An isolated quartic correlation under signed prime dilations

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Abstract

Can a sign sequence reverse under prescribed prime dilations, cancel at every nonzero translate of a quartic correlation, and retain a bias at zero? For any prime set S containing $2, 3$ with $\sum_{p\in S} p^{-1} < \infty$, we construct such a sequence and prescribe its zero-shift mean on a set of positive density. Cancellation is uniform over displacements up to the dyadic interval length and phases of any fixed polynomial degree. If $\sum_{p\in S} p^{-1/4} < \infty$, the centred sum is $O(\sqrt{X \log X})$. We characterize which affine quadruples inherit the bias.

A three-adic quadratic division model has four-sign moment $1/6$ at a three-divisible origin. Its binary spectral density has a second-derivative increment of order $|t| \log(1/|t|)$ at zero. For each prescribed diverging lower cutoff, integer realizations combine the model word laws with local Gowers uniformity and signed dilation in density at every fixed prime. For the single sign, we identify the first order at which the cutoff must be chosen in advance. It is U^2 for the rotation law and U^3 for the quadratic law. For an explicit cutoff a moving prime band has positive dilation defect.

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1 An exceptional translate

Put $\lambda(n) = (-1)^{\Omega(n)}$, where primes are counted with multiplicity. The product $\lambda(n)\lambda(n+1)\lambda(3n+1)\lambda(3n+2)$ is also the product of the four consecutive signs starting at $3n$. This identity uses just the prime 3. Which extra information is supplied by other prime identities, by cancellation at pairs of sites, or by translating the last two sites together?

For a sign sequence x on the positive integers put

$$L(n) = (n, n+1, 3n+1, 3n+2), \qquad Q_h(n) = x(n)x(n+1)x(3n+1+h)x(3n+2+h).$$

Write $I_X = (X, 2X] \cap \mathbb{N}$ and $\mathrm{e}(t) = \exp(2\pi i t)$. Displacements h are integers, and all logarithms are natural. Every argument of Q_h is positive on I_X for $|h| \le X$. We measure a sum before dividing by X.

Theorem 1.1 (An isolated quartic mean)**.** *Let S be a set of primes containing $2, 3$, with $\sum_{p\in S} p^{-1} < \infty$. There is an explicit set $A \subset \mathbb{N}$ of natural density*

$$\delta_S = \frac{1}{132} \prod_{p\in S\setminus\{2,3,11\}} (1 - 4/p) > 0. \tag{1.1}$$

Fix any function $b : A \to [-1, 1]$ and extend it by zero on $\mathbb{N} \setminus A$. There exists one $x : \mathbb{N} \to \{-1, 1\}$, $x(1) = 1$, such that $x(pn) = -x(n)$ for every $p \in S$ and every n, and, for each fixed integer $k \ge 0$,

$$\max_{\substack{h\in\mathbb{Z}\\ |h|\le X}} \sup_{\deg P\le k} \left| \sum_{n\in I_X} (Q_h(n) - \mathbf{1}_{h=0} b(n))\mathrm{e}(P(n)) \right| = o(X). \tag{1.2}$$

The supremum is over real polynomials. The same estimate, with centre zero, holds for any two fixed nonproportional integral affine forms of positive slope and for every fixed odd list of such positive-slope forms, allowing repetitions in the odd list. For each fixed prime $p \notin S$, the identity $x(pn) = -x(n)$ fails on a set of natural density $1/2$.

If $\sum_{p\in S} p^{-1/4} < \infty$, all these discrepancy estimates are $O_{x,S,k}(\sqrt{X \log(2X)})$, with additional dependence on any fixed forms.

*Prepared with AI assistance. Details are given at the end of the article.

The data S and b are fixed before the realization is chosen. One realization then handles every fixed degree; the conclusion does not require a common realization for different prime sets. For constant $b = \tau$ on A, the ordinary and logarithmic means of Q_0 are $\tau\delta_S$. For $S = \{2, 3\}$, taking $\tau = 1$ gives $1/132$, and $\tau = 1/3$ gives $1/396$. For this constant choice, the contributions to these means from $n \equiv a$ and $n \equiv -1 - a \bmod q$ agree for every fixed q. This reflection statement concerns the masked means of Q_0, rather than all joint laws of the sequence. For finite S the support itself is periodic.

The construction starts with four fair signs whose product has the chosen expectation. Every three of them are independent. Section 2 embeds disjoint copies in the integers after the primes of S have been stripped away. For four distinct roots, Proposition 2.1 identifies every affine copy which inherits this expectation. Section 3 allows multiplicative weights and separate observations of the four rough cofactors. For finite S, Proposition 3.2 measures what a moving prime dilation loses while the other three signs are held in the product.

Relation to earlier constructions. Tao gave a compact affine model with signed dilation in a comment on Furstenberg limits of Liouville [17]. Sawin's polynomial division models already have, at degree two, independent triples, vanishing odd moments and a nonzero quartic moment [15, Theorem 3.1 and Remarks 3.3–3.4]. Section 4 recalls the rotation model as a reference and introduces a three-adic modification of the quadratic construction. Its specific feature is the spectral formula in Lemma 4.2 and the logarithmic increment in Corollary 4.3; the qualitative regularity and summability threshold also hold in the unmodulated model.

On the integers, McNamara constructed a sign sequence with signed dilation in density for every fixed prime and a persistent adjacent two-point correlation [6, Theorems 1.8 and 4.2]. Theorem 1.1 programs an isolated quartic mean under an exact, previously specified summable set S, together with all the displacement tests in (1.2). Theorem 5.1 combines a prescribed finite-word law with local Gowers estimates above a prescribed diverging cutoff. Its logarithmic clocks and transitions follow the same general construction principle as McNamara's example; the additional phase labels provide the local estimates. The correction making a chosen set of prime identities exact is a general density-zero modification, as the proof makes explicit. Proposition 5.4 locates the first obstructed order for the single sign in each word law. In the quadratic case the binary spectrum controls every growing interval length at order two, while an explicit quadratic phase detects the sign at order three. The dependence on the cutoff is relevant to Tao's formulation of local Gowers uniformity for Liouville [19, Conjecture 1.6 and Theorem 1.8].

If all prime identities held exactly at every integer, repeated division and $x(1) = 1$ would force $x = \lambda$. Proposition 6.1 quantifies a defect for the integer models on a band of primes selected at each large scale. Sections 7 and 8 give complementary comparisons, with their own hypotheses: a stationary eight-sign test and a coupling of two integers. The estimate for λ itself in *An explicit bound below one half for a quartic Liouville correlation* [7] uses arithmetic information beyond the hypotheses of Theorem 1.1.

2 How four labels acquire a common product

Write $\mathcal{D}_S$ for the positive integers supported on S, including 1. Factor each t uniquely as $d_S(t)u_S(t)$, with $d_S(t) \in \mathcal{D}_S$ and $u_S(t)$ free of primes in S. Once the labels $\xi_u \in \{-1, 1\}$ at the free integers are chosen,

$$x(t) = \lambda(d_S(t))\xi_{u_S(t)} \tag{2.1}$$

enforces every prescribed dilation. The difficulty is to use a label in just one programmed block while allowing all its multiplicative copies to appear in the sums.

2.1 Two residue classes

Use the classes $145, 118 \bmod 264$, and discard m if a prime of $S \setminus \{2, 3, 11\}$ divides $\prod_i L_i(m)$. The remaining positive integers form A; the table records their labels.

$m \bmod 264$	$(d_S(L_i(m)))_{i=1}^4$	$(u_S(L_i(m)) \bmod 11)_{i=1}^4$
145	$(1, 2, 4, 1)$	$(2, 7, 10, 8)$
118	$(2, 1, 1, 4)$	$(4, 9, 3, 1)$

For example $m = 145$ gives the arguments $145, 146, 436, 437$, with free parts $145, 73, 109, 437$ for $S = \{2, 3\}$. Reading a free part modulo 11 recovers both the row type and the site. Within that column the free part is an injective affine function of m. Hence $D_m = \{u_S(L_i(m)) : 1 \le i \le 4\}$, $m \in A$, are pairwise disjoint sets of four elements.

The involution $m \mapsto -1 - m$ exchanges the two classes and sends the four arguments to their negatives in reversed pairs. For $p \ge 5$, the four forbidden roots are distinct modulo p. Finite sieving and the Chinese remainder theorem give (1.1) for finite S. To pass to the limit, each L_i takes distinct positive values at most CX on $[1, 2X]$. Multiples of discarded primes $p > T$ therefore affect at most $4CX \sum_{p \in S,\ p > T} 1/p$ rows. The argument also applies after any fixed residue mask. This proves the density and reflection statements. If $\sum_{p \in S} 1/p$ diverges, the finite upper bounds give zero density for this particular support. For the list of all primes $p \le z$, its density is asymptotic to a positive constant times $(\log z)^{-4}$. Indeed $(1 - 4/p)/(1 - 1/p)^4 = 1 + O(p^{-2})$ and Mertens' product formula applies. The realization is chosen after the prime list has been specified.

For each $m \in A$, draw an independent sign B_m of mean $b(m)$. Assign three independent fair labels in D_m and determine the fourth by

$$\prod_{u \in D_m} \xi_u = B_m \prod_{i=1}^{4} \lambda(d_S(L_i(m))).$$

Choose all the remaining singleton labels independently and fairly. Each proper subset of a programmed block is fair and independent, even conditional on B_m. Independent random inputs in what follows are entire blocks or singletons. A simultaneous flip of all labels preserves their joint law. We apply (2.1) first, then multiply every $x(t)$ by $x(1)$. This last step preserves the even products and the absolute values of all the odd sums.

2.2 Recognizing other affine copies

Let $D(T) = \#(\mathcal{D}_S \cap [1, T])$. A positive rational number is an S-unit if its numerator and denominator are supported on S. Extend λ to positive rationals by complete multiplicativity.

Proposition 2.1 (Affine copies of the four sites)**.** *Fix four positive-slope integral affine forms F_i with distinct rational roots. Match their ordered roots to the ordered roots of L. If a positive affine map $m = \alpha n + \beta$ realizes this match, denote the resulting permutation by σ, and write*

$$F_i(n) = r_i L_{\sigma(i)}(\alpha n + \beta), \qquad r_i \in \mathbb{Q}_{>0}.$$

Put $c_F(n) = \lambda(\prod_i r_i) b(\alpha n + \beta)$ when the map exists, all r_i are S-units, and $\alpha n + \beta \in A \cap \mathbb{N}$. Put $c_F(n) = 0$ otherwise. The realization in Theorem 1.1 may be chosen so that, simultaneously for all such fixed quadruples, the product $\prod_i x(F_i(n))$ has centre $c_F(n)$ with the same phase discrepancy at each fixed degree.

Geometry of the mean. Take X large enough that all arguments are positive on I_X, and at most CX. Two distinct-root forms can have the same free part only at one row for each pair of

specified supported parts d_i, d_j. Remove these $O(D(CX)^2)$ rows. At every remaining row a nonzero moment requires four distinct labels making up one entire D_m.

