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HOW FAST SMALE'S MEAN VALUE CONSTANTS (1981) APPROACH 4, AND OTHER MISCELLANEA

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This is the expanded version of the focused paper D. Goel, How fast Smale's mean value constants (1981) approach 4 [20]. *That paper contains Theorem 1.1, Corollary 1.2 and Proposition 3.4 with their proofs; Theorems 1.3 and 1.4, proved in Sections 4 and 5, and Corollary 1.5 appear only here.*

Abstract. For a complex polynomial p of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$, let $S(p)$ be the minimum of $|p(c)/(c\,p'(0))|$ over the critical points c of p, and let $K(d)$ be the supremum of $S(p)$ over all such p of degree d. In 1981 Smale proved that $K(d) \leq 4$ for every d. It is known that $K(d) < 4$ for every d, and the author recently proved that $K(d) \to 4$ as $d \to \infty$. We study how fast $K(d)$ approaches 4. We prove that $K(d) \leq (4/y)\, e^{-\mu(y^{1/d})}$ for every $0 < y < 1$, where μ is the Grötzsch ring function, and deduce that $K(d) < 4 - 4/\log d$ for every $d \geq 8$; for large d the best earlier bound was $K(d) < 4 - 2.263/\sqrt{d}$, due to Crane. We also prove that $4 - K(d) \geq \min\bigl(1.1 \cdot 10^{-4},\ 0.056/\sqrt{\log d}\bigr)$ for every $d \geq 2$ and, in the other direction, that $4 - K(d) \leq 4.9 \cdot 10^4 \sqrt{\log\log\log d/\log d}$ for every $d \geq \exp(1.04 \cdot 10^{10})$. Thus $4 - K(d)$ is of order $1/\sqrt{\log d}$, up to constant factors and the factor $\sqrt{\log\log\log d}$.[1] Theorem 1.1 and Corollary 1.2 are verified in Lean.[2]

1. Introduction

Let p be a complex polynomial of degree $d \geq 2$ with $p(0) = 0$ and $p'(0) \neq 0$. A *critical point* of p is a zero of p', and a *critical value* of p is its value at a critical point. Put

$$S(p) \;:=\; \min_{p'(c)=0} \left| \frac{p(c)}{c\,p'(0)} \right|.$$

Smale's mean value theorem [34, Theorem, p. 33] says that $S(p) \leq 4$ for every such p. For each $d \geq 2$ let

$$K(d) \;:=\; \sup\{S(p) :\ \deg p = d\},$$

the best constant in Smale's theorem for polynomials of degree d. Fields Medalist Stephen Smale proved $K(d) \leq 4$ in 1981 in a complexity study of Newton's method and asked whether 4 can be replaced by 1, or even by $(d-1)/d$ [34, Problem 1E, p. 33]. Smale listed the question again, as the first addendum (*Mean Value Problem*), in the second version of *Mathematical problems for the next century* [35, p. 21] (see also [36]); for forty-five years it stood alongside Sendov's conjecture as one of the central open problems in the analytic theory of complex polynomials, featured in the monographs of Rahman and Schmeisser [28, Section 7.2] and Sheil-Small [33, Section 10.4] and in surveys by Schmeisser [31] and Dubinin [12]. For $d \leq 4$ it can: $K(d) = (d-1)/d$ (see the introduction of [14]). The same introduction reports a computer-assisted proof, due to Crane, for $d = 5$, and Sendov and Marinov [32] give numerical evidence that $K(d) = (d-1)/d$ for $d \leq 10$. In general it cannot: the existence of a counterexample, of very large and unspecified degree, is proved in [1], and Adamczewski and Colbrook gave an explicit one, of degree 29,984 [2]. Jatar and

[1] This is an expanded version of the following focused version [20]: https://doi.org/10.5281/zenodo.24288566

[2] Proved in Lean 4 from its three standard axioms and four assumptions, and checked with `lake comparator`: https://github.com/david44hello/paper_smale-1981-mean-value-rate/tree/v1.00/lean

Ng proved $\sup_d K(d) > 1.1018$ [23], and the author proved in [18] (the expanded version [19] has the full proof used below), using a reduction of theirs, that the constant 4 cannot be lowered at all:

$$\lim_{d\to\infty} K(d) \;=\; \sup_{d\ge 2} K(d) \;=\; 4.$$

The limit exists because $K(d)$ does not decrease with d ([6], [16, p. 99]). On the other hand, $K(d) < 4$ for every d: Beardon, Minda and Ng proved $K(d) \le 4^{(d-2)/(d-1)}$ [3]; Conte, Fujikawa and Lakic proved $K(d) \le 4(d-1)/(d+1)$ [5]; Fujikawa and Sugawa improved this to $K(d) \le 4\big(1+(d-2)\,4^{-1/(d-1)}\big)/(d+1)$ [16]; Dubinin proved $K(d) \le \frac{d-1}{d}\,4^{(d-2)/(d-1)}$ [13] (stated without proof in [10, (8)]); and Crane proved $K(d) < 4 - 2.263/\sqrt{d}$ for $d \ge 8$ [7].

So the constants $K(d)$ do not decrease, tend to 4, and never reach it. The proof of $K(d) \to 4$ in [18, 19] is not effective: for a given $a < 4$ it gives no bound on the least d with $K(d) > a$. In the other direction, the first four of the bounds above show only that $4 - K(d)$ is at least of order $1/d$, and Crane's shows that it is at least of order $1/\sqrt{d}$. In this paper we study the size of the gap $4 - K(d)$ as a function of d. We show that it is of order $1/\sqrt{\log d}$, up to constant factors and a factor $\sqrt{\log\log\log d}$. Table 1 lists the results.

TABLE 1. The main results of this paper. Here log is the natural logarithm.

	Best result before this paper	New result
Upper bound for $K(d)$, every $d \ge 8$	The bound of Fujikawa and Sugawa [16]; Crane's bound $K(d) < 4 - 2.263/\sqrt{d}$ [7] and Crane's numerical bounds at nine degrees	$K(d) < 4 - 4/\log d$ (Corollary 1.2), from the sharper bound $K(d) \le (4/y)\,e^{-\mu(y^{1/d})}$, $0 < y < 1$ (Theorem 1.1)
Lower bound for the gap $4 - K(d)$, large d	$4 - K(d) > 2.263/\sqrt{d}$ [7]	$4 - K(d) \ge 0.056/\sqrt{\log d}$ for $\log d \ge 259{,}174$, and $4 - K(d) \ge \min\big(1.1\cdot 10^{-4},\, 0.056/\sqrt{\log d}\big)$ for every $d \ge 2$ (Theorem 1.3)
Upper bound for the gap $4 - K(d)$, large d	$K(d) \to 4$, with no bound on how fast [18]	$4 - K(d) \le 4.9\cdot 10^4 \sqrt{\log\log\log d/\log d}$ for $d \ge \exp(1.04\cdot 10^{10})$ (Theorem 1.4)

- *Upper bounds for $K(d)$.* Smale's bound $K(d) \le 4$ [34] was improved by Beardon, Minda and Ng [3], by Conte, Fujikawa and Lakic [5], by Fujikawa and Sugawa [16], by Dubinin [10, 13], and by Crane [7], whose bound $4 - 2.263/\sqrt{d}$ is the smallest of these for large d. We prove $K(d) < 4 - 4/\log d$ for every $d \ge 8$. It follows from a sharper bound in terms of the Grötzsch ring function (Theorem 1.1), whose proof combines the Bernstein–Walsh inequality [29, Theorem 5.5.7(a)] with a theorem of Betsakos [4]; the case of the Bernstein–Walsh inequality that we use is also a special case of an inequality of Dubinin [11, 12] (Remark 3.1).
- *The gap is at least of order $1/\sqrt{\log d}$.* The proof develops Crane's idea [7] of using Smale's inequality at many critical points at once.
- *The gap is at most of order $\sqrt{\log\log\log d/\log d}$.* This makes the proof of $K(d) \to 4$ in [19, Section 9] effective.

The results. For $0 < r < 1$ let

$$\mathcal{K}(r) \;:=\; \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \qquad \mu(r) \;:=\; \frac{\pi}{2}\,\frac{\mathcal{K}\big(\sqrt{1-r^2}\big)}{\mathcal{K}(r)}. \tag{1}$$

Here $\mathcal{K}$ is the complete elliptic integral of the first kind, and μ is the *Grötzsch ring function*. The function μ decreases from ∞ to 0 on $(0,1)$. It satisfies $\mu(r) = \log(4/r) + O(r^2)$ as $r \to 0$ and $\mu(r)\,\mu(\sqrt{1-r^2}) = \pi^2/4$ for $0 < r < 1$, so that $\mu(r) \sim \pi^2/\bigl(2\log\frac{16}{1-r^2}\bigr)$ as $r \to 1$ (Section 2.2).

Theorem 1.1. *For every integer $d \geq 2$ and every $y \in (0,1)$,*

$$K(d) \;\leq\; \frac{4}{y}\, e^{-\mu(y^{1/d})}.$$

For fixed d the right-hand side tends to ∞ as $y \to 0$ (by Lemma 2.1(c) it exceeds $y^{1/d-1}$), and to 4 from below as $y \to 1$. To see that it is below 4 for y near 1, put $L := \log(1/y)$. For $0 < L < d$, Lemma 2.1 and the inequality $1 - e^{-t} \geq t - t^2/2$ bound the right-hand side by

$$4\exp\Bigl(L - \frac{\pi^2}{2\log\bigl(8d/(L(1-L/d))\bigr)}\Bigr)$$

(see Steps 1 and 3 of the proof of Corollary 1.2 in Section 3), and this is below 4 when L is small. So Theorem 1.1 recovers Smale's bound $K(d) \leq 4$, and it gives $K(d) < 4$. For fixed y the right-hand side tends to $4/y > 4$ as $d \to \infty$, so y must tend to 1 with d. When d is large and L is a small multiple of $1/\log d$, the displayed bound is approximately 4 minus a positive multiple of $1/\log d$, since the logarithm in the display is then approximately $\log d$. This is why the bound in Corollary 1.2 is 4 minus a multiple of $1/\log d$. The choice $y = \exp(-1/(4\log d))$ leads to a bound in closed form.

Corollary 1.2. *For every integer $d \geq 8$,*

$$K(d) \;<\; 4 - \frac{4}{\log d}.$$

At every degree $d \geq 8$ this bound is below each of the five bounds quoted above and below the numerical bounds that Crane gives at nine degrees [7] (Remark 3.2); see Table 2 and Figure 1. Theorem 1.1 with the best choice of y gives a smaller bound (Remark 3.3), but not one of better order: if $B(d)$ denotes the infimum over $0 < y < 1$ of its right-hand side, then $(4 - B(d))\log d \to 2\pi^2$ as $d \to \infty$ (Proposition 3.4). At $d = 7$, Theorem 1.1 does not give the bound of Corollary 1.2: $B(7) > 4 - 4/\log 7$ (Remark 3.5).

Corollary 1.2 shows that $K(d)$ can approach 4 only slowly. For example, it gives $K(d) < 3.9$ for $8 \leq d \leq e^{40}$, and the bound of Beardon, Minda and Ng gives $K(d) \leq 4^{5/6} < 3.9$ for $d \leq 7$; so a polynomial p with $S(p) \geq 3.9$ has degree larger than $2 \cdot 10^{17}$.

TABLE 2. Upper bounds for $K(d)$, rounded up (Crane's numerical bounds as printed in [7]). The column "earlier" gives the smallest, at that degree, of the earlier bounds of Remark 3.2: Fujikawa and Sugawa [16] at $d = 8$; the table of numerical bounds in [7] at $d = 100$ and $d = 10^4$; Crane's bound $4 - 2.263/\sqrt{d}$ [7] at $d = 10^6$. The column "Theorem 1.1" gives the bound of Theorem 1.1, computed in interval arithmetic with upper and lower bounds for μ as in Remark 3.5, at $y = e^{-L}$, with L from the last column (close to the best L; the proof of Corollary 1.2 uses $L = 1/(4\log d)$).

d	earlier	$4 - 4/\log d$	Theorem 1.1	L
8	2.6321	2.0765	2.0550	0.12766
100	3.621	3.1315	2.5255	0.05337
10^4	3.955	3.5658	2.9495	0.02156
10^6	3.9978	3.7105	3.1754	0.01194

For very large d the gap $4 - K(d)$ is much larger than $1/\log d$: it is at least a constant times $1/\sqrt{\log d}$.

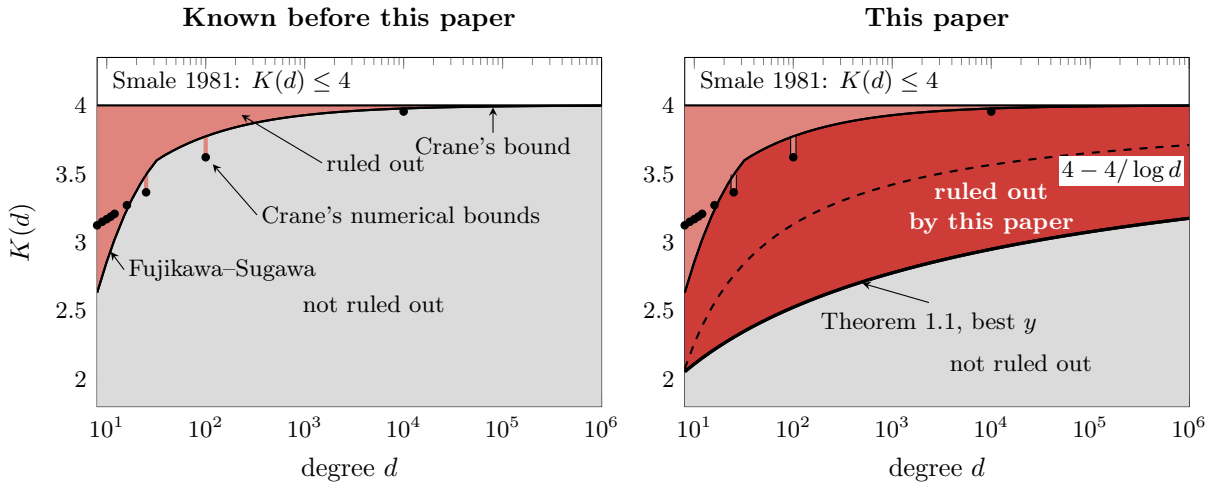

FIGURE 1. Upper bounds for $K(d)$ for $8 \le d \le 10^6$. In both panels the red regions are ruled out and the gray region is where $K(d)$ may still lie. Left: before this paper, $K(d)$ had been proved to lie below the smaller of the bounds of Fujikawa and Sugawa [16] and of Crane [7] (the curve; its corner is where the two bounds cross) and, at nine degrees, below Crane's numerical bounds (the dots). The five dots at $d = 8, \dots, 12$ nearly merge; they and the dot at $d = 16$ lie above the curve, and the three at $d = 25$, 100 and 10^4 lie below it. The light red region, including the short light red strokes above the dots at $d = 25$ and $d = 100$, had been ruled out. Right: Theorem 1.1, with the best value of y at each degree (the lowest curve, computed in floating point; Table 2 gives bounds computed in interval arithmetic at four degrees), rules out the dark red region between the lowest curve and the earlier bounds; the dots and the light red strokes are as on the left. The dashed curve is the closed form $4 - 4/\log d$ of Corollary 1.2.

Theorem 1.3. *For every integer $d \ge 2$,*

$$4 - K(d) \;\ge\; \min\Big(c_1,\; \frac{c_2}{\sqrt{\log d}}\Big), \qquad c_1 := 1.1 \cdot 10^{-4}, \quad c_2 := 0.056.$$

The second term is the smaller one exactly when $\log d > (c_2/c_1)^2 = 259{,}173.5\ldots$, and the bound of Theorem 1.3 exceeds the bound $4/\log d$ of Corollary 1.2 exactly when $\log d > 4/c_1 = 36{,}363.6\ldots$ (Remark 4.2). The next theorem bounds the gap from the other side.

Theorem 1.4. *For every integer $d \ge \exp(1.04 \cdot 10^{10})$,*

$$K(d) \;\ge\; 4 - 4.9 \cdot 10^4 \sqrt{\frac{\log\log\log d}{\log d}}.$$

For $d \ge \exp\big(\exp(7.01 \cdot 10^{18})\big)$ the constant $4.9 \cdot 10^4$ can be replaced by $1.6 \cdot 10^4$ (Remark 5.3).

Corollary 1.5. *For every integer $d \ge \exp(1.04 \cdot 10^{10})$,*

$$\frac{0.056}{\sqrt{\log d}} \;\le\; 4 - K(d) \;\le\; 4.9 \cdot 10^4 \sqrt{\frac{\log\log\log d}{\log d}}.$$

Proof. The upper bound is Theorem 1.4. For these d we have $\log d > 259{,}174$, so the minimum in Theorem 1.3 is its second term; this is the lower bound. □

Thus $4 - K(d)$ is of order $1/\sqrt{\log d}$, up to constant factors and the factor $\sqrt{\log\log\log d}$. This factor grows very slowly: it is less than 2 for every $d < \exp(\exp(e^4))$, and $\exp(e^4) > 10^{23}$. We do not know whether it can be removed (Section 6). Figure 2 shows the bounds for the gap together.

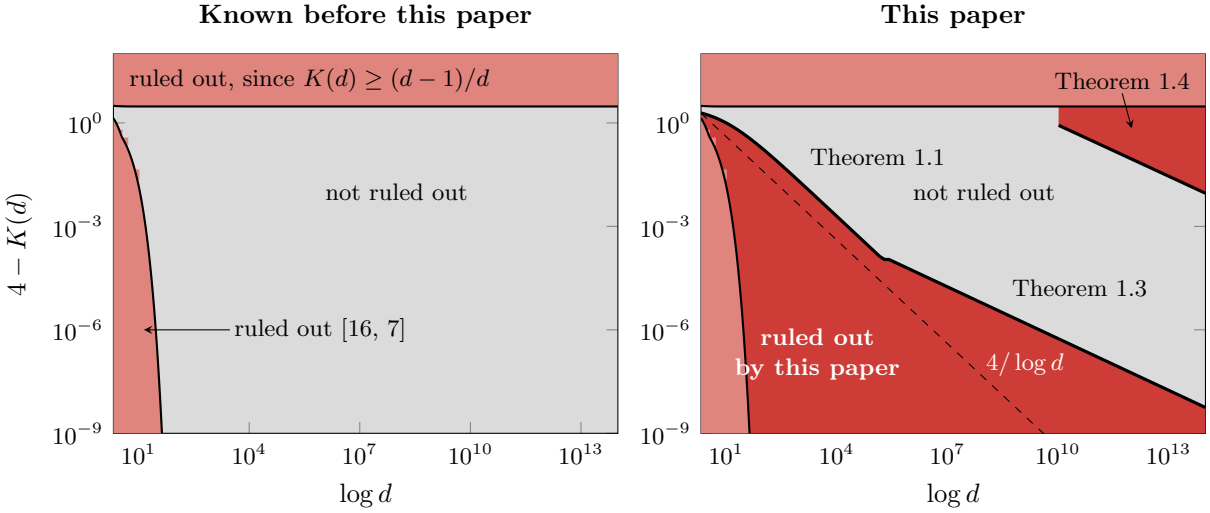

FIGURE 2. The gap $4 - K(d)$ against $\log d$, on logarithmic axes. In both panels the red regions are ruled out and the gray region is where $4 - K(d)$ may still lie. Left: before this paper the light red regions were ruled out: below, by the bound of Fujikawa and Sugawa [16], by Crane's bound $4 - K(d) > 2.263/\sqrt{d}$ [7] and, at the three of Crane's nine degrees where they are the smallest earlier bounds, by Crane's numerical bounds (the three light red ticks); above, because the polynomial $z^d + z$ has $S = (d-1)/d$ (for large d slightly more was known: $4 - K(d) < 2.8982$ [23]). It was known that $4 - K(d) \to 0$ [18], with no bound on how fast. Right: the dark red regions are ruled out by this paper: below, by Theorem 1.1 with the best value of y (computed in floating point) and, for larger d, by Theorem 1.3; above, by Theorem 1.4, for $\log d \geq 1.04 \cdot 10^{10}$ (apart from the thin strip at the top that [23] rules out). The dashed line is the bound $4/\log d$ of Corollary 1.2. Remark 5.3 proves the smaller constant $1.6 \cdot 10^4$ for $\log d \geq \exp(7.01 \cdot 10^{18})$, far to the right.

The ideas of the proofs. We first recall Smale's proof, and then say what we add to it. Dividing p by a constant changes neither its critical points nor $S(p)$; so we may assume that the least modulus of a critical value of p is 1 (if it is 0, then $S(p) = 0$). Let ζ_m be a critical point with $|p(\zeta_m)| = 1$. Since the unit disc $\mathbb{D}$ contains no critical value of p, the branch F of p^{-1} with $F(0) = 0$ is defined and univalent on $\mathbb{D}$, and it omits ζ_m. Since $F'(0) = 1/p'(0)$, the definition of $S(p)$ gives $S(p) \leq |p(\zeta_m)|/|\zeta_m\, p'(0)| = |F'(0)|/|\zeta_m|$. Koebe's one-quarter theorem gives $|\zeta_m| \geq |F'(0)|/4$, so $S(p) \leq 4$: this is Smale's theorem. This argument uses only that F is univalent and omits ζ_m. Our upper bounds for $K(d)$ (Theorems 1.1 and 1.3) come from using, in addition, that p is a polynomial of degree d.

Outline of the proof of Theorem 1.1 (see Figure 3). For $0 < y < 1$ let $V := F(\mathbb{D}_y)$, where $\mathbb{D}_y$ is the open disc of radius y about 0. In the coordinate $x = -\zeta_m/z$, where z is the variable of p, the complement of $\overline{V}$ becomes a bounded Jordan domain E_y that contains the points 0 and -1, the images of ∞ and of ζ_m; the logarithmic capacity $\mathrm{cap}(\overline{E_y})$ of its closure (Section 2.4) is $|\zeta_m|/(y\,|F'(0)|)$, which is at most $1/(y\,S(p))$. Let f map $\mathbb{D}$ conformally onto E_y with $f(0) = 0$, and let $\xi := f^{-1}(-1)$. On ∂V we have $|p| = y$, at ζ_m we have $|p| = 1$, and p has a pole of order d at ∞. So the function $P(w) := w^d\, p(-\zeta_m/f(w))$ is holomorphic in $\mathbb{D}$ (the factor w^d cancels the pole

of order d of $p(-\zeta_m/f(w))$ at $w = 0$), and $\limsup_{|w|\to 1} |P(w)| \le y$; by the maximum principle, $|\xi|^d = |P(\xi)| \le y$ (a case of the Bernstein–Walsh inequality, Remark 3.1). By a theorem of Betsakos [4], $1 = |f(\xi)| \le 4\,\mathrm{cap}(\overline{E_y})\, e^{-\mu(|\xi|)}$; since μ decreases and $|\xi| \le y^{1/d}$, this and the capacity bound above give $e^{\mu(y^{1/d})} \le e^{\mu(|\xi|)} \le 4/(y\,S(p))$, which is the theorem.

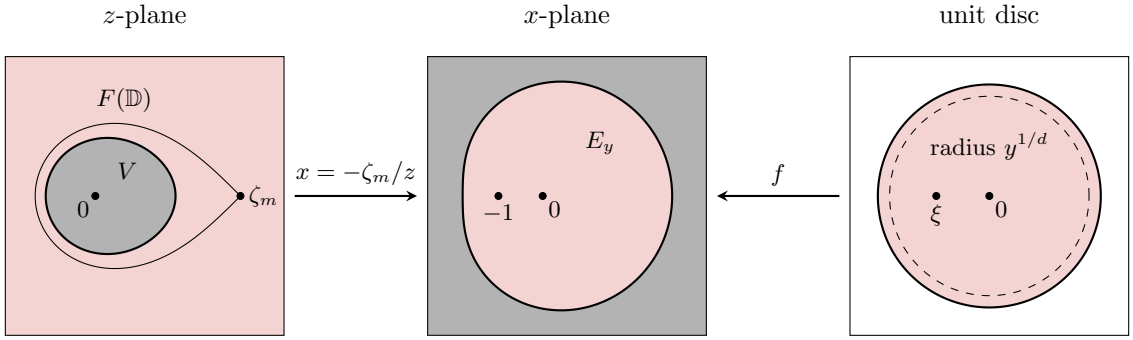

FIGURE 3. The construction in the proof of Theorem 1.1, drawn to scale for $p(z) = 2z - z^2$ (so $d = 2$ and $\zeta_m = 1$) and $y = 0.8$. Left: the domain $V = F(\mathbb{D}_y)$ (dark gray) lies in $F(\mathbb{D})$, the domain inside the thin curve; the critical point ζ_m is not in $F(\mathbb{D})$ (here it lies on its boundary). Middle: in the coordinate $x = -\zeta_m/z$, the complement of $\overline{V}$ becomes the bounded Jordan domain E_y, which contains 0 and -1, the images of ∞ and of ζ_m. Right: a conformal map f of the unit disc onto E_y with $f(0) = 0$ takes a point ξ to -1. The point ξ lies in the closed disc of radius $y^{1/d}$ about 0 (dashed circle). Dark gray marks V and its image $x(V)$; pale shading marks the complement of $\overline{V}$, its image E_y, and the unit disc.

Outline of the proof of Corollary 1.2. With $y = \exp(-1/(4\log d))$, the identity $\mu(r)\,\mu(\sqrt{1-r^2}) = \pi^2/4$ and the inequality $\mu(s) < \log(4/s)$ bound the right-hand side of Theorem 1.1 by an expression in elementary functions. For $d \ge 21$ this expression is below $4 - 4/\log d$ by elementary inequalities; for the thirteen degrees $8 \le d \le 20$ the comparison is a finite computation in interval arithmetic.

Outline of the proof of Theorem 1.3. Let $S(p) = 4e^{-\lambda}$ with λ small; we have to show that $\log d$ is at least a constant times $1/\lambda^2$. Smale's inequality $S(p) \le |p(c)/(c\,p'(0))|$ at a critical point c whose critical value has modulus close to 1 forces c to lie close to ζ_m. The branch F continues univalently across every arc of the unit circle over which there is no such critical point, and it still omits ζ_m; by Koebe's theorem the arcs of this kind that are longer than a constant times λ cover only a part of the unit circle. Hence at least a constant times $1/\lambda$ critical points of this kind cut the boundary of $F(\mathbb{D})$. The curves from these cuts separate at least a constant times $1/\lambda$ disjoint regions of the complement of $F(\mathbb{D})$ ("dead ends"). The boundary of a dead end lies on the boundary of $F(\mathbb{D})$, on curves from the cuts, and on a fixed curve that separates the dead ends from ζ_m; the part on this last curve is its *open side*. The complement of $F(\mathbb{D})$ has area at most $2\pi\lambda$ in a suitable metric; so some dead end has area, in that metric, at most a constant times λ^2. Now let t be a level of order λ. Where $\log|p| > t$, the function $\log|p| - t$ is d times a Green function, and this Green function is bounded on the open sides. At a certain point of the dead end of small area, $\log|p| - t$ equals t, and there is a curve in the dead end from this point to the boundary of $F(\mathbb{D})$ (the image of a radial segment) whose extremal distance from the open side is at least a constant times $1/\lambda^2$. An estimate of Beurling type then bounds the Green function at this point by a constant times e^{-c/λ^2}, and this forces $\log d \ge c'/\lambda^2$.

Outline of the proof of Theorem 1.4. The proof of $K(d) \to 4$ in [19, Section 9] builds, for a small number $\lambda > 0$, a polynomial H with $S(H) \ge 4 - O(\lambda)$, from a Riemann surface with M logarithmic tracts and a plane domain with M thin channels. We carry out that construction with

an explicit bound at every step. The construction needs the conformal radius of the surface at its base point to be at most e^λ, and our bound for this conformal radius gives that once M is at least of order $\lambda^{-1}\log\log(1/\lambda)$. The harmonic measures and the values of Green functions that the degree count uses are at least $e^{-CM/\lambda}$, and the degree of H is at most $e^{CM/\lambda}$. So $\log\deg H$ of order $\lambda^{-2}\log\log(1/\lambda)$ suffices, and solving for λ gives the theorem, since $K(d)$ does not decrease.

Organization. Section 2 collects the notation and the tools. Section 3 proves Theorem 1.1 and Corollary 1.2, Section 4 proves Theorem 1.3, and Section 5 proves Theorem 1.4. Section 6 closes with a discussion of open questions.

2. Notation and tools

2.1. **Notation.** Throughout, log is the natural logarithm, $\mathbb{D}$ is the open unit disc, $\mathbb{T}$ is the unit circle, $\mathbb{D}_y$ is the open disc of radius $y>0$ about 0, and $\widehat{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ is the Riemann sphere. A *Jordan domain* is a domain in $\widehat{\mathbb{C}}$ whose boundary is a Jordan curve. A function is *univalent* if it is meromorphic and injective, and a *conformal map* of a domain D onto a domain D' is a univalent function on D with image D'.

2.2. **The Grötzsch ring function.** Recall the complete elliptic integral $\mathcal{K}$ and the function μ of (1). As usual, $r' := \sqrt{1-r^2}$ is the *complementary modulus* of r, and $\mathcal{K}'(r) := \mathcal{K}(r')$ for $0<r<1$, so that $\mu(r) = \frac{\pi}{2}\,\mathcal{K}'(r)/\mathcal{K}(r)$.

Lemma 2.1. (a) *The function μ is strictly decreasing and maps $(0,1)$ onto $(0,\infty)$.* (b) *For $0<r<1$ we have $\mu(r)\,\mu(r') = \pi^2/4$.* (c) *For $0<s<1$ we have $\mu(s) < \log(4/s)$.*

Proof. (a) For fixed $t\in(0,1)$ the integrand in (1) is continuous and strictly increasing in r. Hence $\mathcal{K}$ is strictly increasing on $(0,1)$. It is continuous, and $\mathcal{K}(r)\to\pi/2$ as $r\to 0$, by dominated convergence (for any $r_1\in(0,1)$, the integrand at r_1 is an integrable majorant on $(0,r_1]$); and $\mathcal{K}(r)\to\int_0^1 dt/(1-t^2)=\infty$ as $r\to 1$, by monotone convergence. So $\mathcal{K}'$ is continuous and strictly decreasing, from ∞ to $\pi/2$, and the quotient $\mu = \frac{\pi}{2}\,\mathcal{K}'/\mathcal{K}$ is continuous and strictly decreasing, from ∞ to 0. (b) The complementary modulus of r' is r, so

$$\mu(r') \;=\; \frac{\pi}{2}\,\frac{\mathcal{K}(r)}{\mathcal{K}(r')} \;=\; \frac{\pi}{2}\,\frac{\mathcal{K}(r)}{\mathcal{K}'(r)};$$

multiplying by $\mu(r)$ gives (b). (c) By [8, (19.9.5)], for $0<k<1$ and $k'=\sqrt{1-k^2}$,

$$\log\frac{(1+\sqrt{k'})^2}{k} \;<\; \frac{\pi\,\mathcal{K}'(k)}{2\,\mathcal{K}(k)} \;<\; \log\frac{2(1+k')}{k}.$$

By [8, (19.2.4), (19.2.8), (19.2.9)], the functions written K and K' there are our $\mathcal{K}$ and $\mathcal{K}'$; so the middle term is $\mu(k)$. Since $k'<1$, we have $2(1+k')<4$, and (c) follows. □

Both bounds in the proof of (c) are $\log(4/k)+O(k^2)$ as $k\to 0$, so $\mu(r)=\log(4/r)+O(r^2)$ as $r\to 0$; with (b), $\mu(r)=\pi^2/(4\mu(r'))\sim\pi^2/\bigl(2\log\frac{16}{1-r^2}\bigr)$ as $r\to 1$.

2.3. **Green functions, harmonic measure and extremal distance.** Let $D\subset\widehat{\mathbb{C}}$ be a domain and $a\in D$. The *Green function* of D with pole at a, when it exists, is the smallest positive harmonic function $g_D(\cdot,a)$ on $D\setminus\{a\}$ for which $g_D(z,a)+\log|z-a|$ is harmonic near a (if $a=\infty$: for which $g_D(z,\infty)-\log|z|$ is harmonic near ∞). It exists for every domain that occurs in this paper. If u is positive and harmonic on $D\setminus\{a\}$, has this singularity at a, and tends to 0 at every point of ∂D, then $u=g_D(\cdot,a)$, by the maximum principle. For example, $g_{\mathbb{D}}(z,0)=\log(1/|z|)$. Green functions are conformally invariant: if f maps D conformally onto D', then $g_{D'}(f(z),f(a))=g_D(z,a)$.

For a Borel set $\Sigma\subset\partial D$ and $z\in D$, the *harmonic measure* $\omega(z,\Sigma,D)$ is the value at z of the Perron solution of the Dirichlet problem in D whose boundary function is 1 on Σ and 0 on $\partial D\setminus\Sigma$.

