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Critical divisor weights and the localisation of parity defects

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Abstract

At smoothing parameter zero the derivative of $\prod_{p|n}(1-p^{-s})$ is the von Mangoldt function. We study its normalised correlation at $n, n+2$ as the two parameters vary on the scale $1/\log X$. Every subsequential defect from the reference profile is the Laplace transform of xyD, where D is supported on the part of $[1/2,1]^2$ satisfying $7\min(x,y)+6\max(x,y)\geq 7$. This bounds the defect at (m,m) by $O((1+m)^C e^{-14m/13})$ and the relative error for two generalised von Mangoldt weights of equal fixed order j by $O((1+j)^C(36/169)^j)$. The separation of this support from its reflection makes the weighted mass on opposite Liouville signs determine all four limiting profiles. With a von Mangoldt weight at the second site, Fourier observations up to index K on the even sector at the first site recover the normalised twin-prime defect with error $O(1/K)$ and coefficient norm at most one. A local estimate at divisor size $X^{1/2}$ removes the ambiguity caused by reflection. Here K is fixed before $X\to\infty$. The observations themselves are arithmetic inputs. Under generalised Elliott–Halberstam up to level one the support reduces to a point, giving an explicit scalar form of Bombieri's parity principle. A separate application gives nonvanishing signed Taylor coefficients on four affine forms.

1 Two sites at critical smoothing

How much of a two-point prime correlation is determined when the prime weights vary through an analytic family of divisor weights? The scale $s=t/\log X$ retains contributions from integers with several large prime factors, while $s=0$ recovers the prime weight itself.

For an integer $m>1$ define

$$H_s(m)=\prod_{p|m}(1-p^{-s})=\sum_{d|m}\mu(d)d^{-s},\qquad W_s(m)=\partial_s H_s(m)=-\sum_{d|m}\mu(d)(\log d)d^{-s}.$$

The derivative acts on the divisor d. The distinction matters if one uses Dirichlet convolutions with a logarithm on the complementary divisor. At $s=0$, $W_0(p^a)=\log p$ for $a\geq 1$, while $W_0(m)=0$ as soon as m has two distinct prime factors. Thus $W_0=\Lambda$. The classical generalised Jordan totient is $\phi_s(m)=m^s H_s(m)$. Hardy and Subbarao [20, p. 276] give the definition for nonintegral s. If $\Lambda_j=\mu*(\log^j)$ denotes the generalised von Mangoldt function, then

$$\phi_s(m)=\sum_{d|m}\mu(d)(m/d)^s=\sum_{j\geq 1}\frac{\Lambda_j(m)}{j!}s^j\qquad(m>1).\tag{1}$$

The equality holds for every complex s. The functions Λ_j are the classical weights of the asymptotic sieve [4, 6]. Our object is their analytic two-site correlation at the critical scale.

On a product pq of distinct primes the two terms in $W_s(pq)$ are

$$(\log p)p^{-s}(1-q^{-s})+(\log q)q^{-s}(1-p^{-s}).$$

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For instance, if $p, q = X^{1/2+o(1)}$, this weight divided by $\log X$ tends to $e^{-t/2}(1-e^{-t/2})$ at $s = t/\log X$. A prime of size $X^{1+o(1)}$ gives e^{-t} instead. At fixed $t > 0$ both populations contribute. We ask how their total weight changes when a second copy, or a prime weight, is placed two integers away.

All logarithms are natural. Let $L = \log X$, $\mathbb{E}_X f = X^{-1}\sum_{X<n\le 2X} f(n)$, and put

$$S = 2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big),$$

$$J_X(z,w) = \frac{L^2}{S}\mathbb{E}_X H_{z/L}(n)H_{w/L}(n+2), \qquad M_X(t,u) = \frac{1}{S}\mathbb{E}_X W_{t/L}(n)W_{u/L}(n+2). \tag{2}$$

The primitive J_X is zero on both axes and $M_X = \partial_z\partial_w J_X$. Write $\mathcal{L}D(z,w) = \langle D, e^{-zx-wy}\rangle$ for the Laplace transform of a compactly supported distribution. Here x, y are logarithmic sizes of divisors, measured in units of L. Distributions occur naturally because the divisor coefficients have signs and their total variations need not be bounded. We use $\langle \delta_c', f\rangle = -f'(c)$, so $\mathcal{L}\delta_c'(z) = ze^{-cz}$.

Put

$$\mathcal{P} = \{(x,y)\in[1/2,1]^2 :\ 7\min(x,y) + 6\max(x,y) \ge 7\}, \tag{3}$$

$$d(t,u) = \min\{7(a+b)/13,\ b/2 + 7a/12\}, \qquad a = \min(t,u), \quad b = \max(t,u). \tag{4}$$

Theorem 1.1 (Localisation and compensation)**.** *From every sequence $X \to \infty$ one can extract a subsequence along which, locally uniformly on $\mathbb{C}^2$ and with all fixed derivatives,*

$$J_X(z,w) \longrightarrow zw + \mathcal{L}D(z,w), \qquad M_X(t,u) \longrightarrow 1 + F(t,u), \qquad F = \mathcal{L}(xyD).$$

Unconditionally $\operatorname{supp} D \subset \mathcal{P}$*. Under generalised Elliott–Halberstam at every fixed level $0 < \theta < \Theta$, in the form given in Section 3, the containing set is $\mathcal{P}\cap[\Theta,1]^2$, where $1/2 < \Theta \le 1$. Take $\Theta = 1/2$ in the unconditional case. Denote the minimum of $tx + uy$ on $\mathcal{P}\cap[\Theta,1]^2$ by $d_\Theta(t,u)$, with $d_{1/2} = d$ as in* (4)*. For real $t, u \ge 0$,*

$$|F(t,u)| \ll (1+t+u)^C e^{-d_\Theta(t,u)}, \tag{5}$$

$$\int_0^\infty F(t,v)\,\mathrm{d}v = 0, \qquad \int_0^\infty F(v,u)\,\mathrm{d}v = 0. \tag{6}$$

The constants and the distribution-order bounds are uniform over the extracted limits. Each fixed derivative satisfies a bound of the form (5)*.*

The two lower bounds on the divisor coordinates come from separate applications of distribution in progressions. To prove $y \ge 1/2$ one decomposes the weight at n into products of primes, counts those products in the class -2 modulo a divisor of $n+2$, and subtracts the local main term. Interchanging n and $n+2$ proves $x \ge 1/2$. In particular the estimate still gives decay when one smoothing parameter is zero. The endpoint value

$$\kappa_X = M_X(0,0) - 1 = \frac{\mathbb{E}_X\Lambda(n)\Lambda(n+2)}{S} - 1$$

can vary with the dyadic interval. The assertion $\kappa_X \to 0$ is the dyadic weighted twin-prime asymptotic. Prime powers of exponent at least two contribute $o(1)$ here by elementary counting.

A direct count of compatible divisor pairs costs $O(L^2X^{x+y-1})$ for divisors up to X^x, X^y. For example, $(3/4, 1/3)$ escapes this estimate but is covered by distribution in the second coordinate. The point $(8/15, 8/15)$ escapes both coordinate estimates. Here a Möbius decomposition of the shorter divisor produces two bounds for the same reciprocal sum. Their

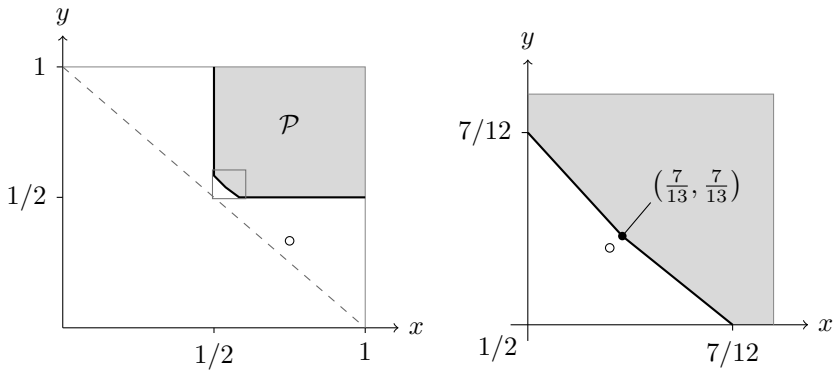

Figure 1: The shaded region contains the support. The dashed line is $x + y = 1$. The open circle on the left is $(3/4, 1/3)$. In the enlargement the open circle is $(8/15, 8/15)$, excluded by the oblique bound, and the filled vertex minimises $x + y$. Whether the support meets that boundary is open.

dependence on the length of the longer bilinear factor has opposite slopes. Averaging the exponents removes that length and excludes a whole neighbourhood of the point (Section 3.2).

The same geometry controls the ordinary weights in (1). For equal orders, Corollary 3.5 gives

$$\limsup_{X\to\infty} \left| \frac{\mathbb{E}_X \Lambda_j(n)\Lambda_j(n+2)}{Sj^2 L^{2j-2}} - 1 \right| \ll (1+j)^C (36/169)^j.$$

On each extraction the residual term at orders j, k is $\langle D, (1-x)^j (1-y)^k \rangle / (jk)$. Equal orders select $(7/13, 7/13)$ in the maximisation. Widely unequal orders select $(1/2, 7/12)$ or its reflection across the diagonal. The orders are fixed before $X \to \infty$.

Reflection in both coordinates sends $\mathcal{P}$ into $x + y \le 12/13$, whereas $\mathcal{P}$ itself has $x + y \ge 14/13$. This gap makes the unsigned defect identifiable from the total weight on opposite signs. At the prime edge the intervals $[1/2, 1]$ and $[0, 1/2]$ still touch. Theorem 5.7 resolves their contact using primes in a fixed residue class to moduli with a divisor in a prescribed interval. It yields a finite observation formula for κ_X. Its arithmetic evaluation, including whether $\kappa_X \to 0$, remains open.

Relation to earlier work. Balasubramanian and Giri [1, Theorem 1 and Corollary 1 in the preprint] study shifted products of multiplicative functions. Their corollary includes the Jordan totients ϕ_k at fixed integer order $k \ge 2$. For the normalised weight H_s the divisor coefficients are $\mu(d)d^{-s}$. At real $s > 0$ their absolute sum with harmonic weight is

$$\sum_{d\ge 1} \frac{|\mu(d)|}{d^{1+s}} = \prod_p (1 + p^{-1-s}) = \frac{\zeta(1+s)}{\zeta(2+2s)} \sim \frac{1}{\zeta(2)s} \qquad (s \downarrow 0).$$

Thus even this single absolute sum grows like $L/(\zeta(2)t)$ at $s = t/L$ for fixed $t > 0$. The critical scale requires cancellation to control the normalised two-site defect.

At the prime edge, Bombieri's *On twin almost primes* [4, Theorem 1], together with its corrigendum [5], gives the relative error $O(k^{1/3}2^{-k/3})$ for $\Lambda(n+2)\Lambda_k(n)$ when the scale is sufficiently large in terms of k. Evans recalls this estimate in [12, p. 304]. Corollary 3.5, with $j = 1$, replaces it by $O((1+k)^C 2^{-k})$ in the regime of fixed order followed by the limit in X. Passing between an initial interval and a dyad preserves this comparison of exponential bases. The containing square $[1/2, 1]^2$ already gives the base 2^{-k} for one growing order. The oblique constraint improves the diagonal rate to $e^{-14m/13}$ and the equal-order base to $36/169$. Each order precedes the limit in X, with no effective threshold $X_0(k)$ asserted. Friedlander and

Iwaniec [14] develop the asymptotic sieve for sequences including $a_n = \Lambda(n+b)$. Ramaré [34] studies related weights concentrated on integers with at most two prime factors and distribution levels close to one; those parameters differ from the order k at the unconditional level considered here.

The principle that one scalar parity parameter remains at full distribution level is already explicit in Bombieri's *The asymptotic sieve* [6, Main Theorem, p. 246]. See also Tao's exposition [35]. Theorem 4.2 realises that principle in this analytic two-site family, through a point-supported distribution and four explicit sign profiles. Ford's examples [13] show why a fixed level below one cannot in general replace the full-level hypotheses of the asymptotic sieve. The extremal supports realised by the pair $n, n+2$ remain unknown.

Correlations of short divisor sums were studied by Goldston and Yıldırım [17, Theorem 8.1 in the 2002 preprint] and, for affine systems, in Green and Tao's divisor-sum estimates [19, Appendix D]. Smoothing and the contribution of rough integers are central in Granville, Koukoulopoulos and Maynard [18]. The prime-product expansion used here first controls either divisor separately. The additional joint constraint in (3) uses the inverse-sum estimates in Irving's *Average bounds for Kloosterman sums over primes* [23, Lemmas 2.1, 2.3 and 3.4]. The boundary estimate for a single prime anchor uses the fixed-class theorem of Maynard [27, Corollary 1.2] and the polynomial-value sieve of Nair and Tenenbaum [29, 21]. The support-from-moments lemma used in Section 5 is a special case of Chen, Garza-Vargas, Tropp and van Handel [8, Lemma 4.9]. The observation result in that section uses transverse absolute averages to retain information that a single projection may cancel.

Two arithmetic applications. *Visible prime factors under neighbouring critical divisor weights* [33] compares the laws of a marked prime and its cofactor, with and without reweighting at $n+2$. For fixed first smoothing and a second parameter tending to infinity more slowly than L, the two cofactor laws approach one another in total variation. In *Reconstruction and sign prediction from sampled prime factors* (DOI `10.5281/zenodo.24443266`) [32], prime labels are sampled with logarithmic weights from an integer of harmonic law. The target there is that integer or its sign. Our prime-edge observations in (19) are dyadic sums with a von Mangoldt weight at $n+2$. Their target is the aggregate κ_X.

Section 6 keeps two prime factors as individual indices and proves uniform estimates over their admissible subsets, including a moving smoothing parameter. Section 7 instead uses the identity $9n(n+1)+2 = (3n+1)(3n+2)$ and positivity of Taylor coefficients on a rough almost-prime family. It transfers Maynard's eleven-factor input [26] to the sufficient orders twelve and seven, independently of the conditional corner formula. The sign product is the same as in *An explicit bound below one half for a quartic Liouville correlation* (DOI `10.5281/zenodo.24443184`) [31]. That paper bounds an unweighted harmonic average. Section 7 treats dyadic averages with divisor weights on a residue class.

2 The divisor distribution and its Fourier bound

Write $u_d = (\log d)/L$. Form the finite signed measure

$$T_X = \frac{L^2}{S}\mathbb{E}_X \sum_{d\mid n,\ e\mid n+2} \mu(d)\mu(e)\delta_{(u_d,u_e)}. \tag{7}$$

Its Laplace transform is J_X, and its support is contained in $[0, 1+O(1/L)]^2$. In Fourier transforms our convention is $\widehat{T}(v,w) = \mathcal{L}T(\mathrm{i}v, \mathrm{i}w)$.

