THE LORENTZIAN CALDERÓN PROBLEM

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Abstract. We prove uniqueness for Calderón's inverse problem on Lorentzian manifolds. More precisely, if two globally hyperbolic Lorentzian metrics agree with the Minkowski metric outside a compact set and have the same exterior measurements for waves, we prove that the metrics are equal up to a diffeomorphism that is the identity in the exterior. A similar result was proved in [ORS26] if one of the metrics was close to Minkowski. More generally, we prove that the Minkowski exterior condition can be replaced by another geometric condition when recovering a metric or a potential. We also give counterexamples showing that some geometric condition is needed. The proof is based on a layer stripping approach, where the local uniqueness step is provided by a localized version of the weighted L^2 estimates for plane waves in [ORS26]. The localization is made possible by a geometric reduction adapted from [KLU18].

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1. Introduction

1.1. **The problem.** Let $\Box_g$ be the wave operator of a Lorentzian metric g. The Lorentzian Calderón problem asks whether measurements of the solutions of $\Box_g u = 0$ on the boundary, or in the exterior, of a region determine g in the region up to a diffeomorphism. For metrics which are independent of time or real-analytic in time the problem is solved by the Boundary Control method [Bel88, BK92, KKL01, Esk07], which uses the unique continuation theorem of Tataru [Tat95]. For metrics depending smoothly on time, unique continuation across timelike hypersurfaces can fail [Ali83]. However, partial results for smooth metrics have recently begun to appear. A version of the Boundary Control method valid under Lorentzian curvature conditions was implemented in [AFO22, AFO25]. Oksanen, Rakesh and Salo proved rigidity when one of the two metrics is the Minkowski metric [ORS26b], uniqueness when one of the two metrics is C^2-close to the Minkowski metric [ORS26, Theorem 1.2], and uniqueness from finitely many families of distorted plane waves when one metric satisfies a directional

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simplicity condition, a global eikonal coordinate condition and a spanning condition [ORS26, Theorem 1.4]. Reconstruction of the conformal type of a Lorentzian metric from the source-to-solution map for a nonlinear wave equation has been considered in [KLU18, Las18]. We point out that nonlinearity of the equation gives a much richer data set for the inverse problem for the nonlinear wave equation than for the corresponding linear wave equation, and thus inverse problems for nonlinear wave equations are surprisingly easier to solve than the corresponding problems for linear equations.

We consider two globally hyperbolic metrics which coincide outside a compact set W of the space-time. The data are the restrictions to $M \setminus W$ of the solutions of the wave equation near W. We prove uniqueness without smallness or simplicity assumptions (Theorem 1.3), derive the corresponding statement for the Cauchy data sets of [ORS26] (Corollary 1.5), and determine a potential on a known background (Theorem 1.6). The proofs are outlined in Section 2.

1.2. **Setting, data and assumptions.** All Lorentzian metrics in this paper have the signature $(-,+,+,\dots,+)$: a vector $v \neq 0$ is timelike, null or spacelike according as $g(v,v) < 0$, $g(v,v) = 0$ or $g(v,v) > 0$, and the Minkowski metric is $g_{\mathrm{Min}} = -dt^2 + |dx|^2$. The wave operator of a Lorentzian metric g is

$$\Box_g u = -|\det g|^{-1/2}\, \partial_j \big(|\det g|^{1/2} g^{jk}\, \partial_k u \big),$$

so that $\Box_{g_{\mathrm{Min}}} = \partial_t^2 - \Delta_x$. Curvature is normalised so that the unit sphere $\mathbb{S}^n$ has scalar curvature $n(n-1)$. A time-oriented Lorentzian manifold is *globally hyperbolic* if it satisfies the strong causality condition and all sets $J^+(p) \cap J^-(q)$ are compact [ONe83, p. 411], [BGP07, Definition 1.3.8]; see Section 4.1 for the notation.

(S) *Setting.* M is a connected smooth manifold without boundary of dimension $1+n$, $n \geq 1$. The set $W \subset M$ is compact and $O = M \setminus W$ is the *exterior*. g and g' are smooth time-oriented Lorentzian metrics on M such that (M,g) and (M,g') are globally hyperbolic, $g = g'$ on O, and the time orientations of g and g' coincide on O.

(D) *Equal exterior data.* $C_g(O) = C_{g'}(O)$, where the *exterior solution set* of a metric g is

$$C_g(O) = \big\{ u|_O : \ u \in C^\infty(M),\ \Box_g u = 0 \text{ in a neighbourhood of } W \big\}.$$

We note that uniqueness in the inverse problem may fail if W is noncompact, see Example 11.1 for a counterexample.

By Proposition 4.1, (D) is equivalent to the equality of the source-to-solution maps $f \mapsto (G_g^{\mathrm{ret}} f)|_O$ and $f \mapsto (G_{g'}^{\mathrm{ret}} f)|_O$, $f \in C_c^\infty(O)$, where G_g^{ret} is the retarded Green operator of $\Box_g$.

The geometric assumption concerns only the exterior, where the metric is known.

Definition 1.1. Let (M, g) be a time-oriented Lorentzian manifold, $W \subset M$ compact and $O = M \setminus W$. A curve $c\colon [0,T) \to M$, $0 < T \leq \infty$, is *future-inextendible* if it is future-directed causal and $c(t)$ has no limit in M as $t \to T$. Let O_+ be the set of all $x \in O$ for which there is a future-inextendible timelike curve $c\colon [0,T) \to O$ with $c(0) = x$, and let

$$S_+ = O \setminus O_+, \qquad W_+ = W \cup S_+ = M \setminus O_+.$$

Thus S_+, the *future-trapped part of the exterior*, consists of the exterior points from which every future-inextendible timelike curve meets W. The sets O_-, S_- and W_- are defined in the same way with past-inextendible past-directed timelike curves.

The sets $O_\pm$, $S_\pm$ depend only on M, W and on the metric and its time orientation on O; in the setting (S) they are the same for g and g'. Inextendibility refers to M and not to O: a timelike curve in O which converges to a point of W is not future-inextendible. With inextendibility understood within O, Lemma 5.5(a) is false (Example 11.8). If $W = \emptyset$, then $O = M$ and $g = g'$.

We consider the following assumptions:

(E) *No future-trapped exterior points:* $S_+ = \emptyset$, that is, every $x \in O$ is the initial point of a future-inextendible timelike curve contained in O; see Figure 1.1,

and the weaker assumption

(H) *One of the two trapped parts of the exterior is bounded:* S_+ is relatively compact in M, or S_- is relatively compact in M.

Observe that the condition (E) implies (H). We have $S_+ \subset J^-(W)$ and $S_- \subset J^+(W)$, and (H) can be checked on a single Cauchy hypersurface (Lemma 4.4). It holds if $M = \mathbb{R}^{1+n}$ and $g = g' = g_{\mathrm{Min}}$ on O (Section 9.1), or more generally if $g = g' = -dt^2 + g_0(t,x)$ on O where $g_0(t,\cdot)$ is a Riemannian metric depending smoothly on t and satisfying an upper bound for large negative times (Lemma 11.10). It fails if W contains a Cauchy hypersurface; see Section 11 for examples. Theorem 1.3 is stated in terms of (H), but its proof reduces to a situation where (E) holds.

Remark 1.2. Condition (E) requires that, for every exterior point $p \in O = M \setminus W$, there exists at least one future-directed timelike curve starting at p which is future-inextendible in the ambient manifold M and does not intersect the unknown part W of the space time. Other future-directed timelike curves from the same point may intersect W; see Figure 1.1. Roughly speaking, an observer starting at a point p in the known part O of the space-time can choose a path that avoids the unknown part W of the space-time.

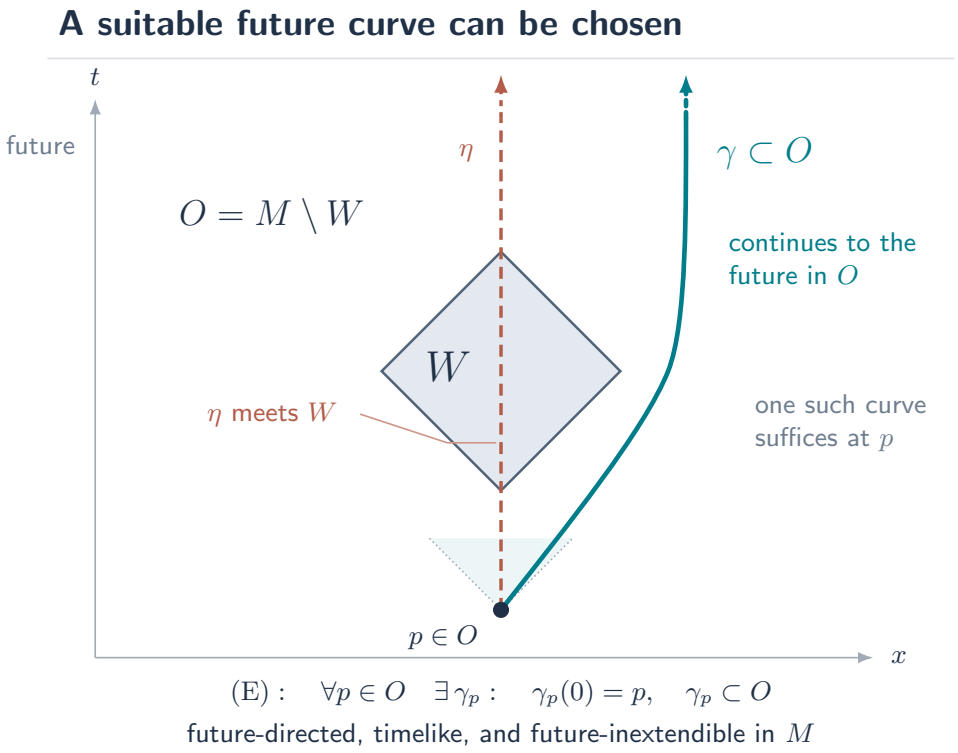

FIGURE 1.1. Illustration of condition (E) in $(M, -dt^2 + dx^2)$ with $M = \mathbb{R}^{1+1}$, with the compact causal diamond

$$W = \{(t,x) : |t| + |x| \leq 1\}.$$

Time increases upwards. From the exterior point p, the solid timelike curve γ remains in $O = M \setminus W$, whereas the dashed timelike curve η intersects W. The upper arrows indicate continuation to $t \to +\infty$, so both curves are future-inextendible in M. Condition (E) requires the existence of a curve of the type γ from every point of O; curves of the type η are also allowed.

1.3. **Main results.**

Theorem 1.3 (Determination of the metric)**.** *Let $n \geq 2$ and assume* (S)*,* (D) *and* (H)*. Then there is a diffeomorphism $\Psi\colon M \to M$ such that*

$$\Psi^* g' = g \qquad \text{and} \qquad \Psi = \mathrm{id} \quad \text{on } O.$$

Moreover, Ψ preserves the time orientation, $\Psi(W) = W$, and $v = v' \circ \Psi$ for all $v, v' \in C^\infty(M)$ with $\Box_g v = 0$, $\Box_{g'} v' = 0$ on M and $v = v'$ on O.

Conversely, if Ψ is a diffeomorphism of M with $\Psi = \mathrm{id}$ on O, then $C_{\Psi^* g'}(O) = C_{g'}(O)$, so the conclusion cannot be improved. For $n = 1$ the wave operator is conformally covariant, which implies that the statement of Theorem 1.3 fails. See Example 11.2 below.

Corollary 1.4 (Minkowski exterior)**.** *Let $n \geq 2$ and let g, g' be smooth globally hyperbolic Lorentzian metrics on $\mathbb{R}^{1+n}$ which are equal to the Minkowski metric $g_{\mathrm{Min}} = -dt^2 + |dx|^2$ outside a compact set W and are time-oriented so that ∂_t is future-directed on $O = \mathbb{R}^{1+n} \setminus W$. If $C_g(O) = C_{g'}(O)$, then $g = \Psi^* g'$ for a diffeomorphism Ψ of $\mathbb{R}^{1+n}$ with $\Psi = \mathrm{id}$ on O.*

Corollary 1.5 (Cauchy data sets; the setting of [ORS26, Theorems 1.1 and 1.2])**.** *Let $n \geq 2$ and let $M_0 \subset \mathbb{R}^{1+n}$ be either the closure of $Q_T = (-T, T) \times B$, where $T > 0$ and $B \subset \mathbb{R}^n$ is the open unit ball, or a compact domain with smooth boundary such that $\mathbb{R}^{1+n} \setminus M_0$ is connected. Let g, g' be smooth globally hyperbolic Lorentzian metrics on $\mathbb{R}^{1+n}$ which are equal to g_{Min} outside M_0. If the Cauchy data sets*

$$C_g^{\mathrm{Hyp}} = \big\{ (u|_{\partial M_0}, \partial_\nu u|_{\partial M_0}) : \ u \in C^\infty(M_0), \ \Box_g u = 0 \text{ in } M_0 \big\}$$

of g and g' coincide, then $g = \Psi^* g'$ for a diffeomorphism Ψ of $\mathbb{R}^{1+n}$ with $\Psi = \mathrm{id}$ outside M_0.

Here ∂_ν is the Euclidean normal derivative on the smooth part of ∂M_0. For $M_0 = \overline{Q_T}$ this is the statement of [ORS26, Theorem 1.2] without the hypothesis $\|g' - g_{\mathrm{Min}}\|_{C^2} \leq \varepsilon$; the other hypotheses of that theorem, namely that g and g' are smooth on $\mathbb{R}^{1+n}$, equal to g_{Min} outside $\overline{Q_T}$ and globally hyperbolic [ORS26, (1.1), (1.2)], are the hypotheses made here. For a domain with smooth boundary it is the statement of [ORS26, Theorem 1.1] with the hypothesis that both metrics are C^2-close to g_{Min} replaced by the hypothesis that their extensions by g_{Min} are globally hyperbolic.

The second main result concerns the operator $\Box_g + q$ on a known background. For $q \in C^\infty(M)$ let

$$C_{g,q}(O) = \big\{ u|_O : \ u \in C^\infty(M), \ (\Box_g + q)u = 0 \text{ in a neighbourhood of } W \big\}.$$

Theorem 1.6 (Determination of a potential)**.** *Let (M, g) be a connected, time-oriented, globally hyperbolic Lorentzian manifold of dimension $1 + n$, $n \geq 1$, let $W \subset M$ be compact, $O = M \setminus W$, and let $q, q' \in C^\infty(M)$ be real-valued with $q = q'$ on O. Assume that*

(N) *for every $p \in W$ the set $J^-(p) \setminus I^-(p)$ meets O, or for every $p \in W$ the set $J^+(p) \setminus I^+(p)$ meets O.*

If $C_{g,q}(O) = C_{g,q'}(O)$, then $q = q'$. Condition (N) *holds if the Cauchy hypersurfaces of (M, g) are not compact, and it holds if* (H) *holds.*

1.4. **Remarks.**

(i) Theorem 1.3 makes no assumption on g and g' in W other than smoothness and global hyperbolicity. The hypotheses (1.3), (1.4), (1.5) of [ORS26, Theorem 1.4] are not needed.

(ii) The proof uses the full data (D): at each point it uses waves of exterior sources which depend on the point and on g'. It does not apply to the formally determined data of [ORS26, Theorem 1.4]. On non-compact domains there are counterexamples, see [OS26] and Example 11.1.

(iii) Hypothesis (H) is used, in the form (E) for the pair (W_+, O_+), in Lemmas 8.2, 8.4, 8.5(b), 5.5(a) and in the proof of Proposition 8.7. We do not know whether Theorem 1.3 holds without (H).

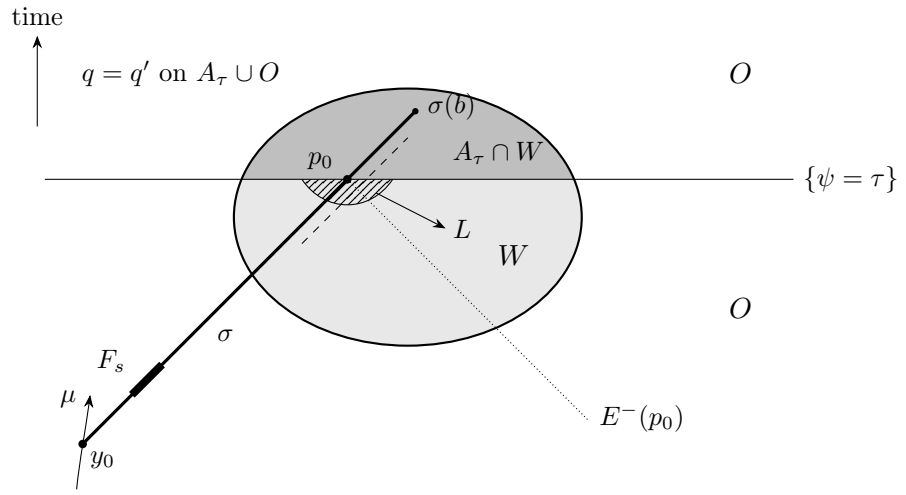

FIGURE 2.1. The setting for the proof of Theorem 1.6. The metric g is the same for both operators, and $q = q'$ on the exterior O of the compact set W. We assume that we have already proved that $q = q'$ on $A_\tau = \{\psi > \tau\}$ (dark part of W). At $p_0 \in \{\psi = \tau\} \cap W$, condition (N), in the form (10.2), gives a point $y_0 \in O$ on the boundary $E^-(p_0)$ of the causal past of p_0 (dotted, and the segment of σ from y_0 to p_0), and a null geodesic σ from y_0 through p_0 which lies on the boundary of the causal future of y_0 up to $\sigma(b)$. Sources F_s supported in O near σ generate waves of $\Box_g + q$ and $\Box_g + q'$ with the same single front (dashed: a neighbouring front). The two waves coincide in A_τ, and $q = q'$ is obtained on the lens L below p_0.

(iv) Condition (N) cannot be dropped in Theorem 1.6: for $M = \mathbb{R} \times S^3$, $W = [-T, T] \times S^3$ and $T > \pi/2$, Example 11.7 gives $q \neq q'$ with $C_{g,q}(O) = C_{g,q'}(O)$. For $T < \pi/2$ condition (N) holds (Example 11.5).

(v) For $n \geq 2$ and W with non-empty interior, the pair (g, q) is not determined by $C_{g,q}(O)$ up to a diffeomorphism since there is a conformal gauge (Remark 11.9).

(vi) Let $n \geq 2$. If only the restrictions to O of the solutions on all of M are given, and (H) holds for the causal hull $J^+(W) \cap J^-(W)$ of W, then the metric is determined up to an isometry which is the identity outside the causal hull (Proposition 9.2).

1.5. **Lean formalization.** We have produced a Lean 4 formalization of the Cauchy data set formulation of Corollary 1.4, which is a representative result for the method. The Lean comparator challenge file has been human curated, and the Lean code is available on GitHub [LOS26] and will be made available on the Palomar registry.

1.6. **Organization.** Section 2 outlines the proofs. Section 3 reviews earlier results. Section 4 contains preliminaries. Section 5 studies first-arrival geodesics. Section 6 constructs one-sided waves. Section 7 contains the weighted energy inequality and the local continuation step. Sections 8, 9 and 10 contain the proofs of Theorem 1.3, of the corollaries and of Theorem 1.6. Section 11 contains examples.

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2. Outline of the proofs

2.1. **Uniqueness for potentials.** The proofs of the main theorems are based on a layer stripping approach, where the layers are level sets of a Cauchy temporal function ψ. To illustrate the main idea, we begin by sketching the proof of Theorem 1.6. There the metric g is fixed, the two potentials $q, q' \in C^\infty(M)$ agree in the exterior $O = M \setminus W$, and the exterior data $C_q(O)$ and $C_{q'}(O)$ are equal.

Consider the statement

$$\mathsf{Q}(\tau)\text{: } q = q' \text{ in } \{\psi > \tau\}.$$

Since $q = q'$ in O, $\mathsf{Q}(\tau_0)$ holds for $\tau_0 > \max_W \psi$. For layer stripping we need to prove:

$$\mathsf{Q}(\tau) \text{ implies } \mathsf{Q}(\tau - \delta) \text{ for some } \delta > 0.$$

Now suppose that $\mathsf{Q}(\tau)$ holds. Fix a point $p_0 \in W \cap \{\psi = \tau\}$ (see Figure 2.1). It is enough to prove that $q = q'$ near p_0. At this point we invoke the geometric condition (N) to find a point $y_0 \in O$, on the boundary of the causal past of p_0, such that there are $(\Box_g + q')$-solutions U_s' generated by sources F_s close to y_0 with U_s' looking like a plane wave locally near p_0. More precisely, condition (N) ensures that there is a future-directed null geodesic σ joining y_0 and p_0 such that, when started at points a bit beyond y_0, there are no conjugate or cut points up to a point that is a bit beyond p_0 (see Figure 2.2). These geometric arguments are adapted from [KLU18, Section 5] and they enable us to avoid the global simplicity condition in [ORS26]; in a sense, we only use simplicity near the fixed null geodesic σ.

Next we employ the plane wave structure of solutions as in [ORS26, FO26], but only locally. The geometric setup implies that near p_0, the solution U_s' corresponding to source F_s near y_0 has the plane wave representation

$$U_s' = u_s' H(\varphi - s)$$

where φ is an eikonal solution for g near p_0 (similar to $t - |x|$ in the Minkowski case), s is a small time delay parameter, H is the Heaviside function, and u_s' is a smooth amplitude in $\{\varphi' \geq s\}$ near p_0.

We now let U_s be the $(\Box_g + q)$-solution for the same source F_s. Since $U_s = U_s'$ in O and $q = q'$ in $\{\psi > \tau\}$, it follows that $U_s = U_s'$ in $\{\psi > \tau\}$. The metric g is fixed, so also U_s has near p_0 the form

$$U_s = u_s H(\varphi - s)$$

with u_s smooth near p_0 and supported in $\{\varphi \geq s\}$. On the interface $\{\varphi = s\}$, the amplitudes u_s and u_s' solve the same transport equation, so $u_s = u_s'$ there. However, $\partial_s(u_s - u_s')$ solves a transport equation involving $q - q'$ on this interface.

At this point we use a localized version of the weighted L^2 estimates from [ORS26] to prove that $q = q'$ near p_0. In effect, we take a small lens $L = \{\tilde{\psi} \geq c\} \cap \{\psi \leq \tau\}$ where $\tilde{\psi}$ is a slightly bent version of ψ (see Figure 2.3). We apply an L^2 estimate for $\Box_g + q$ to the function

$$\partial_s(u_s - u_s')$$

in the small region $L \cap \{\varphi \geq s\}$ near p_0. The L^2 estimate contains exponential weights involving a large parameter σ.

After using the transport equation for $\partial_s(u_s - u_s')$ on $\{\varphi = s\}$, the left hand side of the estimate will have $\sigma^{1/2}$ times a weighted L^2 norm of $q - q'$ on $\{\varphi = s\}$. In turn, using the wave equations for u_s and u_s', the right hand side of the estimate will contain the weighted L^2 norm of $q - q'$ in $L \cap \{\varphi \geq s\}$. After integrating in s and taking σ large enough, the right hand side can be absorbed to the left. This gives $q - q' = 0$ near p_0 and finishes the local uniqueness step.

Repeating the above argument for all points $p_0 \in W \cap \{\psi = \tau\}$, we see that indeed $\mathsf{Q}(\tau)$ implies $\mathsf{Q}(\tau - \delta)$ for some $\delta > 0$. This shows that by layer stripping we indeed obtain $q = q'$ in all of W. This concludes our sketch of the proof of determining a potential when the metric g is fixed.

2.2. **Uniqueness for metrics.** Next we describe how the method in Section 2.1 can be extended to proving that if two metrics g and g' have the same exterior data, and g' satisfies (H), then g and g' are equal up to a diffeomorphism that is identity outside. In fact, after replacing (W, O) by (W_+, O_+), we may suppose that g' satisfies (E).

2.2.1. *The method of* [ORS26]. The proof of [ORS26, Theorem 1.4] compares, for two metrics g and g', solutions of the form

$$U_s = u_s\, H(\varphi - s), \qquad U'_s = u'_s\, H(\varphi' - s), \tag{2.1}$$

where H is the Heaviside function, φ and φ' solve the eikonal equations of g and g', and $s \in \mathbb{R}$. If $\Box_g U_s = 0$, then u_s solves the wave equation in $\{\varphi > s\}$ and a transport equation on $\{\varphi = s\}$. An L^2 estimate with the weight $e^{2\sigma\psi}$, where ψ is a temporal function and σ is large, bounds the difference of the metrics in a common gauge by itself with a small constant. The method requires: (a) solutions of the form (2.1) defined globally, for both metrics and for finitely many directions; (b) $n+1$ eikonal functions which form a global coordinate system; (c) a spanning condition for the differentials of the eikonal functions.

2.2.2. *Induction over the level sets of a temporal function.* Let ψ be a Cauchy temporal function of g, $A_\tau = \{\psi > \tau\}$ and $\Sigma_\tau = \{\psi = \tau\}$. Consider the statement

$\mathsf{P}(\tau)$: there is a smooth map $\Psi_\tau \colon A_\tau \to M$ with $\Psi_\tau^* g' = g$ and $\Psi_\tau = \mathrm{id}$ on $A_\tau \setminus W$.

It holds for $\tau \geq \max_W \psi$ with $\Psi_\tau = \mathrm{id}$. We show that $\mathsf{P}(\tau)$ implies $\mathsf{P}(\tau - \delta)$ for some $\delta > 0$. The weighted estimate is applied on a lens L below Σ_τ near a point $q_0 \in \Sigma_\tau \cap W$ (Figure 2.3, Proposition 7.3). It requires a local gauge and finitely many families of solutions of the form (2.1) near q_0 for g and near $p_0 = \lim \Psi_\tau(q)$ for g', which are related by Ψ_τ above Σ_τ.

2.2.3. *First-arrival waves.* Let $p_0 \in W$ and let $y_0 \in O$ lie on the boundary of the causal past of p_0 with respect to g'. Under (E) such a point exists (Lemma 5.5). There is a null geodesic σ from y_0 to p_0. After y_0 has been moved forward along σ, the geodesic lies on the boundary of the causal future of y_0 up to a point $\sigma(b)$ beyond p_0 (Lemma 5.4, Figure 2.2). In a neighbourhood $\mathcal{T}$ of σ the boundaries of the causal futures of the points $\mu(s)$ of a timelike curve μ through y_0 are the level sets of a smooth eikonal function φ' (Lemma 5.3). A source F_s supported in O on $\{\varphi' = s\}$ generates a retarded wave which has the form (2.1) near p_0 (Lemma 6.3); it vanishes on $\{\varphi' < s\}$ by finite propagation speed. The directions of such geodesics at p_0 form an open subset of the null cone (Lemma 5.6). For $n \geq 2$ this implies (c) (Lemma 5.7).

2.2.4. *Transfer of causal relations.* By uniqueness for the Cauchy problem (Lemma 8.3),

$$G_g^{\mathrm{ret}} F_s = \Psi_\tau^* (G_{g'}^{\mathrm{ret}} F_s) \qquad \text{on } A_\tau.$$

Proposition 7.3 requires the form (2.1) of $G_g^{\mathrm{ret}} F_s$ also below Σ_τ; in particular $G_g^{\mathrm{ret}} F_s$ has to vanish before the front there. This follows from Lemma 8.4: if $y \in O$, $x \in A_\tau$ and $y \leq_g x$, then $y \leq_{g'} \Psi_\tau(x)$ (Lemma 8.6, Step 4). Lemma 8.4 is proved with first-arrival waves of the metric g.

2.2.5. *Gauge, patching, and the trapped part of the exterior.* The local gauge consists of $n+1$ of the local eikonal functions. The local isometries are glued, and are shown to be the identity on O, by means of finitely many solutions v'_i of $\Box_{g'} v' = 0$ whose differentials span the cotangent space at each point of W, and of the solutions v_i of $\Box_g v = 0$ with $v_i = v'_i$ on O (Lemmas 4.3 and 8.3). Under (H) the theorem is first proved for the pair (W_+, O_+), for which (E) holds. Proposition 8.7 shows that the isometry is the identity on S_+.

2.3. **Structure of the proofs and notation.** Figure 2.4 shows the dependencies of the main statements. Table 2.1 lists the notation.

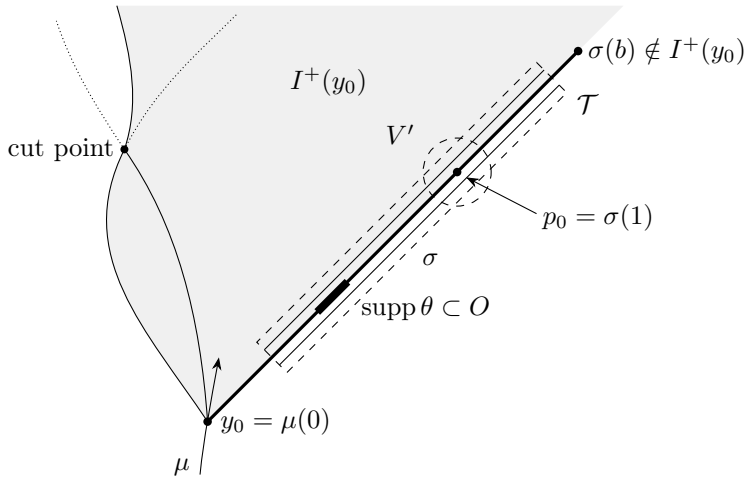

FIGURE 2.2. First arrivals. The null geodesic σ from y_0 lies on the boundary of $J^+(y_0)$ up to $\sigma(b)$, beyond p_0. In the tube $\mathcal{T}$ the sets $\partial J^+(\mu(s))$ are the level sets of a smooth eikonal function (thin lines). The retarded wave of a source supported on such a level set in O has a single front in V' and vanishes before it. Null geodesics from y_0 beyond a cut point (dotted) lie in $I^+(y_0)$.

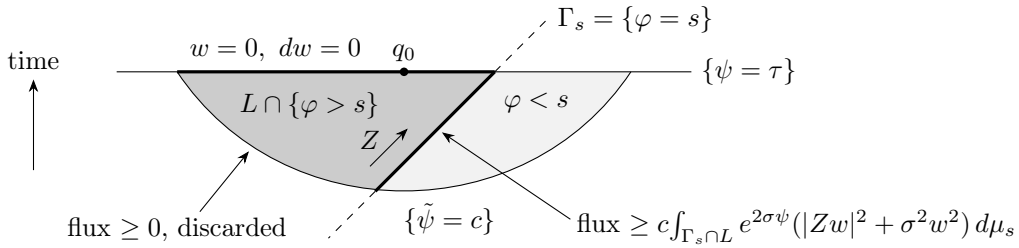

FIGURE 2.3. The weighted energy inequality (Lemma 7.1) on a lens L below a point q_0 of $\{\psi = \tau\}$. The function w has vanishing Cauchy data on the top of L behind the front. The flux through $\{\tilde{\psi} = c\}$ is nonnegative. The flux through the null front Γ_s controls Zw and σw.

3. Earlier results

3.1. **Elliptic and time-independent problems.** The Calderón problem [Cal80] for an isotropic conductivity was solved in dimensions three and higher in [SU87]. For a Riemannian metric, uniqueness up to a diffeomorphism is known in the real-analytic category [LeU89, LU01] and, in a fixed conformal class, for conformally transversally anisotropic manifolds under conditions on the transversal manifold [DKSU09, DKLS16]. For recent results on the smooth anisotropic problem see [Lin26, MTS26, Ope26]. For the wave equation with time-independent coefficients the inverse problem is solved by the Boundary Control method [Bel88, BK92, KKL01], which uses the unique continuation theorem [Tat95]; see [Esk07] for coefficients analytic in time. For coefficients depending smoothly on time, unique continuation across non-pseudoconvex hypersurfaces can fail [Ali83].

3.2. **Time-dependent lower order terms.** For the Minkowski metric, time-dependent potentials were determined from scattering data in [Ste89] and from boundary data in [RaSj91]. On a Lorentzian manifold the Dirichlet-to-Neumann map on a timelike boundary determines the light ray transform of the potential [SY18]. This transform has been inverted in the real-analytic case [Ste17] and in stationary geometries [FIO21, OPS25]; see [KO19, FIKO21] for products of the time axis with a Riemannian manifold. A potential, or a metric in a fixed conformal class, was determined in [AFO22, AFO25] on Lorentzian manifolds satisfying

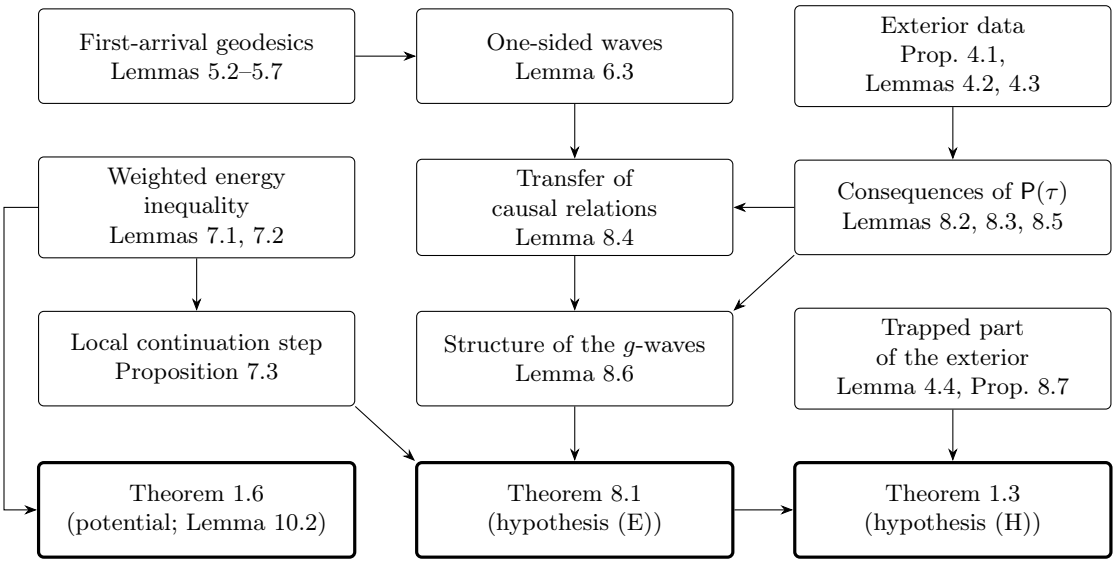

FIGURE 2.4. Dependencies of the main statements. Lemma 6.3 is also used in the proofs of Proposition 8.7 and Theorem 1.6.