Let A and B be subsets of the closure of D in $\widehat{\mathbb{C}}$. A *curve in D from A to B* is a locally rectifiable curve $\gamma\colon (0,1) \to D$ such that $\gamma(t)$ tends to a point of A as $t \to 0$ and to a point of B as $t \to 1$. The sets A and B may lie in D or on its boundary. For a Borel function $\rho \geq 0$ on $D \setminus \{\infty\}$ let L_ρ be the infimum of the lengths $\int_\gamma \rho\,|dz|$ of the curves γ in D from A to B (the integral is over the part of γ in $\mathbb{C}$), and let $A_\rho := \iint_{D\setminus\{\infty\}} \rho^2\,dx\,dy$. The *extremal distance* between A and B in D is

$$\operatorname{ext}_D(A,B) \;:=\; \sup_\rho \frac{L_\rho^2}{A_\rho},$$

the supremum being over all ρ with $0 < A_\rho < \infty$. It does not decrease when A, B or D is replaced by a subset, since fewer curves are then admitted. It is conformally invariant: if f maps D conformally onto D' and extends to a homeomorphism of $\overline{D}$ onto $\overline{D'}$, then $\operatorname{ext}_{D'}(f(A), f(B)) = \operatorname{ext}_D(A,B)$. See [17, Ch. IV] for these notions.

2.4. **Transfinite diameter and capacity.** For a bounded infinite set $A \subset \mathbb{C}$ and $n \geq 2$ let

$$d_n(A) \;:=\; \sup\Big\{ \prod_{1\leq j<k\leq n} |z_j - z_k|^{2/(n(n-1))} :\; z_1,\dots,z_n \in A \Big\}. \tag{2}$$

Since the product is continuous in $z_1,\dots,z_n$, we have $d_n(A) = d_n(\overline{A})$. The sequence $d_n(\overline{A})$ decreases [29, Theorem 5.5.2], and its limit $d_\infty(A) = d_\infty(\overline{A})$ is the *transfinite diameter* of A. By the Fekete–Szegő theorem, the transfinite diameter of an infinite compact set Q equals its logarithmic capacity $\operatorname{cap}(Q)$ ([4, p. 146], [29, Theorem 5.5.2]). If $Q \subset \mathbb{C}$ is compact and connected, has more than one point and has connected complement, and Φ is a conformal map of $\{|u| > 1\} \cup \{\infty\}$ onto $\widehat{\mathbb{C}} \setminus Q$ with $\Phi(\infty) = \infty$, then $\Phi(u) = a\,u + O(1)$ as $u \to \infty$ and $\operatorname{cap}(Q) = |a|$ [29, Theorem 5.2.3 and Corollary 5.2.2].

2.5. **A theorem of Betsakos.** The proof of Theorem 1.1 rests on the following theorem.

Theorem 2.2 ([4, Theorem 1.1])**.** *Let f be a nonconstant bounded holomorphic function in $\mathbb{D}$ with $f(0) = 0$. Then*

$$|f(w)| \;\leq\; 4\,d_\infty(f(\mathbb{D}))\,e^{-\mu(|w|)}, \qquad w \in \mathbb{D}.$$

Equality holds for some $w \in \mathbb{D} \setminus \{0\}$ if and only if f maps $\mathbb{D}$ conformally onto a domain bounded by an ellipse so that 0 and w are mapped onto the foci of the ellipse.

In [4] the transfinite diameter is written d. At $w = 0$ the right-hand side is read as 0. As $|w| \to 0$ the right-hand side is $d_\infty(f(\mathbb{D}))\,|w|\,(1 + O(|w|^2))$, and as $|w| \to 1$ it tends to $4\,d_\infty(f(\mathbb{D}))$. Letting $w \to 0$ gives $|f'(0)| \leq d_\infty(f(\mathbb{D}))$; for a conformal map f this is the classical inequality between the conformal radius $|f'(0)|$ and the capacity of $\overline{f(\mathbb{D})}$.

3. The upper bounds

Proof of Theorem 1.1. Fix $d \geq 2$ and $0 < y < 1$, and let p be a polynomial of degree d with $p(0) = 0$ and $p'(0) \neq 0$. We show that $S(p) \leq (4/y)\,e^{-\mu(y^{1/d})}$; taking the supremum over p gives the theorem.

Normalization. If $p(c) = 0$ for some critical point c of p, then $S(p) = 0$ and there is nothing to prove. Otherwise the least modulus of a critical value of p is positive, and dividing p by it changes neither the critical points nor $S(p)$. So we may assume that this least modulus is 1, and we fix a critical point ζ_m with $|p(\zeta_m)| = 1$. Then $S(p) > 0$, and $\zeta_m \neq 0$ because $p'(0) \neq 0$.

The inverse branch. The disc $\mathbb{D}$ contains no critical value of p, and p is a covering map over the complement of its critical values. Since $\mathbb{D}$ is simply connected and $p(0) = 0$, there is a branch F of p^{-1} on $\mathbb{D}$ with $F(0) = 0$. Since $p(F(w)) = w$, the branch F is univalent and $F'(0) = 1/p'(0)$. By the definition of $S(p)$, at the critical point ζ_m,

$$S(p) \;\leq\; \frac{|p(\zeta_m)|}{|\zeta_m\,p'(0)|} \;=\; \frac{|F'(0)|}{|\zeta_m|}. \tag{3}$$

The domain E_y. Let $V := F(\mathbb{D}_y)$. It is a Jordan domain bounded by the analytic curve $F(\partial\mathbb{D}_y)$, it contains $F(0) = 0$, and its closure $F(\overline{\mathbb{D}_y})$ lies in $F(\mathbb{D})$. The point ζ_m is not in $F(\mathbb{D})$, since $|p(\zeta_m)| = 1$ while $p(F(\mathbb{D})) = \mathbb{D}$; so $\zeta_m \notin \overline{V}$. Let $x(z) := -\zeta_m/z$, a Möbius involution of $\widehat{\mathbb{C}}$, and let

$$E_y \;:=\; x\big(\widehat{\mathbb{C}} \setminus \overline{V}\big).$$

Then E_y is a Jordan domain bounded by $x(\partial V)$. It is bounded, since $x^{-1}(\infty) = 0 \in V$, and it contains $x(\infty) = 0$ and $x(\zeta_m) = -1$.

The capacity of $\overline{E_y}$. The map $\Phi(u) := x(F(y/u))$ is a conformal map of $\{|u| > 1\} \cup \{\infty\}$ onto $\widehat{\mathbb{C}} \setminus \overline{E_y} = x(V)$ with $\Phi(\infty) = \infty$. Since $F(w) = F'(0)\, w + O(w^2)$ as $w \to 0$,

$$\Phi(u) \;=\; -\frac{\zeta_m}{y\, F'(0)}\, u + O(1), \qquad u \to \infty.$$

So the logarithmic capacity of $\overline{E_y}$ is $\operatorname{cap}(\overline{E_y}) = |\zeta_m|/(y\,|F'(0)|)$ (Section 2.4), and by (3)

$$\operatorname{cap}(\overline{E_y}) \;\le\; \frac{1}{y\, S(p)}. \tag{4}$$

A bound for $|\xi|$. Let f be a conformal map of $\mathbb{D}$ onto E_y with $f(0) = 0$, and let $\xi := f^{-1}(-1)$; then $\xi \neq 0$. Put

$$P(w) \;:=\; w^d\, p\big(x(f(w))\big), \qquad w \in \mathbb{D}.$$

The map $x \circ f$ sends $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus \overline{V}$; its only pole is at $w = 0$, and that pole is simple. Since p has degree d, the singularity of P at 0 is removable, so P is holomorphic in $\mathbb{D}$. As $|w| \to 1$, the distance from $f(w)$ to ∂E_y tends to 0. Since the compact set ∂E_y does not contain 0, the map x is uniformly continuous on a neighborhood of ∂E_y; hence the distance from $x(f(w))$ to $x(\partial E_y) = \partial V$ tends to 0. On $\partial V = F(\partial\mathbb{D}_y)$ we have $|p| = y$; as p is uniformly continuous on a compact neighborhood of ∂V and $|w|^d < 1$, we get $\limsup_{|w|\to 1} |P(w)| \le y$. By the maximum principle, $|P| \le y$ in $\mathbb{D}$. At $w = \xi$ we have $x(f(\xi)) = x(-1) = \zeta_m$, so $|\xi|^d = |\xi|^d\,|p(\zeta_m)| = |P(\xi)| \le y$, that is,

$$|\xi| \;\le\; y^{1/d}. \tag{5}$$

Betsakos's theorem. The function f is nonconstant, bounded and holomorphic in $\mathbb{D}$, with $f(0) = 0$. Theorem 2.2 at $w = \xi$ gives

$$1 \;=\; |f(\xi)| \;\le\; 4\, d_\infty(E_y)\, e^{-\mu(|\xi|)}.$$

By Section 2.4, $d_\infty(E_y) = d_\infty(\overline{E_y}) = \operatorname{cap}(\overline{E_y})$, the second equality by the Fekete–Szegő theorem. With (4),

$$e^{\mu(|\xi|)} \;\le\; 4\operatorname{cap}(\overline{E_y}) \;\le\; \frac{4}{y\, S(p)}. \tag{6}$$

Conclusion. The function μ is strictly decreasing (Lemma 2.1(a)), so (5) gives $\mu(y^{1/d}) \le \mu(|\xi|)$. With (6), $\mu(y^{1/d}) \le \log\big(4/(y\, S(p))\big)$, that is, $S(p) \le (4/y)\, e^{-\mu(y^{1/d})}$. □

Remark 3.1 (Related results). Inequality (5) is a case of the Bernstein–Walsh inequality, which is called Bernstein's lemma in [29, Theorem 5.5.7(a)]. For the compact set $\overline{V}$, that inequality reads $|p(\zeta_m)| \le \max_{\overline{V}} |p| \cdot e^{d\, g(\zeta_m)}$, where g is the Green function of $\widehat{\mathbb{C}} \setminus \overline{V}$ with pole at ∞; here $\max_{\overline{V}} |p| = y$ and $|p(\zeta_m)| = 1$. By conformal invariance under x and then under f, we have $g(\zeta_m) = g_{E_y}(-1, 0) = g_{\mathbb{D}}(\xi, 0) = \log(1/|\xi|)$, so the inequality is (5).

Inequality (5) is also a special case of an inequality of Dubinin for meromorphic functions and Green functions [11, Theorem 1], [12, (1.27), pp. 618–619]. In his notation, take the domains D and G to be $\widehat{\mathbb{C}} \setminus \overline{V}$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}_y}$, the point w_0 to be ∞, and his meromorphic function to be p. His hypotheses on this function are that it takes the value w_0 in D and that all its limit values at the boundary of D lie outside G; they hold, since $p(\infty) = \infty$ and $|p| = y$ on ∂V. (They do not require

$p(D) \subset G$, which fails when $d \geq 2$.) As $1/p$ has a zero of order d at ∞ and no other zero, his inequality at $z = \zeta_m$, where $|p| = 1$, reads $\log(1/y) \leq d\, g(\zeta_m)$, which is again (5). According to [12, p. 619], Dubinin's inequality was first obtained by Mityuk [26], under slightly different conditions on the function.

Capacity has entered this problem before. The proof of Dubinin's upper bound for $K(d)$ uses condenser capacities [13]; Crane's bound comes from estimates for the logarithmic capacity of a connected set containing three given points [7]; and Jatar and Ng [22] use a Koebe-type theorem of Cunningham (stated in [22, Theorem 8]) on univalent functions whose image has bounded capacity. What is new in Theorem 1.1 is the combination of the Bernstein–Walsh inequality with the theorem of Betsakos (Theorem 2.2), applied to the domain E_y.

Proof of Corollary 1.2. Let $d \geq 8$ and put

$$\ell := \log d, \quad L := \frac{1}{4\ell}, \quad y := e^{-L}, \quad r := y^{1/d} = e^{-L/d}, \quad r' := \sqrt{1 - r^2} = \sqrt{1 - e^{-2L/d}}.$$

Step 1: a bound without elliptic integrals. Lemma 2.1(a) and (c) at $s = r'$ give $0 < \mu(r') < \log(4/r')$. With Lemma 2.1(b),

$$\mu(r) \;=\; \frac{\pi^2}{4\,\mu(r')} \;>\; \frac{\pi^2}{4\log(4/r')},$$

and Theorem 1.1 with this y gives

$$K(d) \;\leq\; 4e^{L}e^{-\mu(r)} \;<\; M(d) \;:=\; 4\exp\Big(L - \frac{\pi^2}{4\log(4/r')}\Big). \tag{7}$$

It remains to show $M(d) \leq 4 - 4/\ell$.

Step 2: the degrees $8 \leq d \leq 20$. For each of these thirteen degrees, interval arithmetic gives $M(d) < 4 - 4/\log d$ (Table 3); the smallest margin is 0.0173, at $d = 8$.

Step 3: the degrees $d \geq 21$. Since $1 - e^{-t} \geq t - t^2/2$ for $t \geq 0$, the choice $t = 2L/d$ gives $r'^2 \geq (2L/d)(1 - L/d)$. Hence $\log(4/r') = \frac{1}{2}\log(16/r'^2) \leq \Lambda/2$, where

$$\Lambda \;:=\; \log\frac{8d}{L\,(1 - L/d)} \;=\; \ell + \log(32\ell) - \log(1 - L/d),$$

and (7) gives $M(d) \leq 4\exp\big(L - \pi^2/(2\Lambda)\big)$. So $M(d) \leq 4(1 - 1/\ell)$ follows from

$$\frac{\pi^2}{2\Lambda} \;\geq\; \frac{1}{4\ell} - \log\Big(1 - \frac{1}{\ell}\Big). \tag{8}$$

We bound both sides, using $\ell \geq \log 21 > 3.04$. Since $-\log(1 - s) \leq s/(1 - s)$ for $0 < s < 1$, and $\ell \geq 3$, we have $-\log(1 - 1/\ell) \leq 1/(\ell - 1) \leq 1.5/\ell$; hence the right-hand side of (8) is at most $1.75/\ell$. Next, the function $\ell \mapsto \log(32\ell)/\ell$ has derivative $(1 - \log(32\ell))/\ell^2 < 0$ for $\ell \geq 1$, so it decreases there, and its value at $\ell = \log 21$ is $1.5040\ldots < 1.52$; hence $\log(32\ell) \leq 1.52\,\ell$. Moreover, $L/d = 1/(4\ell d) \leq 1/(4 \cdot 3.04 \cdot 21) < 0.0042$, so that $-\log(1 - L/d) \leq -\log(1 - 0.0042) < 0.005 < 0.002\,\ell$. Therefore $\Lambda \leq (1 + 1.52 + 0.002)\,\ell < 2.53\,\ell$, and the left-hand side of (8) is at least $\pi^2/(5.06\,\ell) > 1.95/\ell$. Since $1.95/\ell > 1.75/\ell$, inequality (8) holds. With (7), $K(d) < M(d) \leq 4 - 4/\ell$. □

Remark 3.2 (The earlier bounds). At every degree $d \geq 8$, the bound of Corollary 1.2 is below the bounds of Crane, of Fujikawa and Sugawa, of Conte, Fujikawa and Lakic, of Beardon, Minda and Ng, and of Dubinin, and below Crane's numerical bounds at nine degrees.

(a) Crane proved $K(d) < 4 - 2.263/\sqrt{d}$ for $d \geq 8$ [7]. The function $\sqrt{d}/\log d$ attains its minimum over $d > 1$ at $d = e^2$, where it equals $e/2$. Hence $4/\log d \geq 2e/\sqrt{d} > 2.263/\sqrt{d}$.

TABLE 3. Step 2 of the proof of Corollary 1.2: $M(d)$ of (7) against $4 - 4/\log d$, in interval arithmetic. Upper bounds for $M(d)$ are rounded up and lower bounds for $4 - 4/\log d$ are rounded down, to five decimal places; the margin is the difference of these two columns.

d	$M(d) \le$	$4 - 4/\log d \ \ge$	margin $\ge$
8	2.05907	2.07640	0.01733
9	2.08862	2.17952	0.09090
10	2.11417	2.26282	0.14865
11	2.13661	2.33187	0.19526
12	2.15656	2.39028	0.23372
13	2.17447	2.44051	0.26604
14	2.19069	2.48430	0.29361
15	2.20549	2.52292	0.31743
16	2.21907	2.55730	0.33823
17	2.23162	2.58817	0.35655
18	2.24326	2.61609	0.37283
19	2.25410	2.64150	0.38740
20	2.26424	2.66476	0.40052

(b) The bound of Fujikawa and Sugawa [16, Theorem 1, p. 97] is

$$K(d) \ \le \ \frac{4\big(1 + (d-2)\,q\big)}{d+1} \ = \ 4 - \frac{4\big((d-2)(1-q)+2\big)}{d+1}, \qquad q := 4^{-1/(d-1)}.$$

Since $1 - q \le (\log 4)/(d-1)$, the bracket in the numerator satisfies $(d-2)(1-q)+2 < \log 4 + 2 < 3.39$. The function $d \mapsto (d+1)/\log d$ increases for $d \ge 4$, and at $d = 8$ it equals $9/\log 8 > 4.32 > 3.39$. So the subtracted fraction is less than $4 \cdot 3.39/(d+1) < 4/\log d$ for $d \ge 8$. Hence $4 - 4/\log d$ is smaller than the bound of Fujikawa and Sugawa.

(c) The bounds $4(d-1)/(d+1)$ of Conte, Fujikawa and Lakic [5], $4^{(d-2)/(d-1)} = 4q$ of Beardon, Minda and Ng [3], and $\frac{d-1}{d}\,4q$ of Dubinin [10, 13] are all at least the bound of Fujikawa and Sugawa, since the three differences are $4(d-2)(1-q)/(d+1) \ge 0$ (as $q \le 1$), $4(3q-1)/(d+1) > 0$ (as $q \ge 1/2$ for $d \ge 3$) and $4\big((2d-1)q - d\big)/\big(d(d+1)\big) > 0$ (as $q \ge 4^{-1/7} > 0.8 > d/(2d-1)$ for $d \ge 8$). (Fujikawa and Sugawa compare the first two in [16, Remark (i), p. 97].) Hence, by (b), $4 - 4/\log d$ is also smaller than these three bounds.

(d) The numerical bounds of Crane at the nine degrees $d = 8, 9, 10, 11, 12, 16, 25, 100, 10^4$ [7] are 3.124, 3.150, 3.169, 3.188, 3.208, 3.271, 3.365, 3.621, 3.955. Each exceeds $4 - 4/\log d$ at the same degree by more than 0.389; the smallest difference is at $d = 10^4$.

Remark 3.3 (The best choice of y). For every $d \ge 8$ the proof of Corollary 1.2 gives

$$\inf_{0<y<1} (4/y)\, e^{-\mu(y^{1/d})} < M(d) \le 4 - 4/\log d.$$

Table 2 shows at four degrees how far the bound of Theorem 1.1, at a value of y close to the best one, lies below $4 - 4/\log d$, and Proposition 3.4 below shows that for the best y this difference is asymptotic to $(2\pi^2 - 4)/\log d$ (Figure 4). By that proposition, Theorem 1.1 gives $\liminf_{d\to\infty} (4 - K(d)) \log d \ge 2\pi^2$, while Corollary 1.2 gives $(4 - K(d)) \log d > 4$. Theorem 1.3 gives more for large d: $(4 - K(d)) \log d$ tends to infinity.

Proposition 3.4. *Let $B(d) := \inf_{0<y<1}(4/y)\,e^{-\mu(y^{1/d})}$ be the best bound that Theorem 1.1 gives at degree d. Then $(4 - B(d))\log d \to 2\pi^2$ as $d \to \infty$. More precisely,*

$$(4 - B(d))\log d \;\geq\; 2\pi^2 - O\Big(\frac{\log\log d}{\log d}\Big) \quad (d \to \infty),$$

$$(4 - B(d))\log d \;\leq\; \frac{2\pi^2}{1 - \pi^2/(d\log^2 d)} \quad (d \geq 8).$$

Proof. Let $d \geq 8$. Write $\ell := \log d$ and $y = e^{-L}$ with $L > 0$, and put $r := e^{-L/d}$, $r' := \sqrt{1-r^2}$. Write $\beta_d(L) := 4\exp(L - \mu(r))$ for the bound of Theorem 1.1, so that $B(d) = \inf_{L>0}\beta_d(L)$. By Lemma 2.1(b), $\mu(r) = \pi^2/(4\mu(r'))$.

The lower bound for $(4 - B(d))\log d$. Take $L := \pi^2/(2\ell^2)$. As in Step 3 of the proof of Corollary 1.2, $\mu(r') < \log(4/r') \leq \Lambda/2$ with $\Lambda := \log\big(8d/(L(1 - L/d))\big) = \ell + 2\log\ell + O(1)$. Hence $\log(B(d)/4) \leq L - \pi^2/(2\Lambda) = -\pi^2/(2\ell) + O(\log\ell/\ell^2)$, and $(4 - B(d))\,\ell \geq 2\pi^2 - O(\log\ell/\ell)$.

The upper bound for $(4 - B(d))\log d$. We show that for every $L > 0$, $\beta_d(L)$ is at least $4\exp\big(-\pi^2/(2\ell - 2\pi^2/(\ell d))\big)$; the denominator is positive, since $d\log^2 d \geq 8\log^2 8 > \pi^2$.

For $0 < L < 2$ we first bound $\mu(r')$ from below, and so $\mu(r)$ from above. The left inequality of [8, (19.9.5)] (displayed in the proof of Lemma 2.1(c)) at $k = r'$, whose complementary modulus is r, gives $\mu(r') > \log(4/r') - 2\log\big(2/(1+\sqrt{r})\big) \geq \log(4/r') - L/d$: indeed $\sqrt{r} = e^{-L/(2d)} \geq 1 - L/(2d)$, so $2/(1+\sqrt{r}) \leq 1/(1-\sigma)$ with $\sigma := L/(4d) \leq \frac{1}{2}$, and $-\log(1-\sigma) \leq 2\sigma$ for $0 \leq \sigma \leq \frac{1}{2}$. Also $r'^2 = 1 - e^{-2L/d} \leq 2L/d$, so $\log(4/r') \geq \frac{1}{2}\log(8d/L)$. Hence

$$4\mu(r') \;>\; \Delta_L \;:=\; 2\log\frac{8d}{L} - \frac{4L}{d}.$$

As $0 < L < 2$, we have $\Delta_L > 2\log(4d) - 8/d > 0$. Since $4\mu(r') > \Delta_L > 0$, we get $\mu(r) = \pi^2/(4\mu(r')) < \pi^2/\Delta_L$, and $\beta_d(L)$ exceeds $4\exp(L - \pi^2/\Delta_L)$.

There are three cases.

(a) $L \geq 2$. Lemma 2.1(c) at $s = r$ gives $e^{-\mu(r)} > r/4$, so $\beta_d(L)$ exceeds $e^{(1-1/d)L} \geq e^{7/4} > 4$.

(b) $\pi^2/(2\ell) \leq L < 2$. Then $\Delta_L > 2\log(4d) - 8/d > 2\ell$, so $L - \pi^2/\Delta_L > 0$ and $\beta_d(L)$ exceeds 4.

(c) $0 < L < \min(2, \pi^2/(2\ell))$. Then $\log(8d/L) > \ell + \log(16\ell/\pi^2) \geq \ell$ and $4L/d < 2\pi^2/(\ell d)$, so $\Delta_L > 2\ell - 2\pi^2/(\ell d) > 0$, and $\beta_d(L)$ exceeds $4\exp\big(-\pi^2/(2\ell - 2\pi^2/(\ell d))\big)$.

In cases (a) and (b), $\beta_d(L)$ exceeds 4, so in all three cases $\beta_d(L)$ is at least the last bound of case (c), and so is $B(d)$:

$$B(d) \;\geq\; 4\exp\Big(-\frac{\pi^2}{2\ell - 2\pi^2/(\ell d)}\Big).$$

With $1 - e^{-t} \leq t$ for $t \geq 0$ this gives the second inequality of the proposition, whose right-hand side tends to $2\pi^2$. With the lower bound, $(4 - B(d))\log d \to 2\pi^2$. □

Remark 3.5 (The range $d \geq 8$, and bounds for μ). Theorem 1.1 does not give Corollary 1.2 at $d = 7$: the number $B(7)$ of Proposition 3.4 lies between 2.0190 and 2.0191, while $4 - 4/\log 7 < 1.9445$. It does give $K(7) \leq B(7) < 2.0191$; at $d = 7$ the smallest of the bounds quoted in the Introduction is 2.4842…, that of Fujikawa and Sugawa. At $d = 8$, the number $B(8)$ lies between 2.0549 and 2.0550; so $(4 - B(8))\log 8$, the left end of the curve in Figure 4, is less than 4.045.

The bounds 2.0190 and 2.0191 for $B(7)$, and 2.0549 and 2.0550 for $B(8)$, are computed in interval arithmetic, with upper and lower bounds for μ from the arithmetic-geometric mean. For $0 < k < 1$ let $a_0 = 1$, $b_0 = k$, $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = \sqrt{a_n b_n}$ [8, (19.8.1)]. By the inequality between the arithmetic and the geometric mean, $b_n \leq b_{n+1} \leq a_{n+1} \leq a_n$ and $a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = (a_n - b_n)/2$; so the two sequences have a common limit $\operatorname{agm}(1,k)$, and $b_n \leq \operatorname{agm}(1,k) \leq a_n$ for every n. By [8, (19.8.5)], $\mathcal{K}(k) = \pi/(2\operatorname{agm}(1,k'))$, so $\mu(r) = \pi\operatorname{agm}(1,r')/(2\operatorname{agm}(1,r))$. With $\beta_d(L)$ as in the proof of Proposition 3.4, defined by the same formula for $d = 7$, the upper bounds hold

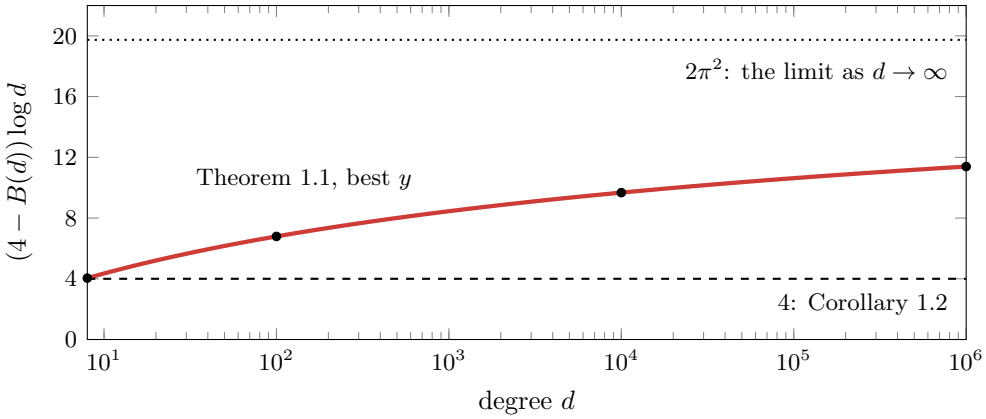

FIGURE 4. The number $(4 - B(d)) \log d$ for $8 \le d \le 10^6$, where $B(d)$ is the infimum over $0 < y < 1$ of the bound of Theorem 1.1 (Proposition 3.4). Since $K(d) \le B(d)$, Theorem 1.1 gives $K(d) \le 4 - h/\log d$, where h is the height of the curve at d. The curve is computed in floating point. The dots are $(4 - T) \log d$, where T is the entry in the column "Theorem 1.1" of Table 2 at $d = 8$, 100, 10^4, 10^6; since $B(d) \le T$, each dot lies on or below the curve. The dashed line is the constant 4 of Corollary 1.2; the curve lies above it at every $d \ge 8$ (Remark 3.3), by less than 0.05 at $d = 8$ (Remark 3.5), and tends to $2\pi^2$ (dotted line) as $d \to \infty$ (Proposition 3.4).

because $\beta_8(0.12766) < 2.0550$ and $\beta_7(0.13621) < 2.0191$. For the lower bounds, first let $L \ge 1$: Lemma 2.1(c) gives $e^{-\mu(e^{-L/d})} > e^{-L/d}/4$, so $\beta_d(L) > e^{(d-1)L/d} \ge e^{6/7} > 2.35$. On an interval $L_1 < L \le L_2$ with $0 \le L_1 < L_2 \le 1$ we have $\beta_d(L) \ge 4e^{L_1} e^{-\mu(e^{-L_2/d})}$, as μ is decreasing. Starting from $(0, 1]$, we halve an interval as long as this lower bound is at most 2.0549 for $d = 8$, or at most 2.0190 for $d = 7$. The halving stops with 1735 intervals for $d = 8$ and 644 for $d = 7$; on every one of them this lower bound exceeds 2.0549 (for $d = 8$) or 2.0190 (for $d = 7$). This proves the two lower bounds.

4. The gap is at least of order $1/\sqrt{\log d}$

In this section we prove Theorem 1.3. It follows from the next proposition. For a polynomial p with $p(0) = 0$, $p'(0) \ne 0$ and $S(p) > 0$ we write

$$\lambda(p) \;:=\; \log \frac{4}{S(p)} \;\ge\; 0.$$

Proposition 4.1. *Let p be a polynomial of degree $d \ge 2$ with $p(0) = 0$, $p'(0) \ne 0$ and $S(p) > 0$, and let $\lambda := \lambda(p)$. If $\lambda < \lambda_0 := 3 \cdot 10^{-5}$, then*

$$\log d \;\ge\; \frac{c_3}{\lambda^2}, \qquad c_3 := 2 \cdot 10^{-4}.$$

Proof of Theorem 1.3 from Proposition 4.1. Let p be as in the definition of $K(d)$. If $S(p) = 0$, then $4 - S(p) = 4$. Otherwise $4 - S(p) = 4(1 - e^{-\lambda})$ with $\lambda = \lambda(p)$. If $\lambda \ge \lambda_0$, then $4 - S(p) \ge 4(1 - e^{-3 \cdot 10^{-5}}) > 1.1 \cdot 10^{-4} = c_1$. If $\lambda < \lambda_0$, then $1 - e^{-\lambda} \ge \lambda(1 - \lambda/2)$ gives $4 - S(p) \ge 3.99\,\lambda$, and Proposition 4.1 gives $\lambda \ge \sqrt{c_3/\log d}$; so $4 - S(p) \ge 3.99\sqrt{c_3}/\sqrt{\log d} > 0.056/\sqrt{\log d} = c_2/\sqrt{\log d}$. In every case $4 - S(p) \ge \min(c_1, c_2/\sqrt{\log d})$, and taking the infimum over p gives the theorem. □

Remark 4.2 (Size of the constants). Theorem 1.3 improves on Corollary 1.2 exactly when $\log d > 4/c_1 = 36{,}363.6\ldots$, and on the sharper bound of Theorem 1.1 (Remark 3.3) only for still larger d; its second term is the smaller one exactly when $\log d > (c_2/c_1)^2 = 259{,}173.5\ldots$. The constants c_1, c_2, c_3, λ_0 can be improved; Remark 4.12 says which steps fix them.

4.1. **The idea of the proof.** The Introduction sketches the proof. Its steps, in order: Smale's inequality places the critical points of small critical value near $x = -1$ (Lemma 4.6); Koebe's theorem makes the cuts dense (Lemma 4.8), and the area identity (Lemma 4.4) makes most angles typical (§4.5); the curves from the cuts separate many dead ends, one of small ρ-area (§4.6, Lemma 4.9); and in that dead end a Green function and an estimate of Beurling type (§4.7, Lemma 4.10) give $\log d \geq c_3/\lambda^2$.

Remark 4.3 (Credits). Using Smale's condition at several critical points at once is due to Crane [7]: his Lemma 2.1 gives many critical values of small modulus, and his Lemmas 2.3 and 3.1 place the corresponding critical points. Lemma 4.6 below is the form of those lemmas used here, and Lemma 4.8 is a variant of the counting idea of his Lemma 2.1. Crane's argument ends in a sum over the reciprocals of the critical points, which gives the order $1/\sqrt{d}$; the argument here ends in a dead end of small ρ-area, in which the function $h - t$ of §4.7 is d times a Green function (19), bounded from above by an estimate of Beurling type (Lemma 4.10); this gives the order $1/\sqrt{\log d}$. The area identity of Lemma 4.4 is formula (1.10) of Grinshpan [21], who derives it from the area theorem of Lebedev [24, p. 229]; the bound for $|X|$ in the proof of Lemma 4.5 (the Cauchy–Schwarz inequality with the area identity) is the one in the proof of [21, Theorem 2.5]; the confocal ellipses of Lemma 4.5 are those of Crane [7, Lemma 3.2].

4.2. **Set-up: the inverse branch and the wall.** Fix p as in Proposition 4.1 and write $S := S(p)$. The quotient $S(p)$ does not change if p is replaced by αp, $\alpha \neq 0$. So we may assume that the least modulus of a critical value of p is 1, and we fix a critical point ζ_m with $|p(\zeta_m)| = 1$. All critical values of p then have modulus at least 1, so the open unit disc $\mathbb{D}$ contains none of them, and there is a branch F of p^{-1} on $\mathbb{D}$ with $F(0) = 0$. Since $p(F(w)) = w$, the map F is univalent and $F(\mathbb{D})$ contains no critical point of p. Then $|F'(0)| = 1/|p'(0)|$. Put

$$S_1 := \frac{|F'(0)|}{|\zeta_m|}, \qquad \lambda := \log \frac{4}{S}.$$

The quotient of p at ζ_m is $|p(\zeta_m)|/(|\zeta_m|\,|p'(0)|) = S_1$, so $S \leq S_1$ and $\lambda \geq \log(4/S_1)$. The point ζ_m is not in $F(\mathbb{D})$, because $|p(\zeta_m)| = 1$. Koebe's one-quarter theorem [15, Theorem 2.3] gives $|\zeta_m| \geq |F'(0)|/4$, that is, $S_1 \leq 4$; and $S_1 < 4$, since equality would make F a rotation of the Koebe function, whereas $F(\mathbb{D})$ is bounded. So $\log(4/S_1) > 0$.