Lemma 2.1 (Two frequency bounds)**.** *The distributions T_X are bounded in a common finite-order seminorm on a fixed compact neighbourhood of $[0,1]^2$. Every sequence going to*

infinity has a distributional limit along a subsequence. For any such limit T and real v, w,

$$|\widehat{T}(v,w)| \ll (1+|v|)^{\alpha}(1+|w|)^{\alpha}, \qquad \alpha = 4/\pi.$$

Proof. Set $g_v(p^j) = |1 - e^{-\mathrm{i}vu_p}|$ for each $j \geq 1$ and extend multiplicatively. This gives exactly $|H_{\mathrm{i}v/L}(m)| = g_v(m)$. Corollary 3 of Nair and Tenenbaum's *Short sums of certain arithmetic functions* [29], for the linear forms $n, n+2$, gives

$$\frac{L^2}{S}\mathbb{E}_X g_v(n) g_w(n+2) \ll \exp\left(\sum_{p \leq 3X} \frac{g_v(p) + g_w(p)}{p}\right). \tag{8}$$

Indeed there are two roots modulo odd primes, and the local density product is $O(L^{-2})$. The higher powers in the two Euler sums cost $O(1)$ in the exponent. The constants are uniform because $g_v(m) \leq 2^{\omega(m)} \ll_\epsilon m^\epsilon$, where ω counts distinct primes.

First use $g_v(p) \leq \min(2, |v|u_p)$. Chebyshev's bound $\sum_{p \leq y} \log p/p \ll \log(2y)$ below $\exp(L/(1+|v|))$ and Mertens' bound above give $\sum_{p \leq 3X} g_v(p)/p \leq 2\log(1+|v|) + O(1)$, uniformly, also when the cutoff is below 2. Hence $|\widehat{T}_X(v,w)| \ll (1+|v|)^2(1+|w|)^2$. Fourier inversion, integrating by parts four times in each large frequency, bounds $|\langle T_X, \phi\rangle|$ by a constant times the supremum of the derivatives $\partial_x^i \partial_y^j \phi$, $0 \leq i, j \leq 4$, on a fixed compact set. Here and below we insert a smooth compact cutoff equal to one near the supports. Extraction on a countable dense set and this seminorm give a limit. The family of exponential tests, and of each of its fixed derivatives, is compact in this seminorm when the parameters vary over a compact set. This gives local uniform convergence of the transforms.

We can now hold v fixed before sending X to infinity. Removing $u_p < \epsilon$ costs $O(|v|\epsilon)$, and Mertens' formula on the remaining range is a Riemann integral. Thus

$$\sum_{p \leq 3X} \frac{g_v(p)}{p} \longrightarrow \int_0^{|v|} \frac{2|\sin(r/2)|}{r}\,\mathrm{d}r = \alpha \log(1+|v|) + O(1).$$

For the last equality, the periodic numerator has mean $4/\pi$ and its centred primitive is bounded. The estimate for T follows from (8). The strict inequality $\alpha < 2$ will bound the order of a distribution concentrated at one point. □

Lemma 2.2 (Sieve estimate with a prescribed divisor)**.** *Fix $K \geq 1$ and put $f_K(p^j) = \min(1, Ku_p)$, extended multiplicatively. For k fixed primitive linear forms F_i of size $\asymp X$ on a dyad, whose roots are distinct outside finitely many primes, one has*

$$\mathbb{E}_X \prod_i f_K(F_i(n)) \ll_K L^{-k}.$$

If $p^a \mid F_i(n)$ is also prescribed, $a \in \{1,2\}$ and $p^a \leq X^{1/4}$, the bound gains a factor $O(f_K(p)/p^a)$. The constants may depend on the forms. The same estimates hold on an interval of length comparable to X.

Proof. On a root class modulo p^a the interval has length $Y \asymp X/p^a$ and contributes the factor $f_K(p)$. Omit p and the fixed exceptional primes from the sieve, including all primes at most $2k$. Retain each remaining prime $\ell \leq Y^\epsilon$ independently with probability $1 - f_K(\ell)$, for a sufficiently small fixed ϵ. The expected indicator of avoiding the retained roots majorises the desired product after the prescribed factor is removed.

For a fixed retained set use the upper-bound fundamental lemma in [15, Lemma 6.8], in the form reproduced in [25, Lemma 9.1]. Take $z = Y^\epsilon$, $D = Y^{1/3}$ and dimension parameter $k+1$. The local density is $g(\ell) = k/\ell$ on retained primes and zero on the others. Its dimension bound

$$\prod_{w_1 \leq \ell < z_1} (1 - g(\ell))^{-1} \leq C_k \left(\frac{\log z_1}{\log w_1}\right)^{k+1} \qquad (2 \leq w_1 < z_1)$$

holds with C_k independent of the retained set, since removing factors decreases the product. Choose ϵ so that $\log D/\log z \geq 9(k+1)+1$. The upper sieve coefficients have absolute value at most one and are supported on squarefree $d \leq D$. Counting the root classes therefore bounds the sifted count by a constant times $Y\prod_{\ell<z}(1-g(\ell))$, with error

$$O\left(\sum_{d\leq D} k^{\omega(d)}\right) = O_k(Y^{1/3}(\log Y)^{O_k(1)}) = O_k(Y^{1/2}).$$

The main-term constant is common to all retained sets. Averaging the density product gives

$$\prod_{\ell\leq Y^\epsilon}^{*}\left(1-\frac{k(1-f_K(\ell))}{\ell}\right) \ll_K L^{-k},$$

since $\sum_{\ell\leq X} f_K(\ell)/\ell \ll_K 1$. Removing one varying prime changes the bound by a bounded factor after the small exceptional primes have been omitted. A primitive linear form has at most one root class modulo p^a, including at the exceptional primes. The argument with $Y \asymp X$ and no prescribed class gives the first estimate. □

We will repeatedly use, for $|z| \leq R$ and $m \leq C_0X^2$,

$$|H_{z/L}(m)| + L^{-1}|W_{z/L}(m)| \ll_{R,C_0} f_K(m) \tag{9}$$

with K large enough in terms of R, C_0. For primes with $Ku_p \leq 1$ use $|1-e^{-zu_p}| \leq |z|u_p e^{|z|u_p}$ and $\sum_{p|m} u_p \leq 2 + O(1/L)$. There are only $O_K(1)$ primes with $Ku_p > 1$. Cauchy's estimate on a larger disc gives the assertion for the derivative.

3 Counting a small divisor at the other site

Two scalar computations determine the main term in (7). The first is the average mass of the unrestricted integer. The second records the exclusion of the residue zero for a prime factor at the other site.

Lemma 3.1. *For fixed real $z > 0$, $L\mathbb{E}_X H_{z/L}(n) \to z$. Put $c_d = \mu(d)\mathbf{1}_{2\nmid d}/\varphi(d)$. For every $\phi \in C_c^\infty(\mathbb{R})$,*

$$L\sum_d c_d\phi(u_d) \longrightarrow -S\phi'(0). \tag{10}$$

Proof. In the first divisor expansion split at $D = X/L^4$. For $d \leq D$, replacing the count of multiples by X/d costs $O(D/X)$ in the mean. For $d > D$, write $n = md$, so $m \leq 2L^4$. The prime number theorem with remainder for the Möbius function and partial summation give an error $O_A((X/m)L^{-A})$ in the inner sum of $\mu(d)d^{-z/L}$ for each m. The tail of $\sum_d \mu(d)d^{-1-z/L}$ from D is $O_A(L^{-A})$ as well, choosing the prime number theorem remainder with a larger logarithmic power. The mean is $1/\zeta(1+z/L) + o(1/L)$. Its translate by two has the same limit.

For the second assertion the Dirichlet series factors as

$$P(s) = \sum_d c_d d^{-s} = \prod_{p>2}\left(1-\frac{p^{-s}}{p-1}\right) = C(s)/\zeta(1+s), \qquad C(0) = S.$$

The p-factor of C for $p > 2$ is $1 - p^{-1-s}/((p-1)(1-p^{-1-s}))$. The coefficients of C are absolutely summable with weight $n^{1/4}$, including its factor at two. Convolution with the coefficients $\mu(n)/n$ of $1/\zeta(1+s)$ yields $\sum_{d\leq y} c_d \ll_A (\log(2y))^{-A}$. Consequently $\sum c_d = 0$ and $\sum c_d \log d = -P'(0) = -S$, with ordinary partial sums. Truncate at $D_0 = \exp(L^{1/4})$ using partial summation, expand $\phi(u_d)$ to first order there, and use $\sum_{d\leq D_0} 1/\varphi(d) \ll \log(2D_0)$. After multiplication by L the quadratic remainder is $O_\phi((\log D_0)^3/L)$. This proves (10). □

The distribution hypothesis. For a finitely supported sequence a and $(b,q)=1$, write

$$\Delta(a;b(q)) = \sum_{n\equiv b\ (q)} a(n) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} a(n).$$

We use GEH[θ] in the formulation of [30, Claim 2.6]. Fix $\epsilon>0$ and let $X^\epsilon \le N, M \le X^{1-\epsilon}$, $NM \asymp X$. Suppose α,β are supported on $[N,2N]$, $[M,2M]$ and bounded in modulus by $\tau(\cdot)^B L^B$ for a fixed B. Require, for every $A>0$, the Siegel–Walfisz estimate

$$|\Delta(\beta \mathbf{1}_{(\cdot,r)=1}; b(q))| \ll_A \tau(qr)^{C_A} M L^{-A} \qquad (q,r\ge 1,\ (b,q)=1).$$

Then GEH[θ] asserts, for every $A>0$,

$$\sum_{q\le X^\theta} \max_{(b,q)=1} |\Delta(\alpha*\beta; b(q))| \ll_A X L^{-A},$$

uniformly over sequences with these fixed bounds. Constants may depend on ϵ,θ,B and the constants in the Siegel–Walfisz estimates. Motohashi's convolution theorem [28], as formulated in [30, Theorem 2.8], gives this conclusion for each $\theta<1/2$. The implication to Elliott–Halberstam for primes is [30, Proposition 2.7]. In the conditional assertions we assume these estimates at each fixed $\theta<\Theta$; no estimate at the endpoint $\theta=\Theta$ is required.

3.1 The two separate divisor bounds

Proof of the coordinate bounds in Theorem 1.1. Let T be a limit from Lemma 2.1. We prove that for $z>0$ and smooth compactly supported ϕ with $\sup\operatorname{supp}\phi<\Theta$,

$$\langle T, e^{-zx}\phi(y)\rangle = -z\phi'(0). \tag{11}$$

Fix $0<\eta<1/8$ for the moment and retain only n all of whose prime factors exceed X^η. For the test $e^{-zx-\mathrm{i}vy}$ at fixed v, (9) and Lemma 2.2 bound the normalised cost of discarding the other n by

$$O_{z,v}\left(\sum_{p\le X^\eta} \frac{f_K(p)}{p}\right) = O_{z,v}(\eta)$$

in the limit, with K fixed. To invert against ϕ, truncate its Fourier integral first. The absolute estimate (8), with f_K at the first site, bounds the tail by a fixed polynomial in v against $|\widehat{\phi}(v)|$, even with the restriction on n. Dominated convergence, first in X as a limsup and then as $\eta\downarrow 0$, removes the restriction with loss $o_{\eta\downarrow 0}(1)$. The one-site mass $L\mathbb{E}_X H_{z/L}$ has the same removal bound. Rough n divisible by a prime square have density $O(X^{-\eta}+X^{-1/2})$. Even a bound $O(L^2\tau(n+2)\|\phi\|_\infty)$ for the expanded summand makes their contribution $o_\eta(1)$ as $X\to\infty$.

For the retained squarefree n the anchor $a(n)=LH_{z/L}(n)$ is a sum over ordered prime products of length $k\le 2/\eta$ with weight

$$\frac{L}{k!}\prod_{j=1}^{k}(1-p_j^{-z/L}), \qquad p_j>X^\eta.$$

We may include products with repeated primes at the preceding cost, since they have bounded multiplicity. Choose $\theta<\Theta$ strictly greater than all nonnegative y in $\operatorname{supp}\phi$ (if there are none the test is zero). For odd $d\le X^\theta$ the test $d\mid n+2$ asks for the unit class $-2 \pmod d$. For each of the finitely many lengths we use

$$\sum_{\substack{d\le X^\theta\\ 2\nmid d}} \left| \sum_{n\equiv -2\ (d)} a_k(n) - \frac{1}{\varphi(d)} \sum_{(n,d)=1} a_k(n)\right| \ll_{A,\eta,z,\theta} X L^{-A} \tag{12}$$

for arbitrarily large fixed A. Here a_k includes the dyadic interval and the product weight above.

For $k = 1$, this is Bombieri–Vinogradov for primes, with partial summation. At higher conjectural levels use Elliott–Halberstam for primes, which follows from its generalised version. For $k \geq 2$ put one prime on the Siegel–Walfisz side of the convolution, and group the other $k - 1$. We give the details because the level in (12) sets the lower bound for each separate divisor coordinate. Partition each prime interval into multiplicative boxes of relative width at most L^{-B}, starting at X^η. Discard boxes straddling a dyadic product endpoint. Their products lie in two intervals of total length $O_k(XL^{-B})$, and each integer supports $O_k(1)$ ordered representations. The sum over moduli of the absolute boundary cost, both in the residue class and in the unit average, is $O_k(L^{O(1)}(XL^{-B} + X^\theta))$, by counting integers in those intervals. Take B large enough. In each remaining box the two factors are supported in dyadic ranges whose sizes exceed fixed powers of X and whose product is comparable to X. Both sequences are divisor-bounded after detaching the outer L. The single-prime sequence satisfies the necessary Siegel–Walfisz estimate on the box by partial summation. Auxiliary coprimality with r removes at most $\omega(r)$ prime terms, absorbed by the allowed divisor factor. Large Siegel–Walfisz moduli are covered by the trivial interval bound. There are only $L^{O_{B,k}(1)}$ boxes, so one chooses the saving in the convolution theorem after B and k. The preceding formulation of Motohashi's theorem applies at every level strictly below $1/2$, and the corresponding GEH hypothesis gives the higher levels. This prime-product decomposition is also used in [30, §4.5]. Throughout this step η is fixed; it is sent to zero only after the progression estimate and the normalised small-prime bound have been combined.

For each rough n one can replace the mass on units in (12) by the entire mass inside the divisor sum. Every prime $p \mid n$ exceeds X^η and

$$\sum_{\substack{d \leq X^\theta \\ p \mid d,\ \mu^2(d)=1}} \frac{1}{\varphi(d)} \leq \frac{1}{p-1} \sum_{e \leq X^\theta} \frac{1}{\varphi(e)} \ll L/(p-1).$$

The normalised error is $O_\eta(X^{-\eta}L^{O(1)})$. Even d give zero actual mass since the rough anchor is odd. Hence the normalised divisor test is the truncated one-site mean of a, times $S^{-1}L \sum c_d \phi(u_d)$, up to the errors just bounded. Lemma 3.1 and removal of η give (11).

Set $D = T - \delta_0' \otimes \delta_0'$. For each such ϕ, the contracted distribution in x has zero Laplace transform on $z > 0$ by (11). Analytic continuation and Fourier uniqueness make it zero. Density of tensor-product tests gives $D = 0$ on $y < \Theta$. Anchoring at $n + 2$ uses the unit class 2 and the same proof, now giving $D = 0$ on $x < \Theta$. We have proved $\operatorname{supp} D \subset [\Theta, 1]^2$ under the stated distribution hypothesis, including $\Theta = 1/2$ unconditionally. □

3.2 A joint restriction from the nonzero frequencies

The two applications of distribution in progressions leave a square. To remove its lower corner we use both Möbius coefficients in (7). Write $\mathrm{e}(s) = e^{2\pi i s}$ and $n \sim N$ for $N < n \leq 2N$. Implicit constants in this subsection can depend on fixed smooth tests. We allow factors X^ϵ, for every fixed $\epsilon > 0$, in bounds for sums before normalisation.