Symbol	Meaning	Introduced in
M, W, $O = M \setminus W$	space-time, compact set, exterior	(S)
g, g'; g	the two metrics; a generic globally hyperbolic metric	(S), Section 4.1
$\Box_g$; $P = \Box_{\mathsf{g}} + q$	wave operator; wave operator with a potential	Sections 1.2, 4.1
$C_g(O)$, $C_{g,q}(O)$	exterior solution sets	(D), Section 1.2
G^{ret}, G^{adv}; L^{ret}, L^{adv}	Green operators; source-to-solution maps on O	(F7), Section 4.3
$x \le y$, $x \ll y$, $J^{\pm}$, $I^{\pm}$	causal and chronological relations, futures and pasts	Section 4.1
$E^{\pm}(x) = J^{\pm}(x) \setminus I^{\pm}(x)$	points of first arrival from x (+), at x (−)	Section 4.1
$O_{\pm}$, $S_{\pm}$, $W_{\pm} = W \cup S_{\pm}$	untrapped and trapped parts of the exterior	Definition 1.1
$Z_{\mathsf{g},\varphi} = -\operatorname{grad}_{\mathsf{g}} \varphi$	null field generating the fronts of an eikonal function φ	Section 4.1
Q; φ	local world function; emission-time function	Lemmas 5.2, 5.3
$F_s = \theta\, \delta(\varphi - s)$; $u_s H(\varphi - s)$	exterior sources; one-sided waves	Lemma 6.3
$\bar{g}$, $\bar{\theta}$, κ	discrepancies of the two wave operators in a gauge	(7.5)
ψ, $A_\tau = \{\psi > \tau\}$, $\Sigma_\tau = \{\psi = \tau\}$	Cauchy temporal function of g, known region, frontier	Section 8
$\mathsf{P}(\tau)$, Ψ_τ	induction statement; the isometry on A_τ	Section 8
$\mathsf{T}v'$; $\mathbf{v}'$, $\mathbf{v} = \mathsf{T}\mathbf{v}'$	transfer of solutions; solution coordinates	Lemmas 4.2, 4.3

TABLE 2.1. Notation.

a curvature bound and on perturbations of the Minkowski geometry; see also [GO26]. Formally determined problems were studied in [RS20a, RS20b, KRS21, KRS23] by adapting the method of [BuKl81]; see [Tak23, FO26] for Lorentzian manifolds. In Theorem 1.6 the data are the exterior solution set of a compact set W of the space-time, and the hypotheses on the background are global hyperbolicity and condition (N). For the Minkowski background related results were proved in [RS20b, KRS21].

3.3. **Determination of the metric.** A metric with the same data as the Minkowski metric is isometric to it [ORS26b]. Uniqueness for Cauchy data sets on the boundary of a compact domain holds when both metrics [ORS26, Theorem 1.1], or one of the two metrics [ORS26, Theorem 1.2], are C^2-close to the Minkowski metric, and for finitely many families of distorted

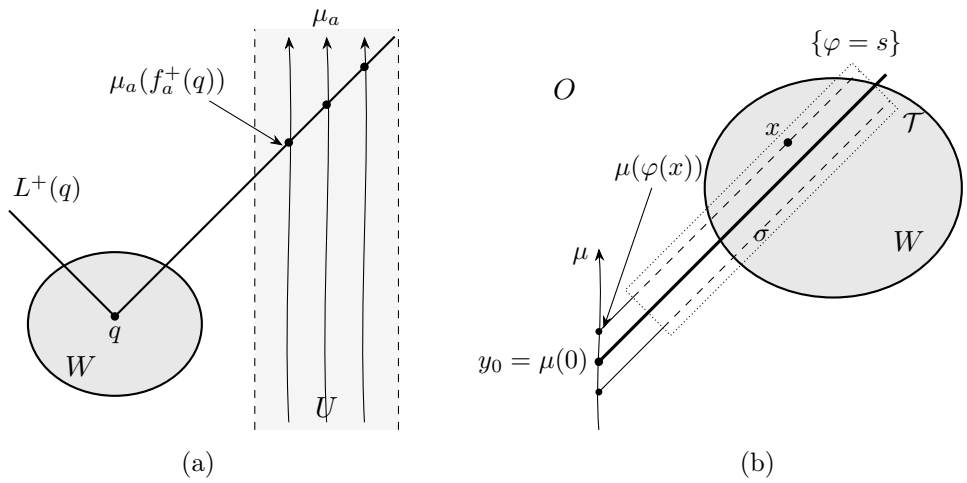

FIGURE 3.1. Observation time functions in one space dimension. (a) The setting of [KLU18]: light from $q \in W$ is observed on timelike paths μ_a in U. (b) This paper: the timelike path $\mu \subset O$ carries the sources, and $\varphi = f_\mu^-$ in the tube $\mathcal{T}$ (Lemma 5.3).

plane waves under directional simplicity, an eikonal coordinate condition and a spanning condition for one of the metrics [ORS26, Theorem 1.4]; see also [ORS26c]. We use the following tools of [ORS26]: the comparison of conormal solutions of the two wave equations, the identities of [ORS26, Lemma 2.7] (Lemma 7.2), and the L^2 estimate with a temporal weight. Proposition 7.3 is a localized form of the argument of [ORS26, Section 2]. There are counterexamples on non-compact domains [OS26] and for partial data [LO22]; they do not apply to a compact set W with data on $M \setminus W$. See also [CJLT26].

3.4. **Other approaches.** For non-linear wave equations the metric is determined up to a conformal factor from light observation sets [KLU18]; see also [FLO21, HU19]. For linear equations, boundary measurements determine the scattering relation of the null geodesic flow [SY18]; the Lorentzian scattering rigidity problem is open in general [Ste24]. These reductions are not used here. The induction of Section 8 is a layer-stripping argument; for layer stripping and invariant imbedding in media with time-independent coefficients see [Syl00, Wes90, Wes92]. Here the layers are bounded by the spacelike hypersurfaces Σ_τ, and the continuation through A_τ is a Cauchy problem (Lemma 8.3).

3.5. **Observation time functions.** Let (M, g) be globally hyperbolic with time separation function τ_{g}, and let $\mu_a \colon [-1,1] \to M$, $a \in \mathcal{A}$, be smooth future-directed timelike paths. The earliest observation time functions of [KLU18, Definition 2.1] are

$$\begin{aligned} f_a^+(x) &= \inf\big(\{s \in (-1,1) : \tau_{\mathsf{g}}(x, \mu_a(s)) > 0\} \cup \{1\}\big), \\ f_a^-(x) &= \sup\big(\{s \in (-1,1) : \tau_{\mathsf{g}}(\mu_a(s), x) > 0\} \cup \{-1\}\big). \end{aligned} \tag{3.1}$$

Let σ, μ, $\mathcal{T}$, η, φ be as in Lemma 5.3. By Lemma 5.2(b) and (5.2), for $x \in \mathcal{T}$ and $|s| < \eta$ we have $\tau_{\mathsf{g}}(\mu(s), x) > 0$ if and only if $s < \varphi(x)$. Hence

$$\varphi(x) = \sup\{s \in (-\eta, \eta) : \tau_{\mathsf{g}}(\mu(s), x) > 0\} = f_\mu^-(x) \qquad (x \in \mathcal{T}), \tag{3.2}$$

where f_μ^- is defined by (3.1) for the path μ with the parameter interval $(-\eta, \eta)$, and $\{\varphi = s\} = E^+(\mu(s)) \cap \mathcal{T}$ (Figure 3.1). Lemmas 5.2 and 5.3 are variants, with reversed time orientation, of [KLU18, Lemma 5.1 and Proposition 5.2]. Step 5 of the proof of Proposition 8.7 follows [KLU18, pp. 801–802]. In [KLU18] the observation time functions of the points of the unknown region are given by the data. Here the sources are placed on first-arrival fronts of g', and the vanishing of the corresponding waves of g before their fronts is proved in Lemma 8.6.

4. Preliminaries

4.1. **Conventions.** All manifolds are smooth, Hausdorff and second countable. Let (M, g) be a time-oriented Lorentzian manifold. Causal curves are piecewise smooth and future-directed unless stated otherwise. We write $x \le y$ (or $x \le_{\mathsf{g}} y$) if $x = y$ or there is a causal curve from x to y, and $x \ll y$ if there is a timelike curve from x to y; $J^+(x) = \{y : x \le y\}$, $I^+(x) = \{y : x \ll y\}$, $J^-(x) = \{y : y \le x\}$, $I^-(x) = \{y : y \ll x\}$, and $J^\pm(A) = \bigcup_{x\in A} J^\pm(x)$. We use the notation

$$E^\pm(x) = J^\pm(x) \setminus I^\pm(x).$$

If $x \ll y \le z$ or $x \le y \ll z$, then $x \ll z$ [ONe83, Corollary 14.1, p. 402]. A function $\psi \in C^\infty(M)$ is *temporal* if $\mathsf{g}(d\psi, d\psi) < 0$ and $-\operatorname{grad}_{\mathsf{g}} \psi$ is future-directed; then ψ is strictly increasing along causal curves, and its level sets are spacelike hypersurfaces. It is *Cauchy temporal* if moreover every level set is a Cauchy hypersurface, that is, is met exactly once by every inextendible timelike curve.

A real function φ with $\mathsf{g}(d\varphi, d\varphi) = 0$ and $d\varphi \neq 0$ is called *eikonal*; it is *eikonal of future type* if the vector field

$$Z_{\mathsf{g},\varphi} := -\operatorname{grad}_{\mathsf{g}} \varphi$$

is future-directed. Then $Z_{\mathsf{g},\varphi}\varphi = 0$, φ is nondecreasing along causal curves, and the integral curves of $Z_{\mathsf{g},\varphi}$ are null geodesics: by the symmetry of the Hessian [ONe83, Lemma 3.49, p. 86], $\mathsf{g}(\nabla_{\operatorname{grad}\varphi} \operatorname{grad}\varphi, Y) = \mathsf{g}(\nabla_Y \operatorname{grad}\varphi, \operatorname{grad}\varphi) = \frac{1}{2} Y\, \mathsf{g}(d\varphi, d\varphi) = 0$ for every vector field Y.

H denotes the Heaviside function. For an eikonal function φ on an open set and a continuous function θ, $\theta\, \delta(\varphi - s)$ denotes the distribution $\chi \mapsto \int_{\{\varphi=s\}} \theta\chi \, d\mu_s$. If M is oriented, μ_s is the smooth measure on the hypersurface $\{\varphi = s\}$ induced by the restriction of the n-form $\iota_N dV_{\mathsf{g}}$, for any vector field N with $N\varphi = 1$ (the restriction does not depend on N); in the nonoriented case μ_s is the smooth positive density given locally, after fixing a local orientation, by $|\iota_N dV_{\mathsf{g}}|$. Then $\int F\, dV_{\mathsf{g}} = \int_{\mathbb{R}} \big(\int_{\{\varphi=s\}} F\, d\mu_s\big) ds$, and $s \mapsto \int_{\{\varphi=s\}} F\, d\mu_s$ is continuous, for continuous F with compact support. Distributions are identified with functions by means of the volume density dV_{g}. For a local diffeomorphism Ψ and a distribution U, $\Psi^* U$ denotes the pull-back; then $\operatorname{supp} \Psi^* U = \Psi^{-1}(\operatorname{supp} U)$, and if $\Psi^* \mathsf{g}' = \mathsf{g}$ then $\Box_{\mathsf{g}} \Psi^* U = \Psi^* \Box_{\mathsf{g}'} U$.

The symbol g always denotes a smooth time-oriented Lorentzian metric on M such that (M, g) is globally hyperbolic; in the proof of Theorem 1.3 the lemmas of Sections 5 and 6 are applied with $\mathsf{g} = g'$ and with $\mathsf{g} = g$. For a real-valued $q \in C^\infty(M)$ we write

$$P = P_{\mathsf{g},q} = \Box_{\mathsf{g}} + q.$$

4.2. **Facts from Lorentzian geometry and from the theory of the Cauchy problem.** Let (M, g) be globally hyperbolic. We recall the following standard facts on Lorentzian geometry:

- (F1) A future-inextendible (past-inextendible) causal curve that starts in a compact set K leaves K and does not return: there is a parameter after which it stays outside K [ONe83, Lemma 14.13, p. 408] (Indeed, the strong causality condition holds on K by global hyperbolicity). A geodesic $\gamma\colon [0, b) \to M$ is extendible as a geodesic if and only if it is continuously extendible [ONe83, Lemma 5.8, p. 130]; hence a causal geodesic defined on its maximal interval is inextendible.
- (F2) If there is a causal curve from p to q which is not a null pregeodesic, then there is a timelike curve from p to q [ONe83, Proposition 10.46, p. 294]. Hence, if $q \in E^+(p)$ and $q \neq p$, every causal curve from p to q is a null pregeodesic, that is, a reparametrization of a null geodesic.
- (F3) If $\sigma\colon [0, b] \to M$ is a null geodesic and $\sigma(r)$, $0 < r < b$, is conjugate to $\sigma(0)$ along σ, then $\sigma(b) \in I^+(\sigma(0))$. This is [ONe83, Proposition 10.48, p. 296], stated there for the focal points of a spacelike submanifold P along a normal null geodesic, in the case $P = \{\sigma(0)\}$, for which focal points are conjugate points [ONe83, p. 283]; in the form used here it is [ONe83, Corollary 14.5, p. 404]. Conjugacy of two points along

a geodesic is a symmetric relation, and the statement holds with the time orientation reversed.

(F4) The relation $\leq$ is closed in $M \times M$ [ONe83, Lemma 14.22, p. 412]; hence $J^{\pm}(K)$ is closed for compact K [BGP07, Lemma A.5.1, p. 173]. The sets $I^{\pm}(p)$ are open and $I^{\pm}(p) = \operatorname{int} J^{\pm}(p)$ [ONe83, Lemma 14.3, p. 403, and Lemma 14.6, p. 404]. The set $J^+(K) \cap J^-(K')$ is compact for compact K, K' [BGP07, Lemma A.5.7, p. 176], and $J^{\pm}(K) \cap \Sigma$ and $J^{\pm}(K) \cap J^{\mp}(\Sigma)$ are compact for compact K and a Cauchy hypersurface Σ [BGP07, Corollary A.5.4, p. 175].

(F5) If $q \in J^+(p)$, $q \neq p$, there is a causal geodesic from p to q whose length is the time separation of p and q [ONe83, Proposition 14.19, p. 411]; it is timelike if $q \in I^+(p)$, since then the time separation is positive [ONe83, Lemma 14.16, p. 410].

(F6) There is a surjective Cauchy temporal function $\psi \colon M \to \mathbb{R}$ [BS05, Theorem 1.1]; see also [BGP07, Theorem 1.3.10, p. 23]. The vector field $N_\psi = \operatorname{grad} \psi / \mathsf{g}(d\psi, d\psi)$ is future-directed timelike with $N_\psi \psi = 1$; its flow is complete and identifies M with $\mathbb{R} \times \Sigma$, where Σ is a level set of ψ, and ψ with the first coordinate. Since M is connected, the level sets are connected. A Cauchy hypersurface is met by every inextendible causal curve [ONe83, Lemma 14.29, p. 415]; hence along a future-inextendible causal curve starting at x, ψ takes every value in $[\psi(x), \infty)$. Any two Cauchy hypersurfaces are homeomorphic [ONe83, Corollary 14.32, p. 417]; in particular they are all compact or all non-compact.

(F7) The results of [BGP07, Chapter 3] are stated for normally hyperbolic operators on vector bundles, that is, for second order operators with principal symbol $-\mathsf{g}^{-1}(\xi, \xi)$; $P = \Box_{\mathsf{g}} + q$ is such an operator on the trivial line bundle.

Cauchy problem [BGP07, Theorems 3.2.11 and 3.2.12, pp. 84–86]. For a spacelike Cauchy hypersurface Σ with future-directed unit normal ν, $f \in C_c^\infty(M)$ and $h_0, h_1 \in C_c^\infty(\Sigma)$ there is a unique $v \in C^\infty(M)$ with $Pv = f$, $v|_\Sigma = h_0$, $\nu v|_\Sigma = h_1$; moreover $\operatorname{supp} v \subset J^+(K) \cup J^-(K)$ with $K = \operatorname{supp} f \cup \operatorname{supp} h_0 \cup \operatorname{supp} h_1$, and the map $(f, h_0, h_1) \mapsto v$ is linear and continuous from $C_c^\infty(M) \oplus C_c^\infty(\Sigma) \oplus C_c^\infty(\Sigma)$ to $C^\infty(M)$. Consequently, if (f, h_0, h_1) depends smoothly on a real parameter s, with supports in a fixed compact set, then $s \mapsto v_s$ is a smooth map into the Fréchet space $C^\infty(M)$, and $(s, x) \mapsto v_s(x)$ is smooth.

Green operators [BGP07, Theorem 3.3.1 and Corollary 3.4.3]; see pp. 87–89 there. There are unique linear operators $G_P^{\mathrm{ret}}, G_P^{\mathrm{adv}} \colon C_c^\infty(M) \to C^\infty(M)$ with $P G_P^{\mathrm{ret}} f = f = G_P^{\mathrm{ret}} P f$ and $\operatorname{supp} G_P^{\mathrm{ret}} f \subset J^+(\operatorname{supp} f)$, and likewise for G_P^{adv} with J^-. (In [BGP07] the operator with support in the causal future is denoted G_+ and called "advanced".) If Σ is a spacelike Cauchy hypersurface with $\Sigma \cap J^+(\operatorname{supp} f) = \emptyset$, then $G_P^{\mathrm{ret}} f$ is the solution of $Pv = f$ with vanishing Cauchy data on Σ; hence $(s, x) \mapsto (G_P^{\mathrm{ret}} f_s)(x)$ is smooth if f_s depends smoothly on s with supports in a fixed compact set. Since P is formally self-adjoint, $\int (G_P^{\mathrm{ret}} f) h \, dV_{\mathsf{g}} = \int f \, (G_P^{\mathrm{adv}} h) \, dV_{\mathsf{g}}$ for $f, h \in C_c^\infty(M)$ [BGP07, Lemma 3.4.4, p. 89]. The operators G_P^{ret}, $G_P^{\mathrm{adv}} \colon C_c^\infty(M) \to C^\infty(M)$ are sequentially continuous [BGP07, Proposition 3.4.8, p. 91]. Hence G_P^{ret} extends to $\mathcal{E}'(M)$ by $\langle G_P^{\mathrm{ret}} \mu, h \rangle = \langle \mu, G_P^{\mathrm{adv}} h \rangle$, $h \in C_c^\infty(M)$, with $P G_P^{\mathrm{ret}} \mu = \mu$, $\operatorname{supp} G_P^{\mathrm{ret}} \mu \subset J^+(\operatorname{supp} \mu)$ and $G_P^{\mathrm{ret}} P U = U$ for $U \in \mathcal{E}'(M)$; likewise for G_P^{adv}. (The extension of the Green operators to distributions with past compact or future compact support, with these properties, and the uniqueness statement below are also stated in [Bär15, Lemma 4.1 and Corollary 4.2], in the numbering of the arXiv version, for the class of Green-hyperbolic operators, which contains the normally hyperbolic ones.) We write $G_{\mathsf{g}}^{\mathrm{ret}} = G_{\Box_{\mathsf{g}}}^{\mathrm{ret}}$ and $G_{\mathsf{g}}^{\mathrm{adv}} = G_{\Box_{\mathsf{g}}}^{\mathrm{adv}}$.

Uniqueness for distributions [BGP07, Theorem 3.1.1, p. 67]. A distribution v on M with $Pv = 0$ whose support is past compact (that is, $\operatorname{supp} v \cap J^-(p)$ is compact for every p) or future compact vanishes. (The hypotheses of that theorem, namely the absence of causal loops, the closedness of $\leq$ and the finiteness and continuity of

the time separation, hold on globally hyperbolic manifolds; see (F4) and [BGP07, Proposition 1.3.15, p. 24].) Consequently, a distribution v with $Pv = 0$ which vanishes on a neighbourhood of a Cauchy hypersurface Σ vanishes identically: the restrictions of v to $I^+(\Sigma)$ and to $I^-(\Sigma)$, extended by zero, are solutions with past compact and with future compact support, respectively.

(F8) *Gauss lemma* [ONe83, Lemma 5.1, p. 127]: $\mathsf{g}(d(\exp_y)_w(w), d(\exp_y)_w(\xi)) = \mathsf{g}(w, \xi)$ for $w \neq 0$ in the domain of $\exp_y$ and $\xi \in T_yM \cong T_w(T_yM)$.

4.3. **Equivalent forms of the data.** We formulate the equivalences for operators with potentials, so that they apply to both main theorems. Assume (S), let $q, q' \in C^\infty(M)$ be real-valued with $q = q'$ on O, and let $P = \Box_g + q$, $P' = \Box_{g'} + q'$. (For Theorem 1.3, $q = q' = 0$; for Theorem 1.6, $g = g'$.) Let

$$C_P(O) = \{u|_O : \ u \in C^\infty(M), \ \operatorname{supp}(Pu) \cap W = \emptyset\},$$

and $L^{\mathrm{ret}}_P f = (G^{\mathrm{ret}}_P f)|_O$, $L^{\mathrm{adv}}_P f = (G^{\mathrm{adv}}_P f)|_O$ for $f \in C^\infty_c(O)$. This implies that $C_{\Box_g}(O) = C_g(O)$ and $C_{\Box_g + q}(O) = C_{g,q}(O)$.

Proposition 4.1 (Equivalent forms of the data)**.** *In this situation the following are equivalent:*

(i) $L^{\mathrm{ret}}_P = L^{\mathrm{ret}}_{P'}$ *on* $C^\infty_c(O)$*;*
(ii) $L^{\mathrm{adv}}_P = L^{\mathrm{adv}}_{P'}$ *on* $C^\infty_c(O)$*;*
(iii) $C_P(O) = C_{P'}(O)$.

If they hold, then $(G^{\mathrm{ret}}_P \mu)|_O = (G^{\mathrm{ret}}_{P'} \mu)|_O$ *and* $(G^{\mathrm{adv}}_P \mu)|_O = (G^{\mathrm{adv}}_{P'} \mu)|_O$ *for every* $\mu \in \mathcal{E}'(O)$.

Proof. (i)⇔(ii): by (F7), $\int_O (L^{\mathrm{ret}}_P f) h \, dV = \int_O f (L^{\mathrm{adv}}_P h) \, dV$ for $f, h \in C^\infty_c(O)$, and the volume densities of g and g' coincide on O. For the last statement let $h \in C^\infty_c(O)$; then $\langle G^{\mathrm{ret}}_P \mu, h\rangle = \langle \mu, G^{\mathrm{adv}}_P h\rangle = \langle \mu, G^{\mathrm{adv}}_{P'} h\rangle = \langle G^{\mathrm{ret}}_{P'} \mu, h\rangle$, because $G^{\mathrm{adv}}_P h = G^{\mathrm{adv}}_{P'} h$ on $O \supset \operatorname{supp} \mu$ by (ii); the advanced case is the same with (i).

(i)⇒(iii). Let $u \in C^\infty(M)$, $Pu = f$, $\operatorname{supp} f \cap W = \emptyset$; then $f = 0$ on an open set $N_1 \supset W$. Choose $\chi \in C^\infty_c(M)$ with $\chi = 1$ on a neighbourhood of W, and shrink N_1 so that $\chi = 1$ on N_1. Then $F := P(\chi u) \in C^\infty_c(M)$ vanishes on N_1, so $F \in C^\infty_c(O)$, and $\chi u = G^{\mathrm{ret}}_P F$. Put $v = G^{\mathrm{ret}}_{P'} F$; by (i), $v = \chi u$ on O. Define $\tilde u = u$ on O and $\tilde u = v$ on N_1; on $O \cap N_1$ both equal u. Then $\tilde u \in C^\infty(M)$, $P'\tilde u = Pu = f$ on O and $P'\tilde u = F = 0 = f$ on N_1. Hence $u|_O = \tilde u|_O \in C_{P'}(O)$. The other inclusion follows by exchanging the roles.

(iii)⇒(i). Let $f \in C^\infty_c(O)$ and $u = G^{\mathrm{ret}}_P f$. Since $\operatorname{supp}(Pu) \cap W = \emptyset$, there is $\tilde u \in C^\infty(M)$ with $\tilde u = u$ on O and $\operatorname{supp}(P'\tilde u) \cap W = \emptyset$; then $P'\tilde u = f$ (on O both sides equal Pu, and near W both vanish). Let ϑ' be a Cauchy temporal function of (M, g'), $K = W \cup \operatorname{supp} f$, $c < \min_K \vartheta'$ and $\Sigma = \{\vartheta' = c\} \subset O$. Then $\Sigma \cap J^+_{g'}(K) = \emptyset$, and also $\Sigma \cap J^+_g(\operatorname{supp} f) = \emptyset$: a g-causal curve from $y \in \operatorname{supp} f$ to $x \in \Sigma$ is g'-causal after its last point w in W (or from y on, if it does not meet W), and $\vartheta'(x) \geq \vartheta'(w) > c$ (respectively $\vartheta'(x) \geq \vartheta'(y) > c$) is a contradiction. As $J^+_{g'}(K)$ and $J^+_g(\operatorname{supp} f)$ are closed, $\tilde u = u$ and $G^{\mathrm{ret}}_{P'} f$ vanish near Σ; the difference $\tilde u - G^{\mathrm{ret}}_{P'} f$ solves the homogeneous equation for P', and by (F7) it vanishes. So $L^{\mathrm{ret}}_{P'} f = \tilde u|_O = L^{\mathrm{ret}}_P f$. □

4.4. **Transfer of solutions and solution coordinates.**

Lemma 4.2 (Transfer of solutions)**.** *Assume* (S) *and* (D). *For every* $v' \in C^\infty(M)$ *with* $\Box_{g'} v' = 0$ *there is a unique* $v \in C^\infty(M)$ *with* $\Box_g v = 0$ *and* $v = v'$ *on* O. *We write this correspondence of* v *and* v' *as* $v = \mathsf{T} v'$.

Proof. Choose $\chi \in C^\infty_c(M)$ with $\chi = 1$ near W. Then $f := \Box_{g'}(\chi v') \in C^\infty_c(O)$ and $G^{\mathrm{ret}}_{g'} f = \chi v'$. Put $v = (1 - \chi)v' + G^{\mathrm{ret}}_g f$. Since $(1 - \chi)v'$ is supported in O, $\Box_g v = \Box_{g'}((1 - \chi)v') + f = \Box_{g'} v' = 0$, and on O, $v = (1 - \chi)v' + L^{\mathrm{ret}}_{g'} f = v'$ by Proposition 4.1.

Next, let us prove uniqueness. If ψ is a Cauchy temporal function of (M, g) and $c > \max_W \psi$, then $\{\psi = c\} \subset O$ is a Cauchy hypersurface, and a solution of $\Box_g v = 0$ vanishing on O vanishes by (F7). □

Lemma 4.3 (Solution coordinates)**.** *Let (M, g) be globally hyperbolic and $K \subset M$ compact. There are $m \in \mathbb{N}$, functions $v_1', \dots, v_m' \in C^\infty(M)$ with $\Box_{\mathsf{g}} v_i' = 0$, and an open set $N' \supset K$ such that $\mathbf{v}' = (v_1', \dots, v_m') \colon N' \to \mathbb{R}^m$ is an immersion.*

Proof. Let $p \in K$, let Σ be the level set through p of a Cauchy temporal function (F6), ν its unit normal, and $\zeta \in T_p^* M$. Choose $h_0, h_1 \in C_c^\infty(\Sigma)$ with $dh_0(p) = \zeta|_{T_p\Sigma}$ and $h_1(p) = \zeta(\nu)$. The solution v' of $\Box_{\mathsf{g}} v' = 0$ with Cauchy data (h_0, h_1) on Σ (F7) has $dv'(p) = \zeta$. Taking ζ in a basis of $T_p^* M$ gives $n+1$ solutions whose differentials are linearly independent at p, hence in a neighbourhood of p. Cover K by finitely many such neighbourhoods and collect the solutions. □

4.5. **The trapped parts of the exterior.** Recall Definition 1.1.

Lemma 4.4 (Properties of O_+)**.** *Let (M, g) be globally hyperbolic, $W \subset M$ compact and $O = M \setminus W$.*

(a) *O_+ is open, and if $c \colon [0, T) \to O$ is a future-inextendible timelike curve, then $c([0,T)) \subset O_+$. Consequently W_+ is closed, and every $x \in O_+$ is the initial point of a future-inextendible timelike curve contained in O_+. Moreover, then hypothesis* (E) *holds for the pair (W_+, O_+) whenever W_+ is compact.*

(b) *$O \setminus J^-(W) \subset O_+$, or equivalently, $S_+ \subset J^-(W)$.*

(c) *Let ψ be a Cauchy temporal function. The set W_+ is compact if and only if $\{\psi \le c\} \subset O_+$ for some $c \in \mathbb{R}$. This holds if, for some $c < \min_W \psi$, the compact set $\{\psi = c\} \cap J^-(W)$ is contained in O_+.*

(d) *S_+ is relatively compact in M if and only if W_+ is compact.*

The corresponding statements hold for O_-, S_-, W_- with the time orientation reversed.

Proof. (a) The second statement holds, since the restriction of c to $[t, T)$ is a future-inextendible timelike curve in O starting at $c(t)$. Let $x \in O_+$ with curve c. Choose $t_0 \in (0, T)$ such that c is smooth on $[0, t_0]$ and $c([0, t_0])$ lies in a coordinate chart contained in O, with coordinates in which $x = 0$; here $c(t_0) \neq x$, since there are no closed causal curves. Let χ be a smooth function with compact support in the chart, $\chi = 1$ near x and $\chi = 0$ near $c(t_0)$. For $v \in \mathbb{R}^{1+n}$ define, in the coordinates, $c_v(t) = c(t) + \chi(c(t))\, v$ for $t \in [0, t_0]$. Then $c_v(0) = v$, $c_v(t_0) = c(t_0)$, and $c_v \to c$ in $C^1([0, t_0])$ as $v \to 0$. Since $c([0, t_0])$ is a compact subset of the open set O and $\dot c$ is future-directed timelike on $[0, t_0]$, the curve c_v is contained in O and is future-directed timelike for all sufficiently small v. Following c_v and then $c|_{[t_0, T)}$ we see that the point with coordinates v belongs to O_+. Hence O_+ is open. The remaining statements of (a) follow.

(b) Let $x \in O \setminus J^-(W)$. The integral curve of N_ψ (F6) starting at x is a future-inextendible timelike curve. It does not meet W, since $x \le w$ for each of its points w. Hence $x \in O_+$.

(c) If W_+ is compact and $c < \min_{W_+} \psi$, then $\{\psi \le c\} \subset O_+$. Conversely, if $\{\psi \le c\} \subset O_+$, then by (b), $W_+ \subset \{\psi \ge c\} \cap J^-(W) = J^+(\Sigma_c) \cap J^-(W)$ with $\Sigma_c = \{\psi = c\}$, which is compact by (F4); as W_+ is closed, it is compact. Now let $c < \min_W \psi$ with $\Sigma_c \cap J^-(W) \subset O_+$. We observe that the set $\Sigma_c \cap J^-(W)$ is compact by (F4). Let $\psi(x) \le c$. The integral curve of N_ψ from x reaches Σ_c at a point x_1, and up to x_1 it lies in $\{\psi \le c\} \subset O$. If $x_1 \in J^-(W)$, then $x_1 \in O_+$ by assumption. Indeed, otherwise we would have $x_1 \in O_+$ by (b). Following this curve up to x_1 and then a future-inextendible timelike curve in O starting at x_1 shows that $x \in O_+$. Thus $\{\psi \le c\} \subset O_+$.

(d) $W_+ = W \cup S_+$ is closed by (a) and W is compact. □

By Lemma 4.4(d), hypothesis (H) says that W_+ is compact or W_- is compact.

5. Geometry of first arrivals

In this section (M, g) is globally hyperbolic and all causal notions refer to the Lorentzian metric g.

Definition 5.1 (First-arrival geodesics)**.** A future-directed null geodesic $\sigma \colon [0, b] \to M$ is a *first-arrival geodesic* if $\sigma(b) \notin I^+(\sigma(0))$.

