hexagon:2611.01216v1

# HOW FAST SMALE'S MEAN VALUE CONSTANTS (1981) APPROACH 4

DAVE GOEL

First version: November 12, 2026. This version (1.00): November 26, 2026.

ABSTRACT. For a complex polynomial $p$ of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$, let $S(p)$ be the minimum of $|p(c)/(c\,p'(0))|$ over the critical points $c$ of $p$, and let $K(d)$ be the supremum of $S(p)$ over all such $p$ of degree $d$. In 1981 Smale proved that $K(d) \leq 4$ for every $d$. It is known that $K(d) < 4$ for every $d$, and the author recently proved that $K(d) \to 4$ as $d \to \infty$. We prove that $K(d) \leq (4/y)\, e^{-\mu(y^{1/d})}$ for every $0 < y < 1$, where $\mu$ is the Grötzsch ring function, and deduce that $K(d) < 4 - 4/\log d$ for every $d \geq 8$; for large $d$ the best earlier bound was $K(d) < 4 - 2.263/\sqrt{d}$, due to Crane. In a companion paper, the author shows that the true order of $4 - K(d)$ is $(\log d)^{-1/2}$, up to a factor $\sqrt{\log\log\log d}$.[1] Theorem 1.1 and Corollary 1.2 are verified in Lean.[2]

## 1. INTRODUCTION

Let $p$ be a complex polynomial of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$. A *critical point* of $p$ is a zero of $p'$, and a *critical value* of $p$ is its value at a critical point. Put

$$S(p) \;:=\; \min_{p'(c)=0} \left| \frac{p(c)}{c\,p'(0)} \right|.$$

Smale's mean value theorem [25, Theorem, p. 33] says that $S(p) \leq 4$ for every such $p$. For each $d \geq 2$ let

$$K(d) \;:=\; \sup\{S(p) :\ \deg p = d\},$$

the best constant in Smale's theorem for polynomials of degree $d$. Fields Medalist Stephen Smale proved $K(d) \leq 4$ in 1981 in a complexity study of Newton's method and asked whether 4 can be replaced by 1, or even by $(d-1)/d$ [25, Problem 1E, p. 33]. Smale listed the question again, as the first addendum (*Mean Value Problem*), in the second version of *Mathematical problems for the next century* [26, p. 21] (see also [27]); for forty-five years it stood alongside Sendov's conjecture as one of the central open problems in the analytic theory of complex polynomials, featured in the monographs of Rahman and Schmeisser [20, Section 7.2] and Sheil-Small [24, Section 10.4] and in surveys by Schmeisser [22] and Dubinin [11]. For $d \leq 4$ it can: $K(d) = (d-1)/d$ (see the introduction of [13]). The same introduction reports a computer-assisted proof, due to Crane, for $d = 5$, and Sendov and Marinov [23] give numerical evidence that $K(d) = (d-1)/d$ for $d \leq 10$. In general it cannot: the existence of a counterexample, of very large and unspecified degree, is proved in [1], and Adamczewski and Colbrook gave an explicit one, of degree 29,984 [2]. Jatar and Ng proved $\sup_d K(d) > 1.1018$ [18], and the author proved in [15], using a reduction of theirs, that the constant 4 cannot be lowered at all:

$$\lim_{d\to\infty} K(d) \;=\; \sup_{d\geq 2} K(d) \;=\; 4.$$

The limit exists because $K(d)$ does not decrease with $d$ ([6], [14, p. 99]). On the other hand, $K(d) < 4$ for every $d$: Beardon, Minda and Ng proved $K(d) \leq 4^{(d-2)/(d-1)}$ [3]; Conte, Fujikawa and Lakic proved $K(d) \leq 4(d-1)/(d+1)$ [5]; Fujikawa and Sugawa improved this to $K(d) \leq$

[1] Expanded version, with further results [16]: https://doi.org/10.5281/zenodo.24288572

[2] Proved in Lean 4 from its three standard axioms and four assumptions, and checked with `lake comparator`: https://github.com/david44hello/paper_smale-1981-mean-value-rate/tree/v1.00/lean

$4\big(1+(d-2)\,4^{-1/(d-1)}\big)/(d+1)$ [14]; Dubinin proved $K(d) \le \frac{d-1}{d}\,4^{(d-2)/(d-1)}$ [12] (stated without proof in [9, (8)]); and Crane proved $K(d) < 4 - 2.263/\sqrt{d}$ for $d \ge 8$ [7].

So the constants $K(d)$ do not decrease, tend to 4, and never reach it. How fast do they approach 4? The first four of these bounds show only that $4 - K(d)$ is at least of order $1/d$, and Crane's shows that it is at least of order $1/\sqrt{d}$. In this paper we prove that $4 - K(d) > 4/\log d$ for every $d \ge 8$ (Corollary 1.2 and Table 1). This follows from a sharper bound in terms of the Grötzsch ring function (Theorem 1.1). Smale's bound follows from Koebe's one-quarter theorem, applied to the branch $F$ of $p^{-1}$ that fixes 0; we apply instead a theorem of Betsakos [4] (Theorem 2.2) to a conformal map built from $F$, and the degree of $p$ enters through the Bernstein–Walsh inequality [21, Theorem 5.5.7(a)] (see the outline of the proof of Theorem 1.1 below). The case of the Bernstein–Walsh inequality that we use is also a special case of an inequality of Dubinin [10, 11] (Remark 3.1).

The companion paper [16] shows that $1/\log d$ is not the true order of the gap $4 - K(d)$. By [16, Theorems 1.3 and 1.4], $4 - K(d) \ge \min\big(1.1 \cdot 10^{-4},\, 0.056/\sqrt{\log d}\big)$ for every $d \ge 2$, and $4 - K(d) \le 4.9 \cdot 10^4 \sqrt{\log\log\log d/\log d}$ for $d \ge \exp(1.04 \cdot 10^{10})$. This lower bound exceeds $4/\log d$ only when $\log d > 36{,}363$; so for $8 \le d \le e^{36{,}363}$ the bound of Corollary 1.2 is the stronger one. Whether the factor $\sqrt{\log\log\log d}$ can be removed, and how large $4 - K(d)$ is for moderate degrees, say $8 \le d \le 10^6$, are open questions; they are discussed in [16, Section 6]. The proofs in this paper are independent of [16].

TABLE 1. The main results of this paper. Here log is the natural logarithm, and $\mu$ is the Grötzsch ring function, defined in (1) below.

| | Best result before this paper | New result |
|---|---|---|
| Upper bound for $K(d)$, every $d \ge 8$ | The bound of Fujikawa and Sugawa [14]; Crane's bound $K(d) < 4 - 2.263/\sqrt{d}$ [7] and Crane's numerical bounds at nine degrees | $K(d) < 4 - 4/\log d$ (Corollary 1.2), from the sharper bound $K(d) \le (4/y)\,e^{-\mu(y^{1/d})}$, $0 < y < 1$ (Theorem 1.1) |

**The results.** For $0 < r < 1$ let

$$\mathcal{K}(r) \;:=\; \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \qquad \mu(r) \;:=\; \frac{\pi}{2}\,\frac{\mathcal{K}\big(\sqrt{1-r^2}\big)}{\mathcal{K}(r)}. \tag{1}$$

Here $\mathcal{K}$ is the complete elliptic integral of the first kind, and $\mu$ is the *Grötzsch ring function*. The function $\mu$ decreases from $\infty$ to 0 on $(0,1)$. It satisfies $\mu(r) = \log(4/r) + O(r^2)$ as $r \to 0$ and $\mu(r)\,\mu(\sqrt{1-r^2}) = \pi^2/4$ for $0 < r < 1$, so that $\mu(r) \sim \pi^2/\big(2\log\frac{16}{1-r^2}\big)$ as $r \to 1$ (Section 2.2).