Lemma 3.2 (Two estimates for the same reciprocal sum)**.** *Let $1 \leq K, T, M, Q \leq X^{O(1)}$, $KT = o(Q)$, $K \ll \sqrt{T}$ and $Q \ll TM^2$. For $|\theta| \in \{1, 2\}$ and three sequences bounded in modulus by one, put*

$$\mathcal{W} = \sum_{q \sim Q} \left| \sum_{\substack{\ell \sim T,\, m \sim M,\, k \sim K \\ (\ell m, q) = 1}} a_\ell b_m c_k \, \mathrm{e}(\theta k \overline{\ell m}/q) \right|.$$

The bar denotes the inverse modulo the denominator. For every $\epsilon > 0$,

$$\mathcal{W} \ll_\epsilon X^\epsilon Q(KT)^{3/4}(Q^{1/4}M^{1/2} + M), \tag{13}$$

$$\mathcal{W} \ll_\epsilon X^\epsilon\{QT(KM)^{1/2} + Q^{1/2}KT^{5/4}M^{3/2}\}. \tag{14}$$

The estimates include composite moduli and numerators sharing a factor with q.

Proof. For (13), fix q and group the pairs (k,ℓ) by $r = (k,\ell)$. At fixed r the reduced fractions $(k/r)/(\ell/r)$ have distinct residues modulo q. Indeed, the nonzero difference of cross-products has absolute value $O(KT) = o(q)$. Multiplication by θ has fibres of size at most two. There are $O(KT/r^2)$ pairs in the group. With $B_q(a) = \sum_{(m,q)=1,\, m\sim M} b_m \mathrm{e}(a\bar{m}/q)$, Hölder and $\sum r^{-3/2} < \infty$ bound the inner sum by

$$\ll (KT)^{3/4}\left(\sum_{a \bmod q} |B_q(a)|^4\right)^{1/4}.$$

Orthogonality bounds the fourth moment by $qJ(q)$, where $J(q)$ counts unit quadruples from the m-interval with $\bar{m}_1 + \bar{m}_2 \equiv \bar{m}_3 + \bar{m}_4 \bmod q$. Irving [23, Lemma 2.3] gives $\sum_{q\sim Q} J(q) \ll_\epsilon X^\epsilon(QM^2 + M^4)$. To see the two terms, an equality of the rational reciprocals has $O_\epsilon(M^{2+\epsilon})$ solutions. For a given reduced sum a/b, each representation $1/m + 1/m' = a/b$ satisfies $(am - b)(am' - b) = b^2$. Every other quadruple contributes only at divisors of a nonzero integer of size $O(M^3)$. Hölder in q now gives (13).

For (14) take absolute values first in ℓ, then use Cauchy–Schwarz in q. In the expanded square the pairs with $k_1m_2 = k_2m_1$ contribute $O_\epsilon(X^\epsilon QKM)$ for each ℓ, by the divisor bound. Otherwise put $b = \theta(k_1m_2 - k_2m_1)$ and $c = \ell m_1m_2$. On $(q,c) = 1$ the phase is

$$\mathrm{e}(b\bar{c}/q) = \mathrm{e}(-b\bar{q}/c)\,\mathrm{e}(b/(cq)).$$

The last factor costs $O(1 + |b|/(cQ)) = O(1)$ by partial summation. Irving's incomplete reciprocal-sum bound [23, Lemma 2.1] bounds each partial interval by $O_\epsilon(X^\epsilon((b,c)(1+Q/c) + \sqrt{c}))$. Here $Q \ll c$ and $(b,c) \le |b| \ll KM \ll \sqrt{T}M$. The off-diagonal contribution under the square root is therefore $O_\epsilon(X^\epsilon K^2\sqrt{T}M^3)$. Taking the root, multiplying by $\sqrt{Q}$ and summing over ℓ gives (14). □

Proposition 3.3. *Every limit of* (7) *vanishes in a neighbourhood of each point of* $[1/2,1]^2$ *with* $7\min(x,y) + 6\max(x,y) < 7$.

Proof. Test by $\phi(\log d/L)\psi(\log e/L)$ in a small rectangle about the point. Both tests are smooth and supported away from zero. Replace the indicator of the n-dyad by $w(n/X)$, smoothing in intervals of relative length L^{-50} with $w^{(i)} \ll_i L^{50i}$. Since the absolute divisor sum per integer is $\ll \tau(n)\tau(n+2)$, Cauchy–Schwarz and $\sum_{n\le 3X}\tau(n)^4 \ll XL^{15}$ make the error $o(X/L^2)$ in the raw sum. This last divisor estimate also follows from $\tau^4 \le \tau_{16}$.

The progression and its zero frequency. Let $g = (d,e)$. The congruences $d \mid n$, $e \mid n+2$ are compatible only for $g \mid 2$. Write $d = ga$, $e = gq$ with $(a,q) = 1$. For $n = ga\ell$ the congruence is $\ell \equiv -(2/g)\bar{a} \bmod q$. Poisson summation in ℓ gives exactly

$$\frac{X}{gaq}\sum_{k\in\mathbb{Z}} \widehat{w}\Big(\frac{kX}{gaq}\Big)\,\mathrm{e}\Big(-\frac{(2/g)k\bar{a}}{q}\Big), \qquad \widehat{w}(\xi) = \int w(s)\mathrm{e}(-\xi s)\,\mathrm{d}s.$$

One may start with $n+2$ instead, translating w and reversing the shift. This lets a be the shorter variable on each pair of dyadic intervals. For $g = 2$ both new variables must be odd. The Möbius factors on them are otherwise separate.

Over the full test, the zero frequency divided by $X\widehat{w}(0)$ has coefficients $b(d,e) = \mu(d)\mu(e)\mathbf{1}_{(d,e)|2}/[d,e]$. Its Dirichlet series is

$$\sum_{d_1,d_2} \frac{b(d_1,d_2)}{d_1^z d_2^{w'}} = \frac{C(z,w')}{\zeta(1+z)\zeta(1+w')} \qquad (\Re z, \Re w' > 0).$$

With $U_p = p^{-1-z}, V_p = p^{-1-w'}$, the local numerator before extracting the two reciprocal zeta factors is $1-U_p-V_p$ at odd p and $1-U_2-V_2+2U_2V_2$ at 2. Thus $C_p = 1-U_pV_p/((1-U_p)(1-V_p))$ at odd primes. Its double series, including the single factor at 2, has coefficients $c(i,j)$ satisfying $\sum |c(i,j)|(ij)^{1/4} < \infty$. After convolution the test factors into

$$\sum_{i,j} c(i,j) \left(\sum_r \frac{\mu(r)}{r} \phi(\log(ir)/L) \right) \left(\sum_s \frac{\mu(s)}{s} \psi(\log(js)/L) \right).$$

If both tests have support in $[\sigma, C_0]$, $\sigma > 0$, the part with $\max(i,j) > X^{\sigma/2}$ costs $O(X^{-\sigma/8}L^2)$. Otherwise the inner variables on the tests exceed $X^{\sigma/2}$. Partial summation with the prime number theorem for μ bounds each parenthesis by $O_A(L^{-A})$ for every fixed A. This disposes of the entire zero frequency, including its exceptional prime.

Splitting the shorter divisor. Write $a \asymp Z$, $q \asymp Q$, with $Z \le Q$ up to a fixed factor, and let $u = \log Z/L$, $v = \log Q/L$. The nonzero frequencies are dyadically split at $K \le X^{\epsilon}\max(1, ZQ/X)$, at total cost $O(X^{-10})$ beyond that cutoff by integration by parts. On a frequency block the factor $\widehat{w}(kX/(gaq))$ separates into twists in $|k|, a, q$ by Fourier inversion in the logarithms. In detail, multiply its function of $\log(|k|/K) - \log(a/Z) - \log(q/Q)$ by a fixed smooth cutoff on a bounded interval. The integral of the modulus of its Fourier transform is $L^{O(1)}$ because $\sup_\xi |(\xi\, d/d\xi)^j \widehat{w}(\xi)| \ll L^{O(1)}$ for $j \le 2$. Higher fixed derivatives obey the same type of bound. The prefactor has size $X/(ZQ)$, and $Z/a, Q/q$ are separate weights. All estimates below are before the outer factor $L^2/(SX)$.

Put $U = X^{1/8}$, $A = \mu\mathbf{1}_{[1,U]}$, $B = \mu - A$. The exact convolution identity

$$\mu = 2A - \mathbf{1} * A * A + \mathbf{1} * B * B$$

follows by expanding $B * B$ and using $\mathbf{1} * \mu = \delta_1$. The term $2A$ vanishes here. In $\mathbf{1} * A * A$, group the two short variables into r. If $r \le U$, the complementary variable is free. For each k, summing the incomplete reciprocal estimate in the free variable over q gives

$$\ll_\epsilon X^\epsilon(Z + Q^{3/2}U). \tag{15}$$

This is Irving [23, Lemmas 3.3–3.4], including a fixed nonunit numerator θk. Explicitly the bound per modulus and r is $X^\epsilon((\theta k, q)(1 + Z/(rq)) + \sqrt{q})$, and $\sum_{q\sim Q}(\theta k, q) \ll_\epsilon QX^\epsilon$. For odd q an odd free variable is imposed by subtracting the even variables, using the inverse of 2 modulo q, which preserves the gcd bound. Product cutoffs restrict an interval of summation. The smooth test and the Poisson factor at fixed q, k, r have polylogarithmic total variation there, so partial summation also gives (15) for the original weights.

For $r > U$ the two grouped factors both exceed a constant multiple of U, since $r \le U^2$ and $Z \gg U^3$. In $\mathbf{1} * B * B$ group $\mathbf{1}$ with one B. After splitting the two factors into dyads, their lengths $T \ge M$ satisfy $TM \asymp Z$ and $\sqrt{Z} \ll T \ll Z/U$. Each coefficient costs at most X^ϵ by a divisor bound. The twists of the product ℓm factor into twists of ℓ and m. To separate a sharp product endpoint on such boxes one can place it at a half-integer and use truncated Perron inversion at height X^{10} on the line $\Re s = 1/L$. Its integral norm is $O(L)$, and the truncation error is negligible since the product is integral and $O(X)$. Partial summation treats a smooth product cutoff by the same bound. Coprimality to g is imposed on both factors separately. After taking absolute values in the outer q sum we can enlarge the set of moduli. Lemma 3.2 now applies.

The common saving. At a point of the square in question, $u \le 7/13$, $v < 7/12$, and $s = u + v < 13/12$. For $s \ge 1$ put $h = s - 1$ and $\tau = \log T/L$. Up to $O(\epsilon) + O(1/L)$ in the exponents, $K \le X^h$, the prefactor is X^{-h}, and $u/2 \le \tau \le u - 1/8$. The hypotheses on the lengths hold with strict margins, since

$$h + \tau - v \le 2u - 9/8 < 0, \qquad h < 1/12 < u/4, \qquad \log(TM^2)/L \ge u + 1/8 + O(1/L) > v.$$

The following exponents include the Poisson prefactor X^{1-s}. In the Type I row they also include the sum over the frequencies. Since $M \ll \sqrt{Q}$, the first term in the parentheses in (13) dominates the second.

Contribution	Exponent, up to $O(\epsilon) + O(1/L)$
Fourth moment (13)	$A = v + u/4 + \tau/4 + 1/4$
Reciprocity (14), off diagonal	$B = v/2 + 3u/2 - \tau/4$
Reciprocity (14), diagonal	$(v + \tau + 1)/2 \le s/2 + 7/16$
Type I (15)	$\max(u, 3v/2 + 1/8)$

For example the off-diagonal exponent before substituting $h = s - 1$ is $-h + v/2 + h + 5\tau/4 + 3(u - \tau)/2$. The diagonal and Type I rows are strictly below one in the open region under consideration. At the boundary point $(1/2, 7/12)$ the Type I exponent reaches one. Both bilinear estimates apply to the same block. Its exponent is therefore bounded by $\max\{\min(A, B), (v + \tau + 1)/2\}$. The dependence on τ drops out of

$$\min(A, B) \le (A + B)/2 = (7u + 6v + 1)/8 < 1.$$

These estimates are continuous at $s = 1$ if the frequency exponent is replaced by $\max(0, s - 1)$ and the prefactor by X^{1-s}. Thus even the point $(1/2, 1/2)$ has an open neighbourhood with a strict saving. At the diagonal choose the orientation separately for each box. On a sufficiently small fixed test rectangle, choosing ϵ below the margins and summing the polylogarithmically many blocks gives $o(X/L^2)$. Density of finite sums of smooth tensor tests completes the localisation. □

For comparison, apply Theorem 1 of Bettin and Chandee [2] directly to the unsplit shorter divisor in the nonzero Poisson sum. On a diagonal box with $Z, Q \asymp X^\xi$, $\xi \ge 1/2$, the largest frequency scale is $X^{2\xi - 1 + O(\epsilon)}$. In their notation the lengths are $M, N \asymp X^\xi$ and $A \asymp X^{2\xi - 1}$ at the level of exponents. The product of the three coefficient norms costs $(AMN)^{1/2}$ and the Poisson factor is $X^{1-2\xi}$. Their two terms then have exponents

$$1 - 2\xi + \tfrac{17}{20}(4\xi - 1) + \tfrac{\xi}{4} = \tfrac{3 + 33\xi}{20}, \qquad 1 - 2\xi + \tfrac{7}{8}(4\xi - 1) + \tfrac{3\xi - 1}{8} = \tfrac{15\xi}{8}.$$

Smaller frequency blocks satisfy the same upper bound. This direct application yields a power saving for $\xi < 17/33$. Splitting the divisor before applying the two reciprocal estimates gives the larger range $\xi < 7/13$.

3.3 Directional decay and ordinary divisor weights

Completion of the proof of Theorem 1.1. Proposition 3.3 gives $\operatorname{supp} D \subset \mathcal{P} \cap [\Theta, 1]^2$. Write $a = \min(t, u)$ and $b = \max(t, u)$. To minimise $tx + uy$ on $\mathcal{P}$, pair the larger coefficient with the smaller coordinate ξ. The lower boundary in that half of the polygon is the segment from $(\xi, \eta) = (1/2, 7/12)$ to $(7/13, 7/13)$. The minimum of $b\xi + a\eta$ is (4). In particular $x + y \ge 14/13$. The conditional minimum can be read from the very same segment. For $1/2 \le \Theta \le 7/13$ the left endpoint moves to $(\Theta, 7(1 - \Theta)/6)$, giving

$$d_\Theta(t, u) = \min\{7(a + b)/13,\ b\Theta + 7a(1 - \Theta)/6\}.$$

For $7/13 \le \Theta \le 1$ the entire smaller square lies in $\mathcal{P}$, so $d_\Theta(t,u) = \Theta(t+u)$.