Fix the class of m and a permutation of its sites. If s_j are the four supported parts in the corresponding table row, each match gives a line

$$\frac{F_i(n)}{d_i} = \frac{L_{\sigma(i)}(m)}{s_{\sigma(i)}}. \tag{2.2}$$

Two different lines in this list have at most one common point. Specifying their *two* site indices and their *two* supported parts already bounds the number of rows, independent of the choices at the other sites. Summing over these choices costs $O(D(CX)^2)$. If all four lines agree, their common positive-slope map sends the four roots to the four reference roots in order. There is at most one such map, and $r_i = d_i/s_{\sigma(i)}$ must be an S-unit. Conversely, when the conditions in the statement hold, $F_i(n)$ and $L_{\sigma(i)}(m)$ have the same free part and the quotient of their two signs is $\lambda(r_i)$. This gives precisely the stated centre. If only rows with $d_i \le M$ are retained, the same exceptional count is $O(D(M)^2)$. All implicit counting constants here can be taken absolute for four sites once C is fixed. □

For Q_h, the roots in the coordinate $3n$ are $0, -3, -h-1, -h-2$. The reference roots are consecutive. If the displaced roots are distinct and form an arithmetic progression, their common difference is a positive integer dividing 1. The distance between -3 and 0 then forces them to be the extremes. This gives $h = 0$, with the identity as matching map. At a repeated root the relation at 3 gives the following exact reductions.

h	-2	-1	1	2
$-Q_h(n)$	$x(n+1)x(3n-1)$	$x(n+1)x(3n+1)$	$x(n)x(3n+2)$	$x(n)x(3n+4)$

All reduced pairs have distinct roots. The collision bounds just proved are uniform in $|h| \le X$. In an odd product the mean before normalization is zero by global sign reversal, including when sites repeat. For $p \notin S$ the free parts at pn, n are pu, u, always distinct. Their moment is exactly zero. As an example of Proposition 2.1, the forms $L_i(2n+1)$ inherit $b(2n+1)$. Multiplying just the first form by a prime outside S removes that centre without changing any of the four roots.

2.3 One sequence for all tests

We finish the proofs of Theorem 1.1 and Proposition 2.1. One free label u can appear $O(D(CX/u))$ times in the forms under test, by injectivity. Changing one random input affects at most four labels. For a deterministic test bounded by one, the sum of the squared changes in either real component of the sum is at most a fixed multiple of

$$\sum_{u \le CX} D(CX/u)^2 \;\le\; CX \sum_{\substack{d,e \in \mathcal{D}_S \\ d,e \le CX}} \frac{1}{\max(d,e)} \;\le\; CX \left(\sum_{\substack{d \in \mathcal{D}_S \\ d \le CX}} d^{-1/2} \right)^2 . \tag{2.3}$$

When $\sum_{p \in S} p^{-1/4} < \infty$, the Euler products $Z_S(a) = \prod_{p \in S} (1 - p^{-a})^{-1}$ converge for $a = 1/4, 1/2$. Thus (2.3) is $O_S(X)$, and $D(CX) \le (CX)^{1/4} Z_S(1/4)$ bounds the total error in the pointwise expectations by $O_S(\sqrt{X})$.

McDiarmid's bounded-differences inequality [5] gives a tail $4\exp(-t^2/(C_S X))$ about the expectation, after adjusting the constant for the forms. Mesh the coefficient of n^j modulo one at spacing X^{-j-2} for $0 \le j \le k$. The sum changes by $O_k(1)$ on interpolating, and there are $X^{O_k(1)}$ choices including the displacement. Taking $t = C' \sqrt{X \log(2X)}$ with C' large enough makes the probabilities summable over integer X. Borel–Cantelli and a countable intersection

give the estimates on one realization. Passing to real X changes only $O(1)$ rows. The fixed-prime assertion follows by dyadic summation of $x(pn)x(n)$. Its failure indicator is $(1+x(pn)x(n))/2$.

When only $Z_S(1) < \infty$ is known, discard all rows with some $d_S(F_i(n)) > M$. With $R_S(M) = \sum_{d\in\mathcal{D}_S,\ d>M} 1/d$, their number is $O(XR_S(M))$, uniformly in the tested displacements. This uses $\lfloor CX/d\rfloor$ and the injectivity at each site. On retained rows (2.3) with $d, e \le M$ is $O_M(X)$, and the error in the expectations is $O(D(M)^2)$. For each fixed integer M the same mesh and concentration argument apply. First let $X\to\infty$, then $M\to\infty$, using $R_S(M)\to 0$. At the four repeated-root shifts one may truncate the reduced binary products. This proves (1.2) and the general affine assertion.

For finite S, write $s = |S|$. On a subinterval of I_X of length $H \ge 1$, the sum of the influences is $O(H+1)$, their maximum $O_S((\log(2X))^s)$. Including all subintervals in the union bound therefore gives discrepancy

$$O_{x,S,k}\big(\sqrt{H}\,(\log(2X))^{(s+1)/2} + (\log(2X))^{2s}\big). \tag{2.4}$$

The second term counts isolated rows. To obtain the fixed-class and logarithmic conclusions of Section 1, expand each fixed residue mask into linear phases, sum dyads down to a bounded endpoint, and use the density calculation for A. Partial summation of a prefix error $o(t)$ gives $o(1)$ after division by the harmonic mass of any window $(u,v]$ for which that mass tends to infinity.

3 Weights chosen after observing the signs

How much of an integer may a test retain before the preceding discrepancy argument ceases to control it? Let $R_y(m) = \prod_{p>y} p^{v_p(m)}$. Write $L_{i,h}$ for the actual four arguments at displacement h.

Proposition 3.1. *One can choose the same x in Theorem 1.1 so that, for each fixed k and every $Y_X\to\infty$,*

$$\sup\ \frac{1}{X}\left|\sum_{n\in I_X}(Q_h(n) - \mathbf{1}_{h=0}b(n))\mathrm{e}(P(n))\prod_{i=1}^{4} g_i(L_{i,h}(n))\,H_i(R_{y_i}(L_{i,h}(n)))\right| \ \longrightarrow\ 0.$$

The supremum permits $h\in\mathbb{Z}$, $|h|\le X$, $\deg P\le k$, $y_i\ge Y_X$, arbitrary unit-modulus completely multiplicative g_i, and arbitrary complex functions $|H_i|\le 1$. All tests may depend on X and on x. The four observations are separate functions of the four cofactors; no arbitrary joint function of all four arguments is asserted.

Proof. Fix a target accuracy $\varepsilon>0$. First truncate supported parts at M so that the deleted rows cost less than εX. For repeated-root shifts use the reduced pair, but keep the original four test arguments. The centring error on retained rows is $O(D(M)^2)$. Concentration at deviation εX costs probability at most $4\exp(-c_M\varepsilon^2 X)$ for a fixed test.

For arguments at most CX, one g_i is specified by $O(X/\log X)$ prime phases. A mesh of spacing $(\log X)^{-2}$ gives uniform $O(1/\log X)$ accuracy since $\Omega(t)\ll\log X$. The logarithm of the number of choices is $o(X)$. If $y_i\ge Y$, any function of R_{y_i} is a function of R_Y. Its possible inputs are Y-rough integers up to CX. For fixed Y their number is at most $CX\prod_{p\le Y}(1-1/p)+O_Y(1)$. A mesh of the closed unit disk at spacing ε now has logarithmic cost $O(X\prod_{p\le Y}(1-1/p)\log(2/\varepsilon))+O_{Y,\varepsilon}(1)$ for all four functions. Choose Y to make this less than $c_M\varepsilon^2X/2$ for large X. The polynomial and displacement meshes cost only $O_k(\log X)$. The remaining failure probabilities are summable, and interpolating the four factors costs $O(\varepsilon X)$. Intersect over a sequence $\varepsilon\downarrow 0$. The dependence on the four cofactors enters precisely in the count of separate function values. □

3.1 A prime step with the three other signs present

An estimate for $x(pn)x(n)$ measures how often signed division fails. To see whether the failure affects the chosen quartic, one can keep its other three signs in the test.

Proposition 3.2. *Let S be finite, $s = |S|$, and $b = \tau$ on A. In Theorem 1.1 the same x can be chosen so that, for every fixed $0 < \alpha < \beta < 1$, uniformly for primes $X^\alpha \le p \le X^\beta$, with $N = X/p$ and $B_p(n) = x(pn+1)x(3pn+1)x(3pn+2)$,*

$$\mathbb{E}_{n\in I_N} x(pn)B_p(n) = \tau\delta_S + o(1),$$
$$\mathbb{E}_{n\in I_N} x(n)B_p(n) = o(1), \qquad \mathbb{E}_{n\in I_N} x(pn)x(n) = o(1).$$

Thus the signed division error $x(pn) + x(n)$, multiplied by the three remaining signs, has mean $\tau\delta_S + o(1)$, while the unweighted failure probability tends to $1/2$.

Proof. The exact period of a residue description of A divides $W = 264\prod_{r\in S\setminus\{2,3,11\}} r$. For $p \nmid W$, pn visits the support on a fraction $\delta_S + O_S(1/N)$ of I_N. The first product is $Q_0(pn)$ and has the affine match $m = pn$ with all scale factors one. For the second, the four forms are $n, pn+1, 3pn+1, 3pn+2$. Their ordered roots give the same unique match. Its first scale factor is $1/p$, which is not an S-unit. Thus the second centre is zero. The two labels in the third product are always distinct. In all cases the exceptional-row proof of (2.2) costs $O_S((\log(2X))^{2s})$, uniformly in p, since every argument is $O(X)$. The sum of influences is $O(N+1)$ and the largest is $O_S((\log(2X))^s)$ by injectivity. Taking a union over primes and integer endpoints up to $2X+1$ in the bounded-differences argument gives error in the sums

$$O_{x,S}\big(\sqrt{N}\,(\log(2X))^{(s+1)/2} + (\log(2X))^{2s}\big).$$

It suffices to take successive powers of two as upper bounds on all arguments, so this also covers real X. Since $N \ge X^{1-\beta}$ and eventually $p \nmid W$, division by $|I_N|$ gives the assertion. □

4 When division changes the state

In Sawin's definition [15, Definition 1.1] a state has a sign F, a shift T and a Haar-distributed origin $J \in \widehat{\mathbb{Z}}$. On $d \mid J$ one may divide the state by d. The division map I_d sends the conditional probability to the original law. It divides the origin, intertwines T^d with T, composes under multiplication, and multiplies the sign by $\lambda(d)$. These identities are available for every integer d. The following examples show why one must specify both the law of the origin and the observable.

The sampling convention determines the value of a four-sign product. For comparison, the constructions below and the one in Section 2 give the following means.

Law	Four sites and origin	Mean
Prescribed prime set S	$L(n)$, ordinary origin	$\tau\delta_S$ for $b = \tau$ on A
Rotation	consecutive, any fixed origin class	$1/3$
Quadratic model	consecutive, $3 \mid J$	$1/6$
Quadratic model	consecutive, unrestricted J	$5/81$

In the last two rows only the sampling of the origin changes. Theorem 5.1 gives the corresponding integer realizations.

4.1 Four evenly spaced phases

The following is the division-map form of Tao's compact affine rotation model [17, comment of 5 March 2017, 6:00 pm]. We include its coordinates and moment calculation to fix the sampling convention. The same skew rotation and semicircle observables occur in the discussion of dynamical obstructions by Frantzikinakis and Host [20, §1.4].

Take independent Haar $J \in \widehat{\mathbb{Z}}$, $u \in \mathbb{T}$ and $\vartheta \in \widehat{\mathbb{Q}}$, the compact dual of discrete $\mathbb{Q}$. Let $s_0(t) = (-1)^{\lfloor 2\{t\} \rfloor}$, so that $F = s_0(u)$. Translation sends the coordinates to $(J+1, u+\vartheta(1), \vartheta)$. On $d \mid J$ put $I_d(J,u,\vartheta) = (J/d, u + \Omega(d)/2, d\vartheta)$. Substitution verifies the identities above. In particular $d\vartheta$ has Haar law, and the half-turn in u accounts for the sign.