**Theorem 1.1.** *For every integer $d \ge 2$ and every $y \in (0,1)$,*

$$K(d) \;\le\; \frac{4}{y}\,e^{-\mu(y^{1/d})}.$$

For fixed $d$ the right-hand side tends to $\infty$ as $y \to 0$ (by Lemma 2.1(c) it exceeds $y^{1/d-1}$), and to 4 from below as $y \to 1$. To see that it is below 4 for $y$ near 1, put $L := \log(1/y)$. For $0 < L < d$, Lemma 2.1 and the inequality $1 - e^{-t} \ge t - t^2/2$ bound the right-hand side by

$$4\exp\Big(L - \frac{\pi^2}{2\log\big(8d/(L(1-L/d))\big)}\Big)$$

(see Steps 1 and 3 of the proof of Corollary 1.2 in Section 3), and this is below 4 when $L$ is small. So Theorem 1.1 recovers Smale's bound $K(d) \le 4$, and it gives $K(d) < 4$. For fixed $y$ the right-hand side tends to $4/y > 4$ as $d \to \infty$, so $y$ must tend to 1 with $d$. When $d$ is large and $L$ is

a small multiple of $1/\log d$, the displayed bound is approximately 4 minus a positive multiple of $1/\log d$, since the logarithm in the display is then approximately $\log d$. This is why the bound in Corollary 1.2 is 4 minus a multiple of $1/\log d$. The choice $y = \exp(-1/(4\log d))$ leads to a bound in closed form.

**Corollary 1.2.** *For every integer $d \geq 8$,*

$$K(d) \;<\; 4 - \frac{4}{\log d}.$$

At every degree $d \geq 8$ this bound is below each of the five bounds quoted above and below the numerical bounds that Crane gives at nine degrees [7] (Remark 3.2); see Table 2 and Figure 1. Theorem 1.1 with the best choice of $y$ gives a smaller bound (Remark 3.3), but not one of better order: if $B(d)$ denotes the infimum over $0 < y < 1$ of its right-hand side, then $(4 - B(d)) \log d \to 2\pi^2$ as $d \to \infty$ (Proposition 3.4). At $d = 7$, Theorem 1.1 does not give the bound of Corollary 1.2: $B(7) > 4 - 4/\log 7$ (Remark 3.5).

Corollary 1.2 shows that $K(d)$ can approach 4 only slowly. For example, it gives $K(d) < 3.9$ for $8 \leq d \leq e^{40}$, and the bound of Beardon, Minda and Ng gives $K(d) \leq 4^{5/6} < 3.9$ for $d \leq 7$; so a polynomial $p$ with $S(p) \geq 3.9$ has degree larger than $2 \cdot 10^{17}$.

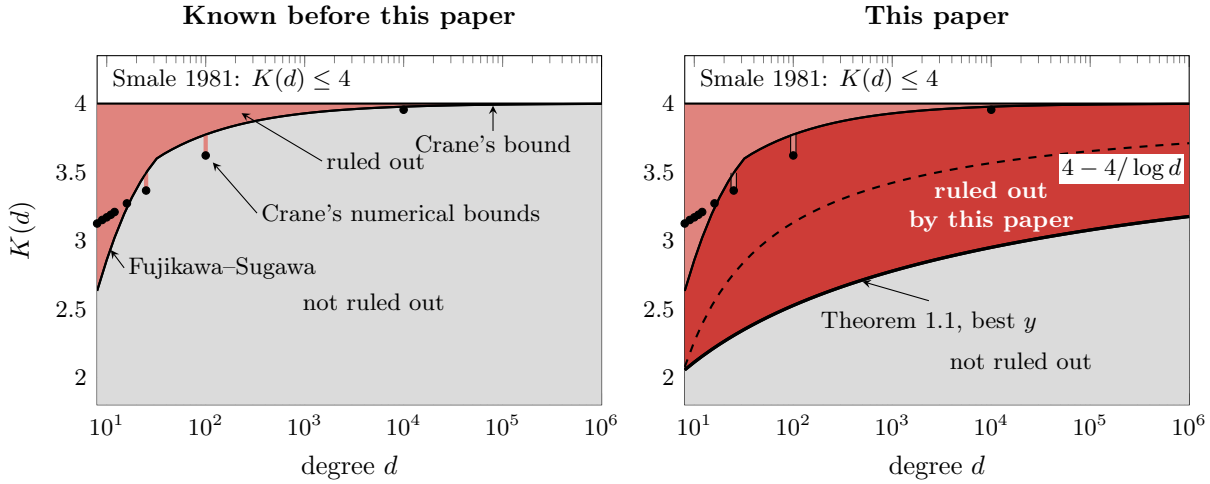


FIGURE 1. Upper bounds for $K(d)$ for $8 \leq d \leq 10^6$. In both panels the red regions are ruled out and the gray region is where $K(d)$ may still lie. Left: before this paper, $K(d)$ had been proved to lie below the smaller of the bounds of Fujikawa and Sugawa [14] and of Crane [7] (the curve; its corner is where the two bounds cross) and, at nine degrees, below Crane's numerical bounds (the dots). The five dots at $d = 8, \ldots, 12$ nearly merge; they and the dot at $d = 16$ lie above the curve, and the three at $d = 25$, 100 and $10^4$ lie below it. The light red region, including the short light red strokes above the dots at $d = 25$ and $d = 100$, had been ruled out. Right: Theorem 1.1, with the best value of $y$ at each degree (the lowest curve, computed in floating point; Table 2 gives bounds computed in interval arithmetic at four degrees), rules out the dark red region between the lowest curve and the earlier bounds; the dots and the light red strokes are as on the left. The dashed curve is the closed form $4 - 4/\log d$ of Corollary 1.2.

**The ideas of the proofs.** We first recall Smale's proof, and then say what we add to it. Dividing $p$ by a constant changes neither its critical points nor $S(p)$; so we may assume that the least

TABLE 2. Upper bounds for $K(d)$, rounded up (Crane's numerical bounds as printed in [7]). The column "earlier" gives the smallest, at that degree, of the earlier bounds of Remark 3.2: Fujikawa and Sugawa [14] at $d = 8$; the table of numerical bounds in [7] at $d = 100$ and $d = 10^4$; Crane's bound $4 - 2.263/\sqrt{d}$ [7] at $d = 10^6$. The column "Theorem 1.1" gives the bound of Theorem 1.1, computed in interval arithmetic with upper and lower bounds for $\mu$ as in Remark 3.5, at $y = e^{-L}$, with $L$ from the last column (close to the best $L$; the proof of Corollary 1.2 uses $L = 1/(4\log d)$).

| $d$ | earlier | $4 - 4/\log d$ | Theorem 1.1 | $L$ |
|---|---|---|---|---|
| 8 | 2.6321 | 2.0765 | 2.0550 | 0.12766 |
| 100 | 3.621 | 3.1315 | 2.5255 | 0.05337 |
| $10^4$ | 3.955 | 3.5658 | 2.9495 | 0.02156 |
| $10^6$ | 3.9978 | 3.7105 | 3.1754 | 0.01194 |

modulus of a critical value of $p$ is 1 (if it is 0, then $S(p) = 0$). Let $\zeta_m$ be a critical point with $|p(\zeta_m)| = 1$. Since the unit disc $\mathbb{D}$ contains no critical value of $p$, the branch $F$ of $p^{-1}$ with $F(0) = 0$ is defined and univalent on $\mathbb{D}$, and it omits $\zeta_m$. Since $F'(0) = 1/p'(0)$, the definition of $S(p)$ gives $S(p) \le |p(\zeta_m)|/|\zeta_m\, p'(0)| = |F'(0)|/|\zeta_m|$. Koebe's one-quarter theorem gives $|\zeta_m| \ge |F'(0)|/4$, so $S(p) \le 4$: this is Smale's theorem. This argument uses only that $F$ is univalent and omits $\zeta_m$. Our upper bound for $K(d)$ comes from using, in addition, that $p$ is a polynomial of degree $d$.

