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Genus Bounds, Symmetric Monodromy, Cover Rigidity, and Formal Algebraic Descent for the Curve of Simultaneous Reality

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Abstract

Assuming the three global companion inputs (G1)–(G3) of Section 1 (bidegree and reduction, smooth-cover ramification and singular-locus data, and generic correction partition with elimination degree bounds) for the simultaneous-reality curve of bidegree $(1156, 269)$, the characteristic-zero genus satisfies $6{,}631 \leq g(C) \leq 16{,}570$. At $p = 1{,}000{,}003$ the special curve E_p has geometric genus 6,631 and total delta invariant 302,909, of which 292,970 units are resolved on the arithmetic normalization: 250,035 ordinary nodes, 1,568 ordinary triple points (4,704 units), and all 38,231 units above $R = 25$ via an explicit integral degree-66 separator whose residues admit a closed-form algebraic description. Every local analytic lemma used in this chain is proved self-contained here, together with a characteristic-zero factorization of the leading elimination coefficient $\ell(R)$ over $\mathbb{Q}$ and a full 1,398-slice deterministic proof of the special-fiber conductor identities. Exact genus 6,631 in characteristic zero and good reduction at p remain conditional on arithmetic integrality lifts for 9,939 delta units at the remaining 125 special base places.

Assuming Global Inputs (G1)–(G2), full arithmetic and geometric monodromy S_{269} and a trivial deck group are proved in both characteristics. The smooth-cover ramification divisor excludes every nonidentity base transformation in the special fiber, and under (G1)–(G3) a semistable contracted-genus bound yields trivial cover-preserving automorphisms in characteristic zero without assuming global good reduction. Using only the lower bound $g(C) \geq 6{,}631$, the Castelnuovo–Severi inequality forces degree at least 26 for every nonconstant map to a smooth projective curve of genus at most seven not factoring through π_R.

Five archived checker invocations spanning three independent codebases across four runtimes (`lean` and `leanchecker` on Lean 4.33.1, together with `con-leche`, its Rust port `con-ron`, and `nanoda_bin` on one exported environment) certify algebraic integral descent over a commutative ring, localization transfer, and a conditional bidegree genus-squeeze implication (with the remaining inequality supplied as a hypothesis). A separate auxiliary open-source dual-route verifier for a restricted exact-rational class of plane curves certifies genera $0, 2, 8$ for singular curves of degrees $5, 7, 7$; the genus-eight curve has gonality three (trigonal) and is non-hyperelliptic. All new derivations, formalizations, symbolic and finite-field audits, and text were produced entirely by AI systems under human direction and supervision.

1 Introduction and the simultaneous-reality curve

In the twin-factor deformation of a Salikhov integral [13], the reality of the first relative saddle correction alters the available estimates for exceptional indices; the motivating integrals are

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studied by Bai and Niedbala Giraudin [1, 20]. Set

$$P(z) = z^2 + 6z + 25, \qquad Q(z) = z^2 - 6z + 25, \qquad \Omega(z) = zP(z)Q(z)(25 - z).$$

For $3/5 \le a, b \le 7/10$ and $0 < t < b$, the logarithmic derivative of the phase and the amplitude are

$$\Phi'(z) = \frac{a}{2z} + (b - t)\frac{P'(z)}{P(z)} + t\frac{Q'(z)}{Q(z)} + \frac{1}{25 - z}, \qquad A(z) = \frac{1}{\sqrt{z}(25 - z)}.$$

The stationary quintic is established in the companion study [14] to have an exterior real zero $R > 25$ and one zero in each open quadrant throughout this parameter range. Let z_-, z_+ denote the two upper zeros in order of increasing real part. At a simple saddle, with $\Phi_j = \Phi^{(j)}(z)$, the first relative coefficient is [21, 14]

$$h = -\frac{A''/A}{2\Phi_2} + \frac{(A'/A)\Phi_3/2 + \Phi_4/8}{\Phi_2^2} - \frac{5\Phi_3^2}{24\Phi_2^3}. \tag{1}$$

In [14], the real-correction arc $\mathcal{S}$ is cut out by $\Im h(z_-) = \Im h(z_+) = 0$ in the box $[69/100, 7/10] \times [3/5, 13/20] \times [7229/10000, 7671/10000]$ in $(a, b, t/b)$, and its image in the (R, b)-plane has primitive irreducible equation $E(R, b) = 0$. Denote the smooth projective normalization by C, the corresponding incidence curve retaining the weight t by Γ, and projection to R by π_R.

Global inputs from the companion study. Three global inputs are imported from the companion study [14] (a preprint in preparation whose frozen E_p coefficient array, discriminant, singular-locus, and ramification tables are shipped directly in this release under `companion_tables/`; the companion text itself is not part of this submission). All results in Sections 3 and 4 are proved as explicit logical implications *assuming Global Inputs (G1)–(G3)*:

(G1) *Bidegree and reduction.* $\mathbb{Q}(\Gamma) = \mathbb{Q}(C) = \mathbb{Q}(R, b)$, the exact bidegree of E is $(1156, 269)$, and at $p = 1{,}000{,}003$ the reduction satisfies $\bar{E} = c_0 E_p$ with $c_0 \in \mathbb{F}_p^\times$ and geometrically integral bihomogeneous special fiber of the same bidegree. Adjunction on $\mathbb{P}^1 \times \mathbb{P}^1$ gives $p_a = 1155 \cdot 268 = 309{,}540$.

(G2) *Smooth-cover ramification divisor and singular-locus breakdown.* On the smooth normalization T of $E_{p,\overline{\mathbb{F}}_p}$, the projection $T \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ is separable of degree 269 with different of degree 13,798; hence $2g(E_p) - 2 = 13{,}798 - 538$, geometric genus $g(E_p) = 6{,}631$, and total delta invariant 302,909. Moreover, the companion Puiseux and blow-up census (`companion_tables/special_places.json`, `companion_tables/table.txt`, `companion_tables/places.json`) enumerates all 131 exceptional closed base places together with their ramification indices and per-stratum plane delta distribution (including the total delta 5,014 across the 122 non-PQ, non-$R = 25, \infty$ special places and maximum fiber delta 1,214 outside $R = 25$); the specific rows invoked in Theorem 4.2 are summarized in Appendix A.

(G3) *Generic correction partition and elimination polynomials.* On Γ, the five stationary roots generically have correction multiplicities $2, 2, 1$; at $R = 65$ the two paired centers lie in $(0.787, 0.788)$ and $(-0.114, -0.113)$, while $h(R) \in (-0.036, -0.035)$, so the equal-value pairing and the singleton R are defined over $\mathbb{Q}(C)$. The two eliminated polynomials $\widehat{\Sigma}_1, \widehat{\Sigma}_2$ defined in Section 3 vanish on Γ and have (b, t)-degrees $41, 68$ and R-degrees at most $237, 356$.

Every local analytic lemma required in the main resolution chain (simple regular and simple pole expansions, double regular collisions, double and triple pole clusters, and the three $R = 25$ integrality regimes), the characteristic-zero factorization of $\ell(R)$, the 1,398-slice conductor proof, and the resolution of $M_2, M_6, R = 25$ are proved and certified self-contained in Section 3. The

companion lower bound $g(C) \geq 6{,}631$ already implies finiteness of $C(\mathbb{Q})$ by Faltings's theorem [10]. Equality asks whether the arithmetic normalization over $\mathbb{Z}_{(p)}$ has a smooth special fiber, separating special-fiber jet-rank computations from arithmetic integrality proofs.

Summary of main results. Assuming Global Inputs (G1)–(G3), the two main results of this paper are: (i) Theorem 3.6, proving the characteristic-zero genus enclosure $6{,}631 \leq g(C) \leq 16{,}570$ by resolving 292,970 of the 302,909 special-fiber delta units on the arithmetic normalization and characterizing exact equality $g(C) = 6{,}631$ as smoothness of the arithmetic normalization over $\mathbb{Z}_{(p)}$; and (ii) Theorem 4.2, establishing full arithmetic and geometric monodromy S_{269} in characteristics zero and p, trivial deck groups, trivial cover-preserving automorphism groups $\mathrm{Aut}(T, \pi_{R,0}) = \{1\}$ and $\mathrm{Aut}_{\overline{\mathbb{Q}}}(C, \pi_R) = \{1\}$ via the contracted-genus bound of Lemma 4.1 without assuming global good reduction, and the Castelnuovo–Severi degree bound $d \geq \lceil 6899/(268 + g_Y) \rceil$ (hence $d \geq 26$ for all $g_Y \leq 7$) for every nonconstant map $C \to Y$ not factoring through π_R.

1.1 Epistemic classification

This manuscript separates mathematical proofs, deterministic computational certificates, and archived historical run logs across three tiers. Tier (A) consists solely of the eight named Lean 4.33.1 declarations in Section 2.2 (including the conditional arithmetic implication `cm_curve_exact_genus_6631`, whose key inequality is an explicit hypothesis). Tier (B) consists of geometric proofs combined with deterministic exact integer, rational, and $\mathbb{F}_p$-computations (all proved assuming Global Inputs (G1)–(G3)), together with explicitly conditional statements for the remaining 125 special base places. Tier (C) consists of independent rational plane-curve verifications requiring no companion data. All Python/FLINT computations and arithmetic-integrality verifications lie outside the Lean formalization. Table 1 records the scope of each claim.

Table 1: Objects, verified conclusions, and epistemic tiers.

Object	Conclusion	Tier
Algebraic local ring and bidegree	Integral closedness in $A[1/\pi]$, localization transfer, and conditional arithmetic implication	A
Special curve E_p	Geometric genus 6,631 and singular-locus census (G2); exact $\mathbb{Q}$-factorization of $\ell(R)$ and 1,398-slice conductor proof	B
M_2 and M_6 strata	250,035 nodes (incl. 58 in 6 orbits) and 1,568 triple points (254,739 units), assuming (G1)–(G3)	B
$R = 25$ fiber	38,231 units resolved via closed-form degree-66 separator, assuming (G1)–(G3)	B
Arithmetic normalization	$6{,}631 \leq g(C) \leq 16{,}570$ assuming (G1)–(G3); conditional good reduction for 9,939 remaining units	B
Degree-269 covers	S_{269} monodromy, trivial deck group, and cover rigidity in char. 0 and p, assuming (G1)–(G3)	B
Non-factoring maps	Degree at least 26 for target genus $g_Y \leq 7$, assuming $g(C) \geq 6{,}631$ from (G1)–(G2)	B
Three plane curves	Self-contained exact genera $0, 2, 8$ via auxiliary dual-route verifier	C

2 Arithmetic normalization, centered valuations, and formal algebraic descent

2.1 Normalization and the finite-jet criterion

Put $S = \operatorname{Spec} \mathbb{Z}_{(p)}$ and let $S^{\mathrm{sh}} = \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{S,\bar{s}}$ be its strict henselization at the closed point $\bar{s}$ with residue field $\overline{\mathbb{F}}_p$. Let $\mathcal{E}$ be the bihomogeneous closure of $E = 0$ over S, $X = \widetilde{\mathcal{E}}$ its normalization, $Y_0 = X_{\overline{\mathbb{F}}_p}$, and T the smooth normalization of $E_{p,\overline{\mathbb{F}}_p}$. At a geometric plane point $Q \in E_{p,\overline{\mathbb{F}}_p}$, write $A_0 = \mathcal{O}_{E_{p,\overline{\mathbb{F}}_p},Q}$, N_Q for its semilocal normalization, and B_Q for the finite A_0-algebra obtained from Y_0 above Q. Write $\delta_Q(Y_0) = \dim_{\overline{\mathbb{F}}_p}(N_Q/B_Q)$.

Proposition 2.1 (Discriminant, centered valuations, and jets)**.** *Assuming Global Inputs (G1)–(G2), the morphism $X \to \mathbb{P}^1_S$ is finite locally free of rank* 269*, and*

$$g(C) = 6{,}631 + \sum_Q \delta_Q(Y_0), \qquad A_0 \subseteq B_Q \subseteq N_Q. \tag{2}$$

Let A be any Noetherian local domain with maximal ideal $\mathfrak{m}_A$ and fraction field K (for example, the local ring of the plane model or of an intermediate finite order over S^{sh} at a closed special point). A rational function $g \in K$ is integral over A if and only if g has nonnegative value in every rank-one valuation of an algebraic closure $\overline{K}$ centered at $\mathfrak{m}_A$. If $g = G_1/G_2$ with $G_1, G_2 \in A$ and G_2 generically nonzero on the special fiber, then whenever g is integral over A its reduction in the special order over A equals $\bar{G}_1/\bar{G}_2$.

At a closed point of E_p of degree d_c over $\mathbb{F}_p$, let A_c be the completed local curve ring and $N = \prod_i F_i[[s_i]]$ its normalization. For $I = \prod_i (s_i^c)$, let A_I be the image of A_c in N/I. If $\dim_{\mathbb{F}_p}((N/I)/A_I) = d_c\delta$, where δ is the delta invariant per geometric point, then $I \subset A_c$. Assuming $I \subset A_c$, if a subalgebra $S_I \subseteq N/I$ is generated over A_I by reductions of functions proved integral over the arithmetic local ring, then the total arithmetic delta gain across the geometric orbit is at least $\dim_{\mathbb{F}_p}(S_I/A_I)$; in particular $S_I = N/I$ gives $B_Q = N_Q$ throughout that orbit.

Proof. Both projective curve fibers have finite projection to the R-line by irreducibility and positive degree in b. Proper quasi-finiteness gives finiteness [22, Tag 02OG], and excellence gives finite normalization. The normal surface X is Cohen–Macaulay by Serre's criterion in dimension two; since $\mathbb{P}^1_S$ is regular, miracle flatness yields local freeness of rank 269 [15], [22, Tag 00R4]. At the generic point of the special plane curve the local ring is a DVR with uniformizer p because E_p is integral, so normalization is an isomorphism there, and finiteness precludes exceptional curves over closed points. The Cartier special fiber of the Cohen–Macaulay surface has no embedded points, so generic reducedness implies reducedness; both persist over $\overline{\mathbb{F}}_p$. Thus Y_0 is integral and finite birational over $E_{p,\overline{\mathbb{F}}_p}$, with normalization $T \to Y_0$.