There is a consistent family at affine forms $an+b$, $a \geq 1$, given by $Y_{a,b} = \lambda(a) s_0(u + \vartheta(b/a))$. Translation sends b to $b+a$, and simultaneous dilation of a, b by d multiplies the sign by $\lambda(d)$. Any two distinct rational offsets give independent fair signs: for $r \neq r'$, the map $(u,\vartheta) \mapsto (u+\vartheta(r), u+\vartheta(r'))$ has injective dual, since $a+b = ar+br' = 0$ forces $a = b = 0$. Odd products vanish by the half-turn. For L, the offsets are $0, 1, 1/3, 2/3$ and the two slope signs cancel. Thus the moment on every origin class is

$$\int_{\mathbb{T}^2} \prod_{j=0}^{3} s_0(u + jv)\, du\, dv = \frac{1}{3}. \tag{4.1}$$

Here $v = \vartheta(1/3)$ is Haar. To check the integral directly, set $a = 2u,\ t = 2v$. Translating a or t by one preserves the four-sign product. On $0 \leq a, t < 1$ it is $(-1)^{\sum_{j=1}^{3} \lfloor a + jt \rfloor}$. Split the a-integral at $\{-jt\}$, $j = 1,2,3$. Their order changes at $t = 1/3, 1/2, 2/3$. Integrating the resulting linear functions on the four intervals gives $1/9, 1/18, 1/18, 1/9$. The geometry and stability of these semicircle profiles are treated in *Sharp inequalities and stability for real profiles on arithmetic progressions* [13].

4.2 A quadratic phase with a three-adic valuation

Sawin's degree-two model already combines independent triples and a nonzero four-sign moment [15, Theorem 3.1 and Remarks 3.3–3.4]. Here the polynomial phase is rescaled by a valuation of the origin. This change retains the division identities and makes the binary spectrum depend explicitly on three-adic distances. The comparison after Corollary 4.3 isolates its effect.

Theorem 4.1. *There exists a dynamical division model with vanishing odd moments and fair independent triples at distinct sites conditional on the entire origin. Its four consecutive signs have moment* $1/6$ *conditional on* $3 \mid J$. *For every* $h \neq 0$ *the global spectral measure of* $D_h = F(F \circ T^h)$ *has a density of class* $C^{2,\gamma}(\mathbb{T})$ *for every* $0 < \gamma < 1$. *Its Fourier coefficients* $\rho_h(k) = \mathbb{E}[D_h(D_h \circ T^k)]$ *satisfy, for* $\sigma \geq 0$,

$$\sum_{k \in \mathbb{Z}} (1+|k|)^{\sigma} |\rho_h(k)| < \infty \qquad \Longleftrightarrow \qquad \sigma < 3. \tag{4.2}$$

Let $K = \widehat{\mathbb{Z}[1/3]}$. Write $\pi(A) = A(1)$ and $w(a/3^l) = a/2 \bmod 1$. This character is well-defined, $2w = 0$, and $3w = w$. On independent Haar coordinates $J \in \widehat{\mathbb{Z}}$, $A, B, C \in K$, use $s(t) = \operatorname{sgn} \cos(2\pi t)$, with an arbitrary sign at its two discontinuities. Put

$$T(J,A,B,C) = (J+1, A+B+C, B+2C, C),$$
$$F(J,A,B,C) = (-1)^{v_3(J)} s\big(\pi(3^{-v_3(J)} A)\big).$$

Valuations are finite almost surely at every integer translate. At time j, the first phase coordinate is $A + jB + j^2 C$. Translation preserves product Haar measure by triangularity. For $d = 3^e u$, $3 \nmid u$, define on $d \mid J$

$$I_d(J,A,B,C) = (J/d,\ 3^{-e}A + \Omega(u)w,\ d3^{-e}B,\ d^2 3^{-e} C).$$

The scalings on K are surjective by injectivity on its dual. Therefore I_d pushes normalized conditional Haar law forward to Haar. The identities $I_d T^d = T I_d$, $I_m I_n = I_{mn}$ hold on their domains by substitution. Division drops the valuation by e, and $3^{-j} w = w$, so $F(I_d x) = \lambda(d) F(x)$. Adding w to A reverses every sign. Evaluation of a quadratic at three distinct integers defines a surjection $K^3 \to K^3$, since its Vandermonde matrix is nonsingular over $\mathbb{Q}$ and hence injective on the dual. The three subsequent valuation scalings are automorphisms. This proves triple independence even with J fixed.

The moment at a multiple of three. Write $J = 3J'$, $a = v_3(J')$, $b = v_3(J'+1)$. The four valuations are $1+a, 0, 0, 1+b$. Haar integration at A, B, C leaves precisely the Fourier indices $t(-3^a, 1, -1, 3^b)$ by the third-difference relation. The nonzero coefficients of s are

$$\widehat{s}(t) = \frac{2\sin(\pi t/2)}{\pi t} \quad (t \text{ odd}), \qquad \widehat{s}(3^r t) = (-1)^r 3^{-r}\widehat{s}(t) \quad (r \geq 0).$$

The valuation signs cancel the Fourier signs. Since $\sum_t |\widehat{s}(t)|^4 = 1/3$ by the sum of reciprocal fourth powers over odd integers, the conditional moment at these valuations is $3^{-a-b}/3$. The event $a = b = 0$ has probability $1/3$, and $a + b = r \geq 1$ has probability $4/(3 \cdot 3^r)$. Averaging gives

$$\frac{1}{3}\left(\frac{1}{3} + \frac{4}{3}\sum_{r\geq 1} 9^{-r}\right) = \frac{1}{6}.$$

Throughout these computations the bounded Fejér means of s may be inserted first. Each marginal phase is Haar, so the product error is bounded by the sum of their L^1 errors. The one-dimensional series left after integration converges absolutely.

4.3 The binary spectrum

Lemma 4.2. *For $k \neq 0, \pm h$, set $q = \gcd(|k-h|, |k+h|)$, $a = (k-h)/q$, $b = (k+h)/q$, and $w_i = (a, -b, b, -a)_i$. For the sites $j_i = 0, h, k, k+h$, put $v_i = v_3(J + j_i)$ and $m = \min_i(v_i + v_3(|w_i|))$. Then*

$$\rho_h(k) = \frac{\mathbf{1}_{\{a,b \text{ odd}\}}}{3a^2b^2}\mathbb{E}_J\, 3^{4m - \sum_i v_i}. \tag{4.3}$$

In particular every coefficient is nonnegative.

Proof. The primitive integer relation annihilating $1, j, j^2$ on the four sites is w_i. After dividing the phases by 3^{v_i}, the primitive vector of Fourier indices is $z_i = 3^{v_i - m} w_i$. Indeed the only new common factor can be a power of three. A surviving term uses tz_i for a common t. If a or b is even there is no term with all four indices odd. Otherwise t must be odd. Multiplication or exact division of an odd argument by a power of three gives the preceding formula for $\widehat{s}$, also for negative exponents whenever its argument is integral. Since a, b occur twice up to sign and $\widehat{s}$ is even, multiplying the four coefficients and the clock signs gives $3^{4m - \sum_i v_i}|\widehat{s}(t)|^4/(a^2b^2)$. Summation gives (4.3). For the other lags, $\rho_h(0) = 1$ and $\rho_h(\pm h) = 0$ by pair independence. □

Proof of the spectral assertion. At least one of a, b is a three-adic unit. If a is a unit, $v_1, v_4 \geq m$ and hence $m \leq v_3(k+h)$. For the pair carrying b the corresponding upper bound, when b is a unit, is $v_3(k-h)$. In either case $\mathbb{P}(m \geq r) \leq 3^{-r}$ using just one of these two valuations. Consequently $4m - \sum_i v_i \leq 2m$, and for the applicable upper bound M_0, $\mathbb{E}3^{2m} \leq 1 + \sum_{r=1}^{M_0} 9^r 3^{-r} \ll 3^{M_0}$. Since $q \mid 2h$, for $|k| > 2|h|$ this implies

$$0 \leq \rho_h(k) \ll_h |k|^{-4}\big(3^{v_3(k+h)} + 3^{v_3(k-h)}\big).$$

On $K_0 < k \leq 2K_0$, $K_0 > 2|h|$, either valuation weight sums to $O_h(K_0 \log(2K_0))$, by counting multiples of 3^r up to $r \leq \log_3(3K_0)$. Summing dyadically proves convergence in (4.2) for $\sigma < 3$. Conversely, when k and h have opposite parity, a, b are odd. With probability at least $1 - 4/27$ all four valuations are at most two, and $3^{4m - \sum v_i} \geq 3^{-8}$ there. Hence $\rho_h(k) \gg_h |k|^{-4}$ on one parity class, so the series diverges at $\sigma = 3$ and above. Absolute Fourier convergence identifies the spectral density. Taking two derivatives in its series and using $|\mathrm{e}(kt) - 1| \ll_\gamma |kt|^\gamma$ proves the stated Hölder regularity. □

For instance, $\rho_1(2)$ is another four-consecutive-site moment. On $J = 0 \bmod 3$ it is $1/6$ as above. On each of the other two classes only one of the middle sites has a positive valuation v, and (4.3) gives $3^{-v}/27$. Since $\mathbb{E}(3^{-v} \mid v \geq 1) = 1/4$, the unconditional value is

$$\rho_1(2) = \frac{1}{3}\left(\frac{1}{6} + \frac{2}{108}\right) = \frac{5}{81}.$$

The distinction concerns the sampling of the first site even for the same consecutive word.

Corollary 4.3 (Behaviour at zero frequency)**.** *Let f_h be the spectral density of $F(F \circ T^h)$, $h \neq 0$, against normalized Haar measure on $\mathbb{T}$. As a real t tends to zero,*

$$f_h''(t) - f_h''(0) \asymp_h |t| \log(1/|t|).$$

In particular the second derivative is not Lipschitz at zero.

Proof. The upper bound from (4.3) already gives $\sum_{K<k\leq 2K} k^2 \rho_h(k) \ll_h K^{-1} \log K$. For a matching lower bound put $u = v_3(|h|)$ and restrict to k of parity opposite to h. For $r = v_3(k+h) \geq u+2$, one has $v_3(k-h) = u$, $v_3(a) = 0$, $v_3(b) = r-u$. Consider the event $v_3(J) = r-1$, of probability $2 \cdot 3^{-r}$. The four valuations are $r-1, u, u, r-1$, with $m = r-1$. Its contribution to $\mathbb{E}3^{4m-\sum v_i}$ is $2 \cdot 3^{r-2-2u}$. Thus $\rho_h(k) \gg_h K^{-4} 3^r$ in the dyad. For every $u+2 \leq r \leq \frac{1}{2}\log_3 K$, the specified parity and exact valuation hold for $\gg K/3^r$ integers there by complete periods modulo $2 \cdot 3^{r+1}$. Summation over these disjoint classes gives

$$\sum_{K<k\leq 2K} k^2 \rho_h(k) \asymp_h K^{-1} \log K.$$

By symmetry and positivity the difference of second derivatives is $8\pi^2 \sum_{k\geq 1} k^2 \rho_h(k)(1-\cos(2\pi k t))$. For the lower bound use $1/(4|t|) < k \leq 1/(2|t|)$. For the upper bound multiply each dyadic estimate by $O(\min(1, (K|t|)^2))$ and sum on both sides of $K = 1/|t|$. □

Comparison with the unmodulated quadratic model. Take independent Haar $A, B, C \in \mathbb{T}$ and $L_j^{(0)} = s(A + jB + j^2 C)$, the quadratic sign process underlying the polynomial model in [15]. With a, b, q as in Lemma 4.2, the same primitive Fourier relation, without the valuation scalings, gives

$$\rho_h^{(0)}(k) := \mathbb{E}[L_0^{(0)} L_h^{(0)} L_k^{(0)} L_{k+h}^{(0)}] = \frac{\mathbf{1}_{\{a,b \text{ odd}\}}}{3a^2b^2}, \qquad k \notin \{0, \pm h\}. \tag{4.4}$$

Indeed the surviving indices are $t(a, -b, b, -a)$, and for odd a, b their Fourier product sums to $(a^2b^2)^{-1}\sum_t |\widehat{s}(t)|^4$. The other values are $\rho_h^{(0)}(0) = 1$ and $\rho_h^{(0)}(\pm h) = 0$. In particular four consecutive signs have moment $1/27$.