We use the coordinate $x := -\zeta_m/z$ on the Riemann sphere. It sends $z = \infty$ to $x = 0$, $z = 0$ to $x = \infty$, and ζ_m to $x = -1$.

The wall. The map F extends continuously to $\overline{\mathbb{D}}$: as $w \to w_0 \in \mathbb{T}$, every limit point of $F(w)$ lies in the finite set $p^{-1}(w_0)$, and the set of limit points is connected, so it is one point. The identity $p \circ F = \mathrm{id}$ persists on $\overline{\mathbb{D}}$, so F is injective there, and $\overline{F(\mathbb{D})}$ is a closed Jordan disc on the sphere. Let $\mathcal{C}$ be the image under x of the complement of $F(\mathbb{D})$ in the sphere, and let E be the interior of $\mathcal{C}$. Then $\mathcal{C}$ is a closed Jordan disc and E a Jordan domain; E contains $x = 0$ (the image of $z = \infty$), and $\mathcal{C}$ contains $x = -1$ (the image of ζ_m). We call the boundary of $\mathcal{C}$, the image of $F(\mathbb{T})$, the *wall*.

The metric and the angle. On $\mathbb{C} \setminus \{0, -1\}$ put

$$\rho(x) := \frac{1}{\sqrt{|x|\,|x+1|}}, \qquad \theta(x) := \arccos\bigl(|x+1| - |x|\bigr) \in [0, \pi].$$

Lengths and areas measured with the length element $\rho(x)\,|dx|$ are called ρ-lengths and ρ-areas. On the ray $x \leq -1$ we have $\theta = \pi$.

4.3. **The logarithmic chart and the area identity.** The equation $x = (j-1)^2/(4j)$ has two solutions j and $1/j$ for each x; they coincide only at $x = 0$ (where $j = 1$) and at $x = -1$ (where $j = -1$). The sphere minus $\mathcal{C}$ is simply connected and contains neither of these points, so j has a single-valued branch there with $j \sim 4x$ as $x \to \infty$. On $0 < |w| < 1$ put

$$Y(w) \;:=\; \log j\bigl(x(F(w))\bigr),$$

a function that is single-valued modulo $2\pi i$. As $w \to 0$ we have $F(w) \sim F'(0)\,w$, so $x \sim -\zeta_m/(F'(0)\,w)$ and $j \sim -4\zeta_m/(F'(0)\,w)$. Hence

$$Y(w) \;=\; \log\frac{1}{w} + c_0 + X(w), \qquad X(w) = \sum_{n\ge 1} c_n w^n, \qquad \operatorname{Re} c_0 = \log\frac{4}{S_1}, \tag{9}$$

with X holomorphic in $\mathbb{D}$. Write $Y = a + ib$. On every branch, $j = e^Y$ gives

$$x = \sinh^2\frac{Y}{2}, \qquad \rho(x)\,|dx| = |dY|, \qquad |x| = \frac{\cosh a - \cos b}{2}, \qquad |x+1| - |x| = \cos b. \tag{10}$$

Indeed $(e^Y-1)^2/(4e^Y) = \sinh^2(Y/2)$, $dx = \frac{1}{2}\sinh Y\,dY$ and $|x|\,|x+1| = |\sinh(Y/2)\cosh(Y/2)|^2 = \frac{1}{4}|\sinh Y|^2$. So $\theta(x) = \operatorname{dist}(b, 2\pi\mathbb{Z})$; in particular θ is 1-Lipschitz for the ρ-length, and for each $\vartheta \in (0,\pi)$ the set $\{\theta \ge \vartheta\}$ is connected and unbounded (it is the image of the strip $\{|b-\pi| \le \pi - \vartheta\}$ and contains the ray $x \le -1$). At $w = re^{i\varphi}$ we have $\theta(x(F(w))) = \operatorname{dist}(\Theta, 2\pi\mathbb{Z})$ with

$$\Theta \;:=\; \varphi + \varphi_0 - \operatorname{Im} X(w), \qquad \varphi_0 := -\operatorname{Im} c_0. \tag{11}$$

Lemma 4.4 (Area)**.** *The ρ-area of $\mathcal{C}$ equals $2\pi\log(4/S_1) - \pi\sum_{n\ge1} n|c_n|^2$. Hence*

$$\rho\text{-area}(\mathcal{C}) \;\le\; A := 2\pi\lambda, \qquad \sum_{n\ge1} n|c_n|^2 \;\le\; 2\log\frac{4}{S_1} \;\le\; 2\lambda.$$

Proof. The map $g := j \circ x \circ F$ is univalent on $\mathbb{D}$ with a simple pole at 0. Let $U_g := g(\mathbb{D})$. The sets U_g and $1/U_g := \{1/u : u \in U_g\}$ are disjoint: if $g(w_1) = 1/g(w_2)$, then $x(F(w_1)) = x(F(w_2))$, because x takes the same value at j and $1/j$; so $w_1 = w_2$ and $g(w_1) = \pm 1$, that is, $x(F(w_1)) \in \{0,-1\} \subset \mathcal{C}$, which is impossible. On the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$ with coordinate $Y = \log j$, the map $Y \mapsto x = \sinh^2(Y/2)$ is two-to-one, it identifies Y with $-Y$, and $|dY| = \rho\,|dx|$. The preimage of $x(F(\mathbb{D}))$ is the disjoint union of $\log U_g$ and $-\log U_g$, so the preimage of $\mathcal{C}$ is the rest of the cylinder, and its area is $2\,\rho$-area$(\mathcal{C})$. By the symmetry $Y \mapsto -Y$, half of this area is the area of the part of the preimage in $\{a > 0\}$, which equals the area of $\{a>0\}\setminus \log U_g$ minus the area of $\log U_g \cap \{a<0\}$ (the mirror image of the latter lies in the former and is not in the preimage). This is the signed area between the line $\{a=0\}$ and the curve $\Gamma := Y(\mathbb{T})$. For $r<1$ the same argument, with $r\mathbb{D}$ in place of $\mathbb{D}$, shows that the signed area between $\{a=0\}$ and the curve $Y(\{|w|=r\})$ is the ρ-area of $\mathcal{C}_r$, the image under x of the complement of $F(r\mathbb{D})$; by Green's theorem and (9) it is $2\pi(\operatorname{Re} c_0 + \log(1/r)) - \pi\sum_n n|c_n|^2 r^{2n}$. The sets $\mathcal{C}_r$ decrease to $\mathcal{C}$ as $r \to 1$ (their intersection is the image under x of the complement of $F(\mathbb{D})$), and each has finite ρ-area (it is a compact set of the x-plane, as $x=\infty$ is the image of $z=0$, and $\rho^2 = 1/(|x|\,|x+1|)$ is integrable on compact sets); so their areas tend to that of $\mathcal{C}$, and the sum increases to $\sum_n n|c_n|^2$. The two inequalities follow, as the area is nonnegative and $\log(4/S_1) \le \lambda$. □

Lemma 4.5 (A thin confocal ellipse)**.** *Let $\beta,\delta>0$ satisfy $2\log(4/S_1)\,\log\bigl(1/(1-e^{-2\beta})\bigr) < \delta^2$. Then every Y_* with $\operatorname{Re} Y_* \ge \operatorname{Re} c_0 + \beta + \delta$ is a value of Y on $0<|w|<1$ (modulo $2\pi i$). Consequently, for every $x_* \in \mathcal{C}$, each of its two values $\pm Y_{x_*}$ in the chart satisfies $|\operatorname{Re} Y_{x_*}| < \log(4/S_1) + \beta + \delta$.*

Proof. For $|w| \le e^{-\beta}$, the Cauchy–Schwarz inequality and Lemma 4.4 give

$$|X(w)|^2 \;\le\; \sum_n n|c_n|^2 \cdot \sum_n \frac{|w|^{2n}}{n} \;\le\; 2\log\frac{4}{S_1}\,\log\frac{1}{1-e^{-2\beta}} \;<\; \delta^2.$$

Put $v_* := Y_* - c_0$, so $\operatorname{Re} v_* \ge \beta + \delta$. On the closed disc $|v - v_*| \le \delta$ we have $\operatorname{Re} v \ge \beta$, so $|X(e^{-v})| < \delta = |v - v_*|$ on its boundary circle. By Rouché's theorem, $v - v_* + X(e^{-v})$ has a zero v in the disc, and then $Y(e^{-v}) = Y_*$ modulo $2\pi i$, with $0 < |e^{-v}| < 1$. If $x_* \in \mathcal{C}$, then $x_* \notin x(F(\mathbb{D}))$, so neither Y_{x_*} nor $-Y_{x_*}$ is a value of Y; hence both have real part less than $\operatorname{Re} c_0 + \beta + \delta = \log(4/S_1) + \beta + \delta$. □

By (10), the condition $|\operatorname{Re} Y| \le a_0$ means $|x| + |x+1| \le \cosh a_0$: an ellipse with foci 0 and -1.

The numbers. From now on $\lambda < \lambda_0 = 3 \cdot 10^{-5}$. Then $\beta = \delta = 0.015$ satisfy the hypothesis of Lemma 4.5, since $2\lambda_0 \log(1/(1-e^{-0.03})) \le 6 \cdot 10^{-5} \cdot 3.5216 < 2.12 \cdot 10^{-4} < 2.25 \cdot 10^{-4}$. Hence, at every point of $\mathcal{C}$,

$$|\operatorname{Re} Y| \;<\; \lambda + 0.03 \;\le\; 0.0301. \tag{12}$$

By (10), $\mathcal{C} \subset \{|x| \le (\cosh 0.0301 + 1)/2\} \subset \{|x| < 2\}$. Moreover, at a point of $\mathcal{C}$ with $|x| \ge 0.9961$ we get $-\cos b = 2|x| - \cosh a \ge 2 \cdot 0.9961 - \cosh 0.0301 > 0.9917$. Put $\theta_e := 7\pi/8$. Then

$$x \in \mathcal{C},\ |x| \ge 0.9961 \quad\Longrightarrow\quad \theta(x) > \arccos(-0.9917) > 3 > \theta_e. \tag{13}$$

4.4. **Continuing the branch; cuts.** The branch F continues along each ray from 0 until the ray meets a critical point of p. For $w = e^{s+i\varphi}$ with $s \ge 0$ we write $f(s,\varphi)$ for the continued value of $F(w)$, in the coordinate x, and Q for the set of (s,φ) reached. Since $p \circ F = \mathrm{id}$, the map f is injective on Q, and conformal on the part of Q with $s > 0$ (as a function of $s + i\varphi$). Its values lie in $\mathcal{C}$, because there $|p| = e^s \ge 1$; so the ρ-area of $f(Q)$, which is $\iint_Q |f'|^2\, ds\, d\varphi$ with $|f'|$ measured in ρ, is at most A. Put

$$h(x) \;:=\; \log\bigl|p(-\zeta_m/x)\bigr|, \qquad \text{so that} \qquad h(f(s,\varphi)) = s.$$

Cuts. A *cut* over the angle φ_c at level $s_c \ge 0$ is a critical point z_c at which the ray of angle φ_c stops at level s_c; we write x_c for it in the coordinate x, and γ_c for the closed arc $f([0,s_c] \times \{\varphi_c\})$ from its foot $f(0,\varphi_c)$ on the wall to x_c. On γ_c we have $h \le s_c$. There are finitely many cuts.

Lemma 4.6 (Smale's inequality at a cut)**.** *Let $\lambda < \lambda_0$. At a cut of level $s_c \le 128\lambda$ we have $|x_c| \ge 0.9961$ and $\theta(x_c) > \theta_e$.*

Proof. The quotient of p at z_c is $|p(z_c)|/(|z_c|\,|p'(0)|) = e^{s_c}|F'(0)|/|z_c|$, and it is at least S. So $|x_c| = |\zeta_m|/|z_c| \ge (S/S_1)\, e^{-s_c} \ge e^{-\lambda - s_c}$, because $S/S_1 \ge S/4 = e^{-\lambda}$. As $\lambda + s_c \le 129\lambda_0$, this is at least $e^{-129\lambda_0} > 0.9961$. The point x_c is in $\mathcal{C}$, since z_c is a critical point and so not in $F(\mathbb{D})$; now apply (13). □

Boundary values. The function Y is continuous on $\overline{\mathbb{D}} \setminus \{0\}$: the branch j is continuous up to the wall away from $x = -1$, both values of j tend to -1 at $x = -1$, and $x = 0$ is an interior point of $\mathcal{C}$.

A maximal inequality. Let $|\cdot|$ be Lebesgue measure on $[0, 2\pi)$ (not normalized). For $\chi \in L^1$ of the circle let $M\chi(\varphi)$ be the supremum of $\frac{1}{|J|}\int_J |\chi|$ over all arcs $J \ni \varphi$ of length $|J| \le 2\pi$.

Lemma 4.7. *For $\chi \in L^1$ of the circle and $\tau > 0$: $\bigl|\{\varphi : M\chi(\varphi) > \tau\}\bigr| \le \frac{2}{\tau}\int_0^{2\pi} |\chi|\, d\varphi$.*

Proof. An arc $J \ni \varphi$ splits at φ into two arcs, one ending and one starting at φ, and the mean of $|\chi|$ over J is at most the larger of the two means. So $\{M\chi > \tau\} \subset \{M_+\chi > \tau\} \cup \{M_-\chi > \tau\}$, where $M_+\chi(\varphi)$ is the supremum of the means over $[\varphi, \varphi + u]$, $0 < u \le 2\pi$, and $M_-\chi$ the same to the left; it suffices to show $|\{M_+\chi > \tau\}| \le \|\chi\|_1/\tau$, and M_- is alike. If $\|\chi\|_1 \ge 2\pi\tau$ there is nothing to prove. Otherwise the continuous function $\Upsilon(\varphi) := \int_0^\varphi (|\chi| - \tau)$ on $\mathbb{R}$ (with $|\chi|$ extended periodically) satisfies $\Upsilon(\varphi + 2\pi) = \Upsilon(\varphi) - \kappa$ with $\kappa > 0$, so $\Upsilon \to -\infty$ as $\varphi \to +\infty$, and $\{M_+\chi > \tau\}$ is contained in the periodic set $\mathcal{O} := \{\varphi :\ \Upsilon(\psi) > \Upsilon(\varphi) \text{ for some } \psi > \varphi\}$. Let φ^* be a point where Υ attains its maximum over $[0,\infty)$. Then $\varphi^* \notin \mathcal{O}$, and $\Upsilon \le \Upsilon(\varphi^* + 2\pi)$ on $[\varphi^* + 2\pi, \infty)$; so $\mathcal{O} \cap (\varphi^*, \varphi^* + 2\pi)$ is the set of $\varphi \in (\varphi^*, \varphi^* + 2\pi)$ with $\Upsilon(\psi) > \Upsilon(\varphi)$ for some $\psi \in (\varphi, \varphi^* + 2\pi]$. By the rising sun lemma of F. Riesz [30] (for a continuous function on a closed interval, this set is a disjoint union of open intervals (a_k, b_k) with $\Upsilon(a_k) \le \Upsilon(b_k)$), we get $\tau(b_k - a_k) \le \int_{a_k}^{b_k} |\chi|$, and summing over k gives $|\mathcal{O} \cap (\varphi^*, \varphi^* + 2\pi)| \le \|\chi\|_1/\tau$. □

Two estimates for X. Let $X^*(\varphi) := \sup_{0<r<1} |X(re^{i\varphi})|$ be the radial maximal function. Then

$$\bigl|\{\varphi : X^*(\varphi) > \eta\}\bigr| \;\le\; \frac{C_H\,\lambda}{\eta^2} \quad (\eta > 0), \qquad C_H := 32\pi, \tag{14}$$

where $|\cdot|$ is Lebesgue measure on $[0,2\pi)$. Indeed, X is the Poisson integral of its boundary function, which lies in L^2 as $\sum_n |c_n|^2 < \infty$. The Poisson kernel $P_r(\psi)$ is even and decreasing in $|\psi| \in [0,\pi]$, so it is an average of the normalized indicator functions of arcs centered at 0; hence $X^* \le M|X|$. By Lemma 4.7 applied to $|X|\mathbf{1}_{\{|X|>\eta/2\}}$ with $\tau = \eta/2$, and since $|X| \le |X|\mathbf{1}_{\{|X|>\eta/2\}} + \eta/2$,

$$\big|\{M|X| > \eta\}\big| \;\le\; \frac{4}{\eta}\int_{\{|X|>\eta/2\}} |X|\,d\varphi \;\le\; \frac{8}{\eta^2}\int_0^{2\pi} |X|^2\,d\varphi \;=\; \frac{16\pi}{\eta^2}\sum_n |c_n|^2 \;\le\; \frac{32\pi\lambda}{\eta^2},$$

by Lemma 4.4. On the circle $|w| = 1/2$, the Cauchy–Schwarz inequality and Lemma 4.4 give

$$|X(w)|^2 \;\le\; \sum_n n|c_n|^2 \cdot \sum_n \frac{4^{-n}}{n} \;\le\; 2\lambda \log\frac{4}{3} \;\le\; 0.58\,\lambda. \tag{15}$$

Length and area. For an arc J of angles and a level $T > 0$, let Area_J be the ρ-area of $f(Q \cap ([0,T]\times J))$, and call the *line* at φ up to level T the image under f of $\{(s,\varphi) : 0 \le s \le T\} \cap Q$. By the Cauchy–Schwarz inequality, the square of its ρ-length is at most $T\int_0^T |f'(s,\varphi)|^2\,ds$, so, by Chebyshev's inequality,

$$\big|\{\varphi \in J:\ \text{the line at } \varphi \text{ up to level } T \text{ has } \rho\text{-length} \ge \Delta\}\big| \;\le\; \frac{T\,\mathrm{Area}_J}{\Delta^2}. \tag{16}$$

4.5. **Cuts are dense.** For $t > 0$, a *gap* is a component of the circle of angles minus the angles of all cuts of level at most t (the whole circle if there is no such cut).

Lemma 4.8 (Cuts are dense)**.** *For every $t > 0$,*

$$\sum_{\text{gaps } G}\int_G \min\big(t, \mathrm{dist}(\varphi,\partial G)\big)\,d\varphi \;\le\; \frac{2\pi}{3-2\sqrt{2}}\log\frac{4}{S_1} \;\le\; 36.63\,\log\frac{4}{S_1}.$$

In particular, the gaps of length at least $2t$ have total length at most $73.3\,\log(4/S_1)/t$.

Proof. Let $U := \mathbb{D} \cup \{re^{i\varphi} : 1 \le r < e^t,\ \varphi \in \text{a gap}\}$, a domain that is star-shaped about 0. The branch F continues along each ray in U, since a ray stops only at a cut; these continuations along rays define one holomorphic function on U, since the continuation along a path from 0 gives the same germ at the end of every path from 0 that stays close enough to it, and so the rays near a given ray give the same germs near its end point; as $p \circ F = \mathrm{id}$, it is univalent on U, and $F(U)$ contains no critical point; in particular it omits ζ_m. Let $\Phi : \mathbb{D} \to U$ be conformal with $\Phi(0) = 0$, and $R := |\Phi'(0)|$. Koebe's one-quarter theorem for $F \circ \Phi$ gives $R\,|F'(0)|/4 \le |\zeta_m|$, that is, $\log R \le \log(4/S_1)$.

Let $g_U(\cdot,0)$ be the Green function of U with pole at 0, and $h_U := g_U(\cdot,0) + \log|\cdot|$. It is harmonic on U, with $h_U(0) = \log R$. Since $|w| \ge 1$ on ∂U, the minimum principle gives $h_U \ge 0$ on U; and $h_U - \log|w| = g_U \ge 0$. In the variable $v = \log w = s + i\varphi$ this says $h_U \ge s$ on $(0,t)\times G$ for each gap G.

Fix φ in a gap G and let $r_0 := \min(t, \mathrm{dist}(\varphi,\partial G))$. The disc of radius r_0 about $v = i\varphi$ is mapped into U by $v \mapsto e^v$: its left half into $\mathbb{D}$, its right half into $(0,t)\times G$. The function $v \mapsto h_U(e^v)$ is harmonic and nonnegative on this disc (no injectivity is needed). Harnack's inequality [29] at the point $ar_0 + i\varphi$, $0 < a < 1$, gives $ar_0 \le h_U \le \frac{1+a}{1-a}\,h_U(e^{i\varphi})$; with $a = \sqrt{2}-1$ this is $h_U(e^{i\varphi}) \ge (3-2\sqrt{2})\,r_0$.

By the mean value property on $|w| = r < 1$ and Fatou's lemma as $r \to 1$ ($h_U \ge 0$, and h_U is continuous at the points $e^{i\varphi}$ with φ in a gap, which lie in U),

$$\log\frac{4}{S_1} \;\ge\; \log R \;=\; h_U(0) \;\ge\; \frac{1}{2\pi}\sum_G\int_G h_U(e^{i\varphi})\,d\varphi \;\ge\; \frac{3-2\sqrt{2}}{2\pi}\sum_G\int_G \min\big(t,\mathrm{dist}(\varphi,\partial G)\big)\,d\varphi.$$

A gap of length $l \ge 2t$ has $\int_G \min(t,\mathrm{dist}) = t(l-t) \ge tl/2$; so does the whole circle ($2\pi t \ge t\cdot 2\pi/2$). This gives the last claim, as $2 \cdot 36.63 \le 73.3$. □

Typical angles and good arcs. Fix

$$I := \{\varphi :\ \varphi + \varphi_0 \in [\pi/8, \pi/2]\}, \qquad |I| = \frac{3\pi}{8}, \qquad \eta := \frac{\pi}{16}, \qquad \Delta := \frac{\pi}{64}, \qquad t := 128\lambda.$$

An angle $\varphi \in I$ is *typical* if (T1) $X^*(\varphi) \le \eta$; (T2) the line at φ up to level $2t$ has ρ-length less than Δ; and (T3) there is no cut of level at most $2t$ over φ itself. By (14) and (16) with $J = I$, $T = 2t$ and $\mathrm{Area}_I \le A$, and since (T3) excludes finitely many angles, the set of angles in I that are not typical has measure at most

$$\frac{C_H\lambda}{\eta^2} + \frac{2tA}{\Delta^2} \ \le\ 2607.6\,\lambda + 667545\,\lambda^2 \ \le\ 0.079.$$

Split I into $M' := \lfloor |I|/(2t) \rfloor$ closed arcs J of equal length. Then $M' \ge 153$, and $2t \le |J| \le \frac{154}{153}\,2t \le 257.7\,\lambda$. Call J *good* if its interior contains the angle of a cut of level at most t and J contains a typical angle. If J is not good, then either its interior lies in a gap, which then has length at least $|J| \ge 2t$, or J contains no typical angle. By Lemma 4.8 with $\log(4/S_1) \le \lambda$ and $t = 128\lambda$, the arcs of the first kind have total length at most $73.3/128 \le 0.5727$; those of the second kind, at most 0.079. So the good arcs have total length at least $3\pi/8 - 0.5727 - 0.079 \ge 0.526$, and their number N_g satisfies

$$N_g \ \ge\ \frac{0.526}{257.7\,\lambda} \ \ge\ \frac{2.04\cdot 10^{-3}}{\lambda} \ \ge\ 68. \tag{17}$$

4.6. **Dead ends.** Let $\varphi_1 < \varphi_2 < \dots$ be the angles in I of all cuts of level at most t, with arcs $\gamma_1, \gamma_2, \dots$ and critical points $x_1, x_2, \dots$. By Lemma 4.6, $\theta(x_k) > \theta_e$ for each k.

For a typical φ let $x_\varphi := f(2t, \varphi)$ (it exists by (T3)), and let $e(\varphi)$ be the image, in the coordinate x, of the ray of angle φ from $|w| = 1/2$ up to $|w| = e^{2t}$ under F and its continuation f; it is a path ending at x_φ. On $e(\varphi)$ we have $\theta \le \pi/2 + \eta + \Delta < \theta_e$: on $1/2 \le |w| \le 1$ by (T1) and (11), since $\varphi + \varphi_0 \in [\pi/8, \pi/2]$ and $|\operatorname{Im} X| \le \eta$; beyond the wall by (T2), since θ is 1-Lipschitz for the ρ-length.

Let $E' := E \setminus \bigcup_k \gamma_k$ and $T_e := \{\theta < \theta_e\}$. For typical φ, the point x_φ lies in $E' \cap T_e$ (we have $h(x_\varphi) = 2t$, while $h = 0$ on the wall and $h \le t$ on each γ_k). Let $\Omega(\varphi)$ be the component of $E' \cap T_e$ that contains x_φ.

$\Omega(\varphi)$ *is simply connected.* The complement of E' in the sphere is $\overline{x(F(\mathbb{D}))}$ (which contains $x = \infty$) together with arcs γ_k attached to it, so it is connected. The complement of T_e is $\{\theta \ge \theta_e\} \cup \{\infty\}$, connected since $\{\theta \ge \theta_e\}$ is connected and unbounded. The two share the point ∞, so their union is connected. Every component of $E' \cap T_e$ other than $\Omega(\varphi)$ has its boundary in this union. Hence the complement of $\Omega(\varphi)$ is connected. The boundary of $\Omega(\varphi)$ lies in the wall, the arcs γ_k, and the curve $\{\theta = \theta_e\}$.

We call the domains $\Omega(\varphi)$ *dead ends*; Figure 5 shows one.

Lemma 4.9 (Separation)**.** *Let $\varphi < \varphi_c < \varphi'$, where $\varphi, \varphi' \in I$ are typical and $\varphi_c \in I$ is the angle of a cut of level at most t. Then $\Omega(\varphi) \ne \Omega(\varphi')$.*

Proof. Suppose $\Omega(\varphi) = \Omega(\varphi')$, and join x_φ to $x_{\varphi'}$ by an arc σ in $E' \cap T_e$. Shorten it: keep σ from its last point on $e(\varphi)$ to the next point on $e(\varphi')$, and cut the paths $e(\varphi)$, $e(\varphi')$ at these points. Let A_0 be the image under F of the arc of $|w| = 1/2$ between the angles φ and φ' inside I; it lies in T_e, by (15) and (11), since $\sqrt{0.58\lambda_0} < 0.0042$ and $\pi/2 + 0.0042 < \theta_e$. Then $\mathcal{J} := e(\varphi) + \sigma + e(\varphi') + A_0$ is a Jordan curve in T_e.

The set $\mathcal{D} := x(F(\{|w| \le 1/2\}))$ is a closed disc, and it meets $\mathcal{J}$ exactly in A_0: the paths $e(\cdot)$ start on its boundary and leave it at once, and $\sigma \subset E$. So $\mathcal{D} \setminus A_0$ is connected and does not meet $\mathcal{J}$; it lies in one complementary component of $\mathcal{J}$, which we call side 2. The point $x = \infty$ is an interior point of $\mathcal{D}$, so it lies in side 2.

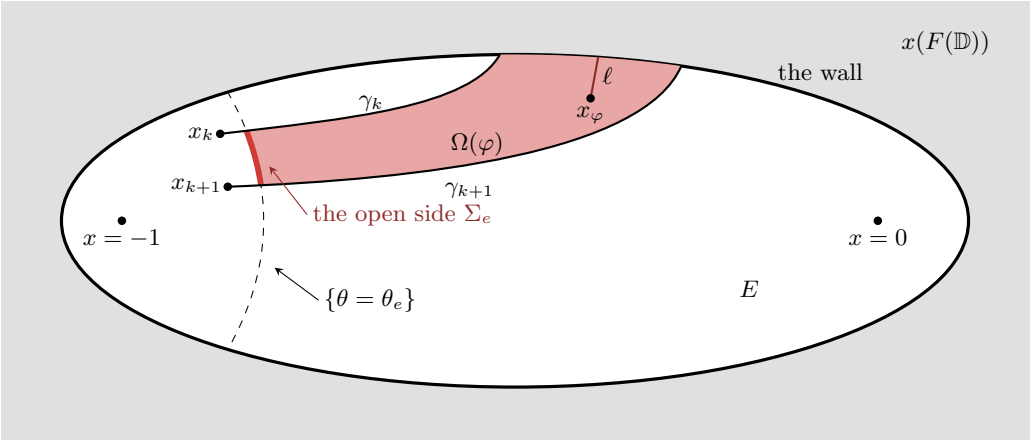

FIGURE 5. A dead end (schematic, not to scale). The set $\mathcal{C}$ (white; its interior is E) is bounded by the wall; in the proof it is very thin, since it lies in the ellipse $|x|+|x+1| \le \cosh 0.0301$. The arcs γ_k and γ_{k+1} run from the wall to the critical points x_k and x_{k+1} of two cuts; these points lie beyond the curve $\{\theta=\theta_e\}$, on the side of $x=-1$. The dead end $\Omega(\varphi)$ (shaded) is bounded by the wall, by the two arcs and by a piece of that curve, its open side Σ_e (in color). Where $h>t$, the function $h-t$ is d times a Green function with pole at $x=0$; at the point x_φ it is at most $d\,M_0$ times the harmonic measure of Σ_e in $\Omega(\varphi)$, by (20), and Lemma 4.10 bounds that harmonic measure by means of the line ℓ from x_φ to the wall.

The image of the ray of angle φ_c, from $|w|=1/2$ to x_c, starts at an interior point of A_0 and leaves $\mathcal{D}$ at once, so it enters side 1. It meets $\mathcal{J}$ nowhere else: F is injective; its part in $x(F(\mathbb{D}))$ is not in E, so not on σ; its part beyond the wall is γ_c, which is not on $\sigma \subset E'$. So x_c lies in side 1.

The set $\{\theta \ge \theta_e\}$ is connected, does not meet $\mathcal{J} \subset T_e$, and is unbounded, so it meets every neighborhood of ∞, which lies in side 2. Hence it lies in side 2. But it contains x_c, which lies in side 1: a contradiction. □

The proof uses only that the γ_k are pairwise disjoint (the function $x \mapsto p(-\zeta_m/x)$ maps γ_k into the ray of argument φ_k, and the φ_k differ), each a Jordan arc from the wall to x_k, or the single point x_k on the wall for a cut of level 0; their shape is otherwise arbitrary.

Counting dead ends. List the good arcs in order, $J_{(1)} < J_{(2)} < \cdots < J_{(N_g)}$, and choose a typical angle $\varphi_i \in J_{(2i)}$ for $1 \le i \le N_g/2$. For $i < i'$, the interior of $J_{(2i+1)}$ lies between φ_i and $\varphi_{i'}$ and contains the angle of a cut of level at most t, so Lemma 4.9 shows that the domains $\Omega(\varphi_i)$ are pairwise different. They are disjoint subsets of $\mathcal{C}$, so their ρ-areas add up to at most A. At most one of them contains $x=0$. So at least $N := \lfloor N_g/2 \rfloor - 1 \ge N_g/2 - 2$ of them do not contain 0; since $N_g \ge 68$, $N \ge 0.47\,N_g$. At least half of these N domains have ρ-area at most

$$\frac{2A}{N} \;\le\; \frac{4\pi\lambda}{0.47 \cdot 2.04 \cdot 10^{-3}/\lambda} \;\le\; 13200\,\lambda^2. \tag{18}$$

4.7. **The Green function in a dead end.** Let $W := \{x : h(x) > t\}$. The function $h-t$ is harmonic on $W \setminus \{0\}$ and vanishes on ∂W, so by the maximum principle every component of W contains the pole $x=0$ of h; hence W is connected. Since $h > t > 0$ on W, the domain W contains no zero of $p(-\zeta_m/x)$ and lies in E. As p has degree d, $h(x) = d\log(1/|x|) + O(1)$ as $x \to 0$. Hence

$$h - t \;=\; d\,g_W(\cdot,0) \quad \text{on } W, \tag{19}$$

where $g_W(\cdot,0)$ is the Green function of W with pole at 0.

Take a dead end $\Omega := \Omega(\varphi)$ that does not contain 0 and has ρ-area at most $13200\lambda^2$. Then $x_\varphi \in W$ and $h(x_\varphi) - t = t$. Let Ω' be the component of $\Omega \cap W$ that contains x_φ. A boundary point of Ω' lies either on ∂W, where $g_W(\cdot, 0) = 0$, or on $\partial\Omega$ inside W. In the second case it is not on the wall ($h = 0$ there) and not on any γ_k ($h \le t$ there), so it lies on $\Sigma_e := \partial\Omega \cap \{\theta = \theta_e\}$. On $\{\theta = \theta_e\}$ we have $|x| \ge \sin^2(7\pi/16)$ by (10), and $W \subset E \subset \{|x| < 2\}$ (the cruder bound $|x| < 2$ suffices here), so there

$$g_W(x, 0) \;\le\; g_E(x, 0) \;\le\; \log\frac{2}{|x|} \;\le\; M_0 := 0.74.$$

Recall the harmonic measure $\omega(z_0, \Sigma, D)$ of $\Sigma \subset \partial D$ in D at z_0 (Section 2.3). The harmonic measure of Σ_e in Ω tends to 1 at every boundary point ξ of Ω' inside W: there $h(\xi) > t$, so the wall and the arcs γ_k stay at positive distance from ξ; near ξ the boundary of Ω lies on $\{\theta = \theta_e\}$ only, and Ω is simply connected. So the harmonic function $M_0\,\omega(\cdot, \Sigma_e, \Omega) - g_W(\cdot, 0)$ on Ω' has nonnegative boundary values at every boundary point of Ω', and by the maximum principle

$$t \;=\; d\,g_W(x_\varphi, 0) \;\le\; d\,M_0\,\omega(x_\varphi, \Sigma_e, \Omega). \tag{20}$$

Lemma 4.10 (Harmonic measure and extremal distance)**.** *Let Ω be a bounded simply connected domain, $z_0 \in \Omega$, ℓ a Jordan arc from z_0 to a point of $\partial\Omega$ that lies in Ω except for its end point, and Σ a closed subset of $\partial\Omega$. Let $\operatorname{ext}_\Omega(\ell, \Sigma)$ be the extremal distance in Ω between $\ell \cap \Omega$ and Σ (§2; its curves are the curves in Ω from a point of $\ell \cap \Omega$ to a point of Σ). Then*

$$\omega(z_0, \Sigma, \Omega) \;\le\; 4\,e^{-\pi \operatorname{ext}_\Omega(\ell, \Sigma)}.$$

Proof. A metric ϱ (a Borel function $\varrho \ge 0$) is *admissible* for a family of curves if each curve of the family has ϱ-length at least 1; the extremal distance of §2 is the reciprocal of the infimum of the areas of the admissible metrics.