Flat projectivity over the DVR S gives $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_{Y_0})$, and the normalization sequence on Y_0 gives $\chi(\mathcal{O}_T) - \chi(\mathcal{O}_{Y_0}) = \sum_Q \delta_Q(Y_0)$, proving (2). Equivalently, formation of the trace discriminant of the finite locally free algebra commutes with base change, yielding

$$2g(C) - 2 + 2 \cdot 269 = 13{,}798 + 2\sum_Q \delta_Q(Y_0)$$

and recovering the residual-genus inequality in this setting (cf. [18]).

For the valuation equivalence, the case where A is a field is trivial (the only valuation centered at (0) is trivial and every element of $K = A$ is integral). Assume therefore that A is not a field. Integral elements have nonnegative order in every valuation centered at $\mathfrak{m}_A$ because valuation rings are integrally closed. Conversely, suppose $g \in K^\times$ is not integral over A. Then the ideal $\mathfrak{m}_A A[g^{-1}] + (g^{-1})$ in $A[g^{-1}]$ is proper: a representation of 1 in it would have constant coefficient in $\mathfrak{m}_A$, and clearing powers of g^{-1} would produce a monic integral equation for g over A after

dividing by a unit. Localizing $A[g^{-1}]$ at a maximal ideal containing this proper ideal yields a Noetherian local domain (not a field, since $g^{-1} \neq 0$ lies in its maximal ideal) dominated by a DVR of K by EGA II, (7.1.7) [12] (see also [22, Tag 00PH]). Extending that DVR to $\overline{K}$ gives a rank-one valuation centered at $\mathfrak{m}_A$ with $v(g^{-1}) > 0$, i.e., $v(g) < 0$. When g is integral, reducing $G_2 g = G_1$ in the integral special order yields $\bar{g} = \bar{G}_1/\bar{G}_2$.

For the jet criterion, standard commutative algebra for complete reduced one-dimensional local rings gives the finite product decomposition $N = \prod_i F_i[[s_i]]$ [15], [22, Tags 032Y and 0C44] (computed explicitly via rational Puiseux expansions [9]). The kernel of the surjection $N/A_c \to (N/I)/A_I$ is $(A_c + I)/A_c$; equality of $\mathbb{F}_p$-dimensions forces $I \subset A_c$. Under $I \subset A_c$, the completed intermediate order $\widehat{B}_c$ contains I, so its image modulo I contains S_I and $\dim_{\mathbb{F}_p}(\widehat{B}_c/A_c) \geq \dim_{\mathbb{F}_p}(S_I/A_I)$, with equality $S_I = N/I$ forcing $\widehat{B}_c = N$. Completion preserves finite lengths [22, Tag 0C44], and dividing by the residue degree d_c recovers the gain per geometric point. □

2.2 The formal declarations

The archived 250-line Lean 4.33.1 module `DVRConductorNoGoingUp.lean` [8, 17] formalizes the elementary algebraic descent lemma underlying [15, Lemma 4.1.18] together with conditional bidegree arithmetic. Neither the EGA II valuation criterion of Proposition 2.1 nor the Python/FLINT computations of Sections 3–5 are formalized in Lean. The eight named verification targets in namespace `DVRConductorNoGoingUp` are:

1. `dvd_of_cleared_monic_relation`. For a commutative ring R_0, reduced $R_0/(\pi)$, and $k > 0$, if $a^d + \sum_{i<d} c_i a^i \pi^{k(d-i)} = 0$, then $\pi \mid a$ (in the degenerate $d = 0$ case, $1 = 0$ forces the zero ring).

2. `dvr_integral_closed_in_away_of_reduced_quotient`. If π is a non-zero-divisor and $R_0/(\pi)$ is reduced, every element of $R_0[1/\pi]$ integral over R_0 lies in the image of R_0.

3. `normal_of_reduced_special_fiber_and_normal_generic_fiber`. Under the same hypotheses, if an injective R_0-algebra map $R_0[1/\pi] \to B$ contains every R_0-integral element of B in its image, then every such element lies in the image of R_0. This algebraic localization descent (used in the proof of Theorem 4.2(b)) requires no Noetherianity or dimension hypothesis.

4. `genus_squeeze_exact`. For natural numbers, $g_p \leq g_C$, $g_C + D \leq p_a$, and $p_a = g_p + D$ imply $g_C = g_p$.

5. `cm_curve_bidegree_pa`. The bidegree arithmetic-genus expression at $(1156, 269)$ equals 309,540.

6. `cm_curve_exact_genus_6631`. Given $6{,}631 \leq g_C$ and $g_C + (250{,}035 + 4{,}704 + 38{,}231 + 4{,}856 + 69 + 5{,}014) \leq 309{,}540$, the conclusion is $g_C = 6{,}631$. The second inequality is an explicit hypothesis representing the conditional arithmetic lift problem.

7. `conic_completing_square`. Univariate quadratic completing-the-square identity bundled in the same module (not invoked in Sections 3–5).

8. `conic_strictly_pos_of_neg_disc`. Strict positivity of a univariate quadratic with positive leading coefficient and negative discriminant, bundled in the same module (not invoked in Sections 3–5).

The five archived checker invocations (spanning three independent codebases across four runtimes: official C++ kernel via `lean` on the source and `leanchecker` on the object file, plus `con-leche` [3], its Rust port `con-ron` [4], and independent Rust checker `nanoda_bin` [2] on one `lean4export` stream) are documented in Section 6.

3 Conductor numerators, local lemmas, and genus bounds

Notation conventions. Throughout this section, $\alpha = a/2$ denotes the half-weight at $z = 0$; h denotes the first relative saddle correction (1); v denotes a rank-one valuation centered above a closed special point with scaling factor ς making the stationary polynomial primitive (so $\Phi^{\sharp\prime} = \Phi'/\varsigma$ and $h^\sharp = \varsigma h$); X_{fer} denotes the indeterminate of the Ferrari resolvent cubic; T denotes exclusively the smooth special curve $T \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$; $b = \beta_2(R)$ denotes the singular section over M_2; $\Lambda = \Delta_* t(b-t)$ denotes the elimination factor; and $S_{25} = t/b + (R-25)\vartheta_{25}$ denotes the separator at $R = 25$.

Eliminating $\alpha = a/2$ by $\Phi'(R) = 0$ gives

$$\alpha = \frac{R}{R-25} - (b-t)\frac{RP'(R)}{P(R)} - t\frac{RQ'(R)}{Q(R)}.$$

Set $k_R = (R-25)P(R)Q(R)$, $F(z) = k_R\Omega(z)\Phi'(z)$, and $F_*(z) = F(z)/(z-R) = \zeta z^4 + f_3 z^3 + f_2 z^2 + f_1 z + f_0$. Writing $d = F'$, $\Omega_0 = \Omega/z$, and $\Omega_{25} = \Omega/(25-z)$, put

$$\begin{aligned} J_F &= F''\Omega - 2d\Omega', & K_F &= F'''\Omega^2 - 3F''\Omega\Omega' + 3d(2(\Omega')^2 - \Omega\Omega''),\\ \mathcal{D} &= 24\Omega d^3, & \mathcal{N} &= -3(3\Omega_0^2 - 4\Omega_0\Omega_{25} + 8\Omega_{25}^2)d^2\\ & & &\quad + 6(-\Omega_0 + 2\Omega_{25})J_F d + 3K_F d - 5J_F^2. \end{aligned}$$

At a simple stationary root away from the poles, differentiating (1) yields $h = k_R\mathcal{N}/\mathcal{D}$. Put $e_i = \zeta^{-2}[Y^i]\operatorname{Res}_z(F_*, \mathcal{D}Y - \mathcal{N})$, $Z = 4e_4e_2 - e_3^2$, $\Delta_* = \operatorname{Disc}_z F_*$, and $\Lambda = \Delta_* t(b-t)$. The two eliminated polynomials of Global Input (G3) are

$$\widehat{\Sigma}_1 = (8e_4^2e_1 - e_3Z)/\Lambda^5, \qquad \widehat{\Sigma}_2 = (64e_4^3e_0 - Z^2)/\Lambda^5.$$

The fourteen finite exceptional classes are the roots of the following irreducible polynomials over $\mathbb{F}_p$ (comprising 21 geometric values, where the last four factor $3R^2-8R-75$ and $25R^4-226R^2+15625$):

$$\begin{gathered} R,\ R+1,\ R-5,\ R+5,\ R-25,\ R^2+25,\ P(R),\ Q(R),\\ R^2+8R-25,\ R^2-8R-25,\ R-214011,\ R-119326,\ R^2+245963,\ R^2+434030. \end{gathered}$$

3.1 Conductor numerators, exact $\mathbb{Q}$-factorization of ℓ, and Ferrari pairing

Lemma 3.1. *Assuming Global Inputs (G1) and (G3), label the roots of F_* so that $h(z_1) = h(z_2) = H_1$ and $h(z_3) = h(z_4) = H_2$. The pairing sum $\tau_\sigma = \zeta(z_1z_2 + z_3z_4)$ belongs to $\mathbb{Q}(C)$. Its monic resolvent cubic and derivative at τ_σ are*

$$\begin{aligned} F_{\mathrm{fer}}(X_{\mathrm{fer}}) &= X_{\mathrm{fer}}^3 - f_2X_{\mathrm{fer}}^2 + (f_1f_3 - 4\zeta f_0)X_{\mathrm{fer}} - (f_0f_3^2 - 4\zeta f_0f_2 + \zeta f_1^2),\\ F'_{\mathrm{fer}}(\tau_\sigma) &= \zeta^2(z_1-z_3)(z_1-z_4)(z_2-z_3)(z_2-z_4). \end{aligned}$$

The barycentric shift $\tau_{\mathrm{bary}} = 3\tau_\sigma - f_2 = \zeta[(z_1-z_3)(z_2-z_4) + (z_1-z_4)(z_2-z_3)]$ and the quartic invariants satisfy

$$\begin{aligned} I &= 12\zeta f_0 - 3f_3f_1 + f_2^2,\\ J &= 72\zeta f_2f_0 + 9f_3f_2f_1 - 27\zeta f_1^2 - 27f_3^2f_0 - 2f_2^3,\\ 4I^3 - J^2 &= 27\operatorname{Disc}_z(F_*) = 27\zeta^6\prod_{i<j}(z_i-z_j)^2. \end{aligned}$$

Let $M = k_RM_{\mathcal{D}}^{-1}M_{\mathcal{N}}$ be the multiplication operator of h in the power basis of the quartic algebra. Then $H_1 + H_2 = \operatorname{Tr}(M)/2$ and $H_1H_2 = ((\operatorname{Tr}M)^2 - 2\operatorname{Tr}(M^2))/8$. Let $\ell(R) = [t^{41}]\widehat{\Sigma}_1$ and put

$L_\pm = R^2 \pm 8R - 25$, $\Xi_2 = 3R^2 - 8R - 75$, *and* $\Xi_4 = 25R^4 - 226R^2 + 15625$. *Over* $\mathbb{Q}$, $\ell(R)$ *has exact degree* 234 *and factors completely as*

$$\ell(R) = 2^{242}\,3^{60}\,5^{126}\,R^3(R{+}1)^8(R{-}5)^6(R{+}5)^6(R{-}25)^{57}(R^2{+}25)^9P(R)^9Q(R)^9L_-^{17}L_+^{17}\Xi_2^8\Xi_4^4. \quad (3)$$

Modulo $p = 1{,}000{,}003$, $\Xi_2 = 3(R - 214011)(R - 119326)$ *and* $\Xi_4 = 25(R^2 + 245963)(R^2 + 434030)$, *so* $\bar{\ell}$ *is supported solely on the fourteen exceptional classes with leading coefficient* $c_\ell = 2^{242}\,3^{68}\,5^{134} \equiv 953{,}287 \pmod p$. *In the special function field* $\mathbb{F}_p(E_p)$,

$$\bar{t} = \frac{N_t^{\mathrm{red}}}{D_0(E_p)_b}, \qquad \bar{\tau}_\sigma = \frac{N_\sigma^{\mathrm{red}}}{D_0(E_p)_b}, \qquad D_0 = (R^2 + 8R - 25)^3(R^2 - 8R - 25)^3, \quad (4)$$

where the stored reduced numerators have R*-degrees* 1168 *and* 1172 *and* b*-degrees at most* 268.

Proof. The matching is invariant under $\mathrm{Gal}(\overline{\mathbb{Q}(C)}/\mathbb{Q}(C))$ because the generic correction centers are distinct (Global Input (G3)). Expanding the product of the three pairing factors yields F_{fer}, its derivative, τ_{bary}, and the invariant identity. Since M has eigenvalues H_1, H_1, H_2, H_2, the trace formulas follow directly.

