Local Compensation of Vaughan Remainders with Prescribed Divisors

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Abstract

Prescribe odd coprime divisors $a \asymp X^{1/4}$, $b \asymp X^{3/5}$ in $X < am = bk - 2 \le 2X$, and give each prime k weight $\log k$. We ask how much of the signed remainder in Vaughan's identity cancels within each pair (a, b). At the common cutoff $z = X^{1/8}(\log X)^{-C}$, three disjoint conditions on the divisors of m give respectively four, minus four and six times the local singular main term. The errors, summed in absolute value over the divisor pairs, save a power of $\log X$. The third contribution covers asymptotically all the negative mass on the rough integers when its positive net value is capped at the available negative mass in each cell and the resulting amounts are summed. After all three families are removed, the mass of each sign still exceeds any fixed power of $\log\log X$ times the local main term in a set of cells of singular-series measure tending to one. The proof uses a type I estimate up to $X^{1/4}$ for an additional divisor and classical estimates for the support and second moment of a truncated Möbius sum. The prime-weighted results also hold uniformly for $am + bk = h$ at even centres $3X \le h \le 4X$.

1 What cancels inside a divisor pair?

The factorization $am + 2 = bk$ places two integers at distance two. If the divisors a, b are prescribed, then m, k run along a single progression. Vaughan's identity writes the prime weight at m as two calculable sums and a remainder with two large divisors. Can one exhibit an appreciable part of this remainder that cancels *before* different divisor pairs are added together? A prime factor just below the cutoff produces a positive net contribution. Splitting a divisor of the same size into two primes reverses its sign, and the two amounts can be matched in the same cells. A third family can pay for the entire negative mass on rough integers. The signs that remain after these three families have been removed still have large total variation.

Write $n \sim Y$ for $Y < n \le 2Y$, and put

$$L = \log X, \quad A = X^{1/4}, \quad B = X^{3/5}, \quad M = X/A, \quad K = X/B, \quad U = K/A = X^{3/20}.$$

A cell is a pair $a \sim A, b \sim B$ of odd coprime integers. We use either

$$X < am \le 2X, \qquad bk = \begin{cases} am + 2 & (h = 2), \\ h - am & (3X \le h \le 4X, \quad 2 \mid h, \quad (ab, h) = 1). \end{cases} \tag{1.1}$$

For either choice let $\mathcal{C}$ be the set of admissible cells and let $\mathcal{B}_{ab}$ consist of the positive solutions (m, k) in the specified cell. We measure the error of a cell array by $\|D\|_1 = \sum_{(a,b)\in\mathcal{C}} |D_{ab}|$. The single-prime and two-prime references are

$$H_{ab} = \frac{X}{b\varphi(a)}, \qquad V_{ab} = \frac{2C_2X}{ab} \prod_{\substack{p \mid abh \\ p>2}} \frac{p-1}{p-2}, \qquad C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

Set $\mathcal{V} = \sum_{\mathcal{C}} V_{ab}$ and $\nu(a,b) = V_{ab}/\mathcal{V}$. We take primes p without multiplicity unless explicitly indicated. The functions $\mu, \Lambda, \lambda = (-1)^{\Omega}$ have their usual meanings, and $\vartheta(k) = \mathbf{1}_{\{k \text{ prime}\}} \log k$. Define

$$R_z(m) = \sum_{\substack{r\ell t = m \\ r>z,\ \ell>z}} \mu(r)\Lambda(\ell), \qquad S(E;F)_{ab} = \sum_{\mathcal{B}_{ab}} E(m) R_z(m) F(k).$$

For an event E we also use its name for its indicator. The variable d in the distribution estimate below will be a divisor of m, additional to the prescribed divisor a of am.

Theorem 1.1 (Matching two signed families)**.** *Write $\delta = 1 - (1 + \log\log 2)/\log 2$, $\rho = \delta/2$. Fix $\gamma, \eta > 0$ with $\gamma + \eta < \rho$, and $0 < \alpha < \beta < 1/2$. Set $T = \exp(L^{\gamma})$, $z = X^{1/8} L^{-C}$. For every sufficiently large fixed C, let E_1 be the event that m has a prime divisor in $(z/T, z]$. Let E_2 be the event that E_1 fails and $pq \mid m$ for primes satisfying*

$$z^{\alpha} < p \le z^{\beta}, \qquad z/T < pq \le z.$$

Set $\kappa = \log\{\beta(1-\alpha)/[\alpha(1-\beta)]\}$. For $F = \vartheta$ in either orientation of (1.1), and for $F = \lambda, \mu$ at shift two, write $\iota_F = \mathbf{1}_{\{F=\vartheta\}}$. Then

$$\|S(E_1;F) - 4\iota_F V\|_1 + \|S(E_2;F) + 4\kappa\iota_F V\|_1 \ll XL^{-\eta}. \tag{1.2}$$

Both estimates also hold with the common restriction $w < P^-(m) \le \sqrt{z}$, where P^- is the least prime factor, $w = (\log L)^j$ and $j > 0$ is fixed before C. For $E_> = \{\exists\, p \mid m :\ z < p \le zT\}$,

$$\sum_{\mathcal{C}} \sum_{\mathcal{B}_{ab}} E_>(m) |R_z(m)| W(k) \ll XL^{-\eta}, \tag{1.3}$$

where $W = \vartheta$ in both orientations or $W = 1$ at shift two. All estimates for reflection are uniform in h. Constants may depend on the fixed parameters.

Taking $\alpha = 1/5$, $\beta = \mathrm{e}/(4+\mathrm{e})$ makes $\kappa = 1$. Thus the two signed sums cancel with error $O(XL^{-\eta})$ in the cell norm. Multiplying each cell by an arbitrary scalar of modulus at most one preserves this conclusion. In (1.3) one can even insert a multiplier depending on the individual point. The difference matters if nearby Liouville signs are observed.

For the third family define $\xi \in (1/200, 1/20)$ by

$$\int_{1/200}^{\xi} \int_{3/50}^{3/25} \frac{\mathrm{d}v\,\mathrm{d}u}{uv(1-u-v)} = \frac{3}{2}.$$

The integral increases continuously from zero to a value exceeding $3/2$. Within $w < P^-(m) \le \sqrt{z}$ and outside $E_1 \cup E_2$, let E_3 require a divisor $pqr \in (z/T, z]$ of m, with p, q, r prime, $z^{1/200} < p \le z^{\xi}$ and $z^{3/50} < q \le z^{3/25}$. Corollary 6.2 evaluates this third contribution.

Theorem 1.2 (Variation after removal of three families)**.** *Fix $\gamma, \eta > 0$ with $\gamma + \eta < \rho$, the matching values of α, β, and $j > 0$. For every fixed $C \ge C_0(\gamma, \eta, j)$, put*

$$G = \{w < P^-(m) \le \sqrt{z}\} \setminus (E_1 \cup E_2 \cup E_3), \qquad B_{ab}^{\pm} = \sum_{\mathcal{B}_{ab}} \mathbf{1}_G(m) (\pm R_z(m))_+ \vartheta(k).$$

For every fixed $s, D > 0$, in either orientation,

$$\nu\Big\{(a,b) : \min(B_{ab}^{+}, B_{ab}^{-}) > DV_{ab}(\log\log X)^s\Big\} \ \longrightarrow\ 1. \tag{1.4}$$

Convergence is uniform in the reflected centre. The same C works for all s, D.

For example $\gamma = 3/100$, $\eta = 1/100$ are admissible in both theorems, since $\rho = 0.04303566\ldots$. Here γ determines the width $\log T$ of the divisor band, and η measures the saving in the sum of cell errors. The auxiliary number of marked factors is chosen after s and is fixed as $X \to \infty$. The measure in (1.4) gives each cell weight $V_{ab}/\mathcal{V}$.

A short example explains how a divisor mark can distinguish signs. At $z = 35$ take a prime $Q > 35$. In $m = 5 \cdot 11 \cdot Q$ the squarefree divisors made of small primes and at most 35 are $1, 5, 11$, whose Möbius sum is -1. In $m' = 3 \cdot 5 \cdot 7 \cdot Q$ the list includes every pair of small primes, giving $1 - 3 + 3 = 1$. The remainders are $+\log Q$ and $-\log Q$, respectively, by (3.3) below. In a cell the other prime factors vary, the prime condition on k changes their divisibility densities, and an integer may receive several marks. These three issues account for the moment bound, the local Euler product and the intersection estimate in the proof.

Relation to other work. Vaughan's identity [20] supplies the remainder. The pairing of divisors by a small prime is due to Erdős and Hall [5, proof of Theorem 1]. Hildebrand [10, §4.3] studied the first absolute moment in natural density, taking the population to infinity with the cutoff fixed. For the moving cutoffs used here, the support estimate of Granville, Koukoulopoulos and Maynard [9, (1.4), §5] and the mean square of Dress, Iwaniec and Tenenbaum [4, 3] give the necessary first moment by Cauchy–Schwarz. Counting divisors in a window on shifted primes and on rough integers is developed by Koukoulopoulos [12] and Ford [7]. Extracting a squarefree divisor and integrating over the logarithms of its prime factors are standard sieve operations [14, §2]. Here one must count each integer only once, retain the sign of its Vaughan coefficient, and compute the local main term before taking absolute errors over the prescribed divisor pairs.

Averaged Vaughan decompositions for a quartic Liouville correlation [17] averages cutoffs in a representation of Liouville. *Joint cofactor signs in prime-factor windows on consecutive integers* [18] studies the distribution of signs after prime-factor observations. The present estimates concern a fixed cutoff for Λ and include an actual prime weight at the other cofactor.

Section 2 proves the required level of distribution. Section 3 identifies the singular normalization and the rough negative part of the coefficient. Section 4 evaluates one prime mark. Section 5 resolves repeated marks, matches the two families and proves the assertion with a least-prime restriction. Section 6 proves the coverage formula and (1.4). The appendix records the positive transfer estimates used to pass from integers to points in cells.

2 A type I estimate with a common coefficient in each residue class

Let I_b be the interval of possible k in (1.1). It has length X/b. Endpoint conventions in reflection follow $X < am \le 2X$. In both orientations, $d \mid m$ reads $bk \equiv h \bmod ad$. Write $D_* = \sqrt{X}/A = X^{1/4}$.

Theorem 2.1. *At shift two, for odd d, set*

$$\Delta_{abd}(F) = \sum_{\substack{k \in I_b \\ bk \equiv 2 \bmod ad}} F(k) - \mathbf{1}_{(b,ad)=1} |I_b| c_{ad}(F).$$

Here $c_q(\vartheta) = 1/\varphi(q)$, $c_q(\mu) = c_q(\lambda) = 0$. For every $P, J > 0$ there exists $c(P, J)$ such that, for $D \le D_* L^{-c(P,J)}$,

$$\sum_{(a,b)\in\mathcal{C}} \left| \sum_{d \le D} w_{a,b \bmod a,d} \Delta_{abd}(F) \right| \ll_{P,J} XL^{-P}, \qquad |w_{a,r,d}| \le L^J \tau(d)^J. \tag{2.1}$$

Even d can be included with reference zero. For $F = \vartheta$ the reflected estimate uses h in the congruence and reference $\mathbf{1}_{(d,h)=1}\mathbf{1}_{(b,ad)=1}|I_b|/\varphi(ad)$ for odd d. In both orientations a common subinterval of the physical range $X < am \le 2X$ is allowed, with the corresponding I_b.

