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HOW FAST SMALE'S MEAN VALUE CONSTANTS (1981) APPROACH 4

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ABSTRACT. For a complex polynomial p of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$, let $S(p)$ be the minimum of $|p(c)/(c\,p'(0))|$ over the critical points c of p, and let $K(d)$ be the supremum of $S(p)$ over all such p of degree d. In 1981 Smale proved that $K(d) \leq 4$ for every d. It is known that $K(d) < 4$ for every d, and the author recently proved that $K(d) \to 4$ as $d \to \infty$. We prove that $K(d) \leq (4/y)\, e^{-\mu(y^{1/d})}$ for every $0 < y < 1$, where μ is the Grötzsch ring function. As a simpler, looser closed-form bound, we deduce that $K(d) < 4 - 4/\log d$ for every $d \geq 8$; for large d the best earlier upper bound was $K(d) < 4 - 2.263/\sqrt{d}$, due to Crane. In a companion paper, the author shows that the true order of $4 - K(d)$ is $(\log d)^{-1/2}$, up to a factor $\sqrt{\log\log\log d}$.[1] Theorem 1.1 and Corollary 1.2 are verified in Lean.[2]

1. INTRODUCTION

Let p be a complex polynomial of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$. A *critical point* of p is a zero of p', and a *critical value* of p is its value at a critical point. Put

$$S(p) \;:=\; \min_{p'(c)=0} \left| \frac{p(c)}{c\,p'(0)} \right|.$$

Smale's mean value theorem [26, Theorem, p. 33] says that $S(p) \leq 4$ for every such p. For each $d \geq 2$ let

$$K(d) \;:=\; \sup\{S(p) :\; \deg p = d\},$$

the best constant in Smale's theorem for polynomials of degree d. Fields Medalist Stephen Smale proved $K(d) \leq 4$ in 1981 in a complexity study of Newton's method and asked whether 4 can be replaced by 1, or even by $(d-1)/d$ [26, Problem 1E, p. 33]. Smale listed the question again, as the first addendum (*Mean Value Problem*), in the second version of *Mathematical problems for the next century* [27, p. 21] (see also [28]). For forty-five years it stood alongside Sendov's conjecture as one of the central open problems in the analytic theory of complex polynomials, featured in the monographs of Rahman and Schmeisser [21, Section 7.2] and Sheil-Small [25, Section 10.4] and in surveys by Schmeisser [23] and Dubinin [11]. For $d \leq 4$ the answer is yes, with $K(d) = (d-1)/d$ (see the introduction of [13]). The same introduction reports a computer-assisted proof, due to Crane, for $d = 5$, and Sendov and Marinov [24] give numerical evidence that $K(d) = (d-1)/d$ for $d \leq 10$. For general d the answer is no: the existence of a counterexample, of very large and unspecified degree, is proved in [1], and Adamczewski and Colbrook gave an explicit one, of degree 29,984 [2]. Jatar and Ng proved $\sup_d K(d) > 1.1018$ [19], the author proved $\sup_d K(d) > 1.1559$ [15], and the author then proved in [16], using a reduction of Jatar and Ng, that the constant 4 cannot be lowered at all:

$$\lim_{d\to\infty} K(d) \;=\; \sup_{d\geq 2} K(d) \;=\; 4.$$

The limit exists because $K(d)$ does not decrease with d ([6], [14, p. 99]). On the other hand, $K(d) < 4$ for every d: Beardon, Minda and Ng proved $K(d) \leq 4^{(d-2)/(d-1)}$ [3]; Conte, Fujikawa and Lakic proved $K(d) \leq 4(d-1)/(d+1)$ [5]; Fujikawa and Sugawa improved this to $K(d) \leq$

[1] Expanded version, with further results [17]: https://doi.org/10.5281/zenodo.24288572

[2] Proved in Lean 4 from its three standard axioms and four assumptions, and checked with `lake comparator`: https://github.com/david44hello/paper_smale-1981-mean-value-rate/tree/v1.00/lean