For (3), set $b = 0$ and write $F = F_0 + tF_1$, $F_* = G_0 + tG_1$, $\mathcal{D} = t^3\mathcal{D}_1 + O(t^2)$, and $\mathcal{N} = t^2\mathcal{N}_1 + O(t)$ in t. Direct expansion over $\mathbb{Q}[R, z]$ gives

$$G_1(z) = [t]F_*(z) = 12(R - 25)(z - 25)(Rz + 25)\big((R^2 - 25)z^2 - 64Rz - 25R^2 + 625\big).$$

Since $\deg_z(Y\mathcal{D}_1 - \mathcal{N}_1) = 18$ and $\deg_t \zeta = 1$, resultant homogeneity gives $\deg_t e_i \le 24 + i$ after dividing by ζ^2. Because $z = 25$ is a common root of G_1 and $\mathcal{D}_1 = 24\Omega(F_1')^3$, $[t^{28}]e_4 = \mathrm{Res}_z(G_1, \mathcal{D}_1)/\zeta_1^2 = 0$, where $\zeta_1 = [z^4]G_1$. Thus $e_4^2e_1$ and $e_4e_3e_2$ have t-degrees at most 79 and 80, and the degree-81 term of $8e_4^2e_1 - e_3(4e_4e_2 - e_3^2)$ comes purely from $+e_3^3$. Writing $G_1 = (z-25)G_{1,\mathrm{rem}}$, $[t^{27}]e_3 = -\mathcal{N}_1(25)\,\mathrm{Res}_z(G_{1,\mathrm{rem}}, \mathcal{D}_1)/\zeta_1^2$, while $[t^{40}]\Lambda^5 = -(\mathrm{Disc}_z\, G_1)^5$ (since $\Lambda = -t^2\Delta_*$ at $b = 0$). Evaluating this exact quotient over $\mathbb{Q}[R]$ (certified symbolically in `verify_paper4_new_checks.py` and independently cross-checked modulo p by 240-point interpolation in `verify_m2_m6.py`) yields (3) and $c_\ell = 953{,}287$.

For $g \in \{t, \tau_\sigma\}$, rescale the defining equation E by a unit in $\mathbb{Z}_{(p)}^\times$ so that $\bar{E} = E_p$ (which leaves the arithmetic model over $\mathbb{Z}_{(p)}$ and all valuations unchanged). The Lagrange interpolant

$$\sum_{E(R,b_i)=0} \ell(R)\, g_i\, \frac{E(R, X)}{X - b_i}$$

has X-degree at most 268 and evaluates to $\ell g E_b$ on the generic curve. At any finite valuation of $\mathbb{Q}(R)$ (including the vertical Gauss valuation above p), put $m_i = \min(0, v(b_i))$. Homogenizing the degree-41 equation $\widehat{\Sigma}_1 = 0$ in (b, t) and clearing $\ell(R)$ shows $v(\ell t_i) \ge m_i$; since F_{fer} is weighted homogeneous in the coefficients of F_*, which are affine in (b, t), the same bound holds for $\ell\tau_{\sigma,i}$. Because the Gauss valuation of $E(R, X)/(X - b_i)$ is at least $-m_i$, the interpolants belong to $\mathbb{Z}_{(p)}[R, X]$. At $R = \infty$, scaling by $s = 1/R$ via $s^{1156}E$, $s^{237}\widehat{\Sigma}_1$, and s^4F_* (each coefficient of F_* has R-degree at most 4) gives R-degree bounds $1393 = 1156 + 237$ and $1397 = 1393 + 4$.

Because the interpolants lie in $\mathbb{Z}_{(p)}[R, X]$ and $\bar{E} = E_p$, their reductions modulo p are uniquely determined by their values modulo $E_p(r, b)$ on any 1,398 distinct slices $r \in \mathbb{F}_p$ (since b-degrees are at most $268 < \deg_b E_p(r, b)$). Note that $D_0 \mid \bar{\ell}$ with quotient $Q_\ell = \bar{\ell}/D_0$ of degree 222, and $Q_\ell N_t^{\mathrm{red}}, Q_\ell N_\sigma^{\mathrm{red}}$ have exact R-degrees $1390 \le 1393$ and $1394 \le 1397$ and b-degrees at most 268. The deterministic script `verify_conductor_1398.py` (`conductor_1398_cert.json`) certifies 1,398 consecutive admissible slices $r \ge 30$: at each slice, $\bar{\ell}(r) \ne 0$, $D_0(r) \ne 0$, $k_R(r) \ne 0$, and $E_p(r, b)$ factors into distinct monic irreducibles $q_{r,m}(b)$ of total degree 269, giving the Chinese Remainder isomorphism $\mathbb{F}_p[b]/(E_p(r, b)) \simeq \prod_m K_{r,m}$ with finite fields $K_{r,m} = \mathbb{F}_p[b]/(q_{r,m})$. Inside every field $K_{r,m}$ (avoiding zero-divisor divisions), the script reconstructs $\widehat{\Sigma}_1, \widehat{\Sigma}_2 \in K_{r,m}[t]$ from

first principles, asserts $\deg_t \widehat{\Sigma}_1 = 41$ with leading coefficient $\bar{\ell}(r)$, and verifies exact Euclidean $\gcd_{K_{r,m}[t]}(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = (t - t_{\mathrm{cand}})^2$ for $t_{\mathrm{cand}} = N_t^{\mathrm{red}}/(D_0(E_p)_b)$, forcing $\bar{t} = t_{\mathrm{cand}}$. At $t = t_{\mathrm{cand}}$ it verifies $\zeta, \mathrm{Disc}_z(F_*), \det(M_{\mathcal{D}}) \neq 0$ and $M^2 - s_1 M + s_2 I = 0$ with $s_1 = \mathrm{Tr}(M)/2$, $s_2 = ((\mathrm{Tr}\, M)^2 - 2\,\mathrm{Tr}(M^2))/8$, and $\Delta_H = s_1^2 - 4s_2 \neq 0$ (forcing eigenvalues H_1, H_1, H_2, H_2 with $H_1 \neq H_2$). With $D_M = 2M - s_1 I$ and spectral projectors $P_\pm = \frac{1}{2}(I \pm D_M/\sqrt{\Delta_H})$ onto the two-dimensional eigenspaces $\mathrm{im}\, P_\pm$, summing the restricted 2×2 determinants $\det(M_z \mid \mathrm{im}\, P_\pm) = \frac{1}{2}(\mathrm{Tr}(P_\pm M_z)^2 - \mathrm{Tr}((P_\pm M_z)^2))$ yields the rational trace identity

$$\bar{\tau}_\sigma = \frac{\zeta}{4}\left(\mathrm{Tr}(M_z)^2 - \mathrm{Tr}(M_z^2) + \frac{\mathrm{Tr}(D_M M_z)^2 - \mathrm{Tr}((D_M M_z)^2)}{\Delta_H}\right),$$

which is verified in every $K_{r,m}$ to equal $\tau_{\mathrm{cand}} = N_\sigma^{\mathrm{red}}/(D_0(E_p)_b)$ (along with $F_{\mathrm{fer}}(\tau_{\mathrm{cand}}) = 0$, $F'_{\mathrm{fer}}(\tau_{\mathrm{cand}}) \neq 0$). CRT reassembly across all 1,398 slices proves $Q_\ell N_t^{\mathrm{red}} = \overline{\ell t E_b}$ and $Q_\ell N_\sigma^{\mathrm{red}} = \overline{\ell \tau_\sigma E_b}$ in $\mathbb{F}_p[R, b]$, establishing (4). □

3.2 Self-contained local expansions and the triple-pole depth lemma

The following two lemmas provide self-contained proofs of every local analytic input used in the resolution of M_2, M_6, and $R = 25$ (assuming Global Inputs (G1)–(G3)).

Lemma 3.2 (Simple roots, regular double collisions, and double pole clusters)**.** *Let v be a rank-one valuation on a characteristic-zero field with residue characteristic $p = 1{,}000{,}003$ (more generally $p > 5$ not dividing the displayed resultants), valuation ring $\mathcal{O}_v$, and maximal ideal $\mathfrak{m}$. Scale $\Phi^{\sharp\prime} = \Phi'/\varsigma$ so that its numerator is primitive over $\mathcal{O}_v$, and put $h^\sharp = \varsigma h$. Assume in each item below that the regular factor (B or ψ) and the regular part of A'/A have integral Taylor coefficients in $\mathcal{O}_v$ at the base point (automatic for the rational functions $\Phi^{\sharp\prime}$ and A'/A away from the remaining poles).*

(L1) At a regular zero $z_0 \in \mathcal{O}_v$ of $\Phi^{\sharp\prime}$ whose reduction $\bar{z}_0$ is a simple zero of the reduced function $\overline{\Phi^{\sharp\prime}}$ (equivalently $\Phi_2^\sharp(z_0) \in \mathcal{O}_v^\times$) away from the poles $\{0, 25, \infty, P = 0, Q = 0\}$, $h^\sharp(z_0)$ is integral with residue

$$\overline{h^\sharp(z_0)} = -\frac{\overline{A''/A}}{2\overline{\Phi_2^\sharp}} + \frac{\overline{A'/A}\,\overline{\Phi_3^\sharp}/2 + \overline{\Phi_4^\sharp}/8}{(\overline{\Phi_2^\sharp})^2} - \frac{5(\overline{\Phi_3^\sharp})^2}{24(\overline{\Phi_2^\sharp})^3}.$$

(L2) At a regular double collision $\Phi^{\sharp\prime}(z) = (z - z_1)(z - z_2)B(z)$ with $x = z_1 - z_2 \in \mathfrak{m}$, $B(z_1) \in \mathcal{O}_v^\times$, and z_1, z_2 away from the poles, put $\chi = (A'/A)/B - B'/(2B^2)$. Then

$$x^3(h^\sharp(z_1) - h^\sharp(z_2)) \equiv -\frac{5}{3\bar{B}(z_1)}, \qquad x^2(h^\sharp(z_1) + h^\sharp(z_2)) \equiv 2\bar{\chi}(z_1) \pmod{\mathfrak{m}}.$$

In particular neither jet is integral and $h^\sharp(z_1) \neq h^\sharp(z_2)$.

(L3) At a simple pole $z_0 \in \mathcal{O}_v$ of $\Phi^{\sharp\prime}$ with small phase residue $\rho \in \mathfrak{m}$, amplitude logarithmic residue γ (so $A'/A = \gamma/(z - z_0) + a_{\mathrm{reg}}$), and $\Phi^{\sharp\prime}(z) = \rho/(z - z_0) + \psi(z)$ with $\psi(z_0) \in \mathcal{O}_v^\times$, the adjacent zero $z_1 = z_0 - \rho/\psi(z_0) + O(\rho^2)$ satisfies

$$h^\sharp(z_1) = \frac{6\gamma^2 + 6\gamma + 1}{12\rho} + \varepsilon,$$

where for $\gamma = 0$ (poles at $PQ = 0$) the finite correction has residue $\bar{\varepsilon} = \overline{\psi'/\psi^2 - (A'/A)/\psi}$ at z_0.

(L4) At a double pole cluster $w\Phi^{\sharp\prime}(w) = (w-\xi_1)(w-\xi_2)B(w)$ in a local coordinate $w = z - z_0$ centered at a pole of amplitude logarithmic residue γ, with $B(0) \in \mathcal{O}_v^\times$ and $\xi_1, \xi_2 \in \mathfrak{m}$,

$$12B(\xi_1)\xi_1(\xi_1-\xi_2)^3 h^\sharp(\xi_1) = q_\gamma(\xi_1,\xi_2) + O((\xi_1,\xi_2)^3),$$
$$q_\gamma(X,Y) = -6\gamma(\gamma-1)(X-Y)^2 - (9-12\gamma)Y(X-Y) - 10Y^2.$$

For $\gamma \in \{-1/2, -1\}$, $h^\sharp(\xi_1)$ and $h^\sharp(\xi_2)$ cannot both be integral.

Proof. Because $h^\sharp = \varsigma h$ is obtained by replacing Φ_j with $\Phi_j^\sharp = \Phi_j/\varsigma$ in (1), (L1) follows immediately: $\Phi_2^\sharp(z_0)$ is a unit and $\Phi_3^\sharp, \Phi_4^\sharp, A'/A, A''/A$ are integral at z_0 (with $2, 8, 24 \in \mathcal{O}_v^\times$).

For (L2), expanding derivatives at z_1 gives $\Phi_2^\sharp = xB$, $\Phi_3^\sharp = 2B + 2xB'$, $\Phi_4^\sharp = 6B' + O(x)$; at $z_2 = z_1 - x$ one has $B(z_2) = B - xB' + O(x^2)$, $\Phi_2^\sharp(z_2) = -x(B - xB') + O(x^3)$, and $\Phi_3^\sharp(z_2) = 2B - 4xB' + O(x^2)$. Substituting into (1) yields

$$h^\sharp(z_1) = -\frac{5}{6x^3B} + \frac{(A'/A)/B - \frac{11}{12}B'/B^2}{x^2} + O(x^{-1}),$$
$$h^\sharp(z_2) = \frac{5}{6x^3B} + \frac{(A'/A)/B - \frac{1}{12}B'/B^2}{x^2} + O(x^{-1}),$$

since $\frac{3}{4} - \frac{5}{3} = -\frac{11}{12}$ and $\frac{3}{4} - \frac{5}{6} = -\frac{1}{12}$. Subtracting and adding gives both congruences (with $5/3 \in \mathcal{O}_v^\times$).

For (L3), put $w = z_1 - z_0$ and write $\psi(z_0 + w) = c + dw + O(w^2)$ with $c = \psi(z_0) \in \mathcal{O}_v^\times$, $d = \psi'(z_0) \in \mathcal{O}_v$. The root equation $\rho/w + \psi(z_0 + w) = 0$ gives $w = -\rho/c - d\rho^2/c^3 + O(\rho^3)$, $\Phi_2^\sharp(z_1) = -\rho/w^2 + \psi'(z_1) = -\psi(z_1)^2/\rho + \psi'(z_1) = -c^2/\rho + 3d + O(\rho)$, $\Phi_3^\sharp(z_1) = 2\rho/w^3 + O(1) = -2c^3/\rho^2 + 6cd/\rho + O(1)$, $\Phi_4^\sharp(z_1) = -6\rho/w^4 + O(1) = -6c^4/\rho^3 + 24c^2d/\rho^2 + O(\rho^{-1})$, $A'/A = -\gamma c/\rho + O(1)$, and $A''/A = \gamma(\gamma-1)c^2/\rho^2 + O(\rho^{-1})$. Substituting into (1) yields leading coefficient $\frac{\gamma(\gamma-1)}{2} + \gamma - \frac{3}{4} + \frac{5}{6} = \frac{6\gamma^2+6\gamma+1}{12}$ times $1/\rho$. At $\gamma = 0$ (where $A'/A = a_{\mathrm{reg}}$ and $A''/A \in \mathcal{O}_v$), collecting the $O(1)$ term gives $\bar{\varepsilon} = \overline{d/c^2 - a_{\mathrm{reg}}(z_0)/c} = \overline{\psi'/\psi^2 - (A'/A)/\psi}$ (certified symbolically in `verify_paper4_new_checks.py`).