Here D_* bounds the additional divisor d, while the full congruence has modulus ad. Distribution results with other coefficient conditions lead to different norms. For example, Bombieri, Friedlander and Iwaniec [2, Theorem 10] use well-factorable weights on moduli and a fixed residue. In (2.1) the sum in d takes place inside each absolute value, and the weight may depend on a and $b \bmod a$.

The order of operations in (2.1) is essential. Take $10U < R \asymp X^{\xi}$ with $3/20 < \xi < 1/4$. For odd a, d, $d \sim R$, choose $1.1K < k < 1.2K$ coprime to ad. There are $\gg K/\log L$ such k, by inclusion–exclusion with error $O(\tau(ad))$. Among the odd $1.1B < b < 1.2B$, the equation $bk \equiv 2 \bmod ad$ gives $\gg B/(ad)$ choices, all admissible. Each such triple a, b, d has at most one k in the full interval I_b, since $ad > |I_b|$. Coefficients chosen separately for each triple can match the sign of $\lambda(k)$. Summing over a, d then gives $\gg X/\log L$. The unrestricted estimate below thus stops at exponent $3/20$, whereas (2.1) goes up to exponent $1/4$.

Lemma 2.2 (Mean square in progressions)**.** *For $Y_1 \asymp K/L^{O(1)}$, a fixed interval $I' \subset [cY_1, c'Y_1]$ and any fixed $t_0, N > 0$, each $F = \vartheta, \mu, \lambda$ satisfies*

$$\sum_{q \le 2AL^{t_0}} \sideset{}{^*}\sum_{r \bmod q} \left| \sum_{k \in I',\ k \equiv r(q)} F(k) - \mathbf{1}_{F=\vartheta} |I'|/\varphi(q) \right|^2 \ll_{t_0,N} Y_1^2 L^{-N}. \tag{2.2}$$

Here and below a star denotes reduced classes, and the estimate is uniform in the two endpoints.

Proof. Use the Barban–Davenport–Halberstam estimate for μ, Λ in the form recalled in [11, §3.1.3, (11) and the adjacent discussion]. The endpoint form follows by subtracting prefixes. Indeed AL^{t_0} is below Y_1 by a fixed power, so one can enlarge the nonnegative modulus sum to $Y_1/(\log Y_1)^{N'}$ at any fixed precision. This first gives the estimate for Λ, μ. Proper prime powers in Λ have total square mass $O(Y_1^{1/2} L^{O(1)})$. For $\lambda(n) = \sum_{v^2 \mid n} \mu(n/v^2)$, handle $v \le L^H$ by Minkowski and the estimate for μ at scale Y_1/v^2. Only v coprime to q contribute and the class is permuted. The rest is supported on $O(Y_1 L^{-H} + \sqrt{Y_1})$ integers and bounded by $\tau(n)$. Cauchy and the fourth divisor moment give square mass $\ll Y_1 L^{O(1)-H/2}$. A square mass E contributes $\ll E \sum_{q \le 2AL^{t_0}} (1 + Y_1/q)$ to (2.2). This handles both tails. An odd restriction on signed weights at odd moduli is imposed by lifting the class to modulus $2q$. □

Lemma 2.3 (Coprimality in a projected residue class)**.** *Fix $t_0, N > 0$, $g \le L^{t_0}$, a common interval $I \subset [cK, c'K]$, and $F = \lambda$ or μ, optionally multiplied by $\mathbf{1}_{2 \nmid k}$. For $a \sim A$, $q = ag$, $(q, h) = 1$, and $x \bmod q$ a unit, put*

$$D_{a,j}(x) = \sum_{\substack{k \in I,\ k \equiv h\bar{x} \bmod q \\ (k,j)=1}} F(k), \qquad \|D_{\cdot,j}\|_{g,2}^2 = \sum_{\substack{a \sim A \\ (ag,h)=1}} \frac{1}{ag} \sideset{}{^*}\sum_{x \bmod ag} |D_{a,j}(x)|^2.$$

Uniformly for $1/2 \le R/g$, $R \le D_*$,

$$\sum_{R/g < j \le 2R/g} \frac{\|D_{\cdot,j}\|_{g,2}}{j} \ll_{t_0,N} \sqrt{A}\, U L^{-N}. \tag{2.3}$$

Proof. Expand $\mathbf{1}_{(k,j)=1} = \sum_{c \mid (k,j)} \mu(c)$. Only squarefree c coprime to q enter on a given row. For fixed c the sum of $1/j$ over the indicated multiples of c is $O(1/c)$, including the last incomplete dyadic interval. Use the triangle inequality in the joint a, x norm before summing this weight.

First let $c \le L^H$ for a fixed H. Write $k = cn$, permuting the reduced classes modulo q. For λ the new factor is $\lambda(c)\lambda(n)$. For μ it is $\mu(c)\mu(n)\mathbf{1}_{(n,c)=1}$. With the odd restriction, even c contribute nothing and n must be odd. For λ use $q' = q$, or $q' = [q, 2]$ if oddness is required. For μ use $q' = [q, c]$, or $q' = [q, 2c]$ with oddness. In each case the sum is over unit

lifts to q', with at most $2c$ lifts of a given reduced class. Cauchy costs $2c$ in the square. As x varies the initial classes are permuted; a given q' can arise from at most $2c$ values of q, since $q'/q \mid 2c$. The factor $1/q$ is $O(1/(Ag))$. Applying Lemma 2.2 at scale K/c, up to moduli $4Agc$, therefore bounds the squared joint norm for this c by

$$\ll \frac{(2c)^2}{Ag}\,(K/c)^2 L^{-N'}$$

with any fixed N'. One increases the fixed modulus exponent in that lemma by $H + O(1)$. The resulting bound still permits summation over the small c with weight $1/c$.

For $c > L^H$ discard the signs and the coprimality left on n. For each row at most $O(K/c)$ integers n occur in total, and at most $O(1 + K/(cq))$ in one class. (Here $c \le 2D_* \ll K$.) Consequently the probability norm over x is $\ll K/(cq) + \sqrt{K/(cq)}$. After summation in c the joint norm on this tail is bounded by a constant times

$$\sqrt{A}\left\{\frac{U}{gL^H} + \sqrt{\frac{U}{gL^H}}\right\}.$$

Take $H > 2N + 2$, then take N' sufficiently large for the small c terms. This proves (2.3). □

Lemma 2.4 (Sampling a segment of the prescribed divisor)**.** *For each $a \sim A$ and $b_0 \bmod a$, let $H_a(b_0 + at)$ have a finite Fourier expansion in t with reduced denominators at most $2R$. Write $\|\cdot\|_{a,2}$ for the probability norm over b_0 and a full common period of t, and $\Pi_{>U}$ for projection onto denominators greater than U. On any common interval I_B of length $H' \ge 2A$,*

$$\sum_{a\sim A}\sum_{b\in I_B}|H_a(b)|^2 \ll (H' + AU^2)\sum_a \|H_a\|_{a,2}^2 + (H' + AR^2)\sum_a \|\Pi_{>U}H_a\|_{a,2}^2.$$

Proof. At fixed a, b_0, the t's form a segment of length at most $H'/a + 1$. Distinct reduced frequencies of denominator at most Y have circular spacing at least Y^{-2}. The dual additive large sieve [16, Theorem 1 and duality] bounds their sampled squared norm by $O(H'/a + Y^2)$ times the sum of squared coefficients. Split at U, using the bound with $Y = U$ below and $Y = 2R$ above (the upper group may be empty). Parseval and summation over b_0 supply the factor a that converts its probability mean into a sum. □

Proof of Theorem 2.1. We first take $|w| \le L^J$, $d \sim R$, and a fixed interval I in $[cK, c'K]$. Initially h is a unit for all moduli retained. For fixed a, write $b = b_0 + at$, $b_0 \bmod a$. The fixed-interval error $H_d(t)$, extended by zero on nonunits $b \bmod ad$, is periodic in $t \bmod d$. Use the probability norm in $b_0 \bmod a$ and t modulo any joint period. With P_g denoting conditional expectation at fixed $b_0, t \bmod g$, the Chinese remainder theorem gives

$$\langle H_d, H_e\rangle = \langle P_g H_d, P_g H_e\rangle, \qquad g = (d, e). \tag{2.4}$$

The same identity holds after multiplication by coefficients depending on b_0. If $d = gj,\ q = ag,\ x = b \bmod q$, the projection at a unit is

$$P_g H_d(x) = \frac{1}{j}\left\{\sum_{\substack{k\in I,\ k\equiv h\bar{x} \bmod q\\ (k,ad)=1}} F(k) - \frac{\varphi(ad)}{\varphi(q)}|I|c_{ad}(F)\right\}. \tag{2.5}$$

Indeed there are j lifts, of which $\varphi(ad)/\varphi(q)$ are units. Inversion permutes the unit lifts. At a nonunit both sides are zero with the convention for H_d. A class modulo q contains $O(1 + K/q)$ integers in I. Counting the occupied classes in the average over all x gives

$$\|P_g H_d\|_2 \ll (L^{O(1)}/j)\{U/g + \sqrt{U/g}\}.$$

Use reduced Fourier denominators in the t variable. Two terms have a common denominator only when it divides their gcd. For $H = \sum_{d\sim R} w_{a,b_0,d} H_d$ the projection to denominators $> Y$ therefore satisfies

$$\|\Pi_{>Y} H\|_2^2 \ll L^{O_J(1)}\{U^2/Y + U\log(2R)\}. \tag{2.6}$$

For each g, the sum of $g^2/(de)$ over the relevant pairs is $O(1)$. Summing $U^2/g^2 + U/g$ gives the display. This argument also bounds the sum of absolute full cross terms with gcd $> Y$.

We claim, with arbitrary fixed precision N, that

$$\sum_{a\sim A} \|H\|_2^2 \ll AU^2 L^{-N}. \tag{2.7}$$

Use (2.6) in its full cross-term form for $g > L^{t_0}$, choosing t_0 large. In (2.5) for $F = \vartheta$ dropping $(k, ad) = 1$ costs at most $\log(ad)$. Summing the mean square of the braces over a gives $\ll K^2(Ag)^{-1}L^{-N'} + AL^2$ by (2.2). The sum of $1/(jj')$ over $d = gj, e = gj'$ in the dyade is bounded for each g. This proves the small-gcd claim for primes.

For either signed weight, the braces in (2.5) are $D_{a,j}(x)$ from Lemma 2.3, since the class already forces $(k, q) = 1$. By (2.4) and Cauchy, the contribution for each small g is bounded by $L^{2J}\big(\sum_{j\sim R/g} j^{-1}\|D_{\cdot,j}\|_{g,2}\big)^2$. Using (2.3) with spare precision to sum $g \le L^{t_0}$ proves (2.7). Restricting the rows or the reduced classes can only decrease the upper bounds for these norms.

It remains to sample an actual segment of b's. Partition $(B, 2B]$ into $\asymp L^S$ intervals of length $H_0' \asymp B/L^S$. Lemma 2.4, (2.7), and (2.6) at $Y = U$ give on each segment

$$\frac{\sum_{a,b}|H(b)|^2}{AH_0'U^2} \ll L^{-N}\left(1 + \frac{AU^2}{H_0'}\right) + L^{O_J(1)}\left(\frac{1}{U} + \frac{AR^2}{H_0'U}\right).$$

The first geometry factor satisfies $AU^2/B = X^{-1/20}$. In the second, $AD_*^2/(BU) = 1$, so the logarithmic reserve in $R \le D_* L^{-c(P,J)}$ pays for the partition and the large-sieve loss. Freezing the endpoints of I_b costs two intervals of length $O(K/L^S + 1)$ per segment. A unit k in either interval specifies one residue of b mod ad with $O(H_0'/(ad))$ representatives, since $ad \ll \sqrt{X} \ll H_0'$. Counting these representatives, and the analogous change of reference, costs $O(L^{O_J(1)}(X/L^S + B))$ per d-dyade on summing the segments. Choose S first, then $N, c(P, J)$. Cauchy, $ABU = X$ and dyadic summation yield (2.1).

