Units in the Ring of Values of Siegel E-Functions in the Broad Sense

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Abstract

We determine the units in the ring of values at algebraic points of Siegel E-functions in the broad sense. They are precisely the numbers αe^{β} with $\alpha \in \overline{\mathbb{Q}}^{\times}$ and $\beta \in \overline{\mathbb{Q}}$. Starting from an inverse relation between two values, we use division by an algebraic zero to obtain a relation in the fibre of a finite differential module. A common tensor comparison preserves this relation and, after a finite Kummer covering, can be chosen globally over $\overline{\mathbb{Q}}$ and normalized at the evaluation point. Its analytic realization produces two reciprocal functions with algebraic Laurent coefficients and controlled growth. We then prove directly that such a function is an algebraic constant times an integral power of the variable times the exponential of a polynomial with algebraic coefficients. Evaluation at the normalized point gives the unit classification. The proof applies directly to the broad class; the corresponding statement for the strict class follows as a corollary.

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Introduction

Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let $\mathscr{E}_{\mathrm{w}}$ denote the class of Siegel E-functions in the broad sense. These are differentially finite series

$$F(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}$$

whose algebraic coefficients satisfy subfactorial bounds for all their conjugates and for their common denominators. The precise conditions are given in Definition 2. The class is closed under addition, multiplication and algebraic changes of scale. Consequently, its values at algebraic points form a ring, which we write as

$$\mathbf{E}_{\mathrm{w}} = \{F(1) : F \in \mathscr{E}_{\mathrm{w}}\}.$$

Using the point 1 entails no restriction: if $a \in \overline{\mathbb{Q}}$, then $F(a)$ is the value at 1 of the function $z \mapsto F(az)$.

The question is which nonzero values have a reciprocal in the same ring. A value belongs to $\mathbf{E}_{\mathrm{w}}^{\times}$ precisely when it is nonzero and its reciprocal is also the value of a function in the same class. The functions $\alpha e^{\beta z}$ and $\alpha^{-1} e^{-\beta z}$ already show that

$$\overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}}) \subseteq \mathbf{E}_{\mathrm{w}}^{\times}.$$

Our main result proves the reverse inclusion.

Theorem 1. *The group of units of* $\mathbf{E}_{\mathrm{w}}$ *is*

$$\mathbf{E}_{\mathrm{w}}^{\times} = \overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}}) = \{\alpha e^{\beta} : \alpha \in \overline{\mathbb{Q}}^{\times},\ \beta \in \overline{\mathbb{Q}}\}.$$

Equivalently, if $f, g \in \mathscr{E}_{\mathrm{w}}$ *satisfy* $f(1)g(1) = 1$*, then there exist* $\alpha \in \overline{\mathbb{Q}}^{\times}$ *and* $\beta \in \overline{\mathbb{Q}}$ *such that* $f(1) = \alpha e^{\beta}$*.*

For E-functions in the strict sense, the asserted description is Conjecture 1.2 of Fischler and Rivoal [2]; their discussion on pp. 3–4 also formulates the broad version. The argument below starts with the broad class itself. In particular, the strict-class statement will follow from Theorem 1.

The main difficulty is that a relation between values is weaker than a relation between functions. If $f(1)g(1) = 1$, one cannot conclude that $f(z)g(z) = 1$; for example, $f(z) = z$ and $g(z) = 1$ satisfy the first equality but not the second. Thus the given function f need not be nowhere zero, and studying its logarithmic derivative does not directly address the unit problem. We instead construct other functions u and v satisfying

$$u(t)v(t) = 1 \quad (t \in \mathbb{C}^{\times}), \qquad u(1) = f(1), \qquad v(1) = g(1).$$

The construction also controls their poles, their Laurent coefficients at the origin, and their growth at infinity. These additional properties are what make the inverse identity useful.

Two arithmetic results of Lepetit [3] enter the construction: division by a nonzero algebraic zero preserves the broad class, and every broad E-function admits an annihilating differential operator with no finite singularity away from the origin. The algebraic comparison uses the Tannakian description in André [1] and Pianzola's triviality theorem for torsors on open subsets of the affine line [4]. Each of these inputs is stated in the form used below, and the passage from them to the required normalized comparison is proved explicitly.

Formal verification.

All results and proofs in this paper have been formalized in Lean 4. The complete formalization is available at `https://github.com/Niklas-Fraccaro/wide-e-values-lean`.

Outline of the Proof

Choose $f, g \in \mathscr{E}_{\mathrm{w}}$ with $f(1)g(1) = 1$. Division by the zero at 1 gives another broad E-function h such that

$$fg - 1 = (z - 1)h.$$

The suitable differential equations for f, g, h allow us to put these functions, the constant function 1, and finitely many derivatives into a finite free module M over $R = \overline{\mathbb{Q}}[z, z^{-1}]$ that is stable under differentiation. The products of elements of M span a second finite differential module N. Multiplication is then a horizontal morphism $\mu : M \otimes_R M \to N$. Since $h \in N$, the displayed identity becomes an exact multiplicative relation in the fibre at $z = 1$.

We next compare this fibre with the underlying modules. The comparison has to be chosen simultaneously for every object in the tensor category generated by M, and it must respect the morphism μ and the unit. Tannakian theory realizes all such comparisons as a torsor under a linear algebraic group over $\overline{\mathbb{Q}}$. To obtain a global comparison, we first pass to the quotient by the identity component of this group. A connected component of the resulting finite *étale* cover of $\mathbb{G}_m$ is a Kummer cover $z = t^q$. On this cover, Pianzola's theorem trivializes the remaining torsor under the connected group. We finally correct the comparison by a constant tensor automorphism so that its specialization at $t = 1$ is the identity.

Applying this normalized comparison to the fibre classes of f and g produces elements $U, V \in M \otimes_R \overline{\mathbb{Q}}[t, t^{-1}]$ whose product, computed in the corresponding base change of N, is 1. Replacing each tensor $a(z) \otimes p(t)$ by the function $a(t^q)p(t)$ gives the desired reciprocal functions u, v. Their values at 1 are preserved by the normalization. Globality of the comparison implies that its coefficients are Laurent polynomials, so the only possible finite pole is at 0. Because the construction is over $\overline{\mathbb{Q}}$, the Laurent coefficients are algebraic. The finite-order growth of the original E-functions supplies a common growth bound for u and v.

It remains to classify these reciprocal functions. After removing an integral power t^m, we obtain a nowhere-zero entire function W whose reciprocal is also entire. We construct an entire logarithm normalized by $W(t) = W(0)e^{L(t)}$ and $L(0) = 0$. The two growth bounds control $\operatorname{Re} L$ on large circles. An explicit Fourier coefficient calculation then shows that all sufficiently high Taylor coefficients of L vanish, so L is a polynomial. The Laurent expansion gives $W(0) \in \overline{\mathbb{Q}}^{\times}$, and the formal logarithmic derivative shows that $L \in \overline{\mathbb{Q}}[t]$. Thus

$$u(t) = \alpha t^m e^{P(t)}, \qquad \alpha \in \overline{\mathbb{Q}}^{\times}, \quad m \in \mathbb{Z}, \quad P \in \overline{\mathbb{Q}}[t], \quad P(0) = 0.$$

Evaluating at $t = 1$ gives $f(1) = \alpha e^{P(1)}$, which proves the required inclusion.

1 Notation and arithmetic preliminaries

Throughout the paper, we fix an algebraic closure $\overline{\mathbb{Q}}$ inside $\mathbb{C}$. The chosen inclusion determines which complex analytic function is associated with a power series over $\overline{\mathbb{Q}}$. We write

$$R = \overline{\mathbb{Q}}[z, z^{-1}], \qquad \partial = \frac{d}{dz}.$$

Thus R is the ring of Laurent polynomials with algebraic coefficients, equipped with its usual derivation. When $K \subset \overline{\mathbb{Q}}$ is a number field, its ring of integers is denoted by $\mathcal{O}_K$. All embeddings of number fields into $\mathbb{C}$ are understood to fix $\mathbb{Q}$.

1.1 The broad class of E-functions

We use the following form of the definition of an E-function in the broad sense; see [3, Definition 1 and pp. 1–2 of the arXiv version]. The normalization by $n!$ is part of the definition and will be retained in every coefficient calculation below.

Definition 2. A power series

$$F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \tag{1.1}$$

is an *E-function in the broad sense* if the following conditions hold.

(i) All coefficients a_n belong to one number field $K \subset \overline{\mathbb{Q}}$.

(ii) The series satisfies a nontrivial homogeneous linear differential equation over $\overline{\mathbb{Q}}(z)$: there are rational functions $p_0, \dots, p_r \in \overline{\mathbb{Q}}(z)$, not all zero, such that

$$\sum_{j=0}^{r} p_j(z) F^{(j)}(z) = 0.$$

(iii) For every $\varepsilon > 0$, there is a constant $C_\varepsilon \geq 1$ such that

$$|\sigma(a_n)| \leq C_\varepsilon (n!)^{\varepsilon} \qquad (n \geq 0,\ \sigma : K \hookrightarrow \mathbb{C}). \tag{1.2}$$

(iv) There are positive integers d_n satisfying

$$d_n a_0, \dots, d_n a_n \in \mathcal{O}_K$$

and, for every $\varepsilon > 0$, a constant $C_\varepsilon \geq 1$ such that

$$d_n \leq C_\varepsilon (n!)^\varepsilon \qquad (n \geq 0). \tag{1.3}$$

We denote this class by $\mathscr{E}_{\mathrm{w}}$.

The constants in (1.2) and (1.3) may initially be different; replacing them by their maximum gives the stated convention. One may take d_n to be the least positive integer clearing $a_0, \dots, a_n$. Such an integer exists because this is a finite set of algebraic numbers. This choice also shows that allowing a separate denominator sequence for each ε gives the same condition: the least clearing integer is bounded by every admissible choice.

Another common formulation requires the bounds $(n!)^\varepsilon$, without a multiplicative constant, only for sufficiently large n. The formulations are equivalent. Eventual bounds imply the displayed bounds by increasing the constant to cover the finitely many remaining indices. Conversely, apply the displayed bounds with $\varepsilon/2$. Since $(n!)^{\varepsilon/2}$ tends to infinity, the fixed constant is at most $(n!)^{\varepsilon/2}$ for all sufficiently large n. Multiplication gives the eventual bound with exponent ε. The same argument applies to the common denominators.

The particular number field in the definition is immaterial once it contains all the coefficients. Indeed, if L/K is a finite extension, every embedding $L \hookrightarrow \mathbb{C}$ restricts to an embedding of K, so a bound for all conjugates over K also bounds those obtained from L. Conversely, every embedding of K into $\mathbb{C}$ extends to L. Also, an element of K is integral over $\mathbb{Z}$ if and only if it belongs to $\mathcal{O}_L$; equivalently, $\mathcal{O}_L \cap K = \mathcal{O}_K$. Thus the same positive integers clear the coefficients whether integrality is tested in K or in L. We may consequently enlarge the coefficient field whenever finitely many additional algebraic numbers occur.

In particular, the rational functions in a chosen differential equation have only finitely many algebraic coefficients. Enlarging K to include them allows that equation to be written over $K(z)$, without changing either arithmetic condition. This does not assert that one number field works for every function in the class. Each function, and each finite collection used in a given argument, has its own finite coefficient field. The bounds on all embeddings are part of the arithmetic definition; bounding only the fixed complex realization would give a weaker condition.

For comparison, an *E-function in the strict sense* satisfies the same coefficient-field and differential-equation conditions, with the last two bounds replaced by

$$|\sigma(a_n)| \leq C^{n+1}, \qquad d_n \leq C^{n+1} \qquad (n \geq 0)$$

for a fixed $C \geq 1$. We shall use the broad definition throughout; the elementary estimate below explains why every strict E-function also belongs to $\mathscr{E}_{\mathrm{w}}$.

1.2 Absorption and differential finiteness

Lemma 3. *Let $A \geq 1$, $b \geq 0$, and $\eta > 0$. There is a constant $C \geq 1$ such that*

$$A^n (n+1)^b \leq C (n!)^\eta \qquad (n \geq 0). \tag{1.4}$$

Proof. For $n \geq 2$, at least $\lfloor n/2 \rfloor$ factors in $n!$ are at least $n/2$. Consequently,

$$\log(n!) \geq \lfloor n/2 \rfloor \log(n/2).$$

On the other hand,

$$\log(A^n(n+1)^b) = n\log A + b\log(n+1).$$

The ratio of this last expression to $\lfloor n/2 \rfloor \log(n/2)$ tends to zero. It is therefore at most η for all sufficiently large n, which proves (1.4) with $C = 1$ in that range. Taking the maximum of 1 and the finitely many ratios $A^n(n+1)^b/(n!)^\eta$ at the remaining indices completes the proof. □

In particular, a fixed exponential bound is smaller than every positive factorial power after adjustment of the constant. Indeed, for each $\varepsilon > 0$, applying the lemma with $A = C$, $b = 0$, and $\eta = \varepsilon$ gives $C^{n+1} \le CC_\varepsilon(n!)^\varepsilon$. Applying this to both the conjugate bounds and the common denominators proves the inclusion of the strict class in the broad class. We will also use Lemma 3 for the polynomial factors introduced by differentiation.