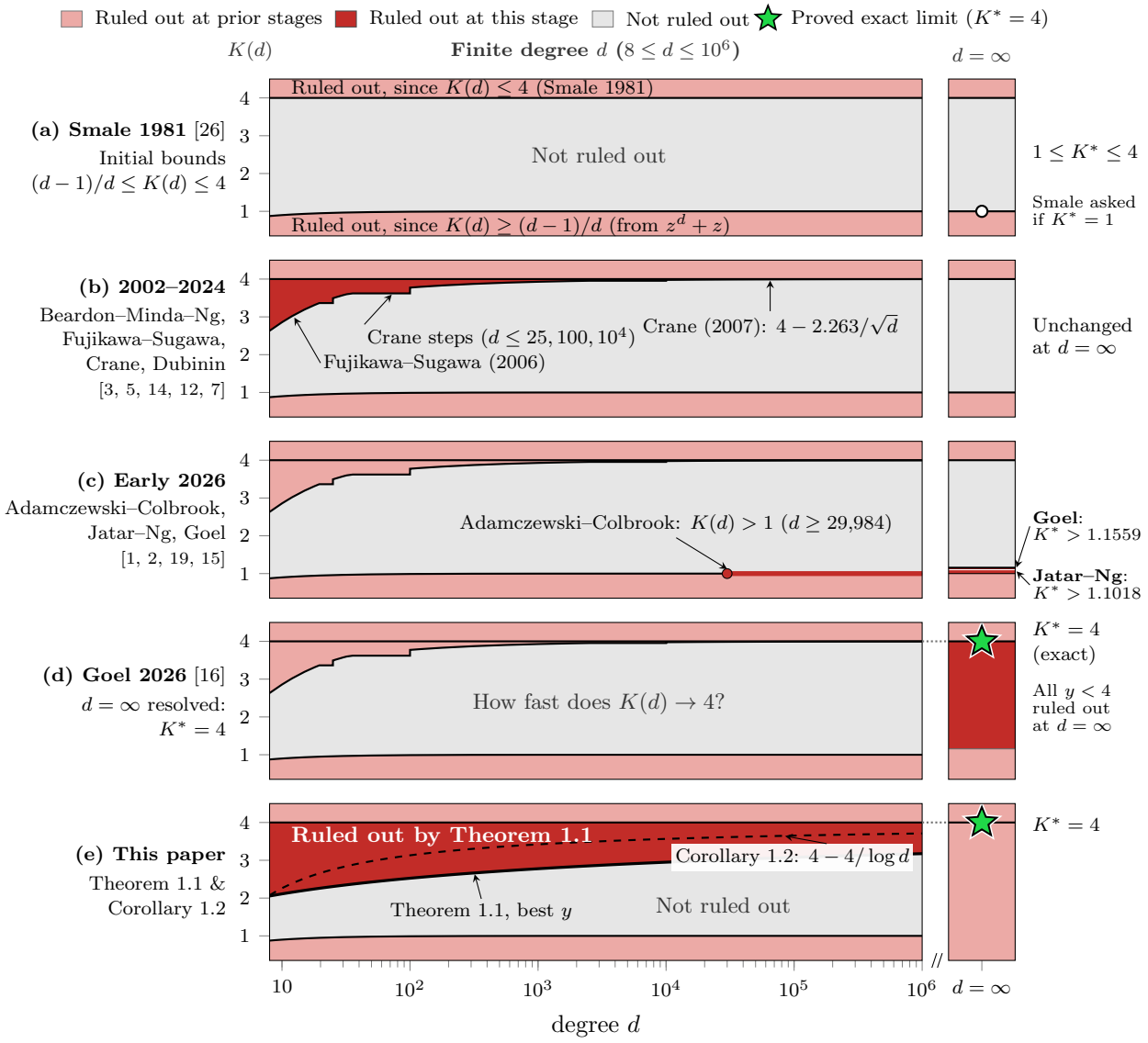


FIGURE 1. History of Smale's mean value problem (1981–2026) for $8 \le d \le 10^6$ (left) and the limiting boundary $d = \infty$ ($K^* := \lim_{d\to\infty} K(d)$, right), to scale. Red is ruled out (bold: this stage; light: prior stages); gray is where $K(d)$ may still lie. **(a)** Smale [26]: $(d-1)/d \le K(d) \le 4$ for all d (exact for $d \le 4$), so $1 \le K^* \le 4$; white dot marks Smale's conjecture $K^* = 1$. **(b)** 2002–2024: upper bounds of Beardon–Minda–Ng [3], Conte–Fujikawa–Lakic [5], Fujikawa–Sugawa [14], Dubinin [9, 12], and Crane [7], with steps implied by monotonicity of $K(d)$ from Crane's numerical bounds at $d = 25, 100, 10^4$; $d = \infty$ untouched. **(c)** Early 2026: $K^* = 1$ disproved [1, 2] (red dot and bold-red segment: $K(d) > 1$ for $d \ge 29{,}984$; visually on $y = 1$ at this scale); $d = \infty$ lower bounds reach 1.1018 [19] and 1.1559 [15]. **(d)** Goel [16] resolves $d = \infty$ completely ($K^* = 4$, bright green star), leaving the finite-d rate open. **(e)** This paper sweeps the finite-d upper bound to Theorem 1.1 (solid, best y, computed in floating point; Table 2 gives certified interval-arithmetic values) and Corollary 1.2 ($4 - 4/\log d$, dashed), independently ruling out the entire bold-red region up to 4 and subsuming all earlier bounds in (b) for $d \ge 8$.

$4\big(1+(d-2)\,4^{-1/(d-1)}\big)/(d+1)$ [14]; Dubinin proved $K(d) \le \frac{d-1}{d}\,4^{(d-2)/(d-1)}$ [12] (stated without proof in [9, (8)]); and Crane proved $K(d) < 4 - 2.263/\sqrt{d}$ for $d \ge 8$ [7].

So the constants $K(d)$ do not decrease, tend to 4, and never reach it. How fast do they approach 4? The first four of these bounds show only that $4 - K(d)$ is at least of order $1/d$, and Crane's shows that it is at least of order $1/\sqrt{d}$. In this paper we prove that $4 - K(d) > 4/\log d$ for every $d \ge 8$ (Corollary 1.2 and Table 1). This follows from a sharper bound in terms of the Grötzsch ring function (Theorem 1.1). Smale's bound follows from Koebe's one-quarter theorem, applied to the branch F of p^{-1} that fixes 0; we apply instead a theorem of Betsakos [4] (Theorem 2.2) to a conformal map built from F, and the degree of p enters through the Bernstein–Walsh inequality [22, Theorem 5.5.7(a)] (see the outline of the proof of Theorem 1.1 below). The case of the Bernstein–Walsh inequality that we use is also a special case of an inequality of Dubinin [10, 11] (Remark 3.1).

The companion paper [17] shows that $1/\log d$ is not the true order of the gap $4 - K(d)$. By [17, Theorems 1.3 and 1.4], $4 - K(d) \ge \min\big(1.1 \cdot 10^{-4},\, 0.056/\sqrt{\log d}\big)$ for every $d \ge 2$, and $4 - K(d) \le 4.9 \cdot 10^4 \sqrt{\log\log\log d/\log d}$ for $d \ge \exp(1.04 \cdot 10^{10})$. This lower bound exceeds $4/\log d$ only when $\log d > 36{,}363$; so for $8 \le d \le e^{36{,}363}$ the bound of Corollary 1.2 is the stronger one. Whether the factor $\sqrt{\log\log\log d}$ can be removed, and how large $4 - K(d)$ is for moderate degrees, say $8 \le d \le 10^6$, are open questions; they are discussed in [17, Section 6]. The proofs in this paper are independent of [17].

TABLE 1. The main results of this paper. Here log is the natural logarithm, and μ is the Grötzsch ring function, defined in (1) below.

	Best result before this paper	New result
Upper bound for $K(d)$, every $d \ge 8$	The bound of Fujikawa and Sugawa [14]; Crane's bound $K(d) < 4 - 2.263/\sqrt{d}$ [7] and Crane's numerical bounds at nine degrees	$K(d) < 4 - 4/\log d$ (Corollary 1.2), from the sharper bound $K(d) \le (4/y)\,e^{-\mu(y^{1/d})}$, $0 < y < 1$ (Theorem 1.1)

The results. For $0 < r < 1$ let

$$\mathcal{K}(r) \;:=\; \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \qquad \mu(r) \;:=\; \frac{\pi}{2}\,\frac{\mathcal{K}\big(\sqrt{1-r^2}\big)}{\mathcal{K}(r)}. \tag{1}$$

Here $\mathcal{K}$ is the complete elliptic integral of the first kind, and μ is the *Grötzsch ring function*. The function μ decreases from ∞ to 0 on $(0,1)$. It satisfies $\mu(r) = \log(4/r) + O(r^2)$ as $r \to 0$ and $\mu(r)\,\mu(\sqrt{1-r^2}) = \pi^2/4$ for $0 < r < 1$, so that $\mu(r) \sim \pi^2/\big(2\log\frac{16}{1-r^2}\big)$ as $r \to 1$ (Section 2.2).

