ and $S \cap \bar{S} = \varnothing$. The character on $(\mathcal{O}_E/\mathfrak{m})^\times$ is the product of the tame characters on $k_w^\times$, pulled back by reduction. The two matched functions may depend on all residue coordinates modulo $\mathfrak{p}_w^{d_w}$. In particular the comparison on columns determines every original covector value over the finite ring. For a primitive row matrix X, take a completion k with X the last rows of k^{-1}, and let Y be the first columns of k. Another completion is $k \left(\begin{smallmatrix} u & v \\ 0 & 1_a \end{smallmatrix}\right)$ with u invertible. Its determinant factor $\chi(\det u)$ cancels the factor $\chi(\det u)^{-1}$ in $\Psi(Yu)$. Thus $\Phi(X) = \chi(\det k)\Psi(Y)$ is independent of the completion. Both functions vanish off the primitive locus. Their pullbacks are locally constant, so the finite spanning formula applies for every $d_w \geq 1$, including the old exponent b_0 in Section 43.4. Outside $S \cup \bar{S}$ the prescribed vacua are identical. In particular the change from the finite ring to the local integral model introduces no level indices or measure ratios.

36.1.6 The equal-rank normalizer

In this subsection, we use the invariant period ratio (4) to normalize the comparison in Section 39.4.

Write $N = 2k+2q(W)$, so that $\mu_\tau \geq k \geq 1$. First take a positive definite space J of rank $2k$ and $Y = H \perp J$. For each ν the equal-rank period $I_{Y,\nu}(z)$ with determinant datum is constant in its unitary variable by the special-unitary invariance and the one-plane Levi calculation of Section 35.1.

Fix a split tame character of the indicated weight. Suppose that every equal-rank period of (Y, Y) with the finite determinant datum is zero. In the large space $X = H^{2k} \perp Y$, with target $J^- \perp Y$, consider an arbitrary real Schwartz vector. First use the self-dual Fourier test with rational Hermitian index J_{J^-} and its prescribed adelic basis. The frame $\binom{1}{0}$ has orthogonal complement Y, and its displacement $\binom{d_c}{0}$ has zero Y-component. On the supported finite cosets the upper triangular determinant in Section 36.1.1 leaves precisely the original square determinant datum on Y, up to the scalar from the frame block. A rational congruence transports the index and the coefficient point together as in Section 35.2.2.

Fix an outer point of the unfolded integral and denote the moved real mixed slice by w. Use the weighted Fock coordinates of Section 34.3.1. For a block of complex dimension d_i, write $du_i = \pi^{-d_i} d\,\mathrm{Leb}(u_i)$. The Gaussian is included in the values of w, and the unit coherent vector is $v_u(z) = \exp(-(|z|^2 + |u|^2)/2 + \bar{u}\cdot z)$. With the inner product linear in the second argument, the reproducing formula in the two blocks gives

$$w(u_1, u_2) = \langle v_{u_1} \boxtimes v_{u_2}, w\rangle, \qquad w = \int w(u_1, u_2)\, v_{u_1} \boxtimes v_{u_2}\, du_1\, du_2.$$

In the real Schrödinger model each v_u is a translate and modulation of the fixed Gaussian, with parameters linear in the real coordinates of u. Translation of the weights and differentiation of the modulation give $p_j(v_u) \ll_j (1 + \|u\|)^j$ for the seminorms of Section 36.1.1. With the first finite vector and test point fixed, the finite-order theta bound therefore gives polynomial growth of $T(v_{u_1})$. All weighted absolute moments of the Schwartz slice w are finite, so the reproducing integral and the contraction below converge in every Schwartz seminorm. In particular, the contraction

$$w_2 = \int w(u_1, u_2) T(v_{u_1}) v_{u_2}\, du_1\, du_2$$

is a real Fock vector with all moments finite. Write $T_{Y,h}(v_{u_2})$ for the second theta evaluation at $h \in [Y]$ with its finite determinant datum. At any chosen $w \in S \neq \varnothing$ its support consists

of invertible square matrices, so it has full row rank in the orientation for integration over $[Y]$. The ordinary period bound (100), with the source point fixed and height exponent zero, bounds $T_{Y,h}(v_{u_2})$ uniformly in h by a fixed Schwartz seminorm of v_{u_2}. Translation and modulation of the Gaussian bound this seminorm by a polynomial in $\|u_2\|$. Thus for some C, M, with the first test and point fixed,

$$\int_{[Y]} \int |w(u_1,u_2) T(v_{u_1}) T_{Y,h}(v_{u_2})|\, du_1\, du_2\, dh$$
$$\leq C\ \mathrm{vol}([Y]) \int |w(u_1,u_2)|(1+\|u_1\|+\|u_2\|)^M\, du_1\, du_2 < \infty.$$

The tensor kernel product and Fubini identify the inner period with the assumed zero period of w_2. For an arbitrary upper vector the original unfolded term is absolutely integrable by Section 36.1.1, so its disintegration over the open orbit integrates this zero inner period.

For the primitive Fourier indices of Section 35.2, the Gram matrix is invertible over E, since their scaled reductions are invertible at the chosen places. At the points used in the finite detection argument, the coefficient of the detector at any remaining T is the coefficient of Ψ_0 times the matrix Gauss sum. If Ψ_0 has a nonzero coefficient, the scaled index is integral at every selected place. At a singular reduction the change $r \mapsto ur$ in that local sum, with $\bar{A}u = \bar{A}$ and $\chi_w(\det u) \neq 1$, gives zero. This applies in particular to every singular rational index. Hence only represented primitive indices require the orbit calculation. For such an index outside the negative real class, choose a real place at which its nondegenerate span has $s_\tau \geq 1$ positive directions. Put this span in standard signed position by a unitary change of coordinates. In the smaller mixed Fock space the source $Y = H \perp J$ has $(p_Y, q_Y) = (2k+1, 1)$. The target has $b + q_Y$ negative coordinates, where $b = 2k$. The span uses $b - s_\tau$ of them, so the compact group on the $q_Y + s_\tau$ unused negative coordinates fixes the entire frame. The real determinant exponent after the source slice is $m_\tau = (\dim Y)/2 - \mu_\tau = k + 1 - \mu_\tau$.

Fix a point of the signed polynomial Fock coordinates, and let Z_- be its q_Y negative-source evaluation columns in the unused negative coordinates. Choose a unit column r there with $r^{\mathsf{t}}\overline{Z_-} = 0$. These are at most q_Y linear conditions in dimension $q_Y + s_\tau > q_Y$. The circle $k_\eta = I + (\bar{\eta} - 1)\bar{r}r^{\mathsf{t}}$, $|\eta| = 1$, lies in the same fixed compact stabilizer. Its determinant is $\bar{\eta}$, and the negative-block vacuum multiplier and the twist give

$$(\det k_\eta)^{-p_Y + m_\tau} = \eta^{p_Y - m_\tau} = \eta^{k + \mu_\tau}.$$

On the signed polynomial coordinates with negative target sign, substitute k_η^{t} on the negative-source columns and k_η^{-1} on the positive-source columns. The first substitution fixes Z_-, since $\bar{r}^{\mathsf{t}} Z_- = 0$. The second multiplies the line component along $\bar{r}$ by η and fixes its complement. If a polynomial vector is invariant under the stabilizer's compact subgroup, its evaluation at this point is therefore the mean on this circle of a polynomial in η all of whose exponents are at least $p_Y - m_\tau > 0$. This mean is zero. As the point was arbitrary, there is no invariant polynomial vector. The compact group preserves each homogeneous polynomial space. Its probability average is a bounded projection on the Fock Hilbert space, so polynomial density makes this projection zero. For a real Schwartz vector the compact orbit map is continuous in the Schwartz topology. Its Haar integral there has the same Hilbert image, hence is zero in Schwartz space. The continuous mixed theta functional may therefore be applied to this average.

Unfold the Fourier term over the rational basis stabilizer as in the self-dual calculation. Disintegrate through the adelic stabilizer and average over the compact subgroup just used,

conjugated back to the given basis. Right multiplication by the compact subgroup preserves the stabilizer Haar, since the modular character is trivial on compact groups. The mixed-model slice commutes with this stabilizer action at the chosen vector, with the lower source rank twist. Thus its compact average vanishes by the Fock calculation. The coefficient therefore vanishes off the negative class for an arbitrary upper real vector. If no rational basis realizes the index, it is an empty sum.

Apply Section 35.2.2 to the large period and the zero functional on the small side. The two cases above give vanishing for every self-dual test. The primitive coefficient transfer then gives vanishing of the large period at every point for every upper real vector and standard covector datum. To descend this vanishing, take any proposed lower period I and the common real bump section on the cone, whose scalar is $B(s) > 0$ for every real s. For this lift of the given lower vector, with its fixed covectors and source point, write $Z(s)$ for the grouped Mellin integral of Section 36.1.4, absorbing the positive unfolding constant. The positive-sign unfolding gives

$$Z(s) = d_\nu |\alpha_0|^s Q_S^{s-1/2} L^S(s,\nu) B(s) I \qquad (\Re s > 1), \qquad d_\nu \neq 0.$$

The positive and Poisson-dual estimates there give an integrable majorant for the grouped integral on each closed vertical strip. Thus $Z(s)$ is entire. Its value at $e+1$ is the constant term of the upper period on this lifted datum.

The right side is holomorphic on $\Re s > -k$ by Tate's continuation theorem [74, Theorem 4.4.1 and §4.5]. The nonzero angular weights exclude a norm character. Holomorphic continuation gives the identity throughout this half-plane. At each needed integer $s = e + 1 > -k$ this Hecke factor is nonzero. For $s > 1$ the Euler product converges absolutely. At $s = 1$ use the nonvanishing argument with $3 + 4\cos\theta + \cos(2\theta)$ above. For $s \leq 0$ the infinity factors are $\Gamma_{\mathbb{C}}(s + \mu_\tau)$ and $\Gamma_{\mathbb{C}}(1 - s + \mu_\tau)$ at the two reflected points. Their arguments are positive, and the Hecke functional equation reduces the assertion to nonvanishing at $1 - s \geq 1$ for the dual character.

The Euler factors removed at $S \cup \bar{S}$ are already the ramified factors in that equation. By the upper vanishing the left side is zero at $s = e + 1$, so $I = 0$. For these removals, use the rational hyperbolic splitting of $J^- \perp J$ given by its diagonal and antidiagonal. In the original finite orthonormal bases of the two copies the Gram matrix is $\operatorname{diag}(I, I)$. At each finite place choose integral u_- and c with $u_- \bar{u}_- = -1$ and $c + \bar{c} = 1$. The unit norm and integral trace are surjective in the split and unramified quadratic algebras, also at residue characteristic two. The two isotropic frames in the original coordinates have columns

$$(u_- u, u), \qquad (-u_- c v, (1 - c) v),$$

for the standard basis vectors u, v of the two coordinate spaces. Their cross pairing is $\bar{u}^{\mathsf{t}} v$. The block matrix with these columns has determinant u_-^{2k}, an integral unit. It therefore preserves the finite lattice and its additive measure. At every split place fix $u_- = (1, -1)$ and $c = (1, 0)$ in the product algebra, independently of the selected set S. Then $u_-^{2k} = (1, 1)$, so the finite determinant character of this basis change has value one on S. Using these rational and finite bases, apply the preceding descent at the successive $2k + 1$ planes from $(H^{2k} \perp Y, J^- \perp Y)$ to $(J, 0)$. After removal $1 \leq i \leq 2k + 1$ the ranks are $6k + 2 - 2i$ and $4k + 2 - 2i$. Thus the positive-model continuation uses $s_i = e_i + 1 = k + 2 - i$, with $e_i = k, k - 1, \ldots, -k$. The smallest argument is $1 - k > -k$. For $s_i \leq 0$ the two gamma arguments satisfy $s_i + \mu_\tau \geq 1$ and $1 - s_i + \mu_\tau \geq 1 + \mu_\tau$. At the last removal the positive-model exponent is $1 - k$ and the dual exponent is k. The conductor of ν^{bc} has exponent one at both members of each pair in

the chosen support and is trivial elsewhere, so its norm is Q_S^2, independently of the depths of the residue tests. Taking absolute values in the completed functional equation gives

$$Q_S^{1/2-k}|L^S(1-k,\nu)| = |\operatorname{disc}(E)|^{k-1/2}\prod_{\tau|\infty}\frac{\Gamma_{\mathbb{C}}(k+\mu_\tau)}{\Gamma_{\mathbb{C}}(1-k+\mu_\tau)}Q_S^{k-1/2}|L^S(k,\nu)|.$$

Here the dual character has the conjugate finite L-value at the real argument k. Both gamma factors are regular, including at $k=1$, where the dual value is taken at one. In the final test take the real scalar vector 1, source point $g=1$, and fixed covector matrix $p=I_{2k}$ at the selected places. The moving matrix has zero rows, so the stacked determinant is $\det p = 1$. Every finite factor equals one and the rational theta sum on the zero Hom space evaluates its scalar at the sole point. The period therefore has absolute value $v_0 = \operatorname{vol}([U(0)]) > 0$, also after transport by a unit phase. This contradiction proves the required nonvanishing of some $I_{Y,\nu}(z)$.

If the family with the fixed infinity type is nonempty, choose one reference character ν_* in it, at any admissible modulus. Fix a single real vector z on (Y,Y) with $R_{Y,\nu_*}(z) > 0$, as just proved. The standard exchange constructed by (1)–(4), allowing its scalar to vanish, applies at $(J,J,0)$ and $(J,0,J)$, with defects k and $-k$, respectively. In the reverse exchange the upper target is $J^- \perp H \perp J$. Use the rational and finite hyperbolic bases above. The $2k+1$ removals end at the zero target also at dyadic places. The split basis choice above applies to the same residue-character test in the reverse comparison. The last step has $e=-k$ and uses $L_\nu^\sharp(k)$ in (3). Use the real image under the first map as the argument of the second, before varying ν. In the forward exchange $b=0$ and $N=2k$. The empty Gram matrix has determinant one. The two frame shifts and the central Fourier group have dimension zero, so the sole Fourier coefficient is the period itself and the real frame is $B \in \mathrm{GL}_0$. The quotient by the full block stabilizer is a point of mass v_0^{-1}, cancelling the inner $U(0)$ mass in the open-orbit disintegration. For $a=0$ the top Fourier index has size $b=N$, and the target first has rank zero at the final removal. Write

$$D(X) = \mathbf{1}_{\mathrm{GL}_N(O_S)}(X)\chi^{-1}(\det X)$$

on integral square matrices, extending the character by zero on nonunits. The forward law $\Phi(k^{-1}) = \chi(\det k)\Psi(0)$ gives $(\Phi,\Psi) = (D,1)$ when $\Psi(0)=1$. In the reverse direction take $p=I_N$. The law $\Phi(0) = \chi(\det k)\Psi(k)$ gives $(\Phi,\Psi) = (1,D)$ when $\Phi(0)=1$. In each case every integral matched pair at the prescribed depth is a scalar multiple, by this law and support on the primitive locus. Both tests have the original vacuum outside the chosen pairs. The adjoint change of variables below uses an isometry of the original quotients and multiplication by a unit scalar. It applies to the preliminary period identity at either sign of the defect, including when the scalar of that identity is zero. With $(\alpha,\alpha') = (1,\gamma)$, write $K_\nu = K_{J,J}^{\gamma,\gamma}$ for the kernel on the fixed real image and the square determinant test D. In the forward exchange, the source character gives

$$K_{J,J}^{\gamma,1}(g,h) = \gamma(\det g)K_\nu(g,h).$$

The zero-source integral is $\gamma(\det g)v_0$. Cancelling this character uses no nonvanishing assertion about c_+. In the reverse exchange the small kernel is $K_\nu(h,g)$; the rank-zero target kernel with characters $(\gamma,1)$ is the fixed scalar coordinate ζ_0. Let $B,C>0$ denote the final normalization factors of (4) in these two exchanges. With the positive zero-group quotient volume v_0, the

identities read

$$\int_{[J]} K_\nu(g,h)\,dh = c_+ v_0, \qquad B|c_+|^2 = M_\nu(k) R_{Y,\nu}(z),$$
$$c_- \int_{[J]} K_\nu(h,g)\,dh = v_0 \zeta_0, \qquad C|c_-|^2 = M_\nu(-k) R_{Y,\nu}(z).$$

The product $[J]^2$ is compact because J is definite. On compact sets of representatives the finite vectors range over a finite set and the real Schwartz seminorms are uniformly bounded. The theta kernel is therefore bounded and continuous on the product. Set $v_J = \mathrm{vol}([J])$. Integrating the first identity in g, then using the second identity with the order of integration reversed, gives

$$c_- c_+ v_0 v_J = c_- \int_{[J]^2} K_\nu(g,h)\,dg\,dh = v_0 \zeta_0 v_J.$$

Thus $c_+ c_- = \zeta_0$. The scalar ζ_0 is the value of the real vector produced by the reverse exchange on the zero Hom space; the point-quotient measure has been accounted for by v_0. Multiplication, using $M_\nu(k) M_\nu(-k) = 1$, gives

$$R_{Y,\nu}(z)^2 = BC|\zeta_0|^2.$$

At ν_* the right side is strictly positive. Since $R_{Y,\nu}(z) = |I_{Y,\nu}(z)|^2 / M_\nu(k+1) \geq 0$ for every admissible character, the equality of squares gives $R_{Y,\nu}(z) = R_{Y,\nu_*}(z) > 0$ for all moduli and characters in the family.

An equal-rank step from rank $2j$ to rank $2j+2$ uses (2) on the standard equal-rank determinant datum. The larger period is constant, hence equals its normalized constant term. For every given lower vector a real isotropic-line bump supplies the same upper cone section before ν, normalized by its positive real integral at $s = j+1$. Equations (3)–(4) say that the ratio at the upper vector is a fixed positive multiple of the lower ratio.

Iterate through $q(W)$ planes. Identify the resulting space with W_0 by the rational Hermitian isometry and changes of adelic basis in the model. The rational group, adelic compact normalization and quotient Haar are transported, and the constant period changes at most by a phase. The rational isometry follows from Landherr's theorem [43, pp. 246–247], since the signatures and the finite local forms agree. This gives a single z_0 for which (5) is a fixed positive R_0'. If the admissible character family is empty the assertion has no instances and a vacuum can be chosen.

Return to the original small quotient. The standard small comparison uses (W, U^-). Write X_U, X_W for the chosen adelic bases. Fix a finite integral unit c_f with $c_f \bar{c}_f = -1$. Such units exist at every finite place since E/F_0 is split or unramified there and the norm on units is onto. For the opposite form $-J_U$ take

$$X_{U^-} = dX_U, \qquad d_\infty = 1, \qquad d_f = \bar{c}_f \bar{\alpha}_0 / \alpha_0.$$

The finite Gram matrix of this basis is the identity because $d_f \bar{d}_f = -1$. For a rational map $A : U \to W$ put

$$T = X_W^{-1} A X_U, \qquad T' = X_{U^-}^{-1} (J_U^{-1} A^* J_W) X_W.$$

At infinity T' is the signed adjoint of T. At finite places the two original Gram matrices in the chosen bases are identities, so $T_f' = d_f^{-1} T_f^*$. In the scaled finite coordinates this is

$$T_f / \alpha_0 = c_f (T_f' / \alpha_0)^*.$$

The rational adjoint is bijective, with inverse $B \mapsto J_W^{-1} B^* J_U$. The opposite Hermitian sign cancels the sign from adjunction in the Heisenberg form. The adjoint preserves the finite vacuum lattice and the real Gaussian, so the two normalized theta distributions differ by a phase. Indeed, in a rational Schrödinger polarization, rational modulation and translation characterize the distribution as a multiple of the sum over the rational points. The normalized mean square on vacuum translates fixes the absolute value of that multiple.

For characters (a_1, a_2) on (U, W), the flipped characters on (W, U^-) are

$$(a_2^{-1} \upsilon^N,\ a_1^{-1} \upsilon^b).$$

With the ranks interchanged, applying this character change a second time gives

$$((a_1^{-1} \upsilon^b)^{-1} \upsilon^b,\, (a_2^{-1} \upsilon^N)^{-1} \upsilon^N) = (a_1, a_2).$$

The real complementary-minor identity and the finite elementary divisors give the multiplier ratio

$$\frac{\upsilon(\det h)^N}{\upsilon(\det g)^b}, \qquad (h, g) \in U(U) \times U(W).$$

The absolute determinant powers cancel because the unitary determinants have norm one. For any finite scaled matrix Z of size b-by-N, let $Z_S^\dagger = (\overline{Z_{\bar{w}}}^{\mathsf{t}})_{w \in S}$, where conjugation identifies $E_{\bar{w}}$ with E_w. The right covariance of the matched column datum gives the pointwise identity

$$\Psi(Z_S^\dagger) = \chi((c_f)_S)^b \, \Psi((c_f)_S Z_S^\dagger).$$

It holds on the full matrix space since the unit scalar preserves the support. Thus the flipped test reads the transpose at the conjugate places and its phase is independent of the matched functions.

The forms $J_U, -J_U$ have the same rational unitary matrices. The basis change $X_{U^-} = dX_U$ is central, so conjugating those matrices into the two adelic models gives the same arithmetic subgroup. The identity on adelic unitary matrices preserves the chosen compact group and its Haar normalization, hence also the quotient measure. Thus the original small integral is related by a unit scalar and no volume ratio. More precisely, starting with (γ, γ) on (U, W), the standard small period obeys

$$\gamma(\det g) \Theta_{\mathrm{small}}(g) = \varepsilon_{\mathrm{ad}} \int_{[U]} K_{U,W}^{\gamma,\gamma}(\Psi, y; h, g)\, dh, \qquad |\varepsilon_{\mathrm{ad}}| = 1.$$

On the standard large side, multiplication by $\gamma(\det g)$ changes the characters from (γ, γ) to $(\gamma, 1)$.

Absorb $\varepsilon_{\mathrm{ad}}$ into the scalar in (4). Since (5) is $R_0' > 0$, the resulting scalar c_ν is nonzero, with

$$|c_\nu|^2 = (R_0' / D_{\mathrm{final}}) M_\nu(t).$$

Finally, the product of the two rank-specific space characters equals $(\nu^{\mathrm{bc}})^{-1}$ precisely when $\alpha\alpha' = \gamma$. This follows by multiplying the two powers of the splitting character and using the surjectivity of $x \mapsto x/\bar{x}$ on the norm-one ideles, by local Hilbert 90, choosing units at the unramified finite places.

Replacing $(1, \gamma)$ by (α, α') on the two middle groups multiplies both kernels by $\alpha(\det g)^{-1}$.

Proposition 36.1 (Normalized period exchange)**.** *Suppose $t \geq 0$. For the fixed spaces W, V, U, Haar measures, infinity weight μ and real Schwartz vector y, one can choose a real Schwartz vector x for (W, V) and $R_0'/D_{\mathrm{final}} > 0$ before the split tame modulus and character ν. For every such modulus $\mathfrak{m}$ with nonempty chosen split support S, and every admissible ν whose tame character on $k_w^\times$ is nontrivial for every $w \in S$, there is a scalar $c_\nu \neq 0$ such that, for all α, α' with $\alpha\alpha' = \gamma$, all matched arrays (Φ, Ψ) over $\mathcal{O}_E/\mathfrak{m}$ for its induced residue character, and every adelic g, both periods are absolutely convergent and*

$$\int_{[V]} K_{W,V}^{\gamma,\alpha}(\Phi, x; g, h)\, dh = c_\nu \int_{[U]} K_{U,W}^{\alpha',\gamma}(\Psi, y; h, g)\, dh,$$
$$|c_\nu|^2 = (R_0'/D_{\mathrm{final}}) M_\nu(t).$$

For the two periods in the proposition, the chosen tests have full row rank on Φ's support and full column rank on Ψ's support at every $w \in S$. Apply (100) to the covector test and to the column test in the conjugate transpose model. At rank zero the integral is over the one-point quotient. The stronger mixed-slice estimate is used for the top central Fourier coefficients, with their adapted frames. For the two one-plane constant terms the ordinary estimate precedes the compact radical average, and on the large side the sum over rational lines is unfolded with the grouped estimate in Section 36.1.4.

36.2 The finite extension of the period identity

In this subsection, after the analytic exchange of Proposition 36.1, finite Fourier inversion gives an identity on the entire matrix space. We determine the resulting next datum, including the inverse determinant character and its scalar.

Let $R = \mathcal{O}_E/(\pi_M Q(\xi))$, a product of finite fields at the chosen split members, and let χ be the character of $R^\times$ of the quotient character. Extend multiplicative characters by zero on nonunits. A paired residue datum on a-by-$(b+a)$ matrices X and $(b+a)$-by-b matrices Y is supported, respectively, on the last a covectors of k^{-1} and the first b columns of k for $k \in \mathrm{GL}_{a+b}(R)$, and satisfies

$$\Phi(X) = \chi(\det k)\Psi(Y)$$

on such pairs. The order of the covectors in this definition is b, a. The matched point data from the measured square is the following. The scalar tested composites $P(X), P'(Y)$ on the prescribed four Hermitian spaces satisfy

$$P(gX) = \chi(\det g)P(X), \qquad P'(Yh) = \chi(\det h)P'(Y),$$
$$\sum_Y \Psi(Y)P'(Y) = c\sum_X \Phi(X)P(X) \qquad \text{for paired data } (\Phi, \Psi).$$

Here the test functional in the top covector is arbitrary; the scalar c is independent of it and of the residue arrays. The first line is the finite determinant covariance. It holds on all residue arrays: the fixed-point tests define individual lifts against the rapid source forms. The coefficient projector argument of Section 34.3.2 applies to each fixed finite datum by the Gaussian theta bound and gives square integrability of the resulting lifts on the middle group. Changing the middle basis in that pairing preserves its quotient measure and gives the displayed determinant multiplier. The second line uses the measured identity only on the full-rank supports of paired data.

For the triangle at the original level take $i, n \geq 0$ and $i + n + 2 \leq m$. In column n the rank is $r + nt$. The two kept spaces at $(i, n + 1)$ and $(i + 1, n + 1)$ have rank $r + (n + 1)t$; the two integrated spaces at $(i + 1, n)$ and $(i, n + 2)$ differ by rank $2t$. The negative counts on the two diagonal pairs agree. For a stem entry with excess parameter a_*, its sign in covector o and column j is

$$(-1)^{o+a_*+\min(j,(a_*-o)_+)}.$$

If $a_* \leq i + n + 1$, the two column changes leave both signs unchanged.

If $a_* \geq i + n + 2$, both signs change and they were opposite. All added entries are positive. The common negative count is at most $2r < r + (n + 1)t$. Moreover the positive weight of the inverse quotient is $(n + 1)\Delta$, greater than $r + (n + 1)t$. These are the rank, sign and weight inequalities for the exchange. In the reversed square interchange the two roles of the diagonal pairs. The two integrated ranks now agree, so the rank defect is zero.

Write $u = v|_{U(1)}$ and $\mu_j = \mu_0(u^{-\Delta}\chi_M)^j$. The horizontal and left-pair determinant twists at ranks p, q in the square respectively have factors

$$\mu_{i+n+2}\xi^{i+n+2}u^{\lfloor p/2 \rfloor}, \qquad \mu_{i+1}^{-1}\xi^{-(i+1)}u^{\lceil q/2 \rceil}.$$

The second is the inverse of the first-column character at rank q, multiplied by u^q. Since $p + q = 2(r + (n + 1)t)$, its product with the first is the rank twist of $(\mu_{i+n+2}/\mu_{i+1})\xi^{n+1}$. In the split Weil characters the two rank corrections therefore cancel, leaving the base change of that quotient. The paired finite determinant character is its inverse. The two lift indices satisfy $1 \leq i+1 < i+n+2 \leq m$, and their difference satisfies $1 \leq n+1 \leq m-1$. The order constraint for powers up to m gives nontriviality at every chosen old or new place. The finite character used above is therefore the character on $R^\times$ of the square.

36.2.1 Fourier transform in the complementary blocks

In this subsection, we calculate the finite transform for the exchange identity in Proposition 36.1.

Write $X = (X_1, X_2)$ and $Y = \binom{Y_1}{Y_2}$ with square blocks X_2, Y_1. For a fixed $w \in M_{a,b}(R)$ use

$$\Phi(X) = \chi^{-1}(\det X_2)\,\mathbf{1}_{\{X_2^{-1}X_1=w\}}, \qquad \Psi(Y) = \chi^{-1}(\det Y_1)\,\mathbf{1}_{\{-Y_2Y_1^{-1}=w\}},$$

defined to be zero if their square block is not invertible. These functions are paired. For supported pairs $XY = 0$. For a given k, Y_1 is invertible if and only if X_2 is invertible, by complementary inverse blocks. When they are invertible, block elimination gives

$$X_2^{-1}X_1 = -Y_2Y_1^{-1}, \qquad \det k = \det Y_1 / \det X_2.$$

Conversely, a supported X is completed to the last covectors of an inverse by adjoining a complementary covector basis, since X_2 is invertible. A supported Y is completed to a column basis in the same way. The verification works over R factor by factor.

Using the covariance on each supported basis, the two weighted sums reduce to

$$|\mathrm{GL}_b(R)|P'\begin{pmatrix} I_b \\ -w \end{pmatrix} = c\,|\mathrm{GL}_a(R)|P(w, I_a). \tag{103}$$

For example $Y = \binom{I_b}{-w}Y_1$; its character cancels $\chi^{-1}(\det Y_1)$ and every Y_1 contributes the same scalar. The left block is similar.

Now take the successive ranks $N, N+t, N+2t$ at the square. In the transposed-lift square Y has N columns, X has $N+2t$ covectors, and the fixed outer blocks adjoined to each datum are the original inclusions. The abstract sizes in the next calculation are $n = N+t$, $a = N$, and $b = t$. Write p for a datum of size n-by-a and define

$$p^{+}\begin{pmatrix} \eta & s \\ 0 & I_b \end{pmatrix} = \frac{p(-\eta)}{|R|^{nb}}, \qquad s \in M_{n,b}(R), \tag{104}$$

with value zero on all other matrices. In the order n, a, b, the two full arguments are

$$Y = \begin{pmatrix} -\eta \\ I_a \\ 0 \end{pmatrix}, \qquad X = \begin{pmatrix} I_n & \eta & s \\ 0 & 0 & I_b \end{pmatrix}.$$

To obtain them in (103), write the basis in the order $a, (n, b)$ and use $w = \binom{\eta}{0}$. Reindex the ambient module from (a, n, b) to (n, a, b) in both the domain and the range of the completion k. Its matrix is conjugated by the block permutation, so its determinant and the matching character are unchanged.

Multiplying the covector argument by $\left(\begin{smallmatrix} I_n & s \\ 0 & I_b \end{smallmatrix}\right)$ introduces the displayed s. This shear has determinant one, so the covariance preserves the tested value. Average over s and then sum with weight $p(-\eta)$ over η. Write $\iota_d = \binom{I_d}{0_{b\times d}}$ for $d = a, n$. With $v = -\eta$ this gives

$$|\mathrm{GL}_a(R)| \sum_{v \in M_{n,a}(R)} p(v) P'\begin{pmatrix} v \\ \iota_a \end{pmatrix} = c\,|\mathrm{GL}_{n+b}(R)| \sum_{Z \in M_{n+b,a+b}(R)} p^{+}(Z) P\big((\iota_n, Z)\big).$$

The sum over s has $|R|^{nb}$ terms, giving precisely the denominator in (104). In column notation the weights are w_n, w_{n+2}.

For this forward square write $k_{i,n}$ for the archimedean projection scalar of Section 34.3.2, and $c_{i,n}(\xi) = c_\xi$ for its nonzero measured partner scalar. The former is independent of ξ. The paired-data scalar is $c = k_{i,n}/c_{i,n}(\xi)$, as in (GC1). The two intermediate ranks differ by $2t$. In Proposition 36.1 the real constant is fixed, the old primes occur in both moduli, and each new place contributes $q_w^{t^2}$ to $M_\nu(t)$. Thus

$$\frac{|c_\xi|^2}{|c_1|^2} = q_\xi^{t^2} Q_{i+1,i+n+2}(\xi)^2,$$

where $Q_{i+1,i+n+2}$ is the product over the critical points $1, \dots, t$ of the absolute ratios for the quotient character at ξ^{n+1}. When the original source $W \neq 0$, (GC2) gives $k_{i,n} \neq 0$. The reciprocal conjugate $\beta'_{i,n}(\xi) = \overline{c_{i,n}(\xi)/k_{i,n}}$ then has this same squared-modulus ratio.

Fourier extension for the reversed square. Write $\mathcal{F}p(Z) = \sum_X p(X)\psi(\mathrm{tr}(ZX))$ for the unnormalized matrix Fourier transform with the original primitive additive character of R. For block $Z = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ compatible with (104), character orthogonality gives

$$\mathcal{F}(p^{+})(Z) = \mathbf{1}_{\{C=0\}} \mathcal{F}p(-A)\,\psi(\mathrm{tr}\,D).$$

Indeed the trace in the Fourier phase is $\mathrm{tr}(A\eta) + \mathrm{tr}(Cs) + \mathrm{tr}(D)$. The mean over s is zero unless $C = 0$, by primitivity and the matrix trace pairing. The change of variable $\eta \mapsto -\eta$ accounts for the sign.

36.2.2 The finite determinant Gauss sum

In this subsection, we evaluate the determinant factor in Section 36.2.1.

We use the following exact determinant Gauss sum over a residue field k of size q. If a function f on $M_b(k)$ transforms by the nontrivial character $\chi(\det g)$ under left multiplication, it is zero on singular matrices. A singular matrix has a left stabilizer with arbitrary determinant, obtained by scaling a line complementary to its image. Thus $f(D) = \chi(\det D) f(I_b)$ on all matrices with the zero extension convention.

The additive Fourier transform of $\chi \circ \det$ transforms by the inverse determinant character on the opposite side. It is also zero on singular matrices, using their right stabilizer. Consequently it equals $\mathcal{G}\, \chi^{-1} \circ \det$, where $\mathcal{G}$ is its value at I_b. For the convention $\mathcal{F}f(Y) = \sum_D f(D)\psi(\mathrm{tr}(YD))$, average the defining sum for $\mathcal{G}$ over left multiplication by upper unitriangular matrices u. This preserves $\chi(\det D)$. The trace is $\mathrm{tr}(uD) = \mathrm{tr}\, D + \sum_{i<j} u_{ij} D_{ji}$. The mean vanishes unless every D_{ji} with $i < j$ is zero. For upper triangular D the strictly upper entries give $q^{b(b-1)/2}$ choices and the nonzero diagonal entries sum independently. Thus for $b \geq 1$

$$\mathcal{G} = q^{b(b-1)/2} \left(\sum_{s \in k^\times} \chi(s)\psi(s) \right)^b ,$$

the full-rank case of [45, arXiv:1105.4513v1, (2.12)–(2.13) and Theorem 2.1]. For $b = 0$ the matrix sum has its single empty term, of value one. Parseval for the unnormalized Fourier transform on the b^2-dimensional matrix space gives

$$|\mathcal{G}|^2 = q^{b^2}.$$

The original modulus is squarefree, and the quotient character is nontrivial in each field factor, because its old and new tame powers have full support. The additive character is primitive in each factor. Taking products proves the same formula with $|R|$ in place of q.

In the reversed square the abstract block sizes are $n = N$, $a = N + t$, $b = t$. Thus the two linear-group orders in (103) are equal after the shear extension, since $a = n + b$. Put $p_* = \mathcal{F}(\bar{p})$. In the sum weighted by $\mathcal{F}(\overline{p^+})$, the last formula leaves only matrices with lower-left block zero. At fixed η, s, the tested value as a function of the lower-right block D transforms by $\chi(\det g)$ under $D \mapsto gD$, by applying $\mathrm{diag}(I_n, g)$ to the whole covector argument.

The sum over D, with its Fourier phase, is therefore $\mathcal{G}$ times the value at $D = I_b$. The extra factor $|R|^{nb}$ converts the unrestricted sum over s into the mean in p_*^+. Hence

$$\sum_X \mathcal{F}(\overline{p^+})(X) P(\iota, X) = \mathcal{G}|R|^{nb} \sum_X p_*^+(X) P(\iota, X),$$

where ι is the fixed inclusion adjoined to the covector argument. The squared modulus of the coefficient here is $|R|^{b(b+2n)}$. Combining it with the paired-data law for p_* gives the reversed commutation coefficient with the additional inverse factor $(\mathcal{G}|R|^{nb})^{-1}$. The remaining paired-data scalar for that square has twist-independent modulus because its two middle ranks agree, and the critical product is empty. Thus the reversed commutation scalar has modulus a fixed constant times a fixed half-integral power of q_ξ.

Matrix reversal in the original two bases. Let S, T have ranks u, v, and let $R = \mathcal{O}_E/\mathfrak{m}$ have the original selected split support. Denote by $K_{S,T}^{a,b}(D, P; g, h)$ the theta kernel with residue datum D and real Fock polynomial P. Write $\psi_0(\bar{x}) = \psi_f(j_{\mathrm{W}} \alpha_f \bar{\alpha}_f x / t_{\mathfrak{m}})$, where $t_{\mathfrak{m}}$ is the chosen

generator on this support. Let $s_{\mathfrak{m}} = \overline{t_{\mathfrak{m}}\alpha_f}/(t_{\mathfrak{m}}\alpha_f)$ at the finite places. With the unnormalized matrix Fourier transform

$$\widehat{\bar{D}}(Z) = \sum_{X \in \mathrm{Mat}_{v\times u}(R)} \overline{D(X)}\, \psi_0(\mathrm{tr}(ZX)),$$

the exact kernel identity is

$$\overline{K^{a,b}_{S,T}(D,P;g,h)} = \tau_{\mathfrak{m}} K^{b,a}_{T,S}(\widehat{\bar{D}}, \mathrm{flip}(\bar{P}); h, g s_{\mathfrak{m}}), \qquad |\tau_{\mathfrak{m}}|^2 = |R|^{-uv}. \tag{MR}$$

For source signs s_j and target signs t_i, the adjoint on the opposite source is $Y = -J_S X^* J_T$. On each coordinate $Y_{ji} = -s_j t_i \overline{X_{ij}}$. The signed Fock coordinate in this position is either the coordinate or its conjugate, with the same choice in the reversed pair. Thus conjugation followed by adjunction sends a polynomial to

$$(\mathrm{flip}(\bar{P}))(Z) = \bar{P}\big((-s_j t_i Z_{ji})_{i,j}\big),$$

where the bar on P conjugates its coefficients. For a minor on row and column sets I, J of size h, the substitution contributes $(-1)^h \prod_{i\in I} t_i \prod_{j\in J} s_j$ times the minor on J, I in the transposed order.

First conjugate with the source replaced by S^-. Negating the Hermitian form negates the symplectic form on $\mathrm{Hom}(S,T)$, so conjugation intertwines the Heisenberg actions with the two opposite central characters. In the chosen coordinates the real sign of every variable changes, so conjugation replaces its holomorphic polynomial coefficients by $\bar{P}$ on the opposite signs. At the finite places the coordinate change is multiplication by u_0^{-1}, for the fixed unit $\bar{u}_0 u_0 = -1$.

Its preservation of the integral boxes changes $D(X)$ to $\overline{D(u_0^{-1}X)}$. The two determinant weights become $a^{-1}\upsilon^u, b^{-1}\upsilon^v$. These powers are the change from each split Weil normalization to its complex conjugate: the underlying splitting character of $E^\times\backslash\mathbb{A}_E^\times$ is conjugated to its inverse, while the fixed standard splitting on a rank h space contributes υ^h. The rational matrices coincide under passage to S^- when the coordinate change is undone. Equivariance for rational Heisenberg translations identifies the transported functional with a scalar multiple of the normalized theta functional. To determine the modulus, write J for the unitary or antiunitary transport. It carries the rational translation lattice onto the new one and preserves probability Haar on the quotient. In the self-dual Schrödinger measures,

$$\int |\theta'(\rho'(x) J v_0)|^2\, dx = \|J v_0\|^2 = 1.$$

Indeed, integration in the modulation variable gives the sum of squared Schrödinger values by Parseval and integration in the displacement variable unfolds this to their squared norm. The same identity holds for the original unit vacuum. Consequently the scalar has modulus one. The adjoint transport below has the same property before the finite dilation.

Next apply the Hermitian adjoint on the opposite-source pair, which returns to the original S as the second space. In rational coordinates it is $X \mapsto J_{S^-}^{-1} X^* J_T$, a bijection of the rational matrix sets. The signed real transpose and the original finite basis scalar are the entrywise transports in the line calculation. On the original source group the basis scalar moves the finite argument by $\bar{\alpha}_f/\alpha_f$. Swapping again inverts the determinant weights and inserts the same respective rank powers, so the two weights are now b, a.

To verify the finite factor, use a point indicator at X_0 after the opposite-source transport. Its pre-dilation vacuum displacement has box $X - X_0/t_{\mathfrak{m}}$ integral at the selected places. The split source dilation by $t_{\mathfrak{m}}^{-1} I_u$ gives the box $(X - X_0)/t_{\mathfrak{m}}$ integral. In rank u the determinant norm of the dilation is $|R|^u$. The split Weil factor is its $v/2$-power, giving squared modulus $|R|^{uv}$ for the multiplier from the displaced vacuum to the point datum.

The Hermitian transpose sends the displacement to the conjugate places, where the Heisenberg relation turns it into multiplication on the original integral box by $\psi_0(\mathrm{tr}(Z(-\bar{u}_0)X_0))$. This is the entrywise trace pairing, so the computation for one column extends to the full matrix without a second measure factor on the conjugate split side. The normalized theta transport under adjoint is a unit phase. The source dilation moves to the group argument and combines with the basis scalar, giving precisely $s_{\mathfrak{m}}$. Solving for the original kernel inverts the displacement multiplier, giving the stated value of $|\tau_{\mathfrak{m}}|^2$. Locally the annihilator of $t_w \mathcal{O}_w$ for $\psi_{E,w}(\delta_w xy)$ is $t_w^{-1}\mathcal{O}_w$. It follows that the pairing on $\mathcal{O}_w/t_w\mathcal{O}_w$ given by $\psi_{E,w}(\delta_w xy/t_w)$ is perfect. For rectangular matrices, testing one matrix unit at a time gives the same conclusion for the trace pairing on the two opposite shapes. At an unselected place the self-dual integral box has volume one and transforms to itself.

Finally sum the point identities with coefficients $\overline{D(u_0^{-1}X_0)}$. The substitution $X_0 = u_0 X$ cancels the shift in the additive character because $-\bar{u}_0 u_0 = 1$. This gives exactly $\widehat{\widehat{D}}$ under the unchanged primitive character ψ_0, proving (MR). On applying the conjugated transform twice, orthogonality for the primitive trace pairing gives

$$\sum_Z \overline{\widehat{\widehat{D}}(Z)}\psi_0(\mathrm{tr}(XZ)) = \sum_{X'} D(X') \sum_Z \psi_0(\mathrm{tr}((X - X')Z)) = |R|^{uv} D(X).$$

The two real signed transpositions also compose to the identity, since $(-s_j t_i)^2 = 1$ at each entry. Put $z = s_{\mathfrak{m}}$ and write $\tau_{S,T}, \tau_{T,S}$ for the scalars in the two orders, with the corresponding characters. Applying (MR) twice gives the original kernel as $\overline{\tau_{S,T}}\tau_{T,S}|R|^{uv}$ times that kernel at (gz, hz). The simultaneous action on $\mathrm{Hom}(S,T)$ is the identity, and the determinant characters contribute

$$K_{S,T}^{a,b}(D, P; gz, hz) = a(z^v)b(z^u)^{-1} K_{S,T}^{a,b}(D, P; g, h).$$

Here $z\bar{z} = 1$, so this factor has absolute value one. The same right translations preserve quotient Haar and every principal level by centrality, including at the two halves of a split place where z need not be integral. Write $J(P) = \mathrm{flip}(\bar{P})$. On edge $j = 0, 1$ let P_j, P_j' be the distinguished original and reversed harmonics, and p_j, p_j' the respective products of the commuting source and target line projectors. In each orientation Section 34.3.2 identifies the source isotypic component with $V_S \widehat{\otimes} V_T$. The two projectors there select the tensor product of the two highest lines. Thus $p_j P_j$ and $p_j' P_j'$ span their respective lines. In a pairing with the two prescribed form types one may insert both projectors on the oscillator datum. Indeed moving each coefficient integral to its form fixes that form, with the inverse coefficient in the bilinear source pairing. The weighted coefficient moments and the two rapid tests justify both interchanges as in Section 34.3.1.

Define the real-place scalars by

$$p_0' J(P_0) = \overline{k_0}\, p_0' P_0', \qquad p_1 J^{-1}(P_1') = \overline{k_1}\, p_1 P_1.$$

On the first edge apply (MR) to P_0 and compress $J(P_0)$ in the reversed kernel, which is in the linear leg. The tested factor is $\tau_{\mathfrak{m}}\bar{k}_0$. On the second edge apply (MR) to $J^{-1}(P_1')$, so that the reversed kernel has polynomial P_1'. Solve for that kernel using $\tau_{\mathfrak{m}} \neq 0$. The original kernel

is conjugated, so its compression gives the tested factor $\tau_{\mathfrak{m}}^{-1}k_1$, with the ranks of the second edge. Both identities hold also for a zero compression. The two scalars k_j depend only on the real data. In (GC3) the two transforms use $R = R_0(\xi)$. With s_0, s_1 these tested factors, γ the reversed-square factor, and η the unitary translation phase in that equation, their product satisfies

$$|s_0\gamma\eta s_1|^2 = C_{i,n}|R_0(\xi)|^{t^2}, \qquad C_{i,n} \geq 0,$$

where $C_{i,n}$ is independent of ξ.

37 The leading coefficient and the four-block test

In this section, we estimate the leading Taylor coefficient on the prescribed Weyl lattice and construct the four-block test. The reference norm determines the normalization of the critical ratios in Section 38.

37.1 A lower bound for a coefficient in twelve coordinates

In this subsection, the four-block test of Section 37.2 requires the leading Taylor coefficient from its first twelve coordinates. We compare this coefficient with its Weyl norm in the prescribed lattice and normalization.

Write $V_0 = s_{\mathrm{vac}}$, $\Lambda_{12} = \mathfrak{a}_0^{12}$, and $\delta = \bar{\pi}_M$. Use the original 12-coordinate Weil vacuum v. The Taylor map $\mathcal{T}$ includes the scale $\pi_M s_{\mathrm{jet}}$. For nonnegative integers λ_τ satisfying $\kappa_\tau \leq \lambda_\tau \leq \kappa_\tau + x^{42}$, put

$$n_{\#} = \lceil 6b/k \rceil, \quad B_\tau = \prod_{j=1}^{n_{\#}} ((\lambda_\tau - n_{\#})_+ + j), \qquad P_p = \sum_\tau S_p(\kappa_\tau, \tau).$$

We construct one nonzero t_0, depending on the datum and the λ_τ, with

$$\log|\sigma t_0| \leq fmb/k \qquad \text{for every } \sigma.$$

At each embedding above a prime p, we will find an exponent α satisfying

$$\sum_c \alpha_{\tau,c} = \lambda_\tau$$

and a residue cell x_0 mod π_M with indicator I_{x_0} such that

$$v_p\left(\frac{V_0^{12}}{t_0}\prod_\tau \lambda_\tau!\,[Z^\alpha]\mathcal{T}I_{x_0}\right) \leq P_p. \tag{105}$$

Lemma 37.1 (Field of definition of the coefficient test)**.** *For every residue cell x_0 and multi-index α, the coefficient $c_{\alpha,x_0} = V_0^{12}[Z^\alpha]\mathcal{T}I_{x_0}$ is algebraic. For any embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbf{C}_p$, its image is the coefficient obtained from the corresponding CM sections using the base-changed invariant differential, torsion translations and finite character sums.*

Proof. Work over a number field defining the CM elliptic curve, its invariant differential, its CM action and the distinguished point $t = -t \neq 0$. Since $t = -t$, the bundle $\mathcal{O}([t])$ is symmetric. Its semicharacter has order at most two, and its Hermitian form satisfies $H(az, w) = H(z, \bar{a}w)$

for $a \in \mathcal{O}_K$. The section has value one at the origin. Consequently its reduced theta series and its normalized torsion translates have algebraic coefficients in the logarithm coordinate z of this differential [8, Prop. 2.1 and Thm. 2.9 in the authors' version]. More precisely, let F be the field of definition above, enlarged to contain K. Since $0 \notin [t]$, the local equation at zero can be taken to be 1; the normalized value there is 1. If $N_u u \in \Lambda_C$, put $n = 2N_u$ and write $\pi_C : \mathbb{C} \to C(\mathbb{C})$ for uniformization. Retain the division point $Q_u = \pi_C(u/(2n)) \in C[4N_u^2]$ and its double $P_u = 2Q_u$. The symmetric bundle $\mathcal{O}([t])$ and these points determine the rational translation function $R_u = f^M_{[t],P_u} \in F(C[4N_u^2])(C)$, with

$$R_u(\pi_C(v)) = \frac{(U_u^M \vartheta_C)(nv)}{\vartheta_C(nv)}, \qquad (U_u^M \vartheta_C)(z) = e^{-\pi(z\bar{u}+|u|^2/2)/A_C}\vartheta_C(z+u).$$

The canonical rational function and the displayed analytic formula are those of [8, Cor. 2.4 and Lemma 2.7 in the authors' version]. The trivialization at Q_u and its inverse cancel their scalar factors in that construction, and its pullback description [8, Prop. 2.8 in the authors' version] identifies the function from the divisor and division points on C^ι with ιR_u. The period law for the semicharacter χ_C gives

$$U^M_{u+\lambda}\vartheta_C = \chi_C(\lambda)\exp\left(\frac{\pi(u\bar{\lambda}-\lambda\bar{u})}{2A_C}\right)U_u^M\vartheta_C, \qquad \lambda \in \Lambda_C.$$

The multiplier for a different representative of the torsion point is a root of unity, since $\mathrm{Im}(u\bar{\lambda})/A_C \in N_u^{-1}\mathbb{Z}$ and $\chi_C(\lambda) \in \{1,-1\}$.

Write y_{W} for the affine coordinate at $t = (0,0)$ on the Weierstrass model with group origin at infinity. Its shifted pullback $y(P) = y_{\mathrm{W}}(P+t)$ satisfies $y(0) = 0$ and $y'(0) = 1$ in the invariant logarithm coordinate by Section 20.1. Take $g_u = y$ if $\pi_C(u) = t$, when the translated divisor has a simple zero at the origin, and take $g_u = 1$ otherwise. Theorem 2.9 of the same reference gives $g_u^{-1}U_u^M\vartheta_C \in F(C[4N_u^2])[[z]]^\times$, where g_u is pulled back by π_C. Multiplication by g_u recovers the translated series, including its zero constant term when $\pi_C(u) = t$.

The forms in Lemma 7.6 satisfy

$$A_{k,i}(\pi_M s_{\mathrm{jet}} Z) = \frac{\pi_M p_F}{q_1}\sum_\tau \overline{\tau(\zeta_{q_1}^k)}Z_{\tau,i}, \qquad p_F = s_0/u \in \overline{\mathbb{Q}}^\times.$$

Here s_0 has a power in K, and u is obtained by taking a root of the product of algebraic isogeny multipliers and discriminants in the same lemma. The finite translates of its rank-one seed are products of the preceding torsion series in these forms, up to roots of unity. This holds at every finite translate by rational approximation modulo the open stabilizer. Tensoring (AV) twelve times therefore gives the asserted algebraicity for every translate of $V_0^{12}v$.

On the residue group D at the prescribed places the identity

$$\mathbf{1}_{\bar{x}=x_0} = |D|^{-1}\sum_{\chi\in\widehat{D}}\chi(-x_0)\chi(\bar{x})$$

expresses the cell indicator times the vacuum as a finite combination of its translates by the dual lattice. The perfect pairing is given in Section 37.1.2. All coefficients and substitutions in this expression lie in a number field. In the chart at t, let $x(Y) \in YF[[Y]]$ solve $x^3+x-a_1xY-Y^2 = 0$. It is unique since the derivative in x at zero is one. The invariant differential gives

$$z = \log_C(y) = \int_0^y \frac{dY}{1+3x(Y)^2-a_1Y} = y + O(y^2).$$

The inverse $y(z)$ reads the point $\pi_C(z)+t$. Accordingly, substitute $y(z/n)$ in $R_u\circ T_{-t}$, where $T_{-t}(P)=P-t$, to obtain $R_u(\pi_C(z/n))$. This and multiplication by $\vartheta_C(z)$ recover the translated series. In each degree the substitutions commute with ι, including when $p\mid n$, since they are over a characteristic-zero field. Coefficient extraction then gives $\iota(c_{\alpha,x_0})$. □

The one-variable tests. Use the integral parameter $y=z+O(z^2)$ and $R=\mathcal{O}_{\mathbf{C}_p}$. The formal group is integral: the translated identity is the section $(0,0)$ of the good Weierstrass model, and the group law transported by this section gives a series in $R[[y_1,y_2]]$. Differentiation preserves $R[[y]]$ since $y'=1+3x^2-a_1y$. Let (π,q) be (p,p) if p splits, (p,p^2) if it is inert, and $(\sqrt{-d},p)$ if $p=d$. The ordinary integral-section calculation gives $y(az)\in R[[y]]$ for $a\in\mathcal{O}_K$ on the formal group at its identity. For $a\neq 0$ let $i(a)$ be the order of the reduced parameter of $[a]$. The CM isogeny extends over the good model, with degree $a\bar a$ on the special fibre: its kernel is contained in the finite kernel of $[a\bar a]$, and the degree is the degree of the pullback of a degree-one line bundle, constant in this smooth proper family. Its local multiplicity at the identity is $i(a)$, so $i(a)\le a\bar a$. For split p, take a principal power $\mathfrak{p}^h=(a)$ of the chosen split prime. Then $a\bar a=p^h$, $\bar a$ is a unit at the chosen embedding, and $i(\bar a)=1$. Composition of reduced parameter series gives $i(p)^h=i(a)i(\bar a)\le p^h$. Thus $i(\pi)\le q$ in every case.

Average $a^{-1}y(az)$ over $a^{q-1}=1$. These Teichmüller values are limits of CM integers at the chosen embedding, so the result $T=y+O(y^2)$ is integral. Root-of-unity covariance gives

$$T(\pi z)=\pi T(z)+b_1T(z)^q+\sum_{j\ge 2}b_jT(z)^{1+j(q-1)},\qquad b_1\in R^\times,$$

where the reduced order bound gives the unit assertion. This also works at $p=2$, when $q-1=1$. For $\varpi^{q-1}=-\pi/b_1$ put $\rho=v_p(\varpi)=v_p(\pi)/(q-1)$. The coefficients of $Y=T(\varpi z)/\varpi$ are integral. Its equation is $Y(\pi z)/\pi=Y-Y^q+\pi E(Y)$ with $E\in X^2R[[X]]$, and the recursion divides in degree $n>1$ by the unit $1-\pi^{n-1}$. Thus $Y\equiv\sum_{j\ge0}z^{q^j}\pmod\pi$.

Put $c_n=\max_{H\in R[[y]]}|n![z^n]H|_p$. For $H=\sum_j h_jy^j$, only $0\le j\le n$ contribute. Since $|h_j|_p\le 1$, the ultrametric inequality gives $c_n=\max_{0\le j\le n}|n![z^n]y^j|_p$. A monomial attains this maximum. Integral differentiation gives $c_n\le 1$, and the integral group law gives $c_{m+n}\le c_mc_n$ by taking the z^mw^n coefficient of $H(z+w)$ and multiplying by $m!n!$. Since $T(z)=\varpi Y(z/\varpi)$ belongs to $R[[y]]$, $[z^n]T=\varpi^{1-n}[z^n]Y$ and $[z^{q^j}]Y$ is a unit. Taking $H=T$ and using Legendre's formula gives

$$c_{q^j}\ge p^{-\gamma_p(q^j-1)},\qquad \gamma_p=\frac{1}{p-1}-\rho.$$

The case $n=0$ gives $c_0=1$. For $n\ge1$, write $q^j=an+b$, $0\le b<n$. Submultiplicativity gives $c_{q^j}\le c_n^a$. Let j tend to infinity to get $c_n\ge p^{-\gamma_pn}$. Consequently in each degree n some $H\in R[[y]]$ has $v_p(n![z^n]H)\le\gamma_pn$.

Put $g=-\log\theta$. The normalized pullback $\exp(a\bar a\,g(z)-g(az))\in R[[y]]$ extends to limits of $(a,a\bar a)$. For $\delta'\in\mathbf{Z}_p$ approximate δ' by ordinary integers, obtaining the same assertion with $\bar aa=(\delta')^2$. There are four elements of $\mathbf{Z}_p$ whose squares sum to -1, by four squares modulo p^u and compactness. Adjoin 1 and 0 to this list to give $\delta_0,\dots,\delta_5$, with $\delta_0=1$, and $\sum_c\delta_c^2=0$. It follows that $\prod_{c=0}^5\theta(\delta_cz)\in R[[y]]^\times$ by multiplying the six pullbacks.

The weighted Gram identity and coefficient calculation. Use vectors w_i in the trace-dual lattice $\mathfrak{a}_0=E\cap\alpha_{\mathrm{W}}\widehat{O}_E$, whose positive integral multiplicities c_i satisfy

$$\sum_i c_i\tau(w_i)\overline{\sigma(w_i)}=M\delta_{\tau,\sigma},\qquad c_i\ge 6fb.$$

For this family take the q_1 powers of ζ_{q_1} and the trace-dual mixed family constructed for ordinary jets, before squaring either scalar. Their scalar constants are q_1 and h respectively, with $(h, q_1) = 1$. They can both be taken at most b. In the mixed family the dual vectors are $\bar{\beta}^{\lceil \log_2 q_1 \rceil} b_i / \Phi'_{q_1}(\zeta)$, with b_i the coefficients of $\Phi_{q_1}(X)/(X-\zeta)$. Write $u = q_1/q_0$. The polynomials satisfy $\Phi_{q_1}(X) = \Phi_{q_0}(X^u)$: if the u-th power of a root of unity has order q_0, its order is q_1, since every prime dividing u divides q_0. The coefficient bound for Φ_{q_0} and polynomial division at $|\zeta| = 1$ give $|b_i| \le q_1 2^{q_0}$ at every embedding. Differentiation gives

$$|\Phi'_{q_1}(\zeta)| = u|\Phi'_{q_0}(\zeta^u)| \ge c(q_0)u,$$

where $c(q_0) > 0$ is the minimum at the primitive q_0-th roots. These are bounds at fixed q_0. Thus h is polynomial in $q_1, d^{\lceil \log_2 q_1 \rceil}$, whereas b exceeds every fixed power of x. Put $m_0 = \lceil 6fb \rceil$. The scale $M \ge b^{20}$ ensures $M - m_0(q_1 + h) \ge q_1 h$. Write the left side as $uq_1 + vh$ with $u, v \ge 0$: choose $0 \le v < q_1$ in the required residue class first. The two multiplicities are $m_0 + u, m_0 + v$.

Write $\mu_i(Z) = p_F \sum_\tau \overline{\tau(w_i)} Z_\tau$. At every coefficient embedding these forms have integral coefficients by the period-factor valuation. At least one μ_i has only unit coefficients. At weight embeddings outside β-primes use the vector 1. At the primes over β use the unchanged vector $\zeta \bar{\beta}^{\lceil \log_2 q_1 \rceil} / \Phi'_{q_1}(\zeta)$; the denominator cancels the period factor in valuation. Either case applies to all weight embeddings together, since their restrictions to K are equal. At an inert prime the μ_i span the coordinate forms integrally: the columns for the powers $1, \zeta, \ldots, \zeta^{f-1}$ give a Vandermonde determinant which is a unit since $p \nmid q_1$ and the f distinct q_1-th roots $\tau(\zeta)$ remain distinct in the residue field.

Put $\Theta(Z) = \prod_i \theta(\mu_i Z)^{c_i}$ and let $\mathcal{A}$ be the integral power series in $y(\mu_i Z)$. For a degree vector ν write $q_N(\nu) \le u$ if some coefficient of combined degree ν in a product of N independent copies $H\Theta$, $H \in \mathcal{A}$, has valuation at most u. Products on distinct copies add bounds. An integral linear substitution of copies within each coordinate preserves the combined degree, and each new coefficient is a finite integral combination of the old ones.

For the six-copy general bound, $U(Z) = \prod_{c=0}^{5} \Theta(\delta_c Z)$ is a unit of $\mathcal{A}$ by the one-variable identity. In the first copy take $H = H_1(\mu_{i_0} Z)/U(Z)$ with μ_{i_0} the unit form, and in the other five take $H = 1$. Substitute $Z_c = \delta_c Z$. The product becomes $H_1(\mu_{i_0} Z)$, whose coefficient is obtained from the one-variable test and multinomial formula. Thus

$$q_6(\nu) + v_p(\nu!) \le \gamma_p |\nu|, \qquad \nu! = \prod_\tau \nu_\tau!. \tag{106}$$

At inert p, the spanning property also gives $q_6(\nu) \le 0$: multiply the integral series with linear terms Z_τ, to powers ν_τ, by Θ. The lowest term in total degree is the desired monomial with unit coefficient.

37.1.1 The inert expansion and valuation

In this subsection, we apply the original expansion of Section 20.3 to the coefficient in twelve coordinates.

The inert normal form has $\Theta(\varpi Z) = W(Z) \exp(K\Xi(Z))$, $K = \varpi^{p+1}/p$, with W an integral unit and Ξ reducing modulo p to

$$\sum_{e \text{ odd}, j \ge 0} \lambda_{e,j} \left(\sum_\tau Z_\tau^{p^e} Z_{-p^e \tau} \right)^{p^j}, \qquad \lambda_{e,0} \in R^\times.$$

Since 2 splits in K, an inert p is odd. Equation (77) gives a unit for the z^{p+1} coefficient. In $Y^{p+1} \bmod p$ each $z^{p^e} z$, for e positive and odd, occurs once: the two Frobenius indices are 0 and e. For integral w_i at this inert prime the odd Frobenius congruence in Section 20.3.1 and the weighted Gram identity give

$$\sum_i c_i \overline{\tau(w_i)}^{p^e} \overline{\sigma(w_i)} \equiv \sum_i c_i((-p^e\tau)(w_i))\overline{\sigma(w_i)} = M\mathbf{1}_{\sigma=-p^e\tau} \pmod p.$$

Each scaled form supplies the additional factor $p_F^{p^e+1}$. Both M and the period factor are units at an inert prime, so the transferred coefficient is a unit.

Let selected starts τ have odd exponents e_τ and ends $u_\tau = -p^{e_\tau}\tau$. For each selected τ, assume that for every selected σ and every positive odd $e \le e_\tau$,

$$-p^e\sigma = u_\tau \quad \Longrightarrow \quad (\sigma, e) = (\tau, e_\tau).$$

Use this coupling j_τ times, with degrees $\nu' = \sum_\tau p^{e_\tau} j_\tau[\tau]$ and $\nu'' = \sum_\tau j_\tau[u_\tau]$; put $J_c = \sum j_\tau$. Apply the substitution using

$$G(Z', Z'') = \Theta(\varpi(Z' + Z'')) \prod_{c=1}^{5} \Theta(\varpi\delta_c Z').$$

The logarithmic derivative in Z''_u, divided by K, is

$$(\partial_u \Xi)(Z' + Z'') + K^{-1}(W^{-1}\partial_u W)(Z' + Z'').$$

Here $v_p(K) = 1/(p-1) - 1 < 0$. If L is integral and $\partial = \partial_{Z''_u}$, then

$$K^{-1}\partial(GL) = G\big(L\,\partial_u\Xi + K^{-1}(\partial L + LW^{-1}\partial_u W)\big),$$

where W and Ξ are evaluated at $Z' + Z''$. Let $\mathfrak{m}_R = \{a \in R : v_p(a) > 0\}$. In the last recursion $K^{-1} \in \mathfrak{m}_R$ since $v_p(K^{-1}) = (p-2)/(p-1)$. Reducing in $R/\mathfrak{m}_R$ and iterating gives $K^{-J_c}\partial_{Z''}^{\nu''} G = GH$, where

$$H \bmod \mathfrak{m}_R = \prod_\tau ((\partial_{u_\tau}\Xi)(Z' + Z'') \bmod \mathfrak{m}_R)^{j_\tau}.$$

The recursion uses ordinary iterated derivatives over characteristic zero, including for $j_\tau \ge p$. The two sets of variables Z', Z'' are independent even if a start equals an end. In the residue field the derivative of a loop monomial $X_u^{p^e+1}$ is $X_u^{p^e}$, with coefficient one.

Set $Z'' = 0$ and all Z'_τ outside the starts to zero. The unique lowest term of each relevant derivative is $\bar\lambda_{e_\tau,0}(Z'_\tau)^{p^{e_\tau}}$. In the residue field the terms with $j > 0$ vanish on differentiation, as does the derivative of every factor $(Z'_\sigma)^{p^e}$. The surviving summands have $-p^e\sigma = u_\tau$; the selection condition leaves just (τ, e_τ) through degree p^{e_τ}. Also $G(Z', 0) = U(\varpi Z')$ is integral with constant term 1, by $T(\varpi z) = \varpi Y(z)$. After restriction to the starts, every other term of a derivative has strictly greater total degree than p^{e_τ}. The product of its prescribed lowest monomials has degree $|\nu'|$. Choosing any higher term, including one from $U(\varpi Z') - 1$, gives larger total degree. Thus the coefficient of $Z'^{\nu'}$ in $(GH)(Z', 0) \bmod \mathfrak{m}_R$ is $\prod_\tau \bar\lambda_{e_\tau,0}^{j_\tau} \neq 0$. Evaluation of the ordinary derivatives at $Z'' = 0$ multiplies the corresponding coefficient by $\nu''!$, including when $p \mid \nu''!$. Thus the coefficient of $Z'^{\nu'} Z''^{\nu''}$ in G has valuation exactly $J_c(1/(p-1) - 1) - v_p(\nu''!)$.

Undo the scale ϖ and use the integral substitution back to the six independent copies. With $\nu = \nu' + \nu''$ this gives

$$q_6(\nu) \le -J_c, \qquad q_6(\nu) + v_p(\nu'!) + v_p(\nu''!) \le \gamma_p|\nu| - J_c.$$

Indeed undoing the common dilation subtracts $\rho|\nu| = |\nu|/(p^2-1)$ from the valuation. For the first bound use $|\nu| \ge (p+1)J_c$. For the second use $|\nu''| = J_c$ and $v_p(\nu'!) \le |\nu'|/(p-1)$. The exact coefficient here is that of the substituted series. On the original independent variables this proves existence at the indicated valuation bound.

We now show

$$q_{12}(\lambda) + \sum_\tau v_p(\lambda_\tau!) \le P_p + 2\sum_\tau v_p(B_\tau). \tag{107}$$

At a split prime $q = p$ and $v_p(\pi) = 1$, giving $\gamma_p = 0$. Also $1 - \chi_K(p^e) = 0$ for every e, so $P_p = 0$. This applies at $p = 2$ because $d \equiv 7 \pmod 8$. Use (106) and the constant coefficients of six further copies. For inert $p > k$ use every start with $e_\tau = 1$, $i_\tau = \lfloor (\lambda_\tau - n_\#)_+/p \rfloor$, and $j_\tau = \min(i_\tau, \lambda_{-p\tau})$. Here 1 is the least positive odd exponent. Multiplication by $-p$ permutes the places, so each end has a unique start. Since $i_\tau \le n_\#$, the outgoing plus incoming degree at each place is at most $(\lambda_\tau - n_\#)_+ + \min(n_\#, \lambda_\tau) = \lambda_\tau$. The remaining six copies give valuation at most zero on unused degree. Put $m_\tau = (\lambda_\tau - n_\#)_+$. Legendre's formula gives

$$v_p(\lambda_\tau!) = i_\tau + v_p(i_\tau!) + v_p(\lambda_\tau!/m_\tau!) \le i_\tau + 2v_p(B_\tau).$$

Indeed $i_\tau! \mid n_\#! \mid B_\tau$, as $B_\tau/n_\#!$ is a binomial coefficient, and $\lambda_\tau!/m_\tau! \mid B_\tau$ also when $\lambda_\tau < n_\#$. Also $i_\tau - j_\tau \le (\kappa_\tau/p - \kappa_{-p\tau})_+$ because $n_\# \ge x$[42]. Summing proves (107).

For fixed d and sufficiently large x we have $d \le k$, so the remaining nonsplit primes all satisfy $p \le k$. Select the pairs with $2e < L_0$ whose two places have weight b, with e among the allowed positive exponents. There are none at d: both endpoints of a weight-b pair would be $-1 \bmod q_0$, forcing $-d^e \equiv 1 \bmod q_0$, whereas $-1 \notin \langle d \bmod q_0 \rangle$ by Lemma 4.1. At inert p the partners are $(-p)^e\tau$. Put $D(\tau) = (-p)\tau$. Since $p \nmid q_1$, this is a permutation. Section 4 excludes three weight-b places $\tau, D^i\tau, D^j\tau$ with $0 < i < j < L_0$, even if the places coincide. Two outgoing edges of different exponents $e < e'$ at the same start give such a triple with $(i,j) = (e,e')$. For two edges ending at the same place, equal exponents give the same start by bijectivity. If $e < e'$, the orbit from the start of the longer edge meets the other start and the common end at $(e'-e, e')$. Finally two successive edges give $(e, e+e')$, with $e + e' < L_0$; this includes a loop followed by itself.

If $-p^{e'}\sigma = u_\tau$ for a selected start and an odd $0 < e' \le e_\tau$, this too is a selected edge, since $2e' < L_0$ and both endpoints have weight b. Uniqueness at the common end gives $(\sigma, e') = (\tau, e_\tau)$. In particular a shorter return from τ would give the prohibited triple (τ, u_τ, u_τ) at exponents (e', e_τ). Take $j_\tau = \lfloor \lambda_\tau/p^e \rfloor$ on these pairs; their ends have degree at least $b \ge j_\tau$. Use the second coupling bound and (106) on the remaining degrees. At a place write $l_1 + l_2 + l_3 = \lambda_\tau$ for the starting, ending and residual degrees. The factorial loss is

$$v_p(\lambda_\tau!) - \sum_{i=1}^3 v_p(l_i!) = \sum_{h \ge 1} \left(\left\lfloor \lambda_\tau/p^h \right\rfloor - \sum_{i=1}^3 \left\lfloor l_i/p^h \right\rfloor \right) \le 3(\log b + 2).$$

Indeed each summand is at most two and vanishes for $p^h > \lambda_\tau$; here $\lambda_\tau \le 2b$ and $p \ge 3$. At the ramified prime d the selected set is empty and the six-copy general bound applies to the entire residual degree.

For each τ,

$$R_\tau = \sum_{e \text{ allowed}} \min(\kappa_\tau/p^e, \kappa_{-p^e\tau})$$

has its selected term at most $j_\tau + 1$; all other terms with $2e < L_0$ are at most a/x_1, and the tail is at most $2b/(kp)$. For the tail, $p^{\lceil L_0/2\rceil - 1} \geq 2^s \geq k$. For inert p the allowed exponents are positive and odd, and for $p = d$ they are all positive. In both cases $v_p(\pi)\sum_{e \text{ allowed}} p^{-e} = \gamma_p$, so subtracting the positive parts defining P_p gives

$$\gamma_p \sum_\tau \kappa_\tau - P_p = v_p(\pi) \sum_\tau R_\tau.$$

Writing $D_\tau = v_p(\lambda_\tau!) - \sum_i v_p(l_i!)$ for the three-factor loss at τ, the two six-copy bounds give

$$q_{12}(\lambda) + \sum_\tau v_p(\lambda_\tau!) - P_p \leq \sum_\tau (\gamma_p(\lambda_\tau - \kappa_\tau) + v_p(\pi)R_\tau + D_\tau) - J_c.$$

The selected contributions to $\sum_\tau v_p(\pi)R_\tau$ are at most $J_c + f$, as $v_p(\pi) \leq 1$. The remaining contributions and the factorial errors are bounded place by place by

$$3(\log b + 2) + x^{42} + 1 + L_0 a/x_1 + 2b/(kp) \leq 3b/(kp) \leq v_p(B_\tau).$$

Here the scales give $3(\log b + 2) + x^{42} + 1 \leq a/x_1$, $(L_0 + 1)a/x_1 \leq b/k^2$, and $b/(kp) \geq 1$. The last inequality follows by counting at least $\lfloor n_\#/p \rfloor$ multiples of p in B_τ. This proves (107) also here.

The exact geometric section and finite cells. At a coefficient furnished by (107), expand each H in monomials in $y(\mu_i Z_c)$. These series have zero constant term, so only the finitely many monomials of parameter degree at most $|\lambda|$ contribute. Their scalar coefficients belong to R. By the ultrametric inequality, some product of unweighted monomials has a coefficient of the required multi-index with no greater valuation. Thus the ensuing geometric section requires only the selected integer exponents. Writing their exponents $u_{i,c}$, the corresponding curve sections are

$$(y\theta^3)^{u_{i,c}}\theta^{c_i - 3u_{i,c}}.$$

The powers are nonnegative since $3|\lambda| \leq 6fb \leq c_i$. The sections have degree c_i and integral twisted torsion values. Evaluate them in the Fock coordinates at the forms $\Omega_0 w_i^\dagger/(\bar\delta\, a_d)$ on the corresponding coordinates. Their weighted pullback Hermitian forms sum to the original Fock form by the weighted Gram identity and $|\bar\delta|^2 = M$.

In the real coordinates $r(\lambda)_{\tau,c} = a_d\tau(\lambda_c)$, trace self-duality gives

$$A_{i,c}(r(\bar\delta\lambda)) = \Omega_0 \operatorname{Tr}_{E/K}(\lambda_c \bar w_i) \in \Omega_0 \mathcal{O}_K \qquad (\lambda \in \Lambda_{12}).$$

Write $H_C(l, z) = \pi\bar l z/A_C$ and let H_{Fock} denote the sum of the pullbacks of $c_i H_C$. The sign $\chi_C(l) = \exp(i \operatorname{Im} H_C(\bar\beta l, l))$ computed for the chosen two-torsion point gives

$$\prod_{i,c} \chi_C(A_{i,c}L)^{c_i} = \exp(i \operatorname{Im} H_{\mathrm{Fock}}(\bar\beta L, L)).$$

This is exactly the prescribed signed period law on $\bar\delta\Lambda_{12} = \pi_M\Lambda_{12}$. To realize this section in the finite Weil model, put $L = \Lambda_{12}$. Self-duality and $E_{\mathrm{alt}}(\bar\delta x, y) = E_{\mathrm{alt}}(x, \delta y)$ for the alternating

polarization give $(\bar{\delta}L)^{\#} = \delta^{-1}L$. The index $[L : \bar{\delta}L] = M^{12f}$ is the product of the polarization elementary divisors on $\bar{\delta}L$. Indeed the polarization kernel is $(\bar{\delta}L)^{\#}/\bar{\delta}L$, of order

$$[\delta^{-1}L : L][L : \bar{\delta}L] = M^{24f}.$$

For the positive polarization in characteristic zero, the dimension of the space of sections is the square root of this order [59, §1, (II)–(IV), p. 289]. For $u \in \delta^{-1}\widehat{L}/\widehat{L}$, the translates $T(u)v$ have this multiplier. The signed translations by L give them distinct commutator characters by the dual-lattice equality. Their entire images are nonzero by injectivity of the finite theta realization in Section 20.1, so they form a basis. The section is therefore the image of a finite-part vector F_*. At $\pi_M s_{\mathrm{jet}}$ its forms are exactly μ_i because $\bar{\delta} = \pi_M$. A finite translate of F_* can be read after rational approximation modulo its open stabilizer. Each factor then evaluates, up to a root of unity, at the twisted value at $-A_{i,c}r(\gamma)$ for the rational displacement γ. Choose an integer $b_0 \geq 1$ with $b_0\gamma \in \Lambda_{12}$. Since $M = \delta\bar{\delta}$,

$$Mb_0A_{i,c}r(\gamma) = \Omega_0\delta\,\mathrm{Tr}_{E/K}(b_0\gamma_c\bar{w}_i) \in \Omega_0\mathcal{O}_K.$$

The twisted torsion values of both θ and $y\theta^3$ are integral by Section 20.1. Thus each theta displacement evaluation of F_* at real argument zero and arbitrary finite displacement is integral.

37.1.2 The finite coefficient pairing

In this subsection, we pair the resulting coefficient of Section 37.1.1 against the finite test.

Put $L = \Lambda_{12}$, $A = L/\bar{\delta}L$ and $B = \delta^{-1}L/L$. The Weil commutator pairs A and B perfectly by $(\bar{\delta}L)^{\#} = \delta^{-1}L$ and $L^{\#} = L$. Both groups have order $\mathrm{N}_{E/\mathbf{Q}}(\pi_M)^{12} = M^{12f}$. Their rational representatives give all classes of the completed lattices, since completion preserves finite quotients. Write $y = \alpha_{\mathrm{W}}^{-1}a$ and $x = \alpha_{\mathrm{W}}^{-1}u$. With the translations of Section 7.4.3, put

$$S(a) = T_{\sharp}(y) = e_0(y)\rho_f(a), \qquad T(u) = \rho_f(u).$$

For $a, a' \in L$ the normalized coordinates y, y' are integral. The multiplier $\psi_0(\bar{c}_0(y')^*y)$ is therefore one, including at places above 2. Thus $S(a)S(a') = S(a + a')$. Rational Heisenberg invariance identifies the signed period law on $\bar{\delta}L$ with $S(a_0)F_* = F_*$ for $a_0 \in \bar{\delta}L$. The ordinary displacement satisfies

$$T(u)S(a)T(-u) = \psi_0(y^*x)S(a).$$

This is the commutator pairing just described. In particular $\bar{\delta}L$ fixes $T(-u)F_*$, so the sum over A is independent of the lifts of its elements. For each u, the sum

$$d_u v = \sum_{a \in A} S(a)T(-u)F_*$$

satisfies the signed period law on L, whose theta space is the vacuum line. Conjugation by $T(u)$ multiplies $S(a)$ by the commutator character at u. Its sum over B is zero for $a \neq 0$ and $|B|$ for $a = 0$. Thus

$$M^{12f}F_* = \sum_{u \in B} d_u\,T(u)v.$$

If the vacuum vanished on the whole finite slice, rational Heisenberg translation would give $v(r_\infty, z_f) = 0$ for every rational $r \in E^{12}$ and every finite argument z_f: translation by $-r$ takes

the real argument to zero and changes the value by a nonzero phase. Since E^{12} is dense in its archimedean vector space, continuity would give $v = 0$. Thus the finite slice has a nonzero value.

Fix such a point z_f^* and put $C_v = V_0^{12} v(0, \alpha_{\mathrm{W}} z_f^*) \neq 0$. Local constancy and finite adelic additive approximation allow the finite translate at this point to be represented by a rational one. Its normalized value is therefore a degree-zero translated CM coefficient, fixed before the coefficient embedding. At this point $d_u v(0, \alpha_{\mathrm{W}} z_f^*)$ is a sum of finite evaluations of F_* multiplied by roots of unity. It is therefore integral also at the primes dividing M. In the normalized finite coordinates $x = \alpha_{\mathrm{W}}^{-1} u$ we have δx integral. The nonintegral components of x are at the conjugates of the places of π_M. Subtracting the other components acts on the vacuum by a root-of-unity phase. Take Z in the chosen Schrödinger lattice. The commutator above makes $T(u)v$ an eigenvector of the integral position translation by Z, with eigenvalue $\psi_0(-Z^* x) = \psi_0(x^* Z)$. Thus on that lattice its additive multiplier relative to the vacuum is $\psi_0(x^* Z)$, up to the fixed phase from the discarded components. It factors through reduction modulo π_M since $\pi_M \bar{x} = \overline{\delta x}$ is integral. The perfect pairing of A and B gives all the residue characters. For the original residue cells write $T(u)v = \sum_x \omega_u(x) I_x$, where each nonzero $\omega_u(x)$ is a root of unity. Set $v_* = v(0, \alpha_{\mathrm{W}} z_f^*)$ and $a_x = \sum_{u \in B} (d_u v_*) \omega_u(x)$. Each product $d_u v_*$ is a sum of the integral algebraic torsion values above with root-of-unity coefficients, so $v_p(a_x) \geq 0$. Multiplying the Fourier identity by v_* and applying the scaled coefficient functional gives

$$M^{12f} C_v [Z^\alpha] \mathcal{T} F_* = \sum_x a_x \, V_0^{12} [Z^\alpha] \mathcal{T} I_x.$$

The jet of F_* on the left is the chosen coefficient of the CM section in the algebraic forms μ_i. It is nonzero. Each scaled cell coefficient on the right is algebraic by Lemma 37.1. Applying the coefficient embedding to this identity and taking the minimum valuation on the right gives a cell x_0 with

$$v_p(V_0^{12} [Z^\alpha] \mathcal{T} I_{x_0}) \leq v_p(M^{12f} C_v) + v_p([Z^\alpha] \mathcal{T} F_*).$$

Set $t_0 = M^{12f} C_v \prod_\tau B_\tau^2$.

It is fixed before the coefficient prime and embedding, and nonzero. Put $C_{\mathrm{int}} = 2 + \log M_K$ and take $C_P = \max\{C_K, C_{\mathrm{int}} + 2d + 6\}$, with the constants of Lemma 7.7. The rank-one conjugate vacuum bound there, applied on the twelve coordinates, gives

$$\log |\sigma t_0| \leq 12 f \log M + 12 C_P q_1^2 + 2 f n_{\#} \log(2b + n_{\#}) \leq f m b / k.$$

Indeed $n_{\#} \leq 7b/k$ eventually and $\log(2b + n_{\#}) \leq B + \log 3 = o(m)$. Also $\log M = O(B)$, $q_1 \leq x^{12}$, and b exceeds every fixed power of x. These bounds hold at one cutoff for the fixed q_0, d, ϵ, independently of the coefficient prime. Now (107) proves (105). The polynomial degree, section, and finite Fourier steps all use the same $\Lambda_{12}, \alpha_{\mathrm{W}}, \pi_M$ as the numerator test.

37.1.3 CM transport of the untwisted numerator

In this subsection, we apply Lemma 7.6 to the coefficient of the original numerator test.

Fix $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Instantiate the exact translated jet identity (AJ) and the vacuum identity (AV) on the standard Hom space of rank $2z$. Their proof depends on the rank only via the coordinate product, so the same construction uses the normalized scale t_σ with $|t_\sigma|^2 = \Omega^{-1}$. Keep the exact identities (JV1) and the finite multiplier law, rather than their subsequent

bounds on coefficient sizes. Write the resulting scalar translate as $b_\sigma = u_\sigma a_\sigma$ where $u_\sigma \bar{u}_\sigma = 1$ and a_σ is an integral unit equal to 1 on the old chosen places. This is possible because $b_\sigma \bar{b}_\sigma$ is a unit. In every split pair choose one member w, using the prescribed member at each old pair. Set $a_{\sigma,w} = 1$ and $a_{\sigma,\bar{w}} = \overline{(b_\sigma \bar{b}_\sigma)_w}$. At a conjugation-fixed place b_σ itself is a unit, so use it as a_σ.

In the untwisted column datum the finite coefficient function is the original chosen-column indicator and has values $0, 1$. After removing the scalar action of u_σ, the exact semilinear operator transports each signed translate by x to the one by $a_\sigma x$, and multiplies the vacuum by one fixed nonzero scalar c_σ. Let D be the additive residue matrix space at the chosen places, and let $Y_0 \in D$ be the prescribed inclusion. On the integral box its indicator has the expansion

$$\mathbf{1}_{Y=Y_0} = |D|^{-1} \sum_{\chi \in \widehat{D}} \chi(-Y_0)\chi(Y).$$

The modulations representing $\chi(Y)$ are supported at the opposite places. Since $a_\sigma = 1$ on the chosen side, (JV1) changes their characters to $\sigma\chi$. Semilinearity changes their coefficients to $(\sigma\chi)(-Y_0)$. Permuting the characters recovers the same indicator times c_σ. At the other places there is only the vacuum.

Write $b_X = \nu_X \upsilon_1^z$, $b_Y = \nu_Y \upsilon_1^z$, where $\upsilon_1 = \upsilon|_{U(1)}$, and let g' denote the inverse source action on the $2z$ Hom coordinates. For a finite norm-one element g, put $D_1 = \Delta_1(g^{-1})$ for the one-coordinate multiplier. The squared-multiplier identity in Section 7.4.3 gives $D_1^2 = |D_1|^2 \upsilon_1(g)$, where $|D_1|^2$ is a rational lattice index. The $2z$ identical coordinates give $\Delta(g') = D_1^{2z} = |D_1|^{2z} \upsilon_1(g)^z$, whence

$$d_\sigma(g') = \frac{\sigma(\Delta(g'))}{\Delta(g')} = \left(\frac{\sigma(\upsilon_1(g))}{\upsilon_1(g)}\right)^z.$$

After writing $\nu_Y(g) = \sigma(\nu_X(g))$ on the finite ideles, the source factor in a corresponding kernel term is

$$\sigma\big(b_X(g)^{-1}\big) d_\sigma(g') = b_Y(g)^{-1} = b_Y(u_\sigma^{-1})\, b_Y(g u_\sigma^{-1})^{-1}.$$

The last inverse character belongs to the transported kernel at $(g u_\sigma^{-1}, h)$. Multiplication by u_σ^{-1} permutes the classes of the norm-one torus at the same old level. Summing over this permutation leaves $b_Y(u_\sigma^{-1})$ outside the class sum.

Let X be the original untwisted datum and Y its transport. Write $X^{\mathrm{raw}}, Y^{\mathrm{raw}}$ for their unscaled class sums, J_X for their common total degree, and h' for the target Hom action at h. Write p_σ for the permutation of the weight embeddings given by $\sigma\Phi_\tau = \Phi_{p_\sigma(\tau)}$. Relabel Z by $(R_\sigma Z)_{\tau,j} = Z_{p_\sigma^{-1}(\tau),j}$, and put $i_\tau^\sigma = i_{p_\sigma^{-1}(\tau)}$. The Schur coordinates are products of polynomials with integer coefficients in these variables. If $P_{X,i}$ and P_{Y,i^σ} are the corresponding coordinate polynomials, then $P_{Y,i^\sigma}(R_\sigma Z) = P_{X,i}(Z)$. Their total degree J_X gives the scalar $t_\sigma^{J_X}$. With $A_X = s_{\mathrm{jet}}^{J_X} X^{\mathrm{raw}}$, the exact transport at the unchanged target point h is

$$\sigma(A_X(h)_i) = t_\sigma^{J_X} c_\sigma b_Y(u_\sigma^{-1}) \frac{\sigma(\Delta(h'))}{\Delta(h')} Y^{\mathrm{raw}}(h)_{i^\sigma}.$$

The scalar c_σ and the Taylor scale were fixed before the character and degree. This proves the transport identity needed for the mixed pairing in the untwisted reference norm. For the twisted numerator the two point tests are constructed separately at κ and $\bar{\kappa}$. The twelve-coordinate cell is evaluated at the identity, and the other block carries the finite twist in (NB2). This keeps the same η, multiplier t_0, and main line at both embeddings.

37.2 The four-block test and the untwisted reference norm

In this subsection, we insert the coefficient of Section 37.1 in the original four-block theta test. The resulting numerator is measured against the untwisted norm on the same space.

Fix a main pair $p = (\nu, z)$, its twist η of exact support Q, and the original lift X from $U(1)$ to $U(2z)$. Let C be the old $U(1)$ class group, $h_1 = |C|$, and $V_0 = s_{\mathrm{vac}}$. Use its original ordered blocks of sizes $12, 4, 4, 2z-20$. Write $\lambda_\tau = c_\tau - z$ for the part sizes and put $J = (\pi_M s_{\mathrm{jet}})^{\sum_\tau \lambda_\tau}$. The normalized main lift is

$$\phi_\eta = V_0^{2z} J \sum_{g \in C} K_X(g),$$

where $K_X(g)$ is the raw kernel form. The class stabilizers in this source sum are trivial.

37.2.1 The derivative estimate

In this subsection, we use the coefficient bound of Section 37.1 for the point test.

The required scalar $t_0 \neq 0$ depends on the part sizes and the fixed datum, with $\log|\sigma t_0| \leq fmb/k$ for every complex automorphism. At each coefficient embedding $\kappa = (\ell, \iota)$, the sharp derivative bound gives an exponent α on the first twelve coordinates with $\sum_i \alpha_{\tau,i} = \lambda_\tau$, and a cell x_0 in the original residue space modulo π_M, such that

$$\left| \frac{V_0^{12}}{t_0} \left(\prod_\tau \lambda_\tau! \right) [Z^\alpha] \mathcal{T} I_{x_0} \right|_\kappa \geq \ell^{-P_\ell}, \qquad P_\ell = \sum_\tau S_\ell(\kappa_\tau, \tau). \tag{NB1}$$

Here I_{x_0} is the point indicator at the chosen places and the original vacuum elsewhere, and $\mathcal{T}$ reads the Taylor series at scale $\pi_M s_{\mathrm{jet}}$. By Lemma 37.1, this coefficient is the image under ι of the normalized algebraic jet. The degree-zero assertion for a translated vacuum gives $C_v \in \overline{\mathbb{Q}}^\times$ and hence $t_0 \in \overline{\mathbb{Q}}^\times$. For a main pair the defining inequalities of Section 38.2 give $w_j(\tau) + r - t \leq c_\tau \leq w_j(\tau)$ and $m < z \leq r$. Since $w_j(\tau) = \kappa_\tau + (2j+1)r^8$ and $t = 5r$, we have

$$\kappa_\tau \leq w_j(\tau) - t \leq c_\tau - z \leq w_j(\tau) \leq \kappa_\tau + x^{42}.$$

The last inequality follows for all $1 \leq j \leq m$ from $(2m+1)r^8 = O(x^{81/2})$ by Section 4.1. Thus Section 37.1 applies at the actual degree $\lambda_\tau = c_\tau - z$; it fixes t_0 before the coefficient embedding and gives (NB1) on the original residue space.

37.2.2 The point test on the original target

In this subsection, we insert the derivative of Section 37.2.1 in the test on four blocks.

Use the empty rank-four pair $p_{\mathrm{aux}} = (\chi_M \upsilon_1^{-2}, 2)$ in the second block at twist η, and two untwisted empty partitions in the third and fourth blocks. Write υ_1 for the restriction to the norm-one torus. For an even rank $2z'$ and an old weight-zero character ψ, the empty partition is the pair $(\psi \upsilon_1^{-z'}, z')$, with rank-twisted character ψ. These even ranks lie between 4 and t, since $m < z \leq r$. The character on the first block is the quotient of the target character of X by the other three characters.

Set $T_\alpha = \prod_\tau \lambda_\tau!/\alpha!$, with $\alpha! = \prod_{\tau,i} \alpha_{\tau,i}!$. Let P_i be the dual Schur basis of products of minors for this weight. Its column minors have size one. In the original Weyl coordinates take

$v_i = [Z^\alpha]P_i$, so v is integral. The Fock polynomial $\mathcal{P}(v)$ in these coordinates satisfies

$$\langle \mathcal{P}(v), P_i\rangle_{\mathrm{Fock}} = \left(\prod_\tau \lambda_\tau!\right) v_i, \qquad \langle Z^\alpha, P_i\rangle_{\mathrm{Fock}} = \alpha![Z^\alpha]P_i.$$

The first identity is the Weyl to dual Schur pairing with the hook product of the one-row partition. The second follows by monomial orthogonality. The pairing with all P_i therefore gives $\mathcal{P}(v) = T_\alpha Z^\alpha$. The contragredient substitution fixes the coefficient functional when the first twelve columns are identity columns. Equivalently the transpose substitution fixes its monomial in the Fock pairing. This gives invariance of v under the last three embedded groups. There is no determinant factor in this Weyl weight.

At the old chosen places write $Y = (x_0, e_2, e_3, e_1)^{\mathsf{t}}$, with e_i the included unit vector in the indicated empty block. Write e_{old} for the original included vector of X and e_{second} for the included vector of the second block. Choose the reduction of the integral point lift A by

$$Ae_{\mathrm{old}} = Y \quad (w \in S_M), \qquad e_{\mathrm{old}}^{\mathsf{t}} A^{-1} = e_{\mathrm{second}}^{\mathsf{t}} \quad (w \in Q).$$

At an old place Y is nonzero by its last component and extends to a basis. At a new place the prescribed nonzero covector extends to an invertible matrix of covectors. Split reduction gives the local integral lifts. They define A by taking the identity at all remaining finite places of F_0.

Put $x = \mathrm{diag}(1, h_2, h_3, h')A$ and let J_Q, χ be the target level and its character. Write the covariance of a form as $f(\gamma y k) = \chi(k)\rho(\gamma)f(y)$, where Γ is the rational group and ρ its coefficient action. Define $F(h', h_2, h_3) = F_x(v)$ by

$$F_x(v)(y) = \sum_{\substack{\gamma\in\Gamma \\ x^{-1}\gamma y\in J_Q}} \chi(x^{-1}\gamma y)\rho(\gamma^{-1})v.$$

The sum is finite by definiteness and discreteness. It has the stated covariance by changing variables and is supported on the class of x. For $k = x^{-1}\gamma y$, transporting the summand to the lattice norm at y acts on v by $k^{-1}x^{-1}$ and a root of unity. The inverse block diagonal fixes v. Both A^{-1} and k^{-1} preserve the local Weyl lattice at every coefficient place above ℓ. Thus each summand has lattice norm at most one there, as does their sum.

At x the sum is over $S_x = \Gamma \cap xJ_Q x^{-1}$. For $\gamma \in S_x$, put $T_\gamma = \chi(x^{-1}\gamma x)\rho(\gamma^{-1})$. It is unitary in the original definite pairing and fixes $f(x)$ by covariance. Since the class weight at x is $|S_x|^{-1}$, the pairing gives

$$\langle F_x(v), f\rangle_{J_Q} = \frac{1}{|S_x|}\sum_{\gamma\in S_x}\langle T_\gamma v, f(x)\rangle = \langle v, f(x)\rangle.$$

Move A from the kernel argument into the finite datum by the split Weil action. The determinant of A occurs in the target variable with the character $\chi_W = 1$ of the definite source line. The local lattice action thus introduces no determinant scalar. The datum now acts on A^{-1} times the residue column. On $\pi_M^2 Q$ the two displayed residue conditions give the prescribed column indicator at S_M and the tame character of the second-block covector at Q.

This is exactly the product of the four block functions, pulled back to that modulus. The archimedean polynomial is $T_\alpha Z^\alpha$ on the first block and 1 on the other three. The rational theta sum and the Weil character both factor across these orthogonal blocks. If K_i are the raw empty kernels and K_0 is the selected first kernel, this proves

$$\langle F(h', h_2, h_3), K_X(g)\rangle = T_\alpha K_0(g)K_2(g, h_2)K_3(g, h_3)K_1(g, h'). \tag{NB2}$$

In this formula the kernel is in the linear leg of the pairing. The same g serves every block.

For an empty block p_i of rank a_i, let $\phi_i = V_0^{a_i}\phi_i^{\mathrm{raw}}$. The denominator $d_i = d_{p_i} = M^{2fa_i}$ is the denominator of Section 30.1 at degree zero. The local jet degree vanishes at each infinite place because the polynomial of this block is 1. Put

$$A_i = C_G d_i^2 \|\phi_i\|^2 / Q_i^{a_i-1}, \quad Q_2 = \mathrm{N}Q \text{ for the twisted block}, \quad Q_1 = Q_3 = 1.$$

Here Q_i in this display denotes only the norm of the auxiliary modulus on that block. The class averaging operator on F weights its value at h by

$$\frac{C_G d_i^2}{|\operatorname{Aut}(h)|\, Q_i^{a_i-1}} \langle 1, \phi_i(h)\rangle_{\mathrm{Weyl}}.$$

These weights are integral at all the embeddings above ℓ, since $\ell \nmid \mathrm{N}Q$, d_i clears the normalized empty lift and C_G clears the class weights.

The weighted point form is independent of the empty-block representatives. For a right level change $h_i \mapsto h_i k_i$ the new point is xk, with $k = A^{-1}\operatorname{diag}(k_i)A$ and the other diagonal blocks unchanged. At the old places $\operatorname{diag}(k_i)$ fixes Y in the residue space. At the new places the defining covector is on the second block, so k belongs to J_Q and its character is the character of the changing block. The point form acquires $\chi(k)^{-1}$ by its definition and the averaging weight acquires $\chi(k)$. Left rational changes act on the point vector through the block action fixing v. The stabilizer weights therefore apply to the actual classes even when several choices give the same target class.

In the pairing against $K_X(g)$ the weights are conjugated. They replace $K_i(g,h)$ in (NB2) by $A_i/(h_1 V_0^{a_i})$. To see this, a normalized central translation by u^{-1} sends $K_i(g)$ to $K_i(gu)$: the simultaneous scalar cancels on the Hom variable, leaving the inverse source twist. This translation is unitary and fixes $\phi_i^{\mathrm{raw}} = \sum_g K_i(g)$, as right translation permutes the source classes. It follows that $\langle \phi_i^{\mathrm{raw}}, K_i(g)\rangle$ is independent of g; summing over g gives its value $\|\phi_i^{\mathrm{raw}}\|^2/h_1$. Combining this with the conjugated scalar $\overline{V_0^{a_i}}$ in the weights gives precisely $A_i/(h_1 V_0^{a_i})$.

37.2.3 The last-block class sum

In this subsection, we evaluate the last block of the test in Section 37.2.2.

Apply the measured period argument in Section 33.1 to the original rank-one descent datum O of the four-block lift X. The source has rank 1, the target has rank $n = 2z$, and the two blocks for this application are the retained first $12+4+4 = 20$ coordinates and the last $n-20$ coordinates. Thus the dimension condition is $2 + 2\cdot 20 + 3 = 45 < n$, since $z > m$ and $m \geq 22$ for all sufficiently large x. At infinity the line parameter is zero at every place, the determinant exponent is zero, and the target space of length λ_τ is matched by the homogeneous Fock polynomials of that degree. Indeed the inverse source scalar acts on that homogeneous degree by its $-\lambda_\tau$-th power, and the inverse character of source weight $-\lambda_\tau$ cancels it. The positive Fock variables give the one-row partition $(\lambda_\tau, 0, \dots, 0)$, with no negative-sector columns or mixed Laplacians. In each degree this is the irreducible symmetric power of the compact target, including its trivial representation in degree zero. The source circle has different characters in different degrees, giving the required correspondence.

Fix a split w with $w, \bar{w}$ off the original chosen and twist places. For this test a false lift y lies in

$$\operatorname{im}(e_{X-1}) \cap \ker \mathcal{P}_{n,w}(X_*) \cap \ker D_O, \qquad X_* = \left(q_w^{n-2}\operatorname{frob}_1(1,w)\right)^{-1},$$

where D_O is the normalized descent for O, and $\mathrm{frob}_1(1,w)$ is the rank-one unramified value of the trivial χ_W with the original splitting character included. If Υ is the original Hecke splitting character of E, then $\mathrm{frob}_1(1,w) = q_w^{1/2}\Upsilon(\varpi_w)^{-1}$. In particular $|X_*| = q_w^{3/2-n}$. The conjugate matrix coefficient of y is killed at $\overline{X_*}$, as in (F1). The coefficient operator here is the one of this rank-one datum, with $\chi_W = 1$, even when η is nontrivial in the other character.

The idempotent e_{X-1} is the product of the means for the normalized actions of the old groups $\mathrm{GL}_1(k_v) = k_v^\times$. Its image consists of the invariant vectors for these actions. The raw stabilizer shear includes the factor $\chi_v(s)^{-1}$ in the calculation of Section 33.1.1. If λ is a nonzero residue covector at a chosen place v, the rank-one set $1 + u\lambda$ with nonzero determinant is all of $k_v^\times$. The action of its hyperplane sum on y is therefore $\left(\sum_{s\in k_v^\times} \chi_v(s)^{-1}\right) y = 0$. Here χ_v is the exact old main-pair tame character, which is nontrivial, while η has trivial old tame components.

The same weighted sum in the product of old local groups is zero if any one of its covectors is nonzero. At the new places the exact-support hypothesis gives nontrivial residue characters. The bottom descent vanishing for the original block is the condition $D_O y = 0$. These are the three conditions for the bottom extension to arbitrary Schwartz data vacuum off the original support. They replace the rank-r argument with selected Green parameters and the orthogonal complement of the projected image.

For the following tower argument, the source at step l, $1 \le l \le 20$, has rank $1 + 2l$. With $k_l = n - (1 + 2l)$, the estimates of (F3) read

$$q_w^{k_l}|\alpha X| = q_w^{1-l} \le 1, \qquad q_w^{1+2l+1}|X'| = q_w^{(6+2l-n)/2} \le 1,$$

where $X = \overline{X_*}$. The latter follows from $n \ge 46$. Hence the same hyperbolic-plane induction gives vanishing of all descents through twenty planes, and the argument from (F4) gives the measured second-block periods with retained dimension 20. The bottom and tower arguments use the fixed O, fixed target point, original finite vacua and original right-coset operators throughout.

For the application to the test, take the fixed first-block point $\mathrm{diag}(1, h_2, h_3)$ of the retained twenty coordinates. Take the additional right translate to be 1. The integral point lift A in $F_x(v)$ then gives the period point $\mathrm{diag}(1, h_2, h_3, h')A$, with finite translation supported at the old and auxiliary split places. The last-block kernel $K_1(g, h')$ is the degree-zero kernel, whose polynomial is 1, whose target dimension is $n - 20$, and whose rank-twisted character is the untwisted old factor ψ_1. Write $b_X = \nu\eta v_1^z$ for the rank-twisted character of the main pair. For its original Schwartz datum the kernel identity

$$K^{1,1}(g,h) = b_X(\det g)K^{1,b_X}(g,h)$$

multiplies the descent at fixed g by a nonzero scalar, conjugated when the kernel is in the conjugate leg. Thus $D_O y = 0$ gives the bottom vanishing with second character 1 on the original datum. The old residue sums and the exact-support condition above extend this vanishing to the Schwartz tests used in the tower. In the block period take the last character $b = \psi_1$ and the auxiliary retained character b^{-1}. Their product is 1 for every allowed ψ_1. The auxiliary retained kernel is chosen only for this vanishing argument.

The finite datum of $K_1(g, h')$ has the original vacuum at the new places and elsewhere off the old support. For each fixed source point g and $h_{\mathrm{ret}} = \mathrm{diag}(1, h_2, h_3)$, Section 33.1.4 constructs an auxiliary twenty-coordinate kernel nonzero at (g, h_{ret}). This scalar is independent of h', so it

cancels from the period for each false lift with the prescribed last-block datum, before summing over g, h_2, h_3. The datum of K_1 can be read on the nominal modulus π_M^2, with the prescribed column indicator depending only on reduction modulo π_M. In the measured period use the conjugate kernel vector $\overline{K_1(g,h')}$ and the conjugate-linear coefficient map $c \mapsto \bar{c}\, v$. This map is equivariant for the real last-block group: the empty weight is trivial, and the retained coefficient vector v is fixed by that group. Since the class pairing is conjugate-linear in its first argument, the resulting scalar integrand is

$$\overline{K_1(g,h')}\,\langle F(h',h_2,h_3), y\rangle.$$

The point-form reproducing identity evaluates the second form at the finite point

$$\operatorname{diag}(1,h_2,h_3,h')A.$$

Use the actual last-block vector-fixer level J_1 of the untwisted degree-zero lift. It is integral and fixes e_1 at the old chosen places, so fixes the last-block datum. At an old place the embedded element fixes the prescribed column $(x_0,e_2,e_3,e_1)^{\mathsf{t}}$. Consequently $A^{-1}\operatorname{diag}(1,k)A$ fixes the originally included column. At a new place this block embedding fixes the prescribed twist covector because that covector lies in the second block of the four-block splitting. Conjugating by A therefore fixes the originally included covector there and has level character 1. Off those places the element is integral. This is the right invariance needed for the period-to-class-sum passage. Rational left invariance follows from the same kernel and form covariance.

The measured period is zero. Compact integration at J_1 multiplies its class sum by the common nonzero factor $[K_{\mathrm{std}} : J_1]^{-1}$ with maximal compact mass one. Retaining the actual stabilizer weights gives

$$\sum_{[h']_{J_1}} \frac{\overline{K_1(g,h')}}{|\operatorname{Aut}_{J_1}(h')|}\langle F(h',h_2,h_3), y\rangle = 0.$$

This is for every original g, h_2, h_3, before the outer three sums. The last-block sum has its vector-fixer level, so there is no need to assert freeness of this class space. Expanding the degree-zero lift in its source class sum now gives the required orthogonality of the averaged test to y. The two remaining scalar averages preserve this orthogonality, and all three averages preserve integrality.

Denote the test after the three averages by Ξ. Substituting the replacement identities into (NB2) gives

$$h_1^3 V_0^{2z-12}\langle \Xi, K_X(g)\rangle = T_\alpha K_0(g) A_1 A_2 A_3. \tag{NB3}$$

37.2.4 Selection of the class character

In this subsection, we use the class sum of Section 37.2.3 to choose characters for the numerator estimate.

Before selecting the two untwisted empty-block characters, put

$$\chi_0 = \frac{\nu\eta\upsilon_1^z}{(\chi_M\upsilon_1^{-2})\eta\upsilon_1^2} = \nu\upsilon_1^z/\chi_M.$$

These are the rank-twisted characters of the main and second blocks, respectively. The common finite twist cancels in the quotient. At $g = 1$ every character has value one, so the twelve-coordinate jet is exactly the untwisted cell coefficient of Lemma 37.1. Let $s(g)$ be the unscaled

first kernel for χ_0, x_0, α. Choose the actual source representatives g_a of C, and let $z_0 = g_{1_C}^{-1}$. The representatives $g_a z_0$ have the identity point in the identity class. For $\psi \in \widehat{C}$ put

$$B_\psi = h_1^{-1} \sum_{a \in C} \psi(\det(g_a z_0)) s(g_a z_0).$$

On these representatives the class of $\det(g_a z_0)$ is a, and $g_{1_C} z_0 = 1$. Since $h_1 = |C|$, character orthogonality gives

$$\sum_{\psi \in \widehat{C}} \psi(\det(g_a z_0)) = h_1 \mathbf{1}_{\{a = 1_C\}}, \qquad \sum_{\psi \in \widehat{C}} B_\psi = s(1).$$

This identity uses only the values of s on the chosen representatives. Applying the ultrametric inequality to the last sum gives a ψ with $|B_\psi|_\kappa \geq |s(1)|_\kappa$, also for $\ell \mid h_1$. The three class weights forming Ξ are integral by the calculation preceding Section 37.2.3.

Let $\mathcal{C}_0$ be the finite group of allowed old tame weight-zero characters. Each $\psi \in \widehat{C}$ belongs to this group by inflation. Indeed, on a norm-one idele take ψ of the class of its finite component. This character is trivial on rational points, on the archimedean torus and on the old principal level. Its finite unit restrictions thus factor through the residue groups at S_M and are trivial elsewhere. Fix $L_{\mathrm{geom}} \neq 0$ by choosing a root

$$L_{\mathrm{geom}}^{|\mathcal{C}_0|} = \prod_{\chi \in \mathcal{C}_0} A_1(\chi) A_3(\chi).$$

These are untwisted degree-zero values on the last and third blocks with the same old places. Proposition 30.1 at $Q = \varnothing$ gives a positive norm for every such character, including the trivial one: the old vector-fixer computation gives $q_w - 1 > 0$ and the omitted Euler product is absolutely convergent and nonzero at $z' \geq 2$. Section 44.4 gives the denominator bound with $q_Q = 1$. In particular $L_{\mathrm{geom}} \neq 0$. As χ runs over $\mathcal{C}_0$, so does $\psi\chi^{-1}$, and hence

$$\prod_{\chi \in \mathcal{C}_0} |A_1(\chi) A_3(\psi\chi^{-1})|_\kappa = |L_{\mathrm{geom}}|_\kappa^{|\mathcal{C}_0|}.$$

At least one factor is at least the geometric mean. Thus one factorization $\psi = \psi_1 \psi_3$ has $|A_1(\psi_1) A_3(\psi_3)|_\kappa \geq |L_{\mathrm{geom}}|_\kappa$. Use these two characters on the corresponding blocks. The first-block target-leg character becomes χ_0/ψ. Its multiplier on the source variable is

$$((\chi_0/\psi)(\det g))^{-1} = \psi(\det g) \chi_0(\det g)^{-1}.$$

Hence $K_0(g) = \psi(\det g) s(g)$.

Write $L_{\mathrm{twist}} = A_2(1)$, and let $\rho_\eta = A_2(\eta)/A_2(1) > 0$. For the empty rank-four pair $p_{\mathrm{aux}} = (\chi_M \upsilon_1^{-2}, 2)$, the exponents $c_v - z$ vanish, so the denominator $d_2 = M^{8f}$ is independent of η. The norm identity (88) gives

$$\frac{\|\phi_2(\eta)\|^2}{\|\phi_2(1)\|^2} = q_Q^3 R_{p_{\mathrm{aux}},+}(\eta), \qquad \rho_\eta = R_{p_{\mathrm{aux}},+}(\eta).$$

On the source, rational equivariance and the trivial source weight give left invariance of the full kernel K_X. The old principal level fixes its old column. At a new place, multiplication of the residue covector value by the unit character cancels the inverse character in the source

Weil action. At all remaining places the units fix the vacuum. Thus K_X descends to the old class group C, and ϕ_η can be evaluated at $g_a z_0$. Sum (NB3) there and take its linear-leg normalization. The result is

$$\frac{h_1^2}{\rho_\eta}\langle \Xi, \phi_\eta\rangle = V_0^{12} J T_\alpha L_{\mathrm{twist}} A_1 A_3 B_\psi. \tag{NB4}$$

Here the third power of h_1 in (NB3) loses one power in passing to the Fourier average.