We next record an algebraic criterion for the differential-equation condition. At this stage, power series can be treated formally inside $\overline{\mathbb{Q}}((z))$; their convergence will follow below.

Lemma 4. *A formal power series $F \in \overline{\mathbb{Q}}[[z]]$ satisfies a nontrivial homogeneous linear differential equation over $\overline{\mathbb{Q}}(z)$ if and only if*

$$V_F = \operatorname{span}_{\overline{\mathbb{Q}}(z)}\{F, F', F'', \ldots\}$$

is finite-dimensional over $\overline{\mathbb{Q}}(z)$.

Proof. The assertion is immediate for $F = 0$, so suppose $F \neq 0$. Choose a differential equation and let r be the largest index with nonzero coefficient. Then $r \ge 1$, since a nonzero rational function cannot annihilate F by multiplication. Dividing by the coefficient of $F^{(r)}$ expresses $F^{(r)}$ as a $\overline{\mathbb{Q}}(z)$-linear combination of $F, \ldots, F^{(r-1)}$. To check stability under differentiation explicitly, if $W = \operatorname{span}_{\overline{\mathbb{Q}}(z)}\{F, \ldots, F^{(r-1)}\}$ and $w = \sum_{j=0}^{r-1} q_j F^{(j)}$, then

$$w' = \sum_{j=0}^{r-1} q_j' F^{(j)} + \sum_{j=0}^{r-1} q_j F^{(j+1)} \in W.$$

All terms except the last possible derivative already lie in the displayed generating set, and the differential equation deals with that remaining term. It follows by induction that every derivative of F belongs to W, proving finite-dimensionality.

Conversely, if V_F has dimension $s < \infty$, the $s+1$ elements $F, F', \ldots, F^{(s)}$ are linearly dependent over $\overline{\mathbb{Q}}(z)$. Their dependence relation is the required nontrivial homogeneous differential equation. □

1.3 Closure properties

Proposition 5. *The class $\mathscr{E}_{\mathrm{w}}$ contains $\overline{\mathbb{Q}}[z]$ and is closed under addition, multiplication, differentiation, and multiplication by algebraic constants. If $F \in \mathscr{E}_{\mathrm{w}}$ and $a \in \overline{\mathbb{Q}}$, then $F(az) \in \mathscr{E}_{\mathrm{w}}$. In particular, every derivative of a broad E-function is again a broad E-function.*

Proof. Whenever several functions or algebraic constants occur, we replace their coefficient fields by their finite compositum. Every embedding of this compositum restricts to an embedding of each original field, so the original conjugate bounds remain available. Algebraic integers in a smaller number field remain algebraic integers in the compositum. Thus neither the conjugate bounds nor the common-denominator conditions are lost by this replacement.

Differential equations. By Lemma 4, the spaces V_F and V_G associated with two functions in $\mathscr{E}_{\mathrm{w}}$ are finite-dimensional and stable under differentiation. The derivatives of $F + G$ belong to

$V_F + V_G$. For products, choose finite spanning sets u_i of V_F and v_j of V_G. The space spanned by the finitely many products $u_i v_j$ contains FG and is stable under differentiation, because

$$(u_i v_j)' = u_i' v_j + u_i v_j'.$$

It therefore contains every derivative of FG. The derivatives of F' already lie in V_F, and scalar multiplication causes no change to the argument.

For the substitution $H(z) = F(az)$ with $a \neq 0$, an equation $\sum_j p_j(z) F^{(j)}(z) = 0$ becomes

$$\sum_j a^{-j} p_j(az) H^{(j)}(z) = 0.$$

At least one transformed coefficient is nonzero, so this is a nontrivial equation over $\overline{\mathbb{Q}}(z)$. When $a = 0$, the function is the algebraic constant $F(0) = a_0$, and its derivative vanishes. This verifies the differential-equation condition for all the claimed operations.

Sums and products. Write

$$F(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}, \qquad G(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!}.$$

Fix $\varepsilon > 0$, and use the coefficient and denominator bounds for F, G with exponent $\delta = \varepsilon/2$. The normalized coefficients of FG are

$$c_n = \sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}. \tag{1.5}$$

For each embedding σ, the inequality $j!(n-j)! \leq n!$ gives

$$\begin{aligned} |\sigma(c_n)| &\leq C_F C_G \sum_{j=0}^{n} \binom{n}{j} (j!)^{\delta} ((n-j)!)^{\delta} \\ &\leq C_F C_G \, 2^n (n!)^{\delta} \leq C_{\varepsilon} (n!)^{\varepsilon}. \end{aligned}$$

The last inequality is Lemma 3 with $A = 2$ and $\eta = \varepsilon/2$.

Let $d_{F,n}$ and $d_{G,n}$ clear the respective coefficients through index n. Their product clears $c_0, \ldots, c_n$: for $m \leq n$, each term in

$$d_{F,n} d_{G,n} c_m = \sum_{j=0}^{m} \binom{m}{j} (d_{F,n} a_j)(d_{G,n} b_{m-j})$$

is an algebraic integer. Moreover,

$$d_{F,n} d_{G,n} \leq C_F C_G (n!)^{2\delta} = C_F C_G (n!)^{\varepsilon}.$$

For $F + G$, the normalized coefficient is $a_n + b_n$, whose conjugates satisfy

$$|\sigma(a_n + b_n)| \leq (C_F + C_G)(n!)^{\delta}.$$

The same product of denominators clears every coefficient through index n, since it clears both summands individually. This proves both arithmetic conditions for sums and products.

Derivatives. The normalized coefficient sequence of F' is a_{n+1}. Using the bounds for F with exponent $\varepsilon/2$, we obtain

$$|\sigma(a_{n+1})| \leq C_{\varepsilon/2} ((n+1)!)^{\varepsilon/2} = C_{\varepsilon/2} (n!)^{\varepsilon/2} (n+1)^{\varepsilon/2} \leq C_{\varepsilon} (n!)^{\varepsilon}.$$

Here Lemma 3 absorbs the final polynomial factor. The integer $d_{F,n+1}$ clears $a_1, \dots, a_{n+1}$, and it obeys precisely the same estimate. Thus $F' \in \mathscr{E}_{\mathrm{w}}$; iteration gives every higher derivative.

Algebraic constants and changes of variable. For $c \in \overline{\mathbb{Q}}$, the conjugates of the coefficients ca_n are bounded by a fixed multiple of those of a_n. Choose an integer $s \geq 1$ such that sc is integral. Then $sd_{F,n}$ clears the coefficients of cF through index n, and the fixed factor s is absorbed into the constant in the denominator bound.

For $F(az)$, denote the normalized coefficients by $\widetilde{a}_n = a^n a_n$. Choose $A \geq 1$ bounding the absolute values of all conjugates of a. With exponent $\varepsilon/2$ in the original bounds,

$$|\sigma(\widetilde{a}_n)| \leq C_{\varepsilon/2} A^n (n!)^{\varepsilon/2} \leq C_\varepsilon (n!)^\varepsilon.$$

Choose an integer $s \geq 1$ with sa integral. For $0 \leq j \leq n$,

$$s^n d_{F,n} \widetilde{a}_j = s^{n-j} (sa)^j (d_{F,n} a_j)$$

is integral. Thus $s^n d_{F,n}$ is a common denominator through index n, and Lemma 3 gives

$$s^n d_{F,n} \leq C_{\varepsilon/2} s^n (n!)^{\varepsilon/2} \leq C_\varepsilon (n!)^\varepsilon.$$

For $a = 0$, the algebraic constant already considered gives the claim directly.

Finally, a polynomial $\sum_{j=0}^m p_j z^j \in \overline{\mathbb{Q}}[z]$ has normalized coefficients $j! p_j$ through index m and zero coefficients thereafter. These coefficients lie in a finite number field, have bounded conjugates, and admit one fixed common integer denominator. The polynomial is annihilated by its $(m+1)$-st derivative. It therefore belongs to $\mathscr{E}_{\mathrm{w}}$, completing the proof. □

1.4 Entire functions and their values

Lemma 6. *Every $F \in \mathscr{E}_{\mathrm{w}}$ defines an entire function. There is a constant $C_F > 0$ such that*

$$|F(z)| \leq C_F \exp(|z|^2) \qquad (z \in \mathbb{C}). \tag{1.6}$$

The same assertion holds for every derivative of F, with a constant depending on the derivative.

Proof. Apply (1.2) with $\varepsilon = 1/2$ and the fixed embedding into $\mathbb{C}$. It gives $|a_n| \leq C\sqrt{n!}$. For $r \geq 0$, Cauchy–Schwarz yields

$$\begin{aligned}
\sum_{n \geq 0} \frac{|a_n| r^n}{n!} &\leq C \sum_{n \geq 0} \frac{r^n}{\sqrt{n!}} \\
&= C \sum_{n \geq 0} \frac{(\sqrt{2} r)^n}{\sqrt{n!}} \, 2^{-n/2} \\
&\leq C \left(\sum_{n \geq 0} \frac{(2r^2)^n}{n!} \right)^{1/2} \left(\sum_{n \geq 0} 2^{-n} \right)^{1/2} = C\sqrt{2}\, e^{r^2}.
\end{aligned}$$

The Cauchy–Schwarz inequality may first be applied to the finite sums through index N. Its right-hand side is bounded by the two convergent series displayed above, independently of N. Letting N increase therefore proves convergence of the nonnegative majorant series; no convergence of the original power series has been assumed in making this estimate.

For any fixed r, this is a summable majorant on $|z| \leq r$. The power series therefore converges uniformly on every closed disc and defines an entire function. The same calculation gives (1.6) with $C_F = C\sqrt{2}$. Proposition 5 places every derivative in $\mathscr{E}_{\mathrm{w}}$, so the argument applies to each derivative separately. □

From now on, we identify the formal series with this entire function. Uniform convergence on compact subsets permits termwise differentiation, so its normalized coefficients satisfy $a_n = F^{(n)}(0)$. In particular, its ordinary Taylor coefficients $a_n/n!$ still lie in the same number field. A formal differential identity is also an analytic identity wherever its rational coefficients are defined: after multiplication by a common polynomial denominator it becomes an identity between entire functions with zero Taylor series at the origin, hence holds everywhere by the identity theorem. We will use these formal and analytic descriptions of a function interchangeably.

Proposition 7. *The set*

$$\mathbf{E}_{\mathrm{w}} = \{F(1) : F \in \mathscr{E}_{\mathrm{w}}\} \tag{1.7}$$

is a $\overline{\mathbb{Q}}$-subalgebra of $\mathbb{C}$. It is also the set of all values of broad E-functions at algebraic arguments:

$$\mathbf{E}_{\mathrm{w}} = \{F(a) : F \in \mathscr{E}_{\mathrm{w}},\ a \in \overline{\mathbb{Q}}\}.$$

Proof. All values are defined by Lemma 6. Proposition 5 gives $F + G, FG, cF \in \mathscr{E}_{\mathrm{w}}$ for $F, G \in \mathscr{E}_{\mathrm{w}}$ and $c \in \overline{\mathbb{Q}}$; evaluating at 1 proves closure under addition, multiplication, and multiplication by $\overline{\mathbb{Q}}$. The constant functions show that $\overline{\mathbb{Q}} \subset \mathbf{E}_{\mathrm{w}}$, including the multiplicative identity.

If $a \in \overline{\mathbb{Q}}$, the function $H(z) = F(az)$ belongs to $\mathscr{E}_{\mathrm{w}}$ and satisfies $H(1) = F(a)$. Hence every algebraic-argument value belongs to $\mathbf{E}_{\mathrm{w}}$. The reverse inclusion follows by taking $a = 1$. When $a = 0$, the value is the algebraic coefficient a_0, consistently with both descriptions. □

The ring property means that a polynomial expression in such values, with algebraic coefficients, can again be represented by one function at 1. For example, if $\xi = F(a)$ and $\eta = G(b)$, then $\xi+\eta = H(1)$ and $\xi\eta = J(1)$ with $H(z) = F(az)+G(bz)$ and $J(z) = F(az)G(bz)$ in $\mathscr{E}_{\mathrm{w}}$. Since $\mathbf{E}_{\mathrm{w}}$ is a subring of $\mathbb{C}$, its units are precisely the nonzero elements whose complex reciprocals also belong to $\mathbf{E}_{\mathrm{w}}$. Thus studying $\mathbf{E}_{\mathrm{w}}^{\times}$ amounts to studying pairs $f, g \in \mathscr{E}_{\mathrm{w}}$ satisfying $f(1)g(1) = 1$, which is the form of the problem used in the next section.

2 Arithmetic input and differential equations

The elementary properties proved above give analytic continuation and growth, but two further facts are needed to pass from a relation between values to a finite differential module over R. We record the precise arithmetic results that provide them.

2.1 Division by an algebraic zero

Theorem 8 (Lepetit)**.** *Let $F \in \mathscr{E}_{\mathrm{w}}$, let $a \in \overline{\mathbb{Q}}^{\times}$, and suppose that $F(a) = 0$. Then the function*

$$H(z) = \frac{F(z)}{z - a},$$

with its holomorphic continuation at a, belongs to $\mathscr{E}_{\mathrm{w}}$.