*Outline of the proof of Theorem 1.1* (see Figure 2). For $0 < y < 1$ let $V := F(\mathbb{D}_y)$, where $\mathbb{D}_y$ is the open disc of radius $y$ about 0. In the coordinate $x = -\zeta_m/z$, where $z$ is the variable of $p$, the complement of $\overline{V}$ becomes a bounded Jordan domain $E_y$ that contains the points 0 and $-1$, the images of $\infty$ and of $\zeta_m$; the logarithmic capacity $\operatorname{cap}(\overline{E_y})$ of its closure (Section 2.3) is $|\zeta_m|/(y\,|F'(0)|)$, which is at most $1/(y\,S(p))$. Let $f$ map $\mathbb{D}$ conformally onto $E_y$ with $f(0) = 0$, and let $\xi := f^{-1}(-1)$. On $\partial V$ we have $|p| = y$, at $\zeta_m$ we have $|p| = 1$, and $p$ has a pole of order $d$ at $\infty$. So the function $P(w) := w^d\, p(-\zeta_m/f(w))$ is holomorphic in $\mathbb{D}$ (the factor $w^d$ cancels the pole of order $d$ of $p(-\zeta_m/f(w))$ at $w = 0$), and $\limsup_{|w|\to 1} |P(w)| \le y$; by the maximum principle, $|\xi|^d = |P(\xi)| \le y$ (a case of the Bernstein–Walsh inequality, Remark 3.1). By a theorem of Betsakos [4], $1 = |f(\xi)| \le 4\operatorname{cap}(\overline{E_y})\, e^{-\mu(|\xi|)}$; since $\mu$ decreases and $|\xi| \le y^{1/d}$, this and the capacity bound above give $e^{\mu(y^{1/d})} \le e^{\mu(|\xi|)} \le 4/(y\,S(p))$, which is the theorem.

*Outline of the proof of Corollary 1.2.* With $y = \exp(-1/(4\log d))$, the identity $\mu(r)\,\mu(\sqrt{1-r^2}) = \pi^2/4$ and the inequality $\mu(s) < \log(4/s)$ bound the right-hand side of Theorem 1.1 by an expression in elementary functions. For $d \ge 21$ this expression is below $4 - 4/\log d$ by elementary inequalities; for the thirteen degrees $8 \le d \le 20$ the comparison is a finite computation in interval arithmetic.

**Organization.** Section 2 collects the notation and the tools. Section 3 proves Theorem 1.1, Corollary 1.2 and Proposition 3.4. Section 4 closes with a brief discussion of open questions.

## 2. Notation and tools

2.1. **Notation.** Throughout, log is the natural logarithm, $\mathbb{D}$ is the open unit disc, $\mathbb{D}_y$ is the open disc of radius $y > 0$ about 0, and $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. A *Jordan domain* is a domain in $\widehat{\mathbb{C}}$ whose boundary is a Jordan curve. A function is *univalent* if it is meromorphic and injective, and a *conformal map* of a domain $D$ onto a domain $D'$ is a univalent function on $D$ with image $D'$.

2.2. **The Grötzsch ring function.** Recall the complete elliptic integral $\mathcal{K}$ and the function $\mu$ of (1). As usual, $r' := \sqrt{1-r^2}$ is the *complementary modulus* of $r$, and $\mathcal{K}'(r) := \mathcal{K}(r')$ for $0 < r < 1$, so that $\mu(r) = \frac{\pi}{2}\,\mathcal{K}'(r)/\mathcal{K}(r)$.

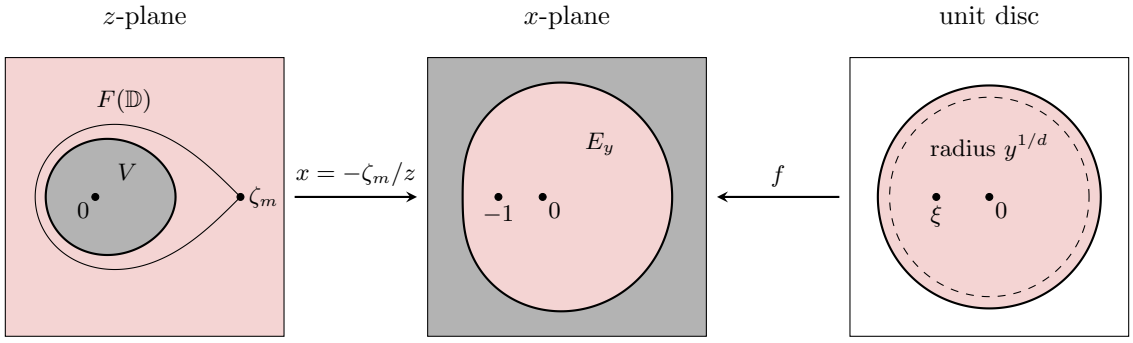


FIGURE 2. The construction in the proof of Theorem 1.1, drawn to scale for $p(z) = 2z - z^2$ (so $d = 2$ and $\zeta_m = 1$) and $y = 0.8$. Left: the domain $V = F(\mathbb{D}_y)$ (dark gray) lies in $F(\mathbb{D})$, the domain inside the thin curve; the critical point $\zeta_m$ is not in $F(\mathbb{D})$ (here it lies on its boundary). Middle: in the coordinate $x = -\zeta_m/z$, the complement of $\overline{V}$ becomes the bounded Jordan domain $E_y$, which contains 0 and $-1$, the images of $\infty$ and of $\zeta_m$. Right: a conformal map $f$ of the unit disc onto $E_y$ with $f(0) = 0$ takes a point $\xi$ to $-1$. The point $\xi$ lies in the closed disc of radius $y^{1/d}$ about 0 (dashed circle). Dark gray marks $V$ and its image $x(V)$; pale shading marks the complement of $\overline{V}$, its image $E_y$, and the unit disc.