For (L4), writing $f(w) = w\Phi^{\sharp\prime}(w)$ gives $\Phi_j^\sharp = P_j/w^{j-1}$ with $P_2 = f'$, $P_3 = wf'' - 2f'$, and $P_4 = w^2f''' - 3wf'' + 6f'$. At $w = \xi_1$ one has $P_2 = (\xi_1 - \xi_2)B(\xi_1)$, $P_3 = 2\xi_2 B(\xi_1) + O(\xi^2)$, and $P_4 = -6\xi_2 B(\xi_1) + O(\xi^2)$. Substituting into (1) and clearing $12B(\xi_1)\xi_1(\xi_1-\xi_2)^3$ yields $q_\gamma(\xi_1,\xi_2) + O((\xi_1,\xi_2)^3)$. Note that $q_{-1}(X,Y) = -12X^2 + 3XY - Y^2$ and $q_{-1/2}(X,Y) = -\frac{9}{2}X^2 - 6XY + \frac{1}{2}Y^2$. If $h^\sharp(\xi_1)$ and $h^\sharp(\xi_2)$ were both integral, unequal valuations $v(\xi_1) \neq v(\xi_2)$ would make the extreme term of q_γ dominate with pole order $-2\min v(\xi_i)$ or $-v(\xi_1) - v(\xi_2) < 0$, while equal valuations $s = \overline{\xi_1/\xi_2}$ would force $q_\gamma(s,1) = q_\gamma(1,s) = 0$, impossible because $\mathrm{Res}_s(q_\gamma(s,1), q_\gamma(1,s))$ equals $19{,}360 = 2^5 \cdot 5 \cdot 11^2$ at $\gamma = -1$ and $-500 = -2^2 \cdot 5^3$ at $\gamma = -1/2$ (both units at p, certified in `verify_paper4_new_checks.py`). □

Lemma 3.3 (Pairing invariants and the triple-pole depth lemma)**.** *Under the valuation and integrality hypotheses of Lemma 3.2, for the two paired root sums and products the invariant*

$$\vartheta = \frac{(z_1+z_2)^2}{z_1z_2} + \frac{(z_3+z_4)^2}{z_3z_4} = \frac{\tau_\sigma^2 - f_2\tau_\sigma + f_3f_1}{\zeta f_0} \tag{5}$$

lies in $\mathbb{Q}(C)$, as do its translates $\vartheta_{25}, \vartheta_R$ formed with $z_i - 25, z_i - R$.

(a) At a double pole cluster $\{\xi_1, \xi_2\}$ with $\gamma \in \{-1/2, -1\}$ and two simple regular remaining roots, the two pole roots are forced to be paired with each other, $v(\xi_1) = v(\xi_2)$, and the ratio $s = \overline{\xi_1/\xi_2}$ satisfies $(s+1)(s^2 - 22s + 1) = 0$ for $\gamma = -1/2$ and $(s+1)(s^2 + 8s + 1) = 0$ for $\gamma = -1$. In particular ϑ is integral.

(b) At a triple pole cluster $\{w_0, w_1, w_2\}$ with $\gamma \in \{-1, -1/2\}$, integral regular parts B, a_{reg} with $B(0) \in \mathcal{O}_v^\times$, if $h^\sharp(w_1)$ and $h^\sharp(w_2)$ are both integral, then

$$v(w_0) = v(w_1) = v(w_2).$$

In particular, if $w_0 = R - 25$ and w_1, w_2 are paired across to two regular roots, then $(R-25)\vartheta_{25}$ is integral.

Proof. Expanding symmetric sums proves (5). For (a), Lemma 3.2(L4) rules out both pole jets being integral, whereas the two regular jets are integral by (L1). Hence at least one pole jet is nonintegral and must match the other pole jet, forcing the two pole roots to form a pair. Comparing pole orders in (L4) gives $-2\min v(\xi_i)$ against $-v(\xi_1)-v(\xi_2)$ whenever $v(\xi_1) \neq v(\xi_2)$, so $v(\xi_1) = v(\xi_2)$. Setting $s = \overline{\xi_1/\xi_2}$, equality of leading terms gives $q_\gamma(s,1) + sq_\gamma(1,s) = 0$ (including $s = 1$, where $q_\gamma(1,1) = -10 \neq 0$), which factors as $\frac{1}{2}(s+1)(s^2-22s+1)$ for $\gamma = -1/2$ and $-(s+1)(s^2+8s+1)$ for $\gamma = -1$. Thus $(\xi_1+\xi_2)^2/(\xi_1\xi_2)$ is integral.

For (b), write $x = w_0$, $y = w_1$, $z = w_2$ in a coordinate centered at the pole, and set $f(w) = w\Phi^{\sharp\prime}(w) = (w-x)(w-y)(w-z)B(w)$ with B a unit. Put $a_0 = (A'/A)(y)$, and evaluate derivatives at y. With $P_2 = f'$, $P_3 = yf''-2f'$, and $P_4 = y^2f'''-3yf''+6f'$, one has $\Phi_j^\sharp = P_j/y^{j-1}$ for $j = 2, 3, 4$. Substitution in (1) yields the exact identity over $\mathbb{Z}$ for $2\gamma \in \{-2, -1, 0\}$:

$$96yP_2^3h^\sharp(y) = 12P_4P_2 - 20P_3^2 + 48(ya_0)P_3P_2 - 48y^2(A''/A)(y)P_2^2. \tag{6}$$

Writing $a_0 = \gamma/y + a_{\mathrm{reg}}(y)$, Taylor expansion at y decomposes the right-hand side as $B(y)^2Q_\gamma(x,y,z)+\Theta$, where Q_γ is the homogeneous quartic obtained by setting $B = 1$, $a_{\mathrm{reg}} = 0$ in (6). With $c = (y-x)(y-z)$ and $s_1 = 2y-x-z$, one has $f' = Bc$, $f'' = 2(Bs_1+B'c)$, $f''' = 6(B+B's_1+B''c/2)$, so every remainder monomial $x^iy^jz^k$ in Θ satisfies $i, k \leq 2$, $j \geq 1$, $i+j+k \geq 5$.

The coefficients of (y^4, x^2z^2) in Q_γ are $(-272, -8)$ at $\gamma = -1$ and $(-164, 4)$ at $\gamma = -1/2$ (spot check at $(s, 3s, 4s)$: $-2000s^4$ and $-2240s^4$). Suppose $v(y) < v(z)$. If $v(x) > v(y)$, then at y the numerator has order $4v(y)$ while $y(y-x)^3(y-z)^3$ has order $7v(y)$, contradicting integrality. If $v(x) \leq v(y)$, evaluate at z (the unique deepest root): the numerator has order $2v(x)+2v(y)$ while the denominator has order $v(z)+3v(x)+3v(y)$; the remainder monomial bounds $i, k \leq 2, j \geq 1$ make every term of Θ strictly deeper. By symmetry, $v(y) = v(z) = e$. If $v(x) > e$, integrality of both jets forces $q_C(s) = Q_\gamma(0, s, 1)/s^2$ and $s^2q_C(1/s)$ to share a root; if $v(x) < e$, the same holds for $q_B(s) = [x^2]Q_\gamma(x, s, 1)$ and $s^2q_B(1/s)$. All four signed resultants listed in Appendix A are units at p (certified in `verify_paper4_new_checks.py`), forcing $v(x) = e$. When $w_0 = R - 25$ and w_1, w_2 pair across to regular roots, each cross-paired summand of ϑ_{25} has valuation at least $-e$, so $(R-25)\vartheta_{25}$ is integral. □

3.3 Ordinary nodes M_2 and triple points M_6

Let $M_2(R) \in \mathbb{F}_p[R]$ denote the monic squarefree polynomial of degree 250,035 cutting out the affine ordinary-node locus of E_p (satisfying $M_2^2 \parallel \mathrm{Disc}_b(E_p)$), with singular section $b = \beta_2(R) \bmod M_2$, and let $M_6(R) \in \mathbb{F}_p[R]$ denote the monic squarefree product of the five irreducible polynomials of degrees $1, 9, 146, 526, 886$ (total degree 1,568) cutting out the ordinary-triple locus of E_p (satisfying $M_6^6 \parallel \mathrm{Disc}_b(E_p)$). Both arrays are archived in `companion_tables/`.

Proposition 3.4. *Assuming Global Inputs (G1)–(G3), the arithmetic normalization resolves all* 250,035 *ordinary nodes over M_2 and all* 1,568 *ordinary triple points over M_6, accounting for* 254,739 *delta units.*

Proof. Exact polynomial verification in `verify_m2_m6.py` (`m2_m6_cert.json`) confirms that M_2 and M_6 are squarefree, satisfy $M_2^2 \parallel \mathrm{Disc}_b(E_p)$ and $M_6^6 \parallel \mathrm{Disc}_b(E_p)$, and are coprime to all fourteen exceptional classes. By (3), $\gcd(\bar{\ell}, M_2) = \gcd(\bar{\ell}, M_6) = 1$, so $\ell(R)$ is a unit at every finite center over M_2 and M_6. Hence t is integral over the plane local ring by its degree-41 equation $\widehat{\Sigma}_1 = 0$, and τ_σ is integral by its monic Ferrari cubic.

On the singular section $b = \beta_2(R) \bmod M_2$, $E_p = (E_p)_R = (E_p)_b = 0$, while both the nodal Hessian and $(E_p)_{bb}$ are units modulo M_2; moreover N_t^{red} vanishes on the section. Along the two distinct nonvertical branches, the limit values of $\bar{t}$ from (4) coincide precisely where the Wronskian determinant

$$W_t = (E_p)_{bb}(N_t^{\mathrm{red}})_R - (E_p)_{Rb}(N_t^{\mathrm{red}})_b$$

vanishes modulo M_2. The exact gcd $\gcd(W_t, M_2)$ has degree 58 (matching `gt58.npy`). Distinct constants separate two smooth completed branches and therefore resolve 249,977 nodes.

The degree-58 gcd factors into six irreducible Frobenius orbits: degrees $1, 2, 26$ carrying a double pole cluster at $z = 0$, and degrees $2, 2, 25$ carrying a double pole cluster at $z = \infty$ (under $z \mapsto 1/z$ with $A\,dz$ transformed as a differential, h and (5) are invariant and the transformed amplitude has logarithmic residue $\gamma = -1/2$ at ∞). On every orbit, t is integral and the residual quadratic factor of F_* is separable with both roots away from $\{0, 25, \infty, R, P = 0, Q = 0\}$ (asserted on all six orbits in `verify_m2_m6.py`); Lemma 3.3(a) at $\gamma = -1/2$ therefore gives integrality of ϑ over the arithmetic local ring. Parametrizing each tangent branch by a local uniformizer ε and expanding the quartic coefficients as $f_i(\varepsilon) = \sum_{j \geq 0} f_{i,j}\varepsilon^j$ (with $f_{2,0}$ a unit; for the $z = \infty$ orbits, applying the same notation to the inverted quartic), exact finite-field expansions over $\mathbb{F}_{p^{2d}}$ in `verify_m2_m6.py` show that on every one of the six orbits the two branches share the same residual t, have $\mathrm{ord}_\varepsilon(f_0) = 1$ on the first branch and $\mathrm{ord}_\varepsilon(f_0) = 2$ on the second, and satisfy $f_{1,1}^2 = 24 f_{0,2} f_{2,0}$ on the second branch. On the first branch, Hensel's lemma over $\overline{\mathbb{F}}_p[[\varepsilon]]$ separates the pole quadratic factor $z^2 + c_1 z + c_0$ from the coprime regular quadratic factor; hence the root sum $\xi_1 + \xi_2 = -c_1 \in \overline{\mathbb{F}}_p[[\varepsilon]]$ has integral ε-valuation whereas the odd valuation $\mathrm{ord}_\varepsilon(c_0) = \mathrm{ord}_\varepsilon(f_0) = 1$ forces half-integer depth $v_\varepsilon(\xi_1) = v_\varepsilon(\xi_2) = 1/2$, requiring leading cancellation and therefore $\bar{s} = \overline{\xi_1/\xi_2} = -1$, so the pole-pair summand $(\xi_1 + \xi_2)^2/(\xi_1\xi_2)$ vanishes. On the second branch, $f_0 \equiv \xi_1\xi_2 f_{2,0}$ and $f_1 \equiv -(\xi_1 + \xi_2) f_{2,0}$ to leading order make the pole-pair summand reduce to $f_{1,1}^2/(f_{0,2} f_{2,0}) = 24$ (matching the factor $s^2 - 22s + 1 = 0$ of Lemma 3.3(a), where $(s+1)^2/s = 24$). Since the regular-pair summand has the same residue on both branches, $\bar{\vartheta}$ differs by $24 \not\equiv 0 \pmod{p}$ on all 58 nodes, resolving 250,035/250,035.

Over each of the five closed places dividing M_6, `verify_m2_m6.py` confirms that E_p vanishes to order 3 with separable cubic tangent cone (nonvertical, since $[b^3] \neq 0$), N_σ^{red} vanishes to order 2, and the discriminant (Hankel determinant of power sums) of the three limit values of $\bar{\tau}_\sigma$ along the three tangent directions is nonzero in the residue field. Because all three branches are smooth and unramified over $u = R - R_0$, their completed normalization is $\overline{\mathbb{F}}_p[[u]]^3$. A single integral element with distinct constant terms generates $\overline{\mathbb{F}}_p^3$ modulo u by Lagrange interpolation and hence generates $\overline{\mathbb{F}}_p[[u]]^3$ over the completed local ring by Nakayama's lemma (preserving finite lengths upon descending to the uncompleted local ring [22, Tag 0C44]), recovering $\binom{3}{2} = 3$ units at each triple point for $3 \cdot 1{,}568 = 4{,}704$ units. □

3.4 Resolution of the fiber at $R = 25$

Proposition 3.5. *Assuming Global Inputs (G1)–(G3), the arithmetic normalization resolves all* $38{,}231 = 36{,}046 + 2{,}185$ *delta units of the fiber above* $R = 25$.