Mixed differentiation gives $F = \mathcal{L}(xyD)$. Enlarge the support by a distance $O((1+t+u)^{-1})$ and insert a smooth cutoff equal to one on a smaller neighbourhood, obtained for instance by convolving the indicator of the enlargement with a bump. The exponential changes by a bounded factor and every fixed cutoff derivative costs a power of $1+t+u$. The common seminorm of Lemma 2.1 now gives (5), also for fixed derivatives. Finally, $G = \mathcal{L}D$ vanishes on both axes. Integrating G_{tu} in u gives $G_t(t,\infty) - G_t(t,0) = 0$. Interchanging t, u gives the second compensation identity. □

Corollary 3.4. *For every $\epsilon > 0$ there is a finite grid $\mathcal{U}_\epsilon \subset (0,\infty)$ and a $c_\epsilon > 0$, common to all limits, such that $F(0,0) \le -\epsilon$ implies $F(t,0) \ge c_\epsilon$ for some $t \in \mathcal{U}_\epsilon$.*

Proof. The uniform derivative bound keeps $F(t,0) \le -\epsilon/2$ on a common initial interval. Truncate the integral in (6) where the uniform tail is smaller than half the missing mass. Some positive value on the remaining interval then exceeds a fixed constant. The same derivative bound transfers this to a sufficiently fine grid. □

Corollary 3.5 (The three ranges of orders)**.** *For fixed $j, k \ge 1$ let $J = \max(j,k)$, $K = \min(j,k)$ and put*

$$H_{J,K} = \begin{cases} (6/13)^{J+K}, & J/K \le 7/6, \\ \left(\dfrac{6J}{7(J+K)}\right)^J \left(\dfrac{K}{J+K}\right)^K, & 7/6 \le J/K \le 7/5, \\ (1/2)^J (5/12)^K, & J/K \ge 7/5. \end{cases}$$

For $\Lambda_j = \mu * (\log^j)$,

$$\limsup_X \left| \frac{\mathbb{E}_X \Lambda_j(n)\Lambda_k(n+2)}{SjkL^{j+k-2}} - 1 \right| \ll (1+J+K)^C H_{J,K}.$$

The orders are chosen before $X \to \infty$. The constants are independent of them. At any fixed level $1/2 < \Theta < 1$ as in Theorem 1.1, one can use the exact maximum on the smaller containing region. For $\Theta \le 7/13$ it is $p_\Theta^J (1 - 7p_\Theta/6)^K$, where

$$p_\Theta = \min\{1-\Theta, \max(6/13,\, 6J/(7(J+K)))\}.$$

For $7/13 \le \Theta < 1$ it is $(1-\Theta)^{J+K}$. The implied constant may depend on Θ.

Proof. By (1), the jth derivative at zero of $m^{z/L} H_{z/L}(m)$ is $L^{-j}\Lambda_j(m)$. On the dyad the prefactor equals $e^z + O_R(1/L)$ for $|z| \le R$. Lemma 2.2 and (9) give $o(1)$ after normalising the two-site primitive, also for each fixed derivative by Cauchy's formula. In the limit $e^{z+w} zw$ contributes jk. The remaining relative term is $\langle D, (1-x)^j (1-y)^k \rangle / (jk)$. Let $p = \max(1-x, 1-y)$ and $q = \min(1-x, 1-y)$. The maximum of the absolute polynomial on $\mathcal{P}$ is the maximum of $p^J q^K$ subject to $p \le 1/2$, $0 \le q \le p$ and $7p + 6q \le 6$. Along the efficient boundary $6/13 \le p \le 1/2$ and $q = 1 - 7p/6$. The stationary point has $p = 6J/(7(J+K))$. Clipping it to this segment proves the three expressions, including continuity at the thresholds.

For completeness, set $\eta = (1+J+K)^{-1}$. At distance at most η from $\mathcal{P}$ the two absolute factors are bounded by $p_0 + \eta, q_0 + \eta$ for a reflected feasible point (using the original coordinate order). Either coordinate smaller than $1/3$ can first be raised to $1/3$ while preserving feasibility, since $7/2 + 6/3 < 6$. Call the resulting coordinates p_1, q_1. Inflation by η now costs at most $(1+3\eta)^{J+K}$, and deleting any fixed number of powers costs at most a fixed power of 3. A cutoff on that scale and the product rule in the common seminorm thus cost only $(1+J+K)^{O(1)} H_{J,K}$. For $\Theta \le 7/13$ the same argument clips the boundary to $p \le 1 - \Theta$, and raising a small coordinate to $1/3$ still respects the extra constraint. For $7/13 \le \Theta < 1$ use cutoff width $(1-\Theta)\eta$ on the smaller square. The point case $\Theta = 1$ is evaluated in Theorem 4.2. □

4 Reflection and the four signs

Write $\lambda(m) = (-1)^{\Omega(m)}$, with multiplicities counted in Ω. For $i, j \in \{0, 1\}$ insert $\lambda(n)^i \lambda(n+2)^j$ in (2), and denote the resulting primitive and mixed derivative by J_X^{ij}, M_X^{ij}. Thus $J_X^{00} = J_X$.

Lemma 4.1. *Along every extraction of Theorem 1.1, with unsigned limit J,*

$$J^{ij}(z, w) = e^{-iz-jw} J((-1)^i z, (-1)^j w).$$

Here i, j in the exponential are the integer sign indices. The convergence, including all fixed derivatives, is locally uniform.

Proof. For squarefree m, $\lambda(m) H_s(m) = m^{-s} H_{-s}(m)$, by factoring $-p^{-s}$ out of each Euler factor. To discard integers with a square at either site use (9) and Lemma 2.2. Squares p^2 for $p \le L^8$ cost $O(\sum f_K(p)/p^2) = O(1/L)$ in the normalised primitive. Larger squares have density $O(L^{-8} + X^{-1/2})$, and $|H| \ll_R 1$. For the remaining integers $m = n, n+2$, replacing $m^{-z/L}$ by e^{-z} costs $O_R(1/L)$ after the same majorisation. Take the limit and apply Cauchy's formula. On distributions the first-coordinate operation is pushforward by $x \mapsto 1 - x$. □

At full distribution level, Bombieri's asymptotic sieve leaves one scalar measuring parity [6, Main Theorem, p. 246]; Tao [35] explains the same principle for the prime and convolution anchors. Here the support theorem and the bound $4/\pi < 2$ give its explicit two-variable analytic form.

Theorem 4.2 (One coefficient at the corner)**.** *If all extracted defects D are supported at $(1, 1)$, put $m_0(t) = 1$, $m_1(t) = (t-1)e^{-t}$. Then, simultaneously for the four pairs of signs and locally uniformly on $\mathbb{C}^2$,*

$$M_X^{ij}(t, u) = m_i(t) m_j(u) + \kappa_X m_{1-i}(t) m_{1-j}(u) + o(1). \tag{16}$$

In particular the conclusion holds under generalised Elliott–Halberstam at every fixed level below one.

Proof. A distribution supported at one point is a finite sum of derivatives of the point mass [22, §2.3]. Thus $J = zw + e^{-z-w} P(z, w)$ for a polynomial P. Lemma 2.1 bounds $P(\mathrm{i}v, \mathrm{i}w)$ by $O((1 + |v|)^\alpha (1 + |w|)^\alpha)$. Fixing one frequency in turn gives degree at most one in each variable. The primitive vanishes on the axes, so $P = \kappa zw$. Its mixed derivative at zero identifies $\kappa = \lim \kappa_X$ on this subsequence. Reflection and differentiation give (16). If the error on a compact set failed to tend to zero, a further extraction would give the just proved limit there, a contradiction. □

In particular $M_X(1, 1) \to 1$ and $M_X^{11}(1, 1) = \kappa_X + o(1)$ under the hypothesis. The weights are nonnegative for real positive parameters, so $|M_X^{11}(1, 1)| \le M_X(1, 1)$. It follows that every limiting κ in this case belongs to $[-1, 1]$.

The corner also has a simple interpretation for ordinary divisor weights. Whenever the hypothesis of Theorem 4.2 holds, for all fixed $j, k \ge 1$ we have

$$\frac{\mathbb{E}_X \Lambda_j(n) \Lambda_k(n+2)}{S j k L^{j+k-2}} = 1 + \mathbf{1}_{j=k=1} \kappa_X + o(1).$$

Indeed $e^{z+w} J(z, w) = zw e^{z+w} + \kappa zw$ on each extraction, and the coefficient calculation in Corollary 3.5 applies. A single order greater than one annihilates the residual polynomial. When one order is one, Bombieri obtains the corresponding higher-order asymptotic from the full-level prime-distribution hypothesis in *On twin almost primes* [4, 5].

Two conditional comparisons locate this scalar more precisely. Wang's preprint [36, Theorem 4.5 and Corollary 4.7] describes, under the relevant Elliott–Halberstam hypotheses,

rough pairs with prescribed numbers of prime factors by a single parity parameter. Its class counts and the four analytic functions in (16) use different observations. In Theorem 4.2 the full-level support and the Fourier growth bound give the analytic formula for this parity principle.

Debouzy [9, Theorem 0.2] studies a prime at $n+2$ and a prime or a factorisation at n, localising one factor to $[n^\beta, n^\gamma]$ under a fixed-residue distribution hypothesis. There is a normalisation issue when comparing the printed statement with an unrestricted smoothing. At $(\beta,\gamma)=(0,1)$, the identity $\Lambda_2 = \Lambda\log + \Lambda * \Lambda$ makes its left side $\sum_{2\le n\le X}\Lambda_2(n)\Lambda(n+2)/\log n$. Under Bombieri's full-level hypothesis, his Theorem 2 with $k=2$ and partial summation give $2SX + o(X)$ in our notation. The right side printed in Debouzy's v1 is $SX(1+o(1))$. The comparison here concerns the localisation of the factors. Our constants are obtained from (2).

Corollary 4.3 (Identification by opposite signs)**.** *Suppose that two extracted limits give the same weighted mass on opposite Liouville signs on a nonempty open rectangle of positive real parameters. Then their four limiting sign profiles agree everywhere. The conclusion also holds if the observed mass is the total on equal signs.*

Proof. The opposite-sign mass is $(M^{00} - M^{11})/2$. Equality on the rectangle extends to all parameters by analyticity, first in one variable and then in the other. Integration from the axes, where all primitives vanish, gives equality of the corresponding primitive differences. Write ΔD for the difference of unsigned defects and $R(x,y) = (1-x, 1-y)$. Lemma 4.1 and Fourier uniqueness give $\Delta D = R_*\Delta D$. Its support lies both in $\mathcal{P}$, where $x+y \ge 14/13$, and in $R(\mathcal{P})$, where $x+y \le 12/13$. Thus $\Delta D = 0$. For equal signs the same argument gives $\Delta D = -R_*\Delta D$ and the same disjoint supports. Reflection then recovers the four character averages, hence the four sign masses by inversion. □

This is uniqueness for the analytic family of weighted observations. A value at one parameter pair supplies no open rectangle.

5 What a prime edge or a thin strip measures

A distribution can cancel completely under projection onto a line. We first give the one-dimensional support test that applies after projection, then show how transverse smoothing recovers information lost on the diagonal. The next lemma is the positive-support case of Chen, Garza-Vargas, Tropp and van Handel [8, Lemma 4.9]. Their proof uses the distributional Cauchy transform of Estrada and Kanwal [11]. We include the argument to specify the form needed here.

Lemma 5.1 (Support from moments)**.** *Let Q be a compactly supported distribution on $(0,\infty)$. If $\limsup_{m\to\infty} |\langle Q, s^m\rangle|^{1/m} \le R$, then $\operatorname{supp} Q \subset (0,R]$. In particular $R=0$ gives $Q=0$.*

Proof. For large $|z|$, the Cauchy transform $\langle Q, (z-s)^{-1}\rangle$ has Laurent coefficients $\langle Q, s^m\rangle$. The series converges on $|z|>R$ and agrees with the upper and lower transforms there by analytic continuation in their respective half-planes. Their boundary values therefore coincide on (R,∞). To see why this kills the restriction of Q, subtract the two kernels at $x \pm \mathrm{i}\epsilon$. Their difference, divided by $-2\pi\mathrm{i}$, is the Poisson kernel. Its convolution with a compact smooth test converges to that test in every fixed derivative on $\operatorname{supp} Q$. The jump thus recovers Q in distributions. □

5.1 With a prime at the second site

Set $B_X(t) = M_X(t,0)$ and $E_m = \limsup_X |B_X(m) - 1|$. The primitive at the prime edge is

$$G_X(z) = \frac{L}{S}\mathbb{E}_X H_{z/L}(n)\Lambda(n+2) = \partial_w J_X(z,0).$$

An extracted limit is $z + \mathcal{L}D_a(z)$, where D_a is the pushforward onto x of $-yD$. Thus $\operatorname{supp} D_a \subset [1/2, 1]$, $\langle D_a, 1\rangle = 0$, and $B - 1 = \mathcal{L}(-xD_a)$. We also have $|\widehat{D}_a(v)| \ll (1 + |v|)^{\alpha}$. Indeed $\Lambda(m) \ll Lf_1(m)$ for $m \asymp X$ (check prime powers), so (8) with this second weight gives the one-variable bound exactly as in Lemma 2.1.

Corollary 5.2. *Unconditionally $E_m \ll (1+m)^C e^{-m/2}$. If $\limsup_m E_m^{1/m} \le e^{-1}$, then locally uniformly in t,*

$$B_X(t) = 1 + \kappa_X(1 - t)e^{-t} + o(1).$$

If $E_m = o(me^{-m})$, then $\kappa_X \to 0$.

Proof. Push $-xD_a$ forward by $s = e^{-x}$. Its moments are the extracted defects at the integers. Lemma 5.1 confines it to $s = e^{-1}$. Since x is nonzero on the support, D_a is concentrated at $x = 1$. Its Fourier growth and zero total mass give $D_a = \kappa\delta_1'$. Differentiate and identify $\kappa = B(0) - 1$. As before, compactness gives the full-sequence formula. For each integer $m \ge 2$ it implies $E_m = (m - 1)e^{-m} \limsup_X |\kappa_X|$, proving the final assertion. □

5.2 A directional rate for the two-variable defect

Fix $a, b > 0$. For positive integers m, let $0 < h_m \le 1$ satisfy $\log(1/h_m) = o(m)$. Define

$$A_m(a, b) = \limsup_{X\to\infty} \frac{1}{h_m} \int_0^{h_m} |M_X(am, bm + v) - 1| \, \mathrm{d}v. \tag{17}$$

The classical relation between the supporting function of a compact support and the regularised growth of its Fourier–Laplace transform is recalled in Boman and Hörmander's *A Paley–Wiener theorem for the analytic wave front set* [3, §1]. Here the observations lie on real segments, sampled at integer multiples of a direction, and each observation is averaged over a width that may shrink.

Write $\mathcal{K}$ for the closure of the union of $\operatorname{supp} D$ over all extracted limits, and use $\inf \varnothing = +\infty$.

Theorem 5.3 (The decay rate in a thin strip)**.** *With these quantifiers,*

$$\limsup_{m\to\infty} A_m(a, b)^{1/m} = \exp\left(- \inf_{(x,y)\in\mathcal{K}} (ax + by) \right).$$

In particular, a bound $\limsup_m A_m(a, b)^{1/m} \le e^{-c}$ is equivalent to $ax + by \ge c$ on every limiting support.

The limit superior in X is taken for each fixed m, before $m \to \infty$. No uniform convergence rate in X is asserted that would permit an immediate substitution $m = m(X)$. Constant widths, negative powers of m and $h_m = e^{-\sqrt{m}}$ satisfy the width condition. Taken over $a, b > 0$, the theorem determines the containing half-planes with positive normals; these data do not in general recover a nonconvex support or its entire convex hull.

Proof. Put $s = ax + by$. For fixed m, the limsup in (17) is the maximum of the same integral over limiting F. This follows from local uniform convergence on the compact segment, by extracting a sequence realising the limsup. If all supports satisfy $s \ge c$, insert a cutoff in s equal to one for $s \ge c - 1/m$ and zero for $s \le c - 2/m$, together with a fixed spatial cutoff. Applying the common seminorm to e^{-ms-vy} gives $A_m(a, b) \ll (1 + m)^C e^{-cm}$. The same bound holds for every fixed derivative in v, with constants depending on its order. In particular we can always take $c_0 = d(a, b)$.