No complex conjugate of V_0 or J occurs in (NB4). The coefficient identity at $g = 1$ is $(J/\alpha!)s(1) = [Z^\alpha]\mathcal{T} I_{x_0}$, since differentiation multiplies the Taylor coefficient by $\alpha!$. In particular $JT_\alpha s(1) = (\prod_\tau \lambda_\tau!)[Z^\alpha]\mathcal{T} I_{x_0}$. Using $v_\kappa(B_\psi) \le v_\kappa(s(1))$ and $v_\kappa(A_1A_3) \le v_\kappa(L_{\mathrm{geom}})$ in (NB4), then (NB1), gives

$$v_\kappa\langle \Xi, \phi_\eta\rangle \le v_\kappa(t_0 L_{\mathrm{twist}} L_{\mathrm{geom}} \rho_\eta) + P_\ell - 2v_\ell(h_1).$$

This bound also applies for $\ell \mid h_1$. For $\ell \in \{2, p_0, p_1\}$, $P_\ell = 0$ because ℓ splits in K, so every factor $1 - \chi_K(\ell^e)$ in S_ℓ vanishes.

Proposition 37.2 (The four-block numerator test)**.** *For each coefficient embedding κ above ℓ and each allowed twist η of exact support Q with $\ell \nmid q_Q$, there is a test $\Xi_{\kappa,\eta}$ integral at every coefficient place above ℓ and orthogonal, for every split off-level w, to $\operatorname{im} e_{X-1} \cap \ker \mathcal{P}_{n,w}(X_*) \cap \ker D_O$, where D_O is descent to the full line form space. It satisfies*

$$|\langle \Xi_{\kappa,\eta}, \phi_\eta\rangle|_\kappa \ge |c_0 \rho_\eta|_\kappa \ell^{-P_\ell}, \qquad c_0 = t_0 L_{\mathrm{twist}} L_{\mathrm{geom}} \ne 0.$$

The constant c_0 is fixed for the original main pair and $\log|\sigma c_0| \le J_*$ at every complex automorphism σ.

At the untwisted levels let $h_{\mathrm{s}}, h_{\mathrm{t}}$ be the numbers of classes of the source line and the target of rank $a_i \le 2r$. Both levels contain the principal subgroup modulo M. Apply the lattice count in the proof of Lemma 7.2 in these ranks. Its integral class bound is $O(u^3 \log x)$ on the logarithmic scale for $u = 2fn$, $n \in \{1, a_i\}$; refinement modulo M adds at most $2fn^2 \log M$. Thus $\log h_{\mathrm{s}} + \log h_{\mathrm{t}} = O(x^{52})$ uniformly in the block. For each of σ and $c\sigma c$, apply the translated-vacuum bound in the proof of Lemma 7.7 in a_i coordinates with polynomial one. The transported $V_0^{a_i}$-scaled vacuum has scalar of modulus at most $\exp(C_{\mathrm{int}} q_1^2 a_i)$. The ordinary Gaussian estimate is $\exp((2d+6)(q_1 a_i + 1)^2)$. The untwisted column test has at most M^{fa_i} finite Fourier terms. Including the degree-zero integral gives

$$U_i = M^{fa_i} \exp\bigl(C_P(q_1 a_i + 1)^2 + f a_i \log 4\bigr)$$

for the absolute value of either conjugated kernel. In the Gram sum use σ on the first kernel and $c\sigma c$ on the kernel whose value is conjugated before applying σ. The stabilizer weights are at most one and the weight-zero Weyl line has norm one. Consequently, for every old character in this sum,

$$|\sigma A_i(\chi)| \le C_G d_i^2 h_{\mathrm{t}} h_{\mathrm{s}}^2 U_i^2.$$

Here $\log C_G \le f(rt)^2 \log M$, $\log d_i = 2fa_i \log M$, and $\log U_i = O((q_1 a_i + 1)^2 + f a_i(1 + \log M))$. The logarithm of the displayed bound is at most x^{53} eventually. It follows that

$$\log|\sigma L_{\mathrm{geom}}| \le 2x^{53}, \qquad \log|\sigma c_0| \le fmb/k + 3x^{53} \le J_*.$$

Indeed $x^{53} = o(fmb/k)$ and $J_*/(fmb/k) = xB^3/m^2 \geq B^3/4$. The cutoff depends only on the fixed q_0, d, ϵ.

For a fixed twist η of exact support Q with $\ell \nmid q_Q$, the construction applies to the same η at κ and $\bar{\kappa}$; only the local exponent, cell, Fourier character and test may change. The fixed multiplier and the main line stay the same.

37.2.5 Untwisted norms at both conjugate embeddings

In this subsection, we control the untwisted norms in the finite comparison of Proposition 37.2.

Let $m < z \leq r$ be the half-rank of the fixed main pair. Take $a = 2z$ and its original level at $Q = \varnothing$. In (NB5)–(NB6) take $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$. Both σ and $\sigma^{\#} = c\sigma c$ then preserve the chosen CM type, the embeddings extending the fixed embedding of K. For its untwisted datum X write $J_X = \sum_\tau \lambda_{X,\tau}$ and $A_X = s_{\mathrm{jet}}^{J_X} X^{\mathrm{raw}}$, with X^{raw} the source class sum. The CM transport of Section 37.1.3, evaluated at the same target point h, gives

$$\sigma(A_X(h)_i) = t_\sigma^{J_X} c_\sigma b_Y(u_\sigma^{-1}) d_\sigma(h')\, Y^{\mathrm{raw}}(h)_{i^\sigma}. \tag{NB5}$$

Here b_Y is the rank-twisted character of the conjugate datum. For σ fixing K, let p_σ be the permutation defined by $\sigma\Phi_\tau = \Phi_{p_\sigma(\tau)}$ on the CM embeddings used for the Fock coordinates. Then $\lambda_{Y,p_\sigma(\tau)} = \lambda_{X,\tau}$ and $i^\sigma_{p_\sigma(\tau)} = i_\tau$, and $d_\sigma(h') = \sigma(\Delta(h'))/\Delta(h')$ for the target Hom action h'. The scale satisfies $|t_\sigma|^2 = \Omega^{-1}$. The vacuum scalar $c_\sigma \neq 0$ depends only on σ, the rank and the original level.

$\sigma^{\#} = c\sigma c$ gives the same datum Y. Its permutation agrees with that of σ since $E/\mathbb{Q}$ is abelian. To check the finite character, write any finite norm-one idele as $y/\bar{y}$ by local Hilbert 90, taking unit preimages at almost all places. Choose a finite ray modulus so that its ray units map into the kernel of the finite character. A power y^h is a principal idele a times a ray unit. Automorphy expresses the corresponding h-th power of the finite character value as a product of integer powers of the embeddings of $a/\bar{a}$. All conjugates of these numbers have absolute value one. Thus for each finite character value v, $\sigma^{\#}(v) = \overline{\sigma(v^{-1})} = \sigma(v)$. Permuting its infinity type and conjugating its finite part therefore gives one automorphic character, on the same level, for both complex automorphisms.

In each Gram term, $D_{h'} = \Delta(h')\overline{\Delta(h')}$ is a positive integral lattice index. Hence

$$d_\sigma(h')\overline{d_{\sigma^{\#}}(h')} = \frac{\sigma(D_{h'})}{D_{h'}} = 1.$$

The class weights and Gram matrix are rational and the place permutation reindexes the basis. Using (NB5) in the two legs gives the equality

$$|\sigma\langle A_X, A_X\rangle| = |c_\sigma c_{\sigma^{\#}}|\, \Omega^{-J_X} \|Y^{\mathrm{raw}}\|^2. \tag{NB6}$$

The two character values in (NB5) have modulus one.

Apply the proof of Proposition 30.1 at the untwisted level to the raw kernel. For the transported datum Y, let $\lambda_{Y,\tau} = c_{Y,\tau} - z$ denote its local degree and put $W_Y = \bigotimes_\tau \mathrm{Sym}^{\lambda_{Y,\tau}}(\mathbb{C}^a)$. Use a highest covector of norm one for the compact invariant Hermitian form. The local monomial has Bargmann square norm

$$\pi^{-a} \int_{\mathbb{C}^a} |Z_1|^{2\lambda} e^{-\|Z\|^2}\, dZ = \lambda!.$$

For a unit highest covector, compact Schur orthogonality gives the scalar coefficient's square integral as the squared Weyl norm divided by $\dim W_Y$. Write $v_j = \mathrm{vol}([U(j)])$ in the original measures of Section 30.1 and $S^0 = S_M \cup \bar{S}_M$. Then

$$C_{a,S_M} = \frac{v_a v_1}{\zeta_{F_0}(2z)} \prod_{w \in S_M} (q_w - 1),$$

$$\|Y^{\mathrm{raw}}\|^2 = C_{a,S_M} \dim W_Y \left(\prod_{\tau} \lambda_{Y,\tau}! \right) L^{S^0}(z, (\nu_Y^{\mathrm{bc}})^{-1}).$$

Indeed, for $q = \mathrm{N}v$ at a split pair $w, \bar{w}$ off S^0, put $A_w = \nu_Y^{\mathrm{bc}}(\varpi_w)$. The two Euler factors of the base change multiply to $|1 - A_w q^{-z}|^{-2}$, with $|A_w| = 1$. At an inert place the base change is trivial on the uniformizer of $F_{0,v}$, and its factor $(1 - q^{-2z})^{-1}$ cancels the local zeta denominator. The finite extension E/F_0 is unramified at every finite place by the inertia calculation in Section 4.2. At an old split pair, even for a ramified tame character, the character is trivial on the principal-unit stabilizer. The local mass, the target index, the square of the source index and the removed zeta factor contribute together

$$\frac{q^{-a}}{q-1}(q^a - 1)(q-1)^2(1 - q^{-a})^{-1} = q - 1.$$

These factors are positive. For $|A| = 1$, $0 < t < 1$, the power series for $\log(1 - At)$ gives $\big|\log|1 - At|\big| \le -\log(1 - t)$. Summing over places yields

$$\left| \log L^{S^0}(z, (\nu_Y^{\mathrm{bc}})^{-1}) \right| \le \log \zeta_E(z) \le [E : \mathbb{Q}] \log \zeta(2) < 4f.$$

The vacuum and jet factors satisfy

$$\|\phi_X\|^2 = |V_0^a|^2 M^{J_X} \langle A_X, A_X \rangle, \qquad |\pi_M|^2 = M \in \mathbb{Q}_{>0}.$$

In particular the common term at σ is

$$C_{\sigma,a,S_M} = \log\Big(\Big| \sigma\Big(|V_0^a|^2 \Big) \Big| \; |c_\sigma c_{\sigma\#}| \, C_{a,S_M} \Big).$$

Equation (NB6) and the permutation of the local degrees give

$$\log|\sigma\|\phi_X\|^2| = C_{\sigma,a,S_M} + r_X + \sum_{\tau} (\log(\lambda_{X,\tau}!) + \lambda_{X,\tau} \log(M/\Omega)),$$

where $r_X = \log \dim W_Y + \log L^{S^0}(z, (\nu_Y^{\mathrm{bc}})^{-1})$. Thus $r_X \ge -4f$, and $r_X \le 4f$ for the empty partition, when $\dim W_Y = 1$.

For the empty reference of this same rank, take $q_{\mathrm{ref}} = (\nu_{\mathrm{ref}}, z)$ with $\nu_{\mathrm{ref}} = v_1^{-z}$, where $v_1 = v|_{U(1)}$. Its old unit characters are trivial, its infinity weights are $-z$, and its rank-twisted character is $\nu_{\mathrm{ref}} v_1^z = 1$. The source retains its old principal level and the target its vector fixer $J_\varnothing$. Since $J_{\mathrm{ref}} = 0$ and the coefficient space has dimension one, the measured identity gives

$$\|\phi_{\mathrm{ref}}\|^2 = |V_0^a|^2 C_{a,S_M} L^{S^0}(z, (\nu_{\mathrm{ref}}^{\mathrm{bc}})^{-1}) > 0.$$

The positive Euler product for the main datum also gives $\|\phi_X\|^2 > 0$. By Section 44.4 at $Q = \varnothing$, the values of $d_{\mathrm{ref}}\phi_{\mathrm{ref}}$, with $d_{\mathrm{ref}} = M^{4fz}$, are integral at every coefficient embedding above each

ℓ. In the class pairing, use the two embeddings κ and $\bar{\kappa}$ for the conjugate factors. The Weyl Gram matrix is integral and every stabilizer order divides C_G, by Section 30.1. Hence

$$I_{\mathrm{ref}} = C_G d_{\mathrm{ref}}^2 \|\phi_{\mathrm{ref}}\|^2 \neq 0, \qquad v_\kappa(I_{\mathrm{ref}}) \geq 0$$

for every κ. The fixed multiplier $H = I_{\mathrm{ref}}/\|\phi_X\|^2$ therefore clears the untwisted norm at every prime. Subtract the two conjugate logarithms. The common constant cancels and the main part sizes are at least κ_τ. Monotonicity of $J(n) = \log(n!) + n\log(M/\Omega)$ gives

$$\log|\sigma H| \leq \log(C_G d_{\mathrm{ref}}^2) + 8f - \sum_\tau J(\kappa_\tau) \leq J_* - \sum_\tau J(\kappa_\tau).$$

Indeed $\log(C_G d_{\mathrm{ref}}^2) \leq f((rt)^2 + 8z)\log M$. Since $z \leq r$, $\log M \leq 22\log b$ and f, r, t, k, m grow at most polynomially in x, this upper bound and $8f$ are $o(J_*)$ for fixed q_0, d, ϵ, where $J_* = fxb(\log b)^3/(mk)$. If σ acts nontrivially on K, then $c\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and $|\sigma H| = |(c\sigma)H|$. Applying the bound to $c\sigma$ on the same old level proves it for every complex automorphism. This is the fixed multiplier estimate for the untwisted term of the reciprocal height.

38 The common twist and bounds on critical values

The two measured norm tests of Section 30 give the signed systems for Proposition 28.3. That proposition selects one twist per coefficient place and bounds the Selmer lengths and signed critical ratios for its prescribed powers. We then apply Proposition 32.1 to those twisted summands to bound the projected pairings.

Fix q_0, d and the exponent of Section 4, and take x sufficiently large for these parameters. Fix the resulting admissible datum. For a coefficient place $\kappa = (\ell, \iota)$ put $\bar{\kappa} = (\ell, \iota \circ \mathrm{conj})$ and normalize $v_\kappa(\ell) = 1$. Write S_M for the old chosen places and $A_{\mathrm{aut}} = \prod_{w \in S_M} |\mathrm{GL}_{rt}(k_w)|$. For a ramified critical pair $p = (\nu, z)$, sign $\epsilon = \pm 1$, and twist η of exact support Q, put $q_Q = \prod_{w \in Q} \mathrm{N}w$, taking one place of each conjugate pair. Define

$$R_{p,\epsilon}(\eta) = \frac{|L^{S_M \cup Q}(z, ((\nu\eta^\epsilon)^{\mathrm{bc}})^{-1})|}{|L^{S_M}(z, (\nu^{\mathrm{bc}})^{-1})|}.$$

The superscript omits the conjugates of the indicated places as well. Let β be either inverse base-change character in the ratio. Omit a conjugation-stable finite set S' containing the ramified places. For real $s > 1$, the Euler product converges absolutely and has the factors at a split pair and an inert place, respectively,

$$|1 - \beta(\varpi_w) q^{-s}|^{-2}, \qquad (1 - q^{-2s})^{-1},$$

where q is the residue norm below the pair or inert place. The first identity uses $\beta^c = \beta^{-1}$ and unitarity. At the inert place, base change is trivial on the uniformizer from F_0. This applies to the two partially omitted products appearing in the ratio as well.

At $z = 1$ the critical pair has $c_v \geq 1$ at each infinite place by Section 2.9. In its chosen complex coordinate the inverse base-change character restricts to the unit circle by $u \mapsto u^{2c_v}$. The finite-order twist leaves this restriction unchanged. Both β and β^2 are therefore nontrivial on the ideles of module one. Hecke continuation gives holomorphy of their L-functions at one [74, Theorem 4.4.1 and §4.5]. For real $s > 1$,

$$\zeta_E^{S'}(s)^3 |L^{S'}(s, \beta)|^4 |L^{S'}(s, \beta^2)| \geq 1$$

by the logarithmic Euler-product argument of Section 30.2.1. A zero at one would make the left side tend to zero, since ζ_E has a simple pole. Removing finitely many Euler factors preserves nonvanishing at every $z \geq 1$. The common twist chosen below controls the valuations of these ratios.

38.1 Signed systems for the critical pairs

In this subsection, we form the systems for the common torsion choice of Proposition 28.3.

Write each of the two coefficient formulas as $F_{p,Q}(\eta) = \sum_i c_i(Q)\eta(u_i)$. Fix any $\ell \nmid q_Q$. The coefficient formulas give integrality at every coefficient embedding. If some $c_i(Q)$ were transcendental, choose a transcendental $u \in \mathbb{Z}_\ell^\times$. A field isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ extending the chosen embedding of the original algebraic data can send $c_i(Q)$ to u/ℓ, by extension of transcendence bases as in Section 23. The negative valuation contradicts integrality. For an algebraic coefficient the same integrality at every place away from q_Q then gives $c_i(Q) \in \overline{\mathbb{Z}}[1/q_Q]$, where $\overline{\mathbb{Z}}$ is the ring of algebraic integers. Use the signed array

$$\Theta_{p,\epsilon,Q} = \sum_i \iota(c_i(Q))[u_i^{-\epsilon}]_Q.$$

For the inverse avatar $\psi = \eta_\ell^{-1}$, evaluate a norm-one class $[u]_Q$ on an Artin preimage of $u = y/\bar{y}$. In the construction of the auxiliary characters, every $\chi_w(\mathrm{Art}\, y)$ is $\theta_w(y/\bar{y})$ for an idele class character θ_w. Changing y by a fixed-field idele therefore leaves the value unchanged, also when $\ell = 2$. Evaluation at the resulting inverse avatar gives $\iota(\eta(u)^{-1})$. Therefore $\psi(\Theta_{p,\epsilon,Q}) = \iota(F_{p,Q}(\eta^\epsilon))$. The sign is in the array, so each sign is evaluated with the same ψ.

The unit-valued character in the Euler norm relation is the avatar ϑ_ϵ of $B_\epsilon = (\nu^{-\epsilon})^{\mathrm{bc}}$. For a finite idele x of E its formula in geometric reciprocity is

$$\vartheta_\epsilon(\mathrm{Art}_E\, x) = \iota(B_{\epsilon,f}(x)) \prod_{\tau \in \Phi} \left(\frac{(\iota\tau)(x_\ell)}{(\iota\bar{\tau})(x_\ell)} \right)^{\epsilon c_\tau},$$

where each embedding reads its induced component above ℓ. On principal ideles the product is the transported infinity component, so it cancels the finite character. The finite values of ν lie in one number field by the norm-one class calculation of Section 4.4. The displayed map is continuous and factors through the compact quotient of the finite ideles by the closure of $E^\times$, by global reciprocity. Its valuation has compact image in $\mathbf{R}$, hence is zero. The algebraic correction in the displayed formula uses only components at ℓ. Away from ℓ and the old level the finite character is trivial on local units, so the avatar is unramified there as required in Section 27. At a split pair outside $S_M \cup \bar{S}_M \cup Q \cup \bar{Q}$ and of residue characteristic different from ℓ, put $A_w = \iota((\nu\eta^\epsilon)^{\mathrm{bc}}(\varpi_w))$. In the geometric Artin convention we have

$$(\vartheta_\epsilon\psi)(\mathrm{Fr}_{w,\mathrm{geom}}) = A_w^{-\epsilon}, \qquad (\vartheta_\epsilon\psi)(\mathrm{Fr}_{w,\mathrm{ar}}) = A_w^{\epsilon}.$$

Indeed $\nu^{-\epsilon}\eta^{-1} = (\nu\eta^\epsilon)^{-\epsilon}$, and the two Frobenius elements are inverses. Each Euler-factor pair is invariant under $A_w \mapsto A_w^{-1}$. On an unramified enlargement of the nominal modulus the rank-one system deletes the pair at exponent z, and the rank-two system deletes the pairs at exponents $0, 1$. The rank-one normalizing power of q_Q cancels the local mass at each new place even when the residue character is trivial. In rank two the local multiplier at $0, 1$ is (92), with the same fixed jet scalar for the two levels. These are multiplication identities even if a deleted factor vanishes.

For clarity, let $\mathcal{C}_Q$ be the finite norm-one class group with maximal integral units off Q and principal units at Q. The old level is part of the fixed norm test and of ν. Write $P : \mathcal{C}_{Q\cup\{w\}} \to \mathcal{C}_Q$ for the quotient and let b_w be the class of $y_w/\bar{y}_w$ where y_w is the idele that is a uniformizer at w and one elsewhere. Put $a_w = \iota(\nu^{\mathrm{bc}}(y_w))$ and $\mathcal{E}_p = \{z\}$ for $z \geq 2$, $\mathcal{E}_p = \{0, 1\}$ for $z = 1$. Then

$$P_*\Theta_{p,\epsilon,Q\cup\{w\}} = \Theta_{p,\epsilon,Q} \prod_{e\in\mathcal{E}_p} (1 - q_w^{-e} a_w^{-\epsilon}[b_w])(1 - q_w^{-e} a_w^{\epsilon}[b_w]^{-1}).$$

For every $j \geq 1$, evaluation on ψ^j sends $a_w^{-\epsilon}[b_w]$ to

$$a_w^{-\epsilon}\iota(\eta(b_w))^{-j} = \iota\big((\nu\eta^{j\epsilon})^{\mathrm{bc}}(y_w)\big)^{-\epsilon} = (\vartheta_\epsilon\psi^j)(\mathrm{Fr}_{w,\mathrm{geom}}).$$

The reciprocal group unit gives the inverse value. Thus only the auxiliary character is raised to j. For $z = 1$, if this Frobenius value is one, the factor at exponent zero makes $F_{p,Q\cup\{w\}}(\eta^{j\epsilon}) = 0$ at this enlarged modulus. The identity still holds for that character. The levels for the two class groups differ only at the split place w. Hence $\ker P$ is the image of the integral norm-one units there. If a power of a character on $\mathcal{C}_{Q\cup\{w\}}$ is trivial on these units, that power descends to $\mathcal{C}_Q$, where the identity applies to it. The characters of $\mathcal{C}_Q$ have trivial residue part at w and separate its group algebra in characteristic zero. Indeed, take a finite extension K_ℓ containing the displayed coefficients and all character values of $\mathcal{C}_Q$. If $D = \sum_c d_c[c]$ is the difference of the two sides, Fourier inversion in K_ℓ gives

$$|\mathcal{C}_Q| d_c = \sum_{\chi\in\widehat{\mathcal{C}_Q}} \chi(c)^{-1}\chi(D) = 0.$$

Thus each $d_c = 0$ in K_ℓ. The identity holds in its valuation ring coefficient by coefficient, including when $\ell \mid |\mathcal{C}_Q|$, since q_w, a_w are units and the arrays are integral. It supplies the norm relation of Section 27.

For $Q = J$ and the auxiliary characters χ_w of that section, choose finite idele representatives y_i with $u_i = y_i/\bar{y}_i$ in $\mathcal{C}_Q$. With N the auxiliary exponent, put $n_{iw} = [-\epsilon\chi_w(\mathrm{Art}\, y_i)]_{\ell^N}$, the representative between 0 and $\ell^N - 1$. The integral polynomial

$$P_{p,\epsilon,Q}(X) = \sum_i \iota(c_i(Q)) \prod_{w\in Q} (1 + X_w)^{n_{iw}}$$

has value $\iota(F_{p,Q}(\eta^{j\epsilon}))$ at $X_w = \zeta_w^j - 1$ whenever $\zeta_w^{\ell^N} = 1$ for each w and $\prod_{w\in Q} \zeta_w^{\chi_w}$ is the inverse avatar of η. The signs therefore use one torsion tuple.

For exact support the two norm formulas give

$$\frac{F_{p,Q}(\eta^\epsilon)}{F_{p,\varnothing}(1)} = \begin{cases} R_{p,\epsilon}(\eta), & z \geq 2, \\ q_Q R_{p,\epsilon}(\eta)^2, & z = 1. \end{cases}$$

At $Q = \varnothing$, the line test prescribes the old column modulo π_M and uses its vector fixer $J_\varnothing$. The rank-two period uses the determinant character modulo $(\pi_M)^2$ and the full standard compact K_2 in (89). The augmentations for both signs are the images under ι of

$$F_{p,\varnothing}(1) = \begin{cases} C_G d_p^2 \|\Phi_1\|_{J_\varnothing}^2, & z \geq 2, \\ |C_G s_p \Lambda_\varnothing(1)|^2, & z = 1. \end{cases}$$

Proposition 30.1 gives positivity in the first case. In the second, (90) and the functional equation there give $\Lambda_{\varnothing}(1) \neq 0$. With $q_{\varnothing} = 1$, the integral coefficient formulas for both tests apply at every coefficient place. Thus $a_p = F_{p,\varnothing}(1)$ is a positive algebraic integer. It is fixed before the auxiliary places and its valuation at a coefficient place may be positive. For a twist of exact support, put $D = F_{p,Q}(\eta^{\epsilon})/F_{p,\varnothing}(1)$. Both evaluations are algebraic by their finite character sums. If $z \geq 2$, the critical ratio is D. If $z = 1$, it is the positive real root of $X^2 - D/q_Q$, hence is algebraic as well. At every coefficient embedding with $\ell \nmid q_Q$ this gives

$$v_{\kappa}(R_{p,\epsilon}(\eta)) = \begin{cases} v_{\kappa}(D), & z \geq 2, \\ \frac{1}{2} v_{\kappa}(D), & z = 1. \end{cases} \tag{108}$$

38.2 Original Hecke systems and their integral models

In this subsection, we define the lift systems corresponding to the arrays of Section 38.1.

Fix the coefficient embedding above ℓ. Let $\mathcal{A}$ be the set of pairs (j, b), where $1 \leq j \leq m$ and b is an outside-level Hecke system in the projected space on W_m. Let $\mathcal{B}$ be the set of (j, p), where $p = (\nu, z)$ is main for j, so that

$$m < z \leq r, \qquad \nu_{\mathrm{tame}} = \chi_M^j, \qquad \mathrm{wt}_v(\nu) = -c_v, \qquad w_j(v) + r - t \leq c_v \leq w_j(v).$$

The spaces of principal-level forms are finite dimensional. For a fixed infinity type and fixed finite level the characters in $\mathcal{B}$ differ by characters of a finite norm-one class quotient. The bounds on c_v, z and j therefore make $\mathcal{B}$ finite.

For each lower system b in $\mathcal{A}$, choose once a nonzero form of system b in the image of a central idempotent for a selected finite type on W_m. This is possible because the sum of the idempotents for the selected types fixes the projected space, and each idempotent commutes with the off-level Hecke operators. The residue action on its algebraic form is the one obtained by reducing at the prescribed level, since the principal kernel fixes the form and the algebraic conversion commutes with right translation.

Choose the attachment R_b of a constituent of this form on W_m. Its local type parameters are those of the chosen constituent. For $(j, b) \in \mathcal{A}$ set

$$T_{j,b} = \theta_j R_b, \qquad \theta_j = \mathrm{av}_{\ell}(u \longmapsto \mu_j^{-1}(u/\bar{u})|u|^{(r-t)/2}),$$

where μ_j is the original descent character of lift j. The local–global comparison gives the scaled lower parameters of $T_{j,b}$. On every nonzero lift, Section 28.1 identifies the normalized upper trace with these parameters and the string χ_{cyc}^{-a}, $0 \leq a < t$. The normalization changes the raw polarization exponent $1 - r$ to $1 - t$, since $\theta_j \theta_j^c = \chi_{\mathrm{cyc}}^{r-t}$. Its jumps are (H1).

For $(j, p) \in \mathcal{B}$, use the constant system on the original norm-one line. The raw attachment is the trivial character, with jump 0 and parameter $X - 1$ at every distinguished place. Put

$$t_p = 2z - 1, \qquad T_{j,p} = \mathrm{av}_{\ell}(u \longmapsto \nu^{-1}(u/\bar{u})|u|^{1-z}).$$

This is the same normalization with lower rank 1 and upper rank $2z$. Its polarization exponent is $1 - t_p$ and its single jump at σ is $z - 1 - \varepsilon_{\sigma} c_v$. In particular

$$|\varepsilon_{\sigma}(z - 1 - \varepsilon_{\sigma} c_v) + w_j(v)| \leq t.$$

Indeed $0 \le w_j(v) - c_v \le t - r$ and $z - 1 < r$. As $w_j(v) > 2(r^8)^2 + t$, there are no small jumps, and the two conjugate jumps add to $2z - 2$. For every allowed twist the lift of the constant line form is nonzero by Proposition 30.1: its squared norm is a positive scalar times an absolutely convergent, nonvanishing Euler product at $z > m \ge 1$. Section 28.1 therefore gives the string of length t_p for its upper system.

The unscaled parameter $X - 1$ is the trivial character of each $\mathrm{GL}_1(k_w)$. It also fixes the upper lift in the normalized residue action. For a tuple g of residue scalars choose the indicated lower integral lift and the upper integral lift of its first block. Their simultaneous Weil action on the column multiplies it by $\prod_{w \in S_M} \nu_w(\det g_w)^{-1}$. In the split matrix coordinates, if i_0 is the original block inclusion, the two lifts act on the support by $X \mapsto k' X k^{-1}$, up to the fixed transpose convention.

Its reduction fixes i_0, since $\mathrm{diag}(g_w, 1) i_0 = i_0 g_w$. The integral matrix lattice is preserved and both determinants are units. The two splitting-character factors have opposite rank powers of the splitting character and leave $\nu_w(\det g_w)^{-1}$ at w. The same identity holds after the auxiliary twist, whose tame part is trivial on S_M. Changing the source variable in the theta integral shows that the upper translation multiplied by $\prod_{w \in S_M} \nu_w(\det g_w)$ acts on the lift as the lower translation acts on the source. The constant vector is fixed. Hence the mean over the $\mathrm{GL}_1(k_w)$ fixes the line, which is the required $X - 1$ idempotent. Its normalized inertial root instead names $(\chi^{j}_{M,w})^{-1}$ in the original parameter convention. It is nontrivial since $1 \le j \le m < q_2$.

At rank r, the same residue equivariance on the vector column gives the corresponding assertion for every component in a selected finite type of a lower eigenform on W_j. Decompose the projected space on W_j into simultaneous eigenspaces for the off-level split spherical Hecke operators and the finite central type projectors. These operators commute and are normal in the original Petersson inner product. Fix the old residue tuple and let b occur on W_j. For each other spherical character b' in the union of the two finite spectra choose a test $h_{b'}$ away from the old level with $b(h_{b'}) \ne b'(h_{b'})$. In the common spherical algebra put

$$h_b = \prod_{b' \ne b} \frac{h_{b'} - b'(h_{b'})\,1}{b(h_{b'}) - b'(h_{b'})},$$

with multiplication by convolution and empty product 1. The spherical operators commute with their adjoints, so the spectrum is semisimple and h_b projects onto the entire b-eigenspace. The trace identity of Section 9.4 at the fixed old tuple gives the same positive rank on W_j and W_m, including all multiplicities.

The hypotheses of that trace argument hold for both source spaces, since the same two finite places with primitive types and the prescribed common regular real parameter occur there. For every nonzero y in such a piece and every admissible ξ, Proposition 34.1 gives

$$\|u_{j,\xi^j} y\|^2_{\mathrm{class}} \ge \tfrac{1}{2} A_j c_Q(\xi) \|y\|^2 > 0.$$

This uses the first-lift estimate on the original single-lift space. Residue equivariance gives the specified upper type. For each lower constituent on W_j with system b, choose a constituent on the definite group W_m realizing its tuple and system by this trace comparison. Its rank-r base change is cuspidal. Indeed at an old split place with primitive type the local component is supercuspidal, and the split base change of [42, Corollaire 5.3] has this component in the chosen coordinate. Two or more global discrete blocks would give a subquotient of induction from a proper local parabolic. A single length-d' Speh block with $d' > 1$ would likewise give a quotient

of induction from d' blocks of rank $r/d' < r$ by [53, theorem of the introduction, pp. 605–606]. Supercuspidality excludes both possibilities. The general discrete-block construction of Section 28.2 also applies at the larger target ranks.

To compare two such definite constituents, choose one prime $\ell' \nmid M$ and one coefficient isomorphism there. Their raw semisimple attachments have equal traces at cofinite split Frobenius classes.

Prime ideals of E of absolute degree at least two have Dirichlet density zero, since for $s > 1$ their sum of inverse norms to the power s is at most $[E : \mathbf{Q}] \sum_p p^{-2s}$. At a degree-one prime away from ramification the underlying rational prime splits completely in the Galois field E. Thus the indicated traces agree on a dense set of Frobenius classes by Chebotarev. Brauer–Nesbitt identifies the two semisimple representations. At every $w \in S_M$ their Frobenius-semisimplified Weil–Deligne parameters agree by Section 28.3. Both use the twist by $|\det|^{(1-r)/2}$ of the local Langlands parameter and geometric reciprocity. Removing that common twist identifies their smooth local components. For the common tame generator of Section 28.3.1, the eigenvalues on the raw inertia representation are the images under the chosen coefficient isomorphism of $\{\jmath(\xi) : c_w(\xi) = 0\}$. Since $\jmath$ is injective and c_w is monic and squarefree, this multiset recovers c_w. Consequently the same system b gives one original tuple, also when the working coefficient prime ℓ divides M, since the comparison of these characteristic-zero types used $\ell' \nmid M$. Return to the prescribed coefficient embedding κ. For each nonzero lifted component of system b, original tuple c and allowed auxiliary twist η, put $\psi_\eta = \mathrm{av}_\ell(\eta^{-1})$. At every split Frobenius outside the two finite exceptional sets, the split polynomial in Section 28.1.1 uses the eigenvalues $b_{w,i}$ of the chosen constituent on W_m. Taking its linear coefficient and using density as above gives

$$\operatorname{tr} \widetilde{R}_{\mathrm{lift}}(g) = \psi_\eta(g) \operatorname{tr} T_{j,b}(g) + \sum_{a=0}^{t-1} \chi_{\mathrm{cyc}}(g)^{-a} \qquad (g \in G_E).$$

Semisimplicity identifies this normalized attachment with $\psi_\eta \otimes T_{j,b} \oplus \bigoplus_{a=0}^{t-1} \overline{\mathbf{Q}}_\ell(-a)$ over $\overline{\mathbf{Q}}_\ell$. The finite twist has jump zero and is unramified at the old places. Thus its main summand has the jumps in (H1) and the same old inertial parameters there, read on potentially crystalline periods at the places over ℓ. For $\eta = \xi_\kappa^j$ the scalar is $\psi_\eta = \psi^j$. Residue equivariance places the component in its own upper type projector for c. The lifts of these components span the original projected lift image. Therefore the finite list supplies a main summand for every generator used in clearing on the original lift image.

Integral models and the full matrix span. Take $K_\ell/\mathbf{Q}_\ell$ to be the normal closure of the finite compositum for the original matrices, prescribed lattices, scalar values and local type eigenvalues. Apply Lemma 28.2 and extend the resulting stable lattices by scalars to this compositum.

Let L be any such invariant lattice over a ring O containing the relevant inertia eigenvalues. At rank r choose a primitive place above an old prime distinct from ℓ. The preceding primitive-parameter calculation gives r mutually incongruent eigenvalues for the specified tame generator s. Their Lagrange idempotents belong to the O-span of the group image and split L into free lines of rank one.

An arithmetic Frobenius ϕ at this place permutes the lines in one cycle and has unit entries between the lines in integral bases, since its inverse preserves the lattice. Thus the products of the idempotents and powers of ϕ give all matrix units up to units. The span is an algebra and is $\mathrm{End}_O(L)$. The operator ϕ^r acts on every line by the same product of the r unit multipliers, with cyclotomic value p_i^r.

At rank one the full span follows from the identity and the same Frobenius is itself scalar, with cyclotomic value p_i. After reduction over any extension of the residue field, the matrix units still span. Consequently every lattice in either rank has absolutely irreducible reduction. A scalar unit-valued twist preserves the span, since each matrix is multiplied by a unit. In particular, after passing to any common finite coefficient DVR O', every stable lattice in this twisted representation is homothetic to the selected one L'. Scale the other lattice L'' so that $L'' \subseteq L'$ but $L'' \not\subseteq \varpi' L'$. Its image in $L'/\varpi' L'$ is a nonzero invariant subspace, hence is the whole reduction. Nakayama's lemma gives $L'' = L'$. This scalar identification preserves the cocycle equations and the normalized local conditions.

The old type calculation bounds the inertial determinants at every original place prime to ℓ, with the same chosen tame generator at a distinguished place and its conjugate inverse. In each summand one may choose the scalar inertia element supplied by that calculation, with scalar ζ satisfying $\zeta^{h\ell^a} - 1 \in O^\times$ for $a \geq 0$, $1 \leq h \leq 4\,\mathrm{rank}(L) + 2$. This element is chosen for the summand and need not be the common reference tame generator g_0. The twists are unramified at the scalar inertia place, so on the twisted lattice the same element acts by ζ. The unit $\zeta - 1$ gives the normalization $z(\gamma) = 0$. Adding the coboundary of $-(\zeta - 1)^{-1} z(\gamma)$ imposes this condition uniquely.

Use one common g_0 and torsion cutoff for $\mathcal{A} \amalg \mathcal{B}$. The joint choice applies every power of the selected character to *every* original summand in this union. The ramified critical pairs and their signs give the separate norm tests. The character is trivial on the full decomposition groups above ℓ, so the selected local lattice is the scalar extension of the original one there.

For a selected twisted lattice L_0 over O_0, the normalized divisible-coefficient Selmer length after any finite coefficient extension is bounded by its value over O_0, by (84). After choosing the torsion tuple and the raised systems, this applies to the common normal closure of Section 31.3.

38.3 The common specialization and its valuations

In this subsection, we use Proposition 28.3 for the signed systems of Section 38.1.

Let $\mathcal{P}_{\mathrm{ram}}$ be the set of ramified critical pairs. The norm tests include each pair with both signs. For the datum fixed at x, their number $2|\mathcal{P}_{\mathrm{ram}}|$ is at most

$$2t(4b)^f |\mathcal{C}_\varnothing| \prod_{w \in S_M} (q_w - 1).$$

Indeed $1 \leq z \leq t$, each integer c_v lies in $[z, 4b]$, and each old residue character is on $k_w^\times$. Two characters with the same infinity weights and residue characters differ by a character of the finite norm-one class group $\mathcal{C}_\varnothing$ of Section 38.1. Thus the family is fixed before the auxiliary level is chosen. In the common specialization there is one chosen Selmer minor for each member of $\mathcal{A} \amalg \mathcal{B}$ and one group-algebra polynomial for each signed norm test. Counting these chosen polynomials at each power $1 \leq h \leq m$ gives

$$m(|\mathcal{A}| + |\mathcal{B}| + 2|\mathcal{P}_{\mathrm{ram}}|)$$

evaluations with multiplicity.

Choose $p_i \in \{p_0, p_1\}$ distinct from ℓ, possible because the two old rational primes are distinct. For the common reference choose one $w_* \in S_M$ over p_i and its designated tame generator g_0. Write $q = \mathrm{N}w_*$, and use the common multiple D of the degrees of the original type factors at w_* in Section 28.3.1. The element $\omega_D(g_0)$ generates $\mathbb{F}_{q^D}^\times$, so its norm $\omega_1(g_0)$ generates $\mathbb{F}_q^\times$. The

old-type determinant bound applies to every original summand at both g_0 and $(g_0^c)^{-1}$ by that same section. A ramified critical pair has nontrivial residue character at every place in S_M. Its inverse has the same property. For each signed system its unit-valued character sends g_0 to a nontrivial root of unity ζ of order $h \mid q_{w_*} - 1$. The identity

$$h = (1-\zeta)\sum_{i=0}^{h-1}(h-1-i)\zeta^i$$

gives $v_\kappa(\zeta - 1) \le v_\ell(q_{w_*} - 1)$. Thus the first-prime bound is the augmentation valuation plus at most $4v_\ell(q_{w_*} - 1)$, as there are at most two Euler exponents.

Fix the norm systems, the coefficient ring for the original Selmer lattices and avatar congruences, the Selmer truncation, and the neighborhood of g_0 before choosing the auxiliary exponent. The uniform bound on the number of auxiliary places permits this exponent to be chosen before the actual auxiliary primes. After selecting that finite prime set, the polynomials for specialization have integral coefficients in $\overline{\mathbb{Q}}_\ell$. They and the finitely many test points lie in a finite extension L of the fraction field K_0 of the fixed Selmer coefficient DVR. The bound ℓ^H in Proposition 28.3 precedes the auxiliary level $\ell^{N_{\mathrm{tor}}}$, where N_{tor} denotes the exponent called N in that proposition. Thus all possible torsion coordinates already belong to $K_0(\mu_{\ell^H})$. After choosing $Q = J$, one may evaluate all the Selmer minors and all signed norm polynomials in $L(\mu_{\ell^H})$, with the same normalized valuation. The witness points for the different polynomials may differ. Lemma 26.1 replaces them by one tuple for all rows, both signs and all powers $1 \le j \le m$. Its tolerance is $2^{-\ell-1}$; the separate witness-point errors are at most $s_0 a_0 \le 2^{-\ell-1}$, giving $2^{-\ell}$ in total. For example, at $\ell = 2$ take $a_0 = 1/(8s_0)$ and an integer

$$N_{\mathrm{tor}} > \max\{H, 8s_0M, \max_i M_i', \max_i(8s_0\beta_i^{\mathrm{norm}} + s_0)\},$$

with the last two maxima over the norm tests. Each selected coordinate of order 2^h has $2^{1-h} \le a_0/m$. Consequently $v_2(\zeta_w^j - 1) = 2^{1-h+v_2(j)} \le a_0$ for every $1 \le j \le m$. For the triangle and turn factors of Section 38.6, the auxiliary powers are respectively $j - i$ and $j + 1$. Their old tame powers are $j-i$ and $i+j+2$, between 1 and $2m < q_2$, and their weight bounds include the required critical points. The same list contains $p_{\mathrm{aux}} = (\chi_M \upsilon_1^{-2}, 2)$ from Section 37.2.2. Its old unit character at each $w \in S_M$ is the character of exact order q_2 from Section 4.4. Its infinity weights are -2, and its new twisting power is the same j as for the main pair.

The forbidden set consists of the places over ℓ, p_0, p_1 and the rational level prime N of Proposition 4.2. This prime is fixed before the exponent of the auxiliary level is chosen. The resulting degree-one support Q is disjoint from its conjugate and avoids this set. The norm-one quotient law identifies the selected product of characters with the inverse avatar of one torus character ξ_κ. At $w \in Q$, if ζ_w has order ℓ^h, then

$$\frac{1}{\varphi(\ell^h)} = v_\ell(\zeta_w - 1) \le \frac{a_0}{m}.$$

Since $a_0 < 1$, we have $\ell^h > m$. For $1 \le j \le m$, $\zeta_w^j \ne 1$ and $v_\ell(\zeta_w^j - 1) = \ell^{v_\ell(j)} v_\ell(\zeta_w - 1) \le a_0$. The other tame characters are trivial on inertia at w. Thus each power has exact support Q on the chosen side and $\bar{Q}$ on the conjugate side.

At the rational prime d the cyclotomic projection of the decomposition group is generated by $d \bmod q_1$. It omits -1 by Lemma 4.1 after reduction modulo q_0, so every place of F_0 over d splits in E. The residue degree over $\mathbb{Q}$ there is at least two since $d \bmod q_0$ has order two.

The prime N splits completely by Proposition 4.2. The places over d, N therefore lie outside $S_M \cup \bar{S}_M \cup Q \cup \bar{Q}$. The same separation holds at each rational prime $p \mid q_0$. Indeed p splits in K, so every place above it splits in E/F_0. The absolute residue degree in E is even at 2 since $\zeta_3 \in E$. At odd $p \mid q_0$ it is a multiple of $\mathrm{ord}_{2^s}(p) > 1$, since $\zeta_{2^s} \in E$ and $1 < p < 2^s$ for all sufficiently large x. Thus no such place belongs to the degree-one old or auxiliary sets. The Euler products retain the two split factors at every place over $q_0 dN$, with the actual unramified character values on uniformizers. Their local conductor exponents are zero in both characters. Consequently only the places in $Q \cup \bar{Q}$ enter the conductor quotient q_Q^2 in (91). This calculation uses the complex Hecke characters also when $\ell = d$ or $\ell = N$.

Write $u_\ell = v_\ell(A_{\mathrm{aut}})$. Proposition 28.3 gives the following bounds in the trace comparison for each selected power $\eta = \xi_\kappa^j$, with ψ the inverse avatar of ξ_κ:

$$\begin{aligned} \mathrm{sel} &\le (4f+3) r u_\ell + 2^{-\ell} = e_\ell, \\ \mathrm{ram} &= \sum_{w \in Q} v_\ell(\psi^j(\gamma_w) - 1) \le |Q| a_0 \le 2^{-\ell-1}. \end{aligned}$$

The first line bounds the normalized Selmer lengths with divisible coefficients, since each main summand has rank at most r. The second sum uses one member of each auxiliary pair. The norm-one quotient $y \mapsto y/\bar{y}$ gives $\psi(cgc^{-1}) = \psi(g)^{-1}$. The conjugate generators have equal valuations since $v_\ell(u^{-1} - 1) = v_\ell(u - 1)$ for units. The character is trivial on each G_{E_v} over ℓ, so the local crystalline lattices there are obtained by scalar extension. At $Q \cup \bar{Q}$ the residue characteristic is different from ℓ, and the contribution is that of the tame character. Changing a tame generator raises its ℓ-power root to a prime-to-ℓ power and preserves the valuation. The coefficient-extension inequality above keeps the Selmer bound in normalized units. The norm tests independently give

$$v_\kappa F_{p,Q}((\xi_\kappa^j)^\epsilon) \le v_\kappa F_{p,\varnothing}(1) + 4 v_\ell(q_{w_*} - 1) + 2^{-\ell}.$$

Since $a_p \neq 0$, the right side is finite, and the numerator is nonzero at the selected specialization. Since $\ell \nmid q_Q$, the exact-support formula (108) gives

$$\begin{aligned} (1 + \mathbf{1}_{z=1}) v_\kappa R_{p,\epsilon}(\xi_\kappa^j) &= v_\kappa F_{p,Q}((\xi_\kappa^j)^\epsilon) - v_\kappa F_{p,\varnothing}(1) \\ &\le 4 v_\ell(q_{w_*} - 1) + 2^{-\ell}. \end{aligned}$$

Consequently

$$v_\kappa R_{p,\epsilon}(\xi_\kappa^j) \le e_\ell. \tag{RT1}$$

More precisely, put $d_z = 1 + \mathbf{1}_{z=1}$ and $b_\ell = 4 v_\ell(q_{w_*} - 1) + 2^{-\ell}$. At $\eta = \xi_\kappa^j$, integrality of $F_{p,Q}(\eta^\epsilon)$ above ℓ gives

$$-\frac{v_\kappa(a_p)}{d_z} \le v_\kappa R_{p,\epsilon}(\eta) \le \frac{b_\ell}{d_z}.$$

The same bounds hold at $\bar{\kappa}$ for this η, since $R_{p,\epsilon}(\eta)$ and a_p are real. For $z = 1$, including at $\ell = 2$, this gives

$$\max_{\lambda \in \{\kappa, \bar{\kappa}\}} \left| R_{p,\epsilon}(\eta)^{-1} \right|_\lambda \le \ell^{b_\ell/2}, \qquad |x|_\lambda = \ell^{-v_\lambda(x)}.$$

Here $q_{w_*} - 1 \mid A_{\mathrm{aut}}$, so $4 v_\ell(q_{w_*} - 1) \le (4f+3) r\, v_\ell(A_{\mathrm{aut}})$. In particular, at the same $\eta = \xi_\kappa^j$ the second block of Section 37.2.4 satisfies

$$\frac{A_2(\eta)}{A_2(1)} = \frac{F_{p_{\mathrm{aux}},Q}(\eta)}{F_{p_{\mathrm{aux}},\varnothing}(1)} = R_{p_{\mathrm{aux}},+}(\eta).$$

38.4 Integral clearing and projected pairings

In this subsection, we apply Proposition 32.1 to the models of Section 38.2.

The first-order comparison on the selected twisted summands gives the denominator-clearing operator on the projected image and the rank-one projected pairing estimate below. With its local defect F_ℓ, odd-class integer X_{odd}, original rank-one level class number h_1, and $Y_{\mathrm{tr}} = r^8$ from Section 29.2, put

$$\begin{aligned} C_\ell &= 4fF_\ell + (1+2t)e_\ell + tv_\ell(X_{\mathrm{odd}}) \\ &\quad + (8ft + 3 + (4t+1)Y_{\mathrm{tr}}^{22} + Y_{\mathrm{tr}})v_\ell(A_{\mathrm{aut}}), \\ B_\ell &= C_\ell + v_\ell(h_1), \qquad J_\ell = B_\ell \log \ell. \end{aligned}$$

On the lift image Z, the operator $C_G D_Z$ clears the orthogonal projections of integral tests perpendicular to all F_w. Its determinant valuation is at most $r' \dim(Z) C_\ell$, where r' is the source rank of this lift. This follows by summing the eigenvalue bound below. For the rank-r source, $\dim Z \le \dim V$. In rank one let Π project to the normalized line $\mathbb{C}u$ for $\eta = \xi_\kappa^j$. The two tests a, a' must be integral at every coefficient embedding over ℓ and perpendicular to the false lifts defined by the kernel of the full descent.

The following pairing and central-average calculation gives the line estimate

$$v_\kappa \langle \Pi a, \Pi a' \rangle \ge -B_\ell. \tag{RT2}$$

Integral pairings after the first-order comparison. Let D_Z be the denominator-clearing operator on the projected image obtained from the first-order coefficients. The original class pairing is the sum of the transported Weyl pairings with denominators the stabilizer orders. The transport and the integral Weyl evaluations use the same original bases. The left test is integral also at the conjugate embedding used for complex conjugation. Each stabilizer order divides C_G by the reduction argument in Section 30.1. Thus

$$v_\kappa \langle a, C_G D_Z \pi_Z a' \rangle \ge 0$$

for the original tests, with the right test perpendicular to the false lifts. The integer multiplication does not affect the integral image property. The valuation per eigenvalue is at most

$$\begin{aligned} &\mathrm{sel}_Z + 2t'\mathrm{ram} + 4fr'F_\ell + \mathrm{err}_\ell + u_\ell \\ &\qquad \le r'(4fF_\ell + (1+2t)e_\ell + \mathrm{err}_\ell + u_\ell). \end{aligned}$$

Here $\mathrm{sel}_Z, \mathrm{ram} \le e_\ell$ since the selected twist bounds each sel_i, and $1 \le r' \le r$, $t' \le t$. This is the allowance $r'C_\ell$ in the determinant on the projected image. The C_G for this pairing is in addition to the C_G^2 in the first-order coefficient.

Rank-one averaging on the original line. Let $T = U(1)$, let K_1 be the original chosen line level, and let s_c represent the h_1 classes of $T(F_0)\backslash T(\mathbf{A}_{F_0,f})/K_1$. Take the representatives integral at ℓ. To do this, write the inverse local component of each proposed representative as $a_v/\bar{a}_v$ by Hilbert 90, using a split pair of factors at a split place. Approximate the a_v at the places above ℓ by a single $a \in E^\times$. Replacing the representative by its product with $a/\bar{a}$ puts the local components in the open integral compact.

For the main pair (ν, z) and its twist η, let $\beta = (\nu\eta)\upsilon_1^z$ be the full target-leg character. For a norm-one scalar x let $T(x)$ act by right translation by xI_{2z}, multiplied by $\beta(x)$. This operator

is unitary, commutes with the type projector and spherical operators, and has adjoint $T(x^{-1})$. For the original line kernel $K(g)$ one has

$$T(x^{-1})K(g) = K(gx).$$

Indeed the simultaneous scalars $(x^{-1}, x^{-1}I_{2z})$ fix the Hom matrix variable. In the Weil action their multiplier is $\beta(x^{-1})^{-1}$, which cancels that of $T(x^{-1})$. The line stabilizers are trivial. With $s = V_0^{2z}J \neq 0$ as in Section 37.2, the raw and normalized lifts satisfy

$$u_{\mathrm{raw}} = \sum_{c} K(s_c), \qquad u = \phi_\eta = s\, u_{\mathrm{raw}}.$$

Translation permutes the classes, so $T(x)u_{\mathrm{raw}} = u_{\mathrm{raw}}$ and $T(x)u = u$.

Here $K(g) = K_X(g)$ is the raw kernel of the four-block test, with the original inclusion column at the old places, the character on the distinguished unit coordinate at Q, and the vacuum elsewhere. In particular the normalized descent to the full line form space is the same D_O as in Proposition 37.2. Indeed contraction with the polynomial of the one-row Weyl type and summation over the original J_Q-classes give

$$(\mathcal{D}y)(g) = (D_O y)(g) = \sum_{[h]_{J_Q}} \frac{\langle K_X(g,h), y(h)\rangle_{\mathrm{Weyl}}}{|\operatorname{Aut}(h, J_Q)|} = \langle K(g), y\rangle_{J_Q}.$$

Set $S = \sum_c T(s_c)$. Commutativity of the torus gives

$$\mathcal{D}(Sy)(g) = \left\langle \sum_c T(s_c^{-1})K(g), y \right\rangle = \langle u_{\mathrm{raw}}, y\rangle = \overline{s}^{-1}\langle u, y\rangle.$$

The equality of the kernel sum uses that gs_c runs over precisely the same line classes. For Proposition 32.1 take $Z = \mathbb{C}u$ and $e_0 = Q = e_{X-1}$. Indeed u is a nonzero multiple of the lift of the original constant line form. Its Hecke system x_0 and its $X-1$ type are those of Section 38.2, and $\pi_Z = \Pi$. At every split off-level place w, the descent identity and the commutation of S with Q and $K_w(X_w)$ give

$$S\big(\operatorname{ran} Q \cap u^{\perp} \cap \ker K_w(X_w)\big) \subseteq \operatorname{ran} Q \cap \ker \mathcal{D} \cap \ker K_w(X_w).$$

The original test a' is perpendicular to the right-hand side. Hence $S^*a' \perp F_w$ for this line at every such w. Also S^*a' is integral at every embedding above ℓ: each scalar translation by s_c^{-1} preserves the integral local Weyl lattice norm. For a finite norm-one idele x integral above ℓ, its class in the original quotient is killed by h_1. Write $x^{h_1} = \gamma k$ with $\gamma \in T(F_0)$ and $k \in K_1$. Then γ is an ℓ-unit and automorphy gives

$$\beta_f(x)^{h_1} = \beta_\infty(\gamma)^{-1}\beta_f(k).$$

The first factor is a monomial in conjugates of an ℓ-unit. The second is a root of unity, since β_f is smooth on the compact K_1. Thus $v_\kappa(\beta_f(x)) = 0$ at every coefficient embedding above ℓ. Since $Su = h_1 u$, orthogonal projection gives $\Pi S^*a' = h_1 \Pi a'$. Let $s_{\mathrm{tr}} = AD_I$ be the Hecke coefficient of Section 32.3 for the system x_0 of u. Here $Qu = u$, $c = C_G = A_{\mathrm{aut}}$, and the eigenvalue of $C_G c^2 s_{\mathrm{tr}} Q$ on $\mathbf{C}u$ is

$$d_u = C_G c^2 s_{\mathrm{tr}}(x_0) \neq 0, \qquad v_\kappa(d_u) \le u_\ell + \mathrm{sel} + 2t'\mathrm{ram} + 4fF_\ell + \mathrm{err}_\ell \le C_\ell.$$

Proposition 32.1 applies to the original integral test $S^{*}a'$ just verified. It gives $c^{2}s_{\mathrm{tr}}Q\Pi S^{*}a' \in L_{\mathrm{orig}}$ and the valuation bound with $r' = 1$ before the extra $u_{\ell} = v_{\ell}(C_G)$. In each class the two transports satisfy $(T^{\#})^{\mathrm{t}}G_{\mathrm{W}}T = G_{\mathrm{W}}$ for the integral Weyl Gram matrix, as in Section 23. Pairing this integral image at κ with a, integral at $\bar{\kappa}$, and clearing the class stabilizers by C_G gives

$$d_u h_1 \langle \Pi a, \Pi a' \rangle = \langle a, C_G c^{2} s_{\mathrm{tr}} Q \Pi S^{*} a' \rangle \in \mathcal{O}_{\kappa}.$$

The scalar acts in the second, linear variable. Taking valuations gives (RT2).

For the class-number factor, put $T(\widehat{\mathcal{O}}_E) = \prod_{v<\infty} T(\mathcal{O}_{E_v})$, with the analogous interpretation at split places. The norm-one ideal map from classes at this maximal compact into $\mathrm{Cl}(E)$ has kernel of order at most 2^{f}. For a principal ideal (a) in the kernel, $a\bar{a}$ is a unit of F_0. Changing a by a unit changes this unit by a norm, and if the resulting unit class is trivial there is a norm-one generator.

Thus the kernel injects into $\mathcal{O}_{F_0}^{\times}/N_{E/F_0}(\mathcal{O}_E^{\times})$. The norm subgroup contains $(\mathcal{O}_{F_0}^{\times})^{2}$. The totally real field F_0 has degree f, so its unit group has rank $f-1$ and torsion $\{\pm 1\}$. Its quotient by squares has size 2^{f}. The index from this maximal compact to K_1 is at most $(MN)^{2f}$, since K_1 contains the principal level MN, and reduction modulo $MN\mathcal{O}_E$ has at most that many values. Hence

$$h_1 \le 2^{f} h_E (MN)^{2f}.$$

By the Minkowski bound [48, Thm. 4.3], each class has an integral representative of norm at most $D = (4/\pi)^{f}(2f)!\sqrt{|\Delta_E|}/(2f)^{2f}$. Thus

$$h_E \le \sum_{\mathrm{N}\mathfrak{a} \le D} 1 \le D^{2}\zeta_E(2) \le 2^{2f}|\Delta_E|\zeta(2)^{2f}.$$

All logarithms on the right-hand side of the bound for h_1 are polynomially bounded in x by the parameter bounds in Section 4.

For every finite set U of rational coefficient primes, factorization gives $\sum_{\ell\in U} v_{\ell}(H)\log \ell \le \log H$ for each positive integer H. Also $C_G = A_{\mathrm{aut}}$ and $\sum_{\ell\in U} 2^{-\ell}\log\ell \le \sum_{n\ge 1} n2^{-n} = 2$. The explicit defect bound of Section 29.1.2 and the expressions for C_{ℓ}, B_{ℓ} give

$$\begin{aligned}\sum_{\ell\in U} J_{\ell} \le{}& 108\cdot 10^{6}\frac{fm}{k}\, b\log b + (1+2t)\big((4f+3)r\log A_{\mathrm{aut}} + 2\big)\\ &+ t\log X_{\mathrm{odd}} + \log h_1\\ &+ \big(8ft + 3 + (4t+1)Y_{\mathrm{tr}}^{22} + Y_{\mathrm{tr}}\big)\log A_{\mathrm{aut}}.\end{aligned}$$

The first term divided by $J_{*} = fxb(\log b)^{3}/(mk)$ is at most $108\cdot 10^{6}m^{2}/(x(\log b)^{2}) \to 0$, since $m^{2} \le 4x$. All other terms are polynomially bounded in x, whereas J_{*} and x_1 each exceed every fixed power of x for fixed q_0, d, ϵ. Each allowance bounds the local term at every coefficient embedding over its rational prime.

For every finite set U of coefficient primes,

$$\sum_{\ell\in U} e_{\ell}\log\ell \le (4f+3)r\log A_{\mathrm{aut}} + 2 \le x_1, \qquad \sum_{\ell\in U} J_{\ell} \le J_{*} = \frac{fa(\log b)^{3}}{mk}. \tag{RT3}$$

Here $\log A_{\mathrm{aut}} \le (rt)^{2} f \log M$, since the original set contains one split half above each place over the two level primes. The allowances are independent of the coefficient embedding, and the twist ξ_{κ} may be chosen at each κ. The first-order comparison is applied at κ on that chosen twist and does not use the twist selected at $\bar{\kappa}$.

38.5 Reciprocal heights of the main critical values

In this subsection, we use the local estimates of Section 38.4 and the numerator test of Proposition 37.2.

For each ramified pair $p = (\nu, z)$, each sign ϵ, and $\eta = \xi_\kappa^j$, equations (RT1) and (RT3) give

$$v_\kappa(R_{p,\epsilon}(\eta)^{-1}) \geq -e_\ell, \qquad \mathrm{Height}((R_{p,\epsilon}(\xi_\kappa^j)^{-1})_\kappa) \leq x_1$$

with multiplier 1. This includes $z = 1$, where the valuation of the squared-period ratio is $2v_\kappa(R_{p,\epsilon}(\eta))$ because q_Q is an ℓ-adic unit.

For the stronger estimate fix a main pair for j, so $m < z \leq r$, and take the positive sign. At both conjugate embeddings use $\eta = \xi_\kappa^j$ and the normalized lift $u = \phi_\eta$ of Section 37.2. Summing the inverse unit residues in the class formula of Section 30.1 gives this same vector Φ_η, with vacuum and jet scalar $V_0^{2z}(\pi_M s_{\mathrm{jet}})^{\sum_\tau (c_\tau - z)}$. Use $u_1 = \phi_1$ at $Q = \varnothing$ and $\rho = R_{p_{\mathrm{aux}},+}(\eta) > 0$. The numerator estimate gives a fixed $c_0 \neq 0$ with $\log|\sigma c_0| \leq J_*$ for every complex automorphism and, for any coprime modulus and any coefficient embedding, an integral test satisfying

$$v_\kappa\langle a, u\rangle \leq v_\kappa(c_0\rho) + P_\ell, \qquad P_\ell = \sum_\tau S_\ell(\kappa_\tau, \tau). \tag{RT4}$$

Here S_ℓ is the local prime sum of Section 7. The test is integral at every coefficient place above this ℓ and has the false-lift orthogonality of (RT2). This is the four-block numerator test on the original rank-one lift.

Apply (RT4) at $\bar\kappa$ to the same η, u and call the resulting test a'. Using that ρ is real and conjugating this inequality gives, for $I = \langle a, u\rangle\langle u, a'\rangle \neq 0$,

$$v_\kappa(I) \leq v_\kappa(c_0\bar c_0) + 2v_\kappa(\rho) + 2P_\ell.$$

Apply (RT2) to these two tests at κ. Proposition 30.1 uses J_Q for u and the old vector fixer $J_\varnothing$ for u_1. At the trivial twist $Q = \varnothing$ and $q_\varnothing = 1$. The vacuum and jet scales agree in both norms. Thus

$$\frac{\|u\|^2}{\|u_1\|^2} = q_Q^{2z-1} R_{p,+}(\eta), \qquad \langle \Pi a, \Pi a'\rangle = \frac{I}{\|u\|^2}.$$

Both norms are nonzero by the positive Euler products. It follows that

$$R_{p,+}(\eta)^{-1} = q_Q^{2z-1}\|u_1\|^2 \frac{\langle \Pi a, \Pi a'\rangle}{I}. \tag{RT5}$$

For algebraic u, one has $v_\kappa(\bar u) = v_{\bar\kappa}(u)$. The numerator test at $\bar\kappa$ uses the same η, whose modulus satisfies $\ell \nmid q_Q$. The projection estimate (RT2) is at κ, with the twist selected there.

For the untwisted term use the integral algebraic norm $I_{\mathrm{ref}} \neq 0$ and the fixed empty reference pair of rank $2z$ from Section 37.2.5. Its multiplier $H = I_{\mathrm{ref}}/\|u_1\|^2$ satisfies

$$H\|u_1\|^2 \text{ integral at every coefficient embedding}, \qquad \log|\sigma H| \leq J_* - \sum_\tau J(\kappa_\tau). \tag{RT6}$$

Put $h = Hc_0\bar c_0$. Since the bound for c_0 applies at σ and $\sigma \circ \mathrm{conj}$,

$$\log|\sigma h| \leq 3J_* - \sum_\tau J(\kappa_\tau).$$

Using $H\|u_1\|^2 = I_{\mathrm{ref}}$, $v_\kappa(q_Q) = 0$ and the bound on $v_\kappa(I)$ in (RT5) gives

$$\begin{aligned} v_\kappa(hR_{p,+}(\eta)^{-1}) &\geq v_\kappa(I_{\mathrm{ref}}) + v_\kappa\langle \Pi a, \Pi a'\rangle - 2v_\kappa(\rho) - 2P_\ell \\ &\geq -B_\ell - 2e_\ell - 2P_\ell \end{aligned}$$

by (RT1), (RT2) and (RT6). Also

$$2\sum_{\ell\in U} P_\ell \log \ell \leq \sum_\tau S(\kappa_\tau, \tau).$$

Indeed $2S_\ell(\kappa_\tau,\tau)\log\ell$ is the sum of the terms of $S(\kappa_\tau,\tau)$ at $n = \ell^e$. Terms with $\ell^e > a$ vanish, since $\kappa_\tau/\ell^e < 1$ whereas every weight is at least one. All terms are nonnegative. By (RT3), the fixed multiplier has total allowance at most

$$3J_* - \sum_\tau J(\kappa_\tau) + J_* + 2x_1 + \sum_\tau S(\kappa_\tau,\tau) \leq 6J_* - \sum_\tau \big(J(\kappa_\tau) - S(\kappa_\tau,\tau)\big),$$

since $x_1 \leq J_*$.

Proposition 38.1 (Reciprocal height of the main critical ratio)**.** *Let $(\xi_\kappa)_\kappa$ be the chosen family. For every main pair p and its lift index j,*

$$\mathrm{Height}\big((R_{p,+}(\xi_\kappa^j)^{-1})_\kappa\big) \leq 6J_* - d_{\mathrm{dir}}, \qquad d_{\mathrm{dir}} = \sum_\tau \big(J(\kappa_\tau) - S(\kappa_\tau,\tau)\big).$$

38.6 Height of the exchange product

In this subsection, we use the same auxiliary twists as in Proposition 38.1 for the exchange factors.

Use the untwisted lift character $\mu_j^0 = \chi_0(\upsilon_1^{-\Delta}\chi_M)^j$. Its weight at τ is $-w_j(\tau)$, where $w_j(\tau) = \kappa_{j,\tau} = \kappa_j(\tau) = \kappa_\tau + (j + \frac{1}{2})\Delta$. The following table lists the characters and endpoints in reverse order of the measured exchanges of Section 39.4.

character	upper point	sign
$\mu_j^0\upsilon^{j+(t-11m-1)/2}$	$7(m/2) + k/2$	$+$
$\upsilon^{-m}\chi_M^j$	m	$+$
$\upsilon^{-m/2}\chi_M^{-j}$	$m/2$	$-$

In this table $\upsilon = \upsilon_1 = \upsilon_E|_{T_1(\mathbb{A}_{F_0})}$. Lowering the point keeps the old character and tame support. For the interval with upper endpoint $z_* = (7m+k)/2 = (r+6m+1)/2$, put $n_j = j + (t-11m-1)/2$. For large x one has $r \geq 6m+1$; hence $z_* \leq r$ and $0 \leq n_j \leq t-r$, since $t = 5r$ and $1 \leq j \leq m$. At the $k/2$ integers $7m/2 < u \leq z_*$ one therefore has $m < u \leq r$, old tame character χ_M^j, and

$$w_j(v) + r - t \leq c_v = w_j(v) - n_j \leq w_j(v), \qquad c_v \geq r \geq u.$$

Here $w_j(v) > 2(r^8)^2 + t$, as in Section 38.2. Thus Proposition 38.1 applies to these points. For the remaining $7m/2 + m + m/2 = 5m$ pairs the original tame powers are $\pm j$ and their weights are at least their respective upper points. Equation (RT1) bounds each reciprocal height by x_1,

using multiplier 1 and (RT3). Write $(\chi_i, z_i, \epsilon_i)$ for the three rows of the table, with $\epsilon_i \in \{1, -1\}$. The three exchange identities of Section 39.4 give

$$\frac{H_j(\eta)}{H_j(1)} = q_Q^{\sum_i z_i^2} \left(\prod_{i=1}^{3} \prod_{1 \le u \le z_i} R_{(\chi_i,u),\epsilon_i}(\eta) \right)^2.$$

Write $M_\chi^S(z)$ for the factor in Section 36.1 with split support S. The character on row i is $(\chi_i \eta^{\epsilon_i})^{-1}$, whose base change is $((\chi_i \eta^{\epsilon_i})^{\mathrm{bc}})^{-1}$. Thus the quotient for this row is

$$\frac{M^{S_M \cup Q}_{(\chi_i \eta^{\epsilon_i})^{-1}}(z_i)}{M^{S_M}_{\chi_i^{-1}}(z_i)} = q_Q^{z_i^2} \prod_{u=1}^{z_i} R_{(\chi_i,u),\epsilon_i}(\eta)^2.$$

At a new pair the numerator omits both Euler factors and the untwisted denominator retains both. The conductor counts the chosen half once, with exponent $\sum_{u=1}^{z_i}(2u-1) = z_i^2$ by (3). The old conductor factors cancel. For $p_{i,u} = (\chi_i, u)$, set

$$U_{i,u} = \frac{F_{p_{i,u},Q}(\eta^{\epsilon_i})}{F_{p_{i,u},\varnothing}(1)}.$$

For exact support Q, Section 38.1 gives $U_{i,u} \in \overline{\mathbf{Q}}^\times$, $R_{p_{i,1},\epsilon_i}(\eta)^2 = U_{i,1}/q_Q$, and $R_{p_{i,u},\epsilon_i}(\eta) = U_{i,u}$ for $u \ge 2$. Each of the three positive endpoints includes $u = 1$, so

$$\frac{H_j(\eta)}{H_j(1)} = q_Q^{z_*^2 + m^2 + (m/2)^2 - 3} \prod_{i=1}^{3} \left(U_{i,1} \prod_{u=2}^{z_i} U_{i,u}^2 \right) \in \overline{\mathbf{Q}}^\times.$$

For each κ the chosen Q avoids ℓ, so the reciprocal conductor power has valuation zero there and requires only multiplier 1. Put $A_j = \{u \in \mathbf{Z} : 7m/2 < u \le z_*\}$. For $u \in A_j$ let $X_{j,u}$ be the original untwisted rank-$2u$ datum for the third measured exchange, the first row of the table. Use its numerator constant $c_{0,j,u}$ from Proposition 37.2 and put

$$h_{j,u} = \frac{C_G M^{8fu} \|\phi_{(v_1^{-u},u)}\|^2}{\|\phi_{X_{j,u}}\|^2}, \qquad h_j = \prod_{u \in A_j} (h_{j,u} c_{0,j,u} \bar{c}_{0,j,u})^2.$$

Both norms use the same old level. The reference in the numerator has row degree zero and denominator M^{4fu}, as in Section 37.2.5. Thus h_j is fixed before κ and Q. At $\eta = \xi_\kappa^j$, the $k/2$ main factors have sign $+$. The remaining $5m$ factors of either sign use multiplier 1. Using (RT1) and (RT5)–(RT6) before summing gives

$$v_\kappa\left(h_j \frac{H_j(1)}{H_j(\eta)} \right) \ge -k(B_\ell + 2e_\ell + 2P_\ell) - 10me_\ell.$$

Since $\log|\sigma h_j| \le k(3J_* - \sum_\tau J(\kappa_\tau))$, the finite-prime sums above give

$$\mathrm{Height}\left(\left(\frac{H_j(\xi_\kappa^j)}{H_j(1)} \right)^{-1} \right) \le 10mx_1 + k(6J_* - d_{\mathrm{dir}}).$$

For $1 \le i < j \le m$ and $0 \le a, b' < m$, respectively, the triangle and turn products on the same family are

$$Q_{ij}(\xi_\kappa) = \prod_{u=1}^{t} R_{(\mu_j^0/\mu_i^0,u),+}(\xi_\kappa^{j-i}),$$
$$T_{ab'}(\xi_\kappa) = \prod_{u=1}^{t} R_{(\mu_{a+1}^0\mu_{b'+1}^0,u),+}(\xi_\kappa^{b'+1}).$$

The local negative weights are $(j-i)\Delta$ and $w_{a+1}(v) + w_{b'+1}(v)$. Both lie in $[t, 4b]$, since $t \le \Delta$, $\kappa_\tau \le b$, and $(m+1)\Delta \le b$ for sufficiently large x. The old tame powers $j-i$ and $a+b'+2$ lie strictly between zero and $q_2 > 2m$, so the old support is exactly S_M. The powers $j-i, b'+1$ lie in $[1, m]$, hence have the same exact new support Q. By (RT1), each of $Q_{ij}(\xi_\kappa)^{-1}$ and $T_{ab'}(\xi_\kappa)^{-1}$ has valuation at least $-te_\ell$. Half-integral powers of q_Q in these formulas also have valuation zero, since their squares are integral powers of q_Q.

39 Determinant factors and the gluing operator

In this section, we track the determinant and measure factors in Section 36 through three period identities. The resulting adjoint is the common period used in Section 41.

39.1 Products of determinants in the period comparison

In this subsection, we check how the product of archimedean determinants transforms in the exchange of Section 36.1. The resulting coefficient is used on the three pairs in Section 39.4.

Let the kept real space have positive and negative dimensions m, ℓ, the old small space dimensions p, ℓ, and the definite target dimension a. The ordered positive basis satisfies $m = p + a$. At each infinite place τ let c_τ be the number of stacked determinants. In the large and small Fock spaces use the polynomials

$$P(\eta) = \prod_{\tau,i} \det[\eta_{\tau i}; Z_\tau], \qquad Q(\eta) = \prod_{\tau} \det(Z_\tau^-)^{c_\tau + 2s_0} \prod_{\tau,i} \det(\eta_{\tau i} Y_\tau), \qquad 2s_0 = \dim W - 2\ell.$$

Here Z_τ is the a-by-m block of positive columns, Y_τ is the m-by-p positive block of the small vector, and Z_τ^- is its ℓ-by-ℓ negative block. Each $\eta_{\tau i}$ is a p-by-m coefficient matrix. The bracket stacks its rows above the a rows of Z_τ in the ordered positive basis. Put $\eta_0 = [I_p \ 0]$ in that basis, at every place and for every index.

Apply the measured partner comparison with the fixed small vector $Q(\eta_0)$. Let v be its large real vector. We show that v can be replaced by $P(\eta)$ for every η, with one scalar κ multiplying the new large polynomial. At this stage κ may be zero. Its nonvanishing in the applications with nonzero projected source will follow from injectivity of the common period. Write α, β for the two determinant characters of the large kernel. For the kernels of (W, I_a) and (V', W) the pairs of characters are (α, β) and $(\alpha/\beta, \alpha)$, respectively. At (g, h) and (x, g), both sides are multiplied by $\beta(\det g)^{-1}$ relative to the choice $\beta = 1$. We cancel this common factor throughout the compact group calculation. For fixed matched finite data and g, the large period is a continuous functional L_g in a real Fock moment seminorm.

The matched support in the first coordinate is primitive, so the kernel estimate at exponent zero in the integrated variable bounds its absolute integral by $C_g p_N(v)$. The functionals L_g are invariant under the definite target real group by right translation in the original quotient.

The standard covariance under right translation in the kept group gives $L_g(\omega(k,u)v) = L_{gk}(v)$. Write S_g for the small period. Proposition 36.1 gives $L_{gk}(v) = c_\nu S_{gk}(Q(\eta_0))$, with $c_\nu \neq 0$.

Let $t(z)$ on the positive kept basis be diagonal with z^{-2} on the p coordinates and z^{-1} on the a coordinates, with $|z| = 1$. Use its identity extension on the negative block, and put $u(z) = z^{-1}I_a$ in the target. In the real representation the target determinant exponent is $\ell - c_\tau$. The vacuum contributes $\det(u_\tau)^{-\ell}$, hence their product contributes $\det(u_\tau)^{-c_\tau}$. For the operator

$$v \longmapsto z^{-a\sum_\tau c_\tau}\omega(t(z),u(z))v$$

these scalar factors cancel. On polynomial vectors the substitution scales all variables outside the a columns by z and fixes the variables in those a columns. Indeed on the positive columns the substitutions use $u(z)$ and $t(z)^{-1}$; on the negative columns the signed Fock coordinate uses the conjugate factor of $u(z)$. Thus averaging over z sets the variables outside the last a positive columns to zero. At each place write $Z_{\tau,0}$ for the remaining a-by-a block.

Under the definite target average this square polynomial is projected to its left determinant weight c_τ. This weight space consists of the multiples of $P(\eta_0) = \prod_\tau \det(Z_{\tau,0})^{c_\tau}$. Indeed, a polynomial with that left covariance under unitary matrices has it under the complexification $\prod_\tau \mathrm{GL}_a(\mathbb{C})$, by differentiation and complex linearity. On invertible square matrices its value is determined by its value at the identity, and polynomial continuation gives the statement.

Average first over the circle and then over the compact target. These bounded operators commute on Fock space. The kept action commutes with the target action, and $u(z) = z^{-1}I_a$ is central in the target. Polynomials are dense, and the resulting line is closed. Hence for the fixed v these two averages give $\kappa P(\eta_0)$ for one scalar κ, independent of all the finite data, the modulus, the character and g. The averages can also be evaluated inside the period.

To obtain polynomial approximation in the moments used by L_g, write $p_k(f;A)$ for the integral of $|f(z)|(1+\|z\|)^k e^{-c\|z\|^2}$ over $A \subset \mathbb{C}^{d_\mathrm{F}}$ in the fixed Fock coordinates, and $p_k(f) = p_k(f;\mathbb{C}^{d_\mathrm{F}})$. For $0 < t_0 \leq t < 1$, the dilation $D_t f(z) = f(tz)$ satisfies

$$p_k(D_t f; \|z\| > R) \leq t_0^{-2d_\mathrm{F}-k} p_k(f; \|z\| > t_0 R).$$

This is the substitution $w = tz$, with the Gaussian decreasing under the substitution. The tail bound and local uniform convergence give $p_k(D_t f - f) \to 0$ as $t \uparrow 1$. Circle extraction gives $p_k(f_j) \leq p_k(f)$ for each homogeneous term f_j. Thus, for fixed $0 < t < 1$,

$$p_k\left(D_t f - \sum_{j=0}^{J} t^j f_j\right) \leq \sum_{j>J} t^j p_k(f_j) \leq \frac{t^{J+1}}{1-t}\, p_k(f).$$

Orthogonality of the degrees gives the two limits in Fock norm as well. The compact averages are bounded in the radial moments because their substitutions are unitary and their character factors have modulus one. Applying the bound for L_g passes the polynomial identities to the period of v on the original finite level.

For $k = (k_\tau) \in \prod_\tau U(m)$ put $\eta_{k,\tau} = \eta_0 k_\tau^{\mathsf{t}}$ and $d(k) = \prod_\tau \det(k_\tau)^{c_\tau}$. On the large side the kept positive block acts by $Z \mapsto Z(k_\tau^{-1})^{\mathsf{t}}$, and

$$\det\begin{bmatrix} \eta_0 \\ Z(k_\tau^{-1})^{\mathsf{t}} \end{bmatrix} = (\det k_\tau)^{-1} \det\begin{bmatrix} \eta_0 k_\tau^{\mathsf{t}} \\ Z \end{bmatrix}.$$

On the small side $Y \mapsto k_\tau^{\mathrm{t}} Y$, the negative block is fixed, and the vacuum and splitting contribute $d(k)^{-1}$. Thus the two real actions on the indicated Fock vectors satisfy

$$\omega_{\mathrm{large}}(k,1)P(\eta_0) = d(k)^{-1}P(\eta_k), \qquad \omega_{\mathrm{small}}(1,k)Q(\eta_0) = d(k)^{-1}Q(\eta_k).$$

Apply the measured comparison at the real point $kt(z)$ and use these covariances. The small basis there is $z^{-2}\eta_k$, so homogeneity contributes $z^{-2p\sum c_\tau}$. The factor $\det(t(z))^{-c_\tau}$ contributes $z^{(2p+a)\sum c_\tau}$, which cancels the preceding power and the additional $z^{-a\sum c_\tau}$. The determinant powers of k_τ cancel between the covariances. Thus the projected identity is

$$\kappa L_g(P(\eta_k)) = c_\nu S_g(Q(\eta_k)).$$

This equality uses no division by κ.

It remains to justify the passage from η_k to arbitrary families of coefficient matrices. At a fixed place consider the polynomial space in m-by-p matrices Y with $F(Yu) = \det(u)^c F(Y)$. For $p = 0$ this is the constant line. For $p > 0$, it is a finite-dimensional representation of GL_m under left substitutions and of $U(m)$ by restriction. It is irreducible. Indeed, in every irreducible summand of this polynomial representation a highest weight line is fixed by the lower unitriangular subgroup, using the corresponding Borel. A polynomial fixed by that subgroup is determined by one scalar.

On the dense set where the first p rows of Y form an invertible block A, a lower block shear sends Y to $(A;0)$. The right covariance then gives $F(Y) = \det(A)^c F((I_p;0))$. Thus the entire unitriangular invariant space has dimension one, and complete reducibility under $U(m)$ gives irreducibility. The finite space of homogeneous degree pc is preserved by the action, and complexified Lie algebra invariance identifies the summands under $U(m)$ and GL_m. Consequently $\prod_i \det(\eta_i Y)$ is a finite linear combination of the corresponding polynomial at the matrices $\eta_0 k^{\mathrm{t}}$. The same holds over all places by tensor products.

The very same coefficients work for the two-sided stacked determinant. At a place with $c_\tau = 0$ both products over i are the constant one, including on a rank-deficient block. At a place with $c_\tau > 0$, for any fixed lower matrix Z, if it has rank smaller than a, the stacked determinant vanishes for every η_i. Otherwise extend its rows to an invertible square matrix. Multiplying on the right by its inverse turns Z into $[0\ I_a]$. It follows that $\det[\eta_i; Z] = d\det(\eta_i Y_Z)$ for a fixed Y_Z, consisting of the first p columns of that inverse, and a scalar d depending on Z alone. Apply the polynomial linear combination at Y_Z, and use the same coefficients in each large evaluation and in the small polynomial. Thus for every family η, every matched pair of finite arrays and every adelic g,

$$\kappa L_g(P(\eta)) = c_\nu S_g(Q(\eta)).$$

Put $t_{\mathrm{ex}} = (\dim I_a - \dim V')/2$ and $A_{\mathrm{ex}} = R_0'/D_{\mathrm{final}} > 0$ as in Proposition 36.1. Since $c_\nu \neq 0$, the coefficient multiplying the large class sum is

$$S_g(Q(\eta)) = \frac{\kappa}{c_\nu} L_g(P(\eta)), \qquad \left|\frac{\kappa}{c_\nu}\right|^2 = \frac{|\kappa|^2}{A_{\mathrm{ex}} M_\nu(t_{\mathrm{ex}})}.$$

For the fixed real data and ordered basis, A_{ex} and κ are chosen before the finite modulus, its character and the family of covectors. These identities include $\kappa = 0$.

39.2 Constants in the three finite comparisons

In this subsection, we compute the finite orbit and stabilizer measures for the three applications of Section 36.1. This identifies the determinant character in each Hecke product and fixes the finite scalar in Section 39.4.

We use the covector, column and theta character conventions of the partner comparison. In the three class-sum applications the matching character is χ^{-1} if χ is the determinant character appearing in the upper datum. Let R be the original residue ring, with its full exponent at each selected place, and write $X = (A, B)$ and $Y = (C; D)$ for the matched blocks of k^{-1} and k, in that order.

Here B, C are square. In the order of Section 36.2, X consists of the last covectors of k^{-1} and Y of the first columns of k. They give a split surjection and injection, respectively. In this order the identities

$$AC + BD = 0, \qquad \det C = \det k \, \det B$$

hold over R, by block multiplication and complementary minors. For any Ψ' invariant under the right action of the integral column basis group, put

$$\Phi(A, B) = \chi(\det B)\Psi'(B^{-1}A), \qquad \Psi(C, D) = \chi(\det C)\Psi'(D),$$

with value zero when the determinant of the corresponding square block is a nonunit of R.

When either block has unit determinant, both do, and $D = -B^{-1}AC$. As $-C$ is invertible, $\Psi'(D) = \Psi'(B^{-1}A)$. Thus $\Phi(X) = \chi^{-1}(\det k)\Psi(Y)$. Conversely B invertible makes the upper array split surjective, and C invertible makes the lower array split injective. This proves support and matching over the actual residue ring, including the chosen prime-square coordinates.

For a definite target $G = U(V_0)$, give G_∞ and the full integral finite compact K_{std} mass one. If F is right invariant under $G_\infty J$ for a compact open $J \subseteq K_{\mathrm{std}}$, disintegration over the finite classes gives

$$\int_{[G]} F(h)\, dh = \frac{1}{[K_{\mathrm{std}} : J]} \sum_{[h]_J} \frac{F(h)}{|G(F_0) \cap hJh^{-1}|}.$$

Here each finite representative h is taken in $G(\mathbb{A}_{F_0,f})$. At $J = K_{\mathrm{std}}$ the index is one. For the stacked determinant, the real target action contributes $\det(u_\tau)^{\ell - c_\tau} \det(u_\tau)^{-\ell} \det(u_\tau)^{c_\tau} = 1$. For $u \in K_{\mathrm{std}}$, write u_S for its component on the chosen split side. The array above satisfies

$$\Phi(u_S^{-1}A, u_S^{-1}B) = \chi(\det u_S)^{-1}\Phi(A, B).$$

The finite target character contributes $\chi(\det u_S)$. The unselected compact factors preserve the vacuum, so the full integral compact fixes the twisted test. Hence these class weights apply to the actual kernels.

For rank zero the quotient is a single point with one automorphism, so its mass is one. The kernel on the zero Hom space with characters $(\alpha/\beta, \alpha)$ is $(\alpha/\beta)(\det g)$ on the retained point. In the first exchange $\beta = \alpha$, so it equals one. Write χ_α for the residue character of the source character α. The matched data are $\chi_\alpha \circ \det$ and 1, since $X = k^{-1}$ and Y has zero columns:

$$\chi_\alpha(\det X) = \chi_\alpha(\det k)^{-1}.$$

In Section 39.3 the source character is $\alpha = c^{-1}$, so $\chi_\alpha = \chi^{-1}$ when χ denotes the residue character of c. The empty small polynomial is one. The normalization of this application is therefore a unit-mass constant, with no unrecorded Haar factor.

For the second and third comparisons, the graph matching above splits the small kernel into its two blocks. The standard determinant factor is the kernel in the proper definite class variable, and the remaining factor is the polynomial kernel in the retained block. Before any trace averaging, each column datum on the small side has the factor $\chi(\det C)$, so it has full column rank on its support even with one trace character in place of the invariant sum. The tensor product of the two real kernel vectors and this column datum is the original small theta kernel on the direct sum. The column estimate (100) therefore bounds its absolute integral for each fixed pair of retained points. In particular this bound applies term by term in the finite class and trace sums. The preceding measured identity can be inserted pointwise and integrated against the remaining factor, retaining each original class weight. This proves the multiplication rule for the exchange constants.

In the third comparison Ψ' is the sum of translates of the trace character by the group over the untwisted base ring $R_0 = \mathcal{O}_E/(\pi_M^2)$, inflated to R. It is invariant under the right basis group over R because its reduction permutes those terms. Each element of $\mathrm{GL}_{3m}(R_0)$ lifts to the original standard level at the chosen split places by lifting entries and testing determinants.

Extend the lift by its conjugate unitary coordinate on the other split half and by identity elsewhere. The splitting character of the retained small kernel has trivial finite tame part, and thus has value one on the determinant of this integral move. The full-level auxiliary lift F° is unchanged by it. At a fixed retained point $g' \in U(W')(\mathbb{A})$, let $K_D(s, g')$ be the trace kernel for one residue translate $D \in \mathrm{GL}_{3m}(R_0)$. The fixed-Schwartz-datum estimate preceding Section 34.3.1 gives $|K_D(s, g')| \le C_{D,g'}\,\mathrm{ht}_S(s)^A$. The first two exchanges give (109); the invertible determinant block there gives $\sup_s \mathrm{ht}_S(s)^A |F^\circ(s)| < \infty$ by Section 41.3. Thus $F^\circ(s)K_D(s, g')$ is integrable on the finite-volume quotient $[U(S)]$. Right translation in this quotient makes the integral of each translated trace character equal to the unshifted integral. The finite sum multiplies by $\mathcal{N} = |\mathrm{GL}_{3m}(R_0)|$. This is the untwisted group order, even if the modulus R has acquired new twist places.

With $e = m/2$, $z_* = 7e + k/2$ and $n_j = j + (t - 11m - 1)/2$, the three defects and period normalization arguments are

kept, large, small	defect	character in the Hecke product
$I_m,\ I_m,\ 0$	e	$c\upsilon^e$
$I_m \perp S,\ I_{3m},\ I_m^-$	m	$c^{-1}\upsilon^m$
$I_{3m} \perp W',\ V'',\ S$	z_*	$(ca_j)^{-1}\upsilon^{3m+z_*}$.

Here m, k are even, W_j has rank $r = k + m - 1$, and $W' = W_j \perp I_j^- \perp I_{3m-j} \perp I_{6m+1}$ has rank $k + 10m = 3m + 2z_*$. The negative rank of each kept space equals that of its small member, respectively $0, m, m$. For the third character, $ca_j = \mu_j^0 \vartheta\, \upsilon^{m+j+\lfloor (t+k-1)/2 \rfloor}$; the exponent left after inversion is

$$3m + z_* - m - j - \lfloor (t + k - 1)/2 \rfloor = -j - (t - 11m - 1)/2 = -n_j.$$

The first conversion to the negative copy of I_m takes the conjugate of its scalar and multiplies by the unit phase $\lambda_0 \chi(\eta)^{-m}$, with η the norm-minus-one residue unit. Thus the squared moduli multiply and the factor from trace averaging is $\mathcal{N}^{-2}$. In particular the coefficient of the third left class sum taking it to the original auxiliary lift period is $\gamma_3\gamma_{12}/\mathcal{N}$, with square modulus equal to a fixed constant divided by the product of the three Hecke factors in the table.

39.3 The rank-zero comparison and the opposite Hermitian form

In this subsection, the first application of Section 36.1 has a rank-zero smaller space. We compute the source-negation factor with the original residue character before applying the comparison to the other two pairs.

Use the modulus $\mathfrak{m} = \pi_M^2 \mathfrak{q}_\vartheta$, the character $c = \chi_M^j \vartheta$ of weight zero, and its residue character χ. Let $R = O_E/\mathfrak{m}$. Write $m = 2e > 0$ and $P = 1$. Apply the measured exchange in Section 36.1 with kept space I_m, positive space I_m, and smaller space of rank zero. Take the characters on the positive kernel to be $\alpha = \beta = c^{-1}$, and its finite function to be $\Psi_+(X) = \chi^{-1}(\det X)$. The two finite functions are matched.

Indeed the block of an inverse matrix $k^{-1} \in \mathrm{GL}_m(R)$ is the entire matrix, and $\Psi_+(k^{-1}) = \chi(\det k)$, the matching factor for the residue character χ^{-1} of α. The other block is the unique empty column, with value one. On singular matrices the extended residue character is zero. In Section 39.1 take $p = \ell = 0$, $a = m$, and $c_\tau = 0$ at every infinite place. Then $P(\eta) = Q(\eta) = 1$. The definite target average sends the real vector supplied by the exchange to κ times the Gaussian with polynomial part 1, where κ is independent of the finite data. On the zero Hom space the only lattice vector is zero and the vacuum has value one. The surviving character is $(\alpha/\beta)(\det g) = 1$, and the zero group has quotient mass one. The projected identity is therefore $\kappa L_g(1) = c_\nu$, with $\nu = c\upsilon^e$. Since $c_\nu \neq 0$, also $\kappa \neq 0$. Set $\gamma_1 = \kappa/c_\nu$. The measured class sum gives

$$\gamma_1 \sum_{[h] \in [U(I_m)]/K_{\mathrm{std}}} \frac{\Theta^{I_m, I_m}_{c^{-1}, c^{-1}}[\Psi_+, 1](g, h)}{|\operatorname{Aut}(h)|} = 1, \qquad |\gamma_1|^2 = \frac{C_1^{\mathrm{ex}}}{\mathcal{L}(c\upsilon^e, e)}.$$

The first equality is the measured exchange, with its class weights unchanged. The second is its scalar formula, since $\alpha^{-1}\upsilon^e = c\upsilon^e$. Here C_1^{ex} depends only on the fixed real and unramified data. In particular the scalar used in the first equality is nonzero. The point of evaluation g is arbitrary on the I_m.

For the source-negation step, take the fixed finite unit u with $u\bar{u} = -1$ and put $\eta = u \bmod \mathfrak{m}$. At split pairs one can choose u with coordinates $1, -1$; at an unramified quadratic place, surjectivity of the norm on units gives a norm-minus-one unit as in Section 10.1. At a place over 2, the two completions are split: $-d \equiv 1 \bmod 8$ makes $K \otimes \mathbb{Q}_2 = \mathbb{Q}_2 \times \mathbb{Q}_2$, and conjugation interchanges the places of E above its two factors. The split choice $(1, -1)$ has norm -1 already in the local integers. Multiplication by the fixed unit preserves the integral boxes and their measures also at these places. Use the original such unit and the original point of the negative space. In chosen coordinates its point matrix is $X_{I_m^-} = (1, u)X_{I_m}$; its rational form is the negative of the original form. On the Hom space, with the target unchanged, the new point matrix is therefore $(1, u^{-1})X_{\mathrm{Hom}}$. Indeed x is a rational Hom vector precisely when $X_T x X_S^{-1}$ is rational in the chosen bases. Replacing X_S by $(1, u)X_S$ therefore identifies the new rational summation set with the old one by $\phi(x) = (1, u^{-1})x$.

On the coefficient realization of the oscillator data, put $Jf(x) = \overline{f(\phi x)}$. This transports the Heisenberg action to the opposite source, because $\beta_{\mathrm{old}}(\phi x, \phi y) = \overline{\beta_{\mathrm{new}}(x, y)}$. At infinity the sign matrix in the bicharacter changes sign and ϕ is the identity. At finite places the norm of u^{-1} is -1. The Gaussian and standard integral box are preserved, so this is a vacuum-preserving transport with the same Haar normalization. The standard holomorphic coordinate of each entry has been conjugated, so P becomes its coefficientwise conjugate. For a residue datum Ψ, reduction of ϕ multiplies every entry by η^{-1} at the selected places. The integral support is

preserved since u is a unit, and the transported residue function is

$$\Psi^{J}(X) = \overline{\Psi(\eta^{-1}X)}.$$

Both functionals are invariant under rational translations and modulations in corresponding Schrödinger polarizations. The uniqueness argument of Section 41.5.1 applies to each: localizing at a rational point in a finite box, use only modulations whose finite component is constant on that box. Their real linear phases span the dual real space, so the localized functional is evaluation, and rational translations identify its multipliers at all rational points. Thus the two functionals are proportional under the transport. Here J matches the rational Heisenberg operators and their vacuum translates. The two normalized theta functionals have the same squared mean, one, on the vacuum translates over their additive adelic quotients. The induced quotient isomorphism preserves probability Haar measure. Consequently for a fixed $|\lambda_0| = 1$,

$$\overline{\theta_{\mathrm{old}}(f)} = \lambda_0 \theta_{\mathrm{new}}(Jf).$$

The scalar depends on the two spaces and their theta normalizations alone.

To track the splitting characters, write $A(x) = hxg^{-1}$. Its matrix stays the same on passing to the opposite source. At a real place the multiplier is the conjugate of the negative principal minor of A. Jacobi's complementary-minor identity and $A^{-1} = J_{\mathrm{sign}} A^{*} J_{\mathrm{sign}}$ give $\overline{\Delta_{\infty,\mathrm{old}}(A)} = \det A\, \Delta_{\infty,\mathrm{new}}(A)$. At the finite places let L be the original standard lattice and choose b with $bL = L \cap A^{-1}L$.

The finite multiplier is $\chi_f(\det b)$, where $\chi_f = (\upsilon_E)_f |\cdot|_f^{-1/2}$ on $\mathbb{A}_{E,f}^{\times}$. The matrix Ab represents $L \cap AL$, so

$$\Delta_f(A^{-1}) = \chi_f(\det A)\Delta_f(A).$$

Since $A^{-1} = A^{*}$, the determinant ideal defining $\Delta_f(A^{-1})$ is the conjugate of the one defining $\Delta_f(A)$, by local Smith form. The character χ_f is unramified on units and takes complex-conjugate values on conjugate ideles. Thus $\Delta_f(A^{-1}) = \overline{\Delta_f(A)}$. Since $\det A$ has idele norm one, the total multiplier under the conjugating transport acquires $\upsilon(\det A)$.

In the oscillator integral this is the only scalar change: the coordinate substitution intertwines the matrices, conjugates the Heisenberg bicharacter, and preserves the Gaussian and Haar measure. As $\det A = (\det h)^m (\det g)^{-m}$, the original pair character $a(\det h)b(\det g)^{-1}$ becomes the pair character with entries $a^{-1}\upsilon^m$ and $b^{-1}\upsilon^m$.

For $a = b = c^{-1}$ and $\Psi_{+} = \chi^{-1} \circ \det$ this gives exactly

$$\Psi_{+}^{J}(X) = \chi(\eta)^{-m} \chi(\det X).$$

The same formula holds at a singular matrix since both sides vanish. The level and class stabilizers do not change, and the class weights are rational. Conjugating the first measured identity gives

$$\overline{\gamma_1}\lambda_0 \chi(\eta)^{-m} \sum_{[h] \in [U(I_m)]/K_{\mathrm{std}}} \frac{\Theta^{I_m^{-}, I_m}_{c\upsilon^m, c\upsilon^m}[\chi \circ \det, 1](g^{-}, h)}{|\operatorname{Aut}(h)|} = 1.$$

In particular the same $\mathcal{L}(c\upsilon^e, e)$ controls the square modulus. The factor λ_0 is one fixed complex phase and $\chi(\eta)^{-m}$ is the residue phase at the actual modulus. This proves the normalization used when inserting the first class sum under the second exchange.

39.4 Three period identities for the gluing operator

In this subsection, we apply the normalized partner comparison to the three pairs defining the operator of Section 34.1. The real factors of Section 39.1 and the finite factors of Section 39.2 identify the resulting common period.

These identities use the partner comparison on the three stated pairs. In particular the scalar formula is used with its original vacuum normalization at the top and after all plane removals. Use the same j, original W_j, V'', V°, and choice set $\mathcal{I}$ as in Section 34.1. Put $I_a = (E^a, 1)$, $S = I_{2m} \oplus I_m^-$ and

$$W' = W_j \oplus I_j^- \oplus I_{3m-j} \oplus I_{6m+1}.$$

The space underlying V'' is the definite space with these indices. Set $e = m/2$ and $z_* = 7e + k/2$, an integer since r is odd and m is even. Then $\dim V'' = 3m + 2z_*$. In these comparisons put $\upsilon = \upsilon_E|_{T_1(\mathbb{A}_{F_0})}$, where υ_E is the splitting Hecke character of Section 4.3. Let $c = \chi_M^j \vartheta$ be the weight-zero character and a_j the twist-free character on the larger pair. Write $\mathfrak{m} = \pi_M^2 \mathfrak{q}_\vartheta$ for the modulus in the twisted intermediate kernels. The finite character of c is the one used in their determinant data and is nontrivial at every place of the prescribed support of $\mathfrak{m}$.

Section 39.3 gives the measured first exchange on the two copies of I_m. Its weighted class sum is normalized to one by γ_1. Its squared modulus is a constant divided by $\mathcal{L}(c\upsilon^e, e)$, where $\mathcal{L}(\nu, z) = M_\nu(z)$ is the normalizing factor of Proposition 36.1 on the split support of $\mathfrak{m}$. Conjugating the first group to I_m^- gives the scalar $\overline{\gamma_1}\lambda_0\chi(\eta)^{-m}$ of that section, with the prescribed reduction and $|\lambda_0| = 1$.

The next exchange uses

$$(I_m \oplus S, I_{3m}; I_m^-)$$

and the source character $c\upsilon^m$. Let γ_2 be the coefficient multiplying the large class sum in this second exchange. Its normalization gives

$$|\gamma_2|^2 = C_2^{\mathrm{ex}} / \mathcal{L}(c^{-1}\upsilon^m, m).$$

Insert the first constant class sum under this second exchange and multiply the kernels for the direct sum $I_m \oplus S$. The remaining kernel on (I_m^-, S), integrated against the constant function on $[U(I_m^-)]$, is the auxiliary lift F°.

Its datum is the original vacuum and the polynomial on the negative block is the $2m$-th power of the determinant minor. For fixed finite and real tests, the convention of Section 34.3 gives

$$\Theta_{\upsilon^m,1}^{S,I_{3m}}[\chi \circ \det, 1](x,h) = c(\det x)\Theta_{\upsilon^m,c}^{S,I_{3m}}[\chi \circ \det, 1](x,h).$$

The small kernel on (I_m^-, S) has the same factor $c(\det x)$ from its first character $c\upsilon^m$. Cancel this factor and substitute the conjugated first class sum to obtain

$$\gamma_{12} = \gamma_2\overline{\gamma_1}\,\lambda_0\chi(\eta)^{-m}.$$

Here λ_0 is the fixed phase of the opposite-form identification, while $\chi(\eta)$ is evaluated at the actual modulus and character. Both have absolute value one. Hence

$$|\gamma_{12}|^2 = \frac{C_{12}^{\mathrm{ex}}}{\mathcal{L}(c^{-1}\upsilon^m, m)\mathcal{L}(c\upsilon^e, e)}.$$

More explicitly, write $\mathcal{C}_a$ for the full-level finite classes of $U(I_a)$, and $e_a(h) = |\operatorname{Aut}(h)|$. Give their representatives real component one. Put

$$A_c(h) = \sum_{g \in \mathcal{C}_m} \frac{\Theta_{c,1}^{I_m, I_{3m}}[1,1](g,h)}{e_m(g)},$$

where the first kernel is K_0 of (I5), with unit modulus and the integral vacuum. The determinant character therefore gives $A_c(h) = c(\det h) A_1(h)$. The upper residue test on $I_m \oplus S$ with $\Psi' = 1$ is $\chi(\det B)$, where B is the square block on S. The polynomial is one. Factoring this tensor and canceling $c(\det s)$ gives the pointwise identity

$$F^{\circ}(s) = \gamma_{12} \sum_{h \in \mathcal{C}_{3m}} \frac{A_c(h)}{e_{3m}(h)} \Theta_{\upsilon^m, c}^{S, I_{3m}}[\chi \circ \det, 1](s,h). \tag{109}$$

For k in the standard finite compact of $U(I_{3m})$, the first kernel gives $A_c(hk) = c(\det k) A_c(h)$. In the second kernel the residue substitution $B \mapsto \bar{k}^{-1} B$ contributes $c(\det k)^{-1}$, since υ^m is unramified there. Hence the product has the asserted full level in the class variable.

For $i = (\Lambda, h^{\flat}, \sigma)$, where σ is the chosen tuple of short covectors, use the third exchange on

$$(I_{3m} \oplus W', V''; S).$$

Put $e_j^{(1)} = j + \lfloor (t + k - 1)/2 \rfloor$. The first-block character and the source character are

$$b''_{j,\vartheta} = \mu_j^0 \vartheta \, \upsilon^{e_j^{(1)}}, \qquad ca_j = b''_{j,\vartheta} \upsilon^m.$$

Thus the third kernel has characters $(ca_j, b''_{j,\vartheta})$. The finite matching pairs the trace character of Λ with the unnormalized invariant sum of the trace character on the ratio of the two square blocks. The left datum is $\chi(\det x'') \psi_{\Lambda}^{\mathrm{inv}}((x'')^{-1} x')$.

39.4.1 Matching the real factors

In this subsection, we form the common real polynomial for the period comparisons of Section 39.4.

For the third comparison the small space S has $2m$ positive columns and m negative columns. Write $r_+ = k + j - 1$ and $d_+ = \dim V''$. At τ take the following ordered $2m$-row matrices $\eta_{\tau i}$ in Section 39.1, extended by zero on the initial I_{3m} block. For every positive column of height $h < r_+$ in the diagram, number the rows by $0 \le a < 2m$. Row a is covector $h + a$ of W_j when $h + a < r_+$, and covector a of I_{3m-j} otherwise. The latter index exists since $3m - j \ge 2m$. For each of the $\Delta' = (\Delta - t - 1)/2$ columns of full positive height, use the ordered $2m$ short rows of $W^{\flat}$ prescribed by σ at that place and column. Thus, with the counts of Section 41.1.4,

$$c_{\mathrm{stack},\tau} = \sum_{h < r_+} \mu_{\tau}^{+}(h) + \Delta' = w_j(\tau) - e_j^{(1)}.$$

Order the negative covectors by $0 \le a < m$, taking row a of I_j^{-} for $a < j$ and row $k + a - 1$ of W_j for $a \ge j$. They exhaust the negative block, including when $j = m$. Its determinant exponent in the small polynomial is

$$c_{\mathrm{stack},\tau} + \dim(I_{3m} \perp W') - 2m = c_{\mathrm{stack},\tau} + d_+ + m.$$

In these orders the small polynomial is exactly P_jQ_σ of Section 41.1.4, independent of the variables of the initial I_{3m}. A fixed alternative order of the negative rows contributes its permutation sign raised to this exponent at each place, a scalar independent of the finite data. The paired large polynomial stacks the same $\eta_{\tau i}$ over the positive variable rows in the indicated order. The third exchange scalar γ_3 has squared modulus C_3^{ex} divided by $\mathcal{L}((ca_j)^{-1}\upsilon^{3m+z_*}, z_*)$. Summing the first block using the identity for the auxiliary lift and then reducing the trace-character data modulo the untwisted π_M^2 gives

$$\gamma_3\gamma_{12}\,\Upsilon_i(g) = \mathcal{N}F_i(g), \qquad \mathcal{N} = |\mathrm{GL}_{3m}(\mathcal{O}_E/(\pi_M^2))|.$$

Indeed the square block C of the matched column array belongs to the (S, I_{3m}) kernel with characters (υ^m, c); its finite factor is $\chi(\det C)$ and its real polynomial is one. The remaining (S, W') kernel has characters (υ^m, a_j) and the polynomial just computed. Substituting $\mu_j^0 = \chi_0(\upsilon^{-\Delta}\chi_M)^j$ and $c = \chi_M^j\vartheta$ gives $a_j = \chi_0\upsilon^{m+e_j^{(1)}-j\Delta}$, independent of ϑ. The two source determinant factors multiply to $(ca_j)(\det s)^{-1}$. Its trace datum is inflated from $R_0 = \mathcal{O}_E/(\pi_M^2)$; at additional twist places the remaining factor is the integral vacuum. For one translate indexed by $D \in \mathrm{GL}_{3m}(R_0)$, write $K_D(s, g')$ for this remaining trace kernel at a fixed point g' of $U(W')$, and write $T_h(s)$ for the (S, I_{3m}) kernel in (109). That finite class expansion gives

$$\int_{[U(S)]} |F^\circ(s)K_D(s, g')|\,ds \le |\gamma_{12}| \sum_{h\in\mathcal{C}_{3m}} \frac{|A_c(h)|}{e_{3m}(h)} \int_{[U(S)]} |T_h(s)K_D(s, g')|\,ds.$$

The product in each inner integral is the small theta kernel on $(S, I_{3m} \perp W')$, up to the fixed unit phase of the product normalization. Its column datum contains $\chi(\det C)$ on the square block at every selected place, so all its supported columns have full rank over the local field. Equation (100) in the conjugate transpose model, with height exponent zero, bounds the inner integral on the finite-volume quotient. Thus right translation in this integral against the full-level auxiliary lift gives the same value for every D.

Their unnormalized sum has $\mathcal{N}$ terms. The integral defining F_i uses $[U(S)]$ with the original quotient measure. Equivalently

$$c'_\vartheta = \gamma_3\gamma_{12}/\mathcal{N}, \qquad |c'_\vartheta|^2 = \frac{C_3^{\mathrm{ex}}C_{12}^{\mathrm{ex}}/\mathcal{N}^2}{H_j(\vartheta)}.$$

Here the product in the denominator is

$$H_j(\vartheta) = \mathcal{L}(c\upsilon^e, e)\,\mathcal{L}(c^{-1}\upsilon^m, m)\,\mathcal{L}((ca_j)^{-1}\upsilon^{3m+z_*}, z_*).$$

To match these factors with the signed critical values of Section 38, write $n_j = j+(t-11m-1)/2$, μ_j^0 for the untwisted lift character, and

$$\nu_+ = \upsilon^{-m}\chi_M^j, \qquad \nu_- = \upsilon^{-e}\chi_M^{-j}, \qquad \nu_* = \mu_j^0\upsilon^{n_j}.$$

Then $c^{-1}\upsilon^m = (\nu_+\vartheta)^{-1}$, $c\upsilon^e = (\nu_-\vartheta^{-1})^{-1}$, and $(ca_j)^{-1}\upsilon^{3m+z_*} = (\nu_*\vartheta)^{-1}$.