Reduction to the disc. Let $\Phi\colon \mathbb{D} \to \Omega$ be the Riemann map with $\Phi(0) = z_0$. As Ω is bounded, Φ has a radial limit $\Phi^*(\zeta)$ at almost every $\zeta \in \mathbb{T}$ (Fatou's theorem [27]). Let Σ' be the set of $\zeta \in \mathbb{T}$ at which $\Phi^*(\zeta)$ exists and lies in Σ; it is a Borel set, as Φ^* is the pointwise limit of the continuous functions $\zeta \mapsto \Phi((1 - 1/k)\zeta)$ on the Borel set where that limit exists. Let m denote normalized length on $\mathbb{T}$, and put $\ell' := \Phi^{-1}(\ell \cap \Omega)$, a curve in $\mathbb{D}$ from 0 along which $|u| \to 1$.

(a) $\omega(z_0, \Sigma, \Omega) \le m(\Sigma')$. For $\epsilon > 0$ let $f_\epsilon\colon \partial\Omega \to [0, 1]$ be continuous, equal to 1 on Σ and to 0 at distance at least ϵ from Σ. Every point of $\partial\Omega$ lies on a continuum in the complement of Ω, so it is a regular point for the Dirichlet problem [29, §4.2]; hence the solution H_ϵ with boundary values f_ϵ is continuous on $\overline{\Omega}$, and $\omega(\cdot, \Sigma, \Omega) \le H_\epsilon$. The bounded harmonic function $H_\epsilon \circ \Phi$ has the radial limit $f_\epsilon(\Phi^*(\zeta))$ wherever $\Phi^*(\zeta)$ exists, so $H_\epsilon(z_0)$ is the mean of $f_\epsilon \circ \Phi^*$ over $\mathbb{T}$, which is at most $m(\{\zeta : \operatorname{dist}(\Phi^*(\zeta), \Sigma) < \epsilon\})$. As $\epsilon \to 0$ these sets decrease to Σ', because Σ is closed.

(b) $\operatorname{ext}_{\mathbb{D}}(\ell', \Sigma') \ge \operatorname{ext}_\Omega(\ell, \Sigma)$. Let ρ_Ω be an admissible metric for the curves in Ω from a point of $\ell \cap \Omega$ to a point of Σ, and let $\epsilon' > 0$. Let γ be a curve in $\mathbb{D}$ from a point of ℓ' to a point $\zeta \in \Sigma'$. If $\Phi \circ \gamma$ has infinite length, then the term $\epsilon'|\Phi'|$ of the metric below alone gives γ infinite length; this is why that term is added. If $\Phi \circ \gamma$ has finite length, it ends at a point $b \in \partial\Omega$; by Lindelöf's theorem [27] the radial limit of Φ at ζ is b, so $b \in \Sigma$, and $\Phi \circ \gamma$ is a curve in Ω from a point of $\ell \cap \Omega$ to a point of Σ, of ρ_Ω-length at least 1. Hence the metric $(\rho_\Omega \circ \Phi + \epsilon')\,|\Phi'|$ is admissible for the curves in $\mathbb{D}$ from a point of ℓ' to a point of Σ', and its area is $\iint_\Omega (\rho_\Omega + \epsilon')^2$, which tends to the area of ρ_Ω as $\epsilon' \to 0$, since Ω is bounded and ρ_Ω has finite area. This gives (b) (if no admissible metric of finite area exists, then $\operatorname{ext}_\Omega(\ell, \Sigma) = 0$ and (b) is trivial).

By (a) and (b) it suffices to show $m(\Sigma') \le 4e^{-\pi \operatorname{ext}_{\mathbb{D}}(\ell', \Sigma')}$, and we may assume $m(\Sigma') > 0$. As Σ' is a Borel set, $m(\Sigma')$ is the supremum of $m(B)$ over the compact sets $B \subset \Sigma'$. Let $B \subset \Sigma'$ be compact with $\nu := m(B) > 0$, and for $0 < r < 1$ let ℓ_r be the piece of ℓ' from 0 to its first point on $|u| = r$. Smaller sets are joined by fewer curves, so $\operatorname{ext}_{\mathbb{D}}(\ell', \Sigma') \le \operatorname{ext}_{\mathbb{D}}(\ell_r, B)$, and it suffices to show $\pi \operatorname{ext}_{\mathbb{D}}(\ell_r, B) \le \mu(s_r)$ with $s_r \to \sin(\pi\nu/2)$ as $r \to 1$: then $\pi \operatorname{ext}_{\mathbb{D}}(\ell', \Sigma') \le \mu(\sin(\pi\nu/2)) <$

$\log\bigl(4/\sin(\pi\nu/2)\bigr)$ by the continuity of μ and Lemma 2.1(c), so $\nu \le \sin(\pi\nu/2) < 4e^{-\pi\,\mathrm{ext}_{\mathbb{D}}(\ell',\Sigma')}$ (as $\nu \le \sin(\pi\nu/2)$ for $0 \le \nu \le 1$), and $m(\Sigma')$ is the supremum of such ν.

The disc. We use the symmetry rule of Garnett and Marshall [17, Ch. IV, (3.5), p. 137]: let D_1 be a domain that is symmetric in a line, let D be the part of D_1 on one side of the line, with D_1 minus the line equal to D together with its reflection, and let $A_1, A_2 \subset \partial D$; then the extremal distance in D_1 between A_1 and A_2, each together with its reflection, is half the extremal distance in D between A_1 and A_2. The curves of the second family lie in the open set D. By a Möbius map, the same holds with a circle in place of the line. We may assume $\nu < 1$ (if $m(\Sigma') = 1$, compact sets $B \subset \Sigma'$ with $\nu < 1$ still have ν arbitrarily close to 1). Let ℓ_r^* be the reflection of ℓ_r in $\mathbb{T}$. A curve in $\mathbb{D}$ from ℓ_r to B contains, from its last point on ℓ_r on, a curve in $\mathbb{D} \setminus \ell_r$ from ℓ_r to B; so $\mathrm{ext}_{\mathbb{D}}(\ell_r, B) = \mathrm{ext}_{\mathbb{D}\setminus\ell_r}(\ell_r, B)$. By the symmetry rule, with $D := \mathbb{D} \setminus \ell_r$ and $D_1 := \widehat{\mathbb{C}} \setminus (\ell_r \cup \ell_r^* \cup B)$ (a domain, as ℓ_r is a Jordan arc and $\mathbb{T} \setminus B \ne \emptyset$), $\mathrm{ext}_{\mathbb{D}}(\ell_r, B)$ is twice the extremal distance in the sphere between the plates $\ell_r \cup \ell_r^*$ and B. We use circular symmetrization of a condenser whose two plates are symmetrized about opposite rays, the first plate about the positive real axis and the second about the negative real axis: this does not increase the capacity of the condenser (see [9, p. 20, (1.16), with (1.12) on p. 17] and [12, §3.3]), so it does not decrease the extremal distance between its plates [38]. The compact set $\ell_r \cup \ell_r^*$ meets every circle $|u| = s$ with $0 \le s \le r$ or $1/r \le s \le \infty$, so its symmetrization contains $[0, r] \cup [1/r, \infty]$; replacing it by that set does not decrease the extremal distance. The set B becomes the arc J of $\mathbb{T}$ of normalized length ν centered at -1. By the symmetry rule again (with $D := \mathbb{D} \setminus [0, r]$, and as before $\mathrm{ext}_{\mathbb{D}}([0, r], J) = \mathrm{ext}_{\mathbb{D}\setminus[0,r]}([0, r], J)$), $\mathrm{ext}_{\mathbb{D}}(\ell_r, B)$ is at most $\mathrm{ext}_{\mathbb{D}}([0, r], J)$. By the symmetry rule in the real axis (with $D_1 := \mathbb{D} \setminus [0, r]$ and D the upper half-disc, using the same equality), this is half the extremal distance in the upper half-disc between $[0, r]$ and the upper half of J. The map $u \mapsto -(u + 1/u)/2$ sends the upper half-disc onto the upper half-plane, $[0, r]$ onto $[-\infty, -a_r]$ with $a_r := (r + 1/r)/2$, and the upper half of J onto $[c, 1]$ with $c := \cos(\pi\nu)$; by the symmetry rule once more (with D the upper half-plane), the extremal distance in the half-plane is twice that of the ring domain $\widehat{\mathbb{C}} \setminus \bigl([-\infty, -a_r] \cup [c, 1]\bigr)$, so the two factors 2 cancel. The Möbius map $u \mapsto (c - u)/(1 - c)$ (note $1 - c > 0$, as $\nu > 0$) sends this ring onto the Teichmüller ring $\widehat{\mathbb{C}} \setminus ([-1, 0] \cup [\Pi_r, \infty])$ with $\Pi_r := (c + a_r)/(1 - c)$, whose extremal distance between its two boundary components is $\frac{1}{\pi}\mu\bigl(s_r\bigr)$, $s_r := 1/\sqrt{1 + \Pi_r}$ [37, (2.13), (2.26)]. As $r \to 1$, $a_r \to 1$, $\Pi_r \to (1 + c)/(1 - c) = \cot^2(\pi\nu/2)$ and $s_r \to \sin(\pi\nu/2)$. □

Remark 4.11. Lemma 4.10 is a form of Beurling's estimate: if Ω is a Jordan domain and Σ a finite union of arcs of $\partial\Omega$, then $\omega(z_0, \Sigma, \Omega) \le \frac{8}{\pi} e^{-\pi\Lambda}$, where Λ is the supremum, over the Jordan arcs σ in Ω from z_0 to $\partial\Omega \setminus \Sigma$, of the extremal distance in $\Omega \setminus \sigma$ between σ and Σ [17, Ch. IV, Theorem 5.3, p. 146]. We give the proof above because here Ω need not be a Jordan domain and Σ is only a closed subset of $\partial\Omega$.

Proof of Proposition 4.1. Let Ω, x_φ be as above, and let ℓ be the line at φ from x_φ down to its foot on the wall. It is a Jordan arc, as f is injective, and it lies in Ω except for its endpoint on $\partial\Omega$, and on it $\theta \le \pi/2 + \eta + \Delta$. Every curve in Ω from ℓ to Σ_e therefore has ρ-length at least

$$L := \theta_e - \frac{\pi}{2} - \eta - \Delta = \frac{19\pi}{64},$$

because θ is 1-Lipschitz for the ρ-length. Testing the extremal distance with the metric ρ on Ω gives $\mathrm{ext}_\Omega(\ell, \Sigma_e) \ge L^2/\rho\text{-area}(\Omega) \ge L^2/(13200\lambda^2)$. By (20) and Lemma 4.10, $t \le 4\,d\,M_0\,e^{-\pi L^2/(13200\lambda^2)}$, that is, with $t = 128\lambda$,

$$\log d \;\ge\; \frac{\pi L^2}{13200\,\lambda^2} - \log\frac{4M_0}{128\,\lambda} \;\ge\; \frac{2.07\cdot 10^{-4}}{\lambda^2} - \log\frac{0.0232}{\lambda}.$$

For $\lambda < \lambda_0$ we have $\log(0.0232/\lambda) \le 7\cdot 10^{-6}/\lambda^2$, so $\log d \ge (2.07\cdot 10^{-4} - 7\cdot 10^{-6})/\lambda^2 = c_3/\lambda^2$. □

4.8. **Remarks.**

Remark 4.12 (Where λ_0 comes from). Two steps fix λ_0: the bound $N_g \geq 67$, needed for $N_g/2 - 2 \geq 0.47 N_g$ in §4.6 ((17) gives $N_g \geq 68$ for $\lambda < \lambda_0$; 67 suffices), holds up to about $\lambda = 3.04 \cdot 10^{-5}$; and the number 0.9961 in Lemma 4.6 holds up to about $\lambda = 3.03 \cdot 10^{-5}$. The measure of the non-typical angles is far from binding. The constants can be improved (the mean value property in place of Harnack's inequality in Lemma 4.8; one dead end in place of half of them; the arc I of length π); none of these changes the order $1/\sqrt{\log d}$.

Remark 4.13 (Where the method stops). The counting of §4.5 works only at the scale λ: Lemma 4.8 says nothing for t below about 12λ, since then $73.3\lambda/t > 2\pi$. The degree enters only as the strength of the pole of h in (19). This is why the method gives the order $1/\sqrt{\log d}$.

5. The gap is at most of order $\sqrt{\log\log\log d/\log d}$

In this section we prove Theorem 1.4. It follows from one construction, Proposition 5.1 below: for a small number $\lambda > 0$ and a number M of tracts it gives a polynomial H with $S(H) \geq 4e^{-7\lambda}$ and $\log\deg H \leq C_* M/\lambda$; with M of order $\log\log(1/\lambda)/\lambda$, as in the proofs below, $\log\deg H$ is of order $\log\log(1/\lambda)/\lambda^2$.

The construction follows the proof of $\lim_{d\to\infty} K(d) = 4$ in [19] (its Section 9, from §9.1 on, and its Appendix B, whose Proposition B.1 is Lemma 5.5 below): a Riemann surface over the plane, mapped onto a plane domain with thin channels by two Riemann maps, and then polynomials. That proof gives no rate: for a given $a < 4$ it gives no bound on the least d with $K(d) > a$ [19, p. 2], because its passage from the domain to polynomials uses Runge's theorem, which bounds no degree, and its Appendix B bounds the conformal radius of the surface in terms of the number of tracts, not $K(d)$ in terms of d. Here every step comes with an explicit bound, and the degree is counted.

Notation. The table lists, by subsection, the symbols that are used beyond the place where they are introduced; there they have the meaning given in the table. Inside one lemma or proof a letter of the table may be used with another meaning; it is then defined on the spot (for example w in the Faber lemma of §5.10 and in §5.11, and r and a in the proof of Lemma 5.14). Some letters keep one meaning throughout this section: g alone is always $2\pi/M$ (the Green functions are g_i and g_K), E alone is always a number (the ends are A_j on Σ and E_j on Ω), and F alone is always the map $s \mapsto k(rs)$ of §5.5, with k the Koebe function (the Faber polynomials are F_n). The letter q alone is $p - v_j$ on the tract T_j in §§5.3–5.4; with an index, $q_k = p - v_k$ throughout. Harmonic measure in a domain X is ω_X. Other letters are introduced where they are used; most of them are local to one subsection or one proof.

where	symbols
§5.1	λ, M, $g = 2\pi/M$; the constant C_*; $\Lambda^+ = 3\log(1/g) + 11.13$
§5.3	θ_j and $v_j = e^{i\theta_j}$; R; $\Lambda = \log(4R/g)$; the number E; the slits σ_j and the base B; the tracts T_j with the chart ℓ; $q_k = p - v_k$; the ends A_j; the surface Σ, its base point o and the projection p; the Riemann map $\zeta\colon \Sigma \to \mathbb{D}$ and the conformal radius r_Σ
§5.5	$T = M/\lambda$; $r = e^{-\lambda}$, $U = k(r\mathbb{D})$, $F(s) = k(rs)$, S_U; the coordinates $x = 1/(4z)$ and $Y = a + ib$; the targets β_j, the positions $b_j = -\tau_j$, the cube $\mathcal{Q}$; J, w, w_m, a_j; the ends E_j of the channels; Ω, its Riemann map Z, r_Ω, $z = Z \circ \zeta$, $\hat{f}$; $z_{\max}$
§5.7	$h_j = \omega_\Omega(\cdot, E_j)$, e_j; the last straight pieces Q_j
§5.8	the hyperbolic distance d_h (on Σ; in the proof of Lemma 5.15, on X_i); the chart radius $r_c(\xi)$; ϵ_0
§5.9	t_i, $K_i = Z(t_i\overline{\mathbb{D}})$, the Green functions g_i of $X_i = \widehat{\mathbb{C}} \setminus K_i$; δ_0; the level β_*
§5.10	for a compact set K: Ψ, $\Phi = \Psi^{-1}$, g_K, the level curves Γ_ρ, the Faber polynomials F_n

5.1. **The construction, and the proof of Theorem 1.4.**

Proposition 5.1 (The construction)**.** *Let* $0 < \lambda \le 0.02$*, let* $M \ge 1000$ *be a multiple of* 4*, and put* $g := 2\pi/M$*. Assume*

$$g \;\le\; \frac{\lambda}{10} \qquad \textit{and} \qquad g\Big(6 + \frac{2}{\pi}\log\Lambda^+\Big) \;\le\; \lambda, \quad \textit{where}\;\; \Lambda^+ := 3\log\frac{1}{g} + 11.13. \tag{21}$$

Then there is a polynomial H *of degree at least* 2 *with* $H(0) = 0$*,* $H'(0) \ne 0$*,*

$$S(H) \;\ge\; 4e^{-7\lambda} \qquad \textit{and} \qquad \log\deg H \;\le\; C_*\,\frac{M}{\lambda}, \qquad C_* := 6.3\cdot 10^4.$$

Proposition 5.1 is proved in §§5.3–5.11; we first derive Theorem 1.4 from it. We use that $K(d)$ does not decrease with d [6], [19, Appendix A], and that for $0 < \lambda \le 0.02$ the function

$$\lambda \;\longmapsto\; \frac{\log\log(1/\lambda)}{\lambda^2}$$

is continuous, strictly decreasing, and tends to $+\infty$ as $\lambda \to 0$.

Proof of Theorem 1.4. Let $d \ge \exp(1.04\cdot 10^{10})$. For $0 < \lambda \le 0.02$ write $L_1 := \log(1/\lambda) \ge \log 50 > 3.912$ and $L_2 := \log L_1 = \log\log(1/\lambda) \ge 1.3640$. At $\lambda = 0.02$,

$$48.1\,C_*\,\frac{L_2}{\lambda^2} \;=\; 48.1\cdot 6.3\cdot 10^4\cdot\frac{1.36405\ldots}{0.0004} \;<\; 1.0334\cdot 10^{10} \;\le\; \log d,$$

so there is a unique $\lambda \in (0, 0.02]$ with $48.1\,C_*L_2/\lambda^2 = \log d$. Let M be the least multiple of 4 with $M \ge 48L_2/\lambda$. We check (21). First, $M \ge 48\cdot 1.364/0.02 > 3273$, so $M \ge 1000$. Next, $g = 2\pi/M \le 2\pi\lambda/(48L_2) \le 0.096\,\lambda \le \lambda/10$. Then, as $4/(2\pi) \le 0.06\,L_2/\lambda$,

$$\frac{1}{g} = \frac{M}{2\pi} \;\le\; \frac{48L_2/\lambda + 4}{2\pi} \;\le\; 7.7\,\frac{L_2}{\lambda}, \qquad \text{so} \qquad \Lambda^+ \;\le\; 3L_1 + 3\log(7.7L_2) + 11.13 \;\le\; 7.7\,L_1;$$

the last inequality holds at $L_1 = \log 50$ (where the two sides are $29.92\ldots$ and $30.12\ldots$), and the difference $4.7L_1 - 3\log(7.7\log L_1) - 11.13$ increases with L_1 for $L_1 \ge \log 50$. Hence $\log\Lambda^+ \le L_2 + \log 7.7$ and

$$\frac{g}{\lambda}\Big(6 + \frac{2}{\pi}\log\Lambda^+\Big) \;\le\; \frac{2\pi}{48L_2}\Big(6 + \frac{2}{\pi}(L_2 + \log 7.7)\Big) \;=\; \frac{12\pi + 4\log 7.7 + 4L_2}{48L_2} \;\le\; 0.784,$$

since the middle term decreases in L_2 and is $0.7838\ldots$ at $L_2 = \log\log 50$. So Proposition 5.1 applies and gives H with $S(H) \ge 4e^{-7\lambda}$ and, as $4 \le 0.1\,L_2/\lambda$,

$$\log\deg H \;\le\; \frac{C_*}{\lambda}\Big(\frac{48L_2}{\lambda} + 4\Big) \;\le\; 48.1\,C_*\,\frac{L_2}{\lambda^2} \;=\; \log d.$$

So $\deg H \le d$, and since K does not decrease,

$$K(d) \;\ge\; K(\deg H) \;\ge\; S(H) \;\ge\; 4e^{-7\lambda} \;\ge\; 4 - 28\lambda \;\ge\; 4 - 28\cdot 0.02 \;=\; 3.44.$$

Finally, $48.1\,C_*L_2 \ge 1$, so $1/\lambda^2 \le \log d$, that is $L_1 \le \frac{1}{2}\log\log d$, and $L_2 \le \log\log\log d - \log 2 < \log\log\log d$. Hence $\lambda = (48.1\,C_*L_2/\log d)^{1/2} \le (48.1\,C_*\log\log\log d/\log d)^{1/2}$ and

$$K(d) \;\ge\; 4 - 28\,(48.1\,C_*)^{1/2}\Big(\frac{\log\log\log d}{\log d}\Big)^{1/2}, \qquad 28\,(48.1\,C_*)^{1/2} = 48741.7\ldots \;\le\; 4.9\cdot 10^4. \quad \square$$

Remark 5.2. The proof gives $K(d) \ge 4 - 28\lambda \ge 3.44$ for every $d \ge \exp(1.04\cdot 10^{10})$, whereas the bound of Theorem 1.4, at $d = \exp(1.04\cdot 10^{10})$, gives only $K(d) \ge 3.148$. There $28\lambda \le 0.56$, while the displayed bound for $4 - K(d)$ is $0.851\ldots$, about 1.52 times larger: the step $L_2 < \log\log\log d$ replaces $L_2 \approx 1.364$ by $\log\log\log d = 3.138\ldots$.

Remark 5.3 (A smaller constant for very large d). For every integer $d \geq \exp\big(\exp(7.01 \cdot 10^{18})\big)$, $K(d) \geq 4 - 1.6 \cdot 10^4 \sqrt{\log\log\log d / \log d}$. The proof is the argument above with 5 in place of 48. Fix $\lambda_1 \in (0, 0.02]$ so small that $L_2 = \log\log(1/\lambda) \geq 12\pi + 5$ for every $\lambda \leq \lambda_1$, and let $d_0 > \exp(\exp(e))$ be so large that $5.01\, C_* \log\log(1/\lambda_1)/\lambda_1^2 \leq \log d_0$. Let $d \geq d_0$, let $\lambda \leq \lambda_1$ be the number with $5.01\, C_* L_2/\lambda^2 = \log d$, and let M be the least multiple of 4 with $M \geq 5L_2/\lambda$. Then $M \geq 1000$, and $g \leq 2\pi\lambda/(5L_2) \leq \lambda/10$ as $L_2 \geq 4\pi$. As $4/(2\pi) \leq 0.2\, L_2/\lambda$, we have $1/g \leq (5L_2/\lambda + 4)/(2\pi) \leq L_2/\lambda$, so $\Lambda^+ \leq 3L_1 + 3\log L_2 + 11.13 = 3L_1\,(1 + (\log L_2 + 3.71)/L_1)$ and $\log \Lambda^+ \leq L_2 + \log 3 + 0.1$, as $L_1 \geq e^{42}$. Hence

$$\frac{g}{\lambda}\Big(6 + \frac{2}{\pi}\log\Lambda^+\Big) \;\leq\; \frac{12\pi + 4\log 3 + 0.4 + 4L_2}{5L_2} \;\leq\; 1,$$

because $12\pi + 4\log 3 + 0.4 < 12\pi + 5 \leq L_2$. Proposition 5.1 gives H with $S(H) \geq 4e^{-7\lambda}$ and, as $4 \leq 0.01\, L_2/\lambda$, $\log\deg H \leq C_*(5L_2/\lambda + 4)/\lambda \leq 5.01\, C_* L_2/\lambda^2 = \log d$. As before, $K(d) \geq 4 - 28\lambda$, $L_2 < \log\log\log d$, and

$$K(d) \;\geq\; 4 - 28\,(5.01\, C_*)^{1/2}\Big(\frac{\log\log\log d}{\log d}\Big)^{1/2}, \qquad 28\,(5.01\, C_*)^{1/2} = 15730.6\ldots \;<\; 1.6 \cdot 10^4.$$

One may take $\lambda_1 := \exp(-e^{12\pi+5})$ and $d_0 := \exp(\exp(7.01 \cdot 10^{18}))$, since $\log\big(5.01\, C_*(12\pi + 5)\big) + 2e^{12\pi+5} < 16.5 + 6.999 \cdot 10^{18} < 7.01 \cdot 10^{18}$; this value is admissible, not the least one.

5.2. **The idea of the proof of Proposition 5.1.** Smale's argument gives $S(p) \leq 4$ by applying Koebe's one-quarter theorem to the inverse branch of p at 0. To get close to 4 we build the inverse branch first. (i) A simply connected Riemann surface Σ over the p-plane, with no branch point, on whose boundary $|p|$ is large except on M short arcs, the ends A_j, where $|p|$ is about 1; the ends sit at the far ends of M logarithmic tracts attached along radial slits from M points v_j spaced $2\pi/M$ apart on the unit circle (§5.3; Figure 6, left). Its conformal radius r_Σ at the base point is at most e^λ: attaching the tracts costs little (Lemma 5.5). (ii) A domain Ω: the image U under the Koebe function of the disc of radius $e^{-\lambda}$, with M thin channels attached, each running from ∂U through the complement of U to near the point of ∂U nearest 0, where $|z|$ is about $1/4$ (Figure 6, right). (iii) The composition z of the two Riemann maps, $\Sigma \to \mathbb{D} \to \Omega$, positioned (by Brouwer's fixed point theorem) so that each end A_j is sent into the far end of channel j. Then the normalized map $\hat f := (r_\Omega/r_\Sigma)\, p \circ z^{-1}$ of (24), with $\hat f'(0) = 1$, has $|\hat f(z)/z|$ near 4 on $\partial\Omega$: on the far ends $|p| \approx 1$ and $|z| \approx 1/4$, and elsewhere $|p|$ is large. (iv) Polynomials $H = P + z\mathcal{V}^m$, with P a Faber partial sum of $\hat f$ and $\mathcal{V}$ a scaled Faber polynomial of a compact subset of Ω, as in Jatar and Ng's passage from such $\hat f$ to polynomials [23, Proposition 4.3].

We give explicit bounds for the quantities that the degree count uses. In the logarithmic chart of Ω the complement of U is a rectangle, so channels of width $w \approx \lambda/M$ fit. The harmonic measures, levels and Green function values that the degree count uses are bounded below by $\exp(-CT)$, with $T := M/\lambda$, and every degree is at most $\exp(CT)$; so $\log\deg H \leq C_* T$. Lemma 5.5 gives $\log r_\Sigma \leq \lambda$ once M is a large multiple of $\log\log(1/\lambda)/\lambda$, and so $\log\deg H$ of order $\log\log(1/\lambda)/\lambda^2$ suffices.

5.3. **The surface.** Let $M \geq 4$ be an integer, $g := 2\pi/M$, and for j modulo M let

$$\theta_j := jg + \theta_*, \qquad v_j := e^{i\theta_j},$$

where the real number θ_* is fixed in §5.5; it only rotates the p-plane, $p \mapsto e^{i\theta_*}p$, so no modulus below depends on it. Let $R \geq 2$, put

$$\Lambda := \log\frac{4R}{g},$$

and let E be an odd multiple of π with $E \geq 2\Lambda^2$.

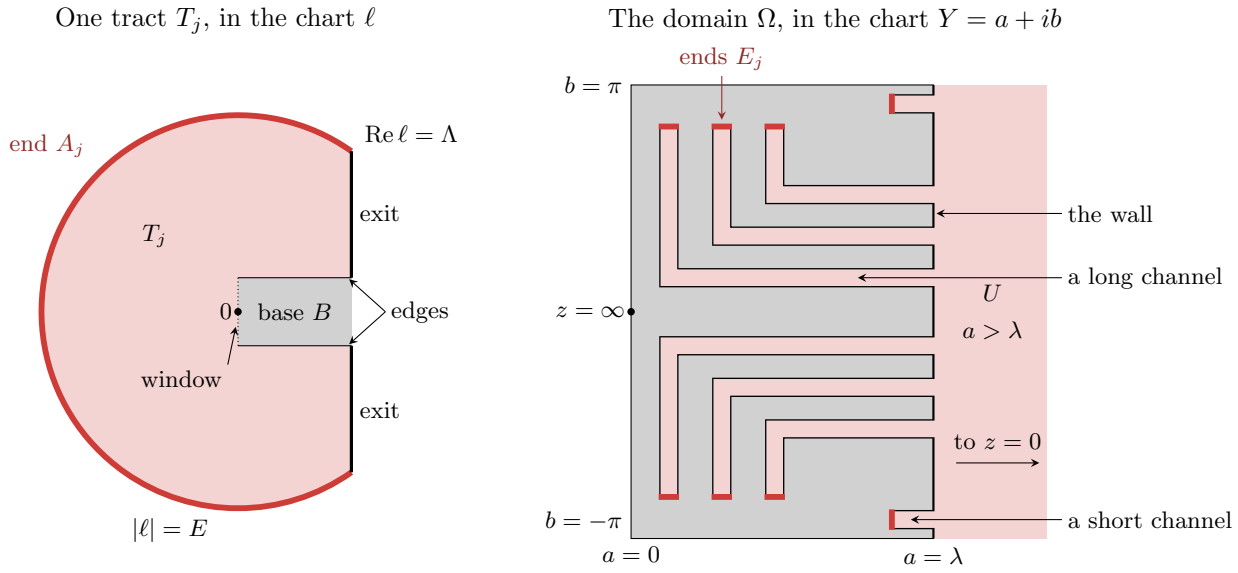

FIGURE 6. The two objects of the construction (schematic, not to scale). Left: one tract T_j of the surface Σ in the chart ℓ, the disc $|\ell| < E$ to the left of the line $\mathrm{Re}\,\ell = \Lambda$, minus the half-strip $\{\mathrm{Re}\,\ell \geq 0,\ |\mathrm{Im}\,\ell| \leq \pi\}$ (gray). Its strip $\{\mathrm{Re}\,\ell < 0,\ |\mathrm{Im}\,\ell| < \pi\}$ is identified with a part of B near v_j; through the window it opens onto the rest of B, and its edges are glued to the two sides of the slit. Its boundary pieces on $\partial\Sigma$ are the two exits, where $|p| \geq R - 1$, and the end A_j (in color), where $|p| \geq 1 - e^{-\sqrt{E}/2}$ (Lemma 5.4). In the proof E is far larger than Λ. Right: the domain Ω in the chart $Y = a + ib$ (three long channels and one short channel are drawn in each half; there are M in all). It consists of $U = \{a > \lambda\}$ and thin channels, attached along the wall $a = \lambda$, that run through the complement of U (gray) to their ends E_j (in color) near $b = \pm\pi$, where $|z| \leq (1 + 1.2\lambda)/4$. The necks and wedges at the mouths of the channels are not drawn. The map $z = Z \circ \zeta$ sends each end A_j of the surface into the end E_j of channel j.

The base. Let $\sigma_j := \{tv_j : 1 \leq t \leq 1 + R\}$ (the j-th slit) and

$$B := \{|p| < 1 + R\} \setminus \bigcup_j \sigma_j,$$

a simply connected domain.

The tracts. Let $\Pi := \mathbb{C} \setminus \{\ell : \mathrm{Re}\,\ell \geq 0,\ |\mathrm{Im}\,\ell| \leq \pi\}$. The j-th tract is a copy of

$$T_j := \{\ell \in \Pi :\ \mathrm{Re}\,\ell < \Lambda,\ |\ell| < E\},$$

on which the projection is

$$p = v_j + q, \qquad q := \frac{g}{4}\, e^{i(\theta_j + \pi)}\, e^{\ell}.$$

So $\arg q = \theta_j$ (modulo 2π) exactly where $\mathrm{Im}\,\ell$ is an odd multiple of π, and $|q| = (g/4)e^{\mathrm{Re}\,\ell} < R$ on T_j, because $e^{\Lambda} = 4R/g$. Wherever p is defined we write $q_k := p - v_k$; so $q_j = q$ on T_j.