This is Proposition 18 of [3], on p. 47 of the arXiv version and pp. 143–144 of the published version. The theorem applies to the subfactorial coefficient and denominator conditions of Definition 2.

The analytic continuation in the statement is easy to describe. A Taylor expansion at a gives

$$F(z) = F'(a)(z - a) + \frac{F''(a)}{2!}(z - a)^2 + \cdots,$$

so $H(a) = F'(a)$ and the quotient is entire. This argument alone does not establish the arithmetic estimates in Definition 2. Their preservation is the content of the cited theorem. In the application below, it ensures that the quotient can itself be included among the functions to which the differential-operator result applies.

2.2 An annihilator with Laurent-polynomial coefficients

Proposition 9 (Consequence of Lepetit's structure theorem)**.** *For every nonzero $F \in \mathscr{E}_{\mathrm{w}}$, there are an integer $r \geq 1$ and $b_0, \ldots, b_{r-1} \in R$ such that*

$$F^{(r)}(z) = \sum_{j=0}^{r-1} b_j(z) F^{(j)}(z). \tag{2.1}$$

Thus a finite list of derivatives of F spans an R-module stable under differentiation.

Proof. Proposition 16 of [3] supplies a nonzero annihilating E-operator in the broad sense. Theorem 7(a) of the same paper states that such an operator has no singularities on $\mathbb{G}_m$. These are on pp. 37 and 39 of the arXiv version, respectively, and pp. 130 and 133 of the published version.

Write the resulting scalar operator as

$$L = a_r(z)\partial^r + a_{r-1}(z)\partial^{r-1} + \cdots + a_0(z), \qquad a_r \neq 0,$$

where $\partial = \mathrm{d}/\mathrm{d}z$ and the coefficients lie in $\overline{\mathbb{Q}}(z)$. Its order cannot be zero: a nonzero rational function times the nonzero function F cannot vanish identically. Hence $r \geq 1$. Dividing the identity $L(F) = 0$ by a_r gives

$$F^{(r)} = -\sum_{j=0}^{r-1} \frac{a_j}{a_r} F^{(j)}.$$

At an ordinary point of a scalar differential equation, the coefficients of its monic form are regular. The absence of singularities on $\mathbb{G}_m$ therefore says precisely that a_j/a_r has no finite pole away from 0.

To identify the allowed coefficients, take a rational function $p(z)/s(z)$ in reduced form, with $p, s \in \overline{\mathbb{Q}}[z]$. If it has no pole in $\overline{\mathbb{Q}}^{\times}$, then every root of s is 0. Since $\overline{\mathbb{Q}}$ is algebraically closed, $s(z) = cz^n$ for some $c \in \overline{\mathbb{Q}}^{\times}$ and $n \geq 0$. Thus $p/s \in \overline{\mathbb{Q}}[z, z^{-1}] = R$. Applying this to the monic coefficients and setting $b_j = -a_j/a_r$ proves (2.1).

Finally, the derivatives of $F, F', \ldots, F^{(r-2)}$ are already in the listed span, and the derivative of $F^{(r-1)}$ is in that span by (2.1). Since $\partial R \subseteq R$, the Leibniz rule then proves stability of the whole R-module. □

The order in Proposition 9 is allowed to exceed the order of the minimal differential equation of F. This freedom is used: a minimal scalar operator may have apparent singularities away from 0, whereas the suitable annihilator above has monic coefficients in the specified ring R. Working over this ring will later ensure that a global comparison introduces no additional finite poles.

3 From an inverse value to a fibre relation

Fix a unit $\xi \in \mathbf{E}_{\mathrm{w}}^{\times}$. By the definition of the ring of values, there exist $f, g \in \mathscr{E}_{\mathrm{w}}$ such that

$$f(1) = \xi, \qquad g(1) = \xi^{-1}. \tag{3.1}$$

Both functions are nonzero. By Proposition 5, $fg - 1 \in \mathscr{E}_{\mathrm{w}}$, and this function vanishes at 1. Theorem 8 therefore gives $h \in \mathscr{E}_{\mathrm{w}}$ with

$$fg - 1 = (z - 1)h. \tag{3.2}$$

If $fg = 1$ identically, we take $h = 0$. We now encode (3.2) in finite modules in a way that keeps both multiplication and differentiation available.

3.1 A finite module of actual functions

Let

$$\mathcal{A}_z = \mathcal{O}(\mathbb{C})[z^{-1}].$$

Here $\mathcal{O}(\mathbb{C})$ is the ring of entire functions. An element of $\mathcal{A}_z$ can be written $z^{-b}F(z)$ with $b \geq 0$ and F entire; it is holomorphic on $\mathbb{C}^\times$ and has at most a pole at 0. Conversely, a function with these properties belongs to $\mathcal{A}_z$, because multiplication by a sufficiently high power of z removes its possible pole. In particular, $\mathcal{A}_z$ is a subring of the field of meromorphic functions on $\mathbb{C}$, and is an integral domain.

For each nonzero F in the finite list f, g, h, choose an equation (2.1), of order denoted by r_F. Define

$$M = R \cdot 1 + \sum_{\substack{F \in \{f,g,h\} \\ F \neq 0}} \sum_{j=0}^{r_F - 1} R \cdot F^{(j)} \ \subseteq \ \mathcal{A}_z. \tag{3.3}$$

The notation denotes the span of the displayed functions, with all relations they actually satisfy. In particular, the listed generators are not being declared linearly independent.

There are only finitely many generators in (3.3). The derivative of 1 is zero, and the derivative of each listed $F^{(j)}$ belongs to M: for $j < r_F - 1$ it is the next generator, and for $j = r_F - 1$ it is given by (2.1). For a general element $a = \sum_i p_i F_i$ of M, the product rule gives

$$a' = \sum_i p_i' F_i + \sum_i p_i F_i' \in M,$$

because $p_i, p_i' \in R$. Hence

$$\partial M \subseteq M. \tag{3.4}$$

Since M is a submodule of the integral domain $\mathcal{A}_z$, it is torsion-free over R: if $p \in R \setminus \{0\}$ and $pa = 0$, then $a = 0$. The ring R is a principal ideal domain, being a localization of $\overline{\mathbb{Q}}[z]$. The structure theorem for finitely generated modules over a principal ideal domain therefore shows that M is free of finite rank. The induced derivation obeys the Leibniz rule, so M is a finite differential module over R.

Every generator in (3.3) is a broad E-function, by Proposition 5. Thus the module consists of finite Laurent-polynomial combinations of functions with algebraic Taylor coefficients and the growth bound of Lemma 6. This concrete description will be retained when the modules are later realized after a change of variable.

3.2 The module containing the products

The module M need not be closed under multiplication. To express the inverse relation without assuming such closure, put

$$N = \mathrm{span}_R\{ab : a, b \in M\} \subseteq \mathcal{A}_z. \tag{3.5}$$

If $m_1, \dots, m_s$ generate M, then the s^2 products $m_i m_j$ generate N, since

$$\left(\sum_i p_i m_i\right)\left(\sum_j q_j m_j\right) = \sum_{i,j} p_i q_j m_i m_j.$$

Thus N is finitely generated. Also

$$(ab)' = a'b + ab' \in N \qquad (a, b \in M),$$

and differentiation of the coefficients in an R-linear combination shows $\partial N \subseteq N$. As a submodule of $\mathcal{A}_z$, it is torsion-free and hence free of finite rank over R.

Because $1 \in M$, every $a \in M$ equals the product $a \cdot 1$, so

$$M \subseteq N. \tag{3.6}$$

In particular, $h \in N$. Multiplication and the constant function 1 give R-linear maps

$$\mu : M \otimes_R M \longrightarrow N, \qquad a \otimes b \longmapsto ab, \tag{3.7}$$

$$\iota : R \longrightarrow N, \qquad p \longmapsto p \cdot 1. \tag{3.8}$$

The map μ is surjective by the definition of N. These maps commute with the differential structures. For μ, this is the product rule, where the differential structure on $M \otimes_R M$ sends $a \otimes b$ to $a' \otimes b + a \otimes b'$. For ι, it is the equality $(p \cdot 1)' = p' \cdot 1$. Thus both maps are horizontal morphisms of differential modules.

3.3 Specialization in the algebraic fibre

For an R-module A, define its fibre at $z = 1$ by

$$A_1 = A/(z-1)A = A \otimes_{R, z \mapsto 1} \overline{\mathbb{Q}}. \tag{3.9}$$

We write $\bar{a}$ for the class of $a \in A$; the ambient module will always be specified. Specialization of (3.7) and (3.8) gives

$$\mu_1 : M_1 \otimes_{\overline{\mathbb{Q}}} M_1 \longrightarrow N_1, \qquad \iota_1 : \overline{\mathbb{Q}} \longrightarrow N_1.$$

The standard identification

$$(M \otimes_R M) \otimes_R \overline{\mathbb{Q}} \simeq M_1 \otimes_{\overline{\mathbb{Q}}} M_1$$

sends $(a \otimes b) \otimes 1$ to $\bar{a} \otimes \bar{b}$. In these terms, $\mu_1(\bar{a} \otimes \bar{b})$ is the class of the actual product ab in N_1.

Since $h \in N$, identity (3.2) implies $fg - 1 \in (z-1)N$. Reducing this identity modulo $(z-1)N$ yields

$$\mu_1(\bar{f} \otimes \bar{g}) = \iota_1(1) \qquad \text{in } N_1. \tag{3.10}$$

This is a relation in a finite-dimensional $\overline{\mathbb{Q}}$-vector space with a specified multiplication morphism. It does not require a ring structure on M_1.

There is also a numerical evaluation map on these fibres, but it plays a different role. Every element of M and N is holomorphic at 1, and evaluation at 1 vanishes on $(z-1)A$ for $A = M, N$. Hence it induces the $\overline{\mathbb{Q}}$-linear map

$$\mathrm{ev}_{A,1} : A_1 \longrightarrow \mathbb{C}, \qquad \bar{a} \longmapsto a(1). \tag{3.11}$$

The map is well defined because two representatives differ by a function $(z-1)b$ holomorphic at 1. We do not need it to be injective. In particular, equation (3.10) was deduced from the identity with the factor $z - 1$ in N, rather than from the numerical identity $f(1)g(1) = 1$ alone. The numerical evaluation maps will be used only after the fibre relation has been lifted, to check that the chosen values are preserved.

4 Differential modules and fibre functors

We now place the modules constructed above in a tensor category. The purpose of this step is to compare their fibres at $z = 1$ with the modules themselves in a way which preserves both the multiplication map μ and the map ι representing the constant function 1. We first establish the exactness properties needed for this comparison.

A differential module over R is an R-module A equipped with a $\overline{\mathbb{Q}}$-linear map $\nabla_A : A \to A$ satisfying

$$\nabla_A(ra) = r'a + r\nabla_A(a) \qquad (r \in R,\ a \in A). \tag{4.1}$$

We call an R-linear map $\phi : A \to B$ horizontal if

$$\nabla_B \circ \phi = \phi \circ \nabla_A. \tag{4.2}$$

The modules M and N of actual functions carry this structure through ordinary differentiation. Their multiplication map is horizontal because differentiation satisfies the product rule.

The algebraic fibre of A at 1 is the vector space

$$A_1 = A/(z-1)A = A \otimes_R \overline{\mathbb{Q}},$$

where $R \to \overline{\mathbb{Q}}$ sends z to 1. If ϕ is R-linear, it induces a linear map $\phi_1 : A_1 \to B_1$. No differentiation operator is imposed on A_1: indeed, the ideal $(z-1)$ is not stable under ∂. The fibre construction will be used as a functor to vector spaces.

4.1 Freeness and exactness

The following elementary property of R prevents torsion from appearing in finite differential modules.

Lemma 10. *The only ideals $I \subseteq R$ satisfying $\partial I \subseteq I$ are 0 and R. Moreover, the elements of R annihilated by ∂ are exactly $\overline{\mathbb{Q}}$.*

Proof. Suppose that $I \neq 0$ is stable under differentiation. Multiplying a nonzero Laurent polynomial in I by a suitable power of z gives a nonzero polynomial belonging to I, since z is a unit of R. Choose a nonzero polynomial $p \in I \cap \overline{\mathbb{Q}}[z]$ of least degree. If $\deg p > 0$, then p' is a nonzero polynomial of smaller degree, because $\overline{\mathbb{Q}}$ has characteristic zero. The inclusion $p' \in I$ contradicts the choice of p. Thus p is a nonzero constant, so $1 \in I$ and $I = R$.

For the assertion about constants, write $r = \sum_{j=a}^{b} r_j z^j$. Then

$$r' = \sum_{j=a}^{b} j r_j z^{j-1}.$$

The Laurent monomials are linearly independent over $\overline{\mathbb{Q}}$, and $j \neq 0$ is invertible in $\overline{\mathbb{Q}}$. Hence $r' = 0$ implies $r_j = 0$ for every $j \neq 0$, as required. □

Lemma 11. *Every differential module over R which is finitely generated as an R-module is free of finite rank.*

Proof. Let A be such a module, and let T be its torsion submodule. We first show that T is stable under ∇_A. If $a \in T$, choose $0 \neq r \in R$ with $ra = 0$. Differentiating this equality gives

$$r\nabla_A(a) = -r'a.$$

Multiplication by r therefore yields $r^2\nabla_A(a) = 0$. Since R is a domain, $r^2 \neq 0$, and consequently $\nabla_A(a) \in T$.