If σ is a first-arrival geodesic, then $\sigma(t) \in E^+(\sigma(0))$ for all $t \in [0,b]$, since $\sigma(0) \ll \sigma(t) \le \sigma(b)$ implies $\sigma(0) \ll \sigma(b)$.

Let $\mathbf{e}\colon \Omega_{\exp} \to M \times M$, $\mathbf{e}(y,w) = (y, \exp_y w)$, be defined on the (open) domain $\Omega_{\exp} \subset TM$ of the extended exponential map. As we see below, Lemmas 5.2 and 5.3 are variants of [KLU18, Lemma 5.1 and Proposition 5.2] for a Lorentzian metric with the reversed time orientation, see Section 3.5.

Lemma 5.2 (The local world function along a first-arrival geodesic)**.** *Let $\sigma\colon [0,b] \to M$ be a first-arrival geodesic (see Definition 5.1), that is, a future-directed null geodesic with*

$$\sigma(b) \notin I^+(\sigma(0)), \tag{5.1}$$

and put $y_0 = \sigma(0)$, $v_0 = \dot\sigma(0)$. Let $0 < t_1 < t_2 < b$ and $\mathcal{K}_0 = \{(y_0, \sigma(t)) : t \in [t_1,t_2]\}$.

(a) *For $0 < t < b$ the map $\exp_{y_0}$ is nonsingular at tv_0, and tv_0 is the only future-directed causal vector $w \in T_{y_0}M$ with $\exp_{y_0} w = \sigma(t)$.*

(b) *There are an open neighbourhood $\mathcal{U}$ of $\{(y_0, tv_0) : t \in [t_1,t_2]\}$ in TM and an open neighbourhood $\mathcal{U}'$ of $\mathcal{K}_0$ in $M \times M$ such that $\mathbf{e}\colon \mathcal{U} \to \mathcal{U}'$ is a diffeomorphism, every causal vector in $\mathcal{U}$ is future-directed, and the smooth function*

$$Q\colon \mathcal{U}' \to \mathbb{R}, \qquad Q(y,p) = \mathsf{g}_y(w,w) \quad \text{with } (y,w) = (\mathbf{e}|_{\mathcal{U}})^{-1}(y,p),$$

has the following properties for $(y,p) \in \mathcal{U}'$:

$$p \in J^+(y) \iff Q(y,p) \le 0, \qquad p \in I^+(y) \iff Q(y,p) < 0.$$

(c) *Let $c(r) = \exp_y(rw)$, $r \in [0,1]$, be the geodesic from y to p belonging to (y,w). Then*

$$d_pQ(y,p) = 2\mathsf{g}(\dot c(1), \cdot), \qquad d_yQ(y,p) = -2\mathsf{g}(\dot c(0), \cdot).$$

Proof. (a) If $\exp_{y_0}$ were singular at tv_0 for some $0 < t < b$, then $\sigma(t)$ would be conjugate to y_0 along σ and $\sigma(b) \in I^+(y_0)$ by (F3), contradicting (5.1). Let w be a future-directed causal vector with $\exp_{y_0} w = \sigma(t)$. The geodesic $r \mapsto \exp_{y_0}(rw)$, $r \in [0,1]$, followed by $\sigma|_{[t,b]}$ is a causal curve from y_0 to $\sigma(b)$. By (5.1) and (F2) it is a null pregeodesic; so its first part is a null geodesic which arrives at $\sigma(t)$ with a velocity positively proportional to $\dot\sigma(t)$, i.e. $\exp_{y_0}(rw) = \sigma(t - \kappa(1-r))$ for some $\kappa > 0$ (if $t - \kappa < 0$, σ denotes the extended geodesic). Then $\sigma(t-\kappa) = y_0 = \sigma(0)$, and since there are no closed causal curves, $\kappa = t$ and $w = tv_0$.

(b) By (a) and the inverse function theorem, $\mathbf{e}$ is a local diffeomorphism at each point (y_0, tv_0), $t \in [t_1,t_2]$, and it is injective on this compact set. Hence it is injective on a neighbourhood of this set, and we obtain $\mathcal{U}$, $\mathcal{U}'$ with $\mathbf{e}\colon \mathcal{U} \to \mathcal{U}'$ a diffeomorphism. By making $\mathcal{U}$ smaller, if necessary, we may assume that $\mathcal{U}$ does not meet the zero section and that every causal vector in $\mathcal{U}$ is future-directed (the vectors tv_0, $t \ge t_1 > 0$, are future-directed null, and the set of past-directed causal vectors is closed in TM minus the zero section).

Let $(y,p) \in \mathcal{U}'$, $(y,w) = \mathbf{e}^{-1}(y,p)$. If $Q(y,p) \le 0$, then w is causal, hence future-directed, and $p = \exp_y w \in J^+(y)$. We claim that, after shrinking $\mathcal{U}'$, and $\mathcal{U}$ accordingly, the converse holds. Otherwise there are $(y_k,p_k) \in \mathcal{U}'$ with $(y_k,p_k) \to (y_0, \sigma(t_*))$, $t_* \in [t_1,t_2]$, $p_k \in J^+(y_k)$ and $Q(y_k,p_k) > 0$. Since $\sigma(t_*) \ne y_0$, we have $p_k \ne y_k$ for large k, and by (F5) there are future-directed causal vectors $w_k \in T_{y_k}M$ with $\exp_{y_k} w_k = p_k$. Fix a Riemannian metric and write $w_k = \lambda_k \hat w_k$ with $|\hat w_k| = 1$; pass to a subsequence with $\hat w_k \to \hat w$, a future-directed causal unit vector at y_0. The geodesic segments $r \mapsto \exp_{y_k}(r\hat w_k)$, $r \in [0,\lambda_k]$, lie in the compact set $K_* = J^+(\overline{Y}) \cap J^-(\overline{P})$, where $\overline{Y}, \overline{P}$ are compact neighbourhoods of y_0 and $\sigma(t_*)$ (F4). If λ_k were unbounded, then (along a subsequence) for every r in the maximal domain $[0, r_{\max})$ of $r \mapsto \exp_{y_0}(r\hat w)$ we would get $\exp_{y_0}(r\hat w) = \lim \exp_{y_k}(r\hat w_k) \in K_*$, so a future-inextendible causal geodesic would be contained in K_*, contradicting (F1). Hence, for a subsequence, $\lambda_k \to \lambda$. The same argument shows $\lambda < r_{\max}$ (if $\lambda \ge r_{\max}$, every $r < r_{\max}$ is smaller than λ_k for large k). So $\lambda\hat w$ lies in the domain of $\exp_{y_0}$ and $\exp_{y_0}(\lambda\hat w) = \lim p_k = \sigma(t_*) \ne y_0$; thus $\lambda > 0$ and, by (a), $\lambda\hat w = t_*v_0$. So $(y_k,w_k) \to (y_0, t_*v_0)$, $(y_k,w_k) \in \mathcal{U}$ for large k, and by the injectivity of $\mathbf{e}$ on $\mathcal{U}$, $(y_k,w_k) = \mathbf{e}^{-1}(y_k,p_k)$. Then $Q(y_k,p_k) = \mathsf{g}(w_k,w_k) \le 0$, a contradiction. This proves the first equivalence.

To prove the second equivalence, observe that $\{p : Q(y,p) < 0\}$ is an open subset of $J^+(y)$. Hence contained in $\operatorname{int} J^+(y) = I^+(y)$ (F4). Conversely, if $Q(y,p) = 0$, then $d_pQ(y,p) \neq 0$ by (c) (whose proof below uses only (F8) and the fact that $\mathbf{e}\colon \mathcal{U} \to \mathcal{U}'$ is a diffeomorphism), so p is a limit of points p' with $(y,p') \in \mathcal{U}'$, $Q(y,p') > 0$, which are not in $J^+(y)$. Thus, $p \notin \operatorname{int} J^+(y) = I^+(y)$.

(c) For fixed y, $Q(y, \exp_y w) = \mathtt{g}(w,w)$ for $(y,w) \in \mathcal{U}$; differentiating in the direction $\xi \in T_w(T_yM)$ and using (F8), $d_pQ\,(d(\exp_y)_w\xi) = 2\mathtt{g}(w,\xi) = 2\mathtt{g}(d(\exp_y)_w w, d(\exp_y)_w\xi)$, and $d(\exp_y)_w w = \dot c(1)$; since $d(\exp_y)_w$ is invertible for $(y,w) \in \mathcal{U}$, this is the first formula. For the derivative in y, let $\tilde w(y,p) = -\dot c(1) \in T_pM$. The point $\tilde w(y,p)$ depends smoothly on $(y,p) \in \mathcal{U}'$, and $\exp_p \tilde w(y,p) = y$, $\mathtt{g}(\tilde w, \tilde w) = Q(y,p)$. Differentiating $\exp_p \tilde w(y,p) = y$ with respect to y gives $d(\exp_p)_{\tilde w} \circ d_y\tilde w = \mathrm{id}_{T_yM}$. Hence, for $X \in T_yM$, by (F8),

$$d_yQ(X) = 2\mathtt{g}(\tilde w, d_y\tilde w\, X) = 2\mathtt{g}\big(d(\exp_p)_{\tilde w}\tilde w, d(\exp_p)_{\tilde w} d_y\tilde w\, X\big) = 2\mathtt{g}\big(d(\exp_p)_{\tilde w}\tilde w, X\big),$$

and $d(\exp_p)_{\tilde w}\tilde w = -\dot c(0)$. □

Lemma 5.3 (The emission-time function)**.** *In the situation of Lemma 5.2, let $\mu\colon (-\eta_0,\eta_0) \to M$ be a smooth future-directed timelike curve with $\mu(0) = y_0$. There are $\eta \in (0,\eta_0)$, an open neighbourhood $\mathcal{T}$ of $\sigma([t_1,t_2])$ and $\varphi \in C^\infty(\mathcal{T})$ with values in $(-\eta,\eta)$ such that:*

(i) *φ is eikonal of future type on $\mathcal{T}$, $\varphi(\sigma(t)) = 0$ and $Z_{\mathtt{g},\varphi}(\sigma(t)) = \lambda(t)\dot\sigma(t)$ with $\lambda(t) > 0$, for $t \in [t_1,t_2]$;*

(ii) *for all $s \in (-\eta,\eta)$: $J^+(\mu(s)) \cap \mathcal{T} = \{x \in \mathcal{T} : \varphi(x) \geq s\}$;*

(iii) *for every $t \in [t_1,t_2]$ and every neighbourhood $\mathcal{C}$ of $\sigma([0,t])$ there is a neighbourhood $\mathcal{C}_1$ of $\sigma(t)$ such that every $x \in \mathcal{C}_1 \cap \mathcal{T}$ with $\varphi(x) = 0$ is the endpoint of a null geodesic segment which starts at y_0 and is contained in $\mathcal{C}$.*

Proof. Let $G(s,p) = Q(\mu(s),p)$, defined for $(\mu(s),p) \in \mathcal{U}'$. Then $G(0,\sigma(t)) = \mathtt{g}(tv_0,tv_0) = 0$ and, by Lemma 5.2(c), $\partial_sG(0,\sigma(t)) = -2\mathtt{g}(tv_0,\dot\mu(0)) > 0$ for $t \in [t_1,t_2]$, because v_0 is future-directed null and $\dot\mu(0)$ future-directed timelike. By compactness there are $\eta > 0$ and an open neighbourhood $\mathcal{T}$ of $\sigma([t_1,t_2])$ such that $(\mu(s),p) \in \mathcal{U}'$ and $\partial_sG(s,p) > 0$ for $|s| < \eta$, $p \in \mathcal{T}$, and such that for every $p \in \mathcal{T}$ the function $G(\cdot,p)$ has a zero $\varphi(p) \in (-\eta,\eta)$. The zero is unique, φ is smooth by the implicit function theorem, $\varphi(\sigma(t)) = 0$, and

$$G(s,p) \leq 0 \iff s \leq \varphi(p) \qquad (|s| < \eta,\ p \in \mathcal{T}). \tag{5.2}$$

This proves Statement (i).

Statement (ii) follows from (5.2) and Lemma 5.2(b). Differentiating $G(\varphi(p),p) = 0$ and using Lemma 5.2(c),

$$d\varphi(p) = -\frac{d_pQ(\mu(\varphi(p)),p)}{\partial_sG(\varphi(p),p)} = -\frac{2\,\mathtt{g}(\dot c(1),\cdot)}{\partial_sG(\varphi(p),p)},$$

where c is the geodesic from $\mu(\varphi(p))$ to p belonging to the branch $\mathcal{U}$. The curve c is a null geodesic as $Q = 0$, and it is future-directed. Hence $\mathtt{g}(d\varphi,d\varphi) = 0$, $d\varphi \neq 0$, and $-\operatorname{grad}\varphi$ is a positive multiple of $\dot c(1)$: φ is eikonal function of future type, and for $p = \sigma(t)$ the geodesic c is $\sigma|_{[0,t]}$ reparametrized, which gives (i). This proves Statement (ii).

Next, we prove Statement (iii). If $x \in \mathcal{T}$, $\varphi(x) = 0$, then $x = \exp_{y_0} w$ with $(y_0,w) \in \mathcal{U}$ null, and $w \to tv_0$ as $x \to \sigma(t)$ (continuity of $\mathbf{e}^{-1}$); the segments $r \mapsto \exp_{y_0}(rw)$, $r \in [0,1]$, then converge uniformly to $\sigma|_{[0,t]}$. □

Lemma 5.4 (First-arrival geodesics through a given point)**.** *Let $(M,\mathtt{g})$ be globally hyperbolic, $O \subset M$ open, $p_0 \in M \setminus O$ and $y \in O \cap E^-(p_0)$. Then there are $b > 1$, $t_1 \in (0,1)$ and a future-directed null geodesic $\sigma\colon [0,b] \to M$ with*

$$\sigma(1) = p_0, \qquad \sigma(b) \notin I^+(\sigma(0)), \qquad \sigma([0,t_1]) \subset O,$$

and $y \leq \sigma(0)$. More precisely, there is a null geodesic $\tilde\sigma\colon [0,1] \to M$ from y to p_0, and for every $t_a \in (0,1)$ with $\tilde\sigma([0,t_a]) \subset O$ the geodesic σ can be chosen as a reparametrization of an extension of $\tilde\sigma|_{[t_a,1]}$, with $\sigma(0) = \tilde\sigma(t_a)$.

Proof. Step 1. We have $y \neq p_0$, since $y \in O$. By (F5) and (F2) there is a null geodesic $\tilde\sigma\colon [0,1] \to M$ from y to p_0. Let $t_a \in (0,1)$ with $\tilde\sigma([0,t_a]) \subset O$, choose $t_b \in (t_a,1)$ with $\tilde\sigma([0,t_b]) \subset O$, and put $y_0 = \tilde\sigma(t_a)$. We extend $\tilde\sigma$ as a geodesic to an interval $[0, 1+\varepsilon_0)$.

Step 2. We claim that $\tilde\sigma(1+\varepsilon) \notin I^+(y_0)$ for some $\varepsilon \in (0,\varepsilon_0)$. To prove this using a contrapositive argument, suppose that no such ε exists. Then there are $\varepsilon_k \downarrow 0$ and, by (F5), future-directed timelike vectors $w_k \in T_{y_0}M$ with $\exp_{y_0} w_k = \tilde\sigma(1+\varepsilon_k)$. As in the proof of Lemma 5.2(b) (the geodesic segments $r \mapsto \exp_{y_0}(rw_k)$, $r \in [0,1]$, lie in the compact set $J^+(y_0) \cap J^-(\tilde\sigma(1+\varepsilon_1))$), a subsequence converges, $w_k \to w$, with w in the domain of $\exp_{y_0}$, future-directed causal, and $\exp_{y_0} w = p_0$. The curve $\tilde\sigma|_{[0,t_a]}$ followed by $r \mapsto \exp_{y_0}(rw)$ is a causal curve from y to $p_0 \notin I^+(y)$, so by (F2) it is a null pregeodesic, and as in the proof of Lemma 5.2(a) this gives $w = w_0 := (1-t_a)\dot{\tilde\sigma}(t_a)$. Now w_k and $\hat w_k := (1+\varepsilon_k - t_a)\dot{\tilde\sigma}(t_a)$ are different vectors (one timelike, one null), both converge to w_0, and $\exp_{y_0} w_k = \exp_{y_0} \hat w_k$. Hence $\exp_{y_0}$ is not injective in any neighbourhood of w_0, so it is singular at w_0: p_0 is conjugate to y_0 along $\tilde\sigma$. Then $y_0 = \tilde\sigma(t_a)$ is conjugate to p_0 along the reversed geodesic from p_0 to $y = \tilde\sigma(0)$, and lies strictly before its endpoint; by (F3) with reversed time orientation, $y \in I^-(p_0)$. This is in contradiction with our assumption and thus a claimed ε exists.

Step 3. With ε from Step 2 put $\sigma(t) = \tilde\sigma(t_a + t(1-t_a))$, $b = 1 + \varepsilon/(1-t_a)$, and $t_1 = (t_b - t_a)/(1-t_a)$. Then $\sigma(0) = y_0 \geq y$, $\sigma(1) = p_0$, $\sigma(b) = \tilde\sigma(1+\varepsilon) \notin I^+(y_0)$ and $\sigma([0,t_1]) = \tilde\sigma([t_a,t_b]) \subset O$. □

Lemma 5.5 (The boundary of the causal past reaches the exterior)**.** *Let (M,g) be globally hyperbolic, $W \subset M$ compact, $O = M \setminus W$ and $p_0 \in W$. Then $O \cap E^-(p_0) \neq \emptyset$ in each of the following cases:*

(a) *every $x \in O$ is the initial point of a future-inextendible timelike curve contained in O (see hypothesis* (E)*);*
(b) *the Cauchy hypersurfaces of (M,g) are not compact.*

Proof. (a) Suppose $O \cap E^-(p_0) = \emptyset$. Then O is the union of the disjoint open sets $O \cap I^-(p_0)$ and $O \setminus J^-(p_0)$, see (F4). If $y \in O \cap I^-(p_0)$, the curve c given by (E) starts at y, is connected and contained in O, hence contained in $O \cap I^-(p_0)$ (it cannot jump to $O \setminus J^-(p_0)$ by connectedness). As the curve c is future-directed, it is also contained in $J^+(y)$. Thus c lies in the set $J^+(y) \cap J^-(p_0)$, which is compact by (F4). This contradicts (F1). So $O \cap I^-(p_0) = \emptyset$, i.e. $I^-(p_0) \subset W$. But a past-inextendible timelike curve starting at p_0 lies in $I^-(p_0) \cup \{p_0\} \subset W$ (such curves exist, e.g. take a maximal past-directed integral curve of the gradient of a Cauchy temporal function), again contradicting (F1).

(b) Let ψ be a Cauchy temporal function (F6), $c < \min_W \psi$ and $\Sigma_c = \{\psi = c\} \subset O$. The set Σ_c is connected and not compact. The set $A = \Sigma_c \cap J^-(p_0)$ is compact by (F4). Hence $A \neq \Sigma_c$, and A is closed in Σ_c. It contains the set $\Sigma_c \cap I^-(p_0)$, which is open in Σ_c and nonempty, since a past-inextendible timelike curve starting at p_0 meets Σ_c by (F6). As Σ_c is connected, A is not open in Σ_c. Let $y \in A$ be a point which is not an interior point of A relative to Σ_c. Then $y \notin I^-(p_0)$, so $y \in E^-(p_0) \cap O$. □

Lemma 5.6 (The admissible directions form an open set)**.** *Let (M,g) be globally hyperbolic, $O \subset M$ open and $p_0 \in M \setminus O$. Let $\mathcal{V}(p_0)$ be the set of future-directed null vectors $\zeta \in T_{p_0}M$ such that the geodesic $\sigma_\zeta(t) = \exp_{p_0}((t-1)\zeta)$ is defined on an interval $[0,b]$, $b > 1$, and satisfies $\sigma_\zeta(b) \notin I^+(\sigma_\zeta(0))$ and $\sigma_\zeta([0,t_1]) \subset O$ for some $t_1 \in (0,1)$. Then $\mathcal{V}(p_0)$ is an open subset of the future null cone of $T_{p_0}M$, and it is nonempty if $O \cap E^-(p_0) \neq \emptyset$.*

Proof. If $O \cap E^-(p_0) \neq \emptyset$, then $\dot\sigma(1) \in \mathcal{V}(p_0)$ for the geodesic σ of Lemma 5.4. Let $\zeta_0 \in \mathcal{V}(p_0)$, $\sigma = \sigma_{\zeta_0}$ with b, t_1 as in the definition, fix $t_2 \in (1,b)$ and let $\mathcal{U}, \mathcal{U}', Q$ be as in Lemma 5.2 for σ and $[t_1,t_2]$. Consider null vectors satisfying $\zeta \to \zeta_0$. Then, the geodesics σ_ζ are defined on $[0,t_2]$ and converge to σ in $C^1([0,t_2])$. Hence, for ζ close to ζ_0 we have $\sigma_\zeta([0,t_1]) \subset O$. With $y_\zeta = \sigma_\zeta(0)$ and $w_\zeta = \dot\sigma_\zeta(0)$ we have $(y_\zeta, tw_\zeta) \in \mathcal{U}$ for $t \in [t_1,t_2]$, so $(y_\zeta, \sigma_\zeta(t_2)) = \mathbf{e}(y_\zeta, t_2 w_\zeta) \in \mathcal{U}'$ and $Q(y_\zeta, \sigma_\zeta(t_2)) = \mathsf{g}(t_2 w_\zeta, t_2 w_\zeta) = 0$. By Lemma 5.2(b), $\sigma_\zeta(t_2) \notin I^+(\sigma_\zeta(0))$. Thus $\zeta \in \mathcal{V}(p_0)$ (with $b = t_2$). □

Lemma 5.7 (Null covectors in an open set span)**.** *Let g_p be a Lorentzian scalar product on a real vector space T of dimension $1+n$, $n \geq 2$, and let $\mathcal{C}$ be a nonempty open subset of the null cone $\{\xi \in T^* \setminus 0 : \mathrm{g}_p(\xi,\xi) = 0\}$. Then $\mathcal{C}$ contains a basis of T^*, and $\{\mathrm{g}_p\} \cup \{\xi \otimes \xi : \xi \in \mathcal{C}\}$ spans the space S^2T^* of symmetric bilinear forms on T. In particular there is a finite subset of $\mathcal{C}$ with both properties.*

Proof. Choose an orthonormal basis, so that null covectors are $\lambda(1,\omega)$, $\lambda \neq 0$, $\omega \in \mathbb{S}^{n-1}$. Let h be a symmetric contravariant 2-tensor with $h(\xi,\xi) = 0$ for all $\xi \in \mathcal{C}$ and $h^{ab}(\mathrm{g}_p)_{ab} = 0$. The function $\omega \mapsto h^{00} + 2h^{0j}\omega_j + h^{jk}\omega_j\omega_k$ is real-analytic on the connected manifold $\mathbb{S}^{n-1}$ and vanishes on a nonempty open subset, hence everywhere. Its odd part gives $h^{0j} = 0$, and then $h^{jk}\omega_j\omega_k = -h^{00}$ on $\mathbb{S}^{n-1}$ gives $h^{jk} = -h^{00}\delta^{jk}$. So $h = -h^{00}\mathrm{g}_p^{-1}$, and $h^{ab}(\mathrm{g}_p)_{ab} = -(n+1)h^{00} = 0$ gives $h = 0$. Hence the annihilator of the set is zero and the set spans the set T^*. Similarly, if $X \in T$ satisfies $\xi(X) = 0$ for all $\xi \in \mathcal{C}$, then $X^0 + \omega \cdot X' = 0$ on an open subset of S^{n-1}, so $X = 0$. □

6. Waves with a single front generated by exterior sources

In this section (M, g) is globally hyperbolic, $q \in C^\infty(M)$ is real-valued and $P = \Box_\mathrm{g} + q$. Distributions of the form $u_s\, H(\varphi - s)$, with φ eikonal and u_s smooth, are called *one-sided waves*.

6.1. **Calculus of conormal waves.** The two lemmas of this subsection are local: they hold for an arbitrary smooth Lorentzian metric g and an arbitrary real-valued smooth function q on the open set under consideration, and global hyperbolicity is not used.

Lemma 6.1 (Calculus of progressing waves)**.** *Let φ be an eikonal function for g on an open set $\mathcal{T}$ and $Z = Z_{\mathrm{g},\varphi}$.*

(a) *For $u \in C^\infty(\mathcal{T})$ and $s \in \mathbb{R}$, in the sense of distributions on $\mathcal{T}$,*

$$P\big(u\, H(\varphi - s)\big) = H(\varphi - s)\, Pu + \big(2Zu + u\,\Box_\mathrm{g}\varphi\big)\,\delta(\varphi - s). \tag{6.1}$$

(b) *Let $B \in C^\infty(\mathcal{T} \times \mathbb{R})$, $B = B(x,r)$, and $u_s(x) = B(x, \varphi(x) - s)$. Then*

$$P\big(u_s\, H(\varphi - s)\big) = \big(2ZB(\cdot,0) + B(\cdot,0)\,\Box_\mathrm{g}\varphi\big)\,\delta(\varphi - s) + R(\cdot, \varphi - s)\, H(\varphi - s),$$

where $R(x,r) = P_x B(x,r) + 2Z_x \partial_r B(x,r) + (\Box_\mathrm{g}\varphi)(x)\,\partial_r B(x,r)$.

Proof. For smooth u and a smooth function h of one variable,

$$\Box_\mathrm{g}(u\, h(\varphi)) = h(\varphi)\Box_\mathrm{g} u + h'(\varphi)(2Zu + u\,\Box_\mathrm{g}\varphi) - h''(\varphi)\, u\, \mathrm{g}(d\varphi, d\varphi), \tag{6.2}$$

and the last term vanishes. (a) Apply (6.2) to smooth nondecreasing functions h_ϵ with $h_\epsilon \to H(\cdot - s)$ pointwise and boundedly on $\mathbb{R}\setminus\{s\}$ and $h_\epsilon' \to \delta_s$. Then $h_\epsilon(\varphi) \to H(\varphi - s)$ in L^1_{loc}, and $h_\epsilon'(\varphi) \to \delta(\varphi - s)$ in $\mathcal{D}'(\mathcal{T})$ by the continuity of $r \mapsto \int_{\{\varphi = r\}} F\, d\mu_r$; adding $q\, u\, h_\epsilon(\varphi)$ and passing to the limit gives (6.1). (b) Since $Z\varphi = 0$, $Zu_s = (Z_x B)(x, \varphi - s)$, which equals $ZB(\cdot,0)$ on $\{\varphi = s\}$. By the chain rule,

$$\Box_\mathrm{g}\big[B(x,\varphi(x) - s)\big] = \big(\Box_{\mathrm{g},x} B + 2Z_x\partial_r B + (\Box_\mathrm{g}\varphi)\,\partial_r B - \mathrm{g}(d\varphi, d\varphi)\,\partial_r^2 B\big)(x, \varphi(x) - s),$$

and the last term vanishes; hence $Pu_s = R(\cdot, \varphi - s)$. Insert this in (6.1). □

Lemma 6.2 (Consequences of the conormal form)**.** *Let $V \subset M$ be open, $\varphi \in C^\infty(V)$ eikonal for g, $\mathcal{J}$ an open interval with $\varphi(V) \subset \mathcal{J}$, $u \in C^\infty(\mathcal{J} \times V)$, $u_s = u(s,\cdot)$, and assume $P(u_s H(\varphi - s)) = 0$ in $\mathcal{D}'(V)$ for all $s \in \mathcal{J}$. Let $Z = Z_{\mathrm{g},\varphi}$. Then:*

(a) *$Pu_s = 0$ on $\{\varphi \geq s\}$ and $2Zu_s + (\Box_\mathrm{g}\varphi)u_s = 0$ on $\{\varphi = s\}$;*

(b) *there are $a, b \in C^\infty(\mathcal{J} \times V)$ with $Pu_s = (\varphi - s)\, a_s$, $a_s = 0$ on $\{\varphi > s\}$, and $2Zu_s + (\Box_\mathrm{g}\varphi)u_s = (\varphi - s)\, b_s$.*

Proof. By (6.1), $0 = H(\varphi - s)Pu_s + (2Zu_s + u_s\Box_{\mathsf{g}}\varphi)\,\delta(\varphi - s)$. The first term is a locally bounded function and the second is a smooth density on the hypersurface $\{\varphi = s\}$, so both vanish: this is (a) on $\{\varphi > s\}$ and on $\{\varphi = s\}$, and on $\{\varphi \geq s\}$ by continuity. For (b), the map $(s, z) \mapsto (y, z)$ with $y = \varphi(z) - s$ is a diffeomorphism of $\mathcal{J} \times V$ onto an open subset of $\mathbb{R} \times V$ whose z-sections are intervals containing 0. The smooth function $F = Pu_s$ vanishes for $y \geq 0$, so $F = y\,\tilde{a}$ with $\tilde{a}(y, z) = \int_0^1 \partial_y F(ty, z)\,dt$ smooth and $\tilde{a} = 0$ for $y > 0$. The function $2Zu_s + (\Box_{\mathsf{g}}\varphi)u_s$ vanishes for $y = 0$ and is treated in the same way. □

6.2. **One-sided retarded waves.**

Lemma 6.3 (One-sided retarded waves)**.** *Let $\sigma\colon [0, b] \to M$ be a future-directed null geodesic with $\sigma(b) \notin I^+(\sigma(0))$, and let μ, $t_1 < t_2$, η, $\mathcal{T}$, φ be as in Lemmas 5.2 and 5.3, with $t_1 < 1 < t_2$. Put $p_0 = \sigma(1)$, let $t_* \in (t_1, 1)$ and let $\mathcal{N}_*$ be an open neighbourhood of $\sigma(t_*)$. Then there are $\theta \in C_c^\infty(\mathcal{N}_* \cap \mathcal{T})$, $\theta \geq 0$, an open neighbourhood $V' \subset \mathcal{T}$ of p_0 and a function $b_0 \in C^\infty(V')$, $b_0 > 0$, all three independent of q, and a function $u \in C^\infty((-\eta, \eta) \times V')$, $u_s = u(s, \cdot)$, such that the distributions*

$$F_s = \theta\,\delta(\varphi - s) \in \mathcal{E}'(\mathcal{N}_*), \qquad |s| < \eta,$$

have the following properties:

(i) $\operatorname{supp} F_s \subset \operatorname{supp}\theta \cap \{\varphi = s\} \subset J^+(\mu(s))$;
(ii) $G_P^{\mathrm{ret}} F_s = u_s\,H(\varphi - s)$ *on* V';
(iii) $u_s = b_0 > 0$ *on* $\{\varphi = s\} \cap V'$;
(iv) $V' \cap \operatorname{supp}\theta = \emptyset$, *and* $\operatorname{supp}\theta$ *can be chosen inside any prescribed neighbourhood of* $\sigma(t_*)$.

Proof. Flow box. Let ϱ be a Cauchy temporal function (F6) and $Z = Z_{\mathsf{g},\varphi}$ on $\mathcal{T}$; then $Z\varrho > 0$, and $\hat{Z} = Z/(Z\varrho)$ satisfies $\hat{Z}\varrho = 1$. By Lemma 5.3(i), $\sigma|_{[t_1,t_2]}$ is a reparametrized integral curve of $\hat{Z}$. Choose $t_1 < t_{\mathrm{in}} < t_* < 1 < t_{\mathrm{out}} < t_{\mathrm{end}} < t_2$ and write $\varrho_{\mathrm{in}} = \varrho(\sigma(t_{\mathrm{in}}))$, and similarly ϱ_*, $\varrho_0 = \varrho(p_0)$, ϱ_{out}, ϱ_{end}; these numbers are increasing in this order. Let $r_{\mathrm{end}} = \varrho_{\mathrm{end}} - \varrho_{\mathrm{in}}$ and let S be a small open neighbourhood of $\sigma(t_{\mathrm{in}})$ in the hypersurface $\{\varrho = \varrho_{\mathrm{in}}\} \cap \mathcal{T}$. If S is small enough, the flow $\beta(\bar{x}, r) = \mathrm{Fl}_r^{\hat{Z}}(\bar{x})$ is defined on $S \times [0, r_{\mathrm{end}}]$ with values in $\mathcal{T}$; it is an injective immersion because $\varrho(\beta(\bar{x}, r)) = \varrho_{\mathrm{in}} + r$. Let $\mathcal{B} = \beta(S \times (0, r_{\mathrm{end}}))$, an open subset of $\mathcal{T}$.