Theorem 1.1. *For every integer $d \ge 2$ and every $y \in (0,1)$,*

$$K(d) \;\le\; \frac{4}{y}\,e^{-\mu(y^{1/d})}.$$

For fixed d the right-hand side tends to ∞ as $y \to 0$ (by Lemma 2.1(c) it exceeds $y^{1/d-1}$), and to 4 from below as $y \to 1$. To see that it is below 4 for y near 1, put $L := \log(1/y)$. For $0 < L < d$, Lemma 2.1 and the inequality $1 - e^{-t} \ge t - t^2/2$ bound the right-hand side by

$$4\exp\Big(L - \frac{\pi^2}{2\log\big(8d/(L(1-L/d))\big)}\Big)$$

(see Steps 1 and 3 of the proof of Corollary 1.2 in Section 3), and this is below 4 when L is small. So Theorem 1.1 recovers Smale's bound $K(d) \le 4$, and it gives $K(d) < 4$. For fixed y the right-hand side tends to $4/y > 4$ as $d \to \infty$, so y must tend to 1 with d. When d is large and L is

a small multiple of $1/\log d$, the displayed bound is approximately 4 minus a positive multiple of $1/\log d$, since the logarithm in the display is then approximately $\log d$. This is why the bound in Corollary 1.2 is 4 minus a multiple of $1/\log d$. The choice $y = \exp(-1/(4\log d))$ leads to a bound in closed form.

Corollary 1.2. *For every integer $d \geq 8$,*

$$K(d) \;<\; 4 - \frac{4}{\log d}.$$

At every degree $d \geq 8$ this bound is below each of the five bounds quoted above and below the numerical bounds that Crane gives at nine degrees [7] (Remark 3.2); see Table 2 and Figure 1. Theorem 1.1 with the best choice of y gives a smaller bound (Remark 3.3), but not one of better order: if $B(d)$ denotes the infimum over $0 < y < 1$ of its right-hand side, then $(4 - B(d))\log d \to 2\pi^2$ as $d \to \infty$ (Proposition 3.4). At $d = 7$, Theorem 1.1 does not give the bound of Corollary 1.2: $B(7) > 4 - 4/\log 7$ (Remark 3.5).

Corollary 1.2 shows that $K(d)$ can approach 4 only slowly. For example, it gives $K(d) < 3.9$ for $8 \leq d \leq e^{40}$, and the bound of Beardon, Minda and Ng gives $K(d) \leq 4^{5/6} < 3.9$ for $d \leq 7$; so a polynomial p with $S(p) \geq 3.9$ has degree larger than $2 \cdot 10^{17}$.

TABLE 2. Upper bounds for $K(d)$, rounded up (Crane's numerical bounds as printed in [7]). The column "earlier" gives the smallest, at that degree, of the earlier bounds of Remark 3.2: Fujikawa and Sugawa [14] at $d = 8$; the table of numerical bounds in [7] at $d = 100$ and $d = 10^4$; Crane's bound $4 - 2.263/\sqrt{d}$ [7] at $d = 10^6$. The column "Theorem 1.1" gives the bound of Theorem 1.1, computed in interval arithmetic with upper and lower bounds for μ as in Remark 3.5, at $y = e^{-L}$, with L from the last column (close to the best L; the proof of Corollary 1.2 uses $L = 1/(4\log d)$).

d	earlier	$4 - 4/\log d$	Theorem 1.1	L
8	2.6321	2.0765	2.0550	0.12766
100	3.621	3.1315	2.5255	0.05337
10^4	3.955	3.5658	2.9495	0.02156
10^6	3.9978	3.7105	3.1754	0.01194

The ideas of the proofs. We first recall Smale's proof, and then say what we add to it. Dividing p by a constant changes neither its critical points nor $S(p)$; so we may assume that the least modulus of a critical value of p is 1 (if it is 0, then $S(p) = 0$). Let ζ_m be a critical point with $|p(\zeta_m)| = 1$. Since the unit disc $\mathbb{D}$ contains no critical value of p, the branch F of p^{-1} with $F(0) = 0$ is defined and univalent on $\mathbb{D}$, and it omits ζ_m. Since $F'(0) = 1/p'(0)$, the definition of $S(p)$ gives $S(p) \leq |p(\zeta_m)|/|\zeta_m\, p'(0)| = |F'(0)|/|\zeta_m|$. Koebe's one-quarter theorem gives $|\zeta_m| \geq |F'(0)|/4$, so $S(p) \leq 4$: this is Smale's theorem. This argument uses only that F is univalent and omits ζ_m. Our upper bound for $K(d)$ comes from using, in addition, that p is a polynomial of degree d.

Outline of the proof of Theorem 1.1 (see Figure 2). For $0 < y < 1$ let $V := F(\mathbb{D}_y)$, where $\mathbb{D}_y$ is the open disc of radius y about 0. In the coordinate $x = -\zeta_m/z$, where z is the variable of p, the complement of $\overline{V}$ becomes a bounded Jordan domain E_y that contains the points 0 and -1, the images of ∞ and of ζ_m. The logarithmic capacity $\mathrm{cap}(\overline{E_y})$ of its closure (Section 2.3) is $|\zeta_m|/(y\,|F'(0)|)$, which is at most $1/(y\,S(p))$. Let f map $\mathbb{D}$ conformally onto E_y with $f(0) = 0$, and let $\xi := f^{-1}(-1)$. On ∂V we have $|p| = y$, at ζ_m we have $|p| = 1$, and p has a pole of order d at ∞. So the function $P(w) := w^d\, p(-\zeta_m/f(w))$ is holomorphic in $\mathbb{D}$ (the factor w^d cancels the pole of order d of $p(-\zeta_m/f(w))$ at $w = 0$), and $\limsup_{|w|\to 1} |P(w)| \leq y$. By the maximum principle,

$|\xi|^d = |P(\xi)| \le y$ (a case of the Bernstein–Walsh inequality, Remark 3.1). By a theorem of Betsakos [4], $1 = |f(\xi)| \le 4\,\mathrm{cap}(\overline{E_y})\, e^{-\mu(|\xi|)}$; since μ decreases and $|\xi| \le y^{1/d}$, this and the capacity bound above give $e^{\mu(y^{1/d})} \le e^{\mu(|\xi|)} \le 4/(y\,S(p))$, which is the theorem.

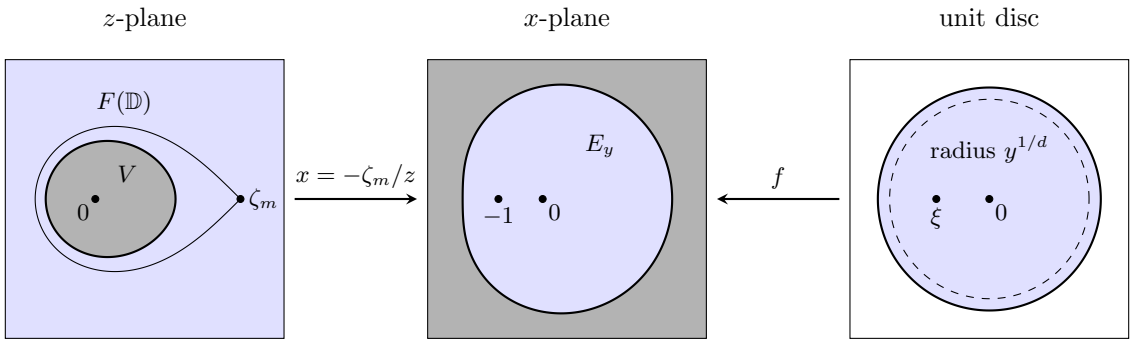


FIGURE 2. The construction in the proof of Theorem 1.1, drawn to scale for $p(z) = 2z - z^2$ (so $d = 2$ and $\zeta_m = 1$) and $y = 0.8$. Left: the domain $V = F(\mathbb{D}_y)$ (dark gray) lies in $F(\mathbb{D})$, the domain inside the thin curve; the critical point ζ_m is not in $F(\mathbb{D})$ (here it lies on its boundary). Middle: in the coordinate $x = -\zeta_m/z$, the complement of $\overline{V}$ becomes the bounded Jordan domain E_y, which contains 0 and -1, the images of ∞ and of ζ_m. Right: a conformal map f of the unit disc onto E_y with $f(0) = 0$ takes a point ξ to -1. The point ξ lies in the closed disc of radius $y^{1/d}$ about 0 (dashed circle). Dark gray marks V and its image $x(V)$; pale shading marks the complement of $\overline{V}$, its image E_y, and the unit disc.