Proof. Set $u = R - 25$, $z_b = 1/b$, $y = z_b/u$, and $v(u) = e > 0$ for any rank-one valuation centered at the plane point $(25, \infty)$.

Step 1: Integrality of y and t/b. Suppose that either $v(y) < 0$ or $v(t/b) < 0$; then $v(b) > \min(-e, v(t))$. Dividing Φ' by the dominant weight eliminates all b-terms from the leading scaled weights (w_0, w_P, w_Q, w_{25}) in three exhaustive regimes (all certified symbolically and over $\mathbb{F}_p$ in `verify_paper4_new_checks.py`):

1. $v(t) > -e$: scaling by α yields reduced weights $(1, 0, 0, 0)$, so the four non-exterior roots approach the four simple poles $P(z)Q(z) = 0$ with equal phase residue $\rho = 0$ to leading order and $\psi(z) = 1/z$. By Lemma 3.2(L3) with $\gamma = 0$, the finite correction is $\varepsilon(z_0) = -\frac{1}{2} -$

$z_0/(25-z_0)$. The differences across conjugate pole pairs are $\varepsilon(-3+4i)-\varepsilon(-3-4i) = -i/4$ and $\varepsilon(3+4i)-\varepsilon(3-4i) = -2i/5$, which satisfy $d_P \neq \pm d_Q$; no cross-pairing or self-pairing can equalize both pairs of jets.

2. $v(t) = -e$: scaling by $1/u$ yields weights $(25-\frac{9}{20}s, -s, s, 0)$ with $s = \overline{ut} \in \overline{\mathbb{F}}_p^\times$, and reduced quartic $G_s(z) = (25-\frac{9}{20}s)P(z)Q(z)+12sz(z^2-25)$. Its leading coefficient is $(500-9s)/20$, $G_s(25) = 10^7 \neq 0$ identically, and its exact discriminant factors over $\mathbb{Q}[s]$ as

$$\operatorname{Disc}_z G_s = \frac{147456}{25}(3s+250)^2(6s-125)^2(40729s^2-81000s+2250000).$$

Forming the binary quartic $\operatorname{Res}_z(G_s, X\mathcal{D}-\mathcal{N}) = \sum_{k=0}^4 A_k(s)X^k$ and imposing the two perfect-square conditions $C_1 = 8A_4^2A_1 - A_3(4A_4A_2-A_3^2) = 0$ and $C_2 = 64A_4^3A_0-(4A_4A_2-A_3^2)^2 = 0$ confirms that every irreducible factor of $\gcd(C_1, C_2)$ divides $A_4(s)$ (zero genuine four-simple-root squares). Thus s must be one of the five degenerate values in $\overline{\mathbb{F}}_p^\times$: $500/9$, $-250/3$, $125/6$, or the two conjugate roots $s_\pm \in \mathbb{F}_{p^2}$ of $40729s^2-81000s+2250000$ (discriminant -600000^2, non-square mod p). At $s = 500/9$, G_s drops to $\frac{2000}{3}z(z-5)(z+5)$, so one root of F_* tends to $z = 0$, one to $z = \infty$, and two to ± 5; the regular jets at ± 5 differ ($4750 \neq 885586 \bmod p$, ruling out $\{5, -5\}$), while by Lemma 3.2(L3) with $\gamma = -1/2$ the root near $z = 0$ has $h^\sharp = -1/(24\rho)+O(1)$ (nonintegral, since $\psi(0) = -12s/25$ is a unit), ruling out cross-pairings. At $s = s_\pm$, G_s has a single regular double root at $z = \pm 5i$ and two simple regular roots away from the poles, ruled out by Lemma 3.2(L2) because the two colliding jets are nonintegral and unequal and therefore pair neither with each other nor with the integral simple-root jets. Finally, at $s = -250/3$ (resp. $s = 125/6$), G_s equals $\frac{125}{2}(z^2-8z-25)^2$ (resp. $\frac{125}{8}(z^2+8z-25)^2$), giving two regular double collisions at conjugate roots $r_1, r_2 \in \mathbb{Q}(\sqrt{41})$. By Lemma 3.2(L2), equality of the two paired jets requires $x_1^3B(r_1) = \pm x_2^3B(r_2)$ and $x_1^{-2}\chi(r_1) = x_2^{-2}\chi(r_2)$, hence $B(r_1)^2\chi(r_1)^3 = B(r_2)^2\chi(r_2)^3$. Exact computation in $\mathbb{Q}(\sqrt{41})$ gives differences $-\frac{661297}{68921000}\sqrt{41}$ and $-\frac{357953}{137842000}\sqrt{41}$, both units at p.

3. $v(t) < -e$: scaling by t yields weights $(-\frac{9}{20}, -1, 1, 0)$; the quintic is divisible by $(z-25)^2$, and the residual cubic $K_3(z)$ is separable with three distinct regular jets (squarefree characteristic polynomial over $\mathbb{F}_p$), leaving only one remaining root of F_* and ruling out a $2+2$ pairing.

Hence y and t/b are integral over the plane local ring A by Proposition 2.1. Dividing F_* by ub yields integral coefficients and shows $\tau_\sigma/(ub)$ is integral as well.

Step 2: First blow-up and structure of the 66*-branch residual center.* Write $E_p = \sum_{j=0}^{269} c_j(R)b^j$. Direct inspection confirms $u^j \mid c_j(25+u)$ for all j and $c_0(25) = 58451 \in \mathbb{F}_p^\times$. Hence the transformed polynomial

$$\widetilde{E}_{25}(u, y) = \sum_{j=0}^{269} u^{-j}c_j(25+u)y^{269-j}$$

is monic in y up to a unit, and adjoining y increases the completed $\overline{\mathbb{F}}_p[[u]]$-order length by $\sum_{j=0}^{268} j = 36{,}046$. Specializing at $u = 0$ factors $\widetilde{E}_{25}(0, y) = y^{66}U_{203}(y)$ with U_{203} squarefree of degree 203 and $U_{203}(0) \neq 0$, resolving 203 smooth points and leaving a single residual center $(u, y) = (0, 0)$ of multiplicity 66 and plane delta 2,185.

At $(u, y) = (0, 0)$, the tangent cone of $\widetilde{E}_{25}$ factors into 58 simple tangents (Frobenius orbit degrees $1, 1, 2, 2, 8, 8, 18, 18$) and a single $\mathbb{F}_p$-rational tangent $\lambda = 0$ of multiplicity 8. Expanding all 66 branches shows that every branch is unramified over u ($u = s$ is a uniformizer and $y(s) = O(s)$ on all 66 branches). Writing $V(u, y) = -\sum_{j=0}^{268} u^{-j}([b^j]N_t^{\mathrm{red}})(25+u)y^{268-j}$, one

has $\overline{t/b} = V/(D_0\partial_y\widetilde{E}_{25})$. Put $\eta = \overline{t/b}$ on each branch, $L_z = (25-z)PQ$, $W_P = z(25-z)P'Q$, $W_Q = z(25-z)Q'P$, and

$$K_\eta(z) = \frac{(-700000 - 180000\eta)L_z + 400000((1-\eta)W_P + \eta W_Q)}{(z-25)^2}.$$

The 66 branches partition cleanly into two strata (certified in `verify_paper4_new_checks.py`):

Orbit degrees	$v_u(D_0\partial_y\widetilde{E}_{25})$	Branches	Reduced roots of $F_*^\sharp$
$1,1,6$ (on $\lambda=0$)	75	8	$\eta = 25/39$; double at 25 + sep. reg. quadratic
$1,1,2,2,8,8,18,18$	65	58	K_η separable cubic, three regular roots

(In particular, on the two $\mathbb{F}_p$-rational simple tangents $\lambda = 982798, 306247$ the residue is $\eta = 84186 \neq 589746 = 25/39$. Because $u(s) = s$ and $y(s) = O(s)$, truncating the coefficient series of $\widetilde{E}_{25}, V, N_\sigma, D_0$ at order 260 introduces zero error modulo s^{140}; with denominator valuation $d_{\mathrm{ord}} \in \{65, 75\}$ and $v_s(V) = d_{\mathrm{ord}}$, $v_s(N_\sigma) = d_{\mathrm{ord}} + 1$, expanding branches to order $140 \geq 75 + 1 + 30 = 106$ determines the exact power series of t/b and $\tau_\sigma/(ub)$ through 30 terms, and the denominator ζg_0 of $u\vartheta_{25}$ has s-order at most 2, giving an exact constant term with a 27-term margin and 10 exact terms with an 18-term margin.)

Step 3: Integrality and closed-form separation of $S_{25} = t/b + u\vartheta_{25}$ over $A_{\mathrm{mid}} = A[y]_{(p,u,y)}$. Pass to the intermediate Noetherian local domain $A_{\mathrm{mid}} = A[y]_{(p,u,y)}$ at the residual center $(u, y) = (0, 0)$ and consider any rank-one valuation v centered at $\mathfrak{m}_{A_{\mathrm{mid}}}$. Since t/b is already proved integral over A_{mid}, the valuation ring $\mathcal{O}_v$ contains the finite integral A_{mid}-order $A_{\mathrm{mid}}[t/b]$; its center on the special fiber lies in the finite birational order $B_{(0,0)}$ above $(u, y) = (0, 0)$, which is dominated by a closed point of the normalization $T \to Y_0$. Reading $\overline{t/b}$ at that point of T places it among the values of η in the two rows above (without assuming Going-Up or good reduction).

- On the 8 branches of Row 1 ($\eta = 25/39$), dividing K_η by $z - 25$ yields a separable regular quadratic whose two roots $r_1, r_2 \in \mathbb{F}_p$ have distinct regular jets, so both regular roots must pair across to the two remaining roots $25 + w_1, 25 + w_2$ of the triple pole cluster $\{u, w_1, w_2\}$ at 25. Lemma 3.3(b) at $\gamma = -1$ gives $v(w_1) = v(w_2) = v(u) = e$, whence $u\vartheta_{25}$ is integral.

- On the 58 branches of Row 2, the three simple regular roots of K_η supply both matching classes, so bH_1, bH_2 are integral. The fourth quartic root $25 + X$ forms a double pole cluster at 25 with $25 + u$ and has integral scaled jet $h^\sharp$. By Lemma 3.2(L4), $q_{-1}(X, u) = -12X^2 + 3Xu - u^2$ has pole order $-2\min(v(X), v(u))$ or $-v(X) - v(u) < 0$ whenever $v(X) \neq v(u)$; hence $v(X) = v(u) = e$ and $u\vartheta_{25}$ is again integral.

Thus $S_{25} = t/b + u\vartheta_{25}$ is integral over A_{mid} by Proposition 2.1, so its reduction $\bar{S}_{25}$ lies in the completed special order at $(u, y) = (0, 0)$.

The 66 branch residues of $\bar{S}_{25}$ admit a closed-form algebraic description verifiable without Puiseux series (and cross-checked identically against the Puiseux expansions in `verify_paper4_new_checks.py`):

- On each of the 58 Row-2 branches, $12\sigma^2 - 3\sigma + 1 = 0$ for $\sigma = \overline{X/u}$ (both roots lie in $\mathbb{F}_p$ since -39 is a square modulo p), two roots of K_η share a common jet while the distinguished third root r_{odd} pairs with $25 + X$, and $\bar{S}_{25} = \eta + (r_{\mathrm{odd}} - 25)/\sigma$; each value of η occurs on exactly two simple tangents, carrying the two distinct roots σ.

- On each of the 8 Row-1 branches, $(\sigma_A, \sigma_B) = (\overline{w_1/u}, \overline{w_2/u})$ runs over the 8 valid solutions (Frobenius orbit degrees $1, 1, 6$) of $Q_{-1}(1, \sigma_A, \sigma_B) = Q_{-1}(1, \sigma_B, \sigma_A) = 0$ with $\sigma_A \neq \sigma_B$ and $\sigma_A, \sigma_B \notin \{0, 1\}$, and $\bar{S}_{25} = 25/39 + (r_1 - 25)/\sigma_A + (r_2 - 25)/\sigma_B$.

The polynomial $\prod_{i=1}^{66}(X-\bar{S}_i)$ (equivalently, the product of one minimal polynomial per Frobenius orbit) is squarefree of degree 66 over $\mathbb{F}_p$ (coefficient digest in Appendix A). Because every branch has uniformizer $u = s$, the completed normalization is $N \simeq \prod_{i=1}^{66} \overline{\mathbb{F}}_p[[u]]$, which is a finite module over the completed special local ring $\widehat{A}_{\mathrm{mid},0}$ at $(u, y) = (0, 0)$. Since $\bar{S}_{25}$ generates $N/uN \simeq \overline{\mathbb{F}}_p^{66}$ by Lagrange interpolation, Nakayama's lemma implies that $\bar{S}_{25}$ generates N over $\widehat{A}_{\mathrm{mid},0}$; because completion preserves finite lengths [22, Tag 0C44], the arithmetic special order equals the full normalization at $(u, y) = (0, 0)$. Together with the 203 smooth points of the first blow-up, this resolves all $36{,}046 + 2{,}185 = 38{,}231$ units. □

3.5 Genus bounds, conditional transfer at the remaining 125 places, and finite-order checks

Table 2 summarizes the delta accounting across all strata of E_p. Rows 1–3 are proved on the arithmetic normalization in Propositions 3.4 and 3.5. Rows 4–6 report special-fiber jet ranks (`all_remaining_jet_v4_cert.json`) at the remaining 125 special base places, computed from candidate rational quotients tested only for nonnegative order along *special-fiber branches* (for instance, at the finite $P = 0$ center of multiplicity 28 and gain 808, both $t/(b-t)$ and $(b-t)/t$ are active, requiring $v(t) = v(b-t)$ at every centered arithmetic valuation). Membership in the special normalization N does not imply arithmetic integrality: in $A = \mathbb{Z}_{(p)}[x]_{(p,x)}$, the function $(x+p)/x$ reduces to 1 on the special fiber yet has negative value at the centered Gauss valuation with $v(x) = 2v(p) > 0$. Accordingly, rows 4–6 remain conditional on arithmetic integrality lifts satisfying Proposition 2.1.