Conversely suppose the upper exponential rate is at most e^{-c} and fix one limit F. For $c \le c_0$ the desired support bound already holds. Otherwise fix $\ell \ge 0$ and $\gamma > 0$. Eventually

$d = e^{-\gamma m} \le h_m$. Choose $N \ge \ell$ with $(N+1-\ell)\gamma > c - c_0$. The moment matrix of $1, v, \dots, v^N$ on $[0,1]$ is positive definite. Hence there is a polynomial K with $\int_0^1 K(v) v^j \, \mathrm{d}v = \ell! \mathbf{1}_{j=\ell}$ for $0 \le j \le N$. Taylor expansion at $v = 0$ yields

$$|\partial_u^\ell F(am, bm)| \ll_{N,\ell} d^{-1-\ell} h_m A_m(a,b) + d^{N+1-\ell}(1+m)^C e^{-c_0 m}.$$

Take mth roots, then let $\gamma \downarrow 0$. The resulting rate is at most e^{-c} for every fixed ℓ.

These derivatives are the moments after pushing $(-y)^\ell xyD$ forward by e^{-s}. Lemma 5.1 therefore gives $\langle xyD, \phi(s) y^\ell \rangle = 0$ for every smooth test supported in $s < c$ and all ℓ. For each fixed ϕ, the resulting compact distribution in y has all moments zero. Its entire Fourier transform has all derivatives zero at the origin and hence vanishes. The map $(x,y) \mapsto (s,y)$ is invertible since $a > 0$. Testing products of functions in these coordinates gives $xyD = 0$ on $s < c$. Division by xy is possible near $\mathcal{P}$. This proves the equivalence for each c. It implies the formula, including zero rate when the union is empty. □

Corollary 5.4. *Write $A_m = A_m(1,1)$. Unconditionally $A_m \ll (1+m)^C e^{-14m/13}$. If $\limsup_m A_m^{1/m} \le e^{-2}$, all four profiles satisfy* (16)*. If $A_m = o(m^2 e^{-2m})$, then $\kappa_X \to 0$.*

Proof. The only point of the square satisfying $x + y \ge 2$ is $(1,1)$. Under the endpoint assumption, (16) yields for $m \ge 2$

$$A_m = (\limsup_X |\kappa_X|)(m-1)e^{-2m} \frac{1}{h_m} \int_0^{h_m} (m+v-1)e^{-v} \, \mathrm{d}v.$$

The multiplier of $\limsup_X |\kappa_X|$ is comparable to $m^2 e^{-2m}$ uniformly for $0 < h_m \le 1$. □

Observing only the diagonal can lose information. For a nonzero smooth bump supported strictly inside $\mathcal{P}$, let J_0 be its Laplace transform multiplied by $tu(t-u)^4$. This is again the transform of a smooth compactly supported distribution, obtained by differentiation. It vanishes on both axes, and $\partial_t \partial_u J_0$ vanishes on the whole diagonal. The example concerns the support test for distributions. The absolute average over a positive transverse width is the ingredient that removes this cancellation in Theorem 5.3; it does not itself provide a stable reconstruction from finitely many noisy data.

Corollary 5.5 (An opposite-sign diagonal criterion)**.** *Write $O_X(t,u) = (M_X^{00}(t,u) - M_X^{11}(t,u))/2$. If*

$$\lim_{m \to \infty} e^{12m/13} \limsup_{X \to \infty} |O_X(m,m) - 1/2| = 0 \qquad (m \in \mathbb{N}),$$

then $\kappa_X \to 0$.

Proof. The reference opposite-sign mass on the diagonal is $[1 - (1-m)^2 e^{-2m}]/2$. With $\pi(x,y) = x + y$, the centred diagonal moments of an extraction are the Laplace transform at m of $C_+ + C_-$, where

$$C_+ = \tfrac{1}{2} \pi_*(xyD), \qquad C_- = -\tfrac{1}{2} \pi_*(xyR_*D).$$

The multiplication by xy comes after reflection. The supports lie respectively in $[14/13, 2]$ and $[0, 12/13]$. Their masses are $\kappa/2$ and $-\kappa/2$, where κ is the extracted limit of κ_X. Indeed $\langle D, (1-x)(1-y) \rangle = \langle D, xy \rangle$ since the two marginals of D vanish by the axis identities. The reference correction and the moments of C_+ are $o(e^{-12m/13})$. Push C_- forward by $s \mapsto e^{-s}$. Its support lies in $[r_0, 1]$, $r_0 = e^{-12/13}$, and the hypothesis makes its mth moment $o(r_0^m)$. Lemma 5.1 puts it at r_0. The moments of a distribution supported at r_0 are r_0^m times a polynomial in m. The little-o bound kills this polynomial and hence the distribution. Its mass gives $\kappa = 0$ on every extraction. □

The unconditional estimate of Corollary 5.4 concerns the unsigned observable. The premise in Corollary 5.5 concerns the reflected contribution and remains open.

5.3 Interpolation on the unrestricted prime edge

Suppose B is an extracted prime-edge profile. The numbers $F_a(j) = B(j) - 1$ are moments of a distribution on $[d_0, d_1] = [e^{-1}, e^{-1/2}]$. Write T_m for a Chebyshev polynomial and set

$$Q_m(s) = \frac{T_m((d_0 + d_1 - 2s)/(d_1 - d_0))}{T_m((d_0 + d_1)/(d_1 - d_0))} = 1 - \sum_{j=1}^{m} w_{j,m} s^j, \qquad \rho = \frac{\sqrt{d_1} - \sqrt{d_0}}{\sqrt{d_1} + \sqrt{d_0}}.$$

Then

$$\left| F_a(0) - \sum_{j=1}^{m} w_{j,m} F_a(j) \right| \ll (1+m)^C \rho^m.$$

In fact the hyperbolic cosine formula bounds Q_m by $O(\rho^m)$ on an m^{-2} enlargement of the interval. The polynomial Markov inequality bounds any fixed derivative there by an additional power of m. A cutoff to that enlargement permits application of the fixed distribution seminorm. All the roots of Q_m are positive and $Q_m(0) = 1$, so its coefficients alternate. Errors of magnitude at most ϵ in the samples can therefore cause an error as large as $\epsilon(Q_m(-1) - 1)$, by choosing their signs. This factor grows exponentially with m.

5.4 What happens at the common endpoint

Keep the prime weight $\Lambda(n+2)$ and observe only the even Liouville sector at n. Define the anchored divisor distribution and the centred Fourier observations by

$$\langle T_{a,X}, f \rangle = \frac{L}{S} \mathbb{E}_X \Lambda(n+2) \sum_{d \mid n} \mu(d) f(\log d / L), \tag{18}$$

$$V_{+,X}(v) = \frac{L}{S} \mathbb{E}_X \mathbf{1}_{\lambda(n)=1} \Lambda(n+2) H_{\mathrm{i}v/L}(n) - \tfrac{1}{2} \mathrm{i}v(1 - e^{-\mathrm{i}v}). \tag{19}$$

The limit of $T_{a,X}$ is $\delta_0' + D_a$. Write $c = 1/2$ and $R_a(x) = 1 - x$. By Lemma 4.1, the limit of $V_{+,X}(v)$ is $\widehat{D}_+(v)$ for $D_+ = (D_a + (R_a)_* D_a)/2$. The centring in (19) is the even part of the reference primitive z. The supports of D_a and $(R_a)_* D_a$ meet at c. At this point a derivative δ_c' changes sign under reflection, so the even observation kills it. We need arithmetic control at the scale of a small neighbourhood of c.

Lemma 5.6 (Local test at the half level)**.** *For $\phi \in C_c^\infty((-1,1))$ and $0 < \rho < 10^{-3}$,*

$$\limsup_{X \to \infty} |\langle T_{a,X}, \phi((\,\cdot - c)/\rho) \rangle| \ll \|\phi\|_{C^5}, \tag{20}$$

with an absolute implied constant. The radius is fixed before $X \to \infty$.

Proof. We can keep just odd squarefree d and genuine primes $n+2$. Proper prime powers at $n+2$ have total cost $O_\epsilon(X^{-1/2+\epsilon})\|\phi\|_\infty$ after using the divisor bound and absorbing logarithms. For the remaining even d the prime $n+2$ would be even.

A factor of intermediate size. Call a modulus in the window good when it has a divisor in $[X^{6\rho}, X^{1/20}]$. On good odd moduli the sum of absolute errors in replacing $X^{-1} \sum_{d \mid n,\, X < n \le 2X} \Lambda(n+2)$ by $1/\varphi(d)$ is $O_{\rho,A}(L^{-A})$ for every A. To check this, apply Maynard's fixed-residue theorem [27, Corollary 1.2] with residue 2, $\delta = 2\rho$, $\eta = \rho$ at each endpoint $s \in [X, 3X]$. The accepted version (Corollary 1.2, p. 3) requires $\delta < 1/42$ and $\eta < (1 - 42\delta)/20$. The arXiv v2 instead prints denominator 4. With $\delta = 2\rho$ and $\eta = \rho$ the stronger inequality is $104\rho < 1$, valid throughout our range. The corollary also requires a modulus at most $s^{1/2+\delta}$ and a divisor strictly between

$$s^{2\delta+\eta} \quad \text{and} \quad \min(s^{1/10 - 7\delta/5 - \eta},\, s^{1/2 - 19\delta - \eta}).$$

The lower exponent is $5\rho < 6\rho$, and both upper exponents $1/10 - 19\rho/5$ and $1/2 - 39\rho$ exceed $1/20$. Also $d \le X^{1/2+\rho} < s^{1/2+2\rho}$. These strict margins cover all $s \in [X, 3X]$ for sufficiently large X at fixed ρ. Partial summation on $[X+2, 2X+2]$ inserts the logarithm, and the prime number theorem replaces the unrestricted logarithmic prime mass by X with arbitrary logarithmic saving. The cost of proper prime powers in the summed discrepancies is covered by the same divisor bound.

Write $d = em$ with all prime factors of e at most $X^{1/40}$ and all those of m larger. If $e \ge X^{6\rho}$, a first product of its prime factors crossing $X^{6\rho}$ lies below $X^{6\rho+1/40} \le X^{1/20}$. Choose a fixed smooth compact cutoff $0 \le \psi \le 1$, equal to one on $[0, 6]$ and zero above 7. Write $\Delta_d = X^{-1} \sum_{X < n \le 2X,\, d \mid n} \Lambda(n+2) - 1/\varphi(d)$ for odd d. To replace Δ_d by $\psi(\log e/(\rho L))\Delta_d$, the discarded multiple $(1-\psi)\Delta_d$ is supported on $e > X^{6\rho}$. Every such d is good by the first-product argument. Its total cost, including the outside factor L/S and the bounded test, is $O_{\rho,A}(\|\phi\|_\infty L^{1-A})$. The full, uncut model tends to zero by Lemma 3.1 because the test vanishes near zero. It remains to bound the arithmetic and model sums with the cutoff separately. The possible m now satisfy

$$X^{1/2-8\rho} \le m \le X^{1/2+\rho}, \qquad P^-(m) > X^{1/40}, \qquad \sum_m \frac{1}{m} \ll \rho + L^{-1}.$$

For the last bound, on each of $O(1+\rho L)$ dyadic blocks $m \asymp M$ the usual upper sieve gives $O(M/L)$ rough integers (sieving to a smaller fixed power of M suffices). Also $1/\varphi(m) \ll 1/m$ since the number of its prime factors is bounded.

Cancellation on the short factor. For fixed m take the one-variable Fourier transform of $g_m(w) = \psi(w)\phi((\log m/L - c)/\rho + w)$. Its modulus is $\ll \|\phi\|_{C^5}(1+|v|)^{-5}$ uniformly in m. With $y = X^{8\rho} < X^{1/40}$ and $n = mk$, inversion turns the e-sum into an integral against $\prod_{\ell \mid k,\, 2<\ell\le y}(1 - e^{\mathrm{i}v\log\ell/(\rho L)})$ (primes ℓ). Indeed g_m kills all $e > X^{7\rho}$ and all eligible e have prime factors at most y. Coprimality with m is automatic. In the model the corresponding product is over all primes $2 < \ell \le y$, with $e^{\mathrm{i}v\log\ell/(\rho L)}$ divided by $\ell - 1$.

Let $G_v(k)$ be the modulus of the arithmetic product. Splitting at $\exp(\rho L/(1+|v|))$ and using $|1 - e^{\mathrm{i}s}| \le \min(2, |s|)$ gives

$$\sum_{2<\ell\le y} \frac{|1 - e^{\mathrm{i}v\log\ell/(\rho L)}|}{\ell} \le 2\log(2+|v|) + O(1).$$

Use $\sum_{\ell \le z} \log\ell/\ell \ll \log(2z)$ below the split and Mertens' estimate above it. If the split is below two, use $\log y \ll 1 + |v|$ directly. The constants are absolute for ρL sufficiently large. For the model, use $-\cos s \le -1 + |1 - e^{\mathrm{i}s}|$ in the logarithm of the product. The extra $-\sum_{\ell \le y} 1/\ell$ gives the bound $O((1+|v|)^2/(\rho L))$.

For the arithmetic term put $Y = X/m$. We claim

$$Y^{-1} \sum_{\substack{Y < k \le 2Y \\ mk+2 \text{ prime}}} \log(mk+2) G_v(k) \ll \frac{(1+|v|)^2}{\rho L}. \tag{21}$$

Use Nair–Tenenbaum [29], in the formulation [21, Theorem 1 in the preprint], on $k, mk+2$. The forms are primitive, have resultant 2, and their product has discriminant 4 and no fixed prime divisor. Its coefficient norm is $\ll m+2$ and $Y \gg (m+2)^{1/4}$ with a power margin. The interval has length Y. Bound the prime indicator by the indicator of integers free of prime factors $\le X^{1/3}$ and the logarithm by $O(L)$. The two multiplicative weights are dominated by 2^ω with uniform growth constants. The local density product is $O(L^{-2})$, since there are two roots outside $2m$ and $\prod_{\ell \mid m}(1 + O(1/\ell)) = O(1)$. The Euler sum for the rough weight is $O(1)$ as the threshold is a fixed power of X. For G_v the primes up to y cost $O((1+|v|)^2)$

by the preceding sum. From y up to $O(X)$ the local values are one, contributing $O(1/\rho)$ by Mertens. Higher prime powers contribute a bounded factor, since each single linear form has at most one root modulo each prime power. Together these bounds give (21).

Fourier inversion now costs $O(\|\phi\|_{C^5}/(m\rho L))$ for each m in either term before the outer L/S. Multiplication by L/S and summation give $O(\|\phi\|_{C^5}(1+1/(\rho L)))$, uniformly in the remaining parameters. Taking $X \to \infty$ gives (20). □

In particular, two anchored limits with the same even sector have equal D_a. Their difference is anti-invariant, hence supported at c. Its Fourier growth of degree at most $4/\pi < 2$ allows only δ_c, δ_c', and anti-invariance leaves only the derivative. Testing with $\phi'(0) = 1$ in (20) costs $O(1)$ uniformly in ρ, whereas δ_c' pairs to $-1/\rho$. The derivative coefficient is zero. The same argument applies when the even primitive is known on a real open interval, by analytic continuation.