Outline of the proof of Corollary 1.2. With $y = \exp(-1/(4\log d))$, the identity $\mu(r)\,\mu(\sqrt{1-r^2}) = \pi^2/4$ and the inequality $\mu(s) < \log(4/s)$ bound the right-hand side of Theorem 1.1 by an expression in elementary functions. For $d \ge 21$ this expression is below $4 - 4/\log d$ by elementary inequalities; for the thirteen degrees $8 \le d \le 20$ the comparison is a finite computation in interval arithmetic.

Organization. Section 2 collects the notation and the tools. Section 3 proves Theorem 1.1, Corollary 1.2 and Proposition 3.4. Section 4 closes with a brief discussion of open questions.

2. Notation and tools

2.1. **Notation.** Throughout, log is the natural logarithm, $\mathbb{D}$ is the open unit disc, $\mathbb{D}_y$ is the open disc of radius $y > 0$ about 0, and $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. A *Jordan domain* is a domain in $\widehat{\mathbb{C}}$ whose boundary is a Jordan curve. A function is *univalent* if it is meromorphic and injective, and a *conformal map* of a domain D onto a domain D' is a univalent function on D with image D'.

2.2. **The Grötzsch ring function.** Recall the complete elliptic integral $\mathcal{K}$ and the function μ of (1). As usual, $r' := \sqrt{1-r^2}$ is the *complementary modulus* of r, and $\mathcal{K}'(r) := \mathcal{K}(r')$ for $0 < r < 1$, so that $\mu(r) = \frac{\pi}{2}\,\mathcal{K}'(r)/\mathcal{K}(r)$.

Lemma 2.1. (a) *The function μ is strictly decreasing and maps $(0,1)$ onto $(0,\infty)$.* (b) *For $0 < r < 1$ we have $\mu(r)\,\mu(r') = \pi^2/4$.* (c) *For $0 < s < 1$ we have $\mu(s) < \log(4/s)$.*

Proof. (a) For fixed $t \in (0,1)$ the integrand in (1) is continuous and strictly increasing in r. Hence $\mathcal{K}$ is strictly increasing on $(0,1)$. It is continuous, and $\mathcal{K}(r) \to \pi/2$ as $r \to 0$, by dominated convergence (for any $r_1 \in (0,1)$, the integrand at r_1 is an integrable majorant on $(0, r_1]$); and $\mathcal{K}(r) \to \int_0^1 dt/(1-t^2) = \infty$ as $r \to 1$, by monotone convergence. So $\mathcal{K}'$ is continuous and strictly

decreasing, from ∞ to $\pi/2$, and the quotient $\mu = \frac{\pi}{2}\,\mathcal{K}'/\mathcal{K}$ is continuous and strictly decreasing, from ∞ to 0. (b) The complementary modulus of r' is r, so

$$\mu(r') \;=\; \frac{\pi}{2}\,\frac{\mathcal{K}(r)}{\mathcal{K}(r')} \;=\; \frac{\pi}{2}\,\frac{\mathcal{K}(r)}{\mathcal{K}'(r)};$$

multiplying by $\mu(r)$ gives (b). (c) By [8, (19.9.5)], for $0 < k < 1$ and $k' = \sqrt{1-k^2}$,

$$\log\frac{(1+\sqrt{k'})^2}{k} \;<\; \frac{\pi\,\mathcal{K}'(k)}{2\,\mathcal{K}(k)} \;<\; \log\frac{2(1+k')}{k}.$$

By [8, (19.2.4), (19.2.8), (19.2.9)], the functions written K and K' there are our $\mathcal{K}$ and $\mathcal{K}'$; so the middle term is $\mu(k)$. Since $k' < 1$, we have $2(1+k') < 4$, and (c) follows. □

Both bounds in the proof of (c) are $\log(4/k) + O(k^2)$ as $k \to 0$, so $\mu(r) = \log(4/r) + O(r^2)$ as $r \to 0$; with (b), $\mu(r) = \pi^2/(4\mu(r')) \sim \pi^2/\bigl(2\log\frac{16}{1-r^2}\bigr)$ as $r \to 1$.

2.3. **Transfinite diameter and capacity.** For a bounded infinite set $A \subset \mathbb{C}$ and $n \geq 2$ let

$$d_n(A) \;:=\; \sup\Bigl\{\prod_{1\leq j<k\leq n} |z_j - z_k|^{2/(n(n-1))} :\; z_1,\dots,z_n \in A\Bigr\}. \tag{2}$$

Since the product is continuous in $z_1,\dots,z_n$, we have $d_n(A) = d_n(\overline{A})$. The sequence $d_n(\overline{A})$ decreases [22, Theorem 5.5.2], and its limit $d_\infty(A) = d_\infty(\overline{A})$ is the *transfinite diameter* of A. By the Fekete–Szegő theorem, the transfinite diameter of an infinite compact set Q equals its logarithmic capacity $\mathrm{cap}(Q)$ ([4, p. 146], [22, Theorem 5.5.2]). If $Q \subset \mathbb{C}$ is compact and connected, has more than one point and has connected complement, and Φ is a conformal map of $\{|u| > 1\} \cup \{\infty\}$ onto $\widehat{\mathbb{C}} \setminus Q$ with $\Phi(\infty) = \infty$, then $\Phi(u) = a\,u + O(1)$ as $u \to \infty$ and $\mathrm{cap}(Q) = |a|$ [22, Theorem 5.2.3 and Corollary 5.2.2].

2.4. **A theorem of Betsakos.** The proof of Theorem 1.1 rests on the following theorem.

Theorem 2.2 ([4, Theorem 1.1])**.** *Let f be a nonconstant bounded holomorphic function in $\mathbb{D}$ with $f(0) = 0$. Then*

$$|f(w)| \;\leq\; 4\,d_\infty(f(\mathbb{D}))\,e^{-\mu(|w|)}, \qquad w \in \mathbb{D}.$$

Equality holds for some $w \in \mathbb{D}\setminus\{0\}$ if and only if f maps $\mathbb{D}$ conformally onto a domain bounded by an ellipse so that 0 and w are mapped onto the foci of the ellipse.