Table 2: Delta accounting across E_p. Rows 1–3 are proved on the arithmetic normalization assuming Global Inputs (G1)–(G3); rows 4–6 record special-fiber jet ranks only (over the singular-locus distribution of (G2)) and remain conditional on arithmetic integrality lifts.

Stratum	Plane delta	Special jet gain	Jet rem.	Arithmetic status
M_2, ordinary nodes	250,035	$249{,}977 + 58$	0	Proved (Prop. 3.4)
M_6, ordinary triples	4,704	$3 \cdot 1{,}568$	0	Proved (Prop. 3.4)
$R = 25$	38,231	$36{,}046 + 2{,}185$	0	Proved (Prop. 3.5)
$P(R)Q(R) = 0$	4,856	4,856	0	Conditional (special-fiber rank)
$R = \infty$	69	69	0	Conditional (special-fiber rank)
Other 122 special places	5,014	5,014	0	Conditional (special-fiber rank)
Total	302,909	302,909	0	292,970 proved / 9,939 cond.

Theorem 3.6 (Genus bounds and conditional good reduction)**.** *Assuming Global Inputs (G1)–(G3), one has*

$$6{,}631 \le g(C) \le 16{,}570.$$

If, in addition, at each of the 125 *remaining special base places in rows 4–6 of Table 2, reductions of functions integral over the arithmetic local ring generate the full special normalization N_Q over A_0 (realizing the full special-fiber jet gains listed in Table 2 via Proposition 2.1), then $\delta_Q(Y_0) = 0$ everywhere, $g(C) = g(E_p) = 6{,}631$, and $X \to S$ is smooth and projective with $Y_0 \simeq T$. Conversely, $g(C) = 6{,}631$ implies $\delta_Q(Y_0) = 0$ everywhere and $X \to S$ is smooth.*

Proof. Propositions 3.4 and 3.5 resolve $254{,}739 + 38{,}231 = 292{,}970$ of the 302,909 plane delta units, so $\sum_Q \delta_Q(Y_0) \le 302{,}909 - 292{,}970 = 9{,}939$. Equation (2) immediately gives $6{,}631 \le g(C) \le 6{,}631 + 9{,}939 = 16{,}570$. Under the additional lift hypothesis all remaining defects vanish; conversely $g(C) = 6{,}631$ forces each nonnegative term $\delta_Q(Y_0)$ to vanish. Then $T \to Y_0$ is an isomorphism stalkwise, so flatness and smoothness of both geometric fibers imply smoothness of $X \to S$. Projectivity follows from finite normalization of the projective model $\mathcal{E}$. □

Remark 3.7 (Independent finite-order obstruction tests). As an independent archived diagnostic on the conditional equisingularity hypothesis (not used in Theorems 3.6 or 4.2), the historical workstation reconstructions `mod_p2_cQ_cert.json` and `mod_p3_c2_and_weierstrass_cert.json` sample 2,320 smooth R-slices: on each slice r, a two-step Hensel–Newton lift produces the monic degree-269 polynomial $M(r,b) \in (\mathbb{Z}/p^3\mathbb{Z})[b]$. Because the b-coefficient rows $c_j(R)$ of E_p are globally coprime in $\mathbb{F}_p[R]$, a Bezout combination of the cross-multiplied residues $c_{269}(r)m_j(r) - c_j(r)$ over the first 2,313 slices (checked on the remaining 7 holdout slices) recovers the leading-coefficient perturbation $\delta_k c_{269}(R)$ uniquely modulo $c_{269}(R)$ with $\deg_R(\delta_k c_{269}) < 1156$ (fixing the global $(\mathbb{Z}/p^3\mathbb{Z})^\times$-scalar ambiguity uniformly across all slices). Using 1,157 interpolation slices then determines all 269 lower rows modulo p^2 and p^3, checked in the archived scripts on 26 strided holdout slices (`range(1157, 2320, 45)`) per row (together with workstation proportionality checks against characteristic-zero anchor slices at $R = 65, 68$, whose external anchor files are not bundled in this release; see `ERRATA_AND_STATUS.md` for a static summary-flag caveat in the archived evaluator). For each plane singularity q of residue degree d with lifted coordinate $\tilde{q} \in W_3(\mathbb{F}_{p^d})^2$ (using the unramified length-3 Witt ring $W_3(\mathbb{F}_{p^d})$, and the projective chart $z_b = 1/b$ at $(25, \infty)$), the first-order obstruction $c_Q = \overline{E(\tilde{q})/p}$ vanishes on all 250,035 nodes of M_2, at $(25,\infty)$, and across 195 additional singular orbits; modulo p^3, the nodal Hessian H is invertible on M_2 and the second-order obstruction $c_2(q) = \overline{E(\tilde{q})/p^2} - \frac{1}{2}\mathbf{a}^\top H^{-1}\mathbf{a}$ (with $\mathbf{a} = \overline{\nabla E(\tilde{q})/p}$) vanishes everywhere on M_2. At 11 rational higher-contact centers where the local Weierstrass preparation has leading quartic $U(u)(k_4u^4 + pk_3u^3 + p^2k_2u^2) \bmod (p^3, u^7)$ (U a unit, $k_4 \in \mathbb{F}_p^\times$), the condition $8k_4k_2 - 3k_3^2 \equiv 0 \pmod p$ holds; equivalently, the quartic is $k_4(u-u_*)^4 \bmod p^3$ with $u_* = -pk_3/(4k_4)$. Finite-order vanishing is consistent with, though weaker than, characteristic-zero equisingularity (as illustrated by $xy = p^3$).

4 Symmetric monodromy S_{269}, smooth-cover rigidity, and quotients

For a finite cover $\pi : C \to \mathbb{P}^1$, denote by $\mathrm{Aut}(C,\pi)$ the group of automorphisms $\sigma \in \mathrm{Aut}(C)$ for which $\pi \circ \sigma = \gamma_{\mathrm{mob}} \circ \pi$ for some $\gamma_{\mathrm{mob}} \in \mathrm{PGL}_2$; surjectivity makes γ_{mob} unique, and deck transformations correspond to $\gamma_{\mathrm{mob}} = 1$.

Lemma 4.1 (Contracted-genus bound on a dominating semistable model)**.** *Let $\mathcal{O}'$ be a complete DVR with algebraically closed residue field $\overline{\mathbb{F}}_p$ and uniformizer ϖ'. Let X' be a normal projective flat relative curve over $\operatorname{Spec}\mathcal{O}'$ whose special fiber Y_0 is integral, and let $\phi : W \to X'$ be a proper birational morphism from a normal semistable model W over $\operatorname{Spec}\mathcal{O}'$ with reduced nodal special fiber W_0. Then for every closed point $q \in Y_0$, the sum of the geometric genera of the irreducible components of W_0 contracted to q satisfies*

$$\sum_{\phi(Z_i)=\{q\}} g(\widetilde{Z}_i) \le \delta_q(Y_0),$$

where $\delta_q(Y_0) = \dim_{\overline{\mathbb{F}}_p}(N_q/\mathcal{O}_{Y_0,q})$ is the local delta invariant of Y_0 at q.

Proof. Because W and X' are normal and ϕ is proper birational, Zariski's main theorem gives $\phi_*\mathcal{O}_W = \mathcal{O}_{X'}$, and every fiber of ϕ is projective and connected. Since X' is regular at the generic point of Y_0, a unique irreducible component of W_0 dominates Y_0. Put $\mathcal{M} = R^1\phi_*\mathcal{O}_W$. Its stalk at q has finite length over $\mathcal{O}_{X',q}$, and $R^2\phi_*\mathcal{O}_W = 0$ because every fiber has dimension at most 1. Applying ϕ_* to the short exact sequence $0 \to \mathcal{O}_W \xrightarrow{\varpi'} \mathcal{O}_W \to \mathcal{O}_{W_0} \to 0$ yields the exact sequence

$$0 \to \mathcal{O}_{Y_0} \to \phi_{0*}\mathcal{O}_{W_0} \to \mathcal{M}[\varpi'] \to 0$$

together with $R^1\phi_{0*}\mathcal{O}_{W_0} \simeq \mathcal{M}/\varpi'\mathcal{M}$. Thus $\phi_{0*}\mathcal{O}_{W_0}/\mathcal{O}_{Y_0} \simeq \mathcal{M}[\varpi']$ and $R^1\phi_{0*}\mathcal{O}_{W_0} \simeq \mathcal{M}/\varpi'\mathcal{M}$ are finite-length $\mathcal{O}_{Y_0,q}$-modules of equal length. Because W_0 is reduced and the fiber $\phi_0^{-1}(q)$

is projective and connected, any section of $\phi_{0*}\mathcal{O}_{W_0}$ near q vanishing on the dominant component restricts to a constant vanishing at the intersection point on every contracted component, hence vanishes identically on W_0. Therefore $\phi_{0*}\mathcal{O}_{W_0}$ embeds into the normalization of Y_0, so $\operatorname{len}(\phi_{0*}\mathcal{O}_{W_0}/\mathcal{O}_{Y_0}) \leq \delta_q(Y_0)$. On the other hand, the normalization sequence $0 \to \mathcal{O}_{W_0} \to \nu_*\mathcal{O}_{\widetilde{W}_0} \to \mathcal{Q} \to 0$ of the nodal curve W_0 has zero-dimensional cokernel $\mathcal{Q}$, so $R^1\phi_{0*}\mathcal{O}_{W_0} \to R^1(\phi_0 \circ \nu)_*\mathcal{O}_{\widetilde{W}_0}$ is surjective; at q, the target is the direct sum of $H^1(\widetilde{Z}_i, \mathcal{O}_{\widetilde{Z}_i})$ over all components Z_i contracted to q, whose $\overline{\mathbb{F}}_p$-dimension is $\sum_{\phi(Z_i)=\{q\}} g(\widetilde{Z}_i)$. □

Theorem 4.2. *Assuming Global Inputs (G1)–(G2) (with (a) in characteristic p requiring only the frozen polynomial E_p, and (b) in characteristic zero also using Global Input (G3) through Step 1 of Proposition 3.5):*

(a) The arithmetic and geometric monodromy groups of π_R in characteristics zero and p are S_{269}. There is no proper intermediate cover of π_R.

(b) Both deck groups are trivial. The smooth special cover $T \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ and the characteristic-zero cover $C \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ satisfy $\operatorname{Aut}(T, \pi_{R,0}) = \{1\}$ and $\operatorname{Aut}_{\overline{\mathbb{Q}}}(C, \pi_R) = \{1\}$.

(c) For every nonconstant map $\psi : C \to Y$ of degree d to a smooth projective curve of genus g_Y over $\overline{\mathbb{Q}}$ not factoring through π_R,

$$d \geq \left\lceil \frac{6899}{268 + g_Y} \right\rceil.$$

In particular, the gonality of C is at least 26, and $d \geq 26$ holds for every target genus $g_Y \leq 7$.

Proof. (a) Let L_0 and L_p be the Galois closures of $\mathbb{Q}(C)/\mathbb{Q}(R)$ and $\mathbb{F}_p(E_p)/\mathbb{F}_p(R)$, respectively. At $R = 37$, $E_p(37, b)$ is irreducible of degree 269, and at $R = 18$, $E_p(18, b)$ factors into distinct irreducibles of degrees $2, 9, 61, 197$; both fibers are squarefree, as certified in `certify_monodromy_S269_smooth.py` and `monodromy_S269_smooth_cert.json`. Over $\mathbb{F}_p$, the Frobenius cycle types at these unramified places lie in the arithmetic monodromy group $\operatorname{Gal}(L_p/\mathbb{F}_p(R))$. For characteristic zero, note the two-stage specialization: because the leading b^{269}-coefficient of $E(r, b) \in \mathbb{Z}[b]$ is a p-adic unit and $\operatorname{Disc}_b E(r, b) \not\equiv 0 \pmod{p}$ at $r \in \{18, 37\}$, Dedekind's theorem places the same Frobenius cycle types inside the Galois group of the splitting field of $E(r, b)$ over $\mathbb{Q}$, which in turn embeds as a decomposition subgroup into the generic arithmetic monodromy group $\operatorname{Gal}(L_0/\mathbb{Q}(R))$. Multiplication of E_p by $c_0 \in \mathbb{F}_p^\times$ does not affect permutations. A 269-cycle of prime degree generates a primitive group, and raising the second permutation to the odd power $\operatorname{lcm}(9, 61, 197) = 108153$ produces a transposition. Conjugates of a transposition form a connected graph on blocks by primitivity and hence generate S_{269}.

For geometric monodromy, at $(r_0, b_0) = (782212, 17723)$ in the frozen E_p scaling, $D_p(r_0) = 0$, $D_p'(r_0) = 775583$, $(E_p)_{bb} = 530499$, and $(E_p)_R = 936426$ (where $D_p = \operatorname{Disc}_b(E_p)$ in `companion_tables/disc.npy`). Dividing $E_p(r_0, b)$ by $(b - b_0)^2$ gives a squarefree degree-267 polynomial nonzero at b_0, giving simple tame ramification and a geometric inertia transposition over $\overline{\mathbb{F}}_p$. In characteristic zero, the system $(E, E_b) = 0$ has invertible Jacobian determinant $E_R E_{bb} \bmod p$ at (r_0, b_0), so Hensel lifting produces an isolated algebraic simple double root over a lift of r_0 with 267 distinct remaining roots and hence a geometric inertia transposition over $\overline{\mathbb{Q}}$. In each characteristic the geometric monodromy group is normal in the arithmetic S_{269} and contains a transposition, so it equals S_{269}. The point stabilizer S_{268} is maximal, ruling out proper intermediate covers.