The first-factor pairing. Write $R = \mathcal{O}_E/\mathfrak{m}$, $R_0 = \mathcal{O}_E/(\pi_M^2)$, and let $\phi : R \longrightarrow R_0$ be reduction. Let T be the coordinate set of V'', $I \subset T$ the coordinates of W_j, and $J_I : R^I \to R^T$ their coordinate inclusion. Write w for the covectors Λ stacked over $\pi_M 1_T$ in the residue coordinates of $\Xi_{\vartheta,i}$. The top covector projection u satisfies $uw = \Lambda$. Define

$$H = \{A \in \mathrm{GL}_T(R) : Ae_b = e_b\ (b \in T \setminus I),\ w\phi(A) = w\}.$$

Here ϕ is applied entry by entry. The inverse image of H under reduction from the full integral subgroup K on V'' is the prescribed level J_Λ. Reduction onto $\mathrm{GL}_T(R)$ is surjective, since each place in the modulus is split in the chosen coordinates and reduction of the integral general linear group onto each local residue ring is surjective.

Let Y, Z be the two residue blocks of the third kernel, with Z square on T and the columns of Y indexed by I_{3m}. The invariant trace function is the unnormalized sum

$$B(Y) = \sum_{D \in \mathrm{GL}_{3m}(R_0)} \psi\big(\operatorname{tr}(\Lambda\phi(Y)D)\big).$$

For $A \in H$, $\Lambda\phi(A) = uw\phi(A) = \Lambda$, so $B(AY) = B(Y)$. The ratio datum $\Phi(Y, Z)$ is $\chi(\det Z)B(Z^{-1}Y)$ for invertible Z, and vanishes otherwise. Put $\Phi_H = 1_{Z \in H}\Phi$. For $\sigma = ZJ_I$ set

$$C_w(\sigma) = \begin{cases} \chi(\det(J_I^{\mathrm{t}}\sigma)), & w\phi(\sigma) = wJ_I, \\ 0, & \text{otherwise.} \end{cases}$$

The character of the residue ring is extended by zero on nonunits. When the columns of Z off I are those of the identity, Z is block triangular in $I \sqcup I^c$ and $\det Z = \det(J_I^{\mathrm{t}} Z J_I)$. Also $w\phi(Z) = w$ is equivalent to $w\phi(ZJ_I) = wJ_I$. On H, $B(Z^{-1}Y) = B(Y)$. Thus

$$\Phi_H(Y, Z) = C_w(ZJ_I)B(Y)1_{Ze_b = e_b \ (b \in I^c)}. \tag{*}$$

For $A \in \mathrm{GL}_T(R)$, define

$$(T_A\Psi)(Y, Z) = \chi(\det A)\Psi(A^{-1}Y, A^{-1}Z).$$

By the finite Weil action, right translation of the kernel by $k \in K$ with residue A replaces Ψ by $T_A\Psi$. The determinant factors give $T_A\Phi = \Phi$. Hence $T_A\Phi_H = 1_{Z \in AH}\Phi$ and

$$\Phi = \sum_{AH \in \mathrm{GL}_T(R)/H} T_A\Phi_H.$$

For a full-level class representative h, put $\Gamma_h = U(V'')(F_0) \cap hKh^{-1}$. The J_Λ-classes above h are the orbits of Γ_h on K/J_Λ under the action through $h^{-1}\Gamma_h h$.

For the restricted kernel $f(hk)$, rational and right J_Λ-invariance give

$$\frac{1}{|\Gamma_h|} \sum_{kJ_\Lambda \in K/J_\Lambda} f(hk) = \sum_{[k] \in (h^{-1}\Gamma_h h)\backslash K/J_\Lambda} \frac{f(hk)}{|\Gamma_h \cap hkJ_\Lambda k^{-1}h^{-1}|}.$$

At a real place $u \in U(V'')$ left multiplies the bottom rows of each stacked determinant. The polynomial contributes $\det(u)^{c_\tau}$; the vacuum and splitting give $\det(u)^{-c_\tau}$. Their product is one in the full kernel. This proves equality of the full-level class sum of Φ and the J_Λ-class sum of Φ_H, with the original class weights. More explicitly, if the Haar measure has $\mathrm{vol}(K) = 1$, the integral of the single J_Λ-invariant kernel f is its weighted J_Λ-class sum divided by $[K : J_\Lambda]$. The full kernel is the sum of all $[K : J_\Lambda]$ translates of this restricted kernel. Integrating that sum restores its weighted class sum exactly, also when the stabilizer orders vary between the full-level classes.

Use the ordered block decomposition of the kept space

$$I_{3m} \oplus W' = W_j \oplus I_j^- \oplus W^{\mathrm{add}},$$

where W^{add} includes I_{3m}. The equality is by permutation of the coordinate parameters. In (*) the columns on W_j carry C_w, those on I_j^- carry the inclusion indicator there, and the remaining columns carry the added-block datum of Section 33.3. Indeed the complementary identity columns split into the negative pad and the fixed positive columns on W^{add}, while B depends only on the I_{3m} columns. For the real polynomial, first expand along the selected coordinate rows. By Section 33.1.1, $\mu_\tau^+(0) = 0$ and $\mu_\tau^+(h) = 0$ for $1 \le h < k-1$, since $\varsigma_{h-1} - \varsigma_h = 1$ in that range. Hence at every contributing height $h < r_+$,

$$1 \le r_+ - h \le j \le m.$$

The $2m$ selected rows of that factor therefore remove exactly the columns of W_j indexed by $h, \dots, r_+ - 1$. The remaining selected rows remove columns of the added positive space. Expanding in the first h columns of W_j gives its height-h minors. In each of the Δ' full-height factors the short rows are supported on $W^\flat$, so expansion in W_j uses all r_+ of its positive columns. Thus the Laplace expansion has precisely the positive Weyl diagram of W_j. On the same decomposition the real polynomial is

$$P(x_{W_j}, x_{\mathrm{add}}) = \sum_\lambda L_\lambda(x_{W_j})\zeta_\lambda(x_{\mathrm{add}}).$$

Here L_λ are the positive Weyl coordinate polynomials, and ζ_λ are the coordinates used in (I5), obtained by expanding the stacked determinants in the W_j columns with coefficients in the added variables. The polynomial is constant on the negative pad.

We use the tensor product of the fixed theta functionals on the standard signed lines, retaining the fixed definite vacuum normalization on the positive lines. For a Hermitian space B with point matrix X_B, let $Y_B = X_B^{-1}E^{\dim B}$ be its rational vectors in these coordinates. Landherr's theorem [43, pp. 246–247] applies as in the plane-removal argument: the finite forms are isometric and the signatures agree at every real place. If J_B, J_{std} are the rational Gram matrices, take a rational isometry $P^* J_B P = J_{\mathrm{std}}$ and write X_{std} for the standard signed point matrix. The point equations imply

$$g_B = X_B^{-1} P X_{\mathrm{std}}, \qquad g_B(X_{\mathrm{std}}^{-1}E^{\dim B}) = Y_B.$$

Here g_B is an adelic isometry in the normalized coordinates. Use the vacuum-normalized Weil action of Lemma 17.1 before the determinant characters of the kernels are inserted. With θ_{std} the product functional just specified, put

$$\theta_B(f) = \theta_{\mathrm{std}}(U_{g_B^{-1}} f).$$

Another rational isometry P' gives $g_B^{-1} g'_B = X_{\mathrm{std}}^{-1}(P^{-1}P')X_{\mathrm{std}}$, the action of a rational unitary element of the standard space. Section 17.1 proves that its action fixes θ_{std}. For a block sum take the direct sum of these maps. The real determinant factors multiply, the finite intersections are direct sums, and the vacuum is a tensor product, so the block Weil action is the tensor product of the two actions. Thus

$$\theta_{B_1 \perp B_2}(f_1 \otimes f_2) = \theta_{B_1}(f_1)\theta_{B_2}(f_2).$$

The same identity applies after the indicated permutation of coordinates. To compare with any previously fixed normalized theta functional θ_B^{orig} on the same model, rational Heisenberg invariance gives $\theta_B^{\mathrm{orig}} = \epsilon_B \theta_B$. The symplectic map g_B carries the standard rational displacement

lattice onto the one for B, and carries the probability Haar on one quotient to that on the other. The mean-square identity of Section 41.5.1 on a unit vector then gives $|\epsilon_B| = 1$. With previously fixed phases the product identity acquires exactly $\epsilon_{B_1 \perp B_2}/(\epsilon_{B_1}\epsilon_{B_2})$. This scalar depends only on the fixed spaces and their normalizations, so it multiplies the same comparison constant for every residue array, modulus and twist. Its squared modulus is one, including in the scalar comparison at the trivial twist. In the indicated product convention, applying the identity twice gives the product of the three kernels. In the Schrödinger model the integral vacuum and the residue functions factor on the three Hom spaces. In the real Fock model the vacuum and the displayed polynomial factor in the same way. The rational summation set is a Cartesian product, and the finite class sums and Schwartz theta series converge absolutely.

In the order $W_j, I_j^-, W^{\mathrm{add}}$, the three character pairs are

$$(\upsilon^{m-j}, b''_{j,\vartheta}), \qquad (\upsilon^{j}, b''_{j,\vartheta}), \qquad (b''_{j,\vartheta}, b''_{j,\vartheta}).$$

Their first characters multiply to ca_j. The three first points are g, the identity of I_j^-, and $\mathrm{diag}(1, h^\flat, h')$ in the added block, with the same second point h in each. Multiply the added kernel by $A_c(h')$ and sum over h' with the weights of (109). The coefficient of L_λ equals the negative-pad kernel at h times this weighted sum with ζ_λ. By (I5) these are the coordinates of $\Upsilon_{\vartheta,i}(h)$ before the scale and conjugation. Thus $\Upsilon_i(g)$ is the J_Λ-class sum of the W_j, V'' kernel with residue function C_w and polynomial vector $\Upsilon_{\vartheta,i}(h)$.

For the class sum, let $k_+ \in J_\Lambda$ be supported at a new split member w. Write its residue in $I \sqcup I^c$ as $A = \left(\begin{smallmatrix} D_w & 0 \\ E_w & 1 \end{smallmatrix}\right)$. Then $J_I^{\mathsf{t}} A^{-1} = D_w^{-1} J_I^{\mathsf{t}}$, $\det A = \det D_w$, and

$$C_w(A^{-1}\sigma) = \chi_w(\det D_w)^{-1} C_w(\sigma).$$

The W_j, V'' kernel is multiplied by this scalar under $h \mapsto hk_+$, since its first splitting character on integral units is trivial. The added and negative-pad inclusion conditions are preserved, and the trace datum is unchanged. Their combined target-leg factor on the residue determinant is $\chi_w(\det D_w)$. Thus the two factors of the class summand have inverse level characters. The conjugated factor $\overline{c''\Upsilon_{\vartheta,i}(h)}$ transforms by $\chi_w(\det D_w)^{-1}$, the level character of the original target lift under $\mathrm{diag}(k_+, 1)$. At an old place above π_M, $k_+ \equiv 1 \bmod \pi_M$ and the residue characters are all trivial.

The conjugate of the first factor is $c''\Upsilon_{\vartheta,i}(h)$ by (I5). Linearity in the polynomial, followed by the third exchange, gives

$$\Xi_{\vartheta,i}(g) = c''\Upsilon_i(g), \qquad c'_\vartheta \Xi_{\vartheta,i}(g) = c'' F_i(g).$$

39.4.2 Ranks and levels in the three comparisons

In this subsection, the applications of Section 36.1 require simultaneous inequalities at every real place and the stated finite levels on each pair. We verify these conditions for the characters just used.

Let $c = \chi_M^j \vartheta$ and let α denote the actual source character in one class-sum comparison. If the kept rank is N, its number of negative signs is ℓ, and the number of real stacked determinants is c_τ, then

$$\mathrm{wt}_\tau(\alpha) = \ell - c_\tau, \qquad \nu = \alpha^{-1}\upsilon^{N/2}, \qquad \mu_\tau = N/2 - \ell + c_\tau.$$

This is the same μ_τ used in the real slice proof and in the normalization. It gives $N \le 2\mu_\tau + 2\ell$. The comparison needs $N \ge 2\ell + 2$, and the small member must have exactly ℓ negative signs since the large target is definite. In order, the small spaces are 0, I_m^-, and $S = I_{2m} \oplus I_m^-$. The third small quotient is therefore the one for the indefinite group $U(2m, m)$, with the Haar measure of Section 36.1.

For the first comparison the ranks (N, a, b) are $(m, m, 0)$, $\ell = 0$, $\alpha = c^{-1}$ and $c_\tau = 0$. The inequality is $m \ge 2$. For the second they are $(4m, 3m, m)$, $\ell = m$, $\alpha = c\upsilon^m$, and again $c_\tau = 0$. For the third they are $(6m + 2z_*, 3m + 2z_*, 3m)$, $\ell = m$, and $\alpha = ca_j$. The $m - j$ negative coordinates of W_j and the j negative pad coordinates give the kept negative rank. The real number of stacked determinants is explicitly

$$c_\tau = w_j(\tau) - j - \lfloor (t + k - 1)/2 \rfloor,$$

the cardinality of the set of determinant indices.

Since ca_j has weight $m + j + \lfloor (t + k - 1)/2 \rfloor - w_j(\tau)$ this is exactly the preceding weight condition. The parameters give m, k even, so $z_* = 7m/2 + k/2$ and all three ranks and defects are integers as written. At each old distinguished place the residue character of χ_M is ω_w, of order q_2. Thus $\omega_w^{\pm j}$ has order $q_2/\gcd(q_2, j) > 1$ for $1 \le j \le m < q_2$. At an additional twist place the character is nontrivial by exact twist support.

The fixed norm-one splitting and fixed a_j have trivial finite tame parts. Thus the same semilocal matching proof applies to the full modulus $\pi_M^2 \mathfrak{q}_\vartheta$ in all three cases. At an old place the determinant character factors through $k_w^\times$, while the test may distinguish classes modulo ϖ_w^2. Proposition 36.1 applies to these locally constant matched tests by the integral matching and prime-power inflation in Section 36.1.5. The reversed matrix Gauss sum in Section 36.2 is taken on its squarefree residue ring. The polynomial transfer uses a fixed real vector before the modulus, and it applies to the same determinant indices for every finite test.

39.4.3 The common period as the full gluing adjoint

In this subsection, with the three comparisons established, we pair their common period against arbitrary tuples in the target of Section 34.1. The identity identifies the adjoint used in the injectivity argument of Section 41.

Let y be in the original projected part of W_j and let $\zeta = (\zeta_{i,c})$ be a tuple in the original second-factor Weyl spaces. Write $K_{\zeta_i}^\circ(g)$ for the sum over the original V°-classes c of the second-block kernel at $(g, (1, c))$, with the polynomial in the negative coordinates associated to $\overline{\zeta_{i,c}}$. This kernel uses the untwisted W_j-character, the indicated power $\upsilon_{U(1)}^{m/2-j}$, modulus π_M^2, and the original inclusion indicator. The class weights and real test of that block are the ones in the gluing map.

Write $\theta_\vartheta(g)$ for the vector-valued kernel in the definite target with its original class pairing. In the second-linear convention the lift is $u_{j,\vartheta} y = \int_{[U(W_j)]} y(g)\theta_\vartheta(g)\, d\mu(g)$ and the normalized descent is $\mathcal{D}_\vartheta v(g) = \langle \theta_\vartheta(g), v \rangle$. Consequently the integrand for $\langle \zeta, G_\vartheta^* u_{j,\vartheta} y \rangle$ is $y(g)\overline{\mathcal{D}_\vartheta(G_\vartheta \zeta)(g)}$. For each (i, c, h) in $G_\vartheta \zeta$, the sum in the point form cancels the reciprocal stabilizer weight at the glued target point, by its defining evaluation identity. The first-block class retains the reciprocal stabilizer weight at J_Λ, which is one here. The second-block level is free, so the remaining sum over c is the plain sum for the arbitrary components $\zeta_{i,c}$. The product formula for the original glued lattice expands this integrand into the finite sum over i and the congruence

group $\bar{J}^1 = \{u \bmod \pi_M^2 : u \equiv 1 \bmod \pi_M\}$ of translates by the fixed lifts $k(u)$ of

$$M^{l_1} c_{L_i}\, y(g) \Xi_{\vartheta,i}(g) K^\circ_{\zeta_i}(g).$$

Let δ_i be the determinant of L_i on the old split members and let $\chi_{W,r}$ be the fixed split Weil character of W_j at rank r. The finite action on the matrix variable of the glued kernel is

$$f(X) \longmapsto c_{L_i} f(L_i^{-1} X), \qquad c_{L_i} = \chi_{W,r}(\delta_i) |\delta_i|^{-r/2}.$$

For a chosen $w \mid \pi_M$, let $s_{i,w}$ be the rank of $\Lambda \bmod w$. The row lattice of $\binom{\Lambda}{\pi_M I_{d_+}}$ has $s_{i,w}$ elementary divisors of valuation zero and $d_+ - s_{i,w}$ of valuation one. Thus $v_w(\det B_+) = s_{i,w}$ for $B_+ = \pi_M D^{-1}$. The other diagonal block $J_z + \pi_M(1 - J_z)$ has $d_- - r$ elementary divisors of valuation one. Hence, with $v_w(\varpi_w) = 1$, $v_w(\delta_{i,w}) = s_{i,w} + d_- - r$. The target character of Section 33.1.4 is $a = \upsilon^{m-j}$ on both blocks. In the convention of (F2), base change contributes $\upsilon_E^{2(m-j)}$, as in Section 20.3.3. The rank-r matrix variable contributes the split Weil factor υ_E^{-r} from Lemma 17.1. Thus $a_r = \chi_{W,r} = \upsilon_E^{2(m-j)-r}$ is unramified at the old split places, and the local factor of c_{L_i} is

$$\big(a_{r,w}(\varpi_w) q_w^{r/2}\big)^{s_{i,w}+d_- - r}.$$

This expression depends only on the original residue choice and the two block ranks. Put $C_i = M^{l_1} c_{L_i}$, the scalar of this summand in the common period. Both the character and L_i are fixed. At the twist places L_i and the lifts $k(u)$ are identity. At an old split place both L_i and $A = \pi_M L_i^{-1}$ are integral by Section 33.2. For $k_+ \in J_\Lambda$, let D_{old} denote the row basis matrix of $\binom{\Lambda}{\pi_M I_{d_+}}$. The two old level conditions give $D_{\mathrm{old}}(k_+ - 1) \equiv 0 \bmod \pi_M^2$. Therefore

$$L_i^{-1} \operatorname{diag}(k_+, 1) L_i - 1 = \pi_M^{-1} A \operatorname{diag}(k_+ - 1, 0) L_i \equiv 0 \bmod \pi_M.$$

This transports the old principal level to the glued point. The condition $L_i^{-1} X \equiv I_{\mathrm{incl}} \bmod \pi_M$ implies X is integral. On integral X it is equivalent to

$$AX \equiv \pi_M I_{\mathrm{incl}} \pmod{\pi_M^2}.$$

The same section splits this condition into the two residue blocks. Let S_+ denote the first-block matrix after the right translation by $k(u)$, with inclusion matrix J_I. The lower rows of w give $\pi_M S_+ \equiv \pi_M J_I \pmod{\pi_M^2}$. Hence $J_I^{\mathrm{t}} S_+ \equiv 1 \pmod{\pi_M}$ at the old places and every old tame determinant multiplier is one there. At each new place $k(u) = 1$, so the corresponding factor in $\chi(\det(J_I^{\mathrm{t}} S_+))$ is exactly the prescribed new determinant weight on the original inclusion block. Thus the first block supplies the entire finite determinant factor. The second block retains its inclusion modulo π_M^2 and character $b_j^\circ = \upsilon^{m/2-j}$. Since $e_j^{(1)} + m/2 - j = (r+t)/2$, the product on the original target is

$$b''_{j,\vartheta} b_j^\circ = \mu_j^0 \vartheta\, \upsilon^{(r+t)/2}.$$

Write $A_i(g) = \Xi_{\vartheta,i}(g) K^\circ_{\zeta_i}(g)$. For any left test for which these integrals converge, right invariance of the quotient measure gives

$$\sum_{u \in \bar{J}^1} \int y(g) A_i(gk(u))\, d\mu(g) = \int \Big(\sum_{u \in \bar{J}^1} y(gk(u)^{-1}) \Big) A_i(g)\, d\mu(g).$$

Every y in the projected source is fixed by the old principal level, which contains the $k(u)$. The parenthesized sum is therefore $|\bar{J}^1| y(g)$ on this source. The integrals converge absolutely: each

class sum is finite and its theta kernels have moderate growth for the fixed Schwartz vectors, while y decreases rapidly on Siegel sets. For the moderate growth use continuity of the lattice sum in the Schwartz seminorms and the polynomial bounds on these seminorms under the real oscillator action. The powers can be absorbed by arbitrarily high rapid-decay powers of y. It follows that

$$\langle \zeta, G_\vartheta^* u_{j,\vartheta} y\rangle = \sum_i M^{l_1} c_{L_i} |\bar{J}^1| \int_{[U(W_j)]} y(g)\Xi_{\vartheta,i}(g) K_{\zeta_i}^\circ(g)\, d\mu(g). \tag{110}$$

At an old place the elements of $\bar{J}^1$ are $1+\varpi_w A$ with $A \in M_r(k_w)$. Thus $|\bar{J}^1| = \prod_{w\in S_M} q_w^{r^2}$. The earlier trace sum has cardinality $\mathcal{N} = \prod_{w\in S_M} q_w^{(3m)^2} |\mathrm{GL}_{3m}(k_w)|$, by reduction of invertible matrices modulo ϖ_w^2. It entered $c'_\vartheta = \gamma_3\gamma_{12}/\mathcal{N}$. Substituting $c'_\vartheta \Xi_{\vartheta,i} = c'' F_i$ now gives

$$c'_\vartheta \langle \zeta, G_\vartheta^* u_{j,\vartheta} y\rangle = \sum_i M^{l_1} c_{L_i} |\bar{J}^1| c'' \int_{[U(W_j)]} y(g) F_i(g) K_{\zeta_i}^\circ(g)\, d\mu(g).$$

The normalized descent is conjugated in this pairing, so the first factor $\overline{c''\Upsilon_{\vartheta,i}}$ contributes c'', and the conjugate descent scalar contributes c_{L_i}. The right side is independent of ϑ. At the trivial twist it is $\langle \zeta, Py\rangle$ for $P = c'_1 G_1^* u_{j,1}$. Equality for all tuples gives the operator identity by nondegeneracy.

Proposition 39.1 (Common period and the gluing adjoint)**.** *On the original projected source, at each allowed twist,*

$$c'_\vartheta G_\vartheta^* u_{j,\vartheta} = P.$$

The period P is independent of ϑ.

In particular $\ker u_{j,\vartheta} \subseteq \ker P$. For the proof of injectivity in Section 41, the integral expression for $\langle \zeta, Py\rangle$ uses the multipliers $M^{l_1} c_{L_i} |\bar{J}^1| c''$, which are nonzero by the formula for c_{L_i} and (I5).

40 Coefficient reproduction on the descent space

In this section, we establish real coefficient separation and the inner-product identities for the original periods of Section 39.4. These give the coefficient tests used in the proof of injectivity in Section 41.

All measures, classes, Fock polynomials and finite lattices in the following argument belong to the original specified space. The sharp inner-product calculation below includes an arbitrary rapidly decreasing left test and the compact-source auxiliary lift. The adelic overlap calculation gives the polarized Fock overlap on the same residue ring, including the unit coefficient at zero displacement.

The original form in the projected space is read in its holomorphic coefficient by the untwisted sharp injection and its adjoint.

40.1 Separation by the circle mean

In this subsection, we prove a reproducing identity by integrating over a circle, then use it to estimate the reference coefficient at a peak. Section 41.1.6 applies the estimate to the common period.

Fix the original signed real product $H = \prod_{\tau\in\Phi} U(J_\tau)$ of rank n, with the product Haar dh, and let $f = |\Phi|$. Use the same injective order o on the basis, with the positive block first. The weights u, u' decrease weakly within each sign block. At every place the difference of any negative entry and any positive entry of each weight is at least $300n_*$, where

$$1 \le n \le t, \qquad n_* = 2ft^3, \qquad |u_{\tau,i}| \le 5b.$$

For either weight v define m_v as the product over the real places of the minor quotients.

In the order o, its local factor is

$$m_{v_\tau}(h) = \det(h)^{v_{\tau,n}} \prod_{a=1}^{n-1} \Delta_a(h)^{v_{\tau,a}-v_{\tau,a+1}}, \qquad \Delta_a(h) = \det(h_{[1,a],[1,a]}). \tag{CM1}$$

The only possible negative exponent on a proper minor is at the end of the positive block. That minor never vanishes on $U(J_\tau)$. Formula (CM1) is also the definition by quotient of two dominant minor products: add $2\sum_i |v_{\tau,i}|$ uniformly to the positive block in the numerator, and divide by the minor product for that shift. In a compact sign group use the same formula with no negative proper-minor exponent.

An elementary majorant. For a mixed signature write $h = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the positive-first order. The equations $h^*Jh = J = hJh^*$ imply that the singular values of A, D are at least one, $\|B\| = \|C\|$, and $\|A\|^2 = \|D\|^2 = 1 + \|C\|^2$. Consequently $\|h\| \le \|A\| + \|C\| \le 2|\det A|$. Every positive-initial segment minor has absolute value at most $|\det A|$, by the singular-value bound for an exterior power of the compression of A. For a longer initial segment, the complementary minor identity and $h^{-1} = Jh^*J$, $|\det h| = 1$, express its absolute value as that of a trailing minor of D. The same singular-value bound applies, and $|\det D| = |\det A|$ by the complementary identity at the end of the positive block. The nonnegative exponents and the one cross-block exponent in (CM1) therefore give

$$|m_{v_\tau}(h)| \le |\det A|^{v_{\tau,1}-v_{\tau,n}} \le \min\{1, (2/\|h\|)^{300n_*}\}. \tag{CM2}$$

In compact signature $\|h\| = 1$ and every proper minor has absolute value at most one, giving the same statement. This argument also proves analyticity locally around each group point by (CM1).

Let $I_\tau = \int |m_{u_\tau}|^2\, dh_\tau$. Take $L = 2\lfloor 5b\rfloor + n$, $\beta = n + L + 1$, and $\delta_1 = (200\beta^5)^{-1}$. For $\|h - 1\| \le \delta_1$, the logarithm series in each principal compression gives $|\log \Delta_a(h)| \le 2n\delta_1$. The sum of absolute exponents in (CM1) is at most $2n \max_i |u_{\tau,i}|$. Hence $|m_{u_\tau}(h)| \ge 1/2$ on that ball. A maximal δ_1/Y-separated set in the group ball $\|h\| \le Y$, $Y \ge 1$, has at most $(2000\beta^6Y^2)^{2n^2}$ points, by packing disjoint ambient matrix balls in dimension $2n^2$. The corresponding right translates of the identity ball cover the group ball, since $\|h^{-1}\| = \|h\|$.

In particular ball volumes have polynomial growth of order $4n^2$. With (CM2) this first proves $0 < I_\tau < \infty$. Then the small-ball estimate gives the sharper bound

$$dh_\tau\{\|h\| \le Y\} \le 4I_\tau(2000\beta^6Y^2)^{2n^2}.$$

Put $c = 10n^3$. Integrating the last bound against $cY^{-c-1}\, dY$ gives

$$\int (2/\|h\|)^c\, dh_\tau \le 8I_\tau(2000\beta^6)^{2n^2}2^c \le I_\tau 2^{13t^3}b^{14t^2}.$$

The first inequality uses $c/(c-4n^2) \le 2$. The second uses the original size condition $2000(n+L+1)^6 \le b^7$ and $n \le t$. No formula for the numerical formal degree is needed for this estimate.

Write $X(h) = \max_\tau \|h_\tau\|$ and $I = \int_H |m_u|^2\, dh = \prod_\tau I_\tau > 0$. For each h, use $6n_*$ extra powers in (CM2) at any place attaining $X(h)$, and c powers at the others. Since $c + 6n_* \le 300n_*$,

$$X(h)^{6n_*}|m_{u'}(h)| \le 2^{6n_*} \prod_\tau (2/\|h_\tau\|)^c.$$

This bound is independent of the choice of maximizing place. Integrating gives

$$\int_H |m_{u'}(h)| X(h)^{6n_*}\, dh \le 2^{6n_*}(2^{13t^3} b^{14t^2})^f I \le b^{n_*} I. \tag{CM3}$$

In the last step use $b \ge 2^{25}$, $t \ge 14$, and $n_* = 2ft^3$.

The circle mean and reproduction. Let a function A on H be holomorphic of weight v in the sense of the model. Its analytic extension to neighborhoods of hR, with R invertible upper triangular in order o, transforms on hR by the product of diagonal characters $\chi_v(R) = \prod_{\tau,i} R_{\tau,ii}^{v_{\tau,i}}$. Use the circle with probability Haar $d\theta$, the simultaneous rotation $r_s = \mathrm{diag}(s^{o(i)})$ at every real place, and let $\chi_v(s) = \chi_v(r_s)$. Then

$$\int_{\mathbb{T}} A(r_s h)\chi_v(s)^{-1}\, d\theta(s) = A(1) m_v(h). \tag{CM4}$$

Here is a direct verification. For h close to the identity write each $h_\tau = L_\tau R_\tau$ with L_τ lower unitriangular and R_τ upper triangular.

For $|s| = 1$,

$$r_s h_\tau = (r_s L_\tau r_s^{-1})(r_s R_\tau).$$

All strictly lower entries of the first factor have positive powers $s^{o(i)-o(j)}$. The factor extends polynomially to the closed disc, is identity at its center, and stays in the analytic neighborhood of the identity for h sufficiently close to it. Write $L_s = r_s L r_s^{-1}$ and $\widetilde{A}$ for the holomorphic extension near the identity. For $|s| = 1$, the expression $L_s = (r_s h)(r_s R)^{-1}$ has first factor in H and second factor upper triangular. The character rule gives

$$\chi_v(s)^{-1} A(r_s h) = \chi_v(R)\widetilde{A}(L_s).$$

The mean value property gives $\chi_v(R)A(1)$, which equals $m_v(h)A(1)$ by the leading-minor formulas for triangular elimination. For arbitrary h join identity to h by a real analytic path in the connected product H.

Such a path is obtained from the compact and positive factors of the polar decomposition. The two sides of (CM4) are real analytic along it. For the left side the local analytic extension is uniform in the circle variable by its compactness. Analytic continuation along the path proves (CM4).

If $A\overline{m_v}$ is integrable, apply (CM4) after the left translations $h \mapsto r_s h$, which preserve the original Haar. The principal minors in (CM1) transform by their diagonal weights, so $m_v(r_s h) = \chi_v(s) m_v(h)$. Joint integrability follows from the same left translations and the probability circle. Hence

$$\int_H A(h)\overline{m_v(h)}\, dh = A(1) \int_H |m_v(h)|^2\, dh. \tag{CM5}$$

The peak estimate. Let F be continuous and bounded on the adelic group, with $h \mapsto F(x(h,1))$ holomorphic of weight u for every x. For $F \neq 0$, call z a peak if $|F(z)| > \|F\|_\infty/2$. Such a point exists by the definition of the supremum, since $0 < \|F\|_\infty < \infty$. Fix one such z. Suppose K_{ref} is bounded on H, and at every peak x some K_x of holomorphic weight $u - u'$ satisfies

$$|K_x(h) - K_{\mathrm{ref}}(h)| \le \delta X(h)^{6n_*}, \qquad \int_H \overline{F(x(h,1))} K_x(h) m_{u'}(h)\, dh = 0. \tag{CM6}$$

Lemma 40.1 (The circle-mean peak estimate)**.** *Under the holomorphic weight, peak, and approximation conditions above, the reference function satisfies*

$$|K_{\mathrm{ref}}(1)| \le (4b^{n_*} + 1)\delta.$$

Proof. Every $z_s = z(r_s, 1)$ is again a peak: its value differs from $F(z)$ by the unit character $\chi_u(s)$. Put $K_* = K_z$ and

$$W_s(h) = \overline{F(z(r_s h, 1))\chi_u(s)^{-1}}.$$

Equation (CM4) at z gives $\int W_s(h) d\theta(s) = \overline{F(z) m_u(h)}$.

Also $|W_s(h)| \le 2|F(z)|$, and (CM6) eliminates the integral with K_{z_s} in place of K_*. Thus (CM3) bounds each fixed slice by

$$\left| \int_H W_s(h) K_*(h) m_{u'}(h)\, dh \right| \le 4\delta |F(z)| b^{n_*} I.$$

Only K_* occurs in the joint integral. It is jointly measurable and integrable by (CM3), since K_* is bounded by a constant times X^{6n_*}. No regularity in the choice of K_{z_s} as a function of s is needed. Average the slice bound and use Fubini and (CM5) on the product $A = K_* m_{u'}$, which is holomorphic of weight u. Since $m_{u'}(1) = 1$, this gives

$$I|F(z)|\,|K_*(1)| \le 4\delta |F(z)| b^{n_*} I.$$

Consequently,

$$|K_{\mathrm{ref}}(1)| \le (4b^{n_*} + 1)\delta. \tag{CM7}$$

Indeed, cancellation gives $|K_*(1)| \le 4\delta b^{n_*}$, and (CM6) at $h = 1$ bounds $|K_{\mathrm{ref}}(1) - K_*(1)|$ by δ, since $X(1) = 1$. □

40.2 The inner product formula at the principal levels

In this subsection, we unfold the inner product with an arbitrary rapidly decreasing left test on the original levels. The calculation extends the use of the Gram identity in Section 34.2 to the left tests needed for injectivity.

Let S, T be the original Hermitian spaces of ranks n, s, with $s > 2n$, and retain their finite unitary bases. Write $G = U(S)$, $H = U(T)$, and use the original quotient measures. On the finite group normalize the integral compact at each place of F_0 to have mass one. On additive compact quotients in the Weil calculation use probability Haar. These conventions leave the factor $V_T = \mathrm{vol}([H])$ when passing from the probability average to the original target pairing. The argument applies to the pairs

$$(n, s) = (r, t - 9m - 1), \quad (3m, 6m + 1), \quad (m, 3m).$$

In the first pair $s - 2n = 3r - 9m - 1 > 0$ for sufficiently large x, since $t = 5r$, $r \geq x^5$ and $m < 2\sqrt{x}$. In the second and third pairs $s - 2n = 1$ and m, respectively. The first two pairs use the original point datum at $\mathfrak{m} = (\pi_M^2)$. The third uses the unit-modulus vacuum and has compact real source.

The sharp datum and its mass. At the chosen member w of each chosen split pair the Schrödinger variable X is an s-by-n matrix. The point datum is the indicator of $X \equiv \iota \pmod{\mathfrak{m}}$ for the inclusion ι, with vacuum at the other places. Write $R = \mathcal{O}_E/\mathfrak{m}$. On the split matrix slice, the unit vacuum is the indicator of the integral box of additive volume one. Its residue classes have equal volume, so the inner product of two tests inflated from $M_{s,n}(R)$ is their average there. The squared norm of the point datum is $|R|^{-sn}$. Let J be the original principal level in the standard compact K^0 at this modulus. The integral action on the slice is

$$\Psi(X) \longmapsto b(\det g)^{-1}\Psi(Xg).$$

Here b is the source-actuating character of the lift. Its tame part is trivial away from the prescribed support, and on J it evaluates to one. In the two point tests its powers are, respectively, $b = \upsilon^{m/2-j}$ and $b = 1$. Both characters are trivial on integral determinants at every finite place, since υ_f is unramified. By Proposition 4.2, $M = p_0p_1$ and $N \notin \{p_0, p_1\}$. The chosen members over p_0, p_1 have exponent two in π_M^2. The opposite members are determined by duality. Every place over N has the unit vacuum and the full integral compact in these reproductions. The additional covector condition at π_N in Section 6 is on the earlier definite source of rank k.

Let g be integral off the pairs and suppose it gives nonzero overlap. At a chosen split place dividing $\mathfrak{m}$ the source acts in the Schrödinger model by $X \mapsto Xg$, times a nonzero determinant scalar. The congruences to the inclusion matrix give integral left inverses of both X and Xg. Applying them to Xg and X, respectively, makes g and g^{-1} integral at the chosen member of every pair. At the opposite member this follows by inverse transpose. The congruences $X \equiv Xg \equiv \iota$, followed by a left inverse of ι in the residue ring, give $g \equiv 1$.

Conversely J fixes the test. The character on the groups therefore gives no factor on its coefficient support. The coefficient on the points integral off the pairs is exactly $|R|^{-sn}\mathbf{1}_J$. Its integral is

$$c_{\mathfrak{m}} = \frac{|R|^{-sn}}{|\operatorname{GL}_n(R)|}.$$

At a selected place w of exact modulus exponent two the local mass is

$$\frac{q_w^{-2sn}}{q_w^{n^2}|\operatorname{GL}_n(k_w)|}.$$

Indeed reduction from exponent two to exponent one is surjective with kernel $1 + \varpi_w M_n(k_w)$. The split group over F_0 is one copy of $\operatorname{GL}_n(E_w)$. The opposite matrix is determined by the inverse transpose and does not give a second volume. At unit modulus $J = K^0$, the finite vacuum is a unit vector, and $c_{(1)} = 1$.

40.2.1 The adelic doubling identity

In this subsection, we identify the doubled kernel in the original quotient measures after the mass calculation of Section 40.2.

Let v be the original real Fock vector and $f = v \otimes F$ the datum with the finite part just specified. Denote the rational Gram matrices by J_S, J_T and the adelic point matrices by X_S, X_T.

Put $u = (1, u_f)$ with $u_f\overline{u_f} = -1$, so the opposite space has $J_{S^-} = -J_S$ and $X_{S^-} = uX_S$. At infinity the coordinate transport is the identity and the alternating form changes sign. The conjugate oscillator datum there is transported by complex conjugation. For an oscillator vector f write $D_f(X) = \theta(\mathcal{R}(X)f)$ on the adelic displacement space. Let $Jf'(Y) = \overline{f'(u^{-1}Y)}$ be the antilinear transport on oscillator data. The probability-mean argument of Section 39.3 for this pair gives $\overline{\theta_{\mathrm{old}}(f')} = \lambda_{S,T}\theta_{\mathrm{new}}(Jf')$ with $|\lambda_{S,T}| = 1$. Set $C = \lambda_{S,T}J$. Moving a Heisenberg translation across J gives

$$D^{\mathrm{new}}_{Cf'}(Y) = \lambda_{S,T}\theta_{\mathrm{new}}\big(J\mathcal{R}_{\mathrm{old}}(u^{-1}Y)f'\big) = \overline{D_{f'}(u^{-1}Y)}.$$

Use the tensor-product normalization of the doubled theta functional from Section 39.4.1. In a direct sum of rational Schrödinger polarizations, summation over the product of the two rational Lagrangians gives the displaced value

$$D_f(X)\overline{D_{f'}(u^{-1}Y)}$$

on $f \otimes Cf'$. If the doubled functional has the previously fixed phase ϵ relative to this product, its theta integral and each graph zero term, obtained by averaging the same displaced functional, both acquire ϵ. Dividing the Siegel–Weil identity by this common phase gives the same pairing. Write $a(\det h)b(\det g)^{-1}$ for the pair character and put $\upsilon_1 = \upsilon_E|_{T_1(\mathbb{A}_{F_0})}$. The determinant of $X \mapsto hXg^{-1}$ on the Hom space is $(\det h)^n(\det g)^{-s}$. The multiplier calculation of Section 39.3 gives the opposite pair $(a^-, b^-) = (a^{-1}\upsilon_1^n, b^{-1}\upsilon_1^s)$. On the diagonal H the product of target twists $\upsilon_1(\det h)^n$ cancels the mixed-model Levi scalar $\upsilon_1(\det h)^{-n}$ of Section 17.2.

In the split polarization the rational Lagrangian

$$L_\delta = \{(\delta^{-1}y, -y) : y \in S\}, \qquad \delta \in G(F_0),$$

is parameterized by $y \mapsto (\delta^{-1}y, -y)$. For a matrix Z in the adelic zero term set

$$M = X_T^{-1}J_T^{-1}ZJ_SX_S, \qquad \widehat{\delta} = X_S^{-1}\delta X_S.$$

The adjoint of this parametrization gives the blocks $(M\widehat{\delta}, uM)$, since $(\delta^{-1})^*J_S = J_S\delta$ and $(-1)^*(-J_S) = J_S$. The second tensor factor therefore has argument M in its original coordinates. The change $Z \mapsto M$ maps the standard rational matrices bijectively onto

$$\{X_T^{-1}WX_S : W \in \mathrm{Mat}_{s\times n}(E)\},$$

because J_T^{-1} and J_S are rational and invertible. The induced isomorphism of additive quotients preserves probability Haar. Rational Weil covariance gives $D_f(M\widehat{\delta}) = D_{\omega_b(\delta)f}(M)$, since b is trivial on rational determinants. Under a rational translation of M, the two displaced values have the same Heisenberg multiplier. Their product below is therefore a function on the additive quotient. The probability mean of Section 41.5.1 gives its value:

$$\int D_{\omega_b(\delta)f}(M)\overline{D_{f'}(M)}\,dM = \langle f', \omega_b(\delta)f\rangle.$$

This is the graph zero term. Translating the two data separately by g, g' and using unitarity gives the coefficient

$$\langle f', \omega_b(g'^{-1}\delta g)f\rangle.$$

Only these graphs contribute. If L is a Lagrangian and its second projection is injective, it is an isomorphism since $\dim L = n$; the isotropy equation then makes the first projection an isometry. Otherwise L contains $(c, 0)$ with $c \neq 0$. In the compact-source case this is impossible by definiteness. For either point test choose a split place $w \mid \mathfrak{m}$, which exists since $M > 1$. The zero-term functional is invariant under the Heisenberg translations dual to this vector, acting on the first tensor factor alone.

Indeed, for $z \in T(\mathbb{A})$, the rank-one map $(v, v') \mapsto z\, h_S(c, v)$ vanishes on L by isotropy. It belongs to the isotropic displacement space averaged in the zero term, so translating by it preserves that mean. It acts only on the first tensor factor. Taking z in the local dual coordinate at the split pair gives a pure modulation on the chosen Schrödinger slice. After moving the source point through the Heisenberg action these modulations act by the multipliers

$$X \longmapsto \psi_w(\mathrm{tr}(ZXc_w)),$$

where Z varies over the dual local vector space and $c_w \neq 0$ is the coordinate in the chosen basis of c.

An invariant functional kills a compactly supported, locally constant test supported where $Xc_w \neq 0$: partition its support into finitely many compact open pieces on each of which one such multiplier is a constant different from one, and apply invariance on each piece. Every matrix in the original point support has an integral left inverse. After any source translate at w, the support is obtained by invertible right multiplication, so every matrix there still has column rank n over E_w. Hence $Xc_w \neq 0$ on that support for every nonzero c_w. This proves vanishing for the translated datum at each pair (g, g'), and the graph terms exhaust the doubled expansion.

The doubled action is identified in Section 17.2. Apply the Siegel–Weil formula [80, Theorem 2, p. 237] to the split polarization of rank n of $S \oplus S^-$, with T as the Hermitian space of dimension s. The doubled splitting uses υ_E^s, whose restriction to $\mathbb{A}_{F_0}^\times$ is ε_{E/F_0}^s. Yamana's division algebra is E, and his Witt index of T is at most $s/2$. Hence the convergent clause $s - \mathrm{Witt}(T) > n$ applies with coefficient one and $s_0 = (s - n)/2 > n/2$.

Let $\Phi = f \otimes Cf'$, $d(g, g') = \mathrm{diag}(g, (g')^-)$, and let Ω be the doubled action with target twist υ^n and trivial source twist. Direct-sum covariance and the opposite character formula above give the separately translated tensor

$$\omega_b(g)f \otimes C\omega_b(g')f' = b(\det g)^{-1} b^-(\det g')^{-1}\, \Omega(d(g, g'))\Phi.$$

The scalar has absolute value one. It multiplies the theta integral and every Siegel section by the same factor. Use the transported Fock polynomial identity of Section 17.2 for the original tensor. Its section at s_0 satisfies

$$f^{(s_0)}_{\Omega(d(g,g'))\Phi}(a) = f^{(s_0)}_{\Phi}(a\, d(g, g')), \qquad f^{(s_0)}_{\Phi}(a) = (\omega(a)\Phi)(0).$$

Thus the formula for the fixed polynomial datum applies at all the translated points. The strict-range formula uses target quotient measure of mass one [78, §52, Theorem 5 and the normalization following (41)]. Multiplying by V_T converts it to the original Haar measure and gives

$$\int_{[H]} \overline{\theta_{f'}(g', h)}\, \theta_f(g, h)\, dh = V_T \sum_{\delta \in G(F_0)} \langle f', \omega_b(g'^{-1}\delta g) f \rangle.$$

Here $dh = V_T\, dh_{\mathrm{prob}}$ on $[H]$, with the probability convention of [79, §1.3, p. 1007]. For the two source integrations we need absolute bounds in the original quotient measures. In the split Schrödinger model and for a doubled datum Φ, set

$$J(\Phi) = \int_{[H]} \sum_{X \in T^n(F_0)} |\Phi(h^{-1}X)|\, dh.$$

Here the rational points of T^n are taken over F_0 as for the restriction of scalars. For a fixed finite factor this is finite on the real Schwartz space.

Indeed the absolute value of the real factor is bounded by a nonnegative Schwartz function, by taking smooth annular cutoffs with rapidly decreasing coefficients. Use that majorant and the absolute value of the finite factor in the convergent theta integral [78, §51, Proposition 8, p. 75]. The target acts by substitution in the split model of Section 17.2. The involution of E is of the second kind, so Weil's type (I) parameter is $1/2$ by [78, §§26–27]. His convergence condition is therefore $s - \mathrm{Witt}(T) > n$, as verified above.

The absolute integral J is bounded by a continuous real Schwartz seminorm with the finite factor fixed. To see this, choose a measurable quotient section and take finite partial sums of the rational evaluations, with measurable multipliers of modulus at most one. After integration over the section these are continuous linear functionals, each bounded by the real supremum norm times a finite constant. Their common pointwise bound is J. Uniform boundedness on the Schwartz Fréchet space gives a common continuous seminorm. Taking the supremum over partial sums and multipliers proves the claim. In particular J bounds the absolute product of the original theta kernels integrated over $[H]$.

At the identity of the doubled group the Iwasawa formula bounds the absolute graph sum by

$$E_{\mathrm{sph}}(1, s_0) \sup_k |(\omega(k)\Phi)(0)|.$$

Here the supremum is over the maximal compact and the spherical Eisenstein series converges absolutely since $s_0 > n/2$. At g, g' apply this same bound to the translated doubled datum at the identity. That supremum is a continuous real Schwartz seminorm up to a fixed constant. The real compact acts continuously on Schwartz data and the finite compact has finitely many translates of the fixed finite factor. The absolute graph sum is rationally invariant in the two source variables by reindexing. The corresponding absolute product on the target side is invariant by rational theta invariance. Thus we may reduce both source representatives as in Section 18.1. Their finite translates range over finite sets.

Multiplication and differentiation operators are conjugated by the real Weil action into linear combinations with coefficients from the symplectic matrix and its inverse. By unitarity their products give polynomial bounds for the translated weighted Sobolev norms, which generate the Schwartz topology. The two seminorm bounds for the doubled translate are therefore at most $C\mathcal{H}(g)^{\nu}\mathcal{H}(g')^{\nu}$ for the original reduction heights. Set $k_{f',f}(a) = \langle f', \omega_b(a) f \rangle$. For rapid y, y', choose decay powers at least ν in each variable. With the original quotient measures this gives

$$\int_{[G]^2} |y(g)y'(g')| \left(\int_{[H]} |\theta_{f'}(g', h)\theta_f(g, h)|\, dh + \sum_{\delta \in G(F_0)} |k_{f',f}(g'^{-1}\delta g)| \right) dg\, dg' < \infty.$$

Indeed the integrand on $[G]^2$ is bounded and the quotient has finite volume. The first term permits interchange of the target and source integrals. The second permits termwise integration

of the rational graph expansion. This argument also applies to the strict-range doubled first arrow.

For rapid y', y, multiply by $\overline{y'(g')}y(g)$ and unfold the rational graph sum. For each fixed g', the sum of the translated source quotient integrals is the $G(\mathbb{A})$-integral in $a = g'^{-1}\delta g$. Indeed the translates by $G(F_0)$ of a measurable fundamental domain tile $G(\mathbb{A})$ up to a null set, and left translation by g'^{-1} preserves Haar measure. This uses counting measure on $G(F_0)$ in the source quotient formula. Left automorphy gives $y(g) = y(g'a)$ on that graph term. Thus

$$\langle L_{f'}y', L_f y\rangle_T = V_T \int_{G(\mathbb{A})} \langle f', \omega_b(a)f\rangle \langle y', R(a)y\rangle_S \, da.$$

In particular the first test need not be fixed by J.

40.2.2 Vacuum coefficients away from the chosen places

In this subsection, we evaluate the local factors in Section 40.2.1.

Let A be the union of conjugation orbits outside the support of $\mathfrak{m}\bar{\mathfrak{m}}$. At A put $\Lambda = \widehat{\mathcal{O}}_E^n$ in the original finite basis. For γ supported on A put $\Delta(\gamma) = (\upsilon_E)_f(\det B)|\det B|^{-1/2}$, where $B\Lambda = \Lambda \cap \gamma^{-1}\Lambda$. This expression is independent of the integral basis of the intersection since $(\upsilon_E)_f$ is unramified there. The index formula is

$$I(\gamma) = |\Delta(\gamma)|^2 = [\Lambda : \Lambda \cap \gamma^{-1}\Lambda] = [\gamma\Lambda : \Lambda \cap \gamma\Lambda].$$

In the normalized finite lattice model the uncharactered Weil operator for an isometry a of the displacement space is

$$\omega(a)u = \Delta_{\mathrm{disp}}(a) \int u(a^{-1}z)\mathcal{R}(z)v_{\mathrm{vac}}\, dz,$$

where z is a finite Heisenberg displacement, $\mathcal{R}(z)$ its operator, and the vacuum lattice has additive measure one.

In the original finite displacement coordinates the vacuum lattice is $D = \alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E^{sn}$. The inverse different identity of Section 8.1 makes D self-dual. Write $z = \alpha_{\mathrm{W}}u$ and $c_{\mathrm{tr}} = (1-j_{\mathrm{W}}^{-1})/2$. The phase on D is

$$\eta(z) = \psi_f\left(\tfrac{1}{2}c_{\mathrm{tr}}j_{\mathrm{W}}\sum_\nu \bar{z}_\nu z_\nu\right) = \psi_0\left(\tfrac{1}{2}c_{\mathrm{tr}}\sum_\nu \bar{u}_\nu u_\nu\right), \qquad \psi_0(t) = \psi_f(j_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}\alpha_{\mathrm{W}}t).$$

Here the finite unitary bases give the identity Hermitian matrix on the Hom coordinates. On $D \cap aD$ the product $\eta(a^{-1}z)\overline{\eta(z)}$ equals one by isometry, including at places above two. The integrand vanishes off this intersection. Since a preserves additive Haar and D has mass one, its integral is

$$\mathrm{vol}(D \cap aD) = [D : D \cap a^{-1}D]^{-1} = |\Delta_{\mathrm{disp}}(a)|^{-2}.$$

It follows that

$$\langle v_{\mathrm{vac}}, \omega(a)v_{\mathrm{vac}}\rangle = \Delta_{\mathrm{disp}}(a)|\Delta_{\mathrm{disp}}(a)|^{-2} = \overline{\Delta_{\mathrm{disp}}(a)}^{-1}.$$

On $\mathrm{Hom}(S,T)$ the action of γ is $X \mapsto X\gamma^{-1}$. The s covectors give $\Delta_{\mathrm{disp}}(a) = \Delta(\gamma^{-1})^s = \overline{\Delta(\gamma)}^s$. For the last equality, the inverse of a unitary matrix is its conjugate transpose. Transposition preserves the elementary divisors, and conjugation takes the idele splitting multiplier to its

complex conjugate in the fixed unramified normalization. The character $b(\det\gamma)^{-1}$ therefore gives the exact coefficient

$$a(\gamma) = \big(b(\det\gamma)\Delta(\gamma)^s\big)^{-1}, \qquad |a(\gamma)| = I(\gamma)^{-s/2}.$$

It multiplies over disjoint conjugation orbits and is bi-invariant by the standard integral group on A. Explicitly, if the elementary divisors at w are $\varpi_w^{e_i}$, the local contribution to $[\Lambda : \Lambda\cap\gamma^{-1}\Lambda]$ is $q_w^{\sum_i \max(0,-e_i)}$. At a split pair the opposite exponents are the negatives of these, so the pair contributes $q_w^{\sum_i |e_i|}$. At an inert place the multiset is its own negative by unitarity, giving $q_w^{\frac{1}{2}\sum_i |e_i|}$. By Section 4.2, the relative extension has no other finite local type.

40.2.3 Summability and injectivity

We use Kneser's method of neighboring lattices to bound the coefficients of Section 40.2.2. For the construction of Hermitian neighbors at primes where the lattice is unimodular, see [32, Propositions 5.3–5.4].

At a split pair a lattice at w determines the opposite lattice by duality. If the moved lattice is not contained in the standard one at w, take a vector x in it with $\varpi_w x$ primitive in the standard lattice. Enlarge the standard lattice by x and use its dual at $\bar{w}$. The intersection with the moved lattice grows by a factor q_w at w, since the new quotient is generated by the class of x with annihilator (ϖ_w). At $\bar{w}$ every element of the old intersection pairs integrally with x, by duality with the moved lattice at w. The opposite intersection is therefore unchanged. If containment is strict in the other direction, choose the vector at $\bar{w}$. Each enlargement at a chosen member is determined by the line of $\varpi_w x$ in k_w^n, giving at most $(q_w^n - 1)/(q_w - 1)$ steps for that member.

At a nonsplit finite place the extension $E_w/F_{0,v}$ is unramified. Take the uniformizer ϖ_w in $F_{0,v}$. Both lattices are self-dual, so containment implies equality. If they differ, scale a vector in the moved lattice outside the standard one by a power of ϖ_w to obtain x with $y = \varpi_w x$ primitive in the standard lattice Λ_w, and put

$$\Lambda_0 = \{z \in \Lambda_w : \langle y, z\rangle \in \varpi_w \mathcal{O}_{E_w}\}.$$

The lattice $\Lambda' = \Lambda_0 + \mathcal{O}_{E_w} x$ is integral: $\langle x, x\rangle$ is integral in the moved lattice and $\langle x, \Lambda_0\rangle \subseteq \mathcal{O}_{E_w}$ by definition. The primitive vector y defines a nonzero residue functional, and $y = \varpi_w x$ belongs to Λ_0. Hence $[\Lambda_w : \Lambda_0] = [\Lambda' : \Lambda_0] = q_w$. The integral lattice Λ' has the covolume of the self-dual lattice Λ_w, so it too is self-dual. Write D_w for the moved lattice and $D_0 = D_w \cap \Lambda_w$. Integrality of D_w gives $D_0 \subset \Lambda_0$, so $D_w \cap \Lambda_0 = D_0$. Reduction gives an isomorphism

$$(D_w \cap \Lambda')/D_0 \longrightarrow \Lambda'/\Lambda_0.$$

Indeed the kernel is zero and $x \in D_w \cap \Lambda'$ maps to a generator. Thus the index is exactly q_w.

For a fixed residue line choose a representative of $\bar{y}$; choices of x then differ by $z \in \Lambda_w$. The integrality of the two Hermitian squares requires $\mathrm{Tr}_{k_w/k_v}\langle \bar{y}, \bar{z}\rangle = 0$. This leaves at most $\sqrt{q_w}$ classes modulo Λ_0 for each residue line. These neighboring self-dual lattices admit integral orthonormal bases. One way to see this last assertion is to find a vector of unit norm by nondegeneracy of the residual Hermitian form, normalize it by surjectivity of the unramified norm on units, and continue on its unimodular orthogonal complement. The residual norm

polynomial is not identically zero on a nondegenerate Hermitian space: if every self-pairing vanished, polarizing after multiplication by all elements of k_w and using the nondegenerate trace would make every pairing zero. This argument also applies in residue characteristic two.

At each orbit, fix for each neighboring lattice an isometry ρ carrying it to the standard lattice. Extend it by the identity off the orbit. At split places the component at the opposite member is the inverse conjugate transpose. The intersection computations at the nonsplit place and the two split places give

$$I(\rho\gamma) = q_w^{-1} I(\gamma), \qquad |a(\gamma)| = q_w^{-s/2} |a(\rho\gamma)|.$$

Indeed I is the index in the moved lattice, and ρ preserves indices. For a union B of whole orbits, take representatives of G_B/K_B supported on B. An index-one moved lattice is contained in the standard lattice, and dualizing gives equality, hence the identity coset. For every other coset choose one reducing step at an orbit where the lattices differ. The positive integral index strictly decreases. If the successive steps are $\rho_1, \dots, \rho_h$, then $\gamma K_B = \rho_1^{-1} \cdots \rho_h^{-1} K_B$. Thus the chosen words distinguish the cosets. The total weight of one-step choices at w is bounded, since $s \geq 2n+1$, by

$$a_w \leq \begin{cases} \dfrac{q_w^n - 1}{q_w - 1} q_w^{-s/2} \leq \dfrac{1}{(q_w - 1)\sqrt{q_w}}, & w \text{ split}, \\ \sqrt{q_w}\, \dfrac{q_w^n - 1}{q_w - 1} q_w^{-s/2} \leq \dfrac{1}{q_w - 1}, & w \text{ nonsplit}. \end{cases}$$

Every $q_w \geq 4$, since E contains both a primitive third and a primitive fourth root of unity. A split orbit has two members of equal norm, so its total is at most $1/3$, as is the weight at a nonsplit orbit. The weights are summable. At split places they are $O(q_w^{-3/2})$, and at nonsplit places $q_w \geq p^2$ for the rational prime p underneath. There are at most $[E : \mathbb{Q}]$ places over p.

For any union B of conjugation orbits of weight $\sum_{w \in B} a_w \leq 1/3$, sum the weights over the descending words. The identity contributes one and the entire remaining mass is at most

$$\sum_{h \geq 1} \Big(\sum_{w \in B} a_w \Big)^h \leq \frac{1}{2}.$$

This proves absolute convergence and injectivity of $\mathcal{V}_B = \sum_{\gamma \in G_B/K_B} a(\gamma) R(\gamma)$ on bounded functions right invariant under K_B. Indeed $\|\mathcal{V}_B u - u\|_\infty \leq \|u\|_\infty / 2$. Assign to every finite conjugation orbit the numerical upper bound in the preceding display, adding the two bounds at a split pair. These majorants depend only on the field and ranks. Choose finitely many orbits so that the sum of the majorants over all remaining orbits is at most $1/3$. Partition the full set of orbits into these single orbits and the tail, then intersect each part with A. Each resulting piece B has weight at most $1/3$, and G_A/K_A is the product of the quotients G_B/K_B. Representatives on disjoint supports commute. The sum of the absolute weights on this product is the product of the convergent factor sums by Tonelli, and multiplicativity gives the same factorization for $\mathcal{V}_A$. Disjoint supports preserve the other factors' levels, and bi-invariance of the coefficient preserves each factor's own level. On the Banach space of bounded functions at the prescribed level, $a(1) = 1$ and the preceding estimate gives

$$\|1 - \mathcal{V}_B\| \leq \tfrac{1}{2}, \qquad \mathcal{V}_B^{-1} = \sum_{d \geq 0} (1 - \mathcal{V}_B)^d, \qquad \|\mathcal{V}_B^{-1}\| \leq 2.$$

An empty piece gives the identity. If the original partition has h parts, the product is invertible with $\|\mathcal{V}_A^{-1}\| \leq 2^h$ for every allowed choice of the omitted places at these ranks and field.

40.2.4 Real convolution and the original measures

In this subsection, we combine the absolutely convergent integrals of Section 40.2.3 with the real projectors.

Take $f' = f$ and y of level J, and put $k_\infty(h) = \langle v, \omega_b(h)v\rangle$. For fixed real h, $\langle y', R(h\gamma)y\rangle$ is right J-invariant in the finite variable, even for a test y' of a different level. On each coset γK_A the compact at A has mass one. Write J_{sel} for the product of the principal subgroups at the selected split pairs, so $J = K_A J_{\mathrm{sel}}$. For the finite part F of the original tensor, multiplicativity of the local Weil coefficients gives, for $k \in K_A$ and an element a_{sel} supported at the selected pairs,

$$\langle F, \omega_b(\gamma k a_{\mathrm{sel}})F\rangle = a(\gamma)|R|^{-sn}\mathbf{1}_{J_{\mathrm{sel}}}(a_{\mathrm{sel}}).$$

At all other places where γ is integral the vacuum coefficient is one. The formula also applies at unit modulus, with the empty product at the selected pairs. As y is fixed by J, the K_A-integral has mass one and the scaled J_{sel}-integral has mass $|R|^{-sn}\operatorname{vol}(J_{\mathrm{sel}}) = c_{\mathfrak{m}}$. The sum and real integral of absolute values after taking the source pairing are bounded by

$$c_{\mathfrak{m}}\|y'\|_2\|y\|_2\,\|k_\infty\|_1 \sum_{\gamma \in G_A/K_A} |a(\gamma)|,$$

by unitarity of right translation. The series of absolute values converges by Section 40.2.3. The real L^1 bound is proved below. Fubini therefore gives

$$\langle Ly', Ly\rangle_T = V_T c_{\mathfrak{m}} \left\langle y', \mathcal{V}_A \left(\int_{G(\mathbb{R})} k_\infty(h) R(h) y \, dh \right) \right\rangle_S .$$

Here $L = L_f$. In each holomorphic test let $M_u = m_u$ be the minor quotient of (CM1), with u the source weight of v for ω_b. The coefficient $h \mapsto \langle v, \omega_b(h)v\rangle$ is holomorphic of weight u: in the highest minor, the strictly upper triangular operators within a sign block annihilate its flag minors, and the mixed second derivatives vanish because only one sign of source variable occurs in each of the two definite-target tests. The diagonal gives the weight u. The extension to the complex upper triangular group is the one proved in Section 40.3. Unitarity and the circle action on the first vector give

$$k_\infty(r_s h) = \chi_u(s)\, k_\infty(h).$$

With the pairing linear in its second variable, the circle average in (CM4) therefore yields $k_\infty(h) = k_\infty(1)m_u(h) = \|v\|^2 M_u(h)$. The opposite-sign gap gives $M_u \in L^1$. On the compact real group of the auxiliary lift source the coefficient is also integrable. Absolute summability of the finite series and integrability on the real group justify interchanging them.

If $Ly = 0$, the formula shows that $z = \mathcal{V}_A \int k_\infty(h) R(h) y \, dh$ is orthogonal to every rapid test. Each right translate of y is such a test. The function z is continuous and bounded by absolute convergence and has finite L^2 norm on the finite-volume quotient. Expressing the second occurrence of z in $\langle z, z\rangle_S$ by this absolutely convergent series and integral of translates gives zero. Thus $z = 0$. The real convolution is right integral-level invariant on A, so the injectivity above gives

$$\int_{G(\mathbb{R})} k_\infty(h) y(xh)\, dh = 0 \qquad \text{for every } x.$$

This is the real convolution implication for both tableau reproductions and the auxiliary lift.

40.3 The adjoint of a single lift and its holomorphic type

In this subsection, we apply the Gram identity of Section 34.2 to the original lift and its adjoint. Tracking the complex conjugation of the real coefficient gives the weight of the projected source used in Section 41.

Fix j and the untwisted descent pair $W = W_j$, $V_j = I_{r+t}$. Write $d_{\mathrm{tgt}} = r + t$, Y^{pr} for the subspace of $L^2([U(W)])$ fixed by the old principal level and the prescribed real-line projector, and $Y = Y_j = e_0 Y^{\mathrm{pr}}$. Every type summand retained by e_0 is cuspidal by the constant-term argument of Section 33.1, so Y is the original projected source. Write $L : Y^{\mathrm{pr}} \to \mathcal{A}(V_j)$ for the single lift into the original algebraic form space and $L_Y = L|_Y = u_{j,1}$. The metric on Y^{pr} is its source Petersson metric, and that on $\mathcal{A}(V_j)$ is the target class metric.

The first-arrow formula of Section 34.2 on this pair gives, for $x, y \in Y$,

$$\langle L_Y x, L_Y y\rangle_{\mathrm{class}} = A_j \langle x, R_j^W y\rangle_W.$$

Here $A_j \neq 0$ is the nonzero real edge constant with its positive class and sharp-level factors. Its real datum is the joint harmonic of the untwisted pair. The first-arrow calculation establishes the nonvanishing before the comparison period is constructed. Exhaust the vacuum series R_j^W by finite unions of double cosets of the standard compact away from the old level. Each partial Hecke sum preserves cuspidality and that compact level. It commutes with the old residue projector and the real line projector, so it sends Y to itself. Since right translation is unitary on the Petersson space, the norm of an omitted sum on y is at most $\|y\|_2 \sum_{\gamma \text{ omitted}} |a(\gamma)|$. The absolute sum converges by Section 40.2.3. Thus $R_j^W y$ belongs to the finite-dimensional, closed space Y. The finite vacuum estimate gives injectivity of R_j^W. Thus $L_Y y = 0$ implies $R_j^W y = 0$ by applying the displayed identity with $x = R_j^W y$, and then $y = 0$. This deduction uses only the two spaces of the single lift.

Let $\mathcal{K}_g \in \mathcal{A}(V_j)$ denote the vector theta kernel at $g \in U(W)(\mathbb{A})$. For a Weyl coordinate vector ζ, its value at the finite target point c is characterized by

$$\langle \zeta, \mathcal{K}_g(c)\rangle_{\mathrm{W}} = \theta_{P_{\bar{\zeta}}, F_{\mathrm{bl}}}(g, (1, c)).$$

Here P_ξ is the mixed Cartan polynomial, F_{bl} is the original untwisted block datum, and the characters are υ^{m-j} on the target determinant and $\mu_j^0 \upsilon^{d_{\mathrm{tgt}}/2}$ on the source determinant in the inverse-character convention of the Weil action.

In particular $P_{\bar{\zeta}}$ is linear in $\bar{\zeta}$. The right reduction actions on the block test and the Weyl action on the polynomial give the stated level and rational covariance of $\mathcal{K}_g$. Explicitly the simultaneous integral source and target lifts preserve the inclusion reduction, and the target rational change is the polynomial's Weyl substitution. Moving the source principal level alone fixes the residue block, and the tame character is trivial on its determinant.

Let C run over the finite target classes at the original level and c_C be their representatives. The integral and the adjoint are

$$\begin{aligned} Ly &= \int_{[U(W)]} y(g) \mathcal{K}_g \, dg, \\ (L^* \Xi)(g) &= \sum_C \frac{\langle \mathcal{K}_g(c_C), \Xi(c_C)\rangle_{\mathrm{W}}}{|\operatorname{Aut}(c_C)|} \\ &= \sum_C \frac{\overline{\theta_{P_{\overline{\Xi(c_C)}}, F_{\mathrm{bl}}}(g, (1, c_C))}}{|\operatorname{Aut}(c_C)|}. \end{aligned}$$

Indeed the scalar expression in the last two lines is the normalized descent at the untwisted block datum. Equation (93) applies at the same old inclusion place to each c_C and each of its polynomial coordinates. Their finite sum is rapid and belongs to $L^2([U(W)])$. The compact target's highest-line projection cuts out its irreducible source partner as in Section 33.1.1. The joint harmonic lies on the highest line of its lowest source compact type, which occurs once. Its source line projector therefore fixes the harmonic. By Cartan equivariance, each $P_{\bar{\zeta}}$ lies in the span of compact target translates of this harmonic. Those translates commute with the source projector. Write $p_S = \int e_S(k)\omega(k,1)\,dk$ for this oscillator projector. At fixed g, c_C, F_{bl} the weighted Fock estimate of Section 34.3.1 and the coefficient moment there allow theta evaluation through this integral, as in Section 34.3.2. Thus

$$R(e_S)\theta_{P_{\bar{\zeta}},F_{\mathrm{bl}}} = \theta_{p_S P_{\bar{\zeta}},F_{\mathrm{bl}}} = \theta_{P_{\bar{\zeta}},F_{\mathrm{bl}}}.$$

Conjugating gives $R(\overline{e_S})\overline{\theta} = \overline{\theta}$, the original source line in the adjoint. Here $\overline{e_S(k)} = e_S(k^{-1})$ in the unitary normalization. Together with the principal-level covariance above, this places the class sum in Y^{pr}. At each of the finitely many target classes the kernel coordinates are in $L^2([U(W)])$ by the same decay estimate. Thus the defining integral for L converges absolutely for every $y \in Y^{\mathrm{pr}}$, by Cauchy–Schwarz. Pairing it with Ξ and moving the finite class sum inside the integral proves the adjoint formula. This pairing has no target level index, because the original metric is the class sum with reciprocal stabilizer weights.

It suffices to use the restriction to the projected space. The orthogonal projection from Y^{pr} to Y is the original unitary finite projector e_0, so $L_Y^* = e_0 L^*$. The injective map L_Y has invertible $L_Y^* L_Y$. Thus for every $y \in Y$ there is $w \in Y$ with

$$y = e_0 L^* L w.$$

Conjugating L^*Lw yields a finite combination of the scalar kernels with their chosen Fock polynomials. The operator obtained by conjugating e_0 is a finite linear combination of finite right translations. It preserves the real coefficient law at every adelic left translate. The real weight can therefore be checked on the scalar kernels of this descent.

At τ let $n(c)$ be the numbering in the descent datum, counting the positive coordinates in their original order and the negative coordinates in reverse. Use the order

$$o(c) = \begin{cases} -1 - n(c), & c \text{ positive}, \\ n(c), & c \text{ negative}. \end{cases}$$

At τ, write $\mu_\tau^{\pm}(h) = k^{\pm}(h)$ for the two column multiplicities of Section 33.1.1, and put $\gamma_\tau = \kappa_{j,\tau} - d_{\mathrm{tgt}}/2$. In the highest polynomial each height-h minor uses the source columns numbered $0, \dots, h-1$ in its sign block. The source diagonal action contributes minus the number of such columns containing c for the positive variables, and plus the corresponding number for the negative variables. The real vacuum adds d_{tgt} on negative source columns, one for each definite target space, and zero on positive ones. The inverse determinant character contributes the common shift $-\mathrm{wt}(\mu_j^0 \upsilon^{d_{\mathrm{tgt}}/2})_\tau = \gamma_\tau$.

The resulting weight is

$$u_\tau(c) = \begin{cases} \gamma_\tau - \displaystyle\sum_{n(c)<h<d_{\mathrm{tgt}}} \mu_\tau^+(h), & c \text{ positive}, \\ \gamma_\tau + d_{\mathrm{tgt}} + \displaystyle\sum_{n(c)<h<d_{\mathrm{tgt}}} \mu_\tau^-(h), & c \text{ negative}. \end{cases}$$

The compact highest weight calculation on the same minors identifies it with the lowest compact weight of the negated doubled source parameter. Equivalently writing $\lambda_{W,\tau}$ for the doubled source parameter, the formula $2u_\tau = -\lambda_{W,\tau} + 2\rho_{\mathrm{n}} - 2\rho_{\mathrm{c}}$ gives the same entries: successive differences within each sign block are the two displayed staircase differences and the vacuum and character shifts give their initial constants.

The strictly upper triangular root operators for o in one sign block kill the joint highest polynomial by the minor rule. In different sign blocks an upper triangular operator is in the positive-to-negative position of this order. Its oscillator action is a sum of mixed second derivatives in a fixed target space. The positive and negative minor diagrams use disjoint target space sets, so every summand is zero. Thus every strictly upper triangular operator kills the highest polynomial and the diagonal acts by u_τ.

For completeness, this infinitesimal statement gives the holomorphic law of the actual matrix coefficients. Let $G_\tau = U(J_\tau)$ and let B_τ be the complex invertible upper triangular group for o. Their product $G_\tau B_\tau$ is open, since $\mathfrak{g}_\tau + \mathfrak{b}_\tau = \mathfrak{gl}_r(\mathbb{C})$ as real spaces. The intersection $G_\tau \cap B_\tau$ is the diagonal unitary torus, by elimination in the ordered form with all diagonal entries nonzero.

For the highest Fock vector v, extend the real orbit map on this product by $\rho(gb)v = \chi_{u_\tau}(b)\rho(g)v$. The torus weight makes the definition independent of the decomposition. Its real differential at 1, on a tangent $Z = Y + X$ with $Y \in \mathfrak{g}_\tau$, $X \in \mathfrak{b}_\tau$, is

$$d\rho(Y)v + d\chi_{u_\tau}(X)v = d\rho_{\mathbb{C}}(Z)v,$$

which is complex linear. The equality uses upper triangular annihilation and the diagonal weight. Left translation by G_τ and right translation by B_τ transport this calculation throughout the open product.

Fix an adelic point x, a target class C and the finite datum. The resulting theta functional $T_{x,C}$ is continuous on the real Schwartz space by the rational lattice-sum estimate. The real oscillator orbit of the polynomial times its Gaussian is smooth in this topology. Indeed, near the identity the symplectic action factors into linear substitutions, multiplication by quadratic phases and their Fourier conjugates. Parameter differentiation acts continuously between Schwartz seminorms, and translation gives the same conclusion near each real group element. Consequently the scalar function on the product of the open sets $G_\tau B_\tau$ given by

$$gb \longmapsto \chi_u(b) T_{x,C}(\rho(g)v)$$

is smooth by local smooth sections of multiplication. Its differential is complex linear by the preceding identity, hence it is holomorphic. The target compact group commutes with the source action, and its translates of the target highest vector span the irreducible Weyl space. The mixed Cartan polynomial map intertwines this target action, with conjugation of target coordinates in the displayed descent formula. It follows that the holomorphic law holds for each prescribed polynomial in the descent. Taking the finite class sum and applying the finite right translations in the conjugate projector gives the law for $h \mapsto \overline{y(x(h,1))}$ at every adelic x.

Finally, put $p_- = m - j$, $d_- = t - 9m - 1$. The complementary kernel weight is

$$u_\tau^\circ(c) = \begin{cases} j - m/2, & c \text{ positive}, \\ j - m/2 + d_- + \displaystyle\sum_{n(c)<h<d_-} \mu_\tau^-(h), & c \text{ negative}. \end{cases}$$

For $p_- > 0$, reading the stem at either side of its j-th gap gives the last negative column count

$$\mu_\tau^-(p_-) = \Delta/2 - \lfloor t/2 \rfloor - 1 \geq 300 n_*.$$

In fact the two stem entries are $\Delta/2-\kappa_{j,\tau}$ and $-\Delta/2-\kappa_{j,\tau}$, while the truncating endpoint for the negative count is $-\lfloor t/2\rfloor-\kappa_{j,\tau}$. With $t=5r$, $n_*=2ft^3$, $\Delta=2r^8$, $f\le x^{12}$ and $r\ge x^5$, one has

$$\frac{300n_*+t}{\Delta/2}=\frac{75000f}{r^5}+\frac{5}{r^7}\le 75000x^{-13}+5x^{-35}<1$$

for sufficiently large x. Since $\lfloor t/2\rfloor+1\le t$, this proves the asserted gap. Every negative numbering is smaller than $p_-<d_-$, so the count is included at every negative index. The two weights are dominant on each sign block because the counts are nonnegative and the positive numbering is reversed in o. They both have the required gap $300n_*$ between a negative and a positive entry. When $p_-=0$ this gap condition is empty. The doubled source parameter has entries of modulus at most $2\kappa_{m+1,\tau}$; the signed sum defining $\rho_{\mathrm{n}}-\rho_{\mathrm{c}}$ contributes at most r in each index. Thus

$$|u_\tau(c)|\le 2\kappa_{m+1,\tau}+r\le 5b.$$

These are the weights and estimates used in the two peak and circle calculations.

41 Injectivity of the common period

In this section, we use the reproducing coefficients of Section 40 to show that the common period of Section 39.4 detects every nonzero vector in the distinguished source subspace. The circle mean at a peak bounds the nonzero reference coefficient by the approximation error.

41.1 Reproduction and separation at a peak

In this subsection, we apply the period identity of Proposition 39.1 at a peak of the descent test.

Fix the j-th descent pair, put $W=W_j$, and let V° be the positive definite complementary space of rank $t-9m-1$. All integrals on the adelic groups below use the original quotient measures. Put $\mathfrak{m}=(\pi_M^2)$. For a class c in $U(V^\circ)$ at this level take its prescribed representative c_f. For a tuple $\zeta=(\zeta_c)_c$ of Weyl vectors of the negative diagram, write $K_\zeta^\circ(g)$ for the sum over c of the kernel of (W,V°) at (g,c_f). Its finite test is the indicator of the inclusion of W modulo $\mathfrak{m}$. Its polynomial is the evaluation in the negative coordinates of $\overline{\zeta_c}$ for the original numbering of W. Its splitting characters are υ^{m-j} and $\upsilon^{m/2-j}$. This pairing is conjugate-linear in ζ.

Write F_i for the twist-free first-factor functions of Section 39.4, with the choices specified below, and P for the common period operator. The seesaw identity gives

$$\langle \eta, Py\rangle=\sum_i C_i^{\mathrm{per}}\int_{[U(W)]} y(g)F_i(g)K_{\eta_i}^\circ(g)\,d\bar g,\qquad C_i^{\mathrm{per}}=C_i|\bar J^1|c_{\mathrm{first}}.$$

Here $C_i=M^{l_1}c_{L_i}$ is the scalar from Section 39.4.3, so $C_i^{\mathrm{per}}\neq 0$. Here c_{first} is the nonzero multiplier c'' chosen for the integral first-factor form, and the divided-lattice Weil scalar c_{L_i} of Section 39.4.3 is nonzero. The separate c'_ϑ is the scalar whose nonvanishing follows from the injectivity of P below. Taking η supported in one choice yields

$$Py=0\quad\Longrightarrow\quad\int_{[U(W)]} y(g)F_i(g)K_\zeta^\circ(g)\,d\bar g=0\quad\text{for every } i,\zeta. \tag{CP1}$$

The class stabilizers at $J(\pi_M^2)$ are trivial by Lemma 6.1 at an old odd split place. Thus a single class component at this original level may be prescribed arbitrarily and has reciprocal stabilizer weight one.

41.1.1 Real reproduction on the complementary block

The sharp-level inner product formula of Section 40.2.4 expresses convolution against the complementary real coefficient in terms of the original class tests. Section 41.1.6 applies the circle mean to the resulting identity at a peak.

Let M_{u° be the holomorphic quotient of principal minors for the complementary kernel weight u° on W. The parameter inequalities give $2r < \dim V^\circ$ and the required holomorphic gap for $M_{u^\circ} \in L^1$. The complementary block reproducing formula says that for each adelic x, there are $a_{c,b}(x)$ for which

$$\int_{U(W)(\mathbb{R})} A(xh) M_{u^\circ}(h)\, dh = \sum_{c,b} a_{c,b}(x) \int_{[U(W)]} A(g) K^\circ_{\zeta(c,b)}(g)\, d\bar{g}. \tag{CP2}$$

Here A can be any rapidly decreasing automorphic function of level $\mathfrak{m}$. Let $\mathbf{b}(X)$ be the column of negative tableau polynomials, G_- their Weyl Gram matrix in the coordinates of Section 33.1, and e_b the b-th coordinate column. The test polynomial at c is $\mathbf{b}(X)^{\mathsf{t}} G_- \overline{\zeta_c}$. Thus take

$$\zeta(c,b)_{c'} = \mathbf{1}_{\{c'=c\}}\, \overline{G_-^{-1} e_b}.$$

This is a vector in the complex tuple space at the original level. Its test is the b-th tableau polynomial since $G_- \overline{\zeta(c,b)_c} = e_b$. When $j = m$, the diagram is empty and the sole polynomial is 1. The definite Fock translates of the highest-weight vector in the negative Weyl space lie in this tableau span at each class. For completeness, the reproducing formula follows by viewing the left side and the finitely many right-side integrals as linear functionals on the rapid space. Vanishing of the latter implies vanishing of the highest-vector theta lift, by the tableau span and the class description at $\mathfrak{m}$. Apply Section 40.2.4 with $y = A$ and the complementary highest datum f. Since $L_f A = 0$ and $k_\infty = \|f_\infty\|^2 M_{u^\circ}$ with $f_\infty \neq 0$, the convolution on the left vanishes at every x. For each fixed x its linear functional therefore factors through the image of the finite class-evaluation map. Extending the induced functional on that image to the full coordinate space gives the coefficients $a_{c,b}(x)$ in (CP2).

Let y belong to the projected space and suppose $Py = 0$. Each yF_i is rapidly decreasing of level $\mathfrak{m}$. By (CP1) its class pairings in (CP2) vanish. Thus for any constants a_i,

$$\int_{U(W)(\mathbb{R})} y(xh) \Big(\sum_i a_i F_i(xh) \Big) M_{u^\circ}(h)\, dh = 0. \tag{CP3}$$

Indeed y is fixed by the old principal level, which contains $J_W(\pi_M^2)$. In the expression for F_i below, an element of $J_W(\pi_M^2)$ acts on the target block of the split residue matrix by the identity modulo π_M^2. It therefore fixes every frequency test ψ_Λ. The target determinant character υ^m is trivial on integral units, and elsewhere the integral compact fixes the vacuum. This proves the asserted level for every i. Write H_S, H_W for the two reduction heights. For each fixed class and polynomial in F_i, the estimate of Section 34.3.1 bounds the kernel at $(s; (g, 1, c_{h^\flat, f}))$ by $B_i H_S(s)^{A_i} H_W(g)^{A_i}$. Indeed the height of the padded target is the maximum of the heights of its orthogonal summands. The fixed definite class contributes a constant. Thus

$$|F_i(g)| \leq B_i H_W(g)^{A_i} \operatorname{vol}([U(S)]) \sup_s H_S(s)^{A_i} |\phi(s)|.$$

The supremum is finite by Section 41.3. For the projected $y = e_0 L^* L w$ of Section 40.3, (93) gives rapid decay of $L^* L w$. The finite integral translations in e_0 preserve the moved lattice and its height. Consequently yF_i is rapid for each i, as required in (CP3).

41.1.2 The finite first-factor choices

In this subsection, we choose the finite test for the real reproduction of Section 41.1.1.

Let S have signature $(2m,m)$, $\widetilde{W}=W\oplus E_-^{j}\oplus E_+^{3m-j}$, and $W^\flat=E_+^{6m+1}$, with the prescribed integral lattices. The auxiliary form ϕ is the theta lift of 1 from E_-^m to S, with trivial finite modulus, finite vacuum, splitting characters $(\upsilon^m,1)$, and $\prod_\tau \det(y_\tau^-)^{2m}$ as Fock polynomial. It is rapidly decreasing of full finite level and holomorphic of weight $(0;3m)$ on S. Its norm can be computed on the constant source. Write $G_0=U(E_-^m)$ and v for the Fock vector of its determinant polynomial. The determinant factors cancel on $G_0(F_\infty)$, as computed in Section 41.3, so the real coefficient is

$$\|v\|^2=\left(\prod_{a=1}^{m}\frac{(2m+a-1)!}{(a-1)!}\right)^{f}.$$

Indeed the Gaussian coordinates outside the determinant have total mass one. In each remaining m-by-m matrix of independent standard complex Gaussians, Gram–Schmidt on the columns writes $|\det Y^-|^2$ as the product of squared residual lengths. If the span of the preceding columns has complex codimension a, the conditional squared residual length is the sum of a independent exponentials of mean one. Its $2m$-th moment is $(2m+a-1)!/(a-1)!$. Successive conditional expectations give the formula at each place. In each finite block B of Section 40.2.3, right translation fixes 1. Thus $\mathcal{V}_B1=\lambda_B1$, where

$$\lambda_B=\int_{G_B}a(\gamma)\,d\gamma,\qquad |\lambda_B-1|\le\tfrac12.$$

The vacuum coefficient satisfies $a(\gamma^{-1})=\overline{a(\gamma)}$ by unitarity. Inversion preserves Haar on G_B, hence λ_B is real and $\lambda_B\ge\frac12$. The finitely many blocks include the tail. At unit modulus $c_{(1)}=1$, and the sharp inner-product formula gives

$$\|\phi\|^2=\operatorname{vol}([U(S)])\operatorname{vol}([G_0])\operatorname{vol}(G_0(F_\infty))\,\|v\|^2\prod_B\lambda_B>0.$$

Use the index $i=(\Lambda,h^\flat,\sigma)$ of Section 39.4.1, with frequency Λ modulo $\mathfrak{m}$, class $h^\flat$ of $W^\flat$ at that level and row choices σ from the short basis of that class. Write $c_{h^\flat,f}$ for its fixed finite representative. Define P_j on $(S,\widetilde{W})$ as the product of the full negative minor and the positive minors, with exponents read from the positive diagram of the untwisted pair. The chosen covectors and the minors are specified in Section 41.1.4. Let Q_σ be the paired-determinant polynomial of the prescribed short covectors in the positive letters of $(S,W^\flat)$. Then $F_i(g)$ is the integral on $[U(S)]$ of ϕ times the theta kernel of $(S,\widetilde{W}\oplus W^\flat)$, at $(g,1,c_{h^\flat,f})$ in the second variable, with polynomial P_jQ_σ, test $\psi_\Lambda(B)=\psi_{\mathfrak{m}}(\operatorname{tr}(B\Lambda))$, and characters $(\upsilon^m,\chi_{\widetilde{W},j})$. Here

$$\chi_{\widetilde{W},j}=\chi_0\upsilon^{\,m+e_j^{(1)}-j\Delta},\qquad e_j^{(1)}=j+\lfloor(t+k-1)/2\rfloor.$$

Thus the second character has trivial tame part and weight $m+e_j^{(1)}-\kappa_{j,\tau}$ at τ. On $R=\mathcal{O}_E/\mathfrak{m}$ we use the additive character

$$\psi_{\mathfrak{m}}(x \bmod \mathfrak{m})=\psi_f(j_{\mathrm{W}}\bar\alpha_{\mathrm{W}}\alpha_{\mathrm{W}}\,e_{\mathfrak{m}}s_0^{-1}x),\qquad x\in\widehat{\mathcal{O}}_E,$$

where s_0 generates $\mathfrak{m}$ in the finite adeles and $e_{\mathfrak{m}}$ is the idempotent of its chosen split places as in Section 41.5.2. By Section 8.1, $j_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}\alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbf{Q}}^{-1}\widehat{\mathcal{O}}_E$. Thus at each selected split member the annihilator of the integers under $\psi_f(j_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}\alpha_{\mathrm{W}}\,\cdot)$ is precisely the integers. After division by s_0 its annihilator on the integers is $\mathfrak{m}$, giving a perfect pairing on the full residue ring at exponent two.

Let θ_{Ψ,P_j} be the $(S,\widetilde{W})$ kernel with any function Ψ modulo $\mathfrak{m}$ and the same two characters. For $(S,W^\flat)$ write $X_{\tau,+}$ for the $(6m+1)$-by-$2m$ matrix of positive source columns, and e_+ for the restriction of the chosen inclusion to those columns. Take the Fock polynomial

$$Q_0(X) = \prod_{\tau} \det\Big(((X_{\tau,+})_{e_+(i),j})_{1\le i,j\le 2m}\Big)^{\Delta'}, \qquad \Delta' = (\Delta - t - 1)/2.$$

It takes the value 1 when the indicated square blocks are identities. A raising operation within the positive block repeats a column. The negative columns do not occur, so the raising operators in the negative block and the mixed second derivatives also annihilate Q_0. Each positive column has degree Δ' and contributes weight $-\Delta'$ under inverse source substitution. The vacuum contributes $6m+1$ on each negative column, and the source determinant character is trivial. Thus the Fock vector is holomorphic of weight $u_1 = (-\Delta'; 6m+1)$. The rank inequality is $2\dim S = 6m < 6m+1 = \dim W^\flat$. The classwise span of its compact target translates is verified below. Apply this reproduction to the rapidly decreasing function

$$x \longmapsto \phi(x)\theta_{\Psi,P_j}(x,(g,1)).$$

Here g is fixed. The same Schwartz seminorm bound gives polynomial growth in x, which the rapid decay of ϕ absorbs. At an old split place $J_S(\pi_M^2)$ acts on the matrix by right multiplication by the identity modulo π_M^2. Hence it fixes every Ψ. Its determinant has trivial $\chi_{\widetilde{W},j}$-factor, and the finite vacuum is integral-compact invariant elsewhere. These source actions commute with the fixed target point, and ϕ has full finite level. This gives precisely the source level of flat reproduction. The resulting kernels on $(S,W^\flat)$ use the indicator δ_{incl} of the inclusion modulo $\mathfrak{m}$ and polynomials Q_σ.

The product of the two kernels is the kernel of the orthogonal sum, with test $\Psi(B_1)\delta_{\mathrm{incl}}(B_2)$ and polynomial P_jQ_σ. Finite Fourier inversion writes that product test as a linear combination of the ψ_Λ on the whole residue matrix space, with its original primitive additive character. Therefore, for each $x_\phi \in U(S)(\mathbb{A})$ and each Ψ, the function of g

$$\int_{U(S)(\mathbb{R})} \phi(x_\phi h')\,\theta_{\Psi,P_j}(x_\phi h',(g,1))M_{u_1}(h')\,dh' \tag{CP4}$$

belongs to the span of the actual F_i. The reproduction coefficients depend on x_ϕ and the finite Fourier coefficients on Ψ. All quotient measures and class representatives are as above.

The class test used in both reproductions can be checked before taking any finite Fourier expansion. Write $K_T = K_T(\pi_M^2)$ and let c_f be the chosen representative of a target class. Every adelic target point a can be written

$$a = \gamma(1,c_f)(1,k)(h,1), \qquad k \in K_T, \quad h = (\gamma^{-1}a)_\infty,$$

using the finite class equality $a_f = \gamma_f c_f k$. For a Fock vector v fix the inclusion indicator and denote the value of its lift at $(1,c_f)$ by $L_c(v)$.

The lift at a is

$$L_c(\omega_\infty(1,h)v).$$

Indeed rational left invariance removes γ. Covariance puts the two right translations into the Weil datum. At the chosen members over M, left multiplication by $k_w^{-1} \equiv 1 \bmod \pi_M^2$ fixes the inclusion residue. Away from these pairs k preserves the vacuum. The character on the target determinant is υ^{m-j} for the complementary block and υ^m for the flat block. Both are trivial on the determinants of these integral matrices. The real translation therefore acts on the Fock vector alone. In the complementary block, writing the translated polynomial in the negative coordinates in the tableau basis uses the same c_f, and the tuple supported at c must take the conjugated Weyl vector as in (CP2).

In the reproduction of the auxiliary form the fixed matrix of short covectors A_c is the transpose of the chosen rational short basis of the lattice at this same c_f. It is invertible at every infinite embedding. For a list σ of $2m$ row indices $I_{\tau,\ell}$ at each τ and $1 \le \ell \le \Delta'$, the corresponding polynomial is

$$Q_\sigma(X) = \prod_{\tau} \prod_{\ell=1}^{\Delta'} \det\left(\tau(A_c)_{I_{\tau,\ell},*}\, X_{\tau,+}\right).$$

At τ the compact target translates the selected rows of Q_0 by a linear substitution, up to its determinant scalar. Express the substituted rows in the basis of rows of $\tau(A_c)$. Multilinearity of each determinant, followed by multiplication over ℓ and τ, puts every translate in the span of these Q_σ. Hence vanishing of all finite class tests kills the lift of the highest vector at every a in either case.

41.1.3 Approximation using a peak of the auxiliary form

In this subsection, we bound the error after applying the test of Section 41.1.2.

Put $n_* = 2ft^3$ and b equal to the large weight scale. Fix a peak x_ϕ of the nonzero bounded auxiliary form ϕ, so $0 < \|\phi\|_\infty < 2|\phi(x_\phi)|$. At this point the first minimum is at least $(5b\sqrt{b})^{-1/2}$ by the bounds for the first minima. If $\lambda_1(W,x) \ge (5b^{24})^{-1/2}$, the lattice at $(x,1)$ is the orthogonal sum of the lattice at x and the two definite identity lattices. For a nonzero vector in a definite identity lattice, its signed Hermitian square is a nonzero integral element of F_0. At least one conjugate has absolute value at least one, so the Euclidean length is at least one. Thus adjoining the two summands preserves the first minimum cut off at one, and

$$\lambda_1(S,x_\phi)\lambda_1(\widetilde{W},(x,1)) \ge (5b^{51/4})^{-1} \ge b^{-13}.$$

We write $(x,1)$ for the block embedding of the full adelic point x with identity on the added definite summands. Let $c_P(h',h)$ be the real Fock coefficient of P_j on $(S,\widetilde{W})$, with h embedded by the identity on the adjoined summands. On this pair and these points the sparse kernel estimate gives a test Ψ_x modulo $\mathfrak{m}$ with

$$\left|\theta_{\Psi_x,P_j}(x_\phi h',(xh,1)) - c_P(h',h)\right| \le b^{-20n_*}|h'|^{6n_*}|h|^{6n_*}|c_P(1,1)|, \tag{CP5}$$

where the absolute values of real points denote the maximum of the operator norms. Here Ψ_x is the probability average of the additive characters modulo $\mathfrak{m}$ weighted by the conjugates of their original kernel values at $(x_\phi,(x,1))$. Section 41.5.3 proves the overlap for this average,

with zero term $c_P(h', h)$. In the overlap sum the surviving rational differences lie in the kernel of reduction on the moved finite lattice from Section 41.4. Multiplication by q_1/π_M^2 takes these differences into the integral Hom lattice, since the only possible denominators of α_{W} divide $D_{\mathrm{rel}} \mid q_1$ by Section 8.1. Apply the bound there and the error estimate of Section 41.5.3 with the minima above, $q = 2n_*$, complex dimension $N = f(3m)(r + 3m) \le ft^2 < q$, and the homogeneous polynomial P_j of degree $d_P \le b\sqrt{b}$ constructed below. The block extension of h has the same maximum operator norm, since $|h| \ge 1$. Thus the resulting bound is (CP5) for every h', h with the same Ψ_x.

Define one reference function

$$F_*(h) = \int_{U(S)(\mathbb{R})} \phi(x_\phi h') c_P(h', h) M_{u_1}(h')\, dh'.$$

It is bounded by the unitary Fock coefficient bound and the integrability of M_{u_1}.

41.1.4 The nonzero reference coefficient

In this subsection, we use the approximation of Section 41.1.3 to construct the comparison function.

In the numbering of W_j, put $r_+ = k + j - 1$. The negative covectors used in $\widetilde{W}$ are indexed by $0 \le a < m$: for $a < j$ take the covector a of the adjoined negative space, and for $a \ge j$ take covector $k + a - 1$ of W_j. The latter covectors lie in the negative block, since $k + a - 1 \ge r_+$, and their indices run through the original negative covectors. The two disjoint pieces give m distinct covectors. For each height $0 \le h < r_+$, take the $2m$ positive covectors indexed by $0 \le a < 2m$: use covector $h + a$ of W_j if $h + a < r_+$, and covector a of the adjoined positive space otherwise. The positive space has dimension $3m - j \ge 2m$. These covectors are distinct, and both choices have positive sign.

Let Y_τ^- be the matrix on the selected negative covectors and all m negative columns of S, and let $Y_{\tau,h}$ be the matrix on the height-h positive covectors and all $2m$ positive columns. Write $n_\tau^\pm(h)$ for the number of columns of height h in the positive and negative diagrams, respectively. Put

$$\begin{aligned} \nu_\tau &= \sum_{h<r_+} n_\tau^+(h), \\ \mu_\tau &= \nu_\tau + \Delta', \\ d_1 &= r + 9m + 1, \\ c_\tau &= \mu_\tau + d_1 + m. \end{aligned}$$

This is precisely the polynomial

$$P_j = \prod_\tau \det(Y_\tau^-)^{c_\tau} \prod_{h<r_+} \det(Y_{\tau,h})^{n_\tau^+(h)}.$$

Each determinant has independent variables in all its positions and is nonzero. The polynomial ring is a domain, so their product is nonzero even when different minors share some variables. Every variable in the product pairs identical signs on the target and source coordinates. It thus has positive oscillator sign, and every contraction involving two opposite oscillator signs kills P_j.

The degree at τ is $mc_\tau + 2m\nu_\tau$. The determinant count on the original positive diagram satisfies

$$\mu_\tau + e_j^{(1)} = \kappa_{j,\tau}.$$

Indeed the number of columns at height h is a successive difference of the shifted parameter $\lambda_{\tau,h-1} + h$ for $h \geq 1$, with the numbering of Section 12.10, so

$$\lambda_{\tau,i} = \begin{cases} k/2 - 1 - i, & i < k-1, \\ \Delta/2 - \kappa_{i+2-k,\tau}, & i \geq k-1. \end{cases}$$

The counts strictly below r_+ telescope to $\nu_\tau = k/2 - (k + j - 1 + \Delta/2 - \kappa_{j,\tau})$. Adding the Δ' columns at full positive height gives $\mu_\tau = \kappa_{j,\tau} - e_j^{(1)}$. The admissible bound $d_1 + m \leq e_j^{(1)}$ now gives $c_\tau \leq \kappa_{j,\tau} \leq 2b$. In particular $2c_\tau \leq 5b$. Since $\nu_\tau \leq c_\tau$, the total degree is at most $10fmb \leq b\sqrt{b}$ on the chosen scale. The second character on the padded pair has weight $e_\tau = m + e_j^{(1)} - \kappa_{j,\tau} = m - \mu_\tau$, and the first has weight m.

Here is the matrix coefficient calculation at one place. For $u \in U(2m, m)$, write $A = \det u_{++}$, $D_- = \det u_{--}$, and $\delta = \det u$. Write $u = k_1 a k_2$, $k_i \in U(2m) \times U(m)$. The Cartan factor a has m two-dimensional boosts with parameters $s_l \geq 0$ and m fixed positive directions. Put $C = \prod_{l=1}^m \cosh s_l$. Both block determinants of a are C, and $\det a = 1$.

In signed Bargmann coordinates a boost on one target row has kernel

$$\frac{1}{\cosh s} \exp\!\left(\varepsilon \tanh s \, (z_1 z_2 - \bar{w}_1 \bar{w}_2) + \frac{z_1 \bar{w}_1 + z_2 \bar{w}_2}{\cosh s} \right).$$

Here the two variables have opposite oscillator signs and $\varepsilon = \pm 1$ depends on the choice of sign for the off-diagonal entry. Indeed the displayed kernel at $s = 0$ is the reproducing kernel, and its s-derivative is $\varepsilon(z_1 z_2 - \partial_{z_1}\partial_{z_2})$ applied to it, the infinitesimal oscillator action of the boost. The Gaussian is in the Bargmann measure in this formula. The polynomial P_j uses only positive oscillator variables. In pairing the kernel with P_j at both ends, every term containing $z_1 z_2$ or $\bar{w}_1 \bar{w}_2$ therefore vanishes by monomial orthogonality. The remaining exponential substitutes the scalar $(\cosh s_l)^{-1}$ on both participating columns. Each positive determinant uses every positive source column once, and each negative determinant uses every negative source column once. Their substitution scalar is $C^{-\nu_\tau - c_\tau}$. The vacuum contributes C^{-1} for each of the $r + 3m$ target rows. Thus the local normalized coefficient at a is $C^{-r-3m-\nu_\tau-c_\tau}$.

For $k = (k_+, k_-)$ put $\eta_\pm = \det(k_\pm)$. The polynomial contributes $\eta_+^{-\nu_\tau}\eta_-^{-c_\tau}$. The vacuum contributes η_+^m from the negative target rows and η_-^{r+2m} from the positive ones. The source twist gives the compact character

$$\chi_P(k) = \eta_+^{m-e_\tau-\nu_\tau}\eta_-^{r+2m-e_\tau-c_\tau}.$$

The inner product is linear in its second variable, so the factor at $k_1 a k_2$ is $\chi_P(k_1)\chi_P(k_2)$ times the factor at a. Also $A = C\det(k_{1,+}k_{2,+})$ and $D_- = C\det(k_{1,-}k_{2,-})$. In particular neither minor vanishes.

It follows that the local multiplier of $c_P(1,1)$ is

$$\delta^{r+3m-e_\tau} A^{-r-2m-\nu_\tau} D_-^{-m-c_\tau}.$$

For a weight constant on the two blocks one has $m_{(a;b)}(u) = \delta^b A^{a-b}$. The complementary identity $A = \delta\overline{D_-}$ and $|\delta| = 1$ therefore give, on multiplying over all places,

$$c_P(u,1) = c_P(1,1) M_{u_2}(u)\overline{M_{u_3}(u)},$$
$$u_2 = (\Delta'; \mu_\tau + r + 2m), \quad u_3 = (0; \mu_\tau + r + 11m + 1).$$

At the identity, write $P_j(z) = \sum_\alpha p_\alpha z^\alpha$ in all signed polynomial coordinates. For N complex coordinates the measure is $\pi^{-N}dz$ and the two Gaussian factors give $e^{-\|z\|^2}$. Monomial orthogonality gives

$$c_P(1,1) = \|v_{P_j}\|^2 = \sum_\alpha \left(\prod_i \alpha_i!\right) |p_\alpha|^2 > 0.$$

Repeated variables in different minors are included in the single coefficient p_α of their common monomial. Unitarity gives $|c_P(u,h)| \le c_P(1,1)$ on both real groups.

The weight of the auxiliary form and the two factors satisfy $(0;3m) + u_1 + u_2 = u_3$, with $u_1 = (-\Delta';6m+1)$. The weights u_1, u_2, u_3 are dominant within blocks. Since $\Delta/2 \ge 300n_* + t$ and $\Delta' = \Delta/2 - (t+1)/2$, the cross-block gaps of u_1, u_3 are at least $300n_*$. Also $|u_{3,i}| \le c_\tau + m \le 2c_\tau \le 5b$. The minor-quotient moment bound thus applies with $(u,u') = (u_3,u_1)$ on S in the original product Haar. In particular $J_3 = \int |M_{u_3}|^2$ is finite and positive. In particular $\mu_\tau = \nu_\tau + \Delta' \ge \Delta'$, so the gap of u_3 follows from $\Delta' + r + 11m + 1 \ge 300n_*$. Apply the holomorphic reproducing formula to the product $h \mapsto \phi(x_\phi h)M_{u_1}(h)M_{u_2}(h)$, of weight u_3. Its pairing with $\overline{M_{u_3}}$ is integrable by the unitary bound on c_P, boundedness of ϕ, and the moment bound. Since $M_{u_i}(1) = 1$, it gives

$$F_*(1) = J_3\,\phi(x_\phi)\,c_P(1,1) \ne 0.$$

All three factors are nonzero at the fixed x_ϕ. This reference value is independent of the subsequent choice of x and Ψ_x.

The weight comparison gives $\int |M_{u_1}(h')||h'|^{6n_*}\,dh' \le b^{n_*}J_3$. Fix x and its single test Ψ_x. For every h, (CP5) and unitarity bound the absolute integrand of (CP4) at $g = xh$ by

$$\|\phi\|_\infty c_P(1,1)|M_{u_1}(h')|(1 + b^{-20n_*}|h'|^{6n_*}|h|^{6n_*}).$$

This is integrable in h', uniformly for h in compact sets. The term 1 bounds the integrand defining $F_*(h)$. Integrating the difference with the same test for every h gives (CP6), since

$$b^{-20n_*}\|\phi\|_\infty|c_P(1,1)|\,b^{n_*}J_3 \le 2b^{-19n_*}|F_*(1)|.$$

Lemma 41.1 (Uniform approximation with a fixed auxiliary form)**.** *For each adelic x with $\lambda_1(W,x) \ge (5b^{24})^{-1/2}$, there are coefficients $a_i(x)$, independent of h, such that*

$$\left|\sum_i a_i(x)F_i(xh) - F_*(h)\right| \le 2b^{-19n_*}|F_*(1)|\,|h|^{6n_*} \quad (h \in U(W)_\infty). \tag{CP6}$$

The reference function F_* is fixed and $F_*(1) \ne 0$. The test and the coefficients may depend on x.

41.1.5 Coefficients and weights of the approximants

In this subsection, we check the weight and source of the approximants in Lemma 41.1.

In the reproduction of the auxiliary form the kernel on $(S, W^\flat)$ has characters $(\upsilon^m, 1)$, while that on $(S, \widetilde{W})$ has $(\upsilon^m, \chi_{\widetilde{W},j})$. The product is the kernel on the target sum with $(\upsilon^m, \chi_{\widetilde{W},j})$. In fact the first character acts separately on each target determinant, and the two factors on the common source determinant multiply. Let $a_r(x_\phi)$ be the coefficients of flat reproduction at the fixed point for the auxiliary lift, with r running over the original class and short-covector

choices. These coefficients work for all rapidly decreasing functions of the given level. With R the residue ring, let

$$f_{\Psi}(\Lambda)=\frac{1}{\#\,\mathrm{Mat}_{d_1\times 3m}(R)}\sum_{B\in \mathrm{Mat}_{d_1\times 3m}(R)}\Psi(B_1)\,\delta_{\mathrm{incl}}(B_2)\overline{\psi_{\Lambda}(B)}.$$

Here B_1 has $r+3m$ rows from $\widetilde{W}$, and B_2 has $6m+1$ rows from $W^{\flat}$. Thus B has $d_1=r+9m+1$ rows and $3m$ columns. Primitivity of $\psi_{\mathfrak{m}}$ identifies its additive dual with the full space of $3m$-by-d_1 matrices Λ through $\mathrm{tr}(B\Lambda)=\sum_{a,b}B_{ab}\Lambda_{ba}$. These are exactly the gluing indices of Section 33.2. Fourier inversion and the product identity give the complex coefficients $a_r(x_{\phi})f_{\Psi}(\Lambda)$ of $F_{(\Lambda,r)}$ in (CP4). Once $\Psi=\Psi_x$ is fixed these coefficients serve for every h on the real translate xh. Indeed the period on the left of (CP3) is $\mathbf{C}$-linear in the first-factor test, so its vanishing on all F_i gives vanishing on this combination at the chosen peak. Put $N_1=|R|^{3m(r+3m)}$ and write K_t for the $(S,\widetilde{W})$ kernel of frequency t at $(x_{\phi},(x,1))$. At $h'=h=1$, the definition of $\Psi_x=N_1^{-1}\sum_t\overline{K_t}\psi_t$ and (CP5) give

$$N_1^{-1}\sum_{t}|K_t|^2\le(1+b^{-20n_*})c_P(1,1).$$

Write $\Lambda=(t,s)$ in the two column blocks and put $N_2=|R|^{3m(6m+1)}$. Orthogonality in the first block and evaluation at the inclusion in the second give

$$f_{\Psi_x}(t,s)=\frac{\overline{K_t}\,\overline{\psi_s(\mathrm{incl})}}{N_1N_2}.$$

All N_2 values of s occur in the original index set. Summation and Cauchy–Schwarz give

$$\sum_{\Lambda}|f_{\Psi_x}(\Lambda)|=N_1^{-1}\sum_{t}|K_t|\le\big((1+b^{-20n_*})c_P(1,1)\big)^{1/2}.$$

This bound holds at every x satisfying the minimum condition of Lemma 41.1. The coefficient bounds at arithmetic embeddings in Section 42.1.3 are applied to the fixed gluing operator G_1.

The scalar c'' of Section 33.3 is nonzero. The untwisted identity $F_i=(c_1'/c'')\Xi_{1,i}$ therefore writes each F_i as a finite combination of (W_j,V'') kernels in the original order, with polynomial in the positive coordinates of the diagram with column counts $n_{\tau}^{+}(h)$. The target is positive definite of rank d_1 and its splitting weight on the source is $e_j^{(1)}-\kappa_{j,\tau}=-\mu_{\tau}$. The highest-weight transformation law is linear in the function. Multiplying $\Xi_{1,i}$ by c_1'/c'' therefore preserves this law also when $c_1'=0$. The polynomial in the positive coordinates gives the holomorphic weight with entries

$$\mu_{\tau}-\sum_{h>n(a)}n_{\tau}^{+}(h)\quad\text{on positive }a,\qquad\mu_{\tau}+d_1\quad\text{on negative }a.$$

Here the positive entries come from the staircase minors and the negative entries from the vacuum and splitting weights. The definite-group translates of this highest polynomial span the Weyl space in the positive coordinates, so the same weight applies to every coefficient in that class sum. Put $d_{-}=t-9m-1$. The complementary weight at τ has entries

$$\begin{cases} j-m/2, & \text{on positive } a,\\ j-m/2+d_{-}+\displaystyle\sum_{n(a)<h<d_{-}}n_{\tau}^{-}(h), & \text{on negative } a.\end{cases}$$

The full staircase shift is $\kappa_{j,\tau}-(r+t)/2$. The numbering $n(a)$ is that of the bottom descent. Since $k=r+1-m$ and $t+k-1$ is even, the first-factor exponent is

$$e_j^{(1)}=(r+t)/2+j-m/2, \qquad \mu_\tau=\kappa_{j,\tau}-e_j^{(1)}.$$

The rank terms add because $d_1+d_-=r+t$, and the positive columns of height at least d_1 and the negative columns of height at least d_- have count zero. Thus the displayed weights add to u on the actual covector. For each fixed peak x, the coefficients $a_i(x)$ in (CP6) are independent of h, so $K_x(h)=\sum_i a_i(x)F_i(xh)$ is holomorphic of weight $u-u^\circ$ in the order of the circle estimate. Its reference function F_* is bounded and has the fixed value $J_3\phi(x_\phi)c_P(1,1)\neq 0$ at the identity.