In [4] the transfinite diameter is written d. At $w = 0$ the right-hand side is read as 0. As $|w| \to 0$ the right-hand side is $d_\infty(f(\mathbb{D}))\,|w|\,(1 + O(|w|^2))$, and as $|w| \to 1$ it tends to $4\,d_\infty(f(\mathbb{D}))$. Letting $w \to 0$ gives $|f'(0)| \leq d_\infty(f(\mathbb{D}))$; for a conformal map f this is the classical inequality between the conformal radius $|f'(0)|$ and the capacity of $\overline{f(\mathbb{D})}$.

3. The upper bounds

Proof of Theorem 1.1. Fix $d \geq 2$ and $0 < y < 1$, and let p be a polynomial of degree d with $p(0) = 0$ and $p'(0) \neq 0$. We show that $S(p) \leq (4/y)\,e^{-\mu(y^{1/d})}$; taking the supremum over p gives the theorem.

Normalization. If $p(c) = 0$ for some critical point c of p, then $S(p) = 0$ and there is nothing to prove. Otherwise the least modulus of a critical value of p is positive, and dividing p by it changes neither the critical points nor $S(p)$. So we may assume that this least modulus is 1, and we fix a critical point ζ_m with $|p(\zeta_m)| = 1$. Then $S(p) > 0$, and $\zeta_m \neq 0$ because $p'(0) \neq 0$.

The inverse branch. The disc $\mathbb{D}$ contains no critical value of p, and p is a covering map over the complement of its critical values. Since $\mathbb{D}$ is simply connected and $p(0) = 0$, there is a branch F of

p^{-1} on $\mathbb{D}$ with $F(0) = 0$. Since $p(F(w)) = w$, the branch F is univalent and $F'(0) = 1/p'(0)$. By the definition of $S(p)$, at the critical point ζ_m,

$$S(p) \;\le\; \frac{|p(\zeta_m)|}{|\zeta_m\, p'(0)|} \;=\; \frac{|F'(0)|}{|\zeta_m|}. \tag{3}$$

The domain E_y. Let $V := F(\mathbb{D}_y)$. It is a Jordan domain bounded by the analytic curve $F(\partial\mathbb{D}_y)$, it contains $F(0) = 0$, and its closure $F(\overline{\mathbb{D}_y})$ lies in $F(\mathbb{D})$. The point ζ_m is not in $F(\mathbb{D})$, since $|p(\zeta_m)| = 1$ while $p(F(\mathbb{D})) = \mathbb{D}$; so $\zeta_m \notin \overline{V}$. Let $x(z) := -\zeta_m/z$, a Möbius involution of $\widehat{\mathbb{C}}$, and let

$$E_y \;:=\; x\big(\widehat{\mathbb{C}} \setminus \overline{V}\big).$$

Then E_y is a Jordan domain bounded by $x(\partial V)$. It is bounded, since $x^{-1}(\infty) = 0 \in V$, and it contains $x(\infty) = 0$ and $x(\zeta_m) = -1$.

The capacity of $\overline{E_y}$. The map $\Phi(u) := x(F(y/u))$ is a conformal map of $\{|u| > 1\} \cup \{\infty\}$ onto $\widehat{\mathbb{C}} \setminus \overline{E_y} = x(V)$ with $\Phi(\infty) = \infty$. Since $F(w) = F'(0)\, w + O(w^2)$ as $w \to 0$,

$$\Phi(u) \;=\; -\frac{\zeta_m}{y\, F'(0)}\, u + O(1), \qquad u \to \infty.$$

So the logarithmic capacity of $\overline{E_y}$ is $\operatorname{cap}(\overline{E_y}) = |\zeta_m|/(y\,|F'(0)|)$ (Section 2.3), and by (3)

$$\operatorname{cap}(\overline{E_y}) \;\le\; \frac{1}{y\, S(p)}. \tag{4}$$

A bound for $|\xi|$. Let f be a conformal map of $\mathbb{D}$ onto E_y with $f(0) = 0$, and let $\xi := f^{-1}(-1)$; then $\xi \neq 0$. Put

$$P(w) \;:=\; w^d\, p\big(x(f(w))\big), \qquad w \in \mathbb{D}.$$

The map $x \circ f$ sends $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus \overline{V}$; its only pole is at $w = 0$, and that pole is simple. Since p has degree d, the singularity of P at 0 is removable, so P is holomorphic in $\mathbb{D}$. As $|w| \to 1$, the distance from $f(w)$ to ∂E_y tends to 0. Since the compact set ∂E_y does not contain 0, the map x is uniformly continuous on a neighborhood of ∂E_y; hence the distance from $x(f(w))$ to $x(\partial E_y) = \partial V$ tends to 0. On $\partial V = F(\partial\mathbb{D}_y)$ we have $|p| = y$; as p is uniformly continuous on a compact neighborhood of ∂V and $|w|^d < 1$, we get $\limsup_{|w|\to 1} |P(w)| \le y$. By the maximum principle, $|P| \le y$ in $\mathbb{D}$. At $w = \xi$ we have $x(f(\xi)) = x(-1) = \zeta_m$, so $|\xi|^d = |\xi|^d\,|p(\zeta_m)| = |P(\xi)| \le y$, that is,

$$|\xi| \;\le\; y^{1/d}. \tag{5}$$

Betsakos's theorem. The function f is nonconstant, bounded and holomorphic in $\mathbb{D}$, with $f(0) = 0$. Theorem 2.2 at $w = \xi$ gives

$$1 \;=\; |f(\xi)| \;\le\; 4\, d_\infty(E_y)\, e^{-\mu(|\xi|)}.$$

By Section 2.3, $d_\infty(E_y) = d_\infty(\overline{E_y}) = \operatorname{cap}(\overline{E_y})$, the second equality by the Fekete–Szegő theorem. With (4),

$$e^{\mu(|\xi|)} \;\le\; 4\operatorname{cap}(\overline{E_y}) \;\le\; \frac{4}{y\, S(p)}. \tag{6}$$

Conclusion. The function μ is strictly decreasing (Lemma 2.1(a)), so (5) gives $\mu(y^{1/d}) \le \mu(|\xi|)$. With (6), $\mu(y^{1/d}) \le \log\big(4/(y\, S(p))\big)$, that is, $S(p) \le (4/y)\, e^{-\mu(y^{1/d})}$. □

Remark 3.1 (Related results). Inequality (5) is a case of the Bernstein–Walsh inequality, which is called Bernstein's lemma in [22, Theorem 5.5.7(a)]. For the compact set $\overline{V}$, that inequality reads $|p(\zeta_m)| \le \max_{\overline{V}} |p| \cdot e^{d\, g(\zeta_m)}$, where g is the Green function of $\widehat{\mathbb{C}} \setminus \overline{V}$ with pole at ∞; here $\max_{\overline{V}} |p| = y$ and $|p(\zeta_m)| = 1$. Write $g_\Omega(\cdot, a)$ for the Green function of a domain Ω with pole at a [22, Definition 4.4.1]; conformal maps preserve Green functions [22, Theorem 4.4.4], and

$g_{\mathbb{D}}(w,0) = \log(1/|w|)$. By conformal invariance under x and then under f, we have $g(\zeta_m) = g_{E_y}(-1,0) = g_{\mathbb{D}}(\xi,0) = \log(1/|\xi|)$, so the inequality is (5).