**Lemma 2.1.** (a) *The function $\mu$ is strictly decreasing and maps $(0,1)$ onto $(0,\infty)$.* (b) *For $0 < r < 1$ we have $\mu(r)\,\mu(r') = \pi^2/4$.* (c) *For $0 < s < 1$ we have $\mu(s) < \log(4/s)$.*

*Proof.* (a) For fixed $t \in (0,1)$ the integrand in (1) is continuous and strictly increasing in $r$. Hence $\mathcal{K}$ is strictly increasing on $(0,1)$. It is continuous, and $\mathcal{K}(r) \to \pi/2$ as $r \to 0$, by dominated convergence (for any $r_1 \in (0,1)$, the integrand at $r_1$ is an integrable majorant on $(0, r_1]$); and $\mathcal{K}(r) \to \int_0^1 dt/(1-t^2) = \infty$ as $r \to 1$, by monotone convergence. So $\mathcal{K}'$ is continuous and strictly decreasing, from $\infty$ to $\pi/2$, and the quotient $\mu = \frac{\pi}{2}\,\mathcal{K}'/\mathcal{K}$ is continuous and strictly decreasing, from $\infty$ to 0. (b) The complementary modulus of $r'$ is $r$, so

$$\mu(r') \;=\; \frac{\pi}{2}\,\frac{\mathcal{K}(r)}{\mathcal{K}(r')} \;=\; \frac{\pi}{2}\,\frac{\mathcal{K}(r)}{\mathcal{K}'(r)};$$

multiplying by $\mu(r)$ gives (b). (c) By [8, (19.9.5)], for $0 < k < 1$ and $k' = \sqrt{1-k^2}$,

$$\log\frac{(1+\sqrt{k'})^2}{k} \;<\; \frac{\pi\,\mathcal{K}'(k)}{2\,\mathcal{K}(k)} \;<\; \log\frac{2(1+k')}{k}.$$

By [8, (19.2.4), (19.2.8), (19.2.9)], the functions written $K$ and $K'$ there are our $\mathcal{K}$ and $\mathcal{K}'$; so the middle term is $\mu(k)$. Since $k' < 1$, we have $2(1+k') < 4$, and (c) follows. □

Both bounds in the proof of (c) are $\log(4/k) + O(k^2)$ as $k \to 0$, so $\mu(r) = \log(4/r) + O(r^2)$ as $r \to 0$; with (b), $\mu(r) = \pi^2/(4\mu(r')) \sim \pi^2/\bigl(2\log\frac{16}{1-r^2}\bigr)$ as $r \to 1$.

2.3. **Transfinite diameter and capacity.** For a bounded infinite set $A \subset \mathbb{C}$ and $n \geq 2$ let

$$d_n(A) \;:=\; \sup\Bigl\{\prod_{1\leq j<k\leq n} |z_j - z_k|^{2/(n(n-1))} :\; z_1,\ldots,z_n \in A\Bigr\}. \tag{2}$$

Since the product is continuous in $z_1,\ldots,z_n$, we have $d_n(A) = d_n(\overline{A})$. The sequence $d_n(\overline{A})$ decreases [21, Theorem 5.5.2], and its limit $d_\infty(A) = d_\infty(\overline{A})$ is the *transfinite diameter* of $A$. By the Fekete–Szegő theorem, the transfinite diameter of an infinite compact set $Q$ equals its logarithmic capacity $\mathrm{cap}(Q)$ ([4, p. 146], [21, Theorem 5.5.2]). If $Q \subset \mathbb{C}$ is compact and connected, has more than one

point and has connected complement, and $\Phi$ is a conformal map of $\{|u|>1\}\cup\{\infty\}$ onto $\widehat{\mathbb{C}}\setminus Q$ with $\Phi(\infty)=\infty$, then $\Phi(u)=a\,u+O(1)$ as $u\to\infty$ and $\operatorname{cap}(Q)=|a|$ [21, Theorem 5.2.3 and Corollary 5.2.2].

2.4. **A theorem of Betsakos.** The proof of Theorem 1.1 rests on the following theorem.

**Theorem 2.2** ([4, Theorem 1.1])**.** *Let $f$ be a nonconstant bounded holomorphic function in $\mathbb{D}$ with $f(0)=0$. Then*

$$|f(w)| \;\le\; 4\,d_\infty(f(\mathbb{D}))\,e^{-\mu(|w|)}, \qquad w\in\mathbb{D}.$$

*Equality holds for some $w\in\mathbb{D}\setminus\{0\}$ if and only if $f$ maps $\mathbb{D}$ conformally onto a domain bounded by an ellipse so that $0$ and $w$ are mapped onto the foci of the ellipse.*

In [4] the transfinite diameter is written $d$. At $w=0$ the right-hand side is read as 0. As $|w|\to 0$ the right-hand side is $d_\infty(f(\mathbb{D}))\,|w|\,(1+O(|w|^2))$, and as $|w|\to 1$ it tends to $4\,d_\infty(f(\mathbb{D}))$. Letting $w\to 0$ gives $|f'(0)|\le d_\infty(f(\mathbb{D}))$; for a conformal map $f$ this is the classical inequality between the conformal radius $|f'(0)|$ and the capacity of $\overline{f(\mathbb{D})}$.

## 3. The upper bounds

*Proof of Theorem 1.1.* Fix $d\ge 2$ and $0<y<1$, and let $p$ be a polynomial of degree $d$ with $p(0)=0$ and $p'(0)\neq 0$. We show that $S(p)\le (4/y)\,e^{-\mu(y^{1/d})}$; taking the supremum over $p$ gives the theorem.

*Normalization.* If $p(c)=0$ for some critical point $c$ of $p$, then $S(p)=0$ and there is nothing to prove. Otherwise the least modulus of a critical value of $p$ is positive, and dividing $p$ by it changes neither the critical points nor $S(p)$. So we may assume that this least modulus is 1, and we fix a critical point $\zeta_m$ with $|p(\zeta_m)|=1$. Then $S(p)>0$, and $\zeta_m\neq 0$ because $p'(0)\neq 0$.

*The inverse branch.* The disc $\mathbb{D}$ contains no critical value of $p$, and $p$ is a covering map over the complement of its critical values. Since $\mathbb{D}$ is simply connected and $p(0)=0$, there is a branch $F$ of $p^{-1}$ on $\mathbb{D}$ with $F(0)=0$. Since $p(F(w))=w$, the branch $F$ is univalent and $F'(0)=1/p'(0)$. By the definition of $S(p)$, at the critical point $\zeta_m$,

$$S(p) \;\le\; \frac{|p(\zeta_m)|}{|\zeta_m\,p'(0)|} \;=\; \frac{|F'(0)|}{|\zeta_m|}. \tag{3}$$

*The domain $E_y$.* Let $V:=F(\mathbb{D}_y)$. It is a Jordan domain bounded by the analytic curve $F(\partial\mathbb{D}_y)$, it contains $F(0)=0$, and its closure $F(\overline{\mathbb{D}_y})$ lies in $F(\mathbb{D})$. The point $\zeta_m$ is not in $F(\mathbb{D})$, since $|p(\zeta_m)|=1$ while $p(F(\mathbb{D}))=\mathbb{D}$; so $\zeta_m\notin\overline{V}$. Let $x(z):=-\zeta_m/z$, a Möbius involution of $\widehat{\mathbb{C}}$, and let

$$E_y \;:=\; x\big(\widehat{\mathbb{C}}\setminus\overline{V}\big).$$

Then $E_y$ is a Jordan domain bounded by $x(\partial V)$. It is bounded, since $x^{-1}(\infty)=0\in V$, and it contains $x(\infty)=0$ and $x(\zeta_m)=-1$.