At shift two an even $d = 2d'$ forces $k = 2k'$ and $bk' \equiv 1$ mod ad'. The rescaled weights are $-\lambda(k')$, $-\mu(k')\mathbf{1}_{2\nmid k'}$. The unit-shift proof applies also to even d'. A prime $k \asymp K$ cannot be even. In reflection a prime $p \mid (d, h)$ would divide bk, and $(b, h) = 1$ would force $k = p$. This is impossible for large X, since $p \le d \le D_* \ll K$ and $k \asymp K$. The zero reference on these moduli is therefore exact. Finally truncate $\tau(d)$ at $L^{J'}$. For unit right-hand side the total absolute contribution at d before its coefficient is $\ll XL^{O(1)}/d$, by counting b mod ad at fixed a, k. For even d rescale as above. Higher divisor moments in $\sum_{d\le D}\tau(d)^m/d \ll_m L^{O_m(1)}$ make the omitted tail $O(XL^{-P})$ by taking J' large. This proves the stated coefficient range. □

The unrestricted range. At shift two with arbitrary coefficients of modulus 1 for each triple a, b, d, the sum of the absolute errors over the triples themselves is $\ll_P XL^{-P}$ for $d \le UL^{-C(P)}$. Here is a direct comparison for the signed and prime tests. For a fixed interval and odd d, put $q = ad \le Q = 2AD$, and write $E_q(r)$ for the residue error in (2.2). Each unit residue of b occurs $O(B/q)$ times and q has at most $\tau(q)$ factorizations. Cauchy bounds the total by

$$B\left(\sum_{q\le Q}\frac{\tau(q)^2\varphi(q)}{q^2}\right)^{1/2}\left(\sum_{q\le Q}\sideset{}{^*}\sum_{r}|E_q(r)|^2\right)^{1/2}.$$

The first factor inside parentheses is $L^{O(1)}$. In the second, Barban–Davenport–Halberstam applies to $Q \le 2KL^{-C(P)}$. The truncations for λ, ϑ in the proof of (2.2) apply at this level

by increasing $C(P)$ after the truncation is chosen. Freeze endpoints on a grid of mesh K/L^S. There are $O(L^{2S})$ interval choices. The omitted ends cost $O(L^{O(1)}B(Q+K/L^S))$, by counting at most $O(1+K/(qL^S))$ integers per class and their corresponding references. Choose S and then the mean-square precision. Even d are reduced as above. This gives the claimed range and, with the single-point example preceding the proof, explains the role of common coefficients.

3 The remainder, its local density and the rough sector

The measurable divisor events are governed by a single multiplicative density. For a fixed admissible cell define $g(1)=1$ and

$$g(p)=\begin{cases}0 & p\mid 2bh,\\ 1/p & p\mid a,\\ 1/(p-1) & p\nmid 2abh,\end{cases}\qquad g(p^v)=p^{1-v}g(p)\quad (v\ge 1). \tag{3.1}$$

The prime reference in Theorem 2.1 for $d\mid m$ is $H_{ab}g(d)$. Indeed for $(d,2bh)=1$ the reference density relative to $d=1$ is $\varphi(a)/\varphi(ad)$. The factor $1/p$ at a prescribed prime of a comes from extending a modulus already divisible by p.

Lemma 3.1 (The singular normalization)**.** *For the density in* (3.1)*, let*

$$\mathcal{H}_g(s)=\prod_p \frac{1-g(p)p^{-s}}{1-p^{-1-s}},\qquad J_g(y)=\sum_{d\le y}\mu(d)g(d)\log(y/d).$$

Then $H_{ab}\mathcal{H}_g(0)=V_{ab}$. For each fixed $c,\sigma>0$, uniformly when $\exp(cL^\sigma)\le y\le X^{O(1)}$,

$$J_g(y)=\mathcal{H}_g(0)+O_N(L^{-N}),\qquad \sum_{d\le y}\mu(d)g(d)=O_N(L^{-N})$$

for any fixed N. These two estimates remain valid after changing $g(p)$ to 0 or $1/p$ on prime divisors of an additional integer of polynomial size in X, using the corresponding modified product. Moreover, uniformly in the admissible centres,

$$\sum_{\mathcal{C}} H_{ab}\ll X,\quad \mathcal{V}=(1+o(1))(\log 2)^2 X\,\frac{\varphi(2h)}{2h}\gg X/\log L. \tag{3.2}$$

Proof. The Dirichlet series for $\mu(d)g(d)$ equals $\mathcal{H}_g(s)/\zeta(1+s)$ for $\Re s>0$. The coefficients h_g of the numerator product are multiplicative, with $h_g(p^v)=p^{1-v}(1/p-g(p))$. With $c_0=1/\log L$, $\sum_n |h_g(n)|n^{c_0}\ll L^{O(1)}$. The generic factors have summable excess $O(p^{-2+c_0})$. For exceptional primes split at L^2, use Mertens below and their total logarithm $O(L)$ above. This also handles the indicated changes. In convolution with $\mu(n)/n$, the part of h_g above $\sqrt{y}$ now saves every power of L. Below $\sqrt{y}$, partial summation of the prime number theorem for μ gives $\sum_{n\le v}\mu(n)/n=O_D((\log v)^{-D})$ and $\sum_{n\le v}(\mu(n)/n)\log(v/n)=1+O_D((\log v)^{-D})$. The constant is the derivative of $1/\zeta(1+s)$ at zero. At $s=0$ an ordinary odd prime contributes $1-1/(p-1)^2$; a prime of a contributes 1, and a prime of $2bh$ contributes $(1-1/p)^{-1}$. Multiplication by $X/(b\varphi(a))$ gives the formula for V, including the prime two.

For the sum of V, fix a, put $D_0=2ah$, and expand the factor depending on b as $\sum_{d\mid b}\xi(d)$, with $\xi(d)=\prod_{p\mid d}(p-2)^{-1}$ on odd squarefree integers and zero otherwise. For $d\le\sqrt{B}$, $(d,D_0)=1$,

$$\sum_{\substack{n\sim B/d\\(n,D_0)=1}}\frac{1}{n}=\frac{\varphi(D_0)}{D_0}\log 2+O(\tau(D_0)d/B).$$

Since $\xi(d) \ll d^{-1+o(1)}$, the discarded tail in the sum over d is $O(B^{-1/2+o(1)})$. The resulting product $\prod_{p\nmid D_0}(1+1/[p(p-2)])$ cancels the ordinary odd factors of C_2. The remaining local factors, including $\varphi(D_0)/D_0$, give 1 after the factor 2 at two. The row sum is $(X/a)(\log 2 + O(B^{-1/2+o(1)}))$. Sum $1/a$ over $(a, 2h) = 1$ by the same elementary inclusion–exclusion. Finally $\sum_{a\sim A} 1/\varphi(a) \ll 1$ follows by expanding $a/\varphi(a) = \sum_{d|a} \mu^2(d)/\varphi(d)$. Summation in b gives the bound for H. □

Proposition 3.2 (Reduction to two large factors)**.** *Put $R_{u,v}(m) = \sum_{r\ell t=m,\ r>u,\ \ell>v} \mu(r)\Lambda(\ell)$, and $P_{ab} = \sum_{\mathcal{B}_{ab}} \vartheta(m)\vartheta(k)$. Given $N, \epsilon_0 > 0$, if $u \geq X^{\epsilon_0}$, $v \geq 1$, $uv \leq D_* L^{-c(N,\epsilon_0)}$, then*

$$\Big\| \sum_{\mathcal{B}_{ab}} R_{u,v}(m)\vartheta(k) - (P - V) \Big\|_1 \ll XL^{-N}.$$

At shift two one also has $\big\|\sum_{\mathcal{B}_{ab}} (\Lambda(m) - R_{u,v}(m))F(k)\big\|_1 \ll XL^{-N}$ for $F = \mu, \lambda$.

Proof. Vaughan's identity with two independent truncations reads

$$\Lambda = \mu_{\leq u} * \log - \mu_{\leq u} * 1 * \Lambda_{\leq v} + \Lambda_{\leq v} + \mu_{>u} * 1 * \Lambda_{>v}.$$

It follows from $1*\Lambda = \log$, $\mu*1 = \mathbf{1}_{\{1\}}$. The isolated term $\Lambda_{\leq v}$ vanishes on $m \asymp M$. The double small term has divisors $d = r\ell \leq uv$ with coefficient at most $\log d$ in modulus. For the first term separate $\log(m/r) = \log(X/(ar)) + \log(am/X)$. Partial summation in the physical interval handles the latter uniformly by Theorem 2.1. Thus all signed references vanish. For the prime reference divide by H_{ab}. The first term becomes $\sum_{r\leq u} \mu(r)g(r)(\log(X/a) + 2\log 2 - 1 - \log r)$, equal to $\mathcal{H}_g(0) + O_N(L^{-N})$ by Lemma 3.1 with spare precision. In the second term fix ℓ with $g(\ell) > 0$. Then $g(r\ell) = g(\ell)g^{(\ell)}(r)$, where the primes of ℓ have density $1/p$. The zero mean from the same lemma applies uniformly and $\sum_{\ell\leq v} \Lambda(\ell)g(\ell) \ll L^{O(1)}$. Proper prime powers on the left cost $\ll AM^{1/2}X^{o(1)}$ by factoring $am + 2$ or $h - am$. On the right they cost $\ll BK^{1/2}X^{o(1)}$ by factoring $bk - 2$ or $h - bk$. These bounds also justify replacing $\vartheta(k)$ by $\Lambda(k)$ in the prime-weighted results in this paper. □

In particular the cutoff for both variables moves from $X^{3/40}L^{-C_0}$ to $z = X^{1/8}L^{-C}$. Subtracting the proposition first with u increased, then with v increased, disposes of the two strips separately in cell norm. The remaining two-large-factor sum has a simple pointwise description away from sparse prime powers.

Put $B_y(n) = \sum_{d|n,\, d\leq y} \mu(d)$ (zero for $y < 1$) and $H_z(n) = \sum_{\ell|n,\, \ell>z} \Lambda(\ell)$. Call n regular if no $p^v \mid n$ has $p \leq z < p^v$. For a regular $m \asymp M$ put $s_m = \prod_{p|m,\, p\leq z} p$, $v_m = \prod_{p>z} p^{v_p(m)}$. Then

$$R_z(m) = \begin{cases} \Lambda(m) - \log m & s_m = 1, \\ 0 & 1 < s_m \leq z, \\ -(\log v_m)B_z(s_m) & s_m > z. \end{cases} \tag{3.3}$$

Indeed $\mu * H_z = \Lambda_{>z}$, and $H_z(m/d) = \log v_m$ for every $d \mid m,\ d \leq z$. With $1 < s_m \leq z$ the full Möbius sum on s_m vanishes and a regular $m \asymp M$ of this kind cannot be a prime power. For each $p \leq z$ take its first power $p^{v(p)} > z$. Splitting at $\sqrt{z}$ gives $\sum_{p\leq z} p^{-v(p)} \ll z^{-1/2}$. Thus $O(Y/\sqrt{z} + z)$ integers up to $2Y$ are irregular. On the full m-range their contribution costs X^{1-c} for some $c > 0$, even with fixed divisor and logarithmic weights, by counting divisors of the two integers of size $\asymp X$.