41.1.6 Injectivity of the common period

In this subsection, we combine Lemmas 40.1 and 41.1 to prove injectivity.

Suppose $0\neq y\in\ker P$. The rapid form $\bar y$ is bounded, so choose z with $|\bar y(z)|>\|\bar y\|_\infty/2$. The compact circle acts by its unitary weight, hence each $z(r_s,1)$ is also a peak. At any peak x of $\bar y$, (PM5) gives the first minimum required by (CP6). At $h=1$ the chosen coefficients satisfy

$$\left|\sum_i a_i(x)F_i(x)\right|\geq(1-2b^{-19n_*})|F_*(1)|>0.$$

In particular the approximant is nonzero at this peak before using any orthogonality consequence of $Py=0$. Use the same coefficients in (CP3). The conjugate weight and the complementary kernel weight have the required dominance and gap, and their difference is the weight of each $F_i(x\,\cdot)$. Hence Lemma 40.1, applied at z, gives

$$|F_*(1)|\leq(4b^{n_*}+1)\,2b^{-19n_*}|F_*(1)|<|F_*(1)|,$$

since $b^{n_*}>10$. This contradiction proves injectivity of P. The first orbital moment in Section 16.2 gives $D>0$ for sufficiently large x, and (4) at the identity gives $\dim Y_j=D$ for the projected automorphic space on W_j. Evaluating $c'_\vartheta G^*_\vartheta u_{j,\vartheta}=P$ on any nonzero vector of Y_j now proves $c'_\vartheta\neq 0$ for every auxiliary twist ϑ at this fixed level. The inclusion $\ker u_{j,\vartheta}\subseteq\ker P$ then proves the injectivity of each original lift $u_{j,\vartheta}$ on Y_j. The same P works for all these twists. The external modulus q_0 and the quadratic discriminant $-d$ are fixed before the large-x estimates.

41.2 First minima at peaks of automorphic forms

In this subsection, we use transvections at the prescribed level to bound the first minimum at points satisfying the half-supremum bound in Section 41.1.6. The estimate separates the nonzero Hom-lattice terms in Section 41.1.3.

Let V have the chosen adelic basis, with Hermitian matrix $J=\operatorname{diag}(\epsilon_i)$ at infinity, and order the positive coordinates before the negative ones. Use the Euclidean norm $\|v\|^2=\sum_\tau\sum_i|v_{\tau,i}|^2$ in the archimedean coordinates of a rational lattice vector. The first minimum $\lambda_1(V,g)$ is cut off at 1. Suppose that F is a bounded automorphic function, adelically holomorphic of weight u_τ in the following sense: at each adelic translate it extends along hB, with $h\in U(J)$, B upper triangular, and with value $\prod_i B_{ii}^{u_{\tau,i}}$ times the value at h. Assume

$$|u_{\tau,i}|\leq A,\quad A\geq 2,\qquad u_{\tau,i}+1\leq u_{\tau,k}\quad(\epsilon_i=1,\ \epsilon_k=-1).$$

Let $p = -\bar{p}$ be an algebraic integer with $0 < \operatorname{Im} \tau(p) \le A'$ at each chosen infinite place, $A' \ge 1$, and assume F is invariant under the principal level $K(p)$. If $C = \sup |F| > 0$ and $|F(g)| \ge C/2$, then

$$\lambda_1(V, g) \ge (AA')^{-1/2}. \tag{PM1}$$

Suppose instead that an original nonzero rational lattice vector v in the basis moved by g has length $< (AA')^{-1/2}$. Its Hermitian square is integral, because the adelic coordinates at every finite place are integral and the finite Hermitian matrices are integral. Its absolute value at every archimedean embedding is at most $\|v\|^2 < 1$. The integral norm therefore forces the square to vanish. If the form is definite at any infinite place, this already contradicts $v \neq 0$. This includes the case $j = m$ of the projected source below.

In the remaining case the moved coordinates v_τ at every real place are nonzero and isotropic. The transvection

$$t = I - pvv^* J$$

in the moved adelic basis is unitary. Its conjugate back to the original basis is the rational transvection defined by the same rational vector and the rational Hermitian form of V. Its finite component is in $K(p)$, since the moved coordinates of v are integral at every finite place. Writing $t = (t_\infty, t_f)$, rational left invariance and right $K(p)$-invariance give

$$F(g(t_\infty h, 1)) = F\big((gtg^{-1})\, g(h, 1)(1, t_f^{-1})\big) = F(g(h, 1)). \tag{PM2}$$

This last equality relates the transvection to the original automorphic function at g.

Here is the triangular factorization and bound for the complexified transvection. At a distinguished place τ write $\tau(p) = ia_\tau$, $a_\tau > 0$, and put $\eta_\tau = a_\tau \|v_\tau\|^2 > 0$. Then $A \sum_\tau \eta_\tau \le 1$, so $\eta_\tau \le 1/2$. For z in the upper half-plane set

$$X_\tau(z) = I - d_\tau(z) v_\tau v_\tau^* J, \qquad d_\tau(z) = ia_\tau(z - i), \qquad h_z = \operatorname{Im} z.$$

The isotropy gives $X_\tau^* J X_\tau = J - 2\operatorname{Re}(d_\tau) J v_\tau v_\tau^* J$. For an initial segment of the ordered coordinates of length j, the determinant of its Gram block, divided by the determinant of the corresponding block of J, is

$$p_j = 1 - 2\operatorname{Re}(d_\tau) \sum_{i \le j} \epsilon_i |v_{\tau,i}|^2.$$

The partial sums in this expression increase to $\|v_\tau\|^2/2$ on the positive block and then return to 0.

The p_j are positive. They start and end at 1, with value $p_* = 1 + \eta_\tau(h_z - 1)$ at the end of the positive block. Thus Hermitian elimination without pivoting gives $X_\tau = h_\tau B_\tau$ with $h_\tau^* J h_\tau = J$ and B_τ upper triangular. Indeed the successive elimination pivots have values $\epsilon_i p_i / p_{i-1}$, and normalizing their absolute values gives $|B_{\tau,ii}| = (p_i/p_{i-1})^{1/2}$.

Write $L_i = \log(p_i/p_{i-1})$ and $q_\tau = 1 + \eta_\tau h_z$. If $h_z \ge 1$, the L_i are nonnegative on the positive block and nonpositive on the negative block. Their total is zero and their positive-block sum is $\log p_*$. The weight gap gives $\sum_i u_{\tau,i} L_i \le -\log p_*$. Moreover $p_* = q_\tau - \eta_\tau \ge (1 - \eta_\tau) q_\tau$ and $-\log(1 - \eta_\tau) \le 2\eta_\tau$. If $0 < h_z < 1$, the signs of the L_i reverse, so

$$\tfrac{1}{2} \sum_i u_{\tau,i} L_i \le -A \log p_* \le 2A\eta_\tau(1 - h_z) \le 2A\eta_\tau - \tfrac{1}{2} \log q_\tau.$$

The last inequality follows from $\log q_\tau \le \eta_\tau h_z$. Both cases imply

$$\left|\prod_i B_{\tau,ii}^{u_{\tau,i}}\right| \le \frac{\exp(2A\eta_\tau)}{\sqrt{1+\eta_\tau \operatorname{Im} z}}. \tag{PM3}$$

These factorizations give one holomorphic function along the upper half-plane. If $hB = h'B'$, then $h^{-1}h' = BB'^{-1}$ is both upper triangular and J-unitary. Such a matrix is diagonal: it preserves the successive nondegenerate coordinate spaces and hence their orthogonal complements. Write $\chi_u(B) = \prod_{\tau,i} B_{\tau,ii}^{u_{\tau,i}}$. The weight rule then gives

$$F(g(h,1))\chi_u(B) = F(g(h',1))\chi_u(B').$$

Near $X_0 = h_0 B_0$ choose continuous factors $h(X), B(X)$ by Hermitian elimination with positive diagonal moduli and a fixed diagonal phase. Then $XB_0^{-1} = h(X)(B(X)B_0^{-1})$ lies in the holomorphic germ neighborhood of h_0. Right translation of that germ by B_0^{-1}, multiplied by $\chi_u(B_0)$, agrees with the factorized value by the weight rule. The local compositions along $z \mapsto (X_\tau(z))_\tau$ therefore give one holomorphic function $f(z)$. Equation (PM3) and $A\sum_\tau \eta_\tau \le 1$ give $\sup|f| \le Ce^2$, and the same bound with the denominators in (PM3) gives $f(z) \to 0$ uniformly as $\operatorname{Im} z \to \infty$. The identity $X(z+1) = t_\infty X(z)$ follows because $(v_\tau v_\tau^* J)^2 = 0$. The triangular character and (PM2) give $f(z+1) = f(z)$. In the variable $q = e^{2\pi i z}$, the function is holomorphic on $0 < |q| < 1$, extends by zero at the origin, and is bounded by Ce^2. Schwarz's lemma at $z = i$ gives $|F(g)| = |f(i)| \le Ce^{2-2\pi} < C/2$, contradicting the peak condition. This proves (PM1) in the moved lattice and the original level.

41.3 The auxiliary automorphic form and its first minimum

In this subsection, we apply the first two exchanges of Section 39.4 to verify the hypotheses of (PM1) for the auxiliary form and for the descent test.

Put $S = I_{2m} \perp I_m^-$, and write $K_0 = U(S)(\widehat{\mathcal{O}}_F)$ for the stabilizer of the stated standard finite lattice. In (109) one may fix $\vartheta = 1$ and $c = \chi_M^j$. At either old chosen place the residue character ω_w^j of χ_M^j is nontrivial, since its order is $q_2/\gcd(q_2, j)$ for $1 \le j \le m < q_2$. Thus $\phi = F^\circ$ is a fixed finite sum of the displayed theta kernels at the original representatives h_a, with coefficients $\gamma_{12}A_c(h_a)/e_{3m}(h_a)$ independent of the point. For $k \in K_0$, the integral Weil action on the finite datum is

$$c(\det k)^{-1}\chi(\det \bar{k}) = 1,$$

where $\bar{k}$ denotes residue reduction. Thus every summand and ϕ itself are right K_0-invariant. In particular they have every principal level contained in K_0. The determinant character extended by zero on nonunits is supported on invertible residue matrices.

Fix one split member of the nonempty support of $\mathfrak{m}$. The square residue matrix there is invertible on the support of the determinant test. In the lower-kernel estimate of Section 34.3.1, S is the varying source and h_a is the fixed definite target point. Thus for a short nonzero source vector v, one has $Yv_w \neq 0$ over E_w for every local matrix Y in the support. This supplies the nontrivial modulation in the local additive pairing. Decomposing the finite determinant test into its residue point indicators and applying that estimate gives, for every $D \ge 0$,

$$\sup_s \operatorname{ht}_S(s)^D \left|\Theta_{v^m,c}^{S,I_{3m}}[\chi \circ \det, 1](s, h_a)\right| \le C_{a,D} < \infty.$$

Here c and the finitely many h_a are fixed once for this presentation of ϕ. Their weighted sum is rapidly decreasing.

Continuity and rational left invariance follow from the theta sum. At infinity the Fock polynomial 1 is killed by the holomorphic lowering operators. Its diagonal weight on S, with weight-zero second splitting character and positive definite rank $3m$ on the other side, is 0 on the positive block and $3m$ on the negative block. For the order numbering the positive block by $-1, -2, \dots$ and the negative block by $0, 1, \dots$, the upper triangular root operators in a single block also annihilate the constant polynomial. The Schwartz-space orbit calculation of Section 40.3 therefore gives its triangular continuation of weight $(0; 3m)$. At fixed (x, h_a) the translated theta functional is continuous on the real Schwartz space. Applying it to this orbit map gives the continuation for each adelic left translate. Every summand uses the same order and weight, so ϕ has these as well.

To prove nonvanishing use the original presentation of ϕ as the lift of 1 from I_m^- to S, with characters $(v^m, 1)$. The finite datum is the integral vacuum at every place. For the sharp convolution formula take the unit modulus, empty prescribed support, and full integral source level. The finite sharp condition at the empty support is vacuous, the vacuum fixes the integral level, its mass there is positive, and the second character is trivial. In the sharp inner-product formula the finite factor is now the vacuum series on all finite places. Its neighbour count applies at each conjugation orbit, so the same absolute convergence and injectivity argument applies. The rank inequality is $3m > 2m$, and the constant source is rapidly decreasing since I_m^- is definite at infinity and its adelic quotient is compact.

Let v be the original Fock vector of $P = \prod_\tau \det(Y_\tau^-)^{2m}$. This polynomial is nonzero. For $u = (u_\tau)$ on the compact negative-definite source, put $\delta_\tau = \det u_\tau$. The $2m$ positive target coordinates give the vacuum factor δ_τ^{2m} from the negative Hom block. On the negative target coordinates the Fock substitution is right multiplication by $(u_\tau^{-1})^{\mathsf{T}}$, so that $\det(Y_\tau^-)^{2m}$ acquires δ_τ^{-2m}. The source splitting character is trivial and the target is evaluated at the identity. Consequently

$$\omega(u, 1)v = v, \qquad k(u) = \langle v, \omega(u, 1)v \rangle = \|v\|^2.$$

If the auxiliary lift vanished, the sharp formula and the injectivity of its finite vacuum series would give

$$0 = (k * 1)(1) = \|v\|^2 \operatorname{vol}(U(I_m^-)(F_\infty)) > 0.$$

The Haar measure is that of the original real convolution. This proves nonvanishing on the same adelic realization as in the identity for the auxiliary lift. In particular ϕ is the single nonzero full-level holomorphic form to which the peak minimum is applied, common to the later source space and class tests.

Apply this first to the auxiliary form ϕ of signature $(2m, m)$. Its nonzero rapid full-level form is holomorphic of weight $(0; 3m)$. Full level contains $K(\sqrt{-d})$; the given CM embeddings read $\sqrt{-d}$ with positive imaginary part $\sqrt{d}$. The admissible scale gives $3m \le 5b$ and $d \le b$. Therefore $A = 5b$, $A' = \sqrt{b}$ give, at the chosen peak x_ϕ,

$$\lambda_1(S, x_\phi) \ge (5b\sqrt{b})^{-1/2}. \tag{PM4}$$

For a nonzero form in the projected space y on W_j, use $F = \bar{y}$ on the original W_j. Its selected finite type is cuspidal and the form is rapidly decreasing. Its original principal level is $J_M = K(\pi_M)$. At each chosen split member $w \mid \pi_M$, one has $v_w(M\sqrt{-d}) = 1$, since M is the product of the two distinct old primes, both prime to d. Consequently $K(M\sqrt{-d}) \subset J_M$. This

containment applies to y and to $\bar{y}$ at every peak. The exponent-two level $K(\pi_M^2)$ enters the reproduction of the products yF_i in (CP2).

The adjoint calculation gives this holomorphic weight on the original projected space. Its proof uses only the Gram operator on the projected source of the single lift and the adjoint. The opposite-sign gap is at least $300n_*$ by the last negative column count, with the definite case having no opposite signs.

Now $p = M\sqrt{-d}$ is purely imaginary in the required sense and

$$0 < \operatorname{Im} \tau(p) = M\sqrt{d} \le b^{22}\sqrt{b} \le b^{23}$$

by the original level-prime interval and $d \le b$. Thus (PM1) with $A = 5b$, $A' = b^{23}$ gives

$$\lambda_1(W_j, g) \ge (5b^{24})^{-1/2} \quad \text{at every peak of } \bar{y}. \tag{PM5}$$

The product of (PM4) with (PM5), after adjoining the fixed definite summands to W_j, is at least $(5b^{51/4})^{-1} \ge b^{-13}$ at the chosen scale. Each fixed definite summand at the identity has first minimum at least 1, so the orthogonal sum has the same truncated first minimum as W_j. At the rotated peak $z_s = z(r_s, 1)$ the finite lattice is the one at z. The real coordinate at τ is multiplied by $r_{s,\tau}^{-1}$, which preserves its Euclidean norm. Hence $\lambda_1(W_j, z_s) = \lambda_1(W_j, z)$ throughout the circle of Lemma 40.1.

41.4 Finite overlaps and Gaussian decay

In this subsection, the error in Section 41 is summed over nonzero maps between two moved lattices. A first-minimum estimate for this Hom lattice gives a summable Gaussian bound in the original normalizations.

The comparison of the first minimum on the Hom space with the two first minima uses the original self-dual Hermitian lattices. Let L be the source lattice in its Euclidean real coordinates, of dimension $D_{\mathbb{R}} = 2f \operatorname{rk} S$. Take $N_c = dq_1$, so $E \subseteq \mathbb{Q}(\zeta_{N_c})$. The integer N_c belongs to the different of E. Indeed it belongs to the different of the cyclotomic field by differentiating $X^{N_c} - 1$ at a primitive root and using the cyclotomic factor. For $C = \mathbb{Q}(\zeta_{N_c})$, transitivity gives $\mathfrak{D}_C = \mathfrak{D}_{C/E}\mathfrak{D}_E\mathcal{O}_C \subseteq \mathfrak{D}_E\mathcal{O}_C$. Contraction to E gives $N_c \in \mathfrak{D}_E$. The Euclidean dual of L is $2J_\infty\mathfrak{D}_E^{-1}L$. The trace of the Hermitian pairing is twice the real Euclidean pairing with the sign matrix J_∞, and the finite Hermitian lattice is self-dual. Consequently the Euclidean dual of $2L/N_c$ is contained in $J_\infty L$, so its nonzero vectors have length at least the cut-off first minimum λ_S.

Banaszczyk's transference theorem [7, Theorem 2.2, p. 632] gives covering radius at most $D_{\mathbb{R}}/(2\lambda_S)$, hence at most $D_{\mathbb{R}}^2/\lambda_S$, for $2L/N_c$. The vectors of a lattice with covering radius at most $R > 0$ and length at most $3R$ span its real space. Otherwise a vector of length $2R$ in their orthogonal complement is farther than R from all those vectors, and a longer lattice vector is also farther than R.

Thus the source lattice contains an E-basis with each norm at most $6N_c(f \operatorname{rk} S)^2/\lambda_S$. A nonzero integral Hom vector sends at least one of these basis vectors to a nonzero vector of the target lattice. If λ_T is the target minimum, the operator norm bound by the Euclidean coordinate norm, even with the weaker factor t^2, gives

$$\lambda_{\mathrm{Hom}(S,T)} \ge \frac{\lambda_S\lambda_T}{6N_c(ft)^2t^2}.$$

At the fixed adelic pair (g,h), use the basis matrices X_S, X_T of Section 40.2.1. Write $\mathcal{O}_f = \widehat{\mathcal{O}}_E$. The rational lattices are

$$L_S = S(E) \cap (X_S g)_f \mathcal{O}_f^{\mathrm{rk}\, S}, \quad L_T = T(E) \cap (X_T h)_f \mathcal{O}_f^{\mathrm{rk}\, T}.$$

The Euclidean realizations use $(X_S g)_\infty^{-1}$ and $(X_T h)_\infty^{-1}$. A rational map $Z\colon S \to T$ has coordinates $z(Z) = (X_T h)^{-1} Z (X_S g)$. For a surviving difference, the integral box and the zero residue condition together give

$$\alpha_{\mathrm{W}}^{-1} z(Z)_f = \pi_M^2 V_f, \qquad V_f \in \mathrm{Mat}_{\mathrm{rk}\, T \times \mathrm{rk}\, S}(\mathcal{O}_f).$$

Thus $w = (q_1/\pi_M^2) Z$ carries L_S into L_T, since its finite matrix $z(w)_f = (q_1 \alpha_{\mathrm{W}}) V_f$ is integral by Section 8.1. Write

$$\|w\|^2 = \sum_{\tau \in \Phi} \sum_{i,j} |z(w)_{\tau,ij}|^2.$$

Since $\pi_M \bar{\pi}_M = M$, every embedding τ satisfies $|\tau(\pi_M^2)| = M$. The same calculation applies to the difference of any two surviving maps, for every choice of g_f, h_f. In the chosen CM type, $j_{\mathrm{W}} = -1/\sqrt{-d}$ from Section 8.1 gives the real Fock scale $(2\pi/\sqrt{d})^{1/2}$. Therefore

$$\|z_{\mathbb{R}}\| = \left(\frac{2\pi}{\sqrt{d}}\right)^{1/2} \frac{M}{q_1} \|w\| \geq \left(\frac{2\pi}{\sqrt{d}}\right)^{1/2} \frac{M \lambda_S \lambda_T}{6 q_1 N_c(ft)^2 t^2} \geq b^6.$$

The last inequality uses $M \geq b^{20}$, $\lambda_S \lambda_T \geq b^{-13}$, and $6 q_1 N_c(ft)^2 t^2 \leq (2\pi/\sqrt{d})^{1/2} b$ for sufficiently large x. The latter follows from polynomial growth in x at fixed q_0, d and the lower bound on b in Part I.

41.5 The normalized overlap on the adelic Hom space

In this subsection, we calculate the overlap of two adelic tests as a product of a residue-ring factor and a polarized real Fock factor. This supplies the coefficient estimates used to compare the leading term and the error in Section 41.

Use the original pair S, T, with $T = \widetilde{W}$ for the sparse test, the modulus $\mathfrak{m} = (\pi_M^2)$, and the Weil parameter $(\psi_f, j_{\mathrm{W}}, \alpha_{\mathrm{W}})$. Put $R = \mathcal{O}_E/\mathfrak{m}$. Let $D = \mathrm{rk}_E \mathrm{Hom}_E(S,T)$. An adelic displacement z is read at the finite places as $a(z) = \alpha_{\mathrm{W}}^{-1} z_f$ and at infinity as $z_{\mathbb{R}} = (\sqrt{2\pi\ \mathrm{Im}\, \tau(j_{\mathrm{W}})}\, z_\tau)_\tau$. Let $\mathcal{R}(z)$ be the Heisenberg operator and v_{vac} the finite unit vacuum.

41.5.1 The probability mean and Fourier inversion

In this subsection, we express the overlap by Fourier inversion in the normalization used to bound the error in Section 41.5.3.

In the rational symplectic displacement space choose a rational Schrödinger polarization $X \oplus X^\vee$ over F_0. Give $X(\mathbb{A})$ the Haar measure for which $X(F_0)$ has covolume one. The annihilator of $X(F_0)$ is $X^\vee(F_0)$. Poisson summation therefore gives covolume one on the dual quotient for the Fourier-dual Haar measure. The theta functional in this realization is rational summation, up to a unit scalar.

Indeed rational modulation invariance of a functional forces its support to be on $X(F_0)$, by adelic self-duality. At a rational point ξ fix a finite box $\xi_f + L_f$, where L_f is a compact open additive subgroup. The rational positions with finite coordinates in the box form a lattice

coset at infinity. Localize at ξ_∞ in a ball meeting this coset only there. Rational modulations from $X^\vee(F_0) \cap L_f^\perp$ are constant on the finite box, and their global values at ξ are one. This intersection is a full lattice in the real dual under the trace pairing. Select modulations with real linear phases ℓ_j forming a basis of that dual. In coordinates t centered at ξ_∞, the localized distribution is killed by each $e^{i\ell_j(t)} - 1$, hence by each $\ell_j(t)$ since the quotient by $\ell_j(t)$ is smooth and nonzero at zero. Every smooth germ vanishing at zero is a sum of the ℓ_j times smooth germs, by integration along radial segments. Thus the distribution acts by evaluation at the point, including on finite tests since they are locally constant. Rational translation invariance gives the same multiplier at every rational point. Denote this common scalar by C. For a Schwartz vector f the translate is

$$\theta(\mathcal{R}(u,v)f) = C \sum_{\xi \in X(F_0)} f(\xi + u)\psi\big(v(\xi + u/2)\big).$$

Multiply by the conjugate of the same expression for f'. The two half-shift factors cancel. Integration over $v \in X^\vee(\mathbb{A})/X^\vee(F_0)$ kills each term whose two rational indices differ. Integration over $u \in X(\mathbb{A})/X(F_0)$ then unfolds the common rational index. Consequently, with dz the probability Haar on the displacement quotient,

$$\int \theta(\mathcal{R}(z)f)\overline{\theta(\mathcal{R}(z)f')}\, dz = |C|^2 \int_{X(\mathbb{A})} f(u)\overline{f'(u)}\, du = |C|^2 \langle f', f\rangle.$$

On compact fundamental sets the finite support confines the rational indices to finitely many lattice cosets. The real Schwartz bounds give absolute uniform convergence there. Taking $f = f'$ equal to the unit vacuum gives $|C|^2 = 1$ by the prescribed mean-square normalization.

Let $A_{f,f'}(z)$ denote the integrand. Rational Heisenberg invariance and cancellation of the common unit phase make it periodic. For a rational displacement $\gamma = (a, b)$ and $z = (u, v)$, composing the displayed Schrödinger actions gives

$$\mathcal{R}(z)\mathcal{R}(\gamma) = \psi\big(b(u) - v(a)\big)\mathcal{R}(\gamma)\mathcal{R}(z), \qquad \psi\big(b(u) - v(a)\big) = \beta(z,\gamma)^2.$$

Applying the last mean identity to $(\mathcal{R}(\gamma)f, f')$, rational invariance gives

$$\int \beta(z,\gamma)^2 A_{f,f'}(z)\, dz = \langle f', \mathcal{R}(\gamma)f\rangle.$$

The inverse characters $\beta(z,\gamma)^{-2}$ are the dual of the quotient.

The series of these Fourier coefficients is absolutely convergent. To see this choose an open finite additive compact group fixing both f, f' under its Heisenberg displacements. The product is invariant under that group because the remaining commutator factors have modulus one and cancel. The quotient by this group is a real torus, by additive approximation. The product descends to a smooth function there, and integration by parts bounds its Fourier coefficients by every inverse power of the frequency norm. Fourier inversion at zero therefore yields

$$\theta(f)\overline{\theta(f')} = \sum_{\gamma \text{ rational}} \langle f', \mathcal{R}(\gamma)f\rangle.$$

In particular the zero-frequency coefficient is $\langle f', f\rangle$ with its original scale.

41.5.2 The finite character and its translate

In this subsection, we compute the finite component of the transform in Section 41.5.1.

Write s_0 for the finite generator of $\mathfrak{m}$ in this paragraph and $e_{\mathfrak{m}}$ for the idempotent of its chosen split places. Define $\psi_0(x) = \psi_f(j_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}\alpha_{\mathrm{W}}x)$. The primitive character on R is the reduction of $\psi_0(e_{\mathfrak{m}}s_0^{-1}x)$. For a residue frequency matrix t, let $\widetilde{t}$ be an integral lift of its transpose, indexed by the Hom coordinates. Its normalized coordinate is

$$y_t = \overline{e_{\mathfrak{m}}s_0^{-1}\widetilde{t}},$$

and its actual adelic displacement is $Y_t = (0, \alpha_{\mathrm{W}}y_t)$. The finite part of the datum of $\psi_t(X) = \psi_{\mathfrak{m}}(\mathrm{tr}(Xt))$ is $\mathcal{R}(Y_t)v_{\mathrm{vac}}$. In fact the projection onto the integral Fourier box fixes v_{vac}. The translation above has support on the conjugate member of each chosen split pair, so in the selected Schrödinger box it is multiplication by the indicated character. The dot product with $\widetilde{t}$ is the trace of Xt. This also shows independence from the integral lift of t.

For an arbitrary adelic displacement z, move $\mathcal{R}(Y_t)$ across the inner product. Its commutator with $\mathcal{R}(z_f)$ contributes $\beta(z_f, Y_t)^2$, with

$$\beta(z_f, Y_t)^2 = \psi_f\left(j_{\mathrm{W}}\sum_{\nu}\bar{\alpha}_{\mathrm{W}}\, e_{\mathfrak{m}}s_0^{-1}\widetilde{t}_{\nu}z_{f,\nu}\right).$$

When $a(z)$ is integral this is $\psi_t(a(z) \bmod \mathfrak{m})$, by $\bar{\alpha}_{\mathrm{W}}z_f = \bar{\alpha}_{\mathrm{W}}\alpha_{\mathrm{W}}a(z)$. The other finite factor is the vacuum coefficient

$$\mathbf{1}_{a(z)\in\widehat{\mathcal{O}}_E^D}\,\psi_f\left(\frac{c_{\mathrm{tr}}j_{\mathrm{W}}}{2}\sum_{\nu}\overline{z_{f,\nu}}z_{f,\nu}\right), \qquad c_{\mathrm{tr}} = (1 - j_{\mathrm{W}}^{-1})/2.$$

Indeed the normalized translation fixes the vacuum on the integral lattice, so the unnormalized coefficient there is the inverse of its quadratic phase. The displayed phase equals its inverse on this support: its square is $\psi_0(c_{\mathrm{tr}}\sum_{\nu}\overline{a(z)_{\nu}}a(z)_{\nu}) = 1$ by integrality and self-duality. This includes places of residue characteristic two. For real Fock vectors v, w the real factor is $\langle v, \mathcal{R}_{\infty}(z_{\mathbb{R}})w\rangle$. Finite character orthogonality is for the perfect trace pairing of the two rectangular matrix spaces over R. The probability average over all t therefore restricts the product of these factors to $a(z) \in \widehat{\mathcal{O}}_E^D$ and $a(z) \equiv 0 \bmod \mathfrak{m}$. At $z = 0$, the finite factor, all commutators, and the normalized average of the constant function are one.

41.5.3 The sparse test and its error

In this subsection, we apply Section 41.5.2 to the test used in the comparison.

Fix the original two adelic points g, h and let p be the induced isometry of $\mathrm{Hom}_E(S, T)$ over the adeles, given by $X \mapsto hXg^{-1}$. Let v be the Fock vector for the polynomial P. Write $K_t(g, h)$ for its kernel with finite test ψ_t. Use the single finite test

$$\Psi(X) = \frac{1}{|M_{\mathrm{rk}\,S,\mathrm{rk}\,T}(R)|}\sum_{t}\overline{K_t(g,h)}\psi_t(X).$$

At $g(u, 1), h(u', 1)$ with real u, u', put $w = \omega_{\infty}^{a,b}(u, u')v$, including the determinant weights of the original characters a, b. Linearity in the finite test writes its kernel as the same average of the product of the theta value at w with the conjugate theta value at v, both moved by p. For

the residue matrix space D_R, write $K_{[B]}$ for the kernel of the indicator of the single residue B, and let $a = (g, h)$, $a' = (g(u, 1), h(u', 1))$. Finite Fourier orthogonality gives

$$\frac{1}{|D_R|}\sum_t K_t(a) = K_{[0]}(a), \qquad \frac{1}{|D_R|}\sum_t \overline{K_t(a)}K_t(a') = \sum_{B\in D_R} \overline{K_{[B]}(a)}K_{[B]}(a').$$

Indeed $K_t = \sum_B \psi_t(B)K_{[B]}$ with the same indicator of the integral matrix box in each summand. At $a' = a$ the second identity is the squared norm of this vector of residue values.

The starting pair contributes the same unit scalar to both theta values and cancels in the product. Move $\omega(p)$ through the Heisenberg coefficient by unitarity and covariance. For a rational difference $Z\colon S \to T$, put $z_0 = X_T^{-1} Z X_S$. Then

$$\omega(p)^*\mathcal{R}(z_0)\omega(p) = \mathcal{R}(p^{-1}z_0), \qquad p^{-1}z_0 = h^{-1}z_0 g = (X_T h)^{-1} Z (X_S g) = z(Z).$$

Use the finite reduction and scaled real coordinate of this same $z(Z)$ at (g, h) as in Section 41.4. The additional real translations u, u' act on the vector w. The Fourier identity just proved applies for each t. The resulting average is thus a sum over precisely the rational differences that satisfy the displayed finite support and congruence, with unit-modulus finite phases. Its zero difference equals $\langle v, \omega_\infty^{a,b}(u, u')v\rangle$. This is the original Fock matrix coefficient with mass one.

For the nonzero differences let N be the number of complex Fock variables. The actual $P = P_j$ is homogeneous of degree $d = \sum_\tau (mc_\tau + 2m\nu_\tau) \le b\sqrt{b}$ by Section 41.1.4. The coherent expression of the normalized real Weil operator is

$$\omega_\infty(\sigma)v = \Delta_\infty(\sigma)\int v(x)\mathcal{R}_\infty(\sigma x)v_{\mathrm{vac},\infty}\,dx.$$

Here $dx = \pi^{-N} d\,\mathrm{Leb}(x)$. To check the scalar, let G denote the unit vacuum. For weighted Fock vectors one has $\langle \mathcal{R}_\infty(x)G, v\rangle = v(x)$. The coherent resolution of the identity follows by integrating against the Fock reproducing kernel with this measure. For this polynomial times the Gaussian, $v \in L^1(dx)$, hence the vector integral below converges in Hilbert norm for every unit Schwartz vector G'. The mixed identity is

$$\int \langle \mathcal{R}_\infty(x)G, v\rangle \mathcal{R}_\infty(x)G'\,dx = \langle G, G'\rangle v.$$

Indeed in a Schrödinger realization Fourier inversion in the modulation variable first identifies the two position arguments, and integration in the translation variable gives $\langle G, G'\rangle$. The scalar in the measure is fixed by $G' = G$. Take $G' = \omega_\infty(\sigma)^{-1}G$ and apply $\omega_\infty(\sigma)$, using Heisenberg covariance. This gives the displayed formula with $\Delta_\infty(\sigma) = 1/\overline{\langle G, \omega_\infty(\sigma)G\rangle}$. Taking the constant polynomial in the boost calculation of Section 41.1.4 gives one inverse hyperbolic cosine for each boost. Compact factors give the inverse determinant on their negative-sign block. Thus in the untwisted unitary normalization the vacuum coefficient is the inverse determinant of the negative-sign block of σ. Each block has operator norm at most $\|\sigma\|$, so $|\Delta_\infty(\sigma)| \le \|\sigma\|^N$.

Inserting the coherent integral into $\langle v, \mathcal{R}_\infty(z_\mathbb{R})\omega_\infty(\sigma)v\rangle$ gives the translate $\mathcal{R}_\infty(z_\mathbb{R} + \sigma x)G$ times the unit Heisenberg phase. Its pairing with v has modulus $|v(z_\mathbb{R} + \sigma x)|$, so the displaced coefficient is bounded by

$$\|\sigma\|^N \int |v(x)|\,|v(\sigma x + z_\mathbb{R})|\,dx.$$

The determinant-character factors are units. If $q > N$, put $A_0 = d + N$ and $c^2 = 1 + q/A_0$. For $G(x) = |c^d v(x/c)|^2$, homogeneity gives

$$\int G \le e^q \|v\|^2, \qquad |v(x)| = G(x)^{1/2} e^{-q\|x\|^2/(2(A_0+q))}.$$

The first inequality uses the dilation Jacobian c^{2N}. Here the norm on displacements is the Euclidean norm after the real scaling above. Passage to the holomorphic coordinates of the signed Fock model conjugates the coordinates with negative oscillator sign. This is a real orthogonal map, so it preserves the Gaussian, dx, and the operator norm of the induced Hom action. The latter preserves real volume by symplectic invariance and has $\|\sigma\| \ge 1$. The inequality

$$\|x\|^2 + \|\sigma x + z_{\mathbb{R}}\|^2 \ge \|z_{\mathbb{R}}\|^2/(2\|\sigma\|^2)$$

follows from the triangle inequality and the two-square bound.

Apply $2ab \le a^2 + b^2$ to the two square roots of G. For $z_{\mathbb{R}} \ne 0$, the remaining Gaussian is at most e^{-qt_z}, where

$$t_z = \frac{\|z_{\mathbb{R}}\|^2}{4(A_0 + q)\|\sigma\|^2}.$$

Since $e^{-qt_z} \le e^{-q} t_z^{-q}$ and σ preserves volume, integration cancels the factor e^q. This gives

$$\|\sigma\|^{N+2q} (4(d + N + q))^q \|v\|^2 \|z_{\mathbb{R}}\|^{-2q}.$$

On Hom, $\sigma(X) = u' X u^{-1}$ and $\|\sigma\| \le |u'|\,|u|$, since the operator norm of a signed unitary matrix equals that of its inverse.

On the reduction kernel the original multiplier q_1/π_M^2 sends every surviving difference into the Hom lattice. When the two given first minima have product at least b^{-13}, the Hom-lattice calculation gives separation $\mathcal{D} = b^6$ for their Fock coordinates, including distance at least $\mathcal{D}$ from zero. Packing balls of radius $\mathcal{D}/2$ bounds the count in the annulus of radii $2^i\mathcal{D}$ and $2^{i+1}\mathcal{D}$ by $(5 \cdot 2^i)^{2N}$. Summing the geometric series and using $q \ge N + 1$ gives

$$\sum_{z_{\mathbb{R}} \ne 0} \|z_{\mathbb{R}}\|^{-2q} \le (5/\mathcal{D})^{2q}.$$

Here $N = f(3m)(r + 3m)$ since $\dim S = 3m$ and $\dim \widetilde{W} = r + 3m$. Both dimensions are at most t for sufficiently large x. Take $q = 2n_*$, so $N \le ft^2 < q$. The error in the sparse test is at most

$$(|u|\,|u'|)^{6n_*} \|v\|^2 \left(\frac{100(d + 2q)}{b^{12}}\right)^q \le b^{-20n_*} (|u|\,|u'|)^{6n_*} \|v\|^2.$$

Indeed $100d \le b^2/2$ for $b \ge 200^2$ and $200q = 400n_* \le b^2/2$ for sufficiently large x with the external modulus, quadratic discriminant and ϵ fixed. This uses the polynomial growth of n_* and $\log b = x^\epsilon + O(1)$. The same finite test works for all u, u' in this estimate. At $u = u' = 1$ the estimate gives

$$\frac{1}{|D_R|} \sum_t |K_t(g, h)|^2 \ge (1 - b^{-20n_*}) \|v\|^2.$$

This is positive for the prescribed nonzero Fock vector. At every real translate the error uses the same fixed norm $\|v\|^2 = c_P(1, 1)$.

42 Compression, separation, and reduction at cusps

In this section, we compress the comparison operator of Section 34 and establish separation on the initial unitary group. The cusp estimates justify the decay needed for the deeper periods of Section 43.

42.1 Compression of the comparison operator

In this subsection, we restrict the comparison of Section 34.1 to the image of the injective lift and project with respect to the Petersson form. We bound the determinant of this operator using the original norm on each factor.

Fix one original descent pair and write $u_\eta = u_{j,\eta}$ for the injective lift on the projected source W. The identity test in (4) gives $\dim W = D$, so $Z_\eta = \operatorname{im} u_\eta$ has dimension D for every allowed twist. Let Π_η be the orthogonal projection onto Z_η for the target class pairing. It is also the Petersson projection, since at a fixed level and Weyl type the two pairings differ by the positive level and compact Schur factor of Section 34.2.1. Use the original gluing operator G_η, its original tuple space, and the operator $U_\eta = n_A h_-^2 G_\eta G_\eta^*$ of Section 34.1. Put

$$\Delta_\eta = \det(\Pi_\eta U_\eta|_{Z_\eta}), \qquad Z_j(\eta) = \det(u_\eta^* u_\eta).$$

42.1.1 The period and the Hecke factor

In this subsection, we use Proposition 39.1 in the compressed pairing.

Lemma 42.1 (Determinant of the compressed comparison)**.** *Set $\rho_\eta = H_j(\eta)/H_j(1)$ for the allowed twists. If $D > 0$, then $\rho_\eta > 0$ and there is a fixed positive definite operator U° on the source such that*

$$u_\eta^* U_\eta u_\eta = \rho_\eta U^\circ, \qquad Z_j(\eta)\Delta_\eta = \det(U^\circ)\rho_\eta^D, \qquad \frac{Z_j(1)}{Z_j(\eta)} = \frac{\Delta_\eta}{\Delta_1}\rho_\eta^{-D}.$$

If $D = 0$, all the determinants and ratios of determinants are 1.

Proof. For $D > 0$, let P be the injective common-period map and $c'_\eta \neq 0$ the scalar in $c'_\eta G_\eta^* u_\eta = P$. The three exchanges of Section 39.4 give

$$|c'_\eta|^2 = C_{\mathrm{ex}}/H_j(\eta),$$

where $C_{\mathrm{ex}} \neq 0$ is independent of η, and H_j is the product of Hecke values including the local conductor factors. In particular

$$\rho_\eta = \frac{H_j(\eta)}{H_j(1)} = \frac{|c'_1|^2}{|c'_\eta|^2} > 0, \qquad u_\eta^* U_\eta u_\eta = \rho_\eta U^\circ, \qquad U^\circ = \frac{n_A h_-^2}{|c'_1|^2} P^* P > 0.$$

Indeed the pairing of the middle operator against $y', y \in W$ is $n_A h_-^2 \langle G_\eta^* u_\eta y', G_\eta^* u_\eta y\rangle$; using the two scalar factors gives the formula. Here $u_\eta^{-1} : Z_\eta \to W$ inverts the lift onto its image, and $u_\eta^*(1 - \Pi_\eta) = 0$ by orthogonality. Hence

$$u_\eta^* U_\eta u_\eta = (u_\eta^* u_\eta)(u_\eta^{-1} \Pi_\eta U_\eta u_\eta).$$

Taking determinants yields

$$Z_j(\eta)\Delta_\eta = \det(U^\circ)\rho_\eta^D, \qquad \frac{Z_j(1)}{Z_j(\eta)} = \frac{\Delta_\eta}{\Delta_1}\rho_\eta^{-D}.$$

For the covectors f_a of (A), transported to the tuple space of G_η as in Section 34.1.1, put $z = u_\eta y$. Orthogonal projection and the rank-one sum give

$$\sum_a |\langle \Pi_\eta G_\eta f_a, z\rangle|^2 = h_-^2 \|G_\eta^* z\|^2 = h_-^2 |c'_\eta|^{-2}\|Py\|^2.$$

Thus the projected vectors span Z_η, and positivity gives the nonzero determinants used above. In dimension zero the determinant is the empty product. □

42.1.2 The projection at complex embeddings

In this subsection, we bound the conjugate projection for the operator of Section 42.1.1.

At each complex automorphism σ, pull back the positive Gram pairing of the coefficientwise images of the form values by σ^{-1}. Use the original rational class weights and Weyl Gram matrices. This pairing is semilinear in its first argument for $\sigma^{-1}c\sigma$, where c is ordinary complex conjugation. Let J_0 be the original target level and J the rational Weyl Gram matrix. For two finite-level forms take a common compact open $K' \subseteq J_0$ fixing both, reducing to the kernels of their finite level characters if necessary. The function $b_\sigma(g) = \langle \sigma f(g), \sigma h(g)\rangle_J$ is locally constant on the finite adelic quotient, by the rational isometry below. The pairing on the coordinatewise images, before pullback, is

$$\frac{1}{\operatorname{vol}(J_0)}\int b_\sigma(g)\,dg = \frac{\operatorname{vol}(K')}{\operatorname{vol}(J_0)}\sum_{[g]_{K'}} \frac{b_\sigma(g)}{|\operatorname{Aut}_{K'}(g)|}.$$

For a rational point γ, each matrix $\sigma\tau(\gamma)$ is unitary, since every embedding of the CM field commutes with conjugation on it. Thus the rational Weyl action preserves the summands under changes of representatives. For two forms of the original level character χ, the equality $\overline{\sigma\chi(j)}\,\sigma\chi(j) = 1$ gives J_0-invariance. For a J_0-class represented by x, write

$$\Gamma_x = x^{-1}U(I_n)(F_0)x \cap J_0.$$

The finer classes are the Γ_x-orbits on J_0/K'. If $(\Gamma_x)_y$ is the stabilizer of a representative y of such an orbit, then

$$\frac{1}{[J_0 : K']}\sum_{y\in\Gamma_x\backslash J_0/K'} \frac{1}{|(\Gamma_x)_y|} = \frac{1}{|\Gamma_x|}.$$

This gives the original weight at level J_0.

For $R(a)f(g) = f(ga)$, right invariance of dg gives $R(a)^* = R(a^{-1})$ on the images with this fixed normalization. In either finite sum one may further intersect the levels of the translated test forms. If a translation carries a tame character factor u, its adjoint carries $\overline{\sigma(u)} = \sigma(u^{-1})$, since u is a root of unity. Pulling back these finite identities by σ^{-1} gives the adjunction for the conjugate pairing. Write $\|\cdot\|_\sigma$ for its positive norm on the images.

The original finite group acts by right translations and finite-order character factors, so its action is unitary at each embedding. Put $\tau_\sigma = \sigma^{-1}c\sigma$. For each ordinary irreducible character

χ of the finite group, $\tau_\sigma(\chi(g^{-1})) = \chi(g)$: the character values are cyclotomic, and complex conjugation commutes with every automorphism of a cyclotomic field. Thus taking adjoints in the finite character sum for e_0 both inverts g and conjugates its coefficient, giving $e_0^{*\sigma} = e_0$. By [54, Corollary 4.12(c)] there is a rational prime $p \equiv 1 \bmod dq_1$ away from the finite level and its conjugate. Every place $w \mid p$ is split over F_0, since $E \subseteq \mathbf{Q}(\zeta_{dq_1})$. Choose one such w. Let $T_{w,i}$ be its degree-i elementary spherical double coset, with $T_{w,0} = 1$. In the notation of Proposition 32.1 put

$$A_w = K_w(X_w), \qquad K_w(X) = \sum_{i=0}^{n} (-X)^i q^{\binom{i}{2}} T_{w,i}, \qquad X_w = (q^{t'-1}\nu_w)^{-1}, \quad q = \mathrm{N}w.$$

Here $n = r' + t'$ is the target rank. On a spherical system with monic Hecke polynomial P_x, this operator has eigenvalue $X_w^n P_x(X_w^{-1})$.

Let $K = \mathrm{GL}_n(\mathcal{O}_w)$, $d_i = \mathrm{diag}(\varpi_w I_i, I_{n-i})$, and $z_w = \varpi_w I_n$. With K of mass one, inversion interchanges the operator of Kd_iK on the original K-fixed space with that of $Kd_i^{-1}K$ in the adjoint. Permuting the diagonal gives

$$T_{w,i}^{*\sigma} = T_{\bar{w},i} = \rho(z_w^{-1}) T_{w,n-i} \qquad (0 \le i \le n).$$

Here ρ is right translation on the split group. The middle identity uses the inverse transpose at the conjugate place; changing the conjugate uniformizer by a unit preserves the double coset. Consequently the adjoint of A_w for this pairing is $K_{\bar{w}}(\tau_\sigma(X_w))$, since all the coefficients $(-1)^i q^{\binom{i}{2}}$ are rational. These two operators commute in the spherical algebra of the split pair. Normality gives $\ker A_w^* = \ker A_w$, so $\operatorname{im} A_w \perp_\sigma \ker A_w$.

The factor $1 - \nu_w q^{t'-1} X$ in the split lift identity of Section 28.1.1 places every lift generator at the trivial twist in $\ker A_w$. Thus $Z_1 \subseteq \operatorname{im} e_0 \cap \ker A_w$. For the same subspace

$$F_w = Z_1^{\perp} \cap \operatorname{im} e_0 \cap \ker A_w$$

used in Proposition 32.1 and (94), the orthogonals in the original metric satisfy $F_w^{\perp} = Z_1 + \ker e_0 + \operatorname{im} A_w$. The last two summands are orthogonal to Z_1 for every conjugate pairing, including the original one. Hence if $v \in F_w^{\perp}$, its decomposition has the form

$$v = z + t, \qquad z = \Pi_1 v \in Z_1, \qquad t \in \ker e_0 + \operatorname{im} A_w.$$

The same two vectors are orthogonal for every conjugate pairing, so

$$\|v\|_\sigma^2 = \|\Pi_1 v\|_\sigma^2 + \|t\|_\sigma^2.$$

Equation (94) with $\vartheta = 1$ gives $U_1 z \in F_w^{\perp}$ for $z \in Z_1$.

42.1.3 The norm at the trivial twist

In this subsection, we combine the projection bound of Section 42.1.2 with the original gluing sum.

Write $A_{\mathrm{glue}} = fmbB^2$, N_{glue} for the number of gluing choices times the number of second-factor classes, and l_1 for the denominator exponent of the splitting. The original gluing sum has the factor M^{l_1}. At the trivial twist, the first-block level J_Λ, the two principal levels in the choice of a tuple coordinate, and the target level all contain $K(\pi_M^2)$ in their respective ranks,

by Section 33.2. Since $\pi_M \bar{\pi}_M = M$, they contain $K(M^2)$. For any of these levels J of rank $n \le r + t$, the lattice count in the proof of Lemma 7.2 gives

$$h(J) \le h(K_{\mathrm{int}})[K_{\mathrm{int}} : K(M^2)] \le h(K_{\mathrm{int}})(M^2)^{2fn^2},$$
$$\log h(J) \ll (2fn)^3 \log x + 4fn^2 \log M < b.$$

Thus each of these class sets has at most e^b elements. For a glued point z, let Γ_z be its target stabilizer. On its single class, $\delta_z(v)$ is the sum of $|\Gamma_z|$ unitary transports of v, including their level characters. The squared class norm has weight $1/|\Gamma_z|$. The transported matrices remain unitary at every σ, so

$$\|\delta_z(v)\|_\sigma^2 \le |\Gamma_z|\, \|v\|_\sigma^2 \le n_A \|v\|_\sigma^2.$$

In the exterior tensor model of Section 33.2, the two block inclusions, signed complement maps and determinant twists preserve the Weyl forms. The normalized Cartan projection is the identity on the included tensor product, as verified there using the highest tensor in the prescribed orders. Write T for this Cartan map. Its coordinates in the integral exterior models are defined over $\mathbf{Z}$, so at every embedding

$$\|\sigma T(y \otimes v)\|^2 = \|\sigma y\|^2 \, \|\sigma v\|^2.$$

By (I5), $\|\phi''_{1,i}(h)\|_\sigma^2 \le e^{A_{\mathrm{glue}}}$ on every original first factor. Apply the triangle inequality to the sum over h on each tuple coordinate. For the sum over coordinates the free second-factor classes give $\sum_{i,c} \|\zeta_{i,c}\|_\sigma \le \sqrt{N_{\mathrm{glue}}}\, \|\zeta\|_\sigma$ by Cauchy–Schwarz. Hence

$$\|G_1 \zeta\|_\sigma^2 \le M^{2l_1} e^{2b} N_{\mathrm{glue}} n_A e^{A_{\mathrm{glue}}} \|\zeta\|_\sigma^2.$$

There are at most e^b choices in each of the two principal-level class sets defining N_{glue}. The residue matrix has $3md_+$ entries, since $\dim_E S = 3m$ and $\dim_E V'' = d_+$. Its residue ring has order at most M^{4f}. At each of the f embeddings the index specifies $\Delta' = (\Delta - t - 1)/2$ ordered tuples of $2m$ covectors from the chosen basis of $W^\flat$, whose rank is $6m + 1$. Consequently

$$N_{\mathrm{glue}} \le e^{2b} M^{4fd_+(3m)} (6m + 1)^{2mf\Delta'}.$$

The column estimate for the denominator exponent is

$$l_1 \le 3fm(b + a/x_1 + \Delta + m\Delta) + ft^2\Delta.$$

Here $B = \log b$, $\log M \le 22B$, $a/x_1 \le b$, and $\log n_A \le f(rt)^2 \log M$. Since f, r, t, m, Δ have polynomial bounds in x, there is a fixed exponent C with

$$2l_1 \log M + 2b + \log(N_{\mathrm{glue}} n_A) \le 264 fmbB + 4b + O(x^C B) = o(A_{\mathrm{glue}}).$$

The same bounds give $n_A h_-^2 \le e^{A_{\mathrm{glue}}}$, since the minus diagram has at most $ft^2\Delta$ boxes and its hook factor is bounded by $L_-^{3L_-}$ for $L_- \ge 2$ bounding that count.

For precision about adjunction, use the full spaces of class values with their rational block Gram matrices. The tuple space of G_1 is the full tuple-coordinate space, since its stabilizers are trivial. The image of G_1 lies in the original target, and its ambient adjoint restricts there to the adjoint. With C_S, C_T the rational Gram matrices, the adjoint matrix is $C_S^{-1} c(G_1)^{\mathsf{t}} C_T$. Therefore $\sigma(G_1^*)$ is the ordinary Gram adjoint of $(c\sigma c)(G_1)$ in the ambient coordinates. On

conjugated class values, the adjoint norm bound follows from the uniform bound for G_1 just proved on its full tuple domain. Thus

$$\|U_1 z\|_\sigma \le n_A h_-^2 e^{2A_{\mathrm{glue}}} \|z\|_\sigma \le e^{3A_{\mathrm{glue}}} \|z\|_\sigma.$$

An eigenvector of $\Pi_1 U_1|_{Z_1}$ with eigenvalue λ now gives $|\sigma\lambda| \le e^{3A_{\mathrm{glue}}}$. Multiplying over the eigenvalues with algebraic multiplicity gives $\log|\sigma\Delta_1| \le 3DA_{\mathrm{glue}}$. This uniform bound implies $\Delta_1 \in \overline{\mathbf{Q}}$ by the transcendence-basis argument of Section 23.

One operator on the full projected lift space. Choose a basis $v_1, \dots, v_s$ from the lift generators of Z. For each i, construct the first-order operator D_i by using the type and original system of v_i. Every generator v_j has its own fixed type parameter. Its projection to the type of v_i is either zero, if the types differ, or v_j itself. In the latter case the spherical system remains the same after inclusion in the raised level used for D_i. Thus $D_i v_j = d_{ij} v_j$, with the bound (RB1) on $v_\ell(d_{jj})$. Put

$$\mathcal{T}_\ell = \{y \text{ integral at the chosen embedding} : y \perp F_w \text{ for every } w\}$$

in the original target at this twist. Equivalently, $\mathcal{T}_\ell = L_\kappa \cap (\sum_w F_w)^\perp$, where L_κ is the transported integral lattice and the sum ranges over the split places outside the original level and its conjugate. Each F_w uses the full projector e_0 and $Z^\perp$. Thus the test space is the same for each individual type and system selected in constructing D_i. Proposition 32.1 gives $D_i \pi_Z y$ integral for each $y \in \mathcal{T}_\ell$.

Write ι for that embedding and, for each j, select $a_j \ne 0$ from the numbers $\iota(d_{ij})$ of maximum absolute value. Reduce the integral ratios $\iota(d_{ij})/a_j$ in the residue field of $\overline{\mathbf{Q}}_\ell$. Each resulting linear form $\sum_i X_i \overline{\iota(d_{ij})/a_j}$ is nonzero. Choose a finite subfield $k' \subset \overline{\mathbf{F}}_\ell$ containing the reduced coefficients with $q = |k'| > s$. The s kernels contain at most $sq^{s-1} < q^s$ points in $(k')^s$. For a tuple in the complement, take $b_i = 0$ at its zero coordinates and the Teichmüller lifts at its other coordinates. Each nonzero b_i satisfies $b_i^{q-1} = 1$, so every b_i is integral and belongs to $\iota(\overline{\mathbf{Q}})$. The operator $D_Z = \sum_i \iota^{-1}(b_i) D_i$ still gives an integral form $D_Z \pi_Z y$ for every $y \in \mathcal{T}_\ell$. On v_j its eigenvalue has absolute value $|a_j|_\ell$, at least $|\iota(d_{jj})|_\ell$. Each original off-level Hecke operator preserves every cofinite spherical isotypic space and commutes with each old type projector, since their local supports are disjoint. The trace coefficients are scalar on those isotypic spaces. Thus the restrictions D_i, and hence D_Z, commute with the off-level Hecke operators on Z, including on joint summands of dimension greater than one. The determinant bound follows by multiplying the eigenvalue bounds. If $Z = 0$, use the identity on the zero space, with determinant one.

42.1.4 Compression on the original local lattice

In this subsection, we apply the integral clearing of Proposition 32.1 to the local determinant.

Fix $\kappa \mid \ell$, normalize $v_\kappa(\ell) = 1$, and take the selected twist η whose auxiliary modulus has norm prime to ℓ. Use the clearing operator $\Lambda = D_Z$ just constructed on $Z = Z_\eta$. The selected-twist bounds in Section 38 give $v_\kappa(\det \Lambda) \le DrC_\ell$. More explicitly, (RB1) and Proposition 32.1 give $v_\kappa(\det D_Z) \le DB_{\mathrm{tr}}$ on this image. The selected twist bounds the Selmer length for every generator in the basis of Z, so $\mathrm{sel}_Z, \mathrm{ram} \le e_\ell$. Using $t' \le t$, $r' = r$, and the definition of C_ℓ in Section 38.4 give $B_{\mathrm{tr}} + v_\ell(C_G) \le rC_\ell$. Clearing the projected vectors uses D_Z itself. The extra factor C_G there clears the denominators of a scalar class pairing. The test for D_Z must

be integral at κ and perpendicular to every F_w. The operator U_η preserves integrality at κ. The vanishing in Section 33.1 gives $U_\eta Z_\eta \subseteq \bigcap_w F_w^{\perp}$. Consequently $\Lambda\Pi_\eta U_\eta|_{Z_\eta}$ preserves integral vectors in Z_η.

For a nonzero eigenvector F of the composite, take one representative x of each original finite class. Write x_ℓ for its components at all the places above ℓ. The Cayley-chart weak approximation of Section 34.1.1 gives a rational unitary element γ such that $\gamma_\ell x_\ell$ is in the standard maximal compact there. Indeed $K_{\mathrm{std},\ell} x_\ell^{-1}$ is a nonempty local open set. For the Weyl representation ρ_{W}, rational equivariance gives

$$\rho_{\mathrm{W}}((\gamma_\ell x_\ell)^{-1}) F(\gamma x) = \rho_{\mathrm{W}}(x_\ell^{-1}) F(x).$$

The matrix on the left preserves the unit Weyl lattice in both directions. Thus the ordinary coordinates at γx test the transported lattice norm at x. Right level changes also preserve it since they act integrally with unit character. The maximum over these finitely many coordinates is positive. Divide F by a coordinate attaining this maximum. The eigenvalue then has nonnegative valuation, because the composite preserves integrality and the normalized vector has a unit coordinate. Taking the product of the eigenvalues gives

$$v_\kappa(\Delta_\eta) \geq -v_\kappa(\det \Lambda) \geq -DrC_\ell, \qquad \log |\Delta_\eta|_\kappa \leq DrC_\ell \log \ell.$$

In particular the finite logarithmic allowance is $\beta_{j,\ell} = C_\ell \log \ell$, or the larger $B_\ell \log \ell$ used for the paired rank-one test. Since $B_\ell = C_\ell + v_\ell(h_1)$ and h_1 is an integer, in either case $\beta_{j,\ell} \leq B_\ell \log \ell = J_\ell$. Thus (RT3) applies also to $\sum_{\ell \in U} \beta_{j,\ell}$ for every finite U. These bounds only use the clearing operator on the same selected twist. The fixed multiplier for the ratio Δ_η/Δ_1 is Δ_1, which does not depend on η and serves as one height multiplier at every coefficient embedding.

42.2 Separation on the initial unitary group

In this subsection, we show that the inclusion at the initial deep level separates the chosen automorphic coefficients. The reproducing coefficient of Section 12.3 gives the local test, with the scale selected in Section 42.3.

Write $V = Y_0$, $Z = V \perp \langle 1 \rangle$, and $\iota(h) = h \oplus 1$ in the original rational basis. In particular, $\iota^{-1}(K_Z(a)) = K_V(a)$ at each principal modulus a. The grid numbers the first r entries of Z exactly as for V. Its last entry has index r and sign $(-1)^{r+1-k} = (-1)^m = +1$. Thus the real group inclusion is the block inclusion of this rational orthogonal sum. Set $a_0 = \pi_M^{b_0}$, $b_0 = \lfloor b \rfloor$, and denote the original principal source level by $J_M = K_V(\pi_M)$. Let W be the space of square-integrable forms at level J_M reproduced by the prescribed real coefficient projector. Define W_Z using the initial true grid projector and level $K_Z(a_0)$. Every element of each space has a bounded continuous representative, by the truncation argument below. We will construct one $\omega \in U(Z)(F_\infty)$ such that

$$y \in W, \quad \int_{[U(V)]} \overline{y(h)}\, \psi(\iota(h)(\omega, 1))\, dh = 0 \quad (\psi \in W_Z) \quad \Longrightarrow \quad y = 0. \tag{DS0}$$

Here F is the maximal real field of E. We use the original quotient Haar measures.

Weights and the archimedean scale. At the infinite place v indexed by τ, put

$$w_v = \kappa_\tau + \Delta/2, \qquad \Delta = 2r^8,$$

with the integer weight κ_τ of Section 2.3. Let l, l_Z denote the numbered doubled grid parameters of V, Z. The actual form types of W, W_Z use $l, -l_Z$, respectively. In ranks $r+z$, $z=0,1$, the numbered doubled entries of the grid are $2\lambda_i + z$. On $0 \le i < k-1$, $\lambda_i = k/2 - 1 - i$; on the tail $i = k-1+j$, $\lambda_i = -w_v - (j+1/2)\Delta$. Both initial blocks have sign $+$, and the tails alternate starting with $+$. The correction from doubled parameter to doubled lowest-weight entry is the signed count of later entries minus earlier entries, signs taken relative to the entry in question.

On the initial block this equals $k-2-2i+z$, since the tail has signed excess z. Thus the lowest-weight entry there is zero. The correction everywhere is bounded in absolute value by $r+1$. Consequently the lowest-weight tails of the unnegated grid parameters l, l_Z are strictly decreasing in $[-3w_v, -w_v]$. Indeed $w_v \ge \Delta^2$, $m\Delta = o(w_v)$, and $\Delta > r+1$. The doubled entries have modulus at most $L_{\mathrm{sep}} = \lceil 20b \rceil$, since $w_v \le 2b$ and $(m+1)\Delta = o(b)$. The opposite-sign gaps for the undoubled parameters are at least Δ before subtracting $500(r+1)^3$. The same gap bounds hold after negating a parameter.

Put $A_{\mathrm{sep}} = r^4$, $D_{\mathrm{sep}} = (2r+1)^2$, $H = 20D_{\mathrm{sep}}$, $\beta = (\sqrt{M})^{\lceil x \rceil}$, $B_d = \beta^{20A_{\mathrm{sep}}(f+1)}$, and let $\kappa = 8fr\,\mathrm{rd}(E) + 48r^6f^2N_D + 1$, where $f = [F:\mathbb{Q}]$ and $N_D = q_1 d$ lies in the different. The parameter bounds $r \asymp x^5$, $m \le 2\sqrt{x}$, $b^{20} \le M \le b^{22}$, and $\log b = x^\epsilon + O(1)$ give

$$\begin{gathered} A_{\mathrm{sep}} > 8(r+1)^2, \quad A_{\mathrm{sep}} \ge 2f(r+1)+2, \quad H < A_{\mathrm{sep}}, \quad \beta \ge \max\{10^{12}, f, 2r+1+L_{\mathrm{sep}}\}, \\ \beta \ge \max\{2000(r+L_{\mathrm{sep}}+1)^6, M\kappa^2\}, \\ (r+1)^4(6w_v)^{2m+2} \le \beta, \qquad 10^{30}(D_{\mathrm{sep}}\log\beta)^3 = O(x^{33+3\epsilon}) \le r^7, \\ |\tau(a_0)|_{\mathbb{C}} = (\sqrt{M})^{b_0} \ge B_d^{3r+2}. \end{gathered} \tag{DS1}$$

Here $|\cdot|_{\mathbb{C}}$ is the Euclidean modulus in the moved lattice. The last equality follows from $\pi_M \bar{\pi}_M = M$. For the first line use the polynomial bounds on $f, q_1 d, \mathrm{rd}(E)$ of the field and $\beta \ge b^{10\lceil x \rceil}$; in particular the admissible bound on f gives $2f(r+1)+2 \le r^4$. For the basis bound, $r+1, 6w_v \le \sqrt{M}$ and $2m+6 \le \lceil x \rceil$. For the logarithmic comparison, $D_{\mathrm{sep}}\log\beta = O(x^{11+\epsilon})$ and $33+3\epsilon < 35$, since $0 < \epsilon < 1/100$. Thus $10^{30}(D_{\mathrm{sep}}\log\beta)^3 = o(r^7)$. Finally $b_0 \ge 20A_{\mathrm{sep}}(f+1)(3r+2)\lceil x \rceil$ since its right side is polynomial in x. The excess gap is at least r^7, and in particular at least $4A_{\mathrm{sep}}$ in the smaller group and $2A_{\mathrm{sep}}$ in the larger group.

The real point in the local period integral. Let μ_i, $0 \le i \le m$, be the decreasing tail of the lowest weight for l_Z, and ν_j, $0 \le j < m$, the increasing tail for $-l$. Then $\mu_i \in [-3w_v, -w_v]$ and $\nu_j \in [w_v, 3w_v]$. Write

$$p(t) = \prod_i (t-\mu_i), \qquad a(t) = \prod_j (t-\nu_j), \qquad [R]_i = \frac{R(\mu_i)}{\sqrt{|p'(\mu_i)a(\mu_i)|}}.$$

In the tail sign models define

$$v_* = [a], \qquad u_j = \sqrt{|p(\nu_j)/a'(\nu_j)|}\,[a(t)/(t-\nu_j)].$$

Since m is even, $p'(\mu_i)a(\mu_i)$ has sign $(-1)^i$. The identity $\sum_i R(\mu_i)/p'(\mu_i) = [t^m]R(t)$ for $\deg R \le m$ gives $(v_*, v_*) = 1$, $(v_*, u_j) = 0$, and $(u_j, u_k) = 0$ if $j \ne k$. The signed square norm of $[a(t)/(t-\nu_j)]$ is $-a'(\nu_j)/p(\nu_j)$: sum the residues of $a(t)/((t-\nu_j)^2 p(t))$.

Its sign is $(-1)^j$. Multiplication by t on the polynomial vectors gives $\mu u_j = \nu_j u_j + \sqrt{|p(\nu_j)/a'(\nu_j)|}\,v_*$. Hence the isometry with columns u_j compresses μ to ν. Extend this isometry by the identity on the common zero-weight positive block X, and append v_* last to obtain

$w \in U(Z_v)$. In these coordinates,

$$|(v_*)_i|^2 = \frac{|a(\mu_i)|}{|p'(\mu_i)|}, \qquad |(u_j)_i|^2 = \frac{|p(\nu_j)a(\mu_i)|}{|a'(\nu_j)p'(\mu_i)|\,|\mu_i - \nu_j|^2}.$$

The distinct integral nodes in each tail give $|a'(\nu_j)|, |p'(\mu_i)| \ge 1$, and the cross distances are in $[2w_v, 6w_v]$. Hence every entry of w has modulus at most $(6w_v)^{m+1}$. For $\omega_v = w^{-1}$, the equality $\|w^{-1}\| = \|w\|$ and (DS1) give $\|\omega_v\|^2 \le \beta$.

Put $g = w(h \oplus 1)w^{-1}$, $h \in U(V_v)$. Then $\operatorname{tr}(g-1) = \operatorname{tr}(h-1)$, and the compression gives $\operatorname{tr}(\mu(g-1)) = \operatorname{tr}(\nu(h-1))$, where both weight matrices are extended by zero on X. The two embeddings contain the diagonal $U(X)$. Let $\theta = c_{l_Z}(g)\overline{c_{-l}(h)}$. To check the conjugations, negate a numbered parameter p while keeping the signed coordinate basis. This reverses its parameter order, so ρ, ρ_K, the minor exponent $a = \lambda + \rho$, and the lowest compact weight $\mu = a - 2\rho_K$ change sign. For $t = k_0 Z$ in polar coordinates the compact numerator in the lowest-type integral of Section 12.7 is $m_\mu(k_0 k)\overline{m_\mu(k)}$. Complementary minors give $m_{-\mu}(k) = \overline{m_\mu(k)}$ on the compact group, so its squared-norm normalization is the same for both parameters. For the positive matrix $D = k^* Z k$,

$$m_{-a}(D) = m_a(D^{-1}) = m_a(JDJ) = m_a(D) > 0.$$

The real denominator is consequently identical. Thus $c_{-p}(t) = \overline{c_p(t)}$, and the integrand in these coordinates is

$$\theta = c_l(h)\overline{c_{-l_Z}(\omega_v^{-1}\iota(h)\omega_v)}.$$

We apply the one-factor coefficient computations of Sections 12.3 and 12.9. Here are the two-factor localization details. Let E_g, E_h be the opposite-sign blocks, $v_0^2 = \|E_g\|_{\mathrm{HS}}^2 + \|E_h\|_{\mathrm{HS}}^2$, $T = D_{\mathrm{sep}} \log \beta$, and $\eta = 10^{-13}/T$. The signed row identity, weighted by $-\mu_i$ and ν_i, bounds the sum of squared row norms of $g - 1$ off X by $6v_0^2$. Explicitly,

$$2\Re(g-1)_{ii} = 2\sum_j |(E_g)_{ij}|^2 - \sum_j |(g-1)_{ij}|^2$$

and likewise for h; add the weighted identities, use the equality of weighted traces, discard the nonnegative terms from the second squared norm, and divide by w_v.

Apply the same argument to the inverse pair for the columns. If $v_0 \le \eta$, the polar decomposition $g_{XX} = up$ gives $\|p - 1\|_{\mathrm{HS}} \le 6\eta^2 \le \eta$, by the signed Gram identity on the X-columns. Left multiply by the diagonal u^*. Both highest vectors are fixed by $U(X)$: the weight-minor thresholds on this zero-weight block cancel its determinant contribution. Thus θ is unchanged. The moved first squared distance to 1 is at most $13\eta^2$.

The unweighted row identity and trace equality give the second squared distance at most $15\eta^2$. Both distances are at most 4η. The diagonal u^* commutes with w, since w fixes X and preserves its orthogonal complement. Thus after moving both arguments we still have $g = w(h \oplus 1)w^{-1}$. The upper-left block of $w^{-1}\mu w$ on V is ν. For every $j \ge 1$,

$$\operatorname{tr}\big(\mu(g - I)^j\big) = \operatorname{tr}\big((w^{-1}\mu w)((h - I)^j \oplus 0)\big) = \operatorname{tr}\big(\nu(h - I)^j\big).$$

The logarithm series is absolutely convergent at both moved arguments. These weighted traces of the logarithms are purely imaginary, since the diagonal entries of the signed unitary Lie algebra are imaginary. The quadratic approximation in Section 12.9 has error at most $a/10 + e^{-100T}$ from a number $a \in [0, 1]$ after removing its imaginary log phase in each factor.

The conjugation on the second coefficient cancels the two phases. Writing $s = e^{-100T}$, the product of the two estimates, with real main terms $a, b \in [0,1]$, gives

$$|\theta - ab| \le 0.21ab + 1.1s(a+b) + s^2.$$

Thus $|\theta| - 2\Re\theta \le -ab + 3|\theta - ab| \le 12s$. If $v_0 \ge \eta$, the two one-factor decay estimates give

$$|\theta|^4 \le [(1 + \|E_g\|_{\mathrm{HS}}^2)(1 + \|E_h\|_{\mathrm{HS}}^2)]^{-r^7} \le (1+\eta^2)^{-r^7} \le e^{-400T}.$$

For this region $|\theta| - 2\Re\theta \le 3|\theta|$ gives the same cone bound. With $J(h) = g \oplus h$ and $Q = (2/\|J(h)\|)^{9D_{\mathrm{sep}}}$, one-factor norm decay at the larger of the two norms gives $|\theta| \le Q^2$. Taking the geometric mean of the two bounds on the positive part, and applying the one-factor small-ball estimate at both matrices, gives

$$|\theta| \le 2\Re\theta + 6\beta^{-50D_{\mathrm{sep}}}Q, \qquad |\theta| \ge \tfrac{1}{4} \quad (\|J(h) - 1\| \le (200\beta^5)^{-1}).$$

The volume of $\|J(h)\| \le R$ is at most $(3R^2/\delta)^{2D_{\mathrm{sep}}}$ times that of $\|J(h) - 1\| \le \delta$ for $R \ge 1$, $0 < \delta \le 1$.

Indeed pack δ/R-separated matrices, and use $\|J(h)^{-1}\| = \|J(h)\|$ to convert each covering ball to a translate of the δ-ball. Let V_b, V_s be the volumes for $R = \beta + 1$, $\delta = (200\beta^5)^{-1}$. Then $V_b \le \beta^{16D_{\mathrm{sep}}} V_s$. The same packing with $\delta = 1$, integrated in R against $9D_{\mathrm{sep}}\, 2^{9D_{\mathrm{sep}}} R^{-9D_{\mathrm{sep}}-1}\, dR$, gives $\int Q \le 16^{9D_{\mathrm{sep}}} V_b$. Thus

$$V_s/4 \le 2\left|\int \theta\right| + 6\beta^{-50D_{\mathrm{sep}}} 16^{9D_{\mathrm{sep}}} V_b, \qquad V_b \le 16\beta^{16D_{\mathrm{sep}}}\left|\int \theta\right|.$$

For the second inequality, multiply the first by $4\beta^{16D_{\mathrm{sep}}}$ and absorb the resulting error into $V_b/2$.

The h-ball of radius 1 about 1 lies in the large ball since $\|w\|^2 \le \beta$. Under its original Haar, write its volume as $v_{1,v}$. The resulting point in the original first-path embedding satisfies

$$\left|\int_{U(V_v)} c_l(h)\overline{c_{-l_Z}(\omega_v^{-1}\iota(h)\omega_v)}\, dh\right| \ge v_{1,v}\beta^{-H}. \tag{DS2}$$

The product $\omega = (\omega_v)_v$ depends only on the arithmetic datum and the real types. It is fixed before choosing y or a test in W_Z.

42.3 Quantitative reduction near the cusp

In this subsection, we quantify the size of a cusp neighborhood in terms of the first minimum of the moved integral lattice. This permits the localization argument of Section 42.2 with a uniform cutoff.

Let W be V moved to the point under consideration. Write $n = \dim_E W$, $f = [E^+ : \mathbb{Q}]$, D_E for the absolute discriminant, and $\mathrm{rd}_E = |D_E|^{1/(2f)}$. Use the rational lattice Λ of vectors with integral finite coordinates in the moved adelic basis, and Euclidean lengths in this basis at infinity. Let $N_D > 0$ be the chosen integer in the different, and set

$$\kappa = 8fn\,\mathrm{rd}_E + 48n^6 f^2 N_D + 1.$$

For $0 < \delta \le 1$ and a positive integer M, suppose the first minimum cut off at one satisfies $\lambda(W) < [\delta/(M\kappa^2)]^n$. We construct $0 \ne A \subset W$ rational isotropic with

$$N_A(\mathbb{A}_{E^+}) \subset N_A(E^+) B_\infty(\delta) J_W(M) \tag{CF1}$$

in this basis. Here $B_\infty(\delta)$ is the product of the operator-norm balls about the identity, and $J_W(M)$ is the original finite standard principal level M in these coordinates.

First let $y_1, \dots, y_n$ be successively shortest lattice vectors independent over E, and let their lengths be $\alpha_1 \le \cdots \le \alpha_n$. These minima exist since the ordinary real lattice is discrete and spans its ambient space. For the ordinary real successive minima μ_j, one has $\alpha_i \le \mu_{2f(i-1)+1}$, by the real dimension $2f(i-1)$ of the archimedean extension of the span of the first $i-1$ vectors. Hence

$$\prod_i \alpha_i^{2f} \le \prod_{j=1}^{2fn} \mu_j.$$

The Euclidean covolume is $2^{-fn}|D_E|^{n/2}$.

In fact the covolume in the archimedean basis multiplies the standard covolume by the inverse of the adelic absolute value of the basis determinant, and the latter is one by the Hermitian basis identity and the product formula for $\det J$. Compute the covolume from an integral basis of O_E, the local indices relative to O_E^n, and the real Jacobian of the inverse infinity basis. Minkowski's second theorem for the Euclidean ball and the contained cube of side $2/\sqrt{2fn}$ give

$$\prod_i \alpha_i^{2f} \le (2fn \operatorname{rd}_E)^{fn}.$$

In particular $\prod_i \alpha_i \le \kappa^n$.

Put $\theta = \delta/(M\kappa^2)$, and choose $0 \ne z \in \Lambda$ of length less than θ^n. The Hermitian pairing of two vectors of Λ is integral in E by the finite basis identity. If their two lengths have product less than one, the pairing has absolute value less than one at every embedding, so is zero. Nondegeneracy of the pairing against the basis y_i therefore gives $1 \le \|z\| \alpha_n$.

If $n = 1$, then $\|z\|\alpha_n < \theta\kappa \le 1$, which is impossible. Thus $n \ge 2$. It follows that

$$\prod_{i<n} \alpha_i \le \|z\| \prod_i \alpha_i < (\theta\kappa)^n.$$

Pair the $n-1$ lengths by $i \leftrightarrow n-i$. At least one pair satisfies $\alpha_i \alpha_{n-i} < \theta\kappa$, since otherwise the square of the last product would be at least $(\theta\kappa)^{n-1}$. Choose its index $i \le n-i$. Then the first i vectors span an isotropic subspace A, orthogonal to the first $n-i$ vectors. The latter span $A^\perp$ by dimension. Write $a = \alpha_i$, $b' = \alpha_{n-i}$, so $a \le b'$ and $ab' \le \delta/(M\kappa)$. The bases consist of rational vectors in the lattice at the point.

For the radical reduction let X, Y be the column matrices of these rational bases of $A, A^\perp$, and let x, y be their adelic coordinates. Write J for the standard adelic Hermitian form matrix in the transported basis. For any matrix z_0 with columns in $A^\perp$, in this basis define

$$\mathcal{R}(z_0) = 1 + 2\big(z_0 x^* - x z_0^* - x z_0^* J z_0 x^*\big) J.$$

It is in the radical and satisfies

$$\mathcal{R}(w_0)\mathcal{R}(z_0) = \mathcal{R}(w_0 + z_0 - x w_0^* J z_0). \tag{CF2}$$

The product follows on expanding and using $x^*Jx = x^*Jz_0 = x^*Jw_0 = 0$.

For surjectivity, use a rational Witt decomposition $A \oplus W_1 \oplus A'$ with X the first basis and the dual basis in A'. Adjoint in the original form becomes adjoint in the Witt Gram matrix in this calculation. The radical matrices in Witt coordinates have middle-right block C, top-middle block $-C^*J_1$, top-right block D with $D + D^* + C^*J_1C = 0$, and identity diagonal. For $z_0 = (Z_1, Z_2, 0)$ the corresponding blocks of $\mathcal{R}(z_0)$ have $C = 2Z_2$ and $D = 2(Z_1 - Z_1^*) - \frac{1}{2}C^*J_1C$.

Take $Z_2 = C/2$ and $Z_1 = (D + \frac{1}{2}C^*J_1C)/4$. Thus every $u \in N_A$ is $\mathcal{R}(z_0)$, and since the columns of Y span $A^\perp$ one may write $z_0 = Myc$ for an adelic matrix c. These identities are in the original basis after transporting back from the Witt coordinates.

There is a rounding function r on adelic scalars such that $c - r(c)$ is finite integral and has archimedean absolute values at most $4f^2N_D$. To see this, approximate the finite adele by $t \in E$ modulo $\widehat{O}_E$ by the Chinese remainder theorem. At infinity use the lattice $\Lambda_0 = (2/N_D)O_E$ with the real Euclidean inner product. Its dual is $N_D\mathfrak{D}_E^{-1} \subset O_E$: the Euclidean bilinear form is half the trace pairing with conjugation.

Its nonzero vectors have length at least one by the product formula. For any Euclidean lattice in dimension d, dual minimum at least one implies covering radius at most d^2. Indeed Minkowski's convex body theorem applied to the lattice and its dual, using reciprocal covolumes and an inscribed cube, gives a nonzero lattice vector of length at most $2d$. Use a shortest vector a, which is primitive.

Orthogonal projection to $a^\perp$ is a lattice since a extends to a lattice basis, and its dual is the intersection of the original dual with $a^\perp$. By induction the projected remainder has length at most $(d-1)^2$. Lift the approximant and round its coefficient along a, adding at most $\|a\|/2 \le d$ to the error. This proves the bound. Apply it to the archimedean residue divided by N_D and multiply back by N_D. The approximant then lies in $2O_E$, so it preserves finite integrality. The archimedean remainder in each coordinate has modulus at most $4f^2N_D$. Apply r entrywise, so the matrix remainder has operator norm at most $4n^2f^2N_D$ at each real factor.

Choose $d = -M(yr(c))^*Jyc$ in adelic coordinates. The entries of $r(c)$ and $r(d)$ in the expression $w_0 = M(xr(d) - yr(c))$ are embedded rational coefficients; thus $\mathcal{R}(w_0)$ corresponds to $\gamma \in N_A(E^+)$. Here d may be nonrational. The rational coefficient is $r(d)$. In fact $x^*Jy = 0$ gives

$$w_0^*Jz_0 = -M^2r(c)^*y^*Jyc = Md, \qquad w_0 + z_0 - xw_0^*Jz_0 = M\{y(c - r(c)) - x(d - r(d))\}.$$

Thus the central cross term in (CF2) contributes only its rounded remainder. Substitution yields

$$\gamma u = 1 + 2(MT - (MT)^* - (MT)^*J(MT))J, \qquad T = (y(c - r(c)) - x(d - r(d)))x^*.$$

The finite entries of T are integral. The operator norms of x, y at infinity are at most n^2a, n^2b', so

$$\|T_v\| \le 4n^6f^2N_D(b' + a)a \le 8n^6f^2N_Dab' \le \delta/(6M).$$

Since every J_v is a sign involution, $\|(\gamma u)_v - I\| \le 2\delta/3 + \delta^2/18 \le \delta$. The finite formula gives $\gamma u \equiv I \bmod M$. This proves (CF1) in the original basis at level M.

42.4 Vanishing of the constant term under the reproducing kernel

In this subsection, the reproducing kernel of Section 12.3 annihilates the relevant constant term. We prove this pointwise and apply it on the neighborhood just constructed in Section 42.3.

Now let e be the small-type reproducing convolution kernel on the real group G_∞ of V. It is integrable, idempotent, and has an integrable first norm moment, by the prescribed large gap. Write $Tf(g) = \int e(u) f(gu)\, du$. We prove the local vanishing for bounded continuous functions invariant under the product of the radicals attached to nonzero isotropic subspaces A_v of the real factors. This applies also to their real conjugates. The proof uses the idempotence and the indicated moment.

For the product of real groups, let $\mathcal{D}$ consist of pairs (a, c) with $c > 0$ such that each a_v multiplies A_v by c and acts as the identity on $A_v^{\perp}/A_v$. Its kernel over $c = 1$ is $N_{A,\infty}$. Each real factor of A is nonzero. Fix one place v_0 and a vector $0 \neq x_{v_0} \in A_{v_0}$. Put $H(g) = \log \|g_{v_0}^{-1} x_{v_0}\|$. Then $H(ag) = H(g) - \log c$ for $(a, c) \in \mathcal{D}$ and

$$|H(gu) - H(g)| \leq \|u_{v_0}\|.$$

The latter follows by submultiplicativity in both directions and $\|u_{v_0}^{-1}\| = \|u_{v_0}\|$.

For each g and each $\epsilon > 0$, take at each place the Euclidean orthogonal projection P onto $g_v^{-1} A_v$, and put $Q = J_v P J_v$. The two projections have orthogonal images by isotropy. The operator $b_v = I + \epsilon P - \epsilon Q/(1+\epsilon)$ is unitary for J_v, acts by $c = 1+\epsilon$ on $g_v^{-1} A_v$, and acts trivially on its orthogonal quotient for J_v. Its distance from I is at most ϵ; hence $(gbg^{-1}, c) \in \mathcal{D}$. Translation is continuous on $L^1(G_\infty)$. Choose ϵ so small that $\|e(b^{-1}\cdot) - e\|_1 < 1/2$ throughout this neighborhood of I.

Let $\mathcal{B}$ consist of the bounded continuous functions fixed by T and left invariant under $N_{A,\infty}$, with the supremum norm. For a fixed $(a, c) \in \mathcal{D}$ with $c = 1 + \epsilon$, left translation $Uf(g) = f(ag)$ preserves $\mathcal{B}$. Given g, the element b just constructed satisfies $gbg^{-1}a^{-1} \in N_{A,\infty}$, hence $f(ag) = f(gb)$. Reproduction and the L^1 translation bound give $\|I - U\| \leq 1/2$ on $\mathcal{B}$. Put $\tau = \log c$, take a large positive integer k, and set $\chi(g) = \exp(-i\pi H(g)/(k\tau))$ and $Sf = T(\chi f)$.

The multiplier is invariant under $N_{A,\infty}$, T commutes with left translations and $T^2 = T$, so S preserves $\mathcal{B}$. The shift in H gives $US = e^{i\pi/k} SU$. It follows that $US^k = -S^k U$; the operator identity $2S^k = (I - U)S^k + S^k(I - U)$ and $\|I - U\| \leq 1/2$ then give $S^k = 0$. On the other hand $Sf = 0$ implies

$$\|f\|_\infty \leq \frac{\pi}{k\tau} \left(\int |e(u)|\, \|u_{v_0}\|\, du \right) \|f\|_\infty,$$

by subtracting $\chi(g) Tf(g)$ in the integral. The moment is finite and the coefficient is less than one for large k. Thus S is injective, and $S^k = 0$ proves $\mathcal{B} = \{0\}$. In particular T kills every bounded continuous left-$N_{A,\infty}$-invariant function.

In original coordinates let N be the radical of the rational isotropic subspace just found at g. In the transported coordinates the two groups are $N^g = g^{-1} N(\mathbb{A}_{E^+}) g$ and $\Gamma_N^g = g^{-1} N(E^+) g$. For $h \in G_\infty$ take the mean of $F(gu(h, 1))$ over $\Gamma_N^g \backslash N^g$. It is well defined because for $\gamma \in N(E^+)$ the argument at $g^{-1}\gamma g u$ is $\gamma g u(h, 1)$. The quotient is compact by (CF1): its representatives lie in the compact intersection of the radical with $B_\infty(\delta) J_W(M)$. Write this probability mean as $F_g(h)$. For $n \in g_\infty^{-1} N(E_\infty^+) g_\infty$, the change $u \mapsto u(n, 1)$ on $\Gamma_N^g \backslash N^g$ gives $F_g(nh) = F_g(h)$. It is bounded and continuous. Since $\|F\|_\infty \|e\|_1 < \infty$, Fubini and reproduction of F give $TF_g = F_g$. Apply the real argument to the isotropic spaces $g_v^{-1} A_v$ in the moved real bases to get $F_g = 0$. Since F is automorphic and right invariant under $J_W(M)$, (CF1) makes every radical value differ from $F(g)$ by no more than the oscillation on the real δ-ball. Averaging against the probability normalization of the same quotient Haar measure gives

$$|F(g)| \leq \sup_{\|h - 1\| \leq \delta} |F(g(h, 1)) - F(g)|.$$

This is the required bound at the cusp.

42.5 Removing the smoothing at the period

In this subsection, we combine the constant term vanishing of Section 42.4 with the quantitative reduction of Section 42.3.

Throughout this subsection write $A = A_{\mathrm{sep}} = r^4$ and $L = L_{\mathrm{sep}} = \lceil 20b \rceil$ as in (DS1). Let G_V, G_Z and Γ_V, Γ_Z be the adelic and rational unitary groups in the given bases. Write $\jmath = \iota$, $c_v = c_l$ and $c'_v = c_{-l_Z}$ at place v. Put $e(u) = \prod_v d_v \overline{c_v(u_v)}$, with d_v the formal degree. Assume the orthogonality in (DS0). For an adelic point a of G_V and an infinite point α, put

$$\Phi_{\alpha,\omega}(g) = \prod_v \overline{c'_v(\alpha_v^{-1} g_v \omega_v)},$$

$$\psi_\alpha = \Phi_{\alpha,\omega} \circ \jmath, \qquad (R_\alpha y)(a) = \int \psi_\alpha(k) y(a(k,1))\, dk.$$

For clarity, fix the center $q = \iota(a)(\alpha, 1)$ in G_Z. The periodization of the conjugate coefficient kernel times $\mathbf{1}_{K_Z(a_0)}$, at arguments $q^{-1}\gamma g$, is the conjugate of a bounded form of the required type and level, divided by the fixed product of formal degrees for normalized coefficients.

To verify this, restrict the supported terms $a_\gamma = q^{-1}\gamma g$ to $\|a_{\gamma,v}\| \le R_v$ at every real place, $R_v \ge 1$. For a fixed one of these terms a_{γ_*}, the matrices $a_\gamma a_{\gamma_*}^{-1} = q^{-1}\gamma\gamma_*^{-1}q$ have integral finite coordinates in the center basis. They are distinct points of the fixed moved rational matrix lattice and have real norms at most R_v^2. Packing its disjoint balls of fixed radius less than half its first minimum gives at most $C_q \prod_v R_v^{4(r+1)^2}$ such points. Indeed the balls fit in the product of local balls whose radii are R_v^2 plus the fixed packing radius.

Dyadic summation converges uniformly in g, since the coefficient decay exponent $2A$ exceeds $4(r+1)^2$. It converges locally uniformly term by term by increasing the real radii by a fixed factor on a compact neighborhood and restricting the finite neighborhood to $gK_Z(a_0)$.

Reindexing gives automorphy and right level invariance. For the type, conjugate and replace γ by γ^{-1}. The conjugate uses arguments $g^{-1}\gamma q$. The idempotent kernel $e_Z = \prod_v d'_v \overline{c'_v}$ gives

$$\int e_Z(u) e_Z(u^{-1} g^{-1}\gamma q)\, du = e_Z(g^{-1}\gamma q)$$

at the real places. Boundedness of the absolute sum and integrability of e_Z justify convolution term by term. Thus the conjugate expression is fixed by right reproduction. Inserting $g = \iota(h)(\omega, 1)$ in this test in W_Z and conjugating the vanishing hypothesis of (DS0) gives zero for the period against $y(h)$. Its kernel is exactly $\Phi_{\alpha,\omega}((\iota(a)^{-1}\gamma\iota(h))_\infty)$ times the finite level indicator.

For a fixed vector $y = R(e)y$ in L^2, let $y_j = y\mathbf{1}_{\{|y| \le j\}}$. Right convolution satisfies

$$\|R(e)y_j\|_\infty \le j\|e\|_1, \qquad \|R(e)y_j - y\|_2 \le \|e\|_1 \|y_j - y\|_2 \longrightarrow 0.$$

Each projected truncation is continuous by continuity of translation in L^1, retains the finite level, and is fixed by $R(e)$. Section 18.1.3 gives cuspidality. At the fixed level, compact type and infinitesimal character, the finiteness argument of Section 34.3.2 places all the projected truncations in one finite-dimensional cusp space. It is closed in L^2, so it contains y. For this bounded continuous representative, $R(e)y = y$ everywhere. Both sides are continuous and Haar

measure has full support on the quotient. This applies to the initial true grid projector at its level as well.

Take a global bound $|y| \le S$ for this representative and set $\lambda(a)$ equal to the first minimum, cut off at 1, in the lattice of V moved to a. If $B_d^2 \le B_d^{3r+2}\lambda(a)$, unfold the period by the cosets Γ_Z/Γ_V of the embedded rational subgroup (equivalently by the orbits of right multiplication). Put $e = e_{\mathrm{last}}$, $K = K_Z(a_0)$, and $b_\zeta = \jmath(a_f)^{-1}\zeta_f$. The finite support condition for the coset $\zeta\Gamma_V$ is

$$\exists h_f \in G_{V,f} :\ b_\zeta \jmath(h_f) \in K \quad \Longleftrightarrow \quad b_\zeta e \in Ke.$$

Indeed $\jmath(h_f)e = e$. Conversely, if $b_\zeta e = ke$ with $k \in K$, then $k^{-1}b_\zeta$ fixes e and preserves $e^\perp$, so it lies in $\jmath(G_{V,f})$. Its inverse gives a suitable h_f. This orbit condition implies the column congruence used below at the chosen split places. The coset itself is determined by the rational column ζe.

Equal columns mean $\zeta^{-1}\zeta'$ fixes the last unit vector and belongs to Γ_V. Write $c(\zeta)$ for this column in the Euclidean real coordinates moved to $\jmath(a)$. The finite coordinates of every supported column are e_{last} modulo a_0. Hence the difference of two supported columns, divided by a_0, is in the moved rational lattice of Z. More explicitly, if $L_{V,f}$ is the original finite lattice (including its chosen basis at the places over N), then the rational vectors with finite coordinates in $a_f L_{V,f}$ form $\Lambda_V(a)$. At $\iota(a)$ this lattice for Z is $\Lambda_V(a) \oplus O_E e_{\mathrm{last}}$. At infinity the sum is orthogonal in the Euclidean coordinates moved to $\iota(a)$. Each nonzero integer of E has modulus at least one at some embedding, so the minimum of this sum is at least $\lambda(a)$. Multiplication by a_0 scales each archimedean length by $M^{b_0/2} \ge B_d^{3r+2}$.

Consequently

$$\|c(\zeta) - c(\zeta')\| \ge B_d^{3r+2}\lambda(a) \ge B_d^2 \quad (\zeta\Gamma_V \ne \zeta'\Gamma_V), \qquad \|c(1)\| = \sqrt{f} \le B_d. \tag{D3}$$

The last column does not vary under multiplication on the right by $\jmath(h)$. At any archimedean point of a contributing term, write g_v for the local group element before the two translations. Its last column is $c(\zeta)_v$. Since $\|g_v\| \ge 1$ and $\|c(\zeta)\|^2 = \sum_v \|c(\zeta)_v\|^2$, one has

$$\prod_v \|g_v\| \ge \max_v \|g_v\| \ge \|c(\zeta)\|/\sqrt{f} \ge \|c(\zeta)\|/B_d.$$

For clarity, the L^1 slice estimate follows from decay and a volume calculation. For $R \ge 1$ and a fixed ambient group element q, take a maximal $1/R$-separated set in the subset of matrices $\jmath(h)q$ of operator norm at most R. Its size is at most $(1 + 2R^2)^{2(r+1)^2}$ by packing in the real vector space of complex matrices, using the operator norm. Two such matrices a, b at distance at most $1/R$ satisfy $\|ab^{-1} - 1\| \le 1$ because $\|b^{-1}\| = \|b\| \le R$.

Thus the measure of the original subset of h's is at most this same number times $v_{1,v}$, by right translation in the subgroup. Sum over $2^j \le \|\jmath(h)q\| < 2^{j+1}$, $j \ge 0$. With $d_{\mathrm{pack}} = 2(r+1)^2$ the sum for the weight $(2/\|\jmath(h)q\|)^A$ is at most

$$v_{1,v}\, 2^A 9^{d_{\mathrm{pack}}} \sum_{j \ge 0} 2^{-j(A - 2d_{\mathrm{pack}})} \le v_{1,v}\, 16^A,$$

using $A > 4d_{\mathrm{pack}}$. The corresponding bound on the left-translated slice follows by inversion of the unitary group and invariance of its Haar measure.

Write

$$C_{\alpha,\omega} = \prod_v [2(\|\alpha_v\|\|\omega_v\|)^2]^A 16^A v_{1,v}.$$

Here and in the following estimates the unit-ball volume in each real-place factor is its own $v_{1,v}$. The local large-type bound $|c_v'(g)| \le (2/\|g\|)^{2A}$ and submultiplicativity give one power of degree A in the slice for integration as above, and one power for $\|c(\zeta)\|/B_d$. For the latter, use the product of the operator norms before applying the translations α^{-1} and ω.

For a supported coset, the set of finite h in its term is $h_0 K_V(a_0)$. Indeed, after choosing one h_0 in the support, $b_\zeta \jmath(h_f) \in K$ is equivalent to $\jmath(h_0^{-1} h_f) \in K$, and $\jmath^{-1}(K) = K_V(a_0)$. Replacing ζ by $\zeta\jmath(\eta)$, $\eta \in \Gamma_V$, replaces this coset by $\eta_f^{-1} h_0 K_V(a_0)$, of the same measure. Thus its total absolute integral is at most

$$\operatorname{vol}(K_V(a_0))\, C_{\alpha,\omega} S (B_d/\|c(\zeta)\|)^A \qquad (\zeta\Gamma_V \ne \Gamma_V).$$

This retains the volume in the unfolded main term.

For the supported cosets in (D3), all nonbase norms and pairwise distances are at least $R_0 = B_d^2/2$. In dimension $2f(r+1)$ the count in $2^j R_0 \le \|c(\zeta)\| < 2^{j+1} R_0$ is at most $(5{\cdot}2^j)^{2f(r+1)}$, by packing balls of radius $R_0/2$. With $d_c = 2f(r+1)$ and $A - d_c \ge 2$, this gives

$$\sum_{\substack{\zeta\Gamma_V \ne \Gamma_V \\ b_\zeta e \in Ke}} (B_d/\|c(\zeta)\|)^A \le (2/B_d)^A \, \frac{5^{d_c}}{1 - 2^{d_c - A}} \le (10/B_d)^A \le 10/B_d.$$

For the absolute unfolding, let $T_\zeta(h)$ be the term at ζ, including $y(h)$ and the finite indicator. Then $T_{\zeta\jmath(\eta)}(h) = T_\zeta(\eta h)$ for $\eta \in \Gamma_V$ by automorphy of y. With one representative per supported orbit, Tonelli and counting measure give

$$\int_{[U(V)]} \sum_{\gamma \in \Gamma_Z} |T_\gamma(h)|\, dh = \sum_{\zeta\Gamma_V \text{ supported}} \int_{G_V} |T_\zeta(h)|\, dh.$$

The nonbase sum is finite by the shell estimate and the base term by the same real slice estimate with the finite compact indicator. This proves the signed unfolding with multiplicity one. For the base coset use $h = ak$; the finite indicator is then that of $K_V(a_0)$ and y is invariant under this group since π_M divides a_0. The base coset contributes precisely $\operatorname{vol}(K_V(a_0))\, R_\alpha y(a)$. For each supported nonbase coset choose a finite solution $h_{\zeta,f}$ to its support condition. After dividing by the same finite volume its exact contribution is

$$I_{\zeta,\alpha}(a) = \int_{G_{V,\infty}} y(h, h_{\zeta,f})\, \Phi_{\alpha,\omega}\big(\jmath(a_\infty)^{-1} \zeta_\infty \jmath(h)\big)\, dh.$$

Indeed every finite argument of this coset is in $h_{\zeta,f} K_V(a_0)$, and $K_V(a_0)$ fixes y. Replacing ζ by $\zeta\jmath(\eta)$ changes $h_{\zeta,f}$ to $\eta_f^{-1} h_{\zeta,f}$ up to this level. The substitution $h \mapsto \eta_\infty^{-1} h$ then gives the same contribution by rational left invariance. Hence the vanishing period is exactly

$$R_\alpha y(a) + \sum_{\zeta\Gamma_V \ne \Gamma_V \text{ supported}} I_{\zeta,\alpha}(a) = 0.$$

The preceding absolute bounds give

$$|R_\alpha y(a)| \le C_{\alpha,\omega} S (10/B_d) \qquad \text{if } B_d^2 \le B_d^{3r+2} \lambda(a). \tag{D4}$$

Removing the smoothing.

Put $\mathfrak{M}(u)=\prod_v d_v(2/\|u_v\|)^{2A}$. The integral of each real factor is at most $8[2000(r+L+1)^6]^{2r^2}2^{2A}$. The Casimir calculation in Section 12.3 gives a translation Lipschitz bound $100(r+L+1)^5$ for each small-type coefficient. In detail, for a unit lowest-type vector and any real Lie algebra element Z it gives $\|d\pi(Z)v\|\le\sqrt{3r}(r+L+1)\|Z\|_{\mathrm{HS}}$. For $\|u-1\|\le 1/2$ apply this to $Z=\log u$ and integrate along $\exp(tZ)$, using $\|Z\|_{\mathrm{HS}}\le 2\sqrt{r}\|u-1\|$.

For larger distances the coefficient difference is at most 2. The bound is uniform in the other argument by unitarity. Now if $K_0=2000(r+L+1)^6$, the small-type coefficient has absolute value at least $1/2$ in the ball $\|u-I\|\le 4/K_0$ by its Lipschitz bound $100(r+L+1)^5$. The volume of that ball is at most $4/d_v$ by the formal degree identity. Packing the ambient matrices as above with gap $(4/K_0)/R$ gives $\mu\{\|u\|\le R\}\le (K_0R^2)^{2r^2}(4/d_v)$ for $R\ge 1$, since $1+K_0R^2/2\le K_0R^2$. Layer integration gives the claimed bound: $2A/(2A-4r^2)\le 2$. In particular (DS1) gives

$$\int \mathfrak{M}(u)\,du\le \frac{B_d}{16\beta\,[\beta^H(2\beta)^{3A}]^f}. \tag{D5}$$

For this use $1+2r^2+2A+H+6A\le 20A$, $16\le\beta$, and the product measure of the real factors.

Suppose $\lambda(a)\ge (B_d^{-2}/(M\kappa^2))^r$. Since $M\kappa^2\le\beta\le B_d$, the hypothesis in (D4) follows from

$$\lambda(a)\ge B_d^{-3r},\qquad B_d^{3r+2}\lambda(a)\ge B_d^2.$$

For a real u, change variables on the subgroup to obtain

$$(R_\omega y)(a(u,1))=(R_{\jmath(u)\omega}y)(a).$$

Average this equality against $e(u)\,du$. Put

$$T_e y(a)=\int e(u)y(a(u,1))\,du,\qquad R_k y(a)=y(a(k,1)).$$

For a bounded form the two convolutions in $T_eR_kT_e$ have absolute integral at most $\|y\|_\infty\|e\|_1^2$. Their kernel at z is

$$\int e(u)e(k^{-1}u^{-1}z)\,du=e(z)\prod_v c_v(k_v)=e(z)\overline{e(k)}/e(1)$$

by the coefficient identity of Section 12.3 with the formal degrees for these same local Haar measures. Since $T_ey=y$, Fubini gives

$$\int e(u)(R_\omega y)(a(u,1))\,du=y(a)\int \psi_\omega(k)\overline{e(k)}/e(1)\,dk=\nu\,y(a),$$
$$|\nu|\ge\beta^{-fH}\prod_v v_{1,v}.$$

Indeed the integral in u,k is bounded absolutely by $\|y\|_\infty\|e\|_1\|\psi_\omega\|_1$, where the slice estimate above gives $\psi_\omega\in L^1$. The scalar is exactly $\prod_v\int c_l(k)\,\overline{c_{-l_Z}(\omega_v^{-1}\iota(k)\omega_v)}\,dk$ from (DS2). At $\alpha=\jmath(u)\omega$,

$$|e(u)|\,C_{\jmath(u)\omega,\omega}\le\mathfrak{M}(u)(2\beta)^{3Af}\prod_v v_{1,v}.$$

Indeed $\|\iota(u_v)\omega_v\|\,\|\omega_v\|\le\|u_v\|\beta$. After division by the local factor of $\mathfrak{M}(u)\,v_{1,v}$, the product on the left is at most

$$(2/\|u_v\|)^{2A}[32\beta^2\|u_v\|^2]^A=(128\beta^2)^A\le(2\beta)^{3A},$$

by the $4A$-decay and $\beta \ge 16$. Consequently the averaged unfolding at this a is

$$\nu\, y(a) = -\sum_{\zeta\Gamma_V \ne \Gamma_V \text{ supported}} \int e(u)\, I_{\zeta, \jmath(u)\omega}(a)\, du.$$

The sum of the absolute integrals is at most $(10S/B_d)\int |e(u)| C_{\jmath(u)\omega,\omega}\, du$ by the coset bounds. This is finite by (D5), justifying the exchange. Dividing by the lower bound for $|\nu|$ gives $|y(a)| \le 10S/(16\beta) \le S/2$.

If instead $\lambda(a) < (B_d^{-2}/(M\kappa^2))^r$, use the quantitative inclusion of Section 42.3 at radius B_d^{-2} and level $K_V(M)$ on V. It gives a rational isotropic subspace whose unipotent radical, read in the basis at a, is covered by its rational translates of the product of the real ball of radius B_d^{-2} and $K_V(M)$. In the moved coordinates, for each $u \in N^a$ choose $\gamma \in \Gamma_N^a$ such that $\gamma u = (u'_\infty, k_f)$, with $\|u'_\infty - I\| \le B_d^{-2}$ at each real place and $k_f \in K_V(M)$. Then $a\gamma a^{-1}$ is rational. Since $K_V(M) \subset J_M$,

$$y(au) = y(a\gamma u) = y(a(u'_\infty, 1)).$$

For the level inclusion, at a chosen place over either old prime $M = \pi_M \bar{\pi}_M$ has valuation one and $J_M = K_V(\pi_M)$. Away from the old pairs J_M is the prescribed integral compact. The larger modulus $\pi_M^{b_0}$ in (D3) is imposed on the target column. By the constant-term vanishing of Section 42.4,

$$|y(a)| \le \sup_{\|u-1\| \le B_d^{-2}} |y(a(u,1)) - y(a)|.$$

Telescoping the product of the small-type coefficients at $u^{-1}z$ and z bounds its difference by

$$100(r+L+1)^5 \sum_v \|u_v - I\| \le 100 f (r+L+1)^5 B_d^{-2} \le \beta^2 B_d^{-2},$$

since each factor has modulus at most one. With $P(z) = \prod_v (2/\|z_v\|)^{2A}$, the same difference is at most $P(u^{-1}z)^2 + P(z)^2$ by the $4A$-decay. Use $\min(U,V) \le \sqrt{UV}$ and $\sqrt{P_1^2 + P_2^2} \le P_1 + P_2$ for nonnegative P_i. Including the formal degrees gives

$$|e(u^{-1}z) - e(z)| \le (\beta/B_d)[\mathfrak{M}(u^{-1}z) + \mathfrak{M}(z)].$$

Applying reproduction of y and left invariance of Haar measure, then (D5), bounds the oscillation by $2S(\beta/B_d)\int \mathfrak{M} \le S/8$. This proves $|y(a)| \le S/2$ for all adelic points a. Repeated halving of any initial bound for y gives $y = 0$.

Compose the deep restriction at this ω with the weighted transpose of the true deep lifts from the definite end of the grid. Section 43.2 deduces $\Theta^\flat(\mathcal{H}_m) = \mathcal{H}_0 = W_Z$ from (TP4) and injectivity of the forward composition. Thus for each nonzero $y \in W$ the separating test in (DS0) is $\Theta^\flat \eta$ for some $\eta \in \mathcal{H}_m$. The level $K_Z(a_0)$ and tuple $\omega = (\omega_v)_{v|\infty}$ are fixed, and the choice of η may depend on y.

43 Theta compositions at deeper principal levels

In this section, we transfer the separation of Section 42.2 along two longer theta compositions. The paired period of Section 34.3 compares them after identifying the opposite Hermitian forms and gives the terminal pairings used in Section 44.

Put $\mathfrak{b} = (\pi_M^{b_0})$ and $R_b = \mathcal{O}_E/\mathfrak{b}$. Use the trivial twist in the true triangle, with this same ring at every node. Its rank in column n is $d_n = r + 1 + n(t-1)$. The identifications j_i of its base copies come from passage to the opposite Hermitian form, with the original finite bases, quotient Haar measures, and principal levels. Use forward triangle forms of the conjugate type to the true-grid forms $\mathcal{H}_j$ of level $K_{Z_j}(\mathfrak{b})$. In particular $\mathcal{H}_0$ consists of the tested forms W_Z of Section 42.2. Write $a_{i,n}$ for the forward point lift and $w_n = |\mathrm{GL}_{d_n}(R_b)|$.

43.1 The first-column return

In this subsection, we compute the first return on the level used in Section 42.2.

At the first arrow put $u = d_0 = r+1$, $v = d_1 = r+t$ and $X_0 = \binom{I_u}{0}$ over R_b. The characters at the two ranks are $\mu_{i+1}^0 \upsilon^{\lceil u/2 \rceil}$ and $\mu_{i+1}^0 \upsilon^{\lceil v/2 \rceil}$. They are unramified off the chosen split pairs. The finite datum is the indicator of X_0 at the chosen places, with the integral vacuum elsewhere. At a chosen split place suppose both X and Xg reduce to X_0. Their first u rows give square matrices congruent to I_u, hence invertible over the local integers. Composing the projection onto those rows with the inverse square matrix gives an integral left inverse for each.

If L_X, L_{Xg} are these left inverses, then $g = L_X(Xg)$ and $g^{-1} = L_{Xg}X$ are integral there. The overlap is then equivalent to $g \equiv I_u \bmod \mathfrak{b}$, and the character of g on the overlap is one. Give the integral matrix box and the integral source compact K_Z° mass one. The squared norm of the residue indicator is $|R_b|^{-uv}$, and its source stabilizer has mass

$$\operatorname{vol} K_Z(\mathfrak{b}) = w_0^{-1}, \qquad w_0 = [K_Z^\circ : K_Z(\mathfrak{b})] = \prod_{w \mid \pi_M} |\mathrm{GL}_u(\mathcal{O}_{E,w}/\mathfrak{b}\mathcal{O}_{E,w})|.$$

Reduction is surjective at each chosen split factor, since every matrix with unit determinant modulo the maximal ideal lifts to an integral invertible matrix. Unitarity determines the conjugate factor. The separate test space W in Section 42.2 has level $K_V(\pi_M)$. The return uses the conjugate of W_Z at level $K_Z(\mathfrak{b})$.