*The capacity of $\overline{E_y}$.* The map $\Phi(u):=x(F(y/u))$ is a conformal map of $\{|u|>1\}\cup\{\infty\}$ onto $\widehat{\mathbb{C}}\setminus\overline{E_y}=x(V)$ with $\Phi(\infty)=\infty$. Since $F(w)=F'(0)\,w+O(w^2)$ as $w\to 0$,

$$\Phi(u) \;=\; -\frac{\zeta_m}{y\,F'(0)}\,u+O(1), \qquad u\to\infty.$$

So the logarithmic capacity of $\overline{E_y}$ is $\operatorname{cap}(\overline{E_y})=|\zeta_m|/(y\,|F'(0)|)$ (Section 2.3), and by (3)

$$\operatorname{cap}(\overline{E_y}) \;\le\; \frac{1}{y\,S(p)}. \tag{4}$$

*A bound for $|\xi|$.* Let $f$ be a conformal map of $\mathbb{D}$ onto $E_y$ with $f(0)=0$, and let $\xi:=f^{-1}(-1)$; then $\xi\neq 0$. Put

$$P(w) \;:=\; w^d\,p\big(x(f(w))\big), \qquad w\in\mathbb{D}.$$

The map $x \circ f$ sends $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus \overline{V}$; its only pole is at $w = 0$, and that pole is simple. Since $p$ has degree $d$, the singularity of $P$ at 0 is removable, so $P$ is holomorphic in $\mathbb{D}$. As $|w| \to 1$, the distance from $f(w)$ to $\partial E_y$ tends to 0. Since the compact set $\partial E_y$ does not contain 0, the map $x$ is uniformly continuous on a neighborhood of $\partial E_y$; hence the distance from $x(f(w))$ to $x(\partial E_y) = \partial V$ tends to 0. On $\partial V = F(\partial \mathbb{D}_y)$ we have $|p| = y$; as $p$ is uniformly continuous on a compact neighborhood of $\partial V$ and $|w|^d < 1$, we get $\limsup_{|w|\to 1} |P(w)| \le y$. By the maximum principle, $|P| \le y$ in $\mathbb{D}$. At $w = \xi$ we have $x(f(\xi)) = x(-1) = \zeta_m$, so $|\xi|^d = |\xi|^d\,|p(\zeta_m)| = |P(\xi)| \le y$, that is,

$$|\xi| \;\le\; y^{1/d}. \tag{5}$$

*Betsakos's theorem.* The function $f$ is nonconstant, bounded and holomorphic in $\mathbb{D}$, with $f(0) = 0$. Theorem 2.2 at $w = \xi$ gives

$$1 \;=\; |f(\xi)| \;\le\; 4\, d_\infty(E_y)\, e^{-\mu(|\xi|)}.$$

By Section 2.3, $d_\infty(E_y) = d_\infty(\overline{E_y}) = \operatorname{cap}(\overline{E_y})$, the second equality by the Fekete–Szegő theorem. With (4),

$$e^{\mu(|\xi|)} \;\le\; 4\operatorname{cap}(\overline{E_y}) \;\le\; \frac{4}{y\,S(p)}. \tag{6}$$

*Conclusion.* The function $\mu$ is strictly decreasing (Lemma 2.1(a)), so (5) gives $\mu(y^{1/d}) \le \mu(|\xi|)$. With (6), $\mu(y^{1/d}) \le \log\bigl(4/(y\,S(p))\bigr)$, that is, $S(p) \le (4/y)\, e^{-\mu(y^{1/d})}$. □

*Remark* 3.1 (Related results). Inequality (5) is a case of the Bernstein–Walsh inequality, which is called Bernstein's lemma in [21, Theorem 5.5.7(a)]. For the compact set $\overline{V}$, that inequality reads $|p(\zeta_m)| \le \max_{\overline{V}} |p| \cdot e^{d\, g(\zeta_m)}$, where $g$ is the Green function of $\widehat{\mathbb{C}} \setminus \overline{V}$ with pole at $\infty$; here $\max_{\overline{V}} |p| = y$ and $|p(\zeta_m)| = 1$. Write $g_\Omega(\cdot, a)$ for the Green function of a domain $\Omega$ with pole at $a$ [21, Definition 4.4.1]; conformal maps preserve Green functions [21, Theorem 4.4.4], and $g_{\mathbb{D}}(w, 0) = \log(1/|w|)$. By conformal invariance under $x$ and then under $f$, we have $g(\zeta_m) = g_{E_y}(-1, 0) = g_{\mathbb{D}}(\xi, 0) = \log(1/|\xi|)$, so the inequality is (5).

Inequality (5) is also a special case of an inequality of Dubinin for meromorphic functions and Green functions [10, Theorem 1], [11, (1.27), pp. 618–619]; the reduction is written out in [16, Remark 3.1]. According to [11, p. 619], Dubinin's inequality was first obtained by Mityuk [19], under slightly different conditions on the function.

Capacity has entered this problem before. The proof of Dubinin's upper bound for $K(d)$ uses condenser capacities [12]; Crane's bound comes from estimates for the logarithmic capacity of a connected set containing three given points [7]; and Jatar and Ng [17] use a Koebe-type theorem of Cunningham (stated in [17, Theorem 8]) on univalent functions whose image has bounded capacity. What is new in Theorem 1.1 is the combination of the Bernstein–Walsh inequality with the theorem of Betsakos (Theorem 2.2), applied to the domain $E_y$.

*Proof of Corollary 1.2.* Let $d \ge 8$ and put

$$\ell := \log d, \quad L := \frac{1}{4\ell}, \quad y := e^{-L}, \quad r := y^{1/d} = e^{-L/d}, \quad r' := \sqrt{1 - r^2} = \sqrt{1 - e^{-2L/d}}.$$

*Step 1: a bound without elliptic integrals.* Lemma 2.1(a) and (c) at $s = r'$ give $0 < \mu(r') < \log(4/r')$. With Lemma 2.1(b),

$$\mu(r) \;=\; \frac{\pi^2}{4\,\mu(r')} \;>\; \frac{\pi^2}{4\log(4/r')},$$

and Theorem 1.1 with this $y$ gives

$$K(d) \;\le\; 4e^{L} e^{-\mu(r)} \;<\; M(d) \;:=\; 4\exp\Bigl(L - \frac{\pi^2}{4\log(4/r')}\Bigr). \tag{7}$$

It remains to show $M(d) \le 4 - 4/\ell$.

*Step 2: the degrees* $8 \le d \le 20$. For each of these thirteen degrees, interval arithmetic gives $M(d) < 4 - 4/\log d$ (Table 3); the smallest margin is 0.0173, at $d = 8$.

*Step 3: the degrees* $d \ge 21$. Since $1 - e^{-t} \ge t - t^2/2$ for $t \ge 0$, the choice $t = 2L/d$ gives $r'^2 \ge (2L/d)(1 - L/d)$. Hence $\log(4/r') = \frac{1}{2}\log(16/r'^2) \le \Lambda/2$, where

$$\Lambda \;:=\; \log \frac{8d}{L\,(1 - L/d)} \;=\; \ell + \log(32\ell) - \log(1 - L/d),$$

and (7) gives $M(d) \le 4\exp\bigl(L - \pi^2/(2\Lambda)\bigr)$. So $M(d) \le 4(1 - 1/\ell)$ follows from

$$\frac{\pi^2}{2\Lambda} \;\ge\; \frac{1}{4\ell} - \log\Bigl(1 - \frac{1}{\ell}\Bigr). \tag{8}$$