Since $q \mid 2h$, these coefficients are $O_h(|k|^{-4})$, with a matching lower bound when k and h have opposite parity. Thus the baseline model already satisfies (4.2) and has a density $f_h^{(0)} \in C^{2,\gamma}$ for $0 < \gamma < 1$. Positivity gives

$$(f_h^{(0)})''(t) - (f_h^{(0)})''(0) = 8\pi^2 \sum_{k\geq 1} k^2 \rho_h^{(0)}(k)(1 - \cos(2\pi k t)) \asymp_h |t|.$$

For the upper bound split at $k = |t|^{-1}$; for the lower bound use the opposite-parity terms between $1/(4|t|)$ and $1/(2|t|)$. This calculation compares the models directly; no spectral assertion is being attributed to [15]. In (4.3), the additional valuation factor accumulates over logarithmically many classes and replaces this order $|t|$ by $|t|\log(1/|t|)$.

5 Realizing the two laws on the integers

In a state space the sign at a divided point is evaluated in a new state. On the integers, n and pn already belong to the same sequence. We now prescribe a lower cutoff on the lengths of local tests and ask for a sequence realizing each law.

McNamara's construction already gives signed dilation in density at every fixed prime and an adjacent binary mean tending to $-1/3$ [6, Theorems 1.8 and 4.2 and the end of §4]. Here the task is to retain an entire conditional word law while also controlling growing local intervals. The sequence is chosen after the lower cutoff has been prescribed.

For a finite interval I, extend $f\mathbf{1}_I$ by zero to a cyclic group of order $(2s+5)|I|$ after translating I to $[1,|I|]$. Define $\|f\|_{U^s(I)}$ as its cyclic Gowers norm divided by that of $\mathbf{1}_I$. In particular $\|f\|_{U^1(I)} = |\mathbb{E}_I f|$. The power 2^s of the normalized expression is the usual cube sum divided by the number of integer cubes in I.

Theorem 5.1. *Prescribe $h_*(X) \to \infty$ and a set S_0 of primes with $\sum_{p\in S_0} 1/p < \infty$. For either law in Section 4 there exist a sign sequence g and a set $S \supset S_0 \cup \{3\}$ with $\sum_{p\in S} 1/p = \infty$ having the following properties.*

1. *For $p \in S$ and all n, $g(pn) = -g(n)$. For every fixed prime the exceptional set has natural density zero. Moreover $g(1) = 1$ and $g(p^j) = (-1)^j$ at every prime power.*

2. *For every fixed list of translates and every fixed residue class of the origin, the finite-word law agrees with the chosen conditional law on ordinary prefixes and every logarithmic window of divergent harmonic mass.*

3. *For $G_B(n) = \prod_{b\in B} g(n+b)$, take B of odd size or any two distinct shifts in the quadratic model. In the rotation model take B of odd size. For all such fixed B and every fixed $s \geq 1$,*

$$\sup_{h_*(X)\leq H\leq X} \frac{1}{X} \sum_{X<x\leq 2X} \|G_B\|_{U^s((x,x+H])} \longrightarrow 0, \tag{5.1}$$

where x, H are integers and an empty range imposes no test.

In particular the integer quartic mean is $1/6$ in the quadratic realization and $1/3$ in the rotation realization.

In the second assertion, the origin class is fixed before the limit. On a class $a \bmod q_0$, counting measure on prefixes or harmonic measure on the window is restricted to that class and normalized to have mass one. The resulting law of $(g(n+b_1),\dots,g(n+b_r))$, for fixed integer shifts, converges to the law of $(F\circ T^{b_1},\dots,F\circ T^{b_r})$ conditional on $J \equiv a \bmod q_0$. Finitely many nonpositive arguments can be omitted. A logarithmic window $(u,v]$ is admissible when $\sum_{u<n\leq v} 1/n \to \infty$.

The order of choice is $\forall h_* \to \infty\ \exists g = g_{h_*}$. The initial summable set S_0 is prescribed, whereas the divergent set S is constructed. The quadratic phase below retains the four-point relation within bounded translates. To have small local norms one must also arrange that the different vertices of a much larger cube typically see different phases. We first construct a sequence G with the word laws, local norms and fixed-prime identities in density, then make a density-zero correction to impose the exact identities.

5.1 A finite list of requirements

At stage l, fix a target error $\varepsilon_l \downarrow 0$, finitely many shift and residue tests, finitely many pairs (B,s) admissible for the chosen law, and finitely many primes $\mathcal{P} \ni 3$. Exhaust the countable lists with l. Write $q = 2$ in the quadratic construction and $q = 1$ in the rotation construction. Let $s_q = s$ or s_0, respectively; here s_q denotes the sign profile, while the integer s indexes a Gowers norm.

First choose a valuation cutoff V and bounded Fejér approximations $s_{q,N}$ of degree N. They use only odd Fourier modes. Truncating higher valuations and replacing the profiles will incur errors made as small as needed for the finite list of tests. Next choose a large clock size m and real frequencies z_r, $r \in \mathcal{P}$, satisfying

$$\|z_r \log p - \mathbf{1}_{p=r}/m\|_{\mathbb{T}} < \delta, \qquad p \in \mathcal{P}.$$

Kronecker approximation applies since the logarithms of distinct primes are rationally independent, and permits all z_r to be arbitrarily large. Put $b_r(n) = \lfloor m\{z_r \log n\} \rfloor$. Given the vector b, the two choices of amplitude D, sign η, and valuation $v(n)$ are

q	$D(b)$	$\eta(b)$	$v(n)$
2	$3^{-b_3}\prod_{r\neq 3} r^{-2b_r}$	$(-1)^{\sum_{r\neq 3} b_r}$	$v_3(n)$
1	$\prod_r r^{-b_r}$	$(-1)^{\sum_r b_r}$	0.

Indices range over $\mathcal{P}$. Multiplying n by a tested prime normally increments just its clock, with no wraparound. Then $D(b)3^{-v(n)}n^q$ is preserved and $(-1)^{v(n)}\eta(b)$ reverses. To bound the exceptional fraction, a clock $z \log n \bmod 1$ lies in a union of k phase intervals of total length u on a fraction $O(u+1/z+k(1+z)/X)$ of a dyad. Integrate over successive periods in $\log n$, then round endpoints to integers. Thus at fixed stage the upper density of bad rows is $O_{\mathcal{P}}(m\delta + 1/m + \sum_r 1/z_r)$.

Include 3^{V+1} and every test modulus as divisors of Q. After discarding sites with valuation above V, put in each retained class modulo Q

$$P(t) = \sum_{j\in B} k_j 3^{-v_j}(t+j)^q, \qquad k_j \text{ odd}, \quad |k_j| \le N, \tag{5.2}$$

where $v_j = v(n+j)$ on that class. There are finitely many nonzero constant values $\xi = D(b)P$. Choose d large enough both for the finite phase tests below and for the repeated-label loss $O_s(1/d)$ in the local-norm argument. Choose real $\beta_0, \dots, \beta_{d-1}$ such that

$$\left| d^{-1} \sum_{a=0}^{d-1} \mathrm{e}(\beta_a \xi) \right| < \varepsilon_l$$

for each of them and $1, \beta_0, \dots, \beta_{d-1}$ are rationally independent. Independent uniform choices on an interval of integer length divisible by all denominators of these rational ξ give second moment $1/d$ for each mean. A union bound over the finite list and the almost-sure rational independence prove existence for arbitrarily large d.

All subsidiary errors are chosen to absorb the finite Fourier coefficient sums and the roots used to recover the norms. Thus the order is: the tests and their accuracy; V, N; the slow-clock data and Q; then d and the β_a. Only after these choices do we decide how far out to begin this stage. The leading coefficients to which Weyl's estimate will be applied then form a fixed finite set.

5.2 Joining bands of different frequencies

Use a positive $R_j \to \infty$ sufficiently slow that $R_{j+1} - R_j = o(1)$, $R_j \le \log\log(j + e^e)$, and eventually $R_j \le \sqrt{\log(\inf_{t\ge e^j} h_*(t))}$. A nondecreasing minorant with these properties exists by a diagonal choice. Put $A_1 = 1$, $A_{j+1} = A_j + 2R_j$. Give the band $A_j \le \log n < A_{j+1}$ a stage $l(j)$ tending to infinity sufficiently slowly. Its branch is defined also on the adjoining bands by

$$a_j(n) = \lfloor d\{t_j \log n\} \rfloor, \qquad G_j(n) = (-1)^{v(n)}\eta(b(n))\, s_q(\beta_{a_j(n)} D(b(n)) 3^{-v(n)} n^q).$$

Take $t_j \in [e^{A_j - 2R_j}, 2e^{A_j - 2R_j}]$, $u_j \in [R_j, e^{R_j}]$, with $d\|t_j \log p\| + \|u_j \log p\|$ smaller than the stage tolerance on $\mathcal{P}$. At a fixed stage, returns to a neighbourhood of zero are relatively dense by compactness of the torus orbit closure. On the first half of the band use G_j when

$$\{u_j \log n\} < (\log n - A_j)/R_j$$

and G_{j-1} otherwise. On the second half use G_j. This defines G, with arbitrary values on a finite initial segment. The linear transition gives the same dominant branch on the two sides of each joining point.

For large X, $[X, 3X + O(1)]$ meets at most two bands. Apart from the indices a_i, the branch and its slow coordinates change $O(e^{R_j})$ times there. Indeed the selector has $O(1+u_j)$ boundaries and each slow clock $O(m(1 + z_r))$, absorbed by postponing the stage. For a branch in use the derivative $\gamma = t_i/n$ satisfies $ce^{-5R_j} \le \gamma \le Ce^{-2R_j}$. The older branch occurs only on the first half of the next band. A tested prime step changes a_i on a fraction $O(d\|t_i \log p\| + 1/t_i + d(1+t_i)/X)$. The branch selector changes on a fraction $O(\|u_j \log p\| + (\log p)/R_j + 1/u_j + u_j/X)$. Both estimates follow by period counting. At a join extend the ramp by its endpoint value 0 or 1. A step crossing it has the same dominant branch, with exceptional phases in an interval of width $O((\log p)/R_j)$. Together with the slow-clock estimate this gives the prime identities in density for G.