Source and amplitudes. Choose S' open with compact closure in S, $\sigma(t_{\mathrm{in}}) \in S'$, numbers $0 < a_1 < \varrho_* - \varrho_{\mathrm{in}} < a_2 < \varrho_0 - \varrho_{\mathrm{in}}$, and $\theta \in C_c^\infty(\beta(S' \times (a_1, a_2)) \cap \mathcal{N}_*)$ with $\theta \geq 0$, $\theta(\sigma(t_*)) > 0$. Define $b_k \in C^\infty(\mathcal{B})$, $k \geq 0$, as the solutions of the linear ordinary differential equations along the curves $r \mapsto \beta(\bar{x}, r)$

$$2Zb_0 + (\Box_{\mathsf{g}}\varphi)\,b_0 = \theta, \qquad 2Zb_{k+1} + (\Box_{\mathsf{g}}\varphi)\,b_{k+1} = -Pb_k, \qquad b_k \to 0 \text{ as } r \to 0. \tag{6.3}$$

Let $K_\theta \subset S'$ be the compact set of all $\bar{x}$ with $\beta(\{\bar{x}\} \times (a_1, a_2)) \cap \operatorname{supp}\theta \neq \emptyset$. All b_k vanish outside $\mathcal{B}_\theta := \beta(K_\theta \times [a_1, r_{\mathrm{end}}])$ (induction on k: the complement of $\mathcal{B}_\theta$ in $\mathcal{B}$ is open and invariant under the backward flow). The equations extend to $r = r_{\mathrm{end}}$, so that the b_k are restrictions of smooth functions with support in the compact set $\beta(K_\theta \times [a_1, r_{\mathrm{end}}])$; in particular all their derivatives are bounded. Moreover $b_0 \geq 0$, $b_0(p_0) > 0$ (the curve through $\sigma(t_{\mathrm{in}})$ passes through $\sigma(t_*)$, where $\theta > 0$, before p_0), and b_0 does not depend on q. Let $\chi_0 \in C_c^\infty(\mathbb{R})$ with $\chi_0 = 1$ near 0. By Borel's lemma with parameters there are $\epsilon_k \in (0, 1]$ such that

$$B(x, r) = \sum_{k \geq 0} b_k(x)\,\frac{r^k}{k!}\,\chi_0(r/\epsilon_k)$$

converges in $C^\infty(\mathcal{B} \times \mathbb{R})$ (choose ϵ_k so small that the k-th term has C^{k-1} norm at most 2^{-k}; cf. [Rau12, Borel's Theorem 4.3.1, p. 128]). Then $\partial_r^k B(x, 0) = b_k(x)$ for all k, and $B(x, r) = 0$ for $x \notin \mathcal{B}_\theta$. By Lemma 6.1(b), with $\tilde{u}_s = B(\cdot, \varphi - s)$,

$$P\big(\tilde{u}_s H(\varphi - s)\big) = \theta\,\delta(\varphi - s) + R(\cdot, \varphi - s)H(\varphi - s) \quad \text{on } \mathcal{B},$$

and $\partial_r^k R(x, 0) = Pb_k + 2Zb_{k+1} + (\Box_{\mathsf{g}}\varphi)b_{k+1} = 0$ for all k. Hence $R_+(x, r) := R(x, r)H(r)$ is smooth on $\mathcal{B} \times \mathbb{R}$.

Cut-off. Let $\chi_b \in C^\infty(\mathbb{R})$ with $\chi_b = 1$ on $(-\infty, \varrho_{\text{out}} - \varrho_{\text{in}}]$ and $\chi_b = 0$ near $[r_{\text{end}}, \infty)$, and $\chi(\beta(\bar{x}, r)) = \chi_b(r)$. Then $\Pi_s := \chi \tilde{u}_s H(\varphi - s)$, extended by zero, is a distribution on M with compact support contained in $\mathcal{B}$, and, since $\chi = 1$ on $\operatorname{supp}\theta$,

$$P\,\Pi_s = F_s + \chi R_+(\cdot, \varphi - s) + E_s, \qquad E_s = [\Box_{\mathsf{g}}, \chi]\big(\tilde{u}_s H(\varphi - s)\big), \qquad \operatorname{supp} E_s \subset \{\varrho \geq \varrho_{\text{out}}\}.$$

The function $(s, x) \mapsto \chi(x) R_+(x, \varphi(x) - s)$, extended by zero, is smooth on $(-\eta, \eta) \times M$ with support in $(-\eta, \eta) \times K_1$ for a compact set K_1. By (F7),

$$G_P^{\text{ret}} F_s = \Pi_s - w_s - G_P^{\text{ret}} E_s, \qquad w_s := G_P^{\text{ret}}\big(\chi R_+(\cdot, \varphi - s)\big) \in C^\infty(M), \tag{6.4}$$

and $(s, x) \mapsto w_s(x)$ is smooth.

Conclusion. (i) holds by Lemma 5.3(ii). Let V' be an open neighbourhood of p_0 with $V' \subset \mathcal{B} \cap \{\varrho_{\text{in}} + a_2 < \varrho < \varrho_{\text{out}}\}$ on which $b_0 > 0$; then (iv) holds, and V' does not depend on q. On V': $\chi = 1$; $G_P^{\text{ret}} E_s = 0$, because $\operatorname{supp} G_P^{\text{ret}} E_s \subset J^+(\{\varrho \geq \varrho_{\text{out}}\}) \subset \{\varrho \geq \varrho_{\text{out}}\}$; and $w_s = 0$ on $V' \cap \{\varphi < s\}$, because the support of $\chi R_+(\cdot, \varphi - s)$ is contained in $\mathcal{T} \cap \{\varphi \geq s\} \subset J^+(\mu(s))$, so that $\operatorname{supp} w_s \subset J^+(\mu(s))$, while $J^+(\mu(s)) \cap V' \subset \{\varphi \geq s\}$ (Lemma 5.3(ii)). Being smooth, w_s vanishes with all derivatives on $\{\varphi \leq s\} \cap V'$, and (6.4) gives $G_P^{\text{ret}} F_s = (\tilde{u}_s - w_s) H(\varphi - s)$ on V'. Put $u(s, x) = B(x, \varphi(x) - s) - w_s(x)$; then $u_s = B(\cdot, 0) = b_0 > 0$ on $\{\varphi = s\} \cap V'$. $\square$

Remark 6.4. Lemmas 5.2–5.6 and 6.3 hold with the time orientation reversed, that is, with past-directed null geodesics, I^-, J^-, E^+ in place of I^+, J^+, E^-, and with G_P^{adv} in place of G_P^{ret}: they are the same statements for the globally hyperbolic manifold (M, g) with the opposite time orientation.

7. The weighted energy inequality and the local continuation step

This section localizes the method of [ORS26, Section 2]. Global hyperbolicity is not used.

7.1. **A weighted energy inequality on lens-shaped regions.**

Lemma 7.1 (Weighted energy inequality)**.** *Let (V, g) be a time-oriented Lorentzian manifold, $\psi, \tilde{\psi} \in C^\infty(V)$ temporal, and $\varphi \in C^\infty(V)$ eikonal of future type. Let $\tau, c \in \mathbb{R}$ be such that $L = \{\psi \leq \tau\} \cap \{\tilde{\psi} \geq c\}$ is compact. Fix a Riemannian metric h on V. Write $\Gamma_s = \{\varphi = s\}$, $Z = -\operatorname{grad}_g \varphi$, and let μ_s be the measure on Γ_s determined by $\int F\, dV_g = \int_{\mathbb{R}} \big(\int_{\Gamma_s} F\, d\mu_s\big) ds$. There are $C > 0$ and $\sigma_0 \geq 1$, depending only on $(g, \psi, \tilde{\psi}, \varphi, \tau, c, h)$, such that for all $s \in \mathbb{R}$, $\sigma \geq \sigma_0$ and all functions w of class C^2 on a neighbourhood of L with $w = 0$ and $dw = 0$ on $\{\psi = \tau\} \cap \{\varphi \geq s\} \cap L$,*

$$\sigma \int_{\Gamma_s \cap L} e^{2\sigma\psi}\big(|Zw|^2 + \sigma^2 w^2\big) d\mu_s + \sigma^2 \int_{L \cap \{\varphi > s\}} e^{2\sigma\psi}\big(|dw|_h^2 + \sigma^2 w^2\big) dV_g \\ \leq C \int_{L \cap \{\varphi > s\}} e^{2\sigma\psi} |\Box_g w|^2 dV_g. \tag{7.1}$$

Moreover, for w of class C^2 on a neighbourhood of L with $w = 0$, $dw = 0$ on $\{\psi = \tau\} \cap L$ (here φ plays no role),

$$\sigma^2 \int_L e^{2\sigma\psi}\big(|dw|_h^2 + \sigma^2 w^2\big) dV_g \leq C \int_L e^{2\sigma\psi} |\Box_g w|^2 dV_g. \tag{7.2}$$

Proof. Step 1 (pointwise inequality). Let $X = -\operatorname{grad}_g \psi$, a future-directed timelike vector field with $g(X, X) = -m^2$, $m > 0$, and put $m_0 = \min_L m$. For $\lambda > 0$ let $T_{ab} = \partial_a w\, \partial_b w - \frac{1}{2} g_{ab}\big(g^{cd} \partial_c w\, \partial_d w + \lambda w^2\big)$. Since $\nabla^a \nabla_a w = -\Box_g w$, one has $\nabla^a T_{ab} = (-\Box_g w - \lambda w) \partial_b w$. Let $J^a = e^{2\sigma\psi} T^a{}_b X^b$. Then

$$\operatorname{div} J = e^{2\sigma\psi}\big[-2\sigma T(X, X) - (\Box_g w)\, Xw - \lambda w\, Xw + T_{ab} \nabla^a X^b\big].$$

Let $h_\psi = g + 2m^{-2} d\psi \otimes d\psi$; this is a Riemannian metric, $|Xw| \leq m |dw|_{h_\psi}$, and $T(X, X) = (Xw)^2 + \frac{1}{2} m^2 (|dw|_g^2 + \lambda w^2) = \frac{1}{2} m^2 (|dw|_{h_\psi}^2 + \lambda w^2)$. On L we have $|T_{ab} \nabla^a X^b| \leq C_1 (|dw|_{h_\psi}^2 + \lambda w^2)$ with C_1 independent of λ, σ. Choose $\lambda = \varepsilon_0^2 \sigma^2$ with $\varepsilon_0 = m_0/2$. Then $|\lambda w Xw| \leq \frac{1}{2} m \sqrt{\lambda} (\lambda w^2 +$

$|dw|^2_{h_\psi}) \le \frac14\sigma m^2(\lambda w^2+|dw|^2_{h_\psi})$ and $|(\Box_g w)Xw| \le \frac14\sigma m^2|dw|^2_{h_\psi}+\sigma^{-1}|\Box_g w|^2$. Hence, for $\sigma\ge\sigma_0:=\max(1,4C_1/m_0^2)$, on L

$$\operatorname{div} J \le e^{2\sigma\psi}\Big[-\tfrac14\sigma m_0^2\big(|dw|^2_{h_\psi}+\varepsilon_0^2\sigma^2w^2\big)+\sigma^{-1}|\Box_g w|^2\Big]. \tag{7.3}$$

Step 2 (fluxes). If Y_1,Y_2 are future-directed timelike vectors, then $T(Y_1,Y_2)\ge 0$: in an orthonormal frame $e_0,\dots,e_n$ with $Y_2=\beta e_0$ and $Y_1=\alpha_0e_0+\alpha_1e_1$, $\alpha_0>|\alpha_1|$, $\beta>0$, one has $T_{00}=\frac12(\sum_{j=0}^n(e_jw)^2+\lambda w^2)$ and $|T_{01}|=|(e_0w)(e_1w)|\le T_{00}$, so $T(Y_1,Y_2)=\beta(\alpha_0T_{00}+\alpha_1T_{01})\ge 0$. Next, $k:=-g(Z,X)>0$, and $\underline{Z}:=\frac2kX-\frac{m^2}{k^2}Z$ is null with $g(Z,\underline{Z})=-2$, so $X=\frac k2\underline{Z}+\frac{m^2}{2k}Z$ and $|dw|^2_g=-(Zw)(\underline{Z}w)+|\nabla\!\!\!/ w|^2$, where $|\nabla\!\!\!/ w|^2\ge0$ is the squared norm of the derivative in the directions orthogonal to $Z,\underline{Z}$. Therefore

$$T(Z,X)=(Zw)(Xw)+\tfrac k2\big(|dw|^2_g+\lambda w^2\big)=\tfrac{m^2}{2k}(Zw)^2+\tfrac k2\big(|\nabla\!\!\!/ w|^2+\lambda w^2\big)\ \ge\ \tfrac{m^2}{2k}(Zw)^2+\tfrac k2\lambda w^2. \tag{7.4}$$

Step 3 (integration). Let $H_1\in C^\infty(\mathbb{R})$ be nondecreasing, $H_1=0$ on $(-\infty,0]$, $H_1=1$ on $[1,\infty)$, and $H_\epsilon(r)=H_1(r/\epsilon)$. Put $\chi_\epsilon=H_\epsilon(\tau-\psi)H_\epsilon(\tilde\psi-c)H_\epsilon(\varphi-s)$; its support is a compact subset of $L\cap\{\varphi\ge s\}$. Integration by parts gives $\int\chi_\epsilon\operatorname{div}J\,dV_g=-\int J(d\chi_\epsilon)\,dV_g=A_\epsilon+B_\epsilon+G_\epsilon$ with

$$A_\epsilon=\int H_\epsilon'(\tau-\psi)H_\epsilon(\tilde\psi-c)H_\epsilon(\varphi-s)\,J(d\psi)\,dV_g,\qquad J(d\psi)=-e^{2\sigma\psi}T(X,X),$$

$$B_\epsilon=-\int H_\epsilon(\tau-\psi)H_\epsilon'(\tilde\psi-c)H_\epsilon(\varphi-s)\,J(d\tilde\psi)\,dV_g,\qquad -J(d\tilde\psi)=e^{2\sigma\psi}T(-\operatorname{grad}_g\tilde\psi,X)\ge0,$$

$$G_\epsilon=-\int H_\epsilon(\tau-\psi)H_\epsilon(\tilde\psi-c)H_\epsilon'(\varphi-s)\,J(d\varphi)\,dV_g,\qquad -J(d\varphi)=e^{2\sigma\psi}T(Z,X)\ge0.$$

On the support of the integrand of A_ϵ we have $\tau-\epsilon<\psi<\tau$, $\varphi\ge s$, $\tilde\psi\ge c$. By compactness and the vanishing of w,dw on $\{\psi=\tau\}\cap\{\varphi\ge s\}\cap L$, $\sup(|w|+|dw|_h)$ over this set tends to 0 as $\epsilon\to0$, while $\int_L H_\epsilon'(\tau-\psi)dV_g$ is bounded; hence $A_\epsilon\to0$. Since $B_\epsilon\ge0$ and $\chi_\epsilon\to1$ pointwise on the interior of $L\cap\{\varphi\ge s\}$, we get $\limsup_{\epsilon\to0}G_\epsilon\le\int_{L\cap\{\varphi>s\}}\operatorname{div}J\,dV_g$. We claim that $\liminf_{\epsilon\to0}G_\epsilon\ge\int_{\Gamma_s\cap L}e^{2\sigma\psi}T(Z,X)\,d\mu_s$. If $\Gamma_s\cap L=\emptyset$ there is nothing to prove. Otherwise, the vector field X/k satisfies $(X/k)\varphi=1$, so its flow Θ_r maps Γ_s into Γ_{s+r}. Let N be a relatively compact open neighbourhood of $\Gamma_s\cap L$ in Γ_s and $r_1>0$ such that Θ_r is defined and injective on N for $0\le r\le r_1$; then $\Theta_r^*\mu_{s+r}=\rho_r\,\mu_s$ on N with a smooth positive density ρ_r, $\rho_0=1$. By the definition of μ_r, $G_\epsilon=\int_0^\epsilon H_\epsilon'(r)\big(\int_{\Gamma_{s+r}}H_\epsilon(\tau-\psi)H_\epsilon(\tilde\psi-c)e^{2\sigma\psi}T(Z,X)\,d\mu_{s+r}\big)dr$, and the integrand is nonnegative. Hence, for $\epsilon<r_1$,

$$G_\epsilon\ge\int_0^\epsilon H_\epsilon'(r)\int_N\big(H_\epsilon(\tau-\psi)H_\epsilon(\tilde\psi-c)e^{2\sigma\psi}T(Z,X)\big)\circ\Theta_r\,\rho_r\,d\mu_s\,dr.$$

The inner integrand is nonnegative and, at every point of N with $\psi<\tau$, $\tilde\psi>c$, converges to $e^{2\sigma\psi}T(Z,X)$ as $\epsilon\to0$ uniformly in $r\in[0,\epsilon]$; since $\int_0^\epsilon H_\epsilon'=1$, Fatou's lemma gives the claim on $\Gamma_s\cap\{\psi<\tau\}\cap\{\tilde\psi>c\}$. The hypersurfaces $\{\psi=\tau\}$, $\{\tilde\psi=c\}$ are spacelike and therefore transversal to the null hypersurface Γ_s, so $\Gamma_s\cap\partial L$ is μ_s-null. Combining with (7.3) and (7.4),

$$\int_{\Gamma_s\cap L}e^{2\sigma\psi}\Big(\tfrac{m^2}{2k}(Zw)^2+\tfrac k2\varepsilon_0^2\sigma^2w^2\Big)d\mu_s+\tfrac14\sigma m_0^2\int_{L\cap\{\varphi>s\}}e^{2\sigma\psi}\big(|dw|^2_{h_\psi}+\varepsilon_0^2\sigma^2w^2\big)dV_g$$
$$\le\sigma^{-1}\int_{L\cap\{\varphi>s\}}e^{2\sigma\psi}|\Box_g w|^2dV_g.$$

Multiplying by σ and using that m,k are bounded above and below on L and that h_ψ and h are comparable on L gives (7.1), with constants independent of s. Inequality (7.2) follows in the same way with $\chi_\epsilon=H_\epsilon(\tau-\psi)H_\epsilon(\tilde\psi-c)$. □

7.2. **Algebra of the comparison.** Let $V, V' \subset M$ be open, $z' = (z'^0, \dots, z'^n)$ coordinates on V', $\Psi\colon V \to V'$ smooth, $\Psi^k = z'^k \circ \Psi$, and $\kappa \in C^\infty(V)$. Define on V

$$\bar g^{\alpha\beta} = g(d\Psi^\alpha, d\Psi^\beta) - \kappa\,(g'^{\alpha\beta} \circ \Psi), \qquad \bar\theta^k = -\Box_g \Psi^k + \kappa\,(\Box_{g'} z'^k) \circ \Psi, \tag{7.5}$$

where $g'^{\alpha\beta} = g'(dz'^\alpha, dz'^\beta)$ and $g(\cdot,\cdot)$, $g'(\cdot,\cdot)$ denote the inner products of covectors. In this subsection and in the proof of Proposition 7.3, Greek indices α, β and the index k are coordinate indices running from 0 to n and are summed when repeated; the letter a is reserved for the labels of the wave families and is never summed implicitly (sums over a are written explicitly).

Lemma 7.2 (cf. [ORS26, Lemma 2.7])**.** *For $v, v_1, v_2 \in C^\infty(V')$,*

$$-\Box_g(v \circ \Psi) + \kappa\,(\Box_{g'} v) \circ \Psi = \bar g^{\alpha\beta}\,(\partial_\alpha \partial_\beta v) \circ \Psi + \bar\theta^k\,(\partial_k v) \circ \Psi, \tag{7.6}$$

$$g\big(d(v_1 \circ \Psi), d(v_2 \circ \Psi)\big) - \kappa\, g'(dv_1, dv_2) \circ \Psi = \bar g^{\alpha\beta}\,(\partial_\alpha v_1\, \partial_\beta v_2) \circ \Psi. \tag{7.7}$$

If Ψ is a local diffeomorphism with $\Psi^* g' = g$ on an open set and $\kappa = 1$ there, then $\bar g = 0$ and $\bar\theta = 0$ there.

Proof. By the chain rule, $\Box_g(v\circ\Psi) = (\partial_k v)\circ\Psi\ \Box_g \Psi^k - (\partial_\alpha\partial_\beta v)\circ\Psi\ g(d\Psi^\alpha, d\Psi^\beta)$ and $\Box_{g'} v = (\partial_k v)\Box_{g'} z'^k - (\partial_\alpha\partial_\beta v) g'^{\alpha\beta}$; this gives (7.6). Identity (7.7) follows from $d(v\circ\Psi) = (\partial_\alpha v)\circ\Psi\, d\Psi^\alpha$. The last statement follows from the naturality of the wave operator and of the inner product under local isometries. □

7.3. **The local continuation step.** Let M be a manifold of dimension $1+n$. In the following proposition Lemma 6.2 is applied, with $q = 0$, to (g, φ_a, u_a) on V and to (g', φ'_a, u'_a) on V'; the functions given by Lemma 6.2(b) for g' are denoted $a'_{a,s}$, $b'_{a,s}$, and we write $Z_a = Z_{g,\varphi_a}$, $Z'_a = Z_{g',\varphi'_a}$.

Proposition 7.3 (Local continuation step)**.** *Let $n \geq 2$, let g, g' be smooth time-oriented Lorentzian metrics on M, and assume:*

- (L0) *$V, V' \subset M$ are open, V' is a coordinate domain, $\Psi\colon V \to V'$ is smooth, $\psi \in C^\infty(V)$ is temporal for g, $q_0 \in V$, $\tau = \psi(q_0)$, $V_+ = \{q \in V : \psi(q) > \tau\}$.*
- (L1) *$\mathcal{A}$ is finite, $\mathcal{J} \subset \mathbb{R}$ an open interval; for $a \in \mathcal{A}$: $\varphi_a \in C^\infty(V)$, $\varphi'_a \in C^\infty(V')$ are eikonal of future type for g, g', with values in $\mathcal{J}$; $u_a \in C^\infty(\mathcal{J}\times V)$, $u'_a \in C^\infty(\mathcal{J}\times V')$ are such that $u_{a,s} H(\varphi_a - s)$ solves $\Box_g(\cdot) = 0$ on V and $u'_{a,s} H(\varphi'_a - s)$ solves $\Box_{g'}(\cdot) = 0$ on V', for all $s \in \mathcal{J}$; and $u_{a,s} > 0$ on $\{\varphi_a = s\}$.*
- (L2) *On V_+: $\Psi^* g' = g$, $\varphi'_a \circ \Psi = \varphi_a$ for all $a \in \mathcal{A}$, and $u_{a,s} = u'_{a,s} \circ \Psi$ on $V_+ \cap \{\varphi_a \geq s\}$ for all $a \in \mathcal{A}$, $s \in \mathcal{J}$.*
- (L3) *There is $I \subset \mathcal{A}$ with $n+1$ elements such that $\varphi'_a \circ \Psi = \varphi_a$ on all of V for $a \in I$, and $(d\varphi'_a(p))_{a\in I}$ is a basis of $T^*_p M$ for every $p \in V'$.*
- (L4) *For every $p \in V'$ the set $\{g'(p)\} \cup \{d\varphi'_a(p) \otimes d\varphi'_a(p)\}_{a\in\mathcal{A}}$ spans $\mathrm{S}^2 T^*_p M$.*

Then there is an open neighbourhood $V^\circ \subset V$ of q_0 on which Ψ is a local diffeomorphism, $\Psi^* g' = g$, $\varphi'_a \circ \Psi = \varphi_a$ for all $a \in \mathcal{A}$, and $u_{a,s} = u'_{a,s} \circ \Psi$ on $V^\circ \cap \{\varphi_a \geq s\}$. Moreover V° can be chosen such that every $w \in C^\infty(V^\circ)$ with $\Box_g w = 0$ on V° and $w = 0$ on $V^\circ \cap V_+$ vanishes identically.

Proof of Proposition 7.3. Step 0 (set-up). By (L2) and continuity, $g'(d\Psi\cdot, d\Psi\cdot) = g$ on $\{\psi \geq \tau\} \cap V$; in particular $d\Psi(q_0)$ is invertible. Let

$$\kappa = \tfrac{1}{n+1}\,(g'_{\alpha\beta} \circ \Psi)\, g(d\Psi^\alpha, d\Psi^\beta) \in C^\infty(V),$$

so that $(g'_{\alpha\beta}\circ\Psi)\bar g^{\alpha\beta} = 0$ on V for $\bar g$ as in (7.5), and $\kappa = 1$ on $\{\psi \geq \tau\} \cap V$. Choose an open neighbourhood $V_1 \subset V$ of q_0, with compact closure in V, on which $d\Psi$ is invertible and $\kappa > \frac12$. By Lemma 7.2, $\bar g = 0$ and $\bar\theta = 0$ on $\{\psi \geq \tau\} \cap V$. Choose coordinates $y = (y^0, y')$ on a neighbourhood $V_2 \subset V_1$ of q_0 with $y(q_0) = 0$, $y^0 = \psi - \tau$, and let $\tilde\psi = y^0 - \mu|y'|^2$ with $\mu > 0$ so small that (after shrinking V_2) $\tilde\psi$ is temporal for g on V_2. For small $r > 0$ the *lens*

$$L = L_r = \{y \in V_2 :\ y^0 \leq 0,\ y^0 - \mu|y'|^2 \geq -\mu r^2\} = \{\psi \leq \tau\} \cap \{\tilde\psi \geq -\mu r^2\}$$

is compact, convex in the coordinates y, contained in V_2, and $L \cup \{y^0 \geq 0\}$ contains a neighbourhood of q_0. We apply Lemma 7.1 on V_2 with these $\psi, \tilde{\psi}, \tau$, $c = -\mu r^2$, and with $\varphi = \varphi_a$. Write $\|F\|_\sigma^2 = \int_L e^{2\sigma\psi}|F|^2 dV_g$. For $a \in \mathcal{A}$ put $\varphi_a'' = \varphi_a' \circ \Psi$, $u_{a,s}'' = u_{a,s}' \circ \Psi$,

$$\bar{\varphi}_a = \varphi_a - \varphi_a'', \qquad \bar{w}_{a,s} = u_{a,s} - u_{a,s}'', \qquad D_{a,s} = L \cap \{\varphi_a \geq s\}, \qquad \Gamma_{a,s} = \{\varphi_a = s\}.$$

By (L2), $\bar{\varphi}_a = 0$ on V_+ and $\bar{w}_{a,s} = 0$ on $V_+ \cap \{\varphi_a \geq s\}$; as these functions are smooth on V, $\bar{\varphi}_a$ has vanishing Cauchy data on $\{\psi = \tau\} \cap L$ and $\bar{w}_{a,s}$ on $\{\psi = \tau\} \cap \{\varphi_a \geq s\} \cap L$ (every point of the latter set is a limit of points of the open set $V_+ \cap \{\varphi_a > s\}$, since $d\psi$ and $d\varphi_a$ are linearly independent). In Steps 1, 2 and 5 only $s \in [\min_L \varphi_a, \max_L \varphi_a]$ occurs (integration in s), and all constants are uniform in such s by compactness; in Step 6 all $s \in \mathcal{J}$ occur, but there no s-dependent constant is used (for $s < \min_L \varphi_a$ one has $D_{a,s} = L$ and $\Gamma_{a,s} \cap L = \emptyset$). In the following C denotes constants independent of σ and s.

Step 1 (gauge directions, $a \in I$). Here $\bar{\varphi}_a = 0$ on V. On $D_{a,s}$, by Lemma 6.2 and (7.6),

$$\Box_g \bar{w}_{a,s} = -\Box_g u_{a,s}'' = \bar{g}^{\alpha\beta}(\partial_\alpha\partial_\beta u_{a,s}') \circ \Psi + \bar{\theta}^k(\partial_k u_{a,s}') \circ \Psi - \kappa\,(\varphi_a'' - s)\, a_{a,s}' \circ \Psi.$$

The last term vanishes on $D_{a,s}$: $a_{a,s}' \circ \Psi \neq 0$ requires $\varphi_a'' = \varphi_a \leq s$, hence $\varphi_a = s$ there. Thus $|\Box_g \bar{w}_{a,s}| \leq C(|\bar{g}| + |\bar{\theta}|)$ on $D_{a,s}$. On $\Gamma_{a,s} \cap L$, using $Z_a = Z_{g,\varphi_a''}$, (7.7), Lemma 6.2 and $\varphi_a'' = s$,

$$\begin{aligned} Z_a \bar{w}_{a,s} &= Z_a u_{a,s} - \kappa\,(Z_a' u_{a,s}') \circ \Psi + \bar{g}^{\alpha\beta}(\partial_\alpha \varphi_a'\, \partial_\beta u_{a,s}') \circ \Psi \\ &= -\tfrac{1}{2}(\Box_g \varphi_a) u_{a,s} + \tfrac{1}{2}\kappa\,(\Box_{g'} \varphi_a') \circ \Psi\ u_{a,s}'' + \bar{g}^{\alpha\beta}(\partial_\alpha \varphi_a'\, \partial_\beta u_{a,s}') \circ \Psi \\ &= \tfrac{1}{2}\big[\bar{g}^{\alpha\beta}(\partial_\alpha\partial_\beta \varphi_a') \circ \Psi + \bar{\theta}^k(\partial_k \varphi_a') \circ \Psi\big] u_{a,s} - \tfrac{1}{2}\kappa\,(\Box_{g'} \varphi_a') \circ \Psi\ \bar{w}_{a,s} \\ &\quad + \bar{g}^{\alpha\beta}(\partial_\alpha \varphi_a'\, \partial_\beta u_{a,s}') \circ \Psi, \end{aligned}$$

where in the last step (7.6) was applied to $v = \varphi_a'$ and $\Box_g \varphi_a = \Box_g(\varphi_a' \circ \Psi)$. Since $u_{a,s} \geq c_0 > 0$ on $\Gamma_{a,s} \cap L$, we get $|\bar{\theta}^k(\partial_k \varphi_a') \circ \Psi|^2 \leq C(|Z_a \bar{w}_{a,s}|^2 + |\bar{w}_{a,s}|^2 + |\bar{g}|^2)$ there. By (7.1) applied to $\bar{w}_{a,s}$,

$$\sigma \int_{\Gamma_{a,s} \cap L} e^{2\sigma\psi} |\bar{\theta}^k(\partial_k \varphi_a') \circ \Psi|^2 d\mu_s \leq C\big(\|\bar{g}\|_\sigma^2 + \|\bar{\theta}\|_\sigma^2\big) + C\sigma \int_{\Gamma_{a,s} \cap L} e^{2\sigma\psi} |\bar{g}|^2 d\mu_s.$$

Integrating in s over $[\min_L \varphi_a, \max_L \varphi_a]$ and summing over $a \in I$, the basis property in (L3) gives $\sigma\|\bar{\theta}\|_\sigma^2 \leq C(\|\bar{g}\|_\sigma^2 + \|\bar{\theta}\|_\sigma^2) + C\sigma\|\bar{g}\|_\sigma^2$, hence for σ large

$$\|\bar{\theta}\|_\sigma \leq C\|\bar{g}\|_\sigma. \tag{7.8}$$

Step 2 (the other directions, $a \in \mathcal{A} \setminus I$). On $D_{a,s}$ the same computation gives $\Box_g \bar{w}_{a,s} = \bar{g}^{\alpha\beta}(\partial_\alpha\partial_\beta u_{a,s}') \circ \Psi + \bar{\theta}^k(\partial_k u_{a,s}') \circ \Psi - \kappa(\varphi_a'' - s) a_{a,s}' \circ \Psi$, and now $a_{a,s}' \circ \Psi \neq 0$ implies $\varphi_a'' \leq s \leq \varphi_a$, so $|\varphi_a'' - s| \leq |\bar{\varphi}_a|$ and $|\Box_g \bar{w}_{a,s}| \leq C(|\bar{g}| + |\bar{\theta}| + |\bar{\varphi}_a|)$ on $D_{a,s}$. On $\Gamma_{a,s} \cap L$ we have $\varphi_a'' - s = -\bar{\varphi}_a$ and

$$\begin{aligned} Z_a \bar{w}_{a,s} &= Z_a u_{a,s} + g(d\varphi_a'', du_{a,s}'') + g(d\bar{\varphi}_a, du_{a,s}'') \\ &= Z_a u_{a,s} - \kappa\,(Z_a' u_{a,s}') \circ \Psi + \bar{g}^{\alpha\beta}(\partial_\alpha \varphi_a'\, \partial_\beta u_{a,s}') \circ \Psi + g(d\bar{\varphi}_a, du_{a,s}'') \\ &= -\tfrac{1}{2}(\Box_g \varphi_a) u_{a,s} + \tfrac{1}{2}\kappa\,(\Box_{g'} \varphi_a') \circ \Psi\ u_{a,s}'' + R_1, \qquad |R_1| \leq C(|\bar{\varphi}_a| + |d\bar{\varphi}_a| + |\bar{g}|), \end{aligned}$$

and, since $\Box_g \varphi_a = \Box_g \bar{\varphi}_a + \Box_g(\varphi_a' \circ \Psi)$, by (7.6)

$$Z_a \bar{w}_{a,s} = -\tfrac{1}{2}(\Box_g \bar{\varphi}_a) u_{a,s} + R_2, \qquad |R_2| \leq C(|\bar{w}_{a,s}| + |\bar{\varphi}_a| + |d\bar{\varphi}_a| + |\bar{g}| + |\bar{\theta}|).$$

Hence $|\Box_g \bar{\varphi}_a|^2 \leq C(|Z_a \bar{w}_{a,s}|^2 + |\bar{w}_{a,s}|^2 + |\bar{\varphi}_a|^2 + |d\bar{\varphi}_a|^2 + |\bar{g}|^2 + |\bar{\theta}|^2)$ on $\Gamma_{a,s} \cap L$, and (7.1) for $\bar{w}_{a,s}$, integrated in s, yields for $\sigma \geq 1$

$$\|\Box_g \bar{\varphi}_a\|_\sigma^2 \leq C\big(\|\bar{g}\|_\sigma^2 + \|\bar{\theta}\|_\sigma^2 + \|\bar{\varphi}_a\|_\sigma^2 + \|d\bar{\varphi}_a\|_\sigma^2\big).$$

By (7.2) applied to $\bar{\varphi}_a$, $\sigma^2\|d\bar{\varphi}_a\|_\sigma^2 + \sigma^4\|\bar{\varphi}_a\|_\sigma^2 \leq C\|\Box_g \bar{\varphi}_a\|_\sigma^2$. For σ large the terms with $\bar{\varphi}_a$ on the right are absorbed, and with (7.8)

$$\sigma^2\|d\bar{\varphi}_a\|_\sigma^2 + \sigma^4\|\bar{\varphi}_a\|_\sigma^2 \leq C\|\bar{g}\|_\sigma^2, \qquad a \in \mathcal{A} \setminus I. \tag{7.9}$$

Step 3 (spanning). For $z \in L$, $\bar{g}(z)$ is a symmetric contravariant 2-tensor at $\Psi(z)$ annihilated by $g'(\Psi(z))$. By (L4) and compactness of $\Psi(L)$, $|\bar{g}| \le C \sum_{a\in\mathcal{A}} |\bar{g}^{\alpha\beta}(\partial_\alpha\varphi'_a\,\partial_\beta\varphi'_a)\circ\Psi|$ on L. By (7.7) and the eikonal equation for g', $\bar{g}^{\alpha\beta}(\partial_\alpha\varphi'_a\partial_\beta\varphi'_a)\circ\Psi = g(d\varphi''_a, d\varphi''_a)$, which vanishes for $a \in I$ and equals $-2g(d\varphi_a, d\bar{\varphi}_a) + g(d\bar{\varphi}_a, d\bar{\varphi}_a)$ for $a \notin I$. Hence $|\bar{g}| \le C\sum_{a\notin I}|d\bar{\varphi}_a|$ on L and by (7.9), $\|\bar{g}\|^2_\sigma \le C\sigma^{-2}\|\bar{g}\|^2_\sigma$. Taking σ large: $\bar{g} = 0$ on L, and then $\bar{\theta} = 0$ and $\bar{\varphi}_a = 0$ on L for all $a \in \mathcal{A}$.