Inequality (5) is also a special case of an inequality of Dubinin for meromorphic functions and Green functions [10, Theorem 1], [11, (1.27), pp. 618–619]; the reduction is written out in [17, Remark 3.1]. According to [11, p. 619], Dubinin's inequality was first obtained by Mityuk [20], under slightly different conditions on the function.

Capacity has entered this problem before. The proof of Dubinin's upper bound for $K(d)$ uses condenser capacities [12]; Crane's bound comes from estimates for the logarithmic capacity of a connected set containing three given points [7]; and Jatar and Ng [18] use a Koebe-type theorem of Cunningham (stated in [18, Theorem 8]) on univalent functions whose image has bounded capacity. What is new in Theorem 1.1 is the combination of the Bernstein–Walsh inequality with the theorem of Betsakos (Theorem 2.2), applied to the domain E_y.

Proof of Corollary 1.2. Let $d \geq 8$ and put

$$\ell := \log d, \quad L := \frac{1}{4\ell}, \quad y := e^{-L}, \quad r := y^{1/d} = e^{-L/d}, \quad r' := \sqrt{1-r^2} = \sqrt{1-e^{-2L/d}}.$$

Step 1: a bound without elliptic integrals. Lemma 2.1(a) and (c) at $s = r'$ give $0 < \mu(r') < \log(4/r')$. With Lemma 2.1(b),

$$\mu(r) \;=\; \frac{\pi^2}{4\,\mu(r')} \;>\; \frac{\pi^2}{4\log(4/r')},$$

and Theorem 1.1 with this y gives

$$K(d) \;\leq\; 4e^{L}e^{-\mu(r)} \;<\; M(d) \;:=\; 4\exp\Big(L - \frac{\pi^2}{4\log(4/r')}\Big). \tag{7}$$

It remains to show $M(d) \leq 4 - 4/\ell$.

Step 2: the degrees $8 \leq d \leq 20$. For each of these thirteen degrees, interval arithmetic gives $M(d) < 4 - 4/\log d$ (Table 3); the smallest margin is 0.0173, at $d = 8$.

Step 3: the degrees $d \geq 21$. Since $1 - e^{-t} \geq t - t^2/2$ for $t \geq 0$, the choice $t = 2L/d$ gives $r'^2 \geq (2L/d)(1 - L/d)$. Hence $\log(4/r') = \frac{1}{2}\log(16/r'^2) \leq \Lambda/2$, where

$$\Lambda \;:=\; \log\frac{8d}{L\,(1-L/d)} \;=\; \ell + \log(32\ell) - \log(1-L/d),$$

and (7) gives $M(d) \leq 4\exp\big(L - \pi^2/(2\Lambda)\big)$. So $M(d) \leq 4(1-1/\ell)$ follows from

$$\frac{\pi^2}{2\Lambda} \;\geq\; \frac{1}{4\ell} - \log\Big(1 - \frac{1}{\ell}\Big). \tag{8}$$

We bound both sides, using $\ell \geq \log 21 > 3.04$. Since $-\log(1-s) \leq s/(1-s)$ for $0 < s < 1$, and $\ell \geq 3$, we have $-\log(1-1/\ell) \leq 1/(\ell-1) \leq 1.5/\ell$; hence the right-hand side of (8) is at most $1.75/\ell$. Next, the function $\ell \mapsto \log(32\ell)/\ell$ has derivative $(1-\log(32\ell))/\ell^2 < 0$ for $\ell \geq 1$, so it decreases there, and its value at $\ell = \log 21$ is $1.5040\ldots < 1.52$; hence $\log(32\ell) \leq 1.52\,\ell$. Moreover, $L/d = 1/(4\ell d) \leq 1/(4 \cdot 3.04 \cdot 21) < 0.0042$, so that $-\log(1 - L/d) \leq -\log(1 - 0.0042) < 0.005 < 0.002\,\ell$. Therefore $\Lambda \leq (1 + 1.52 + 0.002)\,\ell < 2.53\,\ell$, and the left-hand side of (8) is at least $\pi^2/(5.06\,\ell) > 1.95/\ell$. Since $1.95/\ell > 1.75/\ell$, inequality (8) holds. With (7), $K(d) < M(d) \leq 4 - 4/\ell$. □

Remark 3.2 (The earlier bounds). At every degree $d \geq 8$, the bound of Corollary 1.2 is below the bounds of Crane, of Fujikawa and Sugawa, of Conte, Fujikawa and Lakic, of Beardon, Minda and Ng, and of Dubinin, and below Crane's numerical bounds at nine degrees.

(a) Crane proved $K(d) < 4 - 2.263/\sqrt{d}$ for $d \geq 8$ [7]. The function $\sqrt{d}/\log d$ attains its minimum over $d > 1$ at $d = e^2$, where it equals $e/2$. Hence $4/\log d \geq 2e/\sqrt{d} > 2.263/\sqrt{d}$.

TABLE 3. Step 2 of the proof of Corollary 1.2: $M(d)$ of (7) against $4 - 4/\log d$, in interval arithmetic. Upper bounds for $M(d)$ are rounded up and lower bounds for $4 - 4/\log d$ are rounded down, to five decimal places; the margin is the difference of these two columns.

d	$M(d) \le$	$4 - 4/\log d \ \ge$	margin $\ge$
8	2.05907	2.07640	0.01733
9	2.08862	2.17952	0.09090
10	2.11417	2.26282	0.14865
11	2.13661	2.33187	0.19526
12	2.15656	2.39028	0.23372
13	2.17447	2.44051	0.26604
14	2.19069	2.48430	0.29361
15	2.20549	2.52292	0.31743
16	2.21907	2.55730	0.33823
17	2.23162	2.58817	0.35655
18	2.24326	2.61609	0.37283
19	2.25410	2.64150	0.38740
20	2.26424	2.66476	0.40052

(b) The bound of Fujikawa and Sugawa [14, Theorem 1, p. 97] is

$$K(d) \ \le \ \frac{4\big(1 + (d-2)\,q\big)}{d+1} \ = \ 4 - \frac{4\big((d-2)(1-q)+2\big)}{d+1}, \qquad q := 4^{-1/(d-1)}.$$

Since $1 - q \le (\log 4)/(d-1)$, the bracket in the numerator satisfies $(d-2)(1-q)+2 < \log 4 + 2 < 3.39$. The function $d \mapsto (d+1)/\log d$ increases for $d \ge 4$, and at $d = 8$ it equals $9/\log 8 > 4.32 > 3.39$. So the subtracted fraction is less than $4 \cdot 3.39/(d+1) < 4/\log d$ for $d \ge 8$. Hence $4 - 4/\log d$ is smaller than the bound of Fujikawa and Sugawa.