The ring R is noetherian, so T is finitely generated. Its annihilator

$$J = \operatorname{Ann}_R(T) = \{r \in R : ra = 0 \text{ for every } a \in T\}$$

is nonzero. To see this explicitly, choose generators $a_1, \dots, a_s$ of T and nonzero elements r_i annihilating a_i; then $r_1 \cdots r_s$ annihilates all of T. If $T = 0$, there is already nothing to prove.

For $r \in J$ and $a \in T$, the Leibniz rule gives

$$0 = \nabla_A(ra) = r'a + r\nabla_A(a) = r'a,$$

where the last equality uses $\nabla_A(a) \in T$. Thus $r' \in J$, so J is a nonzero differential ideal. Lemma 10 implies $J = R$. In particular 1 annihilates T, and therefore $T = 0$.

It remains to pass from torsion-freeness to freeness. The polynomial ring $\overline{\mathbb{Q}}[z]$ is a principal ideal domain, and its localization $R = \overline{\mathbb{Q}}[z, z^{-1}]$ is again a principal ideal domain. A finitely generated torsion-free module over a principal ideal domain is free. Applying this result to A proves the lemma. □

The finite-generation hypothesis is used at a specific point: it provides a single nonzero element annihilating the whole torsion submodule. The connection then makes its annihilator a differential ideal. This argument applies equally to modules presented abstractly and to submodules of a ring of functions. Thus it also applies to the kernels and quotients introduced by tensor constructions, even when those objects are no longer realized as concrete functions.

Corollary 12. *Kernels, images and cokernels of horizontal maps between finite differential modules are finite free differential modules. Every short exact sequence of finite differential modules remains exact after passage to the fibre at* 1.

Proof. Let $\phi : A \to B$ be horizontal. If $a \in \ker \phi$, then

$$\phi(\nabla_A a) = \nabla_B(\phi(a)) = 0,$$

so the kernel is stable under differentiation. Similarly, $\nabla_B(\phi(a)) = \phi(\nabla_A a)$ shows that the image is stable. This stability makes the formula

$$\nabla_{B/\operatorname{im}\phi}(b + \operatorname{im}\phi) = \nabla_B b + \operatorname{im}\phi$$

well defined on the cokernel. Each resulting operator satisfies the Leibniz rule. The modules are finitely generated because R is noetherian, and Lemma 11 proves their freeness.

Now consider an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow D \longrightarrow 0$$

of finite differential modules. The module D is free, so its basis vectors can be lifted to B. Extending these lifts R-linearly gives a splitting of the sequence as R-modules. Tensoring the split sequence with $R/(z-1)$ preserves exactness and gives

$$0 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow D_1 \longrightarrow 0.$$

The splitting need not be horizontal; its R-linearity is exactly what is required for this argument. □

4.2 Tensor products, duals and the generated category

For finite differential modules A and B, the connection on the tensor product is

$$\nabla_{A\otimes B}(a\otimes b) = \nabla_A(a)\otimes b + a\otimes \nabla_B(b). \tag{4.3}$$

This formula respects the balancing relation $ra\otimes b = a\otimes rb$: applying the Leibniz rule on either side gives the same term $r'a\otimes b$ in addition to the two terms multiplied by r. Thus it defines a connection on $A\otimes_R B$.

The dual module $A^\vee = \mathrm{Hom}_R(A,R)$ is equipped with

$$(\nabla_{A^\vee}\ell)(a) = \partial(\ell(a)) - \ell(\nabla_A a). \tag{4.4}$$

The right-hand side is R-linear in a: replacing a by ra produces two terms $r'\ell(a)$ which cancel. It also satisfies the Leibniz rule in ℓ, so this is a differential module. In particular,

$$\partial(\ell(a)) = (\nabla_{A^\vee}\ell)(a) + \ell(\nabla_A a),$$

which proves that evaluation $A^\vee\otimes_R A\to R$ is horizontal.

Coevaluation is horizontal as well. If $e_1,\dots,e_r$ is a basis of A, with dual basis $e_1^\vee,\dots,e_r^\vee$, it sends 1 to

$$\sum_{i=1}^{r} e_i\otimes e_i^\vee.$$

Write $\nabla_A e_i = \sum_j c_{ji}e_j$. Formula (4.4) then gives

$$\nabla_{A^\vee} e_i^\vee = -\sum_j c_{ij}e_j^\vee.$$

The derivative of the displayed coevaluation tensor is therefore

$$\sum_{i,j} c_{ji}e_j\otimes e_i^\vee - \sum_{i,j} c_{ij}e_i\otimes e_j^\vee = 0,$$

after interchanging the two indices in the first sum. These maps give the usual duality for finite free modules, with all its maps horizontal.

Let $\mathcal{C} = \langle M\rangle^\otimes$ be the full category whose objects are the differential subquotients of finite direct sums of modules

$$M^{\otimes a}\otimes_R (M^\vee)^{\otimes b}, \qquad a,b\geq 0. \tag{4.5}$$

The word with $a=b=0$ is the unit object R, with its ordinary derivative. Morphisms in $\mathcal{C}$ are horizontal R-linear maps. Here a differential subquotient of T means a quotient U/V with $V\subseteq U\subseteq T$ stable under the connection.

We record explicitly the closure properties in this definition. A subobject or quotient of a subquotient of T is another subquotient of T, by taking inverse images under the quotient map $U\to U/V$. Finite direct sums preserve this description. Consequently kernels and cokernels of morphisms in $\mathcal{C}$ remain in $\mathcal{C}$. Corollary 12 shows that they agree with the corresponding module kernels and cokernels. Thus $\mathcal{C}$ is abelian.

To check closure under tensor products, write $A = U/V$ and $B = U'/V'$, where $U\subseteq T$ and $U'\subseteq T'$ lie in finite sums of the words (4.5). All these modules are free, including the relevant quotients. Tensoring their exact sequences therefore gives

$$A\otimes_R B \simeq \frac{U\otimes_R U'}{V\otimes_R U' + U\otimes_R V'}.$$

Moreover, $U \otimes_R U'$ embeds in $T \otimes_R T'$. The latter is a finite direct sum of tensor words, after applying the symmetry which interchanges factors. The displayed quotient is a differential quotient by (4.3). It follows that $A \otimes_R B$ is an object of $\mathcal{C}$.

For duals, the exact sequence $0 \to U \to T \to T/U \to 0$ splits as R-modules. Dualizing therefore shows that $U^\vee$ is a quotient of $T^\vee$. Similarly, dualizing $0 \to V \to U \to A \to 0$ realizes $A^\vee$ as a submodule of $U^\vee$. These maps are horizontal by (4.4). Since the dual of a tensor word is again a tensor word, $A^\vee$ is a subquotient of a finite sum of such words and belongs to $\mathcal{C}$.

The associativity, unit and symmetry isomorphisms for tensor products are horizontal by the product rule. Evaluation and coevaluation satisfy the usual duality identities, as they do for finite free modules. Hence $\mathcal{C}$ is a rigid symmetric $\overline{\mathbb{Q}}$-linear tensor category. Finally,

$$\mathrm{End}_{\mathcal{C}}(R) = \{r \in R : r' = 0\} = \overline{\mathbb{Q}}. \tag{4.6}$$

Indeed, every R-linear endomorphism of R is multiplication by its value at 1, and this multiplication is horizontal precisely when that value has derivative zero.

4.3 The two fibre functors

We use the functors

$$\omega_1 : \mathcal{C} \longrightarrow \mathrm{Vect}_{\overline{\mathbb{Q}}}, \quad A \longmapsto A_1, \qquad \vartheta : \mathcal{C} \longrightarrow \mathrm{Proj}_R, \quad A \longmapsto A. \tag{4.7}$$

Here $\mathrm{Vect}_{\overline{\mathbb{Q}}}$ denotes finite-dimensional vector spaces, and Proj_R denotes finite projective R-modules. The second functor forgets the connection. All its values are in fact free, but the projective-module notation is convenient for the general comparison theorem.

Lemma 13. *Both functors in* (4.7) *are exact faithful tensor functors. In particular, $\mathcal{C}$ is a neutral Tannakian category over $\overline{\mathbb{Q}}$, with fibre functor ω_1.*

Proof. Exactness of ϑ follows from the description of kernels and cokernels above. Exactness of ω_1 is Corollary 12. The tensor constraint for ω_1 is the canonical isomorphism

$$(A \otimes_R B) \otimes_R \overline{\mathbb{Q}} \simeq (A \otimes_R \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} (B \otimes_R \overline{\mathbb{Q}}),$$

and $\omega_1(R) = \overline{\mathbb{Q}}$. The corresponding statements for ϑ follow directly from its definition. These identifications are compatible with associativity, symmetry and the unit.

The forgetful functor is faithful because its action on a morphism does not change the underlying map. For ω_1, let $\phi : A \to B$ be a nonzero horizontal map, and put $I = \mathrm{im}\,\phi$. The module I is free and nonzero, so

$$\dim_{\overline{\mathbb{Q}}} I_1 = \mathrm{rk}_R\, I > 0.$$

Applying the exact fibre functor to $A \twoheadrightarrow I$ and $I \hookrightarrow B$ gives a factorization

$$\phi_1 : A_1 \twoheadrightarrow I_1 \hookrightarrow B_1.$$

Its first map is surjective and its second is injective. Since the intermediate space is nonzero, ϕ_1 cannot vanish. This proves faithfulness.

In particular, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ injects into the finite-dimensional vector space $\mathrm{Hom}_{\overline{\mathbb{Q}}}(A_1, B_1)$, so the morphism spaces are finite-dimensional. Together with rigidity, the abelian structure, (4.6), and the exact faithful functor ω_1, these are the neutral Tannakian properties asserted in the statement. □

Faithfulness concerns horizontal morphisms, rather than arbitrary module maps. For example, multiplication by $z-1$ is a nonzero endomorphism of the underlying module R whose fibre at 1 is zero. It is not horizontal, since

$$\partial((z-1)r) = r + (z-1)r',$$

whereas multiplication by $z-1$ after differentiation gives only $(z-1)r'$. The extra term explains why this map does not contradict Lemma 13. Likewise, when an object consists of functions, the numerical evaluation map $A_1 \to \mathbb{C}$ is separate from the fibre functor. No assertion that such numerical evaluation is injective has been used.

5 The torsor of tensor comparisons

A basis of M identifies its underlying module with $M_1 \otimes_{\overline{\mathbb{Q}}} R$. An independently chosen basis of N gives a second identification, but the two choices need not respect μ. We therefore consider simultaneous comparisons on the whole category $\mathcal{C}$. Their compatibility with morphisms will preserve the multiplication relation, while their tensor compatibility will identify the image of a product with the product of the images.

5.1 Compatible comparisons

Let B be a commutative R-algebra. A tensor comparison over B is a family of B-linear isomorphisms

$$\gamma_A : A_1 \otimes_{\overline{\mathbb{Q}}} B \xrightarrow{\sim} A \otimes_R B, \qquad A \in \mathcal{C}, \tag{5.1}$$

satisfying the following conditions. For every horizontal morphism $\phi : A \to D$, naturality requires

$$(\phi \otimes_R 1) \circ \gamma_A = \gamma_D \circ (\phi_1 \otimes_{\overline{\mathbb{Q}}} 1). \tag{5.2}$$

For every pair of objects A, D, tensor compatibility requires

$$\gamma_{A \otimes_R D} = \gamma_A \otimes_B \gamma_D, \tag{5.3}$$

where the canonical tensor-product identifications are understood. On the unit object we require

$$\gamma_R = \mathrm{id}_B\,. \tag{5.4}$$

These conditions also impose compatibility with duality: applying naturality to evaluation and coevaluation determines the component on $A^\vee$ from that on A.

The functor assigning these families to B is denoted by

$$\mathrm{Isom}^{\otimes}(\omega_1 \otimes_{\overline{\mathbb{Q}}} R, \vartheta).$$

The notation means that ω_1 is first extended from $\overline{\mathbb{Q}}$ to R, and that both resulting functors are then extended to B. Thus the comparison is algebraic in its coefficients; no differential equation is being solved in the coefficient ring B.

We use the following Tannakian comparison theorem in the form given by André [1, Theorem 2.2.1 and §2.3]. The relevant discussion occupies pages 8–10 of the cited preprint and explicitly permits the fibre at an ordinary closed point as the neutral fibre functor.

Theorem 14 (Tannakian comparison)**.** *For the category $\mathcal{C}$ and the functors above, the group*

$$G = \mathrm{Aut}^{\otimes}(\omega_1)$$

is an affine algebraic group over $\overline{\mathbb{Q}}$, and its action on $\omega_1(M)$ identifies it with a closed subgroup of $\mathrm{GL}(\omega_1(M))$. The comparison functor is represented by an affine R-scheme

$$\Sigma = \mathrm{Isom}^{\otimes}(\omega_1 \otimes_{\overline{\mathbb{Q}}} R, \vartheta), \tag{5.5}$$

which is a right torsor under $G_R = G \times_{\overline{\mathbb{Q}}} \mathrm{Spec}\, R$ for the faithfully flat topology.