Gluing. The strip $\{\mathrm{Re}\,\ell < 0,\ |\mathrm{Im}\,\ell| < \pi\}$ of Π is mapped by p bijectively onto $B \cap \{|q| < g/4\}$ (the disc $|q| < g/4$ minus the slit). The points of $B \cap \{|q| < g/4\}$ whose coordinate ℓ in this strip has $|\ell| \geq E$ form the *cap* at v_j,

$$\left\{0 < |q| \leq \tfrac{g}{4}\, e^{-(E^2 - (\mathrm{Im}\,\ell)^2)^{1/2}}\right\}, \qquad \mathrm{Im}\,\ell = \arg q - \theta_j - \pi \in (-\pi, \pi),$$

a closed subset of B around the inner end v_j of the slit σ_j. We remove the caps from B, and the part of the strip with $|\ell| < E$ (which is the part of the strip in T_j) is identified with $B \cap \{|q| < g/4\}$ minus the caps, through p. The lines $\operatorname{Im}\ell = \pm\pi$, $\operatorname{Re}\ell < 0$, lie over the two sides of the slit inside this disc; in T_j they are interior lines. The *window* $\{\operatorname{Re}\ell = 0,\ |\operatorname{Im}\ell| < \pi\}$ lies over the circle $|q| = g/4$ minus its point on the slit, and through it T_j opens onto the rest of B. The *edges* $\{\operatorname{Im}\ell = \pm\pi,\ 0 < \operatorname{Re}\ell < \Lambda\}$ of T_j are glued to the two sides of σ_j where $g/4 < |q| < R$. The two points $\ell = \pm i\pi$ lie over $(1 + g/4)v_j$, one on each side of the slit; we add them as interior points (near each of them, three quadrants of the ℓ-chart lie in T_j and one in B). The surface Σ is B together with all the tracts, glued in this way; its *base point* o is the point of B over $p = 0$.

B stays simply connected when the caps are removed: a cap only thickens the inner end of σ_j, as it is connected and has $v_j \in \sigma_j$ in its closure, so the complement of B minus the caps in the sphere stays connected; and B minus the caps is connected, as moving away from v_j at fixed $\arg q$ never re-enters a cap. Each tract meets the rest along one connected set and all pieces (B minus the caps, and the tracts) are simply connected, so Σ is simply connected. The projection p is one analytic function on both sides of each gluing ($p = v_j + q$), so $p\colon \Sigma \to \mathbb{C}$ is locally univalent: Σ has no branch point. On Σ we have $|p| < 1 + R$. The zeros of p on Σ are o and, on each tract, the points $\ell = \log(4/g) + 2\pi i k$, $k \neq 0$, with $|\ell| < E$.

Symmetries. The rotation $p \mapsto e^{ig}p$, which takes T_j to T_{j+1}, and the reflection $p \mapsto v_j^2\,\overline{p}$ in the line through 0 and v_j, which acts on T_j as $\ell \mapsto \overline{\ell}$ and exchanges T_{j+k} and T_{j-k}, map Σ onto itself and fix o.

Boundary. The boundary of Σ is one Jordan curve. Over each j it consists, in cyclic order, of the upper exit, the end A_j, the lower exit, and an arc of the top:

piece	where	$\lvert p\rvert$	reason
top	$\lvert p\rvert = 1 + R$, between σ_j and σ_{j+1}	$= 1 + R$	definition
exits	$\operatorname{Re}\ell = \Lambda$, $\lvert\ell\rvert < E$, in T_j	$\geq R - 1$	$\lvert q\rvert = R$
end A_j	$\lvert\ell\rvert = E$, $\operatorname{Re}\ell < \Lambda$, in T_j	$\geq 1 - e^{-\sqrt{E}/2}$	Lemma 5.4*

*When $E \geq 100$ and $\Lambda \leq 0.04\sqrt{E}$; the E chosen in (34) satisfies both.

The top and an exit meet tangentially over $(1 + R)v_j$, where the circles $|q| = R$ and $|p| = 1 + R$ touch; an exit and the end meet at a corner. Nothing else lies on the boundary.

Lemma 5.4 (Near the ends)**.** *Assume $E \geq 100$ and $\Lambda \leq 0.04\sqrt{E}$, and put $B_0 := \sqrt{E}/2$. If $\xi \in \overline{T_j}$ satisfies $E - \frac{1}{2} \leq |\ell(\xi)| \leq E$ and $\operatorname{Re}\ell(\xi) \leq \Lambda$, then $|p(\xi)| \geq 1 - e^{-B_0}$. In particular $|p| \geq 1 - e^{-B_0}$ on A_j.*

Proof. Write $\ell = \ell(\xi) = r + is$. If $r \leq \log(4/g) - B_0$, then $|q| = (g/4)e^r \leq e^{-B_0}$ and $|p| \geq |v_j| - |q| \geq 1 - e^{-B_0}$. Otherwise $\log(4/g) - B_0 < r \leq \Lambda$; as $\log(4/g) > 0$, this gives $|r| < B_0 + \Lambda \leq 0.54\sqrt{E}$, so $r^2 \leq 0.2916\,E$. Then $|s| \leq |\ell| \leq E$, and

$$|s| \;=\; \bigl(|\ell|^2 - r^2\bigr)^{1/2} \;\geq\; |\ell| - \frac{r^2}{|\ell|} \;\geq\; E - \frac{1}{2} - \frac{0.2916\,E}{E - 1/2} \;\geq\; E - 0.8.$$

So $s = \pm(E - \rho')$ with $0 \leq \rho' \leq 0.8 < \pi/2$. As E is an odd multiple of π, $\arg q = \theta_j + \pi + s \equiv \theta_j \mp \rho'$ modulo 2π, so $p = v_j\bigl(1 + |q|e^{\mp i\rho'}\bigr)$ and $|p| \geq 1 + |q|\cos\rho' \geq 1$. □

The Riemann map. Σ is a simply connected Riemann surface (with the charts p and ℓ) that carries the non-constant bounded holomorphic function p; by the uniformization theorem it is conformally equivalent to $\mathbb{D}$ (not to $\mathbb{C}$, by Liouville's theorem, nor to the sphere, on which every holomorphic function is constant). Let $\zeta\colon \Sigma \to \mathbb{D}$ be the Riemann map with $\zeta(o) = 0$, normalized so that $p = r_\Sigma\,\zeta + O(\zeta^2)$ near o with $r_\Sigma > 0$; the number r_Σ is the conformal radius of Σ at o. It extends to a homeomorphism of $\overline{\Sigma}$ onto $\overline{\mathbb{D}}$, by Carathéodory's theorem applied locally. Near each

point η of $\partial\Sigma$ (corners included), in the chart p or ℓ, Σ contains a plane Jordan domain N_η whose boundary is an arc α of $\partial\Sigma$ about η together with an arc in Σ. Let ϕ_η map $\mathbb{D}$ conformally onto N_η; by Carathéodory's theorem ϕ_η extends to a homeomorphism of the closures. The univalent map $\zeta_\eta := \zeta \circ \phi_\eta \colon \mathbb{D} \to \mathbb{D}$ has $|\zeta_\eta| \to 1$ at the open arc $\phi_\eta^{-1}(\alpha)$ of $\mathbb{T}$, as ζ is a homeomorphism of Σ onto $\mathbb{D}$; so ζ_η extends analytically across that arc by reflection, and along it the derivative of $\arg\zeta_\eta$ in the angle equals the radial derivative of $\log|\zeta_\eta|$, which is positive by Hopf's lemma ($\log|\zeta_\eta|$ is negative in $\mathbb{D}$ and 0 on the arc); as $|\zeta_\eta'| = |\zeta_\eta|$ times that derivative on the arc, $\zeta_\eta' \neq 0$ there. Hence ζ extends continuously to $\partial\Sigma$, with values in $\mathbb{T}$, as a local homeomorphism of half-neighborhoods; it is injective on $\partial\Sigma$, since two points of $\partial\Sigma$ with one image would give two points of Σ with one image; and its image is $\overline{\mathbb{D}}$, as $\overline{\Sigma}$ is compact. By the uniqueness of the normalized Riemann map, the rotation of Σ becomes $\zeta \mapsto e^{ig}\zeta$, and the reflection in the line through v_j becomes $\zeta \mapsto e^{2i\theta_j}\overline{\zeta}$ (near o, $\zeta \approx p/r_\Sigma$ with r_Σ real). The fixed set of this reflection in Σ contains the curve that runs from o along the radius of B towards v_j and then along $\{\operatorname{Im}\ell = 0,\ -E < \operatorname{Re}\ell < 0\}$ in T_j, to the midpoint $\ell = -E$ of A_j. Its image lies on the diameter of $\mathbb{D}$ through $e^{i\theta_j}$, starts at 0 in the direction of $e^{i\theta_j}$, does not return to 0 (as ζ is injective), and tends to $\mathbb{T}$; so it ends at $e^{i\theta_j}$. As the reflection maps A_j onto itself, $\zeta(A_j)$ is an arc of $\mathbb{T}$ symmetric about $e^{i\theta_j}$:

$$\zeta(A_j) \text{ is an arc of } \mathbb{T} \text{ centered exactly at } e^{i\theta_j}, \text{ for every } E. \tag{22}$$

5.4. **The conformal radius of the surface.**

Lemma 5.5. *If $g \le 0.01$ and $\Lambda \ge 12$, then*

$$\log r_\Sigma \ \le\ g\Big(6 + \frac{2}{\pi}\log\Lambda\Big),$$

for every odd multiple E of π with $E \ge 2\Lambda^2$ and every θ_*.

This is [19, Proposition B.1]; we include the proof to keep this section self-contained; only the statement is used later.

Proof. Thomson's principle. Let $\epsilon > 0$ be small, and let D_ϵ be the component of $\{|p| \le \epsilon\}$ that contains o (the small discs around the other zeros of p are not removed). Let u be the harmonic function on $\Sigma \setminus D_\epsilon$ with $u = 1$ on ∂D_ϵ and $u = 0$ on $\partial\Sigma$. The extremal distance between ∂D_ϵ and $\partial\Sigma$ in $\Sigma \setminus D_\epsilon$ is $1/\|\nabla u\|^2$, where $\|\cdot\|$ is the L^2 norm; in the coordinate ζ, D_ϵ is a near-disc of radius $(1 + O(\epsilon))\epsilon/r_\Sigma$, so this extremal distance is $(2\pi)^{-1}\log(r_\Sigma/\epsilon) + o(1)$ as $\epsilon \to 0$. If V is a divergence-free vector field on $\Sigma \setminus D_\epsilon$ with unit flux out through ∂D_ϵ, all of which leaves through $\partial\Sigma$, then $1 = \left|\int V \cdot \nabla u\right| \le \|V\|\,\|\nabla u\|$ (as $\operatorname{div} V = 0$, the integral is the flux of uV through the boundary, and $u = 1$ on ∂D_ϵ, $u = 0$ on $\partial\Sigma$), and so

$$\frac{1}{2\pi}\log\frac{r_\Sigma}{\epsilon} + o(1) \ \le\ \|V\|^2. \tag{23}$$

The energy $\|V\|^2$ of such a field is conformally invariant (under a conformal change of coordinates, the field that carries the same fluxes across curves has the same energy), so each piece of V below may be computed in its own chart. The field is invariant under the rotation, so the flux through each period is $f_0 := 1/M = g/(2\pi)$.

The flow. (i) In $\{|p| < 1 - g\}$ the field is radial, with uniform density in $\arg p$. Its energy is at most $(2\pi)^{-1}\log(1/\epsilon)$.

(ii) In the cell $\{1 - g < |p| < 1,\ |\arg p - \theta_j| < g/2\}$, minus the part of the disc $\{|q| < g/4\}$ inside $\mathbb{D}$, the field carries the flux f_0 from the inner arc to the part of the semicircle $|q| = g/4$ inside $\mathbb{D}$, with no flux through the arcs of the unit circle (nothing enters the gaps between the slits) and none through the sides $\arg p = \theta_j \pm g/2$. Its energy is $\mathcal{E}_{\text{ii}} f_0^2$, with $\mathcal{E}_{\text{ii}}$ computed below.

(iii) In T_j the field runs from the part of the window that the field of (ii) crosses to two arcs, as described below. Its energy is $\mathcal{E}_{\text{iii}} f_0^2$.

(iv) Then the field runs radially, in a chart described below, out to the exits, which belong to $\partial\Sigma$. Its energy is at most $\big(\frac{2}{\pi}\log\Lambda+\mathcal{E}_{\rm iv}\big)f_0^2$ with $\mathcal{E}_{\rm iv}\le 0$.

The pieces fit (the normal fluxes agree across each interface), so V is divergence-free with unit flux, all of which leaves through the exits. Summing,

$$\|V\|^2 \;\le\; \frac{1}{2\pi}\log\frac{1}{\epsilon}+Mf_0^2\Big(\mathcal{E}_{\rm ii}+\mathcal{E}_{\rm iii}+\mathcal{E}_{\rm iv}+\frac{2}{\pi}\log\Lambda\Big),$$

and $2\pi Mf_0^2=g$. With (23) and $\epsilon\to 0$ this gives $\log r_\Sigma\le g(\mathcal{E}_{\rm ii}+\mathcal{E}_{\rm iii}+\mathcal{E}_{\rm iv}+\frac{2}{\pi}\log\Lambda)$. It remains to construct (ii)–(iv) and to show $\mathcal{E}_{\rm ii}+\mathcal{E}_{\rm iii}\le 6$ for $g\le 0.01$ and $\mathcal{E}_{\rm iv}\le 0$ for $\Lambda\ge 12$.

Cell (ii): a cone. Rotate so that $v_j=1$. Consider the rays $p=1-te^{i\vartheta}$ from v_j, $|\vartheta|\le\vartheta_0$. The ray ϑ meets the circle $|p|=1-g$ at

$$t=\rho(\vartheta):=\cos\vartheta-\big(\cos^2\vartheta-2g+g^2\big)^{1/2},$$

at the argument $-\varphi(\vartheta)$ (with φ odd and increasing), and ϑ_0 is fixed by $\varphi(\vartheta_0)=g/2$. The cone $\{1-te^{i\vartheta}:|\vartheta|\le\vartheta_0,\ g/4\le t\le\rho(\vartheta)\}$ lies in the cell: on the boundary ray $\vartheta=\vartheta_0$ (and symmetrically on $-\vartheta_0$), $|p|^2=1-2t\cos\vartheta_0+t^2$ is convex in t, and at the inner end its derivative in t is $2(1-g)\big(1-g-\cos(g/2)\big)/\rho(\vartheta_0)<0$; so $|p|$ decreases along the whole boundary ray, from below 1 to $1-g$, while $\arg p$ is monotone along it. The cone lies in $|p|<1$ (it has $t\ge g/4$, and $|p|^2<1$ for $0<t<2\cos\vartheta$), and it leaves through the part $|\vartheta|\le\vartheta_0$ of the semicircle $|q|=g/4$, which lies in $\mathbb{D}$, as $\cos\vartheta_0>g/8$ (on $|q|=g/4$, $|p|<1$ exactly where $\cos\vartheta>g/8$). Let the field be radial about v_j in the cone, zero in the rest of the cell, with flux $\psi(\vartheta)\,d\vartheta$ through the angle $d\vartheta$, in units of f_0, where

$$\psi(\vartheta):=\frac{1}{g}\,\varphi'(\vartheta).$$

Then the inflow on the inner arc is uniform in $\arg p$, so it continues (i) exactly, and nothing crosses the sides of the cell or the unit circle. Its energy, in units of f_0^2, is

$$\mathcal{E}_{\rm ii} \;=\; \int_{-\vartheta_0}^{\vartheta_0}\psi(\vartheta)^2\log\frac{4\rho(\vartheta)}{g}\,d\vartheta.$$

(As $g\to 0$, $\rho\sim g\sec\vartheta$ and $\psi\to\sec^2\vartheta$; with the substitution $s=\tan\vartheta$ the limit is $\int_{-1/2}^{1/2}(1+s^2)(\log 4+\frac12\log(1+s^2))\,ds=1.5464\ldots$.)

The bound $\mathcal{E}_{\rm ii}\le 1.5907$ for $0<g\le 0.01$. Let $s:=\varphi(\vartheta)/g\in[-\frac12,\frac12]$, the flux coordinate in units of f_0, so that $\psi\,d\vartheta=ds$ and $\mathcal{E}_{\rm ii}=\int\psi\log(4\rho/g)\,ds$. By symmetry take $s\ge 0$. Put

$$\alpha_1:=\frac{\sin(gs)}{g},\qquad \alpha_2:=\frac{1}{1-g}+\frac{2}{g}\sin^2\frac{gs}{2}.$$

(P1) $s\big(1-(gs)^2/6\big)\le\alpha_1\le s$. (P2) $1+g\le\alpha_2\le 1+1.1352\,g$: here $1/(1-g)=1+g+g^2/(1-g)$ and $1/(1-g)\le 1/0.99<1.0102$, so $1/(1-g)\le 1+g+1.0102\,g^2$; and $(2/g)\sin^2(gs/2)\le gs^2/2\le g/8$; so $\alpha_2\le 1+(1.125+1.0102\,g)\,g\le 1+1.1352\,g$. (P3) In the triangle with vertices 0, $v_j=1$ and $p_\varphi:=(1-g)e^{-i\varphi}$, the angles at 0 and at v_j are φ and ϑ, and the sides opposite them are ρ and $1-g$, while the side opposite the third angle is 1. The law of sines gives $(1-g)\sin(\vartheta+\varphi)=\sin\vartheta$, that is, $\tan\vartheta=\alpha_1/\alpha_2$. Differentiating,

$$\frac{d\vartheta}{ds}=\frac{\cos(gs)/(1-g)-(2/g)\sin^2(gs/2)}{\alpha_1^2+\alpha_2^2},$$

and the numerator is at least $(1-(gs)^2/2)(1+g)-gs^2/2\ge 1+0.87\,g\ge 1$. (P4) Hence $\psi=ds/d\vartheta\le\alpha_1^2+\alpha_2^2\le s^2+(1+1.1352\,g)^2\le 1+s^2+2.285\,g$. (P5) By the law of sines, $\rho=\sin\varphi/\sin(\vartheta+\varphi)$,

and $\sin(\vartheta+\varphi) \ge \sin\vartheta = \alpha_1/(\alpha_1^2+\alpha_2^2)^{1/2}$, as $\vartheta+\varphi < \pi/2$. Since $\sin\varphi = g\alpha_1$, this gives $\rho/g \le (\alpha_1^2+\alpha_2^2)^{1/2} \le (1+s^2+2.285\,g)^{1/2}$. (P6) So, with $\Upsilon(s) := 1.02285 + s^2$,

$$\mathcal{E}_{\rm ii} \;\le\; \int_{-1/2}^{1/2} \Upsilon(s)\Big(\log 4 + \frac{1}{2}\log\Upsilon(s)\Big)ds \;\le\; 1.5907;$$

the integrand increases with $|s|$, so an upper Riemann sum bounds the integral: on $[0,\frac12]$ with 4000 equal steps (right end points), doubled by symmetry, it is below 1.5907 (in interval arithmetic).

Tract (iii): two channels. In T_j use the chart W in the upper half-plane given by

$$\ell \;=\; i\pi + 2\Big(W\sqrt{W^2-1} - \operatorname{arccosh} W\Big), \qquad \frac{d\ell}{dW} = 4\sqrt{W^2-1},$$

with the branches that are positive for $W>1$ ($\sqrt{W^2-1} = \sqrt{W-1}\,\sqrt{W+1}$ with principal roots, $\operatorname{arccosh} W = \log(W+\sqrt{W^2-1})$). It maps the upper half-plane onto Π. On the upper half-plane $\operatorname{Im}\sqrt{W^2-1} > 0$, so $\operatorname{Re}(-i\,d\ell/dW) > 0$, and ℓ is univalent there (for $W_1 \ne W_2$, $-i(\ell(W_2)-\ell(W_1))/(W_2-W_1)$ is the mean of $-i\,d\ell/dW$ over the segment from W_1 to W_2). On the real axis $d\ell/dW = 4i\sqrt{1-W^2}$ on $(-1,1)$ and $d\ell/dW$ is real elsewhere, with $\ell(0)=0$ and $\ell(\pm1) = \pm i\pi$: so the interval $(-1,1)$ goes onto the window, $W>1$ onto the upper edge and $W<-1$ onto the lower edge, each once (as W runs from $-\infty$ to $+\infty$, ℓ runs along the lower edge to $-i\pi$, up the window to $i\pi$ and along the upper edge, with Π on its left); and $\ell \sim 2W^2$ at infinity. By the argument principle the image is Π. The field of (ii) leaves the cone along the ray ϑ, and in the ℓ-chart this ray is the line $\operatorname{Im}\ell = \vartheta$ (on $|q| = g/4$, $\operatorname{Re}\ell = 0$); so the field crosses the window at $\operatorname{Im}\ell = \vartheta$, $|\vartheta| \le \vartheta_0$, with density $\psi(\vartheta)$. On the window put $W = \sin\sigma$; then

$$\operatorname{Im}\ell \;=\; \vartheta(W) := 2\sigma + \sin 2\sigma, \qquad \frac{d\vartheta}{dW} = 4\cos\sigma \le 4,$$

so the field enters the upper half-plane through $(-a,a)$, where $\vartheta(a) = \vartheta_0$. In the coordinates $X+iY := \log W$, each half of the flux (flux $\frac12$ in units of f_0) runs in a channel of height $\pi/8$, along $Y=0$ for $W>0$ (and along $Y=\pi$ for $W<0$), up to $X = \log 2$, and leaves through the arc $\{|W|=2,\ 0<\arg W<\pi/8\}$ (or $7\pi/8<\arg W<\pi$) with uniform density $4/\pi$ per unit angle. On the channel along $Y=0$ the field is

$$V_X = \frac{8}{\pi}\,\mathcal{F}(X), \qquad V_Y = m(X)\Big(1-\frac{8Y}{\pi}\Big), \qquad 0<Y<\frac{\pi}{8},\ X<\log 2,$$

where $m(X)$ is the inflow per unit X through the real axis and $\mathcal{F}(X) := \int_{-\infty}^{X} m$; $\mathcal{F}$ is the flux through $(0,e^X)$, so $\mathcal{F} = s(\vartheta(e^X))$ for $e^X \le a$, and $\mathcal{F} = \frac12$, $m=0$ for $e^X \ge a$. Then $\operatorname{div} V = 0$, $V_Y = 0$ on the top $Y=\pi/8$ of the channel and on the real axis outside $(-a,a)$ (the rest of the window and the edges, which carry no flux), and the energy of one channel is $\int\big(8\mathcal{F}^2/\pi + \pi m^2/24\big)\,dX$. So V is a divergence-free field with exactly the required boundary fluxes; its value outside the two channels is zero. The two sectors $\{|W|\le 2,\ \arg W \in (0,\pi/8)\cup(7\pi/8,\pi)\}$ lie in T_j, below the exits and inside the end: $\ell = 0$ at $W=0$ and $|d\ell/dW| \le 4(|W|^2+1)^{1/2}$, so $|\ell| \le 4\int_0^2 (t^2+1)^{1/2}dt < 11.84 < \Lambda < E$ there.

The bound $\mathcal{E}_{\rm iii} \le 4.3389$ for $0 < g \le 0.01$. (Q1) At $s=\frac12$, $\tan\vartheta_0 = \alpha_1/\alpha_2$ lies in $\big[\frac12(1-0.005^2/6)/(1+0.011352),\ \frac12\big]$, so $\vartheta_0 \in [0.459146,\, 0.463648]$. (Q2) $\vartheta(W)$ increases with $W \in [0,1)$, so $a \in [a_-,a_+] := [0.11504,\, 0.11618]$. (Q3) By (P1)–(P3), $s \le \tan\vartheta\cdot\alpha_2/(1-(gs)^2/6) \le \tan\vartheta\cdot 1.011352/(1-0.005^2/6) \le 1.01136\tan\vartheta$. Further $\tan\vartheta \le (\tan\vartheta_+/\vartheta_+)\,\vartheta$ with $\vartheta_+ := 0.463648$, $\vartheta \le 4\sigma$, and $\sigma \le (\arcsin a_+/a_+)\,W$ for $0 \le W \le a$. So $\mathcal{F} \le C_{\mathcal{F}} W$ with $C_{\mathcal{F}} := 4\cdot 1.01136\,(\tan\vartheta_+/\vartheta_+)(\arcsin a_+/a_+) \le 4\cdot 1.01136\cdot 1.0784054\cdot 1.0022635 \le 4.3725$. (Q4) $m = d\mathcal{F}/dX = \psi(\vartheta)\,(d\vartheta/dW)\,W \le 4(1.25{+}0.02285)\,W =: C_m W$, by (P4) with $|s| \le \frac12$; so $C_m = 5.0914$. (Q5) Per channel, as $\int_{-\infty}^{\log a} e^{2X}dX = a^2/2$, and as $\mathcal{F} = \frac12$ on $\log a \le X \le \log 2$, an interval of length

at most $\log(2/a_-)$,

$$\int \Big(\frac{8}{\pi}\,\mathcal{F}^2 + \frac{\pi}{24}\,m^2\Big)dX \;\le\; \frac{4C_{\mathcal{F}}^2 a_+^2}{\pi} + \frac{\pi C_m^2 a_+^2}{48} + \frac{2}{\pi}\log\frac{2}{a_-} \;\le\; 0.35148 + 1.81795;$$

here, with $C_{\mathcal{F}} \le 4.3725$, the first two terms are at most $0.328574 + 0.022901 \le 0.35148$, and $\log(2/a_-) \le 2.855623$ gives $(2/\pi)\log(2/a_-) \le 1.81795$. So $\mathcal{E}_{\mathrm{iii}} \le 2(0.35148 + 1.81795) \le 4.3389$. With (P6), $\mathcal{E}_{\mathrm{ii}} + \mathcal{E}_{\mathrm{iii}} \le 5.9296 < 6$.

Tract (iv): radially to the exits. In the two sectors $\{|W| \ge 2,\ \arg W \in (0,\pi/8) \cup (7\pi/8,\pi)\}$, of total angle $\pi/4$, the field is radial in W with uniform angular density, out to the exit curve $\operatorname{Re}\ell = \Lambda$. Take $W = \varrho e^{i\phi}$ with $\varrho \ge 2$ and $0 \le \phi \le \pi/8$; the other sector is the mirror image. Write $\sqrt{W^2-1} = W\sqrt{1-W^{-2}}$; then $|\arg\sqrt{1-W^{-2}}| \le \frac{1}{2}\arcsin\frac{1}{4} < 0.13$ and $\frac{\sqrt{3}}{2} \le |\sqrt{1-W^{-2}}| \le \frac{\sqrt{5}}{2}$. Hence

$$\frac{\partial \operatorname{Re}\ell}{\partial\varrho} = 4\varrho\operatorname{Re}\Big(e^{2i\phi}\sqrt{1-W^{-2}}\Big) \;\ge\; 4\varrho\,\frac{\sqrt{3}}{2}\cos\Big(\frac{\pi}{4}+0.13\Big) \;\ge\; 2\varrho,$$

so $\operatorname{Re}\ell(\varrho,\phi) \ge \operatorname{Re}\ell(2,\phi) + \varrho^2 - 4$, strictly increasing along each ray. On $|W| = 2$: $\ell(2) = i\pi + 2\big(2\sqrt{3} - \operatorname{arccosh}2\big)$ has real part $4.29\ldots$, and $|\partial\ell/\partial\phi| \le 8\sqrt{5}$, so over the angle $\pi/8$ the real part stays in $(-3, 11.4)$. So each ray meets $\operatorname{Re}\ell = \Lambda$ exactly once (as $\Lambda \ge 12$), at some ϱ with $\varrho^2 \le \Lambda + 7$. As $|\partial\ell/\partial\varrho| \le 2\sqrt{5}\,\varrho$, there $|\ell| \le \pi + 4.3 + 7.1 + 2.24\,\varrho^2 \le 6\Lambda < E$, since $E \ge 2\Lambda^2 \ge 24\Lambda$. So the field stops at the exits, before the end, and does not depend on E. Its energy is at most that of the radial field of flux f_0 in the full sectors from $|W| = 2$ to $|W| = \sqrt{\Lambda+7}$:

$$\Big(\frac{f_0}{\pi/4}\Big)^2\frac{\pi}{4}\,\log\frac{\sqrt{\Lambda+7}}{2} \;=\; \Big(\frac{2}{\pi}\log\Lambda + \mathcal{E}_{\mathrm{iv}}\Big)f_0^2, \qquad \mathcal{E}_{\mathrm{iv}} := \frac{2}{\pi}\log\frac{1+7/\Lambda}{4} < 0.$$

This completes the proof. □

Remark 5.6. For fixed R, Lemma 5.5 gives $\log r_\Sigma \le g\big(6 + \frac{2}{\pi}\log\log(4R/g)\big)$, which tends to 0 as $g \to 0$.

5.5. **The formula, and the domain.** From now on λ, M and g are as in Proposition 5.1, and

$$T := \frac{M}{\lambda}.$$

Then $g \le \lambda/10 \le 0.002$, $M \ge 1000$ and $T \ge 1000/0.02 = 5\cdot 10^4$.

The formula. Let Ω be a Jordan domain with $0 \in \Omega$, and let $Z\colon \mathbb{D} \to \Omega$ be the Riemann map with $Z(0) = 0$ and $r_\Omega := Z'(0) > 0$. Then $z := Z \circ \zeta\colon \Sigma \to \Omega$ is conformal with $z(o) = 0$, and

$$\hat f \;:=\; \frac{r_\Omega}{r_\Sigma}\,p \circ z^{-1} \tag{24}$$

is holomorphic and locally univalent on Ω, with $\hat f(0) = 0$ and $\hat f'(0) = 1$ (near o, $dp/d\zeta = r_\Sigma$ and $dz/d\zeta = r_\Omega$). For $\xi \in \Sigma$ and $y := z(\xi)$,

$$\Big|\frac{\hat f(y)}{y}\Big| \;=\; \frac{r_\Omega}{r_\Sigma}\cdot\frac{|p(\xi)|}{|z(\xi)|}. \tag{25}$$

The logarithmic chart. Let $k(\eta) := \eta/(1-\eta)^2$ be the Koebe function, $r := e^{-\lambda}$, $U := k(r\mathbb{D})$, and $F(s) := k(rs)$ for $s \in \mathbb{D}$. Then $F\colon \mathbb{D} \to U$ is conformal, $F(0) = 0$, $F'(0) = r$, and the point of ∂U nearest 0 is $k(-r)$, at distance $r/(1+r)^2$; put $S_U := F'(0)/\operatorname{dist}(0,\partial U) = (1+r)^2 \le 4$. On the plane we use the coordinates

$$x := \frac{1}{4z}, \qquad Y = a + ib \quad\text{with}\quad x = \sinh^2\frac{Y}{2};$$

for $z = k(\eta)$, $\eta \in \mathbb{D}$, this is $Y = \log(1/\eta)$, since $1/(4k(\eta)) = (\eta^{-1} - 2 + \eta)/4$. The map $Y \mapsto x$ is one-to-one and conformal on $\{a > 0,\ -\pi < b \le \pi\}$; it identifies $(0, b)$ with $(0, -b)$ and (a, π) with $(a, -\pi)$, and

$$|x| \ = \ \frac{\cosh a - \cos b}{2}. \tag{26}$$

In these coordinates $U = \{a > \lambda\}$, the point $z = 0$ is $a = +\infty$, and the complement of U in the sphere is the image of the rectangle $\{0 \le a \le \lambda,\ |b| \le \pi\}$; the point $z = \infty$ is $Y = 0$, an interior point of that complement. We call the side $a = \lambda$ the *wall*. The wall point $F(e^{i\phi})$ is $Y = \lambda - i\phi$, so the harmonic measure of a piece of the wall in U at $z = 0$ is its b-length divided by 2π.

The low zone. If $|b| \ge \pi - 2\sqrt{\lambda} - g$, then by (26) and $\cos t \ge 1 - t^2/2$,

$$|x| \ \ge \ \frac{1 + \cos(2\sqrt{\lambda} + g)}{2} \ \ge \ 1 - \lambda - \sqrt{\lambda}\, g - \frac{g^2}{4} \ \ge \ 1 - 1.1\lambda, \qquad \text{so} \qquad |z| \ \le \ \frac{1 + 1.2\lambda}{4}; \tag{27}$$

here $\sqrt{\lambda}\, g + g^2/4 \ \le \ 0.1\lambda(\sqrt{\lambda} + 0.025\lambda) \ \le \ 0.1\lambda$, as $g \ \le \ 0.1\lambda$, and $1/(1 - 1.1\lambda) \ \le \ 1 + 1.2\lambda$ for $\lambda \le 0.075$.