(c) The bounds $4(d-1)/(d+1)$ of Conte, Fujikawa and Lakic [5], $4^{(d-2)/(d-1)} = 4q$ of Beardon, Minda and Ng [3], and $\frac{d-1}{d}\,4q$ of Dubinin [9, 12] are all at least the bound of Fujikawa and Sugawa, since the three differences are $4(d-2)(1-q)/(d+1) \ge 0$ (as $q \le 1$), $4(3q-1)/(d+1) > 0$ (as $q \ge 1/2$ for $d \ge 3$) and $4\big((2d-1)q - d\big)/\big(d(d+1)\big) > 0$ (as $q \ge 4^{-1/7} > 0.8 > d/(2d-1)$ for $d \ge 8$). (Fujikawa and Sugawa compare the first two in [14, Remark (i), p. 97].) Hence, by (b), $4 - 4/\log d$ is also smaller than these three bounds.

(d) The numerical bounds of Crane at the nine degrees $d = 8, 9, 10, 11, 12, 16, 25, 100, 10^4$ [7] are 3.124, 3.150, 3.169, 3.188, 3.208, 3.271, 3.365, 3.621, 3.955. Each exceeds $4 - 4/\log d$ at the same degree by more than 0.389; the smallest difference is at $d = 10^4$.

Remark 3.3 (The best choice of y). For every $d \ge 8$ the proof of Corollary 1.2 gives

$$\inf_{0<y<1} (4/y)\, e^{-\mu(y^{1/d})} < M(d) \le 4 - 4/\log d.$$

Table 2 shows at four degrees how far the bound of Theorem 1.1, at a value of y close to the best one, lies below $4 - 4/\log d$, and Proposition 3.4 below shows that for the best y this difference is asymptotic to $(2\pi^2 - 4)/\log d$ (Figure 3). By that proposition, Theorem 1.1 gives $\liminf_{d\to\infty} (4 - K(d)) \log d \ge 2\pi^2$, while Corollary 1.2 gives $(4 - K(d)) \log d > 4$.

Proposition 3.4. *Let $B(d) := \inf_{0<y<1}(4/y)\,e^{-\mu(y^{1/d})}$ be the best bound that Theorem 1.1 gives at degree d. Then $(4 - B(d))\log d \to 2\pi^2$ as $d \to \infty$. More precisely,*

$$(4 - B(d))\log d \;\geq\; 2\pi^2 - O\Big(\frac{\log\log d}{\log d}\Big) \quad (d \to \infty),$$

$$(4 - B(d))\log d \;\leq\; \frac{2\pi^2}{1 - \pi^2/(d\log^2 d)} \quad (d \geq 8).$$

Proof. Let $d \geq 8$. Write $\ell := \log d$ and $y = e^{-L}$ with $L > 0$, and put $r := e^{-L/d}$, $r' := \sqrt{1-r^2}$. Write $\beta_d(L) := 4\exp(L - \mu(r))$ for the bound of Theorem 1.1, so that $B(d) = \inf_{L>0}\beta_d(L)$. By Lemma 2.1(b), $\mu(r) = \pi^2/(4\mu(r'))$.

The lower bound for $(4 - B(d))\log d$. Take $L := \pi^2/(2\ell^2)$. As in Step 3 of the proof of Corollary 1.2, $\mu(r') < \log(4/r') \leq \Lambda/2$ with $\Lambda := \log\big(8d/(L(1 - L/d))\big) = \ell + 2\log\ell + O(1)$. Hence $\log(B(d)/4) \leq L - \pi^2/(2\Lambda) = -\pi^2/(2\ell) + O(\log\ell/\ell^2)$, and $(4 - B(d))\,\ell \geq 2\pi^2 - O(\log\ell/\ell)$.

The upper bound for $(4 - B(d))\log d$. We show that for every $L > 0$, $\beta_d(L)$ is at least $4\exp\big(-\pi^2/(2\ell - 2\pi^2/(\ell d))\big)$; the denominator is positive, since $d\log^2 d \geq 8\log^2 8 > \pi^2$.

For $0 < L < 2$ we first bound $\mu(r')$ from below, and so $\mu(r)$ from above. The left inequality of [8, (19.9.5)] (displayed in the proof of Lemma 2.1(c)) at $k = r'$, whose complementary modulus is r, gives $\mu(r') > \log(4/r') - 2\log\big(2/(1+\sqrt{r})\big) \geq \log(4/r') - L/d$: indeed $\sqrt{r} = e^{-L/(2d)} \geq 1 - L/(2d)$, so $2/(1+\sqrt{r}) \leq 1/(1-\sigma)$ with $\sigma := L/(4d) \leq \frac{1}{2}$, and $-\log(1-\sigma) \leq 2\sigma$ for $0 \leq \sigma \leq \frac{1}{2}$. Also $r'^2 = 1 - e^{-2L/d} \leq 2L/d$, so $\log(4/r') \geq \frac{1}{2}\log(8d/L)$. Hence

$$4\mu(r') \;>\; \Delta_L \;:=\; 2\log\frac{8d}{L} - \frac{4L}{d}.$$

As $0 < L < 2$, we have $\Delta_L > 2\log(4d) - 8/d > 0$. Since $4\mu(r') > \Delta_L > 0$, we get $\mu(r) = \pi^2/(4\mu(r')) < \pi^2/\Delta_L$, and $\beta_d(L)$ exceeds $4\exp(L - \pi^2/\Delta_L)$.

There are three cases.

(a) $L \geq 2$. Lemma 2.1(c) at $s = r$ gives $e^{-\mu(r)} > r/4$, so $\beta_d(L)$ exceeds $e^{(1-1/d)L} \geq e^{7/4} > 4$.

(b) $\pi^2/(2\ell) \leq L < 2$. Then $\Delta_L > 2\log(4d) - 8/d > 2\ell$, so $L - \pi^2/\Delta_L > 0$ and $\beta_d(L)$ exceeds 4.

(c) $0 < L < \min(2, \pi^2/(2\ell))$. Then $\log(8d/L) > \ell + \log(16\ell/\pi^2) \geq \ell$ and $4L/d < 2\pi^2/(\ell d)$, so $\Delta_L > 2\ell - 2\pi^2/(\ell d) > 0$, and $\beta_d(L)$ exceeds $4\exp\big(-\pi^2/(2\ell - 2\pi^2/(\ell d))\big)$.