Write $A_i = a_{i,0} j_i$. The open-graph doubling calculation of Section 34.2 applies at these ranks. The doubled source has Siegel rank $u = r+1$. The target has rank $v = r + t = 6r$, so $v/2 - u = 2r - 1 > 0$ and $v - \iota(T) \geq v/2 > u$. Weil's convergent formula applies to the separately translated Schwartz–Bruhat factors [78, §52, Theorem 5, pp. 76–77]. Its parameter is $(v-u)/2$ and the coefficient is one [80, Theorem 2]. At each prescribed split place every supported matrix has an integral left inverse as above, hence full column rank over the local field. A nongraph orbit would force a rank drop there, so it contributes zero to the global doubling integral. This gives

$$w_0 \langle A_i x, A_i y \rangle = C_i |\operatorname{Mat}_{v \times u}(R_b)|^{-1} \langle x, \mathcal{V}_i y \rangle, \qquad C_i \neq 0. \tag{TP1}$$

Here x, y are in the true forward base $\overline{\mathcal{H}_0}$ at level $K_{Z_0}(\mathfrak{b})$. The projected truncations in Section 42.5 supply bounded cuspidal representatives throughout this base before the doubling. These representatives are rapid by Section 18.1.3. At the fixed finite test, Section 40.2.1 bounds the absolute theta-product integral and the absolute graph sum separately by $CH(g)^B H(h)^B$. Multiplication by $|x(g)y(h)|$ and integration over the two finite-volume source quotients justify the same rearrangements as in Section 34.2.2. Thus the identity applies to the full deep base.

The scalar C_i is the original target quotient volume times the integral of the real joint-harmonic coefficient against $\overline{e_i(h)}/e_i(1)$, with e_i the source projector. We verify the real harmonic at this edge. Fix an infinite place, let L be the doubled forward stem $2\lambda_j + 1$ of Section 42.2. The doubled target parameter L' adds to this stem the block $-2w_{i+1} + t - 2 - 2b$,

$0 \le b < t-1$, with positive signs. Here $w_j = \kappa_\tau + (j+1/2)\Delta$ in the indexing of the grid and $w = w_{i+1}$. The source and target splitting weights are $u/2 - w, v/2 - w$. Both cross shifts are $2w$. At tail stem index $k-1+a$ the common shifted entry is

$$\xi_a = \Delta - 2w_{a+1} + 1 + 2w = \Delta + 1 - 2(a-i)\Delta.$$

The ratio of the two signature signs there is $(-1)^{\min(1,(a-i)_+)} = \operatorname{sgn}\xi_a$. The head has positive shifted entries and unchanged signs. The $v-u = t-1$ inserted entries have positive signature and shifted values $t-2, t-4, \dots, 2-t$. The common shifted entries have modulus at least $\Delta - 1$. The gap between opposite signature signs is therefore at least $(\Delta - t - 2)/2 > 500(u+v)^3 + uv$ in the undoubled parameters.

Use s_a, s'_b for the source and target signs. In the oscillator the doubled source parameter is $-L$ and the target one is L'. After subtracting the vacuum exponents, the lowest compact relative weights are

$$2R_b^{\mathrm{wt}} = \xi_b - s'_b \left(\sum_{b'} s'_{b'} \operatorname{sgn}(\xi_b - \xi_{b'}) + \sum_{a'} s_{a'} \right),$$
$$2C_a^{\mathrm{wt}} = -\xi_a + s_a \left(\sum_{a'} s_{a'} \operatorname{sgn}(\xi_a - \xi_{a'}) + \sum_{b'} s'_{b'} \right).$$

Indeed apply $\lambda + \rho - 2\rho_K$ to the two parameters and subtract the respective vacuum weights $-w + s'_b \sum_a s_a/2$, $w - s_a \sum_b s'_b/2$. At matched indices, the contributions of an index with the same sign of ξ cancel using $s'_a = s_a \operatorname{sgn}\xi_a$. For the opposite sign they cancel using $\operatorname{sgn}(\xi_a - \xi_b) = \operatorname{sgn}\xi_a$. The inserted-block contributions also cancel at matched indices. At an inserted index the matched terms cancel, and the signed sum of the other inserted positions equals its shifted value. It follows that $R_a^{\mathrm{wt}} = -C_a^{\mathrm{wt}}$ at matched indices, $R_b^{\mathrm{wt}} = 0$ at inserted indices, and $\operatorname{sgn} R_a^{\mathrm{wt}} = \operatorname{sgn}\xi_a$.

For a retained target entry, the formula gives $|2R_b^{\mathrm{wt}} - \xi_b| \le u + v - 1 < \Delta - 1$. In either group's decreasing parameter order write L_i^* for the doubled entries and μ_i for the relative compact weights. For indices $i < j$ of the same signature, let c count indices of that signature in $(i, j]$. The vacuum weight is constant on that compact block, so

$$\mu_i - \mu_j = (L_i^* - L_j^*)/2 + (j-i) - 2c \ge 2(j-i-c) \ge 0.$$

Here the successive doubled entries have gaps at least two. In each sign quadrant take the product over $h \ge 1$ of the minors on matched indices of absolute weight at least h, omitting empty factors. On the standard coordinates, weights at least h occupy an initial segment. On the dual coordinates, weights at most $-h$ occupy a final segment, which becomes initial in dual order. Since on matched pairs $R^{\mathrm{wt}} = -C^{\mathrm{wt}}$ and $s'_a = s_a \operatorname{sgn}\xi_a$, the indices matched to each such segment lie in one compact block of the other group. A compact raising operator kills the minor or duplicates an index in it. The row and column weights are $R_b^{\mathrm{wt}}, C_a^{\mathrm{wt}}$. At the matrix with ones on the matched entries and zeros elsewhere each minor is ± 1, so the product is nonzero. Each source or target index is used with one oscillator sign. Thus all mixed derivatives of opposite signs with a common index vanish.

The joint harmonic space contains the indicated pair of compact types with multiplicity one by [61, Theorem 2.2(1) and Lemma 2.5]. Thus this nonzero highest polynomial spans its highest line. Both ranks $u = r+1$ and $v = 6r$ are even. The doubled shifted stem entries

are odd, as are the inserted entries $t-2, t-4, \dots, 2-t$, so both parameters lie in the genuine lattice $\mathbb{Z} + (u+v+1)/2$. The signature calculation and gap above give the discrete-series correspondence of [61, Theorem 2.7(b), p. 641]. Thus the positive Bargmann pairing and the two compact averages used for the first-arrow constant apply to this polynomial. They use the source matrix unit of parameter $-L/2$ and the contragredient scalar-form coefficient. At the real place τ, write $P_{i,\tau}$ for the polynomial, $e_{i,\tau}$ for the source matrix unit, and ω_τ for the Weil action. The preceding positive Bargmann integral and compact Schur averages give

$$Z_{i,\tau} = \int_{U(Z_0)(F_{0,\tau})} \langle P_{i,\tau}, \omega_\tau(h,1) P_{i,\tau} \rangle \overline{\frac{e_{i,\tau}(h)}{e_{i,\tau}(1)}}\, dh > 0.$$

In projection notation this is $\|p_{i,\tau} P_{i,\tau}\|^2 / e_{i,\tau}(1)$, where $p_{i,\tau} = \int \overline{e_{i,\tau}(h)}\, \omega_\tau(h,1)\, dh$ and $e_{i,\tau}(1)$ is the positive formal degree for the original real Haar measure. The product over real places is

$$C_i = \operatorname{vol}([U(T)]) \prod_{\tau \mid \infty} Z_{i,\tau} > 0.$$

The passage to the tensor realization in Section 34.3.2 dualizes the Hom source before applying the common-entry rule of Paul's theorem. The operator $\mathcal{V}_i$ has the original vacuum weights away from the prescribed support. The local lattice count of Section 34.2, with source rank u, gives $O(q_w^u)$ neighbor choices of weight $q_w^{-v/2}$ at each place.

Here $v/2 - u \geq r$ for large x. At a split pair the number of moves is at most $2(1 + \cdots + q_w^{u-1})$, and at a nonsplit place it is at most $\sqrt{q_w}(1 + \cdots + q_w^{u-1})$. Each is at most $4q_w^u$, uniformly in u. The total one-step weight therefore satisfies

$$\varepsilon \leq 4[E:\mathbb{Q}] \sum_{n \geq 2} n^{-r} \ll [E:\mathbb{Q}] 2^{-r} = o(1).$$

The polynomial bound for $[E:\mathbb{Q}]$ in Section 4 gives $\varepsilon < 1/3$ for sufficiently large x, uniformly in the allowed levels. Each right translate is unitary on the ambient Petersson L^2, so the absolute sum over nonidentity words gives

$$\|\mathcal{V}_i - I\|_{2\to 2} \leq \frac{\varepsilon}{1-\varepsilon} < \tfrac{1}{2}$$

on the prescribed level. The series preserves that level by spherical bi-invariance off the prescribed support. Its summands commute with the real projector, hence so does their operator-norm limit. By (TP1), the adjoint on the fixed-type subspace at $K_{Z_0}(\mathfrak{b})$ satisfies

$$A_i^* A_i = \frac{C_i}{w_0 |R_b|^{uv}} \mathcal{V}_i.$$

The scalar is positive, and the preceding operator bound gives

$$\|A_i y\|^2 \geq \frac{C_i}{2 w_0 |R_b|^{uv}} \|y\|^2$$

on this subspace. Thus A_i is injective.

The first descent $d_{i,0}$ is the adjoint return to the next opposite base copy. To realize it with the original moduli, let P_i be the upward joint harmonic and $P_i^\flat$ the opposite real edge harmonic. Conjugation of the Weil theta kernel carries its residue datum Ψ to

$$\Psi_0(X) = \overline{\Psi(u_R^{-1} X)}, \qquad u_R = u_0 \bmod \mathfrak{b}.$$

Here $u_0 \in \widehat{\mathcal{O}}_E^{\times}$ is the fixed unit satisfying $u_0 \bar{u}_0 = -1$ in the matrix reversal (MR). The scalar c_i^{op} in this conjugation is fixed by the normalized theta functionals of the two opposite bases.

The quotient Haar is carried to the same Haar. Replace the conjugate polynomial by $P_i^{\flat}$ in the tested period using the fixed real edge convolution. If its constant is k_i', the identity with the resulting descent B_0 is $\langle x, A_i y\rangle = \overline{c_i^{\mathrm{op}}}\, k_i' \langle B_0 x, j_{i+1} y\rangle$. We use inner products linear in their second argument. Thus the datum $c_i^{\mathrm{op}} \overline{k_i'} \Psi_0$ gives the exact adjoint. This identity uses the opposite-basis conjugation and the real edge convolution of Section 34.3. The conjugation and real replacement take place before any residue expansion, so the initial adjoint datum is a function on R_b.

Let $y^{\flat}$ be the return on the base. Then $\langle x, y^{\flat}\rangle = \langle A_i x, A_i y\rangle$. Let $k_b \in \mathrm{GL}_u(R_b)$ be the reduction of the integral base element k. At the selected split places lift $g_b = \mathrm{diag}(k_b, I_{v-u})$ to the integral target. Then $g_b^{-1} X_0 k_b = X_0$ in the chosen Schrödinger coordinates, and the determinant character contributes a scalar of modulus one. Consequently the source and target translations intertwine A_i up to that unit scalar. Pairing with all x shows that $A_i^* A_i$ commutes with the integral base action. In particular $y^{\flat}$ has the deep level when y does. If $y^{\flat} = 0$, take $x = y$ and use (TP1) to conclude $y = 0$.

43.2 The measured square and transposition

In this subsection, we use the local return of Section 43.1 in the measured square.

At the square (i, n), the kept ranks are d_{n+1}, d_{n+1} and the other ranks are d_n, d_{n+2}. The exchange defect is $t-1$. The stem sign with excess parameter a_* at (o, j) is $(-1)^{o+a_*+\min(j,(a_*-o)_+)}$. Passing from the kept pair to the other pair leaves both signs unchanged if $a_* \le i+n+1$. Otherwise the two changed signs were opposite, so the negative count is preserved. The common negative count q_- satisfies $q_- \le 2(r+1) < d_{n+1}$. The quotient character is

$$\nu = \mu^0_{i+n+2}/\mu^0_{i+1} = (\upsilon^{-\Delta}\chi_M)^{n+1}.$$

Its inverse has positive weight $(n+1)\Delta$, and $d_{n+1} \le (n+1)\Delta$.

At the chosen split places its inverse tame part is the inflation of $\chi_M^{-(n+1)}$, nontrivial by $1 \le n+1 \le m$, and elsewhere it is unramified. The forward and opposite left-pair factors at ranks p, q, $p+q = 2d_{n+1}$, have product

$$\mu^0_{i+n+2}\upsilon^{\lceil p/2\rceil}(\mu^0_{i+1})^{-1}\upsilon^{\lfloor q/2\rfloor} = \nu\upsilon^{d_{n+1}}.$$

The first adds the center $-w_{i+n+2}$. For the left pair, pass to the opposite form and add the center $-w_{i+1}$. Conjugating both parameters replaces the old determinant exponent at rank h by h minus that exponent, giving the floor on the left pair.

The positive-weight character in Section 36.1 is ν^{-1}. Its prescribed splitting character on the kept space is $(\nu^{-1})^{-1}\upsilon^{d_{n+1}} = \nu\upsilon^{d_{n+1}}$, the product just computed. This verifies the remaining character condition for the measured period on this square. Its ratio is $k_{i,n}/c_{i,n}$, where $c_{i,n} \neq 0$ is the measured partner scalar and $k_{i,n}$ comes from the real pairing of the four harmonic vectors. The real replacement is the one in Section 34.3.

We prove that the point extends over R_b. In the decomposition b_1, a of the full matrix, invertible blocks Y_1, X_2 in the first b_1 columns of a matrix and the last a rows of its inverse give

$$X_2^{-1}X_1 = -Y_2 Y_1^{-1}, \qquad \det k = \det Y_1 / \det X_2.$$

These follow by block elimination over each finite local ring. Jacobi's complementary minor identity makes invertibility of either square block equivalent to invertibility of the other. For fixed s, use the determinant-character weighted indicators of $X_2^{-1}X_1 = s = -Y_2Y_1^{-1}$ in the partner period. Covariance cancels the character on the two invertible blocks. Thus the respective point values are multiplied by $|\mathrm{GL}_a(R_b)|$ and $|\mathrm{GL}_{b_1}(R_b)|$. Here only invertible blocks contribute. The argument uses finite groups over R_b itself and requires no Gauss sum with a primitive multiplicative character at the deep power.

For the actual square let $a_* = d_n$, $d_+ = d_{n+1}$, and $b_* = t-1$. In the order d_+, a_*, b_* the two arguments are

$$Y = \begin{pmatrix} -\eta \\ I_{a_*} \\ 0 \end{pmatrix}, \qquad X = \begin{pmatrix} I_{d_+} & \eta & s \\ 0 & 0 & I_{b_*} \end{pmatrix}.$$

They are obtained by choosing the common graph parameter $\binom{\eta}{0}$ in the order $a_*, (d_+, b_*)$ and applying the determinant-one shear with off-diagonal block s. The point extension therefore uses the new descent datum

$$p^+ \begin{pmatrix} \eta & s \\ 0 & I_{b_*} \end{pmatrix} = p(-\eta)|R_b|^{-d_+ b_*}, \qquad p^+ = 0 \text{ on the other matrices.}$$

Indeed the shear fixes the tested covector value by covariance. Average over s and sum over η with weight $p(-\eta)$. Starting with the adjoint datum above and applying this recursion gives descents $d_{i,n}$ with the measured identity

$$w_n a_{i+1,n} d_{i,n} = (k_{i,n}/c_{i,n}) w_{n+2} d_{i,n+1} a_{i,n+1}. \tag{TP2}$$

To test this equality, take a source form and a target test of the indicated real types at a common finite level fixing the residue data. The displayed Y has an identity minor of order a_*, and the first and last column blocks of X give a triangular minor of order $d_+ + b_*$ and determinant one. Thus the primitive coefficient estimate of Section 34.3 applies to each completed pair, before the finite sums in s, η. The two external forms are rapid, and the individual lifts with the prescribed harmonics are rapid at their respective finite levels by that same section. The operator identity follows by testing the difference of the two resulting forms against itself.

Injectivity and transposition. Write $X_{i,h}$ for h weighted forward arrows in sequence i and $X_{i,0} = I$. Iterating (TP2) from left to right gives, for some scalar $\alpha_{i,h}$,

$$X_{i+1,h} d_{i,0} a_{i,0} = \alpha_{i,h} d_{i,h} X_{i,h+1}. \tag{TP3}$$

The initial scalar is $\alpha_{i,0} = w_0^{-1}$. Multiplying on the left by $w_h a_{i+1,h}$ and applying (TP2) gives

$$\alpha_{i,h+1} = \alpha_{i,h} \frac{k_{i,h}}{c_{i,h}} \frac{w_{h+2}}{w_{h+1}}, \qquad X_{i,h+2} = w_{h+1} a_{i,h+1} X_{i,h+1}.$$

If $X_{i,h+1} j_i y = 0$, (TP3) makes $X_{i+1,h} j_{i+1} y^\flat = 0$. Induction on the chain length, starting with $X_{i,0} = I$ and using the injective return, proves $y = 0$. In particular the full true forward chain $F = X_{0,m}$ is injective on the original deep base. This implication uses (TP2) in its displayed direction. By Section 16.2, the projected space in W has positive dimension. Thus (DS0) gives $W_Z \neq 0$. For any nonzero forward base vector y, the return $y^\flat = A_i^* A_i y$ is nonzero. By injectivity at length h, the left side of (TP3) is nonzero on $j_i y$. Therefore $\alpha_{i,h} \neq 0$ for

every admissible i, h. Its recurrence then gives $k_{i,h} \neq 0$ for each square, using $\alpha_{i,h+1} \neq 0$. The argument takes place at the trivial twist and the level $\mathfrak{b}$.

On every forward edge of $F = X_{0,m}$, the residue point is the specified inclusion matrix. Its left and right actions by the two principal subgroups at $\mathfrak{b}$ fix that matrix. Their determinant character is one, since it factors through the prescribed tame reduction. Thus each forward and transposed arrow has the asserted principal level on the target. At an edge let f_1, f_2 be the partial lifts from the two ends in the bilinear pairing. Their prescribed types are cuspidal by Section 34.3. At a definite end the quotient is compact. For the fixed edge datum its kernel is bounded by $C\,\mathrm{ht}(g)^B\,\mathrm{ht}(h)^B$ for some B, by the two-variable theta estimate of that section. The product with $|f_1(g)f_2(h)|$ is therefore integrable on the two quotients. Fubini moves this edge to the other side using the same two quotient measures. At the next edge take the already evaluated partial lifts as the two tests and apply the same two-variable bound. Pullback from a coarser finite level leaves the automorphic function and quotient integral unchanged. This iteration puts the same weight w_j on its transposed edge and gives

$$\int_{[Z_m]} (Fy)\phi = \int_{[Z_0]} y(\Theta^\flat \phi). \tag{TP4}$$

The transpose here is bilinear and has the conjugate endpoint types. If $z \in \mathcal{H}_0$ is perpendicular to its image, take $y = \bar{z}$ and $\phi = \overline{Fz} \in \mathcal{H}_m$. Then (TP4) gives $\|F\bar{z}\|^2 = 0$. Injectivity gives $z = 0$. The fixed-type fixed-level spaces are finite dimensional, so $\Theta^\flat : \mathcal{H}_m \to \mathcal{H}_0$ is onto.

43.3 The first-path array

In this subsection, we write the finite array for the transposed lift of Section 43.2.

Use the point ω from Section 42.2. Let $R_0(\xi) = \mathcal{O}_E/(\pi_M Q(\xi))$ and $R_\xi = \mathcal{O}_E/(\mathfrak{b}Q(\xi))$. The false first column uses its original twist and $R_0(\xi)$. The true first column uses the trivial twist and R_b. Both can be read on R_ξ, since the reduction maps $R_\xi \to R_0(\xi), R_b$ are onto. Indeed, with $R_Q = \mathcal{O}_E/Q(\xi)$, the Chinese remainder theorem gives $R_\xi = R_b \times R_Q$. On the true half the map is projection onto R_b. On the false half it is reduction of each old prime power to its residue field and the identity at the new places, so its kernel is nilpotent. Pullback of an edge datum from either quotient is the same adelic datum: its value on an integral matrix depends on exactly the indicated class, and it has zero value off the integral box at the chosen places. At the extra places on the true side the pulled-back value on the integral box is constant, hence is the vacuum.

For a tuple S of matrices on the two first columns put

$$P_0(S) = \prod_{\substack{z\in\{0,1\}\\ 0\le i<m}} \mathbf{1}_{\{\bar{S}_{z,i}=\iota_{z,i}\}},$$

where $\iota_{z,i}$ is the original inclusion matrix, and the bar denotes reduction to the ring of that column. Put $D_{z,i}$ for the group order at the source of that column edge over $R_0(\xi)$ or R_b, respectively. Let $T_{z,i}(S_{z,i})$ be the unweighted transposed point lift at $S_{z,i}$ over R_ξ.

At an old place each integral residue box for the original column is the disjoint union of its lifts at the deeper exponent. At a new chosen place of the true column the original vector is the vacuum. In its split Schrödinger coordinate this is the indicator of the entire integral matrix lattice, the disjoint union of *all* residue boxes there. The conjugate coordinate belongs to the same split model with its original self-dual measure. Thus the pullback along either reduction

gives the same finite vector in the original model. By linearity, summing the point indicators with their original values gives the weighted arrow as

$$D_{z,i}\sum_{S_{z,i}}\mathbf{1}_{\{\bar{S}_{z,i}=\iota_{z,i}\}}T_{z,i}(S_{z,i}).$$

The matrices in a tuple are independent choices on disjoint edges. Multiplying the two chains and using bilinearity of the diagonal period at ω gives

$$\left(\prod_{z,i<m}D_{z,i}\right)\sum_{S}P_0(S)\mathcal{P}_\omega(S;y',\eta). \tag{TP5}$$

Here $\mathcal{P}_\omega(S;y',\eta)$ is the original diagonal pairing of the two unweighted transposed point chains, at the same real ω, of the false-terminal dual form y' and true-terminal form $\eta\in\mathcal{H}_m$.

Write $G_{z,i}$ for the invertible group in the node's rank over R_ξ. Set $L_{0,i}=\{1\}$ and $L_{1,i}=\ker(G_{1,i}\to \mathrm{GL}_{d_i}(R_b))$. Use the finite set

$$\mathcal{C}=\prod_{z,i<m}G_{z,i}/L_{z,i}.$$

Choose representatives, with identity at the terminal nodes. In the acting family $g(c)$ replace the representative at the true bottom node by the block sum of the false bottom representative and 1. The unused coset coordinate is still counted in $\mathcal{C}$.

For every node (z,a,b) let $g_{z,a,b}$ be an invertible matrix of its rank over R_ξ. The family is admissible if

$$g_{0,0,m}=1,\qquad g_{1,0,m}\equiv 1\pmod{\mathfrak{b}},\qquad g_{1,j,j}=\mathrm{diag}(g_{0,j,j},1)\quad(0\le j\le m)$$

in the prescribed reindexing at each diagonal node. The other node matrices are unrestricted. On the first path the only common diagonal node is $j=0$, which explains the replacement in $g(c)$. At the definite ends these conditions fix the false endpoint modulo $\mathfrak{b}Q(\xi)$ and the true endpoint modulo $\mathfrak{b}$. For an edge $e:p\to q$, put $(S^g)_e=g_q^{-1}S_eg_p$. Let $\chi(g)$ be the product of the edge characters at $\det(g_q)/\det(g_p)$, using the characters of their respective columns. The finite Weil action on the original boxes gives

$$\mathcal{P}_\omega(S^g;y',\eta)=\chi(g)\mathcal{P}_\omega(S;y',\eta)$$

for all admissible families g. Indeed choose integral lifts of these invertible matrices at each node. Moving a lift through the transposed edge contributes its determinant character and replaces the edge matrix by $g_q^{-1}S_eg_p$.

Every interior group translation cancels between the adjacent edges. The bottom translations match by the block relation, commute with ω, and disappear by right invariance of the diagonal quotient measure. At the two ends the translations fix the endpoint forms by their stated levels. All these equalities hold at the given finite points.

Set

$$P_1(S)=\sum_{c\in\mathcal{C}}\chi(g(c))P_0(S^{g(c)}).$$

This array has the required covariance $\chi(g)P_1(S^g)=P_1(S)$. To see this, use $(S^g)^h=S^{gh}$ and send c to gc. The representative of gc differs from $g\,g(c)$ by the node-level subgroups on the

right. These latter groups fix P_0 and have trivial character, since they reduce to identity in the ring of their column. At the true bottom node the discrepancy is identity by the block relation and $L_{0,0} = \{1\}$.

This also verifies independence of representative choices. By reindexing S in each summand and using the preceding period covariance,

$$\sum_S P_1(S)\mathcal{P}_\omega(S; y', \eta) = |\mathcal{C}| \sum_S P_0(S)\mathcal{P}_\omega(S; y', \eta).$$

Each entry contains at most $|\mathcal{C}| \le E_\xi^{2m}$ roots of unity, where E_ξ bounds the number of residue matrices at the allowed grid ranks. If an entry is nonzero, some translate has nonzero indicator.

Fix such a translate g. On a false edge $e : p \to q$, let π project onto the first $\mathrm{rk}(p)$ coordinates and put $C = \pi g_q^{-1} S_e g_p$. Then $C \equiv I$ over $R_0(\xi)$. The reduction $R_\xi \to R_0(\xi)$ has nilpotent kernel, so C is invertible over R_ξ. In the original coordinates the left inverse is

$$(g_p C^{-1} \pi g_q^{-1}) S_e = I.$$

Let $n_i = r + it$ be the rank at the false first column's node i. Its factor in $|\mathcal{C}|$ is $|\mathrm{GL}_{n_i}(R_\xi)|$, while the true factor is the order over R_b, by onto reduction. The order over $R_0(\xi)$ divided by the order over R_ξ is $(|R_0(\xi)|/|R_\xi|)^{n_i^2}$. Indeed a matrix is invertible if and only if its image is, and each matrix has a fiber of size $(|R_\xi|/|R_0(\xi)|)^{n_i^2}$. Thus the ratio of the product in (TP5) to $|\mathcal{C}|$ is the fixed positive number

$$A_{\mathrm{col}} = \prod_{i<m} \left(\frac{\mathrm{Nm}(\pi_M)}{\mathrm{Nm}(\mathfrak{b})} \right)^{n_i^2}. \tag{TP6}$$

For $\eta \in \mathcal{H}_m$, take $r_0\eta$ to be the Riesz representative on the base W of the restriction $(\Theta^\flat \eta)((h \oplus 1)(\omega, 1))$. In the original Petersson metric, linear in the second variable, this means that for $v \in W$,

$$\langle v, r_0\eta \rangle_W = \int_{[U(V)]} \overline{v(h)}\, (\Theta^\flat \eta)(\iota(h)(\omega, 1))\, dh.$$

The integrand is absolutely integrable since v is rapid and $\Theta^\flat \eta \in W_Z$ is bounded. This defines a linear map of η, with no choice involving ξ. Take a test v of the real type of W and level $K_V(\pi_M Q(\xi))$, and pair with its conjugate in the bilinear integral. Its average at $K_V(\pi_M)$ belongs to W, since the finite average preserves the real type. For $k \in K_V(\pi_M)$ write $k = k_b k'$, where k_b is supported at the original places and their conjugate places, and k' is the identity there. Because the twist support is disjoint from these places, k_b fixes every test of level $K_V(\pi_M Q(\xi))$.

The component k' lies in $K_V(\mathfrak{b})$, so its block embedding fixes the true function of deep level, and it commutes with ω. The restriction paired with v therefore equals the restriction paired with its average. Apply the Riesz identity in W to that average, and remove the average from the pairing with $r_0\eta$, which has the base level. If $v \in W$ is perpendicular to $\mathrm{im}\, r_0$, the displayed integral vanishes for every $\Theta^\flat \eta$. By (TP4) these tests fill W_Z, so (DS0) gives $v = 0$. Finite dimensionality gives $\mathrm{im}\, r_0 = W$. To check the real type of the bilinear transpose, take one false edge with joint harmonic P and the oscillator line projectors p_S, p_T of Section 34.3.2. The full source isotypic projection Q_S is central in both commuting von Neumann algebras by [38, Theorem 6.1, p. 550]. Its range is $V_S \widehat{\otimes} V_T$, with V_T irreducible, as proved there. Hence it is the identity on every copy of V_T: a target intertwiner between two copies commutes with Q_S. The matrix-coefficient projector p_T has range in this target isotypic component. Since it commutes

with the source action, $p_T P$ still has the source compact highest weight of P. That highest line in the lowest compact type of V_S occurs once. Consequently

$$p_S p_T P = p_T P = p_T p_S P.$$

Moving the terminal test projector onto the kernel in the bilinear pairing therefore reproduces the conjugate source line. The weighted coefficient and theta bounds of Section 34.3.1 justify this operation on each edge as in the forward direction. Iteration gives the conjugate of a test of the required real type. The source principal subgroup fixes the residue inclusion and has trivial determinant character on each edge. Thus the transpose has level $K_V(\pi_M Q(\xi))$. Applying r_0's identity and (TP5)–(TP6) yields

$$\int_{[Y_m]} y'(h)(T_\xi r_0 \eta)(h)\, dh = A_{\mathrm{col}} \sum_S P_1(S) \mathcal{P}_\omega(S; y', \eta). \tag{TP7}$$

43.4 Transport across the consecutive squares

In this subsection, we transport the arrays of Section 43.3 through the squares.

Parameterize each half path from its definite endpoint by right and down steps toward a common diagonal node (j, j), recording the true half with its coordinates transposed. At a square fix all residue matrices on unchanged edges.

In the ordered direct sum of the two kept node spaces the old two edge blocks form a column Y and the new blocks a block row X. The number of columns of Y is the rank of the removed node, and the number of rows of X is the rank of the added node. At an off-diagonal square there is no line variable. At a diagonal square write v for the removed column of the line and v' for the new covector. In the prescribed covector and column coordinates define

$$P^{\mathrm{F}}(Y, v') = \sum_v \psi(\mathrm{tr}(v' v)) P_{\mathrm{old}}(Y, v). \tag{TP8}$$

The sum for an off-diagonal turn has the one zero line coordinate and is the old value. The character ψ is the one in the line reversal below.

Here is the matched finite array for the partner comparison of Section 36.1. For a completion k in the general linear group of the kept direct sum, write Y for its first columns and X for the complementary rows of k^{-1}. Define

$$P_{\mathrm{new}}(X, v') = \chi(\det k) P^{\mathrm{F}}(Y, v') \tag{TP9}$$

and set it equal to zero for matrices X admitting no completion. This is independent of the completion: another completion with the same last rows of its inverse has the first columns Yg, where g is invertible on $\ker X$, and its determinant is multiplied by $\det g$.

Indeed the change of basis of the two completions has block form $\left(\begin{smallmatrix} g & * \\ 0 & I \end{smallmatrix}\right)$. The old path covariance at the removed node gives $P^{\mathrm{F}}(Yg, v') = \chi(\det g)^{-1} P^{\mathrm{F}}(Y, v')$; that node acts on Y on the right and fixes the line coordinate and the retained edges. Call an edge vertical when it changes the second index on the false half or the first index on the true half, in the coordinates of $n_z(a, b)$ below. On the first path all edges have the form $(z, 0, i) \to (z, 0, i+1)$, so precisely the false edges are vertical. The left inverse just constructed over R_ξ starts the induction, also for $\xi = 1$. At every turn at least one old block is vertical. If (TP8) is nonzero, some old summand is nonzero, hence that block has a left inverse. If the old edge has an extra line

column, restrict its left inverse to the retained columns and project to their coordinates. Over each local factor of R_ξ, complete the residue columns of Y to a basis and lift the extra columns. The resulting determinant is a unit in that factor. The Chinese remainder theorem assembles these completions into one k, allowing a different pivot minor at each place.

The node covariance of (TP9) follows by the same change of completions. Explicitly, the retained-node block element G and the new-node element h change X to $h^{-1}XG$. One may replace k by $G^{-1}k\,\mathrm{diag}(1,h)$, so that Y changes to $G^{-1}Y$ and the determinant acquires $\det h/\det G$. At a diagonal turn the action on the old column and new covector of the line is inverse at their common subdiagonal node.

Changing v in (TP8) therefore preserves $\mathrm{tr}(v'v)$. In detail, let χ_1,χ_2 be the characters of the two old edges, with target determinants d_1,d_2. The removed node determinant is the same on its line extension. Thus $\chi=(\chi_1\chi_2)^{-1}$. If C_w is the product from the retained edges, the Fourier value after the move with the removed element equal to the identity is multiplied by $[C_w\chi_1(d_1)\chi_2(d_2)]^{-1}$.

Write $d_h=\det h$. The completion contributes the factor $\chi(d_h/(d_1d_2))$. Opposite edges have equal characters, so the new edge-character product is $C_w\chi_2(d_h/d_1)\chi_1(d_h/d_2)$, which cancels the product of these two factors. Thus the new array has the required covariance. Each new vertical block is opposite a split injective old block. Over each residue field, if the projection of Y into one retained summand is injective, then its image intersects the other retained summand trivially.

Since $\ker X=\mathrm{im}\,Y$, X on this other summand is injective. Lifting the resulting minor over the finite local factors makes this a split injection over R_ξ. If the new edge also contains the covector v', extend the left inverse by zero on that coordinate. The retained edges keep their support conditions. Each new entry has at most as many root-of-unity terms as the old bound times the number of choices of v, which is at most V_ξ.

For half $z=0,1$, the rank at (a,b) is $n_z(a,b)=r+z+(t+z)a+(t-z)b$. Put $d=n_z(a,b)$. The two retained ranks are $d+t-z,d+t+z$; the two integrated ranks are $d,d+2t$. Thus both sums are $N=2(d+t)$. In the stem sign formula of Section 34.3.2 the offset is zero. The positive excesses run from 1 to $r+z-k=m-1+z$, each once. In column b a stem entry of excess h is negative exactly when $h>b$ and $h-b$ is odd. All added entries are positive. With $s=m-b+z$, each diagonal therefore has

$$q=\lfloor s/2\rfloor+\lfloor(s-1)/2\rfloor=m-b-1+z$$

negative signs, also for $b=m-1$. Since $q\le m<d+t$, the lower bound $2q+2\le N$ holds at every place. The product character before inversion is $\mu^0_{a+1}\mu^0_{b+1}\xi^h$, where $h=b+1$ on the false half and $h=a+1$ on the true half. Its inverse has weight at least $\Delta\ge N/2$. The old tame character is χ_M^{a+b+2} with $2\le a+b+2\le 2m$, and the new one is ξ^h with $1\le h\le m$. Each is nontrivial at its chosen places, including after inflation to the deep old powers. At ranks p,q' the two edge characters have splitting exponents $\lfloor(p+1-z)/2\rfloor$ and $\lfloor(q'+z)/2\rfloor$, respectively. Their sum is $N/2$, since $p+q'=N$ is even. The twist belongs to the vertical edge in each orientation. The true orientation of the square (i,j) uses $(a,b)=(j,i)$, so both use $h=j+1$. The same square calculation applies at (j,j) before the line flip.

Fix a square $s=(i,j)$ and the residue matrices on the unchanged edges. In either half's recorded coordinates the retained nodes are $W_1=(a,b+1)$ and $W_2=(a+1,b)$. The old corner passes through $V'=(a,b)$, and the new corner through $V=(a+1,b+1)$. Let L_B be the lift with point matrix B, and ${}^{\mathrm{t}}L_B$ its transpose for the bilinear period. Equation (TP9) gives

matched arrays supported on split surjections and split injections at the chosen places. For the incoming partial lift f on W_1 and a linear functional λ on the output at W_2, the matched operator identity of Section 34.3.2, obtained from (95), gives

$$\sum_{Y} P^{\mathrm{F}}(Y, v')\, \lambda(L_{Y_2}{}^{\mathsf{t}} L_{Y_1} f) \\ = \frac{k_s}{c_{\mathrm{sq},s}(\xi)} \sum_{X} P_{\mathrm{new}}(X, v')\, \lambda({}^{\mathsf{t}}L_{X_2} L_{X_1} f).$$

Indeed, the two weighted outputs at a common finite level have the prescribed cuspidal real type. Testing their difference against itself in the Petersson pairing proves equality as forms. The measured partner scalar satisfies $c_{\mathrm{sq},s}(\xi) \neq 0$. The real coefficient k_s is fixed before the twist and the forms; it need not be inverted.

Off the diagonal, λ applies the remaining arrows from W_2, then pairs with the unchanged half at the fixed real point. The unchanged true half is the second argument when $i < j$; the unchanged false half is the first when $i > j$. The latter form already has the conjugate type. The diagonal integral is absolutely convergent since the smaller-group form is in L^1 and the translated larger-group form is bounded. All four point lifts have their original quotient measures. Thus $c_s = c_{\mathrm{sq},s}$ off the diagonal. On the diagonal the false block uses the same square identity with v' fixed. The square rank defect is t; the square law is evaluated at the same determinant shifts and common Haar as in the comparison. Put

$$S_\xi = S_M \cup \operatorname{supp} Q(\xi), \qquad \nu_{ij,\xi} = (\mu^0_{i+1}\mu^0_{j+1}\xi^{j+1})^{-1}, \qquad T_{ij}(\xi) = \prod_{k=1}^{t} \frac{|L^{S_\xi}(k, \nu_{ij,\xi})|}{|L^{S_M}(k, \nu_{ij,1})|}.$$

The incomplete base-change series omit each indicated place and its conjugate, as in Section 36.1. Taking the ratio $M_{\nu_{ij,\xi}}(t)/M_{\nu_{ij,1}}(t)$ gives

$$\frac{|c_{\mathrm{sq},s}(\xi)|^2}{|c_{\mathrm{sq},s}(1)|^2} = \mathrm{Nm}(Q(\xi))^{t^2}\, T_{ij}(\xi)^2. \tag{TP10}$$

The factors from the old residue cardinalities cancel. The appearance of $Q(\xi)$ only to t^2 uses that the conductor factor is a product over the support, and the new support is squarefree. The locally constant matched comparison in Section 36.1.5 applies to the pullbacks of the arrays at the old exponent b_0. Thus (TP10) holds for the arrays over R_ξ.

At the diagonal j, the two intermediate spaces are identified in their chosen coordinates. Give this space rank $n_j = r + (2j+1)t$. The old and new lines have signs $-\epsilon$ and ϵ, $\epsilon = (-1)^{j+1}$. Choose the finite idele t_ξ generating the grid modulus with component one outside the selected halves, including at their conjugates. With $\alpha_f = \alpha_{\mathrm{W}}$ the fixed finite Weil scale, the old and new finite line points are u_ξ^j, u_ξ^{j+1} for $u_\xi = \overline{t_\xi \alpha_f}/(t_\xi \alpha_f)$. In the original sign coordinates the polynomial reversal is $X_{v,i,1} \mapsto -\epsilon s_i X_{v,1,i}$, where s_i are the intermediate signs.

In fact the Hermitian adjoint transposes and conjugates, while reversing the line changes its sign and thus the conjugation in the holomorphic Fock coordinate. The Gaussian is unchanged. Let u_0 be the fixed finite norm-minus-one unit and $c_0 = u_0^\epsilon$. On the self-dual coordinates use

$$\psi(\mathrm{red}\, z) = \psi_f(j_{\mathrm{W}}\, \alpha_f \overline{\alpha_f}(-\bar{c}_0) z / t_\xi)\,.$$

The integral box shifts by v/t_ξ and the line dilation changes it to $(X - v)/t_\xi$ integral.

The squared modulus of the split Weil multiplier of this dilation is $|R_\xi|^{n_j}$. The conjugate-place Heisenberg displacement gives the multiplier $\psi(\mathrm{tr}(v'v))$ on the reversed integral box. Before changing the determinant character on the reversed line, write $C_{\mathrm{rev}}(\xi)$ for the scalar of this unnormalized finite Fourier sum. Its squared modulus is $|R_\xi|^{-n_j}$. The rational Hermitian transpose identifies the rational summation spaces. In the finite adelic bases write the displacement as $z = \alpha_f X$. The adjoint change in these bases is

$$-c_0 \frac{\alpha_f}{\overline{\alpha_f}} z^* = \alpha_f(-c_0 X^*).$$

Thus $X \mapsto -c_0 X^*$ preserves the integral matrix box in normalized coordinates, including at the primes of the relative different. The factor $\overline{\alpha_f}/\alpha_f$ links the two line frames in the group argument. The dilation $\bar{t}_\xi / t_\xi$ acts as t_ξ^{-1} on the selected halves and as its inverse conjugate on their partners. Their product is u_ξ. The transport of the normalized theta functional has unit modulus. For the determinant phases put $u = \upsilon|_{U(1)}$ and set

$$H_j(a) = \mu_{j+1}^0 u^{\lceil a/2 \rceil}, \qquad V_j(a) = \mu_{j+1}^0 \xi^{j+1} u^{\lfloor a/2 \rfloor}.$$

The ranks $h_j = r + 2jt$ and h_{j+1} are odd, so $H_j(h_j + 1) = H_j(h_j)$ and $V_j(h_{j+1} + 1) = u V_j(h_{j+1})$. With $k = u_\xi^j$ the determinant factors on the two products are

$$\frac{H_j(h_j)(\det g_{\mathrm{mid}})}{H_j(n_j)(k \det g_{\mathrm{old}})}, \qquad \frac{V_j(n_j)(k u_\xi \det g_{\mathrm{new}})}{(u V_j(h_{j+1}))(\det g_{\mathrm{mid}})}.$$

They are exactly the true-edge factors by the parity identities. Reversing the column replaces its character $H_j(n_j)$ by $H_j(n_j)^{-1} u^{n_j}$. The other character is already u. Thus the ratio to the required line kernel on the new side is

$$\frac{H_j(n_j)^{-1} u^{n_j}}{V_j(n_j)}(k u_\xi) = ((\mu_{j+1}^0)^{-2} \xi^{-(j+1)})(k u_\xi).$$

Denote this value by $m_{j,\xi}$ and put $c_{\mathrm{line}}(\xi) = C_{\mathrm{rev}}(\xi) m_{j,\xi}$. Write $K_{v,p}^-(h)$ for the old line kernel from k to the intermediate point h, and $K_{v',\tau p}^+(h)$ for the reversed kernel from h to $k u_\xi$, with the prescribed characters on both edges. Then

$$K_{v,p}^-(h) = c_{\mathrm{line}}(\xi) \sum_{v' \in R_\xi^{1 \times n_j}} \psi(\mathrm{tr}(v'v))\, K_{v',\tau p}^+(h).$$

The factor $m_{j,\xi}$ has complex absolute value one and is independent of h, v, v' and p. It is included in $c_{\mathrm{line}}(\xi)$. The coefficients of the finite sum in (TP8) are the values of ψ. The permutation of the block and line coordinates preserves the quotient Haar measure. In the split Fourier coordinates the self-dual additive measure gives the integral box volume one. Its pushforward to $R_\xi^{n_j}$ gives each residue class mass $|R_\xi|^{-n_j}$. In the displayed finite pairing the annihilator of the integral box is $t_\xi \widehat{\mathcal{O}}_E$ by self-duality and the unit $-\bar{c}_0$. In particular the finite pairing is perfect: for columns $v, w \in R_\xi^{n_j}$,

$$\sum_{v' \in (R_\xi^{n_j})^\vee} \psi(v'(v - w)) = |R_\xi|^{n_j} \mathbf{1}_{v=w}.$$

Indeed a nonzero coordinate of $v-w$ generates a nonzero ideal on which the primitive character is nontrivial, so the corresponding sum vanishes by translation. Let $F_{v',v} = \psi(v'v)$ in a cyclotomic field containing the additive and residue determinant character values. The identity reads $c(F)^{\mathsf{T}}F = |R_\xi|^{n_j} I$, where c is complex conjugation. At a coefficient embedding ι the two factors are read via $\iota \circ c$ and ι, respectively. For a residue function D, integration of $D(v)\psi(v'v)$ over the integral box gives $|R_\xi|^{-n_j} \sum_v D(v)\psi(v'v)$. The sum in (TP8) uses counting measure. By the displayed orthogonality its unitary normalization for the same measures on both residue spaces is $|R_\xi|^{-n_j/2} \sum_v D(v)\psi(v'v)$. The unitary reversal supplies the reciprocal squared factor $|c_{\mathrm{line}}(\xi)|^2 = |R_\xi|^{-n_j}$ in every residue characteristic. Thus

$$\frac{|c_{\mathrm{line}}(1)|^2}{|c_{\mathrm{line}}(\xi)|^2} = \left(\frac{|R_\xi|}{|R_1|}\right)^{n_j} = \mathrm{Nm}(Q(\xi))^{n_j}.$$

The common old factor cancels before taking coefficient valuations, so the ratio is a unit at each coefficient place of residue characteristic prime to q_ξ.

For the measured diagonal exchange apply the two seesaws on the intermediate groups. On the old side the true-edge datum is the product of its original false block indicator and its line-column indicator. On the new side the factorization is into the false block and the line-covector indicator. The factorizations have the actual line points just computed.

Both seesaws use the original quotient measures. Write $x = (\xi, y', \eta)$ for the entire triple, z for the fixed real point of the intermediate group, and p for a complex polynomial in the archimedean line variables. Denote by $\mathcal{C}_{z,p}(x)$ and $\mathcal{B}_{z,p}(x)$ the old column and new covector periods with this line weight. Let $\widetilde{S}, S$ be the true and false intermediate spaces of rank n_j, and $s_j : U(\widetilde{S}) \to U(S)$ their fixed coordinate reindexing. For each residue tuple let f_+ be the form from the true path and f_- the conjugate-type form from the false path, including its two corner lifts. The term paired against the corresponding old or new array is

$$\int_{[U(\widetilde{S})]} K(g)\, f_+(g)\, f_-\big(s_j(g)(z,1)\big)\, dg.$$

Here $K(g) = K^-_{v,p}(s_j(g))$ in the column period and $K(g) = K^+_{v',p}(s_j(g))$ in the covector period. The line point is fixed at u_ξ^j or u_ξ^{j+1}, respectively. In both integrals the Haar measure is the original quotient measure, and the real translation $(z,1)$ occurs only in the false form. Sum over every residue tuple on the corresponding path, including its line coordinate.

Suppose that the new path period at every real point is a scalar multiple of a function $\Xi(x)$, with the same scalar for all x at that point. The new seesaw implies, for each z, p, that $\mathcal{B}_{z,p}(x) = \beta(z,p)\Xi(x)$ for all x. Here are the details of the passage to real tests. Fix the line polynomial p first. For a polynomial A on the false block, the product Ap, in the reindexed orthogonal blocks, gives the whole-edge kernel whose product factorization is the false block kernel of A times the line kernel of p. The residue point datum on the whole edge is the product of the two point indicators on those blocks. Substituting this identity gives a new whole-edge lift with real datum Ap. Let G_+ be the real true group at $(j+1, j+1)$, and write $\mathcal{P}_k^{\mathrm{new}}(x)$ for the new path period with the true form translated on the right by $k \in G_+$. The partial source form on $\widetilde{S}$ is fixed by the contragredient source-line projector. Apply Section 34.3.2 to this source and the whole true edge. If P_+ is its distinguished harmonic, there is $\alpha_{Ap} \in C_c^\infty(G_+)$ such that $p_{\widetilde{S}}(Ap) = \omega(\alpha_{Ap}) p_{\widetilde{S}} P_+$. Consequently the factored double integral, summed over the new array, is

$$\int_{G_+} \alpha_{Ap}(k) \mathcal{P}_k^{\mathrm{new}}(x)\, dk.$$

Indeed, at the block point the true lift is evaluated at $\operatorname{diag}(h, u_{\xi}^{j+1})(k, 1)$, while the false form is still evaluated at h. Real convolution leaves the finite block and line points fixed. The compactly supported function depends only on the real datum Ap. If $\Xi(x_0) \neq 0$, the proportionality coefficient at each averaging point is the period there at x_0 divided by $\Xi(x_0)$. It is integrable against the smoothing function by the same fixed-triple bound. If $\Xi = 0$, the average vanishes. In either case the integrated expression belongs to $\mathbb{C}\Xi$ for every A.

To pass to the actual real translate of the joint harmonic, regard each summed double integral as a continuous linear functional of its real Fock vector. Fix x, z, p and one residue summand. Write $f_1(g), f_2(h)$ for the forms supplied by the two remaining paths. Successive arrows with the distinguished polynomial have the prescribed cuspidal target type by Section 34.3; at a definite endpoint the quotient at fixed level is compact. The two-variable Schwartz bound for the kernel in that section, followed by the coherent-state estimate of Section 34.3.1, majorizes the absolute integrand by

$$C p_N(v)\, |f_1(g) f_2(h)|\, \mathrm{ht}(g)^B\, \mathrm{ht}(h)^B$$

for some C, B, N. Indeed adjoining the fixed line gives an orthogonal direct sum of lattice norms, and a fixed real translate changes the minimum by bounded factors. The fixed line kernel has polynomial growth by the same kernel estimate at its fixed polynomial, absorbed in B. The bound is uniform on the compact support of the smoothing function. The two forms are rapidly decreasing on their quotients, so the height-weighted product is integrable. There are only finitely many residue summands for fixed x. If $\Xi(x_0) \neq 0$, a family of such integrals $F_v(x)$ is in $\mathbb{C}\Xi$ precisely when

$$\Xi(x_0) F_v(x) - \Xi(x) F_v(x_0) = 0 \qquad \text{for all } x.$$

Each of these equations is continuous in the real vector v. If $\Xi = 0$, use the equations $F_v(x) = 0$. On the false block take v to be its distinguished harmonic translated by z on the intermediate group. Covariance gives the false form at $s_j(g)(z, 1)$, leaving the line weight p at $s_j(g)$. This translated polynomial times a Gaussian has finite moments of all orders: the moved Gaussian has a positive definite real part in its decaying exponent. The Taylor-kernel approximation of Section 34.3.2 therefore converges to v in every p_N. Tensoring with the fixed line polynomial times its Gaussian preserves this convergence, since the Gaussian factors and $(1+|(w_1, w_2)|)^N \leq (1 + |w_1|)^N (1 + |w_2|)^N$. Taking the limit in the continuous equations gives $\mathcal{B}_{z,p} \in \mathbb{C}\Xi$. In particular $\beta(z, p)$ is independent of every component of x.

The line Fourier law gives (TP8). For each fixed new finite covector and each tuple on the unchanged edges, apply (TP9) and the measured partner identity to the false block. The archimedean coefficient k_s is the fixed one of that square and the line reversal $p \mapsto \tau p$ depends on its signed real coordinates only. The measured square denominator $c_{\mathrm{sq}}(\xi) = c_{\mathrm{sq},(j,j)}(\xi)$ and the Fourier scalar $c_{\mathrm{line}}(\xi)$ are independent of the finite covector and the unchanged edges. Summing the resulting identities gives

$$\mathcal{C}_{z,p}(x) = \frac{k_s c_{\mathrm{line}}(\xi)}{c_{\mathrm{sq}}(\xi)} \mathcal{B}_{z,\tau p}(x) = k_s \beta(z, \tau p) \frac{\Xi(x)}{c_s(\xi)}, \qquad c_s(\xi) = \frac{c_{\mathrm{sq}}(\xi)}{c_{\mathrm{line}}(\xi)}.$$

Since $|c_{\mathrm{line}}(\xi)|^2 = |R_\xi|^{-n_j} > 0$ and $|R_\xi|/|R_1| = \mathrm{Nm}(Q(\xi))$, (TP10) gives the full diagonal ratio

$$\frac{|c_{(j,j)}(\xi)|^2}{|c_{(j,j)}(1)|^2} = \mathrm{Nm}(Q(\xi))^{t^2 + n_j}\, T_{jj}(\xi)^2.$$

For the old seesaw, products Ap in the false block and old line variables span all the whole-edge polynomials. Each product has the factorization into the two block point indicators and the two real polynomials on those blocks. This time smooth A on the false edge with the false partial source at the smaller diagonal node. Its target is the intermediate group $G_- = U(S)_\infty$. The same projected edge identity and Fubini give the tested old integral as

$$\int_{G_-} \alpha_A(k)\mathcal{C}_{k,p}(x)\,dk, \qquad \alpha_A \in C_c^\infty(G_-).$$

Here the translated false form occurs at $s_j(g)(k,1)$, and the old line kernel is still evaluated at $s_j(g)$ with its finite point u_ξ^j. The averaging preserves the common proportionality to $\Xi(x)/c_s(\xi)$. The same continuous equations, now on the old side and with this function of x, extend the assertion to the actual translated whole-edge harmonic. This proves that at every fixed real point the old path period is proportional to $\Xi(x)/c_s(\xi)$ by one scalar for all triples. The off-diagonal turn has this property directly by its tested square identity.

43.5 Composition and terminal measure

In this subsection, we compose the transports of Section 43.4 and retain the original terminal measure.

For a pair of paths ending on the diagonal at j, take the cells in the j-by-j diagonal square, together with the cells to the left of each down step in both halves, reflecting the true half. A down step at horizontal coordinate a across sequence b contributes the cells (h,b), $0 \le h < a$, in its half's recorded coordinates. At an off-diagonal turn the down step moves one column to the right, so the union contains one additional cell.

At a diagonal turn the two strips of length j next to the j-by-j square are already included, and the enlarged diagonal square adds just (j,j). Initially the union is empty. A half with a down step before a right step contains an adjacent down–right pair and therefore admits the turn. If neither half has one, all right steps precede the down steps. If $j < m$, both last steps are down steps and give a diagonal turn. If $j = m$, the last path has been reached. Thus an arbitrary successive choice of available corners crosses each square exactly once, ending at the nodes (z,j,m) on each half.

Fix one such order of the m^2 turns for all twists. Start from P_1 at each twist and choose the arrays by (TP8)–(TP9) in that order, before choosing either terminal form. An off-diagonal turn only multiplies a selected entry by a character value. The diagonal turn at j sums over $|R_\xi|^{n_j}$ columns. Thus the number of roots of unity in each terminal entry is at most

$$E_\xi^{2m} \prod_{j=0}^{m-1} |R_\xi|^{n_j} \le E_\xi^{2m} V_\xi^{m^2}.$$

The backward implications compose by multiplying the $c_s(\xi)$. If $c_\xi = \prod_s c_s(\xi)$, then $c_\xi \neq 0$ and

$$\left|\frac{c_\xi}{c_1}\right|^2 = \mathrm{Nm}(Q(\xi))^{m^2t^2+m^2t+mr} \prod_{i,j<m} T_{ij}(\xi)^2. \tag{TP11}$$

Here the sum of the line dimensions is $\sum_{j=0}^{m-1}(r+(2j+1)t) = mr + m^2t$. At the trivial twist all critical ratios are equal to one. This follows directly from the nonzero Hecke denominators and the definition of the normalized ratios.

For the last path let $I_{z,j}$ be the original finite level index at node (z, j, m). Every such index other than $I_{0,0}$ is the general linear group order at $\mathfrak{b}$, because the level there is the kernel of $\mathrm{GL}(R_b) \times \mathrm{GL}(R_Q) \to \mathrm{GL}(R_b)$ in the corresponding rank. At $(0, 0, m)$ the level is the identity over R_ξ and the rank is $r + mt$, so $I_{0,0} = |\mathrm{GL}_{r+mt}(R_\xi)|$. The last nodes are the original definite models.

On each edge with fixed polynomial degree D_e, the scalar analytic image of a source algebraic form is the analytic realization of its jet lift times

$$\frac{\beta_e}{s_e I_{z,j}}. \tag{TP12}$$

In detail, the original jet matrix applied to a source vector is paired with the highest target covector. At this last-path edge the jet is taken at $s_{\mathrm{jet}} = p_F a_d/\omega_0$, as in Section 44.3.1. For each polynomial Q of the edge's homogeneous degree D_e,

$$Q(\partial_z)\Theta(s_{\mathrm{jet}} z)\big|_{z=0} = s_{\mathrm{jet}}^{D_e}\, Q(\partial_z)\Theta(z)\big|_{z=0}.$$

Thus conversion to the normalized analytic joint harmonic gives $a_e s_{\mathrm{jet}}^{-D_e}$, with a_e the fixed harmonic scalar.

The original compact real integral, using the unit highest source vector, gives $1/\dim V_{\mathrm{src}}$ by Schur orthogonality. The finite class integral of the algebraic lift gives the source level mass one, whereas in the analytic measure its mass is $I_{z,j}^{-1}$. Finally the algebraic datum multiplies the same matrix indicator by $s_e = s_{\mathrm{vac}}^{\dim(\mathrm{Hom})}$. These are precisely the original source, target and polynomial conventions in the definite jet lift.

Thus $\beta_e = a_e s_{\mathrm{jet}}^{-D_e}/\dim V_{\mathrm{src}}$ and s_e are fixed independently of the residue point and the twist. Iterating (TP12) gives, after j edges, the algebraic chain's analytic realization times $b_{z,j} I_{z,j}/I_{z,0}$, with

$$b_{z,0} = 1, \qquad b_{z,j+1} = \frac{\beta_e b_{z,j}}{s_e I_{z,j+1}}.$$

The $b_{z,j}$ are fixed, because the last index in this recursion has $j + 1 > 0$. Here the integrals for an individual tuple are Haar integrals, with each node level assigned mass one. A single point-indicator tuple need only be fixed by a finer level. We take the finite array sum before replacing these integrals by sums over the displayed levels.

At the final diagonal the true form's finite point is the block point $\mathrm{diag}(h, u_\xi^m)$ in its given reindexing. The extra line has sign $+1$ since m is even. The two nodes here have ranks $N = r + 2mt$ and $N + 1$. We construct the bilinear branching pairing for their own weights. Fix one infinite place, suppress it from the notation and write $w_i = \kappa_\tau + i\Delta + \Delta/2$ and $s_i = 2\lambda_{k-1+i} = \Delta - 2w_{i+1}$ for $0 \le i \le m$, extending the same arithmetic progression for its final term. The doubled parameters to interlace are the false parameter before duality and the negative of the true grid parameter.

For an integer threshold b', write $\phi(y) = \mathbf{1}_{b'<y}$ and $g_n(y) = \max(0, \min(n, \lfloor (y - b')/2 \rfloor))$. A descending block $d, d-2, \ldots, d-2(n-1)$ contributes $g_n(d+1)$ entries above the threshold. With i pairs of blocks included, the counts on the false and true sides are respectively

$$C_i^- = \sum_{a<k-1+i} \phi(2\lambda_a) + \sum_{j=1}^{i} \{g_t(2w_j + t) + g_t(-2w_j + t)\},$$

$$C_i^+ = \sum_{a<k+i} \phi(-2\lambda_a - 1) + \sum_{j=1}^{i} \{g_{t+1}(-2w_j + t + 1) + g_{t-1}(2w_j + t - 1)\}.$$

These are the base entries and the two progressions inserted at each step of the diagonal grid, with lengths t, t on the false side and $t+1, t-1$ on the true side. We claim that

$$\phi(s_i) \le C_i^+ - C_i^- \le \phi(-s_i - 1).$$

For $i = 0$, the central false progression is $k-2, k-4, \dots, 2-k$. The central true progression consists of the same entries diminished by one, together with $-s_0-1$ above it. Their counts differ by at most one, and all central entries exceed s_0, which gives both endpoints. For the induction use the following block count. The difference between the counts of the centered progressions of lengths $n+1, n$ at center c is $g_{n+1}(c+n+1) - g_n(c+n)$. Their entries alternate, so this is zero or one, equals one below $c-n$, and vanishes at or above $c+n$. At $c = -2w_{i+1}, n = t$, this difference lies between $\phi(s_{i+1})$ and $\phi(s_i)$. At $c = 2w_{i+1}, n = t-1$ it lies between $\phi(-s_i - 1)$ and $\phi(-s_{i+1} - 1)$. Indeed $w_{i+2} - w_{i+1} = \Delta$ and $t + 1 \le \Delta/2$. In passing from i to $i+1$, add the first difference, subtract the second, and add $\phi(-s_{i+1} - 1) - \phi(s_i)$ from the two new base entries. This proves the claim. At $i = m$, since $r = k - 1 + m$, the counts of the actual terminal parameters differ by zero or one at every threshold.

Let the descending highest weights of the larger module and the smaller primal module be $\Lambda_1, \dots, \Lambda_{N+1}$ and $\nu_1, \dots, \nu_N$, respectively. The doubled parameters L_i^+ and L_i^- of the two lists have opposite parity. The base entries of the true list are odd and those of the false list are even. Their inserted blocks have those same parities since t is odd. Thus the count inequality gives

$$L_i^+ \ge L_i^- + 1, \qquad L_i^- \ge L_{i+1}^+ + 1.$$

Subtracting the two rank shifts gives $\Lambda_i = (L_i^+ - N - 2 + 2i)/2$ and $\nu_i = (L_i^- - N - 1 + 2i)/2$, hence $\Lambda_i \ge \nu_i \ge \Lambda_{i+1}$. Put $e = \Lambda_{N+1}$ and $e_0 = \nu_N$. Of the $\Lambda_i - \Lambda_{i+1}$ upper columns at height i, retain $\nu_i - \Lambda_{i+1}$ and contract $\Lambda_i - \nu_i$ against the last unit vector. For $i < N$, the retained columns at i and contracted columns at $i+1$ give exactly $\nu_i - \nu_{i+1}$ lower columns. At $i = N$, the retained full columns contribute $e_0 - e$ to the lower determinant power. The extra line acts with exponent

$$L_\tau = e + \sum_{i=1}^{N} (\Lambda_i - \nu_i).$$

This also follows by subtracting the two central weights. The sum of the highest weights at each rank is half the sum of its doubled entries. The false primal list therefore has sum of highest weights $\sum_{a<r} \lambda_a$; the true list has sum $-\sum_{a<r+1} \lambda_a - (r+1)/2 - 2\sum_{i=1}^{m} w_i$. The central head has sum zero and $\lambda_{k-1+i} = \Delta/2 - w_{i+1}$. Their difference in the order true minus false gives

$$L_\tau = w_{m+1} - (2m+1)\Delta/2 - (r+1)/2 = \kappa_\tau + \Delta - (r+1)/2.$$

Let E_+, E_- be the integral inclusions of the upper and lower Weyl lattices into their exterior tensor models. Let C be the tensor of the retained-column projections and the last-covector contractions with the above column counts. Reorder columns and absorb the retained full lower columns into the lower determinant line. Laplace expansion gives one last-line factor per contraction. With $G_- = E_-^{\mathsf{t}} E_-$ and c_- the Fock square of the full lower highest polynomial, take

$$b_{\mathrm{grid}} = c_- G_-^{-1} E_-^{\mathsf{t}} C E_+.$$

The Fock clearing calculation of Section 7.5, applied to the full lower diagram, gives $c_- G_-^{-1}$ as the Fock matrix on the integral dual polynomials. Its entries are integers. The other factors in

this formula are integral, and the Weyl projection and the column operations commute with the smaller group. The map is nonzero. Pairing the contracted evaluation tensor of a generic upper flag with the unit lower highest tensor gives a product of nonzero minors. A retained column uses its leading rows, and a contracted column uses its shortened leading rows and the last row. The number of lower boxes after removing determinant powers is at most the upper count H summed over all places, since $\nu_i - e_0 \le \Lambda_i - e$. The minor and factorial bounds give $c_- \le H^{3H}$, with value one for $H = 0$. Before multiplication by c_- the map is the composition of the exterior operations and an orthogonal Weyl projection. Thus $\|b_{\mathrm{grid}}\| \le H^{3H}$, and $|L_\tau| \le H + |e|$.

Compose b_{grid} with the integral invariant duality of the false node to obtain the integer bilinear matrix $B_{\mathrm{grid}} = J_{\mathrm{diag}} b_{\mathrm{grid}}$. Here complementary coordinate wedges, with the volume sign, identify the primal representation isometrically with the conjugate of its dual; the dual tensor Gram matrix composed with this signed identification gives J_{diag}, as in Section 44.3.4 at this rank. Cauchy–Schwarz bounds the resulting bilinear pairing by H^{3H} times the product of the two Weyl norms. Let K be the compact real false group, with probability Haar measure, and $\iota(k) = \mathrm{diag}(k, 1)$ in the prescribed reindexing. Write ρ_- for its action on the primal false module and ρ_+ for the true action. If h_0 is the unit highest vector of the primal false type and ℓ_g is the true highest covector after translation by g^{-1}, the real integral inserts the linear map

$$A_g = \int_K \rho_-(k) h_0 \left(\ell_g \circ \rho_+(\iota(k)^{-1}) \right) dk.$$

The false dual coefficient evaluates on $\rho_-(k)h_0$. Left invariance gives $\rho_-(k_0) A_g = A_g \rho_+(\iota(k_0))$ for every $k_0 \in K$. The branching line has multiplicity one. After removing the common determinant power, the Schur character formula on the union of an N-variable alphabet and the last variable has one contribution for each horizontal strip [46, Ch. I, (5.8)–(5.9), pp. 71–72]. The strip condition is the interlacing just proved, and the skew Schur polynomial in one variable is a single monomial. Thus the average is $a(g) b_{\mathrm{grid}}$. It is independent of the twist and the finite point and can be zero.

At each finite point this leaves the bilinear pairing $F_-(h)^{\mathsf{t}} B_{\mathrm{grid}} F_+(\mathrm{diag}(h, u_\xi^m))$. The compact finite measure converts to the class integral of level mass one with coefficient $I_{0,m}^{-1}$. For an individual locally constant summand, take an open subgroup $J' \subset J$ of the displayed level that fixes it. A J'-coset has mass $[J : J']^{-1}$ in the Haar normalization giving J mass one, and mass $[K_{\mathrm{std}} : J']^{-1}$ in the original finite measure. Dividing the latter mass by the former gives $[K_{\mathrm{std}} : J]^{-1} = I_{0,m}^{-1}$, also when the individual summand has smaller stabilizer than J. After multiplying by the array, invariance under right translations at the internal nodes follows by reindexing their residue matrices.

The determinant phases cancel by (TP9). More explicitly, write J for the product of the levels of the $2m + 1$ independently integrated class variables, counting the common terminal variable once. Its two actions at that variable are u and $\mathrm{diag}(u, 1)$, the latter commuting with $\mathrm{diag}(1, u_\xi^m)$. Write F for the sum of all the point-indicator integrands with their array coefficients. At a change of the common terminal representative by k,

$$\mathrm{diag}(hk, u_\xi^m) = \mathrm{diag}(h, u_\xi^m)\, \mathrm{diag}(k, 1).$$

Hence the extra-line coordinate stays at its original value. For fixed outer tests the remaining kernel multipliers are the node characters in the array covariance, and reindexing its residue tuples gives $F(hk) = F(h)$ for $k \in J$. Choose an open $J' \subset J$ fixing the individual summands and let $\mathcal{C}(J')$, $\mathcal{C}(J)$ be the respective sets of class tuples. Each fiber over $\mathcal{C}(J)$ has cardinality

$[J : J']$: every component of J lies in the old principal subgroup at the split place above p_0, so Lemma 6.1 applies to each definite group. With J of mass one, we therefore obtain

$$\frac{1}{[J : J']} \sum_{h \in \mathcal{C}(J')} F(h) = \sum_{h \in \mathcal{C}(J)} F(h).$$

In particular no inverse level index occurs in the final class sum, including at coefficient primes that divide $[J : J']$. The two outer levels are respected: the false outer deep principal level is contained in the original $K(\pi_M Q(\xi))$, and the true outer level is $K(\mathfrak{b})$.

There is one common terminal group variable. Its measure gives one factor $I_{0,m}^{-1}$. With both chain conversions the scalar is

$$I_{0,m}^{-1} \prod_{z=0}^{1} \frac{b_{z,m} I_{z,m}}{I_{z,0}} = \frac{b_{0,m} b_{1,m}}{I_{0,0}} \frac{I_{1,m}}{I_{1,0}}.$$

Equivalently, before cancelling indices the scalar is

$$\frac{1}{I_{0,m}} \prod_{z=0}^{1} \prod_{j=0}^{m-1} \frac{\beta_{e(z,j)}}{s_{e(z,j)} I_{z,j}},$$

where $e(z, j)$ starts at (z, j, m). The common diagonal variable occurs once. Of these $2m + 1$ measure factors, only $I_{0,0}$ depends on the twist. By the same calculation as (TP6), $I_{0,0}$ differs from $I_\xi = |\mathrm{GL}_{r+mt}(R_0(\xi))|$ by the fixed ratio $I_\xi / I_{0,0} = (\mathrm{Nm}(\pi_M)/\mathrm{Nm}(\mathfrak{b}))^{(r+mt)^2}$. Consequently the last-period function at any real g is a scalar depending only on g and the datum times $\Xi_\xi(y', \eta)/I_\xi$, where Ξ_ξ is the final array paired against those algebraic class sums. Apply the composed backward implication to that function and evaluate at ω in (TP7). With one scalar A_* independent of the twist and both forms, we obtain the following identity.

Proposition 43.1 (The measured terminal period)**.** *Let ξ be a twist for the full chain as in Section 2.6, let y' be a dual test at the original full principal terminal level, and let $\eta \in \mathcal{H}_m$. With the original paths, real points and quotient measures, one scalar A_*, independent of ξ, y', η, gives*

$$I_\xi \int_{[Y_m]} y'(h)(T_\xi r_0 \eta)(h)\, dh = A_* \, \Xi_\xi(y', \eta)/c_\xi. \tag{TP13}$$

Here $c_\xi \neq 0$ and I_ξ is the index of the original false terminal test level, computed by surjectivity of split reduction.

44 Chain determinants and completion of the lattice

In this section, we compute the determinant by composing the individual lifts, using the terminal periods of Section 43. The same calculation gives the auxiliary vectors and the clearing scalar needed in Section 23.

44.1 Determinants of the composition

In this subsection, we compute the determinant of the weighted composition by reducing it to the determinants of the individual lifts. This gives the ratio whose height is estimated in Section 44.2.

Fix q_0, d, ϵ as in Section 4, take a sufficiently large integer x, and fix the resulting admissible datum. Write $\kappa = (\ell, \iota)$ for a coefficient embedding. Let W be the original full source of level $K(\pi_M)$, and $V \subset W$ its projected space of dimension D. Write $T_\xi = L_\xi$ for the full weighted chain of Section 17.3 on W, and $\Theta_\xi = T_\xi|_V$ for its restriction to the projected space at the allowed m-fold twist ξ. The source weights below are $w_n = A_{\xi,n}$. The terminal algebraic conversion is that of the same chain. The target inner product is the terminal class pairing. Write

$$Z_\xi = \det(\Theta_\xi^* \Theta_\xi), \qquad Z_j(\xi^j) = \det(u_{j,\xi^j}^* u_{j,\xi^j}),$$

where u_{j,ξ^j} is the j-th single lift on the projected space of the specified space W_j, also with its target class pairing. The same source metrics are used at all twists. Set

$$B = \log b, \qquad J_* = \frac{faB^3}{mk}, \qquad L = faB^4, \qquad A = mk\, d_{\mathrm{dir}}.$$

Here Section 38 gives

$$d_{\mathrm{dir}} = \sum_\tau \big(\log(\kappa_\tau!) + \kappa_\tau \log(M/\Omega) - S(\kappa_\tau, \tau)\big).$$

Thus $A = mk\, d_{\mathrm{dir}}$ is the same quantity as in Section 5. Each height bound uses one multiplier for all coefficient embeddings.

Finite factors in the chain. For a twist ξ put $q_\xi = \mathrm{N}Q(\xi)$ and let w_n be the order of $\mathrm{GL}_{r+nt}(\mathcal{O}_E/(\pi_M Q(\xi)))$. Let $a_{i,n}$ be the horizontal arrow and $d_{i,n}$ the downward arrow from column $n+1$ of sequence i to column n of sequence $i+1$. Denote the composite of the first n arrows of sequence i, weighted by $w_0, \dots, w_{n-1}$, by $X_{i,n}$. Thus $X_{i,0} = 1$ and $X_{i,n+1} = w_n a_{i,n} X_{i,n}$ for $i + n + 1 \le m$. Let j_i be the isometric base identification of sequence i.

Injectivity before dividing by the archimedean scalar. The first-source calculation in Section 34.2 applies to every $y \in W$. In fact y is invariant under the point stabilizer at $\pi_M Q(\xi)$, which is contained in $K(\pi_M)$, and is invariant under the integral finite group off the support. With $A_i = a_{i,0} j_i$, Proposition 34.1 gives

$$w_0 \langle A_i x, A_i y\rangle = C_i B_\xi^{-1} \langle x, \mathcal{V}_i y\rangle \qquad (x, y \in W).$$

Put $R_0 = R_0(\xi) = \mathcal{O}_E/(\pi_M Q(\xi))$ and $B_\xi = |R_0|^{r(r+t)}$. The constant C_i is nonzero, and $\mathcal{V}_i$ is the original convergent spherical operator. Its off-identity operator norm is less than one, by the neighbor count in that section. On the projected subspace it is R_{i+1}. In particular the first arrow is injective on W. Indeed, if $A_i y = 0$, taking $x = y$ gives $\langle y, \mathcal{V}_i y\rangle = 0$, whereas $\Re\langle y, \mathcal{V}_i y\rangle \ge (1 - \|\mathcal{V}_i - 1\|)\|y\|^2 > 0$ for $y \ne 0$.

For the descents at the original level pick the first datum by adjointness on the two isometric copies, and propagate it by the finite shear formula (104) on R_0. The square gives the equality

$$w_n a_{i+1,n} d_{i,n} = u_{i,n} w_{n+2} d_{i,n+1} a_{i,n+1}, \qquad u_{i,n} = k_{i,n}/c_{i,n}, \quad c_{i,n} \ne 0. \tag{GC1}$$

Here the measured partner scalar $c_{i,n}$ is nonzero from the start. The archimedean pairing factor $k_{i,n}$ is allowed to be zero in (GC1). At the original triangle modulus $\pi_M Q(\xi)$, the first-column calculation in Section 34.2.3 gives $R(u)^{-1} a_{i,0}^* a_{i,0} R(u) = a_{i,0}^* a_{i,0}$ for every $u \in K_0$. The isometry j_i intertwines these right translations. Hence $A_i^* A_i$ preserves the J_M-fixed space W and commutes there with $p = \rho(e_0)$, a finite linear combination of such translations. The first adjoint identity gives

$$B_i = j_{i+1}^{-1} d_{i,0} a_{i,0} j_i = A_i^* A_i \qquad \text{on } W.$$

The spherical series preserves W as well. Its complete double cosets are bi-invariant at the unprescribed places and act by the identity factor at the old and twist places. Testing the first-lift identity against every $x \in W$ therefore gives

$$B_i = \frac{C_i}{w_0 B_\xi} \mathcal{V}_i|_W, \qquad B_i|_V = \frac{C_i}{w_0 B_\xi} R_{i+1}, \qquad V = pW.$$

In particular B_i is injective on W. The adjoint here is represented by the original transposed kernel on the indicated types, with the original Haar.

If $X_{i,n}$ is the first n weighted arrows, iterating (GC1) from $n = 0$ gives

$$X_{i+1,n} d_{i,0} a_{i,0} = \frac{w_{n+1}}{w_0 w_1} \left(\prod_{s<n} u_{i,s} \right) d_{i,n} X_{i,n+1}. \tag{GC2}$$

For $n = 0$ this is $d_{i,0} a_{i,0} = w_0^{-1} d_{i,0} X_{i,1}$. At the induction step multiply the identity on the left by $w_n a_{i+1,n}$. Use (GC1) and the definition of the weighted composite to obtain

$$\begin{aligned} w_n a_{i+1,n} d_{i,n} &= u_{i,n} w_{n+2} d_{i,n+1} a_{i,n+1}, \\ w_{n+1} a_{i,n+1} X_{i,n+1} &= X_{i,n+2}. \end{aligned}$$

In particular the implication $X_{i,n+1} j_i y = 0 \Rightarrow X_{i+1,n} j_{i+1} B_i y = 0$ involves no inversion of $k_{i,n}$. Induct simultaneously over all admissible i on the chain length. Injectivity at length n gives $B_i y = 0$ from the right-hand vanishing, and the injectivity of B_i gives $y = 0$. If $W \neq 0$, the left side of (GC2) is nonzero on $j_i y$ for every $y \neq 0$. Taking $n = s + 1$ shows $u_{i,s} \neq 0$ for every square. We may therefore use $\beta'_{i,n} = \overline{c_{i,n}/k_{i,n}}$ in the Gram computation. When $D = 0$, the empty determinants give the ratio formula directly, since the individual projected spaces have the same dimension. In the calculation of the ratio below we can thus assume $D > 0$, so $W \neq 0$.

The terminal level of the chain image. For an integral terminal element at level places, let g be its reduction, upper block triangular of sizes $r, t, \dots, t$, and g_j its upper left corner of rank $r + jt$. For the coordinate inclusion I_j from rank $r + jt$ to rank $r + (j + 1)t$, the following identities hold over R_0:

$$g_{j+1} I_j = I_j g_j, \qquad \det(g_{j+1}) / \det(g_j) = \det(g^{(j+1)}),$$

where $g^{(j+1)}$ denotes the last diagonal block of g_{j+1}. After lifting each corner integrally by split reduction, the first identity holds modulo $\pi_M Q(\xi)$, and the second determines the tame determinant character on the lifted units. The simultaneous Schrödinger action on the original point indicator therefore moves the terminal translation through arrow j, multiplying by the $j + 1$st lift character at $\det(g^{(j+1)})$.

This is the determinant ratio of the target and source actions, with the same tame character at the two ranks. Its other local factors are trivial on integral units. Iterate this equality. At old places the terminal element is the identity after reduction, so g_0 fixes the base form. At each new place g_0 is integral, so it again fixes the base form. The resulting product of the diagonal-block characters is the original terminal level character. The passage from scalar forms to the algebraic compact model intertwines these translations. Thus the terminal algebraic map with this character is the very same injective weighted chain.

Write $I_{\mathrm{par}}(\xi)$ for the index of this terminal level. Its factor at a twist place is the index of the block upper triangular group. The compact real Schur average and the finite quotient volume

convert the Petersson norm of the forward chain to its algebraic class norm by $d_Y I_{\mathrm{par}}(\xi)$. This differs from the dual-delta test conversion $d_Y I_\xi$, where $I_\xi = |\mathrm{GL}_{r+mt}(R_0)|$. For that period the dual test has the full principal level at the twist places. More explicitly, write $K_{\mathrm{pr}} \subset K_{\mathrm{par}}$ for the two terminal levels and $G_{\mathrm{pr}}(F), G_{\mathrm{par}}(F)$ for the two class Gram matrices of a tuple $F = (T_\xi v_i)_i$. An old principal factor makes the rational stabilizer trivial, as proved in Section 23. Hence each K_{par}-class splits into exactly $[K_{\mathrm{par}} : K_{\mathrm{pr}}] = I_\xi / I_{\mathrm{par}}(\xi)$ principal classes. For $k \in K_{\mathrm{par}}$, every $F_i(hk)$ differs from $F_i(h)$ by the same unitary level character. Its two factors cancel in each Hermitian matrix entry, giving

$$G_{\mathrm{pr}}(F) = \frac{I_\xi}{I_{\mathrm{par}}(\xi)} G_{\mathrm{par}}(F).$$

The determinant Z_ξ uses G_{par} in an orthonormal source basis. The principal-level delta identity (113) tests a single value $F_i(h)$, so its integrality assertion applies directly at the parabolic class representatives, including at primes dividing the index.

The adjoint square and the Gram scalar. The tested adjoint of the upper point arrow is the arrow with conjugate kernel and opposite determinant characters, by Fubini on the two original quotients. Use the original reversed local kernels with Fourier datum $\mathcal{F}(\bar{p})$, and let $\mathcal{R}_n$ be right translation by the inverse of the finite central reversal scalar on rank $r + nt$. In the notation where D_0, D_1 are the two descents and B_0', B_1' their reversed lifts, the opposite-kernel identities in Section 36.2 read

$$\langle D_0 x', y\rangle = s_0 \langle x', B_0' \mathcal{R}_n y\rangle,$$
$$\langle x, B_1' \mathcal{R}_{n+1} z\rangle = s_1 \langle D_1 x, z\rangle.$$

They are tested identities and the real polynomial replacement constants in s_0, s_1 may vanish.

Put $N = r + nt$. The reversed finite square on R_0 gives $a_{\mathrm{up}} B_0' = \gamma B_1' a_{\mathrm{low}}$. Its integrated nodes $(i+1, n+1)$ and $(i, n+1)$ both have rank $N + t$. Thus Proposition 36.1 applies with defect zero. The defining products give $M_\nu(0) = 1$, so $|c_\nu|^2 = R_0'/D_{\mathrm{final}} > 0$ for this square, with the same ratio for every admissible tame modulus and character. The remaining finite factor in γ is $(\mathcal{G}|R_0|^{Nt})^{-1}$. For the lower lift let a, b be its splitting characters in the order $K_{S,T}^{a,b}$, and put $z = s_{\pi_M Q(\xi)}$. The source and target ranks are $N, N + t$. Simultaneous translation by the finite central scalar z fixes the Hom variable and the original residue indicator. The determinant characters give

$$K(gz, hz) = \eta K(g, h), \qquad \eta = a_{\mathrm{f}}(z^{N+t}) b_{\mathrm{f}}(z^N)^{-1}, \qquad |\eta| = 1.$$

Centrality preserves both right levels. Changing the source variable to $g' = gz^{-1}$ in the original quotient Haar gives $a_{\mathrm{low}} \mathcal{R}_n = \eta \mathcal{R}_{n+1} a_{\mathrm{low}}$, where both translations are by z^{-1}. The deeper line period of Section 43.4 uses its own generator for $\mathfrak{b}Q(\xi)$. For $x' = a_{\mathrm{up}}^* x$ this gives

$$\langle D_0 a_{\mathrm{up}}^* x, y\rangle = s_0 \gamma \eta s_1 \langle D_1 x, a_{\mathrm{low}} y\rangle. \tag{GC3}$$

After removing the fixed real factors, the squared moduli of s_0, γ, s_1 have powers of $|R_0|$

$$-N(N+t), \qquad -t(t+2N), \qquad (N+t)(N+2t).$$

The first and last come from (MR). For the middle factor the rank defect is zero and the determinant block has size $b = t$. The inverse square modulus $|\mathcal{G}|^{-2}|R_0|^{-2Nt} = |R_0|^{-t(t+2N)}$ comes from Section 36.2.2.

Indeed,

$$-N(N+t)-t(t+2N)+(N+t)(N+2t)=t^2.$$

The adjoint square has no critical ratio. Its fixed real factor may vanish during the iteration. The conjugate kernel represents the adjoint functional tested on the original cusp types.

In the forward square the integrated ranks differ by $2t$. In Proposition 36.1 the modulus factor in $M_\nu(t)$ is $\prod_{w\in S} q_w^{t^2}$; at the trivial twist only the old primes occur. The quotient of the remaining L-value factors is $Q_{i+1,i+n+2}(\xi)^2$, where $Q_{ij}(\xi)$ is the product of the t positive critical ratios for the quotient character of lifts i,j at ξ^{j-i}. Thus $|\beta'_{i,n}| = |\beta'_{i,n}(1)|\, q_\xi^{t^2/2} Q_{i+1,i+n+2}(\xi)$ when $W\neq 0$. Put $\beta_{i,n}=\prod_{s<n}\beta'_{i,s}$ and $\epsilon'_{i,n}=s_0\gamma\eta s_1$ as in (GC3).

The corresponding adjoint law, starting with the first-column adjointness, is

$$\langle v, a_{i,n}X_{i,n}j_i y\rangle = \epsilon_{i,n}\langle d_{i,n}v, X_{i+1,n}j_{i+1}y\rangle, \qquad \epsilon_{i,n}=\prod_{s<n}\epsilon'_{i,s}.$$

At the step from n to $n+1$, moving $a_{i,n+1}$ to the first slot and applying the preceding length-n law gives

$$w_n\epsilon_{i,n}\langle d_{i,n}a^*_{i,n+1}v, X_{i+1,n}j_{i+1}y\rangle.$$

By (GC3) this equals $\epsilon_{i,n}\epsilon'_{i,n}\langle d_{i,n+1}v, X_{i+1,n+1}j_{i+1}y\rangle$, since $w_n a_{i+1,n}X_{i+1,n} = X_{i+1,n+1}$. Let $\mathcal{B}_{i,n}(x,y)=\langle X_{i,n}j_i x, X_{i,n}j_i y\rangle$. Take $v=X_{i,n+1}j_i x$ in the adjoint law, retaining the real weight w_n in the second argument. By (GC2),

$$d_{i,n}v=\frac{w_0w_1}{w_{n+1}}\overline{\beta_{i,n}}\,X_{i+1,n}j_{i+1}B_i x.$$

The first argument of the inner product is conjugate-linear, hence

$$\mathcal{B}_{i,n+1}(x,y)=\epsilon_{i,n}\beta_{i,n}\frac{w_0w_1w_n}{w_{n+1}}\,\mathcal{B}_{i+1,n}(B_ix,y).$$

For $W\neq 0$, the left side is positive at $x=y\neq 0$; in particular the scalar on the right is nonzero whenever it is used.

Carry out the recursion with first argument in W and second argument in V. All B_i preserve W, and all R_j preserve V. For $x\in W$, $v\in V$, self-adjointness and the first-lift formula give

$$w_0\langle B_ix,v\rangle = w_0\langle A_ix, A_iv\rangle = C_iB_\xi^{-1}\langle x, R_{i+1}v\rangle.$$

Consequently the terminal Petersson Gram form is

$$g_\xi\frac{w_0w_1^m}{w_mB_\xi^m}\langle x, R_1\cdots R_m y\rangle.$$

Indeed each substitution of the first-lift formula contributes $C_i/(w_0B_\xi)$ and left-multiplies the remaining operator by R_{i+1}. The residual weight in the step at n is w_1w_n/w_{n+1}, and its product over $0\le n<m$ is $w_0w_1^m/w_m$. At the turn from sequence i the squares are indexed by $0\le s\le m-i-2$. The map $(i,s)\mapsto(a,b)=(i+1,i+s+2)$ lists each pair $1\le a<b\le m$ once, with twist power $b-a=s+1$. For each such square, positivity of the recursion at lengths $s+1, s+2$ gives $\epsilon_{i,s}\epsilon_{i,s+1}\neq 0$, hence $\epsilon'_{i,s}\neq 0$, at both twists. The three squared-modulus powers in (GC3) therefore give $|\epsilon'_{i,s}(\xi)/\epsilon'_{i,s}(1)| = q_\xi^{t^2/2}$, using $|R_0(\xi)| = |R_0(1)|q_\xi$ by the Chinese

remainder theorem. Multiplication by the corresponding ratio for $\beta'_{i,s}$ gives $q_\xi^{t^2} Q_{i+1,i+s+2}(\xi)$. Thus

$$\left|\frac{g_\xi}{g_1}\right| = q_\xi^{t^2\binom{m}{2}} \prod_{1\le i<j\le m} Q_{ij}(\xi).$$

The Rallis operators act in the indicated order. The squared norm at a nonzero $y \in V$ is positive by injectivity. Thus the product of archimedean factors in the final Gram form and its untwisted value are nonzero on $V \neq 0$.

At $w \in Q(\xi)$ put $G_n = \mathrm{GL}_{r+nt}(k_w)$, $g_n = |G_n|$ and $g'_t = |\mathrm{GL}_t(k_w)|$. With $P_n \subseteq G_n$ the block upper triangular subgroup of sizes $r, t, \dots, t$, its unipotent radical has $nrt + \binom{n}{2}t^2$ free entries. Hence

$$|P_n| = g_0 (g'_t)^n q_w^{nrt+\binom{n}{2}t^2}, \qquad \delta_w = [G_1 : P_1] g_0 q_w^{rt} = g_1/g'_t.$$

It follows that

$$[G_m : P_m] = q_w^{-mrt-\binom{m}{2}t^2} \delta_w^m \frac{g_m}{g_0 g_1^m}.$$

At the fixed chosen places the terminal index is the full general linear group order.

The Petersson-to-class factor is $d_Y I_{\mathrm{par}}(\xi)$, with d_Y fixed. Put $B_w = q_w^{r(r+t)}$ and $F_w = \prod_{t<h\le t+r}(1-q_w^{-h})$. At a new place the node weights, terminal index, square factors and matrix counts give

$$\frac{g_0 g_1^m}{g_m}[G_m : P_m] q_w^{t^2\binom{m}{2}} B_w^{-m} = q_w^{-mrt}(\delta_w/B_w)^m = q_w^{-mrt}(q_w^{rt}F_w)^m.$$

Indeed $\delta_w = g_1/g'_t$ and $|\mathrm{GL}_a(k_w)| = q_w^{a^2}\prod_{h=1}^a(1-q_w^{-h})$. At an old place the pointwise fixer of the inclusion has index equal to the number of full-rank matrices, by transitivity.

The sharp mass has the additional inverse source index g_0^{-1} (with $g_0 = |\mathrm{GL}_r(k_w)|$ at that place), so the fixed old factor for each single lift is $g_0^{-1}\prod_{t<h\le t+r}(1-q_w^{-h})$. With $c_Q(\xi) = \prod_{w\in Q(\xi)} q_w^{rt}F_w$, the Gram operator of the chain is

$$C_\xi c_Q(\xi)^m R_1 \cdots R_m, \qquad |C_\xi| = C q_\xi^{-mrt} \prod_{1\le i<j\le m} Q_{ij}(\xi), \qquad C > 0.$$

The single-lift identities for this comparison are the Gram operators $C_j c_Q(\xi) R_j^W$, where C_j is independent of the twist. The sharp datum at the chosen places fixes the inclusion matrix, and the datum at the twist places has invertible first block and arbitrary lower block. The Haar index and the squared residue mass therefore give exactly the same $c_Q(\xi)$ above.

It is the same at every ξ^j since these powers have the full modulus. For the Rallis series put A equal to the complement of the old and twist supports, including their conjugates. Write $K = K_A$. For $\gamma \in G_A$, let $H_\gamma = K \cap \gamma K \gamma^{-1}$ and $m_\gamma = [K : H_\gamma]$. Writing $K = \bigsqcup_{i=1}^{m_\gamma} k_i H_\gamma$ gives $K\gamma K = \bigsqcup_i k_i \gamma K$. For each K-fixed vector y, integration over these cosets gives

$$R\left(\frac{\mathbf{1}_{K\gamma K}}{\mathrm{vol}(K)}\right) y = \sum_{i=1}^{m_\gamma} R(k_i\gamma)y = m_\gamma P_K R(\gamma) y,$$

where P_K is the probability average over K. By the bi-invariance of u_j proved in Section 34.2.2, for each finite $\mathcal{D} \subset K_A\backslash G_A / K_A$ the partial series on both spaces is the action of the same finite Hecke test

$$h_{\mathcal{D}} = \sum_{K_A\gamma K_A \in \mathcal{D}} u_j(\gamma) \frac{\mathbf{1}_{K_A\gamma K_A}}{\mathrm{vol}(K_A)},$$

extended by normalized integral identity factors at the twist places.

This expression uses every finite place of A, with the same original bases and vacuum coefficient. The norm of each normalized double-coset operator is at most m_γ by unitarity. Thus, on either projected space, the operator-norm error is bounded by

$$\sum_{K\gamma K\notin\mathcal{D}} m_\gamma |u_j(\gamma)| = \sum_{gK:\, KgK\notin\mathcal{D}} |u_j(g)|.$$

This is the tail of the absolutely convergent series of Section 34.2.2. Hence the same tests converge to R_j and R_j^W, respectively. For every positive integer s, apply (4) to the convolution power $h_{\mathcal{D}}^{*s}$, and take the limit. Continuity of powers and traces in finite dimensions gives

$$\operatorname{tr}(R_j^s) = \operatorname{tr}((R_j^W)^s).$$

Taking the integral identity test in (4) gives the common dimension D. Newton's identities give the same characteristic polynomial and hence $\det R_j = \det R_j^W$. This limit compares the series in the original complex Hilbert spaces. All subsequent coefficient embeddings are applied to the fixed height multipliers or to the scalar ratios after the Rallis determinants cancel.

Set $W_\xi = q_\xi^{-mrt} \prod_{i<j} Q_{ij}(\xi)$. The two factorizations just proved give

$$Z_\xi = C^D W_\xi^D c_Q(\xi)^{mD} \prod_j |\det R_j(\xi)|,$$

$$\prod_j Z_j(\xi^j) = \left(\prod_j |C_j|\right)^D c_Q(\xi)^{mD} \prod_j |\det R_j^W(\xi)|.$$

The factor $c_Q(\xi)^{mD}$ is identical on the two sides because $Q(\xi^j) = Q(\xi)$ for all $1 \le j \le m$. At $w \in Q(\xi_\kappa)$ the numbers $q_w^h - 1$ in F_w can be divisible by ℓ. The common factor cancels in the complex equality before taking valuations. For $D > 0$, $Z_j(\xi^j) > 0$ by injectivity of the single lifts, and $Z_\xi > 0$ by (GC2). The same holds at the trivial twist. The two factorizations and $\det R_j = \det R_j^W$ give

$$\frac{Z_\xi}{\prod_j Z_j(\xi^j)} = \left(\frac{C}{\prod_j |C_j|}\right)^D W_\xi^D.$$

Compare this quotient at ξ and at 1, where $W_1 = 1$.

Proposition 44.1 (The chain determinant ratio)**.** *For a twist ξ for the full chain as in Section 2.6, the weighted chain and its single lifts satisfy*

$$\frac{Z_1}{Z_\xi} = \left(q_\xi^{-mrt} \prod_{1\le i<j\le m} Q_{ij}(\xi)\right)^{-D} \prod_{j=1}^m \frac{Z_j(1)}{Z_j(\xi^j)}. \tag{111}$$

The formula includes the empty determinant.

44.2 Height of the chain determinant ratio

In this subsection, we apply the critical-value comparisons of Section 38 to Proposition 44.1.

Use the family ξ_κ of Section 38 for the finite list of critical inverse ratios and all single lifts. Its modulus is prime to ℓ and every elementary critical inverse ratio in the square and turn lists has height at most x_1. On this same family, Section 42.1 gives operators $U_j(\xi)$ with

$$u_{j,\xi^j}^* U_j(\xi) u_{j,\xi^j} = \rho_j(\xi) U_j^\circ$$

for a fixed positive definite U_j°. The ratio in Lemma 42.1 is

$$\rho_j(\xi) = \rho_{\xi^j} = \frac{H_j(\xi^j)}{H_j(1)}.$$

The exchange-product comparison in Section 38.6 bounds its reciprocal on this family. Write $u = u_{j,\xi^j}$ and let Π be orthogonal projection onto its image.

Set $\Delta_j(\xi) = \det(\Pi U_j(\xi)|_{\operatorname{im} u})$. On the source, conjugate the compression by the isomorphism u onto its image. Multiplying the conjugated operator by u^*u gives $u^*\Pi U_j(\xi)u = u^*U_j(\xi)u$. Taking determinants gives

$$Z_j(\xi^j)\Delta_j(\xi) = C_j' \rho_j(\xi)^D, \qquad C_j' = \det U_j^\circ \neq 0.$$

The required bounds for this comparison are

$$\begin{aligned} \log|\sigma\Delta_j(1)| &\le 3Dfmb B^2 \quad \text{for all } \sigma, \\ \log|\Delta_j(\xi_\kappa)|_\kappa &\le Dr\beta_{j,\ell}, \qquad \sum_{\ell\in U} \beta_{j,\ell} \le J_* \quad (U \text{ finite}), \\ \operatorname{Height}(\rho_j(\xi_\kappa)^{-1}) &\le 10mx_1 + k(6J_* - d_{\mathrm{dir}}). \end{aligned}$$

The second line is a logarithmic allowance, independent of the embedding above ℓ. Taking positive parts of these allowances preserves the finite-sum bound: apply that bound to the subset of primes with positive allowance. These are respectively the untwisted compression estimate, the tested clearing estimate, and the fixed numerator and inverse ratio bound on the same regular family.

Dividing the comparison identity by its untwisted value gives

$$\frac{Z_j(1)}{Z_j(\xi_\kappa^j)} = \frac{\Delta_j(\xi_\kappa)}{\Delta_j(1)} \rho_j(\xi_\kappa)^{-D}.$$

For the first ratio use $\Delta_j(1)$ as its fixed height multiplier. Multiplication gives the allowance

$$D\{3fmbB^2 + rJ_* + 10mx_1 + k(6J_* - d_{\mathrm{dir}})\}.$$

On dividing the four positive terms inside the braces by L/m, using $a = xb$, their sum becomes

$$\frac{3m^2}{xB^2} + \frac{r}{kB} + \frac{10m^2x_1}{fxbB^4} + \frac{6}{B}.$$

It tends to zero under the admissible choices $m^2 \le 4x$, $r/k \le 2$, $\log b \sim x^\epsilon$, and $\log x_1 \le x^{\epsilon/6}$. This gives height at most $D(L/m - kd_{\mathrm{dir}})$. For each lift index j, the third measured exchange of Section 39.4 has $k/2$ points $7m/2 < u \le (7m+k)/2$ in its upper interval. At each point

Proposition 38.1 gives the term $-d_{\mathrm{dir}}$, with multiplicity two in the squared exchange product of Section 38.6. These terms inside the factor D sum over all lifts to

$$2\sum_{j=1}^{m}\sum_{i=1}^{k/2}(-d_{\mathrm{dir}}) = -mkd_{\mathrm{dir}} = -A.$$

For $D>0$ let $h_j \neq 0$ be the fixed multiplier of Section 38.6. The determinant identity at 1 has $Z_j(1)>0$, $C_j' \neq 0$ and $\rho_j(1)=1$, so $\Delta_j(1)\neq 0$. A single multiplier for the entire ratio is

$$H = \prod_{j=1}^{m}(\Delta_j(1)h_j^{D}).$$

For $D=0$ both determinants are empty and the multiplier is one. Indeed, with $\xi=\xi_\kappa$,

$$H\frac{Z_1}{Z_\xi} = q_\xi^{mrtD}\prod_{i<j}Q_{ij}(\xi)^{-D}\prod_{j=1}^{m}\left[\Delta_j(\xi)(h_j\rho_j(\xi)^{-1})^{D}\right].$$

For $D>0$ and $\kappa \mid \ell$, equations (RT1)–(RT2) and the signed product bound of Section 38.6 give

$$\begin{aligned}\frac{1}{D}v_\kappa\left(H\frac{Z_1}{Z_{\xi_\kappa}}\right) \geq\ & -mrC_\ell - mk(B_\ell + 2e_\ell + 2P_\ell)\\ & -(10m^2 + t\tbinom{m}{2})e_\ell.\end{aligned}$$

Here Q_{ij} uses ξ_κ^{j-i} and ρ_j uses ξ_κ^{j}. The conjugated coefficients of the main ratios are evaluated at $\bar\kappa$ on the same ξ_κ^{j}, as in (RT4)–(RT5). The right side involves only the datum at x and the rational prime ℓ. The archimedean bound for H uses $\log|\sigma H| = \sum_j(\log|\sigma\Delta_j(1)| + D\log|\sigma h_j|)$. For each $\kappa\mid\ell$, the integer q_{ξ_κ} is prime to ℓ. There are exactly $t\binom{m}{2}$ elementary inverses in $W_{\xi_\kappa}^{-1}$. By (RT1),

$$v_\kappa(W_{\xi_\kappa}^{-1}) = -\sum_{i<j}v_\kappa Q_{ij}(\xi_\kappa) \geq -t\binom{m}{2}e_\ell.$$

Thus (RT3) bounds the height of the triangular factor by $t\binom{m}{2}x_1$, with multiplier 1, before the D-th power. The upper bound is

$$Dt\binom{m}{2}x_1 + \sum_{j=1}^{m}D(L/m - kd_{\mathrm{dir}}) \leq D(b+L-A), \qquad A = mk\,d_{\mathrm{dir}}.$$

Proposition 44.2 (Uniform height of the chain ratio)**.** *Fix q_0, d, ϵ as in Section 4. For every sufficiently large integer x, choose the common twists $(\xi_\kappa)_\kappa$ as in Section 38. Then*

$$\mathrm{Height}\big((Z_1/Z_{\xi_\kappa})_\kappa\big) \leq D(b+L-A). \tag{112}$$

Here m^2t grows at most polynomially in x, whereas $B=\log b \sim x^{\epsilon}$ and $\log x_1 \leq x^{\epsilon/6}$. Thus $2m^2tx_1 \leq b$ eventually. The cutoff in x may depend on q_0, d, ϵ; it is chosen before any coefficient embedding or auxiliary twist.

44.3 Spanning vectors from the terminal period

In this subsection, we use Proposition 43.1 to bound the norms and denominators of the vectors used in Section 23.

Let W be the full principal-level source, with projected subspace V, and T_ξ its chain. Let $\mathcal{H}$ be the algebraic space at the true outer node, of level $K(\pi_M^{b_0})$. Use the map $r_0 : \mathcal{H} \to W$, array Ψ_ξ, and scalar c_ξ of Section 43. The deep transpose and separation prove surjectivity of r_0, with the same period at all the false levels $K(\pi_M Q(\xi))$. The terminal dual delta identity below gives, for a fixed β_0,

$$v^{\mathsf{t}} J_{\mathrm{W}}\big((c_\xi/c_1) T_\xi r_0 \eta\big)(x) = \beta_0\, \Xi_\xi(\delta_x(v), \eta).$$

The terminal delta and η have the original false and true outer levels. By (TP11),

$$|c_\xi/c_1| = q_\xi^{(m^2t^2+m^2t+mr)/2} \prod_{0\le i,j<m} T_{ij}(\xi).$$

The conductor exponent is integral since m is even. Each T_{ij} consists of the t critical ratios at power $j+1$ on that original square. For $i \neq j$ the reflected square (j,i) has the same untwisted character and uses power $i+1$.

44.3.1 Bounds for the two terminal jets

In this subsection, we apply the jet estimates of Section 7.4 to both factors in the measured period.

Put $D_0 = M^{b_0}$, $\mathfrak{b} = (\pi_M)^{b_0}$, and $\mathfrak{m} = \mathfrak{b}Q(\xi)$. Use the original edge in either half, with source and target ranks n, n', respectively, and fix its residue matrix $S \in \mathrm{Mat}_{n'\times n}(\mathcal{O}_E/\mathfrak{m})$. The indicated point test is the characteristic function $\mathbf{1}_{\{S\}}$ on this matrix ring. On its original standard definite Hom space, the edge datum Φ_S is this test on the vacuum already scaled by $s_{\mathrm{vac}}^{nn'}$. Its jet scale is s_{jet}. The π_M-factor in the earlier forward-chain jet scale does not enter the last-path kernel.

Write $z = 0, 1$ for the false and true halves and put

$$p_j = r + z + j(t+z) + m(t-z), \qquad \delta_j = (1-z)((j+m) \bmod 2), \qquad 0 \le j \le m.$$

At each τ, let $A(c,h) = \{c + h - 1 - 2a : 0 \le a < h\}$ as a multiset. With the base entries λ_a of Section 42.2, the doubled parameters of the algebraic path module, in decreasing order, are the entries of

$$-\left(\{2\lambda_a + z : 0 \le a < r+z\} \sqcup \bigsqcup_{i=1}^{j} A(2w_i, t+z) \sqcup \bigsqcup_{i=1}^{m} A(-2w_i, t-z)\right) + \delta_j.$$

Indeed (z, j, m) has j up blocks and m down blocks of the respective lengths $t+z, t-z$ in Section 34.3.2; its lowering is δ_j. The definite up edge uses the negative of that lowered parameter. In particular at $z = j = 0$ this module is the dual of the original outer Y_m-module, of rank $r + mt$. At $j = m$ the false primal module for the branching pairing has the opposite parameter, while the true module retains the displayed parameter, in ranks $r+2mt, r+2mt+1$, respectively. All entries have absolute value at most $P = \lceil 20b \rceil$: $1 \le \kappa_\tau \le b$, $w_i = \kappa_\tau + (i+1/2)\Delta$, and $(m+2)\Delta + t + k = o(b)$.

For either sign of the parameter, write its sorted list as $L_1 > \cdots > L_p$, with rank $p = p_j$. The highest weight, determinant exponent and polynomial row lengths are

$$h_i = \tfrac{1}{2}(L_i - p - 1 + 2i), \quad e = h_p = \tfrac{1}{2}(L_p + p - 1), \quad h_i - e = \tfrac{1}{2}(L_i - L_p) - (p - i) \in [0, P].$$

The entries have a common parity, so each successive difference is at least two. Since $p \le 2n_m \le P$, also $|e| \le P$.

For the displayed path parameter, in passing from j to $j+1$ the old entries acquire the common shift $\delta_{j+1} - \delta_j$. The new block lies below them since $w_1 \gg k + t$ and $w_{j+1} - w_j = \Delta > t + 1$. Thus the column counts $(L_i - L_{i+1})/2 - 1$ below height p_j are unchanged and the counts strictly between p_j and p_{j+1} vanish. To check the full-height factors, put $c_j(p) = \lfloor (p+1-z)/2 \rfloor - w_{j+1}$. At $j = 0$, the last entry is $-(k-2+z)$ and $e_0 = m(t+1-z)/2$. At $j+1$ the last entry is $-2w_{j+1} - (t+z-1) + \delta_{j+1}$, giving $e_{j+1} = c_j(p_j)$. At the joining height p_j, the difference of the two adjacent weights is $e_j - c_j(p_{j+1})$. Writing $n = p_j$, $n' = p_{j+1} = n + t + z$, this number of height-n minors is

$$e_j - c_j(n') = \begin{cases} w_1 - (k + t + 2z - 1)/2, & j = 0, \\ \Delta - (t + z), & 1 \le j < m. \end{cases}$$

Both are positive. More explicitly, let $h_i^{(j)}$ be the source weights. On the retained indices $i \le n$ the change of weight is

$$h_i^{(j+1)} - h_i^{(j)} = (\delta_{j+1} - \delta_j - t - z)/2 = c_j(n) - c_j(n').$$

On the new indices $n < i \le n'$, every weight equals $c_j(n)$. Thus the rectangular Schur map uses one partition $\lambda_i = h_i^{(j)} - c_j(n')$ for $i \le n$, padded by zeros in rank n'. The source representation is its Schur module times $\det^{c_j(n')}$, and the target representation its padded Schur module times $\det^{c_j(n)}$. This gives both determinant powers on the actual edge.

Every nonzero Schur jet entry is a linear combination of products with exactly the prescribed number of minors of each height below $n' = p_{j+1}$, including the minors of height $n = p_j$. Its total degree, summed over all places, therefore satisfies

$$H = \sum_{\tau} \sum_{i=1}^{n'} (h_{\tau,i} - e_\tau) \le f(2n_m) P \le b_0 r^4, \qquad n_m = k + m(t+1) = r + mt + 1.$$

Indeed $f \le x^{12}$, $n_m \le x^6$, $r \ge x^5$ and $b_0 \ge b/2$ eventually.

The dual has the weights in reverse order and with opposite sign. At each place put $c = h_1 - h_p \le P$ and $\lambda_i = h_i - h_p$. The dual determinant exponent is $-h_1$ and its row lengths satisfy

$$\lambda_i^\vee = c - \lambda_{p+1-i}, \qquad \sum_{i=1}^{p} \lambda_i^\vee = pc - \sum_{i=1}^{p} \lambda_i \le pP.$$

Equivalently, each column of height d_i is replaced by one of height $p - d_i$, in reverse order. Summing over the places gives at most $f(2n_m)P \le b_0 r^4$ boxes in the dual terminal diagram of Section 44.3.4, including the case $c = 0$.

Fix a prime $\ell \nmid q_\xi$ and an arbitrary coefficient embedding. The integer $q_\xi D_0$ belongs to $\mathfrak{m}$. Since $D_0 \mathcal{O}_E \subseteq \mathfrak{b}$ and $(\mathfrak{b}, Q(\xi)) = 1$, the Fourier denominator of the point test is

$$|\operatorname{Mat}_{n' \times n}(\mathcal{O}_E/\mathfrak{m})| = (\operatorname{Nm}(\mathfrak{b}) q_\xi)^{nn'}, \qquad \operatorname{Nm}(\mathfrak{b}) \mid D_0^{2f}.$$

Its ℓ-part therefore divides $D_0^{2fnn'}$ for every $\ell \nmid q_\xi$, independently of the size of q_ξ. For $B_\mathfrak{m} = \mathrm{Mat}_{n'\times n}(\mathcal{O}_E/\mathfrak{m})$ Fourier inversion gives

$$D_0^{2fnn'}\mathbf{1}_{\{S\}}(X) = \frac{D_0^{2fnn'}}{|B_\mathfrak{m}|}\sum_{\chi\in\widehat{B}_\mathfrak{m}} \chi(-S)\chi(X).$$

The prefactor is integral above ℓ and the character values are roots of unity. Each $\chi(X)$ on the vacuum is the modulation in the dual half of the Heisenberg polarization, up to a root of unity in the signed convention of Section 44.4. Thus $D_0^{2fnn'}\Phi_S$ is an integral combination of these finite translates of the normalized vacuum. For every translating vector a one has $D_0 a_w \in \mathcal{O}_{E,w}^{nn'}$ at each $w \mid \ell$, and the ℓ-part of the Fourier denominator divides $D_0^{2fnn'}$. The residue test is on the original common modulus $\mathfrak{m} = \mathfrak{b}Q(\xi)$. In these normalized coordinates the commutator of the translations by x, y is $\psi(y^*x)$: its two half-pairing factors are $\psi(y^*x/2)$ and $\psi(x^*y/2)^{-1}$. Above $\ell \nmid q_\xi$ the Fourier character is trivial on $D_0\Lambda_\ell$; self-duality therefore gives $D_0 x \in \Lambda_\ell$, also above 2. The point indicator has values zero and one at every coefficient embedding.