We bound both sides, using $\ell \ge \log 21 > 3.04$. Since $-\log(1 - s) \le s/(1 - s)$ for $0 < s < 1$, and $\ell \ge 3$, we have $-\log(1 - 1/\ell) \le 1/(\ell - 1) \le 1.5/\ell$; hence the right-hand side of (8) is at most $1.75/\ell$. Next, the function $\ell \mapsto \log(32\ell)/\ell$ has derivative $(1 - \log(32\ell))/\ell^2 < 0$ for $\ell \ge 1$, so it decreases there, and its value at $\ell = \log 21$ is $1.5040\ldots < 1.52$; hence $\log(32\ell) \le 1.52\,\ell$. Moreover, $L/d = 1/(4\ell d) \le 1/(4 \cdot 3.04 \cdot 21) < 0.0042$, so that $-\log(1 - L/d) \le -\log(1 - 0.0042) < 0.005 < 0.002\,\ell$. Therefore $\Lambda \le (1 + 1.52 + 0.002)\,\ell < 2.53\,\ell$, and the left-hand side of (8) is at least $\pi^2/(5.06\,\ell) > 1.95/\ell$. Since $1.95/\ell > 1.75/\ell$, inequality (8) holds. With (7), $K(d) < M(d) \le 4 - 4/\ell$. □

TABLE 3. Step 2 of the proof of Corollary 1.2: $M(d)$ of (7) against $4 - 4/\log d$, in interval arithmetic. Upper bounds for $M(d)$ are rounded up and lower bounds for $4 - 4/\log d$ are rounded down, to five decimal places; the margin is the difference of these two columns.

| $d$ | $M(d) \le$ | $4 - 4/\log d \;\ge$ | margin $\ge$ |
|---|---|---|---|
| 8 | 2.05907 | 2.07640 | 0.01733 |
| 9 | 2.08862 | 2.17952 | 0.09090 |
| 10 | 2.11417 | 2.26282 | 0.14865 |
| 11 | 2.13661 | 2.33187 | 0.19526 |
| 12 | 2.15656 | 2.39028 | 0.23372 |
| 13 | 2.17447 | 2.44051 | 0.26604 |
| 14 | 2.19069 | 2.48430 | 0.29361 |
| 15 | 2.20549 | 2.52292 | 0.31743 |
| 16 | 2.21907 | 2.55730 | 0.33823 |
| 17 | 2.23162 | 2.58817 | 0.35655 |
| 18 | 2.24326 | 2.61609 | 0.37283 |
| 19 | 2.25410 | 2.64150 | 0.38740 |
| 20 | 2.26424 | 2.66476 | 0.40052 |

*Remark* 3.2 (The earlier bounds). At every degree $d \ge 8$, the bound of Corollary 1.2 is below the bounds of Crane, of Fujikawa and Sugawa, of Conte, Fujikawa and Lakic, of Beardon, Minda and Ng, and of Dubinin, and below Crane's numerical bounds at nine degrees.

(a) Crane proved $K(d) < 4 - 2.263/\sqrt{d}$ for $d \ge 8$ [7]. The function $\sqrt{d}/\log d$ attains its minimum over $d > 1$ at $d = e^2$, where it equals $e/2$. Hence $4/\log d \ge 2e/\sqrt{d} > 2.263/\sqrt{d}$.

(b) The bound of Fujikawa and Sugawa [14, Theorem 1, p. 97] is

$$K(d) \;\le\; \frac{4\bigl(1 + (d-2)\,q\bigr)}{d+1} \;=\; 4 - \frac{4\bigl((d-2)(1-q) + 2\bigr)}{d+1}, \qquad q := 4^{-1/(d-1)}.$$

Since $1-q \le (\log 4)/(d-1)$, the bracket in the numerator satisfies $(d-2)(1-q)+2 < \log 4 + 2 < 3.39$. The function $d \mapsto (d+1)/\log d$ increases for $d \ge 4$, and at $d=8$ it equals $9/\log 8 > 4.32 > 3.39$. So the subtracted fraction is less than $4 \cdot 3.39/(d+1) < 4/\log d$ for $d \ge 8$. Hence $4 - 4/\log d$ is smaller than the bound of Fujikawa and Sugawa.

(c) The bounds $4(d-1)/(d+1)$ of Conte, Fujikawa and Lakic [5], $4^{(d-2)/(d-1)} = 4q$ of Beardon, Minda and Ng [3], and $\frac{d-1}{d}\,4q$ of Dubinin [9, 12] are all at least the bound of Fujikawa and Sugawa, since the three differences are $4(d-2)(1-q)/(d+1) \ge 0$ (as $q \le 1$), $4(3q-1)/(d+1) > 0$ (as $q \ge 1/2$ for $d \ge 3$) and $4\big((2d-1)q-d\big)/\big(d(d+1)\big) > 0$ (as $q \ge 4^{-1/7} > 0.8 > d/(2d-1)$ for $d \ge 8$). (Fujikawa and Sugawa compare the first two in [14, Remark (i), p. 97].) Hence, by (b), $4 - 4/\log d$ is also smaller than these three bounds.

(d) The numerical bounds of Crane at the nine degrees $d = 8, 9, 10, 11, 12, 16, 25, 100, 10^4$ [7] are 3.124, 3.150, 3.169, 3.188, 3.208, 3.271, 3.365, 3.621, 3.955. Each exceeds $4 - 4/\log d$ at the same degree by more than 0.389; the smallest difference is at $d = 10^4$.

*Remark* 3.3 (The best choice of $y$). For every $d \ge 8$ the proof of Corollary 1.2 gives

$$\inf_{0<y<1} (4/y)\, e^{-\mu(y^{1/d})} < M(d) \le 4 - 4/\log d.$$

Table 2 shows at four degrees how far the bound of Theorem 1.1, at a value of $y$ close to the best one, lies below $4 - 4/\log d$, and Proposition 3.4 below shows that for the best $y$ this difference is asymptotic to $(2\pi^2 - 4)/\log d$ (Figure 3). By that proposition, Theorem 1.1 gives $\liminf_{d\to\infty} (4 - K(d)) \log d \ge 2\pi^2$, while Corollary 1.2 gives $(4 - K(d)) \log d > 4$.

**Proposition 3.4.** *Let $B(d) := \inf_{0<y<1} (4/y)\, e^{-\mu(y^{1/d})}$ be the best bound that Theorem 1.1 gives at degree $d$. Then $(4 - B(d)) \log d \to 2\pi^2$ as $d \to \infty$. More precisely,*

$$(4 - B(d)) \log d \;\ge\; 2\pi^2 - O\Big(\frac{\log\log d}{\log d}\Big) \quad (d \to \infty),$$

$$(4 - B(d)) \log d \;\le\; \frac{2\pi^2}{1 - \pi^2/(d \log^2 d)} \quad (d \ge 8).$$

*Proof.* Let $d \ge 8$. Write $\ell := \log d$ and $y = e^{-L}$ with $L > 0$, and put $r := e^{-L/d}$, $r' := \sqrt{1-r^2}$. Write $\beta_d(L) := 4\exp(L - \mu(r))$ for the bound of Theorem 1.1, so that $B(d) = \inf_{L>0} \beta_d(L)$. By Lemma 2.1(b), $\mu(r) = \pi^2/(4\mu(r'))$.

*The lower bound for $(4 - B(d)) \log d$.* Take $L := \pi^2/(2\ell^2)$. As in Step 3 of the proof of Corollary 1.2, $\mu(r') < \log(4/r') \le \Lambda/2$ with $\Lambda := \log\big(8d/(L(1 - L/d))\big) = \ell + 2\log\ell + O(1)$. Hence $\log(B(d)/4) \le L - \pi^2/(2\Lambda) = -\pi^2/(2\ell) + O(\log\ell/\ell^2)$, and $(4 - B(d))\,\ell \ge 2\pi^2 - O(\log\ell/\ell)$.