Step 4 (conformality). On $L \subset V_1$, $d\Psi$ is invertible and $\kappa > \frac{1}{2}$, and $\bar{g} = 0$ means that the Lorentzian metric $g'' := \Psi^* g'$ on V_1 satisfies $g'' = \kappa g$ on L, i.e. $g = c\,g''$ with $c = \kappa^{-1}$, $c = 1$ on $\{\psi = \tau\} \cap L$.

Step 5 ($c \equiv 1$). By Step 3, $\bar{g} = 0$ and $\bar{\theta} = 0$ on L, so that (7.6) gives $\Box_g(v\circ\Psi) = \kappa\,(\Box_{g'}v)\circ\Psi$ at the points of L, for every smooth function v defined near the image point (the identity (7.6) is pointwise). On the interior of L the map Ψ is a local diffeomorphism, so that every smooth function is locally of the form $v\circ\Psi$, and $(\Box_{g'}v)\circ\Psi = \Box_{g''}(v\circ\Psi)$ by naturality. Hence

$$\Box_g w = \kappa\,\Box_{g''} w \qquad \text{for all smooth } w, \text{ locally on the interior of } L.$$

On the other hand, for $g = cg''$ in dimension $1+n$,

$$\Box_g w = c^{-1}\Box_{g''}w - \tfrac{n-1}{2}\,c^{-2}g''(dc, dw), \qquad Z_{g,\varphi} = c^{-1}Z_{g'',\varphi}, \tag{7.10}$$

and $c^{-1} = \kappa$ by Step 4. Comparing the two formulas we obtain $(n-1)\,g''(dc, dw) = 0$ for all smooth w. As $n \ge 2$, this gives $dc = 0$ on the interior of L. Since L is the closure of its interior, which is connected, and $c = 1$ on the top of L, we conclude $c \equiv 1$: $\Psi^* g' = g$ on L.

Step 6 (amplitudes). Now $\Box_g \bar{w}_{a,s} = -(\Box_{g'}u'_{a,s})\circ\Psi = -(\varphi_a - s)a'_{a,s}\circ\Psi = 0$ on $D_{a,s}$, and (7.1) gives $\bar{w}_{a,s} = 0$ on $D_{a,s}$.

Finally let V° be the interior of $L \cup (V_2 \cap \{\psi \ge \tau\})$. It is an open neighbourhood of q_0, and the asserted identities hold on L (Steps 3–6), on $V_2 \cap V_+$ by (L2), and on $\{\psi = \tau\}$ by continuity.

The additional property of V°. Let $w \in C^\infty(V^\circ)$ with $\Box_g w = 0$ on V° and $w = 0$ on $V^\circ \cap V_+$. For $0 < \rho < r$ the lens L_ρ is contained in V°: its points with $y^0 < 0$ satisfy $y^0 - \mu|y'|^2 \ge -\mu\rho^2 > -\mu r^2$ and are interior points of L_r, and its points with $y^0 = 0$ satisfy $|y'| \le \rho < r$ and are interior points of $L_r \cup \{y^0 \ge 0\}$. On $\{\psi = \tau\} \cap L_\rho$ we have $w = 0$ and $dw = 0$ by continuity. By (7.2) on L_ρ, $w = 0$ on L_ρ. As $V^\circ \subset \{\psi \ge \tau\} \cup \bigcup_{\rho<r} L_\rho$, $w = 0$ on V°. □

7.4. **Rigidity of local isometries.**

Lemma 7.4 (Rigidity of local isometries)**.** *Let (N_1, g_1), (N_2, g_2) be semi-Riemannian manifolds of the same dimension, N_1 connected, and $F, \tilde{F}\colon N_1 \to N_2$ smooth maps with $F^* g_2 = \tilde{F}^* g_2 = g_1$. If $F = \tilde{F}$ on a nonempty open set, then $F = \tilde{F}$.*

Proof. The set $\{q : F(q) = \tilde{F}(q),\ dF(q) = d\tilde{F}(q)\}$ is closed and nonempty. It is open: F and $\tilde{F}$ map geodesics to geodesics, so $F(\exp_q v) = \exp_{F(q)}(dF(q)v)$ for v in a neighbourhood of 0 in $T_q N_1$, and likewise for $\tilde{F}$. (This is the argument of [ONe83, Proposition 3.62, p. 91], which is stated there for local isometries of a connected manifold with equal differentials at one point.) □

8. Proof of Theorem 1.3

We first prove the theorem under the stronger hypothesis (E) (Sections 8.1–8.3) and then reduce the general case to it (Section 8.4).

Theorem 8.1. *Let $n \ge 2$ and assume* (S), (D) *and* (E)*. Then the conclusions of Theorem 1.3 hold.*

In Sections 8.1–8.3 we assume (S), (D) and (E); the hypothesis $n \ge 2$ is used only in Section 8.3. ψ is a fixed Cauchy temporal function of (M, g) (F6), $A_\tau = \{\psi > \tau\}$ and $\Sigma_\tau = \{\psi = \tau\}$. By (F6), each A_τ is open, connected (it corresponds to $(\tau, \infty)\times\Sigma$), a future set

for g (that is, $J_g^+(A_\tau) \subset A_\tau$), and $A_\tau \backslash W \supset \{\psi > \max(\tau, \max_W \psi)\}$ is a nonempty open set. By Lemma 4.3 (for $\mathsf{g} = g'$ and $K = W$) we fix $v_1', \dots, v_m'$ with $\Box_{g'} v_i' = 0$ and an open $N' \supset W$ such that $\mathbf{v}' = (v_1', \dots, v_m') \colon N' \to \mathbb{R}^m$ is an immersion, and put $\mathbf{v} = (\mathsf{T}v_1', \dots, \mathsf{T}v_m') \in C^\infty(M; \mathbb{R}^m)$ (Lemma 4.2); thus $\mathbf{v} = \mathbf{v}'$ on O. Consider the statement

$\mathsf{P}(\tau)$: there is a smooth map $\Psi_\tau \colon A_\tau \to M$ with $\Psi_\tau^* g' = g$ on A_τ and $\Psi_\tau = \mathrm{id}$ on $A_\tau \setminus W$.

By the rigidity of local isometries (Lemma 7.4), Ψ_τ is unique and $\Psi_\tau|_{A_{\tau_1}} = \Psi_{\tau_1}$ for $\tau_1 > \tau$.

8.1. Consequences of $\mathsf{P}(\tau)$.

Lemma 8.2 (No escape)**.** *If $\mathsf{P}(\tau)$ holds, then Ψ_τ preserves the time orientation and $\Psi_\tau(A_\tau \cap W) \subset W$.*

Proof. Write $\Psi = \Psi_\tau$. Let Y, Y' be future-directed timelike vector fields for g, g'. The function $g'(d\Psi(Y), Y' \circ \Psi)$ on A_τ is continuous, nowhere zero, and negative on $A_\tau \backslash W$; as A_τ is connected it is negative everywhere.

Let $q \in A_\tau \cap W$ and suppose $x = \Psi(q) \notin W$. By (E) there is a future-inextendible timelike curve $c' \colon [0, \infty) \to O$ with $c'(0) = x$. Since Ψ is a local diffeomorphism there is a maximal lift $c \colon [0, s_1) \to A_\tau$, $c(0) = q$, $\Psi \circ c = c'$; it is a future-directed g-timelike curve. If $s_1 < \infty$ and c had no limit in M as $s \to s_1$, then c would be future-inextendible, hence by (F1) outside W for s close to s_1, where $\Psi = \mathrm{id}$; thus $c(s) = c'(s) \to c'(s_1)$, a contradiction. So $c(s) \to q_1 \in M$, and $\psi(q_1) \geq \psi(q) > \tau$, i.e. $q_1 \in A_\tau$; as Ψ is a local diffeomorphism at q_1 the lift extends beyond s_1, contradicting maximality. Hence $s_1 = \infty$. If $c(s)$ converged to some $q_\infty \in M$ as $s \to \infty$, then $q_\infty \in A_\tau$ and $c'(s) = \Psi(c(s)) \to \Psi(q_\infty)$, contradicting the inextendibility of c'; so c is future-inextendible, and by (F1) there is s_2 with $c(s) \notin W$, hence $c(s) = c'(s)$, for $s \geq s_2$. Let $s_3 = \inf\{s \geq 0 : \ c = c' \text{ on } [s, \infty)\}$. If $s_3 > 0$, then $y = c(s_3) = c'(s_3) \in A_\tau \setminus W$, $\Psi = \mathrm{id}$ near y, and for $s < s_3$ close to s_3 both $c(s)$ and $c'(s)$ are close to y and have the same image $c'(s)$ under Ψ, which is injective near y; so $c(s) = c'(s)$, contradicting the definition of s_3. Hence $s_3 = 0$ and $q = c(0) = c'(0) = x$, which is impossible since $q \in W$ and $x \notin W$. □

Lemma 8.3 (Propagation of identities)**.** *Assume $\mathsf{P}(\tau)$.*

(a) *For every $F \in \mathcal{E}'(O)$: $G_g^{\mathrm{ret}} F = \Psi_\tau^*(G_{g'}^{\mathrm{ret}} F)$ on A_τ.*

(b) *For every $v' \in C^\infty(M)$ with $\Box_{g'} v' = 0$: $\mathsf{T}v' = v' \circ \Psi_\tau$ on A_τ. In particular $\mathbf{v} = \mathbf{v}' \circ \Psi_\tau$ on A_τ.*

Proof. Write $\Psi = \Psi_\tau$. (a) Let $U = G_g^{\mathrm{ret}} F$, $U' = G_{g'}^{\mathrm{ret}} F$ and $z = U|_{A_\tau} - \Psi^* U' \in \mathcal{D}'(A_\tau)$ (the pull-back by the local diffeomorphism Ψ). By naturality, $\Box_g z = F|_{A_\tau} - \Psi^* F$. By Lemma 8.2 and $\Psi = \mathrm{id}$ on $A_\tau \cap O$, $\Psi^{-1}(\operatorname{supp} F) = \operatorname{supp} F \cap A_\tau$; hence $\Psi^* F = 0 = F$ on $A_\tau \setminus \operatorname{supp} F$, and $\Psi^* F = F$ on the open set $A_\tau \cap O \supset A_\tau \cap \operatorname{supp} F$. So $\Box_g z = 0$ on A_τ. Moreover $z = 0$ on $A_\tau \setminus W$, by Proposition 4.1 and $\Psi = \mathrm{id}$ there. Let $\vartheta \in C_c^\infty(A_\tau)$ and $\chi = G_g^{\mathrm{ret}} \vartheta$. Then $\operatorname{supp} \chi \subset J_g^+(\operatorname{supp} \vartheta) \subset A_\tau$ and $J_g^+(\operatorname{supp} \vartheta) \cap W$ is a compact subset of A_τ. Choose $\zeta \in C_c^\infty(A_\tau)$ with $\zeta = 1$ near this set. Then $(1 - \zeta)\chi$ vanishes near $A_\tau \cap W \supset \operatorname{supp} z$, so the compactly supported function $\vartheta - \Box_g(\zeta\chi) = \Box_g((1 - \zeta)\chi)$ vanishes near $\operatorname{supp} z$, and $\langle z, \vartheta \rangle = \langle z, \Box_g(\zeta\chi) \rangle = \langle \Box_g z, \zeta\chi \rangle = 0$. Hence $z = 0$.

(b) $z = \mathsf{T}v' - v' \circ \Psi \in C^\infty(A_\tau)$ satisfies $\Box_g z = 0$ on A_τ and $z = 0$ on $A_\tau \setminus W$; the same argument gives $z = 0$. □

Lemma 8.4 (Transfer of causal relations)**.** *Assume $\mathsf{P}(\tau)$. If $x \in A_\tau$, $y \in O$ and $y \leq_g x$, then $y \leq_{g'} \Psi_\tau(x)$.*

Proof. Write $\Psi = \Psi_\tau$. Let $c \colon [0, \infty) \to O$ be a future-inextendible timelike curve with $c(0) = y$ (E); it is timelike for g and for g'. The set $\{t \geq 0 : c(t) \leq_g x\}$ is closed (F4), contains 0, and is bounded, since otherwise c would stay in the compact set $J_g^+(y) \cap J_g^-(x)$ (since $c(t) \leq_g x$ implies $c(t') \leq_g x$ for $t' < t$), contradicting (F1). Let t^* be its maximum and $z = c(t^*) \in O$. Then $y \leq_{g'} z$, and it suffices to show $z \leq_{g'} \Psi(x)$.

If $z = x$, then $x \in A_\tau \cap O$, $\Psi(x) = x$, and we are done. Otherwise $z \notin I_g^-(x)$: indeed $I_g^-(x)$ is open, so $z \in I_g^-(x)$ would give $c(t) \leq_g x$ for some $t > t^*$. By (F5) there is a g-causal

geodesic $\gamma\colon [0,1] \to M$ from z to x; it is null, because $x \notin I_g^+(z)$. For large k the point $x_k = \gamma(1-1/k)$ lies in A_τ. Fix such k and apply Lemmas 5.2, 5.3 and 6.3 to the metric $\mathsf{g} = g$, to $\sigma(t) = \gamma((1-1/k)t)$, $t \in [0,b]$, $b = k/(k-1)$, for which $\sigma(0) = z$, $\sigma(1) = x_k$ and $\sigma(b) = x \notin I_g^+(z)$, and to an arbitrary timelike curve μ through z. Let $\mathcal{C} \subset O$ be an open neighbourhood of z and choose $0 < t_1 < t_* < 1 < t_2 < b$ with $\sigma([0,t_*]) \subset \mathcal{C}$. Let $\mathcal{C}_1$ be the neighbourhood of $\sigma(t_*)$ given by Lemma 5.3(iii) for $t = t_*$, and let θ, $V' \ni x_k$, u and $F := F_0 = \theta\,\delta(\varphi)$ be given by Lemma 6.3 with $\mathcal{N}_* = \mathcal{C}_1 \cap O$. Then $F \in \mathcal{E}'(O)$, and every point of $\operatorname{supp} F \subset \mathcal{C}_1 \cap \{\varphi = 0\}$ is the endpoint of a g-null geodesic segment from z contained in $\mathcal{C} \subset O$; such a segment is a g'-causal curve, so

$$\operatorname{supp} F \subset J_{g'}^+(z).$$

By Lemma 6.3(ii),(iii), $G_g^{\mathrm{ret}} F = u_0 H(\varphi)$ near x_k with $u_0(x_k) > 0$ and $\varphi(x_k) = 0$, $d\varphi(x_k) \neq 0$; hence $x_k \in \operatorname{supp} G_g^{\mathrm{ret}} F$. By Lemma 8.3(a), and since $\operatorname{supp}(\Psi^* U') = \Psi^{-1}(\operatorname{supp} U')$ for the local diffeomorphism Ψ and a distribution U', $\Psi(x_k) \in \operatorname{supp} G_{g'}^{\mathrm{ret}} F \subset J_{g'}^+(\operatorname{supp} F) \subset J_{g'}^+(z)$. Letting $k \to \infty$ and using that $J_{g'}^+(z)$ is closed, $z \leq_{g'} \Psi(x)$. □

Lemma 8.5 (Behaviour at the frontier)**.** *Assume* $\mathsf{P}(\tau)$ *and let* $q_0 \in \Sigma_\tau \cap W$*. Then:*

(a) *there is* $p_0 \in W$ *with* $\Psi_\tau(q) \to p_0$ *as* $q \to q_0$, $q \in A_\tau$*; and* $\mathbf{v}'(p_0) = \mathbf{v}(q_0)$*;*
(b) *if* $q_0 \in \partial W$*, then* $p_0 = q_0$*;*
(c) *there are an open neighbourhood* B *of* q_0 *and* $\hat\Psi \in C^\infty(B; M)$ *with* $\hat\Psi = \Psi_\tau$ *on* $B \cap A_\tau$*; moreover* $g'(d\hat\Psi\,\cdot, d\hat\Psi\,\cdot) = g$ *on* $B \cap \{\psi \geq \tau\}$*, and* $d\hat\Psi(q_0)$ *is invertible.*

Proof. Write $\Psi = \Psi_\tau$. (a) Choose a decreasing neighbourhood basis (B_k) of q_0 with $\overline{B_k} \subset N'$ compact and $B_k \cap A_\tau$ connected (coordinate balls in coordinates in which ψ is a coordinate function). By Lemma 8.2, $\Psi(B_k \cap A_\tau) \subset W \cup B_k$. The sets $C_k = \operatorname{cl} \Psi(B_k \cap A_\tau)$ are compact, connected and decreasing, so $C = \bigcap_k C_k$ is nonempty, compact and connected, and $C \subset W$. If $p \in C$, then $p = \lim \Psi(q_j)$ with $q_j \to q_0$, $q_j \in A_\tau$, and by Lemma 8.3(b), $\mathbf{v}'(p) = \lim \mathbf{v}'(\Psi(q_j)) = \lim \mathbf{v}(q_j) = \mathbf{v}(q_0)$. Thus C is a connected subset of a fibre of the immersion $\mathbf{v}'|_{N'}$, which is discrete: $C = \{p_0\}$.

(b) Let $q_0 \in \partial W$ and $z_j \in O$, $z_j \to q_0$. Let c_j be the curves given by (E) with $c_j(0) = z_j$; along c_j the function ψ is increasing and takes every value in $[\psi(z_j), \infty)$ (F6). Let x_j be the point of c_j with $\psi(x_j) = \tau_j := \max(\psi(z_j), \tau) + 1/j$. Then $x_j \in O \cap A_\tau$, $z_j \leq_g x_j$, and the x_j lie in the compact set $J_g^+(\overline{B_1}) \cap J_g^-(\Sigma_{\tau+1})$ for large j (F4). If x_∞ is a limit point, then $q_0 \leq_g x_\infty$ and $\psi(x_\infty) = \lim \tau_j = \tau = \psi(q_0)$, hence $x_\infty = q_0$. So $x_j \to q_0$, and $p_0 = \lim \Psi(x_j) = \lim x_j = q_0$.

(c) Since $\mathbf{v}'$ is an immersion at p_0, after renumbering the functions v_i' the map $\mathbf{v}_\flat' = (v_1', \dots, v_{n+1}')$ has invertible differential at p_0. By the inverse function theorem there is an open neighbourhood $V_0' \subset N'$ of p_0 which $\mathbf{v}_\flat'$ maps diffeomorphically onto an open set $\mathcal{N}_\flat \subset \mathbb{R}^{n+1}$. Let $\mathsf{p}\colon \mathbb{R}^m \to \mathbb{R}^{n+1}$ be the projection to the first $n+1$ coordinates, $\mathcal{N}_0 = \mathsf{p}^{-1}(\mathcal{N}_\flat)$ and $\varpi = (\mathbf{v}_\flat'|_{V_0'})^{-1} \circ \mathsf{p}\colon \mathcal{N}_0 \to V_0'$. Then $\varpi \circ \mathbf{v}' = \mathrm{id}$ on V_0'; in particular $\mathbf{v}'$ is injective on V_0'. By (a), $\mathbf{v}(q_0) = \mathbf{v}'(p_0) \in \mathcal{N}_0$, and there is an open neighbourhood B of q_0 with $\mathbf{v}(B) \subset \mathcal{N}_0$ and $\Psi(B \cap A_\tau) \subset V_0'$. Put $\hat\Psi = \varpi \circ \mathbf{v}$ on B. For $q \in B \cap A_\tau$, $\mathbf{v}(q) = \mathbf{v}'(\Psi(q))$ and $\Psi(q) \in V_0'$, so $\hat\Psi(q) = \Psi(q)$. The identity $g'(d\hat\Psi\,\cdot, d\hat\Psi\,\cdot) = g$ holds on $B \cap A_\tau$ and, by continuity, on $B \cap \{\psi \geq \tau\}$; in particular $d\hat\Psi(q_0)$ is injective, hence invertible. □

8.2. **The waves of the first metric near a frontier point.**

Lemma 8.6 (Structure of the g-waves)**.** *Assume* $\mathsf{P}(\tau)$*, let* $q_0 \in \Sigma_\tau \cap W$ *and let* p_0 *be as in Lemma 8.5. Let* $\sigma\colon [0,b] \to M$ *be a future-directed* g'*-null geodesic with* $b > 1$*,* $\sigma(1) = p_0$*,* $\sigma(b) \notin I_{g'}^+(\sigma(0))$ *and* $\sigma(t_*) \in O$ *for some* $t_* \in (0,1)$*. Let* μ*,* $t_1 < t_* < 1 < t_2$*,* η*,* $\mathcal{T}$*,* φ' *be as in Lemmas 5.2, 5.3 for* $\mathsf{g} = g'$*, and* θ*,* V'*,* u'*,* $F_s = \theta\,\delta(\varphi' - s)$ *as in Lemma 6.3 for* $\mathsf{g} = g'$*,* $q = 0$ *and* $\mathcal{N}_* = O$*. Then there are an open neighbourhood* V *of* q_0*,* $\eta_2 \in (0, \eta]$*,* $\varphi \in C^\infty(V)$ *and* $u \in C^\infty((-\eta_2, \eta_2) \times V)$ *such that:*

(i) φ *is eikonal of future type for* g *on* V*;* $\Psi_\tau(V \cap A_\tau) \subset V'$*;* $\varphi = \varphi' \circ \Psi_\tau$ *on* $V \cap A_\tau$*; and* $\varphi(q_0) = \varphi'(p_0) = 0$*;*

(ii) $G_g^{\mathrm{ret}} F_s = u_s\, H(\varphi - s)$ *on* V *for* $|s| < \eta_2$, *and* $V \cap \operatorname{supp}\theta = \emptyset$;
(iii) $u_s = u_s' \circ \Psi_\tau$ *on* $V \cap A_\tau \cap \{\varphi \geq s\}$, *and* $u_s > 0$ *on* $\{\varphi = s\} \cap V$.

Proof. Write $\Psi = \Psi_\tau$ and $U_s = G_g^{\mathrm{ret}} F_s$, $U_s' = G_{g'}^{\mathrm{ret}} F_s$; recall $U_s' = u_s' H(\varphi' - s)$ on V', $\varphi'(p_0) = 0$.

Step 1 (the phase). Let B, $\hat\Psi$ be as in Lemma 8.5(c), with B so small that $\hat\Psi(B) \subset V'$ and $\overline{B} \cap \operatorname{supp}\theta = \emptyset$ (possible since $q_0 \in W$ and $\operatorname{supp}\theta$ is a compact subset of O); thus $\Box_g U_s = F_s = 0$ on B. Put $\varphi^\natural = \varphi' \circ \hat\Psi \in C^\infty(B)$. By Lemma 8.5(a) and the continuity of $\hat\Psi$, $\hat\Psi(q_0) = \lim_{q \to q_0,\, q \in A_\tau} \Psi(q) = p_0$; hence $\varphi^\natural(q_0) = \varphi'(p_0) = 0$. On $B \cap A_\tau$, $\varphi^\natural = \varphi' \circ \Psi$ is eikonal of future type for g (Lemma 8.2); by continuity $g(d\varphi^\natural, d\varphi^\natural) = 0$ and $d\varphi^\natural \neq 0$ on $B \cap \{\psi \geq \tau\}$ (after shrinking B). Let $\mathsf{H}(x,\xi) = \frac{1}{2} g_x^{-1}(\xi,\xi)$ on T^*M, Φ_r its Hamilton flow and $\pi\colon T^*M \to M$ the projection; $r \mapsto \pi\Phi_r(x,\xi)$ is the geodesic with initial velocity $g^{-1}\xi$. Let $S_0 \subset \Sigma_\tau \cap B$ be a small open neighbourhood of q_0 in Σ_τ, diffeomorphic to a ball, and

$$\Lambda = \{\Phi_r(\bar x, d\varphi^\natural(\bar x)) :\ \bar x \in S_0,\ |r| < r_0\}, \qquad \mathfrak{F}(\bar x, r) = \pi\Phi_r(\bar x, d\varphi^\natural(\bar x)).$$

Since $\operatorname{grad}_g \varphi^\natural(\bar x)$ is a past-directed null vector, it is transversal to the spacelike hypersurface Σ_τ; hence, for S_0 and r_0 small, $\mathfrak{F}$ is a diffeomorphism of $S_0 \times (-r_0, r_0)$ onto an open neighbourhood $B_2 \subset B$ of q_0, with $\mathfrak{F}(S_0 \times (-r_0, 0]) = B_2 \cap \{\psi \geq \tau\}$. The submanifold Λ is Lagrangian and contained in $\mathsf{H}^{-1}(0)$: H is invariant under the flow and vanishes on $\Lambda_0 = \{(\bar x, d\varphi^\natural(\bar x)) : \bar x \in S_0\}$; the symplectic form vanishes on pairs of vectors tangent to $\Phi_r(\Lambda_0)$ (because Λ_0 lies in the graph of the closed form $d\varphi^\natural$ and Φ_r is symplectic) and on pairs consisting of the Hamilton field and a vector tangent to Λ (because $d\mathsf{H}$ vanishes on $T\Lambda$). As $\pi|_\Lambda$ is a diffeomorphism onto B_2, Λ is the graph of a 1-form ξ on B_2. The pull-back of the symplectic form by the section $\xi\colon B_2 \to T^*M$ is, up to sign, $d\xi$; hence $d\xi = 0$ because Λ is Lagrangian. Since B_2 is diffeomorphic to $S_0 \times (-r_0, r_0)$, hence to a ball, the Poincaré lemma gives $\varphi \in C^\infty(B_2)$ with $d\varphi = \xi$, which we normalize by $\varphi(q_0) = \varphi^\natural(q_0)$; then $g(d\varphi, d\varphi) = 0$ on B_2. (This is the classical solution of the eikonal equation by the method of characteristics; for the corresponding statements in $\mathbb{R}^n$ see [Rau12, Appendix 5.I], in particular the Covering Lemma 5.I.3, the Main Theorem 5.I.4 and the Tangency Lemma 5.I.5.)

We claim $\varphi = \varphi^\natural$ on $B_2 \cap \{\psi \geq \tau\}$. Over the open set $B \cap A_\tau$ the graph $\Lambda^\natural$ of $d\varphi^\natural$ is a Lagrangian submanifold contained in $\mathsf{H}^{-1}(0)$, so the Hamilton field is tangent to it (for a vector X tangent to $\Lambda^\natural$, the symplectic form evaluated on the Hamilton field and X equals $\pm d\mathsf{H}(X) = 0$, and a Lagrangian subspace contains every vector that is symplectically orthogonal to it) and $\Phi_r(x, d\varphi^\natural(x)) \in \Lambda^\natural$ as long as the base point stays in $B \cap A_\tau$. For $x \in B \cap A_\tau$ near $\bar x \in S_0$ and $r \in (-r_0, 0]$ the base curve is future-directed as r decreases, hence stays in the future set A_τ (and in B, if S_0, r_0 are small). Letting $x \to \bar x$ we get $\Phi_r(\bar x, d\varphi^\natural(\bar x)) = (z, d\varphi^\natural(z))$ with $z = \mathfrak{F}(\bar x, r)$, for $r \in (-r_0, 0]$. Hence $d\varphi = d\varphi^\natural$ on the connected set $B_2 \cap \{\psi \geq \tau\}$, and the claim follows. In particular $d\varphi \neq 0$ and $-\operatorname{grad}_g \varphi$ is future-directed at q_0, hence on B_2 after decreasing S_0 and r_0 (which keeps the product structure of B_2): φ is eikonal of future type for g on B_2, $\varphi = \varphi' \circ \Psi$ on $B_2 \cap A_\tau$, and $\varphi(q_0) = 0$.

Step 2 (above the frontier). By Lemma 8.3(a), on $B_2 \cap A_\tau$

$$U_s = \Psi^* U_s' = u_s^\natural\, H(\varphi - s), \qquad u^\natural(s,x) := u'(s, \hat\Psi(x)), \quad u^\natural \in C^\infty((-\eta,\eta) \times B_2). \tag{8.1}$$

Step 3 (lenses). Choose coordinates $y = (y^0, y')$ on an open neighbourhood $V_2 \subset B_2$ of q_0 with $y(q_0) = 0$ and $y^0 = \psi - \tau$, and $\mu_0 > 0$ such that (after shrinking V_2) the functions $y^0 - \mu|y'|^2$, $0 \leq \mu \leq \mu_0$, are temporal for g on V_2. Write $\tilde\psi = y^0 - \mu_0|y'|^2$. For small $r, \delta > 0$ let

$$D = \{y \in V_2 :\ y^0 \leq \delta,\ \tilde\psi \geq -\mu_0 r^2\},$$
$$L^\flat = \{y \in V_2 :\ \delta/4 + \mu_2|y'|^2 \leq y^0 \leq \delta\}, \qquad \mu_2 = \frac{\delta}{4(r^2 + \delta/\mu_0)}.$$

For r, δ small both are compact subsets of V_2, q_0 is an interior point of D, $L^\flat \subset A_\tau$, and $D \cap \{y^0 \geq \delta/2\} \subset L^\flat$. We use:

(∗) a future-directed g-causal curve which starts in D stays in D until it meets $\Sigma_{\tau+\delta}$, and the same holds for $L^\flat$;

indeed ψ, $\tilde\psi$ and $y^0-\mu_2|y'|^2$ increase along it while it is in V_2, and D, $L^\flat$ are compact subsets of V_2.

Step 4 (one-sidedness). We claim that $\operatorname{supp} U_s\cap D\subset\{\varphi\geq s\}$ for $|s|<\eta$. Let $x\in D$ with $\varphi(x)<s$ and suppose $x\in\operatorname{supp}U_s\subset J_g^+(\operatorname{supp}F_s)$. Recall that $D\subset V_2\subset B_2$, where φ is eikonal of future type. The integral curve of $Z_{g,\varphi}$ through x is a future-directed null curve along which φ is constant and ψ increases at a rate bounded below on D; by (∗) it reaches a point $x'\in D\cap\Sigma_{\tau+\delta}\subset B_2\cap A_\tau$, with $\varphi(x')=\varphi(x)<s$ and $x'\in J_g^+(x)\subset J_g^+(\operatorname{supp}F_s)$; moreover $\Psi(x')=\hat\Psi(x')\in V'$ and $\varphi(x')=\varphi'(\Psi(x'))$ by Step 1. So there is $y\in\operatorname{supp}F_s\subset O$ with $y\leq_g x'$. By Lemma 8.4, $y\leq_{g'}\Psi(x')$, hence by Lemma 6.3(i) and Lemma 5.3(ii)

$$\Psi(x')\in J_{g'}^+(\operatorname{supp}F_s)\cap V'\subset J_{g'}^+(\mu(s))\cap\mathcal{T}=\{\varphi'\geq s\},$$

that is, $\varphi(x')=\varphi'(\Psi(x'))\geq s$, a contradiction.