5.3 Recovering the finite-word laws

Partition each dyad into blocks of length $B_j = \lfloor e^{R_j} \rfloor$. Blocks meeting a branch or slow-coordinate change cost $O(e^{R_j} B_j)$ rows. Those on which a tested site crosses a fast boundary have proportion $O(dB_j(1+t_i)/X) = o(1)$. On other blocks restrict to classes modulo Q. The cost of the valuation truncation is $O_B(3^{-V} + B_j^{-1})$. For nonconstant P in (5.2) the polynomial $\beta_a D(b)P$ has irrational leading coefficient. Weyl equidistribution on each progression is uniform in its initial point. For degree two, differencing removes the lower coefficients from the upper bound, and for degree one this is the geometric sum bound. Applying the same argument to a single site and continuous majorants of $|s_q - s_{q,N}|$ bounds the mean approximation error by $\|s_q - s_{q,N}\|_1 + o(1)$.

For nonzero constant P, average over a whole compartment of fixed branch and slow coordinates. A bounded periodic function of $t_i \log n$ differs from its phase mean by $O(X/t_i)$ after integration on one compartment, by integration by parts with a bounded periodic primitive. On a fixed residue class the rounding error is $O_Q(1)$ per discontinuity. Across the dyad there are $O(e^{R_j})$ compartments but only $O(d(1 + t_i))$ rapid boundaries per active branch. Both total errors are $o(X)$. We may thus use the chosen d-point mean. Finally $P \equiv 0$ imposes exactly Haar orthogonality on the coefficients of a degree-q polynomial. For $q = 2$ these are the equations at A, B, C conditional on the valuations. For $q = 1$ they are the equations at two independent Haar phases $u, \vartheta(1)$. In an odd product the leading equation fails by parity after clearing powers of three, and in an even product the common η disappears. This proves the finite-word limits on dyads, including dyads cut short at either end with error $o(X)$. Dyadic summation and partial summation give the prefix and logarithmic assertions.

5.4 Local boxes and separation of the phase labels

The word limits use fixed shifts. The larger cubes in (5.1) require a second averaging scale. The following form of Gowers–Cauchy–Schwarz reduces that task to boxes much shorter than the tested interval.

Lemma 5.2 (Local boxes control a global norm)**.** *Let Γ be a finite abelian group, $s \ge 1$, and μ a probability distribution on Γ. Put $F_t(v_1, \dots, v_s) = f(t + v_1 + \cdots + v_s)$. Then*

$$\|f\|_{U^s(\Gamma)}^{2^s} \le \mathbb{E}_{t\in\Gamma} \|F_t\|_{\square(\mu^s)}^{2^s}. \tag{5.3}$$

The box seminorm is defined by

$$\|F\|_{\square(\mu^s)}^{2^s} = \mathbb{E}_{v_i^{(0)}, v_i^{(1)} \sim \mu} \prod_{\omega\in\{0,1\}^s} \mathcal{C}^{|\omega|} F(v_1^{(\omega_1)}, \dots, v_s^{(\omega_s)}),$$

with all draws independent and $\mathcal{C}$ denoting complex conjugation.

Proof. Let $x_i^{(0)}, x_i^{(1)}$ be independent uniform elements of Γ. The global cube average equals

$$\mathbb{E}_x \prod_{\omega\in\{0,1\}^s} \mathcal{C}^{|\omega|} f\left(\sum_i x_i^{(\omega_i)}\right).$$

Replacing each $x_i^{(e)}$ by $x_i^{(e)} + v_i^{(e)}$, with independent $v_i^{(e)}$ of law μ, preserves the joint uniform law. For fixed x, put $t_\omega = \sum_i x_i^{(\omega_i)}$. Gowers–Cauchy–Schwarz in the v-variables bounds the absolute value of the inner mixed average by $\prod_\omega \|F_{t_\omega}\|_{\square(\mu^s)}$. Average in x and apply Hölder with exponent 2^s. Each t_ω is uniform in Γ, so the product of the resulting identical factors is the right side of (5.3). □

For a dyad in the j-th band put $B_j = \lfloor e^{R_j} \rfloor$ as above and $M_j = \lfloor e^{6R_j} \rfloor$. The scale choices give, uniformly for every branch in use,

$$\frac{B_j}{M_j} = O(e^{-5R_j}), \qquad \gamma B_j + \frac{1}{M_j\gamma} = O(e^{-R_j}), \tag{5.4}$$

$$\frac{e^{R_j}(M_j+1)}{h_*(X)} \ll e^{7R_j - R_j^2} \longrightarrow 0, \qquad \frac{M_j^2}{X} \ll e^{12R_j - A_j + O(1)} \longrightarrow 0. \tag{5.5}$$

The first two bounds use $ce^{-5R_j} \le \gamma \le Ce^{-2R_j}$. For the third, $X \ge e^j$ eventually and $\log h_*(X) \ge R_j^2$; for the fourth, $A_j \ge j$ eventually and $R_j \le \log\log(j + e^e)$. Adjacent bands change the bounds by harmless fixed factors, since $R_{j+1} - R_j = o(1)$. These four margins pay for incomplete boxes, changes and coincidences of fast labels, slow boundaries uniformly in $H \ge h_*(X)$, and the Taylor error, respectively.

Apply Lemma 5.2 to the zero extension on the cyclic group for $(x, x+H]$, with μ uniform on $[1, M_j]$. The normalization by $\|\mathbf{1}_{(x,x+H]}\|_{U^s}$ costs only a constant depending on s: the interval has a fixed proportion of the ambient group and contains $\asymp_s H^{s+1}$ integer cubes. Except for $O_s(e^{R_j}(M_j+1)) = o(H)$ starting points, each box lies wholly outside the support or in one slow compartment, including the tested shifts. Write $M = M_j$. At vertices $z_\omega = t + \sum_i v_i^{(\omega_i)}$, Taylor's formula gives

$$t_i \log z_\omega = t_i \log t + \gamma \sum_i v_i^{(\omega_i)} + O_s(M^2/X)$$

on the active branch, with $\gamma = t_i/t$. Divide the $2s$ independent variables into blocks of length B_j, discarding proportions $O_s(B_j/M)$. The discrepancy of M successive steps of size γ is $O(\gamma + 1/(M\gamma))$, by grouping full turns. Thus the fraction of small boxes where a vertex or one of its sites crosses a fast boundary is

$$O_{s,B}\big(d(\gamma B_j + M^2/X + \gamma + 1/(M\gamma))\big) = o(1)$$

at a fixed stage. For two distinct vertices one variable occurs at just one of them. With the others fixed, equality of their labels has probability at most $1/d + O_s(\gamma + 1/(M\gamma) + M^2/X)$. The loss from repeated labels is $O_s(1/d) + o(1)$.

On retained boxes the labels at different vertices are distinct and constant. Truncation, fixed residue classes and Fejér approximation apply in any variable of a vertex as for finite words. Each permitted vertex polynomial P_ω has positive degree. For an odd list, clearing powers of three in the leading coefficient leaves a sum of an odd number of odd integers. For a pair at distinct shifts b_1, b_2 in the quadratic model, simultaneous cancellation of the quadratic and linear coefficients would give

$$k_1 3^{-v_1} + k_2 3^{-v_2} = 0, \qquad b_1 k_1 3^{-v_1} + b_2 k_2 3^{-v_2} = 0,$$

which forces both modes to vanish, contrary to their being odd.

For completeness, write the total Fourier phase on such a box as

$$\Phi(v) = D \sum_{\omega\in\{0,1\}^s} (-1)^{|\omega|}\beta_{a_\omega} P_\omega\left(t + \sum_i v_i^{(\omega_i)}\right), \qquad P_\omega \in \mathbb{Q}[t].$$

Let $r \geq 1$ be the largest degree among these polynomials, let $c_{\omega,r}$ be their degree-r coefficients, and choose ω_* with $c_{\omega_*,r} \neq 0$. For any coordinate i, set $e = \omega_{*,i}$. The coefficient of $(v_i^{(e)})^r$ in the entire phase is

$$D \sum_{\substack{\omega\in\{0,1\}^s \\ \omega_i = e}} (-1)^{|\omega|}\beta_{a_\omega} c_{\omega,r}. \tag{5.6}$$

It is a rational linear combination of distinct β_a, with a nonzero coefficient at a_{ω_*}. Rational independence of $1, \beta_0, \ldots, \beta_{d-1}$ makes it irrational. It is independent of all the other variables; passing to a progression modulo Q only multiplies it by Q^r. Weyl's estimate on a progression of length at least $B_j/Q - 1$ is consequently uniform in all lower coefficients and in the other variables.

There are only finitely many coefficients (5.6) at a fixed stage. After its data have been chosen, postpone that stage until its Weyl bounds, clock returns, boundary estimates and (5.4)–(5.5) all hold. Selecting $l(j) \to \infty$ only beyond these thresholds implements the diagonal choice. The repeated-label loss is absorbed by the choice of d, and the remaining errors by the subsidiary accuracies. This proves (5.1) for G, at this point uniformly even in x.

5.5 Stability under a density-zero modification

The correction of the prime identities will preserve the averaged local norms, even though a density-zero exceptional set need not be small in every short interval.

Lemma 5.3. *Let $|f|, |f'| \leq 1$ agree outside a set $E \subset \mathbb{N}$ of natural density zero. For every fixed $s \geq 1$,*

$$\sup_{\substack{H\in\mathbb{N} \\ 1\leq H\leq X}} \frac{1}{X} \sum_{X<x\leq 2X} \Big| \|f\|_{U^s((x,x+H])} - \|f'\|_{U^s((x,x+H])} \Big| \longrightarrow 0.$$

The same holds for any fixed products of translates of the two sequences.

Proof. Put $p = 2^s$. Telescoping the normalized cube products on an interval I of length H gives

$$\Big| \|f\|_{U^s(I)}^p - \|f'\|_{U^s(I)}^p \Big| \leq C_s \frac{|E \cap I|}{H}.$$

For any chosen vertex and its fixed value there are $O_s(H^s)$ possible cubes, whereas the normalization counts $\asymp_s H^{s+1}$ cubes. This proves the bound for all $H \geq 1$, including $s = 1$. Since $|a - b|^p \leq |a^p - b^p|$ for nonnegative a, b, take a p-th root and average in x. Each integer belongs to at most H of the intervals in question, so

$$\frac{1}{XH} \sum_{X<x\leq 2X} |E \cap (x, x + H]| \leq \frac{|E \cap (X, 3X]|}{X} = o(1), \qquad 1 \leq H \leq X. \tag{5.7}$$

Hölder proves the assertion uniformly in H. For a fixed product of translates, replace E by the finite union of the corresponding translates of E, which also has density zero. □

5.6 The exact primes

For any prime set E put $G_E(n) = \lambda(d_E(n))G(n/d_E(n))$. This enforces $G_E(pn) = -G_E(n)$ for $p \in E$. If $\sum_{p\in E} 1/p < \infty$, it differs from G on a density-zero set. To see this quantitatively, let

$\mathcal{D}_E$ be the supported semigroup and put

$$R_E(M) = \sum_{\substack{d \in \mathcal{D}_E \\ d > M}} \frac{1}{d} \longrightarrow 0, \qquad \sum_{d \in \mathcal{D}_E} \frac{1}{d} = \prod_{p \in E} (1 - 1/p)^{-1} < \infty.$$

Rows with $d_E(n) > M$ number at most $XR_E(M)$ up to X. For each of the finitely many remaining supported parts, iteration of the fixed-prime identities in density gives $G(du) = \lambda(d)G(u)$ outside a density-zero set. Thus

$$\limsup_{X \to \infty} \frac{1}{X} \#\{n \le X : G_E(n) \ne G(n)\} \le R_E(M).$$

Let $M \to \infty$. This correction uses only the fixed-prime identities in density and applies equally to McNamara's sequence [6]; here it also preserves the prescribed word laws and, by Lemma 5.3, the local estimates.