All hypotheses have been checked in Section 4. In particular, the use of one tensor generator M gives an algebraic group of finite type. This finite-type assertion is needed when we consider its identity component and its finite component group below.

To describe the role of the tensor generator more concretely, put $W = \omega_1(M)$. A tensor automorphism is determined on every tensor word by its component on W: tensor products use tensor powers of this component, and the component on $W^\vee$ is its inverse transpose. Naturality with the canonical inclusions and projections determines the components on direct sums. For a subquotient, naturality then determines the induced restriction and quotient maps. Since every object of $\mathcal{C}$ is obtained in this way, restriction to W is faithful.

There are algebraic conditions on a matrix in $\mathrm{GL}(W)$ for these constructions to define a tensor automorphism. For instance, a differential submodule $U \subseteq T$ of a tensor construction determines a subspace $\omega_1(U) \subseteq \omega_1(T)$. The matrix acting on $\omega_1(T)$ must preserve this subspace. In a basis adapted to it, this says that a block of matrix entries is zero. Entries of the induced action on a tensor construction are polynomials in the entries of the original matrix and its inverse determinant. Horizontal maps impose analogous equations through the requirement that the corresponding diagrams commute. The theorem identifies the simultaneous solution of these compatibility conditions with G. Although all objects and morphisms occur in the definition, their equations generate an ideal in the noetherian coordinate ring of $\mathrm{GL}(W)$, hence an ideal generated by finitely many equations. The existence of the comparison torsor, including its faithfully flat local triviality, is the additional assertion of Theorem 14.

5.2 The torsor and its sections

We spell out the part of the torsor description that will be used. If $X = \mathrm{Spec}\, R$ and P is a right torsor under a group scheme G_X, then $P \to X$ is faithfully flat and locally trivial for a faithfully flat covering. Its right action satisfies the isomorphism

$$P \times_X G_X \xrightarrow{\sim} P \times_X P, \qquad (p, g) \longmapsto (p, p \cdot g). \tag{5.6}$$

This identity is the scheme-theoretic expression of free and transitive action in each fibre. It remains valid after every base change.

For $P = \Sigma$, the action has a concrete meaning. A point g of G is a tensor automorphism with components g_A on the fibres A_1. Its action on a comparison is given by precomposition:

$$(\gamma \cdot g)_A = \gamma_A \circ g_A, \tag{5.7}$$

after extension of scalars to the coefficient ring. If γ and γ' are two comparisons over the same ring, their componentwise quotient $\gamma_A^{-1} \circ \gamma'_A$ is a tensor automorphism of the source functor. Hence it supplies the unique g with $\gamma' = \gamma \cdot g$, in agreement with (5.6).

A global section $s : X \to P$ trivializes a torsor: the map

$$G_X \longrightarrow P, \qquad g \longmapsto s \cdot g$$

is an isomorphism by (5.6). Conversely, a trivialization gives a section by evaluating at the identity of the group. For Σ, such a section is exactly a family (5.1) over R satisfying (5.2)–(5.4).

This last assertion can be read entirely in terms of matrices for each individual object. Choose an R-basis $e_1, \dots, e_r$ of A. Its classes form a $\overline{\mathbb{Q}}$-basis of A_1, and a comparison over B has the form

$$\gamma_A(\overline{e}_j \otimes 1) = \sum_{i=1}^{r} c_{ij}(e_i \otimes 1), \qquad (c_{ij}) \in \mathrm{GL}_r(B).$$

The inverse comparison has coefficients in the same ring B. Thus a section over an affine base gives actual module isomorphisms over its coordinate ring, rather than a family which is only invertible at generic points. Under a change of basis the matrices transform by the usual left and right multiplication, so the maps themselves are intrinsic. Naturality imposes identities between these matrices and the matrices of horizontal morphisms; tensor compatibility imposes the corresponding identities for tensor-product matrices.

Theorem 14 supplies local sections in the faithfully flat topology. The next section produces a global section after the particular finite covering $z = t^q$. Its coefficients will then lie in $R_q = \overline{\mathbb{Q}}[t, t^{-1}]$. This control of the coefficient ring is needed later: a comparison defined only over $\overline{\mathbb{Q}}(t)$ could have a denominator $t - a$ with $a \neq 0$, and its analytic realization could consequently acquire a pole at a.

The comparisons do not have to be horizontal maps from a trivial connection. For example, let $A = Re$ with $\nabla_A(e) = e$. The module isomorphism $\overline{e} \otimes 1 \mapsto e$ identifies $A_1 \otimes_{\overline{\mathbb{Q}}} R$ with A, but it does not identify the trivial connection with ∇_A, since the derivatives of these two displayed generators are respectively 0 and e. What is required in (5.2) is compatibility with the horizontal morphisms already present in $\mathcal{C}$. This is sufficient to preserve μ and ι.

6 A global comparison after a Kummer covering

We first treat a torsor under an arbitrary algebraic group over $\overline{\mathbb{Q}}$. The argument separates its finite component group from its connected identity component. The former is removed by a covering of the punctured line; the latter is handled by the following theorem of Pianzola.

Theorem 15 (Pianzola)**.** *Let k be an algebraically closed field of characteristic zero, let H be a smooth connected linear algebraic group over k, and let Y be a nonempty open subscheme of $\mathbb{A}^1_k$. Every étale-locally trivial torsor under the constant group scheme $H \times_k Y$ has a global section.*

This is [4, Proposition 5(ii)(b)], on page 5 of the cited preprint. The statement applies to all smooth connected linear groups, so no reductivity assumption will be needed. The qualification "constant" refers to the fact that the group scheme comes by base change from k.

6.1 Finite coverings of the punctured line

Lemma 16. *Let $Y \to \mathbb{G}_m$ be a nonempty connected finite étale covering over $\overline{\mathbb{Q}}$, of degree q. There is a coordinate t identifying Y with the punctured line for which the covering is*

$$z = t^q. \tag{6.1}$$

Proof. The curve Y is smooth. Its irreducible components are disjoint: a point where two components met would have a local ring with more than one minimal prime and hence would not be regular. Because there are finitely many components, each is both open and closed. Connectedness therefore implies that Y is integral.

Let $\overline{Y}$ be the normalization of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ in the function field $\overline{\mathbb{Q}}(Y)$. Normalization in a finite extension of the function field of a smooth curve is finite. In dimension one a normal curve is regular; since $\overline{\mathbb{Q}}$ has characteristic zero, $\overline{Y}$ is smooth. It is projective, and the covering extends to a finite separable morphism

$$\pi : \overline{Y} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}$$

of degree q. Over the punctured line the normalization is exactly Y, because Y is already finite and normal with this function field. Thus every ramification point of π lies over 0 or ∞.

Let r_0 and r_∞ be the numbers of distinct points over 0 and ∞. Both are positive, and all these points are $\overline{\mathbb{Q}}$-rational. For each of the two fibres, the sum of the ramification indices is q. The characteristic-zero Riemann–Hurwitz formula [5, Tag 0C1B] consequently gives

$$\begin{aligned} 2g(\overline{Y}) - 2 &= -2q + \sum_{P|0}(e_P - 1) + \sum_{P|\infty}(e_P - 1) \\ &= -2q + (q - r_0) + (q - r_\infty) \\ &= -r_0 - r_\infty. \end{aligned} \tag{6.2}$$

The left-hand side is at least -2, while the right-hand side is at most -2. Hence equality holds, and

$$g(\overline{Y}) = 0, \qquad r_0 = r_\infty = 1.$$

Write P_0 and P_∞ for the two points above 0 and ∞. Each has ramification index q.

A smooth projective genus-zero curve over an algebraically closed field is isomorphic to $\mathbb{P}^1$; see [5, Tag 0C6L]. In the present situation, the coordinate can also be obtained directly from Riemann–Roch. The space of rational functions with at most a simple pole at P_∞ and no other pole has dimension two, while its subspace of regular functions has dimension one. Choose a nonconstant function in the former space. It has a simple pole at P_∞, and the resulting map to $\mathbb{P}^1$ has degree one, so it is an isomorphism. Subtracting its value at P_0 gives a coordinate s which vanishes at P_0. Its zero is simple, because its pole divisor has degree one. Thus s has a simple zero at P_0 and a simple pole at P_∞. The divisors of $\pi^* z$ and s^q are both

$$q[P_0] - q[P_\infty].$$

Their quotient therefore has no zero or pole on $\overline{Y}$. It is a nonzero constant, since regular functions on an integral projective curve are constant. Thus $\pi^* z = c s^q$ for some $c \in \overline{\mathbb{Q}}^\times$. Choose $b \in \overline{\mathbb{Q}}^\times$ with $b^q = c$ and put $t = bs$. Then $\pi^* z = t^q$. Removing P_0 and P_∞ identifies Y with $\operatorname{Spec} \overline{\mathbb{Q}}[t, t^{-1}]$ and proves (6.1). □

For later use, write

$$R \longrightarrow R_q = \overline{\mathbb{Q}}[t, t^{-1}], \qquad z \longmapsto t^q. \tag{6.3}$$

This is a finite free extension of degree q. Indeed, every exponent $n \in \mathbb{Z}$ has a unique expression $n = aq + i$ with $0 \le i < q$, so $1, t, \dots, t^{q-1}$ form an R-basis. The extension is étale because the derivative $q t^{q-1}$ of $t^q - z$ is a unit in R_q. It is also faithfully flat, being free of positive rank.

6.2 Removing the component group

Proposition 17. *Let G be a linear algebraic group over $\overline{\mathbb{Q}}$, and let $P \to X = \mathbb{G}_m$ be a right torsor under G_X. For some integer $q \ge 1$, the pullback of P along* (6.3) *has a global section.*

Proof. The identity component. Every algebraic group over a characteristic-zero field is smooth [5, Tag 047N]. Its identity component $H = G^\circ$ is a smooth connected normal subgroup, and

the quotient $\Pi = G/H$ is a finite étale group scheme. Over the algebraically closed field $\overline{\mathbb{Q}}$, Π is constant.

The morphism $P \to X$ is smooth and surjective. These properties can be checked after a faithfully flat trivializing base change, where it becomes $G_X \to X$. A smooth surjective morphism has sections étale-locally on its base [5, Tag 055U]. Each such section trivializes the torsor by (5.6). Hence P may be treated as an étale-locally trivial torsor.

The finite quotient. Form the quotient

$$Q = P/H_X.$$

For completeness, it can be constructed by descent from trivializations of P. On a trivializing étale cover $U \to X$, the quotient is $\Pi \times_{\overline{\mathbb{Q}}} U$. Transition maps for a right G-torsor are left multiplications by elements of G. They induce left multiplications on G/H, and these satisfy the same cocycle identities. Descent for finite étale schemes therefore produces Q, finite étale over X. Since H is normal, the quotient group Π acts on the right, making Q a Π_X-torsor.

The map $P \to Q$ is itself a torsor under the constant group H_Q. To verify the group and the action explicitly, restrict (5.6) to pairs with the same image in Q. This gives

$$P \times_{\overline{\mathbb{Q}}} H \xrightarrow{\sim} P \times_Q P, \qquad (p, h) \longmapsto (p, p \cdot h). \tag{6.4}$$

The assertion can be checked on a trivialization of P, where it is

$$G \times_{\overline{\mathbb{Q}}} H \xrightarrow{\sim} G \times_{G/H} G.$$

Its inverse sends (g, g') to $(g, g^{-1}g')$, and $g^{-1}g' \in H$ is exactly the condition that the two right cosets agree. Also, $G \to G/H$ is a faithfully flat H-torsor, as is seen by restricting to its cosets. These properties descend to $P \to Q$. Thus the group in (6.4) is the base change of the original H over $\overline{\mathbb{Q}}$.

One can also see this constancy in transition functions after fixing a section of Q. If p_i are local sections of P and $p_j = p_i g_{ij}$ on an overlap, represent the chosen quotient section locally by cosets $a_i H$. Such representatives can be chosen after refining the trivializing cover: Π is finite constant, and every coset has a $\overline{\mathbb{Q}}$-point. Equality of the quotient sections says

$$a_i H = g_{ij} a_j H.$$

Consequently $h_{ij} = a_i^{-1} g_{ij} a_j$ belongs to H, and the corresponding lifts satisfy

$$p_j a_j = p_i a_i h_{ij}.$$

These are transition functions for a torsor with the usual right action of H. This is precisely the reduction used below, and it shows directly why Theorem 15 applies to the constant group H.

A connected covering. Choose a connected component Q_0 of Q. Because Q is finite étale over the noetherian scheme X, its connected components are open and closed. The image of Q_0 in X is closed by finiteness and open by étaleness. It is nonempty, so connectedness of X implies that the image is all of X. Therefore $Q_0 \to X$ is a nonempty connected finite étale covering, of some degree $q \geq 1$.