Targets and the cube. We take $\theta_* := g/2$ in §5.3, so $\theta_j = (j + \frac{1}{2})g$, and we put $\beta_j := -\theta_j$, taken in $(-\pi, \pi)$. As j runs modulo M, the *targets* β_j run through the M numbers $\pm(k - \frac{1}{2})g$, $1 \le k \le M/2$; the target β_j is the b-coordinate of the wall point $F(e^{i\theta_j})$. Channel j will be attached to the wall at $b = b_j := -\tau_j$, where $\tau = (\tau_j)_j$ ranges over the cube

$$\mathcal{Q} := \prod_j I_j, \qquad I_j := [\theta_j - g/4,\ \theta_j + g/4],$$

so that $|b_j - \beta_j| \le g/4$. Channel j is *long* if $|\beta_j| \le \pi - 2\sqrt{\lambda}$ and *short* otherwise. In each of the halves $\{b > 0\}$ and $\{b < 0\}$ there are J long channels, and at most $2\sqrt{\lambda}/g + 1 = (\sqrt{\lambda}/\pi)M + 1$ short ones; as $\lambda \le 0.02$,

$$J \ \ge \ \Big(\frac{1}{2} - \frac{\sqrt{\lambda}}{\pi}\Big)M - 1 \ \ge \ 0.4549\, M - 1, \qquad \text{so} \qquad 4J + 4 \ \ge \ 1.8\, M. \tag{28}$$

The channels. Put

$$w := \frac{\lambda}{4J + 4}, \qquad w_m := \frac{g^3}{10^4}.$$

By (28) and $J \le M/2$, $\lambda/(2M + 4) \le w \le \lambda/(1.8M) \le \lambda g/\pi$; in particular $w \le g/150$. We describe channel j for $b_j = \beta_j > 0$; for $\beta_j < 0$ it is the mirror image in $\{b = 0\}$, and for general $b_j \in [\beta_j - g/4, \beta_j + g/4]$ it is the translate by $i(b_j - \beta_j)$ of this drawing, its end included. All channels have width w after a short neck:

- *neck*: $\{\lambda - w_m < a < \lambda,\ |b - b_j| < w_m/2\}$; its side on the wall, $\{a = \lambda,\ |b - b_j| < w_m/2\}$, is the *mouth*;
- *wedge*: widening from width w_m to width w across a further depth w, with straight sides;
- a *long* channel then has a *horizontal part* $\{a_j - w/2 < a < \lambda - w_m - w,\ |b - b_j| < w/2\}$, where $a_j := 2wk'$ if channel j is the k'-th long channel of its half in the order of $|\beta_j|$ ($1 \le k' \le J$); an *elbow*, the corner rounded on the inner side by a quarter circle of radius $w/2$; a *vertical part* $\{|a - a_j| < w/2,\ b_j \le b < b_e\}$, $b_e := \pi - \sqrt{\lambda}/2$; and the *end* $E_j := \{b = b_e,\ |a - a_j| \le w/2\}$;
- a *short* channel has a horizontal part only, $\{\lambda - w_m - w - \lambda/8 < a < \lambda - w_m - w,\ |b - b_j| < w/2\}$, and its end E_j is its side $a = \lambda - w_m - w - \lambda/8$.

Let $\Omega = \Omega(\tau)$ be U together with all necks, wedges and channels, in the x-plane, and $z = 1/(4x)$.

Checks. For every $\tau \in \mathcal{Q}$ (we write $b_{(k')}$ and $a_{(k')}$ for b_j and a_j of the k'-th long channel of a half): (a) The long channels of one half are disjoint: if $k' < k''$, channel k'' has its horizontal part at $b \ge b_{(k')} + g/2$ and its vertical part at $a \ge a_{(k')} + 2w$, so it lies in $\{a > a_{(k')} + w/2,\ b > b_{(k')} + w/2\}$, inside the L of channel k', with gaps at least w. (b) Short channels lie in $\{a > 7\lambda/8 - w_m - w\}$

and long vertical parts in $\{a < a_J + w/2 < \lambda/2 + w\}$; so they are disjoint, and the horizontal parts of different channels are at least $g/2 - w$ apart. (c) No channel meets $\{a < 3w/2\}$ (as $a_j - w/2 \geq 3w/2$), nor $\{|b| < g/5\}$ (as $|b_j| - w/2 \geq g/4 - w/2 \geq g/5$). (d) The complement of Ω in the sphere is connected: from any of its points one reaches the free strip $\{0 \leq a < w\}$, which contains $Y = 0$ ($z = \infty$), along the corridors between channels (of width at least w), through the top region $\{|b| > \pi - \sqrt{\lambda}/2 + g/4\}$, which no long channel meets, or through the region below the first long channel. The channels are disjoint fingers on the wall, so $\partial\Omega$ is a Jordan curve. (e) On Ω, either $a > \lambda$, and then $|x| \geq \sinh^2(\lambda/2) \geq g^2/101$, or $|b| \geq g/5$, and then $|x| \geq (1 - \cos b)/2 \geq \sin^2(g/10) \geq g^2/101$. So

$$|z| \;\leq\; z_{\max} := \frac{26}{g^2} \qquad \text{on } \Omega. \tag{29}$$

(f) Every point of every end, and of the last straight piece Q_j before it (defined in §5.7), lies in the low zone (27): long ends lie at $|b| \geq \pi - \sqrt{\lambda} - g/4$, and short channels lie entirely in $|b| \geq \pi - 2\sqrt{\lambda} - g/4 - w/2$.

The sizes R, Λ, r_Σ. In §5.3 we now take

$$R := 2 + \frac{80\, S_U\, z_{\max}}{F'(0)}. \tag{30}$$

As $S_U \leq 4$ and $F'(0) = e^{-\lambda} \geq 1/2$, $R \leq 2 + 16640/g^2 \leq 17000/g^2$, so $\Lambda = \log(4R/g) \leq 3\log(1/g) + 11.13 = \Lambda^+$. As $S_U/F'(0) = (1+r)^2/r \geq 4$, $R \geq 8320/g^2$, so $\Lambda \geq \log(33280/g^3) > 24$. So Lemma 5.5 applies ($g \leq 0.01$, $\Lambda \geq 12$), and with the second hypothesis in (21),

$$\log r_\Sigma \;\leq\; g\Big(6 + \frac{2}{\pi}\log\Lambda\Big) \;\leq\; g\Big(6 + \frac{2}{\pi}\log\Lambda^+\Big) \;\leq\; \lambda. \tag{31}$$

Note that R is a polynomial in M, and Λ depends on M only through a logarithm. Lemma 5.5 holds for every R, so there is no circularity.

5.6. **Positioning the channels.** For $\tau \in \mathcal{Q}$ let $Z_\tau \colon \mathbb{D} \to \Omega(\tau)$ be the Riemann map with $Z_\tau(0) = 0$, $Z_\tau'(0) > 0$, and let $\Theta_j(\tau)$ be the center of the arc $Z_\tau^{-1}(E_j)$ of $\mathbb{T}$. We want τ with $\Theta_j(\tau) = \theta_j$ for every j.

Lemma 5.7 (Displacement)**.** *Let Ω be a Jordan domain that consists of U, finitely many disjoint channels (each with its neck and wedge) and their mouths, each channel attached to U along one mouth arc of ∂U, and let $Z \colon \mathbb{D} \to \Omega$ be its Riemann map with $Z(0) = 0$, $Z'(0) > 0$. Let ν_0 be the sum of the harmonic measures in U at 0 of the mouths, and assume $\nu_0 \leq 1/2$. Then $G := Z^{-1} \circ F$ satisfies $|\arg G(s) - \arg s| \leq 2\pi\nu_0 + 4\sqrt{\nu_0}$ for every $s \in \mathbb{T}$ outside the mouth arcs F^{-1}(mouths), and, if the mouth of channel j is the image under F of an arc containing $e^{i\tau_j}$, then the center Θ_j of $Z^{-1}(\alpha_j)$, for any arc α_j of the part of $\partial\Omega$ that bounds channel j (its neck and wedge included), satisfies*

$$|\Theta_j - \tau_j| \;\leq\; 6\pi\nu_0 + 4\sqrt{\nu_0}.$$

Proof. The map $G \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is continuous and injective, $G(0) = 0$ and $G'(0) > 0$; it maps the arcs of $\mathbb{T}$ off the mouths into $\mathbb{T}$, and the mouth arcs into $\mathbb{D}$. (a) $L(s) := \log(G(s)/s) = u + iv$ is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, with $v(0) = 0$ and $u(0) = \log G'(0) =: -c_G$, $c_G \geq 0$ by Schwarz's lemma. (b) On $\mathbb{T}$, $u = \log|G| = 0$ off the mouth arcs. On the mouth arcs, $-\log 4 - c_G \leq u < 0$, by Koebe's growth bound $|G(s)| \geq G'(0)|s|/(1+|s|)^2$ [15, §2.3]. (c) The mean of u over $\mathbb{T}$ is $u(0) = -c_G$, so $c_G \leq \nu_0(\log 4 + c_G)$, and $c_G \leq 2\nu_0 \log 4$ as $\nu_0 \leq 1/2$. (d) In the L^2 norm for normalized length on $\mathbb{T}$, Parseval's identity and $v(0) = 0$ give $\|v\|_2 \leq \|u\|_2 \leq \sqrt{\nu_0}(\log 4 + c_G) \leq 2.8\sqrt{\nu_0}$. (e) Let $s_1, s_2 \in \mathbb{T}$ lie off the open mouth arcs, let I be the positive arc from s_1 to s_2, and I' the positive arc from $G(s_1)$ to $G(s_2)$. Then $v(s_2) - v(s_1) = |I'| - |I|$ (lengths): along I, $\arg G$ changes by exactly $|I'|$, since

each crosscut $G(\text{mouth arc of channel } k)$, together with the arc $Z^{-1}(\partial(\text{channel } k))$ of $\mathbb{T}$ between its ends, bounds a region that does not contain 0, so the two are homotopic in $\overline{\mathbb{D}} \setminus \{0\}$. Moreover

$$\frac{|I'|-|I|}{2\pi} \;=\; \text{exc}+\omega_\Omega(0,\text{channels in } I)-\omega_U(0,\text{mouths in } I), \qquad \text{exc} := \omega_\Omega(0,W_I)-\omega_U(0,W_I) \geq 0,$$

where $W_I := F(I)$ minus the mouths, and exc ≥ 0 by Carleman's principle, as $U \subset \Omega$. Over the whole circle, the total excess plus $\omega_\Omega(0,\text{all channels})$ equals ν_0. So $|I'|-|I| \in [-2\pi\nu_0, 2\pi\nu_0]$, and off the mouth arcs v oscillates by at most $2\pi\nu_0$. (f) The set off the mouth arcs has normalized length at least $1-\nu_0 \geq 1/2$; take s_* in it with $|v(s_*)|$ least. By (d), $|v(s_*)|^2(1-\nu_0) \leq \|v\|_2^2$, so $|v(s_*)| \leq 4\sqrt{\nu_0}$. With (e), $|v| \leq 2\pi\nu_0 + 4\sqrt{\nu_0}$ off the mouth arcs, which is the first claim. (g) Let the mouth arc of channel j run from s_1 to s_2. The arc $Z^{-1}(\alpha_j)$ lies in the arc from $G(s_1)$ to $G(s_2)$, whose length is $2\pi\omega_\Omega(0,\text{channel } j) \leq 2\pi\nu_0$, and $e^{i\tau_j}$ lies in the mouth arc, of length $2\pi\omega_U(0,\text{mouth } j) \leq 2\pi\nu_0$. So $|\Theta_j - \tau_j| \leq 2\pi\nu_0 + |v(s_1)| + 2\pi\nu_0 \leq 6\pi\nu_0 + 4\sqrt{\nu_0}$. □

For our channels each mouth has b-length w_m, so

$$\nu_0 \;=\; \frac{M w_m}{2\pi} \;=\; \frac{w_m}{g} \;=\; \frac{g^2}{10^4}, \qquad 6\pi\nu_0 + 4\sqrt{\nu_0} \;\leq\; 0.0019\,g^2 + 0.04\,g \;\leq\; \frac{g}{8}. \tag{32}$$

In particular $\nu_0 \leq 1/2$, and the domain $\Omega(\tau)$ has the other properties that Lemma 5.7 uses, by the Checks (a)–(d). So, for every $\tau \in \mathcal{Q}$: $|\Theta_j(\tau) - \tau_j| \leq g/8$ for all j, and $|\arg G(s) - \arg s| \leq g/8$ off the mouth arcs.

Lemma 5.8 (Continuity)**.** *The map $\tau \mapsto \Theta(\tau) = (\Theta_j(\tau))_j$ is continuous on $\mathcal{Q}$.*

Proof. Fix $\tau^0 \in \mathcal{Q}$ and let $\tau \in \mathcal{Q}$, with $\Delta b_j := b_j(\tau) - b_j(\tau^0)$. For each j let χ_j be a smooth function of Y, equal to 1 on a neighborhood of the union V_j of all translates by at most $g/4$ in b of the drawing of channel j with $b_j = \beta_j$ (neck, wedge, horizontal part, elbow, vertical part and end), and equal to 0 outside a slightly larger neighborhood; the taper lies within $w/3$ of V_j. The sets V_j do not depend on τ, and they are at least w apart, by the proof of Check (a). Channels of different halves are at least $g/2 - w$ apart in b, since the targets of the two halves are at least g apart modulo 2π and the vertical parts lie in $g/4 \leq |b| \leq b_e + g/4 < \pi - g$. Within one half: two vertical parts are at least w apart in a; the horizontal part of a later long channel of a half and the vertical part of an earlier one are at least w apart in a; the horizontal part of an earlier long channel and the vertical part of a later one are at least $g/2 - w/2$ apart in b; horizontal parts, wedges and necks of different channels are at least $g/2 - w$ apart in b; short channels are apart from the rest by Check (b). So the supports of the χ_j are disjoint, and by Check (c) they do not meet $\{a < w\}$. Let Ξ_τ be the time-one flow of the vector field $\sum_j \Delta b_j\, \chi_j\, \partial/\partial b$. It has no $\partial/\partial a$ part, so it keeps every line $a = \text{const}$; it preserves $U = \{a > \lambda\}$ and the wall, it is the identity near $z = 0$ and near the fold $a = 0$, and it moves each channel rigidly by Δb_j, end included, since $b_j(\tau^0)$ and $b_j(\tau)$ both lie in $[\beta_j - g/4, \beta_j + g/4]$, so that the channel stays in V_j, where $\chi_j = 1$, during the motion. So $\Xi_\tau(\Omega(\tau^0)) = \Omega(\tau)$ and $\Xi_\tau(E_j(\tau^0)) = E_j(\tau)$. As $\tau \to \tau^0$, $\Xi_\tau \to \text{id}$ in C^1, so

$$\psi_\tau \;:=\; Z_\tau^{-1} \circ \Xi_\tau \circ Z_{\tau^0} \colon \mathbb{D} \to \mathbb{D}$$

is K_τ-quasiconformal with $K_\tau \to 1$, and $\psi_\tau(0) = 0$; it extends to the sphere by reflection in $\mathbb{T}$. By Mori's theorem (see [25]) the family is equicontinuous on $\overline{\mathbb{D}}$, so every sequence $\tau \to \tau^0$ has a subsequence along which ψ_τ converges uniformly on $\overline{\mathbb{D}}$. The limit is not constant: each ψ_τ maps $\mathbb{T}$ onto $\mathbb{T}$, and so does the uniform limit (a point $\zeta = \psi_{\tau_n}(\zeta_n)$ of $\mathbb{T}$ is the image of a limit point of the ζ_n). A uniform limit of K_n-quasiconformal maps with $K_n \to 1$ that is not constant is 1-quasiconformal [25], hence conformal; so the limit is a conformal automorphism of $\mathbb{D}$ fixing 0, a rotation. Near 0, $\psi_\tau = Z_\tau^{-1} \circ Z_{\tau^0}$ is conformal with positive derivative at 0, so the rotation is the identity. Hence $\psi_\tau \to \text{id}$ uniformly on $\overline{\mathbb{D}}$, and as $Z_\tau^{-1}(E_j(\tau)) = \psi_\tau\big(Z_{\tau^0}^{-1}(E_j(\tau^0))\big)$, $\Theta(\tau) \to \Theta(\tau^0)$. □

Brouwer. The map $\tau \mapsto \tau - (\Theta(\tau) - \theta)$, $\theta := (\theta_j)_j$, is continuous and maps $\mathcal{Q}$ into itself, since $|\Theta_j(\tau) - \tau_j| \le g/8$. By Brouwer's fixed point theorem it has a fixed point. *From now on τ is such a fixed point*, $\Omega := \Omega(\tau)$, $Z := Z_\tau$, and

$$Z^{-1}(E_j) \text{ is an arc of } \mathbb{T} \text{ centered at } e^{i\theta_j}, \quad \text{for every } j. \tag{33}$$

As $U \subset \Omega$, Schwarz's lemma gives $r_\Omega \ge F'(0) = e^{-\lambda}$. None of this depends on E.

5.7. **Harmonic measure.** Write $h_j := \omega_\Omega(\cdot, E_j)$ for the harmonic measure of the end E_j in Ω. We compute in the Y-chart (a cylinder: b is taken modulo 2π), which is conformal and one-to-one on $\Omega \setminus \{0\}$, by Check (c). We use Harnack's inequality in this form: if $u > 0$ is harmonic on a disc $B(c, \rho)$ and $|c' - c| \le \rho/2$, then $u(c)/3 \le u(c') \le 3u(c)$.

Lemma 5.9 (Through a channel)**.** *For every j and every $\tau \in \mathcal{Q}$,*

$$\omega_\Omega(0, E_j) \;\ge\; \exp\Big(-5.5\,\frac{L_j}{w} - 10.8 \log\frac{1}{w_m} - 27\Big),$$

where L_j is the total length of the straight parts of channel j ($L_j \le \pi + \lambda$ for long channels, $\lambda/8$ for short ones). Consequently $e_j := \pi\,\omega_\Omega(0, E_j)$ (the half-length of the arc $Z^{-1}(E_j)$ of $\mathbb{T}$) satisfies $e_j \ge e^{-39T}$.

Proof. (a) *Base to mouth.* On the half-cylinder $\{a > \lambda'\}$, $\lambda' := \lambda + w_m/2$, which lies in U, the function h_j is harmonic and bounded, and $z = 0$ is its point $a = +\infty$; in the coordinate $\eta = e^{-Y}$ the circles $a = \text{const}$ are circles about $\eta = 0$. So $h_j(0)$ is the mean of h_j over the circle $a = \lambda'$, and $h_j(0) \ge (w_m/(4\pi)) \min h_j$ over $\{a = \lambda',\ |b - b_j| \le w_m/4\}$, which is at least $(w_m/(12\pi))h_j(c_0)$, $c_0 := \lambda' + ib_j$, by Harnack's inequality on $B(c_0, w_m/2) \subset U$. (b) *A chain of discs.* Take centers $c_0, c_1, \dots, c_N$ on the axis of the channel, with $c_{i+1} \in B(c_i, \rho_i/2)$ and $B(c_i, \rho_i) \subset \Omega$; then $h_j(c_i) \ge h_j(c_{i+1})/3$. Mouth and neck: $\rho = w_m/2$, steps $w_m/4$, from $a = \lambda + w_m/2$ to $a = \lambda - w_m$: 6 steps. Wedge: at depth t below the neck the half-width is $H(t) = w_m/2 + t(w - w_m)/(2w)$; take $\rho = H/2$ (the disc lies in the wedge, whose sides make an angle α with the axis, $\tan\alpha \le 1/2$, and $H/2 \le H\cos\alpha$; it also clears the neck's sides) and steps $H/4$, so that H grows by a factor at least 1.12 per step: at most $1 + 8.9\log(w/w_m)$ steps. Straight parts (at most two): $\rho = 0.4w$, steps $0.2w$: at most $5L_j/w + 2$ steps. Elbow (the axis is a quarter circle of radius w, at distance $w/2$ from both walls): at most 8 steps. The last wedge center is the axis point c_W where the wedge meets the horizontal part; the disc of radius $0.4w$ about c_W lies in the channel, as its distance to the sides of the wedge is at least $(w/2)\cos\alpha \ge 0.44w$, so the straight chain starts at c_W. The last center of the chain lies on the axis at distance $0.4w$ from E_j, and the point c_N of (c) is within $0.15w \le 0.2w$ of it. We also count 2 junction steps, which only weakens the bound. (c) *The end.* Let c_N be the point of the axis at distance $w/4$ from E_j. The half-disc of radius $w/2$ on E_j lies in the channel, and in it the harmonic measure of its diameter at c_N is $1 - (4/\pi)\arctan(1/2) \ge 0.4$. By Carleman's principle, $h_j(c_N) \ge 0.4$. Hence $\log(1/h_j(0)) \le \log(95/w_m) + (5L_j/w + 8.9\log(w/w_m) + 19)\log 3$, and as $w < 1$, $\log 95 + 19\log 3 \le 25.5$, $5\log 3 \le 5.5$ and $1 + 8.9\log 3 \le 10.8$, this gives the bound. For the consequence: $L_j/w \le (\pi+\lambda)(2M+4)/\lambda \le 7T$, and $\log(1/w_m) = 3\log(M/2\pi) + \log 10^4 \le 3\log M + 4 \le T/100$; so the exponent is at most $38.5T + 0.11T + 27 \le 39T$, and $e_j \ge \pi e^{-39T} \ge e^{-39T}$. □

Lemma 5.10 (Off the last straight piece)**.** *Let Q_j be the last straight piece before E_j: for a long channel, the part of the vertical part beyond the crosscut $\gamma_j := \{b = \pi - \sqrt{\lambda} + (b_j - \beta_j)\}$ (for $\beta_j > 0$; mirrored for $\beta_j < 0$), of length $l_j := \sqrt{\lambda}/2$; for a short channel, the last half of the horizontal part, of length $l_j := \lambda/16$. Then*

$$h_j \;\le\; 32\,e^{-\pi l_j/w} \;<\; \frac{1}{2} \qquad \text{on } \Omega \setminus Q_j.$$

Proof. For a short channel $l_j/w = (J+1)/4 \geq 0.1137\,M > 11$, by (28); for a long one $l_j/w \geq 0.9\,M/\sqrt{\lambda}$. So $32e^{-\pi l_j/w} < 32e^{-11\pi} < 1/2$. Use coordinates (s,t) on Q_j: $s \in (0, l_j)$ from γ_j (or from the start of Q_j) to E_j, and $t \in (0,w)$ across. In the half-strip $\{s < l,\ 0 < t < w\}$ the harmonic measure of the end $\{s = l\}$ is

$$\sum_{n \text{ odd}} \frac{4}{\pi n}\, e^{-n\pi(l-s)/w} \sin\frac{n\pi t}{w} \ \leq\ 1.2756\, e^{-\pi(l-s)/w} \qquad \text{for } l - s \geq w.$$

Let $\gamma^+ := \{s = w\}$ and $\sigma^+ := \sup_{\gamma^+} h_j$. By the maximum principle $h_j \leq \sigma^+$ on Ω minus the part $\{s > w\}$ of Q_j, in particular on γ_j; and on Q_j, $h_j \leq \omega_{Q_j}(\cdot, E_j) + \sigma^+ \omega_{Q_j}(\cdot, \gamma_j)$. By domain monotonicity, with the half-strips $\{s < l_j\}$ and $\{s > 0\}$, both containing Q_j: on γ^+, $\omega_{Q_j}(\cdot, E_j) \leq 1.2756\, e^{\pi} e^{-\pi l_j/w}$ and $\omega_{Q_j}(\cdot,\gamma_j) \leq 1.2756\, e^{-\pi} \leq 0.0552$. So $\sigma^+ \leq 29.6\, e^{-\pi l_j/w} + 0.0552\,\sigma^+$, that is, $\sigma^+ \leq 31.4\, e^{-\pi l_j/w}$. □

Lemma 5.11 (The end of a tract is small)**.** *If $E \geq 2\Lambda e^{8\pi/3}$, then $\omega_\Sigma(o, A_j) \leq 32\,(2\Lambda/E)^{3/4}$.*

Proof. In the ℓ-chart of T_j let $s_0 := 2\Lambda$ and $\gamma_{\mathrm{s}} := \{|\ell| = s_0,\ \operatorname{Re}\ell < \Lambda\}$. The part of Σ beyond γ_{s} (the points of T_j with $|\ell| > s_0$, including those of the strip near v_j) meets the rest of Σ only along γ_{s}, since the window and the edges lie in $|\ell| \leq (\Lambda^2 + \pi^2)^{1/2} < s_0$. In the coordinates $(\log|\ell|, \arg\ell)$ this part lies in a strip of width $w_{\mathrm{s}} := \pi + 2\arcsin\frac12 = \frac{4\pi}{3}$ (at $|\ell| = s \geq s_0$, the condition $\operatorname{Re}\ell < \Lambda$ leaves an arc of angle $\pi + 2\arcsin(\Lambda/s)$), from $\log s_0$ to $\log E$, with A_j in its end; its other boundary is the exit, where the harmonic measure of A_j vanishes. Repeat the proof of Lemma 5.10 in that strip, with w_{s} in place of w and γ_{s}^+ at distance w_{s} beyond γ_{s}: on γ_{s}^+ the end has harmonic measure at most $1.2756\, e^{\pi}(s_0/E)^{\pi/w_{\mathrm{s}}} \leq 29.6\,(s_0/E)^{3/4}$ in the half-strip, and γ_{s} at most 0.0552. So $\sigma^+ \leq 31.4\,(2\Lambda/E)^{3/4}$, and $\omega_\Sigma(o, A_j) \leq \sigma^+$ by the maximum principle on Σ minus the part beyond γ_{s}^+. □

The choice of E. By (22), $\zeta(A_j)$ is the arc of $\mathbb{T}$ centered at $e^{i\theta_j}$ of half-width $\pi\,\omega_\Sigma(o, A_j)$; by (33), $Z^{-1}(E_j)$ is the arc centered at $e^{i\theta_j}$ of half-width $e_j \geq e^{-39T}$ (Lemma 5.9). So $\zeta(A_j) \subset Z^{-1}(E_j)$, that is $z(A_j) \subset E_j$, once $32\pi(2\Lambda/E)^{3/4} \leq e^{-39T}$. We take

$$E := \text{the least odd multiple of } \pi \text{ above } \max\bigl(2\Lambda^2,\ 2000\,\Lambda\, e^{52T}\bigr), \tag{34}$$

for which $32\pi(2\Lambda/E)^{3/4} \leq 32\pi \cdot 10^{-9/4} e^{-39T} < e^{-39T}$. Then $E \geq 100$, $\Lambda \leq 0.04\sqrt{E}$ (so Lemma 5.4 applies), $E \geq 2\Lambda e^{8\pi/3}$ (so Lemma 5.11 applies: then the end of the strip in its proof lies at distance at least w_{s} beyond γ_{s}^+, as the series bound needs), and $\log E \leq 52T + \log(2000\Lambda) + 1 \leq 53T$, as $\log(2000\Lambda) + 1 \leq T$.

Lemma 5.12 (Near the end of a tract)**.** *If $\xi \in T_j$, $E - \frac12 \leq |\ell(\xi)| < E$ and $\operatorname{Re}\ell(\xi) \leq \Lambda - 3$, then $\omega_\Sigma(\xi, A_j) \geq 0.68$.*

Proof. Let $\ell^\dagger := E\,\ell(\xi)/|\ell(\xi)|$, the nearest point of the circle $|\ell| = E$. Then $\operatorname{Re}\ell^\dagger < \Lambda$, so $\ell^\dagger \in A_j$. The set $\mathcal{N} := B(\ell^\dagger, 2) \cap \{|\ell| < E\}$ lies in T_j: on it $\operatorname{Re}\ell < \Lambda - 0.9$, and it misses the half-strip $\{\operatorname{Re}\ell \geq 0,\ |\operatorname{Im}\ell| \leq \pi\}$, since on it either $\operatorname{Re}\ell < -1$ (if $\operatorname{Re}\ell^\dagger < -3$) or $|\operatorname{Im}\ell| > (E^2 - \Lambda^2)^{1/2} - 2 > \pi$. The disc $\{|\ell| < E\}$ lies on one side of its tangent at $\ell^\dagger$, so $\mathcal{N}$ lies in the half-disc HD of radius 2 on that tangent, and the circular part of $\partial\mathcal{N}$ lies in the semicircle of HD. The point ξ lies on the axis of HD at height $t := E - |\ell(\xi)| \leq \frac12$ above the tangent. By the maximum principle, $\omega_{\mathcal{N}}(\xi, \text{circular part}) \leq \omega_{HD}(\xi, \text{semicircle}) = \frac{4}{\pi}\arctan\frac{t}{2} \leq \frac{4}{\pi}\arctan\frac14 \leq 0.32$. The rest of $\partial\mathcal{N}$ is an arc of A_j, so by Carleman's principle $\omega_\Sigma(\xi, A_j) \geq \omega_{\mathcal{N}}(\xi, A_j \cap \partial\mathcal{N}) \geq 0.68$. □

Transfer. The map $z = Z \circ \zeta \colon \Sigma \to \Omega$ is conformal and $z(A_j) \subset E_j$. So for ξ as in Lemma 5.12, $h_j(z(\xi)) = \omega_\Sigma(\xi, z^{-1}(E_j)) \geq \omega_\Sigma(\xi, A_j) \geq 0.68 > \frac12$, and by Lemma 5.10 the point $z(\xi)$ lies in Q_j.

By Check (f) and (27),

$$|z(\xi)| \;\le\; \frac{1+1.2\lambda}{4} \qquad \text{for every } \xi\in T_j \text{ with } E-\tfrac12 \le |\ell(\xi)| < E,\ \operatorname{Re}\ell(\xi)\le \Lambda-3. \tag{35}$$

5.8. **The outer layer.** We now bound $|\hat f(y)/y|$ from below near $\partial\Omega$. On Σ we use the hyperbolic distance d_h pulled back by ζ from the density $2|d\zeta|/(1-|\zeta|^2)$ on $\mathbb{D}$; then $1-|\zeta(\xi)| \ge e^{-d_h(o,\xi)}$, since $d_h(o,\xi) = \log\frac{1+|\zeta(\xi)|}{1-|\zeta(\xi)|}$. For $\xi\in\Sigma$ and each k put $q_k(\xi) := p(\xi) - v_k$.

Charts. For ξ in a tract T_j with $\operatorname{Re}\ell(\xi) < 0$ (that is, $|q_j(\xi)| < g/4$: the strip and the deeper sheets; the gluing identifies the part of the strip with $|\ell| < E$ with $B\cap\{|q_j| < g/4\}$ minus the caps, so that part is in both B and T_j) the chart is ℓ, with $dp/d\ell = q_j$. For every other ξ the chart is p: either $\xi\in B$ and $|q_k(\xi)| \ge g/4$ for all k (we call such ξ a *B-point*), or ξ lies on a sheet of a tract T_j with $|q_j(\xi)| \ge g/4$ (there p may be close to another v_k, which is an ordinary point of the sheets of T_j). A chart disc about ξ *lifts* if the inverse of the chart extends to it as a map into Σ. Let the *chart radius* $r_{\mathrm c}(\xi)$ be the largest radius, at most $g/8$, of a chart disc about ξ that lifts. If a chart disc of radius r about ξ lifts, the hyperbolic density of Σ at ξ in that chart is at most $2/r$ (Schwarz–Pick: Σ contains the lifted disc).

Lifting. (l1) An ℓ-disc with center in $\{\operatorname{Re}\ell\le 0\}$ (or on the end circle there) and radius at most $\frac12$ lies in $\{\operatorname{Re}\ell < \frac12\}$. Its part in $\{\operatorname{Re}\ell < 0\}$ lies in T_j (the lines $\operatorname{Im}\ell = \pm\pi$ are interior there). Its part in $\{0\le\operatorname{Re}\ell<\frac12,\ |\operatorname{Im}\ell|\le\pi\}$ lies over B in the annulus $g/4 \le |q_j| < (g/4)e^{1/2} < 0.42g$, where B contains no v_k but v_j (the others are about g away) and no slit but σ_j; its part with $|\operatorname{Im}\ell| > \pi$ lies on T_j. So the disc lifts unless it meets the end $|\ell| = E$. (l2a) A p-disc with center in B and radius $r \le \min_k|q_k|/2$ (so $r\le g/8$ is allowed at B-points) contains no v_k; each slit it meets crosses it as a chord (neither end of the slit is inside), and lifting along radii from the center puts the part beyond a chord on the sheet of that tract, which has no slits. (l2b) A p-disc with center on a sheet of T_j and radius $r\le|q_j|/2$ (so $r\le g/8$ is allowed when $|q_j|\ge g/4$): on the sheets of T_j the only singular point is v_j, which is outside the disc; if the disc crosses an edge, the part beyond lies over B next to σ_j with $\operatorname{Re}(p/v_j) \ge 1+\operatorname{Re}(q_j/v_j) > 1$, so in $|p|>1$, where there is no v_k, and it meets slits only as chords, as in (l2a). In (l2a) and (l2b) the disc lifts unless it meets the top $|p| = 1+R$, an exit, or an end.

Two classes. (i) $\xi\in T_j$ with $|\ell(\xi)| \ge E-\frac12$ and $\operatorname{Re}\ell(\xi)\le\Lambda-3$. (ii) $|p(\xi)|\ge R/40$. On a tract, $\operatorname{Re}\ell > \Lambda-3$ gives $|q_j| \ge Re^{-3}$, hence $|p| \ge Re^{-3}-1 \ge R/40$, that is (ii), as $R\ge 81$.

Lemma 5.13 (Paths)**.** *If ξ is in neither class, then $d_h(o,\xi)\le 40\log(4ER/g)$. So, with*

$$\epsilon_0 \;:=\; \frac12\Big(\frac{g}{4ER}\Big)^{40},$$

every ξ with $1-|\zeta(\xi)|\le\epsilon_0$ is in class (i) or (ii).

Proof. We bound d_h by the cost $\int 2\,|dc|/r$ of a path, where r is the radius of a lifted chart disc about the moving point c, measured in the same chart.