Outside $S_0 \cup \{3\}$, reserve a set of primes of divergent reciprocal mass whose complement there also has divergent mass. Start with $S_0 \cup \{3\}$ and add finite packets $\mathcal{P}_l$ from this complement. Before packet l, choose N_l beyond every earlier packet and so large that the previous correction differs from G on at most $1/l$ of every prefix of length at least N_l. Select $\mathcal{P}_l$ above N_l with reciprocal mass between $1/l$ and $2/l$; the next threshold is beyond the whole packet. For $N_l \le X < N_{l+1}$, future packets affect no integer up to X, and only multiples of primes in $\mathcal{P}_l$ can add to the previous error. Therefore

$$\frac{1}{X} \#\{n \le X : G_S(n) \ne G(n)\} \le \frac{1}{l} + \sum_{p \in \mathcal{P}_l} \frac{1}{p} \le \frac{3}{l}. \tag{5.8}$$

The union S has divergent reciprocal mass, while (5.8) gives the density-zero modification.

Replace $G_S(n)$ by $\lambda(n)$ wherever $n/d_S(n)$ equals one or a prime power. This condition is invariant under multiplication by an S-prime. Such n have at most one distinct prime divisor from the reserved set. For a finite reserved subset of reciprocal mass $H_0 > 1$, the count of its prime divisors has mean H_0 and variance at most H_0 on uniform residues, by the Chinese remainder theorem. Chebyshev bounds the upper density of having at most one such divisor by $H_0/(H_0 - 1)^2$. Letting $H_0 \to \infty$ proves that this modification also has density zero.

The resulting g has the required exact identities and prime-power values. A density-zero change preserves fixed-prime identities in density, since its pullback by $n \mapsto pn$ still has density zero for fixed p. It also preserves the word laws in each fixed residue class. Lemma 5.3 preserves (5.1), with exactly its stated average over origins. Finally $Q_0(n) = \prod_{j=0}^{3} g(3n + j)$ by exact division at 3, so the word law sampled on $3 \mid J$ gives the two quartic means. This completes the proof of Theorem 5.1.

More generally, the fixed affine family in the rotation realization has the law of $Y_{a,b}$ on every fixed origin class. Take D a common multiple of the slopes. Outside a density-zero set one may replace $g(a_i n + b_i)$ by $\lambda(D/a_i)g(Dn + Db_i/a_i)$. The origin Dn lies in a fixed progression where the linear phase pair remains Haar. Since $\vartheta(1/D)$ is Haar, the offset phases have the asserted joint law, and the common factor $\lambda(D)$ is absorbed by a half-turn in u.

5.7 Which local norms need a prescribed cutoff?

Consider a single sign, without any product of translates. For a bounded sequence g write

$$A_s(g; X, H) = \frac{1}{X} \sum_{X < x \le 2X} \|g\|_{U^s((x, x+H])}.$$

We take X, H to be positive integers. A finite-word law prescribes this average at each *fixed* length in the limit $X \to \infty$. We ask which estimates at growing lengths it forces on every

realization. The detection of a degree-q phase by U^{q+1} is classical [18, §§1.5–1.6]. Here the quadratic phase is rescaled differently on each valuation stratum, and the lengths available to an integer observer grow with X.

Proposition 5.4 (The first order requiring a cutoff)**.** *Suppose that the ordinary prefix frequencies of every finite word of $g : \mathbb{N} \to \{-1, 1\}$ agree in the limit with the unconditioned word law of one of the models in Section 4. Put $q = 1$ for the rotation and $q = 2$ for the three-adic quadratic model. Then for $1 \le s \le q$,*

$$\lim_{K\to\infty} \limsup_{X\to\infty} \sup_{K\le H\le X} A_s(g; X, H) = 0. \tag{5.9}$$

For each $s > q$ there is a length $H(X) \to \infty$, with $H(X) = o(X)$, such that

$$\liminf_{X\to\infty} A_s(g; X, H(X)) \;\ge\; \frac{c_q}{2s+5}, \qquad c_1 = \frac{2}{\pi}, \qquad c_2 = \frac{3}{2\pi}.$$

The constants c_q are the magnitudes of the phase correlations computed below. The displayed lower bound for the norm uses the interval normalization of Section 5.

Thus in the quadratic model the single-sign U^2 estimate holds for every diverging lower cutoff on the same realization. At U^3 the order of choice in Theorem 5.1 is necessary. In the rotation model this change already occurs at U^2.

Proof. From fixed words to all long intervals. Use Lemma 5.2 with an independent law μ_i in each coordinate. Its proof is unchanged because translating each uniform variable by an independent increment still gives uniform variables, and box Cauchy–Schwarz holds for a product of probability spaces. Take μ_i uniform on $[1, m_i]$. Set $p_s = 2^s$, $M = \sum_i m_i$, and

$$W_{\boldsymbol{m}}(t) = \mathbb{E}_{v_i^{(0)}, v_i^{(1)} \in [1,m_i]} \prod_{\omega\in\{0,1\}^s} g\left(t + \sum_i v_i^{(\omega_i)}\right).$$

This real number belongs to $[0, 1]$, as it is a box norm to the power p_s. For each interval $(x, x+H]$, apply the lemma to the zero extension. A box lies outside the interval, lies inside it, or meets an endpoint. The last possibility costs $O_s(M/H)$, including wraparound in the cyclic group. In the interior the box average is $W_{\boldsymbol{m}}(t)$. The cyclic norm of the interval indicator is at least its mean $1/(2s + 5)$. Averaging over x and counting each interior origin at most H times gives, for $H \ge 2M$,

$$\frac{1}{X} \sum_{X<x\le 2X} \|g\|^{p_s}_{U^s((x,x+H])} \ll_s \frac{M}{H} + \frac{1}{X} \sum_{X<t\le 3X} W_{\boldsymbol{m}}(t). \tag{5.10}$$

For fixed $\boldsymbol{m}$, the last average tends to $2\,\mathbb{E}_\mu W_{\boldsymbol{m}}(0)$, by subtraction of two prefix averages. Here μ is the model word law. It suffices, by Hölder's inequality, to make this expectation arbitrarily small before letting K tend to infinity.

At $s = 1$ the expectation is $1/m_1$ in both models, by pairwise independence. At $s = 2$ in the quadratic model write $m_1 = D,\ m_2 = L$. Let $w_m(d) = (m - |d|)_+/m^2$, the difference law of two uniform draws on $[1, m]$. Stationarity gives the identity

$$\mathbb{E}_\mu W_{D,L}(0) = \sum_{d,k\in\mathbb{Z}} w_D(d) w_L(k) \rho_d(k).$$

For $d = 0$ put $\rho_0(k) = 1$. For each fixed $d \ne 0$, Theorem 4.1 gives $\sum_k |\rho_d(k)| < \infty$, so its weighted mean in k is bounded in absolute value by $L^{-1} \sum_k |\rho_d(k)|$. First let $L \to \infty$ at fixed D. Only $w_D(0) = 1/D$ remains. Now take D large. This use of two side lengths requires no spectral estimate uniform in d and proves (5.9).

A polynomial phase on the model word. Write $F_j = F \circ T^j$. In the rotation model take $P(j) = u + j\vartheta(1) \bmod 1$. For almost every state the slope is irrational. Consequently $H^{-1}\sum_{j=1}^{H} F_j \mathrm{e}(-P(j))$ tends to the first Fourier coefficient of s_0, of magnitude $2/\pi$.

In the quadratic model take $P(j) = \pi(A + jB + j^2C) \bmod 1$ and $P_v(j) = \pi(3^{-v}(A + jB + j^2C)) \bmod 1$. Thus $3^v P_v = P$ on the circle. Almost surely $\pi(C)$ is irrational, so every P_v is equidistributed on each fixed arithmetic progression by Weyl's theorem [18, §1.1]. For fixed J, the stratum $v_3(J + j) = v$ is the union of two classes modulo 3^{v+1}. Its contribution to the limiting correlation is

$$\frac{2}{3^{v+1}}(-1)^v \int_{\mathbb{T}} s(t)\mathrm{e}(-3^v t)\, dt = \frac{2}{3^{v+1}}\, 3^{-v}\widehat{s}(1).$$

The omitted strata $v \geq V$ have upper density 3^{-V}. The semicircle has only finitely many discontinuities, so ordinary equidistribution suffices on each retained class. Since $\widehat{s}(1) = 2/\pi$, the full correlation exists and equals

$$\lim_{H\to\infty} \frac{1}{H}\sum_{j=1}^{H} F_j \mathrm{e}(-P(j)) = \widehat{s}(1)\sum_{v\geq 0} \frac{2}{3^{v+1}} 3^{-v} = \frac{3}{2\pi}. \tag{5.11}$$

All valuation strata therefore contribute with the same sign.

For $s > q$, twisting by a polynomial phase of degree q preserves the normalized interval U^s norm. Indeed its alternating difference on each integer s-cube is zero, and the zero extension counts exactly the cubes contained in the interval. Apply $|\mathbb{E}_\Gamma f| \leq \|f\|_{U^s(\Gamma)}$ to the twisted zero extension and use $\|\mathbf{1}_I\|_{U^s(\Gamma)} \leq 1$. On any model word of length H this gives

$$\|F_\bullet\|_{U^s([1,H])} \geq (2s+5)^{-1}\left|H^{-1}\sum_{j=1}^{H} F_j \mathrm{e}(-P(j))\right|.$$

Fatou's lemma gives the claimed lower bound in the limit inferior of the expected model norm. For each fixed H, ordinary word convergence identifies the latter expectation with $\lim_{X\to\infty} A_s(g; X, H)$. Choose increasing $X_j \geq j^2$ so that this approximation at length j has error at most $1/j$ for every $X \geq X_j$. Taking $H(X) = j$ on $[X_j, X_{j+1})$ proves the assertion. □

In the rotation case the first obstructed order also has an exact fourth moment. At every fixed H, the limiting spatial average of $\|g\|^4_{U^2((x,x+H])}$ is the integral of $\|j \mapsto s_0(u + jv)\|^4_{U^2([1,H])}$ over $\mathbb{T}^2$. As $H \to \infty$ this tends to $1/3$. For irrational v, the three phases of the integer cube parameters j, h_1, h_2 equidistribute on the region where the four vertices belong to $[1, H]$. Subdivide its dilate into boxes of side ϵH, use geometric sums in each coordinate, and let $\epsilon \downarrow 0$. The boundary and the sign discontinuities have measure zero. The Haar cube integral is $\sum_k |\widehat{s_0}(k)|^4 = 1/3$.