By Lemma 16, identify this covering with

$$X_q = \operatorname{Spec} R_q \longrightarrow X, \qquad z = t^q.$$

Let $j : X_q \to Q$ be the resulting inclusion of the component. It determines a section of the pulled-back finite quotient,

$$s : X_q \longrightarrow Q \times_X X_q, \qquad y \longmapsto (j(y), y).$$

Take its inverse image in $P \times_X X_q$:

$$T = (P \times_X X_q) \times_{Q \times_X X_q,\, s} X_q.$$

Equivalently, $T = P \times_{Q,\, j} X_q$. Base change of (6.4) gives

$$T \times_{\overline{\mathbb{Q}}} H \simeq T \times_{X_q} T.$$

Since faithful flatness is preserved by base change, $T \to X_q$ is a torsor under $H \times_{\overline{\mathbb{Q}}} X_q$.

Trivializing the connected torsor. The group H is smooth, connected and linear, and X_q is the punctured affine line. As above, smoothness makes T étale-locally trivial. All hypotheses of Theorem 15 therefore hold, and $T \to X_q$ has a global section. Composing that section with $T \to P \times_X X_q$ gives a global section of the pulled-back original torsor. □

6.3 Normalization at the evaluation point

We apply Proposition 17 to Σ. For an object $A \in \mathcal{C}$, write $A_q = A \otimes_R R_q$.

Theorem 18. *There exist an integer $q \geq 1$ and natural tensor-compatible isomorphisms*

$$\gamma_A : A_1 \otimes_{\overline{\mathbb{Q}}} R_q \xrightarrow{\sim} A_q, \qquad A \in \mathcal{C}, \tag{6.5}$$

whose component on R is the identity and whose specialization at $t = 1$ is the identity of A_1 for every A.

Proof. Theorem 14 represents the comparisons by the G_R-torsor Σ. Proposition 17 gives a global section of $\Sigma \times_R R_q$ for some q. Let γ^0_A be the resulting family of comparisons. It remains to adjust its value at $t = 1$.

The substitution $z = t^q$ sends $t = 1$ to $z = 1$. Accordingly, there are canonical identifications

$$A_q \otimes_{R_q} \overline{\mathbb{Q}} \simeq A \otimes_R \overline{\mathbb{Q}} = A_1, \qquad (A_1 \otimes_{\overline{\mathbb{Q}}} R_q) \otimes_{R_q} \overline{\mathbb{Q}} \simeq A_1. \tag{6.6}$$

Here the R_q-module structure on $\overline{\mathbb{Q}}$ is evaluation at $t = 1$. Under these identifications, the specialization of γ^0_A is an invertible $\overline{\mathbb{Q}}$-linear map

$$a_A : A_1 \longrightarrow A_1.$$

Specializing naturality gives $\phi_1 a_A = a_D \phi_1$ for every horizontal $\phi : A \to D$. Specializing tensor compatibility gives $a_{A \otimes D} = a_A \otimes a_D$, and the unit component gives $a_R = \mathrm{id}_{\overline{\mathbb{Q}}}$. Thus the entire family $a = (a_A)$ is a single element of $G(\overline{\mathbb{Q}}) = \mathrm{Aut}^{\otimes}(\omega_1)(\overline{\mathbb{Q}})$.

Define

$$\gamma_A = \gamma^0_A \circ (a_A^{-1} \otimes_{\overline{\mathbb{Q}}} 1). \tag{6.7}$$

This is the action of the constant element $a^{-1} \in G(\overline{\mathbb{Q}})$ on the global section, so it remains a tensor comparison over R_q. Explicitly, naturality of a^{-1} gives

$$(\phi_1 \otimes 1)(a_A^{-1} \otimes 1) = (a_D^{-1} \otimes 1)(\phi_1 \otimes 1),$$

which, combined with naturality of γ^0, proves naturality of γ. Likewise, $a^{-1}_{A \otimes D} = a_A^{-1} \otimes a_D^{-1}$ proves tensor compatibility, and $a_R^{-1} = \mathrm{id}$ preserves the unit component.

Finally, specialization of (6.7) is $a_A a_A^{-1} = \mathrm{id}_{A_1}$. This proves the required normalization simultaneously on every object. □

In particular, define

$$\mathrm{sp}_{A,1} : A_q \longrightarrow A_1, \qquad a \otimes p(t) \longmapsto p(1)\overline{a}.$$

The conclusion of Theorem 18 can then be written

$$\mathrm{sp}_{A,1}(\gamma_A(x \otimes 1)) = x \qquad (x \in A_1). \tag{6.8}$$

The maps γ_A are defined over $\overline{\mathbb{Q}}$, and their coefficients in chosen bases are Laurent polynomials in t. Consequently they are global on the punctured line and preserve algebraic Laurent coefficients under analytic realization. Equation (6.8) will ensure that the realized lift has the prescribed value at 1.

7 Lifting the inverse relation

We return to the modules M, N constructed in Section 3. Let $\mathcal{C} = \langle M \rangle^{\otimes}$ be their ambient tensor category. The surjective horizontal map $\mu : M \otimes_R M \to N$ identifies N with a quotient of an object of $\mathcal{C}$, so $N \in \mathcal{C}$. The unit object R also belongs to $\mathcal{C}$, and the horizontal map $\iota : R \to N$ is a morphism of this full subcategory. Thus one tensor comparison applies simultaneously to M, $M \otimes_R M$, N, R, and the two maps that encode equation (3.10).

Choose $q \geq 1$ and a normalized comparison from Theorem 18. Write

$$R_q = \overline{\mathbb{Q}}[t, t^{-1}], \qquad z \longmapsto t^q, \qquad A_q = A \otimes_R R_q.$$

The comparison consists of isomorphisms

$$\gamma_A : A_1 \otimes_{\overline{\mathbb{Q}}} R_q \xrightarrow{\sim} A_q \qquad (A \in \mathcal{C}).$$

They are natural for the horizontal morphisms in $\mathcal{C}$, compatible with tensor products, equal to the identity on the unit, and equal to the identity after specialization at $t = 1$.

7.1 Preservation of the product identity

Define

$$U = \gamma_M(\bar{f} \otimes 1), \qquad V = \gamma_M(\bar{g} \otimes 1) \quad \text{in } M_q. \tag{7.1}$$

Here $\bar{f}, \bar{g}$ are the algebraic fibre classes in M_1. They are vectors over $\overline{\mathbb{Q}}$ even though their numerical evaluations may be transcendental. Denote the base changes of μ and ι by

$$\mu_q : M_q \otimes_{R_q} M_q \longrightarrow N_q, \qquad \iota_q : R_q \longrightarrow N_q.$$

Naturality for μ gives the commutative square

$$\begin{array}{ccc} (M_1 \otimes_{\overline{\mathbb{Q}}} M_1) \otimes_{\overline{\mathbb{Q}}} R_q & \xrightarrow{\gamma_{M \otimes_R M}} & (M \otimes_R M) \otimes_R R_q \\ \downarrow{\scriptstyle \mu_1 \otimes 1} & & \downarrow{\scriptstyle \mu_q} \\ N_1 \otimes_{\overline{\mathbb{Q}}} R_q & \xrightarrow[\gamma_N]{} & N_q. \end{array}$$

In this diagram, the upper right corner is identified with $M_q \otimes_{R_q} M_q$. Tensor compatibility identifies the upper horizontal arrow with $\gamma_M \otimes_{R_q} \gamma_M$. Applying the square to $(\bar{f} \otimes \bar{g}) \otimes 1$ therefore gives

$$\begin{aligned} \mu_q(U \otimes V) &= \gamma_N(\mu_1(\bar{f} \otimes \bar{g}) \otimes 1) \\ &= \gamma_N(\iota_1(1) \otimes 1) \\ &= \iota_q(\gamma_R(1)) = \iota_q(1). \end{aligned} \tag{7.2}$$

The second equality uses (3.10). The third is naturality for the morphism ι, and the final equality uses preservation of the tensor unit. Each equality is in N_q. In particular, this argument does not equip M_q with an algebra structure or replace the product morphism by an unspecified product.

7.2 Realization by functions on the covering

Put

$$\mathcal{A}_t = \mathcal{O}(\mathbb{C})[t^{-1}].$$

For $A = M, N$, define an R_q-linear map

$$\rho_A : A_q \longrightarrow \mathcal{A}_t, \qquad a(z) \otimes p(t) \longmapsto a(t^q)p(t). \tag{7.3}$$

There are two points to check in this definition. First, the expression belongs to $\mathcal{A}_t$. If $a(z) = z^{-b}F(z)$ with F entire and $b \geq 0$, then

$$a(t^q)p(t) = t^{-qb}F(t^q)p(t).$$

The composition $F(t^q)$ is entire, and multiplication by the Laurent polynomial $t^{-qb}p(t)$ produces at most a pole at 0. Since every element of A belongs to $\mathcal{A}_z$, this proves the assertion.

Second, the formula must respect the balancing relation of the tensor product. For $r \in R$, $a \in A$, and $p \in R_q$, it gives

$$\rho_A((ra) \otimes p) = r(t^q)a(t^q)p(t) = \rho_A\big(a \otimes r(t^q)p(t)\big).$$

Thus the formula is R-balanced and induces the asserted map. Its R_q-linearity follows directly by multiplication in the second factor.

The realization respects the product morphism. For pure tensors $X = a \otimes p$ and $Y = b \otimes r$, one has

$$\begin{aligned} \rho_N\big(\mu_q(X \otimes Y)\big) &= \rho_N(ab \otimes pr) \\ &= a(t^q)b(t^q)p(t)r(t) \\ &= \rho_M(X)\rho_M(Y). \end{aligned}$$

Every tensor is a finite sum of pure tensors, so bilinearity extends the identity to all $X, Y \in M_q$. Also $\rho_N(\iota_q(1)) = 1$. Accordingly, define

$$u = \rho_M(U), \qquad v = \rho_M(V). \tag{7.4}$$

Applying ρ_N to (7.2) gives

$$u(t)v(t) = 1 \qquad (t \in \mathbb{C}^\times). \tag{7.5}$$

No injectivity assertion about ρ_M or ρ_N is needed: the argument uses only their well-definedness and the displayed compatibility with multiplication.

7.3 Preservation of the numerical values

The specialization map for any object $A \in \mathcal{C}$ is

$$\mathrm{sp}_{A,1} : A_q \longrightarrow A_1, \qquad a \otimes p \longmapsto p(1)\bar{a}. \tag{7.6}$$

Indeed, specializing R_q at $t = 1$ specializes R at $z = 1$, since $1^q = 1$. For $A = M, N$, the numerical evaluation maps from (3.11) satisfy

$$\rho_A(X)(1) = \mathrm{ev}_{A,1}\big(\mathrm{sp}_{A,1}(X)\big) \qquad (X \in A_q). \tag{7.7}$$

For $X = a \otimes p$, both sides are $a(1)p(1)$; linearity then proves the formula for a general element.

The normalized comparison satisfies

$$\mathrm{sp}_{A,1}(\gamma_A(x \otimes 1)) = x \qquad (x \in A_1).$$

Taking $A = M$ and $x = \bar{f}, \bar{g}$, respectively, and using (7.7), gives

$$u(1) = \mathrm{ev}_{M,1}(\bar{f}) = f(1) = \xi, \qquad v(1) = \mathrm{ev}_{M,1}(\bar{g}) = g(1) = \xi^{-1}. \tag{7.8}$$

Thus the normalization controls the actual values without identifying the algebraic fibre with a subspace of $\mathbb{C}$.

The construction so far proves the following lifting statement.

Proposition 19. *Let $f, g \in \mathscr{E}_{\mathrm{w}}$ satisfy $f(1)g(1) = 1$, and let M be the module in* (3.3)*, with h supplied by* (3.2)*. There exist an integer $q \geq 1$ and elements $U, V \in M_q$ whose realizations $u = \rho_M(U)$ and $v = \rho_M(V)$ belong to $\mathcal{A}_t$ and satisfy*

$$uv = 1, \qquad u(1) = f(1), \qquad v(1) = g(1).$$

If $F_1, \dots, F_s$ is the finite list of generators used to define M, then there are Laurent polynomials $p_j, r_j \in \overline{\mathbb{Q}}[t, t^{-1}]$ such that

$$u(t) = \sum_{j=1}^{s} p_j(t) F_j(t^q), \qquad v(t) = \sum_{j=1}^{s} r_j(t) F_j(t^q). \tag{7.9}$$

Every F_j belongs to $\mathscr{E}_{\mathrm{w}}$.

To obtain the last assertion, use that the displayed generators of M also generate M_q over R_q. Thus $U = \sum_j F_j \otimes p_j$ and $V = \sum_j F_j \otimes r_j$ for suitable coefficients in R_q. Applying (7.3) gives (7.9). This reasoning does not require the list $F_1, \dots, F_s$ to be a basis.