B-points. Use p-discs of radius $\frac12\min_k|p-v_k|$, which lift by (l2a) unless they meet the top. (b1) If $|p(\xi)|\le 1$: the segment $[0,\xi]$. On it $\min_k|p-v_k| \ge \max\big(1-|p|,\ \min(1,\min_k|q_k(\xi)|)/\sqrt2\big)$ (for each k, the point of the segment nearest v_k is ξ, or the foot of the perpendicular, at distance $\sin|\arg\xi-\theta_k| \ge |q_k(\xi)|/\sqrt2$, or else the distance is at least 1), and $\min_k|q_k(\xi)|\ge g/4$; the cost is at most $4\log(4\sqrt2/g)+4 \le 4\log(6/g)+4$. (b2) If $1<|p(\xi)|\le 2$: out from 0 along the ray midway between the two slits next to $\arg\xi$ to $|p|=2$; on it $\min_k|p-v_k| \ge \max(g/4,\ \big||p|-1\big|)$, and the cost is at most $2\big(4+4\log(4/g)\big) = 8\log(4/g)+8$. Then along $|p|=2$ inside that sector to $\arg\xi$ (an arc of angle less than g that meets no slit), with discs of radius $\frac12$: cost at most 2. Then radially in to ξ; on this segment the point nearest each v_k is ξ itself, so $\min_k|p-v_k| \ge \max(g/4, |p|-1)$, and the cost is at most $4\log(4/g)+4$. (b3) If $2<|p(\xi)|<R/40$: as in (b2) to $|p|=2$ and along the arc, then radially out with discs of radius $|p|/4$ (they contain no v_k, meet slits only as chords,

and stay below the top and the exits): cost at most $8\log R$. In all three cases the cost is at most $12\log(8/g)+8\log R+50$.

Points of a tract T_j, not in (i) or (ii); so $\operatorname{Re}\ell(\xi)\le\Lambda-3$ and $E-|\ell(\xi)|\ge\frac12$. Put Conv $:=\{\operatorname{Re}\ell<-\frac12,\ |\ell|<E\}$, a convex set; ℓ-discs centered in Conv with radius at most half the distance to $\partial\,$Conv lie in $\{\operatorname{Re}\ell<0,\ |\ell|<E\}$, which is part of T_j, so they lift. (t0) From o to the window by (b1) along the segment towards v_j: cost at most $4\log(6/g)+4$. From the window to $\ell=-1$ with ℓ-discs of radius $\frac12$ (by (l1)): cost at most 4. Then along the negative real axis to $c_0:=-E/2$, with radius half the distance to $\partial\,$Conv: cost at most $8\log E+8$. (t1) If $\operatorname{Re}\ell(\xi)<-1$: the segment from $\ell(\xi)$ to c_0 lies in Conv, and the distance to $\partial\,$Conv along it is at least the linear interpolation of its end values (a concave function on a convex set), which are at least $\frac12$ and $E/2-\frac12$; the cost is at most $12.1\log(2E)$. (t2) If $-1\le\operatorname{Re}\ell(\xi)\le\Lambda-3$ and $|\operatorname{Im}\ell(\xi)|\ge\pi+1$: move along $\operatorname{Im}\ell=\text{const}$ to $\operatorname{Re}\ell=-1$ with p-discs of radius $|q_j|/2$; in the ℓ-chart they lie in $B(\ell,0.7)$, so they stay on the sheets of T_j, away from the half-strip $\{\operatorname{Re}\ell\ge0,\ |\operatorname{Im}\ell|\le\pi\}$, the exit and the end (if $E-|\ell|<1$ near ξ, first step radially inwards by $\frac12$, at cost at most 4); the ℓ-length is at most $\Lambda+1$ and the cost at most $5(\Lambda+2)$; then continue as in (t1). (t3) If $|\operatorname{Im}\ell(\xi)|<\pi+1$ and $\operatorname{Re}\ell(\xi)\ge-1$ (ξ on the first sheets next to an edge, or near the window): first turn around v_j at fixed $|q_j|$, with p-discs of radius $\frac12\min_k|q_k|$, until $|\operatorname{Im}\ell|=\pi+1$. These discs lift by (l2a) at centers in the strip, which is part of B, and by (l2b) on the sheets of T_j. Along this arc $\min_k|q_k|\ge0.54|q_j|$: if $\operatorname{Re}\ell(\xi)\ge0$, then $\pi<|\operatorname{Im}\ell|\le\pi+1$ on the arc, so $\operatorname{Re}(p/v_j)=1-|q_j|\cos(\operatorname{Im}\ell)\ge1+0.54|q_j|$, and $|q_k|\ge\operatorname{Re}(p/v_j)-\operatorname{Re}(v_k/v_j)\ge0.54|q_j|$ as $\operatorname{Re}(v_k/v_j)\le1$; if $\operatorname{Re}\ell(\xi)<0$, then $|q_j|<g/4$ on the arc while $|q_k|\ge|v_k-v_j|-|q_j|>g/4$ for $k\ne j$. So the ℓ-radius is at least $\log1.27>0.22$; the ℓ-length is at most $2\pi+2$, and the cost at most 80. Then continue as in (t2).

The largest total is that of case (t3), whose path is (t0) followed by the paths of (t1), (t2) and (t3), run backwards from c_0 to ξ: its cost is at most

$$\begin{aligned}&\big(4\log\tfrac6g+4+4+8\log E+8\big)+80+\big(4+5(\Lambda+2)\big)+12.1\log(2E)\\&\quad=\ 20.1\log E+4\log\tfrac6g+5\Lambda+110+12.1\log2,\end{aligned}$$

which is at most $20.1\log E+4\log(6/g)+5\Lambda+118.4$; cases (t1) and (t2) cost less. So the total is at most this for points of a tract and $12\log(8/g)+8\log R+50$ for B-points. As $\log(8/g)\le\Lambda$, $\log R\le\Lambda$ and $\Lambda\ge24$, both are at most $40(\log E+\Lambda)=40\log(4ER/g)$. Finally, $1-|\zeta(\xi)|\ge e^{-d_h(o,\xi)}\ge(g/(4ER))^{40}=2\epsilon_0$. □

Lemma 5.14 (Chart radius)**.** *For every $\xi\in\Sigma$, $r_c(\xi)\ge g\,(1-|\zeta(\xi)|)^2/1000$.*

Proof. If $r_c(\xi)=g/8$ there is nothing to prove. Otherwise write $r_c:=r_c(\xi)<g/8$. We find the point η_b of $\partial\Sigma$ that stops the chart disc and a disc $B(\eta_b,r)$ about it (Step 1), show that Σ lies in a thin disc segment S inside this disc (Step 2), bound the harmonic measure $\omega_\Sigma(\xi,\partial\Sigma\setminus B(\eta_b,r))$ from above by $7r_c/r$ (Step 3) and from below by $(1-|\zeta(\xi)|)/4$ (Step 4), and compare the two (Step 5).

Step 1: the obstruction η_b, and the disc $B(\eta_b,r)$. By (l1), (l2) the obstruction is a point η_b of $\partial\Sigma$ at chart distance r_c from ξ: in the ℓ-chart (where $\operatorname{Re}\ell<0$) only the end circle $|\ell|=E$, with the domain inside it; in the p-chart the top circle, an exit circle $|q_j|=R$, an end, the corner of an exit and an end, or the point where the top meets an exit over $(1+R)v_j$. In the p-chart an end is, inside the disc $B(\eta_b,r)$ chosen below, a curve of curvature at most $\kappa_E:=6.2/(gE)$: with $\ell=Ee^{it}$ we have $dp/dt=i\ell q_j$ and $p''/p'=i(1+\ell)$, so its signed curvature is $\operatorname{Re}(1+\ell)/(E|q_j|)=4(1+a)e^{-a}/(gE)$, $a:=\operatorname{Re}\ell$; in $B(\eta_b,r)$ we have $|q_j|>g/16$, that is $a>-\log4$, and $|1+a|e^{-a}\le1$ for $a\ge0$, $|1+a|e^{-a}<1.55$ for $-\log4<a<0$, so the curvature is less than $4\cdot1.55/(gE)=\kappa_E$. Put $\kappa:=\max(1/R,\kappa_E)\le1/g$ and $r:=\min\big(g/16,\ (r_c/\kappa)^{1/2}\big)$ (in the p-chart, $|q_j(\eta_b)|>g/8$ at an end point, and the disc of radius $g/16$ stays in $|q_j|>g/16$, so it does not wind around v_j). Assume $r_c\le r/10$ (otherwise the inequality of Step 5 holds trivially). We use the *lift* of $B(\eta_b,r)$ that contains ξ (it lies over one tract and the part of the base next to it). Below, $\partial\Sigma\cap B(\eta_b,r)$ and

$\Sigma \cap B(\eta_b, r)$ mean the parts of $\partial\Sigma$ and of Σ on this lift, not on the whole preimage of the plane disc, which meets many sheets.

Step 2: $\Sigma \cap B(\eta_b, r)$ lies in the disc segment $S := B(\eta_b, r) \cap H'$, where H' is the tangent half-plane at η_b moved out by r_c. That is, inside $B(\eta_b, r)$ the domain lies beyond the tangent line at η_b (a supporting line at the corner) by at most $\kappa r^2 \le r_c$. By the kind of η_b, the pieces of $\partial\Sigma$ in $B(\eta_b, r)$ are these. (e) η_b on an end: in the ℓ-chart only the end circle, with the domain inside it, so no bulge; in the p-chart the end, which bulges by at most $\kappa_E r^2$, and, if a corner is within r, the exit beyond it. (ex) η_b on an exit: the exit, an arc of the circle $|q_j| = R$ with the domain of T_j inside it, so no bulge; if a corner is within r, the end beyond it; and, if the *junction*, the point over $(1+R)v_j$ where the top meets this exit, is within r, the top there and the part of the base across σ_j. That part is not inside the exit circle, but it has $\mathrm{Re}(q_j/v_j) \le R$, as $|p| \le 1+R$, and it lies on the side of the line $\{\mathrm{Im}(q_j/v_j) = 0\}$ (which carries σ_j) away from η_b; so, with $\psi \in [0, \pi/2]$ the angle of η_b from the junction seen from v_j (say $q_j/v_j = Re^{i\psi}$ at η_b, and $\mathrm{Im}(q_j/v_j) \le 0$ on that part), it lies at height $\mathrm{Re}(e^{-i\psi} q_j/v_j) - R = \cos\psi\,\mathrm{Re}(q_j/v_j) + \sin\psi\,\mathrm{Im}(q_j/v_j) - R \le R(\cos\psi - 1) \le 0$ over the tangent of the exit at η_b: no bulge. (t) η_b on the top: the circle $|p| = 1+R$, with the domain inside it (also the part of T_j near the junction, as $|q_j| < R$ there gives $|p| < 1+R$), so no bulge; and, near the junction, the exit there. In (e), (t), and in (ex) for the end beyond a corner, a second piece only removes more from the domain, as the domain lies on the inner side of each piece. (c1) η_b a corner of an exit and an end: the two pieces meet at the angle $\pi/2 + \arcsin(\Lambda/E) < \pi$ on the side of the domain (in the ℓ-chart; the p-chart keeps angles), so a supporting line at η_b exists, and the domain lies beyond it by at most the bulge $\kappa_E r^2$ of the end. (c2) η_b the junction: the two circles, tangent there, and the part of the base across σ_j; this is the case $\psi = 0$ of (ex), so no bulge. So, in every case, $\Sigma \cap B(\eta_b, r)$ lies in S.

Step 3: $\omega_\Sigma(\xi, \partial\Sigma \backslash B(\eta_b, r)) \le 7r_c/r$. By Step 2 and the maximum principle, $\omega_\Sigma(\xi, \partial\Sigma \backslash B(\eta_b, r)) \le \omega_S(\xi, \text{arc of } S)$. If r_c were 0, S would be a half-disc; for the unit half-disc, the harmonic measure of the semicircle at a point is $2 - \frac{2}{\pi}\theta$, where θ is the angle at that point under which the diameter is seen. For S itself we use a Möbius map. The Möbius map that sends the ends of the chord of S to 0 and ∞ takes S onto a sector of angle at most $\pi/2 + 2r_c/r$, with the chord on one side: the angle of the sector is the angle of S at an end of its chord, which is $\pi/2 + \arcsin(r_c/r)$, as the chord lies at distance r_c from the center η_b (on the far side from the domain), and $\arcsin(r_c/r) \le 2r_c/r$; $\omega_S(\cdot, \text{arc})$ is the angle from that side divided by the angle of the sector; and ξ lies within r_c of η_b, at height at most $2r_c$ over the chord, where that angle is at most $4.5r_c/r$: it is π minus the angle under which the chord is seen from ξ, that is, the sum of the two base angles of the triangle formed by ξ and the ends of the chord, and the foot of ξ on the line of the chord is at distance at least $(r^2 - r_c^2)^{1/2} - r_c \ge 0.89r$ from each end, so each base angle is at most $2r_c/(0.89r) < 2.25r_c/r$. So $\omega_\Sigma(\xi, \partial\Sigma \setminus B(\eta_b, r)) \le 7r_c/r$.

Step 4: $\omega_\Sigma(\xi, \partial\Sigma \setminus B(\eta_b, r)) \ge (1 - |\zeta(\xi)|)/4$. The *period* j is the part of $\partial\Sigma$ over j in the table of §5.3 (the upper exit, A_j, the lower exit and the following arc of the top); the M periods are rotated copies of one another, they cover $\partial\Sigma$, and two of them meet at most at an end point. The rotation fixes o, so the M periods have the same harmonic measure at o, namely $1/M$. The piece $\partial\Sigma \cap B(\eta_b, r)$ (on the lift of Step 1) meets at most two neighboring periods. Indeed, by Step 1 its exits and end are those of one tract T_j, of period j. On the top circle the arcs of periods $k-1$ and k meet at the outer end $(1+R)v_k$ of σ_k, and these ends are more than g apart; so the piece, of diameter at most $2r \le g/8$, meets the top arcs of at most two neighboring periods, and these are $j-1$ and j if it also meets T_j, as the lift reaches the base from T_j only across the slit σ_j (the edges of T_j are glued to its sides, and the strip meets the base only in $\{|q_j| < g/4\}$), so its points on the top circle lie within $2r \le g/8$ of σ_j, that is, near its outer end $(1+R)v_j$. So the harmonic measure of the piece at o is at most $2/M \le \frac{1}{2}$. Harnack's inequality in $\mathbb{D}$ for $u := \omega_\Sigma(\cdot, \partial\Sigma \setminus B(\eta_b, r)) \circ \zeta^{-1}$, with $u(0) \ge \frac{1}{2}$, gives $u(\zeta) \ge (1 - |\zeta|)/4$; at $\zeta = \zeta(\xi)$ this is the claim.

Step 5: the conclusion. By Steps 3 and 4, $1-|\zeta(\xi)| \le 28 r_{\mathrm{c}}/r$. If $r = g/16$ this gives $r_{\mathrm{c}} \ge g(1-|\zeta(\xi)|)/448$; if $r = (r_{\mathrm{c}}/\kappa)^{1/2}$, it gives $r_{\mathrm{c}} \ge (1-|\zeta(\xi)|)^2/(784\kappa) \ge g(1-|\zeta(\xi)|)^2/784$. Both are at least $g(1-|\zeta(\xi)|)^2/1000$. □

The ratio on the outer layer. Recall $r_\Omega \ge e^{-\lambda}$ and $r_\Sigma \le e^{\lambda}$ (§5.6, (31)). Let ξ satisfy $1-|\zeta(\xi)| \le \epsilon_0$, and let $y := z(\xi)$. In class (i), Lemma 5.4 gives $|p(\xi)| \ge 1 - e^{-B_0} \ge 1-\lambda$ (as $E \ge e^{52T}$), and (35) gives $|y| \le (1+1.2\lambda)/4$; so by (25)

$$\Big|\frac{\hat f(y)}{y}\Big| \;\ge\; 4e^{-2\lambda}\,\frac{1-\lambda}{1+1.2\lambda} \;\ge\; 4e^{-5\lambda},$$

as $(1-\lambda)/(1+1.2\lambda) \ge e^{-3\lambda}$ for $\lambda \le 0.3$. In class (ii), $|p(\xi)| \ge R/40 \ge 2S_U z_{\max}/F'(0)$ by (30), and $|y| \le z_{\max}$; so $|\hat f(y)/y| \ge (F'(0)/e^{\lambda})\cdot 2S_U/F'(0) = 2S_U e^{-\lambda} \ge 4$. Put $t_1 := 1-\epsilon_0$ and $K_1 := Z(t_1\overline{\mathbb{D}})$. Every $y \in \Omega \setminus K_1$ is $z(\xi)$ for some ξ with $1-|\zeta(\xi)| < \epsilon_0$, so by Lemma 5.13

$$\Big|\frac{\hat f(y)}{y}\Big| \;\ge\; 4e^{-5\lambda} \qquad \text{on } \Omega \setminus K_1. \tag{36}$$

5.9. **Green functions.** For $i = 1,2,3,4$ put $t_i := 1-\epsilon_0 2^{1-i}$, $K_i := Z(t_i\overline{\mathbb{D}})$, and let g_i be the Green function of $X_i := \widehat{\mathbb{C}} \setminus K_i$ with pole at ∞, extended by 0 on K_i. Then $K_1 \subset K_2 \subset K_3 \subset K_4 \subset \Omega$, so $g_4 \le g_3 \le g_2 \le g_1$.

Distances. For $|\zeta| \le t < 1$, Koebe's theorems [15, §2.3] give $\operatorname{dist}(Z(\zeta), \partial\Omega) \ge |Z'(\zeta)|(1-|\zeta|^2)/4$ and $|Z'(\zeta)| \ge r_\Omega(1-|\zeta|)/(1+|\zeta|)^3$. So $\operatorname{dist}(K_i, \partial\Omega) \ge r_\Omega(1-t_i)^2/16$, and a curve from ∂K_i to ∂K_{i+1} has length at least $r_\Omega(1-t_{i+1})(t_{i+1}-t_i)/8$. As $r_\Omega \ge e^{-\lambda} \ge \frac12$, both are at least

$$\delta_0 \;:=\; \epsilon_0^2/2048 \qquad (i \le 4,\ \text{resp. } i \le 3).$$

Lemma 5.15. *For $i \le 4$, $g_i \ge \exp(-3\cdot 10^4\, T)$ on $\partial\Omega$.*

Proof. For $y \in \partial\Omega$, $g_i(y) = \log\coth(d_h/2) \ge 2e^{-d_h}$, where d_h is the hyperbolic distance in X_i from ∞ to y (the density $2/(1-|\cdot|^2)$ of $\mathbb{D}$, after a Riemann map of X_i onto $\mathbb{D}$ sending ∞ to 0). We bound d_h by the cost of a path from y to ∞, with density at most $2/\varrho$ at the center of a chart disc of radius ϱ inside X_i. Every point y' outside Ω has $\operatorname{dist}(y', K_i) \ge \max(\delta_0, \operatorname{dist}(y', \partial\Omega))$, since a segment from y' to K_i crosses $\partial\Omega$. We work in the Y-chart. On the Y-neighborhood of width w of $\partial\Omega$ we have $|x| \ge g^2/260$ (the worst place is below the first long channel, at $|b| \ge g/4 - 1.5w \ge g/8$) and $0.12g \le |\sinh Y| \le 3$, so $|dz/dY| = |\sinh Y|/(8|x|^2)$ lies in $[g/600,\ 10^5/g^4]$; this is the only place where z-distances are converted, and we put $\delta_Y := \delta_0\, g^4/(2\cdot 10^5)$. The path has four parts, drawn for one point y in Figure 7: (a) away from $\partial\Omega$, (b1) a first stretch away from the mouths, (b2) along the *lanes* to the free strip, and (c) down the free strip to $z=\infty$. For each part we state the distance to $\partial\Omega$ that it keeps, its Y-length and its cost.

(a) From y to a point P_0 at Y-distance $w/4$ from $\partial\Omega$. The Y-distance to $\partial\Omega$ is at least one third of the Y-length traveled, so the Y-length is at most $3w/4$; the cost is at most $6+6\log(w/(4\delta_Y))$. The path goes from y into the complement of Ω. Such a path exists: in the Y-chart the complement of Ω is bounded by straight sides and the quarter circles of the elbows (at such an arc the complement is convex), with all corner angles at least $\pi/2$ (wall–neck $\pi/2$, neck–wedge $\pi-\alpha$, wedge–horizontal part $\pi+\alpha$, ends and elbow at least $\pi/2$), so a cone of aperture $\pi/4$ leaves every boundary point. By the kind of y: on a straight side of a horizontal or a vertical part, on the quarter circle of an elbow, or on the wall at distance at least w from every mouth, the path is the segment along the normal into the complement, where the corridor has width at least w, so the distance grows at rate 1; at the outer corner of an elbow, at a corner of an end, or at a corner between a wedge and a horizontal part (angle at least π in the complement), it is the segment along the bisector of the corner, and the distance again grows at rate 1; on an end it is the segment along the outward normal (increasing $|b|$ for a long channel, decreasing a for a short one), into the region beyond the

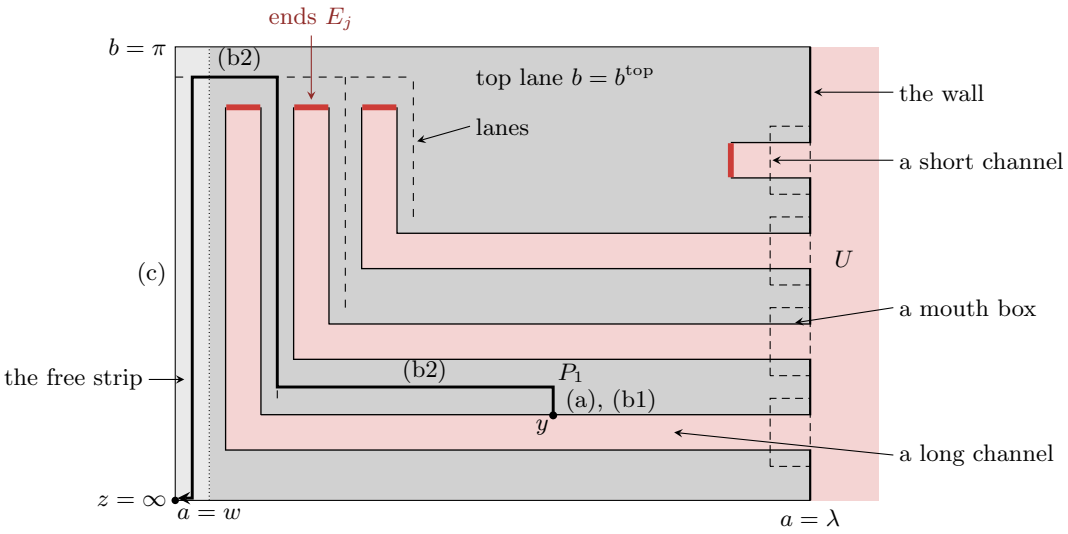

FIGURE 7. The path in the proof of Lemma 5.15 (schematic, not to scale). The upper half $\{b > 0\}$ of the chart $Y = a + ib$ (the lower half is its mirror image): three long channels and one short channel, tinted like Ω, with their ends E_j in color; the complement of Ω is gray, and the free strip $\{0 < a < w\}$ is lighter. Dashed: the lanes, at distance $w/2$ from the vertical parts of the long channels, the top lane $b = b^{\mathrm{top}}$ above the ends, and the mouth boxes at the wall. The path from a point y on the upper side of the first long channel: (a) and (b1) lead away from $\partial\Omega$ to P_1; (b2) runs in the direction of decreasing a to the first lane, up that lane, and along the top lane to the line $a = w/2$; (c) runs down the free strip to $Y = w/2$ and on to $z = \infty$.

end, which no channel meets within w; near a mouth (on the wall within w of the mouth, on the neck or the wedge, or at their corners) we take the straight segment from y in the direction of the segment from the wall–neck corner to $(\lambda - w/2,\ b_j \pm 3w/4)$, at the angle $\arctan(3/2)$ from the inward normal of the wall. Along it the distance to the wall grows at rate $\cos(56.3^\circ) \geq 0.55$ and the distance to the side of the neck or wedge at rate at least $\sin(56.3^\circ - 26.6^\circ) \geq 0.49$, both more than $\frac{1}{3}$ (the side of the wedge has slope at most $\frac{1}{2}$), and at its end the distance to $\partial\Omega$ is at least $0.44w$. Elsewhere the path goes into a corridor (of width at least w) or beyond an end. With discs of radius $\max(\delta_Y, s/3)$ at Y-length s, the cost is at most $6 + 6\log(w/(4\delta_Y))$.

Lanes and mouth boxes. Parts (b1) and (b2) lead from P_0 to the mid-line $a = w/2$ of the free strip $\{0 < a < w\}$. We describe the half $\{b > 0\}$; the other half is its mirror image. Put $b^{\mathrm{top}} := \pi - \sqrt{\lambda}/2 + g/4 + w/2$; then $b^{\mathrm{top}} < \pi$, and every long end lies in $\{b \leq b^{\mathrm{top}} - w/2\}$, as $|b_j - \beta_j| \leq g/4$. The *lane* of the k'-th long channel ($1 \leq k' \leq J$) is the segment $\{a = a_{(k')} + w,\ b_{(k')} + w \leq b \leq b^{\mathrm{top}}\}$, and the *top lane* is $\{0 \leq a \leq a_J + w,\ b = b^{\mathrm{top}}\}$. Every point of a lane is at distance at least $w/2$ from $\partial\Omega$: the lane of the k'-th channel runs at distance $w/2$ from its vertical part, end included, and its lowest point is the center of the quarter circle of its elbow; the later long channels lie in $\{a \geq a_{(k')} + 3w/2\}$, as $a_{(k'')} \geq a_{(k')} + 2w$ for $k'' > k'$, the earlier ones lie in $\{a \leq a_{(k')} - 3w/2\} \cup \{b \leq b_{(k')} - g/2 + w/2\}$ by Check (a), and the short channels lie in $\{a > 7\lambda/8 - w_m - w\}$ (Check (b)), while $a_J + w = (2J+1)w < \lambda/2$; the top lane passes at least $w/2$ above every long end and to the left of every short channel. The *mouth boxes* are the rectangles $\{a > \lambda - w_m - w,\ |b - b_j| < w\}$.

(b1) First stretch: from P_0 to a point P_1 at distance at least $w/2$ from $\partial\Omega$, outside the mouth boxes. The distance to $\partial\Omega$ stays at least $w/4$, the Y-length is at most w, the discs have radius $w/4$,

and the cost is at most $2w/(w/4) = 8$. Its points lie near the wall or within $w/2$ of a channel, so $a \geq w$ by Check (c), and the discs lie in $\{a > 0\}$. We continue the last segment of (a) in its own direction to P_1. By the kind of y in (a): on a side, an elbow, an end, the wall at distance at least w from every mouth, or a corner other than that between a wedge and a horizontal part, the distance to the piece of y grows at rate 1, and after a further Y-length $w/4$ the point is at distance $w/2$ from $\partial\Omega$, as the corridor has width at least w, no channel meets the region beyond an end within w, and the vertical parts next to a long end are at distance w from it; this point lies outside the mouth boxes. At a corner between a wedge and a horizontal part we continue along the bisector until $|b - b_j| = w$, which takes Y-length at most $0.52w$ from the corner; the corner, of angle $\pi + \alpha$ in the complement, stays the nearest point of $\partial\Omega$. Near a mouth we continue until the point lies in $\{a \leq \lambda - w/2,\ |b - b_j| \geq w\}$ on the side of y, which is at distance at least $w/2$ from $\partial\Omega$, as the half-widths of neck, wedge and horizontal part are at most $w/2$ and the other channels are at least $g/2 - w$ away (Check (b)); the segment has slope $3/2$ in the coordinates $\lambda - a$ and $|b - b_j|$, so this takes Y-length at most $1.21w$ from y, of which (a) used at least $w/4$, the distance growing at rate at most 1. On the way the distance to $\partial\Omega$ does not decrease. Take y above the axis $b = b_j$ (otherwise the mirror image). Near the mouth let Ω^+ be $\{a > \lambda\}$ together with $\{b - b_j < \varphi(\lambda - a)\}$, where the half-width φ of neck, wedge and horizontal part does not decrease and has slope at most $\frac{1}{2}$ in the same coordinates. Then $\Omega \subset \Omega^+$ there, and a point of the complement of Ω above the channel has the same distance to Ω^+ as to Ω, because a segment from it to a point of $\Omega^+ \setminus \Omega$ crosses the channel or $\{a > \lambda\}$. Every translate of Ω^+ backwards along the segment lies in Ω^+, as the segment has slope $3/2 > \frac{1}{2}$; so, as the point moves along the segment, its distance to Ω^+, which is its distance to Ω, does not decrease.

(b2) Second stretch: from P_1 along the lanes to the line $a = w/2$. The distance to $\partial\Omega$ stays at least $w/2$, the Y-length is at most $\pi + 2\lambda$, and the cost is at most $5(\pi + 2\lambda)/w + 4\log 1.6$. We use discs of radius $0.4w$. The path: (i) if P_1 lies in $\{|a - a_j| < w\}$ above the end of a long channel j, up to $b = b^{\mathrm{top}}$ (the vertical parts next to it are at least $w/2$ away, and the distance to the end grows), then along the top lane; (ii) otherwise in the direction of decreasing a to the first lane that the path meets, up that lane to b^{top}, and along the top lane; if it meets no lane, straight on to the free strip. On the first leg of (ii) the distance stays at least $w/2$: a channel j with $|b(P_1) - b_j| \geq w$ is passed at least $w/2$ from its neck, wedge and horizontal part, whose half-widths are at most $w/2$; a channel with $|b(P_1) - b_j| < w$ lies in $\{a > a(P_1)\}$, since P_1, outside the mouth boxes and at distance at least $w/2$ from $\partial\Omega$, cannot lie above or below it; and if a long vertical part, end included, comes within $w/2$ of the leg, then its channel, say the k'-th, has $b_{(k')} + w \leq b(P_1) \leq b^{\mathrm{top}}$ and $a_{(k')} + w \leq a(P_1)$ (otherwise P_1 would be within $w/2$ of that channel, or in case (i)), so the leg meets the lane of k' first. The legs have Y-length at most λ, π and λ, so (b2) has Y-length at most $\pi + 2\lambda$. With discs of radius $0.4w$, except on its last piece from $a = 0.8w$ down to $a = w/2$, where the radius is $a/2$ (the chart folds at $a = 0$), it costs at most $5(\pi + 2\lambda)/w + 4\log 1.6$. So (b1) and (b2) together cost at most $5(\pi + 2\lambda)/w + 4\log 1.6 + 8 \leq 42/w$.

(c) Down the free strip from the line $a = w/2$ to $Y = w/2$ ($b = 0$), then to $x = 0$ ($z = \infty$). The first piece has Y-length at most π; the cost is at most $26/w + 2$. With discs of radius $w/4$ (no channel meets $\{a < 3w/2\}$, by Check (c)), the first piece costs at most $8\pi/w \leq 26/w$. There $|x| = \sinh^2(w/4) \leq w^2/15 < g^2/200$, and the disc $\{|x| < g^2/110\}$ lies in X_i by Check (e), so the last step to $x = 0$ ($z = \infty$) costs at most $\log \frac{1+110/200}{1-110/200} < 2$.

Total. Adding (a), (b1), (b2) and (c), as $w/4 < 1$, $d_h \leq 68/w + 6\log(1/\delta_Y) + 10$. Here $68/w \leq 68(2M + 4)/\lambda \leq 136T + 272/\lambda$ and $272/\lambda = (272/M)T \leq 0.3T$. Further $\log(1/\delta_Y) = \log(2 \cdot 10^5 \cdot 2048) + 2\log(1/\epsilon_0) + 4\log(1/g)$ and $\log(1/\epsilon_0) = 40(\log E + \Lambda) + \log 2 \leq 2080T + 40(\Lambda + \log(2000\Lambda) + 1) + \log 2$, by (34). So

$$d_h \ \leq\ 136T + 0.3T + 24960T + 480\big(\Lambda + \log(2000\Lambda) + 1\big) + 24\log\frac{1}{g} + 138.$$

With $\Lambda \le 3\log(M/2\pi) + 11.13$ and $T \ge 50M$, the last three terms are at most $0.4T$ for every $M \ge 1000$: they are below 18700 at $M = 1000$, where $0.4T \ge 20M = 20000$, and their derivative with respect to $\log M$ is at most $480(3 + 3/24) + 24 \le 1524$, while that of $20M$ is $20M \ge 20000$. So $d_h \le 25800T \le 3 \cdot 10^4 T$, and $g_i(y) \ge 2e^{-d_h}$. □

Lemma 5.16. *For $i \le 3$, $g_i \ge 0.49 \min_{\partial\Omega} g_i$ on ∂K_{i+1}.*

Proof. The function g_i is harmonic on $\Omega \setminus K_i$ and vanishes on ∂K_i, so it is at least $\min_{\partial\Omega} g_i$ times the harmonic measure of $\partial\Omega$ in $\Omega \setminus K_i$. By conformal invariance (the annulus $t_i < |\zeta| < 1$), that harmonic measure equals $\log(t_{i+1}/t_i)/\log(1/t_i) \ge 0.49$ on ∂K_{i+1}, as $\epsilon_0 \le 0.01$. □

The level β_*. Put $\beta_* := 0.49 \min_{\partial\Omega} g_4$. By Lemma 5.15, $\beta_* \ge 0.49\, e^{-3\cdot 10^4 T} \ge \exp(-3.1 \cdot 10^4 T)$. Also $\beta_* \le 0.49 \log(1/t_4) < \epsilon_0/8 < 1$. Indeed, $u := g_4 \circ Z$ is harmonic on $t_4 < |\zeta| < 1$, vanishes on $|\zeta| = t_4$, and has flux 2π (that of g_4 around K_4), so its mean over $|\zeta| = r$ is $\log(r/t_4)$. For each $r < 1$ some point of that circle has $u \le \log(r/t_4) < \log(1/t_4)$; the images of these points under Z leave every compact subset of Ω as $r \to 1$ and stay bounded, so they accumulate at a point of $\partial\Omega$, where g_4 is continuous (it is harmonic off K_4). So $\min_{\partial\Omega} g_4 \le \log(1/t_4)$, and with $t_4 = 1 - \epsilon_0/8$, $\log(1/t_4) \le (\epsilon_0/8)/(1 - \epsilon_0/8) \le \frac{8}{7} \cdot \frac{\epsilon_0}{8}$, so $\beta_* \le 0.56\, \epsilon_0/8$.