6 A defect on a moving band of primes

For g in Theorem 5.1, each fixed prime has an exceptional set of density zero. We can measure what is lost on taking this limit before choosing the prime. Write

$$H_X = \sum_{n\leq X} 1/n, \qquad \langle f\rangle_X = H_X^{-1}\sum_{n\leq X} f(n)/n, \qquad \delta_p(X) = \langle \mathbf{1}_{\{g(pn)\neq -g(n)\}}\rangle_X.$$

Proposition 6.1. *Use $h_*(X) = \sqrt{\log\log(10 + X)}$ in Theorem 5.1. Let $c = 1/6$ for the quadratic realization and $c = 1/3$ for the rotation realization. For every $0 < \beta \leq 10^{-3}$ and $\varepsilon > 0$ there is $0 < \alpha < \beta$ such that every sufficiently large X admits a dyadic P with*

$$(\log X)^\alpha \leq P \leq (\log X)^\beta, \qquad \mathbb{E}_{P<p\leq 2P}\delta_p(X) \geq c/8 - \varepsilon. \tag{6.1}$$

Here and below the average on a prime band is uniform over its primes.

The numerical lower bounds are $1/48-\varepsilon$ and $1/24-\varepsilon$, respectively. The band $P=P(X)$ may change with X. This is compatible with $\delta_p(X)\to 0$ for each fixed prime: the proposition takes the prime from a growing range after the scale has been chosen.

Proof. Put $\ell=\log X$, $H=\lfloor\log(2+\ell)\rfloor$, and define

$$T(p)=\left\langle \prod_{j=0}^{3} g(3n+jp)\right\rangle_X, \qquad T^*(p)=\left\langle \prod_{j=0}^{3} g(p(3n+j))\right\rangle_X .$$

Then $T(1)\to c$. The two prime steps are compared by entropy, while the mean of $T(p)$ itself is controlled by the local fourth norm.

Removing the divisibility weight. For $0<\eta<1/4$, we claim

$$\mathbb{E}_{2^m<p\le 2^{m+1}}|T^*(p)-T(p)|\ll\eta \tag{6.2}$$

for $P=2^m\le \ell^{1/1000}$, apart from indices of total reciprocal sum $O_\eta(1)$. This is the entropy-decrement step in the proof of Tao and Teräväinen [16, Theorem 3.1 and Propositions 4.2–4.3]. We detail the adaptation, since the given sequence has a separate dilation error.

Substituting $t=pn$ shows that the difference is the average of $(p\mathbf{1}_{p|t}-1)\prod_{j=0}^{3} g(3t+jp)$ up to $O((1+\log p)/\ell)$, by moving the endpoints of the harmonic sum. Discard $t\le X^\eta$ at cost $O(\eta)$, also for $p\mathbf{1}_{p|t}$ by summing over multiples of p. Use the normalized harmonic law on the remaining interval. Translating t by $i=1,\dots,P$ changes its law in total variation by $O_\eta(PX^{-\eta})$, which absorbs even the factor p. Average over these translations. Let W_m be the word of $C2^m$ successive triples $(g(3t+j),0\le j<3)$, starting at t, where a fixed integer $C>12$ covers the offsets and translations. Let Y_m be the residues of t at primes in the band and $Y_{<m}$ those at earlier dyadic bands.

The conditional mutual informations satisfy

$$\sum_{1\le m,\ 2^m\le \ell^{1/1000}} 2^{-m} I(W_m;Y_m\mid Y_{<m})=O(1).$$

Indeed $H(W_{m+1}\mid Y_{\le m})\le 2H(W_m\mid Y_{\le m})+o(1)$ by splitting the word in half. Translation acts on residues by a permutation. The total variation cost $O_\eta(PX^{-\eta})$ changes the entropies by $o(1)$, since the logarithms of the word and residue alphabet sizes are $O(P)$; these errors remain negligible after summing over the $O(\log\log X)$ bands. More explicitly, if $a_m=2^{-m}H(W_m\mid Y_{<m})$, then

$$a_{m+1}\le a_m-2^{-m}I(W_m;Y_m\mid Y_{<m})+o(1).$$

Telescope and use $a_m\ge 0$, starting from $H(W_1)=O(1)$.

Fix the word, and temporarily sample Y_m independently and uniformly at its primes. In the translated fluctuation, the summand for p has mean zero and absolute value bounded by 3, since among P consecutive translates there is at most one multiple of p. Different prime summands are independent. Multiply by deterministic signs selecting the absolute values in (6.2). Hoeffding's inequality gives probability $O(\exp(-c_0\eta^2P/m))$ that their mean exceeds η in absolute value. For the actual joint law versus the law retaining $(W_m,Y_{<m})$ with independent uniform Y_m, the relative entropy is $I(W_m;Y_m\mid Y_{<m})+o(1)$. In fact the product of all primes up to $2P$ is $\exp(O(P))$, and the harmonic law on $t>X^\eta$ has residue distribution within $O_\eta(\exp(O(P))X^{-\eta})$ of uniform, by summation on complete periods. Entropy continuity on this alphabet bounds the residue contribution by $o(1)$. The inequality $\Pr(E)\log(1/\Pr_0(E))\le D(\Pr\|\Pr_0)+\log 2$ transfers the tail. For large m with $I\le c_1\eta^3P/m$ the transferred expectation is $O(\eta)$ by boundedness. The remaining m have $\sum 1/m=O_\eta(1)$ by the telescoping estimate, as claimed.

Removing the prime step. We next show $\mathbb{E}_{P<p\le 2P}T(p)=o(1)$ uniformly for $\ell^\alpha\le P\le \ell^\beta$ when $\alpha,\beta>0$ are fixed in the stated range. Average $n\mapsto n+t$ on $1\le t\le P$, at cost $O((1+\log P)/\ell)$. The prime weight can now be replaced by the average over reduced classes $p=Wu+r$, where $W=\prod_{p\le w}p$, with uniform counting measure in u. The error has limsup tending to zero as $w\to\infty$. More explicitly, pass to the von Mangoldt weight by the prime number theorem, $\log p=\log P+O(1)$, and the negligible prime powers. Split into classes and subtract one from the normalized weight $(\varphi(W)/W)\Lambda(Wu+r)$. Apply [16, Lemma 5.2] in the shift t and the step u to the four functions with directions $3t+jWu$, $0\le j\le 3$. Its constant is uniform in W,r and in the shifts of the functions. One may use the difference of two prefix averages for the prime interval and take the translation range of length W times the step range in each prefix. All such translations have the same negligible harmonic cost. Lemma 5.3 there bounds the resulting Gowers norm of the weight in the successive limits $P\to\infty,\ w\to\infty$.

At fixed W divide the two ranges for t,u into intervals of length $b=\lfloor H/(20(W+1))\rfloor$. The omitted boundary strips cost $O_W(H/P)$. In each full box the four arguments are $m+a_j+3t'+jWu'$ with $m=3n$, $a_j=O(P)$, and $1\le t',u'\le b$. They lie in intervals $(m+a_j,m+a_j+H]$. Extend each site function by zero there and embed its translated interval in a prime cyclic group of size between $100H$ and $210H$. Encode the restriction on t' in the factor with $j=0$, using the inverse of 3 in the group, and take the indicator of u' as a fifth function. All five coefficient vectors are pairwise nonproportional for sufficiently large H at fixed W. Four applications of Cauchy–Schwarz, along directions annihilating the other four linear forms, bound the normalized box average by $O_W(\|g\|_{U^4((m+a_j,m+a_j+H])})$ for any $j\neq 0$. The size ratio between box and cyclic group is bounded at fixed W. Counting cubes in the interval gives the same normalized interval norm for either ambient group.

The positive harmonic mean at $m=3n$ costs at most $3+o(1)$ times that over all origins up to $3X$. Translating by $a_j=O(P)$ costs $O((1+\log P)/\ell)$, uniformly. On dyads $(Y,2Y]$ above H, we have $h_*(Y)\le H\le Y$ for $Y\le 4X$ and large X. Equation (5.1) for $B=\{0\}$ and $s=4$ therefore controls all such averages, uniformly once Y is large. The smaller dyads cost $O(\log(2H)/\ell)$. This proves the estimate for the prime step.

Recovering the numerical defect. For each p the difference $|T(1)-T^*(p)|$ is at most twice the sum of the four harmonic probabilities of a dilation error at $3n+j$. For $j=0,1,2$, grouping residue classes gives total $3\delta_p(3X)+O(1/\ell)$, since $1/n=3/(3n+j)+O(n^{-2})$. For $j=3$, exact division at three gives $\delta_p(X)+O(1/\ell)$ by shifting $n+1$ to n. Also $\delta_p(3X)=\delta_p(X)+O(1/\ell)$, uniformly in p. Thus the cost is at most $8\delta_p(X)+O(1/\ell)$. Fix η for the desired ε, then choose $\alpha<\beta$ so that the sum of $1/m$ over $\ell^\alpha\le 2^m\le\ell^\beta$ exceeds the exceptional bound in (6.2). Combining both prime-step estimates on any remaining band proves (6.1). □

There is a further restriction on combining exact dilation with agreement on integers of prescribed factor count. It has a direct proof independent of the phase construction.

Corollary 6.2. *Suppose a sign sequence g has exact signed dilation at every prime in S, with $\sum_{p\in S}1/p=\infty$. If $g(m)\neq\lambda(m)$ and $r=\Omega(m)$, then*

$$\sum_{\substack{\Omega(n)=r+1\\ g(n)\neq\lambda(n)}}\frac{1}{n}=\infty.$$

Consequently, agreement with λ outside a set of finite reciprocal mass in each *fixed factor-count class forces $g=\lambda$. For either realization in Proposition 6.1 at least one such class has divergent reciprocal error.*

Proof. For every $p\in S$, both $g(pm)$ and $\lambda(pm)$ reverse their respective signs at m. The distinct integers pm all lie in the class $r+1$ and have reciprocal sum $m^{-1}\sum_{p\in S}1/p$. Finally (6.1) with $\varepsilon<c/8$ forces at least one erroneous prime identity for either realization, hence at least one disagreement with λ. □

7 Seven independent signs and an eight-point rectangle

For the adjacent product of a stationary sign process L, two tests look similar. One tests its Fourier sum at a preassigned frequency. The other takes a product of four adjacent pairs whose centres form an additive rectangle. A rectangle with disjoint edges uses eight signs. The following example separates the two tests even when every seven signs have exactly the fair product law.

Finite-order independence within stationary processes is classical; see, for example, Bradley and Pruss [21]. The point here is the simultaneous behaviour of the genuine adjacent product, its maximum over frequencies, and its rectangle under every diffuse sampling law. The process is nonergodic, as explained below the proof.