A global comparison for the rough part. Let ω denote Buchstab's function, $W(u) = u\omega(u)$ for $u \geq 1$. Thus $W = 1$ on $[1, 2]$ and $W'(u) = \omega(u - 1)$ for $u > 2$. For fixed $t \geq 1$, $y = z^{1/t}$, define the locally finite signed measure

$$\sigma = \delta_0 + \sum_{i\geq 1} \frac{(-1)^i}{i!} (\mathbf{1}_{[1,\infty)}(s)\, \mathrm{d}s/s)^{*i}$$

with additive convolution. In either orientation,

$$\mathcal{V}^{-1}\sum_{\mathcal{C}}\sum_{\mathcal{B}_{ab}}\mathbf{1}_{P^-(m)>y}R_z(m)\vartheta(k)\ \longrightarrow\ C(t)=1-\int_{[0,t]}W(5t+1-s)\,\mathrm{d}\sigma(s). \tag{3.4}$$

Here t is fixed before X tends to infinity. To verify this, apply the prime row estimate (A.1) in the appendix. On a row $Y=X/a$, the prime term from $\Lambda_{>z}$ contributes $Y+o(Y)$. Open the term $\mu_{\le z}*H_z$, retaining $P^-(d)>y,\ P^-(m/d)>y$, with $P^-(1)=\infty$. Write $Y'=Y/d$. Uniformly for $d\le z$,

$$\frac{1}{Y'}\sum_{\substack{n\sim Y'\\ P^-(n)>y}}H_z(n)=W\Big(1+\frac{\log(Y'/z)}{\log y}\Big)+o_t(1).$$

Proper powers contribute $o_t(Y')$ before normalization, by counting multiples of $p^2,\ p>y$. Marking a prime $p>z$ writes $n=vp$. For $v\le Y'/z$ the prime number theorem gives $Y'/v+O_N(Y'/(vL^N))$. The boundary $Y'/z<v\le 2Y'/z$ has rough harmonic mass $O_t(1/L)$. Partial summation of the rough number estimate of Buchstab [19, Ch. III.6] gives $\sum_{v\le y^u,\ P^-(v)>y}1/v=1+\int_1^u\omega(s)\,\mathrm{d}s+o_t(1)=W(1+u)+o_t(1)$ on the compact range in use. Finally expand squarefree $d\le z$ into at most t primes $>y$. Their harmonic measures in $\log p/\log y$ converge to $\mathrm{d}s/s$. Repeated primes have vanishing weight and the boundary has zero continuous measure. The total harmonic weight is $O_t(1)$. Since $\log(Y/z)/\log y=5t+o(1)$, this gives (3.4) after (3.2).

On $P^->z$ the coefficient is nonpositive. Denote its negative mass in a cell by

$$N_{ab}=\sum_{\mathcal{B}_{ab}}\mathbf{1}_{P^-(m)>z}(\log m-\Lambda(m))\vartheta(k).$$

Then $\sum N_{ab}\sim(W(6)-1)\mathcal{V}$. $W(6)>1$ follows already by integrating W' above two. On $\sqrt{z}<P^-\le z$, a regular m with two or more small primes has $B_z(s_m)=1-\omega_0(s_m)\le 0$, with ω_0 counting distinct primes, since every product of two exceeds z. This sector is nonnegative up to the sparse error in (3.3). Its total signed mass is $(C(2)-C(1)+o(1))\mathcal{V}$. Here $C(1)=1-W(6)$ and $C(2)=1-W(11)+\int_1^2 W(11-u)\,\mathrm{d}u/u$. For scale, $2.36874<-C(1)<2.36878$ and $1.45653<-C(2)<1.45667$. These enclosures follow by integrating the delay equation with upper and lower sums on a mesh of width $1/120\,000$. The exact rational bounds are reproduced in the source archive. Applying (A.1) to $\vartheta(m)$ also gives $\sum_{\mathcal{C}}P_{ab}\sim\mathcal{V}$ by the prime number theorem and the same removal of primes dividing h. Thus the global prime-pair asymptotic is available even though its cell-norm counterpart is open.

4 Extracting a prime at the cutoff

A prime just below z and a prime just above z occur with comparable harmonic frequency. The term that distinguishes them is a Möbius sum with cutoff z/p. Write $U_{s,z}(n)=\sum_{d\mid n,\ d\le s}\mu(d)H_z(n/d)$. If $p\nmid n,\ n>1$, then separating the divisors divisible by p in $R_z=\Lambda_{>z}-U_{z,z}$ gives exactly

$$R_z(pn)=-U_{z,z}(n)+U_{z/p,z}(n)-\mathbf{1}_{p>z}\log p\,B_z(n).$$

When n is regular this becomes

$$R_z(pn)=H_z(n)B_{z/p}(n)-(H_z(n)+\mathbf{1}_{p>z}\log p)B_z(n). \tag{4.1}$$

The last term has small first moment on both bands. The first term vanishes on the upper band and survives on the lower one.

Lemma 4.1 (Classical first-moment bounds)**.** *Let* $B_u(n) = \sum_{d|n,\, d\le u} \mu(d)$*,* $u \ge 3$*, and put* $\rho = \delta/2$*. For* $2Y \ge u^{5/2}$*,*

$$\sum_{n\le 2Y} |B_u(n)| \ll \frac{Y}{(\log u)^{\rho}(1+\log\log u)^{3/4}}. \tag{4.2}$$

On the wider range $u^2 \le Y$ *one has* $\sum_{n\le 2Y} |B_u(n)| \ll Y(\log u)^{-\delta/4}$*.*

Proof. With $x = 2Y$, $R = u$, the upper bound in [9, (1.4), §5] gives $\#\{n \le 2Y : B_u(n) \ne 0\} \ll Y(\log u)^{-\delta}(1+\log\log u)^{-3/2}$ in the first range. Expanding the square gives

$$\sum_{n\le 2Y} B_u(n)^2 = 2Y \sum_{d,e\le u} \frac{\mu(d)\mu(e)}{[d,e]} + O(u^2) \ll Y + u^2.$$

Here $[d,e]$ is the least common multiple. The bounded double sum is due to Dress, Iwaniec and Tenenbaum [4], see also [3, (1.1)–(1.5)]. Cauchy–Schwarz on the support proves (4.2).

For the wider range one can use the earlier small-prime pairing. Take $v = \exp((\log u)^{\delta/2})$. The elementary upper-bound sieve for integers with no prime factor up to v gives $O(Y/\log v)$ such integers. Otherwise take $p \mid n$, $p \le v$, write $n = p^i n_0$, $(n_0, p) = 1$, and use the pairing of Erdős and Hall [5, proof of Thm. 1] to obtain

$$B_u(n) = B_u(n_0) - B_{u/p}(n_0).$$

A nonzero value forces a divisor of n in $(u/v, u]$. Cover this by $O(1+\log v)$ intervals $(t, 2t]$ with $\log t \asymp \log u$. Ford's divisor-in-an-interval estimate [6, Thm. 1, case $z = 2y$] bounds the population in each by $O(Y/(\log u)^{\delta})$. The range $u \le \sqrt{Y}$ suffices. Consequently $\#\{n \le 2Y : B_u(n) \ne 0\} \ll Y(\log u)^{-\delta/2}$. The same second moment and Cauchy–Schwarz now give the second assertion. □

We use the following two transfer bounds, proved in Appendix A. For a common nonnegative f supported on $[M/2, 2M]$ with $f(m) \le L^{j_0}\tau(m)^{j_0}$ for fixed j_0, they give

$$\sum_{\mathcal{C}} \sum_{\mathcal{B}_{ab}} f(m) W(k) \ll A\, c_W \sum_{m} f(m) + O_N(XL^{-N}), \qquad c_\vartheta = 1, \quad c_1 = \log L. \tag{4.3}$$

Lemma A.1 gives this for $W = \vartheta$ in either orientation, and Lemma A.2 gives it for $W = 1$ at shift two. Using the trivial divisor bound $X^{o(1)}$ instead of (4.3) would exhaust the logarithmic saving of Lemma 4.1.

Let $\mathcal{P}_- = (z/T, z] \cap \mathbb{P}$, $\mathcal{P}_+ = (z, zT] \cap \mathbb{P}$, and use $\mathcal{P}$ for either. Partial summation of the prime number theorem gives $\sum_{p\in\mathcal{P}} 1/p \ll \log T/L$. For $m = pn \asymp M$, $Y' \asymp M/p = X^{5/8+o(1)}$, whereas $z^{5/2} \le X^{5/16}$. Thus every invocation with cutoff at most z and quotient length Y' satisfies the stronger range of (4.2), on either side of z. The sum over $m \asymp M$ of $\sum_{p|m,\ p\in\mathcal{P}} L|B_z(m/p)|$ is at most $O(M(\log T)L^{-\rho})$. To bound the exceptional points with $p \mid n$ or irregular n, observe that each mark has $O(Y'/p + 1 + Y'/\sqrt{z} + z)$ exceptional n, then at most $X^{o(1)}$ choices and fixed divisor weights at fixed a, m. Summation over p, a gives $O(X^{1-c})$ for some $c > 0$. By (4.3) the last term of (4.1) costs $O(XL^{-\eta})$, since $\gamma + \eta < \rho$. For $\mathcal{P}_+$ this proves (1.3).

For the lower band retain

$$Z_1(m) = \sum_{\substack{p\in\mathcal{P}_- \\ p|m}} H_z(m/p) B_{z/p}(m/p).$$

A number with two distinct marks p, q was counted twice. The absolute cost of the surviving terms on all such numbers is bounded at each m by $O(L\sum_{p\ne q\in\mathcal{P}_-;\, pq|m} \sum_{d\le T,\ d|m/p} 1)$.

Since $T < \min \mathcal{P}_-$, one has $pqd \mid m$, $pqd \le z^2T = o(M)$. Summing in m costs $\ll ML(\log T/L)^2 \log(2T)$. At a point with $t \ge 2$ marks, each of its t summands occurs in an ordered pair. By (4.3),

$$\sum_{\mathcal{C}} \sum_{\mathcal{B}_{ab}} |E_1(m)R_z(m) - Z_1(m)|W(k) \ll XL^{-\eta}. \tag{4.4}$$

This replacement in absolute mass will also permit a restriction on $P^-(m)$.