Step 5 (the amplitude). Let $X=-\operatorname{grad}_g\psi$ and $\mathcal{N}=X/(X\varphi)$ on V_2; here $X\varphi=-g(X,Z)>0$ with $Z=Z_{g,\varphi}$. Choose coordinates (ς,z), $z=(z^1,\dots,z^n)$, on a neighbourhood $V_3\subset V_2$ of q_0 with $\varsigma=\varphi$ and $\partial_\varsigma=\mathcal{N}$. Since $g^{\varsigma\varsigma}=g(d\varphi,d\varphi)=0$,

$$\Box_g=2Z\partial_\varsigma+(\Box_g\varphi)\,\partial_\varsigma+P,\qquad Z=-g^{\varsigma j}\partial_{z^j},$$

where P is a differential operator of order two containing only the derivatives ∂_{z^j}, with coefficients depending smoothly on (ς,z). For a smooth function w write $j_k=\partial_\varsigma^k w|_{\varsigma=s}$, a function of z. Then, by Leibniz' rule,

$$\partial_\varsigma^k(\Box_g w)\big|_{\varsigma=s}=2Z_sj_{k+1}+(\Box_g\varphi)_sj_{k+1}+\sum_{i\leq k}P_{k,i}^{(s)}\,j_i,\tag{8.2}$$

where the subscript s denotes restriction of the coefficients to $\varsigma=s$ and $P_{k,i}^{(s)}$ are differential operators in z whose coefficients are smooth in (s,z). Consequently (Lemma 6.1 with $h=H(\cdot-s)$): $\Box_g(wH(\varphi-s))=R\,H(\varphi-s)$ with R vanishing to infinite order on $\{\varphi=s\}$ if and only if

$$2Z_sj_0+(\Box_g\varphi)_sj_0=0,\qquad 2Z_sj_{k+1}+(\Box_g\varphi)_sj_{k+1}=-\sum_{i\leq k}P_{k,i}^{(s)}\,j_i\quad(k\geq0).\tag{8.3}$$

By (8.1) the functions $j_k^\natural(s,z)=\partial_\varsigma^k u_s^\natural|_{\varsigma=s}$ satisfy (8.3) at all points of A_τ. The equations (8.3) are linear ordinary differential equations along the integral curves of Z_s in $\{\varsigma=s\}$, which are transversal to the spacelike surface $\Sigma_{\tau+\delta}$. The domains are arranged as follows. Take $V_3=(-a,a)\times Z_0$ a product in the coordinates (ς,z) with $0<a\leq\eta$ (possible since $\varphi(q_0)=0$), let $\eta_2=a$, and choose r,δ so small that $D\cup L^\flat\subset V_3$. Let V_4 be the set of points of V_3 whose integral curve of Z in V_3 meets $\Sigma_{\tau+\delta}$; it is open, and $D\cup L^\flat\subset V_4$ by (∗). The equations (8.3), with $s=\varphi(x)$ at the point x, are linear ordinary differential equations along these curves for functions J_k on V_4 (so that $j_k(s,\cdot)=J_k|_{\{\varsigma=s\}}$); let $J_k\in C^\infty(V_4)$ be the solutions which equal $j_k^\natural(\varphi(x),z(x))$ on $\Sigma_{\tau+\delta}$. A segment of an integral curve of Z which joins a point of A_τ to $\Sigma_{\tau+\delta}$ lies in A_τ (a future set); by uniqueness for (8.3) (induction on k), $J_k(x)=j_k^\natural(\varphi(x),z(x))$ on $V_4\cap A_\tau$. Choose $\rho\in C_c^\infty(V_4)$ with $\rho=1$ on an open set $V_5\supset D\cup L^\flat$. Write $(\rho J_k)(s,z)$ for the value of ρJ_k at the point with coordinates $(\varsigma,z)=(s,z)$; these are smooth functions with compact support in $(-a,a)\times Z_0$. As in the proof of Lemma 6.3 (Borel's lemma with parameters) there are $\epsilon_k\in(0,1]$ such that

$$\tilde u(s,(\varsigma,z))=\sum_{k\geq0}(\rho J_k)(s,z)\,\frac{(\varsigma-s)^k}{k!}\,\chi_0\big((\varsigma-s)/\epsilon_k\big)$$

converges in $C^\infty((-\eta_2,\eta_2)\times V_3)$; then $\partial_\varsigma^k\tilde u_s|_{\varsigma=s}=(\rho J_k)|_{\{\varsigma=s\}}$ for all k and $|s|<\eta_2$. Then:

(1) $\Box_g(\tilde u_sH(\varphi-s))=f_s$ on V_5, where $f_s=R_sH(\varphi-s)$, $R_s=\Box_g\tilde u_s$, is smooth, jointly in (s,x);

(2) $Y_s := (u_s^\natural - \tilde u_s)H(\varphi - s)$ is smooth on $V_5 \cap A_\tau$, jointly in (s,x), and $\Box_g Y_s = -f_s$ there, by (8.1) and (1).

For the joint smoothness, use the coordinates (s,y,z) with $y = \varsigma - s$: the smooth function R satisfies $\partial_y^k R(s,0,z) = 0$ for all k and all (s,z) with $(s,z) \in V_5$, by (8.2) and (8.3); differentiating these identities in s and z shows that R vanishes to infinite order on $\{y=0\}$, so that $R\,H(y)$ is smooth in (s,y,z). The same argument applies to $u^\natural - \tilde u$ on A_τ, whose ∂_y-derivatives of all orders vanish on $\{y=0\}$. Choose $\zeta \in C_c^\infty(V_5)$ with $\zeta = 1$ near $D \cup L^\flat$ and $\zeta_1 \in C_c^\infty(V_5 \cap A_\tau)$ with $\zeta_1 = 1$ near $L^\flat$. Let $v_s \in C^\infty(M)$ be the solution of $\Box_g v_s = -\zeta f_s$ whose Cauchy data on the Cauchy hypersurface $\Sigma_{\tau+\delta}$ are those of $\zeta_1 Y_s$ (F7); $(s,x) \mapsto v_s(x)$ is smooth. Put

$$\Delta_s = U_s - \zeta\,\tilde u_s H(\varphi - s) - v_s \in \mathcal{D}'(M).$$

(α) $\Box_g \Delta_s = 0$ on a neighbourhood of $D \cup L^\flat$, because $F_s = 0$ and $\zeta = 1$ there.

(β) $\Delta_s = 0$ on $L^\flat$: near $L^\flat$ we have $\Delta_s = Y_s - v_s$ by (8.1); this function is smooth, solves $\Box_g(Y_s - v_s) = 0$ near $L^\flat$, and its Cauchy data vanish on $\Sigma_{\tau+\delta} \cap L^\flat$. By the energy inequality (7.2) of Lemma 7.1, applied on the lens $L^\flat$ (with the temporal functions ψ and $y^0 - \mu_2|y'|^2$), $Y_s - v_s = 0$ on $L^\flat$.

(γ) $\Delta_s = 0$ on $\operatorname{int} D$. Let $\vartheta \in C_c^\infty(\operatorname{int} D)$ and $\chi = G_g^{\mathrm{ret}}\vartheta$; choose $\epsilon > 0$ with $\operatorname{supp}\vartheta \subset \{\psi < \tau + \delta - 2\epsilon\}$, $2\epsilon < \delta/2$, and $\varkappa \in C^\infty(\mathbb{R})$ with $\varkappa = 1$ on $(-\infty, \tau+\delta-2\epsilon]$ and $\varkappa = 0$ on $[\tau+\delta-\epsilon, \infty)$. By $(*)$, $\operatorname{supp}(\varkappa(\psi)\chi) \subset J_g^+(\operatorname{supp}\vartheta) \cap \{\psi \le \tau+\delta-\epsilon\}$ is a compact subset of D, and $\Box_g(\varkappa(\psi)\chi) = \vartheta + [\Box_g, \varkappa(\psi)]\chi$, where the last term is smooth and supported in $D \cap \{\tau+\delta-2\epsilon \le \psi \le \tau+\delta-\epsilon\} \subset L^\flat$. By ($\alpha$) and ($\beta$), $0 = \langle \Box_g\Delta_s, \varkappa(\psi)\chi\rangle = \langle \Delta_s, \vartheta\rangle + \langle \Delta_s, [\Box_g, \varkappa(\psi)]\chi\rangle = \langle \Delta_s, \vartheta\rangle$.

Thus $U_s = \tilde u_s H(\varphi - s) + v_s$ on $\operatorname{int} D$. By Step 4, $v_s = 0$ on $\operatorname{int} D \cap \{\varphi < s\}$; being smooth, v_s vanishes to infinite order on $\{\varphi = s\} \cap \operatorname{int} D$, and $U_s = (\tilde u_s + v_s)H(\varphi - s)$ on $\operatorname{int} D$. Let $V = \operatorname{int} D$ and $u = \tilde u + v$ on $(-\eta_2, \eta_2) \times V$. This proves (ii). By (8.1), $u_s = u_s^\natural = u_s' \circ \Psi$ on $V \cap A_\tau \cap \{\varphi > s\}$, hence on its closure in $V \cap A_\tau$. On $\{\varphi = s\} \cap V$ we have $u_s = j_0(s,\cdot)$, which solves the first equation in (8.3) along the integral curves of Z and equals $u_s' \circ \Psi > 0$ on $\Sigma_{\tau+\delta}$ (Lemma 6.3(iii)); hence $u_s > 0$ there. Statement (i) holds by Step 1, since $V \subset B_2 \subset B$ and $\Psi(B \cap A_\tau) = \hat\Psi(B \cap A_\tau) \subset V'$. □

8.3. **Proof of Theorem 8.1.** *Step A (start).* If $W = \emptyset$, then $g = g'$ and $\Psi = \mathrm{id}$ has all the required properties; let $W \ne \emptyset$ and $\tau_{\max} = \max_W \psi$. For $\tau \ge \tau_{\max}$, $A_\tau \cap W = \emptyset$ and $\Psi_\tau = \mathrm{id}$ satisfies $\mathsf{P}(\tau)$ by (S).

Step B (closedness). If $\mathsf{P}(\tau)$ holds for all $\tau > \tau_\bullet$, then $\mathsf{P}(\tau_\bullet)$ holds: by uniqueness the maps Ψ_τ define a smooth map on $A_{\tau_\bullet} = \bigcup_{\tau > \tau_\bullet} A_\tau$ with the required properties.

Step C (extension). We show: if $\mathsf{P}(\tau)$ holds, then $\mathsf{P}(\tau - \delta)$ holds for some $\delta > 0$. Then the set of τ with $\mathsf{P}(\tau)$, an up-set containing $[\tau_{\max}, \infty)$, is all of $\mathbb{R}$: otherwise its infimum $\tau_\bullet$ is finite, $\mathsf{P}(\tau_\bullet)$ holds by Step B, and $\mathsf{P}(\tau_\bullet - \delta)$ holds, a contradiction. Write $\Psi = \Psi_\tau$. If $\Sigma_\tau \cap W = \emptyset$, then $W \cap \{\tau - \delta \le \psi \le \tau\} = \emptyset$ for small $\delta > 0$ and Ψ extended by the identity satisfies $\mathsf{P}(\tau - \delta)$. Assume now $\Sigma_\tau \cap W \ne \emptyset$ and fix $q_0 \in \Sigma_\tau \cap W$; let p_0 be as in Lemma 8.5.

C1 (directions). By Lemma 5.5(a) and Lemma 5.6 for $\mathsf{g} = g'$ (and the open set O), the set $\mathcal{V}(p_0)$ is a nonempty open subset of the future null cone at p_0; its image under $\zeta \mapsto g'(\zeta,\cdot)$ is a nonempty open subset of the null cone of $T^*_{p_0}M$. By Lemma 5.7 there is a finite set $\mathcal{A} \subset \mathcal{V}(p_0)$ and $I \subset \mathcal{A}$ with $n+1$ elements such that $\{g'(p_0)\} \cup \{g'(\zeta,\cdot) \otimes g'(\zeta,\cdot)\}_{\zeta\in\mathcal{A}}$ spans $\mathrm{S}^2T^*_{p_0}M$ and $(g'(\zeta,\cdot))_{\zeta\in I}$ is a basis of $T^*_{p_0}M$. For $a = \zeta \in \mathcal{A}$ let $\sigma_a = \sigma_\zeta$ be the geodesic of Lemma 5.6, and let $b_a > 1$ and $t_{1,a} \in (0,1)$ be as in the definition of $\mathcal{V}(p_0)$ for ζ. Choose a future-directed timelike curve μ_a through $\sigma_a(0)$ and numbers $0 < t_1' < t_* < t_{1,a}$ and $1 < t_2 < b_a$ (they depend on a; the index is suppressed; here $\sigma_a(t_*) \in O$), and let φ_a', θ_a, V_a', u_a', $F_{a,s} = \theta_a\delta(\varphi_a' - s) \in \mathcal{E}'(O)$ be given by Lemmas 5.3 and 6.3 (for $\mathsf{g} = g'$, $q = 0$, $\mathcal{N}_* = O$, the geodesic σ_a, the curve μ_a and the interval $[t_1', t_2]$). By Lemma 5.3(i), $d\varphi_a'(p_0) = -\lambda_a\, g'(\zeta,\cdot)$ with $\lambda_a > 0$, and $\varphi_a'(p_0) = 0$. Let V_a, φ_a, u_a, $\eta_{2,a}$ be given by Lemma 8.6.

C2 (local gauge). Let $V'_{\mathrm{inj}} \ni p_0$ be an open set on which $\mathbf{v}'$ is injective (see the proof of Lemma 8.5(c)). Choose $0 < \eta_3 \le \min_a \eta_{2,a}$, $\mathcal{J} = (-\eta_3, \eta_3)$, and a coordinate neighbourhood $V' \subset V'_{\mathrm{inj}} \cap \bigcap_a V'_a$ of p_0 such that $\varphi'_a(V') \subset \mathcal{J}$, the spanning and basis properties of C1 hold with $d\varphi'_a(p)$ at every $p \in V'$, and $\Phi'_I = (\varphi'_a)_{a\in I}$ is a diffeomorphism from V' onto an open subset of $\mathbb{R}^{1+n}$. Choose an open neighbourhood $V \subset \bigcap_a V_a$ of q_0 with $\varphi_a(V) \subset \mathcal{J}$, $\Phi_I(V) \subset \Phi'_I(V')$ where $\Phi_I = (\varphi_a)_{a\in I}$, and $\Psi(V \cap A_\tau) \subset V'$ (possible since $\Phi_I(q_0) = 0 = \Phi'_I(p_0)$ and by Lemma 8.5(a)); in addition $V \subset \operatorname{int} W$ if $q_0 \in \operatorname{int} W$, and $V \subset V'$ if $q_0 \in \partial W$ (possible since then $p_0 = q_0$). Define

$$\Psi_{q_0} = (\Phi'_I|_{V'})^{-1} \circ \Phi_I \colon V \to V'.$$

For $q \in V \cap A_\tau$ we have $\Phi'_I(\Psi(q)) = \Phi_I(q)$ (Lemma 8.6(i)) and $\Psi(q) \in V'$, hence $\Psi_{q_0} = \Psi$ on $V_+ = V \cap A_\tau$.

C3 (application of the local step). The hypotheses of Proposition 7.3 hold for Ψ_{q_0} on V: (L1) by Lemma 6.3(ii),(iv) and Lemma 8.6(ii),(iii), since $F_{a,s} = 0$ on V and on V'; (L2) by $\Psi_{q_0} = \Psi$ on V_+ and Lemma 8.6(i),(iii); (L3) by construction; (L4) by C2. Hence there is an open neighbourhood $V^\circ(q_0) \subset V$ of q_0, with the additional property stated in Proposition 7.3, on which Ψ_{q_0} is a local diffeomorphism with $\Psi_{q_0}^* g' = g$.

C4 (identity outside W). We claim $\Psi_{q_0} = \mathrm{id}$ on $V^\circ(q_0) \setminus W$. If $q_0 \in \operatorname{int} W$ this set is empty. Let $q_0 \in \partial W$. For each i the function $w = \mathsf{T} v'_i - v'_i \circ \Psi_{q_0}$ is smooth on $V^\circ(q_0)$, $\Box_g w = 0$ there (as Ψ_{q_0} is a local isometry on $V^\circ(q_0)$), and $w = 0$ on $V^\circ(q_0) \cap A_\tau$ by Lemma 8.3(b). Hence $w = 0$, i.e. $\mathbf{v} = \mathbf{v}' \circ \Psi_{q_0}$ on $V^\circ(q_0)$. For $x \in V^\circ(q_0) \setminus W$ we also have $\mathbf{v}(x) = \mathbf{v}'(x)$, so $\mathbf{v}'(x) = \mathbf{v}'(\Psi_{q_0}(x))$ with $x \in V \subset V' \subset V'_{\mathrm{inj}}$ and $\Psi_{q_0}(x) \in V' \subset V'_{\mathrm{inj}}$; by injectivity $\Psi_{q_0}(x) = x$.

C5 (patching). Let $\mathcal{N}_\psi = \operatorname{grad}_g \psi / g(d\psi, d\psi)$, so that $\mathcal{N}_\psi \psi = 1$, and let $\Xi(r, y)$ be the value at time r of the integral curve of $\mathcal{N}_\psi$ starting at $y \in \Sigma_\tau$; then $\psi(\Xi(r, y)) = \tau + r$, and Ξ is an injective local diffeomorphism on its open domain in $\mathbb{R} \times \Sigma_\tau$. For each $q_0 \in \Sigma_\tau \cap W$ choose a connected open neighbourhood $S(q_0)$ of q_0 in Σ_τ and $\delta(q_0) > 0$ with $\Xi((-\delta(q_0), \delta(q_0)) \times S(q_0)) \subset V^\circ(q_0)$. By compactness finitely many $S_i = S(q_i)$, $i = 1, \dots, N$, cover $\Sigma_\tau \cap W$. Let $\delta_0 = \min_i \delta(q_i)$ and $B_i = \Xi((-\delta_0, \delta_0) \times S_i)$. Each connected component of $B_i \cap B_j = \Xi((-\delta_0, \delta_0) \times (S_i \cap S_j))$ is of the form $\Xi((-\delta_0, \delta_0) \times S)$ with S a component of $S_i \cap S_j$; it is connected and meets A_τ, where $\Psi_{q_i} = \Psi = \Psi_{q_j}$. By Lemma 7.4, $\Psi_{q_i} = \Psi_{q_j}$ on $B_i \cap B_j$. Since the compact sets $W \cap \{\tau - \delta \le \psi \le \tau\}$ decrease to $W \cap \Sigma_\tau \subset \bigcup_i B_i$ as $\delta \downarrow 0$, there is $\delta \in (0, \delta_0)$ with $W \cap \{\tau - \delta \le \psi \le \tau\} \subset \bigcup_i B_i$. Define Ψ^{new} on $A_{\tau-\delta}$ by

$$\Psi^{\mathrm{new}} = \Psi \text{ on } A_\tau, \qquad \Psi^{\mathrm{new}} = \Psi_{q_i} \text{ on } B_i \cap A_{\tau-\delta}, \qquad \Psi^{\mathrm{new}} = \mathrm{id} \text{ on } A_{\tau-\delta} \setminus W.$$

These open sets cover $A_{\tau-\delta}$, and the definitions agree on overlaps: $\Psi_{q_i} = \Psi$ on $B_i \cap A_\tau$ by C2; $\Psi_{q_i} = \Psi_{q_j}$ on $B_i \cap B_j$; $\Psi_{q_i} = \mathrm{id}$ on $B_i \setminus W$ by C4; and $\Psi = \mathrm{id}$ on $A_\tau \setminus W$. Thus Ψ^{new} is smooth, $(\Psi^{\mathrm{new}})^* g' = g$, and $\Psi^{\mathrm{new}} = \mathrm{id}$ outside W: $\mathsf{P}(\tau - \delta)$ holds.

Step D (conclusion). By Steps A–C, $\mathsf{P}(\tau)$ holds for all τ, and the maps Ψ_τ define a smooth $\Psi \colon M \to M$ with $\Psi^* g' = g$ and $\Psi = \mathrm{id}$ on O. By Lemma 8.2, $\Psi(W) \subset W$ and Ψ preserves the time orientation. Ψ is a local diffeomorphism, and it is proper: for compact K the closed set $\Psi^{-1}(K)$ is contained in $K \cup W$. We show that Ψ is bijective. For $y \in M$ the fibre $\Psi^{-1}(y)$ is compact and discrete, hence finite; let $N(y)$ be its cardinality. The function N is locally constant: let $\Psi^{-1}(y) = \{x_1, \dots, x_k\}$ and choose disjoint open neighbourhoods U_i of x_i which Ψ maps diffeomorphically onto open sets. There is a neighbourhood V_y of y with $\Psi^{-1}(V_y) \subset \bigcup_i U_i$: otherwise there are $z_j \to y$ and $x'_j \in \Psi^{-1}(z_j) \setminus \bigcup_i U_i$; by properness a subsequence of (x'_j) converges to a point $x' \notin \bigcup_i U_i$ with $\Psi(x') = y$, a contradiction. Hence $N = k$ on $V_y \cap \bigcap_i \Psi(U_i)$ (on V_y if $k = 0$). As M is connected, N is constant. M is not compact (F6), so there is $y \in O$, and $\Psi^{-1}(y) = \{y\}$ because $\Psi(W) \subset W$ and $\Psi = \mathrm{id}$ on O. Hence $N \equiv 1$, and Ψ is a bijective local diffeomorphism, that is, a diffeomorphism; then $\Psi(W) = W$. (This is a special case of the fact that a proper local diffeomorphism is a covering map.) The last assertion of the theorem is Lemma 8.3(b), which holds for every τ. This proves Theorem 8.1. $\square$

8.4. **The general case: proof of Theorem 1.3.** We now assume (S) with $n \geq 2$, (D) and (H).

Reduction to the case that W_+ is compact. The hypotheses (S), (D) and the conclusions of Theorem 1.3 do not change if the time orientations of g and g' are both reversed (the set $C_g(O)$ does not depend on the time orientation), and this exchanges S_+ and S_-. Hence we may assume that S_+ is relatively compact, that is, that W_+ is compact (Lemma 4.4(d)).

Enlarging W. The setting (S) holds with W_+, O_+ in place of W, O. By Proposition 4.1, (D) gives $L_g^{\mathrm{ret}} = L_{g'}^{\mathrm{ret}}$ on $C_c^\infty(O)$; restricting to sources and observations in $O_+ \subset O$ and applying Proposition 4.1 to the pair (W_+, O_+) we obtain $C_g(O_+) = C_{g'}(O_+)$. By Lemma 4.4(a), (E) holds for (W_+, O_+). Hence Theorem 8.1 gives a diffeomorphism $\Psi\colon M \to M$ with $\Psi^* g' = g$, $\Psi = \mathrm{id}$ on O_+, which preserves the time orientation, and such that $v = v' \circ \Psi$ whenever $\Box_g v = 0$, $\Box_{g'} v' = 0$ and $v = v'$ on O_+. It remains to show that $\Psi = \mathrm{id}$ on $S_+ = O \setminus O_+$; then $\Psi = \mathrm{id}$ on O, $\Psi(W) = W$, and Theorem 1.3 is proved. This is the content of the following proposition.

Proposition 8.7 (The isometry is the identity on the trapped part of the exterior)**.** *Assume* (S) *with $n \geq 2$ and* (D)*, and assume that W_+ is compact. Let $\Psi\colon M \to M$ be a diffeomorphism with $\Psi^* g' = g$ which preserves the time orientation and satisfies $\Psi = \mathrm{id}$ on O_+. Then $\Psi = \mathrm{id}$ on O.*

For points of a connected component of O which meets O_+ this follows from Lemma 7.4, since Ψ and the identity are local isometries of (O, g) into (M, g'). The following proof covers all points of S_+, including components of O enclosed by W.

Proof. Let $x \in S_+$ and $\tilde{x} = \Psi(x)$. As $\Psi(O_+) = O_+$, we have $x, \tilde{x} \in W_+$. Recall that (E) holds for the pair (W_+, O_+) (Lemma 4.4(a)).

Step 1 (an identity for advanced waves). Let $F \in \mathcal{E}'(O)$ and $\Psi_* F = (\Psi^{-1})^* F$. The distribution $U = \Psi^*\big(G_{g'}^{\mathrm{adv}}(\Psi_* F)\big)$ satisfies $\Box_g U = F$ and $\operatorname{supp} U = \Psi^{-1}\big(\operatorname{supp} G_{g'}^{\mathrm{adv}}(\Psi_* F)\big) \subset \Psi^{-1}\big(J_{g'}^-(\Psi(\operatorname{supp} F))\big) = J_g^-(\operatorname{supp} F)$, because Ψ is an isometry preserving the time orientation. Thus $U - G_g^{\mathrm{adv}} F$ solves the homogeneous equation and has future compact support (F4), so $U = G_g^{\mathrm{adv}} F$ by (F7). By Proposition 4.1, $G_g^{\mathrm{adv}} F = G_{g'}^{\mathrm{adv}} F$ on O. Since $\Psi = \mathrm{id}$ on the open set O_+, we conclude

$$G_{g'}^{\mathrm{adv}}(\Psi_* F) = G_{g'}^{\mathrm{adv}} F \quad \text{on } O_+, \qquad F \in \mathcal{E}'(O). \tag{8.4}$$

Step 2. Claim: if $y \in O_+ \cap E_{g'}^-(x)$, then $y \leq_{g'} \tilde{x}$. Apply Lemma 5.4 with $\mathtt{g} = g'$, the open set O_+, $p_0 = x$ and y: there are $b > 1$, $t_1 \in (0,1)$ and a g'-null geodesic $\sigma\colon [0,b] \to M$ with $\sigma(1) = x$, $\sigma(b) \notin I_{g'}^+(\sigma(0))$, $\sigma([0,t_1]) \subset O_+$ and $y \leq_{g'} \sigma(0)$. Fix $t_c \in (0, t_1)$ and put $p_1 = \sigma(t_c) \in O_+$. The curve

$$\rho(t) = \sigma\big(b - t(b - t_c)\big), \qquad 0 \leq t \leq b' := \frac{b}{b - t_c},$$

is a past-directed g'-null geodesic with $\rho(0) = \sigma(b)$, $\rho(1) = p_1$, $\rho(b') = \sigma(0) \notin I_{g'}^-(\rho(0))$, and $\rho(t_*) = x$ for $t_* = (b-1)/(b-t_c) \in (0,1)$. Let $\mathcal{N}_* \subset O$ be an open neighbourhood of x (recall $x \in O$). By Lemmas 5.2, 5.3 and 6.3 with the time orientation reversed (Remark 6.4), applied to $\mathtt{g} = g'$, $q = 0$, the geodesic ρ, a past-directed timelike curve through $\rho(0)$ and the point $\rho(t_*) = x$, there are an eikonal function φ near p_1 with $\varphi(p_1) = 0$, a function $\theta \in C_c^\infty(\mathcal{N}_*)$, and a neighbourhood V' of p_1 such that $F = \theta\, \delta(\varphi) \in \mathcal{E}'(O)$ satisfies $G_{g'}^{\mathrm{adv}} F = u_0 H(\varphi)$ on V' with $u_0 > 0$ on $\{\varphi = 0\} \cap V'$. Since $d\varphi(p_1) \neq 0$, every neighbourhood of p_1 contains points where $u_0 H(\varphi) \neq 0$; thus $p_1 \in \operatorname{supp} G_{g'}^{\mathrm{adv}} F$. By (8.4) and $p_1 \in O_+$,

$$p_1 \in \operatorname{supp} G_{g'}^{\mathrm{adv}}(\Psi_* F) \subset J_{g'}^-\big(\Psi(\operatorname{supp} F)\big) \subset J_{g'}^-\big(\Psi(\overline{\mathcal{N}_*})\big).$$

Letting $\mathcal{N}_*$ run through a basis of neighbourhoods of x with compact closures and using that the relation $\leq_{g'}$ is closed (F4), we get $p_1 \leq_{g'} \Psi(x) = \tilde{x}$. As $y \leq_{g'} \sigma(0) \leq_{g'} p_1$, the claim follows.

Step 3. Claim: if $y \in O_+ \cap E^-_{g'}(\tilde{x})$, then $y \leq_{g'} x$. The hypotheses of the proposition are symmetric under the exchange of (g, g', Ψ) with (g', g, Ψ^{-1}). Step 2 in the exchanged situation says: if $y \in O_+ \cap E^-_g(x)$, then $y \leq_g \Psi^{-1}(x)$. Now let $y \in O_+ \cap E^-_{g'}(\tilde{x})$. Since Ψ is an isometry preserving the time orientation with $\Psi(y) = y$ and $\Psi(x) = \tilde{x}$, we have $y \in E^-_g(x)$, hence $y \leq_g \Psi^{-1}(x)$, and applying Ψ, $y \leq_{g'} x$.

Step 4. Claim: $O_+ \cap E^-_{g'}(x) = O_+ \cap E^-_{g'}(\tilde{x})$. From now on all causal notions refer to g'. Let $y \in O_+ \cap E^-(x)$. By Step 2, $y \leq \tilde{x}$. Suppose $y \in I^-(\tilde{x})$. Let $c\colon [0,T) \to O_+$ be a future-inextendible timelike curve with $c(0) = y$. The set $\{t \in [0,T) : c(t) \leq \tilde{x}\}$ is closed in $[0,T)$ (F4), contains a neighbourhood of 0 because $I^-(\tilde{x})$ is open, is an interval containing 0, and is not all of $[0,T)$, since otherwise c would stay in the compact set $J^+(y) \cap J^-(\tilde{x})$, contradicting (F1). Let $t^* > 0$ be its maximum and $z = c(t^*) \in O_+$. Then $z \leq \tilde{x}$ and $z \notin I^-(\tilde{x})$ (otherwise $c(t) \leq \tilde{x}$ for some $t > t^*$), so $z \in O_+ \cap E^-(\tilde{x})$, and $z \leq x$ by Step 3. But $y \ll z$, hence $y \ll x$, which contradicts $y \in E^-(x)$. Therefore $y \in E^-(\tilde{x})$. The other inclusion is proved in the same way, with the roles of Steps 2 and 3 exchanged.

Step 5 (conclusion). The following argument is, with the time orientation reversed, the argument used in [KLU18, pp. 801–802] to show that a point is determined by its earliest light observation set. Suppose $\tilde{x} \neq x$. Let $\mathcal{V}(x)$ be the set of Lemma 5.6 for $\mathfrak{g} = g'$, the open set O_+ and $p_0 = x$. By Lemma 5.5(a), applied to the compact set W_+, and Lemma 5.6, $\mathcal{V}(x)$ is a nonempty open subset of the future null cone of T_xM. Let $\zeta \in \mathcal{V}(x)$, and let σ_ζ, b, t_1 be as in the definition of $\mathcal{V}(x)$. For $t \in [0, t_1]$ the point $\sigma_\zeta(t)$ belongs to O_+, satisfies $\sigma_\zeta(t) \leq x$, and does not belong to $I^-(x)$, because $\sigma_\zeta(0) \leq \sigma_\zeta(t) \ll x \leq \sigma_\zeta(b)$ would give $\sigma_\zeta(b) \in I^+(\sigma_\zeta(0))$. Thus $\sigma_\zeta(t) \in O_+ \cap E^-(x) = O_+ \cap E^-(\tilde{x})$ for $t \in [0, t_1]$, by Step 4. Fix $t_a \in (0, t_1]$ and let $y' = \sigma_\zeta(t_a)$; then $y' \neq \tilde{x}$, since $y' \in O_+$ and $\tilde{x} \in W_+$. By (F5) and (F2) there is a null geodesic ρ_ζ from y' to $\tilde{x}$. The curve $\sigma_\zeta|_{[0,t_a]}$ followed by ρ_ζ is a causal curve from $\sigma_\zeta(0)$ to $\tilde{x}$, and $\sigma_\zeta(0) \in E^-(\tilde{x})$; by (F2) it is a null pregeodesic. Hence ρ_ζ is a reparametrization of a segment of the maximally extended geodesic σ_ζ which begins at the parameter t_a: there is $a_\zeta > t_a$ in the domain of the maximally extended geodesic, again denoted by σ_ζ, with

$$\tilde{x} = \sigma_\zeta(a_\zeta).$$

Since σ_ζ is injective (there are no closed causal curves) and $\tilde{x} \neq x = \sigma_\zeta(1)$, we have $a_\zeta \neq 1$.

Since $n \geq 2$, the future null cone of T_xM is a manifold of dimension $n \geq 2$, in which a ray has empty interior; hence the open set $\mathcal{V}(x)$ contains two linearly independent vectors ζ_1, ζ_2. Write $\sigma_i = \sigma_{\zeta_i}$, $a_i = a_{\zeta_i}$ and $y_i = \sigma_i(0)$; recall that $y_i \in E^-(x) \cap E^-(\tilde{x})$ and $\tilde{x} = \sigma_i(a_i)$. We use the following consequence of (F2): a causal curve consisting of two null geodesic segments whose velocities at the junction are not proportional is not a null pregeodesic (the image of a reparametrized geodesic is, near each of its points, an embedded curve to which both one-sided velocities are tangent), so its endpoints are chronologically related.

Case $a_1 > 1$. Then $x \leq \tilde{x}$ and $x \neq \tilde{x}$, hence $\tilde{x} \not\leq x$, because there are no closed causal curves; therefore also $a_2 > 1$. The curve $\sigma_1|_{[0,1]}$ from y_1 to x followed by $\sigma_2|_{[1,a_2]}$ from x to $\tilde{x}$ is causal, and its velocities at the junction x are ζ_1 and ζ_2. Hence $y_1 \ll \tilde{x}$, which contradicts $y_1 \in E^-(\tilde{x})$.