(b) Any deck transformation centralizes S_{269} and is therefore trivial. On the smooth special cover $T \to \mathbb{P}^1_{\overline{\mathbb{F}}_p}$, every ramification index satisfies $e_P \leq 269 < p$, so all ramification is tame and the pushforward of the different assigns to each $x \in \mathbb{P}^1_{\overline{\mathbb{F}}_p}$ the coefficient $f_x = \sum_{P|x}(e_P - 1)$. From

the complete smooth ramification table of Global Input (G2) (`companion_tables/special_places.json`; relevant rows in Appendix A), the values $f_x = 5$ (attained uniquely at $R = 5$) and $f_x = 6$ (attained uniquely at $R = -5$) force γ_{mob} to fix 5 and -5 individually, while the unique four points with $f_x = 120$ are the roots of $P(R)Q(R) = 0$. In the coordinate $w = (R-5)/(R+5)$, γ_{mob} acts as $w \mapsto \lambda w$ preserving the four-element set $\{i/2, -i/2, 2i, -2i\}$ over $\mathbb{F}_{p^2}$ ($p \equiv 3 \pmod 4$); testing all four images of $i/2$ leaves only $\lambda = \pm 1$, i.e., the identity and the involution $R \mapsto 25/R$. At any finite R where $\deg_b E_p(R, b) = 269$ (no branch meets $b = \infty$, so the affine discriminant agrees with the binary discriminant), the local order index $\delta_R = \sum_{Q|R} \dim_{\overline{\mathbb{F}}_p}(N_Q/A_Q)$ of the affine b-model satisfies $v_R(D_p) = f_R + 2\delta_R$. Now $R_1 = 381924$ and $R_2 = 348525$ have $\deg_b E_p(R_i, b) = 269$ and odd discriminant valuation $v_R(D_p) = 141$, so both fibers must ramify on T independently of the Puiseux table (the table confirms ramification $(4, 4, 2)$ and $f = 7$). Their images under $R \mapsto 25/R$ are 889355 and 53081, where E_p has full degree 269 and nonzero discriminants $433249, 854844$. Thus $\gamma_{\mathrm{mob}} = 1$, and triviality of the deck group gives $\mathrm{Aut}(T, \pi_{R,0}) = \{1\}$.

For characteristic zero, pass first to the completion $\widehat{\mathbb{Z}^{\mathrm{sh}}_{(p)}}$ of the strict henselization and then to a finite (possibly ramified) extension $\mathcal{O}'$, which is again a complete DVR with algebraically closed residue field $\overline{\mathbb{F}}_p$. On the base-changed arithmetic model $X_{\mathcal{O}'}$, affine coordinate rings A_{ext} are flat with reduced special fiber $A_{\mathrm{ext}}/\varpi' A_{\mathrm{ext}} \simeq (A/pA) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ (since $\mathbb{F}_p$ is perfect) and normal generic fiber, so the formalized localization-descent lemma `normal_of_reduced_special_fiber_and_normal_generic_fiber` of Section 2.2 ([15, Lemma 4.1.18]) shows that $X_{\mathcal{O}'}$ remains normal. After a further finite extension of $\mathcal{O}'$, there exists a normal semistable model $\phi : W \to X_{\mathcal{O}'}$ dominating $X_{\mathcal{O}'}$ with reduced nodal special fiber W_0 (by Temkin's stable modification theorem [24, Theorem 1.5]; cf. [6, 15, 16]), and the unique irreducible component of W_0 dominating Y_0 has normalization T.

By Lemma 4.1, every component of W_0 contracted to a closed point $q \in Y_0$ has geometric genus at most $\delta_q(Y_0)$. At $R = 25$, merely the first blow-up (y-integrality in Steps 1–2 of Proposition 3.5) already bounds $\delta_q(Y_0) \le 2{,}185 < 6{,}631$; over M_2 and M_6, each plane singularity already has plane delta 1 or $3 < 6{,}631$; and over every other point the singular-locus delta distribution supplied by Global Input (G2) (Table 2; `companion_tables/table.txt`) bounds $\delta_q(Y_0) \le \max(4856, 69, 5014) = 5{,}014 < 6{,}631$ (in fact at most $4856/4 = 1{,}214$ per geometric fiber). Consequently every contracted component has geometric genus strictly less than 6,631. Passing to the unique stable model [7] contracts rational components, leaving T as the unique normalized component of genus 6,631 on the stable fiber. Any $\sigma \in \mathrm{Aut}_{\overline{\mathbb{Q}}}(C, \pi_R)$ extends to the stable model and therefore preserves the divisorial valuation of T, whose restriction to the R-line is the Gauss valuation (since R is nonconstant on T). Representing the induced base transformation $\gamma_{\mathrm{mob}} \in \mathrm{PGL}_2(\overline{\mathbb{Q}})$ by an integral matrix with a unit entry, singular reduction would render $\gamma_{\mathrm{mob}} \circ \pi_R$ constant at the generic point of T, whereas $\pi_R \circ \sigma$ reduces dominantly. Hence γ_{mob} has good reduction and specializes to the identity by the smooth-cover argument above.

Finally, $\mathrm{Aut}_{\overline{\mathbb{Q}}}(C)$ is finite (since $g(C) \ge 6{,}631 \ge 2$) and has no element of order p: even with the arithmetic-genus bound $2g(C) + 1 \le 619{,}081 < p$, cyclic Riemann–Hurwitz $2g - 2 = p(2h_0 - 2) + r(p-1)$ gives $r \ge 3$ if $h_0 = 0$, $r \ge 1$ if $h_0 = 1$, and $g \ge p+1$ if $h_0 \ge 2$, all impossible. A finite-order matrix in $\mathrm{PGL}_2(\overline{\mathbb{Q}})$ of order prime to p is projectively semisimple, and roots of unity of order prime to p inject under reduction; thus reducing to the identity forces $\gamma_{\mathrm{mob}} = 1$, and triviality of the deck group forces $\sigma = 1$.

(c) If $\psi : C \to Y$ does not factor through π_R, the product morphism $C \to \mathbb{P}^1 \times Y$ is birational onto its image by (a), meeting fibers over $\mathbb{P}^1$ in 269 points and fibers over Y in d points. Adjunction on $\mathbb{P}^1 \times Y$ gives arithmetic genus $dg_Y + 268(d-1)$. The Castelnuovo–Severi inequality [5], [23, Theorem 3.11.3] together with $g(C) \ge 6{,}631$ yields $6631 \le dg_Y + 268(d-1)$, i.e., $d \ge \lceil 6899/(268 + g_Y) \rceil$. For all $0 \le g_Y \le 7$, $\lceil 6899/(268 + g_Y) \rceil = 26$. □

5 An auxiliary dual-route plane-curve genus verifier

As an auxiliary self-contained open-source tool (not a general new genus algorithm), the program `compute_exact_genus.py` uses exact rational arithmetic and SymPy [19] with no dependence on the companion curve data. Its supported input class consists of reduced $\mathbb{Q}$-irreducible plane curves $f(x, y) \in \mathbb{Q}[x, y]$ of degree d supplied with a smooth $\mathbb{Q}$-point (which certifies absolute irreducibility, since geometrically irreducible components are Galois conjugate and a smooth rational point lies on a unique component) such that every singular point of the projective closure, every multiple tangent direction met during blow-ups, and every multiple Newton residual root in Route B is $\mathbb{Q}$-rational, and every irrational branch fiber of the generic projection is simple ($v(\mathrm{Disc}) = 1$). Unsupported geometric configurations (including non-reduced curves, nonrational singular points, nonrational multiple tangent directions, nonrational multiple residual roots, nonsimple irrational branch fibers, or exhaustion of the coordinate search below) raise `Unsupported` cleanly rather than guessing, whereas failed irreducibility or smooth-witness checks (such as a reducible reduced curve or an off-curve or singular witness) return `all_agree` equal to `false`; command-line execution exits with status 1 in both cases.

Route A checks reducedness, homogenizes f to $F(X, Y, Z)$, locates affine singularities by lexicographic Gröbner elimination of (f, f_x, f_y), factors tangent cones over $\mathbb{Q}$, blows up each multiple tangent, and evaluates

$$\delta_P = \sum_{Q \text{ infinitely near } P} \binom{m_Q}{2}, \qquad g_A = \frac{(d-1)(d-2)}{2} - \sum_P \delta_P,$$

raising `Unsupported` whenever a nonrational singularity or a nonrational multiple tangent direction (an irreducible tangent-cone factor of degree greater than one with multiplicity greater than one) is encountered. On the line at infinity $Z = 0$, `infinity_singular_points` inspects $[0 : 1 : 0]$ directly and takes the univariate gcd of all nonzero restricted partial derivatives among $(F_X(1, y, 0), F_Y(1, y, 0), F_Z(1, y, 0))$ (raising `Unsupported` if all three vanish identically on $Z = 0$); by Euler's identity $dF = XF_X + YF_Y + ZF_Z$ in characteristic zero, this captures every singular point $[1 : Y_0 : 0]$ even when one or two restricted partials vanish identically (as for $x^2 - y^5 = 0$, where $F_X = F_Z = 0$ on $Z = 0$; this case, its projective shift $(x - y)^2 = (2x + y)^5$, and a quartic with two distinct nodes on $Z = 0$ are all included as regression tests in `test_new_curves.py`, alongside direct completeness verification for C_3 in both `test_new_curves.py` and `verify_paper4_new_checks.py`).

Route B runs a finite fixed-seed search testing up to 20,000 integer matrices M with entries in $[-2, 2]$ and $0 < |\det M| \le 6$ (raising `Unsupported` if this search budget is exhausted) until the transformed homogeneous polynomial G satisfies $G(0, 1, 0) \neq 0$, $G(1, 0, 0) \neq 0$, and $\mathrm{Disc}(G(X, Y, 0)) \neq 0$, certifying that the projection $(x, y) \mapsto x$ has full degree d, constant leading coefficient in y, and is unramified over $x = \infty$. For every rational branch point $x_0 \in \mathbb{Q}$ it extracts exact Newton–Puiseux ramification indices e_P (verifying the fiber-degree identity $\sum_{P|x_0} e_P[\kappa(P) : \kappa(x_0)] = d$), while every irrational branch factor of $\mathrm{Disc}_y f'$ is required to be simple ($v_{x_0}(\mathrm{Disc}) = 1$, contributing $e = 2$ per geometric branch point, and raising `Unsupported` otherwise). The Riemann–Hurwitz genus $g_B = 1 - d + \frac{1}{2}\sum_P (e_P - 1)$ is computed strictly from these ramification indices without using local delta invariants; `local_delta` is invoked separately in Route B only to cross-check projective invariance of $\sum \delta_P$ and the fiber discriminant identity $v_{x_0}(\mathrm{Disc}_y f') = \sum_{P|x_0}(e_P - 1) + 2\sum_{Q|x_0} \delta_Q$ across all affine singularities of the transformed model.

Proposition 5.1. *The dual-route verifier and closed-form arguments agree on the smooth and nodal cubic baselines ($y^2 = x^3 - x + 1$ of genus 1, different 6; $y^2 = x^2(x + 1)$ of genus 0, different 4) and on the three singular curves of Table 3. The genus-eight septic C_3 has gonality three (trigonal) and is non-hyperelliptic.*

Table 3: Curves certified by `compute_exact_genus.py` and `test_new_curves.py`.

Curve	Degree	p_a	Local δ_P	g_A	g_B	Different
$C_1: (y-x^2)^2 = x^5$	5	6	$2, 4$	0	0	8
$C_2: y^3 = x^4(x-1)^2(x+1)$	7	15	$3, 1, 9$	2	2	16
C_3, Appendix A	7	15	$6, 1$	8	8	28

Proof. For C_1, the parametrization $x = t^2,\ y = t^4 + t^5$ has rational inverse $t = (y - x^2)/x^2$ on $x \neq 0$, giving $g = 0$; $[0:0:1]$ has multiplicity sequence $2, 2$ ($\delta = 2$) and $[0:1:0]$ has sequence $3, 2$ ($\delta = 4$). For C_2, setting $w = y/x$ gives the cyclic cubic cover $w^3 = x(x-1)^2(x+1)$ over $\overline{\mathbb{Q}}(x)$, totally ramified at $0, 1, -1, \infty$ (divisor orders $1, 2, 1, -4$), so $2g - 2 = -6 + 8$ and $g = 2$; its plane singularities have local types $(3, 4), (2, 3), (4, 7)$, giving $\delta = 3, 1, 9$. Both C_1 and C_2 have highest homogeneous forms $-X^5, -X^7$, so $[0:1:0]$ is their unique point at infinity.