The local average of the surviving term. We give the calculation for a general squarefree mark j' in $(z/T, z]$ whose primes all exceed T. This includes the prime $j' = p$, the two-prime case of Section 5, and the marks in Section 6. For a specified collection $\mathcal{J}$ write

$$Z_{\mathcal{J}}(m) = \sum_{\substack{j' \mid m \\ j' \in \mathcal{J}}} H_z(m/j')B_{z/j'}(m/j').$$

Using $H_z(n) = \log n - \sum_{\ell \le z,\, \ell \mid n} \Lambda(\ell)$, the divisor conditions in the expansion are

$$v = j'd \mid m, \quad j'd \le z; \qquad v = j'[d, \ell] \mid m, \quad j'd \le z, \quad \ell \le z, \quad v \le z^2. \tag{4.5}$$

The factor j' occurs only once, even where ℓ shares a prime with j'. At a fixed grouped modulus v, each j' is a divisor of v, $d \mid v$, and the possible ℓ have total Λ-weight at most $\log v$. The coefficients are $O(L^{O(1)}\tau(v)^2)$, common to all b. The term $\log(am/X)$ is handled as in Proposition 3.2. Choosing C sufficiently large gives $z^2 \le D_*L^{-c}$. Thus (2.1) gives zero reference for both signed tests at shift two. For the prime test and $g(j') > 0$ put $g^{(j')}(p) = 1/p$ at $p \mid j'$, preserving the other factors of g and the prime-power rule. Then $g(j'v) = g(j')g^{(j')}(v)$. Write $l_{aj'} = \log(X/(aj')) + 2\log 2 - 1$, the physical mean of $\log(m/j')$. At the precision of (2.1), the reference divided by H_{ab} is

$$\sum_{\substack{j' \in \mathcal{J} \\ g(j') > 0}} g(j') \sum_{d \le z/j'} \mu(d) \left\{ g^{(j')}(d) l_{aj'} - \sum_{\ell \le z} \Lambda(\ell) g^{(j')}([d, \ell]) \right\}. \tag{4.6}$$

For squarefree $d \le T$, $g^{(j')}(d) = g(d)$, and uniformly

$$\sum_{\ell \le z} \Lambda(\ell) g^{(j')}([d, \ell]) = g(d)\{\log z + O(\log(2d) + \log L)\}. \tag{4.7}$$

Indeed if $g(d) = 0$, so are all the terms on the left. Otherwise, after division by $g(d)$, prime powers of a prime dividing d cost an extra $O(\log p)$ by their geometric series. At the other primes compare with $\sum_{\ell \le z} \Lambda(\ell)/\ell = \log z + O(1)$. The generic errors sum as $\sum_p O(\log p/p^2)$. The errors at primes dividing $2abhj'$ sum to $O(\log L)$ by splitting at L^2. Also $l_{aj'} - \log z = L/2 + O_C(\log L + \log T)$.

For the prime mark, uniformly for $1 \le d \le T$, partial summation gives

$$\sum_{z/T < p \le z/d} g(p) = \frac{\log(T/d)}{\log z} + O_N\left(\frac{(1 + \log T)^2}{L^2} + L^{-N} \right). \tag{4.8}$$

The main term is the first-order expansion of $\log(\log(z/d)/\log(z/T))$. Generic $g(p) - 1/p$ and exceptional prime divisors of $2abh$ cost $O(LT/z)$ on this band. Use $\sum_{d \le T} |\mu(d)|g(d) \ll \log(2T)$, by the Euler product majorant. Combining (4.6)–(4.8), and using $(L/2)/\log z = 4 + O_C(\log L/L)$, the local reference becomes

$$4H_{ab}J_g(T) + O_C\left(H_{ab} \frac{(\log(2T) + \log L)^3}{L} \right). \tag{4.9}$$

In particular the factor four is $\lim \log(M/z^2)/\log z$. The local product comes from a *logarithmically weighted* Möbius sum. By Lemma 3.1 its value in (4.9) is $4V_{ab}$ to the required precision. Sum the errors using (3.2) and $3\gamma+\eta<1$. This proves the E_1 part of (1.2).

For a fixed ratio $t>1$ in place of T, the same argument stops at $4H_{ab}J_g(t)$, with error $O(XL^{-\eta})$. For $t=2$, $J_g(2)=\log 2$, since the potential term $d=2$ has log weight zero. At that width the principal summand $H_z(m/p)B_{z/p}(m/p)$ is nonnegative for each prime mark. Equation (4.4) therefore bounds the total negative mass in that lower band by $O(XL^{-\eta})$. This finer sign assertion uses the fixed ratio.

5 Semiprime divisors and disjoint compensation

For $z^\alpha<p\le z^\beta$, $z/T<pq\le z$, the prime q is larger than $z^{1-\beta}/T>\sqrt{z}$, whereas $p<\sqrt{z}$. Both are less than z/T for large X. The order of the two factors is therefore determined. Let $\mathcal{J}$ be the set of products pq with these conditions.

More generally, if j' is squarefree with all prime factors at most z, $(j',n)=1$, $n>1$ regular, then

$$R_z(j'n)=-H_z(n)\sum_{e\mid j'}\mu(e)B_{z/e}(n)\qquad (j'>1). \tag{5.1}$$

In fact $j'n$ is composite and regular. Every squarefree divisor $\le z$ splits uniquely as ed, $e\mid j'$, $d\mid n$, and $H_z(j'n/(ed))=H_z(n)$. For $j'=pq$ the four cutoffs are $z, z/p, z/q, z/(pq)$. The first three are at least z^α/T for large X (indeed $z/q\ge p$), and at most z. Lemma 4.1 suppresses each of them in absolute mass. Only the last one survives and it carries the sign $-\mu(pq)=-1$.

Here are the estimates with multiplicities. The harmonic mass of all marks pq is $O_{\alpha,\beta}(\log T/L)$ because for each p the q's lie in a multiplicative window of ratio T at polynomial size, and the p-range has bounded harmonic mass. As $M/j'=X^{5/8+o(1)}$ and $(z/e)^{5/2}\le X^{5/16}$ for every $e\mid j'$, the absolute error from the proper subsets in (5.1), transferred by (4.3), is $O(XL^{\gamma-\rho}\log L)$. The events that j' shares a prime with n, or that n is irregular, have power-saving cost. One may bound their union in m by the exceptional set for regularity and the integers divisible by p^2 for a prime $p\gg X^{c_\alpha}$. The multiplicity of marks is at most $\tau(m)$, so fixed divisor moments or the bound $X^{o(1)}$ suffice here.

Lemma 5.1 (Counting points with several marks)**.** *With the sign convention of* (5.1)*, one has*

$$\sum_{\mathcal{C}}\sum_{\mathcal{B}_{ab}}|E_2(m)R_z(m)+Z_{\mathcal{J}}(m)|W(k)\ll XL^{-\eta}. \tag{5.2}$$

The same estimate holds after inserting any further indicator on the points.

Proof. Two distinct semiprime marks have either distinct prime factors throughout, or share exactly one prime. In the first case the sum of the inverse least common multiples is bounded by the square of the single harmonic mass. If they share p, fix it and sum the two q-windows. If they share q, fixing this common q places each p in a window of ratio T at polynomial size, and all possible q's have bounded harmonic mass. In all cases the bound is $O((\log T/L)^2)$. A marked prime of $\mathcal{P}_-$ and a semiprime mark have three distinct factors, so their union obeys the same bound.

The surviving summand for a specified j' is bounded by $O(L)\sum_{d\le T,\ d\mid m/j'}1$. All such d are coprime to all marked primes. A second mark j'' forces $d[j',j'']\mid m$, and $d[j',j'']\le Tz^2=o(M)$. Thus the single-variable absolute cost is at most

$$O\left(ML\sum_{d\le T}\frac{1}{d}\sum_{j',j''}\frac{1}{[j',j'']}\right)\ll M(1+\log T)^3/L.$$

The same argument counts the forbidden overlap with $\mathcal{P}_-$. To pass to indicators, choose one surviving mark per point. Its absolute error is bounded by the sum of the single-mark errors, and the extra principal summands are charged to ordered pairs. On points excluded by E_1, charge every principal summand to a pair with a prime mark. Transfer by (4.3) proves the claim. □

It remains to average $Z_{\mathcal{J}}$. The support and coefficients are exactly those of (4.5). Formula (4.8) now reads

$$\sum_{\substack{j'\in\mathcal{J}\\ j'\le z/d}} g(j') = \frac{\log(T/d)}{\log z}\int_\alpha^\beta \frac{\mathrm{d}u}{u(1-u)} + O_{\alpha,\beta,N}\left(\frac{(1+\log T)^2}{L^2} + L^{-N}\right). \tag{5.3}$$

Indeed, first sum $g(q)$ over $z/(Tp) < q \le z/(dp)$, obtaining $\log(T/d)/\log(z/p)$ with that error. The partial summation over p sends $1/p$ to $\mathrm{d}u/u$, $u = \log p/\log z$, and $\log z/\log(z/p) = (1-u)^{-1}$. Primes dividing $2abh$ cost $O(LX^{-c})$ in these fixed ranges. Equations (4.6), (5.3) and Lemma 3.1 give $4\kappa V$ for $Z_{\mathcal{J}}$. Lemma 5.1 supplies its negative for E_2. For a prescribed ratio $t > 0$ of the two signed main terms, the choice

$$\beta = \frac{\alpha \mathrm{e}^t}{1-\alpha+\alpha\mathrm{e}^t}, \qquad 0 < \alpha < \frac{1}{1+\mathrm{e}^t},$$

gives $\kappa = t$ and keeps the prime ranges distinct. The matching in Theorem 1.1 uses $t = 1$.

Keeping both contributions below $\sqrt{z}$ and above small primes. A prime mark in $\mathcal{P}_-$ might occur with no other small prime. If $P^-(m) > \sqrt{z}$, then $\Omega(m) = O(1)$, so $|R_z(m)| \ll L$. For each $p \in \mathcal{P}_-$ the number of $n \asymp M/p$ with $P^-(n) > \sqrt{z}$ is $O(M/(pL))$ by the upper sieve. The harmonic cost of the marks and (4.3) give $O(X(\log L)\log T/L)$. For a semiprime mark $p < \sqrt{z}$ is forced.

To remove $P^-(m) \le w = (\log L)^j$, apply the mask to (4.4) or (5.2) *before* expanding it. For a large fixed J, put $r = \lfloor J\log L/(j\log\log L)\rfloor$. The truncated inclusion–exclusion is a sum $\sum_{1<e\le L^J} a_e \mathbf{1}_{e|m}$, $|a_e| \le 1$, over squarefree products of primes $\le w$. Its error is at most $\binom{\omega_w(m)}{r}$, with ω_w counting these primes. Write $S_w = \sum_{p\le w} 1/p$. Classifying two r-subsets by the size of their union gives

$$\sum_{n\le 2X}\binom{\omega_w(n)}{r}^2 \ll X\,8^r \sum_{i=r}^{2r}\frac{S_w^i}{i!} \ll XL^{-J/j+o(1)}.$$

For a given union there are at most 8^r configurations (choose the first subset and its intersection with the second). At $n = am$ fixed the number of representations is at most $\tau(n)\tau(n+2)$ or $\tau(n)\tau(h-n)$. Since $\omega_w(m) \le \omega_w(n)$ and $|Z_{\mathcal{J}}(m)| \ll L\tau(m)^2$, Cauchy and fixed divisor moments give a truncation cost $\ll XL^{O(1)-J/(2j)+o(1)}$, uniformly in the centre. The distinct arguments $n, h-n \asymp X$ can be separated by Cauchy. The fixed divisor powers in this estimate determine an absolute constant b in the exponent $b - J/(2j) + o(1)$. Choose $J > 2j(b+\eta+1)$. Then choose the precision P in (2.1) and a fixed coefficient exponent J_0 large enough for the grouped weights. We may now require $2C \ge J + c(P, J_0) + 1$, so $L^J z^2$ is within the distribution range.