Case $a_1 < 1$. Then $\tilde{x} \leq x$ and $\tilde{x} \neq x$, hence $a_2 < 1$ as well. The curve $\sigma_1|_{[0,a_1]}$ from y_1 to $\tilde{x}$ followed by $\sigma_2|_{[a_2,1]}$ from $\tilde{x}$ to x is causal. Its velocities $\dot\sigma_1(a_1)$, $\dot\sigma_2(a_2)$ at the junction $\tilde{x}$ are not proportional: otherwise, by the uniqueness of geodesics, $\sigma_2(t) = \sigma_1(\alpha t + \beta)$ with constants $\alpha > 0$ and β; then $\sigma_1(\alpha + \beta) = \sigma_2(1) = x = \sigma_1(1)$ and the injectivity of σ_1 give $\alpha + \beta = 1$, so that $\zeta_2 = \dot\sigma_2(1) = \alpha\,\zeta_1$, contradicting the linear independence. Hence $y_1 \ll x$, which contradicts $y_1 \in E^-(x)$.

Both cases are impossible. Hence $\tilde{x} = x$. □

9. Proofs of the corollaries

9.1. **Proof of Corollary 1.4.** The setting (S) holds with $M = \mathbb{R}^{1+n}$, and (D) holds by assumption. We verify (H) by showing that W_+ is compact. Let $T_0, R_0 > 0$ with $W \subset$

$[-T_0,T_0]\times\overline{B_{R_0}}$, where $B_{R_0}\subset\mathbb{R}^n$ is the open ball of radius R_0. On O the metric is g_{Min} and ∂_t is future-directed. Points $(t,x)\in O$ with $|x|>R_0$ or $t>T_0$ belong to O_+: the curve $s\mapsto(t+s,x)$, $s\geq 0$, is timelike, future-inextendible and contained in O. If $|x|\leq R_0$ and $t<-T_0-4R_0$, let e be a unit vector with $e\cdot x\geq 0$; the timelike curve $s\mapsto(t+s,x+\frac{1}{2}se)$, $s\geq 0$, is future-inextendible, and $|x+\frac{1}{2}se|\geq\frac{1}{2}s$ is nondecreasing in s; so the curve is in $\{t<-T_0\}$ for $s<-T_0-t$ and in $\{|x|>R_0\}$ for $s\geq -T_0-t>4R_0$, hence in O. Thus $W_+\subset[-T_0-4R_0,T_0]\times\overline{B_{R_0}}$, and W_+ is compact because it is closed (Lemma 4.4(a)). Theorem 1.3 gives the diffeomorphism Ψ. □

The verification of (H) uses the time orientation only on the set $\{|x|>R_0\}\cup\{|t|>T_0\}$, which is connected and contained in O, and on which the time orientation of g_{Min} is that of ∂_t or that of $-\partial_t$. In the second case the same argument, applied after the reflection $t\mapsto -t$, gives $S_+\subset[-T_0,T_0+4R_0]\times\overline{B_{R_0}}$. Hence (H) holds whenever $M=\mathbb{R}^{1+n}$ and $g=g'=g_{\mathrm{Min}}$ on O.

9.2. **Proof of Corollary 1.5.** Let $W=M_0$ and $O=\mathbb{R}^{1+n}\setminus M_0$, which is connected in both cases. On O each of the two metrics equals g_{Min}, and its time orientation is that of ∂_t or the opposite one. Reversing the time orientation of a metric changes neither its global hyperbolicity nor its Cauchy data set; hence we may assume that ∂_t is future-directed on O for both metrics. By Corollary 1.4 and Proposition 4.1 it suffices to show that $C_g^{\mathrm{Hyp}}=C_{g'}^{\mathrm{Hyp}}$ implies $L_g^{\mathrm{ret}}=L_{g'}^{\mathrm{ret}}$ on $C_c^\infty(O)$.

Let $f\in C_c^\infty(O)$ and $u=G_g^{\mathrm{ret}}f$. Then $u|_{M_0}\in C^\infty(M_0)$ solves $\Box_g u=0$ in M_0, so there is $\tilde u\in C^\infty(M_0)$ with $\Box_{g'}\tilde u=0$ in M_0 and with the same Cauchy data $(\tilde u|_{\partial M_0},\partial_\nu\tilde u|_{\partial M_0})=(u|_{\partial M_0},\partial_\nu u|_{\partial M_0})$ on the smooth part $\partial^{\mathrm{reg}}M_0$ of ∂M_0 (which is all of ∂M_0 in the second case, and the complement of the edges $\{\pm T\}\times\partial B$ in the first case). Since the restriction to $\partial^{\mathrm{reg}}M_0$ determines the tangential derivatives, the full differentials coincide there: $d\tilde u=du$ on $\partial^{\mathrm{reg}}M_0$. Let $\hat u=u$ on $\mathbb{R}^{1+n}\setminus M_0$ and $\hat u=\tilde u$ on M_0; this is a continuous function. We claim that $\Box_{g'}\hat u=f$ in $\mathcal{D}'(\mathbb{R}^{1+n})$. Let $\vartheta\in C_c^\infty(\mathbb{R}^{1+n})$. For a metric g, a bounded Lipschitz domain D with piecewise smooth boundary and functions w,ϑ which are smooth up to the boundary, the divergence theorem gives Green's formula

$$\int_D\big(w\,\Box_{\mathsf{g}}\vartheta-\vartheta\,\Box_{\mathsf{g}}w\big)\,dV_{\mathsf{g}}=\int_{\partial D}\big(\vartheta\,\mathsf{g}^{-1}(\mathsf{n},dw)-w\,\mathsf{g}^{-1}(\mathsf{n},d\vartheta)\big)\,|\det\mathsf{g}|^{1/2}\,dS,$$

where n is the Euclidean outward unit conormal and dS the Euclidean surface measure. We apply this with $D=\operatorname{int}M_0$, $\mathsf{g}=g'$, $w=\tilde u$, and with $D=B_\varrho\setminus M_0$ for a large ball $B_\varrho\subset\mathbb{R}^{1+n}$ containing $M_0\cup\operatorname{supp}\vartheta$ in its interior, $\mathsf{g}=g_{\mathrm{Min}}$, $w=u$. On ∂M_0 we have $g'=g_{\mathrm{Min}}$, the two outward conormals are opposite, and $d\tilde u=du$ on $\partial^{\mathrm{reg}}M_0$, whose complement in ∂M_0 has surface measure zero. Hence the boundary terms cancel, and

$$\langle\hat u,\Box_{g'}\vartheta\rangle=\int_{M_0}\vartheta\,\Box_{g'}\tilde u\,dV_{g'}+\int_{\mathbb{R}^{1+n}\setminus M_0}\vartheta\,\Box_{g_{\mathrm{Min}}}u\,dV_{g_{\mathrm{Min}}}=\langle f,\vartheta\rangle,$$

since $\Box_{g'}\tilde u=0$ in M_0 and $\Box_{g_{\mathrm{Min}}}u=\Box_g u=f$ outside M_0.

As in the proof of Proposition 4.1, (iii)⇒(i), there is a Cauchy hypersurface $\Sigma\subset O$ of $(\mathbb{R}^{1+n},g')$ near which both $G_{g'}^{\mathrm{ret}}f$ and u, hence $\hat u$, vanish. By (F7) (uniqueness for distributions), $\hat u=G_{g'}^{\mathrm{ret}}f$, and therefore $L_{g'}^{\mathrm{ret}}f=\hat u|_O=u|_O=L_g^{\mathrm{ret}}f$. □

Remark 9.1. Under the hypotheses of Theorem 1.3, all objects defined by the metrics correspond under Ψ: for instance $G_g^{\mathrm{ret}}=\Psi^*\circ G_{g'}^{\mathrm{ret}}\circ(\Psi^{-1})^*$, and the causal relations, null geodesics, conjugate points and null cut loci of g and g' are mapped to each other. In particular, properties such as directional simplicity [ORS26, Definition 3.1] hold for g if and only if they hold for g'; this is a consequence, not an ingredient, of the proof.

9.3. **Data consisting of global solutions only.** Consider the restrictions to the exterior of the solutions on all of M:

$$C_g^0(O)=\big\{u|_O:\ u\in C^\infty(M),\ \Box_g u=0\text{ on }M\big\}\subset C_g(O).$$

These data determine the source-to-solution map for sources in O and observations outside the causal hull of W.

Proposition 9.2 (Global solutions as data)**.** *Assume* (S) *and* $W \neq \emptyset$*, and let* $\widehat{W} = J_g^+(W) \cap J_g^-(W)$ *and* $\widehat{O} = M \setminus \widehat{W}$*.*

(a) $J_g^\pm(W) = J_{g'}^\pm(W)$*. The set* $\widehat{W}$ *is compact and contains* W*, and* (S) *holds for the pair* $(\widehat{W}, \widehat{O})$ *in place of* (W, O)*.*

(b) (D) *implies* $C_g^0(O) = C_{g'}^0(O)$*.*

(c) *If* $C_g^0(O) = C_{g'}^0(O)$*, then* $G_g^{\mathrm{ret}} f = G_{g'}^{\mathrm{ret}} f$ *on* $\widehat{O}$ *for every* $f \in C_c^\infty(O)$*; in particular* $C_g(\widehat{O}) = C_{g'}(\widehat{O})$*, that is,* (D) *holds for the pair* $(\widehat{W}, \widehat{O})$*.*

(d) *Let* $n \geq 2$*,* $C_g^0(O) = C_{g'}^0(O)$*, and assume that* (H) *holds for the pair* $(\widehat{W}, \widehat{O})$*. Then there is a diffeomorphism* $\Psi \colon M \to M$ *with* $\Psi^* g' = g$ *which preserves the time orientation and is the identity on* $\widehat{O}$ *and on every connected component of* O *that meets* $\widehat{O}$*. In particular,* (M, g) *and* (M, g') *are isometric, and if* $\widehat{W} = W$ *the conclusions of Theorem 1.3 hold.*

Hypothesis (H) for the pair $(\widehat{W}, \widehat{O})$ holds if $M = \mathbb{R}^{1+n}$ and $g = g' = g_{\mathrm{Min}}$ on O, by Section 9.1 applied to the compact set $\widehat{W}$. We have $\widehat{W} = W$ if and only if W is causally convex, that is, if every causal curve with endpoints in W is contained in W; the closed cylinder of Corollary 1.5 is not causally convex.

Proof. Let ϑ' be a Cauchy temporal function of (M, g'), $m_- = \min_W \vartheta'$ and $m_+ = \max_W \vartheta'$. We use the following observation, already made in the proof of Proposition 4.1. Let ρ be a g-causal curve from y to x. If ρ does not meet W, it is g'-causal, and $\vartheta'(x) \geq \vartheta'(y)$. If ρ meets W, its part after the last point in W and its part before the first point in W are g'-causal (their one-sided velocities at these points are non-zero limits of future-directed g'-causal vectors), hence

$$\vartheta'(x) \geq m_- \qquad \text{and} \qquad \vartheta'(y) \leq m_+. \tag{9.1}$$

Consequently, for compact $K_0 \subset M$,

$$J_g^+(K_0) \subset \{\vartheta' \geq \min(m_-, \min_{K_0} \vartheta')\}, \qquad J_g^-(K_0) \subset \{\vartheta' \leq \max(m_+, \max_{K_0} \vartheta')\}, \tag{9.2}$$

and if $K_0 \subset \{\vartheta' > m_+\}$, then no g-causal curve starting in K_0 meets W, so that

$$J_g^+(K_0) = J_{g'}^+(K_0) \subset \{\vartheta' \geq \min_{K_0} \vartheta'\} \subset O \tag{9.3}$$

(a g'-causal curve starting in K_0 stays in $\{\vartheta' > m_+\} \subset O$ and is g-causal).

(a) Let $x \in J_g^+(W) \setminus W$ and let ρ be a g-causal curve from a point of W to x. Its part after the last point in W is a g'-causal curve from a point of W to x; hence $x \in J_{g'}^+(W)$. By symmetry $J_g^+(W) = J_{g'}^+(W)$, and in the same way $J_g^-(W) = J_{g'}^-(W)$. The set $\widehat{W}$ is compact by (F4) and contains W; as $\widehat{O} \subset O$, the metrics and their time orientations agree on $\widehat{O}$.

(b) Let $u \in C^\infty(M)$ with $\Box_g u = 0$ on M. By (D) there is $\tilde{u} \in C^\infty(M)$ with $\tilde{u} = u$ on O and $\Box_{g'} \tilde{u} = 0$ near W. On O we have $\Box_{g'} \tilde{u} = \Box_g u = 0$. Hence $\Box_{g'} \tilde{u} = 0$ on M, and $u|_O \in C_{g'}^0(O)$. The other inclusion is symmetric.

(c) Let $f \in C_c^\infty(O)$ and $d = G_g^{\mathrm{ret}} f - G_{g'}^{\mathrm{ret}} f \in C^\infty(M)$. On O we have $\Box_{g'} d = \Box_g G_g^{\mathrm{ret}} f - f = 0$.

Step 1. By (9.2), $d = 0$ on $\{\vartheta' < m_0\}$, where $m_0 = \min(m_-, \min_{\mathrm{supp}\, f} \vartheta')$.

Step 2: $d = 0$ *on* $\{\vartheta' > m_+\}$. Let $h \in C_c^\infty(M)$ with $\mathrm{supp}\, h \subset \{\vartheta' > m_+\}$, and let $b = \max_{\mathrm{supp}\, h} \vartheta'$. The function $z = G_{g'}^{\mathrm{ret}} h$ is supported in $J_{g'}^+(\mathrm{supp}\, h) = J_g^+(\mathrm{supp}\, h) \subset O$, by (9.3). Hence $\Box_g z = \Box_{g'} z = h$ on O, and $z = 0 = h$ on the open set $M \setminus \mathrm{supp}\, z$, which together with O covers M; so $\Box_g z = h$ on M. As $\mathrm{supp}\, z \subset J_g^+(\mathrm{supp}\, h)$ is past compact for g (F4), the distribution $z - G_g^{\mathrm{ret}} h$ vanishes by (F7):

$$G_g^{\mathrm{ret}} h = G_{g'}^{\mathrm{ret}} h = z \quad \text{on } M.$$

The function $v = G_g^{\mathrm{adv}} h - G_g^{\mathrm{ret}} h$ satisfies $\Box_g v = 0$ on M. By assumption there is $v' \in C^\infty(M)$ with $\Box_{g'} v' = 0$ on M and $v' = v$ on O. Let $w' = G_{g'}^{\mathrm{adv}} h - G_{g'}^{\mathrm{ret}} h$. By (9.2), $G_g^{\mathrm{adv}} h$ and $G_{g'}^{\mathrm{adv}} h$ vanish on $\{\vartheta' > b\} \subset O$, so that $v' = v = -z = w'$ there. Thus the solution $v' - w'$ of the g'-wave equation vanishes on a neighbourhood of the Cauchy hypersurface $\{\vartheta' = b + 1\}$ of (M, g'), and $v' = w'$ on M by (F7). On O this gives $G_g^{\mathrm{adv}} h - z = v = v' = w' = G_{g'}^{\mathrm{adv}} h - z$, that is,

$$G_g^{\mathrm{adv}} h = G_{g'}^{\mathrm{adv}} h \quad \text{on } O.$$

Since $\operatorname{supp} f \subset O$, $\operatorname{supp} h \subset O$, and the volume densities of g and g' agree on O, (F7) gives

$$\int (G_g^{\mathrm{ret}} f)\, h \, dV = \int f\, (G_g^{\mathrm{adv}} h)\, dV = \int f\, (G_{g'}^{\mathrm{adv}} h)\, dV = \int (G_{g'}^{\mathrm{ret}} f)\, h \, dV.$$

Hence $\int d\, h \, dV = 0$ for all such h, and $d = 0$ on $\{\vartheta' > m_+\}$.

Step 3: $d = 0$ on $\widehat{O}$. Let $x \in \widehat{O}$. Then $x \in O$, and $x \notin J^-(W)$ or $x \notin J^+(W)$; by (a) these sets are the same for g and g'. Assume first $x \notin J_{g'}^-(W)$. There is an open neighbourhood B of x with compact closure such that $J_{g'}^+(\overline{B}) \cap W = \emptyset$: otherwise there are $x_k \to x$ and $w_k \in W$ with $x_k \le_{g'} w_k$, and a limit point $w \in W$ of the w_k satisfies $x \le_{g'} w$ by (F4). Let $\phi \in C_c^\infty(B)$ and $\zeta = G_{g'}^{\mathrm{ret}} \phi$, so that $\operatorname{supp} \zeta \subset J_{g'}^+(\overline{B})$. Choose $c > \max(m_+, \max_{\overline{B}} \vartheta')$. The set $L = J_{g'}^+(\overline{B}) \cap \{\vartheta' \le c\}$ is compact (F4) and contained in O. Choose $\chi \in C_c^\infty(O)$ with $\chi = 1$ on a neighbourhood of L. Every point of $\operatorname{supp} \zeta \setminus L$ lies in $\{\vartheta' > c\}$, where $d = 0$ by Step 2. Therefore $d \Box_{g'} \zeta = (\chi d)\, \Box_{g'} \zeta$ on M, and $\Box_{g'}(\chi d) = 0$ on a neighbourhood of $\operatorname{supp} \zeta$: near L it equals $\Box_{g'} d = 0$, and on $\{\vartheta' > c\}$ the function χd vanishes. Since $\chi d \in C_c^\infty(M)$ and $\Box_{g'}$ is formally self-adjoint,

$$\int d\, \phi \, dV_{g'} = \int (\chi d)\, \Box_{g'} \zeta \, dV_{g'} = \int \Box_{g'}(\chi d)\, \zeta \, dV_{g'} = 0.$$

As $\phi \in C_c^\infty(B)$ is arbitrary, $d = 0$ on B. If $x \notin J_{g'}^+(W)$, the same argument applies with the time orientation reversed: with a neighbourhood B such that $J_{g'}^-(\overline{B}) \cap W = \emptyset$, with $\zeta = G_{g'}^{\mathrm{adv}} \phi$, $c < \min(m_0, \min_{\overline{B}} \vartheta')$, $L = J_{g'}^-(\overline{B}) \cap \{\vartheta' \ge c\}$, and with Step 1 in place of Step 2.

Thus $L_g^{\mathrm{ret}} = L_{g'}^{\mathrm{ret}}$ for sources and observations in $\widehat{O}$, and Proposition 4.1, applied to the pair $(\widehat{W}, \widehat{O})$, gives $C_g(\widehat{O}) = C_{g'}(\widehat{O})$.

(d) By (a) and (c), the hypotheses (S), (D) and (H) hold for the pair $(\widehat{W}, \widehat{O})$, and Theorem 1.3 gives a diffeomorphism Ψ with $\Psi^* g' = g$ and $\Psi = \mathrm{id}$ on $\widehat{O}$, preserving the time orientation. Let O_1 be a connected component of O which meets $\widehat{O}$. The restriction of Ψ to O_1 and the inclusion $O_1 \to M$ are smooth maps F with $F^* g' = g$ on O_1 (because $g = g'$ on O), and they agree on the nonempty open set $O_1 \cap \widehat{O}$. By Lemma 7.4, $\Psi = \mathrm{id}$ on O_1. □

10. Proof of Theorem 1.6

In this section (M, g), W, O, q, q' are as in Theorem 1.6, and $P = \Box_g + q$, $P' = \Box_g + q'$. The setting (S) holds with $g' = g$ and any $n \ge 1$, so that Proposition 4.1 applies: the hypothesis $C_{g,q}(O) = C_{g,q'}(O)$ is equivalent to

$$(G_P^{\mathrm{ret}} F)|_O = (G_{P'}^{\mathrm{ret}} F)|_O \qquad \text{for all } F \in \mathcal{E}'(O), \tag{10.1}$$

and to the same statement for the advanced Green operators.

10.1. **Condition (N).**

Lemma 10.1. *Condition* (N) *of Theorem 1.6 holds if the Cauchy hypersurfaces of (M, g) are not compact, and it holds if S_+ or S_- is relatively compact in M.*

Proof. If the Cauchy hypersurfaces are not compact, then $O \cap E^-(p) \neq \emptyset$ for every $p \in W$ by Lemma 5.5(b). If S_+ is relatively compact, then W_+ is compact and (E) holds for the pair (W_+, O_+) (Lemma 4.4); by Lemma 5.5(a), applied to W_+, $O_+ \cap E^-(p) \neq \emptyset$ for every

$p \in W \subset W_+$, and $O_+ \subset O$. If S_- is relatively compact, the same argument with the time orientation reversed gives $O \cap E^+(p) \neq \emptyset$ for every $p \in W$. □

The hypotheses and the conclusion of Theorem 1.6 are invariant under reversal of the time orientation, which exchanges the two alternatives in (N). Hence we assume from now on:

$$O \cap E^-(p) \neq \emptyset \qquad \text{for every } p \in W. \tag{10.2}$$

10.2. **The equations for the difference of the amplitudes.**

Lemma 10.2. *Let $V \subset M$ be open, $\varphi \in C^\infty(V)$ eikonal for g, $Z = Z_{g,\varphi}$, $\mathcal{J}$ an open interval with $\varphi(V) \subset \mathcal{J}$, and let $u, u' \in C^\infty(\mathcal{J} \times V)$ be such that, for all $s \in \mathcal{J}$,*

$$P\big(u_s H(\varphi - s)\big) = 0, \qquad P'\big(u'_s H(\varphi - s)\big) = 0 \quad \text{in } \mathcal{D}'(V), \qquad u_s = u'_s \ \text{ on } \{\varphi = s\}.$$

Let $w_s = u_s - u'_s$ and $\omega_s = \partial_s w_s$. Then, for all $s \in \mathcal{J}$,

(a) $P\omega_s = -(q - q')\,\partial_s u'_s$ *on* $\{\varphi \geq s\}$;
(b) $2Z\omega_s + (\Box_g \varphi)\,\omega_s = (q - q')\,u'_s$ *on* $\{\varphi = s\}$.

Proof. Write $\Gamma_s = \{\varphi = s\}$. By Lemma 6.2(a), $Pu_s = 0$ and $P'u'_s = 0$ on $\{\varphi \geq s\}$; hence $Pw_s = -(P' + (q - q'))u'_s = -(q - q')u'_s$ on $\{\varphi \geq s\}$. Since $w_s = 0$ on Γ_s and Z is tangent to Γ_s (because $Z\varphi = 0$), also $Zw_s = 0$ on Γ_s. By (6.1),

$$P\big(w_s H(\varphi - s)\big) = H(\varphi - s)\,Pw_s + (2Zw_s + w_s \Box_g \varphi)\,\delta(\varphi - s) = -(q - q')\,u'_s\,H(\varphi - s) \quad \text{in } \mathcal{D}'(V). \tag{10.3}$$

We differentiate (10.3) with respect to s. Let $\chi \in C_c^\infty(V)$. For a function $f \in C^\infty(\mathcal{J} \times V)$, $f_s = f(s, \cdot)$, we have $\langle f_s H(\varphi - s), \chi \rangle = \int_s^\infty \big(\int_{\Gamma_r} f_s \chi \, d\mu_r\big) dr$, and this is a C^1 function of s with derivative

$$\frac{d}{ds}\langle f_s H(\varphi - s), \chi \rangle = \langle (\partial_s f_s) H(\varphi - s), \chi \rangle - \int_{\Gamma_s} f_s \chi \, d\mu_s,$$

because $I(s, r) = \int_{\Gamma_r} f_s \chi \, d\mu_r$ and $\partial_s I$ are continuous in (s, r): $f\chi$ is smooth in (s, x) with support in $\mathcal{J} \times \operatorname{supp} \chi$, and with a partition of unity and flow boxes of a vector field N with $N\varphi = 1$, $I(s, r)$ becomes an integral of a jointly smooth compactly supported function over a fixed slice. For $f = w$ the last term vanishes, and for $f = (q - q')u'$ it equals $\langle (q - q')u'_s \delta(\varphi - s), \chi \rangle$. Since P is formally self-adjoint, (10.3) says $\langle w_s H(\varphi - s), P\chi \rangle = -\langle (q - q')u'_s H(\varphi - s), \chi \rangle$ for all s; differentiating in s we obtain

$$P\big(\omega_s H(\varphi - s)\big) = -(q - q')\,(\partial_s u'_s)\,H(\varphi - s) + (q - q')\,u'_s\,\delta(\varphi - s) \quad \text{in } \mathcal{D}'(V).$$

By (6.1) the left-hand side equals $H(\varphi - s)P\omega_s + (2Z\omega_s + \omega_s \Box_g \varphi)\delta(\varphi - s)$. Comparing the locally bounded functions and the smooth densities on Γ_s on both sides, as in the proof of Lemma 6.2, gives (a) on $\{\varphi > s\}$, hence on $\{\varphi \geq s\}$ by continuity, and (b). □

10.3. **The continuity argument.** Let ψ be a Cauchy temporal function of (M, g) (F6), $A_\tau = \{\psi > \tau\}$, $\Sigma_\tau = \{\psi = \tau\}$, and consider the statement

$\mathsf{Q}(\tau)$: $q = q'$ on A_τ.

$\mathsf{Q}(\tau)$ holds for $\tau \geq \max_W \psi$, and if it holds for all $\tau > \tau_\bullet$ it holds for $\tau_\bullet$. As in Section 8.3, Step C, it therefore suffices to show: *if $\mathsf{Q}(\tau)$ holds, then $\mathsf{Q}(\tau - \delta)$ holds for some $\delta > 0$.* Assume $\mathsf{Q}(\tau)$. If $\Sigma_\tau \cap W = \emptyset$, then $W \cap \{\tau - \delta \leq \psi \leq \tau\} = \emptyset$ for small $\delta > 0$ and $\mathsf{Q}(\tau - \delta)$ holds. Assume $\Sigma_\tau \cap W \neq \emptyset$ and fix $q_0 \in \Sigma_\tau \cap W$.

Step 1 (the waves). By (10.2) and Lemma 5.4 (for $\mathsf{g} = g$, the open set O and $p_0 = q_0$) there are $b > 1$, $t_1 \in (0, 1)$ and a null geodesic $\sigma\colon [0, b] \to M$ with $\sigma(1) = q_0$, $\sigma(b) \notin I^+(\sigma(0))$ and $\sigma([0, t_1]) \subset O$. Let μ be a timelike curve through $\sigma(0)$, choose $0 < t'_1 < t_* < t_1$ and $t_2 \in (1, b)$, and let η, $\mathcal{T}$, φ be given by Lemma 5.3 for σ, μ and $[t'_1, t_2]$. Apply Lemma 6.3 with $\mathcal{N}_* = O$ to the operators P and P'. Since θ, V' and b_0 do not depend on the potential, we obtain $\theta \in C_c^\infty(O)$, an open neighbourhood $V' \subset \mathcal{T}$ of q_0 with $V' \cap \operatorname{supp} \theta = \emptyset$, a function $b_0 > 0$ on V', and $u, u' \in C^\infty((-\eta, \eta) \times V')$ such that, with $F_s = \theta\,\delta(\varphi - s) \in \mathcal{E}'(O)$,

$$U_s := G_P^{\mathrm{ret}} F_s = u_s H(\varphi - s), \qquad U'_s := G_{P'}^{\mathrm{ret}} F_s = u'_s H(\varphi - s) \quad \text{on } V',$$

and $u_s = u_s' = b_0$ on $\{\varphi = s\} \cap V'$. As $F_s = 0$ on V', the hypotheses of Lemma 10.2 hold on $V = V'$ with $\mathcal{J} = (-\eta, \eta)$.

Step 2 (the waves coincide above Σ_τ). Let $z = U_s - U_s' \in \mathcal{D}'(M)$. Then $Pz = F_s - P'U_s' - (q - q')U_s' = -(q - q')U_s'$, which vanishes on A_τ by $\mathsf{Q}(\tau)$, and $z = 0$ on O by (10.1). Let $\vartheta \in C_c^\infty(A_\tau)$ and $\chi = G_P^{\mathrm{ret}}\vartheta$. Then $\operatorname{supp}\chi \subset J^+(\operatorname{supp}\vartheta) \subset A_\tau$, and $J^+(\operatorname{supp}\vartheta) \cap W$ is a compact subset of A_τ (F4). Choose $\zeta \in C_c^\infty(A_\tau)$ with $\zeta = 1$ near this set. Then $(1-\zeta)\chi$ vanishes on a neighbourhood of $W \cap A_\tau \supset \operatorname{supp} z \cap A_\tau$, so the compactly supported function $\vartheta - P(\zeta\chi) = P((1-\zeta)\chi)$ vanishes near $\operatorname{supp} z$, and $\langle z, \vartheta\rangle = \langle z, P(\zeta\chi)\rangle = \langle Pz, \zeta\chi\rangle = 0$. Hence $z = 0$ on A_τ, that is,

$$w_s := u_s - u_s' = 0 \quad \text{on } V' \cap A_\tau \cap \{\varphi > s\}, \qquad |s| < \eta. \tag{10.4}$$

Step 3 (Cauchy data of $\omega_s = \partial_s w_s$). The set of all $(s, x) \in (-\eta, \eta) \times (V' \cap A_\tau)$ with $\varphi(x) > s$ is open, and w vanishes on it by (10.4). Hence ω and its first derivatives vanish on the closure of this set in $(-\eta, \eta) \times V'$. This closure contains every (s, x) with $x \in \Sigma_\tau \cap V'$ and $\varphi(x) \geq s$, because $d\psi$ and $d\varphi$ are linearly independent ($d\psi$ is timelike and $d\varphi$ is null). Thus

$$\omega_s = 0 \ \text{ and } \ d\omega_s = 0 \quad \text{on } \Sigma_\tau \cap V' \cap \{\varphi \geq s\}, \qquad |s| < \eta. \tag{10.5}$$

Step 4 (the estimate on a lens). As in Step 0 of the proof of Proposition 7.3, choose coordinates $y = (y^0, y')$ on a neighbourhood $V_2 \subset V'$ of q_0 with $y(q_0) = 0$, $y^0 = \psi - \tau$, and $\mu_0 > 0$ such that $\tilde\psi = y^0 - \mu_0|y'|^2$ is temporal on V_2, and let, for small $r > 0$,

$$L = \{y \in V_2 :\ y^0 \leq 0,\ \tilde\psi \geq -\mu_0 r^2\},$$

a compact subset of V_2 such that $L \cup \{y^0 \geq 0\}$ contains a neighbourhood of q_0. Let $[s_-, s_+] = \varphi(L) \subset (-\eta, \eta)$ and write $\Gamma_s = \{\varphi = s\}$. In the following C denotes constants independent of $\sigma \geq 1$ and of $s \in [s_-, s_+]$. By (10.5), the inequality (7.1) of Lemma 7.1 (on V_2, with ψ, $\tilde\psi$, φ, τ, $c = -\mu_0 r^2$) applies to ω_s:

$$\sigma \int_{\Gamma_s \cap L} e^{2\sigma\psi}\big(|Z\omega_s|^2 + \sigma^2\omega_s^2\big)d\mu_s + \sigma^4 \int_{L \cap \{\varphi > s\}} e^{2\sigma\psi}\omega_s^2\, dV_g \leq C \int_{L \cap \{\varphi > s\}} e^{2\sigma\psi}|\Box_g\omega_s|^2 dV_g.$$

By Lemma 10.2(a), $\Box_g\omega_s = -q\,\omega_s - (q - q')\partial_s u_s'$ on $\{\varphi \geq s\}$, hence $|\Box_g\omega_s|^2 \leq C(\omega_s^2 + |q - q'|^2)$ on $L \cap \{\varphi \geq s\}$. For σ large the term with ω_s^2 is absorbed by the left-hand side, and we get

$$\sigma \int_{\Gamma_s \cap L} e^{2\sigma\psi}\big(|Z\omega_s|^2 + \omega_s^2\big)d\mu_s \leq C \int_L e^{2\sigma\psi}|q - q'|^2 dV_g.$$

By Lemma 10.2(b) and $u_s' = b_0 \geq \min_L b_0 > 0$ on $\Gamma_s \cap L$, we have $|q - q'|^2 \leq C(|Z\omega_s|^2 + \omega_s^2)$ on $\Gamma_s \cap L$. Therefore

$$\sigma \int_{\Gamma_s \cap L} e^{2\sigma\psi}|q - q'|^2 d\mu_s \leq C \int_L e^{2\sigma\psi}|q - q'|^2 dV_g, \qquad s \in [s_-, s_+].$$

Integrating over $s \in [s_-, s_+]$ and using $\int_L F\, dV_g = \int_{s_-}^{s_+}\big(\int_{\Gamma_s \cap L} F\, d\mu_s\big)ds$ we obtain $\sigma \int_L e^{2\sigma\psi}|q - q'|^2 dV_g \leq C(s_+ - s_-)\int_L e^{2\sigma\psi}|q - q'|^2 dV_g$. For σ large this gives $q = q'$ on L.

Step 5 (conclusion). Let $\mathcal{N}(q_0)$ be the interior of $L \cup (V_2 \cap \{\psi \geq \tau\})$; it is an open neighbourhood of q_0, and $q = q'$ on $\mathcal{N}(q_0)$ by Step 4 and $\mathsf{Q}(\tau)$. Finitely many of the sets $\mathcal{N}(q_0)$, $q_0 \in \Sigma_\tau \cap W$, cover the compact set $\Sigma_\tau \cap W$. Since the compact sets $W \cap \{\tau - \delta \leq \psi \leq \tau\}$ decrease to $W \cap \Sigma_\tau$ as $\delta \downarrow 0$, there is $\delta > 0$ such that $W \cap \{\tau - \delta \leq \psi \leq \tau\}$ is contained in the union of these finitely many sets. As $q = q'$ on O, $\mathsf{Q}(\tau - \delta)$ holds.