7.4 Laurent coefficients and growth

Proposition 20. *The functions u, v in Proposition 19 have the following properties.*

(i) *They are holomorphic on $\mathbb{C}^\times$ and meromorphic at 0.*

(ii) *Their Laurent expansions at 0 have coefficients in a common number field contained in $\overline{\mathbb{Q}}$ and only finitely many negative powers.*

(iii) *There are constants $C \geq 1$ and $D \geq 0$ such that*

$$|u(t)|,\ |v(t)| \leq C(1 + |t|)^D \exp(C|t|^{2q}) \qquad (|t| \geq 1). \tag{7.10}$$

Proof. Write u, v as in (7.9). Each composition $F_j(t^q)$ is entire. A Laurent polynomial has no finite pole except possibly at 0, where its principal part is finite. Finite sums and products preserve these properties, proving (i) and the finiteness of the negative powers in (ii).

Choose a number field K_j containing the normalized Taylor coefficients of F_j, and write

$$F_j(z) = \sum_{n \geq 0} a_{j,n} \frac{z^n}{n!}, \qquad a_{j,n} \in K_j.$$

Then

$$F_j(t^q) = \sum_{n \geq 0} \frac{a_{j,n}}{n!} t^{qn}.$$

Choose one finite number field K containing all the finitely many fields K_j and all coefficients of p_j, r_j. Such a field exists because a compositum of finitely many number fields is finite over $\mathbb{Q}$. If $p_j(t) = \sum_{\ell=L_j}^{U_j} p_{j,\ell} t^\ell$, the coefficient of t^b in $p_j(t)F_j(t^q)$ is

$$\sum_{\substack{L_j \le \ell \le U_j \\ b-\ell \in q\mathbb{Z}_{\ge 0}}} p_{j,\ell} \frac{a_{j,(b-\ell)/q}}{((b-\ell)/q)!}. \tag{7.11}$$

It is a finite sum in K. Summing over the finite index set of j therefore gives Laurent coefficients of u in K; the identical calculation with r_j gives those of v. The series converge on punctured discs because they arise from convergent entire series multiplied by Laurent polynomials. This proves (ii).

By Lemma 6, there are constants $C_j > 0$ with

$$|F_j(t^q)| \le C_j \exp(|t|^{2q}) \qquad (t \in \mathbb{C}).$$

Choose a nonnegative integer D at least as large as every positive exponent appearing in any p_j or r_j. For $|t| \ge 1$ and an exponent $\ell \le D$, one has $|t^\ell| \le (1+|t|)^D$, also when $\ell < 0$. It follows, for example, that

$$|p_j(t)| \le \left(\sum_{\ell=L_j}^{U_j} |p_{j,\ell}| \right) (1+|t|)^D.$$

The same bound holds for r_j with its own coefficient sum. Multiply by the estimates for $F_j(t^q)$ and sum over j. Increasing a common constant to be at least 1 gives (7.10), proving (iii). □

For $q > 1$, a function $F_j(t^q)$ need not be an E-function in the variable t. Proposition 20 records exactly the properties that will be used in Section 8: the possible finite pole is confined to 0, the Laurent coefficients are algebraic, and both reciprocal functions satisfy a finite-order growth bound.

7.5 The lifting calculation in coordinates

The following coordinate description makes explicit the compatibility conditions carried by the tensor comparison. Choose R-bases $e_1, \dots, e_r$ of M and $n_1, \dots, n_s$ of N. Their elements are actual functions in $\mathcal{A}_z$. There are unique structure coefficients $c_{ij}^\ell, d_\ell \in R$ such that

$$e_i e_j = \sum_{\ell=1}^{s} c_{ij}^\ell(z) n_\ell, \qquad 1 = \sum_{\ell=1}^{s} d_\ell(z) n_\ell. \tag{7.12}$$

Their specializations at $z = 1$ give the coordinate descriptions of μ_1 and ι_1.

Represent γ_M and γ_N by the invertible matrices

$$\mathsf{A}(t) \in \mathrm{GL}_r(R_q), \qquad \mathsf{B}(t) \in \mathrm{GL}_s(R_q),$$

using the column convention

$$\gamma_M(\bar{e}_i \otimes 1) = \sum_{a=1}^{r} \mathsf{A}_{ai}(t)(e_a \otimes 1),$$
$$\gamma_N(\bar{n}_m \otimes 1) = \sum_{\ell=1}^{s} \mathsf{B}_{\ell m}(t)(n_\ell \otimes 1).$$

The specialization condition reads

$$\mathsf{A}(1) = I_r, \qquad \mathsf{B}(1) = I_s. \tag{7.13}$$

Expand both sides of the naturality identity for μ on $\bar{e}_i \otimes \bar{e}_j$. On one side, tensor compatibility first applies the two columns of A, and then multiplication uses $c_{ab}^{\ell}(t^q)$. On the other side, specialization first gives $c_{ij}^{m}(1)$, and then γ_N applies B. Comparing the coefficients of $n_\ell \otimes 1$ gives

$$\sum_{a,b=1}^{r} \mathsf{A}_{ai}(t)\mathsf{A}_{bj}(t)c_{ab}^{\ell}(t^q) = \sum_{m=1}^{s} c_{ij}^{m}(1)\mathsf{B}_{\ell m}(t) \qquad (i,j,\ell). \tag{7.14}$$

Naturality for the unit morphism similarly gives

$$\sum_{m=1}^{s} d_m(1)\mathsf{B}_{\ell m}(t) = d_\ell(t^q) \qquad (1 \le \ell \le s). \tag{7.15}$$

These are simultaneous constraints on the two comparison matrices. They are the specific multiplication and unit constraints required by the proof.

Write

$$\bar{f} = \sum_{i=1}^{r} x_i \bar{e}_i, \qquad \bar{g} = \sum_{j=1}^{r} y_j \bar{e}_j, \qquad x_i, y_j \in \overline{\mathbb{Q}}.$$

Equation (3.10) is equivalent to

$$\sum_{i,j=1}^{r} x_i y_j c_{ij}^{m}(1) = d_m(1) \qquad (1 \le m \le s). \tag{7.16}$$

The coefficient of $n_\ell \otimes 1$ in $\mu_q(U \otimes V)$ is therefore

$$\begin{aligned} &\sum_{i,j=1}^{r} x_i y_j \sum_{a,b=1}^{r} \mathsf{A}_{ai}(t)\mathsf{A}_{bj}(t)c_{ab}^{\ell}(t^q) \\ &\quad = \sum_{m=1}^{s} \mathsf{B}_{\ell m}(t) \sum_{i,j=1}^{r} x_i y_j c_{ij}^{m}(1) \\ &\quad = \sum_{m=1}^{s} \mathsf{B}_{\ell m}(t) d_m(1) = d_\ell(t^q). \end{aligned}$$

The three equalities use (7.14), (7.16), and (7.15), in that order. By (7.12), the last expression is precisely the coefficient of the unit in N_q. This recovers (7.2) coefficient by coefficient.

The corresponding analytic function is

$$u(t) = \sum_{a,i=1}^{r} \mathsf{A}_{ai}(t) x_i e_a(t^q).$$

At $t = 1$, equation (7.13) yields

$$u(1) = \sum_{i=1}^{r} x_i e_i(1) = \mathrm{ev}_{M,1}(\bar{f}) = f(1).$$

The coefficients x_i and those of $\mathsf{A}(t)$ are algebraic; the values $e_i(1)$ are permitted to be transcendental. This is the coordinate form of the distinction between algebraic coefficients of the comparison and numerical values of the functions it compares.

8 Classification of the lifted functions

The lifting proposition produces two functions whose product is identically equal to one. Proposition 20 gives additional information of two different kinds: a growth bound on large circles, and algebraicity of the Laurent coefficients at the origin. We now show how these properties determine the form of the functions. The growth bound forces a normalized entire logarithm to be a polynomial. The Laurent expansion then determines the multiplicative constant and proves that the coefficients of this polynomial are algebraic. We give both arguments explicitly.

Lemma 21. *Let u and v be holomorphic functions on $\mathbb{C}^\times$, meromorphic at the origin, such that*

$$u(t)v(t) = 1 \qquad (t \in \mathbb{C}^\times). \tag{8.1}$$

Assume that the Laurent coefficients of u at the origin belong to $\overline{\mathbb{Q}}$. Suppose, moreover, that there are an integer $d \geq 1$ and real constants $C \geq 1$, $D \geq 0$ such that

$$|u(t)|,\ |v(t)| \leq C(1+|t|)^D \exp(C|t|^d) \qquad (|t| \geq 1). \tag{8.2}$$

Then there are uniquely determined elements

$$m \in \mathbb{Z}, \qquad \alpha \in \overline{\mathbb{Q}}^\times, \qquad P \in \overline{\mathbb{Q}}[t], \qquad P(0) = 0,$$

with $\deg P \leq d$, such that

$$u(t) = \alpha t^m e^{P(t)}, \qquad v(t) = \alpha^{-1} t^{-m} e^{-P(t)} \qquad (t \in \mathbb{C}^\times). \tag{8.3}$$

The degree condition includes the case $P = 0$.

The coefficient hypothesis is imposed only on u. Algebraicity of the Laurent coefficients of v follows from the product identity and will also appear in the proof. In contrast, the two growth bounds will be used separately. We do not require any growth estimate in a punctured neighbourhood of zero, since the meromorphicity assumption already specifies the singularities there.

Proof. Removal of the factor at the origin. Equation (8.1) implies that neither function vanishes on $\mathbb{C}^\times$. In particular, neither is identically zero. Since u is meromorphic at the origin, its Laurent expansion has a first nonzero term. Write

$$u(t) = a_m t^m + a_{m+1} t^{m+1} + \cdots, \qquad a_m \neq 0,$$

in a sufficiently small punctured disc, where m is an integer. Thus $m = \operatorname{ord}_0(u)$; a negative value of m represents a pole. The hypothesis on the Laurent coefficients gives $a_m \in \overline{\mathbb{Q}}^\times$. Here meromorphicity means that multiplication by a sufficiently large nonnegative power of t gives a holomorphic function near zero. The resulting holomorphic function has a finite order of vanishing: otherwise all of its Taylor coefficients would be zero, and the function would vanish identically near zero. The identity theorem would then force u to vanish on $\mathbb{C}^\times$, contrary to (8.1). This justifies the existence of the finite integer m in every case, including when zero was originally a removable singularity.

Let $n = \operatorname{ord}_0(v)$, and let b_n be the first nonzero Laurent coefficient of v. Multiplying the two convergent Laurent series, the first nonzero term of their product is $a_m b_n t^{m+n}$. Its coefficient is nonzero, so it cannot be cancelled by terms of higher degree. Comparing with the constant function one in (8.1) gives

$$n = -m, \qquad a_m b_{-m} = 1.$$

Consequently the functions

$$W(t) = t^{-m}u(t), \qquad Z(t) = t^m v(t), \tag{8.4}$$

initially defined away from zero, have removable singularities at zero. Their extensions have values $W(0) = a_m$ and $Z(0) = a_m^{-1}$.

Both extensions are entire, because the original functions have no other finite singularities. Their product equals one on $\mathbb{C}^\times$, and continuity gives the same identity at zero. We therefore have

$$WZ = 1 \quad \text{on } \mathbb{C}, \qquad W(0) = \alpha := a_m \in \overline{\mathbb{Q}}^\times. \tag{8.5}$$

In particular, W is nowhere zero on the whole plane, and its reciprocal is the entire function Z. The algebraicity of α has already been established, before any logarithm is introduced.

Construction of a normalized entire logarithm. Since W is entire and nowhere zero, the quotient W'/W is entire. It has an entire primitive on $\mathbb{C}$. More explicitly, define

$$L(t) = \int_0^t \frac{W'(s)}{W(s)}\, ds. \tag{8.6}$$

The integral may be taken along any piecewise smooth path from zero to t. It is independent of the path: the difference of two such integrals is the integral of an entire function around a closed curve in the simply connected plane, and is zero by Cauchy's theorem. Local differentiation of the integral gives

$$L'(t) = \frac{W'(t)}{W(t)}, \qquad L(0) = 0.$$

Thus L is entire. The product rule now yields

$$\frac{d}{dt}(W(t)e^{-L(t)}) = e^{-L(t)}(W'(t) - W(t)L'(t)) = 0.$$

An entire function with zero derivative is constant. Evaluating at zero and using (8.5), we obtain

$$W(t) = \alpha e^{L(t)} \qquad (t \in \mathbb{C}). \tag{8.7}$$

This construction does not involve a choice of a complex logarithm of α. The normalization $L(0) = 0$ separates the value of W at zero from the variable part of its exponential representation.