In cases (a) and (b), $\beta_d(L)$ exceeds 4, so in all three cases $\beta_d(L)$ is at least the last bound of case (c), and so is $B(d)$:

$$B(d) \;\geq\; 4\exp\Big(-\frac{\pi^2}{2\ell - 2\pi^2/(\ell d)}\Big).$$

With $1 - e^{-t} \leq t$ for $t \geq 0$ this gives the second inequality of the proposition, whose right-hand side tends to $2\pi^2$. With the lower bound, $(4 - B(d))\log d \to 2\pi^2$. □

Remark 3.5 (The range $d \geq 8$, and bounds for μ). Theorem 1.1 does not give Corollary 1.2 at $d = 7$: the number $B(7)$ of Proposition 3.4 lies between 2.0190 and 2.0191, while $4 - 4/\log 7 < 1.9445$. It does give $K(7) \leq B(7) < 2.0191$; at $d = 7$ the smallest of the bounds quoted in the Introduction is $2.4842\ldots$, that of Fujikawa and Sugawa. At $d = 8$, the number $B(8)$ lies between 2.0549 and 2.0550; so $(4 - B(8))\log 8$, the left end of the curve in Figure 3, is less than 4.045.

The bounds 2.0190 and 2.0191 for $B(7)$, and 2.0549 and 2.0550 for $B(8)$, are computed in interval arithmetic, with upper and lower bounds for μ from the arithmetic-geometric mean. For $0 < k < 1$ let $a_0 = 1$, $b_0 = k$, $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = \sqrt{a_n b_n}$ [8, (19.8.1)]. By the inequality between the arithmetic and the geometric mean, $b_n \leq b_{n+1} \leq a_{n+1} \leq a_n$ and $a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = (a_n - b_n)/2$; so the two sequences have a common limit $\operatorname{agm}(1,k)$, and $b_n \leq \operatorname{agm}(1,k) \leq a_n$ for every n. By [8, (19.8.5)], $\mathcal{K}(k) = \pi/(2\operatorname{agm}(1,k'))$, so $\mu(r) =$

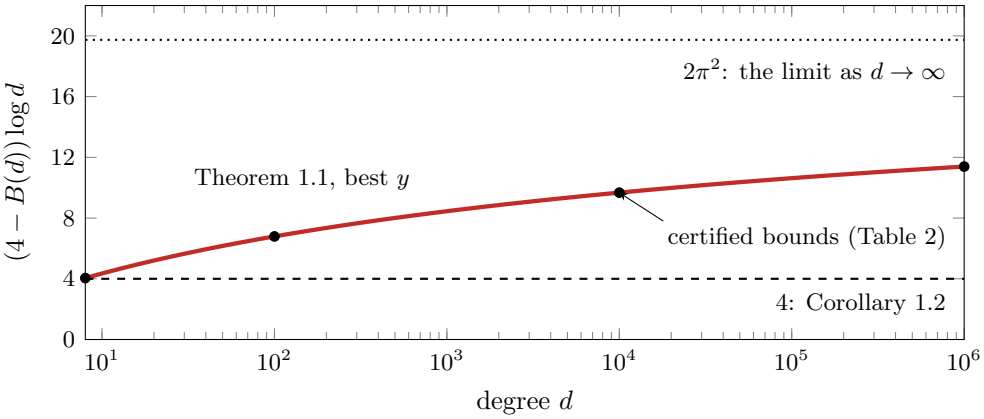


FIGURE 3. The number $(4 - B(d)) \log d$ for $8 \le d \le 10^6$, where $B(d)$ is the infimum over $0 < y < 1$ of the bound of Theorem 1.1 (Proposition 3.4). Since $K(d) \le B(d)$, Theorem 1.1 gives $K(d) \le 4 - h/\log d$, where h is the height of the curve at d. The curve is computed in floating point. The dots are $(4 - T) \log d$, where T is the entry in the column "Theorem 1.1" of Table 2 at $d = 8$, 100, 10^4, 10^6; since $B(d) \le T$, each dot lies on or below the curve. The dashed line is the constant 4 of Corollary 1.2; the curve lies above it at every $d \ge 8$ (Remark 3.3), by less than 0.05 at $d = 8$ (Remark 3.5), and tends to $2\pi^2$ (dotted line) as $d \to \infty$ (Proposition 3.4).

$\pi \operatorname{agm}(1, r')/(2 \operatorname{agm}(1, r))$. With $\beta_d(L)$ as in the proof of Proposition 3.4, defined by the same formula for $d = 7$, the upper bounds hold because $\beta_8(0.12766) < 2.0550$ and $\beta_7(0.13621) < 2.0191$. For the lower bounds, first let $L \ge 1$: Lemma 2.1(c) gives $e^{-\mu(e^{-L/d})} > e^{-L/d}/4$, so $\beta_d(L) > e^{(d-1)L/d} \ge e^{6/7} > 2.35$. On an interval $L_1 < L \le L_2$ with $0 \le L_1 < L_2 \le 1$ we have $\beta_d(L) \ge 4e^{L_1} e^{-\mu(e^{-L_2/d})}$, as μ is decreasing. Starting from $(0, 1]$, we halve an interval as long as this lower bound is at most 2.0549 for $d = 8$, or at most 2.0190 for $d = 7$. The halving stops with 1735 intervals for $d = 8$ and 644 for $d = 7$; on every one of them this lower bound exceeds 2.0549 (for $d = 8$) or 2.0190 (for $d = 7$). This proves the two lower bounds.

4. Discussion

Before this paper the best upper bound for $K(d)$ at large degree was Crane's $K(d) < 4 - 2.263/\sqrt{d}$ [7]. Theorem 1.1 and Corollary 1.2 replace that $1/\sqrt{d}$ gap below 4 by $4/\log d$ for every $d \ge 8$, improving on each of the five earlier upper bounds cited in Section 1 and on Crane's numerical bounds at nine degrees at every degree $d \ge 8$ (Remark 3.2). Several natural questions remain open.

The exact order of $4 - K(d)$. The companion paper [17, Theorems 1.3 and 1.4] shows that $4 - K(d)$ is of order $1/\sqrt{\log d}$ up to constant factors and a factor $\sqrt{\log\log\log d}$ for very large d. Can the factor $\sqrt{\log\log\log d}$ be removed, and does $(4 - K(d))\sqrt{\log d}$ have a limit as $d \to \infty$?

Moderate degrees. Corollary 1.2 and Theorem 1.1 give the strongest upper bounds of this paper and its companion for $8 \le d \le e^{36{,}363}$, where the gap has order $1/\log d$ (Proposition 3.4). How large is $4 - K(d)$ for moderate degrees, say $8 \le d \le 10^6$?

The smallest degree where Smale's conjecture fails. In full strength Smale's conjecture asks whether $K(d) = (d - 1)/d$; this holds for $d \le 4$ [13], while $K(d) > 1$ holds at degree 29,984 [2]. What is the least degree d with $K(d) > (d - 1)/d$?

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