*The upper bound for $(4 - B(d)) \log d$.* We show that for every $L > 0$, $\beta_d(L)$ is at least $4\exp\big(-\pi^2/(2\ell - 2\pi^2/(\ell d))\big)$; the denominator is positive, since $d \log^2 d \ge 8 \log^2 8 > \pi^2$.

For $0 < L < 2$ we first bound $\mu(r')$ from below, and so $\mu(r)$ from above. The left inequality of [8, (19.9.5)] (displayed in the proof of Lemma 2.1(c)) at $k = r'$, whose complementary modulus is $r$, gives $\mu(r') > \log(4/r') - 2\log\big(2/(1+\sqrt{r})\big) \ge \log(4/r') - L/d$: indeed $\sqrt{r} = e^{-L/(2d)} \ge 1 - L/(2d)$, so $2/(1+\sqrt{r}) \le 1/(1-\sigma)$ with $\sigma := L/(4d) \le \frac12$, and $-\log(1-\sigma) \le 2\sigma$ for $0 \le \sigma \le \frac12$. Also $r'^2 = 1 - e^{-2L/d} \le 2L/d$, so $\log(4/r') \ge \frac12 \log(8d/L)$. Hence

$$4\mu(r') \;>\; \Delta_L \;:=\; 2\log\frac{8d}{L} - \frac{4L}{d}.$$

As $0 < L < 2$, we have $\Delta_L > 2\log(4d) - 8/d > 0$. Since $4\mu(r') > \Delta_L > 0$, we get $\mu(r) = \pi^2/(4\mu(r')) < \pi^2/\Delta_L$, and $\beta_d(L)$ exceeds $4\exp(L - \pi^2/\Delta_L)$.

There are three cases.

(a) $L \ge 2$. Lemma 2.1(c) at $s = r$ gives $e^{-\mu(r)} > r/4$, so $\beta_d(L)$ exceeds $e^{(1-1/d)L} \ge e^{7/4} > 4$.

(b) $\pi^2/(2\ell) \le L < 2$. Then $\Delta_L > 2\log(4d) - 8/d > 2\ell$, so $L - \pi^2/\Delta_L > 0$ and $\beta_d(L)$ exceeds 4.

(c) $0 < L < \min(2, \pi^2/(2\ell))$. Then $\log(8d/L) > \ell + \log(16\ell/\pi^2) \geq \ell$ and $4L/d < 2\pi^2/(\ell d)$, so $\Delta_L > 2\ell - 2\pi^2/(\ell d) > 0$, and $\beta_d(L)$ exceeds $4\exp\bigl(-\pi^2/(2\ell - 2\pi^2/(\ell d))\bigr)$.

In cases (a) and (b), $\beta_d(L)$ exceeds 4, so in all three cases $\beta_d(L)$ is at least the last bound of case (c), and so is $B(d)$:

$$B(d) \;\geq\; 4\exp\Bigl(-\frac{\pi^2}{2\ell - 2\pi^2/(\ell d)}\Bigr).$$

With $1 - e^{-t} \leq t$ for $t \geq 0$ this gives the second inequality of the proposition, whose right-hand side tends to $2\pi^2$. With the lower bound, $(4 - B(d))\log d \to 2\pi^2$. □

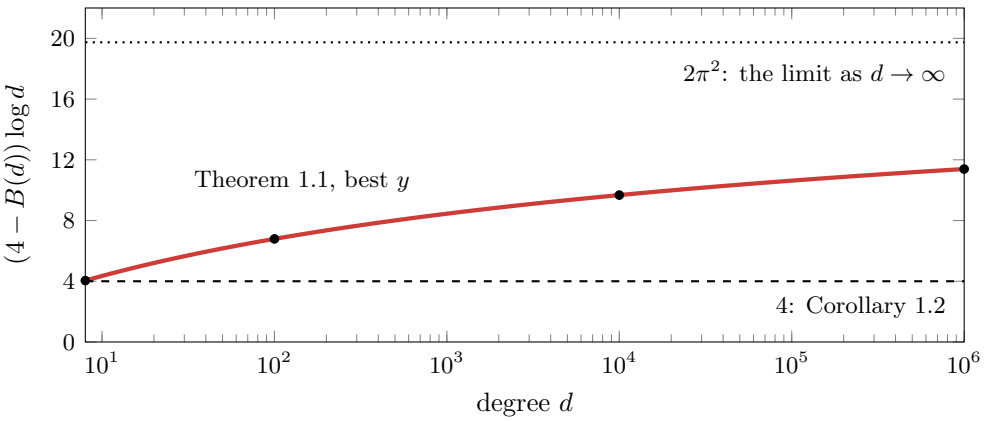


FIGURE 3. The number $(4 - B(d))\log d$ for $8 \leq d \leq 10^6$, where $B(d)$ is the infimum over $0 < y < 1$ of the bound of Theorem 1.1 (Proposition 3.4). Since $K(d) \leq B(d)$, Theorem 1.1 gives $K(d) \leq 4 - h/\log d$, where $h$ is the height of the curve at $d$. The curve is computed in floating point. The dots are $(4 - T)\log d$, where $T$ is the entry in the column "Theorem 1.1" of Table 2 at $d = 8$, 100, $10^4$, $10^6$; since $B(d) \leq T$, each dot lies on or below the curve. The dashed line is the constant 4 of Corollary 1.2; the curve lies above it at every $d \geq 8$ (Remark 3.3), by less than 0.05 at $d = 8$ (Remark 3.5), and tends to $2\pi^2$ (dotted line) as $d \to \infty$ (Proposition 3.4).

*Remark* 3.5 (The range $d \geq 8$, and bounds for $\mu$). Theorem 1.1 does not give Corollary 1.2 at $d = 7$: the number $B(7)$ of Proposition 3.4 lies between 2.0190 and 2.0191, while $4 - 4/\log 7 < 1.9445$. It does give $K(7) \leq B(7) < 2.0191$; at $d = 7$ the smallest of the bounds quoted in the Introduction is 2.4842…, that of Fujikawa and Sugawa. At $d = 8$, the number $B(8)$ lies between 2.0549 and 2.0550; so $(4 - B(8))\log 8$, the left end of the curve in Figure 3, is less than 4.045.

The bounds 2.0190 and 2.0191 for $B(7)$, and 2.0549 and 2.0550 for $B(8)$, are computed in interval arithmetic, with upper and lower bounds for $\mu$ from the arithmetic-geometric mean. For $0 < k < 1$ let $a_0 = 1$, $b_0 = k$, $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = \sqrt{a_n b_n}$ [8, (19.8.1)]. By the inequality between the arithmetic and the geometric mean, $b_n \leq b_{n+1} \leq a_{n+1} \leq a_n$ and $a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = (a_n - b_n)/2$; so the two sequences have a common limit $\operatorname{agm}(1, k)$, and $b_n \leq \operatorname{agm}(1, k) \leq a_n$ for every $n$. By [8, (19.8.5)], $\mathcal{K}(k) = \pi/(2\operatorname{agm}(1, k'))$, so $\mu(r) = \pi\operatorname{agm}(1, r')/(2\operatorname{agm}(1, r))$. With $\beta_d(L)$ as in the proof of Proposition 3.4, defined by the same formula for $d = 7$, the upper bounds hold because $\beta_8(0.12766) < 2.0550$ and $\beta_7(0.13621) < 2.0191$. For the lower bounds, first let $L \geq 1$: Lemma 2.1(c) gives $e^{-\mu(e^{-L/d})} > e^{-L/d}/4$, so $\beta_d(L) > e^{(d-1)L/d} \geq e^{6/7} > 2.35$. On an interval $L_1 < L \leq L_2$ with $0 \leq L_1 < L_2 \leq 1$ we have $\beta_d(L) \geq 4e^{L_1}e^{-\mu(e^{-L_2/d})}$, as $\mu$ is decreasing. Starting from $(0, 1]$, we halve an interval as long as this lower bound is at most 2.0549 for $d = 8$, or at most 2.0190 for $d = 7$.