Let the two finite points be integral at all places above ℓ, and let g be their induced isometry of the finite Hom space in the normalized coordinates. For the vacuum lattice L put $H = L \cap g^{-1}L$. The finite vacuum expansion of Section 19.3 expresses $\omega(g)T(x)v_L$ as translated vacua with shifts $g(x+z)$, $z \in L/H$, coefficients differing by roots of unity, and common coefficient of modulus $[L : H]^{-1/2}$ in the original complex normalization. At the coefficient embedding we use its actual value $\Delta(g)/[L : H]$. Since $H_\ell = L_\ell$, the index is prime to ℓ and all the z may be chosen zero above ℓ. Thus $D_0 g(x+z)$ is integral there whenever $D_0 x$ is.

In the formula for $\Delta(g)$ the intersection determinant is a finite idele t that is a unit above ℓ. Its absolute norm is an ℓ-unit. Its splitting factor $\upsilon_f(t)$ is an ℓ-unit as well: raise t to a ray class exponent and write $t^h = ak$ with $a \in E^\times$ and k in the ray units. Then every conjugate of a is a unit above ℓ, and the square of the archimedean angular character on a is a monomial in those conjugates and their inverses. Automorphy gives the assertion. For each endpoint determinant, which is a norm-one idele integral above ℓ, the same unit assertion is exactly the character calculation in Section 38.4. Hence the full adelic coefficients are integral at the chosen embedding, including when the points move the vacuum at other places. This proves preservation of the standard data of level D_0. At the scale $D_0 d\, s_{\mathrm{jet}}$, the ordinary integral jet calculation applies to each Schur matrix entry, since the Schur polynomials have integral coefficients and degree H.

At the actual scale s_{jet} homogeneity introduces $|D_0 d|_\ell^{-H}$. Multiplication by $D_0^{2fnn'}(D_0 d)^H$ therefore makes the edge matrix send the source unit lattice into the target unit lattice. For arbitrary two finite points move the source and target separately by rational unitary matrices into the open sets of points integral above ℓ. The Cayley chart gives both weak approximations. Rational equivariance of the jet changes its argument and its value by the same weight representations at the two endpoints.

These changes cancel exactly in the respective transported lattice coordinates. This proves the bound at every pair of points. In particular the common integer

$$D_{\mathrm{jet}} = D_0^{2f(2n_m)^2}(D_0 d)^{b_0 r^4}$$

depends only on the original datum. Indeed D_0, d and the rank and degree bounds in its exponents precede the choices of S, ξ, ℓ and the embedding. Thus the same integer works at

both ι and $\iota \circ \mathrm{conj}$ whenever $\ell \nmid q_\xi$. Iterating gives the factor D_{jet}^{m} on each half, on the original outer levels and in their transported Weyl lattices.

For the complex bounds take the trivial twist. Fourier inversion of the single indicator on the finite matrix ring of cardinality J has J coefficients of absolute value $1/J$ at every complex embedding. Thus the total coefficient norm of the translated integral vacua is one. The pointwise conjugate vacuum bound for Hom, of rank nn', is

$$\exp(C_K(q_1 nn' + 1)^2).$$

Apply Lemma 7.6 at the rank of the Hom space and the Gaussian Schur calculation of Section 7.4.4 with $c = L = 1$.

In particular the squared scale in the integral is Ω^{-1}. Write $\lambda_{\tau,i} \geq 0$ for the part sizes in the edge Schur polynomial. They sum to H. The conjugated Weyl operator norm is at most the exponential of

$$C_K(q_1 nn' + 1)^2 + fnn' \log 4 + \frac{1}{2} \sum_{\tau,i} (\lambda_{\tau,i} + \log(\lambda_{\tau,i}!) + \lambda_{\tau,i} \log(\Omega^{-1})).$$

In fact a matrix coefficient against a covector of dual Weyl norm one is bounded by the product of the singular values with the indicated covector exponents.

The factorial estimate with $L = 1$ in the Gaussian integral leaves the factor $4^{fnn'}$. This proof therefore bounds the operator directly between the conjugated Weyl norms. Using $\mathrm{rd}(E) \leq q_1 d$ and the explicit formula for Ω, we have $|\log \Omega| \ll_d 1 + \log q_1$. The sum of the factorial logarithms is at most $H \log \max(1, H)$. With $q_1, f \leq x^{12}$, $n_m \leq x^6$, $r < 3x^5$ and $\log b \leq x$, the exponent is $O_d(x^{48} + bx^{21})$, hence at most $x^{99} b^2$. This bound is uniform over the original edge points and holds at every complex embedding.

At the trivial twist each last-path level contains the principal congruence subgroup of rational level D_0. Apply the counting argument of Section 7.2 at each rank $n \leq 2n_m$, with D_0 as level. There $u = 2fn \ll x^{18}$ and the maximal-level count has logarithm $O(u^3 \log x) = O(x^{54} \log x)$. Refining to the path level contributes at most $2fn^2 \log D_0 = O(x^{25} b)$ in the logarithm. Thus each class count and each unscaled edge norm have the common bound $G = \exp(x^{99} b^2)$ used below.

44.3.2 Local transport and terminal branching

In this subsection, we use the bounds of Section 44.3.1 in the terminal comparison.

Let t_ξ generate the finite ideal of $\mathfrak{m}$ and write $u_\xi = \overline{t_\xi \alpha_{\mathrm{W}}}/(t_\xi \alpha_{\mathrm{W}})$. In the original terminal charts the false point is g and the true one is $\mathrm{diag}(g, u_\xi^m)$. Write $u' = u_\xi^{-m}$. If v', v at these points have lattice norm at most one, their Weyl-coordinate transports w', w by g^{-1} and $\mathrm{diag}(g^{-1}, u')$ have integral entries. Use the bilinear matrix B_{grid} constructed for the two terminal grid nodes in Section 43.5. Its equivariance for the common diagonal group and the extra line gives

$$(w')^{\mathsf{t}} B_{\mathrm{grid}} w = \left(\prod_{\tau} \iota_\tau(u')^{L_\tau} \right) \iota((v')^{\mathsf{t}} B_{\mathrm{grid}} v),$$

with the weight embeddings read on the corresponding finite components. The original pairing is bilinear in these two already-dual Weyl models, as in the displayed coordinate product.

The scalar u_ξ and its inverse become integral after multiplication by $q_\xi D_0 q_1$. For u_ξ, write this product as

$$(q_\xi D_0/t_\xi)\,\overline{t_\xi(q_1\alpha_{\mathrm{W}})}\,\alpha_{\mathrm{W}}^{-1}.$$

Here $q_\xi D_0/t_\xi$ is integral since $q_\xi D_0 \in \mathfrak{m}$; the choices of the adelic Hermitian basis give $q_1\alpha_{\mathrm{W}}$ and α_{W}^{-1} integral. Thus the three factors are integral on the finite adeles. For the inverse conjugate this expression, since $\bar{u}_\xi = u_\xi^{-1}$. Above $\ell \nmid q_\xi$, either sign of L_τ is therefore cleared by $(D_0 q_1)^{m|L_\tau|}$. Because the left-hand side of the equivariance identity is integral, the original pairing is cleared by

$$M_0 = \prod_\tau (D_0 q_1)^{m|L_\tau|}.$$

Combining with the two half paths gives $N_{\mathrm{path}} = M_0 D_{\mathrm{jet}}^{2m}$ on every summand of the original finite expansion.

The true terminal diagram has at most $b_0 r^4$ boxes by Section 44.3.1. Its determinant exponent is the last highest-weight entry; the bound $\lceil 20b \rceil$ for the doubled parameters and $N+1 \le 2n_m \le \lceil 20b \rceil$ give $|e| \le \lceil 20b \rceil$ after removing the rank shift. Hence the exponents of this pairing satisfy $|L_\tau| \le b_0 r^4 + \lceil 20b \rceil$. The rank and box bounds and $\log M = O(\log b)$ give $\log N_{\mathrm{path}} \le x^{99} b^2$. By (TP13), the original internal integrals are free class sums. Reindexing the combined array at the internal nodes cancels the adjacent kernel characters against the array character. At the linked terminal node the basis changes are u and $\mathrm{diag}(u, 1)$. Each array entry is a sum of roots of unity. Thus the edge and branching bounds give integrality of $N_{\mathrm{path}} \Xi_\xi(\delta_x(v), \eta)$ on integral outer data at all embeddings above $\ell \nmid q_\xi$.

At the trivial twist, put $G = \exp(x^{99} b^2)$ and $h_0 = b_0 r^4$. Use the bounds G for each class count and each point-indicator edge operator in the conjugate Weyl metrics, and $h_0^{3h_0}$ for the terminal branching pairing. In the residue ring of the grid modulus, put

$$N_b = |\mathcal{O}_E/(\pi_M)^{b_0}|, \qquad E_0 = N_b^{(2n_m)^2}, \qquad V_0 = N_b^{2n_m}.$$

There are at most E_0^{2m} edge arrays. The first path supplies at most E_0^{2m} root-of-unity terms per entry and each of the m^2 turns multiplies that bound by at most V_0.

Hence at most $V_0^{m^2} E_0^{4m} G^{2m+1}$ terms remain after summing over the $2m+1$ original class variables. Each term has absolute value at most

$$h_0^{3h_0} G^{2m} \|\sigma v\|_{\mathrm{W}} \sqrt{B_\eta},$$

by the two compositions of m edge operators, where B_η bounds the squared norm of every σ-conjugate value of η. The numerical bound

$$V_0^{m^2} E_0^{4m} h_0^{3h_0} G^{4m+1} \le \exp(x^{200} b^2)$$

now gives the corresponding bound for $|\sigma \Xi_1(\delta_x(v), \eta)|$. Indeed $f, q_1 \le x^{12}$, $n_m \le x^6$, $r \le 3x^5$, $m \le 2\sqrt{x}$ and $\log M \ll \log b \le x$ for large x. Thus $\log N_b \le x^{13} b$. With $h_0 = b_0 r^4$, each term in the logarithm of the count bound is $o(x^{200} b^2)$. The same estimates give $\log(M_0 D_{\mathrm{jet}}^{2m}) = o(x^{99} b^2)$ using the bound on $|L_\tau|$. This estimate concerns the same path period as the finite calculation.

44.3.3 The dual vector on the terminal lattice

In this subsection, we construct the dual vector for the terminal pairing of Section 44.3.2.

Write $K_\xi = K(\pi_M Q(\xi))$ on the original terminal definite group Y_m, and let K_{std} denote its maximal integral group. Put $I_\xi = [K_{\mathrm{std}} : K_\xi]$, and let d_Y be the dimension of the terminal Weyl module. Reduction at the chosen split places is onto, with kernel K_ξ. Hence I_ξ is the order of the general linear group of the false terminal node over the residue ring of $\pi_M Q(\xi)$. In particular this is the terminal divisor appearing in the last-path period. The measures in this calculation are the original quotient measures, with the compact real group and the integral finite group at the terminal space each of mass one.

Use the dual module of type obtained by reversing and negating the terminal parameter. Write $B(v,z) = v^{\mathsf{t}} J_{\mathrm{W}} z$ for its normalized invariant bilinear pairing with the terminal module. Let $\delta_x(v)$ be the dual form of level K_ξ whose value at x is v, extended by equivariance on the class of x and zero elsewhere. The stabilizer is finite by definiteness. At an old split place above the odd prime p_0 it is conjugate into $1 + p_0 M_{r+mt}(\mathbb{Z}_{p_0})$, which is torsion-free: the matrix logarithm is injective there, and $\log(u^a) = a \log u$ for every positive integer a. Thus the stabilizer is trivial and the equivariant extension has the prescribed value at the representative. Use its conjugate-type real realization in the period.

Let w be the tensor of unit highest wedges in the flags ordered by the terminal parameter, and let ρ_∞ denote the compact action on this module in the scalar realization. The negative parameter orders each flag in reverse. With the sign of the complement map fixed below, the coefficient of the dual test at k is $B(v_h, \rho_\infty(k)w)$, where $v_h = \delta_x(v)(h)$. Indeed this equality holds at the identity by the normalization on the opposed flags, and at k by invariance of B. The primal coefficient at h, with $z_h = (T_\xi y)(h)$, is $\langle \rho_\infty(k)w, z_h \rangle_{\mathrm{W}}$. For compact Haar measure of mass one, Schur orthogonality now gives

$$\begin{aligned} \int B(v_h, \rho_\infty(k)w) \langle \rho_\infty(k)w, z_h \rangle_{\mathrm{W}} \, dk &= B\Big(v_h, \int \rho_\infty(k)w \langle \rho_\infty(k)w, z_h \rangle_{\mathrm{W}} \, dk \Big) \\ &= d_Y^{-1} B(v_h, z_h). \end{aligned}$$

The averaged operator commutes with the irreducible compact action and has trace $\|w\|_{\mathrm{W}}^2 = 1$. The rational left actions cancel in this expression by invariance of B. Both forms are fixed by K_ξ, and the finite mass of the class at x is I_ξ^{-1} because its rational stabilizer is trivial. Thus

$$I_\xi \int_{[U(Y_m)]} \delta_x(v)(h)(T_\xi y)(h) \, dh = d_Y^{-1} B(v, (T_\xi y)(x)). \tag{113}$$

This formula pairs the algebraic form with its dual bilinearly. The dual real type has already absorbed the complex conjugation.

In Proposition 43.1, use $y' = \delta_x(v)$. Its right stabilizer contains the false outer deep subgroup $K(\pi_M^{b_0} Q(\xi))$. The true form η has level $K(\pi_M^{b_0})$. Combining (TP13) with (113) gives

$$B(v, ((c_\xi/c_1) T_\xi r_0 \eta)(x)) = \beta_0 \, \Xi_\xi(\delta_x(v), \eta), \qquad \beta_0 = \frac{d_Y A_*}{c_1}. \tag{114}$$

The finite index has canceled and the compact dimension is retained in β_0. In particular $\beta_0 \neq 0$ if $D > 0$ and T_1 is injective on V. In that case take $y \in V \setminus \{0\}$, lift it by r_0, and choose a dual vector separating a nonzero value of $T_1 y$.

44.3.4 Integral dual coordinates and pointwise bounds

In this subsection, we estimate the coordinates of the dual vector of Section 44.3.3.

At each infinite place let $d_1 \geq \cdots \geq d_c$ be the column heights of the polynomial diagram in rank n. The dual diagram has column heights $n - d_c, \ldots, n - d_1$. In the exterior tensor model send each coordinate wedge to the complementary wedge with the sign determined by the unit volume wedge, then reverse the order of its c tensor factors. This is an integral isometry between the two prescribed tensor products, also for $c = 0$. For an invertible basis matrix g, the complement identity on each exterior power moves g to $(g^{-1})^{\mathsf{t}}$ and supplies one factor of $\det g$ per column. Indeed, for a highest weight $h_1 \geq \cdots \geq h_n$, the polynomial diagram has $c = h_1 - h_n$ columns and the determinant exponent is h_n. For the dual, that exponent is $-h_1$. Hence the total factor $(\det g)^{c+h_n}$ equals $\det((g^{-1})^{\mathsf{t}})^{-h_1}$, including when either determinant exponent is negative. At each place, the complement of an initial segment of length j in the primal ordered flag is the initial segment of length $n - j$ in the reversed flag. Their ordered coordinate wedges agree up to a sign. Multiply the tensor complement by the product of these fixed signs, so that it sends the unit highest tensor to the prescribed unit highest tensor of the dual. This fixes its sign also when the diagram is empty, in which case the product is one.

Applying this identity to invertible bases proves that the tensor isometry restricts to the Weyl spans, since the evaluation vectors of those bases span. The Weyl coordinates are integral on these tensors: the standard-tableau coefficient matrix has an integral left inverse, obtained from its unitriangular minor submatrix. The same holds for the inverse complement map. Denote this unimodular isometry by J and the dual Weyl Gram matrix by $G_\vee$. The pairing has matrix $J_{\mathrm{W}} = G_\vee J$, so in the convention linear in the second argument

$$B(v, z) = \langle \overline{Jz}, v \rangle_{\mathrm{W}}, \qquad \|\overline{Jz}\|_{\mathrm{W}} = \|z\|_{\mathrm{W}}.$$

For the unit highest tensor w in the real coordinate wedges, put $w^\vee = Jw$. Then $B(w^\vee, w) = 1$, as in (113).

The Fock pairings of the integral dual minor basis form an integral matrix $F_\vee$ with $F_\vee^{\mathsf{t}} G_\vee = H_\vee I$, where $H_\vee > 0$ is the Fock square of the dual highest minor polynomial. Indeed the standard minor polynomials are the coordinate functions of the evaluation vector in the dual Weyl module. The Hermitian metric of these coordinate functionals induced by the Weyl metric has matrix $G_\vee^{-1}$ in the dual basis, since $G_\vee$ is real symmetric. This metric and the Fock metric are invariant under the compact action on the irreducible polynomial module, so they are proportional. The highest coordinate functional is the inner product with the unit highest tensor, by its torus weight and its value one on that tensor. Its squared norm is one in the induced metric and $H_\vee$ in the Fock metric. This gives $F_\vee^{\mathsf{t}} G_\vee = H_\vee I$. Thus

$$C_{\mathrm{W}} J_{\mathrm{W}} = H_\vee I, \qquad C_{\mathrm{W}} = J^{-1} F_\vee^{\mathsf{t}} \in M(\mathbb{Z}).$$

There are at most $h_0 = b_0 r^4$ boxes in the dual diagram. The determinant expansion bounds the coefficient ℓ^1-norm of its highest polynomial by $h_0^{h_0}$, and each monomial factorial by $h_0^{h_0}$. Consequently $H_\vee \leq h_0^{3h_0}$.

If x is integral above ℓ and v is an integer dual coordinate vector, $\delta_x(v)$ has lattice norm at most one. Indeed x^{-1} is integral at those places, as it is the Hermitian adjoint of x, and the polynomial representation and determinant units preserve the unit lattice. Equivariance gives this lattice norm at every point of the class. At any coefficient automorphism σ, its pointwise Weyl norm is at most $\|\sigma v\|_{\mathrm{W}}$. Indeed every embedding $\lambda : E \to \mathbb{C}$ satisfies $\lambda(\bar{a}) = \overline{\lambda(a)}$ for $a \in E$, since E is CM. Thus $(\sigma \circ \lambda)(\gamma)$ is unitary for every rational unitary transport γ, and its Weyl action preserves the corresponding rational Gram form. This argument applies to the transport entries before pairing with the arbitrary vector σv.

Let $N_{\mathrm{path}} = M_0 D_{\mathrm{jet}}^{2m}$, the integer fixed above for every allowed twist and every coefficient prime off its modulus. Set $s_0 = H_\vee N_{\mathrm{path}}/\beta_0$ when $\beta_0 \neq 0$ and $s_0 = 1$ otherwise. To see the cancellation in the first case, put $F = (c_\xi/c_1)T_\xi(s_0 r_0 \eta)$ and take v_i to be the transpose of row i of C_{W}. At an x integral above ℓ, use $C_{\mathrm{W}} J_{\mathrm{W}} = H_\vee I$ in (114). This gives the equality

$$H_\vee F_i(x) = B(v_i, F(x)) = H_\vee N_{\mathrm{path}}\, \Xi_\xi(\delta_x(v_i), \eta).$$

The right-hand period multiplied by N_{path} is integral. Cancellation in characteristic zero gives the integral coordinate $F_i(x) = N_{\mathrm{path}} \Xi_\xi(\delta_x(v_i), \eta)$, also when $\ell \mid H_\vee$. For $\beta_0 = 0$, nondegeneracy in (114) gives $F(x) = 0$. In both cases $s_0 \neq 0$ is independent of ξ and ℓ.

For arbitrary x, use the Cayley-chart weak approximation of Section 34.1.1 to choose $\gamma \in U(Y_m)(F_0)$ with $\gamma_\ell x_\ell \in K_{\mathrm{std},\ell}$, where the subscript ℓ denotes all local components above ℓ. Apply the integral estimate at γx. Writing $F = (c_\xi/c_1)T_\xi(s_0 r_0 \eta)$ and ρ_{W} for the terminal Weyl action at each coefficient embedding above ℓ, rational equivariance gives

$$\rho_{\mathrm{W}}((\gamma_\ell x_\ell)^{-1}) F(\gamma x) = \rho_{\mathrm{W}}(x_\ell^{-1}) F(x).$$

Thus the transported coordinates are integral at the original point.

At the empty twist, if the path bound is

$$|\sigma \Xi_1(\delta_x(v), \eta)| \le e^{x^{200} b^2} \|\sigma v\|_{\mathrm{W}} \sqrt{B_\eta},$$

then the dual isometry and (114) give the squared norm bound

$$\left(H_\vee N_{\mathrm{path}} e^{x^{200} b^2}\right)^2 B_\eta \le e^{2x^{201} b^2} B_\eta.$$

The last bound uses $\log N_{\mathrm{path}} \le x^{99} b^2$ and the estimate for $H_\vee$ just proved. Indeed the period identity for the normalized vector has scalar $s_0 \beta_0 = H_\vee N_{\mathrm{path}}$ when $\beta_0 \neq 0$, hence is fixed by every σ. The identity $\sigma B(v, F(x)) = B(\sigma v, \sigma F(x))$ follows from the integral pairing matrix. Taking $\sigma v = \overline{J(\sigma F(x))}$ in the complex dual space gives the stated norm bound, with the zero value immediate. If $\beta_0 = 0$, all of these terminal values vanish by (114).

44.3.5 Integral generators at the outer node

In this subsection, we use the dual coordinates of Section 44.3.4 to bound the original generators.

At the true outer node $(1, 0, m)$ the rank is $n_H = r + 1 + m(t-1) \le 2n_m$ and the level is $K(\mathfrak{b})$. For each free class g at that level write $L(g) = E^{n_H} \cap g\widehat{\mathcal{O}}_E^{n_H}$. The short-basis calculation at the source supplies $Y(g) \in \mathrm{GL}_{n_H}(E)$ whose columns lie in $L(g)$ and whose entries have absolute value at most $R = 6(q_1 d)(f n_H)^2$ at every complex embedding. Here the columns form an E-basis, and may span a proper $\mathcal{O}_E$-sublattice of $L(g)$. Only invertibility over E and $g^{-1}Y(g) \in M_{n_H}(\widehat{\mathcal{O}}_E)$ are used. For each actual determinant exponent e_τ apply the same construction at g^{e_τ}.

At the representative g, multiply the polynomial Weyl action of $(\iota_\tau Y(g))_\tau$ on the evaluation vector by

$$\prod_\tau \det \iota_\tau Y(g^{e_\tau}).$$

After moving into the local lattice norm at g, the determinant at any coefficient embedding is evaluated on

$$\det(g)^{-e_\tau} \det Y(g^{e_\tau}) = \det\left((g^{e_\tau})^{-1} Y(g^{e_\tau})\right).$$

The identity also holds for negative e_τ. Its right-hand side and the polynomial Weyl action of $g^{-1}Y(g)$ are integral. For root-of-unity evaluation arrays all initial coordinates are integral at every coefficient embedding.

The rational left action and right level action show that this is the lattice norm at every class point. Writing such a point as $\gamma g k$, with k in the deep compact group and γ rational, equivariance cancels the weight factor of γ against transport by $(\gamma g k)^{-1}$. The remaining k^{-1} acts integrally, with inverse also integral. The free class is supported at a single class and thus incurs no stabilizer sum. A corresponding conjugate value has the same norm as at g, because the rational definite unitary action remains unitary in the simultaneous conjugated Weyl coordinates, and the class level has trivial character.

The polynomial Weyl actions of the $Y(g)$ are invertible, and all the determinant factors are nonzero, so they preserve the span of the evaluation vectors. Apply the exterior-power bound in Lemma 7.5 with rank n_H. Each column of an evaluation matrix with root-of-unity entries has norm at most $\sqrt{n_H}$, and the operator norm of $Y(g)$ is at most $n_H R$. Each exterior tensor of total degree at most $b_0 r^4$ is bounded by the corresponding power of $n_H^{3/2} R$. Each determinant factor is at most $(\sqrt{n_H} R)^{n_H}$ by Hadamard's inequality. Since $R \geq n_H^2$, the squared norm is bounded by

$$(n_H^{3/2} R)^{2 b_0 r^4} (\sqrt{n_H} R)^{2 f n_H} \leq R^{6(f n_H + b_0 r^4)} \leq \exp(x^{99} b^2).$$

Transport to the analytic form space is by the same coefficient-type isomorphism at the actual deep level.

For each free outer class these values span its whole fiber. In fact put $H_0 = b_0 r^4$ and choose any integer $l > H_0$. Take one independent matrix of variables for each infinite place and let every entry range over μ_l. At this fixed class the factors $\det \iota_\tau Y(g^{e_\tau})$ are nonzero constants for every sign of e_τ. The variable part consists of products of minors with total degree at most H_0. A functional vanishing on the indicated Cartesian grid therefore gives a polynomial $P(Y)$ of degree less than l in each variable. Writing k_0 for the number of variables, for any multi-index $0 \leq a_i < l$ orthogonality of roots of unity gives

$$[Y^a] P = l^{-k_0} \sum_{Y \in \mu_l^{k_0}} P(Y) \prod_{i=1}^{k_0} Y_i^{-a_i} = 0.$$

This coefficient extraction takes place in $\mathbb{C}$. The integral generators themselves use only polynomial evaluations, including at singular arrays. The evaluation vectors span the Weyl space by its minor construction, and the fixed invertible transforms preserve their span. Taking the union of the generators supported on individual classes gives a spanning set of $\mathcal{H}$; since r_0 is onto and $s_0 \neq 0$, the $s_0 r_0 \eta$ span the full source W.

44.3.6 Projection to the projected space

In this subsection, we project the spanning set of Section 44.3.5 to the projected space, retaining the bounds needed in Section 45.

Use the integral spanning set of $\mathcal{H}$ just constructed. Its forms have integral values in the transported lattices and squared norms at most $e^{x^{99} b^2}$ at every complex embedding. Let $z_\eta = s_0 r_0 \eta$. The z_η span W. Section 44.3.4 proves that $\gamma_\xi T_\xi z_\eta$, with $\gamma_\xi = c_\xi / c_1$, is integral at every embedding over $\ell \nmid q_\xi$, and that each conjugated value of $T_1 z_\eta$ has squared norm at most

$e^{x^{202}b^2}$. Set

$$\mathcal{Z} = \{\, |\mathcal{G}| p z_\eta : \eta \in \mathcal{S}_H \,\} \subseteq V,$$

where $\mathcal{G} = G_0 \times G_1 = G$ is the original finite group and $p = \rho(e_0)$ projects the principal-level space W onto V as in Section 2.4.

The operator $|\mathcal{G}|p : W \to V$ is surjective and is a sum of integral right translations with coefficients algebraic integral at every coefficient embedding. Indeed its coefficient at a group element is the sum of dimension times the character value of the retained simple types. Its conjugate absolute value is at most $|\mathcal{G}|$, by the bound of character values by dimension and the sum-of-squares identity. Apply Lemma 7.3 with the identity translate. Its first-block lifts $k_m(t)$ give

$$(\gamma_\xi T_\xi(|\mathcal{G}| p z_\eta))(h) = \sum_{t \in \mathcal{G}} |\mathcal{G}| e_0(t) (\gamma_\xi T_\xi z_\eta)(h k_m(t)).$$

Each lift and its inverse are integral above ℓ; transporting from $hk_m(t)$ to h preserves the unit lattice. Thus the sum is integral at every coefficient embedding above $\ell \nmid q_\xi$. For $\xi = \xi_\kappa$, apply this at ι and $\iota \circ \mathrm{conj}$ to the same vector $\gamma_\xi T_\xi z$, for every $z \in \mathcal{Z}$. The latter application gives the conjugate coordinates used in the mixed Gram pairing at κ. For any σ, the sum of the absolute values of the conjugated coefficients is bounded by $|\mathcal{G}|^2$; the triangle inequality contributes at most $|\mathcal{G}|^4$ to the squared pointwise bound. The class count at the original terminal level from Section 7.2 and $|\mathcal{G}|^4$ multiply the squared pointwise bound by at most $e^{x^{99}}$. The total conjugate squared norm of each projected form is therefore at most $e^{2x_1^2 b^2}$, by $x^{99} + x^{202}b^2 \le 2x_1^2 b^2$.

For $\xi = \xi_\kappa$ put $p_{ij,k} = (\mu_{i+1}^0 \mu_{j+1}^0, k)$ and

$$U_{ij,k} = \frac{F_{p_{ij,k}, Q(\xi)}(\xi^{j+1})}{F_{p_{ij,k}, \varnothing}(1)}.$$

The support of ξ^{j+1} is $Q(\xi)$ for $0 \le j < m$. The two norm formulas of Section 38.1 give $q_\xi^{-1} U_{ij,1}$ for the square of the first factor of $T_{ij}(\xi)$ and $U_{ij,k}$ for each factor $k \ge 2$. Since $\mathrm{Nm}(Q(\xi)) = q_\xi$, equation (TP11) gives

$$\gamma_\xi \bar{\gamma}_\xi = q_\xi^{m^2 t^2 + m^2 t + mr - m^2} \prod_{0 \le i,j < m} \left(U_{ij,1} \prod_{k=2}^{t} U_{ij,k}^2 \right) \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}.$$

Consequently its valuations at κ and $\bar{\kappa}$ agree for this twist. As $\ell \nmid q_\xi$, (RT1) gives

$$v_\kappa\big((\gamma_\xi \bar{\gamma}_\xi)^{-1}\big) = -2 \sum_{0 \le i,j < m} v_\kappa T_{ij}(\xi) \ge -2m^2 t e_\ell.$$

Hence (RT3) with multiplier 1 gives

$$\mathrm{Height}\big(((\gamma_{\xi_\kappa} \bar{\gamma}_{\xi_\kappa})^{-1})_\kappa\big) \le 2m^2 t x_1 < b.$$

Indeed $m^2 \le 4x$, $t < 15x^5$ and $\log(2m^2 t x_1) \le \log 120 + 6 \log x + x^{\epsilon/6} < x^\epsilon < \log b$ for sufficiently large x.

Proposition 44.3 (Spanning vectors for the lattice completion)**.** *Choose $(\xi_\kappa)_\kappa$ as in Section 38. There is a set $\mathcal{Z} \subset V$ with*

$$\operatorname{span} \mathcal{Z} = V, \qquad \|\sigma \Theta_1 z\|^2 \le e^{2x_1^2 b^2} \quad (z \in \mathcal{Z},\ \sigma \in \mathrm{Aut}(\mathbb{C})).$$

For each κ, the scalar $\gamma_{\xi_\kappa} = c_{\xi_\kappa}/c_1$ is nonzero and $\gamma_{\xi_\kappa} \Theta_{\xi_\kappa} z$ is integral at both κ and $\bar{\kappa}$ for all $z \in \mathcal{Z}$. Moreover,

$$\mathrm{Height}\big(((\gamma_{\xi_\kappa} \bar{\gamma}_{\xi_\kappa})^{-1})_\kappa\big) \le b.$$

44.4 Integral derivatives in the two norm formulas

In this subsection, we prove the integrality used by the two norm constructions in Sections 30.1 and 30.2. Fourier inversion first reduces the point indicators to translated vacuum vectors. At coefficient primes outside the auxiliary support, the same denominator clears each expansion.

Fix a coefficient prime $\ell \nmid q_Q$ and any embedding above it. For a finite modulus $\mathfrak{m}$ with $q_Q D \in \mathfrak{m}$, Fourier inversion on the finite matrix coordinates expresses its integral indicator datum as a combination of translates, with roots of unity in the numerator. The Fourier denominator divides

$$D^{2fn'n}$$

in the chosen valuation, since $\mathcal{O}_E/\mathfrak{m}$ at ℓ is a quotient of $\mathcal{O}_E/(D)$ at ℓ and $[E : \mathbb{Q}] = 2f$. The Fourier dual translations are integral after multiplication by D above ℓ, by the self-dual translation pairing. Thus multiplying by $D^{2fn'n}$ gives the standard datum of level $\delta = D$, once the vacuum has been normalized by $s_{\mathrm{vac}}^{n'n}$. This argument uses coprimality with q_Q alone. For finite unitary points integral invertible above ℓ, the lattice index in Section 7.4.3 is prime to ℓ. For each norm-one determinant u of these points, write $u^h = ak$, with a a rational norm-one scalar and k an integral norm-one idele, as in that section. For every endpoint character α of integral infinity weight,

$$\alpha_{\mathrm{f}}(u)^h = \alpha_{\mathrm{f}}(k)\alpha_\infty(a)^{-1}.$$

The first factor is a root of unity. Every conjugate of a is a unit above ℓ, so the infinity weight monomial in the second factor is an ℓ-adic unit as well. Applied to the splitting character, the square identity for Δ_g in that section now gives $v_\ell(\Delta_g) = 0$. The Heisenberg phases are roots of unity. The finite vacuum formula therefore preserves the integral span of translates with Dx integral above ℓ, also at $\ell = 2$.

Put $\mathfrak{q} = \prod_{w \in Q} \mathfrak{p}_w$, so that $q_Q \in \mathfrak{q}$. In the rank-one test the old column is prescribed modulo π_M. At each new place only its distinguished entry is prescribed modulo $\mathfrak{p}_w$. Thus this indicator factors through the column space modulo $\mathfrak{m}_1 = (\pi_M)\mathfrak{q}$, including when written on the larger modulus $(\pi_M^2)\mathfrak{q}$. On the smaller space it is a sum of distinct cell indicators with coefficient one. Since $M = \pi_M \bar{\pi}_M$, we have $q_Q M \in \mathfrak{m}_1$. Apply Fourier inversion at $\mathfrak{m}_1$ with $D = M$ and $(n', n) = (a_0, 1)$. Since $\mathrm{Nm}_{E/\mathbb{Q}}(\pi_M) = M^f$, the full column space there has cardinality $(M^f q_Q)^{a_0}$. The column polynomial has integral coefficients and degree $u = \sum_v (c_v - z)$. Using the ordinary jet bound at the integral scale $M d_K s_{\mathrm{jet}}$ and homogeneity at the actual scale $\pi_M s_{\mathrm{jet}}$ shows that the Fourier-cleared column kernel has lattice norm at most $|\pi_M/(M d_K)|_\ell^u$ at its actual target point. Here the column kernel already includes the vacuum scalar $s_{\mathrm{vac}}^{a_0}$. Thus

$$M^{2fa_0}(M d_K)^u K_Q(g, h)$$

has lattice norm at most one in the Weyl coordinates transported by h. The source vector is the constant 1 in the trivial source weight. Rational equivariance and weak approximation justify the lattice norm at arbitrary g, h. In particular the bound applies to both coefficient embeddings in the Gram pairing and to the old level primes and 2.

The rank-two determinant point test prescribes the identity matrix modulo $\mathfrak{m}_2 = (\pi_M^2)\mathfrak{q}$. Its matrix space has cardinality $(M^{2f} q_Q)^4$. Here $q_Q M^2 \in \mathfrak{m}_2$, so take $D = M^2$, $(n', n) = (2, 2)$, and $Q(Z) = \prod_v \det Z_v^{k_v}$, where $k_v = c_v - 1$. This polynomial has integer coefficients and degree $l = 2\sum_v k_v$; the datum is the indicator of the identity residue matrix. The point kernel uses

the unscaled vacuum and the unscaled homogeneous jet. The normalized kernel is exactly the jet of the scalar-cleared datum at the scale $c = M^2 d_K s_{\mathrm{jet}}$. By homogeneity,

$$s_p K^{\mathrm{pt}}_{Q,1}(h) = Q(\partial)\Theta_{F_h}(cz)\big|_{z=0}, \qquad s_p = (M^2)^{8f} s_{\mathrm{vac}}^4 c^l.$$

Here F_h is the paired Weil translate at $(1,h)$ of the identity-residue datum multiplied by $(M^2)^{8f} s_{\mathrm{vac}}^4$. Choose $\gamma \in H(F_0)$ by weak approximation so that γh is integral invertible at all places above ℓ. By rational invariance of the adelic theta functional, left multiplication by γ in the finite argument acts on the real test by the inverse real action. At $\tau = 1$ the endpoint character is $\alpha_1 = \nu\upsilon|_{U(1)}$ of weights $-k_v$, so its factor at γ_∞^{-1} is $A_\gamma = \prod_v \Phi_v(\det \gamma)^{k_v}$. The bare determinant polynomial contributes A_γ^{-1} by the real action in Section 30.2. Thus $K^{\mathrm{pt}}_{Q,1}(\gamma h) = K^{\mathrm{pt}}_{Q,1}(h)$. The datum $F_{\gamma h}$ is standard of level $\delta = M^2$. The ordinary jet estimate at scale $M^2 d_K s_{\mathrm{jet}}$ gives $s_p K^{\mathrm{pt}}_{Q,1}(h) \in \mathcal{O}$. The multiplier s_p is independent of the point representative and of Q. The statement applies before the squared-modulus coefficient system is formed and supplies integrality at both embeddings.

45 Proof of the determinant estimate

In this section, we combine the determinant-ratio height of Proposition 44.2 with the spanning vectors of Proposition 44.3. At each coefficient place, the same auxiliary twist works in both factors of the terminal Hermitian pairing. This permits the use of the product formula in Section 23 and proves Proposition 5.2.

Proof of Proposition 5.2. The proof applies to the original datum of Section 16.1 at every sufficiently large integer x for fixed q_0, d, ϵ. Use the weighted chain $T_\xi = L_\xi$ of Section 17.3 and restrict it to V. The injectivity argument in Section 44.1 uses the first-arrow return and (GC1) before division by the real square factor. The single lifts are injective by Section 41.1.6 and Proposition 39.1.

On this datum the real scalar in (72) is nonzero. The unscaled diagonal moment of Proposition 9.2 and the source codimension bound of Proposition 6.2 give

$$\|pBP_{C(S)}\|_{\mathrm{HS}}^2 \geq (1 - b^{-8} - 5b^{-4}) d_X > 0.$$

Since $B = (v_G \nu)^{-1/2} \Pi C^{-1}$, some $\phi \in S$ has $p\Pi\phi \neq 0$ in V. If $c_{\mathrm{meas}} = 0$, (72) at the trivial twist gives $L_1 \Pi\phi = 0$. Lemma 17.2 then gives $L_1(p\Pi\phi) = 0$, contrary to injectivity of the weighted chain on V. Therefore

$$c_{\mathrm{meas}} \neq 0. \tag{115}$$

The diagonal moment and the chain injectivity here concern the unscaled maps B and L_1.

For each κ use the twist ξ_κ of Section 38. Its modulus is prime to the rational prime under κ. It supplies the same signed critical ratios and main Selmer tests for all the original lift indices. The compression in Section 42.1 therefore uses the regular twist and its powers on the original projected subspaces. The finite spherical trace identity (4) identifies the corresponding Rallis determinants. Equation (112) gives $j_Z \leq D(b + L - A)$ on this family.

Proposition 44.3 supplies a spanning set with

$$b_{\mathrm{short}} = 2x_1^2 b^2, \qquad j_\gamma \leq b.$$

At each κ, the one scalar γ_{ξ_κ} clears these forms at both κ and $\bar{\kappa}$. The height multipliers are independent of the coefficient place.

Let $n = \#y$ for an independent tuple as in the proposition. The images of $\mathcal{Z}$ span $V/\operatorname{span}(y)$; choose representatives $z_1, \dots, z_s \in \mathcal{Z}$ of a basis of this quotient. Then $s = D - n \le 4D/b^2$ and (y, z) is a basis chosen before κ. For fixed κ, the index hypothesis allows a clearing matrix for y at κ and one at $\bar{\kappa}$, both valid for the same ξ_κ since its modulus is prime to their common residue characteristic. For a tuple of translated images, Proposition 22.2 supplies both clearing matrices for the fixed tuple, each valid for every allowed twist of level prime to that residue characteristic. The free terminal classes give integral pairings at κ, as proved in Section 23. Proposition 23.1 gives

$$\begin{aligned} 0 &\le c + 2\delta + s(2x_1^2 b^2 + b) + D(b + L - A) \\ &\le c + 2\delta + D(2b + L - A) \\ &\le c + 2\delta + D(3L - A). \end{aligned}$$

Here $L = faB^4 \ge b$, and

$$s(2x_1^2 b^2 + b) \le D(8x_1^2 + 4/b) \le Db$$

by $s \le 4D/b^2$ and $x_1^{10^6} \le b$ for sufficiently large x. This gives $D(A - 3L) \le c + 2\delta$ with a cutoff depending only on the fixed q_0, d, ϵ. □

Part VI The analytic conclusion

46 The analytic criterion

In this section, we give an analytic criterion in terms of a one-sided prime sum. We choose the auxiliary quadratic twist so that the resulting Dirichlet L-functions have no zeros in a prescribed rectangle. Section 47 derives the estimate used in the Mellin transform argument from Proposition 5.3.

46.1 The one-sided Mellin criterion

In this subsection, we use the prime sums of Section 2.1 to rule out zeros by positivity.

Fix $8 \mid q_0$, a reduced class $d_0 \equiv 7 \bmod 8$, and $1/2 < \beta < 1$. The required bound is $R_{d,q_0}(x) \le x^{\beta-1}$ for all sufficiently large real x, for each prime $d \equiv d_0 \bmod q_0$. The threshold may depend on d. Here, with $\varphi = \varphi(q_0)$,

$$\lambda(n) = \mathbf{1}_{n \equiv 1 \bmod q_0} \Lambda(n)(1 - (n/d)), \qquad R_{d,q_0}(x) = \sum_{n \le x} \lambda(n)(n^{-1} - x^{-1}) - \frac{\log x}{\varphi} - c.$$

For a character χ, let F_χ be $L(s, \chi)$ with its pole removed by multiplication by $s - 1$ when χ is principal.

Dirichlet nonvanishing gives $F_\chi(1) \ne 0$ for nonprincipal χ [54, Theorem 4.9]. In the principal case, $F_\chi(1) = \prod_{p \mid \operatorname{mod}(\chi)} (1 - p^{-1})$. Set

$$D(s) = \sum_{\chi \bmod q_0} \left(\frac{F'_{\chi(\cdot/d)}(s)}{F_{\chi(\cdot/d)}(s)} - \frac{F'_\chi(s)}{F_\chi(s)} \right), \qquad c = \frac{D(1) - 1}{\varphi}.$$

The same definitions with modulus $q = 1$ give $R_{d,1}$ and $c_{d,1}$. The character set is closed under conjugation, so $D(1)$ is real. Since $(d, q_0) = 1$, the Chinese remainder theorem identifies the restriction of each twist to $(\mathbb{Z}/d\mathbb{Z})^\times$ with the nontrivial quadratic character. Thus the untwisted principal character contributes the only pole of an unregularized factor at 1.

Lemma 46.1 (One-sided remainder criterion)**.** *Fix $1/2 < \beta < 1$, $H_0 > 0$, and a prime d in the prescribed class. Suppose the twisted factors have no zeros in $\Re s \ge \beta$, $|\Im s| \le H_0$ and $R_{d,q_0}(x) \le x^{\beta-1}$ for all sufficiently large real x. Then the untwisted factors F_χ have no zeros at*

$$\Re s > \beta, \qquad |\Im s| \le H_0.$$

Proof. For integers $N \ge 2$, the divisor identity $\sum_{n|m} \Lambda(n) = \log m$ gives

$$\frac{N}{2} \sum_{n \le N} \frac{\Lambda(n)}{n} \le \sum_{n \le N} \Lambda(n) \lfloor N/n \rfloor = \log(N!).$$

Since $0 \le \lambda(n) \le 2\Lambda(n)$, it follows that $R_{d,q_0}(x) = O_{d,q_0}(1 + \log x)$. For $\sigma = \Re s > 1$, character orthogonality and the Euler products give

$$\sum_n \lambda(n) n^{-s} = \frac{D(s) + (s-1)^{-1}}{\varphi}, \qquad \int_n^\infty (n^{-1} - x^{-1}) x^{-s}\, dx = \frac{n^{-s}}{s(s-1)}.$$

At powers of d the twisted character vanishes, as required by $1-(n/d)$. The kernel is nonnegative for real $x \ge n$. The second identity at $s = \sigma$ and $\sum_n \lambda(n) n^{-\sigma} < \infty$ justify the exchange of sum and integral.

The transforms of $\log x$ and 1 are $(s-1)^{-2}$ and $(s-1)^{-1}$, respectively. Therefore

$$\mathcal{M}_R(s) = \int_1^\infty R_{d,q_0}(x) x^{-s}\, dx = \frac{D(s) - 1 - s(D(1) - 1)}{\varphi s(s-1)} \qquad (\Re s > 1). \tag{116}$$

The right side is meromorphic on $\Re s > 0$. Its value at the removable singularity $s = 1$ is $(D'(1) - D(1) + 1)/\varphi$. Each logarithmic derivative has at most a simple pole at a zero. Thus for $\Re\rho > 0$, $\rho \ne 1$,

$$\operatorname{Res}_{s=\rho} \mathcal{M}_R(s) = \frac{\sum_{\chi \bmod q_0} (\operatorname{ord}_\rho F_\chi - \operatorname{ord}_\rho F_{\chi(\cdot/d)})}{\varphi \rho(1-\rho)}.$$

An Euler factor omitted when passing from a primitive character to its induced character has the form $1 - \alpha p^{-s}$, $|\alpha| = 1$, and is nonzero in this half-plane. In the prescribed rectangle every twisted order vanishes, including on $|\Im\rho| = H_0$. The numerator is then the total untwisted multiplicity. At a real $\beta < \rho < 1$ it gives a strictly positive residue whenever there is a zero. The Euler products are nonzero on $\Re s > 1$.

For $x \ge X_0$ sufficiently large put $h(x) = x^{\beta-1} - R_{d,q_0}(x) \ge 0$. For $\Re s > 1$, its Mellin transform is

$$\int_{X_0}^\infty h(x) x^{-s}\, dx = \frac{1}{s-\beta} - \mathcal{M}_R(s) - \int_1^{X_0} (x^{\beta-1} - R_{d,q_0}(x)) x^{-s}\, dx. \tag{117}$$

The last integral is entire because its interval is compact and R_{d,q_0} is locally bounded. Denote the tail transform by H and its real abscissa of convergence by $a \in [-\infty, 1]$. Positivity gives

absolute convergence and holomorphy on $\Re s > a$. By the meromorphic identity theorem, (117) holds on $\Re s > \max(a, \beta)$.

Suppose $a > \beta$. Landau's argument [54, Lemma 15.1] gives a singularity at a. Indeed, if H were holomorphic across a, the Taylor series about $b = a + \delta$ would reach $b - 3\delta/2$ for sufficiently small $\delta > 0$. Differentiation is justified by convergence at $a + \delta/2$. Since $X_0 \geq 1$, Tonelli's theorem gives

$$\int_{X_0}^{\infty} h(x) x^{-a+\delta/2}\, dx = \sum_{k\geq 0} \frac{(3\delta/2)^k}{k!} \int_{X_0}^{\infty} h(x) x^{-b} (\log x)^k\, dx = \sum_{k\geq 0} \frac{(-3\delta/2)^k}{k!} H^{(k)}(b) < \infty,$$

contradicting the definition of a. At $a = 1$ the right side of (117) is regular. If $\beta < a < 1$, its only possible singularity is a simple pole of strictly negative residue. Such a pole gives $H(\sigma) \to -\infty$ as $\sigma \downarrow a$, whereas $H(\sigma) \geq 0$. Thus $a \leq \beta$. On the tail we have $|R_{d,q_0}(x)| \leq x^{\beta-1} + h(x)$, so $\mathcal{M}_R$ itself converges absolutely and is holomorphic on $\Re s > \beta$. Its meromorphic formula agrees with this integral throughout that half-plane. A zero of an untwisted factor there at height $|\Im \rho| \leq H_0$ would give a nonzero residue there. □

46.2 Choosing a twist with no zeros in the rectangle

In this subsection, we choose the prime needed for the Mellin criterion of Section 46.1.

Proposition 46.2 (A quadratic twist avoiding a fixed box)**.** *Let* $8 \mid q_0$, $(d_0, q_0) = 1$, *and* $d_0 \equiv 7 \pmod 8$. *Fix* $1/2 < \beta < 1$ *and* $H_0 > 0$. *There is a prime* $d \equiv d_0 \bmod q_0$ *such that*

$$L(s, \chi(\cdot/d)) \neq 0 \quad (\chi \bmod q_0,\ \Re s \geq \beta,\ |\Im s| \leq H_0).$$

Proof. For simultaneous twist selection set

$$\mathcal{P} = \{d \text{ prime} : d \equiv d_0 \pmod{q_0}\}.$$

Fix a character $\xi \bmod q_0$, and let ξ^* be its primitive inducing character. Since d_0 is a reduced class, $d \nmid q_0$ for $d \in \mathcal{P}$. The two primitive characters in $\chi = \xi^*(\cdot/d)$ have coprime conductors, so their product is primitive with conductor $Q = \operatorname{cond}(\xi^*)d > 1$ [54, Lemma 9.3]. The Euler factors removed from $L(s, \chi)$ in passing to the twist modulo $q_0 d$ are nonzero on $\Re s > 0$. The Euler product gives nonvanishing for $\Re s > 1$, so it suffices to prove the assertion on the closed rectangle $\beta \leq \Re s \leq 1$, $|\Im s| \leq H_0$. For $d \leq X$ put $T_{\mathrm{an}} = (q_0 X)^{1/2} \geq \sqrt{Q}$, and let $j \in \{0, 1\}$ be the parity of χ. Since $d \equiv 3 \pmod 4$, $(-1)^j = -\xi^*(-1)$, which depends only on ξ. For $a = \chi, \mu\chi$ set

$$\theta_a(y) = y^{1/2} \sum_{n\geq 1} a(n)(ny)^j e^{-\pi(ny)^2}, \qquad J_d = \int_{1/T_{\mathrm{an}}}^{\infty} \left(|\theta_\chi(y)| + |\theta_{\mu\chi}(y)|\right) \frac{dy}{y}.$$

Here $J_d = J_{d,\xi}(T_{\mathrm{an}})$ depends on both the character and the cutoff. Put $\Gamma_j(s) = \pi^{-(s+j)/2}\Gamma((s+j)/2)/2$. With $\tau(\chi)$ the Gauss sum, the theta transformation [54, Theorem 10.6, (10.16)–(10.17)] at Qy^2 gives

$$\theta_\chi(y) = \varepsilon_\chi \theta_{\bar\chi}(1/(Qy)), \qquad \varepsilon_\chi = \frac{\tau(\chi)}{i^j \sqrt{Q}}, \qquad |\varepsilon_\chi| = 1.$$

On $\Re s > 1$, termwise integration with $t = \pi(ny)^2$ gives the full Mellin integral $\Gamma_j(s)L(s,\chi)$. The theta transformation gives exponential decay at zero and infinity, so the integral continues this product. Splitting at $Q^{-1/2}$ and substituting $y \mapsto 1/(Qy)$ gives

$$\Gamma_j(s)L(s,\chi) = I_\chi(s) + \varepsilon_\chi Q^{1/2-s} I_{\bar\chi}(1-s), \qquad I_a(s) = \int_{Q^{-1/2}}^{\infty} \theta_a(y) y^{s-1/2} \frac{dy}{y}.$$

On $\Re s = 1/2$ the power of Q has modulus one and $|\theta_{\bar\chi}| = |\theta_\chi|$. Since $T_{\mathrm{an}} \geq \sqrt{Q}$, each integral is bounded by J_d there. Let $M(s)$ be the Mellin integral of $\theta_{\mu\chi}(y)y^{s-1/2}$ from $1/T_{\mathrm{an}}$ to ∞ with measure dy/y. Then $K(s) = \Gamma_j(s)L(s,\chi)M(s) - \Gamma_j(s)^2$ satisfies $|K| \ll 1 + J_d^2$ on $\Re s = 1/2$, since $|M(s)| \leq J_d$ there and Γ_j is bounded on that line for both parities. On $\Re s = 2$, absolute convergence and $\sum_n \mu(n)\chi(n)n^{-s} = 1/L(s,\chi)$ identify the full reciprocal integral with $\Gamma_j(s)/L(s,\chi)$. For $0 < y \leq 1$, $\sum_{n\geq 1}(ny)^j e^{-\pi(ny)^2} \ll 1/y$, by comparison with a decreasing Gaussian of smaller exponent. Hence $|\theta_{\mu\chi}(y)| \ll y^{-1/2}$, and the removed tail is bounded by

$$\int_0^{1/T_{\mathrm{an}}} y^{3/2} O(y^{-1/2}) \, \frac{dy}{y} = O(T_{\mathrm{an}}^{-1}).$$

The Euler integral for Γ_j and absolute convergence of the Dirichlet series bound $\Gamma_j(s)L(s,\chi)$ uniformly.

Thus $|K| \ll T_{\mathrm{an}}^{-1}$ there. For fixed χ, theta reflection gives exponential decay at both ends of the completed Mellin integral. On $1/2 \leq \Re s \leq 2$ the integrals defining $\Gamma_j L$ and M have integrable majorants independent of $\Im s$. The majorants remain integrable after multiplication by any fixed power of $|\log y|$; hence the integrals are holomorphic in the interior and continuous and bounded on the closed strip. The Euler integral gives the same strip bound for Γ_j. The two boundary estimates for K are uniform in the character and cutoff. Suppose $J_d \leq T_{\mathrm{an}}^{\kappa}$ with $\kappa = (\beta - 1/2)/8$.

For $\lambda = (\Re s - 1/2)/(3/2)$, the three-lines theorem gives

$$|K(s)| \ll T_{\mathrm{an}}^{2\kappa(1-\lambda)-\lambda} \leq T_{\mathrm{an}}^{-5(\beta-1/2)/12} \quad (\beta \leq \Re s \leq 1).$$

In particular $K = o(1)$ for the fixed range $\beta \leq \Re s \leq 1$, $|\Im s| \leq H_0$. On a zero in this box $K = -\Gamma_j(s)^2$, whose absolute value has a positive lower bound depending only on β, H_0 for the two parities. Thus, for sufficiently large T_{an}, even a real zero forces $J_{d,\xi}(T_{\mathrm{an}}) > T_{\mathrm{an}}^{\kappa}$.

For $d \equiv 7 \bmod 8$, quadratic reciprocity gives $(n/d) = (-d/n_{\mathrm{odd}})$, where n_{odd} is the odd part of n, with Jacobi symbol equal to one for denominator one. For each of $b(n) = \xi^*(n)$, $\mu(n)\xi^*(n)$, let

$$G_j(t) = t^j e^{-\pi t^2}, \qquad \Theta_b(h,y) = y^{1/2} \sum_{n\geq 1} b(n) \Big(\frac{-h}{n_{\mathrm{odd}}}\Big) G_j(ny).$$

For $h = d \in \mathcal{P}$ these give the two theta sums in $J_{d,\xi}$. They converge absolutely and locally uniformly for $y > 0$, hence are continuous in y. Positivity gives

$$\sum_{\substack{d\leq X \\ d\in\mathcal{P}}} |\Theta_b(d,y)|^2 \leq \sum_{1\leq h\leq X} |\Theta_b(h,y)|^2.$$

For fixed $y > 0$ absolute Gaussian convergence permits expansion of the square on the right. At $u = (nm)_{\mathrm{odd}}$ the character sum is $\sum_{h\leq X}(-h/u)$. It is bounded by X for square u. Otherwise a prime divides u to odd order, so $(\cdot/u)$ is a nonprincipal character modulo u by the Chinese

remainder theorem. Pólya–Vinogradov [54, Theorem 9.18], which also covers imprimitive characters, bounds the sum by $O(\sqrt{u}\log(2u)) \ll_\eta (nm)^{1/2+\eta}$. For $n \le A,\ m \le B$, the square case gives

$$\#\{(n,m) : (nm)_{\mathrm{odd}} \text{ is a square}\} \le \sum_{a,b\ge 0} \sum_{k \le \min(A,B)} \frac{\sqrt{AB}}{k\, 2^{(a+b)/2}} \ll \sqrt{AB}\log(2\min(A,B)).$$

Here $n = 2^a k h_1^2,\ m = 2^b k h_2^2$ with k odd and squarefree. For $y \le 1$, split ny, my into $[0,1]$ and $(2^{h-1}, 2^h]$, $h \ge 1$. On the interval with index $h \ge 0$, $|G_j| \ll e^{-c4^h}$ for some $c > 0$. With $A = 2^h/y,\ B = 2^{h'}/y$, the contributions before multiplication by y are bounded by

$$X \sum_{h,h'\ge 0} e^{-c(4^h+4^{h'})}\sqrt{AB}\log(2\min(A,B)) \ll Xy^{-1}\log(2/y),$$

$$\sum_{h,h'\ge 0} e^{-c(4^h+4^{h'})}(AB)^{3/2+\eta} \ll_\eta y^{-3-2\eta}.$$

Consequently

$$\sum_{1\le h\le X} |\Theta_b(h,y)|^2 \ll_\eta X\log(2/y) + y^{-2-2\eta} \qquad (1/T_{\mathrm{an}} \le y \le 1).$$

For $y > 1$, the termwise absolute bound $y^{1/2}\sum_{n\ge 1}|G_j(ny)| \ll e^{-c'y^2}$ suffices. Integrating Cauchy–Schwarz in the finite sum over h gives

$$\sum_{\substack{d\le X\\ d\in\mathcal{P}}} J_d \ll_\eta X\left(1 + \int_{1/T_{\mathrm{an}}}^1 \sqrt{\log(2/y)}\,\frac{dy}{y}\right) + \sqrt{X}\int_{1/T_{\mathrm{an}}}^1 y^{-2-\eta}\,dy \ll_{q_0,\eta} T_{\mathrm{an}}^{2+\eta}. \tag{118}$$

Let $\mathcal{B} \subseteq \mathcal{P}$ consist of the primes for which at least one prescribed twist has a zero in the fixed box. For each sufficiently large X, use the same $T_{\mathrm{an}} = \sqrt{q_0 X}$ in (118) for every ξ. With $\eta = \kappa/2$,

$$\#\{d \le X : d \in \mathcal{B}\} \le T_{\mathrm{an}}^{-\kappa} \sum_{\xi \bmod q_0} \sum_{\substack{d\le X\\ d\in\mathcal{P}}} J_{d,\xi}(T_{\mathrm{an}}) \ll_{q_0,\beta,H_0} X^{1-\kappa/4}.$$

Here the outer sum runs over characters. Partial summation gives $\sum_{d\in\mathcal{B}} d^{-1} < \infty$. The reciprocal sum of primes in the prescribed reduced class diverges [54, Corollary 4.12(c)], so some prime from the class avoids all the bad zeros. □

For $q_0 = 18480q$ and $d_0 = q_0/2 - 1$, the hypotheses on the residue class follow from Lemma 4.1. The bound for this progression modulus is derived in the next section.

47 The bound for the prime remainder

In this section, we apply the cancellation estimate of Proposition 5.3 to the prime sum of Section 46. The error from the auxiliary scale x_1 is estimated with the fixed-modulus prime number theorem. We then pass from integer to real arguments before applying the criterion.

Proposition 47.1 (The one-sided prime-remainder bound)**.** *Fix q_0 with $18480 \mid q_0$ and set $d_0 = q_0/2 - 1$. For each prime $d \equiv d_0 \pmod{q_0}$ and $0 < \epsilon < 1/100$, there is a constant $C_{q_0,d,\epsilon}$ with*

$$R_{d,q_0}(y) \le C_{q_0,d,\epsilon} y^{-1/2+6\epsilon}$$

for all sufficiently large real y.

Proof. Write $B = \log b$, $u = \#\Phi_a$, $v = |T| > 0$, and $\mathcal{R}$ for the total remainder defined in Section 5. The exact expansion is

$$\frac{\mathcal{R}}{av} = R_{d,q_0}(x) - R_{d,q_0}(x_1) + \frac{u}{v} R_{d,1}(x_1).$$

The datum has $u/v \le B$ and $f = \varphi(q_0)u$. Thus $m\mathcal{R} \le faB^5$ gives

$$R_{d,q_0}(x) \le \frac{\varphi(q_0)B^6}{m} + |R_{d,q_0}(x_1)| + B|R_{d,1}(x_1)|.$$

For either $q \in \{q_0, 1\}$, put

$$S_q(t) = \sum_{n \le t} \lambda_{d,q}(n), \qquad E_q(t) = S_q(t) - t/\varphi(q).$$

The prime number theorem in arithmetic progressions [54, Corollary 11.19] applies at the fixed modulus qd. Let $b_a \bmod qd$ be the class $1 \bmod q$, $a \bmod d$, and write $\psi(t; qd, b_a) = \sum_{n \le t,\ n \equiv b_a \ (qd)} \Lambda(n)$. With $o_1 = 1$ and $o_{q_0} = 2$, Lemma 4.1 gives

$$S_q(t) = \sum_{a \in (\mathbb{Z}/d\mathbb{Z})^\times} (1 - \chi_d(a))\psi(t; qd, b_a) + \left\lfloor \frac{\log t}{o_q \log d} \right\rfloor \log d.$$

Indeed $d^e \equiv 1 \bmod q$ precisely when $o_q \mid e$; these powers have weight one since $\chi_d(d^e) = 0$. If $p \mid q_0$, then p splits in K by Lemma 4.1. Hence $1 - \chi_d(p^e) = 0$ for every $e \ge 1$, also in the modulus-one sum. The reduced-class weights sum to $d - 1$, so their main terms sum to $(d-1)t/\varphi(qd) = t/\varphi(q)$. For $t \ge e^{q_0 d}$ both progression moduli satisfy $qd \le \log t$. Applying the cited corollary with this exponent one and bounding the powers of d gives, for an absolute $c_1 > 0$,

$$|E_q(t)| \ll (d-1)te^{-c_1\sqrt{\log t}} + \log t.$$

Absorbing the remaining bounded interval into the constant gives, for every $A > 0$,

$$E_q(t) = O_{q,d,A}\big(t(\log t)^{-A-1}\big) \qquad (t \ge 2).$$

For real $\sigma > 1$, the Dirichlet series identity in Section 46.1 at modulus q gives

$$\int_1^\infty E_q(t) t^{-\sigma-1}\, dt = \frac{D_q(\sigma) + (\sigma-1)^{-1}}{\sigma\varphi(q)} - \frac{1}{\varphi(q)(\sigma-1)} = \frac{D_q(\sigma) - 1}{\sigma\varphi(q)}.$$

The bound on E_q supplies an integrable majorant at $\sigma = 1$. Letting $\sigma \downarrow 1$ identifies $c_{d,q} = \int_1^\infty E_q(t)t^{-2}\, dt$. Integrating each term in the finite sum for S_q up to y now gives the exact tail and its bound

$$R_{d,q}(y) = -\int_y^\infty E_q(t) t^{-2}\, dt = O_{q,d,A}\big((\log y)^{-A}\big).$$

Choose $0 < \epsilon < 1/100$ and use a datum at each large integer x, with $B \le 2x^{\epsilon}$, $m \ge \sqrt{x}$, and $\log x_1 \ge x^{\epsilon/12}$. Take $A > 24/\epsilon$ in the fixed-modulus bound. Both remainders at x_1 are at most x^{-2} in absolute value for sufficiently large x. Consequently

$$R_{d,q_0}(x) \le (64\varphi(q_0) + 3)x^{-1/2+6\epsilon}.$$

For real $y \ge 2$, the exact tail gives

$$R_{d,q_0}(y) - R_{d,q_0}(\lfloor y \rfloor) = \int_{\lfloor y \rfloor}^{y} E_{q_0}(t)t^{-2}\,dt = O_{q_0,d}(y^{-1}).$$

Put $\alpha = -1/2 + 6\epsilon$. As $-1 < \alpha < 0$ and $\lfloor y \rfloor \ge y/2$, we have $\lfloor y \rfloor^{\alpha} \le 2^{-\alpha} y^{\alpha}$. The integer bound gives

$$R_{d,q_0}(y) \le R_{d,q_0}(\lfloor y \rfloor) + O_{q_0,d}(y^{-1}) \le C_{q_0,d,\epsilon} y^{-1/2+6\epsilon}. \qquad \square$$

Proof of Theorem 1.1. Let χ be a character modulo q. Induce it to $\widetilde{\chi}$ modulo $q_0 = 18480q$. The Euler product gives

$$L(s,\widetilde{\chi}) = L(s,\chi) \prod_{\substack{p \mid q_0 \\ p \nmid q}} (1 - \chi(p)p^{-s}).$$

The finite factors are nonzero for $\Re s > 0$. Suppose $L(\rho,\chi) = 0$ with $1/2 < \Re\rho < 1$. Choose

$$\tfrac{1}{2} < \beta < \Re\rho, \qquad H_0 > |\Im\rho|.$$

Proposition 46.2, with $d_0 = q_0/2 - 1$, gives a prime d for which all the level-q_0 quadratic twists are zero-free in the prescribed rectangle. Fix d and take $0 < \epsilon < 1/100$ with $6\epsilon < \beta - 1/2$. Proposition 47.1 gives a cutoff $X_R = X_R(q_0, d, \epsilon)$ and a constant $C = C_{q_0,d,\epsilon} \ge 1$. Since $\beta - 1/2 - 6\epsilon > 0$,

$$x \ge \max\{X_R, C^{1/(\beta-1/2-6\epsilon)}\} \quad \Longrightarrow \quad R_{d,q_0}(x) \le Cx^{-1/2+6\epsilon} \le x^{\beta-1}.$$

Hence Lemma 46.1 gives $L(\rho,\widetilde{\chi}) \ne 0$, a contradiction. For $\Re s > 1$, use the Euler product. At $s = 1$, use [54, Theorem 4.9] and the simple pole for the principal character. At $s = 1 + it$, $t \ne 0$, the nonnegative identity $3 + 4\cos\theta + \cos(2\theta) = 2(1 + \cos\theta)^2$ applied termwise to the Euler logarithms gives

$$\zeta(\sigma)^3 |L(\sigma + it, \chi)|^4 |L(\sigma + 2it, \chi^2)| \ge 1 \qquad (\sigma > 1).$$

Primes dividing q contribute only the positive zeta term. As $\sigma \downarrow 1$, the last factor stays bounded because its only possible pole is at 1. A zero of the middle factor at $1 + it$ would make the product $O(\sigma - 1)$, since ζ has a simple pole. This contradicts the lower bound.

Let ψ be the primitive character inducing χ, of conductor q_*. The extra Euler factors relating their L-functions [54, (10.20)] can vanish only on $\Re s = 0$, by taking absolute values in $\psi(p)p^{-s} = 1$ at a deleted prime. For $q_* > 1$ put

$$\Lambda_{\psi}(s) = (q_*/\pi)^{(s+j)/2}\Gamma((s+j)/2)L(s,\psi), \qquad \psi(-1) = (-1)^j.$$

The primitive functional equation [54, Corollary 10.8, (10.19)] relates $\Lambda_{\psi}(s)$ to $\Lambda_{\bar{\psi}}(1-s)$ by a nonzero constant. For $q_* = 1$, the inducing function is ζ and its completed function has the corresponding reflection [54, Corollary 10.3]. In the open strip the gamma factors are finite and

nonzero. Thus a zero with $0 < \Re s < 1/2$ would give a zero at $1 - s$ for the primitive conjugate, already excluded.

For $q_* > 1$ and $\Re s \le 0$, the reflected completed function is finite and nonzero by the nonvanishing on $\Re(1-s) \ge 1$. The only zeros of $L(s,\psi)$ there come from the reciprocal gamma factor, at $s = -j - 2n$ for $n \ge 0$, each simple. For $q_* = 1$ use [54, Corollary 10.4, (10.9)] in the form

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

For $\Re s \le 0$ and $s \ne 0$ this gives precisely the negative even zeros. At $s = 0$ the sine cancels the pole of $\zeta(1-s)$ and gives $\zeta(0) = -1/2$. For imprimitive characters the deleted Euler factors account for all additional boundary zeros. For each prime $p \mid q$ with $p \nmid q_*$, the deleted factor contributes a simple zero at each $i(\arg \psi(p) + 2\pi k)/\log p$, $k \in \mathbb{Z}$. At a nonzero ordinate t, this can occur for at most one prime. Indeed, $t \log p/(2\pi)$ is a nonzero rational number because $\psi(p)$ is a root of unity, whereas $\log p/\log p'$ is irrational for distinct primes by unique factorization. At the origin the order is

$$\operatorname{ord}_{s=0} L(s,\chi) = \mathbf{1}_{\{q_*>1,\ \psi(-1)=1\}} + \#\{p \mid q : p \nmid q_*,\ \psi(p) = 1\}. \qquad \square$$

A Period rings and integral comparison matrices

In this appendix, we construct the coordinate maps and comparison matrices for the local lattices of Section 29.1. The approximation and divisibility results give the integral period bases used in Appendix B.

Use the local representations, jump convention and normalized defect of Section 29.1. Here p is the residue characteristic at the coefficient place.

A.1 Integral coordinates for the period basis

In this subsection, we construct coordinate maps from A_{inf} to the power series ring S. The construction uses the graded modules for the ideal (p, z), and it extends to the bounded period ring. Section A.5 uses these maps to preserve divisibility at the de Rham period.

Write $A = A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p^\flat})$, $S = W(k)[[z]] \subset A$, and $I = (p, z)A$. Use the chosen Teichmüller coordinate z and generator $E = p + z^e u$ of $\ker \theta$ from Section 29.1.1, or $E = p - z^e$ in the tame chart constructed below. Here K is the finite extension of $\mathbf{Q}_p$ defining the chart, k its residue field, e its absolute ramification index, and $u \in S^\times$. For the comparison matrix assume that $T[1/p]$ is crystalline over K. If T is defined over the integers O of a finite coefficient field, take its underlying $\mathbf{Z}_p$-lattice of rank $d = \operatorname{rank}_{\mathbf{Z}_p}(O) \operatorname{rank}_O T$. The coefficient action is then by $\mathbf{Z}_p$-linear endomorphisms. The invariant crystalline and de Rham bases below are over $K_0 = W(k)[1/p]$ and K, respectively.

Lemma A.1 (Graded coordinates of the period ring)**.** *The ring A is complete and separated for $I = (p, z)A$. For each $n \ge 0$, there is an isomorphism*

$$\bigoplus_{i=0}^{n} A/I \xrightarrow{\sim} I^n/I^{n+1}, \qquad (\bar{a}_i)_i \longmapsto \left[\sum_{i=0}^{n} a_i p^i z^{n-i}\right].$$

Proof. The reduction of z in the tilt is nonzero and is a nonunit. The ring A is p-torsion free and A/p is a complete rank-one valuation ring. Multiplication by z is injective on A/p^m for each m, by induction using A/p. Also A/p^m is complete and separated for its z-adic topology. To see the induction, use $p^{m-1}A/p^m \simeq A/p$, with its induced topology: if $z^h a$ is in $p^{m-1}A$, then $a \in p^{m-1}A$ by z-injectivity modulo p^{m-1}. Thus the induced topology on the kernel of $A/p^m \to A/p^{m-1}$ is its own z-adic topology. Thus reduction modulo z^h preserves the short exact sequence with these three terms. The transition maps on the kernels $(A/p)/z^h$ are surjective, so the inverse limit over h is still exact. Comparing the two short exact sequences proves completeness and separatedness of A/p^m by induction. Finally

$$I^{m+h-1} \subset (p^m, z^h)A \subset I^{\min(m,h)}, \qquad \varprojlim_{m,h} A/(p^m, z^h) = \varprojlim_{m} A/p^m = A.$$

For each n, the monomials $p^i z^{n-i}$, $0 \le i \le n$, identify I^n/I^{n+1} with $(A/I)^{n+1}$. Surjectivity follows by expansion. For injectivity reduce a relation modulo p and cancel z^n to see that its coefficient of z^n lies in I. Subtract that term. The intersection $pA \cap I^{n+1} = pI^n$ follows by reducing a generating expression for I^{n+1} modulo p: its pure z-term has coefficient divisible by p. Divide the remaining relation by p and induct on n. This proves the asserted graded formula. □

Choose a k-basis of A/I containing 1, and lift its vectors to A, using 1 for the distinguished lift. The graded formula shows that the resulting map $\bigoplus_i S/(p,z)^n \to A/I^n$ is an isomorphism for each n. Taking inverse limits identifies A with the families $(s_i)_i \in \prod_i S$ for which, at each n, all but finitely many s_i lie in $(p,z)^n S$. The inverse sends $(s_i)_i$ to the I-adic sum $\sum_i b_i s_i$, where b_i are the chosen lifts. The projections $F_i : A \to S$ are S-linear and continuous for these adic topologies. They separate elements and preserve p- and z-divisibility. The coordinate of 1 satisfies $F_0|_S = \mathrm{id}_S$.

For the bounded period ring, realize $B^+ = A_{\max}[1/p]$ with $A_{\max}$ the p-adic completion of $A[E/p]$. Equip B^+ with the ring topology whose neighborhoods of zero are the additive lattices $p^n A_{\max}$, $n \ge 0$. All closures and bounded-ring approximations below refer to this topology. For any S-linear $F : A \to A$, first extend by inverting p. Since $E \in S$, the extended map sends $(E/p)^j a$ to $(E/p)^j F(a)$. For a polynomial relation multiply by a power of p clearing the denominators, use S-linearity in A, and cancel that power. The extension preserves $A[E/p]$, hence extends continuously to $A_{\max}$ and B^+. It commutes with multiplication by the closure S_B of $S[1/p] = S[E/p][1/p]$ in B^+, by continuity. For the coordinate maps $F_i : A \to S$ the extension takes values in S_B. This also identifies S_B with the scalars whose multiplication commutes with every S-linear endomorphism of B^+ carrying $A_{\max}$ into itself. Indeed such endomorphisms are continuous and commute with $S[1/p]$, hence with its closure S_B. Conversely, for any scalar b in this commutant, the distinguished coordinate satisfies $b = b\,\widehat{F}_0(1) = \widehat{F}_0(b) \in S_B$. In particular one denominator suffices at all precisions:

$$S_B = (\overline{S[E/p]}^{\,A_{\max}})[1/p].$$

Indeed, for $b \in S_B$ take m with $p^m b \in A_{\max}$. Approximate $p^m b$ by elements of $A[E/p]$. The continuous map $\widehat{F}_0$ fixes $p^m b$ and sends these approximations to $S[E/p]$. The reverse inclusion follows from the definition of S_B. Frobenius preserves S_B: it preserves S, and $\varphi(E)/p = 1 + (z^e)^p \varphi(u)/p$ if $E = p + z^e u$. The relation $z^e/p = (E/p - 1)/u$ puts this expression in $S[E/p]$. Since $\varphi(A) \subset A$ and $\varphi(p) = p$, Frobenius preserves $A[E/p]$ and extends continuously to $A_{\max}$

and B^+. For any finite tuple y over B^+ and any precision $p^j A_{\max}$, approximate its entries by polynomials in E/p after clearing p-denominators.

Expand the finitely many coefficients in the lifted basis $(b_i)_i$ modulo a high power of (p,z). Since $(p,z)^e A \subset pA_{\max}$, these truncations approximate every entry, to the required precision, by a finite combination $y_0 = \sum_{i\in J_0} b_i s_i$ with coefficients in S_B. Use the union of the supports for all entries. For a finite $J \supseteq J_0$ put $P_J = \sum_{i\in J} b_i \widehat{F}_i$. Each P_J preserves $A_{\max}$ and fixes y_0, since $\widehat{F}_i(b_k s) = \delta_{ik} s$ for $s \in S_B$. Thus, entrywise,

$$y - y_0 \in p^j A_{\max} \quad \Longrightarrow \quad y - P_J y = (1 - P_J)(y - y_0) \in p^j A_{\max}.$$

The extended coordinates separate elements of B^+ by separatedness of its lattice topology.

Suppose $U \in \mathrm{Mat}_d(B^+)$ is invertible over $\mathrm{Frac}(B^+)$. If $Uy_j = E^h x_j$ with $y_j \in S_B^d$, an A-linear relation among the x_j is the same as the relation among the y_j. Applying each extended coordinate to a relation $\sum_j a_j y_j = 0$ gives $\sum_j F_i(a_j) y_j = 0$, since $y_j \in S_B^d$. Thus also $\sum_j F_i(a_j) x_j = 0$. If the x_j are S-independent, then $F_i(a_j) = 0$ for every i, j. The coordinates separate elements of A, so all $a_j = 0$. This bounds the cardinality of the family by d.

Let $\mathcal{P}$ be the S-submodule of A^d of vectors x for which $Uy = E^h x$ has a solution $y \in S_B^d$. It is (p,z)-adically separated inside A^d, and p-saturated since $1/p \in S_B$. In particular $\mathcal{P} \cap pA^d = p\mathcal{P}$. For $x \in \mathcal{P}$ with $zx \in p\mathcal{P}$, reduction in the domain A/p gives $x \in pA^d$, hence $x \in p\mathcal{P}$. Thus z acts injectively on $\mathcal{P}/p\mathcal{P}$. Put $J = (p,z)S$. In an element of $p\mathcal{P} \cap J^{n+1}\mathcal{P}$, the coefficient of the pure term z^{n+1} is zero modulo $p\mathcal{P}$, by this injectivity. Thus $p\mathcal{P} \cap J^{n+1}\mathcal{P} = pJ^n\mathcal{P}$. Cancelling p in the remaining terms as in the graded calculation for A gives the monomial formula for $J^n\mathcal{P}/J^{n+1}\mathcal{P}$ with coefficient space $\mathcal{P}/J\mathcal{P}$. Any independent family of residue vectors lifts to an independent family over S, by taking the least (p,z)-order in a presumed relation. So the residue space has dimension at most d. Lift a basis, and express any element of $\mathcal{P}$ successively modulo $(p,z)^n$ in those lifts x_i, correcting the coefficients at step n by elements of $(p,z)^n S$. The coefficients converge to $s_i \in S$. Then

$$x - \sum_i s_i x_i \in (p,z)^n \mathcal{P} \subset I^n A^d \qquad (n \geq 1).$$

Separation of A^d gives equality. Thus the x_i form a finite S-basis of the solution module.

Finally a principal ideal (δ) with $p \nmid \delta$ is closed in A for the topology of (p,z). There is $k > 0$ such that $\bar{\delta} \mid \bar{z}^k$ in the rank-one valuation ring A/p. Write $\delta u = z^k + pv$. Modulo p^m, the geometric sum gives

$$\delta u \sum_{i=0}^{m-1} z^{k(m-1-i)} (-pv)^i \equiv z^{km} \pmod{p^m A}.$$

If δa_j is a Cauchy sequence in A for (p,z), then modulo p^m it is Cauchy for z. Multiplying any difference in $z^{L+km}(A/p^m)$ by the displayed multiplier and cancelling the nonzero divisor z^{km} puts the difference of the a_j's in $z^L(A/p^m)$. This works for every L, m. The a_j therefore converge in A, and their product with δ has limit in (δ). The topology of (p,ξ) is the same here, since ξ is associated to $E = z^e a + p$ with a a unit of the chart. This supplies the closedness used to pass from the finite coordinate partial sums to divisibility in the basis proof.

A.2 Integral approximation in the fixed period chart

In this subsection, we establish divisibility for the coordinate maps of Section A.1 and use the Ax–Sen–Tate theorem in the tower of roots of the uniformizer to approximate fixed tilt vectors.

The approximation gives equality in the prescribed tilt component modulo p. It is used to prove that the determinant in Section B.1 is nonzero modulo p.

For the chart $S \subset A_{\mathrm{inf}}$ of the preceding construction and each integer $j \geq 0$, put $\theta_j = \theta \circ \varphi^{-j}$ and $E_j = \varphi^j E$. The map $\theta_j : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ is surjective, with kernel (E_j), since Frobenius is an automorphism of A and E generates $\ker \theta$. Moreover $\varphi(S) \subset S$, so $E_j \in S$. For any S-linear map $F : A_{\mathrm{inf}} \to S$, $\alpha \in S$, $r \in A_{\mathrm{inf}}$, the implication

$$\theta_j(\alpha) \mid \theta_j(r) \quad \Longrightarrow \quad F(r) - \alpha x \in E_j S \quad \text{for some } x \in S$$

follows by lifting an integral quotient to $a \in A_{\mathrm{inf}}$. Then $r - \alpha a = E_j b$ and one can take $x = F(a)$. This gives the preservation of divisibility by p in the residue at the j-th component for every chart coordinate map used in the period basis.

Set $K_0 = W(k)[1/p]$ and $\pi_j = \theta_j(z)$. The Teichmüller coordinate gives $\pi_{j+1}^p = \pi_j$, with π_0 the chosen uniformizer of K. If $H(X)$ is the Eisenstein polynomial of π_0 over K_0, then $H(X^{p^j})$ is Eisenstein of degree $d_j = ep^j$. Thus $L_j = K_0(\pi_j)$ has degree ep^j over K_0, with $L_0 = K$, and has ring of integers $W(k)[\pi_j]$. Indeed, in the basis $1, \ldots, \pi_j^{d_j - 1}$ the nonzero terms $c_i \pi_j^i$ have valuations $v_p(c_i) + i/d_j$ with distinct fractional parts. Their sum is integral exactly when all $c_i \in W(k)$. Now θ_j maps $W(k)$ onto itself by inverse Frobenius, and every series in S converges upon evaluation at π_j. Consequently

$$R_j = \theta_j(S) = \mathcal{O}_{L_j}, \qquad L_\infty = \bigcup_{j \geq 0} L_j, \qquad \mathcal{O}_{L_\infty} = \bigcup_{j \geq 0} R_j.$$

For $\theta_j(r) \neq 0$, take $n = \lfloor d_j v_p(\theta_j(r)) \rfloor$ and $\alpha = z^n$ in the divisibility implication. Since $v_p(\pi_j) = 1/d_j$,

$$v_p(\theta_j(F(r))) \geq \frac{\lfloor d_j v_p(\theta_j(r)) \rfloor}{d_j}.$$

If $\theta_j(r) = 0$, then $r \in E_j A_{\mathrm{inf}}$ and $\theta_j(F(r)) = 0$. In particular all divisibilities by powers of p are preserved in the residue. The order retraction in Section B.2.2 uses exact vanishing on the corresponding de Rham grades. The pointwise fixer of S fixes each R_j by Galois equivariance of θ_j. Conversely, fixing L_∞ fixes $W(k)$ and every π_j, hence the tilt sequence defining z. Such an element fixes $S = W(k)[[z]]$ by continuity.

Let $y_1, \ldots, y_l \in \mathcal{O}_{\mathbb{C}_p^\flat}$ be fixed by this group, and fix $m \geq 0$. The elements $t_i = \theta_m([y_i])$ are integral and fixed. The algebraic extension $L_\infty / \mathbf{Q}_p$ is Henselian with its induced valuation and has characteristic zero. The Ax–Sen–Tate fixed field theorem [6, theorem, p. 417] gives

$$\mathbb{C}_p^{\mathrm{Gal}(\overline{\mathbf{Q}}_p / L_\infty)} = \widehat{L_\infty}.$$

Choose $x_i \in L_\infty$ with $v_p(x_i - t_i) > 1$ by density. The x_i are integral. Choose $j_0 \geq m$ with $x_1, \ldots, x_l \in R_{j_0}$. For every $j \geq j_0$ the inclusions $R_{j_0} \subset R_j$ give $s_{j,i} \in S$ satisfying

$$\theta_j(s_{j,i}) = x_i \equiv \theta_m([y_i]) \pmod{p} \qquad (1 \leq i \leq l).$$

Thus at every sufficiently large common level j the j-th tilt components of the $s_{j,i}$ mod p equal the prescribed m-th components of all the y_i.

A.3 Uniform denominator clearing in period coordinates

In this subsection, we combine division in $A_{\max}$ with a Frobenius relation to obtain a bound for integral denominators in the solution module of Section A.1. The bound applies uniformly to all its coordinate maps and is used in Section A.6.

In $A = A_{\inf}$ choose the cyclotomic generator of $\ker\theta$ as in [14, §3.3, before Definition 3.22]. Explicitly,

$$\tau = [\epsilon^{1/p}] - 1, \qquad \xi = \sum_{i=0}^{p-1}(1+\tau)^i, \qquad \varphi(\tau) = [\epsilon] - 1 = \mu = \xi\tau.$$

Thus $E/\xi \in A^\times$. Put $R = A[X]/(pX-\xi)$. If $pF = (pX-\xi)G$ in $A[X]$, then $\bar{\xi}\,\bar{G} = 0$ in $(A/p)[X]$. Since $\bar{\xi} \neq 0$ in the tilt domain, $G = pG_1$. Cancelling p shows that R is p-torsion free. It therefore embeds in $A[1/p]$ by $X \mapsto \xi/p$, since this map becomes an isomorphism after inverting p. Its p-adic completion $A_{\max}$ is also p-torsion free: an element killed by p has component modulo p^{m+1} in $p^m R/p^{m+1}R$ for every m, hence is zero. Dividing compatible components by p at one higher precision also gives $A_{\max}/pA_{\max} = R/pR = (A/(p,\xi))[X]$. Multiplication by a monic polynomial is injective in a polynomial ring over any coefficient ring. Hence if $P \in A_{\max}$ reduces to a monic polynomial and $Pw \in p^m A_{\max}$, successive reduction modulo p and cancellation show $w \in p^m A_{\max}$.

Write $\iota : A \to A_{\max}$ for the natural map. If $\iota(a) \in p^m A_{\max}$, then $a \in (p,\xi)^m A$. The map θ extends continuously to $A_{\max}$ by $X \mapsto 0$. For $m > 0$ we have $\theta(a) \in p^m \mathcal{O}_{\mathbb{C}_p}$; lift the quotient under θ to $c \in A$. Then $a = \xi w + p^m c$ with $w \in A$. Using $\xi = pX$ and cancelling p shows $Xw \in p^{m-1}A_{\max}$. The preceding regularity and induction put w in $(p,\xi)^{m-1}$. Conversely, $\iota((p,\xi)^m A) \subset p^m A_{\max}$ by $\xi = pX$. Hence $\ker\iota \subset \bigcap_m (p,\xi)^m A = 0$ by the separation in Section A.1. Also X is a nonzero divisor in $A_{\max}$ by its monic reduction and p-adic separation, so $\xi = pX$ can be cancelled in both $A_{\max}$ and B^+.

Since $\xi - p \in \tau A$, we have $p(X-1) = \tau u$ for an integral u. For $c = 0, 1$, every $y \in A_{\max}$ can be written $a + (X-c)b$ with $a \in A$ and $b \in A_{\max}$. Indeed, reduce y to $P(X) \in (A/(p,\xi))[X]$. Lift $P(c)$ to $a_0 \in A$ and lift the polynomial $(P(X) - P(c))/(X-c)$ to $b_0' \in A[X]$. Then $y - a_0 - (X-c)b_0' \in pA_{\max}$. Repeat on the quotient by p; the scalar digits converge in A and the polynomial digits converge in $A_{\max}$ after multiplication by their successive powers of p. Taking $c = 1$ and using $p(X-1) = \tau u$ gives $pA_{\max} \subset A + \tau A_{\max}$, and iteration gives $p^j A_{\max} \subset A + \tau^j A_{\max}$. Taking $c = 0$ gives $pA_{\max} \subset A + \xi A_{\max}$.

Let $V \subset A_{\max}$ and suppose for each $v \in V$ there is $w \in V$ such that $\xi^h v = u_0 \varphi(w)$ with $u_0 \in A$ fixed. Then for every $m \geq 0$ there exists $a_m \in A$ with

$$p^{h+1}v - a_m \in \tau^{h+1} p^m A_{\max}.$$

The case $m = 0$ follows from the preceding polynomial division. For the induction step, if $p^{h+1}w = a_m(w) + \tau^{h+1}p^m b_m$, $b_m \in A_{\max}$, then

$$\xi^h p^{h+1} v = u_0\varphi(a_m(w)) + \xi^{h+1}\tau^{h+1}p^m u_0 \varphi(b_m).$$

Here $u_0\varphi(a_m(w)) \in A \cap \xi^h B^+ = \xi^h A$. Indeed for $a \in A \cap \xi B^+$ evaluation by θ gives $a \in \ker(\theta|_A) = \xi A$, and one cancels ξ and iterates. Dividing the display by ξ^h leaves an error in $\tau^{h+1}p^{m+1}A_{\max}$, since $\xi = pX$. This gives the next integral approximation. The difference of two approximations at precision m lies in $p^m A_{\max} \cap A = (p,\xi)^m A$. They converge in A, by Section A.1. Their limit in $A_{\max}$ is $p^{h+1}v$. Thus $p^{h+1}V$ has integral A-coordinates.

For the period matrix U, apply this to the coordinates of the solutions w of

$$Uy = E^h x, \quad x \in A^d, \qquad U\widehat{F}(y) = E^h w$$

as y and the S-linear maps $F : A \to A$ vary. There is one power p^m putting all such w in $A_{\max}$: the inverse of U up to $(E\tau)^h$ first bounds a denominator of y, and integrality of the extended coordinate maps gives the same bound for $F(y)$.

Applying U/E^h requires one further fixed power. For the denominator claim, note that $\xi \equiv \tau^{p-1} \bmod p$, so $\tau^{p-1} = p(X - b_0)$ with $b_0 \in A$. If $\mu z \in A_{\max}$ then multiplication by τ^{p-2} gives $p^2 X(X - b_0)z \in A_{\max}$. Multiplication by the monic polynomial $X(X - b_0)$ is p-saturated, by its reduction modulo p. Clearing an initial power of p in B^+ and using this saturation yields $p^2 z \in A_{\max}$. Iteration gives $p^{2h} z \in A_{\max}$ if $\mu^h z \in A_{\max}$; the same conclusion holds when $d^h z \in A_{\max}$ with $d \mid \mu$ in A. For the individual factors, X and $X - b_0$ are both monic modulo p. Hence, for $z \in B^+$,

$$\xi z \in A_{\max} \text{ or } \tau z \in A_{\max} \quad \Longrightarrow \quad pz \in A_{\max}.$$

In the first case use $\xi z = X(pz)$. In the second case multiply by τ^{p-2} to get $(X - b_0)pz \in A_{\max}$. At $p = 2$ this multiplier is 1.

Section A.6 verifies Frobenius descent for the retracted solutions and applies the contraction with a uniform exponent.

A.4 The two period matrices over the bounded ring

In this subsection, we construct the crystalline comparison matrix and its partial inverse over B^+. The key scalar assertion is divisibility by μ in the common kernel of the Fontaine evaluations at nonnegative Frobenius powers. This gives the matrix equations used in Section B.1.

For the same chart write $\mu = \xi\tau$. In $A = A_{\mathrm{inf}}$, the common kernel of $\theta\varphi^j$, $j \geq 0$, is μA. The individual kernels are generated by the distinct primes $\varphi^{-j}\xi$. For $0 \leq i < j$, $\theta\varphi^j(\varphi^{-i}\xi) = p$, since $\theta\varphi^{j-i}([\epsilon^{1/p}]) = 1$. On $\varphi^{-j}\xi$ the same evaluation is zero. Thus these prime ideals are pairwise distinct. The intersection of any finite initial set of these principal prime ideals is their product. Its generator is

$$\prod_{j=0}^{m-1} \varphi^{-j}(\xi) = \mu/\varphi^{-m}(\mu).$$

The intersection of these products for $m \geq 1$ is μA by [14, Lemma 3.23]. Applying this result to $\varphi(a)$ gives the kernel τA for $\theta\varphi^j$, $j \geq 1$.

For $y \in B^+$ choose $c \geq 0$ such that $r = p^c y \in A_{\max}$. Section A.3 gives

$$pr = a + \tau v, \qquad pv = a_1 + \xi w', \qquad a, a_1 \in A, \quad v, w' \in A_{\max}.$$

Thus $p^{c+2} y = pa + \tau a_1 + \mu w'$. Write $N = c+2$ and $a' = pa + \tau a_1 \in A$. If $\theta\varphi^j(y) = 0$ for all $j \geq 0$, the same evaluations kill a', whence $a' \in \mu A$ and $y \in \mu B^+$. With $j \geq 1$ the same proof gives $y \in \tau B^+$. In particular, if $\varphi(y) \in \varphi(\tau)B^+ = \mu B^+$, then $\theta\varphi^j(y) = 0$ for $j \geq 1$ and $y \in \tau B^+$. If $\varphi(y) = \mu^h c$ with $h \geq 1$ and $c \in B^+$, write $y = \tau y_1$ by this implication. Cancellation gives $\varphi(y_1) = \mu^{h-1} c$. Induction yields $y \in \tau^h B^+$. For the single evaluation, $p^N y = a' + \mu w'$ gives $\theta(a') = 0$ if $\theta(y) = 0$. Since $\ker(\theta|_A) = \xi A$ and $\mu = \xi\tau$, this proves $\ker(\theta|_{B^+}) = \xi B^+ = EB^+$.

For comparison with the usual period ring, take the tilt element $p^\flat$ from compatible p-power roots of p, with $v_\flat(p^\flat) = 1$. Since $[p^\flat] - p$ also generates $\ker\theta$ [13, Proposition 10.2],

one has $A[\xi/p] = A[q]$ for $q = [p^\flat]/p$. In the Witt coordinates of $A[1/p]$, Berger's valuation at $r_0 = (p-1)/p$ is

$$V\left(\sum_n p^n[x_n], r_0\right) = \inf_n (n + v_\flat(x_n)).$$

This is the valuation defining $B^+_{\max}$ in [13, §13]. Its unit ball is the indicated algebra. Indeed, a term $p^{-j}[x]$ of negative index satisfies the unit-ball condition exactly when $v_\flat(x) \geq j$, in which case it belongs to $([p^\flat]/p)^j A$. The finitely many negative indices and the integral tail give the assertion. Completing the unit ball gives $A_{\max}$ and inverting p gives Berger's $B^+_{\max} = B^+$. In particular,

$$p^m A_{\max} = \{x \in B^+ : V(x, r_0) \geq m\} \qquad (m \in \mathbf{Z}).$$

For $P = X$ or $X - b_0$ from Section A.3, the reduction of P^n modulo p is monic for every $n \geq 1$. Thus $0 \leq nV(P, r_0) < 1$ for every n, so $V(P, r_0) = 0$. The equalities $\xi = pX$ and $\tau^{p-1} = p(X - b_0)$ give

$$V(E, r_0) = V(\xi, r_0) = 1, \qquad V(\tau, r_0) = \frac{1}{p-1}, \qquad V(\mu, r_0) = \frac{p}{p-1}.$$

Here E/ξ and its inverse belong to A. We use the embedding $B^+ \hookrightarrow B^+_{\mathrm{dR}}$ of [13, Theorem 13.1] for the de Rham tests below. Also $t = \log(1+\mu)$ and t/μ is a unit of B^+. In fact $\mu = pX\tau$, so the logarithm quotient converges and its nonconstant terms lie in $pA_{\max}$, using $j - 1 - v_p(j) \geq 1$ for $j \geq 2$ except $p = j = 2$. For $p = 2$ one has $\xi = 2 + \tau = 2X$, whence $\mu/2 = X\tau = 2X(X-1) \in 2A_{\max}$. Hence $B^+[1/\mu] = B^+[1/t] = B_{\max}$.

Let $a \geq 0$ be an integer. If $y \in B_{\max}$ satisfies $\varphi^j y \in t^{-a} B^+_{\mathrm{dR}}$ for every $j \geq 0$, then $\mu^a y \in B^+$. Write $y = b/\mu^k$ with $b \in B^+$. For $j \geq 0$,

$$\varphi^j(\mu) = \mu \sum_{i=0}^{p^j - 1} (1+\mu)^i, \qquad \theta\left(\sum_{i=0}^{p^j-1} (1+\mu)^i\right) = p^j.$$

Thus every $\varphi^j(\mu)$ has de Rham order one. For either $D = A$ or $D = B^+$ this proves the exact test, for $n \geq 0$,

$$\{c \in D : \varphi^j(c) \in \mathrm{Fil}^n B^+_{dR} \text{ for all } j \geq 0\} = \mu^n D.$$

Indeed the common-kernel calculation in D gives the first factor of μ; after division all orders decrease by one, and induction gives the remaining factors. If $k > a$, apply this test to b with $n = k - a$ to obtain $\mu^a y \in B^+$.

For a crystalline lattice on this chart choose an invariant matrix $M \in \mathrm{GL}_d(B_{\max})$ whose columns are the coordinates of a K_0-basis of $D_{\mathrm{cris}}(T[1/p])$ in a $\mathbf{Z}_p$-basis of T, where $K_0 = W(k)[1/p]$. Crystalline comparison makes M invertible and gives $\rho(g)g(M) = M$ for $g \in G_K$ [13, §§4, 13–14]. For $j \geq 0$ put $C_j = M^{-1}\varphi^j(M)$. Both C_j and C_j^{-1} have entries fixed by G_K, since Frobenius commutes with G_K and fixes the entries of $\rho(g)$. Thus $C_j \in \mathrm{GL}_d(K_0)$ by [13, §13, corollary to Theorem 13.1]. The columns of M form a K-basis of D_{dR} by crystalline comparison. Choose $Y \in \mathrm{GL}_d(K)$ so that the columns of MY form a filtration-adapted K-basis d_j of D_{dR}. Write the rescaled matrix as

$$M = W\ \mathrm{diag}(t^{k_j})\, Y^{-1} \quad \text{in } B_{\mathrm{dR}}, \qquad W = (t^{-k_j} d_j)_j \in \mathrm{Mat}_d(B^+_{\mathrm{dR}}).$$

Here k_j is the filtration index $d_j \in \mathrm{Fil}^{k_j} \setminus \mathrm{Fil}^{k_j+1}$. The Tate twist $\mathbf{Q}_p(1)$ has index -1 in this convention.

For every finite $K/\mathbf{Q}_p$ one has

$$\mathbf{C}_p^{G_K} = K, \qquad \mathbf{C}_p(n)^{G_K} = 0 \quad (n \in \mathbf{Z} \setminus \{0\}).$$

The first assertion is Tate's Theorem 1. To apply his Theorem 2, put $K_a = K(\mu_{p^a})$ and $K_\infty = \bigcup_a K_a$. Since $\mathbf{Q}_p(\mu_{p^a})/\mathbf{Q}_p$ is totally ramified, the absolute residue degrees of the K_a are bounded by $[K : \mathbf{Q}_p]$. Choose $L = K_a$ where they have stabilized and $\chi_{\mathrm{cyc}}(G_L)$ is torsion-free, taking $a \geq 2$ when $p = 2$. The latter image is open in $\mathbf{Z}_p^\times$: each finite $L/\mathbf{Q}_p$ gives a subgroup of finite index. For every $n \neq 0$, the kernels of χ_{cyc}^n and χ_{cyc} agree on G_L. Their common fixed field is K_∞, totally ramified over L with group $\mathbf{Z}_p$. Theorem 2 now gives $\mathbf{C}_p(n)^{G_K} \subseteq \mathbf{C}_p(n)^{G_L} = 0$ [75, §3.3, Theorems 1 and 2, p. 176].

The columns $w_j = t^{-k_j} d_j$ of W are integral. Their residues are independent over $\mathbf{C}_p$. Indeed, normalize one nonzero coefficient in a relation of least support to $a_i = 1$. Since $g(w_j) = \chi_{\mathrm{cyc}}(g)^{-k_j} w_j$ for the diagonal action of G_K, minimality gives $g(a_j) = \chi_{\mathrm{cyc}}(g)^{k_j - k_i} a_j$. Thus a_j belongs to $\mathbf{C}_p(k_i - k_j)^{G_K}$, so $k_j = k_i$ and $a_j \in K$ for every nonzero coefficient. This would give $\sum_j a_j d_j \in \mathrm{Fil}^{k_i+1}$, contrary to independence in $\mathrm{gr}^{k_i} D_{\mathrm{dR}}$. Thus $\det W$ has nonzero residue and W^{-1} is integral. Since $C_j^{-1} Y$ is invertible over B_{dR}^+, the identity

$$\varphi^j(M) C_j^{-1} Y = M Y = W \,\mathrm{diag}(t^{k_i})$$

gives elementary divisors t^{k_i} for each $\varphi^j(M)$. Thus for jumps in $[-a, b]$ each $\varphi^j M$ has entries in $t^{-a} B_{\mathrm{dR}}^+$ and its inverse has entries in $t^{-b} B_{\mathrm{dR}}^+$. The bounded-ring criterion and $E\tau \sim \mu$ give the matrices

$$U = (E\tau)^a M, \qquad W' = (E\tau)^b M^{-1} \quad \text{in } \mathrm{Mat}_d(B^+).$$

They satisfy $UW' = (E\tau)^{a+b} I$.

The entries of C_1 and its inverse belong to $K_0 \subset S_B$. Also $\varphi(E)$ is a unit in that bicommutant: $E = z^e u + p$ with u a chart unit, and $E = pX'$ in $A_{\max}$. Thus $z^e \in p A_{\max}$ and, more precisely,

$$\varphi(E)/p = 1 + p^{p-1} (z^e/p)^p \varphi(u), \qquad z^e/p = (E/p - 1)/u \in S[E/p].$$

The geometric series for $(\varphi(E)/p)^{-1}$ therefore converges in S_B, where p is invertible. Writing $v_0 = \mu/(E\tau) \in A^\times$, we get

$$\varphi(U) = v_0^a U (\varphi(E)^a C_1)$$

with an invertible last factor over the bicommutant.

For the Galois equation, E/gE is a unit since both elements generate $\ker \theta$. The ideal μA is independent of the chosen compatible primitive roots by [14, Lemma 3.23]. Applying φ^{-1} shows that $g\tau A = \tau A$, so $\tau/g\tau \in A^\times$. The integral representation matrix, scaled by the unit $(E\tau/g(E\tau))^a$, takes $g(U)$ back to U. Finally the divisibility condition under coordinate retraction is proved in Section A.5. This gives the period-matrix hypotheses of the integral basis for each lattice in the modular cup product, on the crystallizing extension in the common local class.

A.5 Divisibility under the coordinate retraction

In this subsection, we prove that the retraction of Section A.1 respects the Hodge filtration relevant to the equation $Uy = E^h x$. Approximation of the change-of-basis coefficients in the bounded ring reduces the assertion to divisibility by a power of E.

The corresponding period matrix of height h over B^+ is $U = (E\tau)^a M$.

Use the usual embedding of K in B^+_{dR}, on which θ is the identity. Each $u \in K$ is approximable at every ξ-adic precision by elements of S_B. First, $\theta(S_B) = K$. The chart constants and the image of its uniformizer generate K, and p is invertible in S_B. Conversely, the uncompleted ring $S[E/p][1/p]$ has image in K, since $\theta(E) = 0$ and series in the uniformizer converge in K. Continuity of θ in the bounded-ring p-adic topology and completeness of K give the assertion after completion. Start with $b \in S_B$ having residue u. For a polynomial $P \in \mathbb{Q}_p[X]$ having a simple root u, lift $P'(u)^{-1}$ to $c \in S_B$ by this same residue map. If $b \equiv u \bmod \xi^j$, Taylor expansion gives $b - cP(b) \equiv u \bmod \xi^{j+1}$. This yields the required approximations.

For $r \in A_{\max}$ write $r = a + (\xi/p)b$, with $a \in A$ and $b \in A_{\max}$, by polynomial division at zero in Section A.3. If $\theta(r) = 0$, then $a \in \xi A$, so $r \in (\xi/p)A_{\max} = (E/p)A_{\max}$. Since E/p has de Rham order one, iteration gives

$$A_{\max} \cap \mathrm{Fil}^n B^+_{\mathrm{dR}} = (E/p)^n A_{\max} \qquad (n \geq 0).$$

After inverting p, the corresponding identities are

$$B^+ \cap \mathrm{Fil}^n B^+_{\mathrm{dR}} = E^n B^+, \qquad S_B \cap \mathrm{Fil}^n B^+_{\mathrm{dR}} = E^n S_B.$$

The first identity follows by localization of the integral one. For the second, apply the S_B-linear retraction $\widehat{F}_0$ to $s = E^n y$ with $s \in S_B$. For a shifted divisor $E_j = \varphi^j E$ the integral test is

$$a \in E_j^n A \quad \Longleftrightarrow \quad \varphi^{-j}(a) \in \mathrm{Fil}^n B^+_{\mathrm{dR}} \qquad (a \in A,\ j \geq 0).$$

Here one iterates $\ker(\theta|_A) = EA$ on $\varphi^{-j}(a)$ and transports the quotient by φ^j in A. For $a \in S$, applying F_0 to an integral quotient gives $E_j^n A \cap S = E_j^n S$.

For $y_i \in B^+$ and $\beta_i \in K$, let $\widehat{F}$ be the extension of any S-linear $F : A \to A$ constructed in Section A.1. If $\sum_i \beta_i y_i$ is divisible by ξ^j in B^+_{dR}, approximate each β_i modulo ξ^j by $b_i \in S_B$. The first intersection formula then gives $\sum_i b_i y_i \in E^j B^+$. Apply $\widehat{F}$, which commutes with S_B and multiplication by E^j, and replace b_i by β_i again modulo ξ^j. It follows that

$$\mathrm{ord}_\xi\left(\sum_i \beta_i \widehat{F}(y_i)\right) \geq \mathrm{ord}_\xi\left(\sum_i \beta_i y_i\right).$$

If the latter sum is zero, this says that the former is also zero, by separatedness.

Since $Y^{-1} \in \mathrm{GL}_d(K)$, each of its rows has coefficients in the field just approximated. There are $s_j \in B^+_{\mathrm{dR}}$ such that $(Y^{-1}\widehat{F}(y))_j = s_j (Y^{-1} y)_j$. Since the two diagonal factors commute, we obtain

$$U\widehat{F}(y) = W\ \mathrm{diag}(s_j)\, W^{-1} U y.$$

Consequently divisibility of Uy by E^h implies the same for $U\widehat{F}(y)$, first in B^+_{dR} and then in B^+ as above. Section A.6 applies this retraction to the bounded-ring solutions.

A.6 Divisibility of the period matrix against integral covectors

In this subsection, we use the denominator bound of Section A.3 and the retraction of Section A.5 to extend divisibility from the basis solutions to all bounded-ring solutions. Section B.1 uses this assertion to prove full rank and then to obtain the integral span.

Work on the same chart and take the symmetric height bound b. Put $h = 2b$, $A = A_{\mathrm{inf}}$, and denote the invariant matrix M of Section A.4 by P, so that $U = (E\tau)^b P$. For this same P put

$$W = (E\tau)^b P^{-1}, \qquad v = (\xi/E)^b, \qquad F = \varphi(E)^b P^{-1} \varphi(P).$$

The comparison in Section A.4 gives $W \in \mathrm{Mat}_d(B^+)$, $v \in A^\times$, $F \in \mathrm{GL}_d(S_B)$, and

$$UW = WU = (E\tau)^h I, \qquad \varphi(U) = vUF.$$

Write $\widehat{f}$ for the extension of an S-linear $f : A \to A$ to B^+. For any solution $Uy = E^h x$, $x \in A^d$, the evaluation divisibility in Section A.5 gives

$$U\widehat{f}(y) = E^h w.$$

A single power p^N, depending on the chart and matrix, puts every such w in A^d.

Choose $c_W, c_U \geq 0$ clearing the entries of W, U into $A_{\max}$, respectively. Cancelling E^h in $(E\tau)^h y = WUy = E^h Wx$ gives $\tau^h y = Wx$. The single-factor estimates of Section A.3 apply iteratively to τ and to E, which is a unit multiple of ξ. The first gives $p^{c_W + h} y \in A_{\max}^d$. Since $\widehat{f}$ preserves $A_{\max}$ and $E^h w = U\widehat{f}(y)$, the second gives

$$p^{N_2} w \in A_{\max}^d, \qquad N_2 = c_W + c_U + 2h$$

for every solution and every map.

For Frobenius contraction set $x' = \varphi^{-1}(vE^h x) \in A^d$. Apply φ to $UWx' = (E\tau)^h x'$ and cancel vU to get

$$F\varphi(Wx') = \varphi(E\tau)^h y.$$

The inverse of F is defined over B^+, so $\varphi(Wx')$ is divisible there by $\varphi(\tau)^h$. The cyclotomic kernel argument of Section A.4 gives $y' = \tau^{-h} Wx' \in (B^+)^d$, with $Uy' = E^h x'$. Put $f' = \varphi^{-1} f \varphi$ on A. It is S-linear because $\varphi(S) \subset S$. Write $U\widehat{f'}(y') = E^h w'$. The equality $\varphi\widehat{f'} = \widehat{f}\varphi$ holds on A and then on $A[E/p]$ by clearing a power of p from each input. Both sides preserve $A_{\max}$ and its p-powers, giving equality on the completion and on B^+. The S_B-linearity permits $\widehat{f}$ to pass through F and $\varphi(E)^h$. Since $F\varphi(y') = \varphi(E)^h y$, we get

$$\varphi(E)^h \varphi(w') = vUF\widehat{f}(\varphi(y')) = v\varphi(E)^h U\widehat{f}(y), \qquad \varphi(w') = vE^h w.$$

The entries of $p^{N_2} w$ thus belong to a single subset of $A_{\max}$ closed under the descent relation $\xi^h a = u\varphi(a')$, with the fixed unit $u = (\xi/E)^h v^{-1}$. The contraction of Section A.3 yields $N = N_2 + h + 1 = c_W + c_U + 3h + 1$, independent of x, y, f.