Moreover,

$$\{g_i < \beta_*\} \subset Z(t_{i+1}\mathbb{D}) \quad (i \le 3), \qquad \{g_4 < \beta_*\} \subset \Omega. \tag{37}$$

Indeed, outside Ω, $g_i \ge \min_{\partial\Omega} g_i \ge \min_{\partial\Omega} g_4 > \beta_*$ (minimum principle on the complement of $\overline{\Omega}$, where $g_i \to \infty$ at the pole); and on $\Omega \setminus Z(t_{i+1}\mathbb{D})$ both boundary curves carry values at least β_*, by Lemma 5.16 and $g_i \ge g_4$.

An upper bound. By Koebe's one-quarter theorem K_1 contains the disc of radius $r_\Omega t_1/4 \ge 1/10$ about 0, so its logarithmic capacity is at least $1/10$. As $g_1(z) = \int \log|z - y'|\, d\nu_1(y') - \log \operatorname{cap} K_1$, with ν_1 the equilibrium measure of K_1 [29, §4.4], and $\overline{\Omega} \subset \{|z| \le z_{\max}\}$,

$$g_1 \;\le\; \log(2z_{\max}) + \log 10 \;=\; \log(20 z_{\max}) \qquad \text{on } \overline{\Omega}, \text{ in particular on } K_4. \tag{38}$$

5.10. **Faber polynomials.** Let $K \subset \mathbb{C}$ be a closed Jordan domain. Let Ψ be the conformal map of $\{|w| > 1\}$ onto $\mathbb{C} \setminus K$ with $\Psi(w)/w \to \operatorname{cap} K > 0$ as $w \to \infty$ (it exists by the Riemann mapping theorem for the simply connected domain $\widehat{\mathbb{C}} \setminus K$, with $\infty \mapsto \infty$), and $\Phi := \Psi^{-1}$. The Green function of $\widehat{\mathbb{C}} \setminus K$ with pole at ∞ is $g_K = \log|\Phi|$; we put $g_K := 0$ on K, so that g_K is continuous on $\mathbb{C}$ (Ψ extends to a homeomorphism of the closures, by Carathéodory's theorem). For $\rho > 1$ the level curve $\Gamma_\rho := \Psi(\{|w| = \rho\})$ is an analytic Jordan curve, and its interior is $\{g_K < \log\rho\}$. The *Faber polynomial* F_n ($n \ge 0$) is the polynomial part of the Laurent expansion of Φ^n at ∞: $\Phi^n = F_n + \mathcal{R}_n$, with $\mathcal{R}_n$ holomorphic on $\widehat{\mathbb{C}} \setminus K$ and $\mathcal{R}_n(\infty) = 0$. It has degree n.

For $\rho > 1$, $n \ge 0$ and $z \notin \Gamma_\rho$,

$$\frac{1}{2\pi i} \oint_{|w|=\rho} \frac{w^n\, \Psi'(w)}{\Psi(w) - z}\, dw \;=\; \begin{cases} F_n(z) & \text{if } z \text{ lies inside } \Gamma_\rho, \\ F_n(z) - \Phi(z)^n & \text{if } z \text{ lies outside } \Gamma_\rho. \end{cases} \tag{39}$$

Indeed, with $\eta = \Psi(w)$ the left side is $\frac{1}{2\pi i} \oint_{\Gamma_\rho} \big(F_n(\eta) + \mathcal{R}_n(\eta)\big)\, d\eta/(\eta - z)$. The term with F_n is $F_n(z)$ or 0, by Cauchy's formula. The function $\mathcal{R}_n(\eta)/(\eta - z)$ is $O(\eta^{-2})$ at ∞, so by the residue theorem in the outside of Γ_ρ the term with $\mathcal{R}_n$ is 0 if z is inside Γ_ρ, and $-\mathcal{R}_n(z) = F_n(z) - \Phi(z)^n$ if z is outside. If $z \in K$, the function $w \mapsto \Psi'(w)/(\Psi(w) - z)$ is holomorphic on $\{|w| > 1\}$ and vanishes at ∞, and by (39) its Laurent coefficients are the numbers $F_n(z)$:

$$\frac{\Psi'(w)}{\Psi(w) - z} \;=\; \sum_{n \ge 0} F_n(z)\, w^{-n-1} \qquad (z \in K,\ |w| > 1). \tag{40}$$

Lemma 5.17. *Let K be a closed Jordan domain, with Ψ, Φ, g_K and F_n as above.*

(a) If $z \in K$ and $|w| > 1$, then $|w\Psi'(w)/(\Psi(w) - z)| \le (|w| + 1)/(|w| - 1)$. Hence $|F_n(z)| \le \rho^n(\rho + 1)/(\rho - 1)$ for $z \in K$, $\rho > 1$ and $n \ge 0$.

(b) If $\rho > 1$ and $|\Phi(z)| \ge 2\rho - 1$, then $F_n(z) = \Phi(z)^n(1 + \varepsilon)$ with $|\varepsilon| \le (\rho/|\Phi(z)|)^n\, 4\rho/(\rho - 1)$.

(c) Let $0 < \beta \le 1$, and let f be holomorphic on $\{g_K < \beta\}$ with $|f| \le f_{\max}$ there. For $n \ge 0$ let $a_n := \frac{1}{2\pi i} \oint_{|w|=e^{\beta/2}} f(\Psi(w))\, w^{-n-1}\, dw$. Then for every integer $D \ge 0$ the polynomial $P_D := \sum_{n=0}^{D} a_n F_n$, of degree at most D, satisfies $|f - P_D| \le 64 f_{\max} \beta^{-2} e^{-D\beta/4}$ on K.

(d) (Bernstein–Walsh) If P is a polynomial of degree at most n, then $|P| \le \max_K |P| \cdot e^{n g_K}$ on $\mathbb{C}$.

Proof. (a) Fix $z \in K$, and put $h(t) := 1/(\Psi(1/t) - z)$ for $0 < |t| < 1$ and $h(0) := 0$. Then h is univalent on $\mathbb{D}$, and $th'(t)/h(t) = w\Psi'(w)/(\Psi(w) - z)$ for $w = 1/t$. The Koebe transform $\tilde{h}(u) := \big(h(\frac{u+t}{1+tu}) - h(t)\big)/\big((1 - |t|^2)h'(t)\big)$ is univalent on $\mathbb{D}$ with $\tilde{h}(0) = 0$ and $\tilde{h}'(0) = 1$, and $\tilde{h}(-t) = -h(t)/((1 - |t|^2)h'(t))$. The growth theorem $|\tilde{h}(u)| \ge |u|/(1 + |u|)^2$ [15, §2.3] at $u = -t$ gives $|th'(t)/h(t)| \le (1 + |t|)^2/(1 - |t|^2) = (1 + |t|)/(1 - |t|)$, the first claim. The second follows from (39), as z lies inside Γ_ρ: $|F_n(z)| \le \rho^n \max_{|w|=\rho} |w\Psi'(w)/(\Psi(w) - z)|$.

(b) Here $|\Phi(z)| \ge 2\rho - 1 > \rho$, so z lies outside Γ_ρ, and by (39)

$$F_n(z) - \Phi(z)^n \;=\; \frac{1}{2\pi i} \oint_{|w|=\rho} \frac{w^n\, \Psi'(w)}{\Psi(w) - z}\, dw.$$

For $|w| = \rho$, Ψ is univalent on $B(w, \rho - 1)$, and $\Phi(z) \notin B(w, \rho - 1)$ as $|\Phi(z) - w| \ge |\Phi(z)| - \rho \ge \rho - 1$. So $z \notin \Psi(B(w, \rho - 1))$, which contains the disc of radius $|\Psi'(w)|(\rho - 1)/4$ about $\Psi(w)$ by Koebe's one-quarter theorem. Hence $|\Psi(w) - z| \ge |\Psi'(w)|(\rho - 1)/4$, so $|F_n(z) - \Phi(z)^n| \le \rho^n \cdot 4\rho/(\rho - 1)$; divide by $|\Phi(z)|^n$.

(c) Put $\rho := e^{\beta/2}$ and $\rho_1 := e^{\beta/4}$; the closed interior of Γ_ρ lies in $\{g_K < \beta\}$. For $z \in K$, Cauchy's formula on Γ_ρ, the substitution $\eta = \Psi(w)$ and (40) give

$$f(z) \;=\; \frac{1}{2\pi i} \oint_{|w|=\rho} f(\Psi(w))\, \frac{\Psi'(w)}{\Psi(w) - z}\, dw \;=\; \sum_{n \ge 0} a_n F_n(z);$$

the series may be integrated term by term, since $|F_n(z) w^{-n-1}| \le (\rho_1/\rho)^n(\rho_1 + 1)/(\rho(\rho_1 - 1))$ on $|w| = \rho$, by (a). As $|a_n| \le f_{\max}\rho^{-n} = f_{\max} e^{-n\beta/2}$, (a) with ρ_1 (and $\rho = \rho_1^2$) gives

$$|f(z) - P_D(z)| \;\le\; f_{\max}\, \frac{\rho_1 + 1}{\rho_1 - 1} \sum_{n > D} \rho_1^{-n} \;=\; f_{\max}\, \frac{\rho_1 + 1}{(\rho_1 - 1)^2}\, \rho_1^{-D}.$$

As $\rho_1 = e^{\beta/4}$, $(\rho_1 + 1)/(\rho_1 - 1)^2 = \cosh(\beta/8)/\big(2e^{\beta/8} \sinh^2(\beta/8)\big) \le 32/\beta^2$, since $\cosh(\beta/8) \le e^{\beta/8}$ and $\sinh(\beta/8) \ge \beta/8$. So $|f - P_D| \le 32 f_{\max} \beta^{-2} e^{-D\beta/4}$ on K, which is less than the bound $64 f_{\max} \beta^{-2} e^{-D\beta/4}$ of (c).

(d) The function $u := \log|P| - n g_K$ is subharmonic on $\mathbb{C} \setminus K$ and bounded above near ∞, since $g_K(z) - \log|z|$ has a finite limit there; so it extends to a subharmonic function on $\widehat{\mathbb{C}} \setminus K$. As g_K is continuous and vanishes on ∂K, $\limsup u \le \log \max_K |P|$ at ∂K, and the maximum principle gives $u \le \log \max_K |P|$. On K the claim is clear. This is the Bernstein–Walsh inequality [29]. □

5.11. **The polynomial H.** We now prove Proposition 5.1. Let Ω, Z, $\hat{f}$, K_i, g_i, ϵ_0, δ_0 and β_* be as in §§5.5–5.9, and put

$$f_{\max} := 4 z_{\max}(1 + R), \qquad N' := (f_{\max} + 1)/\delta_0.$$

Sizes. Put $\mathcal{L} := \log T$. As $M = \lambda T \le 0.02\, T$ and $M \ge 1000$, we have $1/g = M/(2\pi) \le T/(100\pi)$ and $1/\lambda = T/M \le T/1000$; and $T \ge 5 \cdot 10^4$, so $\mathcal{L} \ge 10.8$ and $500\mathcal{L} + 100 \le 0.2\, T$ (this holds at

$T = 5\cdot 10^4$, and the left side grows more slowly). By (29), (30) (so $\Lambda \le \Lambda^+$ and $1+R \le 17001/g^2$), (34) (so $\log E \le 52T + \log(2000\Lambda) + 1$), Lemma 5.13, the definition of δ_0 and the level β_* (§5.9),

$$\log\frac{1}{g} \le \mathcal{L} - 5.74, \qquad \Lambda \le 3\mathcal{L} - 6.1, \qquad \log z_{\max} \le 2\mathcal{L} - 8.2, \qquad \log f_{\max} \le 4\mathcal{L} - 8.6,$$

$$\log E \le 52T + 2\mathcal{L}, \qquad \log\frac{1}{\epsilon_0} \le 2080T + 200\mathcal{L} + 1 \le 2121T, \qquad \log\frac{1}{\beta_*} \le 3.1\cdot 10^4 T, \tag{41}$$

$$\log\frac{1}{\delta_0} \le 4160T + 400\mathcal{L} + 10, \qquad \log N' \le 4160T + 404\mathcal{L} + 10.$$

Here $f_{\max} \le 4\cdot 26\cdot 17001\, g^{-4}$, $\log(1/\epsilon_0) = 40(\log E + \Lambda) + \log 2$, $\log(1/\delta_0) = 2\log(1/\epsilon_0) + \log 2048$, and $\log(2000\Lambda) + 1 \le \log(6000\mathcal{L}) + 1 \le 2\mathcal{L}$. In particular each of the nine quantities on the left of (41) is at most $3.2\cdot 10^4 T$, which is less than $10^{-3}E$, as $E \ge e^{52T}$. Also $\beta_* < \epsilon_0/8$, so $\log(1/\beta_*) \ge \log(1/\epsilon_0) \ge 40\log E$.

The derivative of $\hat f$. On Σ, $|p| < 1+R$ (on a tract, $|q_j| = (g/4)e^{\operatorname{Re}\ell} < (g/4)e^{\Lambda} = R$); $r_\Omega \le z_{\max}$ by Schwarz's lemma, as $\Omega \subset \{|z| \le z_{\max}\}$; and $r_\Sigma \ge 1 - g/4 \ge 0.99$ by Schwarz's lemma, as Σ contains the disc $\{|p| < 1 - g/4\}$ of the base about o (the slits lie in $|p| \ge 1$, the caps in $\{|q_j| < g/4\}$). So $|\hat f| \le z_{\max}(1+R)/0.99 \le f_{\max}$ on Ω, by (24). We claim that

$$|\hat f'| \;\ge\; e^{-2E} \qquad \text{on } K_3. \tag{42}$$

As $\hat f'$ has no zero on Ω, it suffices to prove this on $\partial K_3 = Z(t_3\mathbb{T})$, where $1 - |\zeta| = \epsilon_0/4$. Let $y = Z(\zeta_0)$ with $|\zeta_0| = t_3$, let $\xi := z^{-1}(y)$, and let $\mathfrak{c}$ (p or ℓ) be the chart at ξ (§5.8), with a lifted chart disc of radius $r_{\mathrm{c}}(\xi) \ge g\epsilon_0^2/16000$ (Lemma 5.14). The map $\mathfrak{c} \mapsto \zeta$ is univalent on that disc, with values in $\mathbb{D}$, so by Koebe's one-quarter theorem $|d\zeta/d\mathfrak{c}| \le 4(1-|\zeta_0|)/r_{\mathrm{c}}(\xi)$ at ξ. Next, $|dp/d\mathfrak{c}|$ is 1 in the p-chart and $|q_j| = (g/4)e^{\operatorname{Re}\ell} \ge (g/4)e^{-E}$ in the ℓ-chart; and $|Z'(\zeta_0)| \le z_{\max}/(1-|\zeta_0|)$ by the Schwarz–Pick lemma for $Z/z_{\max}$. By (24), $\hat f \circ Z = (r_\Omega/r_\Sigma)\, p\circ\zeta^{-1}$ and $r_\Omega/r_\Sigma \ge e^{-2\lambda}$, so

$$|\hat f'(y)| \;\ge\; e^{-2\lambda}\,\frac{g}{4}e^{-E}\cdot\frac{r_{\mathrm{c}}(\xi)}{4(1-|\zeta_0|)}\cdot\frac{1-|\zeta_0|}{z_{\max}} \;\ge\; \frac{g^2\epsilon_0^2\, e^{-E}}{5\cdot 10^5\, z_{\max}} \;\ge\; e^{-2E},$$

since $\log(5\cdot 10^5) + \log z_{\max} + 2\log(1/\epsilon_0) + 2\log(1/g) \le 14 + 5\cdot 10^{-3}E \le E$, by (41).

The polynomial P. By (37), $\hat f$ is holomorphic on $\{g_4 < \beta_*\} \subset \Omega$, with $|\hat f| \le f_{\max}$ there, and $0 < \beta_* < 1$. Lemma 5.17(c), for $K := K_4$, $\beta := \beta_*$ and

$$D \;:=\; \Big\lceil \frac{4}{\beta_*}\Big(3E + \log\frac{64 f_{\max}}{\beta_*^2}\Big)\Big\rceil,$$

gives a polynomial P_D of degree at most D with $|\hat f - P_D| \le e^{-3E}$ on K_4. For $y \in K_3$ the disc $B(y,\delta_0)$ lies in K_4, since a segment from y to a point outside K_4 contains a curve from ∂K_3 to ∂K_4, of length at least δ_0 (§5.9); so by Cauchy's estimate $|P_D' - \hat f'| \le e^{-3E}/\delta_0$ on K_3. Put $P(y) := P_D(y) - P_D(0) - (P_D'(0) - 1)\, y$. Then $\deg P \le D$, $P(0) = 0$ and $P'(0) = 1$; and since $\hat f(0) = 0$, $\hat f'(0) = 1$, $0 \in K_3$ and $|y| \le z_{\max}$ on K_4,

$$|P - \hat f| \;\le\; (2 + z_{\max}/\delta_0)\, e^{-3E} \;\le\; e^{-2.9E} \text{ on } K_4, \qquad |P' - \hat f'| \;\le\; 2e^{-3E}/\delta_0 \;\le\; e^{-2.8E} \text{ on } K_3,$$

by (41). With (42), Cauchy's estimate $|\hat f'| \le f_{\max}/\delta_0$ on K_3, and (36) on $K_4 \setminus K_1$, where $|y| \ge 1/10$ (§5.9), this gives

$$\tfrac{1}{2}e^{-2E} \;\le\; |P'| \;\le\; N' \quad \text{on } K_3, \qquad \Big|\frac{P(y)}{y}\Big| \;\ge\; 4e^{-5\lambda} - e^{-2.7E} \quad \text{on } K_4 \setminus K_1. \tag{43}$$

The polynomial $\mathcal{V}$. Let now Ψ, Φ and F_n be those of $K := K_1$ (§5.10), so that $g_K = g_1$ and $\Gamma_\rho = \{g_1 = \log\rho\}$. Put $\gamma := \beta_*/8 \le 1/8$ and

$$s := 1+\gamma, \quad s_1 := 1+2\gamma, \quad s_2 := 1+2.5\gamma, \quad s_* := 1+3\gamma, \qquad n := \Big\lceil \frac{2}{\gamma}\log\frac{32}{\gamma}\Big\rceil, \qquad \mathcal{V} := s_*^{-n} F_n.$$

By Lemma 5.17(b) with $\rho := s$ (so $2\rho - 1 = s_1$), $\tilde{F}_n := F_n \circ \Psi$ satisfies $\tilde{F}_n(w) = w^n(1 + \varepsilon(w))$ on $\{|w| \geq s_1\}$, with $|\varepsilon(w)| \leq (s/|w|)^n \, 4s/(s-1)$. As $\log(s_1/s) = \log\bigl(1 + \frac{\gamma}{1+\gamma}\bigr) \geq \frac{\gamma}{1+2\gamma} \geq 0.8\gamma$ and $n\gamma \geq 2\log(32/\gamma)$, we have $(s/s_1)^n \leq (\gamma/32)^{1.6}$, and so

$$|\varepsilon| \;\leq\; \frac{4(1+\gamma)\,\gamma^{0.6}}{256} \;\leq\; 0.0051 \qquad \text{on } \{|w| \geq s_1\}.$$

For $|w| \geq s_2$ the disc $B(w, \gamma/2)$ lies in $\{|w'| \geq s_1\}$, so Cauchy's estimate gives $|w\varepsilon'(w)| \leq (2|w|/\gamma)\,(s/(|w|-\gamma/2))^n \, 4s/(s-1)$; this decreases in $|w|$, and at $|w| = s_2$ it is at most $2s_2 \cdot 0.0051/\gamma \leq 0.014/\gamma$. As $n \geq (2/\gamma)\log 256 \geq 11/\gamma$, we get $|w\tilde{F}_n'(w)/\tilde{F}_n(w)| = |n + w\varepsilon'/(1+\varepsilon)| \geq n - 0.015/\gamma \geq n/2$ on $\{|w| \geq s_2\}$. For $z = \Psi(w)$, $z\mathcal{V}'(z)/\mathcal{V}(z) = \bigl(\Psi(w)/(w\Psi'(w))\bigr) \cdot w\tilde{F}_n'(w)/\tilde{F}_n(w)$, and Lemma 5.17(a) with $z = 0 \in K_1$ gives $|\Psi(w)/(w\Psi'(w))| \geq (|w|-1)/(|w|+1) \geq (s_2-1)/(s_2+1) \geq \gamma$. So:

(V1) If $|\Phi(z)| \geq s_2$, then $|z\mathcal{V}'(z)/\mathcal{V}(z)| \geq n\gamma/2$; in particular $\mathcal{V}$ has no zero and no critical point there.

(V2) On Γ_{s_2}, $|\mathcal{V}| \leq 1.0051\,(s_2/s_*)^n \leq 0.02$, since $\log(s_*/s_2) \geq 0.5\gamma/(1+3\gamma) \geq 0.36\gamma$ gives $(s_2/s_*)^n \leq (\gamma/32)^{0.72} \leq 256^{-0.72}$. By the maximum principle $|\mathcal{V}| \leq 0.02$ on the closed interior of Γ_{s_2}, which contains K_1 and, by (V1), all zeros and critical points of $\mathcal{V}$.

(V3) Hence $\{|\mathcal{V}| > 3/4\}$ lies in $\{|\Phi| > s_2\}$, where $|\mathcal{V}| \geq 0.9949\,(|\Phi|/s_*)^n$. So $|\mathcal{V}| \leq 5/4$ implies $g_1 \leq \log s_* + 1/n \leq 3\gamma + \gamma/11 < \beta_*$ (also on the closed interior of Γ_{s_2}), and by (37), $\{|\mathcal{V}| \leq 5/4\} \subset \{g_1 < \beta_*\} \subset Z(t_2\mathbb{D}) \subset K_3$.

(V4) By Lemma 5.17(d), (V2) and (38), $|\mathcal{V}| \leq e^{ng_1} \leq (20z_{\max})^n$ on K_4; so $|\mathcal{V}'| \leq (20z_{\max})^n/\delta_0$ on K_3, by Cauchy's estimate.

The polynomial H. Let m be the least integer with

$$m \;\geq\; \max\Bigl(\frac{4}{n\gamma},\; \frac{4D}{n},\; 5\Bigl(\log N' + 9E + 3\log\frac{64 f_{\max}}{\beta_*^2} + 1\Bigr),\; \frac{N'}{\lambda},\; 4\bigl(2E + n\log(20z_{\max}) + 3.6\cdot 10^4 T + 2\bigr)\Bigr),$$

and put $H := P + z\mathcal{V}^m$, so that $H' = P' + \mathcal{V}^{m-1}(\mathcal{V} + mz\mathcal{V}')$. Below we show that $\log(mz_{\max}/\delta_0) \leq 3.6 \cdot 10^4 T$; so

$$m \;\geq\; 4\bigl(2E + n\log(20z_{\max}) + \log(mz_{\max}/\delta_0) + 2\bigr). \tag{44}$$

No critical point in $\{|\mathcal{V}| \leq 3/4\}$. This set lies in K_3 by (V3), and there, by (V4), (44) and (43),

$$|\mathcal{V}^{m-1}(\mathcal{V} + mz\mathcal{V}')| \;\leq\; (3/4)^{m-1}\bigl(1 + mz_{\max}(20z_{\max})^n/\delta_0\bigr) \;<\; \tfrac{1}{2}e^{-2E} \;\leq\; |P'|.$$

For the middle inequality, put $X := 2E + n\log(20z_{\max}) + \log(mz_{\max}/\delta_0)$: the logarithm of the left side is at most $-(m-1)\log\frac{4}{3} + \log 2 + X - 2E$, and $(m-1)\log\frac{4}{3} \geq (4X+7)\log\frac{4}{3} > X + 2\log 2$.

No critical point in $\{|\mathcal{V}| \geq 5/4\}$. There $|\Phi| > s_2$, so by (V1) and $m \geq 4/(n\gamma)$, $|\mathcal{V} + mz\mathcal{V}'| \geq |\mathcal{V}|(mn\gamma/2 - 1) \geq |\mathcal{V}|$; thus $|\mathcal{V}^{m-1}(\mathcal{V} + mz\mathcal{V}')| \geq |\mathcal{V}|^m$, and $|\mathcal{V}| \geq \frac{3}{4}e^{n(g_1 - \log s_*)}$. By Lemma 5.17(d) for P' (of degree less than D) on K_3, (43) and $g_3 \leq g_1$, $|P'| \leq N'e^{Dg_1}$. If $g_1 \geq 2\log s_*$, then $m\log|\mathcal{V}| \geq mng_1/2 - 0.29m \geq Dg_1 + mn\log s_*/2 - 0.29m$ (as $m \geq 4D/n$), and $n\log s_* \geq 6(1 - 1.5\gamma)\log(32/\gamma) \geq 27$; so $m\log|\mathcal{V}| \geq Dg_1 + 13.2m > Dg_1 + \log N'$. If $g_1 < 2\log s_* \leq 6\gamma$, then $m\log|\mathcal{V}| \geq m\log\frac{5}{4} \geq 0.22m$, while $Dg_1 + \log N' \leq 6\gamma D + \log N' \leq 9E + 3\log(64 f_{\max}/\beta_*^2) + 1 + \log N' < 0.22m$. In both cases $|H'| \geq |\mathcal{V}|^m - |P'| > 0$.

The critical points. So every critical point c of H lies in $\{3/4 < |\mathcal{V}| < 5/4\}$, which lies in $K_3 \setminus K_1$ by (V2), (V3), and in $\{|\Phi| > s_2\}$. There $H'(c) = 0$ reads $\mathcal{V}(c)^{m-1}(\mathcal{V}(c) + mc\mathcal{V}'(c)) = -P'(c)$, and $|c\mathcal{V}'(c)| \geq (n\gamma/2)|\mathcal{V}(c)| \geq 3n\gamma/8 \geq 2$ by (V1). So $|\mathcal{V}(c) + mc\mathcal{V}'(c)| \geq 2m - 5/4 \geq m$, and $|\mathcal{V}(c)|^m \leq \frac{5}{4}|\mathcal{V}(c)|^{m-1} \leq \frac{5}{4}N'/m \leq 1.25\lambda$, by (43) and $m \geq N'/\lambda$. By (43), and as $e^{-2.7E} \leq 0.05\lambda$ ($\lambda \geq 1000/T$),

$$\Bigl|\frac{H(c)}{c}\Bigr| \;\geq\; \Bigl|\frac{P(c)}{c}\Bigr| - |\mathcal{V}(c)|^m \;\geq\; 4e^{-5\lambda} - e^{-2.7E} - 1.25\lambda \;\geq\; 4e^{-5\lambda} - 1.3\lambda \;\geq\; 4e^{-6\lambda},$$

the last because $4e^{-5\lambda}(1 - e^{-\lambda}) \geq 3.5\lambda$ for $\lambda \leq 0.02$ (the quotient by λ decreases in λ).

Conclusion. $H(0) = 0$, and $H'(0) = 1 + \mathcal{V}(0)^m$ with $|\mathcal{V}(0)| \le 3/4$ by (V2), as $0 \in K_1$. So $H'(0) \neq 0$ and

$$S(H) \;\ge\; \frac{4e^{-6\lambda}}{1+(3/4)^m} \;\ge\; 4e^{-7\lambda},$$

as $(3/4)^m \le e^\lambda - 1$ (since $m \ge 45E \ge 4\mathcal{L}$ and $\lambda \ge 1000/T$). Since $mn \ge 4D > \deg P$, $\deg H = mn + 1 \ge 2$.

The degree. By (41), $n \le (16/\beta_*)\log(256/\beta_*) + 1 \le (17/\beta_*)\big(5.6 + \log(1/\beta_*)\big)$, so $\log n \le 3.1 \cdot 10^4 T + \mathcal{L} + 13.2$. As $\log(64 f_{\max}/\beta_*^2) + 1 \le E/4$, $D \le 13E/\beta_*$. Of the five lower bounds for m, the first two are at most E (as $n \ge (16/\beta_*)\log 256$), the third is at most $50E$, the fourth at most e^{4170T} (as $\log(1/\lambda) \le \mathcal{L} - 6.9$), and the fifth at most $4(3E + 2\mathcal{L}n) \le 12\mathcal{L}n$ (as $\log(20 z_{\max}) \le 2\mathcal{L}$, and $n \ge 1/\beta_* \ge E^{40}$). Hence $m \le 2\max(12\mathcal{L}n, e^{4170T})$, and

$$\log m \;\le\; 3.1 \cdot 10^4 T + 2\mathcal{L} + 16.4.$$

So $\log(m z_{\max}/\delta_0) \le 3.516 \cdot 10^4 T + 404\mathcal{L} + 18.2 \le 3.6 \cdot 10^4 T$, as claimed above, and

$$\log \deg H \;=\; \log(mn+1) \;\le\; \log m + \log n + 1 \;\le\; 6.2 \cdot 10^4 T + 3\mathcal{L} + 30.6 \;\le\; 6.3 \cdot 10^4\, T \;=\; C_* \frac{M}{\lambda}.$$

This proves Proposition 5.1.

Remark 5.18 (Sources). The passage from $\hat{f}$ to H in §§5.10–5.11 is the proof of Jatar and Ng [23, Proposition 4.3] (see also [19, Proposition 2.1]), with every choice made explicit and a Faber partial sum in place of Runge's theorem. Adamczewski and Colbrook [2, Theorem 5.2] give a quantitative passage with Faber polynomials H_n of an analytic Jordan domain G: from a polynomial f with $f(a) = 0$, $f'(a) = 1$, $f' \neq 0$ on $\overline{G}$ and $|f(z)/(z-a)| \ge A$ on ∂G, to the polynomials $f + (H_n - H_n(a))/n$ of degree n, whose quotient at a is at least $A - O(1/n)$; there the constants depend on the domain, on f and on a. Lemma 5.17 collects classical estimates for Faber polynomials and the Bernstein–Walsh inequality [29]. Lemma 5.9 is a crude form of the lower bound in Beurling's estimate of harmonic measure by extremal distance [17, Ch. IV, Theorem 5.2], and Lemmas 5.10 and 5.11 are upper bounds of the same kind (compare [17, Ch. IV, Theorem 5.3]). For the surface of §5.3 and its background see [19, Section 9 and Remark 9.8].

Remark 5.19 (The factor $\sqrt{\log\log\log d}$). In Theorem 1.4 and Remark 5.3 the factor $\sqrt{\log\log\log d}$ comes from Lemma 5.5. The exits lie at $\operatorname{Re}\ell = \Lambda = \log(4R/g)$, of order $\log(1/g)$, and the bound $\log r_\Sigma \le g(6 + \frac{2}{\pi}\log\Lambda)$ is at most λ only when $M = 2\pi/g$ is at least of order $\log\log(1/\lambda)/\lambda$. Then $\log\deg H \le C_* M/\lambda$ is of order $\log\log(1/\lambda)/\lambda^2$, and solving $\log d = \log\log(1/\lambda)/\lambda^2$ for λ gives λ of order $(\log\log\log d/\log d)^{1/2}$. We do not know whether the factor can be removed (see Section 6).

6. Discussion

Before this paper the best upper bound for $K(d)$ at large degree was Crane's $K(d) < 4 - 2.263/\sqrt{d}$ [7], and the proof of $K(d) \to 4$ in [18, 19] was not effective. This paper improves the upper bound to $K(d) < 4 - 4/\log d$ for every $d \ge 8$ (Corollary 1.2) and pins down the asymptotic order of the gap $4 - K(d)$ to $1/\sqrt{\log d}$ up to constant factors and a factor $\sqrt{\log\log\log d}$ (Corollary 1.5). Three main questions remain open.

(1) *The order of the gap.* By Corollary 1.5, the quotient of the upper and the lower bound for $4 - K(d)$ is a constant times $\sqrt{\log\log\log d}$. We do not know whether this factor can be removed, that is, whether $4 - K(d) \le C/\sqrt{\log d}$ for some constant C and all large d. See Remark 5.19.

(2) *The constants.* The constants 0.056 and $4.9 \cdot 10^4$ of Corollary 1.5 are far apart, and Theorem 1.4 is proved only for $d \ge \exp(1.04 \cdot 10^{10})$. If the factor $\sqrt{\log\log\log d}$ can be removed, does $(4 - K(d))\sqrt{\log d}$ have a limit as $d \to \infty$?

(3) *Moderate degrees.* Theorem 1.3 improves on Corollary 1.2 only when $\log d > 36{,}363$ (Remark 4.2), and on Theorem 1.1 only for still larger d; it improves on both for all sufficiently large d (Remark 3.3). So for $d \geq 8$ with $\log d \leq 36{,}363$ the strongest bound of this paper is Theorem 1.1, whose gap is of order $1/\log d$ (Proposition 3.4); it is below the earlier bounds listed in Remark 3.2. What is the size of $4 - K(d)$ for, say, $8 \leq d \leq 10^6$?

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