For fixed e in this expansion, the divisibility moduli in (4.5) become $j'[d,e]$ and $j'[d,e,\ell]$, because primes of j' exceed L^J, T. They are at most $L^J z^2$, and the coefficients at a given modulus acquire at most one more factor $\tau(v)$. The zero references stay zero. For the prime reference the expression $J_g(T)$ in (4.9) is replaced by

$$J_{g,e}(T) = \sum_{d\le T}\mu(d)g([e,d])\log(T/d).$$

For this assertion apply (4.6) with $[e,d]$ in the densities and keep d in the cutoff. The bound in the braces gains $O(\log e)$ in the error. Summing all such errors costs at most the factor furnished by

$$\sum_{e,d\le T} |a_e\mu(d)|\, g([d,e]) \ll_J \log(2T)(\log L)^{O(1)}.$$

Indeed the local factor is at most $1+3g(p)$ for $p\le L^J$ and $1+g(p)$ thereafter. Thus the analogues of (4.9) have total error $\ll XL^{-1+3\gamma}(\log L)^{O_J(1)}$, besides the type I errors. Finally for $g(e)>0$ let $g_{\perp e}$ have zero density on the primes of e. Splitting d by its common primes with e gives exactly

$$J_{g,e}(T) = g(e)\sum_{u\mid e}\mu(u)J_{g_{\perp e}}(T/u).$$

Here $e\le L^J$, so Lemma 3.1 applies uniformly. The common leading term is annihilated by $\sum_{u\mid e}\mu(u)=0$, since $e>1$. Errors sum with weight $\sum|a_e|g(e)\tau(e)\ll(\log L)^{O(1)}$. If $g(e)=0$ the whole contribution vanishes. This proves all the restrictions in Theorem 1.1.

For clarity, the losses for fixed windows are recorded below. The bounds are divided by X; fixed powers of $\log L$ and constants depending on the windows are suppressed.

Source of loss	Bound	Sufficient margin
Proper divisors of a mark, by (4.2)	$L^{\gamma-\rho}$	$\gamma+\eta<\rho$
Repeated marks and expansion of the local density	$L^{-1+3\gamma}$	$3\gamma+\eta<1$
Prime mark with $P^-(m)>\sqrt{z}$	$L^{-1+\gamma}$	$\gamma+\eta<1$
Truncated small-prime condition	$L^{b-J/(2j)+o(1)}$	$J>2j(b+\eta+1)$
Grouped divisor remainders	L^{-P}	$L^Jz^2\le D_*L^{-c(P,J_0)}$

The first margin implies the next two because $3\rho<1$. These bounds apply also to the three fixed prime ranges of Corollary 6.2. For the upper band only the absolute first-moment estimate is needed once repeated and irregular factors are discarded.

A conditional wider range. Suppose that for $3\le u\le\sqrt{2Y}$ the ordinary-integer estimate

$$\sum_{n\le 2Y}|B_u(n)| \ll \frac{Y}{(\log u)^{\delta}(1+\log\log u)^{3/2}} \tag{5.4}$$

holds uniformly. This is the $s=1$ ordinary upper bound announced in Aquila's *Fractional moments of the sharp Möbius divisor sum on integers and reflected primes* [1, Theorem 1], where $\min(u,2Y/u)=u$.[1] Under (5.4), Theorem 1.1 and Corollary 6.2 hold for $\gamma+\eta<\delta$. Indeed every proper cutoff is in this range and only the first row of the table changes, to $L^{\gamma-\delta}$; $3\delta<1$. The transfer to prime partners is still supplied by Lemma A.1. The unconditional assertions throughout the paper use (4.2).

A finite cell with repeated marks. At shift two use bandwidth 2 and take

$$z=1584,\qquad X=z^8,\qquad a=2\,509\,057,\qquad b=2\,970\,329\,053\,142\,991.$$

The progression has 5 317 terms, 208 of them with k prime. Direct enumeration gives the following unweighted counts. A point can carry more than one divisor mark.

Divisor condition	Points	Marks	$R_z>0$	$R_z=0$	$R_z<0$
Prime in $(z/2,z]$	27	30	19	8	0
Prime in $(z,2z]$	24	25	4	13	7
$pq\in(z/2,z]$, outside first row	16	17	3	9	4

[1] In the indexed version, (4) of [1] gives power $2s$ on a sum of two L^2-norms. The local Sobolev and Hölder estimate for $|F|^s$ is homogeneous of degree s, the power used in the ensuing (5). We use (5.4) only in the conditional assertions.

The last row uses $z^{1/5} < p \le z^{2/5}$, which here gives the same prime list as the matching value of β. For $k = 19\,710\,219\,707$ the mark $11 \cdot 97$ gives $\exp(R_z(m)) = 1/4\,373\,721\,012\,067\,333$. Computing the exponential as a rational product of prime powers makes every sign test exact. This cell exhibits the difference between counts of points and of marked divisors. The logarithmic saving in Theorem 1.1 is an assertion about a growing family of cells.

6 Both signs after compensation

The same divisor calculation yields two different statements about how much has cancelled. One measures overlap with a known negative family in each cell and then sums the overlaps. The other finds large masses of both signs among the points that remain after the covering family itself has also been removed.

Corollary 6.1 (Overlap with the rough negative mass)**.** *Let $D = S(E_1 \cap \{w < P^- \le \sqrt{z}\}; \vartheta)$. With N_{ab} as in Section 3, in either orientation,*

$$\liminf_{X\to\infty} \frac{\sum_{\mathcal{C}} \min\{N_{ab}, (D_{ab})_+\}}{\sum_{\mathcal{C}} N_{ab}} \;\ge\; \frac{2}{3}.$$

Proof. Each point of N contributes at most $\log(2M)\,\vartheta(k)$. We use this to prove the envelope $\|(N - 6V)_+\|_1 = o(\mathcal{V})$. Take $D_0 = D_* L^{-c}$ with enough spare precision in (2.1), and sieve m to $y_0 = \min(z, \sqrt{D_0})$ by the universal upper linear-sieve weights of level D_0. Take the same set of sieving primes in every cell, including those with $g(p) = 0$. For squarefree d the coefficients are then common in b. The dimension condition follows, uniformly, from $g(2) = 0$, $g(p) \le 1/(p-1)$ for odd primes, by Mertens' product. The upper function at $s = \log D_0/\log y_0 = 2 + o(1)$ tends to $F(2) = \mathrm{e}^{\gamma_E}$, where γ_E is Euler's constant [8, Chs. 11–12], [21, Lemma 2.2]. Moreover

$$H_{ab} \prod_{p \le y_0} (1 - g(p)) = (1 + o(1)) \frac{\mathrm{e}^{-\gamma_E}}{\log y_0} V_{ab}.$$

This is the Euler product of Lemma 3.1 times Mertens' product. The generic tail is $O(1/y_0)$ and the exceptional tail $O(L/y_0)$. Now $\log(2M)/\log y_0 \to 6$. The sieve majorization is pointwise, its total remainder after summing absolute errors in the cells is $o(\mathcal{V})$ by (2.1). More explicitly fix an arbitrary sieve tolerance first, apply the universal weights and then send the tolerance to zero. The choice of type I precision pays the one factor $\log(2M)$. This proves the envelope.

By Theorem 1.1, $\|D - 4V\|_1 = o(\mathcal{V})$. Since the map $d \mapsto \min(n, d_+)$ is 1-Lipschitz, replacing D by $4V$ costs $o(\mathcal{V})$ in the numerator. For $n, v \ge 0$, $\min(n, 4v) \ge (2/3)(n - (n-6v)_+)$. Sum this inequality and use $\sum N \sim (W(6) - 1)\mathcal{V}$ from (3.4). □

The fraction refers to the sum of the separately truncated overlaps. No positive lower bound on N_{ab}/V_{ab} in every cell enters the proof. In particular, for E_1, E_2 matched as above, Proposition 3.2 and Theorem 1.1 give

$$\|S(1 - E_1; \vartheta) - (P - 5V)\|_1 \ll XL^{-\eta}, \qquad \|S(1 - E_1 - E_2; \vartheta) - (P - V)\|_1 \ll XL^{-\eta}.$$

Here $S(1 - E_1 - E_2; \vartheta)$ includes the full remaining range of $P^-(m)$.

Corollary 6.2 (A fixed three-prime family covering the rough mass)**.** *With the parameters and event E_3 of Theorem 1.2, write $D_3 = S(E_3; \vartheta)$. In either orientation,*

$$\|D_3 - 6V\|_1 \ll XL^{-\eta}, \qquad \frac{\sum_{\mathcal{C}} \min\{N_{ab}, (D_{3,ab})_+\}}{\sum_{\mathcal{C}} N_{ab}} \longrightarrow 1.$$

Here N_{ab} is the negative mass at $P^-(m) > z$ in the same cell.

Proof. For $b_0 \in [1/200, 1/20]$, let

$$I(b_0) = \int_{1/200}^{b_0} \int_{3/50}^{3/25} \frac{\mathrm{d}v\,\mathrm{d}u}{uv(1-u-v)}.$$

This is continuous and strictly increasing from zero. At $1/20$ it exceeds $\log(10)\log(2) > 3/2$. Define ξ by $I(\xi) = 3/2$. The enclosure $0.03488219 < \xi < 0.03488221$ follows by expanding $(1-u-v)^{-1}$ through total degree 15 and bounding the positive tail by $(b_0 + 3/25)^{16}/(1 - b_0 - 3/25)$ before integration against $\mathrm{d}u\,\mathrm{d}v/(uv)$. The third prime has logarithm to base z at least $1 - 1/20 - 3/25 - o(1) = 0.83 - o(1)$, and at most $1 - 1/200 - 3/50 < 1$. It is separated from the first two prime ranges, from $\mathcal{P}_-$, and from both individual prime ranges of the semiprime family.

Here are the changes in the proof of Theorem 1.1. In (5.1) the seven proper subsets leave cutoffs between $z^{1/200}$ and z, and the sign of the full subset is $+1$. The harmonic mass of the marks is $O(\log T/L)$, by summing over r first. Lemma 4.1 thus bounds the proper-subset errors as in Section 5. For two distinct marks, fix a slot in which they differ. The two different primes in that slot each have harmonic cost $O(\log T/L)$ after the other primes in their union are fixed. The remaining fixed exponent ranges have bounded harmonic mass. This gives the same inverse-lcm bound as Lemma 5.1. Overlaps with E_1, E_2 have separate prime factors in the two marks, so the same bound applies. The least-prime mask uses the same $L^J z^2$ modulus. One factor is already below $\sqrt{z}$.

In (4.6), summing r first gives the factor $\log(T/d)/\log(z/(pq))$. The two remaining harmonic prime sums give $I(\xi)/\log z$ as the coefficient of $\log(T/d)$, by partial summation on fixed polynomial ranges. The error is as in (5.3), since $1 - u - v \geq 0.83$. Hence the signed reference is $4I(\xi)V = 6V$ with cell error $O(XL^{-\eta})$. Finally

$$\sum_{\mathcal{C}} \left(N_{ab} - (D_{3,ab})_+\right)_+ \leq \|(N - 6V)_+\|_1 + \|D_3 - 6V\|_1 = o(\mathcal{V}).$$

The denominator is asymptotic to $(W(6) - 1)\mathcal{V}$. This proves the asserted comparison of net masses. The underlying points are disjoint since E_3 requires $P^-(m) \leq \sqrt{z}$, whereas N requires $P^-(m) > z$. The excess above N_{ab} is discarded in each cell. A signed evaluation of the full defect $P - V$ would also have to account for that excess. □

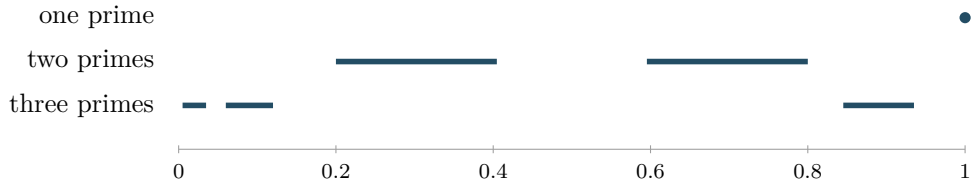

Figure 1: Limiting positions of the individual prime factors on the axis $\log p/\log z$. Each complete mark lies in $(z/T, z]$, so its exponents sum to $1 + o(1)$. The dot represents the limit of the entire one-prime band. The intervals in a row give its marginal ranges. The three rows have signed prime-weighted main terms $4V, -4V, 6V$, respectively, with the exclusions in Theorem 1.1 and Corollary 6.2.