The halving stops with 1735 intervals for $d = 8$ and 644 for $d = 7$; on every one of them this lower bound exceeds 2.0549 (for $d = 8$) or 2.0190 (for $d = 7$). This proves the two lower bounds.

## 4. Discussion

Before this paper the best upper bound for $K(d)$ at large degree was Crane's $K(d) < 4-2.263/\sqrt{d}$ [7]. Theorem 1.1 and Corollary 1.2 replace that $1/\sqrt{d}$ gap below 4 by $4/\log d$ for every $d \geq 8$, improving on each of the five earlier upper bounds cited in Section 1 and on Crane's numerical bounds at nine degrees at every degree $d \geq 8$ (Remark 3.2). We end with three open questions.

*The exact order of* $4 - K(d)$. The companion paper [16, Theorems 1.3 and 1.4] shows that $4 - K(d)$ is of order $1/\sqrt{\log d}$ up to constant factors and a factor $\sqrt{\log\log\log d}$ for very large $d$. Can the factor $\sqrt{\log\log\log d}$ be removed, and does $(4 - K(d))\sqrt{\log d}$ have a limit as $d \to \infty$?

*Moderate degrees.* Corollary 1.2 and Theorem 1.1 give the strongest upper bounds of this paper and its companion for $8 \leq d \leq e^{36{,}363}$, where the gap has order $1/\log d$ (Proposition 3.4). How large is $4 - K(d)$ for moderate degrees, say $8 \leq d \leq 10^6$?

*The smallest degree where Smale's conjecture fails.* In full strength Smale's conjecture asks whether $K(d) = (d-1)/d$; this holds for $d \leq 4$ [13], while $K(d) > 1$ holds at degree 29,984 [2]. What is the least degree $d$ with $K(d) > (d-1)/d$?

## Acknowledgments

This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

## References

[1] T. Adamczewski, Lean 4 disproof of Smale's mean value conjecture ($K = 1$), found by GPT-6 Astra, GitHub repository, 2026, https://github.com/tadamcz/mean-value-problem
[2] T. Adamczewski and M. J. Colbrook, From boundary geometry to counterexamples in Smale's mean value problem, preprint, arXiv:2610.36158v1, 2026.
[3] A. F. Beardon, D. Minda and T. W. Ng, Smale's mean value conjecture and the hyperbolic metric, *Math. Ann.* **322** (2002), no. 4, 623–632.
[4] D. Betsakos, Holomorphic functions with image of given logarithmic or elliptic capacity, *J. Aust. Math. Soc.* **94** (2013), no. 2, 145–157.
[5] A. Conte, E. Fujikawa and N. Lakic, Smale's mean value conjecture and the coefficients of univalent functions, *Proc. Amer. Math. Soc.* **135** (2007), no. 10, 3295–3300.
[6] E. Crane, Extremal polynomials in Smale's mean value conjecture, *Comput. Methods Funct. Theory* **6** (2006), no. 1, 145–163.
[7] E. Crane, A bound for Smale's mean value conjecture for complex polynomials, *Bull. Lond. Math. Soc.* **39** (2007), no. 5, 781–791.
[8] *NIST Digital Library of Mathematical Functions*, Release 1.2.8 of 2026-09-15, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl and M. A. McClain, eds., https://dlmf.nist.gov/
[9] V. N. Dubinin, Inequalities for critical values of polynomials, *Sb. Math.* **197** (2006), no. 8, 1167–1176.
[10] V. N. Dubinin, Majorization principles for meromorphic functions, *Math. Notes* **84** (2008), no. 5–6, 751–755.
[11] V. N. Dubinin, Methods of geometric function theory in classical and modern problems for polynomials, *Russian Math. Surveys* **67** (2012), no. 4, 599–684.
[12] V. N. Dubinin, Blaschke products and Smale's conjecture on complex polynomials, *Math. Notes* **116** (2024), no. 1–2, 252–257.
[13] V. N. Dubinin and T. Sugawa, Dual mean value problem for complex polynomials, *Proc. Japan Acad. Ser. A Math. Sci.* **85** (2009), no. 9, 135–137.
[14] E. Fujikawa and T. Sugawa, Geometric function theory and Smale's mean value conjecture, *Proc. Japan Acad. Ser. A Math. Sci.* **82** (2006), no. 7, 97–100.
[15] D. Goel, Smale's mean value constant (1981) equals 4, 2026, Zenodo https://doi.org/10.5281/zenodo.24069884 and Hexagon https://hexagonmath.org/2611.00282

[16] D. Goel, How fast Smale's mean value constants (1981) approach 4, and other miscellanea, 2026, Zenodo https://doi.org/10.5281/zenodo.24288572
[17] A. Jatar and T. W. Ng, Smale's mean value conjecture and its dual conjecture for complex polynomials, preprint, arXiv:2608.27047v1, 2026.
[18] A. Jatar and T. W. Ng, Smale's mean value constant and admissible univalent Blaschke pairs, preprint, arXiv:2610.21472v2, 2026.
[19] I. P. Mityuk, Estimates of the inner radius (capacity) of some domain (condenser), *Izv. Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly Estestv. Nauk* 1983, no. 3, 36–38 (Russian).
[20] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, London Math. Soc. Monogr. (N.S.) **26**, Oxford Univ. Press, Oxford, 2002.
[21] T. Ransford, *Potential Theory in the Complex Plane*, London Math. Soc. Student Texts **28**, Cambridge Univ. Press, Cambridge, 1995.
[22] G. Schmeisser, The conjectures of Sendov and Smale, in *Approximation Theory: A Volume Dedicated to Blagovest Sendov* (B. D. Bojanov, ed.), DARBA, Sofia, 2002, pp. 353–369.
[23] B. Sendov and P. Marinov, Verification of Smale's mean value conjecture for $n \leq 10$, *C. R. Acad. Bulgare Sci.* **60** (2007), no. 11, 1151–1156.
[24] T. Sheil-Small, *Complex Polynomials*, Cambridge Stud. Adv. Math. **75**, Cambridge Univ. Press, Cambridge, 2002.
[25] S. Smale, The fundamental theorem of algebra and complexity theory, *Bull. Amer. Math. Soc. (N.S.)* **4** (1981), no. 1, 1–36.
[26] S. Smale, Mathematical problems for the next century, second version, August 7, 1998, https://www.fim.uni-passau.de/fileadmin/dokumente/fakultaeten/fim/lehrstuhl/muller/SmaleProblems1998.pdf
[27] S. Smale, Mathematical problems for the next century, in *Mathematics: Frontiers and Perspectives* (V. Arnold, M. Atiyah, P. Lax and B. Mazur, eds.), Amer. Math. Soc., Providence, RI, 2000, pp. 271–294.