Let $r = \mathrm{rank}_S \mathcal{P} \leq d$ and write $Q \in \mathrm{Mat}_{d \times r}(A)$ for the matrix of an S-basis of the integral vectors admitting a solution over S_B, as in Section A.1. For every row $l \in A^{1 \times d}$ and $\delta \in A$ with $p \nmid \delta$, use entrywise divisibility for lQ. If $x \in A^d$ admits a solution $y \in (B^+)^d$, then

$$\delta \mid lQ \quad \Longrightarrow \quad \delta \mid lx \qquad (Uy = E^h x). \tag{1}$$

Choose the topological chart basis $(b_i)_i$. For $f = \mathrm{coord}_i$, the entries of $\widehat{f}(y)$ are in S_B.

The uniform exponent gives $x_i = p^N w_i \in A^d$ and

$$U\widehat{(p^N \mathrm{coord}_i(y))} = E^h x_i.$$

Thus $x_i = Qs_i$ with $s_i \in S^r$. Take $k = c_U + 2h$ to clear the entries of U and divide by E^h in $A_{\max}$. For any m choose one finite subset I_m of the basis with

$$y - \sum_{i \in I_m} b_i \widehat{\mathrm{coord}_i}(y) \in p^{m+k} A_{\max}^d.$$

This is the simultaneous finite approximation of the finitely many entries.

Set

$$t = p^N x - \sum_{i \in I_m} b_i Q s_i \in A^d.$$

Applying $p^N U$ to the approximation gives $E^h t \in p^{m+2h} A_{\max}^d$. Since $E \mid \mu$, the denominator estimate applied to $p^{-m-2h} t$ gives $t \in p^m A_{\max}^d$. The identity $p^m A_{\max} \cap A = (p, \xi)^m$ puts lt in $(p, \xi)^m$. If $lQ = \delta g$, then the multiples $\delta \sum_{i \in I_m} b_i g s_i$ approximate $p^N l x$ to every (p, ξ)-adic order. By closedness of δA from Section A.1, $p^N l x \in \delta A$. Since p is prime in A and $p \nmid \delta$, repeatedly cancel p in this divisibility. This proves (1) for every bounded-ring solution on the given chart.

B Integral crystalline bases and filtration orders

In this appendix, we use the comparison matrices of Appendix A to construct an integral period basis and describe its filtration under the divided Galois action. The resulting orders are used in the cup calculation of Appendix C and the lifts of Appendix D.

B.1 The integral period basis

In this subsection, we construct an integral basis from the solution module of Section A.1. The fixed-vector approximation of Section A.2 rules out a kernel modulo p. We then obtain the two partial inverses and the equations for Frobenius and Galois.

Use the S-basis of the solution module from Section A.1, with its columns in Q and solutions in Z. Initially the number of columns is at most $d = \mathrm{rank}_{\mathbf{Z}_p} T$. For a free O-lattice this is $\mathrm{rank}_O(T)[\mathrm{Frac}(O) : \mathbf{Q}_p]$, as in Section A.1. We take determinants on the underlying $\mathbf{Z}_p$-lattice. Multiplication by $o \in O$ has a matrix H_o over $\mathbf{Z}_p$ and induces a matrix C_o over K_0 on the invariant crystalline periods. Thus $H_o U = U C_o$, and $Uy = E^h x$ gives $U(C_o y) = E^h H_o x$. Since $K_0 \subset S_B$, the integral solution module is stable under O. We have $UZ = E^h Q$ with Z over S_B. The divisibility against covectors of Section A.6 holds before asserting that Q is square. If the rank of Q were less than d, clearing denominators in its left kernel over $\mathrm{Frac}\, A$ would give $0 \neq l \in A^{1 \times d}$ with $lQ = 0$. Apply (1) with $\delta = z^k$, $y = We_j$, $x = \tau^h e_j$. Since $p \nmid z$, this gives $\tau^h l_j \in z^k A$ for every j and $k \geq 1$. The intersection of these ideals is zero by (p, z)-adic separatedness. Since A is a domain, this contradicts $l \neq 0$. The covector argument proves the following rank statement.

Lemma B.1 (Full rank of the integral solution basis)**.** *The solution matrices Q, Z are square of size $d = \mathrm{rank}_{\mathbf{Z}_p} T$, and $\det Q \neq 0$.*

If $Qs = px$ for $s \in S^d$ and $x \in A^d$, then $U(Zs/p) = E^h x$ and $Zs/p \in S_B^d$. It follows that $x = Qs'$ for $s' \in S^d$, and $s = ps'$ by injectivity of Q. Thus the chart-coordinate covectors of Q separate vectors over S/p: if all coordinate covectors kill a vector modulo p, the z-adic

coordinate expansion implies $Qs = 0 \bmod p$, hence $s = 0 \bmod p$. Over the fraction field of $S/p = k[[z]]$, these covectors therefore span the full dual space. Choose d rows indexed by pairs (i_ν, j_ν) of a coordinate map and a row of Q, and put

$$N_{\nu t} = F_{i_\nu}(Q_{j_\nu t}), \qquad (Nu)_\nu = F_{i_\nu}((Qu)_{j_\nu}) \quad (u \in S^d), \qquad \overline{\det N} \neq 0.$$

The equation $\varphi U = vUF$ and the invertibility of $\varphi(E)$ in S_B give

$$U\left((E/\varphi E)^h F \varphi Z\right) = E^h v^{-1} \varphi Q.$$

By the solution basis, $QB = v^{-1}\varphi Q$ for some $B \in M_d(S)$. For nonzero $a \in A$, the order $\mathrm{ord}_p(a) = \max\{j : a \in p^j A\}$ is finite by separatedness and additive on products since A/p is a domain. Frobenius preserves this order and v is a unit. Taking determinants gives $\mathrm{ord}_p(\det B) = 0$. Throughout, $S/p \hookrightarrow A/p$ by the chart retraction.

Suppose $\det Q = 0 \bmod p$. Over the valuation ring A/p choose a nonzero kernel vector a of minimal support and divide by a coordinate of smallest valuation to make that coordinate 1. If g fixes S pointwise, it belongs to G_K since $K = W(k)[1/p](\theta(z))$, and fixes Z by continuity on $S_B = \overline{S[1/p]}$. Since $gE = E$, the Galois equation of Section A.4 gives

$$U = R_g\, g(U), \qquad R_g = (\tau/g\tau)^b \rho(g) \in \mathrm{GL}_d(A).$$

Applying g to $UZ = E^h Q$ and cancelling E^h therefore gives $Q = R_g\, g(Q)$. Thus $g(a)$ is again in the kernel. Subtracting a gives a kernel vector of strictly smaller support unless $g(a) = a$. The normalized vector a is consequently fixed by the pointwise fixer of S.

Put $C_n = B\varphi(B)\cdots\varphi^{n-1}(B)$ and $D_0 = \overline{\det B}\,\overline{\det N}$. The Frobenius equation and the determinant product imply

$$\bar{Q}\bar{C}_n a^{p^n} = 0, \qquad D_0\,\overline{\det C_n} \mid D_0^{p^n}.$$

Indeed $\varphi^n Q = (\prod_{j<n} \varphi^j v) Q C_n$, and $\overline{\det C_n} = (\overline{\det B})^{(p^n-1)/(p-1)}$ with $(p^n-1)/(p-1) \leq p^n - 1$.

Let $\pi_j : A/p \to \mathcal{O}_{\mathbb{C}_p}/p$ be reduction of θ_j. Thus $\pi_{m+n}(x^{p^n}) = \pi_m(x)$. Fix m. By Section A.2, each coordinate of a admits the required approximation at every sufficiently large j. Choose a common $j = m + n$ and $s \in S^d$ such that $\pi_j(\bar{s}) = \pi_m(a)$. The first identity above gives $\pi_j(\overline{QC_n s}) = 0$. Applying the coordinate divisibility of Section A.2 with $\alpha = p$ to its j_ν-th entry gives $\pi_j(\overline{NC_n s})_\nu = 0$, since $C_n s \in S^d$. One coordinate of $\pi_j(\bar{s})$ is 1. Multiplication by the adjugate over $\mathcal{O}_{\mathbb{C}_p}/p$ gives $\pi_j(\overline{\det(NC_n)}) = 0$. Since

$$\overline{\det(NC_n)} \mid D_0\,\overline{\det C_n} \mid D_0^{p^n},$$

we obtain $\pi_m(D_0) = \pi_j(D_0^{p^n}) = 0$. All components vanish, contradicting $D_0 \neq 0$ in the tilt. Hence $p \nmid \det Q$: the reduction of Q has rank d over $\mathrm{Frac}(A/p)$. In particular $Qt \in pA^d$ implies $t \in pA^d$ for $t \in A^d$, by reduction and the adjugate. Iteration gives

$$QA^d \cap p^s A^d = p^s Q A^d \qquad (s \geq 0).$$

Thus for each $s \geq 1$, reduction of the inclusion of the integral column span identifies QA^d/p^sQA^d with its image in $(A/p^s)^d$. In particular,

$$QA^d = \bigcap_{s \geq 1} (QA^d + p^s A^d).$$

Indeed the coefficient vector of a residue in this image is unique modulo p^s since Q is injective modulo p^s. For a vector in the intersection the coefficient vectors are therefore compatible in s, and lift in the p-adically complete ring A. In $W(\mathbb{C}_p^\flat)$ the residue $\overline{\det Q}$ is a nonzero field element, so Q is invertible over this Witt ring and over all its quotients by p^s. The two assertions use respectively injectivity and invertibility of the reduced determinant over the indicated rings.

Change the integral basis by $H \in \mathrm{GL}_d(\mathbf{Z}_p)$ and the crystalline period basis by $C \in \mathrm{GL}_d(K_0)$. The period matrices and the solution module become

$$P_1 = H^{-1} P C, \qquad U_1 = H^{-1} U C, \qquad \mathcal{P}_1 = H^{-1}\mathcal{P}.$$

Indeed $U_1 y = E^h x$ is equivalent to $U(Cy) = E^h H x$, and both C and C^{-1} preserve S_B^d. For any S-basis Q_1 of $\mathcal{P}_1$ and its solution matrix Z_1 there is therefore a $V \in \mathrm{GL}_d(S)$ with

$$Q_1 = H^{-1} Q V, \qquad Z_1 = C^{-1} Z V.$$

The formula for Z_1 follows from the injectivity of U_1. Thus the determinant of the integral solution basis changes by a unit of A, including when O is ramified.

For $\sigma \in G_{\mathbf{Q}_p}$, transport the representation to $G_{\sigma K}$ and the chart to $(\sigma S, \sigma E)$. Using the same τ, put $u = (\tau/\sigma\tau)^b \in A^\times$. The transported period matrix is $u\sigma U$, and its solution module is $u\sigma\mathcal{P}$, since σS_B is the bounded scalar ring of the new chart. Thus the transported solution basis $u\sigma Q$ has determinant $u^d \sigma(\det Q)$, again nonzero modulo p.

For $Uy = E^h x$, $x \in A^d$, $y \in (B^+)^d$, apply (1) with $\delta = \det Q$ and the rows of $\mathrm{adj}(Q)$. This gives $\mathrm{adj}(Q)x = (\det Q)t$ for $t \in A^d$. Multiplication by Q and cancellation give $x = Qt$. Conversely $U(Zt) = E^h Q t$, so the integral solution vectors over B^+ form precisely

$$QA^d = A \otimes_S QS^d = \{x \in A^d : \mathrm{adj}(Q)x \in (\det Q)A^d\}.$$

The tensor product embeds via its independent basis columns. The last equality follows by multiplying the adjugate equation by Q and cancelling $\det Q$. The principal ideal $(\det Q)A$ is closed in the (p,z)-adic topology by Section A.1, since $p \nmid \det Q$. Its inverse image under the continuous map $x \mapsto \mathrm{adj}(Q)x$ is therefore closed in A^d. With $y = We_j$ we get $QQ' = \tau^h I$ for $Q' \in M_d(A)$.

Put $v_0 = \mu/(E\tau) \in A^\times$. To construct the second partial inverse, take $y \in B^+$ and $s, k \geq 0$. For some $c \geq 0$, $a \in A$, and $b \in A_{\max}$,

$$p^c y = a + p^{c+k}\mu^s b.$$

Indeed choose c_0 with $p^{c_0} y \in A_{\max}$. Approximate by a polynomial in ξ/p modulo $p^{c_0+k+2s}A_{\max}$, then multiply by p^{c-c_0} to put all of its evaluated terms in A. The resulting error is $p^{c+k+2s}b_0$. For $X = \xi/p$, polynomial division in Section A.3 gives $A_{\max} = A + (X-1)A_{\max} = A + XA_{\max}$ by division at 1 and 0, respectively. Iterating and using $p(X-1) \in \tau A$, $pX = \xi$, gives

$$p^s b_0 = a_1 + \tau^s b_1, \qquad p^s b_1 = a_2 + \xi^s b, \qquad a_1, a_2 \in A.$$

The remaining error after absorbing integral terms is $p^{c+k}\tau^s\xi^s b = p^{c+k}\mu^s b$.

Apply this entrywise to U with $k = 0, s = h$. Take one common c and obtain $p^c U = A_0 + p^c\mu^h V_0$ with integral A_0 and V_0 over $A_{\max}$. Then

$$A_0 = p^c U(I - v_0^h W V_0), \qquad p^c\mu^h V_0 = p^c(Ev_0)^h QQ' V_0.$$

Taking $y = p^c E^h (I - v_0^h W V_0)$ gives $Uy = E^h A_0$. Hence $A_0 = QA_1$ for $A_1 \in M_d(A)$, and

$$Y = p^{-c} A_1 + (Ev_0)^h Q' V_0 \in M_d(B^+), \qquad U = QY.$$

Since B^+ is a domain containing A, cancellation of Q in $QYZ = E^h Q$ gives $YZ = E^h I$.

Finally let g fix the chart field, so it fixes the Witt constants and $\theta(z)$. If $U = A_g g U$ is the period equation, then $UgZ = (gE)^h A_g g Q$, so $UgZ = E^h R_g g Q$ with $R_g = (gE/E)^h A_g$ integral. The integral span gives $R_g g Q = Q G_g$ for G_g integral. Cancel U in the resulting equality to obtain $gZ = ZG_g$. Altogether the basis satisfies

$$QQ' = \tau^h I, \qquad UZ = E^h Q, \qquad YZ = E^h I, \qquad \varphi Q = vQB, \qquad gZ = ZG_g.$$

B.2 The period basis, divided action, and tensor orders

In this subsection, we use the de Rham evaluations of the integral basis in Section B.1 to control division of Galois differences by $z\tau$. The retraction preserves filtration orders, and the tensor estimates give the orders in Section C.2.

Keep the chart of Sections A.1 and A.3, and the original crystalline $\mathbf{Z}_p$-lattice T. Write $A = A_{\mathrm{inf}}$, $\mathrm{Fil}^j = t^j B_{\mathrm{d}R}^+$, $E_a = \varphi^a E$, and let K' be the local fixed field of the chart subgroup. We use the symmetric height bound b for the Hodge jumps. For an invariant crystalline period matrix P, use the matrices of Section B.1 with $h = 2b$, so

$$U = (E\tau)^b P, \qquad UZ = E^{2b} Q, \qquad QQ' = \tau^{2b} I, \qquad YZ = E^{2b} I.$$

The columns n_i of $\tau^{-b} Q$, in the original lattice basis, satisfy

$$E_a^b \lambda\Big(\sum_i c_i \varphi^a n_i\Big) = (\lambda_{\mathrm{lat}} \varphi^a P) \varphi^a(Z) c \qquad (a \geq 0). \tag{1}$$

For the integral tests c has entries in $A_{(p)} = (A \setminus pA)^{-1} A \subset B_{\mathrm{d}R}$. The formula holds for every $c \in B_{\mathrm{d}R}^d$ after extending the chart-equivariant covector λ by linearity. The row λ_{lat} gives its values on the original lattice basis. The matrix identity underlying (1) is $E^b n = PZ$. Write $\rho(g)$ for the matrix on T and $H = G_{K'}$. Equivariance gives $g(\lambda_{\mathrm{lat}}) = \lambda_{\mathrm{lat}} \rho(g)$. Since $\rho(g)$ has $\mathbf{Z}_p$-entries, $g(\varphi^a P) = \rho(g)^{-1} \varphi^a P$. Thus $\lambda_{\mathrm{lat}} \varphi^a P$ has entries in $B_{\mathrm{d}R}^H = K'$. For a crystalline functional both $\lambda(T)$ and the entries of $\varphi^a P$ lie in $B_{\max}$, so the invariant products belong to $B_{\max}^H = K_0 \subset S_B$ [13, §13, corollary to Theorem 13.1]. For each standard covector e_j^{T}, define $(\lambda_{a,j})_{\mathrm{lat}} = e_j^{\mathsf{T}} (\varphi^a P)^{-1}$. Its entries are crystalline periods. Since $g((\varphi^a P)^{-1}) = (\varphi^a P)^{-1} \rho(g)$, its row defines a chart-equivariant crystalline functional. Its product with $\varphi^a P$ is e_j^{T}. For $x_a = \sum_i c_i \varphi^a n_i$, put $L_a(x_a) = (\lambda_{a,j}(x_a))_j$. Substitution in (1) and $YZ = E^{2b} I$ give

$$\varphi^a(Z) c = E_a^b L_a(x_a), \qquad E_a^{2b} c = \varphi^a(Y)\, E_a^b L_a(x_a). \tag{2}$$

In particular, for any B^+-submodule $J \subset B_{\mathrm{d}R}$, $E_a^b L_a(x_a) \in J^d$ implies $E_a^{2b} c \in J^d$, since $\varphi^a(Y)$ has entries in B^+.

The A-span M of the n_i is exactly the part of $A[1/\tau] \otimes T$ specified by the de Rham conditions

$$\lambda(\varphi^a x) \in \mathrm{Fil}^0 B_{\mathrm{dR}} \qquad (a \geq 1,\ \lambda \text{ a chart-equivariant form}).$$

In one direction (1) proves the integrality, since E_a has de Rham order zero for $a > 0$. In the other direction $QQ' = \tau^{2b} I$ first gives a presentation $x = \tau^{-m} \sum_i c_i n_i$, $c_i \in A$. Take m least. If $m > 0$, for every $a > 0$ the values of $\varphi^a(\tau^m x)$ have order at least one, since $\theta \varphi^a(\tau) = 0$.

Apply (2) at that level to $\varphi^a(c_i)$ and use $\theta(E_a) \neq 0$. Then $\theta\varphi^a(c_i) = 0$ for all $a > 0$. The cyclotomic intersection of these kernels is τA. This contradicts the least choice of m. The columns of $n = \tau^{-b}Q$ are independent by $QQ' = \tau^{2b}I$. Since $nQ' = \tau^b I$ and $\tau^b n = Q$ is integral, they give

$$\tau^b(A \otimes T) \subset M \subset \tau^{-b}(A \otimes T). \tag{3}$$

The positive Frobenius tests defining M are independent of the chosen bound. Hence one may take b to be the maximum absolute value of the jumps, with $b = 0$ for the zero lattice. Every element acting on T and normalizing the chart subgroup preserves these tests, since it permutes the equivariant forms and commutes with Frobenius. This applies to the tame generator because the tame chart field is Galois over $\mathbf{Q}_p$.

Put $N = \sum_i Sn_i$ and invert all E_a to obtain N_{loc} and M_{loc} from N and M. The integral change of basis is

$$E^b\varphi(n) = nB, \qquad B \in M_d(S). \tag{4}$$

Indeed the scalar in $\varphi Q = vQB$ is $v = (\varphi(\tau)/(E\tau))^b$ by Section A.4, so $E^b\varphi(\tau)^{-b}\varphi Q = \tau^{-b}QB$. Since n and $\varphi(n)$ are invertible over $W(\mathbb{C}_p^\flat)$, by reduction of $\det Q$ and τ in its residue field, and $E \neq 0$, we have $\det B \neq 0$. Iterating (4) gives, with the integral product $C_a = B\varphi(B)\cdots\varphi^{a-1}(B)$ from Section B.1,

$$\varphi^a(n) = n\mathcal{B}_a, \qquad \mathcal{B}_a = \left(\prod_{j=0}^{a-1} E_j^b\right)^{-1} C_a \in \mathrm{GL}_d(\mathrm{Frac}\, S), \qquad \mathcal{B}_0 = I.$$

Define the value field

$$\mathcal{K}_a = K'(S_B, \varphi^{-a}S) \subset B_{\mathrm{dR}}.$$

By (1), forms take values in $\mathcal{K}_0$ on n_i; by the identity $\varphi^{-a}(n) = n\varphi^{-a}(\mathcal{B}_a^{-1})$, they take values in $\mathcal{K}_a$ on $\varphi^{-a}(N_{\mathrm{loc}})$. The inverse Frobenius here acts on Witt coefficients, before taking their de Rham images.

B.2.1 Integral divided operators

We use the basis and the partial inverse $QQ' = \tau^h I$ constructed in Section B.1 to bound the divided Galois operators.

For the scalar integral cancellation, let $y = z$ or $y = \tau$. In each case $p \nmid y$, $\theta(y) \neq 0$, and $\xi - pu \in yA$ for some $u \in A$. For z use $E = p + z^e u'$ and $E \sim \xi$. For τ use $\xi \equiv p \bmod \tau$. The map $A[X]/(pX - \xi) \to A/(y, p^j)$ defined by $X \mapsto u$ extends to the p-adic completion. If $a \in A \cap yB^+$, clear p^m from its quotient by y in B^+ and apply these maps.

It follows that $p^m a \in yA + p^j A$ for every j. The ideal yA is closed for the p-adic topology: since $\bar{y} \neq 0$ in the tilt domain, divisibility of yc by p^j implies divisibility of c by p^j. The approximating quotients therefore converge. The same regularity cancels p^m and gives $a \in yA$. For $a \in A$ and $r \geq 0$, contract $E^r a \in yB^+$ to $E^r a = yc$ with $c \in A$. Since E is prime and $\theta(y) \neq 0$, E^r divides c. This proves

$$E^r a \in yB^+ \implies a \in yA. \tag{5}$$

For a product $y_1 \cdots y_k$ of copies of z, τ, suppose $E^r a \in y_1 \cdots y_k B^+$. Equation (5) gives $a = y_1 a_1$ with $a_1 \in A$. Cancelling y_1 in B^+ gives $E^r a_1 \in y_2 \cdots y_k B^+$. Induction yields $a \in y_1 \cdots y_k A$, including all $y = \tau^m z^n$.

Write $\delta = z\tau$ and let H be the subgroup fixing the chart field K', so $g \in H$ fixes the Witt constants and $\theta(z)$. For $g \in H$ the chart action satisfies

$$gs - s \in \delta EA \quad (s \in S), \qquad gy - y \in \delta B^+ \quad (y \in S_B). \tag{6}$$

For the first inclusion $gz/z = [v]$ is a Teichmüller lift with $v^\sharp = 1$. For every $j \geq 0$, $\theta\varphi^j([v]-1) = (v^\sharp)^{p^j} - 1 = 0$. The common kernel calculation in Section A.4 gives $[v]-1 \in \mu A$, with $\mu = \varphi(\tau)$. Thus $gz - z \in z\mu A = \delta EA$. For $s = \sum_n a_n z^n$ with its Witt coefficients fixed,

$$gs - s = (gz - z) \sum_{n \geq 1} a_n \sum_{j=0}^{n-1} (gz)^j z^{n-1-j}.$$

The inner sum lies in $z^{n-1}A$, so the series converges in the (p, z)-adic topology of A.

For the second inclusion put $R = A_{\max}$ and $\partial_g = (g-1)/(z\tau)$. The action preserves R, since g sends the generator ξ of $\ker\theta$ to a unit multiple of ξ. We have $\partial_g(S) \subset EA$. Writing $E = p + z^e u$, the inclusions $gz/z - 1 \in \mu A$ and $gu - u \in z\mu A$ give $gE - E \in z^e \mu A$. Hence

$$\partial_g(E/p) \in z^{e-1}(\xi/p)A \subset R, \qquad \partial_g(vw) = g(v)\partial_g(w) + \partial_g(v)w.$$

Consequently $\partial_g(S[E/p]) \subset R$. For $y \in S_B \cap R$, write $y = \sum_{m \geq 0} p^m H_m$ with $H_m \in A[E/p]$ by p-adic completeness. The extended coordinate $\widehat{F}_0$ from Section A.1 fixes y and sends H_m to $h_m \in S[E/p]$. Therefore

$$gy - y = z\tau \sum_{m \geq 0} p^m \partial_g(h_m) \in z\tau R.$$

The series on the right converges in R. For arbitrary $y \in S_B$, apply this to $p^m y \in S_B \cap R$ for a sufficiently large m. This gives $gy - y \in z\tau B^+$. For $x \in N$, (1) gives $h = E^b \lambda(x) \in S_B$ for each crystalline λ.

Equivariance and (6) give

$$E^{2b}\lambda(gx - x) = \frac{E^b}{(gE)^b}(E^b(gh - h) + (E^b - (gE)^b)h) \in \delta B^+,$$

since gE/E is an integral unit. Write $gx - x = \sum_i c_i n_i$, with $c_i \in A$ by stability of M. At $a = 0$, (2) yields

$$E^{3b}c = Y E^{2b} L_0(gx - x) \in (\delta B^+)^d.$$

Equation (5) gives $c_i \in \delta A$ for each i, hence $(gx - x)/\delta \in M$. For $x = y/D \in N_{\mathrm{loc}}$, each factor E_a of D satisfies $gE_a/E_a = \varphi^a(gE/E) \in A^\times$. Thus $gx \in M_{\mathrm{loc}}$, and

$$\frac{gx - x}{\delta} = D^{-1}\left(\frac{gy - y}{\delta} - \frac{gD - D}{\delta}\, gx\right) \in M_{\mathrm{loc}}$$

by (6) applied to $D \in S$. Write $\pi_a = \theta\varphi^{-a}(z)$. At every $a \geq 0$,

$$\theta\varphi^{-a}(\delta) = \pi_a(\zeta_{p^{a+1}} - 1) \neq 0.$$

At $p = 2, a = 0$ this value is $-2\pi_0$. Thus $\varphi^{-a}(\delta)$ is a unit in B^+_{dR} at every level. Division by it preserves the de Rham orders of the difference. The coefficient projector commutes with this scalar and with g, which fixes the Witt constants.

B.2.2 Retraction at each order

In this subsection, we construct a retraction that preserves the specified orders and depths after applying the divided differences of Section B.2.1.

Let $F : A \to S$ be any S-linear coordinate map, $F_a = \varphi^{-a} F \varphi^a : A \to A$. On A it commutes with multiplication by $\varphi^{-a}(s)$ for every $s \in S$, since F is S-linear. In particular F_a is S-linear by $\varphi^a S \subset S$. Extend it to B^+ by applying it to coefficients of divided polynomials in E/p and then taking limits. Each scalar $\varphi^{-a}(s)$ belongs to A, so its commutation survives this extension. The extended map is also S_B-linear. The image of $S[1/p]$ under θ is K', generated by the Witt constants and $\theta(z)$. This field is complete and θ is continuous on B^+, so $\theta(S_B) = K'$. Successive Newton approximation lifts each constant of K' to S_B modulo any specified positive de Rham order. For $c \in K'$ take its minimal polynomial $P \in \mathbf{Q}_p[X]$. Choose $b, d \in S_B$ with $\theta b = c$ and $\theta d = 1/P'(c)$. If $b - c \in \mathrm{Fil}^j$ for $j \geq 1$, Taylor expansion gives $b - dP(b) - c \in \mathrm{Fil}^{j+1}$.

Consequently, for $u_i \in B^+$, $c_i \in K'$, replacing u_i by $\widehat{F}_a(u_i)$ can only increase the order of $\sum_i c_i u_i$. For positive precision, approximate the c_i by elements of S_B, use $B^+ \cap \mathrm{Fil}^j = E^j B^+$, and apply S_B-linearity. Order zero is automatic. For the values $v_i \in \mathcal{K}_a$, clear a common nonzero denominator and expand each numerator as a finite sum of $c\, b\, \varphi^{-a}(s)$ with $c \in K', b \in S_B, s \in S$. Applying the preceding argument to the resulting single finite sum and using both scalar actions gives

$$\mathrm{ord}_t\Big(\sum_i \varphi^{-a}(F(r_i))v_i\Big) \;\geq\; \mathrm{ord}_t\Big(\sum_i \varphi^{-a}(r_i)v_i\Big), \qquad r_i \in A. \tag{7}$$

The clearing factor shifts both orders equally. Separatedness includes the case of an exact relation.

Take $F = \mathrm{coord}_0$, which fixes 1. Then, inside $B_{\mathrm{d}R}$, for every $a \geq 0$,

$$A \cap \mathcal{K}_a = \varphi^{-a} S.$$

For b in the intersection, use $v_1 = b, v_2 = -1$ with $r_1 = 1, r_2 = \varphi^a b \in A$ in (7). The right-hand sum is zero. Thus $b = \varphi^{-a} F(\varphi^a b) \in \varphi^{-a} S$. The reverse inclusion follows from $\varphi^{-a} S \subset A$ and the definition of $\mathcal{K}_a$.

Write $x = \sum_i r_i n_i$ in M. If all its form values lie in $\mathcal{K}_0$, apply (7) with $a = 0$ to the values $-\lambda(x), \lambda(n_i)$ and coefficients $1, r_i$. The exact relation survives the replacement $r_i \mapsto F(r_i)$. Crystalline forms separate vectors by the inverse period matrix, so $x = \sum_i F(r_i) n_i \in N$. For every $l \geq 0$ the same argument gives

$$\{x \in \varphi^l M : \lambda(x) \in \mathcal{K}_0 \text{ for every chart-equivariant } \lambda\} = \sum_i S \varphi^l(n_i).$$

Indeed the basis values $\lambda(\varphi^l n_i)$ lie in $\mathcal{K}_0$ by (1); the coefficients $\varphi^l(r_i)$ of x may therefore be replaced by $F(\varphi^l(r_i)) \in S$. The reverse inclusion uses $S \subset \mathcal{K}_0$ and $\varphi^l A = A$. Since $\varphi^l(S) \subset S$, the right side is $S \cdot \varphi^l(N_T)$.

Let D be a finite product of the E_j, allowing repetitions and the empty product. On the localized modules define

$$R_T\left(D^{-1} \sum_i r_i n_i\right) = D^{-1} \sum_i F(r_i) n_i, \qquad r_i \in A.$$

Passing to a common denominator gives the same expression, by the independence of the n_i and S-linearity of F. Under the changes of basis in Section B.1, the period columns become

$n_1 = H^{-1}nV$. For a column of coefficients $r \in A^d$, $F(V^{-1}r) = V^{-1}F(r)$ entrywise. Hence the new retracted column is $D^{-1}n_1F(V^{-1}r) = H^{-1}D^{-1}nF(r)$. This is the same vector in the new integral coordinates, so its orders on all equivariant forms are preserved. In particular $R_T(ay) = F(a)y$ for $a \in A$ and $y \in N_{\mathrm{loc}}$, and R_T fixes N_{loc}. At each fixed denominator D its integral bound is

$$R_T(p^m D^{-1} M_T) = p^m D^{-1} N_T \qquad (m \geq 0).$$

The inclusion follows from $F(p^m A) \subset p^m S$; the restriction to the right-hand side is the identity. For a chart-equivariant form λ, put $v_a(y) = \lambda(\varphi^{-a}y)$. For $y \in N_{\mathrm{loc}}$ we have $v_a(y) \in \mathcal{K}_a$ by (1) and $\varphi^{-a}(n) = n\varphi^{-a}(\mathcal{B}_a^{-1})$. If $D = \prod_j E_j^{d_j}$, then $\mathrm{ord}_t(\varphi^{-a}D) = d_a$. Indeed E generates $\mathrm{Fil}^1 B_{\mathrm{dR}}^+$; for $j > a$ the value $\theta\varphi^{j-a}(\xi) = p$ is nonzero, and for $s = a - j > 0$ one has

$$\theta\varphi^{-s}(\xi) = \frac{\zeta_{p^s} - 1}{\zeta_{p^{s+1}} - 1} \neq 0.$$

The distinction between the three cases is preserved by $E/\xi \in A^\times$. Write $x = \sum_i r_i(n_i/D)$. Applying (7) to the values $v_a(n_i/D)$ and the additional value $-v$ with coefficient $1 = F(1)$ gives, for every $a \geq 0$ and $v \in \mathcal{K}_a$,

$$\mathrm{ord}_t(v_a(R_T x) - v) \geq \mathrm{ord}_t(v_a(x) - v).$$

In particular $v = 0$ preserves every lower order bound. For $H = G_{K'}$ and $d \in \mathbf{Z}$, the depth condition on a value $y \in B_{\mathrm{d}R}$ is equivalent to a congruence

$$(g-1)y \in \mathrm{Fil}^d \quad (g \in H) \qquad \Longleftrightarrow \qquad y - c \in \mathrm{Fil}^d \quad \text{for some } c \in K'.$$

Indeed, if $m = \mathrm{ord}_t\, y < d$, its leading class in $\mathrm{gr}^m B_{\mathrm{d}R} = \mathbf{C}_p(m)$ is nonzero and fixed by H. Tate's fixed-point theorems give $m = 0$ and $\theta(y) \in K'$. Subtract this constant. Any further nonzero leading class below d would have positive degree and be fixed by H, which is impossible [75, §3.3, Theorems 1 and 2, p. 176]. The converse follows from the H-stability of Fil^d. Apply the preceding inequality to $v = c \in K' \subset \mathcal{K}_a$ for each level and each form. Thus the same $R_T x$ retains all prescribed order and depth bounds simultaneously.

An equivariant lattice map $f : T \to T'$ carries M_T to $M_{T'}$ and N_T to $N_{T'}$ by pulling back the forms in their defining tests. In the two chart bases, write $f(n_i) = \sum_j s_{ji} n'_j$, $s_{ji} \in S$. Using the same scalar coordinate F for both lattices gives

$$R_{T'} f(x) = D^{-1} \sum_j F\Big(\sum_i s_{ji} r_i\Big) n'_j = D^{-1} \sum_{i,j} s_{ji} F(r_i) n'_j = f R_T(x).$$

This applies to the lattice projections. It also gives commutation with their integral spectral idempotents, which are polynomials in the coefficient action over $W(k)$.

Let γ be a $\mathbf{Z}_p$-automorphism of the coefficient ring O. On the same chart put $T^\gamma = O_\gamma \otimes_O T$, where O_γ has its usual left action and right action $b \cdot a = b\gamma(a)$. The map $f_\gamma(x) = 1 \otimes x$ is Galois-equivariant and $\mathbf{Z}_p$-linear, with $f_\gamma(ax) = \gamma(a) f_\gamma(x)$. Extending it by the identity on the chart scalars gives

$$R_{T^\gamma} f_\gamma = f_\gamma R_T, \qquad f_\gamma e_i(\zeta) = e_i(\gamma(\zeta)) f_\gamma.$$

Here $e_i(\zeta)$ is the projector onto the $[i]$-eigenspace of multiplication by the coefficient root ζ, as in Section B.3. Its projector polynomial has coefficients in $W(k)$, which the extended map fixes, while it carries multiplication by ζ to multiplication by $\gamma(\zeta)$.

In particular the coordinate retraction is O-linear on the lattice with its given coefficient action. For a finite extension O'/O put $T' = O' \otimes_O T$, on the same chart. An O-basis of O' gives equivariant inclusions and projections exhibiting T' as a direct sum of copies of T over O. Applying the preceding identities to these maps gives, under the natural maps,

$$M_{T'} = O' \otimes_O M_T, \qquad N_{T'} = O' \otimes_O N_T, \qquad R_{T'} = 1 \otimes R_T.$$

The last identity holds after inverting the E_a as well. For $l \geq 0$, let $P_l = \varphi^l(\tau^b M_T) \subset A \otimes T$. Since $\tau^b M_T = QA^d$ and φ is bijective on A, the saturation of the integral column span in Section B.1 gives $P_l \cap p^m(A \otimes T) = p^m P_l$. If ϖ is a uniformizer of O, write $p = u\varpi^{e_O}$ with $u \in O^\times$. For $y \in A \otimes T$ with $\varpi^s y \in P_l$, choose m such that $e_O m \geq s$. The O-action on T commutes with Frobenius and preserves M_T, so $p^m y = u^m \varpi^{e_O m - s}(\varpi^s y) \in P_l$. Thus $y \in P_l$, which gives saturation at every power of ϖ as well.

For $\sigma \in G_{\mathbf{Q}_p}$, transport the lattice and chart as in Section B.1. Taking $F^\sigma = \sigma F \sigma^{-1} : A \to \sigma S$ in the transported basis σn_i gives

$$R^\sigma(\sigma x) = (\sigma D)^{-1} \sum_i F^\sigma(\sigma r_i)\, \sigma n_i = \sigma R_T(x), \qquad x = D^{-1} \sum_i r_i n_i.$$

Indeed $\sigma E_j = \varphi^j(\sigma E)$, so these are the divisors in the transported chart. For translates by the subgroup H with the original coordinate map fixed, we use the following residue calculation.

For $v \in \mathcal{K}_a \cap \mathrm{Fil}^0$, choose $r \in A$ and $m \geq 0$ with $p^m v \equiv r \bmod \mathrm{Fil}^1$, using the surjectivity of $\theta : A \to \mathcal{O}_{\mathbb{C}_p}$. In (7) use the two values $p^m v, -1$ and the two coefficients $1, \varphi^a(r)$. This replaces r by $\varphi^{-a} F \varphi^a(r)$ modulo Fil^1. With $\pi_a = \theta(\varphi^{-a} z)$ this gives

$$\theta(v) = p^{-m} \theta \varphi^{-a} F(\varphi^a r) \in K'(\pi_a).$$

The integral tower calculation in Section A.2 gives the first equality below. The preceding inclusion gives one direction of the second, and the residue field contains the residues of K' and $\varphi^{-a} z$, hence contains $K'(\pi_a)$. Thus

$$\theta \varphi^{-a}(S) = W(k)[\pi_a] = \mathcal{O}_{K'(\pi_a)}, \qquad \theta(\mathcal{K}_a \cap B^+_{\mathrm{dR}}) = K'(\pi_a).$$

More explicitly, let $g_b \in H$, $m_b \in \mathbf{Z}$, and suppose $\sum_b m_b g_b(s) \in E_a A$ for all $s \in S$. For $v \in \mathcal{K}_a \cap \mathrm{Fil}^0$ the residue calculation above supplies $s \in S$ with $p^m v \equiv \varphi^{-a} s \bmod \mathrm{Fil}^1$ for some m. Commuting each g_b with φ^{-a} and reducing gives $\sum_b m_b g_b(v) \in \mathrm{Fil}^1$. In particular $\pi_0 \in K'$, so an integral form value at level zero of $x \in N_{\mathrm{loc}}$ has g-difference of order at least one for every $g \in H$. This is the extra order needed for the level-one division by E below. For the tame generator acting nontrivially on $\theta(z)$, Section C.2.2 proves divisibility by z using the annihilating polynomial of its linear period action.

B.2.3 Tensor pairings and orders

In this subsection, we apply the order estimates of Section B.2.2 to a pairing.

Finally let $b_0 : T_1 \times T_2 \to T_3$ be an equivariant pairing, bilinear over the coefficient ring. The localized coefficients and all their Frobenius translates lie in $A_{(p)}$, since each inverted period has nonzero reduction modulo p. The inclusions

$$W(\mathbb{C}_p^\flat) \longleftarrow A_{(p)} \longrightarrow B_{\mathrm{dR}}$$

give Witt reduction modulo p^m on the left and the de Rham value on the right. For the value at level a, apply φ^{-a} before taking the right inclusion. The Witt inclusion commutes with Frobenius and the chart Galois action, and scalar extension preserves the pairing. The coefficient idempotents for an index i commute with this base change, and bilinearity gives $b_0(e_i y, x) = e_i b_0(y, x) = b_0(y, e_i x)$. In particular $b_0(e_i y, e_{i'} x) = 0$ for $i \neq i'$. At level a the relevant index in both factors and in the target is $j = i^{p^{-a}}$. A form of type u on T_s takes the lattice projected to j into Fil^{-u}.

For each projected factor, choose a K'-basis of the invariant de Rham dual adapted to its filtration. Write $-u_s$ for the filtration index of λ_s, so its type is u_s. The functionals $t^{u_s}\lambda_s$ form an integral de Rham dual basis, as proved in Section B.3. On the two parts indexed by j, expand the pullback of a form λ of type at most v as

$$\lambda \circ b_0 = \sum_{s,h} \alpha_{sh} \lambda_s \otimes \lambda'_h, \qquad \alpha_{sh} \in K'.$$

Indeed the tensor products give a de Rham basis, and its coefficients are fixed by the chart subgroup. Bilinearity on the original lattices makes $t^v(\lambda \circ b_0)$ integral; in the rescaled dual bases this means $t^{v-u_s-u'_h}\alpha_{sh} \in B^+_{\mathrm{dR}}$. A nonzero scalar in K' has de Rham order zero, so $\alpha_{sh} = 0$ if $u_s + u'_h > v$. For two vectors whose orders are bounded below by c_1, c_2, it follows that

$$\mathrm{ord}_t\, \lambda(\varphi^{-a} b_0(x_1, x_2)) \;\geq\; \min_{u+u' \leq v} (c_1(a,u) + c_2(a,u')),$$

where u, u' run through the jumps at j, and the minimum of an empty set is $+\infty$.

For $x_s \in M_{T_s}$, bilinearity on the original lattices first gives $b_0(x_1, x_2) \in A[1/\tau] \otimes T_3$. For every $n > 0$, expand the pullback of any invariant form on T_3 in products of invariant dual bases of the two factors. All coefficients lie in K'; both evaluations on $\varphi^n x_s$ lie in B^+_{dR} by the defining test for M_{T_s}. This proves $b_0(x_1, x_2) \in M_{T_3}$. For localized vectors with denominators D_1, D_2, the product $D_1 D_2$ clears the paired vector, giving the same conclusion in $M_{T_3,\mathrm{loc}}$.

For a saturated inclusion $T_0 \subset T$ of lattices crystalline on this chart, restriction of equivariant de Rham forms from T to T_0 is surjective. After extending scalars from K' to B_{dR}, the comparison isomorphisms identify this restriction with ordinary dual restriction. Thus it is already surjective over K'. With $A_{\mathrm{loc}} = A[1/E_a : a \geq 0]$, the positive Frobenius tests and the value-field test give

$$M_{T_0,\mathrm{loc}} = M_{T,\mathrm{loc}} \cap (A_{\mathrm{loc}}[1/\tau] \otimes T_0),$$
$$N_{T_0,\mathrm{loc}} = N_{T,\mathrm{loc}} \cap (A_{\mathrm{loc}}[1/\tau] \otimes T_0).$$

Indeed, clear a common finite product of the E_a and apply those tests before localization. Saturation makes T_0 a direct summand of the underlying $\mathbf{Z}_p$-module, so the cleared vector lies in $A[1/\tau] \otimes T_0$. For the orders at each individual jump on the kernel of a cocycle extension we use the type-preserving extension of forms in Section C.2.

B.3 Standard orders and projections on the chart

In this subsection, we identify the lattices defined by the filtration in Section B.2 with $\varphi^l M_T$. Frobenius transports the projection cokernel along the residue-field embeddings, as used in Section D.2.

For a lattice T crystalline on the chart subgroup use the two bases $M_T = \sum A n_i$ and $N_T = \sum S n_i$ just constructed. Write $E_a = \varphi^a E$ for $a \geq 0$, and localize both bases at the

E_a. A residue index is a nonzero residue-field element i. Write $m = |k^\times|$ and let Z_ζ be the endomorphism of T given by the chosen coefficient root ζ, of order dividing m. Its integral projector for $i \in k^\times$ is

$$e_i = \frac{1}{m} \sum_{r=0}^{m-1} [i]^{-r} Z_\zeta^r.$$

Indeed $m \in W(k)^\times$, $Z_\zeta^m = 1$, and the distinct Teichmüller roots have unit differences. These projectors commute with R_T by $W(k)$-linearity and naturality for Z_ζ. After extending the de Rham scalars to $\mathbf{C}_p$, e_i selects exactly the coefficient embeddings γ with $\gamma(\zeta) = [i]$, since Z_ζ acts on this summand by $\gamma(\zeta)$. For a further coefficient embedding γ' extending γ, this eigenvalue is unchanged. At level a Frobenius on the Witt coefficients gives $\varphi^{-a}(e_i) = e_{i^{p^{-a}}}$.

Let λ be a chart-equivariant de Rham form of type u on that latter part, meaning that its restriction to the original lattice, after projection, takes values in $t^{-u} B_{\mathrm{d}R}^+$. The Hodge filtration is split after choosing an adapted basis of invariant forms. In particular an element of the de Rham vector space belongs to the original $B_{\mathrm{d}R}^+$-lattice precisely when its values on the basis forms of types u are in $t^{-u} B_{\mathrm{d}R}^+$.

Indeed the reductions of the rescaled forms $t^u \lambda$ are independent over $\mathbf{C}_p$. In a minimal dependence, write $a_i = 1$ for the normalized coefficient and u_j for the type of λ_j. The residue of $t^{u_j} \lambda_j$ transforms by $\chi_{\mathrm{cyc}}^{u_j}$. Uniqueness of the normalized relation gives

$$h(a_j) = \chi_{\mathrm{cyc}}(h)^{u_i - u_j} a_j, \qquad a_j \in \mathbf{C}_p(u_j - u_i)^H.$$

By the fixed-point calculation in Section A.4, every nonzero coefficient has $u_j = u_i$ and belongs to K'. Lifting this relation with these constants gives a relation in the corresponding graded space, contrary to the adapted basis.

The rescaled forms therefore give a basis of the $B_{\mathrm{d}R}^+$-dual: their number is the dimension of the projected space by de Rham comparison. For each residue index j, let λ_s be this adapted dual basis and write

$$D_j = D_{\mathrm{dR}}(T[1/p])_j, \qquad u_s = -\max\{k : \lambda_s \in \mathrm{Fil}^k(D_j^\vee)\}.$$

Here $D_j^\vee = \mathrm{Hom}_{K'}(D_j, K')$ carries the decreasing dual filtration. The u_s are the jumps of the original representation on that part. Fix a finite set $\mathcal{J}_j$ containing all these types. For an invariant form $\lambda = \sum_s a_s \lambda_s$, the coordinates of $t^u \lambda$ in the integral dual basis are $a_s t^{u - u_s}$. Every nonzero $a_s \in K'$ has de Rham order zero. Hence the forms of type at most u are precisely the K'-span of the λ_s with $u_s \le u$. A vector $x \in e_i M_{T,\mathrm{loc}}$ has order bounds c when for all $a \ge 0$, $j = i^{p^{-a}}$, $u \in \mathcal{J}_j$, and all chart-equivariant forms of type at most u on the part at j,

$$\mathrm{ord}_t\, \lambda(\varphi^{-a} x) \ge c(a, u).$$

Equivalently, in the adapted dual basis for $j = i^{p^{-a}}$ the exact tests at level a are

$$\mathrm{ord}_t\, \lambda_s(\varphi^{-a} x) \ge \max_{\substack{u \in \mathcal{J}_j \\ u_s \le u}} c(a, u) \qquad (\text{every } s).$$

Each λ_s occurs in every test indexed by $u \ge u_s$. Conversely, expanding any form of type at most u in the indicated K'-span gives its order bound by the ultrametric inequality. In particular the standard bounds 0 and $-u$ give respectively 0 and $-u_s$ on λ_s. Its intersection with $N_{T,\mathrm{loc}}$ is the module of chart vectors with those orders. The coordinate retraction maps the first module onto

the latter, preserves congruences of the values to constants, and commutes with an equivariant projection of lattices.

The shifted divisors can be distinguished by their Fontaine values. For every integer j we have $v_p(\theta\varphi^j z) = p^j/e$. The two terms of $\varphi^j E = p + (\varphi^j z)^e \varphi^j(u_E)$, where $E = p + z^e u_E$ and $u_E \in S^\times$, have valuations $1, p^j$. For $j \neq 0$ these valuations are distinct, so

$$v_p(\theta\varphi^j E) = \min\{1, p^j\}.$$

Since E_a generates $\ker(\theta\varphi^{-a}|_A)$ and E is a uniformizer of B^+_{dR}, this gives $\mathrm{ord}_t(\varphi^{-a}E_b) = \mathbf{1}_{a=b}$. Also

$$\theta\varphi^j\tau = \begin{cases} 0, & j > 0, \\ \zeta_{p^{1-j}} - 1, & j \leq 0. \end{cases}$$

In particular $\varphi^{-a}(z\tau)$ has de Rham order zero for all $a \geq 0$, including $p = 2, a = 0$ where $\theta(\tau) = -2$.

Define $\mathrm{std}_l(a, u) = 0$ for $a < l$, and $\mathrm{std}_l(a, u) = -u$ for $a \geq l$. The part at i having these orders is precisely the part at i of $\varphi^l M_T$. To prove the direct inclusion, $\varphi^l M_T$ is contained in $M_{T,\mathrm{loc}}$ by $E^b\varphi(N_T) \subset N_T$ and induction. Write a vector there as $\varphi^l y$. At $a < l$, positive Frobenius integrality of y gives order at least 0. At $a \geq l$, a power of τ makes y an original integral vector. That power has order zero under every $\theta\varphi^{-(a-l)}$, so the lattice condition gives order at least $-u$. Conversely, clear from x a finite product of the E_a and a power of τ so that all its original coordinates are integral. For $a \geq l$ the adapted-basis test puts every original coordinate of $\varphi^{-a}x$ in B^+_{dR}. If an original coordinate of x is written $c/(E_a^h D')$, with $c \in A$, $h > 0$, and D' a product of τ and the other shifted divisors, then $\theta\varphi^{-a}D' \neq 0$. Since E has de Rham order one, $\varphi^{-a}c \in \mathrm{Fil}^h$, in particular $c \in E_a A$. Division by E_a reduces h by one. Repeating for every $a \geq l$ removes precisely the denominators excluded by these tests.

Write the remaining common denominator as $\tau^m \prod_{a<l} E_a^{m_a}$, with nonnegative exponents. Since E_a is associated to $\varphi^a\xi$ and $\varphi^l\tau = \tau\prod_{a<l}\varphi^a\xi$, this denominator divides $(\varphi^l\tau)^h$ for $h \geq m$ and $h \geq m_a$ for all $a < l$. Thus $\tau^h\varphi^{-l}x \in A \otimes T$. Put $y = \varphi^{-l}x$. To test $\varphi^n y$ for $1 \leq n \leq l$, take $a = l - n$ in the standard orders: all basis forms have integral values. If $n > l$, use $x \in M_{T,\mathrm{loc}}$ to choose a product D' of the E_a with $D'x \in M_T$. The positive Frobenius test makes every tested value of $\varphi^{n-l}(D'x)$ integral. Each factor of $\varphi^{n-l}D'$ has de Rham order zero, since $\theta\varphi^{n-l}(E_a) \neq 0$ for every $a \geq 0$. Dividing gives integral values on $\varphi^n y$. Thus $y \in M_T$, proving the equality. For a fixed index, the corresponding chart lattice is

$$e_i\varphi^l M_T \cap N_{T,\mathrm{loc}} = e_i \sum_j S\,\varphi^l(n_j).$$

Indeed the values of every chart vector lie in $\mathcal{K}_0$ at level zero, so the coefficient extraction of Section B.2.2 gives the inclusion from left to right. The reverse follows from (4) and preservation of $N_{T,\mathrm{loc}}$ by e_i.

Use one scalar coordinate $F = F_0 : A \to S$, with $F(1) = 1$, for all lattices on this chart. For a trivial finite free parameter lattice C over O, choose a $\mathbf{Z}_p$-basis of C. At the positive Frobenius powers its coordinate rows are equivariant de Rham forms. If $\tau^{-m}a$, $a \in A$, is one coefficient of a point of M_C and $m > 0$, then $\theta\varphi^j(a) = 0$ for every $j > 0$. The cyclotomic intersection gives $a \in \tau A$. Iterating on m proves $M_C = A \otimes_{\mathbf{Z}_p} C$. The intrinsic chart test at level zero puts each coordinate of an element of N_C in $\mathcal{K}_0$. The exact relation in (7) fixes each such coefficient under F, so the coefficients belong to S. Conversely, all vectors of $S \otimes_{\mathbf{Z}_p} C$ pass

the test since the values on constant basis vectors lie in K'. Thus $N_C = S \otimes_{\mathbf{Z}_p} C$, with $e_i N_C$ generated over S by $e_i(1 \otimes c)$, $c \in C$. On the parameter R_C acts coefficientwise by F, also after localization. The assertion applies both to a single-cocycle parameter $C = O$ and to the finite free module of crystalline cocycles in Section E.1.

For an extension on any of the three sides, write $\pi : X \to C$ for its parameter projection. Write $V_l(i)$ for the level-l standard-order submodule of the part at i of $M_{X,\mathrm{loc}}$. The multiplier ideal is the following ideal of A:

$$I_l(i) = \{f \in A : f e_i(A \otimes C) \subset \pi(V_l(i))\}.$$

For $c \in C$ put $c_i = e_i(1 \otimes c)$. These projected constants span $e_i(A \otimes C)$ over A; for the trivial lattice each lies in its chart module, since the constant $1 \otimes c$ is Frobenius fixed with integral constant form values and the projector preserves the chart module. Thus for $q = p^\nu$, membership $f \in I_l(i) + (q)$ is equivalent to

$$\exists w \in A \quad \forall c \in C \quad \exists x_c \in V_l(i), \qquad \pi(x_c) = (f - qw)c_i.$$

For $f \in S$, retraction replaces w by the single element $F(w) \in S$ and x_c by a lift in $V_l(i) \cap N_{X,\mathrm{loc}}$. Indeed R commutes with π, fixes the projected constants and satisfies $R(ac_i) = F(a)c_i$.

Witt Frobenius identifies $V_l(i) = e_i \varphi^l M_X$ bijectively with $V_{l+1}(i^p)$ and sends c_i to c_{i^p}. Since it fixes q, for each $c \in C$ we have

$$\pi(x_c) = (f - qw)c_i \quad \Longleftrightarrow \quad \pi(\varphi x_c) = (\varphi(f) - q\varphi(w))c_{i^p}.$$

The witness $\varphi(w)$ works simultaneously for all c. In the reverse direction apply φ^{-1} to the common witness and every lift over $A = W(\mathcal{O}^\flat_{\mathbf{C}_p})$. Thus

$$\varphi\big(I_l(i) + (q)\big) = I_{l+1}(i^p) + (q).$$

For $f \in S$ this gives an equivalence of membership for f and $\varphi(f)$, with chart lifts obtained by retraction at the corresponding level. More explicitly, for $f \in S$, $\pi x = (f - qw)c_i$ and $a \geq 0$, put $i_a = i^{p^a}$. Then

$$\pi R_X(\varphi^a x) = \big(\varphi^a f - qF(\varphi^a w)\big)c_{i_a},$$
$$\pi\varphi^a(R_X x) = \big(\varphi^a f - q\varphi^a(F(w))\big)c_{i_a}.$$

Subtracting the two parameter projections gives

$$\pi(R_X\varphi^a - \varphi^a R_X)x = q\big(\varphi^a(F(w)) - F(\varphi^a w)\big)c_{i_a}.$$

Thus both projections reduce to $(\varphi^a f)c_{i_a}$ modulo q. For exact lifts $w = 0$ they agree integrally at every level. These identities apply componentwise to the parameter maps of the two cocycle extensions and the extension of the pairing. At level a the same scalar coordinate is used on each lattice in the form $F_a = \varphi^{-a} F \varphi^a$ of (7).

Multiplication by E_a raises the level-a order by one and leaves all other orders unchanged, since $\theta\varphi^{-j}(E_a) \neq 0$ for $j \neq a$. If $\omega \geq 0$ and $u \leq \omega$ for every $u \in \mathcal{J}_i$, multiplication by E^ω carries level-zero lifts to level-one lifts: at level zero the new bound is $\omega - u \geq 0$, and at every positive level the required bound is still $-u$.

For g in the chart subgroup and $x \in N_{T,\mathrm{loc}}$ of standard order l, Section B.2.1 gives $(gx - x)/(z\tau) \in M_{T,\mathrm{loc}}$ with the same lower bounds on its orders. When $l = 1$, we may also divide

by E while preserving these bounds. To see this, apply at level zero the residue calculation of Section B.2.2. If $\lambda(x) \in \mathrm{Fil}^0$, then $\theta(\lambda(x)) \in K'$; hence $\lambda(gx - x) = g\lambda(x) - \lambda(x) \in \mathrm{Fil}^1$. The other levels are unaffected by division by E. On the trivial parameter, the same quotient of a chart scalar $w \in S$ is integral by $gw - w \in z\tau EA$. Thus division and subsequent retraction give the integral lift-ideal operations with divisor $z\tau$ at level zero and $z\tau E$ at level one, for the original parameter, index, and modulus.

C The tame cup-product estimate

In this appendix, we apply the filtration orders of Appendix B on the tame chart. The level-zero estimates control the four terms in the cup calculation and give the bound at the tame level primes in Section 29.1.

C.1 Level-zero steps for tame periods

In this subsection, we use the divided difference in Section B.3 to raise the order of a Galois difference at $E = 0$. The valuation of the scalar $(\omega^j - 1)/(z\tau)$ determines the power of p required at each step. The operator preserves every filtration bound already attained.

Let $H = G_{K'}$ for the finite tame chart field K' over which the lattices are crystalline. At $p \mid M$, Section C.2.2 takes $K' = K_0(\pi)$ with $K_0/\mathbf{Q}_p$ unramified of degree $r!$, $e = \mathrm{lcm}_{1 \le b \le r}(p^b - 1)$, and $\pi^e = p$. Write $E = p - z^e$, $\Delta_z = z\tau$, and $\mu = E\tau v_0$ with $v_0 \in A^\times$. Choose $g \in H$ such that $gz = [\epsilon]z$. Indeed, $X^{p^a} - \theta(z)$ is Eisenstein over K', so at each finite level such a conjugate of the chosen root exists. The conditions are nested and H is compact. Write $gE = E\omega$, where $\omega \in A^\times$ since g preserves $\ker\theta$. Then

$$\frac{\omega - 1}{\Delta_z} = -v_0 z^{e-1} \sum_{i=0}^{e-1} [\epsilon]^i.$$

Its value under θ has normalized p-adic valuation $(e-1)/e$, since e is prime to p. Let $\delta = 1$ for $e > 1$ and $\delta = 0$ for $e = 1$. For $0 \neq j \in \mathbf{Z}$ put $X_j = (\omega^j - 1)/\Delta_z$. This lies in A by the geometric sum and the invertibility of ω. Since $v_p(\theta z) = 1/e$ and $v_p(\theta\tau) = 1/(p-1)$, we obtain

$$v_p(\theta\omega - 1) = \frac{p}{p-1}, \qquad v_p(\theta X_j) = \frac{e-1}{e} + v_p(j).$$

Indeed, if $v_p(a) > 1/(p-1)$, the unique term of least valuation in $(1+a)^p - 1$ is pa. Taking a power prime to p preserves $v_p(a)$ by the geometric sum, and inversion preserves it since $1 + a$ is a unit. These calculations also apply at $p = 2$, where $v_2(\theta\omega - 1) = 2$. Every nonzero step in the tame level-prime application satisfies $p \nmid j$ by the arithmetic radius. For these j the ratio $p^\delta/\theta(X_j)$ is integral. Surjectivity of θ therefore gives $v_j \in A$ with $v_j X_j \equiv p^\delta \bmod E$.

If $y \in B_{\mathrm{dR}}$ satisfies $(h-1)y \in \mathrm{Fil}^j$ for all $h \in H$, then $y \in K' + \mathrm{Fil}^j$. Indeed, if its order n is below j, the nonzero leading class belongs to $\mathbf{C}_p(n)^H$. This space vanishes for $n \neq 0$, and for $n = 0$ it equals K'. In the latter case subtract the representative in K' and apply the same argument to the difference. The fixed-point calculation for every integer n is given in Section A.4.

Let x be a localized chart point over a constant parameter, with selected form value $y = \lambda(x)$ of depth j. Choose $k \in K'$ with $y - k \in \mathrm{Fil}^j$. By Section B.2.2, $y, E \in \mathcal{K}_0$ and $K' \subset \mathcal{K}_0$. Thus

$h = E^{-j}(y-k)$ belongs to $\mathcal{K}_0 \cap B_{\mathrm{dR}}^{+}$ for either sign of j. Its residue is in $K'(\pi_0) = K'$ by the same section. Consequently $gh - h \in \mathrm{Fil}^{1}$ and $g(y-k) - \omega^{j}(y-k) = \omega^{j} E^{j}(gh-h) \in \mathrm{Fil}^{j+1}$. Since $gk = k$,

$$\theta\Big(E^{-j}(gy-y)/\Delta_z\Big) = \theta(X_j)\,\theta(h) \qquad (j \neq 0).$$

Consequently

$$x \longmapsto R\Big(p^{\delta}x - \upsilon_j(gx-x)/\Delta_z\Big)$$

moves the values to the next filtration step modulo the scaled constants. The divided difference and its retraction preserve the old orders and lie over zero. This follows from Section B.3, since g fixes the parameter constant and Δ_z has order zero at all places of the chart. At $j = 0$ a chart value that is integral at E has residue fixed by the chart group, so $p^{\delta}x$ already advances this step. Thus the contribution of each step is at most δ.

Perform the steps in increasing order of j. For a line whose target depth is $j+1$, all earlier needed steps have then been performed, while the operator preserves previously obtained depths at every line. Indeed a value congruent to a constant modulo Fil^{h} has Galois difference of order at least h. Division by Δ_z has order zero, and the retraction preserves the congruence. This advances the orders over the finite union of step intervals. If a form value originally had positive order, the constant in its higher-depth congruence must be zero. Otherwise the new guaranteed order is the minimum of 0 and the new depth. In both cases the operator also retains the initially imposed orders. In the tame application the starting order is $-u$, and the final depth at zero is $\eta - u$. The final order is at least $-u$. On the side whose far jumps are negative we have $u \le \eta$. Its increased depth is nonnegative, so its final order is at least $\max(0, -u)$, in particular at least $\max(0, -u - \eta)$.

C.2 The four terms of the crystalline cup calculation

In this subsection, we prove the tame cup estimate by comparing the two pairs of crystalline lifts before and after Frobenius. The order estimates of Section C.1 place four bilinear terms in the image of the Witt vectors of the maximal ideal of the tilt. The Artin–Schreier–Witt equation then produces a continuous cochain whose differential is the required cup product.

On both sides choose the resulting points at each index with the same constant multiplier, multiplying by extra powers of p to reach the common exponent. Their orders are at least $-u$ at every level and their depths are $\eta - u$ at zero and $-u$ above. A difference of two points over the same parameter belongs to the kernel lattice with those orders.

For strictness of the filtration, take an invariant form λ of type u on the kernel and a crystalline primitive $b(c)$ for the cocycle of a basis element c of the trivial parameter lattice over $\mathbf{Z}_p$. Since

$$(g-1)\lambda(b(c)) = \lambda(z(c)(g)) \in \mathrm{Fil}^{-u},$$

the invariant calculation in Section C.1 gives $k(c) \in K'$ with $\lambda(b(c)) - k(c) \in \mathrm{Fil}^{-u}$. Extend λ on (v, c) by adding this last value to $\lambda(v)$, and extend linearly over that parameter basis. The resulting form is equivariant and has type at most u. The integral localized basis of the kernel is the intersection with its ambient subspace, by clearing the E_a and applying the positive Frobenius tests. Hence the order bounds on a difference apply to its original kernel components.

At index i, write x, x' for the two current points and y, y' for the Frobenius translates from the preceding index. These old points have order at least 0 at level zero and at least $-u$ at every positive level. Their differences under Galois have order at least 1 at level zero: for a chart

vector in N_{loc}, the values of its Frobenius translate under θ belong to the chart field, by the change of basis, the invariance of the period covector, and the residue map on S_B. At higher levels the old differences inherit the depth at the preceding index.

Suppose the first side is the negative-far side. Take the path with $w = x, w' = x'$. Its first two moves are $(y, y') \to (x, y') \to (x, x')$. The first move $y - x$ has order at least zero at level zero because x and y both do. Use the Frobenius translate of the preceding third period, of order at least zero at level zero. The second-side resting difference at y' has order at least 1 there, giving total at least 1.

At positive levels the two orders sum to at least $-u-u'$. In the expansion of the trilinear form with Tate target from Section B.2.3, a third jump v can contribute only when $u+u'+v \le -1$. If there is no such triple, the component of the trilinear form is zero. Otherwise each contributing third jump has standard order $-v \ge u+u'+1$. In the second move $y' - x'$, the moving side has $u' \ge -\eta$, so its order is at least $-u' - \eta$ at zero. The resting difference at x has depth $\eta - u$, and the current third period has order at least $u+u'+1$ on each contributing term. Thus total order is at least 1, also at higher levels using the standard orders.

The last two moves are zero. If the second side has the negative far jumps, take $w = y, w' = x'$. The two active moves $(y, y') \to (y, x') \to (x, x')$ are now the second and third, respectively. They use the preceding and current third periods in that order. The same order bounds apply with the sides interchanged. The pair of cocycle differences at two current points also passes the order test with the current third period, since both depths dominate their respective standard orders.

In each active test the factor Δ_z comes from the Galois difference of the resting point, or one of the two cocycle differences. Dividing by Δ_z preserves its orders and the integral localized span by the basis calculation. This applies also to a Galois translate of the difference: $g(z)/z \in A^\times$, and $g(\tau)A = \tau A$, since g preserves every $\ker(\theta\varphi^j|_A)$, $j > 0$, whose intersection is τA. Thus $g(\Delta_z)/\Delta_z \in A^\times$. The third extension form is equivariant, so orders add by the filtered tensor rule. The resulting Tate value has its standard orders 1 at every level. It is therefore in the Tate basis at level zero by Section B.3. The Tate basis bound says that multiplication by τ puts it in $A \otimes \mathbf{Z}_p(1)$. The remaining factor z places the reduced pairing in the small Witt ideal. This verifies all the order tests for the shared Frobenius cochain calculation. Summing over indices preserves the tests because the equivariant bilinear pairing is O-balanced, so distinct spectral idempotents have zero cross pairing. Since $\sum_i e_i = 1$ and $\varphi(e_i) = e_{i^p}$, summing the points at all indices puts each point on side j over $p^{e_j}(1 \otimes 1)$, and summing the translates gives the Frobenius translate of the same point.

C.2.1 The Frobenius cup calculation

In this subsection, we use the four terms and depth bounds in Section C.2 to construct the cochain.

Fix two extensions $Y \oplus O, D \oplus O$ for z_1, z_2 and the third period of the pairing. Work over the chart subgroup H. Write x, w for first-side points over p^{e_1}, and x', w' for second-side points over p^{e_2}. The two endpoints and intermediate points give the path

$$(\varphi x, \varphi x') \longrightarrow (w, \varphi x') \longrightarrow (w, w') \longrightarrow (x, w') \longrightarrow (x, x').$$

In the following computation the order tests for the four moves mean that their four cross

pairings, with the resting difference as the other argument, lie in the subgroup

$$I = \ker\Big(W_m(\mathcal{O}_{\mathbf{C}_p^\flat}) \longrightarrow W_m(\overline{\mathbf{F}}_p)\Big) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1) \ \subset\ W_m(\mathbf{C}_p^\flat) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1).$$

Membership means that all m Witt coordinates have positive valuation in a Tate basis. This subgroup is stable under H.

Fix a $\mathbf{Z}_p$-linear functional $\lambda : O \to \mathbf{Z}_p$. The integral O-bilinear pairing ψ is equivariant modulo p^m, so

$$\psi_m = ((\lambda \otimes 1) \circ \psi) \bmod p^m : (Y/p^m Y) \times (D/p^m D) \longrightarrow (\mathbf{Z}/p^m)(1)$$

is equivariant and bilinear over $\mathbf{Z}/p^m$. It satisfies $\psi_m(by, d) = \psi_m(y, bd)$ for $b \in O$. Let B be the Witt-bilinear extension of $p^{e_3}\psi_m$ on the two kernel components. For points in $Y \oplus O$ and $D \oplus O$, use the first-component projections pr_Y and pr_D of these module splittings. A difference over zero already belongs to its kernel. The pairing takes values in $W_m(\mathbf{C}_p^\flat) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)$. All localized evaluations are defined there. Indeed on a chart $E = p + z^e u$ with $u \in S^\times$, one has

$$E_a \bmod p = \bar{z}^{ep^a}\overline{\varphi^a(u)} \neq 0.$$

Thus E_a is a unit in $W_m(\mathbf{C}_p^\flat)$, including on the tame chart $u = -1$, and τ likewise has nonzero reduction. Thus the map $A_{\mathrm{inf}} \to W(\mathbf{C}_p^\flat) \to W_m(\mathbf{C}_p^\flat)$ extends to the localization used for the points. On the underlying lattices it is scalar extension over $\mathbf{Z}_p$. In particular the functional on Witt coefficients is $1 \otimes \lambda$; its integrality follows from $\lambda(O) \subseteq \mathbf{Z}_p$. The order estimates place the indicated pairings in I. The third period over p^{e_3} gives this multiplier by its extension form. The smallness order test is applied to points over zero. The sum of the three orders must be at least 1, the Tate standard order. A difference of Galois translates has the additional factor $z\tau$, which puts the integral Tate value in I: τ clears the Tate basis. Write $z = [z_0]$ with $v_\flat(z_0) > 0$. In ordinary Witt coordinates

$$z(w_0, \dots, w_{m-1}) = (z_0^{p^r} w_r)_{0 \le r < m},$$

so every coordinate of the product has positive valuation.

Solve

$$(\varphi - 1)y = B(\mathrm{pr}_Y w, \varphi x' - w') + B(\mathrm{pr}_Y x, w' - x').$$

Frobenius acts on the Witt scalars and fixes a Tate basis. It commutes with Galois. In each successive quotient for the filtration by initial zero Witt coordinates, $\varphi - 1$ acts as $v \mapsto v^p - v$ on $\mathbf{C}_p^\flat$. This map is onto because the tilt is algebraically closed [14, p. 223]. Induction over the m coordinates therefore gives y. Write $a = \mathrm{pr}_Y x$ and $b = \mathrm{pr}_D x'$. For $g \in H$ set $A_g = gx - x$, $D_g = gx' - x'$ and $c_{i,g} = p^{e_i} z_i(g)$, so that

$$A_g = ga - a + c_{1,g}, \qquad D_g = gb - b + c_{2,g}.$$

Define

$$F(g) = B(\mathrm{pr}_Y(gx), gb) - g\big(B(a, b) - y\big) - B(a, D_g) - y = B(c_{1,g}, gb) - B(a, D_g) + (g - 1)y.$$

Here gb is the linear action on the kernel of the second extension. Since φ fixes $c_{i,g}$ and commutes with g, expanding $(\varphi - 1)F(g)$ and substituting the equation for y gives the four terms

$$\begin{aligned}(\varphi - 1)F(g) = {} & B(gw - w, g(\varphi x' - w')) \\ & - B(\varphi x - w, g\varphi x' - \varphi x') \\ & - B(w - x, gw' - w') + B(gx - x, g(w' - x')).\end{aligned}$$

Each term lies in I by the indicated path.

For every d with $(\varphi-1)d \in I$ there is a unique $\gamma \in (\mathbf{Z}/p^m)(1)$ with $d-\gamma \in I$. In a Tate basis, filter I by the number of initial zero Witt coordinates. On each successive quotient $\varphi-1$ acts as $v \mapsto v^p - v$ on the maximal ideal of $\mathcal{O}_{\mathbf{C}_p^\flat}$. For u in this ideal, Hensel's lemma gives a unique root of $v^p - v = u$ reducing to zero. Induction on the finite filtration proves that $\varphi - 1 : I \to I$ is bijective. Subtract from d the unique element of I with the same Frobenius difference. The remainder lies in the fixed submodule $W_m(\mathbf{F}_p)(1) = (\mathbf{Z}/p^m)(1)$, since each of its Witt coordinates is fixed by $v \mapsto v^p$. Apply this to $d = F(g)$ and denote the resulting constant by $\gamma(g)$.

Use additive notation on the Tate module and the inhomogeneous differential $(dF)(g,h) = gF(h) - F(gh) + F(g)$. The relations $D_{gh} = D_g + gD_h$ and $c_{1,gh} = c_{1,g} + gc_{1,h}$ give

$$\begin{aligned}(dF)(g,h) &= B(c_{1,g}, gD_h) - B(c_{1,g}, g(hb-b)) - B(A_g, gD_h)\\ &= B(c_{1,g}, gc_{2,h}) - B(A_g, gD_h).\end{aligned}$$

The first term on the last line is $p^{e_1+e_2+e_3}(\psi_m)_*(\bar{z}_1 \cup \bar{z}_2)(g,h)$ by the definition of B. The same order test applied to the two cocycle differences puts the final term in I. As I is stable under the Galois action, the identity modulo I is exactly $p^{e_1+e_2+e_3}(\psi_m)_*(\bar{z}_1 \cup \bar{z}_2) = d\gamma$ on the chart subgroup. Since $F(1) = 0$, all m Witt coordinates of $F(g)$ tend to zero as $g \to 1$. To check the action on an integral tilt element v, let $\pi_n(v) \in \mathcal{O}_{\mathbf{C}_p}/p$ be its n-th component. Approximate a lift of this component modulo p by an algebraic integer over $\mathbf{Z}_p$. Its stabilizer is open and fixes $\pi_n(v)$. On this subgroup $\pi_n(gv - v) = 0$, so $v_\flat(gv - v) \geq p^n$, with $v_\flat(p^\flat) = 1$. This proves continuity on integral tilt elements, and passing to quotients gives it on $\mathbf{C}_p^\flat$. The input cocycles are continuous, the finite Witt operations are polynomial, and the Tate action modulo p^m is locally constant. The maximal ideal of the tilt is open, so $F(g) \in I$ on a common neighborhood of 1. There $\gamma(g) = 0$ by uniqueness. Choose an open subgroup in this neighborhood on which $\bar{z}_2 = 0$. For each g, the differential identity gives $\gamma(gh) = \gamma(g)$ for h in this subgroup. Hence γ is a continuous cochain. For a $\mathbf{Z}_p$-basis (b_i) of O and its dual coordinate forms λ_i, write γ_{λ_i} for these scalar cochains. On the chart subgroup the continuous cochain $\Gamma_O = \sum_i b_i \gamma_{\lambda_i}$ satisfies

$$d\Gamma_O = p^{e_1+e_2+e_3}(\psi \bmod p^m)_*(\bar{z}_1 \cup \bar{z}_2) \quad \text{in } (O/p^m O)(1).$$

Indeed the Galois action is trivial on the coefficients b_i, and evaluating the equality at each λ_i gives the scalar calculation.

C.2.2 The three extensions at a prescribed level prime

In this subsection, we apply Section C.2.1 on the original local group and bound the common exponent.

For the chosen infinite place τ_∞, put $p = \ell \mid M$, $\beta = S_j(\tau_\infty)$, $W_* = 4t$, $N = (r+t)r$, $k_0 = r^2(N+1)^2$, and $n_0 = 6(\eta + W_*)$. Both level primes split completely in E by Section 6, so $E_v = \mathbf{Q}_p$. Proposition 4.2 gives $p \geq b^{10}$. Here $\beta = \kappa_{\tau_\infty} + \delta_j \leq b + \delta_j$. Since $t = 5r$, $\eta = r^{16} + 1$, $N = 6r^2$ and $\delta_j \leq (2m+1)r^8$, we have $\delta_j + n_0 + k_0 + 2 = O(r^{16}) = o(b)$, uniformly for $1 \leq j \leq m$. Thus $p > \beta + n_0 + k_0 + 2$ for sufficiently large x. In particular the dyadic place lies outside M and is treated in Appendix E. Set $D = L^\dagger$ for the original summand in the colength estimate. Fix $Y \in \mathcal{Q}_v$, crystalline cocycles z_1, z_2 of Y, D, and an integral bilinear form ψ giving one of the pairings modulo p^m. The third lattice is $J = (Y \otimes_O D)^\vee(1)$. Its cocycle is

$$z_3(g) = (g\psi - \psi)/p^m.$$

It is integral by finite-level equivariance, and crystalline with period primitive ψ/p^m. On these three lattices $V_1 = Y, V_2 = D, V_3 = J$, use the split-as-modules extensions

$$X_i = V_i \oplus O, \qquad g(v, c) = (gv + c\, z_i(g), c).$$

The constructions are O-linear in c. For X_3 keep this parameter even if $z_3 = 0$. Over $\mathrm{Frac}(O)$ the map $(\lambda, c) \mapsto (\lambda + c\psi/p^m, c)$ identifies $X_3[1/p]$ equivariantly with $J[1/p] \oplus \mathrm{Frac}(O)$, by the formula for z_3. Thus its rational jump multiset and its linear period inertia are independent of m for the fixed Y, D. On the original lattice the extension form is

$$E_3((\lambda, c), y \otimes d) = p^m \lambda(y \otimes d) + c\, \psi(y \otimes d).$$

The integral map $e_m : X_3 \to J$, given by $e_m(\lambda, c) = p^m\lambda + c\psi$, is equivariant because $p^m z_3(g) = g\psi - \psi$. Its restriction to J is multiplication by p^m. For a chart-equivariant de Rham form f of type at most u on J, the form $p^{-m} f \circ e_m$ on X_3 extends f with the same type bound: $e_m(X_3) \subseteq J$ and p is a unit of B^+_{dR}. Thus the positive Frobenius tests and the order and depth bounds on kernel vectors apply at every shift with the original form f. Using the same scalar coordinate map on both lattices, Section B.2.2 gives $R_J e_m = e_m R_{X_3}$ on the localized modules. In $W_m(\mathbf{C}_p^\flat)$ all chart denominators used in the three points are units. Take $1 \in O$ as the third parameter. Modulo p^m the $p^m\lambda$ term vanishes, so a standard lift over $p^{k_0} \cdot 1$ evaluates to $p^{k_0}\psi$. When $z_3 = 0$, the Galois-equivariant section $c \mapsto (0, c)$ gives a standard lift of 1 in the trivial summand, which we multiply by p^{k_0}. In both cases Frobenius fixes the projected parameter. Composing with the chosen $\mathbf{Z}_p$-linear functional therefore gives the same $B = p^{k_0}\psi_m$ at each move.

Put $e = \mathrm{lcm}_{1 \le a \le r}(p^a - 1)$, $q = p^{r!}$, and $K_0 = W(\mathbf{F}_q)[1/p]$. The unramified field of degree $\mathrm{lcm}(1, \dots, r)$ in Section 28.3.2 lies in K_0. Since $a \mid r!$ for $1 \le a \le r$, we have $e \mid q - 1$ and $\mu_e \subset K_0$. Take $K' = K_0(\pi)$ with $\pi^e = p$ and set $H = G_{K'}$. The Eisenstein polynomial $X^e - p$ has all its roots in $\mu_e\pi \subset K'$. Since $K_0/\mathbf{Q}_p$ is Galois, so is $K'/\mathbf{Q}_p$, and its maximal unramified subfield is K_0. Thus H is the pointwise stabilizer in G_{E_v} of $W(\mathbf{F}_q)$ and π. Replacing π by another root gives the same field since their ratio lies in K_0. In particular

$$[G_{E_v} : H] = [K' : K_0][K_0 : \mathbf{Q}_p] = e\, r!, \qquad (p, e\, r!) = 1,$$

since $p \nmid e$ and $p > r$. On inertia the character

$$I_p \longrightarrow \mu_e, \qquad g \longmapsto g(\pi)/\pi$$

is surjective with kernel $I_p \cap H$. Indeed I_p fixes K_0 and $X^e - p$ remains Eisenstein over every finite unramified extension of K_0.

On the potentially crystalline periods of Y and L, the type polynomials of Section 28.3.2 and the definition of $\mathcal{Q}_v$ in Section 29.1 give trivial wild inertia and tame inertia eigenvalues in μ_e. On $D = L^\dagger$ these eigenvalues are inverted. For a basis v_{cyc} of $\mathbf{Q}_p(1)$ the crystalline period $t^{-1} \otimes v_{\mathrm{cyc}}$ is Galois invariant, so the Tate factor contributes the trivial period character. The finite linear inertia operators are semisimple in characteristic zero and satisfy $X^e - 1 = 0$. The kernel of the displayed character in the procyclic tame quotient consists of the e-th powers. Consequently $I_p \cap H$ acts trivially on the potentially crystalline periods of both Y and D.

For $V = Y[1/p]$ or $D[1/p]$, take the Galois closure over $\mathbf{Q}_p$ of the composite of K' with a crystallizing field of V. The resulting finite Galois extension $L'/\mathbf{Q}_p$ contains K', and V

remains crystalline over L'. Let L'_0 be its maximal unramified subfield over $\mathbf{Q}_p$. The inertia subgroup of $\mathrm{Gal}(L'/K')$ acts trivially on $D_{\mathrm{cris,L'}}(V)$ by the preceding calculation. Restriction to L'_0 identifies the quotient by inertia with $\mathrm{Gal}(L'_0/K_0)$, since $K'L'_0/K'$ is the maximal unramified subextension. The quotient acts semilinearly with invariants $D_{\mathrm{cris,K'}}(V)$. Galois descent gives

$$L'_0 \otimes_{K_0} D_{\mathrm{cris,K'}}(V) \simeq D_{\mathrm{cris,L'}}(V).$$

Thus Y, D and their tensor dual J are crystalline over H. Each $z_i|_H$ is a coboundary over the crystalline period ring and splits the corresponding extension there. Hence all three X_i are crystalline over H. Use the same Kummer coordinate z with $\theta(z) = \pi$, $E = p - z^e$. Choose an inertia element s inducing a generator of the order-e inertia quotient of this chart and satisfying $s(z) = \zeta z$, where ζ has order e. Inertia fixes the unramified chart coefficients. Here $\zeta \bmod p$ has that same order.

To choose s, fix compatible roots π_a of π of degrees p^a, and let $\zeta_a^{p^a} = \zeta$ in μ_e. The polynomial $X^{ep^a} - p$ is Eisenstein over every finite unramified field, so inertia takes π_a to $\zeta_a \pi_a$. The closed conditions at successive a are nested; compactness gives one s for all a. Their Teichmüller ratios give $s(z) = \zeta z$. The unramified part has trivial different. On adjoining π, the different is generated by $e\pi^{e-1}$, of normalized valuation $(e-1)/e$. Its ceiling is the δ used above.

Take the first two annihilating polynomials q_i at s over $\mathbf{Z}_p$. Each inertia eigenvalue of these types has order dividing $p^a - 1$ for some $1 \le a \le r$. The value and its inverse are Teichmüller units in an unramified extension of degree a, hence have monic minimal polynomials over $\mathbf{Z}_p$ of degree at most a. Multiplying these polynomials over the original upper type gives degree at most $N = (r+t)r$. Finite inertia is semisimple in characteristic zero. This gives the indicated linear-period annihilators, with the inverse type for D. To pass to the underlying $\mathbf{Q}_p$-period space, put $F_O = \mathrm{Frac}(O)$. Over $\overline{\mathbf{Q}}_p$ the scalar algebra $K_0 \otimes_{\mathbf{Q}_p} F_O$ splits into factors indexed by pairs (τ, ι) of embeddings. The element s acts linearly over this algebra. Applying any $g \in G_{\mathbf{Q}_p}$ to the matrix identity on one factor gives the identity on $(g\tau, g\iota)$, since $q_i \in \mathbf{Z}_p[X]$. For fixed $g\iota$, crystalline Frobenius transports that identity to every embedding of K_0: it commutes with s and acts on K_0 by a generator of its unramified Galois group. All pairs are therefore covered, and faithful scalar extension gives the identity on the original period space. For the third side take annihilators of Y, D at s^{-1}. The characteristic polynomial of multiplication by $X \otimes X$ on the tensor product of their two polynomial quotient algebras is monic of degree at most N^2. By Cayley–Hamilton it kills the tensor period operator at s^{-1}, hence kills the dual operator at s.

The Tate factor has trivial linear inertia on its crystalline period. This is the polynomial q_3. On X_i the polynomial $Q_i = (X-1)q_i$ kills the period action, by the exact sequence with trivial quotient. The possible jump sets for the extensions are the two original intervals with 0 adjoined and $\{0\} \cup \{-u - u' - 1\}$ for the third side. Here u, u' range over the two jump sets at the corresponding indices, already with 0 adjoined. If both jumps are low then $|u + u'| \le 2\eta$. If only one is far then $|u + u'| \le \beta + W_* + \eta$, and if both are far, their opposite signs give $|u + u'| \le W_*$. Consequently

$$|-u-u'-1| \le 1 + \max(2\eta, \beta + W_* + \eta) \le \omega, \qquad \omega = \beta + n_0.$$

The two original jump sets also lie in $[-\omega, \omega]$. Use this symmetric height for the three integral period bases. All coefficient embeddings restrict to the same chosen embedding of the local field $E_v = \mathbf{Q}_p$. Thus the union of their jump sets still has the specified far sign and center on each

side, even if the coefficient ring is ramified. Choose $1 \in O^\times$ as the coefficient root of unity on all three extensions. For $i \in \mathbf{F}_q^\times$, the integral projector of Section B.3 acts by the scalar

$$e_i = \frac{1}{q-1}\sum_{n=0}^{q-2}[i]^{-n} = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Thus all coefficient embeddings occur at $i = 1$, which Frobenius fixes. The primitive root $\zeta \in W(\mathbf{F}_q)^\times$ of order e gives the inertia action $s(z) = \zeta z$ on the chart coordinate.

The exact cokernel calculation on the basis. Use the integral crystalline basis on each X_i, and the coordinate retraction on its localization by all the $\varphi^a E$. At level l, the standard orders are 0 at $a < l$, and $-u$ at $a \geq l$, where a is Frobenius level and u the jump. At an index let $I_l \subset A_{\mathrm{inf}}$ be the annihilator of the quotient of the trivial parameter module by the image of the standard-order module. Put $J_l = I_l + (p^{k_0+1})$. The basis includes $\tau^\omega(A_{\mathrm{inf}} \otimes X_i)$ in its standard module for the symmetric height ω, where $\tau = [\underline{\epsilon}^{1/p}] - 1$.

Thus $\tau^\omega \in J_0$ at every index. Modulo p the tilt is a valuation ring with rank-one valuation, and z and τ both have strictly positive value. Consequently z is in the radical of (p, τ). There is a least integer $\nu \geq 0$ for which $z^\nu \in J_0$ at every index. Multiplying by E^ω passes to level one. Since $z^e \in (E, p)$, binomial expansion puts $z^{\nu+e(\omega+k_0)}$ into J_1. Indeed a term of the added power has either E-degree at least ω or p-degree at least $k_0 + 1$.

For level-one lifts use the divided Kummer difference with divisor $z\tau E$. For $j > 0$, put $v = v_p(j)$. The cyclotomic factorization of (LI1) leaves v factors after dividing by $z\tau E$, namely

$$\frac{(g-1)z^j}{z\tau E} = u_j z^{j-1}\prod_{a=1}^{v}\varphi^a E, \qquad u_j \in A_{\mathrm{inf}}^\times.$$

An empty product is one. Since $\varphi^a E = p - z^{ep^a}$, the product is $p^v + zv_j$ with $v_j \in A_{\mathrm{inf}}$. After multiplying the divided lift by u_j^{-1} we may subtract the correction $v_j z^j \in J_1$. The chart error modulo p^{k_0+1} stays integral because $(g-1)S \subset z\tau E A_{\mathrm{inf}}$. Thus $z^j \in J_1$ gives $p^v z^{j-1} \in J_1$. Every step down to the nearest multiple $p\lfloor n/p \rfloor$ has $v = 0$. Inverse Frobenius passes from the level-one lift ideal to the level-zero lift ideal at the preceding index and divides this last exponent by p. The Frobenius permutation runs over all indices. Minimality gives

$$\nu \leq \left\lfloor \frac{\nu + e(\omega + k_0)}{p} \right\rfloor, \qquad (p-1)\nu \leq e(\omega + k_0) < e(p-1).$$

In particular $\nu < e$.

Suppose $c \geq 0$, $1 \leq j \leq \nu$, and $p^c z^j \in J_0$. Choose $u \in A_{\mathrm{inf}}$ with $p^c z^j - p^{k_0+1}u \in I_0$ before choosing the parameter. For each constant parameter d take a standard-order point with projection $(p^c z^j - p^{k_0+1}u)d$. By Section B.2.2, retracting this point preserves its orders and changes its projection to $(p^c z^j - p^{k_0+1}F(u))d$. The scalar coordinate $F = \mathrm{coord}_0$ is fixed on this chart, so $F(u) \in S$ is independent of d. Write $F(u) = w + zv$, with $w \in W(\mathbf{F}_q)$, $v \in S$. The inertia element s fixes w and the trivial parameter d. Apply $Q_i(s)$ to the retracted point x. For a crystalline form λ, put $\lambda_a = s^{-a}\lambda s^a$. In an H-invariant period basis $(P_b)_b$, write $x = \sum_b h_b P_b$. The values $\lambda(s^a P_b)$ lie in the unramified constant field, which s fixes. Thus $\lambda_a(x) = \sum_b h_b \lambda(s^a P_b)$ evaluates the a-th power of the linear period operator. The coefficients b_a of $Q_i = \sum_a b_a X^a$ are in $\mathbf{Z}_p$ and commute with the covector. The annihilator identity gives $\sum_a b_a \lambda_a(x) = 0$ on the lifted point x. Thus the value on $Q_i(s)x$ is $\sum_a b_a(s^a - 1)\lambda_a(x)$.

The retracted standard-order lift belongs to $M \cap N_{\mathrm{loc}} = N$. For a crystalline covector the formula (1) at level zero puts $E^{\omega}\lambda_a(x)$ in S_B. Indeed its row on the invariant period basis has entries in the unramified fixed field. To divide the difference on S_B, first take $f \in S$. Since $s^a(z) = \zeta^a z$ and s fixes the Witt constants, $(s^a f - f)/z \in S$. Also $s(E/p) = E/p$, so the same operator preserves $S[E/p]$. By Section A.1, $S_B \cap A_{\max}$ is the closure of $S[E/p]$ in $A_{\max}$. Expand any element of that closure as a sum $\sum_{n \geq 0} p^n h_n$ with $h_n \in S[E/p]$, by applying the extended first coordinate to an expansion in $A[E/p]$. The divided differences converge term by term in $A_{\max}$, as each $(s^a h_n - h_n)/z$ belongs to $S[E/p]$. After inverting p, this yields $E^{\omega}\lambda(Q_i(s)x) \in zB^{+}$. Apply this to the rows L_0 in (2). Since s preserves M, the basis coordinates of $Q_i(s)x$ are integral. Multiplying by the matrix Y and using (5) shows that each is divisible by z in A. Hence $Q_i(s)x/z \in M$. Division by z preserves each order since $\theta\varphi^{-a}(z) \neq 0$. Using $Q_i(1) = 0$ and $s^a(z) = \zeta^a z$, the projection of $Q_i(s)x/z$ is

$$\left(p^c z^{j-1} Q_i(\zeta^j) - p^{k_0+1} \sum_{a=0}^{\deg Q_i} b_a \zeta^a s^a(v) \right) d.$$

The sum belongs to S and is the same for every d. Since these constants generate the parameter module over A_{inf}, the multiplier in parentheses belongs to I_0. Therefore $p^c z^{j-1} Q_i(\zeta^j) \in J_0$. If $Q_i(\bar{\zeta}^j) \neq 0$, then $Q_i(\zeta^j)$ is a unit in $W(\mathbf{F}_q)$, so $p^c z^{j-1} \in J_0$. In all cases the Kummer difference $g(z) = [\underline{\epsilon}]z$ also gives

$$p^c z^j \in J_0 \quad \Longrightarrow \quad p^{c+v_p(j)+1} z^{j-1} \in J_0.$$

Indeed modulo z one has $E \equiv p$, and $\tau(p^{v_p(j)+1} + zB) = A([\underline{\epsilon}]^j - 1)$ with $A, B \in A_{\mathrm{inf}}$. Multiply this identity by $p^c z^j$ and divide the lifted Kummer difference by $z\tau$; the correction proportional to the old multiplier lies in the lift ideal.

The monic reduction of Q_i vanishes on at most $\deg q_i + 1$ of the distinct $\bar{\zeta}^j$ for $1 \leq j \leq \nu < e$. Also

$$j < e \leq \prod_{a=1}^{r} (p^a - 1) < p^{r(r+1)/2} \leq p^{r^2}.$$

Thus $v_p(j) + 1 \leq r^2$. On the three extensions the total costs are at most $r^2 \deg Q_i$, respectively, with

$$r^2 \deg Q_1,\ r^2 \deg Q_2 \leq r^2(N+1),$$
$$r^2 \deg Q_3 \leq r^2(N^2+1) \leq r^2(N+1)^2 = k_0.$$

Multiplying each resulting scalar by the unused power of p gives $p^{k_0} \in J_0$ for each of the three sides. Nakayama applied to the cyclic image of (p^{k_0}) modulo I_0 removes the modulus p^{k_0+1}. Explicitly, for some $w \in A$ we have $p^{k_0}(1 - pw) \in I_0$, and the geometric series in pw inverts $1 - pw$ in A. This proves

$$p^{k_0} \text{ kills each of the three standard-order projection cokernels at each index.}$$

Application of the chart depth and cup calculation. Use the three cokernel annihilators with E-exponent 0. The negative-jump exclusion $[-0, 0)$ is empty. There is no nonzero multiple of p within n_0 of 0 or $\pm\beta$, since $p > \beta + n_0$. The level-zero depth increase from $-u$ to $\eta - u$ uses the integers $-u \leq j < \eta - u$. For $|u| \leq \eta$ these lie in $[-\eta, 2\eta]$. For $|u| \in [\beta, \beta + W_*]$ they lie within $\eta + W_*$ of one of $\pm\beta$. Thus each required step lies within n_0 of one of the three centers and every nonzero step is prime to p. The contribution at a step $j \neq 0$ is $v_p(j) + \delta$, where $\delta \leq 1$

is the tame different allowance, and step 0 contributes δ. Summing over at most four intervals, each of length $2n_0+1$, contributes at most $(8n_0+4)\delta$ per side.

The two points can therefore be taken over the same $p^{k_0+(8n_0+4)\delta}$ at every index. A third period over p^{k_0} already has the standard orders needed in Section C.2. Take $e_1 = e_2 = k_0 + (8n_0+4)\delta$ and $e_3 = k_0$. In Section C.2.1 the two cocycles are scaled by p^{e_1}, p^{e_2} and the reduced pairing by p^{e_3}. Their product therefore multiplies the restricted cup class by $p^{e_1+e_2+e_3}$, with

$$e_1+e_2+e_3 \le 3k_0 + 2(8n_0+4)\delta \le 3k_0 + 96(\eta + W_*) + 8 \le 3k_0 + 150(\eta + W_* + 1).$$

For $G = G_{E_v}$, put $\alpha = p^{e_1+e_2+e_3}[(\psi_m)_*(\bar{z}_1 \cup \bar{z}_2)] \in H^2(G, \mu_{p^m})$. The cochain just constructed gives $\operatorname{res}^G_H \alpha = 0$. The cocycle for α and the cochain for its restriction take values in the finite module μ_{p^m}. By compactness and continuity, choose a common open normal subgroup of G, contained in H, through whose cosets both factor and which acts trivially on the module. On the resulting finite quotient, [50, Ch. II, Proposition 1.30, p. 70] gives, after inflation,

$$[G:H]\alpha = \operatorname{cor}^G_H \operatorname{res}^G_H \alpha = 0.$$

Here $[G:H] = er!$ is prime to p since $p > r$ and $p \nmid e$. As $p^m\alpha = 0$, this proves $\alpha = 0$. For the applications at $p \nmid M$, Section E.3 works on $H = G = G_{E_v}$ itself. This includes all $p \mid q_1 d$, since the two primes of M are 1 mod q_2^2 and $q_1 d \mid q_2^2$.

Proposition C.1 (The tame cup-product exponent)**.** *Under the tame chart and annihilator hypotheses above, let z_1, z_2 be the crystalline cocycles of an allowed cup test defining $\Delta_v(L)$, with pairing ψ and depth m. For each $\mathbf{Z}_p$-linear functional $\lambda : O \to \mathbf{Z}_p$, let ψ_m be the reduction of $(\lambda \otimes 1) \circ \psi$ modulo p^m. Then*

$$p^{3k_0+150(\eta+W_*+1)}[(\psi_m)_*(\bar{z}_1 \cup \bar{z}_2)] = 0 \quad \text{in } H^2(E_v, \mu_{p^m}).$$

D Depth operators and crystalline lifts

In this appendix, we extend the divided operators from Appendix C to the higher Frobenius levels. We use the projection ideals and the period basis to construct lifts with prescribed filtration orders. The construction gives uniform cup exponents for the colength estimates in Appendix E.

D.1 Depth operators on the integral chart

In this subsection, we extend the divided operators of Section C.1 to $E_a = \varphi^a(E)$, including at $p = 2$. Each operator raises the order of the Galois difference at E_a by one and preserves the earlier orders. Its factor of p enters the annihilator bounds of Section D.2.

Keep the integral localized basis of Section B.3. For the side of any pairing let x be a chart point over a constant at index j, and put $v_{a,\lambda}(x) = \lambda(\varphi^{-a}x)$. Depth $d(a,u)$ means that for each equivariant form λ on the jump line u at $\varphi^{-a}j$,

$$v_{a,\lambda}(gx - x) \in \mathrm{Fil}^{d(a,u)} \qquad (g \in H = G_{K'}).$$

By Section C.1 this is equivalent to $v_{a,\lambda}(x) - c \in \mathrm{Fil}^{d(a,u)}$ for some $c \in K'$, which may depend on a, λ, x. If the value already has order $l > 0$ and $d(a,u) > 0$, then $c \in K' \cap \mathrm{Fil}^1 = 0$. The

new lower bound on the order of the value is therefore $d(a,u)$ when $l>0$ and $\min(0,d(a,u))$ otherwise, in addition to its previous bound.

Write $e=e(K'/\mathbf{Q}_p)$ for the absolute ramification index of the chart field. In the off-level application $K'=E_v$ for each finite coefficient extension. Put $D_E=0$ if $e=1$ and $D_E=1+v_p(e)$ otherwise. A depth step from n to $n+1$ at level a requires

$$C(a,n)=v_p(n)+D_E+3a, \qquad v_p(0)=0.$$

Here is a construction including the case $p=2$. Put $\pi_b=\theta\varphi^{-b}(z)$. For every integer j one can choose $g_j\in H$ with $g_jz=[\underline{\epsilon}]^jz$. Indeed $X^{p^b}-\pi_0$ is Eisenstein over K'. Hence the condition $g_j\pi_b=\zeta_{p^b}^j\pi_b$ cuts out a nonempty closed subset of H. These subsets are nested in b by compatibility of the roots, so compactness gives g_j. For $n\neq 0$ take $g=g_{p^a}$. Write $gE=E\omega$; then $gE_a=E_a\varphi^a(\omega)$. Put $\Delta_z=z\tau$. The chart action gives $\Delta_z\mid\omega-1$, and $\Delta_z\mid\varphi^a(\Delta_z)$. Thus $X=(\varphi^a(\omega)^n-1)/\Delta_z$ is integral, also for negative n.

Let F be the Eisenstein minimal polynomial of $\theta(z)$ over the unramified coefficient field. Weierstrass division in S identifies the kernel of specialization at $\theta(z)$ with ES. Thus $F(z)=EG$ for $G\in S$. The reductions of $F(z)$ and E both have order e in z, so $G\in S^\times$. The subgroup H fixes $\theta(G)$. Taylor expansion of $F(gz)$ at z, followed by division by E and Fontaine evaluation, therefore gives

$$\theta(\omega-1)\theta(G)=F'(\theta z)\,\theta((gz-z)/E).$$

The derivative has valuation at most D_E. If $e>1$, the nonzero derivative terms have distinct valuations modulo $\mathbf{Z}$, and the term of degree $e-1$ has valuation $v_p(e)+(e-1)/e$; for $e=1$ the derivative is a unit.

Since $gz=[\underline{\epsilon}]^{p^a}z$,

$$\frac{gz-z}{E\Delta_z}=\frac{\mu}{E\tau}\sum_{i=0}^{p^a-1}[\underline{\epsilon}]^i.$$

The first factor is a unit of A_{inf}, and the sum has Fontaine value p^a. Thus the valuation of the displayed quotient after applying θ is a. Hence

$$v_p(\theta(\omega-1))\le D_E+a+v_p(\theta\Delta_z).$$

Moreover $v_p(\theta(\omega-1))\ge v_p(\theta\Delta_z)>1/(p-1)$, since $\theta(\tau)=\zeta_p-1$ and $\theta(z)$ has positive valuation. If $v_p(y)>1/(p-1)$, the linear term py has uniquely smallest valuation in $(1+y)^p-1$. Iterating and using the geometric sum for a prime-to-p exponent gives $v_p((1+y)^n-1)=v_p(y)+v_p(n)$ for $n>0$. For $n=-m<0$, use $(1+y)^{-m}-1=-((1+y)^m-1)/(1+y)^m$, with unit denominator. At $p=2$ this uses $v_2(\theta\Delta_z)=1+1/e>1$. Moreover, $\theta\varphi^{-a}(z)=\pi_a$ and $\theta\varphi^{-a}(\tau)=\zeta_{p^{a+1}}-1$, so

$$v_p(\theta\Delta_z)-v_p(\theta\varphi^{-a}\Delta_z)=(1-p^{-a})\left(\frac{1}{e}+\frac{1}{p-1}\right)\le 2a.$$

Since $\theta\varphi^{-a}(\Delta_z)\neq 0$, we have

$$\begin{aligned}v_p(\theta\varphi^{-a}X)&=v_p(\theta(\omega)^n-1)-v_p(\theta\varphi^{-a}\Delta_z)\\&\le v_p(n)+D_E+a+v_p(\theta\Delta_z)-v_p(\theta\varphi^{-a}\Delta_z)\le C(a,n).\end{aligned}$$

More precisely, G and $\mu/(E\tau)$ are units, so the derivative identity and the valuation of the geometric sum give

$$v_p(\theta\varphi^{-a}X)=v_p(F'(\pi_0))+a+v_p(n)+(1-p^{-a})\left(\frac{1}{e}+\frac{1}{p-1}\right).$$

In particular $\theta\varphi^{-a}X \neq 0$, and $p^{C(a,n)}/\theta\varphi^{-a}X \in \mathcal{O}_{\mathbf{C}_p}$. Lift this quotient to $\upsilon \in A_{\mathrm{inf}}$ by $\theta\varphi^{-a}$. Since its kernel is $E_a A_{\mathrm{inf}}$, we have $\upsilon X \equiv p^{C(a,n)} \pmod{E_a}$.

At level a the selected g fixes $K'(\pi_a)$, since $\theta\varphi^{-a}([\underline{\epsilon}]^{p^a}) = 1$. Write $v_{a,\lambda}(x) = c + y$, with $c \in K'$ and $y \in \mathrm{Fil}^n \cap \mathcal{K}_a$. The element $\varphi^{-a}(E_a) = E$ has filtration order one. The residue of $y/E^n \in \mathcal{K}_a \cap \mathrm{Fil}^0$ lies in $K'(\pi_a)$ by Section B.2.2. Hence $(g-1)(y/E^n) \in \mathrm{Fil}^1$. Multiplying by $g(E^n)$ gives

$$(g-1)y \equiv (\varphi^{-a}(g(E_a)/E_a)^n - 1)y \pmod{\mathrm{Fil}^{n+1}}.$$

Division by $\varphi^{-a}(\Delta_z)$ has order zero. Thus the divided difference has value $\varphi^{-a}(X)y$ modulo the next grade, for either sign of n. Therefore the operator

$$p^{C(a,n)}x - R(\upsilon(gx - x)/\Delta_z)$$

advances the depth. For $n \neq 0$, put $y_1 = p^{C(a,n)}x - \upsilon(gx-x)/\Delta_z$ and $c_1 = p^{C(a,n)}c \in K'$. Since $R(x) = x$, the displayed vector is $R(y_1)$. The relative estimate of Section B.2.2 gives

$$\mathrm{ord}_t(v_{a,\lambda}(R(y_1)) - c_1) \geq \mathrm{ord}_t(v_{a,\lambda}(y_1) - c_1) \geq n + 1.$$

The selected form thus has Galois differences of order at least $n+1$, since c_1 is fixed. The correction projects to zero by naturality of R, and retains the earlier order bounds on the other lines by the same retraction inequality.

For $n = 0$ set $C_0 = D_E + 3a$ and $y = \sum_{j=0}^{p^{C_0}-1} g_j x$. A value of depth zero is integral since it differs from a constant in K' by an element of Fil^0. Write its residue in $K'(\pi_a)$ as $\sum_{i=0}^{p^a-1} b_i \pi_a^i$, with $b_i \in K'$. In $\mathbf{C}_p$ its conjugates under the g_j are obtained by putting $\zeta_{p^a}^j \pi_a$ in place of π_a. As $C_0 \geq a$, each embedding occurs p^{C_0-a} times. Thus

$$\theta(v_{a,\lambda}(y)) = \sum_{i=0}^{p^a-1} b_i \pi_a^i \sum_{j=0}^{p^{C_0}-1} \zeta_{p^a}^{ij} = p^{C_0} b_0 \in K'.$$

Take the coordinate retraction after this sum. Since $R(x) = x$,

$$R(y) = p^{C_0}x + \sum_{j=0}^{p^{C_0}-1} R(g_j x - x).$$

The retraction preserves congruence of the value to the constant $p^{C_0}b_0$ modulo Fil^1, so all its Galois differences have depth at least one at this step. Each correction lies over zero and preserves the old order bounds, as in the nonzero step (multiply the divided difference by Δ_z). The projection of $R(y)$ is p^{C_0} times the original parameter.

For each (a, n) fix the Kummer elements used above and, for $n \neq 0$, the lift υ, independently of the chart vector and of the equivariant form. Each resulting operator is $\mathbf{Z}_p$-linear, since the Galois action and the coordinate retraction are $\mathbf{Z}_p$-linear. For finitely many required depth increases take the union of their steps (a, n) over the jump lines and perform them in increasing order of n, in any order among ties. If x has orders o and depth d, every difference $gx - x$ has orders at least both o and d by equivariance. At every level $\varphi^{-a}(z\tau)$ has de Rham order zero. The divided difference belongs to the localized integral module by Section B.2.1, and the retraction preserves its orders and its zero projection by Section B.2.2. Thus the correction in either step has orders at least both o and d. Its own Galois differences have orders at least d.

The resulting point keeps the old orders and depths on all lines and raises the selected depth to $n+1$ on every line where that step is required. If a line is already deeper, it retains its bound. The projection is multiplied by $p^{C(a,n)}$ once for the whole step. Induction gives the sum of the costs of the distinct steps, independent of the rank.

D.2 Ideals controlling the crystalline lifts

In this subsection, we bound the annihilator of the cokernel of projection to the trivial parameter module in Section B.3. The divided operators of Section D.1 give a power of p in the sparse case and an E-power under the unramified Hodge bounds.

We give the calculation of the two types of annihilator used at a cut. Write $p = \ell$, $\mathfrak{S} = W(\mathbf{F}_{p^f})[[z]]$, $E_a = \varphi^a E$, $\tau = [\underline{\epsilon}^{1/p}] - 1$, and $D_E = 0$ if $e = 1$ and $\mathrm{ord}_p(e) + 1$ otherwise. The coefficient ring of the chart is a subring of A_{inf}. Keep the extension $0 \to Y \to X \xrightarrow{\pi} C \to 0$ of one fixed side, in particular any of the two universal sides or the Tate-dual side. Localize the period modules M_X, N_X at all E_a. By Section B.3 the A_{inf}-module of order r at j is $e_j \varphi^r M_X$, and its chart vectors form $e_j \varphi^r M_X \cap N_{X,\mathrm{loc}}$. On each tested form of type at most u at $j^{p^{-a}}$, the order required at E_a is ≥ 0 for $a < r$ and $\geq -u$ for $a \geq r$.

For a nonnegative integer R let $J_r(j; R) \subset A_{\mathrm{inf}}$ be the ideal of scalars lifting constants at order r, plus (R). The integer R in this notation is the modulus. Write R_X, R_C for the coordinate retractions. Concretely, a member s means that for some $t \in A_{\mathrm{inf}}$ and every $c \in C$, the constant $(s - Rt)c_j$ is $\pi(x)$ for some order-r vector x in component j of the A_{inf}-module. The following properties of the period basis are used here. The basis spans the order-zero module over A_{inf}, and each way between that module and $A_{\mathrm{inf}} \otimes X$ is bounded by a power of τ.

Section B.2.2 gives $R_C \pi = \pi R_X$. Applying this to $\pi(x) = (s - Rt)c_j$ with $s \in \mathfrak{S}$ gives

$$\pi(R_X x) = (s - RF(t))c_j, \qquad F(t) \in \mathfrak{S}.$$

The same $F(t)$ works for every c, and each $R_X x$ retains its prescribed orders. Frobenius transports the standard-order module separately, as in Section B.3.