Theorem 5.7 (The prime defect from even Fourier observations)**.** *Fix a smooth even h supported in $[-1,1]$, equal to one near zero and with $0 \le h \le 1$. For each integer $K \ge 1$,*

$$\limsup_X \left| \kappa_X - \frac{8}{\pi^2} \sum_{\substack{1 \le j \le K \\ j \text{ odd}}} \frac{h(j/K)}{j^2} \Re\Big(e^{\mathrm{i}\pi j/4} V_{+,X}(\pi j/2) \Big) \right| \ll_h K^{-1}. \tag{22}$$

Errors of modulus at most ϵ per complex observation change the reconstructed value by at most ϵ.

Proof. The test for $\kappa = \lim \kappa_X$ is $g(x) = -x$ against D_a. Extend it symmetrically from the right of c to a periodic triangle $\Phi(x) = -c - |x - c|$ on $[c-2, c+2]$ of period four. With $v_j = \pi j/2$, its coefficients in the basis $e^{-\mathrm{i}v_j x}$, by direct integration, are

$$c_0 = -3/2, \qquad c_j = \frac{2(1-(-1)^j)}{\pi^2 j^2} e^{\mathrm{i}\pi j/4} \quad (j \neq 0).$$

Let P_K multiply each coefficient by $h(j/K)$. Since P_K is symmetric, $\langle D_a, P_K \rangle = \langle D_+, P_K \rangle$, and both distributions have mass zero.

We record the cutoff argument because a finite-order distribution need not act on a function with a corner. The convolution kernel of the filter is the periodisation of a Schwartz kernel at scale $1/K$. On the line it has mass one and first moment zero. Thus it reproduces $-x$. Subtracting $-x$ inside the convolution and differentiating the kernel gives, for $c \le x \le 1 + 1/10$,

$$|(P_K - g)^{(i)}(x)| \ll_{i,N,h} K^{i-1}(1 + K(x-c))^{-N}. \tag{23}$$

Indeed $\Phi - g$ vanishes on $[c, c+2]$ and grows at most linearly with distance to this interval. For $c - 2/K \le x \le c$ the same estimate holds without the decay factor.

Cut off the smooth error just to the left of $c - 1/K$ and just to the right of 1. In the central interval of radius $O(1/K)$, the rescaled C^5 norm is $O(K^{-1})$. On a smooth dyadic shell $x - c \asymp 2^j/K$ that norm at scale $C2^j/K$ is $O_{N,h}(K^{-1}(1+2^j)^{5-N})$ by (23) and the product rule. Apply (20) to the central piece and each shell at radius below 10^{-3} and sum with $N \ge 8$. On the remaining part the common finite-order seminorm costs $O_A(K^{-A})$ for any A. Small K can be absorbed in the constant. Hence $|\langle D_a, P_K - g \rangle| \ll_h K^{-1}$ uniformly over limits. Passing back to fixed finite sums, joining conjugate frequencies and using $\sum_{j>0,\, j \text{ odd}} j^{-2} = \pi^2/8$ proves both assertions. □

Here $\lambda(n) = 1$ excludes genuine primes at the first site. The second site retains its von Mangoldt anchor. Formula (22) expresses the unknown prime-pair mass through these observations for K fixed before the limsup in X.

5.5 Recovering the entire prime edge

For $t \geq 0$ put $q(t) = (1 - t/2)e^{-t/2}$. Choose an even $\chi \in C_c^\infty((-3/2, 3/2))$ equal to one on $[-3/4, 3/4]$. Set $f_t(x) = -xe^{-tx} + q(t)x$. For $x \geq 1/2$ in one period take $\Phi_t(x) = \chi(x - 1/2)f_t(x)$, reflect about $1/2$ to the other half, and extend with period four. For $v_j = \pi j/2$ put

$$c_j(t) = \tfrac{1}{4} \int_{-3/2}^{5/2} \Phi_t(x) e^{\mathrm{i}v_j x} \, \mathrm{d}x.$$

For fixed $t \geq 0$, $K \geq 1$,

$$\limsup_X \left| B_X(t) - 1 - q(t)\kappa_X - \sum_{|j| \leq K} c_j(t) V_{+,X}(v_j) \right| \ll_t (1 + K)^{\alpha - 3}. \tag{24}$$

Let $\mathcal{F}_{K,X}$ denote the sum in (22), including its factor $8/\pi^2$. Combining (24) with Theorem 5.7 gives, for every fixed $t \geq 0$,

$$\limsup_X \left| B_X(t) - 1 - q(t)\mathcal{F}_{K,X} - \sum_{|j| \leq K} c_j(t) V_{+,X}(v_j) \right| \leq C_t(1 + K)^{\alpha - 3} + C_h |q(t)| K^{-1}.$$

Thus the reconstruction uses only the even sector at the first site. At $t = 0$ it reduces to the recovery of κ_X and at $t = 2$ only the faster truncation error survives. Absolute errors bounded by ϵ in the frequencies cost at most $(|q(t)| + \sum_j |c_j(t)|)\epsilon$, uniformly in K. Appendix A gives the corresponding bounds for real integer samples.

To verify (24), let $R(x) = 1 - x$. By differentiating the reflected primitive at $w = 0$, the limiting V_+ is the Fourier transform of $D_+ = (D_a + R_* D_a)/2$. It has growth $O((1 + |v|)^\alpha)$. Because $f_t'(1/2) = 0$, the reflected function is C^2 and its third derivative has bounded variation. Thus $|c_j(t)| \ll_t (1 + |j|)^{-4}$. Its Fourier series pairs with D_+, and truncation costs $O_t((1 + K)^{\alpha - 3})$. By symmetry the full pairing equals the one with D_a. On $[1/2, 1]$, Φ_t agrees with f_t. To justify this equality for the distribution at the endpoint, subtract the smooth periodic extension of $\chi(x - 1/2)f_t(x)$. The difference vanishes to the right of $1/2$ throughout the support, and has absolutely summable Fourier coefficients with weight $(1 + |j|)^\alpha$. Translate it slightly left and mollify on a still smaller scale. These tests vanish on a neighbourhood of $[1/2, 1]$ and converge to the difference in that weighted norm by dominated convergence. Hence the full pairing equals $\langle D_a, f_t \rangle = B(t) - 1 - q(t)(B(0) - 1)$. Taking all extractions proves the stated limsup.

Smooth truncation gives a stronger estimate at the parameter where q vanishes. Multiply $c_j(t)$ in (24) by $h(j/K)$. The corresponding symmetric test is the filtered reflection of f_t. Its first derivative vanishes at $1/2$. The proof of Theorem A.1 gives a truncation error $O_{t,h}(K^{-3})$ for $t > 0$ and $O_h(K^{-5})$ for the centred profile at $t = 6$. In particular,

$$\limsup_X \left| B_X(2) - 1 - \sum_{|j| \leq K} h(j/K) c_j(2) V_{+,X}(v_j) \right| \ll_h K^{-3}.$$

The coefficients still have bounded total modulus. At a general t, using (22) for the missing $q(t)\kappa_X$ adds at most $O_h(|q(t)|/K)$ to the corresponding bound.

6 Keeping two prime factors fixed

We now impose $n = pq$ and $n + 2 = rz$. Keep the actual primes p, r as indices. The free prime q ranges over an interval of length X/p, which sets the available distribution level. For a prime $X^{.490} < p \leq X^{.495}$ let

$$A_p = \#\{q \text{ prime} : X < pq \leq 2X\}.$$

For a prime $X^{.15} < r \le X^{.16}$ put

$$M_{pr}(v) = \frac{r-1}{A_p} \sum_{\substack{X<pq\le 2X,\ q \text{ prime} \\ r\mid pq+2}} W_{v/L}((pq+2)/r).$$

Write $P_{pr}(v)$ for the same sum with $\lambda((pq+2)/r)$ inserted. The factor A_p is the actual number of primes in this row. A divisor d of the cofactor imposes a modulus rd on q. The exponent below is the margin left by the classical half-level for primes in an interval of size X/p, after accounting for r.

Proposition 6.1. *Fix $B > 0$. For each subset $\mathcal{R}$ of the r-range put $H = \sum_{r\in\mathcal{R}} 1/(r-1)$ and require $H \ge L^{-B}$. Let $\vartheta = (1-.495)/2 - .16 = .0925$. For fixed $v \ge 1$,*

$$\limsup_{X\to\infty} \sup_{p,\mathcal{R}} \frac{1}{H} \sum_{r\in\mathcal{R}} \frac{|M_{pr}(v) - S|}{r-1} \ll (1+v)^4 e^{-\vartheta v}. \tag{25}$$

A single implied constant works in (25) *for all fixed $B > 0$ and $v \ge 1$. For each such B, v and any positive tolerance, the threshold in X for the corresponding finite-X bound may depend on all three. For $v = v(X) \to \infty$, $v = o(L)$, the mean on the left tends uniformly to zero. For every fixed $0 < a < b$ and $B_0 > 0$, for each allowed p and each dyadic $(R, 2R]$ contained in the r-range, all but $O_{a,b,B_0}(R/L^{B_0})$ primes r satisfy*

$$\int_a^b |P_{pr}(v)|\,\mathrm{d}v \gg_{a,b} 1.$$

Proof. Choose $1 \le c < r$ with $pc \equiv -2 \pmod r$ and set $c' = (pc+2)/r$. On writing $q = rk + c$ the second form is $z = pk + c'$. Their determinant is $rc' - pc = 2$, and the product polynomial has discriminant four and coefficient norm $O(pr)$. The interval for k has location and length comparable to $Y = X/(pr) \gg X^{.345}$. For $v_0 \ge 1$ define $f(\ell^j) = \min(2, v_0 \log \ell/L)$ multiplicatively. A bound uniform in the two primes and in v_0 is

$$\sum_{\substack{k:\ X<p(rk+c)\le 2X \\ rk+c \text{ prime}}} f(pk+c') \ll (1+v_0)^2 Y/L^2. \tag{26}$$

Apply Nair–Tenenbaum in the form recalled as Theorem 1 of Henriot's *Nair–Tenenbaum bounds uniform with respect to the discriminant* [21]; see also [29, Corollary 3]. The constant in this recalled theorem may depend on the discriminant. Here that discriminant is exactly 4, so this dependence remains fixed as p, r vary. The forms are primitive. Modulo 2 their roots coincide because $rc' - pc = 2$. Away from $2, p, r$ they have two distinct roots, leaving at least one admissible residue even at 3. At p and r one form is constant and nonzero, again leaving an admissible residue. A prime $q \asymp X/p$ is free of prime factors $\le X^{.4}$, so its indicator is majorised by the multiplicative indicator of that condition. Its Euler sum is bounded, since the summation length in this bound is comparable to $Y < X^{.4}$ by a power. The roots give $O(L^{-2})$ in the density product, with only $2, p, r$ exceptional. The other Euler sum costs $O((1+v_0)^2)$ by the argument of (8). All growth conditions hold uniformly since $f(m) \le 2^{\omega(m)}$. With coefficient exponent $1/4$, the theorem's length condition has the margin

$$Y \gg X^{.345}, \qquad \|(rk+c)(pk+c')\|^{1/4} \ll (pr)^{1/4} \le X^{.16375}.$$

Here the norm is the largest absolute coefficient of the polynomial. Covering by comparable intervals accommodates endpoints.

For arbitrary phases $|\alpha_r| \le 1$ define a divisor distribution and an averaged phase by

$$T(\phi) = \frac{L}{A_p H} \sum_{r\in\mathcal{R}} \alpha_r \sum_{\substack{q \text{ as above} \\ r\mid pq+2}} \sum_{d\mid (pq+2)/r} \mu(d)\phi(u_d), \qquad c_X = \frac{1}{H} \sum_{r\in\mathcal{R}} \frac{\alpha_r}{r-1}.$$

The prime number theorem gives $A_p \asymp X/(pL)$ uniformly. Taking $v_0 = 1 + |w|$ in (26) yields $|\widehat{T}(w)| \ll (1+|w|)^2$; its support lies in $[0, .85 + O(1/L)]$. The normalisation here uses $\sum_{r\in\mathcal{R}} 1/r \le H$. The common seminorm with four derivatives is independent of B, the phases, the row and the subset.

Fix $\epsilon > 0$. If $d \le X^{\vartheta-\epsilon}$, then $d < r < p$ and $rd \le (X/p)^{1/2}X^{-\epsilon}$. The condition is one unit class $q \equiv -2\bar{p} \pmod{rd}$ when d is odd, and the even case gives no primes in this interval. Bombieri–Vinogradov for primes replaces the count by $A_p/\varphi(rd)$, with arbitrary summed logarithmic savings in X/p. Each modulus rd occurs at most once, since $d < X^{.15}$ cannot contain another prime from the column range. Every prime in the row exceeds rd, so the coprime total in the progression main term is exactly A_p. Equivalently one may pass from logarithmic integrals to that total by the prime number theorem and $\sum_{m\le X} 1/\varphi(m) \ll L$. Finally $\varphi(rd) = (r-1)\varphi(d)$ for odd d in this range. Consequently for bounded tests supported below $\vartheta - \epsilon$ the error in replacing T by $c_X L \sum_d c_d \delta_{u_d}$ is $O_{A,B,\epsilon}(L^{-A}\|\phi\|_\infty)$, with c_d from Lemma 3.1. The factor $L/(A_pH)$ is $O(L^{B+2}/(X/p))$. Choosing the progression saving with exponent $A + B + 4$ suffices. Its threshold may depend on B. The limiting seminorm above does not.

For fixed v, take any convergent extraction of T and $c_X \to c$. Lemma 3.1 identifies its limit as $cS\delta_0' + D'$, with $\operatorname{supp} D' \subset [\vartheta, .85]$. The test $-xe^{-vx}$ gives the average of $\alpha_r M_{pr}(v)$ against $1/(H(r-1))$. A cutoff on a $(1+v)^{-1}$ enlargement of the support bounds the defect by $O((1+v)^4 e^{-\vartheta v})$. Choose α_r to give the absolute deviations in (25). Since the extraction was arbitrary, the bound is uniform in the chosen rows and subsets.

For a moving v a finite-X argument is necessary. Take a fixed spatial cutoff equal to one near $[0,\gamma]$, with $0 < \gamma < \vartheta$, supported below ϑ. The common seminorm bounds the complementary part of the actual test by $O((1+v)^4 e^{-\gamma v})$. On the cutoff use the progression estimate directly. Restoring the rest of the model costs $o(1)$ by partial summation: beyond $d \ge X^\gamma$, the partial sums of c_d have arbitrary logarithmic saving and xe^{-vx} times the complementary cutoff has bounded variation uniformly for $v \ge 1$. The full model on the test is $c_X P'(v/L) = c_X(S + O(v/L))$. Applying the phases again proves the moving assertion.