Proposition 7.1. *For each $0 < d < 1/2$ there is a stationary sign process $(L_j)_{j\in\mathbb{Z}}$ whose distinct-site marginals up to order seven are fair independent and whose odd moments vanish. Write $A_j = L_jL_{j+1}$ and $M_H = \sup_{\theta\in\mathbb{T}} |H^{-1}\sum_{j=1}^{H} A_j\mathrm{e}(j\theta)|$. Then*

$$\mathbb{E}\left|H^{-1}\sum_{j=1}^{H} A_j\mathrm{e}(j\theta)\right|^2 = 1/H \quad (\theta \text{ fixed}), \qquad \lim_{H\to\infty}\mathbb{E}M_H^2 = \frac{4d^2}{\pi^2}.$$

For four independent integer draws p, p', q, q' of common law ν, independent of the process,

$$\mathbb{E}[A_{p+q}A_{p+q'}A_{p'+q}A_{p'+q'}] \longrightarrow d^4/3 \qquad \text{as} \quad \sup_{n\in\mathbb{Z}}\nu(n) \to 0. \tag{7.1}$$

Proof. Put $v = d/(1-d)$. Let D_j be a stationary Markov chain on $\{0,1\}$, with transitions $\mathbb{P}(1 \mid 0) = v$, $\mathbb{P}(1 \mid 1) = 0$. The occupied edges $(j, j+1)$, $D_j = 1$, form a matching of density d. With independent fair signs ξ_j, and independent Haar $\alpha, \phi \in \mathbb{T}$, all independent of D, set

$$L_j = \begin{cases} \xi_j, & D_{j-1} = 0, \\ \xi_{j-1}s(\phi + (j-1)\alpha), & D_{j-1} = 1. \end{cases}$$

The shift sends ϕ to $\phi + \alpha$, giving stationarity. Integration over ξ kills a product at distinct vertices unless they form a union of completed matching edges. This kills all odd products. With at most seven vertices a surviving union consists of one, two or three edges. A half-turn of ϕ kills an odd number of edge signs. At two distinct edges the phases are independent Haar by the rank-two evaluation map on (ϕ, α). This proves fair independence by expansion of cylinder indicators. At eight consecutive sites the only completed matching begins at $0, 2, 4, 6$ after translation. It has probability dv^3, and the four phases form an arithmetic progression with Haar step 2α. Thus $\mathbb{E}\prod_{j=0}^{7} L_j = dv^3/3$ by (4.1). For $j \neq k$, $\mathbb{E}A_jA_k = 0$, including overlapping pairs, so the fixed-frequency formula follows.

Conditional on α, ϕ, the mean of A_j is $d\,s(\phi + j\alpha)$. Fix an irrational α. Rotation by α times the mixing chain and the independent sign shift is ergodic. The remainder $E_j = A_j - d\,s(\phi + j\alpha)$ has covariance $O(v^{|k|})$ for $|k| \geq 3$ on this component. Indeed only simultaneous occupation of both separated edges survives integration over ξ, and $\mathbb{P}(D_0 = D_k = 1) - d^2 = d(1-d)(-v)^{|k|}$. Van der Corput's inequality, followed by the ergodic theorem at finitely many lags, gives almost surely

$$\limsup_{H}\sup_{\theta}\left|H^{-1}\sum_{j\leq H,\ j\geq 1} E_j\mathrm{e}(j\theta)\right|^2 \ll K^{-1} + K^{-1}\sum_{k=1}^{K}|\mathbb{E}(E_0E_k \mid \alpha)|.$$

Let $K \to \infty$. This is the elementary summable-covariance case of the uniform Wiener–Wintner theorem [1, Theorem 21.8]. For the rotation part, a fixed trigonometric polynomial has limiting maximal coefficient equal to its largest Fourier amplitude. The frequencies $k\alpha$ in its finite expansion are distinct, so at any θ all but at most one geometric sum are uniformly $o(1)$. Approximate s by bounded Fejér polynomials and use equidistribution to control the L^1 error on

the orbit. The limiting maximum is $\max_k |\widehat{s}(k)| = 2/\pi$. This gives the asserted limit by bounded convergence.

For (7.1), two of the four centres lie within distance K with probability $O((K+1)\sup\nu)$, by fixing all but one of the differing draws. Away from this event, for $K \geq 3$, integration over ξ requires all four edges to be occupied. Their probability is $d^4 + O(v^K)$, by the second eigenvalue $-v$ of the chain. Approximate the four remaining s's by bounded Fejér polynomials of degree N, with error at most $4\|s - s_N\|_1$. For modes a, b, c, e in rectangle order, Haar integration in α requires

$$(a+b)p + (c+e)p' + (a+c)q + (b+e)q' = 0.$$

If any coefficient is nonzero the probability is at most $\sup\nu$. The identically zero choices are $(k, -k, -k, k)$, which also survive integration in ϕ. First send $\sup\nu \to 0$ at fixed N, K, then send $N, K \to \infty$. The resulting Fourier sum is $\sum_k |\widehat{s}(k)|^4 = 1/3$. □

Why the process is nonergodic. The frequency α is invariant in the construction, and the conditional mean $d\,s(\phi + j\alpha)$ retains its rotation frequencies. Averaging over α cancels the off-diagonal covariances. Nonergodicity also holds for the observed sign process itself: if it were ergodic, van der Corput's inequality and Birkhoff's theorem would give, for every K,

$$\limsup_{H\to\infty} M_H^2 \ll \frac{1}{K} + \frac{1}{K}\sum_{k=1}^{K} |\mathbb{E}(A_0 A_k)| = \frac{1}{K} \quad \text{almost surely.}$$

Letting $K \to \infty$ and using bounded convergence would contradict the positive limit in Proposition 7.1. The separation between the two Fourier tests therefore relies on averaging over components; it is not a separation within an ergodic law with the same zero covariances.

The periodic matching in *Bernoulli components and single-pair tests in logarithmic Liouville systems* [9, §6.2] also has seven-site independence and fixed-frequency second moment $1/H$. Four independent uniform primes in a growing dyad give even pair origins and a limiting rectangle $1/6$. For fair independent signs the same rectangle is exactly $2/m - 1/m^2$, where m is the number of primes in the dyad. Only a repeated row or column contributes. These are unconditioned translate rectangles of stationary laws. In Proposition 7.1, mixing of the occupied edges gives $d^4/3$ for any four independent draws with common law ν and $\|\nu\|_\infty \to 0$. The spectral maximum also depends explicitly on d. The transfer to an eight-sign mean conditioned on actual prime factors is a separate arithmetic result in [9, Proposition 6.2].

8 An affine identity and a coupling at displacement two

Two direct comparisons locate the arithmetic content of a proposed transfer. First, $G(n) = \lambda(2n+1)$ satisfies $G(3n+1) = -G(n)$ at every integer since $2(3n+1)+1 = 3(2n+1)$. Yet

$$\lim_{X\to\infty} \|G\|_{U^s([1,X])} = 0 \qquad (s \geq 1).$$

This uniformity is a consequence of the known theory. The prime number theorem in progressions gives aperiodicity of λ (also by $\lambda(n) = \sum_{d^2|n} \mu(n/d^2)$ and truncation). Apply *Higher order Fourier analysis of multiplicative functions and applications* [2, Theorem 2.5]. Equivalently one may transfer the nilsequence orthogonality in *The Möbius function is strongly orthogonal to nilsequences* [3] through the same square-divisor identity and use *An inverse theorem for the Gowers $U^{s+1}[N]$-norm* [4].

For the progression restriction, work in a sufficiently large even cyclic group with the interval zero extension of λ up to $2X+1$. Multiplying by $\mathbf{1}_{\text{odd}} = (1-(-1)^n)/2$ preserves the small upper bound for $s \geq 2$, by the triangle inequality and invariance under multiplication by a linear character. Integer cubes in its odd support correspond, by $n = 2m+1$, to the cubes for

the reindexed sequence, with a bounded normalization factor depending only on s. The single endpoint at $m = 0$ is negligible. The $s = 1$ case also follows from the prime number theorem in progressions. Thus a phase cancellation estimate on a single G of the kind discussed in *Polynomial phases in ordinary two-point correlations of multiplicative functions* [11] can coexist with perfect signed dependence between $G(n)$ and $G(3n + 1)$.

Proposition 8.1. *Given any $H(X) \to \infty$, for every sufficiently large integer X there exist exactly uniform marginals U on I_X and V on $I_X + 2$, coupled so that*

$$|V - U - 2| \le H(X), \qquad V - U \equiv 2 \bmod Q(X), \qquad \mathbb{P}(U \text{ and } V \text{ both prime}) = 0.$$

Every fixed positive integer divides $Q(X)$ for all sufficiently large X. One may choose $V - 2$ as a permutation of U.

Proof. Take $L = \lfloor \min(H(X)/3, \sqrt{X}) \rfloor$ and partition I_X into consecutive intervals of lengths between L and $2L$, merging a short last remainder. Fix Q and split each interval by its classes modulo Q. In every resulting cell C, the proportions of primes among $u \in C$ and among $u + 2$ for $u \in C$ both tend to zero uniformly in the cell. To see this, choose a finite set of sieving primes coprime to Q. The fraction avoiding them all is at most $\prod(1 - 1/p) + O_{Q,\{p\}}(1/L)$ by complete periods. Actual primes coinciding with sieving primes contribute at most $O_{Q,\{p\}}(1/L)$. The product can be made arbitrarily small, also after translation by two.

The sum of the two prime counts is thus eventually at most $|C|$. Map the prime inputs injectively to positions $w \in C$ for which $w + 2$ is composite, then extend to a permutation of the cell. The union of these permutations gives σ with all the required properties for $V = \sigma(U) + 2$. A diagonal choice of factorial moduli growing sufficiently slowly gives $Q(X)$. □

Consequently these couplings reproduce the exact-shift law for every fixed periodic test of the two coordinates, eventually. They can suppress all simultaneous prime outcomes even at a prescribed arbitrarily slow growing displacement tolerance. Thus fixed congruence tests and this approximation to displacement two do not by themselves preserve simultaneous primality. The exact integer relation is one way to retain the divisor weights in *Critical divisor weights and the localisation of parity defects* [10] and the common observations in *Visible prime factors under neighbouring critical divisor weights* [14]. A different coupling could suffice if it directly controlled the arithmetic events or weights being transferred; Proposition 8.1 only identifies the insufficiency of the tests it preserves.

The hypotheses also distinguish the present constructions from *Prime noise, polynomial observations and trace-preserving completion of multiplicative functions* [12], where one preserves traces of genuinely completely multiplicative functions, and from *Averaged Vaughan decompositions for a quartic Liouville correlation* [8], where the remaining sums carry specific arithmetic weights. Here the isolated shift, the measured prime defects and the eight-sign rectangle identify separate pieces of information that an estimate for the Liouville quartic must recover.

Properties belonging to each construction. The following table records which conclusions hold for the same object. In the first row $b = \tau$ on A; the integer laws in the third row are chosen separately for the two models and for each cutoff h_*.

Construction	Dilation or sampling information	Retained tests and bias
Prescribed S, §§2–3	Exact signed dilation at the prescribed summable S; fixed-prime defect $1/2$ outside S	Moving displaced quartics and fixed-degree phases; the sole programmed centre is $\tau\delta_S$ at $h = 0$
Division models, Section 4	Conditional state division for every integer	Rotation moment $1/3$; quadratic moment $1/6$ on $3 \mid J$, $5/81$ without conditioning
Integer realizations, §§5–6	Every fixed prime in density; exact identities on a constructed divergent S	Prescribed word laws and local norms above h_*. For the single sign the first order that needs this choice is U^2 for rotation and U^3 for the quadratic law. Quartic means $1/3$ and $1/6$, respectively
Stationary matching, Section 7	Nonergodic stationary law; no prime-division assertion	Seven-site independence; rectangle $d^4/3$ for diffuse draws and positive Fourier maximum
Integer coupling, Section 8	Uniform marginals, fixed congruences and a diverging displacement tolerance	Every fixed periodic test of the pair is retained; simultaneous primality has probability zero

Assistance. OpenAI's ChatGPT assisted with the exploration and exposition of these constructions. OpenAI's Codex assisted with the extension to summable prime sets, the spectral calculation, the comparison of references, and the preparation and execution of the finite checks supplied with the sources. ChatGPT and Codex also assisted with the local-norm proofs, the calculation of the quadratic phase correlation and the comparison of the two matching models. The source archive includes the check programs and their outputs; these finite calculations complement the written proofs and do not constitute a formal verification.

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