Proof of Theorem 1.2. We construct a further family with r distinct prime divisors, for each fixed integer $r \geq 2$. Its sign alternates with r, and its singular-series coefficient has order $(\log L)^{r-1}$. Given $\gamma + \eta < \rho$, choose

$$0 < \epsilon < \min\left\{\frac{1}{8},\ 1 - \frac{\gamma+\eta}{\rho},\ \frac{1-3\gamma-\eta}{2},\ 1-\gamma\right\}. \tag{6.1}$$

Every entry is positive. This choice leaves slack for powers of $\log L$ both in extraction and in counting intersections. Choose $r-1$ disjoint closed intervals $[\alpha_i, \beta_i]$ strictly inside $(0, \epsilon]$, with positive gaps. Put

$$u_i = L^{-\beta_i}, \quad v_i = L^{-\alpha_i} \quad (i < r), \qquad j' = p_1 \cdots p_r \in (z/T, z], \quad z^{u_i} < p_i \le z^{v_i} \quad (i < r).$$

The last prime is determined as the factor outside the first $r-1$ ranges. It is $z^{1-o(1)}$, but is below z/T, since each small marked prime exceeds T. The other marked primes are $z^{o(1)}$, all larger than $P_0 = \exp(cL^{1-\epsilon})$ for a fixed $c > 0$. For sufficiently large X, the first $r-1$ primes are smaller than every prime slot of the fixed two- and three-prime families. The last prime is larger than all those slots, since their largest upper endpoint is $z^{187/200}$, whereas $p_r = z^{1-o(1)}$. It is below the one-prime band because $j' \le z$ and each removed factor exceeds T. These separations concern individual factors. An integer may still contain a second divisor from a fixed family.

Let E_r' mean that $m \in G$ has at least one such j'. We prove

$$\left\| S(E_r'; \vartheta) - (-1)^{r+1} 4\kappa_r(X) V \right\|_1 \ll_r XL^{-\eta}, \qquad \kappa_r(X) = \int_{\prod_{i<r}[u_i, v_i]} \frac{\prod_{i<r}(\,\mathrm{d}t_i/t_i)}{1 - \sum_{i<r} t_i}. \tag{6.2}$$

In particular,

$$\kappa_r(X) \sim (\log L)^{r-1} \prod_{i<r} (\beta_i - \alpha_i). \tag{6.3}$$

The latter follows from $\sum v_i = o(1)$ and the lengths of the intervals in the measure $\mathrm{d}t/t$.

Extraction. In (5.1) each proper divisor $e \mid j'$ leaves $P_0 \le z/e \le z$. The sum of $1/j'$ over the marks is $O_r((\log T)(\log L)^{r-1}/L)$, by summing over the last prime first. Since $2Y \asymp M/j' = X^{5/8+o(1)}$ and $(z/e)^{5/2} \le X^{5/16}$, (4.2) applies down to P_0. With $\log(z/e) \gg L^{1-\epsilon}$, it and (4.3) give a cost from all proper divisors

$$\ll_r XL^{\gamma-(1-\epsilon)\rho} (\log L)^{O(r)} \ll_r XL^{-\eta}.$$

The first strict margin is $(1-\epsilon)\rho > \gamma + \eta$ by (6.1). The full divisor $e = j'$ has sign $-\mu(j') = (-1)^{r+1}$ and cutoff $z/j' \le T$. Noncoprimality of j' and its quotient requires a square $p^2 \mid m$, $p \ge P_0$. Such m have density $O(P_0^{-1} + M^{-1/2})$ on the full range. Irregular quotients are contained in the irregular set of density $O(z^{-1/2} + z/M)$. Before transfer, Cauchy with fixed divisor moments makes these costs smaller than every fixed power of L^{-1}. Using a bound $X^{o(1)}$ against the first exceptional set would be insufficient. For fixed r the majorants have the form $O_r(L^{j_0}\tau(m)^{j_0})$ with j_0 absolute, since the number of marks is at most $\tau(m)$, the subsets of one mark at most $\tau(m)$, and the remaining divisors at most $\tau(m)$.

Intersections. Two distinct mobile marks differ in a slot. Once the other factors in their union and their assignments to the two marks have been fixed, the two different primes in that slot must both lie in windows of ratio T. Either harmonic sum is $O(\log T/L^{1-\epsilon})$ by the prime number theorem at size at least P_0. For all other slots there are at most two factors per slot, each with harmonic sum $O(\log L)$, including the possible last primes. For fixed r there are finitely many patterns of shared slots. As in Lemma 5.1, the divisor $d \le T$ is coprime to their union and $d[j', j''] \le Tz^2$. The resulting absolute cost is

$$O_r\Big(XL^{-1+2\epsilon+3\gamma} (\log L)^{O(r)}\Big).$$

The second margin in (6.1) makes this $O_r(XL^{-\eta})$. For an intersection with any of E_1, E_2, E_3, forget the exclusions in the fixed event and count every pair of marks. Their factors are disjoint, so the inverse least common multiple is exactly $1/(j'j'')$. Each fixed family has harmonic mass $O(\log T/L)$. The product with the mobile mass bounds the harmonic sum of pairs by $O_r((\log T)^2(\log L)^{r-1}/L^2)$. Multiplying by $O(L)\sum_{d\le T} 1/d$ and transferring costs

$O_r(XL^{-1+3\gamma}(\log L)^{O(r)})$. This allows replacement by one occurrence per integer and removal of all three fixed families. Every mobile mark already includes a factor below $\sqrt{z}$.

Averaging the full divisor. In (4.6), sum over p_r first. Conditional on the other primes, $\log(T/d)/\log(z/(p_1\cdots p_{r-1}))$ is the main term. All excluded prime factors of $2abh$ cost $O_r(L^{O(1)}/P_0)$ times powers of $\log L$. Partial summation over the other variables replaces $1/p_i$ by $\mathrm{d}t_i/t_i$ with errors $O_{r,N}(L^{-N})$, by the prime number theorem above P_0. To see uniformity, use the reciprocal prime measure on each range. The test $(1-\sum t_i)^{-1}$ and its fixed-order derivatives are bounded there for large X. The other harmonic masses cost $O_r((\log L)^{r-1})$. Thus (5.3) holds with $\kappa_r(X)$, with an additional factor $O_r((\log L)^{r-1})$ in its errors. The argument for (4.9) and for excluding $P^- \le w$ in Section 5 gives (6.2). In particular, after the small-prime truncation, the maximal modulus is still $L^J z^2$. At a given modulus v, choosing $j' \mid v$ determines its slots uniquely. Choosing the two additional divisors and summing $\Lambda(\ell)$ costs at most $\tau(v)^3 \log v$, independently of the number of slots. The corresponding pointwise bounds for truncation were also independent of r. Consequently J, the type I precision, and then C may be chosen using only γ, η, j. After s, D have been specified, the ranks r and their intervals may be fixed. Constants and the threshold for X may depend on r.

Finally choose one odd and one even r with $r-1 > s$. For either choice the set where the absolute error in (6.2) exceeds V_{ab} has ν-measure $\ll_r XL^{-\eta}/\mathcal{V} = o(1)$. Outside the union of these two sets, B^+ dominates the odd signed sum and B^- dominates the negative of the even signed sum. By (6.3) both exceed $DV_{ab}(\log L)^s$ for sufficiently large X. This proves (1.4), also uniformly in the reflected centre. □

The same moving-window proof has a conditional consequence. If (5.4) holds, then (1.4) holds throughout $\gamma + \eta < \delta$. In (6.1) replace ρ by δ. The proper-divisor error becomes $O_r(XL^{\gamma-(1-\epsilon)\delta}(\log L)^{O(r)})$, since $P_0 \le z/e \le z \le \sqrt{2Y}$. The two intersection bounds and the choice of C are unchanged. The strict inequalities are compatible because $3\delta < 1$.

The unresolved signed term. Theorem 1.2 controls the two nonnegative parts separately. Their signed difference on G in individual cells, and the full defect $P - V$ in cell norm, are still undetermined. At shift two the tests $\mu(k), \lambda(k)$ are available for the signed lower bands, whereas the three neighbouring Liouville signs in a quartic correlation vary within the cell. Transferring all three requires an additional estimate. In reflection the prescribed factors give two cofactor variables in $am + bk = h$. The corresponding asymptotic for prime representations of the centre also remains open here.

At shift two an exact second-moment identity isolates the remaining cancellation. Write $J_{ab} = |\mathcal{B}_{ab}| = X/(ab) + O(1) \asymp U$. Indeed, m runs through one residue class modulo b in an interval of length X/a. List its solutions in increasing order as (m_t, k_t), $1 \le t \le J_{ab}$, and set

$$d_{ab}(t) = \vartheta(m_t)\vartheta(k_t) - V_{ab}/J_{ab}.$$

Extend this sequence by zero off the cell. Expanding the square and grouping by the rank difference gives

$$\sum_{(a,b)\in\mathcal{C}} \frac{(P_{ab}-V_{ab})^2}{J_{ab}} = \sum_{(a,b)\in\mathcal{C}} \frac{1}{J_{ab}} \sum_{|\ell|<J_{ab}} \sum_{t\in\mathbb{Z}} d_{ab}(t)d_{ab}(t+\ell).$$

Since $\sum J_{ab} \ll ABU = X$, an $o(X)$ bound for the right side would imply $\|P - V\|_1 = o(X)$ by Cauchy–Schwarz. The gaps $|\ell| \le U/L^5$ already contribute $O(X/L)$ in absolute value. In fact $V_{ab}/J_{ab} \ll L$ by Mertens' product, since $(p-1)/(p-2) = (1-1/p)^{-1}(1+1/(p(p-2)))$ for $p > 2$. Hence $|d_{ab}(t)| \ll L^2$, and each gap contributes at most $O(ABL^4)$. The longer gaps, up to the length of each cell, retain both prime weights.

The cell norm also has the exact dual form $\|P - V\|_1 = \sup_{|s_{ab}|\le 1} |\sum s_{ab}(P_{ab} - V_{ab})|$, by choosing the sign of each cell error. A bound restricted to arrays constant on a few intervals

in b for each a would require an approximation to these signs to give the full norm. Pointwise multipliers of modulus at most one preserve bounds for sums of absolute errors. They can destroy cancellation within a signed sum, as the choice of the sign of each summand shows.

A Positive transfer and a weighted sieve application

A saving for the average of a common function of m must survive sampling on the lines $bk = am + 2$ or $bk = h - am$. This is needed precisely when an error is taken in absolute value *before* the sum inside a cell. We prove both versions of (4.3).