For the signed conclusion we need control of squares at the individual r for which the unsigned value is large. On any admissible subset the harmonic mean of the normalised weight of z containing a square is $o(1)$, with $|W_{v/L}|$ in the sum, uniformly for v in a fixed compact disc. Here are the two estimates. For $\ell \le L^K$ and $\ell^2 \mid z$, retain the factor $f_C(\ell)$ from (9). On the progression in k the forms q and z/ℓ^2 have slopes $r\ell^2, p$ and determinant two. Their coefficient norm is $O(pr\ell^{O(1)})$ and their interval has length $\asymp Y/\ell^2$. For odd $\ell \ne p, r$ they are primitive: a simultaneous zero modulo a common slope prime would divide the determinant. In particular q is nonzero modulo ℓ. At $\ell = 2$ the class cannot contain an odd q, and $p, r > L^K$ for large X. Use (26) with the ℓ factor of f_C set to one, so it is counted only once. The normalised individual loss is $O(\sum_\ell f_C(\ell)/\ell^2) = O(1/L)$ after (9). For $\ell > L^K$ use $\ell^2 \mid pq + 2$ and allow all q in the interval. The count is $O((X/p)L^{-K} + \sqrt{X})$, by counting a class modulo each ℓ^2. Each $pq + 2$ is divisible by at most seven of the r in our band. Normalisation and $|W| \ll L$ cost $O(L/(A_pH))$. Since $X/p \ge X^{.505}$ and $H \ge L^{-B}$, the resulting loss is

$$O\Big(L^{B+2-K} + L^{B+2}X^{-.005}\Big) = o(1) \qquad (K > B + 4).$$

Take v_* so large that the right-hand bound in (25), with its common constant, is smaller than $S/4$. This choice precedes B_0 and B. A set E of primes in $(R, 2R]$ has $H_E \ge |E|/(2R)$. If $|E| \ge R/L^{B_0}$, then $H_E \ge L^{-(B_0+1)}$ for large X. Apply (25) to this very set with $B = B_0 + 1$. It cannot consist entirely of columns with $M_{pr}(v_*) \le S/2$. This gives the claimed exceptional-set bound. The same subset argument removes $O(R/L^{B_0})$ further columns at any fixed positive square-loss threshold. On squarefree z the marked-prime expansion gives $\lambda(z)W_{-v_*/L}(z) = -|W_{-v_*/L}(z)| \le -W_{v_*/L}(z)$. Using the square estimate at both parameters

leaves $P_{pr}(-v_*) \le -S/4$ on the good columns for large X. By (9) and (26) the individual P_{pr} form a locally bounded entire family. If their integrals of the absolute value on $[a,b]$ could tend to zero, a locally uniform limit would vanish on that interval, contradicting its value at $-v_*$. This gives the common positive lower bound. □

The same subset argument gives an individual unsigned statement. In each such $(R, 2R]$, for fixed $v \ge 1$ all but $O_{B_0,v}(R/L^{B_0})$ primes satisfy $|M_{pr}(v) - S| \ll (1+v)^4 e^{-\vartheta v}$. For a prescribed moving $v(X)$ in the above range the bound is ϵ outside $O_{B_0,\epsilon}(R/L^{B_0})$ exceptions, for every fixed $\epsilon > 0$ and sufficiently large X depending also on the chosen sequence.

The absolute value inside (25) also permits any joint test of modulus at most one depending on (p, r), followed by any probability average over p, with each admissible row normalised as stated. At $v = 0$ the fourth factor z must be a prime power for $W_0(z)$ to contribute. The signed integral on a positive interval does not evaluate that individual parameter.

7 Signed Taylor coefficients for four affine forms

Let $C(n) = \lambda(n)\lambda(n+1)\lambda(3n+1)\lambda(3n+2)$ and $M = 9n(n+1)$. The identity $M + 2 = (3n+1)(3n+2)$ allows us to place the two weights at distance two, even though C involves four forms. Restrict to $n \equiv 1 \pmod{24}$ and keep the dyadic denominator X, writing $\mathbb{E}_X^* f = X^{-1} \sum_{X < n \le 2X,\, n \equiv 1 \ (24)} f(n)$. Put

$$(P_0, P_1, P_2, P_3) = \big(n, (n+1)/2, (3n+1)/4, 3n+2\big).$$

We consider the two analytic functions

$$\begin{aligned} U_X(t) &= L^5 \mathbb{E}_X^* C(n) W_{t/L}(M) W_{t/L}(M+2), \\ A_X(t) &= \mathbb{E}_X^* C(n) \prod_{i=0}^{3} W_{t/L}(P_i). \end{aligned} \tag{27}$$

Here A_X is the four-weight observable, and $A_m(a,b)$ in (17) denotes a strip average.

Theorem 7.1. *For all sufficiently large X, $[t^j]U_X = 0$ for $0 \le j < 5$, and*

$$[t^5]U_X = 12(\log 2)^2 \log 3 \, A_X(0). \tag{28}$$

For each fixed j in the stated range,

$$\begin{aligned} \liminf_{X\to\infty} (-1)^j [t^j] U_X &> 0 \qquad && (j \ge 12), \\ \liminf_{X\to\infty} (-1)^{j+1} [t^j] A_X &> 0 && (j \ge 7). \end{aligned} \tag{29}$$

There are $b_0, c > 0$ such that on every sufficiently large dyad each function has at most seven real zeros in $(0, b_0]$, counted with multiplicity. For every $0 < h \le b_0/8$, including choices varying with X,

$$\max_{1\le k\le 8} |U_X(kh)| \ge c h^{12}, \qquad \max_{1\le k\le 8} |A_X(kh)| \ge c h^7. \tag{30}$$

The positive constants and the threshold in X implicit in (29) may depend on the fixed order j. The constants b_0, c and the threshold for (30) are common to all admissible steps h. The orders twelve and seven are sufficient orders obtained from the almost-prime input below; no optimality of these thresholds is asserted.

7.1 A rough subfamily of Maynard's almost primes

Writing $n = 24v + 1$ gives

$$24v + 1, \quad 12v + 1, \quad 18v + 1, \quad 72v + 5.$$

Call their product $\Pi(v)$. Each slope has prime divisors $2, 3$, and neither divides any form. The six pairwise determinants in this order are $12, 6, 48, -6, -12, 18$. Consequently distinct forms are coprime at every integer and have distinct roots modulo every prime ≥ 5. The four roots omit at least one residue even for $p = 5$. This checks admissibility and Hypothesis 1 in Maynard's notation.

Lemma 7.2. *For some fixed $\eta > 0$, on every large dyad $X < n \leq 2X$ there are $\gg X/L^4$ integers $n = 24v + 1$ such that $\Pi(v)$ is squarefree, has at most eleven prime factors, and each of them exceeds X^η.*

Proof. We extract the roughness and the count from Maynard's *Almost-prime k-tuples* [26]. We give the argument because a statement about infinitely many values of $\Omega(\Pi)$ alone would not control our Taylor coefficients. All formula numbers in this proof refer to arXiv:1205.4610v1. Write $N = X/24$. The dyad $N < v \leq 2N$ differs from the required one by $O(1)$ endpoints, which may be removed from the final count.

Denote Maynard's nonnegative Selberg weight $(\sum_{d|\Pi(v),\ d\leq R_2} \lambda_d)^2$ by $w_N(v)$. Use his parameters $h = 3$, $R_1 = N^{r_1}$, $R_2 = N^{r_2}$ with $r_1 = 1/2 + \epsilon$ and $r_2 = 1/4 - \epsilon$, and his polynomial $P(x) = 1 + 22x$. Choose first $c_* = 47/4$, strictly between the limiting threshold $11.653\ldots$ in Table 3 and twelve, then choose $\epsilon > 0$ sufficiently small to preserve this margin. His formulae (5.7), (5.25)–(5.28) give, for some fixed $c_1 > 0$,

$$\sum_{\substack{N<v\leq 2N \\ \Pi(v)\ \text{squarefree}}} (c_* - \Omega(\Pi(v)))w_N(v) \ \geq \ c_1 N(\log N)^4. \tag{31}$$

Indeed the free parameter ν in (5.26) can be chosen with a fixed positive margin while $\nu + \log \Pi(v)/\log R_1 < c_*$ on the dyad for large N. The positive main term then exceeds its error by a fixed multiple of $N(\log N)^4$. Formula (5.7) already restricts the left side to squarefree values.

To remove small prime factors, take a smooth $0 \leq W_0 \leq 1$ equal to one on $[0, \delta]$ and supported in $[0, 2\delta]$, with $2\delta < 1$. Proposition 5.1 of [26] evaluates

$$\sum_{N<v\leq 2N} w_N(v) \sum_{p|\Pi(v)} W_0(\log p/\log R_2).$$

In its main-term integral J_{01} the squared difference of the polynomial at arguments separated by y is $O(y^2)$ and the outer measure is $\mathrm{d}y/y$ on $(0, 2\delta)$. In J_{02} the inner interval has length y and the integrand before the outer measure is $O(y)$. The third integral is zero. More explicitly, for $P(x) = 1 + 22x$, $J_{01} \leq 968\delta^2$ and $J_{02} \leq 8\delta(1 + 44\delta)^2$, by integrating up to 2δ and using $1 - x \leq y$ in the second inner integral. Thus the displayed sum is $O(\delta N(\log N)^4) + o_\delta(N(\log N)^4)$. The constant in the main bound is independent of δ, and δ is fixed before $N \to \infty$. The positive part of $c_* - \Omega(\Pi)$ is bounded by c_*. Choosing δ sufficiently small therefore leaves a positive proportion of (31) on squarefree values with all prime factors exceeding R_2^δ.

Only $\Omega(\Pi) \leq 11$ can contribute positively. On these values $w_N(v) \ll (\log N)^8$. For details, there are at most 2^{11} possible divisors. In (4.10)–(4.15) one has $|y_r| \ll 1$, and for squarefree d coprime to the product of the slopes,

$$|\lambda_d| \ll \prod_{p|d} \frac{p}{p-4} \prod_{5\leq p\leq R_2} \left(1 + \frac{4}{p-4}\right) \ll (\log N)^4 \qquad (\omega(d) \leq 11).$$

The products run over primes and the first is uniformly bounded with this restriction. Dividing the remaining mass by the pointwise bound proves the count. Choose finally a fixed $0 < \eta < r_2\delta$; then $X^{\eta} \le R_2^{\delta}$ for all large X. The choices are in the order c_*, the sieve parameters and their positive margin, δ, and η, all before $X \to \infty$. □

Booker and Browning's *Square-free values of reducible polynomials* [7, Theorem 1.1, Table 1 and §3] gives twelve factors for a degree-four product of four linear forms. Its weighted lower sieve sifts out primes below a fixed power and applies on this dyad using the local root counts above. That input alone gives degrees thirteen and eight below. Lemma 7.2 gains one order in each case through the estimate on removed Selberg mass.

Lewulis [24, Theorem 12 and Table D in the arXiv version] obtains eleven factors unconditionally and eight under GEH for four forms, using an extended simplex for the latter bound. To transfer the conditional eight-factor result to the present observables would require the quantitative, squarefree and rough version on every large dyad supplied here by Lemma 7.2, together with control of the presieving classes. We use Maynard's input throughout this section.

7.2 Why the signs agree

For any $m > 1$ with $r = \omega(m)$,

$$(-1)^{r-1} W_{-t/L}(m) = \sum_{p \mid m} (\log p) e^{t u_p} \prod_{q \mid m,\ q \ne p} (e^{t u_q} - 1). \tag{32}$$

Every coefficient on the right is nonnegative and its first nonzero degree is $r - 1$. Directly at the positive parameter the coefficient of t^{r-1} in $W_{t/L}(m)$ is $r L^{1-r} \prod_{p \mid m} \log p$, as follows by differentiating the leading term of H.

Here $M = 18 P_0 P_1$, $M + 2 = 4 P_2 P_3$, and $C(n) = -\prod_i \lambda(P_i)$. Set $k = \sum_i \omega(P_i) \ge 4$. The first grouped degree is $k + 1$, because the two sites together have three extra distinct-prime occurrences ($2, 3$ and 2). The first direct degree is $k - 4$. Grouped degree five requires every P_i to be a prime power. The two numbers of distinct primes are then $4, 3$. The leading coefficients just computed give exactly (28), including the case of nontrivial powers. For example, $n = 97$ gives $(97, 49, 73, 293)$ and $C = +1$, whereas $n = 2857$ gives the four primes $(2857, 1429, 2143, 8573)$ and $C = -1$. Those powers contribute $o(1)$ to $A_X(0)$: on each form there are $O(\sqrt{X} \log X)$ such values, each summand being $O(L^4)$. On the simultaneous-prime population $C = -1$. Thus the scale of degree five involves a four-prime count on this progression.

If $\Pi(v)$ is squarefree, $C = (-1)^{k+1}$. The sign in the grouped product of (32) is $(-1)^{k+1} = C$; in the direct product it is $(-1)^{k-4} = -C$. Hence every squarefree value contributes in the orientation of (29), in every degree in which it contributes.

We control the remaining values on compact parameter discs. By Lemma 2.2 the mean of $\prod_i f_K(P_i)$ is $O_K(L^{-4})$. With a square in one form it is $O_K(L^{-5})$: for $p \le L^{10}$ sum $f_K(p)/p^2 \ll_K L^{-1}$ after forcing p^2; larger squares have density $O(L^{-10} + X^{-1/2})$. For the grouped primitives the two sites contribute three fixed factors $1 - 2^{-z/L}$ or $1 - 3^{-z/L}$, giving an extra $O_R(L^{-3})$ on fixed discs; the variable factors are bounded with (9). Multiply the primitive by L^7 and differentiate once in each of two independent parameters, then restrict to the diagonal. This gives compact-uniform bounds for U_X and a square contribution of $O_R(1/L)$. Applying the same argument with four independent primitive parameters and a factor L^4 gives these bounds for A_X. In particular every fixed Taylor coefficient has square error $O_j(1/L)$.

On the population in Lemma 7.2, mark one large prime at each grouped site in (32). The two marked logarithms contribute $\gg L^2$, the three fixed unmarked factors supply $\gg L^{-3}$ in their first degrees, and all the other unmarked logarithmic proportions are at least η. The

first coefficient thus has size $\gg_\eta L^{-1}$. The exponential of a marked prime supplies any fixed extra degree with a positive lower bound depending on that degree and η. Here $k+1 \le 12$. Multiplying by L^5 and by the density $\gg L^{-4}$ gives the first line of (29). For the direct product mark one large prime in each of the four forms. The marked logarithms contribute $\gg L^4$, and $k-4 \le 7$, giving the second line. This proves both lower bounds after restoring the negligible square contribution.

7.3 Zeros and moving observations

The quotient $V_X(t) = U_X(t)/t^5$ extends through zero. By (29), $V_X^{(7)}(0) = 7![t^{12}]U_X$ is bounded below by a fixed positive constant for large X. The same is true of $A_X^{(7)}(0) = 7![t^7]A_X$. Cauchy's estimate gives uniform bounds for their eighth derivatives on $[0,1]$, using a larger disc and the removable zero of U_X. Thus both seventh derivatives are bounded below on a common $[0, b_0]$. Repeated Rolle's theorem bounds the number of real zeros there by seven for each of V_X, A_X.

For either $f = V_X$ or $f = A_X$ and $0 < h \le b_0/8$ the seventh forward difference starting at h equals

$$\sum_{j=0}^{7} (-1)^{7-j} \binom{7}{j} f((j+1)h) = \int_{[0,h]^7} f^{(7)}(h + s_1 + \cdots + s_7)\, \mathrm{d}\mathbf{s} \gg h^7.$$

The sum of the absolute coefficients is 2^7, so some sample has size $\gg h^7$. Restoring $(kh)^5$ proves (30) and finishes Theorem 7.1. More generally, for $0 \le a < b \le b_0$,

$$\int_a^b |U_X(t)|\, \mathrm{d}t \gg b^5 (b-a)^8, \qquad \int_a^b |A_X(t)|\, \mathrm{d}t \gg (b-a)^8.$$

Take step $(b-a)/16$ and average the difference over starting points in $[a+(b-a)/4, a+(b-a)/2]$. Every evaluation lies between $b/4$ and b. On any other fixed positive interval the integral of the absolute value of either function still has a positive liminf: a locally uniform limit with zero integral would be identically zero and would contradict its fixed coefficient in (29). None of these positive lower bounds determines the unweighted quartic correlation.