For such a localized chart vector y, the divided difference $(gy - y)/(z\tau)$ lies in the localized A_{inf}-module by Section B.2.1. The divisor has order zero at every E_a; equivariance therefore preserves the prescribed lower bounds after division. At order 1 we may also divide by E. At $a = 0$ the residues of the integral form values lie in K' by Section B.2.2. The chart subgroup fixes K', so each value of $gy - y$ at this level is in Fil^1. At $a > 0$ the factor E has order zero. These properties apply to each of the three extensions. They are the properties of the basis used here. Suppose $F \in \mathfrak{S} \cap J_r(j; R)$ and $\alpha, \beta, s \in A_{\mathrm{inf}}$ satisfy

$$\alpha(gF - F) = D(s + \beta F)$$

for $D = z\tau$ at order 0, or $D = z\tau E$ at order 1.

Then $s \in J_r(j; R)$. Indeed lift $(F - Rt)c_j$ over $\mathfrak{S}$ and apply $\alpha(g-1)/D$. As g fixes c_j, the new lift has projection $(s + \beta F - R\alpha(gt - t)/D)c_j$. Here $t \in \mathfrak{S}$ by the preceding retraction, so $gt - t \in z\tau E A_{\mathrm{inf}}$ by (6). Thus $s + \beta F \in J_r(j; R)$, and subtracting $\beta F \in J_r(j; R)$ proves the assertion.

Choose a chart element g with $gz = [\underline{\epsilon}]z$. The cyclotomic factorization for $n > 0$ gives

$$\alpha\big([\underline{\epsilon}]^n - 1\big) = \tau E\big(p^{\mathrm{ord}_p(n)} + \beta z\big). \tag{LI1}$$

Indeed $[\epsilon]^n - 1$ is a unit times $\tau \prod_{a=0}^{\mathrm{ord}_p(n)} E_a$. For the prime-to-$p$ part of n the geometric sum is a unit. For its p-part the consecutive cyclotomic factors give the displayed generators. Each $E_a \equiv p \bmod z$, which proves (LI1). Applying the difference operator just established to $F = p^c z^n$ gives the two implications

$$p^c z^n \in J_1(j;R) \implies p^{c+\mathrm{ord}_p(n)} z^{n-1} \in J_1(j;R),$$
$$p^c z^n \in J_0(j;R) \implies p^{c+\mathrm{ord}_p(n)+1} z^{n-1} \in J_0(j;R).$$

The additional factor p in the second line is $E \bmod z$. Put $d_{n,m} = \mathrm{ord}_p(n!) - \mathrm{ord}_p(m!)$ for $0 \le m \le n$. Iterating down from n to m increases the exponent of p by at most $d_{n,m}$ at level one and $n - m + d_{n,m}$ at level zero.

For the order-one operation on the scalar ideal, take $a, c \ge 0$ and $n \ge 1$ integral. Use $gE = \omega E$, $z\tau X = \omega^n - 1$, $\upsilon X \equiv p^{\mathrm{ord}_p(n)+D_E} \pmod E$. The chart supplies this by choosing $gz = [\underline{\epsilon}]z$. To see the valuation, write $F_0(z) = EG_0$ with F_0 the minimal polynomial of the chosen uniformizer over the unramified field. As in Section D.1, $G_0 \in S^\times$, so g fixes $\theta(G_0)$. Write $gz - z = Ew$. Taylor expansion and reduction by Fontaine's map θ give

$$\theta(\omega - 1)\theta(G_0) = F_0'(\theta z)\theta(w), \qquad \theta(w) = \theta(z\tau) \cdot \text{unit}.$$

The different bound is $v_p(F_0'(\theta z)) \le D_E$.

For $e > 1$ the nonzero derivative terms have distinct valuations modulo $\mathbf{Z}$; the leading term has valuation $v_p(e) + (e-1)/e$. For $e = 1$ the derivative is a unit. The derivative calculation of Section D.1 applies also at $p = 2$: $v_p(\theta(z\tau)) > 1/(p-1)$ and $\omega - 1$ is divisible by $z\tau$. Hence $v_p(\theta X) \le D_E + \mathrm{ord}_p(n)$. Lift a quotient $p^{D_E+\mathrm{ord}_p(n)}/\theta X$ by the surjectivity of θ. Inserting υ in the difference operator at order 1 trades E^n for zE^{n-1} at contribution $D_E + \mathrm{ord}_p(n)$. In detail, if $F = p^c z^a E^n$, then $gz^a/z^a = 1 + \tau E b$ for some $b \in A_{\mathrm{inf}}$. Write $\upsilon X = p^{D_E+\mathrm{ord}_p(n)} + Ek$. With $\beta = zk + \upsilon b \omega^n \in A_{\mathrm{inf}}$, we obtain

$$\frac{z\upsilon(gF - F)}{z\tau E} = p^{c+D_E+\mathrm{ord}_p(n)} z^{a+1} E^{n-1} + \beta F.$$

This is the asserted trade on $J_1(j;R)$.

Frobenius carries the order-zero ideal at j to the order-one ideal at φj. Thus $p^c z^{pa} \in J_1(\varphi j;R)$ is equivalent to $p^c z^a \in J_0(j;R)$. No integer between $p\lfloor n/p \rfloor + 1$ and n is divisible by p. Thus lowering to that multiple of p is free, and subsequent inverse Frobenius gives $p^c z^{\lfloor n/p \rfloor}$ in the preceding ideal.

Put $I = \bigcap_j J_0(j; p^{\kappa+1})$, and let b_X bound the absolute jumps of X. For every $c \in C$, take $x_c \in X$ with $\pi(x_c) = c$. By (3) the component at j of $\tau^{b_X}(1 \otimes x_c)$ has standard orders and maps to $\tau^{b_X} c_j$. Hence $(\tau^{b_X}, p^{\kappa+1}) \subseteq I$. In the rank-one valuation ring A_{inf}/p, the reduction $\bar{z}$ has positive tilt valuation since z comes from compatible roots of a uniformizer. Choose $m_0 \ge 1$ with $m_0 v^\flat(\bar{z}) \ge b_X v^\flat(\bar{\tau})$. Then $z^{m_0} \in (\tau^{b_X}, p)$, so

$$z^{m_0(\kappa+1)} \in (\tau^{b_X}, p^{\kappa+1}) \subseteq I.$$

Let a be the least nonnegative integer with $z^a \in I$. If ω_0 bounds all jumps above, multiplying the lifts by E^{ω_0} puts them at order 1. The relation $z^e = uE + vp$ with $u, v \in \mathfrak{S}$ then puts $z^{a+e(\omega_0+\kappa)}$ in every order-one ideal modulo $p^{\kappa+1}$. Indeed, every term of its binomial expansion

contains E^{ω_0} or $p^{\kappa+1}$. The preceding Frobenius step therefore puts $z^{\lfloor (a+e(\omega_0+\kappa))/p \rfloor}$ in every order-zero ideal. Minimality of a yields

$$(p-1)a \le e(\omega_0+\kappa). \tag{LI2}$$

For an unramified chart suppose $p \ge 5$, $E = z+p$, and $\omega_0 \le (p-3)(s+t)$. Set $\kappa = s+t$. Then

$$(p-1)a \le (p-2)\kappa, \qquad a + \operatorname{ord}_p(a!) \le \kappa,$$

by $(p-1)\operatorname{ord}_p(a!) \le a$. The order-zero lowering operation puts $p^{\kappa-m}z^m$ in $J_0(j;p^{\kappa+1})$ for $0 \le m \le \kappa$: when $m < a$ use the accumulated contribution formula, and when $m \ge a$ use ideal closure. In particular the ideal contains p^κ. Modulo $J_0(j;0)$ this means $p^\kappa = p^{\kappa+1}h$. Since $1-ph$ is a unit, $p^\kappa \in J_0(j;0)$, so $J_0(j;p^{\kappa+1}) = J_0(j;0)$. Expand $p^sE^t = p^s(z+p)^t$. Each term is one of those monomials times its integer binomial coefficient. Thus p^sE^t kills the constant cokernel.

For a cut followed by J indices of jump bound ω_1, assume $8e \le p^{J+1}$. Use (LI2) with the global jump bound ω_0. Start with z^a at the J-th successor of the cut, and work backwards through the successors. At each successor multiply the lifts by E^{ω_1} and trade its factors for z. Put $d = D_E\omega_1 + \operatorname{ord}_p(\omega_1!)$ and $R = p^{\kappa+1}$. For each preceding index i this gives

$$p^cz^n \in J_0(\varphi i;R) \implies p^{c+d}z^{n+\omega_1} \in J_1(\varphi i;R) \implies p^{c+d}z^{\lfloor (n+\omega_1)/p \rfloor} \in J_0(i;R).$$

The second implication first lowers to the nearest multiple of p at level one and then applies inverse Frobenius. After J steps the monomial p^cz^n in the ideal at the cut satisfies

$$c \le J\big(D_E\omega_1 + \operatorname{ord}_p(\omega_1!)\big), \qquad n \le a/p^J + \omega_1/(p-1).$$

Since $p \ge 2$, (LI2) gives $a/p^J \le 2e\omega_0/p^{J+1} + \kappa/4$.

Removing the remaining z^n requires $n + \operatorname{ord}_p(n!) \le 2n$. Therefore p^κ lies in the ideal modulo $p^{\kappa+1}$ if $c + 2n \le \kappa$. As above, $1-ph$ is a unit, so the same power lies in the exact ideal. Indeed Legendre's inequality and the preceding bounds give

$$c + 2n \le JD_E\omega_1 + (J+2)\frac{\omega_1}{p-1} + \frac{4e\omega_0}{p^{J+1}} + \frac{\kappa}{2}.$$

Thus it suffices that

$$2JD_E\omega_1 + (2J+4)\frac{\omega_1}{p-1} + \frac{8e\omega_0}{p^{J+1}} \le \kappa. \tag{LI3}$$

For the stated cut parameters the global bound is $2b_0$ and the bound for a thin next component is G_1 in the cut-size conditions. They impose $D_EG_1 \le X_0$, $G_1/(p-1) \le X_0$, and $2eb_0/p^{J+1} \le D_0$. The left side of (LI3) is at most $(4J+4)X_0 + 8D_0$, below $\kappa = 12((J+1)X_0 + D_0) + 1$.

At a shifted cut the unramified argument applies with $s = \kappa$, $t = s_j$, since the global jump bound is at most $(p-3)(\kappa + s_j)$. This proves the two annihilator assertions on all three sides. It also gives the annihilator with no p-factor E^A in the separated-interval regime, using the jump bound $(p-3)A$.

The basis used for these divisibilities is built in Sections B.1 and B.3. Over the finite crystallizing field K', whose residue field is perfect, the underlying $\mathbf{Z}_p$-lattice of a crystalline representation is the étale realization of a prismatic F-crystal on $\operatorname{Spf}\mathcal{O}_{K'}$, also when $p = 2$. The chosen generator $E = p + z^eu$ differs from the Eisenstein polynomial of $\theta(z)$ by a unit of $\mathfrak{S}$, as in

Section D.1. Thus $(\mathfrak{S},(E))$ is the Breuil–Kisin prism of the chosen uniformizer. Evaluation gives a finite free $\mathfrak{S}$-module $\mathfrak{M}$ with $(\varphi^*\mathfrak{M})[1/E]\simeq\mathfrak{M}[1/E]$ [15, Examples 2.6(1), 4.3, Theorem 5.6 and §7]. Evaluation on this prism is fully faithful on crystalline lattices [15, Theorem 7.9, p. 31 of the authors' version]. For finite free Breuil–Kisin modules with linearization invertible over $\mathfrak{S}[1/E]$, base change to $\mathfrak{S}[1/E]^{\wedge}_{p}$ is fully faithful by [15, Theorem 7.2, p. 28 of the authors' version]. Kisin states this for integral linearizations with cokernel killed by a power of E [41, §§2.1.1, 2.1.3 and Proposition 2.1.12]. His completed ring is $\mathfrak{S}[1/z]^{\wedge}_{p}$. Modulo p^m, $m\geq 1$, the nilpotence of p and $E=p+z^eu$, $u\in\mathfrak{S}^{\times}$, give the same localization upon inverting E or z. Taking inverse limits identifies the two completed rings. For a pair of modules, multiplying both linearizations by the same sufficiently high power of E gives that formulation and preserves the Frobenius intertwining equation.

One can identify the integral model of T in these statements directly. Let $N_T=\sum_j\mathfrak{S}n_j$ be the chart lattice and $\Phi:\varphi^*N_T\to N_T[1/E]$ its linearized Frobenius. If the jumps lie in $[-b,b]$, then

$$E^bN_T\subseteq\Phi(\varphi^*N_T)\subseteq E^{-b}N_T.$$

The right inclusion is (4). For $x\in N_T$, the adapted dual basis of Section B.3 gives $\operatorname{ord}_t\lambda(x)\geq -u_s\geq -b$ for every basis form of type u_s. Multiplication by E^b therefore gives the level-one standard orders at $a=0$. At $a>0$ the level-zero and level-one bounds coincide and $\theta\varphi^{-a}E\neq 0$. Thus E^bx lies in $\varphi M_T\cap N_{T,\mathrm{loc}}$, which on summing the projected equalities in that section is $\sum_j\mathfrak{S}\,\varphi(n_j)=\Phi(\varphi^*N_T)$. In particular $E^b\Phi$ has integral cokernel killed by E^{2b}.

Put $K'_{\infty}=\bigcup_{a\geq 0}K'(\theta(\varphi^{-a}z))$ and $W_{\flat}=W(\mathbb{C}_p^{\flat})$. The group $G_{K'_{\infty}}$ fixes z, the Witt constants, and hence S_B by continuity. In the column identity $E^bn=PZ$ of Section B.2, P is invariant for the diagonal action and Z has entries in S_B. Hence $G_{K'_{\infty}}$ fixes each n_j for that action. The column matrix n is invertible over $W_{\flat}$, so the inclusion of its columns identifies

$$W_{\flat}\otimes_{\mathfrak{S}}N_T=W_{\flat}\otimes_{\mathbf{Z}_p}T$$

with its Frobenius and $G_{K'_{\infty}}$-action. Its Frobenius-fixed vectors form the original $T|_{G_{K'_{\infty}}}$, since $W_{\flat}^{\varphi=1}=\mathbf{Z}_p$. For the convention $\widetilde{\theta}=\theta\varphi^{-1}$ on the perfect prism, use $\widetilde{\iota}=\varphi\circ\iota$ for the chart embedding $\iota:\mathfrak{S}\to A_{\mathrm{inf}}$ [15, §5 and Notation 7.1]. The map sends the chart coordinate to z^p and acts on the Witt constants by Frobenius, as in [14, p. 281, before Lemma 4.30]. Thus $\widetilde{\theta}\circ\widetilde{\iota}=\theta\circ\iota$. Let $j:N_T\to W_{\flat}\otimes_{\mathbf{Z}_p}T$ be the inclusion given by the columns n_j in the chart ι. In the shifted chart the $W_{\flat}$-linear identification is

$$W_{\flat}\otimes_{\widetilde{\iota},\mathfrak{S}}N_T\longrightarrow W_{\flat}\otimes_{\mathbf{Z}_p}T,\qquad a\otimes y\longmapsto a\,\varphi(j(y)),$$

where φ acts on the coefficients of T. Indeed $\varphi(j(sy))=\widetilde{\iota}(s)\varphi(j(y))$ for $s\in\mathfrak{S}$, and its column matrix $\varphi(n)$ is invertible over $W_{\flat}$. The map intertwines Frobenius since j does after inverting E, and intertwines $G_{K'_{\infty}}$ since this group fixes the columns n_j and commutes with φ. It commutes with the O-action since the endomorphisms of T giving that action are $\mathbf{Z}_p$-linear. This is the integral identification of [14, Theorem 4.4, p. 265]. The equivalence for the completed étale ring used in [15, Corollary 3.8, proof of Corollary 7.10 and Remark 7.11] therefore identifies the étale realization of N_T with that of the evaluated prismatic crystal. Theorem 7.2 lifts this identification and its inverse uniquely to $\mathfrak{S}$. The resulting Breuil–Kisin–Fargues module is

$$A_{\mathrm{inf}}\otimes_{\widetilde{\iota},\mathfrak{S}}N_T=\varphi^*(A_{\mathrm{inf}}\otimes_{\iota,\mathfrak{S}}N_T).$$

Its linearization is invertible after inverting $\widetilde{\xi} = \varphi(\xi)$, since $\widetilde{\iota}(E)A_{\mathrm{inf}} = \widetilde{\xi}A_{\mathrm{inf}}$. This is the scalar extension of [14, Definition 4.22, p. 276, and Proposition 4.32, p. 281]. Its integral étale realization is described in [14, Lemma 4.26, p. 278]. The numerical bounds (LI2)–(LI3) follow from the coordinate divisibilities above.

D.3 The convergence radius of the period basis

In this subsection, we prove a generation bound for positive Frobenius translates of the basis in Section B.1. The stronger de Rham orders at $E = 0$ supply the intermediate lifts in Section D.4.

Use the chart and period comparison of Sections A.1 and A.3. Write $\Lambda = W(\mathbf{F}_{p^f})$, $S = \Lambda[[z]]$, $A = A_{\max}$, $B^+ = A[1/p]$, and $S_B \subset B^+$ for the closure of $S[E/p][1/p]$ in the bounded-ring topology. The extended first coordinate $\mathcal{R} : B^+ \to S_B$ is the identity on S_B and is S_B-linear. It restricts to $\mathrm{coord}_0 : A_{\mathrm{inf}} \to S$ on A_{inf}. It is compatible with approximation by finite chart coordinates.

Here are two scalar consequences. If $\nu \in S$ and $\xi - pb_0 \in \nu A_{\mathrm{inf}}$ for some $b_0 \in A_{\mathrm{inf}}$, then for $r \in A$ and $M \geq 0$ there are $s \in S$, $y \in S_B$ with

$$p^M \mathcal{R}(r) = s + \nu^M y. \tag{119}$$

To see this, express r as a p-convergent sum of polynomials in ξ/p with integral coefficients and expand in $\xi/p - b_0$. After multiplication by p^M a term of degree $k < M$ becomes a multiple of $p^{M-k}(\xi - pb_0)^k$ in A_{inf}. For $k \geq M$ use

$$p^M(\xi/p - b_0)^k = (\xi - pb_0)^M(\xi/p - b_0)^{k-M}.$$

Writing the original expansion as $\sum_{i \geq 0} p^i h_i(\xi/p)$, with $h_i \in A_{\mathrm{inf}}[X]$, retains the factor p^i in each of the two sums, including after extracting $(\xi - pb_0)^M$ in the second sum. Apply $\mathcal{R}$. This applies to $\nu = E$ with $b_0 = 0$. It applies to $\nu = z$ because $E = p + z^e u$ and E is associated to ξ. It follows, using $\nu = z$, that every element of S_B is constant in $\Lambda[1/p]$ modulo zS_B.

For g fixing Λ and $\theta(z)$, one has

$$z\tau \mid gy - y \quad \text{in } B^+, \qquad y \in S_B. \tag{120}$$

Indeed the bounded integral calculation in Section B.2.1 gives $(g-1)(S_B \cap A_{\max}) \subset z\tau A_{\max}$. Inverting p gives (6) on S_B. The cancellation (5) gives the contraction used below: if $z^r\tau \mid E^m x$ in B^+ for $m, r \geq 0$ and $x \in A_{\mathrm{inf}}$, then $z^r\tau \mid x$ in A_{inf}.

Suppose Z has entries in S_B, $Z'Z = E^h I$, with $Z' \in M_d(B^+)$, and $tZ = ZG$ for $G \in M_d(A_{\mathrm{inf}})$, where t fixes $K' = \theta(S_B)$ and $tz = [\underline{\epsilon}]z$. Apply $\mathcal{R}$ entrywise to Z', which preserves $Z'Z = E^h I$. The constant term of $p^{-h}\mathcal{R}(Z')$ modulo z gives a matrix B_0 over $\Lambda[1/p]$ with

$$Y = B_0 Z = I + zW, \qquad W \in M_d(S_B).$$

Indeed $p^{-h}E^h \equiv 1 \bmod z$, and multiplication by Z preserves the discarded ideal. Moreover $E^h(G - I) = Z'(tZ - Z)$, hence $G - I$ is divisible by $z\tau$ over A_{inf}, by (120) and contraction. The relation $tY = YG$ holds since $tB_0 = B_0$.

For clarity the successive approximation proving the radius starts with the period coefficients Y_i. Suppose

$$A_n = q_n Y_i - z^{n+1} W_n \in A_{\mathrm{inf}}^d, \qquad W_n \in S_B^d, \quad q_n \in \mathbf{N}.$$

The initial values satisfy $q_0 = 1,\ W_0 = W_i,\ A_0 = I_i$. The period term cancels from $A_nG - tA_n$, leaving

$$z^{n+1}(([\underline{\epsilon}]^{n+1} - 1)W_n + [\underline{\epsilon}]^{n+1}(tW_n - W_n) - W_n(G - I)).$$

The last two terms in parentheses are divisible by $z\tau$. For $v_n = v_p(n+1)$, the cyclotomic factorization (LI1) and $E = p \bmod z$ give a multiplier $\alpha \in A_{\mathrm{inf}}$ with

$$\alpha([\underline{\epsilon}]^{n+1} - 1) = \tau(p^{v_n+1} + zu), \qquad u \in A_{\mathrm{inf}}.$$

The numerator $\alpha(A_nG - tA_n)$ has entries in A_{inf}. Its divisibility by $z^{n+1}\tau$ in B^+ therefore puts the quotient in A_{inf}^d by contraction. In B^+ it equals $p^{v_n+1}W_n + zV_n$. Applying $\mathcal{R}$ produces $s_n \in S^d$ and $W_{n+1} \in S_B^d$ such that

$$p^{v_n+1}W_n = s_n + zW_{n+1}.$$

Consequently $A_{n+1} = p^{v_n+1}A_n + z^{n+1}s_n$ and $q_{n+1} = p^{v_n+1}q_n$ obey the same recurrence.

Take $N = p^l \geq 3e$. Since $z^e/p = (E/p - 1)u^{-1} \in S[E/p]$, the value $X = z^N/p^2 = pz^{N-3e}(z^e/p)^3$ is in $pS[E/p] \subset pA$. Thus evaluation $\mathrm{ev} : S[[X]] \to A$ is defined by p-adic convergence. Put $a_n = n - v_p(n!) + 2$. Applying φ^l to the recurrence for the period coefficients gives

$$p^{a_n}\varphi^l W_n = p^{a_{n+1}-2}\varphi^l s_n + X\, p^{a_{n+1}}\varphi^l W_{n+1}.$$

Indeed $v_p((n+1)!) \leq n+1$ and $a_{n+1} + v_n + 1 = a_n + 2$. In particular at $p = 2$, $a_{n+1} - 2$ is the sum of the binary digits of $n+1$, so the coefficients still lie in S. Put $c_n = p^{a_{n+1}-2}\varphi^l s_n \in S^d$. For each $m \geq 1$, finite iteration gives

$$p^2\varphi^l W_0 - \sum_{n=0}^{m-1} X^n c_n = X^m p^{a_m}\varphi^l W_m \in X^m(B^+)^d.$$

The tail of $\sum_{n\geq 0} X^n c_n$ belongs to $X^m A^d$, since $X \in pA$. Thus the difference between the full sum and $p^2\varphi^l W_0$ lies in $X^m(B^+)^d$ for every m.

To see that B^+ is z-adically separated, write $\xi = u_0 E$ with $u_0 \in A_{\mathrm{inf}}^\times$ and set $X_0 = \xi/p$. In B^+ the element z^e is a unit multiple of $X_0 - u_0$, since $E = p + z^e u$. If a nonzero element were divisible by every power of z, rescale by a power of p to obtain an element of $A_{\max} \setminus pA_{\max}$ with the same divisibility. Each power of $X_0 - u_0$ has monic reduction in $(A_{\mathrm{inf}}/(p,\xi))[X_0]$. By the p-saturation proved in Section A.3, division by that power takes place in $A_{\max}$. Reduction would make a nonzero polynomial divisible by monic polynomials of arbitrarily high degree. This proves separatedness.

Applying this construction to every row of W gives a matrix F over $S[[X]]$ and

$$\varphi^l(B_0)\varphi^l(Z) = \mathrm{ev}(I + XF).$$

Also $\varphi^l(E) = p\,\mathrm{ev}(\varepsilon)$ with $\varepsilon = 1 + pX\varphi^l((E-p)/z)$. The series $I + XF$ and ε are invertible over $S[[X]]$, with constant terms I and 1. Thus for any $b \geq 0$,

$$\varphi^l(Z) = \varphi^l(E)^b\, C\, \mathrm{ev}(V), \qquad C = (p^b\varphi^l(B_0))^{-1}, \qquad V = \varepsilon^{-b}(I + XF).$$

The matrix C is fixed by every g fixing Λ. Its inverse exists because $\varphi^l(B_0)\varphi^l(Z)$ is invertible over B^+.

For integers $A_0, \delta \geq 0$ with $eA_0 \leq N\delta$, approximate each formal series $H \in S[[X]]$ by its truncation of degree $< \delta$. After multiplying by $p^{2\delta}$, the truncated part evaluates into S, and the tail is $z^{N\delta}$ times the evaluation of another series, so lies in $p^{A_0}A$. In fact $\mathrm{ev}(H) \in S_B$, since $X \in pS[E/p]$ and the partial sums converge in the p-adic completion of that ring. Write the tail after clearing the denominator as $p^{A_0}r$, with $r \in A$. This tail is fixed by $\mathcal{R}$, as both the full sum and the truncated polynomial are in S_B. Hence $p^{A_0}r = p^{A_0}\mathcal{R}(r)$. Applying (119) with $M = A_0$ and $\nu = E$ gives

$$p^{2\delta}\mathrm{ev}(H) \equiv s \pmod{E^{A_0}B^+}, \qquad s \in S.$$

Apply this entrywise to V^{-1} and V, obtaining matrices $M, M' \in M_d(S)$ congruent modulo $E^{A_0}B^+$ to $p^{2\delta}\,\mathrm{ev}(V)^{-1}$ and $p^{2\delta}\,\mathrm{ev}(V)$, respectively. In that quotient,

$$\mathrm{ev}(V)M = p^{2\delta}I, \qquad \mathrm{ev}(V)(p^{4\delta}I - MM') = 0.$$

The columns of M and $p^{4\delta}I - MM'$ have images $p^{2\delta}e_i$ and 0, respectively, under $\mathrm{ev}(V)$ in this quotient, where e_i is the corresponding coordinate vector. For every $y \in S^d$,

$$p^{4\delta}y = (p^{4\delta}I - MM')y + M(M'y),$$

so these columns span a module containing $p^{4\delta}S^d$.

For the period basis of a lattice, take b to be the nonnegative uniform jump bound used in choosing the invariant period matrix P. The basis construction in Section B.1 supplies $Z'Z = E^{2b}I$ and $tZ = ZG_t$ with G_t integral. Choose the element t with $tz = [\underline{\epsilon}]z$ as in Section D.1. The basis vectors $n_\nu = \tau^{-b}Q_\nu$ satisfy the formula for every equivariant de Rham form λ and column y of basis coordinates:

$$\lambda\left(\sum_\nu y_\nu \varphi^l n_\nu\right) = (\lambda_{\mathrm{lat}}\varphi^l P)\,\frac{\varphi^l Z}{\varphi^l(E)^b}\,y.$$

This follows from $UZ = E^{2b}Q$ and $U = (E\tau)^b P$, in the given integral basis. Write $b_\lambda = \lambda_{\mathrm{lat}}\varphi^l(P)C$. Its entries belong to $K' \subset B^+_{\mathrm{dR}}$ by (1) and the invariance of C. For a column y with $\mathrm{ev}(V)y = r + E^{A_0}w$, $r \in \mathbb{Z}^d$, $w \in (B^+)^d$, we obtain

$$\lambda\left(\sum_\nu y_\nu \varphi^l n_\nu\right) = b_\lambda r + E^{A_0}b_\lambda w \equiv b_\lambda r \in K' \pmod{\mathrm{Fil}^{A_0}B_{\mathrm{dR}}}.$$

Here $\mathrm{Fil}^{A_0}B_{\mathrm{dR}} = E^{A_0}B^+_{\mathrm{dR}}$, since the image of E has de Rham order one. The span statement therefore gives the precise $p^{4\delta}$ radius property at level l used in the long-gap coefficient extraction. For an index j, apply e_j to the resulting span. It preserves the translated chart module by Section B.3. For a generator w, the value $\lambda(e_j w) = (\lambda \circ e_j)(w)$ is again constant modulo Fil^{A_0}, since the chart group fixes the Witt scalars of e_j and its coefficient action is equivariant. Thus the projected generators span a module containing $p^{4\delta}e_j \sum_\nu S\varphi^l(n_\nu)$.

D.4 Crystalline lifts and termination of the construction

In this subsection, we apply the annihilator estimates of Section D.2 to construct lifts of constants at selected coefficient summands. Between successive summands we transport the lifts

by Frobenius. The radius estimate from Section D.3 supplies the additional depth needed for the longer intervals.

Use the period basis and its depth operators on each of the three sides of the pairing. At an index j let β_j be the far start, let $\eta \geq 1$ be the low radius, let V be the width of the separated intervals, and fix $W_0 \geq V + 4\eta$, $J \geq 1$, n_0, b_0, G_1, X_0 and D_0. The numerical requirements, verified for the two schemes below, are

$$\beta_j + s_j + W_0 \leq 2b_0, \quad s_j + W_0 \leq pn_0, \quad G_1 \leq (p-1)n_0, \quad 2eb_0 \leq p^{J+1}D_0,$$
$$n_0 + \lfloor \log_p(2b_0) \rfloor + (J + 1 + D_E)W_0 + D_E G_1 \leq X_0,$$
$$\kappa = 12((J+1)X_0 + D_0) + 1.$$

At a cut we apply one of the two estimates in Section D.2. Either its shift $s_j = 0$ and all three sides at the next J indices have upper jump bound G_1, with $8e \leq p^{J+1}$, or the chart is unramified, $p \geq 5$ and $s_j < \beta_j$ with all jumps bounded above by $(p-3)(\kappa + s_j)$. In either case $p^\kappa E^{s_j}$ kills all three standard projection cokernels. The third-side jumps are 0 and $-u - v - 1$, for a jump u from the first side and v from the second side. If both are in their low intervals then $|u + v + 1| \leq 2\eta + 1$. If one is far and one low then $|u + v + 1| \leq \beta_j + V + \eta + 1$. If both are far, their opposite signs give $|u + v + 1| \leq V + 1$. Since $W_0 \geq V + 4\eta$ and $\eta \geq 1$, all three sides have absolute jump bound $\beta_j + W_0$ at index j.

At a cut j let $l \leq 2J+1$ be the distance to the preceding cut and s_j its shift. The annihilator $p^\kappa E^{s_j}$ gives a point over $p^\kappa c_j$ by lifting the multiple of the constant, applying the coordinate retraction, and dividing by E^{s_j}. The retraction fixes the projected multiple. The resulting orders and depths are at least $-u - s_j \mathbf{1}_{a=0}$. Increase the depth to

$$D(a, u) = -u + s_j \mathbf{1}_{a=0} + \eta \mathbf{1}_{a \leq l}.$$

Call the improved orders $C_j(a, u)$, obtained from $-u - s_j \mathbf{1}_{a=0}$ and $D(a, u)$ by the constant-term rule of Section D.1. Explicitly, if $I_j(a, u) = -u - s_j \mathbf{1}_{a=0}$, then

$$C_j(a, u) = \begin{cases} D(a, u), & I_j(a, u) > 0, \\ \min(0, D(a, u)), & I_j(a, u) \leq 0. \end{cases}$$

At the cut j and level $0 \leq a \leq l$, put $i = F^{-a}j$, where $F(i) = i^p$, and $r_0 = s_j \mathbf{1}_{a=0}$. On either moving side a filtration step $n \mapsto n + 1$ on a line of jump u has

$$-u - r_0 \leq n < -u + r_0 + \eta.$$

For $|u| \leq \eta$ this gives $|n| \leq r_0 + 2\eta$. For $u \in \epsilon_i[\beta_i, \beta_i + V]$, with ϵ_i the sign of the far interval at i, it gives $|n + \epsilon_i \beta_i| \leq r_0 + \eta + V$. Thus the two integer intervals have centers $0, -\epsilon_i \beta_i$ and common radius $R = r_0 + W_0 \leq pn_0$, since $W_0 \geq V + 4\eta$. Their absolute endpoint bound is $\beta_j + s_j + W_0 \leq 2b_0$ at $a = 0$ and $\beta_i + W_0 \leq 2b_0$ at $a > 0$. There are no steps for $a > l$, and $l \leq 2J + 1$. If $r_0 > 0$, then $a = 0$ and $D_E = 0$. The sum of $v_p(n)$ for nonzero n in either interval of radius R is at most $2R/(p-1) + \lfloor \log_p(2b_0) \rfloor$, where $2b_0$ bounds all absolute endpoints.

Indeed, for either such interval I this sum equals

$$\sum_{q=1}^{\lfloor \log_p(2b_0) \rfloor} \#\{n \in I \cap \mathbb{Z} : n \neq 0,\ p^q \mid n\}.$$

Each inner count is at most $2R/p^q+1$. Thus the two intervals contribute at most $4R/(p-1)+2\lfloor\log_p(2b_0)\rfloor$ from the valuations. Since $R\le pn_0$ and $p\ge 2$, the first term is at most $8n_0$. If $r_0>0$, the constant part D_E+3a of the cost is zero. Thus both intervals on one side at one level have total exponent at most

$$8n_0+2\lfloor\log_p(2b_0)\rfloor+30(J+1+D_E)W_0\ \le\ 30X_0.$$

For the last term use $4W_0+2\le 5W_0$ and $D_E+3a\le 6(J+1+D_E)$ when the shift is zero. The size conditions

$$n_0+\lfloor\log_p(2b_0)\rfloor+(J+1+D_E)W_0+D_EG_1\le X_0,\qquad 12((J+1)X_0+D_0)<\kappa$$

give $\kappa+4D_0+(2J+2)30X_0<25\kappa$. Hence on each moving side the standard points may be taken over $p^{25\kappa}c_j$, using any unused powers of p as scalars. The Hodge intervals determine the cuts and their distances on the finite Frobenius cycle. Fix the two parameters and choose one such point over each scaled parameter at every cut. Each segment uses these chosen points at its two ends. The Tate-dual side uses its own projection annihilator as in the order criterion below. At $a=0$, the absence of a jump in $[-s_j,-\eta)$ shows that $C_j(0,u)\ge -u-\eta$: for $u<-s_j$, the initial order is positive and improves to $D(0,u)$; otherwise $u\ge-\eta$, the required order is nonpositive and is bounded by $\min(D(0,u),0)$. At all other levels $C_j(a,u)\ge -u$.

D.4.1 Termination at a plain cut

In this subsection, we use the ideal bounds of Section D.2 for a plain cut.

For the stop at a plain cut, choose a sign such that for every index at distance $a<l$ the far interval has that sign or lies inside the low interval. The side with negative separated interval for the chosen sign has all its jumps $v\le\eta$ at those levels; the moving side has all $u\ge-\eta$. Use the former side's standard point as the stop. Its depth remains $D(a,v)=-v+s_j\mathbf{1}_{a=0}+\eta\mathbf{1}_{a\le l}$, and it has orders at least

$$O_l(a,v)=\begin{cases}\max(0,-v-\eta), & a<l,\\ -v, & a\ge l.\end{cases}$$

At level 0, if $v\ge-\eta$, then $v\le\eta$ and $D(0,v)=-v+s_j+\eta\ge 0$, so the improved order is nonnegative. If $v<-\eta$ and $s_j=0$, the initial order is $-v\ge 1$. If $v<-\eta$ and $s_j>0$, then $v\le-\beta_j$ and the shifted cut has $s_j<\beta_j$. Thus $-v-s_j\ge 1$ also in this case. In both cases the improved order is $D(0,v)$.

At the other levels the asserted orders follow from the depth increase. Let x_0 be a standard point at the preceding cut with constant parameter, and take $g\in G_{K'}$. By Section B.2.1,

$$w_g=(gx_0-x_0)/(z\tau)\in M_{T,\mathrm{loc}}\cap\ker\pi.$$

Here π is the projection to the parameter. We have

$$(g-1)\varphi^l x_0=\varphi^l(z\tau)\,\varphi^l w_g.$$

At level $a<l$, put $n=l-a>0$. Choose a finite product $D=\prod_b E_b^{m_b}$ such that $Dw_g\in M_T$. For every chart-equivariant form λ,

$$\lambda(\varphi^n w_g)=\frac{\lambda(\varphi^n(Dw_g))}{\prod_b E_{b+n}^{m_b}}\in B_{\mathrm{dR}}^+.$$

The positive Frobenius test makes the numerator integral, and $\theta(E_{b+n}) \neq 0$ since $b+n>0$. Further, $\varphi^n\tau = [\epsilon]^{p^{n-1}} - 1$ has Fontaine value zero. Thus the difference has order at least one. At $a \geq l$, $\theta\varphi^{-(a-l)}(z\tau) \neq 0$; the factor has order zero and the preceding depth at $a-l$ is preserved. The order of the difference of two points over the same constant is bounded below by the minimum of their two orders. For the move from the preceding cut to the present point on the moving side, that minimum is at least $\min(0,-u-\eta)$ at level 0, $\min(0,-u)$ at $0<a<l$, $-u-\eta$ at $a=l$, and $-u$ above. At $a=0$ we have $u \geq -\eta$, so adding $D(0,v)$ gives at least $s_j - u - v$. For $0<a<l$, use $\min(0,-u) \geq -u-\eta$. At $a=l$ the extra η in $D(l,v)$ cancels the possible loss η in the order. The resulting sum is at least $-u-v$ for every $a>0$. Thus the annihilator of the Tate-dual side at the current cut satisfies the order criterion below.

For a long gap, the preceding cut has shift 0. A pivot j_* occurs $L = J+1$ steps after it, with the rest of the indices at distance $< l$ of opposite sign or bounded by η. Let g_0 bound the positive jumps of the resting side at the L indices up to the pivot and let E^{s_0} kill the pivot's cokernel. The gap imposes $N = s_0 + g_0 \sum_{0\leq a<L} p^a < \beta_{j_*}$.

On the unramified chart take $A = 2b_0$, with $N < A \leq p^L D_0$. Start with the point y at j_* over c_{j_*} obtained from E^{s_0}. Multiply it by

$$f(E) = E^{s_0} \prod_{0\leq a<L} E_a^{g_0}, \qquad E_a = (E-p)^{p^a} + p.$$

This is monic of degree N in E. Since E_a has order one at its own level and zero at the others, the orders of $f(E)y$ are at least $g_0 - u$ at $0 \leq a < L$ and $-u$ at $a \geq L$. The former are nonnegative by the upper bound at these L indices. Write T for the lattice of the extension on the resting side. Since $y \in N_{T,\mathrm{loc}}$ and $f(E) \in \mathfrak{S}$, every equivariant form has value in $\mathcal{K}_0$ on $f(E)y$. Its standard orders place it in $\varphi^L M_T$ by Section B.3. The value-field test of Section B.2.2 therefore puts $f(E)y$ in $\sum_i \mathfrak{S}\varphi^L(n_i)$. Since $A \leq p^L D_0$ and $3 \leq p^L$ on this unramified chart, Section D.3 expresses $p^{4D_0} f(E)y$ as an $\mathfrak{S}$-linear combination of vectors of this module whose form values are fixed modulo Fil^A. Project this equality to the summand at j_*. The projector fixes its left side, preserves the module and preserves the filtration condition by pullback of forms. Since $E = z+p$, expand the $\mathfrak{S}$-coefficients in powers of E over $\Lambda = W(\mathbf{F}_{p^f})$. Indeed evaluation at $z=-p$ maps $\mathfrak{S}$ onto Λ with kernel (E). Repeated division supplies the first $N+1$ coefficients and a remainder in $E^{N+1}\mathfrak{S}$. The subgroup fixes those coefficients, giving

$$p^{4D_0} f(E)y = \sum_{n=0}^{N} E^n \xi_n + E^{N+1} y_1$$

Each ξ_n and y_1 belongs to the level-L standard module, since its coefficients after division by E still lie in $\mathfrak{S}$. At level zero the value of every tested form λ on each ξ_n is fixed modulo Fil^A. By Section C.1 this value is congruent modulo $E^A B_{\mathrm{dR}}^+$ to a constant in K'.

Fix a form on a line of jump $u < -N$ at the pivot. The value of the left side has order at least $g_0 - u \geq N+1$. After removing the first k terms and dividing by E^k, the residual value has order at least $N+1-k$. For $k \leq N$, its reduction modulo E is the constant of ξ_k, because the later coefficients and the remainder have order at least zero at the pivot. The constant lies in $K' \cap \mathrm{Fil}^1 B_{\mathrm{dR}} = 0$, so ξ_k has order at least A on this form. Since $A \geq N+1$, subtraction of its value and division by E give the induction at $k+1$. On the trivial quotient write $f_n = [X^n]f(X)$ and $w_n = \pi(\xi_n) - p^{4D_0} f_n c_{j_*}$. Since $\pi(y) = c_{j_*}$,

$$\sum_{n=0}^{N} E^n w_n = -E^{N+1}\pi(y_1).$$

The right side has order at least $N+1$ at level zero, and each coordinate of w_n is congruent to an element of K' modulo Fil^A. The same induction gives order at least A for each w_n. Since $f_N = 1$, this yields $\pi(\xi_N) = p^{4D_0} c_{j_*} + w$. In a $\mathbf{Z}_p$-basis of the trivial parameter lattice C, all coordinates of w belong to S: the coefficient ξ_N is an S-linear combination of $\varphi^L n_i$, and $\pi(\varphi^L n_i) = \varphi^L \pi(n_i) \in S \otimes C$ by $\pi(N_T) \subset N_C = S \otimes C$ and $\varphi^L(S) \subset S$. The quotient calculation gives each coordinate de Rham order at least A. On this unramified chart $\ker(\theta|_S) = (z+p)S = ES$. Divide successively by E in S, decreasing the de Rham order by one each time. Hence $w/E^A \in S \otimes C$. It is supported at j_* and has the standard orders at level L, since the only jump of C is zero.

Over A_{inf}, φ^L carries the preceding summand and its level-zero standard module to the pivot summand and its level-L standard module. Projection to the trivial quotient commutes with φ^L, which fixes p^κ. The trivial quotient has only jump zero, so its standard module at level L is $e_{j_*}(A_{\mathrm{inf}} \otimes C)$. The projected parameters $e_{j_*}(1 \otimes c)$, $c \in C$, generate this module over A_{inf}. Thus the transported multiplier p^κ lifts every p^κ-multiple of this module, in particular $p^\kappa w/E^A$. Apply the coordinate retraction of Section D.2 to a lift. It preserves the standard orders and fixes w/E^A in the localized $\mathfrak{S}$-module of the trivial quotient. This gives a point x_1 of the same orders with $\pi(x_1) = p^\kappa w/E^A$. Put $q_* = p^\kappa \xi_N - E^A x_1$. This point lies over the exact constant $p^{\kappa+4D_0} c_{j_*}$, has depth A at the pivot, and has order at least A there for $u < -N$. Both terms lie in $N_{T,\mathrm{loc}} \cap \varphi^L M_T$. Put $t = l - L = l - J - 1$ and take $x = \varphi^t q_* \in \varphi^l M_T$ at the current cut. Its pivot is at level t.

For $a < t$ write $h = t - a > 0$. For each g fixing the chart field, Section B.2.1 gives $y_g = (gq_* - q_*)/(z\tau) \in M_{T,\mathrm{loc}}$. Since g commutes with Frobenius, every tested form λ satisfies

$$\lambda(\varphi^{-a}(gx - x)) = \varphi^h(z\tau)\, \lambda(\varphi^h y_g).$$

Clear a product D of shifted divisors E_j so that $Dy_g \in M_T$. The positive Frobenius test gives $\lambda(\varphi^h(Dy_g)) \in B^+_{\mathrm{dR}}$, and each factor of $\varphi^h D$ is E_{j+h}, a de Rham unit by Section B.3. Thus $\lambda(\varphi^h y_g) \in B^+_{\mathrm{dR}}$. The scalar $\varphi^h z$ has order zero. Moreover

$$\varphi^h \tau = (1+\mu)^{p^{h-1}} - 1 \equiv p^{h-1}\mu \pmod{\mu^2 B^+_{\mathrm{dR}}}.$$

Here $\mu = \xi\tau$ has order one, and p^{h-1} is a de Rham unit. Hence $\varphi^h(z\tau)$ has order one and the depth of x at every $a < t$ is at least one.

D.4.2 Raising depth before termination

In this subsection, we apply the depth step to the remaining indices before checking the two schemes in Section D.4.3.

At the levels $t < a < l$ raise depth from 0 to 1 by the step $n = 0$. The chart is unramified, so its cost is $C(a, 0) = 3a$. Since $l \le 2J + 1$ and $l - t - 1 = J$,

$$\sum_{a=t+1}^{l-1} 3a = \frac{3J(l+t)}{2} \le 3J(2J+1) < \kappa.$$

The strict inequality follows from $X_0 \ge (J+1)W_0$, $W_0 \ge 4\eta \ge 4$ and the definition of κ. The multiplier exponent is therefore at most $\kappa + 4D_0 + \sum_{a=t+1}^{l-1} 3a < 3\kappa$. Scale the point to the common exponent 25κ. The resting side has $v \ge -\eta$ off the pivot below l. At the pivot either $v \ge -\eta$ or $v < -N$. In the latter case the order is at least A, and $-v \le \beta_{j_*} + V \le 2b_0 = A$.

Thus the point has order bounds $O_l(a, v)$ and depth bounds 1 below l off the pivot, A at the pivot, and $-v$ at $a \geq l$.

At a long gap use the preceding cut's Tate-dual annihilator. Off the pivot the moving side has $u \leq \eta$ below l, so $C_j(a, u) \geq 0$ there. Its old point carried from the preceding cut has order ≥ 0 below l. At $a = t$ the order of their difference is at least $\min(0, \eta - u)$, and $u < \eta + A$. At $a \geq l$ the order is at least $-u$; the shift of the old cut is zero. Adding the depth of the stop yields at least 1 below l and $-u - v$ thereafter, as required for the move.

Here is the order criterion used in both cases. The dual jumps $-u-v-1$ can lie in both far intervals. At the cut l_1 steps before the current index use the Tate-dual projection annihilator $p^{\kappa}E^{s}$. Lift the multiple $p^{\kappa}E^{s}$ of the constant, divide by E^{s}, and carry the resulting point over p^{κ} by Frobenius. Denote its order bound on a dual jump w by $P_a(w)$. This is zero for $a < l_1$ and $-w - s\mathbf{1}_{a=l_1}$ for $a \geq l_1$, hence is nonincreasing in w. The Tate order 1 follows if the mover's order $o_a(u)$ and the resting point's depth $d_a(v)$ satisfy

$$o_a(u) + d_a(v) \geq 1 - P_a(-u - v - 1) = \begin{cases} 1, & a < l_1, \\ s - u - v, & a = l_1, \\ -u - v, & a > l_1. \end{cases}$$

A tensor covector of type at most r is a sum of products of covectors with $u+v \leq r$, by the tensor filtration. Then $P_a(-r-1) \geq P_a(-u-v-1)$, so the tensor has order at least $1 - P_a(-r-1)$ on weight r. In the final pairing a contributing dual weight w satisfies $w + r \leq -1$, the Tate jump. Hence $P_a(w) \geq P_a(-r-1)$, giving total order at least one. The bounds hold for sums of terms since each filtration step is a subspace. In the Galois difference of the resting point divide by $z\tau$ before taking this contraction.

This keeps the orders by the chosen basis. The contraction has τ times its value in the integral Tate lattice. Returning the removed $z\tau$ places the value in $W(\mathfrak{m}^{\flat})$, since z lies in that ideal. Reduction modulo the pairing exponent kills the component of the Tate-dual extension over zero, so the remaining value is precisely p^{κ} times the desired reduced pairing. For a move on the second side the test pairs the first difference with the Galois translate of the move. Equivariance preserves its orders, and conjugating the group element gives the same depth bound. This proves the required pairing condition for either side.

The moves to and from a stop also satisfy this criterion. The move from the old cut to the stop uses $l_1 = l$ and that cut's shift s_{old}. Below l its order is ≥ 0 and the other old point's depth is ≥ 1. At l the lower bounds are $-u - \eta$ and $-v + s_{\mathrm{old}} + \eta$, and above l their sum is at least $-u - v$. The move from the stop to the new point uses $l_1 = 0$ and $s = s_j$.

At 0 its order is at least $-u - \eta$ and the depth is at least $-v + s_j + \eta$; above 0 their sum is at least $-u - v$. Finally the pair of cocycle differences of two standard points transported from a cut satisfies the criterion with that cut's annihilator. Their depths are at least 1 below the transport length, at the transport length the order of each increases by the shift of the cut, and above it each is at least its standard order. The cochain calculation now applies to either cut scheme.

D.4.3 Termination in the two filtration schemes

We check the hypotheses of Section D.4 for the two distributions of Hodge jumps in Section 29.1. In the first case the large jumps occur sparsely in the Frobenius cycle. In the second case we use the two possible magnitudes of the centers of their intervals.

Work in the Frobenius cycle of one original chart. Let B_0, G_0 be the larger and smaller starting points of the separated intervals, of width V. Index a segment by $0, 1, \dots, l$ in the Frobenius direction, with the preceding cut at 0 and the current cut at l. An index at either end is counted in this direction even if the segment wraps around the cycle.

Sparse cuts. Here $\eta_0 \geq 1$ and $G_0 + V \leq \eta_0$. Extend the Frobenius cycle periodically to $\mathbf{Z}$, and let T be the positions with the larger start B_0. These positions are thick and their complement is thin. The sparsity hypothesis is

$$\{k, k+a, k+a+b\} \not\subseteq T \qquad (k \in \mathbf{Z}, \quad 1 \leq a, b \leq J).$$

The three positions may repeat residues on the cycle. The signed ratios in Section 29.1.1 exclude these triples as well. Assume either that the chart is unramified, $p \geq 7$ and $10p^J\eta_0 \leq B_0$, or that the signs are constant and $8e \leq p^{J+1}$. In both cases require

$$2e(B_0 + 5\eta_0) \leq p^{J+1}D_0, \qquad 20(J + 1 + D_E)\eta_0 + \lfloor \log_p(2(B_0 + 5\eta_0)) \rfloor \leq X_0.$$

Take $s_j = 0$ at every sparse cut. A cut is an index whose next J indices are thin. If a thick index w fails the cut condition, some thick index w' occurs a steps after it with $1 \leq a \leq J$. The J indices immediately before w must all be thin, since any thick one would give a forbidden triple ending at w'. Therefore $w - (J + 1)$ is a cut.

For any index i, either $i - (J + 1)$ is a cut or there is a thick w in the interval from $i - J$ to $i - 1$. In the second case one of w and $w - (J + 1)$ is a cut. Thus the nearest strictly preceding cut has $l \leq 2J + 1$. Within its segment, every thick intermediate index occurs exactly $J + 1$ steps after that cut. It occurs at least $J + 1$ steps later by the definition of cut. If it occurred more than $J + 1$ steps later, the preceding argument would put another cut strictly inside the segment.

Suppose all thick indices in $1, \dots, l$ have the same sign. The plain stop applies since every thin separated interval is contained in the low interval. This includes the constant-sign alternative. Otherwise there are exactly two thick indices in $1, \dots, l$. Take the earlier one, at step $L = J + 1$, as the pivot. The only other thick index is the current cut, whose sign is opposite to the pivot sign.

The opposite signs force the unramified alternative, so $p \geq 7$ and $10p^J\eta_0 \leq B_0$. Both cut shifts are zero. The low radius satisfies $\eta_0 \geq 1$. For the resting side take $g_0 = \eta_0$, including at the pivot, where its separated interval is negative. The integer

$$s_0 = \left\lfloor \frac{B_0 + 10\eta_0}{p - 3} \right\rfloor + 1$$

is an E-annihilator exponent for the resting side at the pivot. Indeed, at every index its jumps are at most $\max(\eta_0, B_0 + V) \leq B_0 + \eta_0 \leq (p - 3)s_0$. This is the global upper bound required by the unramified lift-ideal estimate. The long-stop degree satisfies

$$N = s_0 + \eta_0 \sum_{a=0}^{J} p^a \leq (B_0 + 10\eta_0)/4 + 1 + 2p^J\eta_0 < B_0.$$

For the last inequality, $10p^J\eta_0 \leq B_0$, $p^J \geq 7$, and $B_0 \geq 70$. Set $b_0 = B_0 + 5\eta_0$. The radius parameter in the long stop is $A = 2b_0 \leq p^L D_0$, by the sparse depth hypothesis with $e = 1$. In particular $N < A$. This proves the geometric and numerical hypotheses of the long stop at every sparse cut.

For the sparse lift ideals and the step bound take the auxiliary width $W_0 = 5\eta_0$, and take $G_1 = n_0 = 10\eta_0$. The J successors of a cut satisfy $\beta + W_0 \le G_1$. Also $V + 4\eta_0 \le W_0$, $W_0 \le pn_0$, $G_1 \le (p-1)n_0$, and $\beta + W_0 \le 2b_0$. The depth hypothesis gives $2eb_0 \le p^{J+1}D_0$. The ramification inequality $8e \le p^{J+1}$ is assumed in the constant-sign case and follows from $e = 1$, $p \ge 7$, $J \ge 1$ in the other. Finally

$$n_0 + \lfloor \log_p(2b_0) \rfloor + (J + 1 + D_E)W_0 + D_E G_1 \le 20(J + 1 + D_E)\eta_0 + \lfloor \log_p(2b_0) \rfloor \le X_0.$$

Thus the lift and step estimates apply at every sparse cut, where all J succeeding indices are thin.

Two far starts. Assume an unramified chart with $p \ge 7$, $\eta \ge 1$, $0 < G_0 \le B_0 < Kp^2$, $2G_0 < Kp$, and $\widetilde{W} = V + 4\eta \le K$. Put $q = \lfloor B_0/p \rfloor$ and $s_0 = q + 12K$. At every index, independently of its shift and sign,

$$(p-3)s_0 - (\beta_j + \widetilde{W}) \ge -3q + 12K(p-3) - (p-1) - \widetilde{W} > 0.$$

Indeed $B_0 \le pq + p - 1$, $q < Kp$, $\widetilde{W} \le K$, $K \ge 1$ and $p \ge 7$. This bounds the jumps on all three sides wherever the unramified lift-ideal estimate is used. Let the shift at an index be $\max(0, q - 7K)$ if its successor is thick, and zero otherwise. An index is a cut when its shift is smaller than its far start. Every thick index is a cut, since its shift is at most $q < B_0$. Every index with thin next component is a cut, since its shift is zero. Therefore if the immediate predecessor is no cut, it is thin and the second predecessor is a cut. Thus $l \in \{1, 2\}$.

For $l = 1$, or $l = 2$ with equal signs on the indices at steps $1, 2$, use the plain stop. When $l = 1$, the preceding cut may be thick with the opposite sign. Only the current index enters the sign condition at $a < l$. At $a = l$ the two jump indices are both $F^{-l}j$, as in Section D.4.1. Otherwise $l = 2$, the index at step 1 is thin and the current index is thick. The preceding cut has shift zero. The no-cut condition at the thin index gives $G_0 + 7K \le q$, since $G_0 > 0$. The pivot is the current index. Take $L = 2$ (the long-stop parameter is $J = 1$), and $g_0 = G_0 + \widetilde{W}$. The global jump bound gives the annihilator E^{s_0} at the pivot. The bound on g_0 uses the thin start at step 1 and the negative separated interval of the resting side at the pivot. Since $G_0 + \widetilde{W} \le q - 6K$,

$$\begin{aligned} N &= s_0 + g_0(1 + p) \\ &\le (p + 2)q - (6p - 6)K \le B_0 + 2q - (6p - 6)K < B_0. \end{aligned}$$

With $b_0 = B_0 + \widetilde{W}$, one has $N < 2b_0 \le 4Kp^2$, so the radius parameter $A = 2b_0$ meets the long-stop bound with $D_0 = 4K$.

For the step bound use $n_0 = K$, $G_1 = G_0 + \widetilde{W}$, $J = 1$, $D_0 = 4K$, and $X_0 = 3K + \lfloor \log_p(2b_0) \rfloor$. We have $G_1 \le (p-1)K$ by $2G_0 < Kp$. Also $s + \widetilde{W} \le pK$ by $q < pK$ and the two cases of $\max(0, q - 7K)$, and $\beta + s + \widetilde{W} \le 2b_0$. Because $D_E = 0$, the step-bound follows from $K + 2\widetilde{W} \le 3K$. For a cut with a thin next component use the lift-ideal estimate for a thin next component with shift zero. For a cut with thick successor (even if its shift happens to be zero), $s_0 - s \le 19K < \kappa$, so the unramified lift-ideal estimate gives $p^{\kappa}E^s$. All cuts and stops use the original signs and coefficient intervals of the original components.

The total exponent. For each index j, let $l(j)$ be the distance to the nearest preceding cut strictly before it, and let $d(j) = 0$ at a cut and $d(j) = l(j)$ elsewhere. In a cycle of length h_c with a single cut j, one has $l(j) = h_c$. The preceding point at this cut is consequently transported

by φ^{h_c}, even though its residue index returns to j. Transport the two selected points at the last cut by $\varphi^{d(j)}$. The points chosen at the cuts have the improved orders C_j and the lower orders needed at a plain cut. Write $Fj = j^p$. The identity $l(j) = 1 + d(F^{-1}j)$ gives, for either family (x_i) of points at the cuts,

$$\varphi\left(\sum_j \varphi^{d(j)} x_{F^{-d(j)}j}\right) = \sum_j \varphi^{l(j)} x_{F^{-l(j)}j}.$$

Reindex the left sum by the permutation F. On either moving side, for $a = d(j)$ or $a = l(j)$, the projection to its parameter lattice sends the transported point to

$$\pi(\varphi^a x_{F^{-a}j}) = p^{25\kappa}\varphi^a(e_{F^{-a}j}(1 \otimes c)) = p^{25\kappa} e_j(1 \otimes c).$$

Here φ acts on the scalar ring and fixes $1 \otimes c$. Summing over j gives $p^{25\kappa}(1 \otimes c)$. Off a cut the transported points agree since $d(j) = l(j)$. At a cut the two points have equal projections even when the preceding cut has the same residue index. Write (y, y') for the transported pair and (x, x') for the chosen pair there. In every summand use the same order of changes,

$$(y, y') \longrightarrow (w, y') \longrightarrow (w, w') \longrightarrow (x, w') \longrightarrow (x, x').$$

If the first side has the stopping point q, take $(w, w') = (q, x')$; the fourth change is zero. If the second side has the stopping point q', take $(w, w') = (y, q')$; the first change is zero. In each case the other three changes satisfy the order inequalities just proved. Thus each position in this sequence can be summed across coefficient summands.

For the cocycle-difference test, take the third period from the cut supplying the endpoints and transport it by $\varphi^{d(j)}$ from $j^{p^{-d(j)}}$. At a cut this is the current third period. Off a cut it is the transported preceding period. Write $d = d(j)$ and let s be the shift of that cut. Below level d, each transported Galois difference has order at least one by the factor $\varphi^{d-a}(z\tau)$ at level $a < d$. At $a = d$ the two differences have orders at least $-u+s+\eta$ and $-v+s+\eta$, whose sum is at least $s-u-v$. At $a > d$ their sum is at least $-u-v$. These are the bounds $1-P_a(-u-v-1)$ for the third period transported through these same d steps. At the fixed pairing depth m, the third period is over $p^\kappa e_j$, where e_j is the projector to index j. The reduction of its extension form is $p^\kappa\psi(e_j y, y')$, followed by the same $\mathbf{Z}_p$-linear functional on O for every test. On index j this is $p^\kappa\psi(y, y')$.

The cross pairings of distinct indices vanish: balance for the coefficient root of unity multiplies such a pairing by the difference of two Teichmüller lifts, which is a Witt unit. Thus the sum over all indices satisfies the four move tests and the two-cocycle difference test of Section C.2.1. Including the two moving sides and the Tate-dual period gives the factor

$$(p^{25\kappa})^2 p^\kappa = p^{51\kappa}.$$

In the sparse regime $\kappa = 12((J+1)X_0 + D_0) + 1$, so $51\kappa \le 10^3((J+1)X_0 + D_0)$. In the two-start regime $\kappa = 12(2X_0 + 4K) + 1$, where $X_0 = 3K + \lfloor \log_p(2b_0) \rfloor$. Consequently

$$51\kappa \le 10^4(K + \lfloor \log_p(2b_0) \rfloor).$$

Here $K \ge 1$.

Proposition D.1 (Uniform cup annihilation in the two filtration schemes)**.** *Use the Kummer chart of E_v specified in Section 29.1.1, with working group $H = G_{E_v}$. Suppose the allowed partner Y and the fixed Tate-dual lattice $L^\dagger$ are crystalline over E_v. Let $z \in Z_f(Y)$, $z' \in Z_f(L^\dagger)$, with the pairing ψ and depth m of Section D.4. For any $\mathbf{Z}_p$-linear functional $\lambda : O \to \mathbf{Z}_p$, put $\psi_m = ((\lambda \otimes 1) \circ \psi) \bmod p^m$. Use one of the two filtration schemes under the interval and cut hypotheses of Section D.4.3. Then*

$$p^{51\kappa}[(\psi_m)_*(\bar{z} \cup \bar{z}')] = 0 \quad \text{in } H^2(E_v, \mu_{p^m}), \qquad 51\kappa \le \begin{cases} 10^3((J+1)X_0 + D_0), & \text{sparse regime,} \\ 10^4(K + \lfloor \log_p(2b_0) \rfloor), & \text{two-start regime.} \end{cases}$$

E Local colength away from the original level

In this appendix, we apply the lifts of Appendix D to the local coefficient lattices. We handle separated Hodge–Tate weights and the cases of exact integrality, then combine the bounds at the primes away from the original level in Section E.3.

E.1 Colength for separated Hodge–Tate weights

In this subsection, we treat the unramified case in which the nonzero multiples of p avoid the intervals of filtration orders used by the divided operators. Using the E-power annihilator of Section D.2, we construct one submodule of crystalline cocycles on which all the requisite cup products vanish. Its colength gives the local estimate.

Use an unramified chart with residue degree f and $p \ge 5$. At each residue-field index j, the jumps of the two sides lie in $[-\eta, \eta]$ or in their respective far intervals $\pm\varepsilon_j[\beta_j, \beta_j + W]$, with opposite signs as in Section 29.1.1. Here $\eta \ge 1$ and $W \ge 0$ are integers. Let $A \ge 0$ be an integer with

$$\beta_j + W + \eta \le (p-3)A, \qquad 6(A + \eta + W) < \beta_j$$

at every index j. Require that no nonzero multiple of p lie within $6(A + \eta + W)$ of $0, \pm\beta_j$. For jumps u, v of the two sides at the same index, one of them is at least $-\eta$, since their separated intervals have opposite signs. Both are at least $-(\beta_j + W)$, as $\beta_j > \eta$. It follows that

$$-u - v - 1 \le \beta_j + W + \eta - 1 \le (p-3)A.$$

The trivial quotient contributes only the jump 0. On each original side the jumps are at most $\beta_j + W$. Thus the unramified lift-ideal estimate in Section D.2 applies to all three extensions, with annihilators $p^s E^{A-s}$ for $0 \le s \le A$. Write $R(t)$ for the orders $R(t)(a, u) = -u + t\mathbf{1}_{a=0}$.

Let L be the coefficient lattice of the fixed side, so $C = Z^1_f(L)$ is its module of crystalline cocycles. Use the single extension on $L \oplus C$, with trivial action on C and with

$$g(y, c) = (gy + c(g), c).$$

The cocycle equation gives the action law. The module C is finite free: integral H^1 is finitely generated, the coboundaries have finite rank, and C is torsion free over O. A crystalline primitive for each element of an O-basis of C splits this extension over B_{cris}. In the de Rham grading it adds copies of jump zero. The nonzero-jump estimates in the extension thus use the multiplicities of L; the depth estimates over zero use the kernel L itself.

Let ρ_j be the projection of $e_j N_{L\oplus C,\mathrm{loc}}$ to its trivial quotient at index j, and put $\iota_j(c_0) = e_j(1 \otimes c_0)$ for $c_0 \in C$. Define $\mathcal{D}_j(t,c) \subseteq \iota_j(C)$ by

$$\iota_j(c_0) \in \mathcal{D}_j(t,c) \quad \Longleftrightarrow \quad \exists x \in e_j N_{L\oplus C,\mathrm{loc}}, \quad \rho_j(x) = \iota_j(c_0), \quad x \text{ has orders } R(t) \text{ and depth } c.$$

The retraction of Section B.2.2 fixes each such constant and preserves the orders of the form values. It also preserves depth, since a value of depth d is congruent modulo Fil^d to a scalar in the chart field. Thus $\mathcal{D}_j(t,c)$ is the image under ρ_j of these points with the specified orders and depth. All preimages under ι_j are taken in the same C, before choosing the partner, its cocycle, the finite level or the pairing. For each element of the intersection the lifts at every index therefore project to the components of that element itself, including when it is divisible by p in C. For a partner or a Tate-dual period in an individual cup test one may use the extension by O defined by that single cocycle. Let n_j, r_j bound f times the low-jump and all-jump multiplicities of L at index j.

We first prove the integral linear algebra for one order step. Suppose a system of orders c is being raised at (a,u), and $c(a,v) > c(a,u)$ for $v < u$ at the same index and level. Take $fm_u + 1$ vectors of orders c in the localized chart module, where m_u is the multiplicity of the jump at the index $\varphi^{-a}j$ read at level a from the current index j. Consider the leading coefficient covectors on a basis of the m_u-dimensional graded space of de Rham forms at u. Lower lines make no contribution by the strict inequality on orders. Over C_p choose a maximal-rank minor with largest absolute determinant among minors of that rank. Cramer's rule then writes every covector in terms of its $d \le m_u$ selected covectors with coefficients in $\mathcal{O}_{C_p}$. For each coefficient α choose $\widetilde{\alpha} \in A_{\mathrm{inf}}$ with $\theta(\widetilde{\alpha}) = \alpha$. Use $\varphi^a\widetilde{\alpha}$ in the relation at E_a, since $(\theta\varphi^{-a})(\varphi^a\widetilde{\alpha}) = \alpha$. The difference is a vector whose order is raised at (a,u).

Write $M(c)$ for the submodule of chart vectors of orders c. Let x_j denote the original vectors and $x_{s(k)}$ the d selected ones. The retraction fixes these vectors. It preserves the raised order of their difference and sends a product $bx_{s(k)}$, $b \in A_{\mathrm{inf}}$, to $F(b)x_{s(k)}$, with $F(b) \in \mathfrak{S}$. Hence

$$x_j - \sum_{k=1}^{d} b_{jk} x_{s(k)} \in M(c + \mathbf{1}_{(a,u)}), \qquad b_{jk} \in \mathfrak{S}.$$

Modulo E, the d-tuples of coefficients are in $W(\mathbf{F}_{p^f})^d$, of $\mathbf{Z}_p$-rank $fd \le fm_u$. Divide a nonzero integral relation by a coefficient of smallest valuation. The resulting coefficients a_j include 1 and satisfy

$$\sum_j a_j b_{jk} \in E\mathfrak{S}, \qquad \sum_j a_j x_j \in M(c + \mathbf{1}_{a=0}) + M(c + \mathbf{1}_{(a,u)}).$$

Here E raises exactly the level-zero orders. This argument is also valid for the actual extension lattice. Its multiplicity at any nonzero u is that of the sublattice L, since the parameter quotient is trivial.

E.1.1 Uniform comparison of the depth steps

In this subsection, we compare successive depths using the lift construction of Section D.4.

The depth version compares $\mathcal{D}_j(0,c)$ with the constants for the stronger depth c^+ at (a,u). Assume all current depth orders are $\ge -u$, so the order module of vectors over zero with those orders is contained in the no-pole order module. Denote this kernel order module by M. Let $X(c)$ consist of the pole-free points of depth c whose projection under ρ_j belongs to $\iota_j(C)$. Then

$\rho_j(X(c)) = \mathcal{D}_j(0,c)$ by definition. Every vector of M lies in $X(c)$: its order bounds imply the corresponding bounds on its Galois differences. Its projection is zero. Consequently ρ_j induces a surjection

$$Q := X(c)/(X(c^+) + M) \longrightarrow \mathcal{D}_j(0,c)/\mathcal{D}_j(0,c^+).$$

Put $k = C(a, c(a,u))$. The one-step operator sends $x \in X(c)$ to a point of $X(c^+)$ congruent to $p^k x$ modulo M; in particular $p^k Q = 0$.

Choose $fm_u + 1$ classes in $Q[p]$ and representatives x_j. By the definition of Q, write $px_j = x'_j + m_j$ with $x'_j \in X(c^+)$ and $m_j \in M$. The inclusion $L \subset L \oplus C$ is saturated since C is free. The localized kernel is its intersection with the ambient chart lattice by Section B.2.3. Every form of type at most u on this kernel extends to one of the same type bound on the extension by Section C.2. Thus each m_j lies in the kernel chart lattice with all the specified orders. Applying the preceding relation there, with the multiplicity m_u of L, gives a primitive combination x of the x_j for which $px - w$ has the raised depth and w is over zero with all level-zero orders raised by 1.

Suppose further that no multiple of p lies strictly between $c(0,v)$ and $c(0,v) + p^a$ at any line v. All the depth operations that raise w from the first level-zero order increment to p^a increments are free. Use their common composite B on $x, w, px - w$. By Section D.1, its factors are $\mathbf{Z}_p$-linear, each has scalar multiplier 1, and its correction on the original system lies in M. Thus $B(px - w) = pBx - Bw$ and $Bx - x \in M$. Since w lies over zero and has the increased orders, it has at least those depths as well.

The free steps and improvement of orders give $Bw = E^{p^a} w_1$ with $w_1 \in M$. In checking this improvement, either $c(0,v) + 1 > 0$ or $c(0,v) + p^a \le 0$, because 0 is absent from the open interval. In the former case the positive initial order forces the improved order to equal or exceed the target. In the latter case the target is nonpositive, so congruence to a constant to that depth already gives the target order. This includes the endpoint $c(0,v) + p^a = 0$. Use $E^{p^a} = E_a + pF$ with $F \in \mathfrak{S}$. Then

$$p(Bx - Fw_1) = B(px - w) + E_a w_1.$$

The first term has the raised depth because the free operator preserves depths already acquired. The second term has that depth because w_1 has orders c and the multiplication raises every order at a by 1.

At each period prime E_b, the scalar p has order zero, so $Bx - Fw_1$ has the raised depth as well. It differs from x by an element of M. It remains pole-free, since M is contained in the no-pole module. The resulting primitive relation is a nontrivial relation in $Q[p]$, so $\dim_{\mathbf{F}_p} Q[p] \le fm_u$. Multiplication by p^{i-1} injects $Q[p^i]/Q[p^{i-1}]$ into $Q[p]$ for $1 \le i \le k$. Hence

$$\operatorname{length}_{\mathbf{Z}_p}(\mathcal{D}_j(0,c)/\mathcal{D}_j(0,c^+)) \le \operatorname{length}_{\mathbf{Z}_p} Q \le kfm_u.$$

One may iterate this estimate toward a target depth that is antitone in the jumps at the level being raised. At each step take the least jump u whose current depth is short of the target d. For every smaller jump $v < u$,

$$c(a,u) < d(a,u) \le d(a,v) \le c(a,v).$$

Thus the primitive relation applies even when two target depths are equal. Write $i = \varphi^{-a} j$ and let $m_{i,u}$ be the multiplicity at u there. With the other levels fixed and the open-interval

condition valid at each step, the total loss in length is at most

$$f \sum_{u \in \mathcal{J}_i} m_{i,u} \sum_{c(a,u) \le n < d(a,u)} \big(v_p(n) + 3a\big), \qquad v_p(0) = 0.$$

Each increase removes one pair (u, n) from this sum. Equal step values on different jumps occur separately with their respective multiplicities. For level 0 the open intervals of length $p^0 = 1$ are always empty of integers. For level 1 we retain the same depth at level 0 on every line, a multiple of p, so the intervals of length p contain no multiple in their interiors.

Fix an index j. For each integer $0 \le s \le A$, the unramified lift-ideal bound with $t = A - s$ and $\kappa = s + t = A$ gives the annihilator $p^s E^{A-s}$ on all three extensions. On either original side let $d = e_j(1 \otimes c)$ for an element c of its trivial parameter lattice. Choose a standard-order lift of $p^s E^{A-s} d$. Its coordinate retraction preserves these orders and fixes the projection, since d is a chart vector and $E \in S$. Dividing by E^{A-s} gives a localized chart point over $p^s d$ with orders $R(-(A - s))$. Indeed E has order one at level zero and order zero at every positive level.

Raise its depth to $R(2A + \eta)$. Every level-zero step is within $6(A + \eta + W)$ of 0 or $\pm\beta_j$ and costs no factor of p, so the projection stays $p^s d$. At $s = 0$ the gap $[-A, -\eta)$ gives improved orders at least $R(-\eta)$ over d. At $s = A$ they are at least $R(0)$ over $p^A d$. The depth $R(A)$ for the tensor-dual extension is verified for its own jumps in Section E.1.2.

E.1.2 Pole removal and the final integral lift

In this subsection, we apply the comparison of Section E.1.1 at each pole order.

Remove the pole one step at a time. For $t = -\eta, \dots, -1$, compare $\mathcal{D}_j(t, R(2A + \eta))$ with $\mathcal{D}_j(t + 1, R(2A + \eta))$. Their quotient is killed by p^A, since $p^A \iota_j(C)$ already has lifts of order $R(0)$ and the prescribed depth. For a jump $v < t$, the old order $t - v > 0$ and the retained depth $2A + \eta - v > 0$ force the approximating constant to be zero. Thus the value has order at least $2A + \eta - v \ge t + 1 - v$. For $v > t$, its order is at least $\min(0, 2A + \eta - v) \ge t + 1 - v$, since $v \ge t + 1$ and $2A + \eta \ge t + 1$. If $m_t = 0$, this covers every basis form, so the two modules of constants coincide. If $m_t > 0$, take $fm_t + 1$ representatives and apply the single-order relation at t, using $R(t)(0, v) = t - v > 0 = R(t)(0, t)$ for $v < t$. Both terms in the resulting primitive combination have order at least one at $(0, t)$. The combination retains its depth by $\mathbf{Z}_p$-linearity, so the bounds at $v \ne t$ apply to it as well. It therefore lies in the smaller module. If Q_t denotes this quotient of constants, any $fm_t + 1$ classes in Q_t/pQ_t therefore satisfy a nontrivial relation. Thus $\dim_{\mathbf{F}_p} Q_t/pQ_t \le fm_t$, and each layer $p^i Q_t/p^{i+1}Q_t$ is a quotient of Q_t/pQ_t. Since $p^A Q_t = 0$, its length is at most Afm_t. The trivial parameter quotient adds no jump at $t < 0$. All these jumps are low, so pole removal has total $\mathbf{Z}_p$-colength at most

$$Af \sum_{-\eta \le t < 0} m_t \le An_j.$$

At an index where the separated interval of the fixed side is positive, retain the depth $R(2A + \eta)$. At a negative-interval index seek the depths

$$\begin{aligned} d_j(0, u) &= \beta_j + W + A + \eta, \\ d_j(1, u) &= W + 1 - u + 2A\mathbf{1}_{|u| \le \eta}, \\ d_j(a, u) &= -u \qquad (a \ge 2). \end{aligned}$$

At level zero first increase the depth to the least multiple T of p at least $\beta_j + W + A + \eta$. First replace each larger old requirement by T, enlarging the module of admissible constants. The old and new requirements are both at least $-u$, since $-u \le \beta_j + W$ here. Where the old requirement is smaller, the exact step range is $2A + \eta - u \le y < T$; otherwise there is no step. Intersecting the resulting lattice with the original one costs at most the length bound for these increases. For $|u| \le \eta$ the steps lie in $[0, T)$, with $C(0, 0) = 0$. The choice of T gives $T \le \beta_j + W + A + \eta + p - 1 \le (p - 2)A + p - 1$. By Legendre's formula,

$$\sum_{2A+\eta-u \le y < T} C(0, y) \le v_p((T - 1)!) \le (T - 1)/(p - 1) \le A + 1.$$

For a far negative jump, a multiple of p below T is at most $T - p$. If such a multiple y occurs among the steps, then

$$\beta_j + 2A + \eta \le -u + 2A + \eta \le y \le T - p < \beta_j + W + A + \eta.$$

The exclusion of nonzero multiples near β_j makes these steps free. By the depth version of the order estimate the contribution in length is at most $(A + 1)n_j$.

Let $i = \varphi^{-1} j$ and keep T at level zero. As $p \mid T$, the open interval $(T, T + p)$ contains no multiple of p. For each jump u at i the first-level steps are

$$-u \le n < -u + W + 1 + 2A\mathbf{1}_{|u| \le \eta}.$$

For a low jump, $|n| \le \eta + W + 2A$. For a far jump, n is within $2W$ of one of $\pm\beta_i$. The exclusion at i therefore makes each step zero or prime to p, of cost 3 at level one. The target is antitone in u on each interval and across the gap since $\beta_i - \eta > 2A$. The total first-level contribution is at most $3(W + 1)r_i + 6An_i$. Finally the depth at level zero may be lowered from T to $d_j(0, u)$ without any loss. The contribution at j is therefore at most

$$(2A + 1)n_j + 3(W + 1)r_{\varphi^{-1} j} + 6An_{\varphi^{-1} j}.$$

Let $D_j \subseteq \iota_j(C)$ be the resulting lattice of constants. Set $V = \bigcap_j \iota_j^{-1}(D_j)$. The map

$$C/V \hookrightarrow \bigoplus_j \iota_j(C)/D_j, \qquad c_0 + V \longmapsto \big(\iota_j(c_0) + D_j\big)_j$$

is injective by the definition of V. Since Frobenius permutes the indices, summing gives

$$\operatorname{length}_{\mathbf{Z}_p}(C/V) \le (8A + 1) \sum_j n_j + 3(W + 1) \sum_j r_j \le 11 \sum_j \big((A + 1)n_j + (W + 1)r_j\big).$$

Thus V has full $\mathbf{Z}_p$-rank in C. The multiplicities in this bound are those of the fixed lattice L. The same V is used for every allowed partner, including at a cut where the partner point stays fixed during the middle change.

For any allowed partner and pairing, the constructed points give safe moves, with a cut at every index. On the partner take points over every constant of depth $R(2A + \eta)$, starting from E^A and using the free steps. Their orders are the improvement of $R(-A)$ by that depth. In particular their orders at level zero are at least $-u - \eta$. Suppose the fixed side has a positive separated interval at j. Write u for a partner jump and v for a fixed-side jump at the index

read at the level in question. Put $i = \varphi^{-1} j$, and denote the chosen partner and fixed-side points at k by x_k, y_k, respectively. The path is

$$(\varphi x_i, \varphi y_i) \longrightarrow (x_j, \varphi y_i) \longrightarrow (x_j, y_j).$$

These are the first two changes of the four-move path.

For the first change use the Tate-dual point obtained from the E^A-annihilator at i, transported once. The required order-plus-depth bounds of Section D.4.2 are 1 at level zero, $A-u-v$ at level one and $-u-v$ above. At zero, $u \le \eta$ and the improved order of x_j is nonnegative. The transported partner point also has nonnegative order, and φy_i has depth at least one by the transported $z\tau$ divided difference. At level one the partner difference has order at least $\min(-u, -u-\eta) = -u-\eta$. At i the old fixed depth at level zero is $2A+\eta-v$ for the positive sign and $\beta_i + W + A + \eta$ for the negative sign. Both are at least $A+\eta-v$, since $-v \le \beta_i + W$. At levels $a \ge 2$ the partner order is at least $-u$ and the fixed depth at least $-v$.

For the second change use the Tate-dual point from the E^A-annihilator at j. The required sum is $A-u-v$ at zero and $-u-v$ above. At zero, $v \ge -\eta$, so the fixed-side order and current partner depth satisfy

$$\min(0, -v) + 2A + \eta - u \;\ge\; A - u - v.$$

At every positive level both fixed-side points have order at least $-v$, including at level one by pole removal at i. The current partner has depth at least $-u$, giving the required sum.

Suppose now that the fixed side has a negative separated interval at j. With x_k the partner point and y_k the fixed-side point, change the fixed side and then the partner along

$$(\varphi x_i, \varphi y_i) \longrightarrow (\varphi x_i, y_j) \longrightarrow (x_j, y_j), \qquad i = \varphi^{-1} j.$$

They occupy the second and third positions of the four-move path. Continue to write u for a partner jump and v for a fixed-side jump at the index read at level a.

For the initial change use the once-transported Tate-dual lift from the E^A-annihilator at i. At level zero the depth $d_j(0, v) = \beta_j + W + A + \eta > 0$ puts each tested form value of y_j in $K' + \mathrm{Fil}^1 \subset \mathrm{Fil}^0$. The transported φy_i also has nonnegative order, and φx_i has depth at least one by the transported $z\tau$ divided difference. At level one both fixed points have order at least $-v$, using pole removal at i for φy_i. The resting depth there is $2A+\eta-u$ for either sign at i. Thus the sum is at least $A-u-v$. At $a \ge 2$ the fixed difference has order at least $-v$ and the resting depth is at least $-u$. These are the three required bounds of Section D.4.2, also when $i = j$.

The partner now moves with y_j at rest. On the fixed side, $-v \le \beta_j + W$ at level zero, so $d_j(0, v) \ge A + \eta - v$; at level one, $d_j(1, v) \ge -v$. Its pole-free orders dominate $R(-\eta)$, and the partner has orders at least $R(-\eta)$ and depth $R(2A+\eta)$. For this move use a Tate-dual point at the predecessor $i = \varphi^{-1} j$, of depth $R(A)$ and orders given by the improvement of $R(-A)$ by $R(A)$. Its jumps are 0 and $-u-v-1$ for jumps u, v at i. If both are low or both are far, the dual jump has absolute value at most $2\eta + W + 1$ because the two far intervals have opposite signs. If only one is far, the dual jump is within $\eta + W + 1$ of one of $\pm\beta_i$. Thus the exclusion at i supplies the free steps. After Frobenius transport from i to j the level-zero dual order is at least zero. For the moving order write $o(u)$ for the improvement of $-A-u$ by $2A+\eta-u$. If $u < -A$, then $o(u) > 0$. If $u \ge -A$, then $o(u) = \min(0, 2A+\eta-u)$. Since $u \le \beta_j + W$, it follows in both cases that $\min(0, o(u)) + \beta_j + W + A + \eta > 0$. At levels > 1 the three orders sum to the standard Tate order 1.

At level 1, both jumps are read at $\varphi^{-1}j$, where the two far intervals have opposite signs and the same start. Write

$$E_q(w) = \begin{cases} q - w, & w < -A, \\ \min(q - w, 0), & w \geq -A. \end{cases}$$

The sum of the three orders there is at least

$$E_A(-u - v - 1) + \min(-u, E_{2A+\eta}(u)) + W + 1 - v + 2A\mathbf{1}_{|v|\leq\eta}.$$

This is positive. For $|v| \leq \eta$, use $E_A(-u - v - 1) \geq u + v + 1 - A$ and the moving order $\geq -u - A$, giving $W + 2$. If v is far and u is low, the gap gives $E_A(-u - v - 1) \geq A + u + v + 1$, again giving $W + 2$. If both are far, then $|u + v| \leq W$, the moving order equals $-u$, and $E_A(-u - v - 1) \geq u + v - W$. The sum is at least 1. The safe-move criterion, with the dual point in place of its standard lift, now applies. Fix a partner cocycle z and $c \in V$. Let e_j denote the coefficient idempotent at j. The partner point x_j and the fixed-side point y_j lie over $e_j(1 \otimes 1)$ and $e_j(1 \otimes c)$. Put $x = \sum_j x_j = (a, 1)$ and $y = \sum_j y_j = (b, c)$. With $i(j) = \varphi^{-1}j$,

$$\sum_j \varphi x_{i(j)} = \varphi x, \qquad \sum_j \varphi y_{i(j)} = \varphi y, \qquad \varphi(e_{i(j)}) = e_j.$$

For the intermediate first point use $w_j = x_j$ at a positive fixed sign and $w_j = \varphi x_{i(j)}$ at a negative one; the intermediate second point is $w'_j = y_j$ for both signs. Their sums lie over the same parameters as x, y. The four changes therefore add in the slots of Section C.2.1. In a second-side change the second argument is the Galois translate of the difference. The group H fixes the idempotents and preserves the order bounds. Differences over a common parameter have the asserted kernel orders by Section C.2.

There is also the endpoint test for every $g, h \in H$. At j, pair $gx_j - x_j$ with $g(hy_j - y_j)$. At level zero the sum of their depths is at least

$$(2A + \eta - u) + (A + \eta - v) \geq A - u - v$$

for either fixed sign. At every positive level the sum is at least $-u - v$. Thus the current Tate-dual lift from the E^A-annihilator gives total order at least one at every level. Dividing the first difference by $z\tau$ and returning this factor after the contraction puts the value in the small Witt ideal, as in Section D.4.2.

At depth m the integral pairing ψ gives the third cocycle $(g\psi - \psi)/p^m$. Each third point used at j projects to $e_j(1 \otimes 1)$, also after transport from $i(j)$. Its extension form therefore reduces to ψ mod p^m on the j-components by Section C.2.2. Balance over O gives $\psi(e_j v, e_k v') = 0$ for $j \neq k$ after extension to Witt scalars. Hence the four changes and the endpoint test still hold after summation and composition with any $\mathbf{Z}_p$-linear $\lambda : O \to \mathbf{Z}_p$. The two kernel differences are $ga - a + z(g)$ and $gb - b + c(g)$. Section C.2.1 with $e_1 = e_2 = e_3 = 0$ now gives a continuous coboundary for $((\lambda \otimes 1)\psi \bmod p^m)_*(\bar{z} \cup \bar{c})$ on $H = G_{E_v}$. This holds for every $c \in V$, partner, depth and test pairing. The annihilator is O-linear: multiplying a cocycle by a coefficient scalar composes each integral coefficient functional in the test with multiplication by that scalar. This allows the constructed $\mathbf{Z}_p$ sublattice to bound the colength of the O-annihilator.

Write $K_O = \mathrm{Frac}(O)$ and let d_j count its coefficient embeddings at index j. These are counted in the de Rham grading over $\mathbb{C}_p$, so each embedding occurs at one index. Restriction of scalars in the order steps is over $\mathbf{Z}_p$: the multiplicity at a jump is the sum of the graded multiplicities of those embeddings, and $\mathfrak{S}/(E)$ has rank f over $\mathbf{Z}_p$. Thus $n_j = fd_j\mu$, $r_j = fd_j\rho$

suffice for the respective bounds per embedding μ, ρ, with $\sum_j d_j = [K_O : \mathbf{Q}_p]$. For the coefficient field of Section 29.1, the root ζ_v generates the unramified subfield of degree $f = f_v$. Each of its f embeddings extends to $[K_O : \mathbf{Q}_p]/f$ embeddings of K_O. Hence d_j equals this quotient on the Frobenius orbit and vanishes elsewhere. These counts are over $\mathbf{C}_p$. Let $\mathcal{A} \subset C$ be the annihilator on the integral O-lattice. Since $V \subseteq \mathcal{A}$, the map $C/V \to C/\mathcal{A}$ is a surjection over $\mathbf{Z}_p$. The residue degree of O converts O-length to $\mathbf{Z}_p$-length, and $v_O(p)$ is the ramification index. Thus

$$\frac{\operatorname{length}_O(C/\mathcal{A})}{v_O(p)} = \frac{\operatorname{length}_{\mathbf{Z}_p}(C/\mathcal{A})}{[K_O : \mathbf{Q}_p]} \leq \frac{\operatorname{length}_{\mathbf{Z}_p}(C/V)}{[K_O : \mathbf{Q}_p]} \leq 11f(\mu(A+1) + \rho(W+1)).$$

This estimate persists under finite extensions O'/O of coefficient DVRs with the prescribed jump supports. Set $V_O = O \cdot V \subset C$, which is still in the cup annihilator. Choose an O-basis (b_i) of O', with coordinate maps $h_i : O' \to O$. Applying these maps to a cocycle and to its crystalline primitive gives

$$Z^1_f(O' \otimes_O L) = \bigoplus_i b_i Z^1_f(L) = O' \otimes_O C.$$

An allowed partner Y' over O' restricts to an allowed O-lattice: at each coefficient embedding the graded spaces are the direct sum over its extensions, and the class allows arbitrary multiplicities on the prescribed support. For an integral O'-pairing ψ' equivariant modulo p^m, restrict the fixed input to L. Each $h_i \circ \psi'$ is an integral O-pairing equivariant modulo the same p^m. For any $\lambda' : O' \to \mathbf{Z}_p$ linear over $\mathbf{Z}_p$,

$$\lambda'(x) = \sum_i \lambda_i(h_i(x)), \qquad \lambda_i(s) = \lambda'(b_i s).$$

Write $\mathcal{A}_O$ and $\mathcal{A}_{O'}$ for the full annihilators in C and $O' \otimes_O C$. The coordinate identity proves that every $c \in \mathcal{A}_O$ passes the tests against Y'. Absorbing a scalar of O' into λ' gives $O' \otimes_O \mathcal{A}_O \subseteq \mathcal{A}_{O'}$. Conversely, write $c' = \sum_i b_i c_i \in \mathcal{A}_{O'}$. For an allowed O-partner Y, cocycle z, integral pairing ψ equivariant modulo p^m, and functional λ, take $O' \otimes_O Y$, $1 \otimes z$, $1 \otimes \psi$ in the extended class. Each extended embedding has the jumps prescribed at its restriction. Composing the pairing with $\lambda \circ h_i$ gives exactly the original cup against c_i at that depth. It vanishes, so $c_i \in \mathcal{A}_O$ for every i. Thus

$$\mathcal{A}_{O'} = O' \otimes_O \mathcal{A}_O, \qquad (O' \otimes_O C)/\mathcal{A}_{O'} = O' \otimes_O (C/\mathcal{A}_O).$$

In particular $O' \otimes_O V_O$ lies in this annihilator. If $e' = e(O'/O)$, then

$$\operatorname{length}_{O'}(O' \otimes_O (C/V_O)) = e' \operatorname{length}_O(C/V_O), \qquad v_{O'}(p) = e' v_O(p),$$

so this lattice has the same normalized colength. The same identities with $\mathcal{A}_O$ in place of V_O give equality of the normalized defects for the fixed lattice and its scalar extension.

E.2 Exact integrality in two special cases

In this subsection, in two special cases the filtration bounds in Section E.1 let us use the full crystalline cocycle module. We verify the order conditions for the coboundary calculation of Section C.2.1 with lifts over the unscaled constants.

Keep the unramified chart and separated-interval hypotheses of Section E.1. At each index i the two jump sets fit opposite signed intervals of far start β_i. Put $R = 6(A + \eta + W)$. The required arithmetic hypothesis is that no nonzero multiple of p lies within R of 0 or $\pm\beta_i$. The three upper jump bounds $(p-3)A$ give annihilators E^A for both side cokernels and the Tate-dual cokernel. Use the integral lift-ideal estimate of Section D.2. Write P_s for the order function $-u - s\mathbf{1}_{a=0}$, where a is Frobenius level, and put

$$D_{s,h,h_1}(a,u) = -u + (s+h)\mathbf{1}_{a=0} + h_1\mathbf{1}_{a=1},$$
$$O_h(a,u) = \begin{cases} \max(0, -u-h), & a = 0, \\ -u, & a > 0. \end{cases}$$

A point of order P_s has a pole of order at most s at level zero. All points below lie over the given constant in the given index, with no p-multiplier.

If neither side has a jump in $[-A, 0)$, lift using E^A and divide by E^A. The initial orders and depths are P_A. Raise the depth to $D_{A,\eta,0}$. The steps lie at level zero in $[-u-A, A+\eta-u)$. A low jump gives a step within $A+2\eta$ of zero and a far jump gives one within $A+\eta+W$ of its opposite far start. Both are within R. The contribution is zero by the unramified depth calculation. If $u < -A$, the initial order $-u-A > 0$ eliminates its constant term and the new order is at least $A + \eta - u$. Otherwise $u \geq 0$, and the improved order is at least $\min(A + \eta - u, 0) \geq -u$. Thus the new points have order P_0.

At each index let the side with the negative separated interval rest. Its jumps $v \leq \eta$ imply that its just-constructed point has order O_η: for $v < -\eta$ use P_0, and otherwise use the nonnegative depth $A + \eta - v$ to rule out a pole. The moving side has $u \geq -\eta$ at level zero and both its current and preceding points have order P_0. The difference $\varphi x_{i^{p^{-1}}} - x_i$ has orders at least $\min(0, -u)$ at level zero and $-u$ at levels ≥ 1. Adding the resting depth gives at least $A - u - v$ at zero and $-u - v$ at the other levels. These are exactly the order bounds required for the Tate-dual annihilator E^A at the current index. At index i, put $i_- = i^{p^{-1}}$. The path from $(\varphi x_{i_-}, \varphi y_{i_-})$ to (x_i, y_i) passes through $(x_i, \varphi y_{i_-})$ when the first side has negative far sign and through $(\varphi x_{i_-}, y_i)$ when the second side has negative far sign.

Now let the second side have jump set in $[-\eta, 0]$, and put $h = A + \eta$. Its lift ideal contains (1), by the small-jump estimate with maximum jump 0. Give the points on the first side orders P_h and depth $D_{A,h,0}$ by starting at P_A and raising the depth for free. Give the points on the second side orders P_0 and the same depth by starting at P_0. The steps in either case are within $A + h + \eta + W \leq R$ of the three centers. For the first side the improved orders dominate P_h because $h \geq A$: if $-u - A > 0$ the order becomes $A + h - u$, and otherwise the improved order $\min(0, A + h - u)$ is at least $-u - h$. For the second side $u \leq 0$, so the same constant-term rule retains P_0.

Where the first side has negative separated interval, its point has stop orders O_h by the constant-term rule and $u \leq \eta \leq h$. The moving second side has pole-free preceding and current points, with $v \geq -h$. The sum of the moved order and resting depth is at least $A - u - v$ at level zero and $-u - v$ above.

Where the first side has positive separated interval, transport the preceding point of the second side by φ. Its depth is initially 1 at level zero, $A + h - v$ at level one, and $-v$ above. Its orders are 0 at zero and $-v$ above. Raise the depth at zero to $A + h - v$ by free steps; these lie between 1 and $A + h + \eta$. The result has order at least O_h, since $v \geq -\eta \geq -h$. Its depth dominates $D_{A,h,h}$, including the extra h at level one. The moving first side has $u \geq -h$ at level

zero. The difference between its preceding translated point and its present point has orders bounded below by

$$-u-h \quad (a=0,1), \qquad -u \quad (a>1).$$

The resting depth therefore gives $A-u-v$ at zero and $-u-v$ at all other levels. This explains the additional first-level depth required when the moving point has a pole at the preceding index. Let i precede j under Frobenius, write x_i, y_i and x_j, y_j for the chosen first- and second-side points, and write $\widehat{y}_j$ for the level-zero improvement of φy_i. At a positive far index on the first side the moves are

$$(\varphi x_i, \varphi y_i) \longrightarrow (\varphi x_i, \widehat{y}_j) \longrightarrow (x_j, \widehat{y}_j) \longrightarrow (x_j, y_j).$$

In the middle move $\widehat{y}_j$ retains the transported depth $A+h-v$ at level one. At a negative far index on the first side use the two moves through $(x_j, \varphi y_i)$.

In the two cases put $h_0 = \eta$ and $h_0 = h$, respectively. Every index is a cut of shift A and the transport length is one. Write s, t for a moving and a resting jump. For the move from the preceding point to the stop, both moving points have nonnegative order at zero and the transported resting point has depth at least one. At level one the moving order is at least $-s-h_0$, and the transported resting depth at least $A+h_0-t$. Above level one their sum is at least $-s-t$. These are the bounds for the transported third point from the preceding cut. For a move from the stop to the current point, the bounds at zero are $-s-h_0$ and $A+h_0-t$; at positive levels their sum is at least $-s-t$. Use the third point at the current cut for this move.

At the endpoints both depths at zero are at least $A+h_0$ minus the respective jump, and both positive-level depths dominate minus that jump. The pair of endpoint cocycle differences thus passes the test against the current third point, also after translating the second difference by an element of H. The constants on all three sides are unscaled. The four-move calculation gives an actual cup coboundary on the chart. In the off-level application below its group already equals G_{E_v}, so the cochain gives the required coboundary on the original group. This yields zero normalized defect.

E.3 The bound for the original coefficient lattices

In this subsection, we apply the preceding estimates to the coefficient primes away from the original level. The bounds from Sections D.4, E.1 and E.2 yield the required normalized colength on the local class of Section 29.1.

Use the original local class in Section 29.1, with $p = \ell \nmid M$, fixed local dual $L^\dagger$, and $d_v = [E_v : \mathbf{Q}_\ell] = e_v^{\mathrm{ram}} f_v$. Write $\rho = \operatorname{rank}_O L$; write $c_* = 1$ for its main summand and $c_* = t$ for each additional character. Only the actual jumps of $L^\dagger$ are used on the fixed side. The test-side jump set is the full original upper set. The Kummer chart is that of E_v, with working group G_{E_v} itself. Take the common coefficient DVR large enough to contain the residue-field root ζ_v of Section 29.1.1. Fix the lift and its descent pair before varying the upper lattice. Denote the lift number by j_0, and the variable residue-field index by j. Then $\delta_{j_0} = j_0\Delta + h - 2t$ is fixed, while the start β_j can vary along the Frobenius cycle. At a residue parameter i write U_i, V_i for the unions of the partner and fixed-side jump sets over coefficient embeddings with residue index i, with 0 adjoined to both. Put $T_1 = Y$, $T_2 = L^\dagger$, $T_3 = (Y \otimes_O L^\dagger)^\dagger$. Off M, the first two are crystalline over E_v by Section 29.1, and the tensor product and Tate dual give crystallinity of T_3. At depth ν, write c_1, c_2 for the first two cocycles and $c_3(g) = (g\psi - \psi)/p^\nu$. The last is integral by equivariance modulo p^ν. Each has a primitive b_i over B_{cris}. For $i = 1, 2$, apply

$\varphi \otimes 1$ to a $B_{\max}$-primitive. Frobenius commutes with g, fixes $1 \otimes c_i(g)$, and $\varphi(B_{\max}) \subset B_{\mathrm{cris}}$. For $i = 3$, take $b_3 = \psi/p^{\nu}$. Under $g(v, s) = (gv + sc_i(g), s)$ the vector $(-b_i, 1)$ is an invariant lift of 1, so the rational representation of each extension is crystalline. For the extension by the entire cocycle module in Section E.1, choose primitives on an O-basis and extend O-linearly. The parameter quotient contributes only the jump 0. The jump set for the third extension at i lies in $\{0\} \cup \{-u - v - 1 : u \in U_i,\ v \in V_i\}$, since an O-bilinear pairing pairs only equal coefficient embeddings.

Small coefficient primes. Suppose $\ell \le Q_*$. Apply the sparse case of Proposition D.1 with

$$\eta_0 = 2G, \quad B_0 = b + \delta_{j_0}, \quad G_0 = G + \delta_{j_0}, \quad J = \lfloor \log_\ell h^3 \rfloor, \quad D_0 = 2K_\ell, \quad X_0 = K_\ell.$$

The actual far width is W_*. The thin intervals fit inside $[-\eta_0, \eta_0]$, since $G_0 + W_* \le 2G$. The thick-index sparsity for successive gaps at most J is the one proved for these coefficient parameters. Here $Q_* = kt = (r + 1 - m)5r = R_{\mathrm{prog}}$, so the signed-ratio exclusion applies throughout this range of coefficient primes. If $\ell \nmid q_1 d$, the chart is unramified, $\ell > 7$, and $10\ell^J \eta_0 = 20\ell^J G \le 20h^3 G < B_0$. On a segment with opposite thick signs use the pivot at step $J + 1$ of Section D.4.3. If $\ell \mid q_1 d$, the thick signs are constant at the fixed completion and $8e_v^{\mathrm{ram}} < h^3 < \ell^{J+1}$. The thin intervals of either sign lie in $[-\eta_0, \eta_0]$, including at $\ell = d$. On every segment with a common thick sign choose the side with negative far interval there as the resting side, and choose either side if the segment is entirely thin. On a segment of length l, the resting jumps satisfy $v \le \eta_0$ and the moving jumps satisfy $u \ge -\eta_0$ at levels $0 \le a < l$. At $a = l$ the plain stop uses the depth transported from the preceding cut. All sparse cut shifts are zero, so these segments have the plain stop of Section D.4.1. At the dyadic prime, $\lambda_2 \ge v_2((2b)!) \ge b$, by the b even integers in the factorial, so $K_2 = \lfloor 2\lambda_2/Q_* \rfloor \ge 1$ for large x. At a prime dividing q_1, use the splitting of K and put $a_\ell = v_\ell(q_1)$. The roots of order prime to ℓ give an unramified extension, and $\Phi_{\ell^{a_\ell}}(1 + X)$ is Eisenstein. Thus $e_v^{\mathrm{ram}} = \varphi(\ell^{a_\ell})$ and $D_E = 1 + v_\ell(e_v^{\mathrm{ram}}) = a_\ell$. Here $e_v^{\mathrm{ram}} > 1$ also at $\ell = 2$, since $16 \mid q_0$. For the fixed progression modulus q_0, the definition $q_1 = 3L_0\, 2^s q_0$ gives

$$a_2 = s + v_2(q_0), \qquad a_3 = \lfloor \log_3 s \rfloor + 5 + v_3(q_0), \qquad a_\ell = v_\ell(q_0) \quad (\ell \ge 5,\ \ell \mid q_0).$$

At $\ell = d$ the cyclotomic part is unramified and the quadratic part gives $e_v^{\mathrm{ram}} = 2$, $D_E = 1$. In the dyadic depth calculation of Section D.1, $v_2(\theta\Delta_z) = 1 + 1/e_v^{\mathrm{ram}} > 1$. The plain stop applies at all these ramified primes. At $\ell = 2$, let j be a thick position on the periodically extended cycle. Section 29.1.1 excludes another thick position at positive distance less than L_0 in either direction, and $J < L_0$. Therefore j and $j - J - 1$ are cuts. Every position strictly between them has j among its next J positions. The preceding cut is exactly $J + 1$ steps before j, and both shifts are zero. If j is a thin cut, the positions $j, \dots, j + J - 1$ are thin, so $j - 1$ is a cut and the preceding distance is one. This includes every position on a cycle without thick positions. The plain-cut argument in both cases uses the divisor $z\tau$ on the same localized kernel.

The sparse depth inequality follows from

$$B_0 + 5\eta_0 < 2B_0, \quad 2B_0 < h^2 K_\ell, \quad e_v^{\mathrm{ram}} \le h, \quad h^3 < \ell^{J+1}.$$

These give $2e_v^{\mathrm{ram}}(B_0 + 5\eta_0) < \ell^{J+1}(2K_\ell)$. For the other sparse condition, $J + 1 + D_E \le h$ and $\lfloor \log_\ell(2(B_0 + 5\eta_0)) \rfloor < m \le h$. Indeed h^3, e_v^{ram}, and $2(B_0 + 5\eta_0)$ are at most $4b$, and $\log_\ell(4b) \le \log(4b)/\log 2 < m$. Thus $J \le m - 1$, $D_E \le m$ and $J + 1 + D_E \le 2m \le h$. The second sparse condition is

$$20(J + 1 + D_E)\eta_0 + \lfloor \log_\ell(2(B_0 + 5\eta_0)) \rfloor \le 40hG + h \le 41hG < K_\ell.$$

Here $\eta_0 = 2G$, $G \geq 1$, and the last inequality follows from $K_\ell > 2B_0/h^2 \geq 2b/h^2$ and $b \geq 99G(h^3+1)$. The uniform exponent is at most $10^3 K_\ell(J+3) \leq 10^4 K_\ell m$. At $\ell = 2$, the parameters above give $\kappa = 12(J+3)K_2 + 1$. For each allowed partner, pairing, functional and finite depth, Proposition D.1 gives the same annihilating power $2^{51\kappa}$. The chosen points may depend on the test. By bilinearity and the definition in Section 29.1, the cup annihilator therefore contains $2^{51\kappa} Z_f(L^\dagger)$.

Large coefficient primes with large loss. All rational primes ramified in E divide $q_1 d$. Since $q_1/q_0 = 3L_0 \cdot 2^s$ has prime divisors only $2, 3$, and $6 \mid q_0$, this set is contained in the prime support of the fixed integer $q_0 d$. The cutoff $Q_* = kt$ tends to infinity with x. For sufficiently large x, every $\ell > Q_*$ therefore gives an unramified chart and satisfies $\ell \geq 7$. Suppose $\ell > Q_*$ and $\lambda_\ell \geq h^3$. Use the two-start case of Proposition D.1 with $B_0 = b + \delta_{j_0}$, $G_0 = G + \delta_{j_0}$. Here $\lambda_\ell \geq h^3 > Q_*$, so $K_\ell = \lfloor 2\lambda_\ell/Q_* \rfloor \geq \lambda_\ell/Q_*$. Since $\lambda_\ell \geq 1$ and $\ell(\lambda_\ell+1) > 2b$, we have $\ell\lambda_\ell \geq \ell(\lambda_\ell+1)/2 > b$. The required sizes follow from

$$\begin{aligned} K_\ell \ell^2 - b &> (\ell\lambda_\ell/Q_*)(\ell - Q_*) \geq h^3 > \delta_{j_0}, \\ K_\ell \ell &> b/Q_* > 2(G + \delta_{j_0}), \\ K_\ell &\geq h^3/Q_* > W_* + 4\eta. \end{aligned}$$

The argument of the logarithm in the two-start estimate is at most $4b$. Since $2\log(4b) < m$ and $\log \ell > 1/2$, the integer $\lfloor \log_\ell(2(B_0 + W_* + 4\eta)) \rfloor$ is at most $m-1$. With $K_\ell \geq 1$, the resulting uniform exponent is at most $10^4 K_\ell m$.

Large coefficient primes with small loss. For $\ell > Q_*$ and $\lambda_\ell < h^3$, put $A = 2(\lambda_\ell + 1)$. The missing-multiple radius $6(A + \eta + W_*)$ is exactly the one proved above. It is smaller than G, so in particular smaller than both far starts. Also

$$\beta_j + W_* + \eta < 2b < \ell A/2 \leq (\ell - 3)A.$$

The last inequality uses $\ell \geq 7$. For any jumps u, v at the same index, the opposite signs of the separated intervals give the estimate of Section E.1 for the Tate-dual jump,

$$-u - v - 1 \leq \beta_j + W_* + \eta - 1 < (\ell - 3)A.$$

Thus the required upper bound on the jumps holds on all three sides of the cup pairing. For $0 \leq a < t'$, the dual of the Tate character lattice $O(-a)$ is $O(a+1)$, of jump $-a - 1 \in [-\eta, 0]$. Section E.2 therefore applies to its full crystalline cocycle module.

Equation (86) gives zero defect for the full original sum of characters. For the main summand, if $\lambda_\ell < \lfloor h/5 \rfloor$, then $A < h/2$ for the scales. Every negative jump on either side is at most $-h/2$, so the same section gives zero defect by the gap below zero. In particular $K_\ell = 0$ implies $2\lambda_\ell < Q_*$ and $\lambda_\ell < \lfloor h/5 \rfloor$ by the scale inequalities, so both defects vanish in that case.

It remains to treat a main summand with $\lfloor h/5 \rfloor \leq \lambda_\ell < h^3$. Use the common submodule from Section E.1, with at most $\mu = \lfloor \rho m/(kc_*) \rfloor$ low jumps per embedding. Its normalized colength is at most

$$11 f_v\big(\mu(A+1) + \rho(W_* + 1)\big) \;\leq\; 11 d_v\big(6\mu\lambda_\ell + \rho K_\ell\big).$$

Here $A+1 \leq 6\lambda_\ell$ and $W_*+1 \leq K_\ell$. For the second inequality use $\lambda_\ell \geq \lfloor h/5 \rfloor$ and $99(W_*+1)Q_* \leq h$ to see $2\lambda_\ell \geq Q_*(W_*+1)$. The bound comes from the colength of this single submodule, whose cup vanishes unscaled for every partner and every pairing.

On the two branches with a uniform exponent the rank bound for the crystalline cocycle lattice gives normalized defect at most $2d_v\rho\cdot 10^4K_\ell m$. In the remaining branch the bound is $11d_v(6\mu\lambda_\ell+\rho K_\ell)$. Both estimates are bounded by the expression below, since $m\ge 1$.

Proposition E.1 (Off-level crystalline colength)**.** *Fix q_0,d,ϵ and take the arithmetic datum at a sufficiently large scale x as in Section 4. For every $\ell\nmid M$ and every $v\mid\ell$, fix the lift, descent pair, common coefficient ring and local class as above. Let L be a main or additional character lattice in this class, and let $\mu=\lfloor \rho m/(kc_*)\rfloor$. Then the normalized defect of the cup annihilator in $Z_f(L^\dagger)$ satisfies*

$$\frac{\Delta_v(L)}{v_{\mathrm{ord}}(\ell)}\le 10^5 d_v(\rho K_\ell m+\mu\lambda_\ell).$$

For $\ell>Q_*$, the defect vanishes for an additional character if $\lambda_\ell<h^3$, and for a main summand if $\lambda_\ell<\lfloor h/5\rfloor$.

The comparison with $d_v\rho F_\ell/c_*$ in Section 29.1.2 now applies.

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