Lemma A.1 (Prime-weighted transfer along rows)**.** *Let f be a common function supported on $[M/2, 2M]$, with $|f(m)| \ll L^j \tau(m)^j$ for fixed j. Sum over odd rows $a \sim A$, $(a, h) = 1$, and put $H_a(m) = am + 2$ at shift two, $H_a(m) = h - am$ in reflection. Write $T_a(f)$ for the sum of $f(m)\vartheta(k)$ over $X < am \le 2X$, $bk = H_a(m)$, and all odd $b \sim B$. For every $N > 0$, uniformly in the allowed centres,*

$$\sum_a \left| T_a(f) - (\log 2) \sum_{\substack{X/a < m \le 2X/a \\ 2 \nmid m}} f(m) \right| \ll_{N,j} XL^{-N}. \tag{A.1}$$

For $f \ge 0$ the sum over admissible cells is at most $O(A \sum f) + O_{N,j}(XL^{-N})$. If $P^-(m) > X^\sigma$ throughout the support for a fixed $\sigma > 0$, the signed row estimate also holds with $(b, h) = 1$ in reflection, with a constant allowed to depend on σ.

Proof. The row equation and $(a, h) = 1$ imply $(a, b) = 1$. For primes $k \asymp K$ with $k \nmid h$, the residue classes $m \equiv -2\bar{a}$ or $h\bar{a} \bmod k$ are units and distinct as a varies, since $2A < k$. Since a, k are odd and h even, b is odd exactly when m is odd. Write $E(I; k, r)$ for the difference of the sum of $f\mathbf{1}_{\text{odd}}$ on I in the residue r and the mean over reduced classes. Orthogonality of characters at a prime and the multiplicative large sieve give

$$\sum_{k \asymp K,\ k \text{ prime}} \sideset{}{^*}\sum_{r \bmod k} \sup_I |E(I; k, r)|^2 \ll L^{O_j(1)} \frac{M + K^2}{K} M.$$

Indeed all the nonprincipal characters are primitive, and the orthogonality factor is $1/(k-1)$. For the supremum, decompose an interval into at most two dyadic pieces per scale. Cauchy over scales and the disjoint square masses within one scale cost only logarithms. The character inequality follows from the large sieve [16], [19, Ch. I.4, §4.5] by expansion in Gauss sums. The actual I for a, k is the intersection of the physical m-interval and $B < H_a(m)/k \le 2B$, with endpoints in either order. Cauchy over (a, k), insert $\log k$ and use the injectivity above. The total error is $\ll L^{O_j(1)} \sqrt{AM(M + K^2)} = X^{9/10+o(1)}$. Primes $k \mid h$ in reflection number $O(1)$ at this size and force $k \mid m$, with cost $\ll A(1 + M/K)X^{o(1)}$, including their model contribution. In the unit mean, restoring m with $k \mid m$ costs $O(AMK^{-1}X^{o(1)})$ in all. For each $m \ll M$ only $O(1)$ primes $k \asymp K$ divide m, and each has weight $(\log k)/(k-1) \ll L/K$. The same estimate allows the missing $k \mid h$ to be included in the mean. For each fixed a, m, the coefficient from the remaining primes is

$$\sum_{\substack{H_a(m)/(2B) \le k \le H_a(m)/B \\ k \text{ prime}}} \frac{\log k}{k-1} = \log 2 + O_N(L^{-N})$$

with the corresponding endpoint conventions. This is the prime number theorem on an interval of fixed ratio at size $\asymp K$, and proves (A.1).

For nonnegative f lifting $(b, h) = 1$ gives the first bound in (4.3). For the signed claim assume $P^-(m) > X^\sigma$. An excluded prime $p \mid (b, h)$ divides m, as $(a, h) = 1$. The centre has

$O_\sigma(1)$ such prime divisors. Counting m divisible by p, then the divisors of $h - am$, costs $O_\sigma(A(MX^{-\sigma}+1)X^{o(1)})$. This restores the reflected admissibility condition. In (3.4) one can take any sufficiently small $\sigma > 0$ depending on the fixed t. □

Lemma A.2 (Unweighted positive transfer at shift two)**.** *Let $h = 2$ and let $0 \le f(m) \ll L^j\tau(m)^j$ be common to the cells and supported on $[M/2, 2M]$. With k ranging over all positive integer partners, and every fixed $N > 0$,*

$$\sum_{(a,b)\in\mathcal{C}} \sum_{\mathcal{B}_{ab}} f(m) \ll_j A \log L \sum_m f(m) + O_{N,j}(XL^{-N}).$$

Proof. For $f \ge 0$, enlarge to $k \asymp K,\ a \sim A,\ (a,k) = 1$, retaining only $m \equiv -2\bar{a} \bmod k$, and sum over the support of f. The coprimality $(a,k) = 1$ holds in actual cells because a is odd and divides $bk - 2$. Put

$$\widehat{f}(\theta) = \sum_m f(m)\mathrm{e}^{2\pi i m\theta}, \qquad P_d(r) = \frac{1}{d} \sideset{}{^*}\sum_{t \bmod d} \widehat{f}(t/d)\mathrm{e}^{-2\pi i t r/d}.$$

Additive orthogonality gives the exact sum in a residue r mod k as $\sum_{d|k}(d/k)P_d(r)$, including nonunit r. On $d \sim Q$, Parseval in r and the additive large sieve give

$$\sum_{d\sim Q} \sum_{r \bmod d} |P_d(r)|^2 \ll L^{O_j(1)}M(M+Q^2)/Q.$$

For each $d \mid k$, a class $-2\bar{a}$ mod d has $O(1 + A/Q)$ preimages among admissible a, since $(2,d) \le 2$. Also $\sum_{k\asymp K,\ d|k} d/k \ll 1$. Cauchy over a, d, k with weights d/k therefore bounds this dyade by

$$L^{O_j(1)}\sqrt{A(1+A/Q)M(M+Q^2)}.$$

Relative to $X = AM$ this is $\ll L^{O_j(1)}(Q^{-1/2} + \sqrt{Q/M})$ for $Q \le A$, and $\ll L^{O_j(1)}(A^{-1/2} + K/\sqrt{X})$ for $A < Q \ll K$. Hence $Q > L^S$ costs $O_N(XL^{-N})$ for large enough S.

For $d \le L^S$, each unit class a mod d occurs among $(a,k) = 1$ with the same main multiplicity $(A/d)\prod_{p|k,\,p\nmid d}(1 - 1/p)$ and with error $O(\tau(k))$. In the main term the sum of the phases over these units is a Ramanujan sum $c_d(2)$ of absolute value at most two. Since $|\widehat{f}| \le \sum f$, the main term at fixed d, k, before d/k, is $O((A/d)\sum f)$. The error is $O(d\tau(k)\sum f)$. Summing over k with $d \mid k$ and then $d \le L^S$ costs $O(A\log L\sum f)$ for the main term and $X^{o(1)}\sum f$ for the error. This proves the second line of (4.3). The small Ramanujan sum here uses the fixed shift two. The initial enlargement is available because f is nonnegative. □

A weighted sieve consequence of the same level. As an illustration of the coefficient condition in Theorem 2.1, in either orientation all but $O_N(AB/L^N)$ admissible cells contain $\gg U/L^2$ points with k prime and m squarefree with at most four prime factors. In fact the factors of m may all be required to be at least $X^{49/600}$. The weights are the classical ones [8, Ch. 25], [21, Lemma 2.2]. Meng [15, Theorem 1] averages a coefficient in a prime–almost-prime equation with comparable coefficient scales. Huixi Li [13, Theorem 1] gives representations $ap + bP_2 = N$ for coprime squarefree coefficients under local conditions. Here both coefficients range in specified unequal dyadic intervals and the exceptional set is counted in that two-parameter family. In terms of the size of the sifted cofactor, our level is $D = M^{49/150}$. The switching theorem of Matomäki and Zuniga–Alterman [14, Theorem 3.2] requires distribution estimates for both the original and the switched sequence. Theorem 2.1 supplies the original-sequence input for the four-factor argument below.

To see this, take $D = X^{49/200}$, $y_1 = D^{1/3}$, $y_2 = D^{9/10}$, $\eta_0 = 3/2$, and write $V_g(v) = \prod_{p<v}(1-g(p))$. In the positive measure $\vartheta(k)$ take the Richert weight

$$\mathbf{1}_{P^-(m)\ge y_1}\left\{\eta_0 - \sum_{\substack{y_1\le p<y_2\\ p\mid m}} (1-\log p/\log y_2)\right\}.$$

Use the lower linear sieve at level D for the first term and the upper sieve at level D/p for the marked terms, sifting the latter to $\min(y_1, \sqrt{D/p})$. Primes p dividing the augmented modulus lie above the smaller sieving range. On putting $t = \log p/\log D$, the resulting upper coefficient divided by $g(p)H_{ab}V_g(y_1)$ is at most $2\mathrm{e}^{\gamma_E}/(3(1-t)) + o(1)$, up to an arbitrarily small fixed sieve tolerance. For the branch with smaller sieving range this uses $V_g(\sqrt{D/p})/V_g(y_1) \le (1+o(1))\log y_1/\log\sqrt{D/p}$. Mertens' formula holds uniformly on these polynomial-sized ranges. The finitely many exceptional prime divisors of the parameters there cost $o(1)$ in the harmonic sum. With $\beta_0 = 9/10$ the lower coefficient of $H_{ab}V_g(y_1)$ is therefore at least

$$\frac{2\mathrm{e}^{\gamma_E}}{3}\left\{\frac{3}{2}\log 2 - \int_{1/3}^{\beta_0}\frac{1-t/\beta_0}{t(1-t)}\,\mathrm{d}t\right\} + o(1) > 0.30$$

up to the fixed sieve tolerance, which can be chosen smaller than the margin. Evaluation of the integral gives $\log(3\beta_0) - (\beta_0^{-1}-1)\log((2/3)/(1-\beta_0))$. For given modulus $d \le D$ only $O(1)$ marked prime divisors $p \ge y_1$ are possible, and the remaining sieve coefficients are universal. The whole linear combination of remainders is therefore covered by (2.1), to any fixed logarithmic precision. Since $H_{ab}V_g(y_1) \gg U/L$, Markov gives the stated exceptional count. Points with $p^2 \mid m$, $p \ge y_1$ cost $\ll A(M/y_1 + \sqrt{M})X^{o(1)}$ globally. They can be discarded with the same exceptional count. For a remaining point of positive weight, $\Omega(m) < \eta_0 + \log(2M)/\log y_2 < 5$, by adding the nonpositive penalties for $p \ge y_2$. Each contributes at most $\eta_0 \log k \ll L$, yielding the count.

Inputs used in the proof. The table identifies the estimates used in the proofs and the step where each enters.

Input and location	Application in this paper
Mean square in progressions, [11, §3.1.3, (11)]	Small-gcd energy in (2.7), with the square-divisor reduction for λ.
Large sieve, [16, Thm. 1]	Sampling b after (2.7) and both bounds in (4.3).
Sharp Möbius sum, [9, (1.4), §5], [4, 3]	Support and mean square for (4.2). All proper cutoffs in the fixed and mobile windows satisfy its size condition.
Small-prime pairing, [5, Theorem 1, proof], [6, Theorem 1]	The second, wider-range bound in Lemma 4.1.
Prime number theorem, Mertens estimates and Buchstab's estimate, [19, Chs. I.1, II.4, III.6]	The convolution in Lemma 3.1, harmonic prime windows and the rough mass in (3.4).
Rosser–Iwaniec sieve and Richert weights, [8, Chs. 11–12, 25], [21, Lemma 2.2]	The upper envelope in Corollary 6.1 and the four-factor consequence above.

AI assistance and reproducibility

OpenAI's ChatGPT assisted with the mathematical arguments and exposition. Codex assisted with the proofs, the comparison with published estimates, and the accompanying exact computations. The source includes code for the residue-class masks, the rational enclosures, and the full progression in Section 5. The estimates as $X \to \infty$ rest on the proofs above. The author assumes responsibility for the manuscript.

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