A Integer samples on the even prime edge

The Fourier formula (22) has bounded amplification of measurement error. For real positive smoothings the observations are moments in $z = e^{-x}$. In the classical Hausdorff problem, Gerth, Hofmann, Hofmann and Kindermann [16, Theorem 1 and Proposition 13] obtain logarithmic conditional stability for full recovery in a Sobolev class and exclude a Hölder rate there. Here the unknowns are specified functionals of a signed distribution. The first nonzero odd derivative of their test at the reflection point determines the power in our estimates. For $s \ge 0$ set

$$Z_X(s) = S^{-1} \mathbb{E}_X \mathbf{1}_{\lambda(n)=1} \Lambda(n+2) W_{s/L}(n) - \tfrac{1}{2}[1 + (s-1)e^{-s}].$$

Its extracted limit is $\langle D_+, -xe^{-sx} \rangle$. In particular the real data here use only one sign at n. The Chebyshev interpolant in Section 5 uses the unrestricted B_X.

Theorem A.1. *For each integer $N \ge 1$ one can construct real coefficients $w_{m,N}$, $1 \le m \le N$, with*

$$\limsup_X \left| \kappa_X - \sum_{m=1}^{N} w_{m,N} Z_X(m) \right| \ll (1+N)^{-1}, \qquad \sum_{m=1}^{N} |w_{m,N}| \ll e^{2N}.$$

For errors ϵ per sample, choosing $N = \max(1, \lfloor \frac{1}{4}\log(1/\epsilon)\rfloor)$ gives total error $O(1/\log(1/\epsilon))$ as $\epsilon \downarrow 0$, with $X \to \infty$ first for each N. In particular, $Z_X(m) \to 0$ for every positive integer m implies $\kappa_X \to 0$.

For fixed $t > 0$ one can reconstruct $C_X(t) = B_X(t) - 1 - q(t)\kappa_X$ from the same samples with coefficients of norm $O_t(e^{2N})$ and intrinsic error $O_t((1+N)^{-r})$, where $r = 3$ for $t \neq 6$ and $r = 5$ for $t = 6$.

Proof. The smooth test against D_a is $g(x) = -x$ for κ, and $g(x) = -xe^{-tx} + q(t)x$ for $C(t)$. The first nonzero odd derivative at $c = 1/2$ has order $r = 1$ in the former case. In the latter $g'(c) = 0$, and for odd $j \geq 3$, $g^{(j)}(c) = t^{j-1}e^{-t/2}(t/2 - j)$. This gives the two stated exponents.

Extend g by $\Phi(x) = \chi(x-c)g(c + |x-c|)$ with the cutoff and period of Section 5.5. Smoothly truncate its Fourier series by $h(j/K)$, obtaining P_K. Then $|\langle D_a, g - P_K\rangle| \ll_g K^{-r}$. To check the improved power, subtract from Φ the smooth periodic extension of $\chi(x-c)g(x)$. The difference vanishes to the right of c and is $O_g(|x-c|^r)$ to its left near c. Convolution with the ith derivative of the periodised filter thus costs $O_{A,i,g}(K^{i-r}(1 + K(x-c))^{-A})$ on $[c, 1 + 1/10]$, plus $O_{A,i,g}(K^{-A})$ for the smooth part. The same bound without the decay factor holds on $[c - 2/K, c]$. Rescaling the central interval and the dyadic shells exactly as in (23) gives the claim by Lemma 5.6.

The symmetry of P_K and D_+ implies $\langle D_a, P_K\rangle = \langle D_+, P_K\rangle = 2\langle xD_+, P_K\rangle$, since $x + (1 - x) = 1$. Put $z = e^{-x}$ and $K = \max(1, \lfloor N/30\rfloor)$. In the Fourier expansion of P_K, replace $e^{-\mathrm{i}v_j x} = z^{\mathrm{i}v_j}$ by its Taylor polynomial of degree N centred at $a = 7/10$, using the principal logarithm. On $|z - a| = 3/5$ the modulus is at most $e^{\pi|v_j|/2}$. A fixed complex neighbourhood of $0 \leq x \leq 1$ has $|e^{-x} - a| < 2/5$. Cauchy's estimate on a slightly smaller neighbourhood bounds the remainder and its first four x-derivatives by $O((2/3)^N e^{\pi|v_j|/2})$. The norm of its polynomial coefficients in the monomial basis is at most

$$e^{\pi|v_j|/2} \sum_{k=0}^{N} ((1+a)/(3/5))^k.$$

The coefficients of Φ are absolutely summable, by two piecewise integrations by parts. The choice $K \leq N/30$ for large N therefore makes the approximation error exponentially small and the coefficient norm $O_g(e^{2N})$. The Fourier bound of xD_+ gives a uniform C^4 seminorm on a fixed neighbourhood of its support in $[0, 1]$. Indeed multiplication by a cut-off coordinate convolves the Fourier transform with a rapidly decreasing function. We may therefore use this real polynomial $\sum_{m=0}^{N} b_m z^m$ in the pairing. Take $w_{m,N} = -2b_m$. The constant term drops out, since $\langle xD_+, 1\rangle = \langle D_+, 1\rangle/2 = 0$. The construction is finite once the cutoff is chosen, and the error estimates are uniform over extractions. □

In particular $C_X(6) = B_X(6) - 1 + 2e^{-3}\kappa_X$. Adding the single unrestricted sample $B_X(6)$ therefore improves the error in recovering κ_X to order $\log(1/\epsilon)^{-5}$ for even-sector noise ϵ, plus $e^3/2$ times the error in this extra sample.

Why a logarithmic rate appears

The preceding powers of the logarithm are sharp under the bounds used here as constraints on *abstract distributions*. The following example uses measures of bounded variation. Their realisation as limits of the arithmetic divisor sums is unknown.

Choose $b = 1/4$ and a nonnegative nonzero $\eta \in C_c^\infty((1,2))$. Push

$$K_n(\theta)\frac{\mathrm{d}\theta}{2\pi}, \quad K_n(\theta) = \sum_{k\geq 1} \eta(k/n)\cos(k\theta), \quad -\pi \leq \theta \leq \pi,$$

forward by $x = c + y$, $y = b|\sin(\theta/2)|$, to a signed measure D_n. Its mass is zero. Repeated finite differences give $|K_n(\theta)| \ll_A n(1 + n|\theta|)^{-A}$. Hence the variation is bounded, $\int y^j \,\mathrm{d}|D_n| \ll_j n^{-j}$,

and both the Fourier bound and (20) hold uniformly. The symmetrised measure, tested by e^{-sx}, integrates K_n against $e^{-sc}\cosh(sb\sin(\theta/2))$. For $-xe^{-sx}$ differentiate in s. The two test functions are periodic and holomorphic in a fixed strip $|\Im\theta| \le \sigma$ with $b\cosh(\sigma/2) < c$, uniformly bounded there for $s \ge 0$. Their Fourier coefficients decay exponentially. All the real observations, even on the entire half-line, thus have modulus $O(e^{-c_1 n})$ for some $c_1 > 0$.

For either target g in Theorem A.1, expand $g(c+y)$ to degree r. Each even power of y is a trigonometric polynomial of fixed degree and the lower odd coefficients vanish. The coefficient of order r is nonzero. The positive-frequency Fourier coefficients of $|\sin(\theta/2)|^r$ are asymptotic to a nonzero constant times k^{-r-1}, since its distributional derivative of order $r+1$ has a nonzero atom at zero and an integrable remainder. Consequently the target pairing is $c_g n^{-r} + o(n^{-r})$, $c_g \ne 0$, including the $O_g(n^{-r-1})$ Taylor remainder. At noise of order $e^{-c_1 n}$, zero and these measures cannot be distinguished. A two-scale combination of these measures will also keep an extra scalar observation fixed.

Several unrestricted smoothings

There is a systematic choice when $d \ge 1$ unrestricted parameters are available. Let x_i, v_i be the nodes and weights of d-point Gauss–Laguerre quadrature for $x^2 e^{-x}\,\mathrm{d}x$ [10, §§3.5, 18.12]. Put $t_i = 2x_i$, $a_i = v_i e^{x_i}/x_i^2$. The linear combination $\sum_i a_i C_X(t_i)$ can be reconstructed from $Z_X(1), \dots, Z_X(N)$ with intrinsic error $O_d((1+N)^{-(2d+3)})$ and coefficient norm $O_d(e^{2N})$. The coefficient needed to recover κ_X from the additional observation $\sum_i a_i B_X(t_i)$ is

$$\beta_d = \sum_i a_i q(t_i) = -\frac{8d}{3(d+1)(d+2)}.$$

Indeed the combined target's odd derivative of order $2k+1$, $1 \le k \le d$, is $2^{2k}\sum_i v_i x_i^{2k-2}(x_i - 2k - 1)$. Quadrature is exact here through degree $2d-1$, and the integral vanishes by integration by parts. The first derivative vanishes as before. The cutoff proof above applies whenever all odd derivatives through order $2d+1$ vanish, giving the stated error.

For the coefficient, normalise $P(x) = L_d^{(2)}(x)/\binom{d+2}{2}$ and put $p_1 = P'(0) = -d/3$. At the nodes $(1-P(x))/x$ equals $1/x$ and $(1-(1-p_1 x)P(x))/x^2$ equals $1/x^2$. Both polynomials are in the exactness range. The generating function for $L_d^{(2)}$ gives $\int e^{-x}P(x)\,\mathrm{d}x = (d+1)/\binom{d+2}{2}$ and $\int xe^{-x}P(x)\,\mathrm{d}x = 1/\binom{d+2}{2}$ on $[0,\infty)$. Integrating the difference of the two polynomials against $x^2 e^{-x}\,\mathrm{d}x$ yields β_d. The next proposition quantifies the cost of measuring these smoothings separately.

Proposition A.2 (Errors in the additional real observations)**.** *Fix $d \ge 1$. For the above rule, let $G(x) = \sum_i a_i[-xe^{-t_i x} + q(t_i)x]$. Its first nonzero odd derivative at $c = 1/2$ has order $2d+3$. Suppose the observed value of each $B_X(t_i)$ has deterministic absolute error at most ϵ_i. The exact worst-case contribution of those errors to the reconstruction of κ_X is*

$$\frac{\sum_i |a_i|\epsilon_i}{|\beta_d|}. \tag{33}$$

For a common tolerance ϵ_B this is $A_d\epsilon_B$, where

$$A_d = \frac{\sum_i a_i}{|\beta_d|} > \frac{3(d+1)(d+2)}{4}.$$

If $\sum_i a_i B_X(t_i)$ is itself supplied with a single error bound ϵ_{sum}, its contribution is $\epsilon_{\mathrm{sum}}/|\beta_d|$.

Proof. Write $Q_d(F) = \sum_i v_i F(x_i)$ and $\mathrm{d}\mu = x^2 e^{-x}\,\mathrm{d}x$ on $[0,\infty)$. The odd derivatives through $2d+1$ vanish as above. For the next derivative,

$$G^{(2d+3)}(c) = 2^{2d+2} Q_d(x^{2d}(x - 2d - 3)).$$

Let $p(x) = \prod_i (x - x_i)$ be the monic Laguerre polynomial and $\sigma = \sum_i x_i > 0$. The polynomial $x^{2d}(x - 2d - 3) - (x - 2d - 3 + 2\sigma)p(x)^2$ has degree at most $2d - 1$, so quadrature integrates it exactly. The first summand has integral zero. Also $\int xp^2 \, \mathrm{d}\mu = (2d + 3) \int p^2 \, \mathrm{d}\mu$. Indeed, integrate the derivative of $x^3 e^{-x} p^2$ and use the orthogonality of p to $xp' - dp$. At the nodes p vanishes. It follows that $G^{(2d+3)}(c) = -2^{2d+3}\sigma \int p^2 \, \mathrm{d}\mu < 0$.

The identity

$$\beta_d \kappa_X = \sum_i a_i (B_X(t_i) - 1) - \sum_i a_i C_X(t_i)$$

gives (33) by the triangle inequality, with equality for errors whose weighted signs agree. All a_i are positive. For $x > 0$, the positive Taylor terms of e^x give $e^x/x^2 > \sum_{m=0}^{2d-1} x^m/(m + 2)!$. Applying Q_d and using its exactness and $\int x^m \, \mathrm{d}\mu = (m + 2)!$ yields $\sum_i a_i > 2d$. Division by $|\beta_d|$ proves the lower bound. □

For example, the same already-combined gain $9/4$ occurs with one or two unrestricted parameters. The individual gains are shown below.

d	Parameters t_i	Intrinsic error	Gain A_d per common tolerance
1	6	$O(N^{-5})$	$e^3/2$
2	$4, 12$	$O(N^{-7})$	$(27e^2 + e^6)/32$

For $d = 2$ the weights are $3e^2/8$ and $e^6/72$. The error in the finite real even-sector data is additional. At tolerance ϵ for each $Z_X(m)$, the full bound after $\limsup_{X\to\infty}$ is

$$O_d\Big((1 + N)^{-(2d+3)} + e^{2N}\epsilon\Big) + \frac{\sum_i |a_i|\epsilon_i}{|\beta_d|}.$$

Here d, N are fixed in the limit over X. The quadratic lower bound concerns the specified quadrature formula.

Corollary A.3 (Sharpness for one exact combined observation)**.** *Fix $d \geq 1$ and write $b(x) = \sum_i a_i(-xe^{-t_i x})$. Let D_a range over signed measures of mass zero on $[1/2, 1]$ with total variation at most one, and let $D_+ = (D_a + (R_a)_* D_a)/2$. Suppose $\langle D_a, b\rangle$ is given exactly and $\langle D_+, -xe^{-sx}\rangle$ is observed with absolute error at most ϵ at every real $s \geq 0$. The smallest worst-case error in estimating $\langle D_a, -x\rangle$ is of order $(\log(1/\epsilon))^{-(2d+3)}$ as $\epsilon \downarrow 0$, with constants depending on d.*

Proof. The finite estimator above applies to this class: unit total variation bounds the pairing error by the uniform approximation error on $[1/2, 1]$, and zero mass removes the constant term. Choose $N = \max(1, \lfloor \frac{1}{4} \log(1/\epsilon) \rfloor)$ in the full bound with the combined observation exact.

For the lower bound, put $r = 2d + 3$. Since $G = b + \beta_d x$, we have $b'(c) = -\beta_d \neq 0$. The preceding construction gives $\langle D_n, b\rangle \sim c_b n^{-1}$ and $\langle D_n, G\rangle \sim c_G n^{-r}$ with both constants nonzero, by the first nonzero odd derivative of each test. Set

$$E_n = D_n - \gamma_n D_{2n}, \qquad \gamma_n = \langle D_n, b\rangle / \langle D_{2n}, b\rangle \longrightarrow 2.$$

Then $\langle E_n, b\rangle = 0$ exactly. All its real even-sector observations are $O_d(e^{-c_1 n})$, whereas

$$-\beta_d \langle E_n, -x\rangle = \langle E_n, G\rangle \sim c_G(1 - 2^{1-r})n^{-r}.$$

The total variations of E_n are uniformly bounded. Rescale by a fixed constant to make them at most one and choose the least sufficiently large n with $C_d e^{-c_1 n} \leq \epsilon$. Zero data are then compatible with both the zero measure and the rescaled E_n, whose target values differ by $\gg_d (\log(1/\epsilon))^{-r}$. □

These witnesses keep one weighted sum fixed. Their d separate readings may differ, and their realisation by arithmetic divisor sums is unknown.

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