A bound for the real part of the logarithm. Fix $r \geq 2$ and let $|t| = r$. For either sign of m,

$$|t^{-m}| \leq r^{|m|}, \qquad |t^m| \leq r^{|m|}.$$

Combining (8.2) and (8.4), and using $1 + r \leq 2r$, gives

$$|W(t)|,\ |Z(t)| \leq Ar^{D+|m|}\exp(Cr^d), \qquad A := C2^D \geq 1. \tag{8.8}$$

Set

$$B(r) = \log A + (D + |m|)\log r + Cr^d.$$

The first bound in (8.8) implies $\log|W(t)| \leq B(r)$. Since $Z = 1/W$, the second implies $-\log|W(t)| \leq B(r)$. Together they yield

$$-B(r) \leq \log|W(t)| \leq B(r).$$

All logarithms in these inequalities are ordinary real logarithms of positive real numbers. Taking absolute values in (8.7) gives $|W(t)| = |\alpha|e^{\operatorname{Re} L(t)}$, and hence

$$\operatorname{Re} L(t) = \log|W(t)| - \log|\alpha|.$$

It follows that

$$|\operatorname{Re} L(t)| \le B(r) + |\log|\alpha||.$$

Because $d \ge 1$, we have $\log r \le r^d$ and $1 \le r^d$ for $r \ge 2$. The constants in the last bound can therefore be absorbed into a single constant $C_1 > 0$, independent of r and t, giving

$$|\operatorname{Re} L(t)| \le C_1 r^d \qquad (|t| = r \ge 2). \tag{8.9}$$

The estimate on the reciprocal is used precisely here: it supplies the lower bound for $\log|W|$, and therefore controls both signs of $\operatorname{Re} L$.

Vanishing of the high Taylor coefficients. Since L is entire and $L(0) = 0$, write

$$L(t) = \sum_{j=1}^{\infty} c_j t^j.$$

For each fixed $r > 0$, this power series converges absolutely and uniformly on the circle $|t| = r$. Indeed, convergence on any larger closed disc implies uniform convergence on this circle. We may thus take real parts and integrate term by term in the identity

$$\operatorname{Re} L(re^{i\theta}) = \frac{1}{2} \sum_{\ell=1}^{\infty} \left(c_\ell r^\ell e^{i\ell\theta} + \overline{c_\ell} r^\ell e^{-i\ell\theta} \right).$$

The asserted interchange can also be checked by uniform remainders. If L_N is the Taylor polynomial of degree N, then $\sup_\theta |L(re^{i\theta}) - L_N(re^{i\theta})|$ tends to zero. The same is true after taking real parts and multiplying by $e^{-ij\theta}$, whose modulus is one. The difference between the corresponding integrals has modulus at most 2π times this supremum, and hence tends to zero. Thus the computation with finite sums passes to the full series without any additional convergence assumption. For a fixed integer $j \ge 1$, multiply by $e^{-ij\theta}$ and integrate over $0 \le \theta \le 2\pi$. In the first part of the sum the integral of $e^{i(\ell-j)\theta}$ is zero unless $\ell = j$, when it equals 2π. In the second part every frequency is $-(\ell + j) \ne 0$, so every integral is zero. Consequently

$$c_j r^j = \frac{1}{\pi} \int_0^{2\pi} \operatorname{Re} L(re^{i\theta}) e^{-ij\theta}\, d\theta. \tag{8.10}$$

In particular, the real part controls each complex Taylor coefficient; no separate bound for the imaginary part is needed.

For $r \ge 2$, equations (8.9) and (8.10) give

$$|c_j| r^j \le \frac{1}{\pi} \int_0^{2\pi} C_1 r^d\, d\theta = 2C_1 r^d, \qquad |c_j| \le 2C_1 r^{d-j}.$$

If $j > d$, the right-hand side tends to zero as $r \to \infty$. The coefficient c_j is independent of r, so $c_j = 0$. This holds separately for every integer $j > d$. We have proved

$$L(t) = P(t) := \sum_{j=1}^{d} c_j t^j \in \mathbb{C}[t], \qquad P(0) = 0. \tag{8.11}$$

The possibility that some or all of these coefficients vanish is included.

Algebraicity of the polynomial. It remains to locate the coefficients of P in $\overline{\mathbb{Q}}$. Multiplication of the Laurent series of u by t^{-m} gives the Taylor expansion

$$W(t) = \sum_{n=0}^{\infty} w_n t^n, \qquad w_n \in \overline{\mathbb{Q}}, \qquad w_0 = \alpha \ne 0.$$

We first check directly that the Taylor coefficients of $Z = 1/W$ also belong to $\overline{\mathbb{Q}}$. Write $Z(t) = \sum_{n\geq 0} b_n t^n$ near zero. The constant coefficient of $WZ = 1$ gives $b_0 = w_0^{-1}$, and comparison of the coefficient of t^n for $n \geq 1$ gives

$$w_0 b_n + \sum_{j=1}^{n} w_j b_{n-j} = 0.$$

Equivalently,

$$b_0 = w_0^{-1}, \qquad b_n = -w_0^{-1} \sum_{j=1}^{n} w_j b_{n-j} \quad (n \geq 1). \tag{8.12}$$

Induction on n proves $b_n \in \overline{\mathbb{Q}}$ for every n: each new coefficient is obtained from earlier coefficients and the w_j by field operations in $\overline{\mathbb{Q}}$. This also proves directly that the formal inverse of W lies in $\overline{\mathbb{Q}}[[t]]$. Its coefficients agree with those of the analytic reciprocal, because both satisfy the same recursion.

Differentiation and multiplication of these Taylor series now give

$$\frac{W'}{W} = W'Z \in \overline{\mathbb{Q}}[[t]].$$

For example, the coefficient of t^n in this series is the finite sum

$$\sum_{j=0}^{n} (j+1) w_{j+1} b_{n-j},$$

which belongs to $\overline{\mathbb{Q}}$. On the other hand, (8.6) and (8.11) imply the analytic identity $W'/W = P'$. Equality of analytic functions near zero gives equality of their Taylor coefficients. Thus, for $1 \leq j \leq d$, the coefficient jc_j of t^{j-1} in P' belongs to $\overline{\mathbb{Q}}$. As $\overline{\mathbb{Q}}$ has characteristic zero, division by the nonzero integer j shows that $c_j \in \overline{\mathbb{Q}}$. Since $P(0) = 0$, this accounts for every coefficient, and proves $P \in \overline{\mathbb{Q}}[t]$.

More explicitly, comparison of the coefficient of t^{j-1} gives

$$c_j = \frac{1}{j} \sum_{\ell=0}^{j-1} (\ell+1) w_{\ell+1} b_{j-1-\ell} \qquad (1 \leq j \leq d). \tag{8.13}$$

Together with (8.12), this formula recovers c_j using only $w_0, \ldots, w_j$. For instance, the first two coefficients, when present, are

$$c_1 = \frac{w_1}{\alpha}, \qquad c_2 = \frac{w_2}{\alpha} - \frac{w_1^2}{2\alpha^2}.$$

These identities make the arithmetic conclusion independent of any choice of a logarithm of an algebraic number. They also show that, once the degree bound is known, only finitely many Taylor coefficients are needed to recover the entire polynomial P. The preceding analytic argument is what proves that no further coefficients of P can occur.

The condition on the constant term is essential to this last conclusion. The logarithmic derivative determines the nonconstant coefficients of P, while the value at zero determines the prefactor α. Our normalization assigns the entire constant factor to α; there is no additional constant in the exponent to determine.

Combining (8.4), (8.7), and (8.11) proves the formula for u in (8.3). The formula for v follows from $uv = 1$.

Uniqueness. In any representation of the asserted form, $e^{P(t)}$ is holomorphic and nonzero at zero, with value one. The order of u at zero is therefore m, and its first nonzero Laurent

coefficient is α. Both are determined by u. If two normalized polynomials P and Q give the same remaining exponential factor, then $e^{P-Q} = 1$. Differentiation gives $(P' - Q')e^{P-Q} = 0$, hence $P' = Q'$. Their difference is constant, and $P(0) = Q(0) = 0$ makes that constant zero. Thus $P = Q$, completing the proof. □

For the functions constructed in Proposition 19, Proposition 20 supplies the hypotheses of Lemma 21 with $d = 2q$. The resulting representation can be evaluated directly at $t = 1$:

$$u(1) = \alpha e^{P(1)} \in \overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}}). \tag{8.14}$$

Here $1^m = 1$ for every integer m, including negative integers, and $P(1) \in \overline{\mathbb{Q}}$ because P has finitely many coefficients in $\overline{\mathbb{Q}}$. Thus the pullback variable causes no difficulty in recovering the prescribed value. Neither a descent of u to the original variable nor an E-function property for u is needed in this step.

9 Proof of the unit classification

We now combine the preceding statements. We prove the inclusion supplied by exponential functions directly, so that both directions use the definition of the same ring of values.

Proof of Theorem 1. Let $\alpha \in \overline{\mathbb{Q}}^{\times}$ and $\beta \in \overline{\mathbb{Q}}$. Consider

$$F(z) = \alpha e^{\beta z}, \qquad G(z) = \alpha^{-1} e^{-\beta z}.$$

They satisfy $F' = \beta F$ and $G' = -\beta G$, so both are differentially finite. Their normalized coefficient sequences are

$$a_n = \alpha \beta^n, \qquad b_n = \alpha^{-1}(-\beta)^n,$$

and lie in the number field $K = \mathbb{Q}(\alpha, \beta)$. This also covers $\beta = 0$, with the usual constant coefficient at $n = 0$ and zero coefficients for $n > 0$.

There is a constant $A \geq 1$ bounding the absolute values of every conjugate of α, α^{-1}, and β. It follows that

$$|\sigma(a_n)|,\ |\sigma(b_n)| \leq A^{n+1} \qquad (\sigma : K \hookrightarrow \mathbb{C}).$$

Choose positive integers s_+, s_-, s_β such that $s_+\alpha$, $s_-\alpha^{-1}$, and $s_\beta\beta$ are algebraic integers. For $0 \leq j \leq n$,

$$(s_+ s_\beta^n)\alpha\beta^j = (s_+\alpha)s_\beta^{n-j}(s_\beta\beta)^j$$

is integral. Thus $s_+ s_\beta^n$ clears $a_0, \ldots, a_n$. Similarly, $s_- s_\beta^n$ clears $b_0, \ldots, b_n$, because the factor $(-1)^j$ is integral. These common denominators are exponentially bounded. Lemma 3 turns each exponential bound into a bound by a constant times $(n!)^\varepsilon$ for every $\varepsilon > 0$. Hence $F, G \in \mathscr{E}_{\mathrm{w}}$. Since

$$F(1) = \alpha e^{\beta}, \qquad G(1) = \alpha^{-1} e^{-\beta}, \qquad F(1)G(1) = 1,$$

we have proved

$$\overline{\mathbb{Q}}^{\times} \exp(\overline{\mathbb{Q}}) \subseteq \mathbf{E}_{\mathrm{w}}^{\times}. \tag{9.1}$$

Conversely, let $\xi \in \mathbf{E}_{\mathrm{w}}^{\times}$, and choose $f, g \in \mathscr{E}_{\mathrm{w}}$ with $f(1) = \xi$ and $g(1) = \xi^{-1}$. Theorem 8 gives $h \in \mathscr{E}_{\mathrm{w}}$ satisfying $fg - 1 = (z-1)h$. Proposition 9 then permits the construction of the finite differential modules M, N in Section 3. Their fibres satisfy $\mu_1(\bar{f} \otimes \bar{g}) = \iota_1(1)$.

The results of Sections 4–6 produce a global tensor comparison over $R_q = \overline{\mathbb{Q}}[t, t^{-1}]$, for some integer $q \geq 1$, normalized at $t = 1$. Proposition 19 applies this comparison to the fibre relation and produces u, v with

$$uv = 1 \quad \text{on } \mathbb{C}^{\times}, \qquad u(1) = \xi, \qquad v(1) = \xi^{-1}.$$

By Proposition 20, both functions are holomorphic on $\mathbb{C}^\times$ and meromorphic at 0, their Laurent coefficients are algebraic, and they satisfy (7.10). All hypotheses of Lemma 21 therefore hold with $d = 2q$. It follows that

$$u(t) = \alpha t^m e^{P(t)}$$

for some $\alpha \in \overline{\mathbb{Q}}^\times$, $m \in \mathbb{Z}$, and $P \in \overline{\mathbb{Q}}[t]$ with $P(0) = 0$.

Evaluating at $t = 1$, and using $1^m = 1$, gives

$$\xi = u(1) = \alpha e^{P(1)} \in \overline{\mathbb{Q}}^\times \exp(\overline{\mathbb{Q}}), \tag{9.2}$$

because $P(1)$ is a finite sum of algebraic numbers. Since ξ was an arbitrary unit, (9.2) proves the reverse inclusion. Together with (9.1), this proves the theorem. □

Corollary 22. *Let $\mathbf{E}_{\mathrm{str}}$ be the ring of values at algebraic points of E-functions in the strict sense. Then*

$$\mathbf{E}_{\mathrm{str}}^\times = \overline{\mathbb{Q}}^\times \exp(\overline{\mathbb{Q}}).$$

Proof. Every strict E-function belongs to $\mathscr{E}_{\mathrm{w}}$ by Lemma 3. If ξ is a unit of $\mathbf{E}_{\mathrm{str}}$, both ξ and its reciprocal belong to $\mathbf{E}_{\mathrm{w}}$, so $\xi \in \mathbf{E}_{\mathrm{w}}^\times$. Theorem 1 therefore gives $\xi \in \overline{\mathbb{Q}}^\times \exp(\overline{\mathbb{Q}})$. Conversely, the functions $\alpha e^{\beta z}$ and $\alpha^{-1} e^{-\beta z}$ used above already satisfy exponential bounds for their conjugates and common denominators. They are strict E-functions and their values are mutual inverses. This proves the other inclusion. □

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