Hence $\mathsf{Q}(\tau)$ holds for all τ, that is, $q = q'$ on M. Together with Lemma 10.1 this proves Theorem 1.6. □

Remark 10.3. Steps 1–4 show the following local statement: if $\mathsf{Q}(\tau)$ holds and $q_0 \in \Sigma_\tau \cap W$ satisfies $O \cap E^-(q_0) \neq \emptyset$, then $q = q'$ in a neighbourhood of q_0. Condition (N) is used only to reach every point of W.

11. Examples for the condition on W and the hypotheses (E), (H) and (N)

In the following, we give several examples of spacetimes that illustrate the hypotheses (E), (H) and (N) and show that the compactness assumption on W cannot be removed.

Example 11.1 (A counterexample with non-compact W)**.** Next, we construct a counterexample when W is a non-compact subset of the spacetime. As inverse problems where the metric is unknown in a cylindrical subset have been extensively studied under various assumptions, see e.g. [AFO22] and references therein, we consider the case when W is a cylindrical subset of the spacetime $\mathbb{R}^{1+3}$ with the Minkowski metric tensor that is perturbed in W.

Let $M = \mathbb{R}^{1+3}$, $W = \mathbb{R} \times \overline{B_1}$, where $B_R \subset \mathbb{R}^3$ is a ball centered at zero of radius $R > 0$. Moreover, let $O = M \setminus W$ and $U = \mathbb{R} \times B_{1/4}$. Choose $a \in C^\infty([0,\infty); [0,2])$ with $a = 0$ on $[0, 1/4] \cup [3/4, \infty)$ and $a = 2$ on $[3/8, 5/8]$. Put $v(x) = -a(|x|)x/|x|$ for $x \neq 0$, $v(0) = 0$, and, for a real $\phi \in C_c^\infty(U)$ to be chosen below, set

$$g = -dt^2 + \sum_{j=1}^{3} (dx^j - v^j(x)\, dt)^2, \qquad g' = e^{2\phi} g.$$

These metrics are smooth and Lorentzian, and both equal to the Minkowski metric g_{Min} for $|x| \geq 3/4$. Time-orient them by $\partial_t + v \cdot \partial_x$. Since $g^{-1}(dt, dt) = -1$, every future causal curve $\gamma(t)$ can be parametrized as $(t, x(t))$, and $g(\dot\gamma, \dot\gamma) \leq 0$ gives

$$|\dot x - v(x)| \leq 1, \qquad |\dot x| \leq 3,$$

where $\dot x(t) = \frac{d}{dt} x(t)$. A finite endpoint of its t-range would therefore give a spacetime endpoint and allow extension. Thus every slice $t =$ constant is a Cauchy surface, and both metrics are globally hyperbolic. Moreover, their causal cones coincide. On $3/8 \leq r = |x| \leq 5/8$ we have $\dot r \leq -a(r) + 1 = -1$. No future causal curve can cross the annulus $\{(t, x) :\ 3/8 \leq |x| \leq 5/8\}$ outward, so

$$J_g^+(U) \cap O = J_{g'}^+(U) \cap O = \emptyset.$$

For any $u \in C^\infty(M)$, the function $F = (\Box_g - \Box_{g'})u$ has compact support contained in $\operatorname{supp} \phi \subset U$. By (F7), $u' = u + G_{g'}^{\mathrm{ret}} F$ satisfies $\Box_{g'} u' = \Box_g u$ and $u' = u$ on O. Applying this to solutions near W, and then interchanging the metrics, gives $C_{g'}(O) = C_g(O)$ directly. For $u = G_g^{\mathrm{ret}} f$ with $f \in C_c^\infty(O)$, we also have $\operatorname{supp} F \subset J_g^+(\operatorname{supp} f)$. As the causal cones for g and g' are the same, the uniqueness of the Cauchy problem for the wave equation gives

$$L_{g'}^{\mathrm{ret}} f = L_g^{\mathrm{ret}} f \qquad (f \in C_c^\infty(O)).$$

Finally, $C_0 := \sup_M |\operatorname{Scal}(g)| < \infty$, since g is stationary and Minkowskian outside a spatially compact set. Choose $\chi \in C_c^\infty(U)$ equal to 1 near $p = (0, 0)$ and take $\phi(t, x) = At^2 \chi(t, x)$ with $12A > C_0$. Since $g = g_{\mathrm{Min}}$ near p and $\phi(p) = d\phi(p) = 0$, the conformal scalar-curvature formula [Wal84, Appendix D] gives

$$\operatorname{Scal}(g')(p) = 6\Box_g \phi(p) = 12A > C_0.$$

Thus g and g' are not isometric, even without requiring the isometry to fix O. The exterior satisfies (E) by vertical timelike rays, but W is noncompact and the pocket U cannot send signals to O. Other, related examples with non-compact unknown sets W are discussed in [OS26].

Example 11.2 (Conformal invariance in spacetime dimension two)**.** Let $M = \mathbb{R}^{1+1}$, $g = g_{\mathrm{Min}}$ be the Minkowski metric, $W = \{(t, x) : |t| + |x| \leq 1\}$ and $O = M \setminus W$. Choose $0 \neq \phi \in C_c^\infty(\operatorname{int} W; \mathbb{R})$ and set $g' = e^{2\phi} g$, with future direction ∂_t. The causal structure is Minkowskian, so both metrics are globally hyperbolic and $M = I^+(O) \cap I^-(O)$. Every exterior point admits a future-inextendible timelike ray in O with outward speed sufficiently close to one; thus (E) and (H) hold. In dimension two,

$$\Box_{g'} = e^{-2\phi} \Box_g, \qquad G_{g'}^{\mathrm{ret}} f = G_g^{\mathrm{ret}} f \quad (f \in C_c^\infty(O)).$$

The first identity gives $C_{g'}(O) = C_g(O)$; the second follows from $e^{2\phi} f = f$ and the uniqueness of the Cauchy problem for the wave equation. However, $\mathrm{Scal}(g') = 2e^{-2\phi}\Box_g \phi$ is not identically zero. Thus g' is not isometric to the flat metric g.

Example 11.3 ((H) holds, (E) fails)**.** Let $M = \mathbb{R}^{1+n}$ with the Minkowski metric and let $W = \{(t,x) : t^2 + |x|^2 \leq 1\}$. The point $p = (t_0, 0)$ with $-\sqrt{2} < t_0 < -1$ belongs to S_+: at the time $t = -1/\sqrt{2}$ every timelike curve starting at p is at a point (t,x) with $|x| < t - t_0 < 1/\sqrt{2} = \sqrt{1-t^2}$, which lies in W. Thus (E) fails, while (H) holds by Section 9.1. More generally (E) fails whenever W has a cavity, that is, a relatively compact connected component of $M \setminus W$.

Example 11.4 (Compact Cauchy hypersurfaces; (H) holds)**.** Let $M = \mathbb{R} \times S^n$ with $g = -dt^2 + g_{S^n}$, let $D \subset S^n$ be a closed geodesic ball of radius $\rho < \pi$ and $W = [-1,1] \times D$. A point $(t,x) \in O$ with $x \notin D$ or $t > 1$ belongs to O_+ (use the curve $s \mapsto (t+s, x)$). A point (t,x) with $x \in D$ and $t < -1 - \rho$ belongs to O_+: move radially away from the centre of D with a constant speed smaller than 1 until a point outside D is reached, and then stay at this point of S^n; since the distance to ∂D is at most $\rho < -1 - t$, the speed can be chosen so that this happens before the time -1. Hence $S_+ \subset [-1-\rho, -1] \times D$ is relatively compact, and (H) holds.

Example 11.5 ((H) fails because W contains a Cauchy hypersurface)**.** Let $M = \mathbb{R} \times S^n$ with $g = -dt^2 + g_{S^n}$ and $W = [-T, T] \times S^n$. Then $O_+ = \{t > T\}$ and $O_- = \{t < -T\}$, so that $S_+ = \{t < -T\}$ and $S_- = \{t > T\}$ are not relatively compact and (H) fails. The null geodesics through a point $p = (0,x)$ are $s \mapsto (\pm s, \gamma(s))$ with unit speed geodesics γ of S^n starting at x; they all meet again at the antipodal point at $s = \pi$, and $E^{\pm}(p) = \{(\pm s, y) : d(x,y) = s,\ 0 \leq s \leq \pi\}$; similarly for $p = (t_p, x)$. Thus $E^-(p)$ meets O if and only if $t_p - \pi < -T$, and $E^+(p)$ meets O if and only if $t_p + \pi > T$. Hence condition (N) holds if and only if $T < \pi/2$. For $T < \pi/2$ Theorem 1.6 applies and Theorem 1.3 does not; for $T \geq \pi/2$ neither applies.

Example 11.6 ((H) fails although the Cauchy hypersurfaces are not compact)**.** Let $M = (0,3) \times \mathbb{R}^n$ with the Minkowski metric; (M,g) is globally hyperbolic with Cauchy hypersurfaces $\{t = c\}$. Let $W = [1,2] \times \overline{B_4}$, where $B_4 \subset \mathbb{R}^n$ is the ball of radius 4. Every point (t,x) with $0 < t < 1$ and $|x| \leq 3 + t$ belongs to S_+, since a timelike curve starting there is at the time 1 at a point with $|x| < 4$; and every point (t,x) with $2 < t < 3$ and $|x| \leq 6 - t$ belongs to S_-. These sets are not relatively compact in M. Hence (H) fails, while (N) holds by Lemma 10.1. As the sets $S_\pm$ depend only on the conformal class of the metric, the same holds for the metrics $a(t)^2(-dt^2 + |dx|^2)$ on M, with $a > 0$ smooth.

Example 11.7 (Condition (N) cannot be dropped in Theorem 1.6)**.** Let $M = \mathbb{R} \times S^3$ with $g = -dt^2 + g_{S^3}$, where $S^3 \subset \mathbb{R}^4$ is the unit sphere, let $T > \pi/2$, $W = [-T, T] \times S^3$ and $O = \{|t| > T\}$; by Example 11.5, (N) fails. Choose $t_0 < t_1$ with $-T < t_0$ and $t_1 < T - \pi$, a real-valued $\mathsf{q} \in C_c^\infty((t_0, t_1) \times S^3)$, $\mathsf{q} \not\equiv 0$, and put

$$q = 1 + \mathsf{q}, \qquad q' = 1 + \mathsf{q}', \qquad \mathsf{q}'(t,x) = \mathsf{q}(t - \pi, -x).$$

Then $q = q' = 1$ on O, $q \neq q'$ (if $\mathsf{q} = \mathsf{q}'$, then $\mathsf{q}(t,x) = \mathsf{q}(t - 2\pi, x)$ for all (t,x), which is impossible for a nonzero function with compact support), and

$$C_{g,q}(O) = C_{g,q'}(O).$$

Thus the conclusion of Theorem 1.6 fails for every $T > \pi/2$, while it holds for $T < \pi/2$, where (N) holds.

Proof. Let $P_1 = \Box_g + 1 = \partial_t^2 - \Delta_{S^3} + 1$, $P = \Box_g + q$ and $P' = \Box_g + q'$. The eigenvalues of $-\Delta_{S^3}$ are $k(k+2)$, $k = 0, 1, 2, \ldots$, and the eigenfunctions Y with eigenvalue $k(k+2)$ are the restrictions of homogeneous harmonic polynomials of degree k, so that $Y(-x) = (-1)^k Y(x)$. If $u \in C^\infty(M)$ solves $P_1 u = 0$, the coefficient $c(t)$ of $u(t,\cdot)$ with respect to such an eigenfunction solves $c'' + (k+1)^2 c = 0$, hence $c(t+\pi) = (-1)^{k+1} c(t)$. Therefore all coefficients of the function

$u(t+\pi,-x)+u(t,x)$ vanish:

$$u(t+\pi,-x)=-u(t,x) \qquad \text{for every } u\in C^\infty(M) \text{ with } P_1u=0. \tag{11.1}$$

For a function u on an open set of M let $(\mathsf{R}u)(t,x)=-u(t-\pi,-x)$. Since $(t,x)\mapsto(t-\pi,-x)$ is an isometry of (M,g), we have $P'\mathsf{R}u=\mathsf{R}(Pu)$, and $\mathsf{R}u=u$ if $P_1u=0$ on M, by (11.1).

By Proposition 4.1 it suffices to show $L^{\mathrm{ret}}_P=L^{\mathrm{ret}}_{P'}$. Let $f\in C^\infty_c(O)$, $f=f_-+f_+$ with $\operatorname{supp} f_-\subset\{t<-T\}$ and $\operatorname{supp} f_+\subset\{t>T\}$. The function $G^{\mathrm{ret}}_{P_1}f_+$ is supported in $\{t>T\}$, where $P=P'=P_1$; hence $G^{\mathrm{ret}}_P f_+=G^{\mathrm{ret}}_{P'}f_+=G^{\mathrm{ret}}_{P_1}f_+$. Let $u=G^{\mathrm{ret}}_P f_-$ and $u_1=G^{\mathrm{ret}}_{P_1}f_-$. Then $u=u_1$ on $\{t<t_0\}$, where $P=P_1$. Let u_- and u_+ be the solutions of $P_1u_\pm=0$ on M with $u_-=u_1$ on $\{t>-T\}$ and $u_+=u$ on $\{t>t_1\}$ (on these sets $P_1u_1=0$ and $P_1u=0$, respectively). Define

$$u'=u_1 \ \text{ on } \{t<t_0+\pi\}, \qquad u'=\mathsf{R}u \ \text{ on } \{t>-T+\pi\}.$$

On the overlap $\{-T+\pi<t<t_0+\pi\}$ both definitions give u_-: indeed $u_1=u_-$ there, and $(\mathsf{R}u)(t,x)=-u(t-\pi,-x)=-u_-(t-\pi,-x)=u_-(t,x)$, because $u=u_1=u_-$ on $\{-T<t<t_0\}$. We have $P'u'=P_1u_1=f_-$ on $\{t<t_0+\pi\}$, since $\mathsf{q}'=0$ there, and $P'u'=\mathsf{R}(Pu)=0=f_-$ on $\{t>-T+\pi\}$. As u' vanishes for $t\ll 0$, $u'=G^{\mathrm{ret}}_{P'}f_-$. Finally $u'=u_1=u$ on $\{t<-T\}$, and for $t>T$, where $t-\pi>t_1$, we get $u'(t,x)=-u_+(t-\pi,-x)=u_+(t,x)=u(t,x)$. Hence $L^{\mathrm{ret}}_{P'}f=L^{\mathrm{ret}}_P f$. □

Example 11.8 (Inextendibility refers to M; first arrivals on the Einstein cylinder)**.** Let $M=\mathbb{R}\times S^n$, $n\geq 1$, with $g=-dt^2+g_{S^n}$, and recall that $(t_1,x_1)\leq(t_2,x_2)$ if and only if $d_{S^n}(x_1,x_2)\leq t_2-t_1$, and $(t_1,x_1)\ll(t_2,x_2)$ if and only if $d_{S^n}(x_1,x_2)<t_2-t_1$.

(a) Let $W=[-\pi,0]\times S^n$, which is the set of Example 11.5 with $T=\pi/2$, translated in time. Then $S_+=\{t<-\pi\}$, $S_-=\{t>0\}$, and (H) fails. Every point (t,x) with $t<-\pi$ is the initial point of the timelike curve $s\mapsto(t+s,x)$, $0\leq s<-\pi-t$, which lies in O and cannot be continued within O; but it converges to the point $(-\pi,x)\in W$ and is not future-inextendible in the sense of Definition 1.1. If inextendibility were understood within O, hypothesis (E) would hold in this example, while the conclusion of Lemma 5.5(a) fails: for $p=(0,N)$, $N\in S^n$,

$$E^-(p)=\{(-s,x):0\leq s\leq\pi,\ d_{S^n}(x,N)=s\}\subset W,$$

because $d_{S^n}\leq\pi$. The past-directed null geodesics from p meet again at the point $(-\pi,-N)$ of W and enter $I^-(p)$ afterwards. The proof of Lemma 5.5(a) uses the inextendibility in M through (F1).

(b) The same space-time illustrates why Lemma 5.4 moves the initial point. Let $\gamma(r)=(t_0+r,x(r))$ be a null geodesic, where x is a unit speed geodesic of S^n. The point $\gamma(r)$ belongs to $E^+(\gamma(0))$ for $0\leq r\leq\pi$ and to $I^+(\gamma(0))$ for $r>\pi$. If $p_0=\gamma(\pi)$, then $\gamma(0)\in E^-(p_0)$, but no extension of γ beyond p_0 is a first arrival from $\gamma(0)$, and the first-arrival condition (5.1) cannot be satisfied with $\sigma(0)=\gamma(0)$ and $b>1$. For $0<a<\pi$ and $0<\epsilon\leq a$, however, $\gamma(\pi+\epsilon)\in E^+(\gamma(a))$: starting from the later point $\gamma(a)$, the geodesic is a first arrival up to a point beyond p_0. This is the geodesic σ of Lemma 5.4, with $\sigma(0)=\gamma(a)$, $\sigma(1)=p_0$ and $\sigma(b)=\gamma(\pi+\epsilon)$ (for instance with W as in (a), $-2\pi\leq t_0<-\pi$ and $a<-\pi-t_0$, so that $p_0\in W$ and $\gamma([0,a])\subset O$).

Remark 11.9 (A gauge for the joint problem of metric and potential)**.** Let $n\geq 2$, let M, W, O be as in (S), let g be a globally hyperbolic metric on M and $q\in C^\infty(M)$ real-valued. Let $\Omega\in C^\infty(M)$ be positive with $\Omega=1$ on O, and put

$$h=\Omega^{\frac{4}{n-1}}g, \qquad q_h=\Omega^{-\frac{4}{n-1}}\big(q+\Omega^{-1}\,\Box_g\Omega\big). \tag{11.2}$$

Then

$$(\Box_h+q_h)(\Omega^{-1}u)=\Omega^{-\frac{n+3}{n-1}}\,(\Box_g+q)u \qquad \text{for all } u\in C^\infty(M). \tag{11.3}$$

Indeed, for $h=e^{2\omega}g$ in dimension $1+n$ one has $\Box_h v=e^{-2\omega}\big(\Box_g v-(n-1)g(d\omega,dv)\big)$ (this is the formula used in Step 5 of the proof of Proposition 7.3, with $c=e^{2\omega}$); here $e^\omega=\Omega^{2/(n-1)}$,

so that $(n-1)\,d\omega = 2\Omega^{-1}d\Omega$. By the product rule $\Box_g(fv) = f\,\Box_g v + v\,\Box_g f - 2g(df,dv)$ and by $\Box_g(\Omega^{-1}) = -\Omega^{-2}\Box_g\Omega - 2\Omega^{-3}g(d\Omega,d\Omega)$,

$$\Box_g(\Omega^{-1}u) - 2\Omega^{-1}g\big(d\Omega, d(\Omega^{-1}u)\big) = \Omega^{-1}\Box_g u - \Omega^{-2}(\Box_g\Omega)\,u,$$

and multiplication by $e^{-2\omega} = \Omega^{-4/(n-1)}$ gives $\Box_h(\Omega^{-1}u) = \Omega^{-\frac{n+3}{n-1}}\big(\Box_g u - \Omega^{-1}(\Box_g\Omega)u\big)$, which is (11.3). For $q_h = 0$ this is the gauge transformation by which the determination of a conformal factor is reduced to that of a potential; it is used in this way in [FO26, Remark 1.8], where it is attributed to [AFO22, Section 1].

Consequences. (1) The metric h has the same causal structure as g, so it is globally hyperbolic, and $h = g$, $q_h = q$ on O. By (11.3), $u \mapsto \Omega^{-1}u$ maps the solutions of $(\Box_g + q)u = 0$ near W onto the solutions of $(\Box_h + q_h)v = 0$ near W without changing them on O. Hence

$$C_{h,q_h}(O) = C_{g,q}(O).$$

(2) If $q = 0$ and $\Omega \not\equiv 1$, there is no diffeomorphism Ψ of M with $\Psi = \mathrm{id}$ on O, $\Psi^* h = g$ and $q_h \circ \Psi = q$: the last condition would give $q_h = 0$, that is, $\Box_g\Omega = 0$, and $\Omega - 1$ has compact support, so that $\Omega = 1$ by (F7). Thus $(g,0)$ and (h,q_h) have the same exterior data and are not related by a diffeomorphism fixing O. (3) For $q = q_h = 0$ the same argument shows that the transformation (11.2) is the identity, in accordance with Theorem 1.3; and in Theorem 1.6, where the metric is fixed, $h = g$ forces $\Omega = 1$. (4) For $q = \frac{n-1}{4n}R_g$ the transformation law of the scalar curvature under conformal changes of the metric [Wal84, Appendix D] gives $q_h = \frac{n-1}{4n}R_h$: the exterior data of the conformal wave operator are the same for all metrics $\Omega^{4/(n-1)}g$ with $\Omega = 1$ on O, and at most the conformal class of the metric can be determined from them.

Lemma 11.10 ((H) holds for suitable exteriors)**.** *Let $n \geq 1$, $M = \mathbb{R}^{1+n}$ with coordinates (t,x), $W \subset M$ compact and $O = M \setminus W$. Let g be a time-oriented Lorentzian metric on M such that*

$$\mathsf{g} = -dt^2 + g_0(t,x) \quad \text{on } O, \qquad g_0(t,x) = \sum_{j,k=1}^{n} g_{0,jk}(t,x)\,dx^j\,dx^k,$$

where the $g_{0,jk}$ are smooth functions on O and the matrices $(g_{0,jk}(t,x))$ are positive definite, and such that ∂_t is future-directed on O. Let $T_0, R_0 > 0$ with $W \subset [-T_0,T_0] \times \overline{B_{R_0}}$, let $R_1 > R_0$, and assume that there is a continuous function $A\colon (-\infty,-T_0] \to (0,\infty)$ such that

$$g_0(t,x)(v,v) \leq A(t)^2\,|v|^2 \qquad \text{for } t \leq -T_0,\ |x| \leq R_1,\ v \in \mathbb{R}^n, \tag{11.4}$$

$$\int_{-\infty}^{-T_0} \frac{dt}{A(t)} > R_1. \tag{11.5}$$

Then there is $t_* < -T_0$ such that

$$S_+ \subset [t_*, T_0] \times \overline{B_{R_0}}.$$

In particular S_+ is relatively compact in M, and hypothesis (H) *holds. This is the case if $g_0(t,x)(v,v) \leq C|v|^2$ for $t \leq -T_0$, $|x| \leq R_1$ and $v \in \mathbb{R}^n$, with a constant $C > 0$. The same holds with the time direction reversed: if* (11.4) *holds for $t \geq T_0$, $|x| \leq R_1$, with a continuous $A\colon [T_0,\infty) \to (0,\infty)$ such that $\int_{T_0}^{\infty} dt/A(t) > R_1$, then $S_- \subset [-T_0, t^*] \times \overline{B_{R_0}}$ for some $t^* > T_0$.*

Proof. Step 0 (a criterion). Let $(t,x) \in O$ and let $c(s) = (t+s, y(s))$, $s \in [0,\infty)$, be a continuous, piecewise smooth curve contained in O such that $g_0(c(s))(\dot y(s),\dot y(s)) < 1$ for all s, with one-sided derivatives at the break points. Then $\mathsf{g}(\dot c,\dot c) = -1 + g_0(\dot y,\dot y) < 0$ and $\mathsf{g}(\dot c,\partial_t) = -1 < 0$, so $\dot c$ is timelike and lies in the same time cone as the future-directed vector ∂_t. Moreover the t-coordinate of $c(s)$ tends to ∞, so $c(s)$ has no limit in M as $s \to \infty$. Hence c is a future-inextendible timelike curve in O in the sense of Definition 1.1, and $(t,x) \in O_+$.

Step 1 (points outside the cylinder). Let $(t,x) \in O$ with $|x| > R_0$ or $t > T_0$. The curve $c(s) = (t+s,x)$, $s \geq 0$, does not meet $W \subset [-T_0,T_0] \times \overline{B_{R_0}}$, and $\dot y = 0$. By Step 0, $(t,x) \in O_+$.

Step 2 (a speed profile). By (11.5) and monotone convergence there is $t_* < -T_0$ with $\int_{t_*}^{-T_0} dt/A(t) > R_1$. The function $1/A$ is continuous and positive on $[t_*, -T_0]$, so its lower Darboux sums converge to its integral: there is a partition $t_* = \tau_0 < \tau_1 < \cdots < \tau_N = -T_0$ such that

$$\sum_{i=1}^{N} m_i \, (\tau_i - \tau_{i-1}) > R_1, \qquad m_i = \min_{[\tau_{i-1},\tau_i]} \frac{1}{A}.$$

Choose $\theta \in (0,1)$ such that the left-hand side multiplied by θ is still larger than R_1, and let $\rho\colon [t_*, -T_0] \to [0,\infty)$ be the continuous function with $\rho(t_*) = 0$ which is affine with slope θm_i on $[\tau_{i-1}, \tau_i]$, $i = 1, \dots, N$. Then ρ is nondecreasing, $\rho(-T_0) > R_1$, and on each interval $[\tau_{i-1}, \tau_i]$

$$0 \le \rho' = \theta m_i \le \frac{\theta}{A}. \tag{11.6}$$

Step 3 (points below the cylinder). Let $(t,x) \in M$ with $t < t_*$ and $|x| \le R_0$; then $(t,x) \in O$, since $t < -T_0$. Let $e = x/|x|$ if $x \neq 0$, and let e be any unit vector if $x = 0$; then $|x + re| = |x| + r$ for $r \ge 0$. The continuous function $\tau \mapsto |x| + \rho(\tau)$ is at most $R_0 < R_1$ at $\tau = t_*$ and larger than R_1 at $\tau = -T_0$. Let $\tau_x \in (t_*, -T_0)$ be the smallest τ with $|x| + \rho(\tau) = R_1$. Define $c(s) = (t + s, y(s))$, $s \ge 0$, by

$$y(s) = \begin{cases} x, & t + s \le t_*, \\ x + \rho(t+s)\, e, & t_* \le t + s \le \tau_x, \\ x + \rho(\tau_x)\, e, & t + s \ge \tau_x. \end{cases}$$

This curve is continuous and piecewise smooth. It is contained in O: for $t + s \le \tau_x$ its t-coordinate is smaller than $-T_0$, and for $t + s \ge \tau_x$ we have $|y(s)| = |x| + \rho(\tau_x) = R_1 > R_0$. On the first and on the last part $\dot y = 0$. On the middle part $t + s \le -T_0$ and $|y(s)| = |x| + \rho(t+s) \le R_1$, so (11.4) and (11.6) give

$$g_0(c(s))(\dot y, \dot y) = \rho'(t+s)^2 \, g_0(c(s))(e,e) \le \frac{\theta^2}{A(t+s)^2} \, A(t+s)^2 = \theta^2 < 1.$$

By Step 0, $(t,x) \in O_+$.

Step 4 (conclusion). By Steps 1 and 3, every point of O which is not in $[t_*, T_0] \times \overline{B_{R_0}}$ belongs to O_+. Hence $S_+ = O \setminus O_+ \subset [t_*, T_0] \times \overline{B_{R_0}}$, which is compact. If $g_0(t,x)(v,v) \le C|v|^2$, then (11.4) holds with the constant function $A = \sqrt{C}$, for which the integral in (11.5) is infinite.

Time reversal. Let $\mathsf{R}(t,x) = (-t,x)$. On $\mathsf{R}^{-1}(O) = M \setminus \mathsf{R}(W)$ the metric $\mathsf{R}^*\mathsf{g}$ equals $-dt^2 + g_0(-t,x)$, and we give it the time orientation in which ∂_t is future-directed there; then R is an isometry from $(M, \mathsf{R}^*\mathsf{g})$ onto (M, g) which reverses the time orientation. It maps the future-inextendible timelike curves of $(M, \mathsf{R}^*\mathsf{g})$ contained in $\mathsf{R}^{-1}(O)$ onto the past-inextendible past-directed timelike curves of (M, g) contained in O, so that $S_- = \mathsf{R}(\tilde S_+)$, where $\tilde S_+$ is the future-trapped part of the exterior of $\mathsf{R}(W)$ for $\mathsf{R}^*\mathsf{g}$. The assumptions for $t \ge T_0$ are the assumptions (11.4), (11.5) for $\mathsf{R}^*\mathsf{g}$, $\mathsf{R}(W) \subset [-T_0, T_0] \times \overline{B_{R_0}}$ and the function $t \mapsto A(-t)$. The first part gives $\tilde S_+ \subset [t_*, T_0] \times \overline{B_{R_0}}$, hence $S_- \subset [-T_0, -t_*] \times \overline{B_{R_0}}$. □

Remark 11.11. (1) For $g_0 = |dx|^2$ and $A = 1$, Lemma 11.10 contains the verification of (H) in Section 9.1. Global hyperbolicity is not used in the lemma. In the setting (S) the sets $S_\pm$ depend only on the metric and its time orientation on O, so it suffices that $g = g' = -dt^2 + g_0$ on O with g_0 as in the lemma.

(2) Only an upper bound for g_0 enters. Condition (11.5) says that in the metric $-dt^2 + A(t)^2|dx|^2$ light emitted at sufficiently early times travels a Euclidean distance larger than R_1 before the time $-T_0$, that is, that there is no particle horizon of size at most R_1 in the past of W. Condition (H) can fail without such an upper bound (Example 11.12).

Example 11.12 ((H) can fail if g_0 grows in time)**.** Let $M = \mathbb{R}^{1+n}$, $n \ge 1$, with the metric $\mathsf{g} = -dt^2 + (1+t^2)^2|dx|^2$ on all of M, time-oriented so that ∂_t is future-directed, and let

$W = [-1,1] \times \overline{B_3}$. Here $g_0 = (1+t^2)^2|dx|^2 \geq |dx|^2$. Then (M, g) is globally hyperbolic, its Cauchy hypersurfaces are not compact, and

$$(-\infty, -1) \times \overline{B_2} \subset S_+, \qquad (1, \infty) \times \overline{B_2} \subset S_-.$$

Thus (H) fails, while (N) holds by Lemma 10.1. This is consistent with Lemma 11.10: any admissible choice there has $R_0 \geq 3$, $T_0 \geq 1$ and $A(t) \geq 1 + t^2$, so that $\int_{-\infty}^{-T_0} dt/A(t) \leq \int_{-\infty}^{-1} dt/(1+t^2) = \pi/4 < R_1$, and likewise for $t \geq T_0$.

Proof. The map $\Phi(t, x) = (\arctan t, x)$ is a diffeomorphism of M onto $N = (-\pi/2, \pi/2) \times \mathbb{R}^n$. With $\tau = \arctan t$ we have $d\tau = dt/(1+t^2)$, hence

$$\mathsf{g} = (1+t^2)^2\big(-d\tau^2 + |dx|^2\big).$$

Thus Φ is a conformal diffeomorphism of (M, g) onto the open subset N of Minkowski space with coordinates (τ, x), and it preserves the time orientations. Causal and timelike curves depend only on the conformal class and the time orientation. Since τ is nondecreasing along causal curves, N is causally convex in Minkowski space; hence for $p, q \in N$ the causal diamond of p, q in N is the Minkowski diamond, which is compact, and N is strongly causal as an open subset of Minkowski space. So N, and with it (M, g), is globally hyperbolic. The function t is temporal, and the hypersurface $\{t = c\}$ corresponds to $\{\tau = \arctan c\}$: an inextendible timelike curve in N is, as a function of τ, a curve $x(\tau)$ with $|dx/d\tau| < 1$, and if its τ-range had an endpoint $\tau_1 \in (-\pi/2, \pi/2)$, then $x(\tau)$ would converge as $\tau \to \tau_1$ and the curve would have a limit in N. Hence $\{t = c\}$ is a non-compact Cauchy hypersurface.

Let $t_0 < -1$ and $|x_0| \leq 2$; then $(t_0, x_0) \in O$. Let c be a future-inextendible timelike curve starting at (t_0, x_0). It meets the Cauchy hypersurface $\{t = -1\}$; let $(-1, x_1)$ be its first point there. Before this point, in the coordinates (τ, x), c is a future-directed timelike curve of Minkowski space, so τ increases strictly along it and, as a function of τ, $|dx/d\tau| < 1$. Hence

$$|x_1 - x_0| \leq \arctan(-1) - \arctan t_0 < -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4},$$

and $|x_1| < 2 + \pi/4 < 3$. Thus $(-1, x_1) \in W$, so every future-inextendible timelike curve from (t_0, x_0) meets W, and $(t_0, x_0) \in S_+$. The metric and W are invariant under $(t, x) \mapsto (-t, x)$, which gives the statement for S_-. In the coordinates (τ, x) this is a situation of the type of Example 11.6: the sets $S_\pm$ depend only on the conformal class, and (M, g) is conformal to a slab of Minkowski space. □

AI statement. We have used ChatGPT 6.2 and Claude Fable 5.2. These models were given results from our published work, and from unpublished documents on the Lorentzian travel time problem and focusing techniques for linear waves. In discussion with the authors, they were guided to produce a layer stripping type proof to recover the metric tensor or the potential from its exterior data set. This resulted in the first draft of this article. Sections 1 and 2 include material written by the authors, and all sections have been edited by the authors. The AI models have also been used in language processing. Claude Opus 5.5 was used for the comparator challenge to produce the Lean code that was used for the verification of Corollary 1.4. The Lean comparator challenge file has been human curated.

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