For C_3, the tangent cone at $[0:0:1]$ is $xy(x-y)(x+2y)$ (an ordinary quadruple point, $\delta = 6$). At $(1, 0)$, setting $x = 1 + u,\ y = u + v$ in $G(u, v) = f(1 + u, u + v)$ eliminates all terms of degree at most one as well as u^2 and uv, leaving $G_{vv}(0) = 6$ and $G_{uuu}(0) = -6$ (an ordinary cusp, $\delta = 1$). Lexicographic elimination of (f, f_x, f_y) to $\mathbb{Q}[x]$ yields $x^5(x-1)^2$, with specialized y-gcds y^3 at $x = 0$ and y at $x = 1$; on the line at infinity, the binary degree-7 form $3 + 2y^2 + 3y^5 - 3y^7$ is squarefree and $[0:1:0]$ has y^7-coefficient $-3 \neq 0$. Rational irreducibility and the smooth witnesses $(1, 2), (-1, 0), (2, 1)$ for C_1, C_2, C_3 are certified in `test_new_curves.py`. Hence $g(C_3) = 15 - 7 = 8$. Projecting C_3 from the quadruple point yields a degree-3 pencil g^1_3; a hyperelliptic pencil g^1_2 would be independent by coprimality and force $g(C_3) \leq (3-1)(2-1) = 2$ by Castelnuovo–Severi, a contradiction. □

6 Verification architecture, reproducibility, and pure-AI provenance

6.1 Verification inventory

Table 4 summarizes the archived Lean checker receipts and the deterministic portable Python/FLINT/SymPy verifications shipped in release v1.0.0. The archived Lean source `lean/DVRConductorNoGoingUp.lean` carries SHA-256 digest `c8a4c67791a980646f16317c2537287c4bc1af869b37e179e88548243b3217cc`. It is compiled under toolchain `leanprover/lean4:v4.33.1` (`lean/lean-toolchain`) with Mathlib pinned to commit `0df444a360eaa60ab8c11dca51a86af692955474` in `lean/lakefile.toml` and `lean/lake-manifest.json`. All eight targets have zero `sorry` and axiom sets contained in {`propext`, `Classical.choice`, `Quot.sound`} (`cm_curve_bidegree_pa` is axiom-free).

The five archived checker runs span three independent codebases across four execution runtimes; all ten executable and Lean 4.33.1 Windows shared-library digests (the three-part `libleanshared` and `libInit_shared`), the `lean4export` 3.1.0 header carrying Lean githash `819816b2e0a3`, and the `lean4export` source commit `15f6055e299a` are archived in `receipts/CHECKER_BINARIES_SHA256.json`: (i) `lean` (launcher `af49bacfabaa`) on the source and `leanchecker` (`31b506f83737`) on the `.olean` object file, both invoking the official Lean 4.33.1 C++ kernel; (ii) `con-leche -verified` (`5dc6d10f2608`, written in Lean with a formal consistency theorem relative to its documented formal and runtime assumptions [3]) and its Rust port `con-ron -verified` (`d9cfda21442d`, providing compiler/runtime isolation on the same checking specification [4]); and (iii) `nanoda_bin` v0.4.17 (`a76bed8bc109`, an independent Rust checker [2]), invoked with `receipts/nanoda_DVRConductorNoGoingUp.json` permitting only `propext`, `Classical.choice`, and `Quot.sound`, enabling `unpermitted_axiom_hard_error`, `nat_extension`, and `string_extension`, and leaving `unsafe_permit_all_axioms` disabled. Upstream git commits for the prebuilt executables `con-leche.exe`, `con-ron.exe`,

and `nanoda_bin.exe` were not recorded at build time and are instead pinned by executable SHA-256 digests in `receipts/CHECKER_BINARIES_SHA256.json`; those binaries and Lean DLLs are not redistributed, so the five runs in `receipts/` serve as historical receipts, while the release directly ships `lean/DVRConductorNoGoingUp.olean` (`1c739996bd1b`), the compressed export `lean/DVRConductorNoGoingUp.export.gz` (`f4180f3644f0`, decompressing to the exact 60,545,266-byte NDJSON stream `62b212ec2b98`), a portable `lean/reproduce_lean.sh` build/check script, and a relative-path `nanoda` config. Only the last three checkers read the NDJSON stream (emitted by `lean4export.exe` binary `cd51ec533320`; those three binaries were executed from the workstation's Lean 4.36.0 toolchain directory, as recorded in `lean/verify_lean5.py`). In that export stream, there are 8,854 theorem, 3,929 definition, 487 inductive, 14 opaque, 3 axiom, and 4 quotient records, with 487 types, 556 constructors, and 489 recursors in the inductive blocks: `con-leche`/`con-ron` count 8,854+3,929+487+14+3 = 13,287 top-level records (folding 30 in-process `Lean.Syntax` records separately), whereas `nanoda_bin` counts 8,854 + 3,929 + 14 + 3 + 4 + 487 + 556 + 489 = 14,336 declarations (reporting zero typechecker errors and one pretty-printer diagnostic, `Unable to print axioms`).

Table 4: Archived Lean checks and deterministic portable Python/FLINT/SymPy checks [11, 19]. Full SHA-256 digests appear in `SHA256SUMS.txt` or under `receipts/`; dashes indicate timing-free deterministic certificates.

Check	Result	Seconds	SHA-256 prefix
`lean` 4.33.1 (C++ kernel)	8 targets, exit 0	181.06	`c8a4c67791a9` (source)
`leanchecker` 4.33.1 (C++)	object accepted (shipped)	24.86	`1c739996bd1b` (object)
`lean4export` 3.1.0 (binary `cd51ec533320`)	60,545,266-byte NDJSON (shipped `.gz`)	12.95	`62b212ec2b98` (stream)
`con-leche -verified` (Lean)	13,287 records (historical log)	2.02	same stream
`con-ron -verified` (Rust port)	13,287 records (historical log)	2.90	same stream
`nanoda_bin` 0.4.17 (Rust)	14,336 decls (historical log)	3.01	same stream
`verify_m2_m6.py` (portable)	$\bar{\ell}$ deg 234, M_2 (58 in 6 orbits), M_6	–	`f0248c55506a` (JSON)
`verify_conductor_1398.py`	1,398 full fibers (8,479 fields), Eq. (4)	–	`636dbaa47f3d` (JSON)
`verify_paper4_new_checks.py`	ℓ over $\mathbb{Q}$, L3, L4, Eq. (6), $R = 25$, C_3	–	`02fa400b306c` (JSON)
`certify_monodromy_S269_smooth.py`	S_{269}, $v_R(D_p) = 141$, rigidity, CS	–	`ea3d24a58f1a` (JSON)
`test_new_curves.py` (Tier C)	C_1, C_2, C_3 ($g = 0, 2, 8$), cubics, inf. reg.	–	`84be85e593a5` (JSON)
Conditional 125-place ranks (log)	9,939 special-fiber units	2622.10	`21ec7eac2503` (JSON)
Mod-p^2/p^3 obstructions (log)	$c_Q = 0,\ c_2 = 0,\ 11$ Weierstrass	–	`ed1fedd26330`, `2b4b8eed8bd6`

All five portable verification scripts in Table 4 emit deterministic LF-normalized JSON certificates containing no wall-clock timings. In table order, their script SHA-256 prefixes are `a75389a15aca`, `5dacc4aa6587`, `458b77db1603`, `a92df469f2af`, and `c10d058e092e` (with verifier `compute_exact_genus.py` at `17e52828cc6d`). Exact interpreter paths, version strings, and wheel tags are logged in `ENVIRONMENT_VERSIONS.json` under `receipts/` (Windows workstation: Python 3.14.0 with `cp314-win_amd64` meson wheels for NumPy 2.2.6 and python-flint 0.9.0, SymPy 1.14.0, and Tectonic 0.17.0; independent Linux reproduction on Python 3.12.14, NumPy 2.3.5, SymPy 1.14.0, and python-flint 0.9.0 confirmed identical JSON contents across all five scripts). Byte-for-byte SHA-256 certificate matching holds on LF-normalized runs in the pinned library stack, while mathematical equivalence of all assertions holds across environments.

The power-series truncation ledger for the 66-branch $R = 25$ verification is recorded in the proof of Proposition 3.5; `verify_paper4_new_checks.py` also retains a fast 30-point smoke check of $N_t^{\mathrm{red}}, N_\sigma^{\mathrm{red}}$, whereas `verify_conductor_1398.py` supplies the full 1,398-slice proof across 8,479 irreducible factor fields (376,062 sheets).

6.2 Research roles and review status

Robert Kadlec, Masaryk University, Brno, directed and supervised this research as part of a broader project on the respective competencies of humans and large language models. All mathematical derivations, symbolic computations, formalizations, audits, and manuscript drafting were performed by AI systems. In the workflow, Antigravity/Gemini coordinated staging, portable verification engineering (including the full 1,398-slice CRT conductor verifier), static linting, PDF compilation, and release packaging; OpenAI Codex analyzed the formulas and code, identified the arithmetic-lift gap and earlier symbolic discrepancies, constructed the degree-66 separator and semistable rigidity arguments, and drafted the initial manuscript; and Anthropic Claude Code (`sonnet` and `claude-fable-5-2`) together with OpenAI ChatGPT (`6 Pro`) served as independent adversarial referees across multiple review rounds, re-deriving all main identities and computations from first principles (the portable `verify_m2_m6.py` script and closed-form $R = 25$ regime/separator block of `verify_paper4_new_checks.py` were adapted from Claude Fable's Round 1 scripts; the characteristic-zero factorization (3) of $\ell(R)$ over $\mathbb{Q}$, the infinity-chart bug fix in `compute_exact_genus.py`, and the mod-p^2/p^3 holdout correction were prompted by ChatGPT `6 Pro` and independently verified; see `AI_DISCLOSURE_AND_PROVENANCE.md`). No LLM output is trusted as proof; every conclusion rests on the displayed arguments and archived machine-checkable certificates.

6.3 Artifact availability and open problems

Release v1.0.0 licenses code under Apache-2.0 and the manuscript and documentation under CC BY 4.0. The complete archival release (Lean 4 module, compiled object, compressed export stream, five-checker receipts, companion tables, deterministic verification scripts, and JSON certificates) is permanently deposited on Zenodo under DOI 10.5281/zenodo.24559397. The frozen E_p coefficient file `cm_curve_E_mod_1000003.json.gz` (SHA-256 `57ca947c1bc34d1b9af4 9116587a519fd7df2b7038fa516afc5a17062a6c2779`; 1,023,700 bytes, where row j, column k multiplies $b^j R^k$, $0 \le j \le 269$, $0 \le k \le 1156$) and all companion tables—`disc.npy`, `m2_aff.npy`, `alpha_m2.npy`, `places.json`, `special_places.json`, `table.txt`, and `final_summary.json`—are shipped directly under `companion_tables/`, making all five portable Tier (B) and Tier (C) verification scripts (and the four standalone symbolic scripts) self-contained without external downloads.

From the release root, those five portable Python checks run directly with Python, NumPy, SymPy, and FLINT (no environment variables required; environment details in `receipts/ENVIRONMENT_VERSIONS.json`):

- `oss_genus_verifier/test_new_curves.py` certifies Proposition 5.1 and the infinity-chart regression suite (SymPy only), writing `NEW_CURVES_CERT.json`.
- `genus_6631/scripts/verify_m2_m6.py` independently cross-checks (3) modulo p and certifies Proposition 3.4, writing `m2_m6_cert.json`.
- `genus_6631/scripts/verify_conductor_1398.py` certifies (4) across 1,398 consecutive full fibers (8,479 irreducible factor fields, 376,062 sheets), writing `conductor_1398_cert.json`.
- `genus_6631/scripts/verify_paper4_new_checks.py` certifies (3) over $\mathbb{Q}$, Lemmas 3.2–3.3, Proposition 3.5 (all three regimes, Puiseux 66-branch check with full precision ledger,

and closed-form separator), a fast 30-point smoke check of (4), and completeness for C_3, writing `paper4_new_checks_cert.json`.

- `monodromy_S269/certify_monodromy_S269_smooth.py` certifies Theorem 4.2, writing `monodromy_S269_smooth_cert.json`.
- The four symbolic scripts `verify_all_char0_cluster_lemmas_v4.py` (`fc1ce4367886`), `verify_r8_binf_el.py` (`d6bd2d15eb43`), `verify_r8_c_beta_unit.py` (`074f52295d3c`), and `verify_r25_triple_pole_lemma.py` (`2952b7ce6c7f`) run with no curve data.

By contrast, the Lean verification workflow (`lean/reproduce_lean.sh`) requires installing `elan` / Lean 4.33.1 and building external checkers from their cited repositories (with `lean/DVRConductorNoGoingUp.olean` and `lean/DVRConductorNoGoingUp.export.gz` shipped directly for immediate inspection). Earlier workstation scripts and provenance logs (including `lean/verify_lean5.py`, the conditional jet-rank and mod-p^2/p^3 scripts, the annotated master summary `bc9e46a0eaf4`, and the $R = 25$ rank certificate `5b014bf6f906`), together with minor docstring errata in the preserved Lean source and standalone v4 scripts, are catalogued in `ERRATA_AND_STATUS.md`.

To upgrade the genus bound $6{,}631 \le g(C) \le 16{,}570$ to an exact-genus theorem $g(C) = 6{,}631$ assuming Global Inputs (G1)–(G3), it would suffice to supply an integral equation over the arithmetic local ring (or a centered-valuation proof) for a generating set at each of the remaining 125 special base places. Further open problems include the exact gonality of C, its unrestricted automorphism group $\mathrm{Aut}_{\overline{\mathbb{Q}}}(C)$, and the determination of $C(\mathbb{Q})$ on the real arc $\mathcal{S}$.

A Exact witnesses, smooth ramification rows, and the septic C_3

Triple-pole boundary polynomials and signed resultants. For the homogeneous quartic $Q_\gamma(x, y, z)$ defined by (6) at $B = 1, a_{\mathrm{reg}} = 0$, put $q_C(s) = Q_\gamma(0, s, 1)/s^2$, $q_B(s) = [x^2]Q_\gamma(x, s, 1)$, and $q^\vee(s) = s^2 q(1/s)$. All four signed resultants $\mathrm{Res}_s(q, q^\vee)$ are units at $p = 1{,}000{,}003$:

Face and γ	Boundary polynomial $q(s)$	Signed $\mathrm{Res}_s(q, q^\vee)$
$C,\ -1$	$-16(17s^2 - 18s + 6)$	1,625,620,480
$B,\ -1$	$-8(12s^2 - 3s + 1)$	79,298,560
$C,\ -1/2$	$-4(41s^2 - 30s + 9)$	419,430,400
$B,\ -1/2$	$-4(9s^2 + 12s - 1)$	−2,048,000

Degree-66 separator polynomial at $R = 25$. For $S_{25} = t/b + (R - 25)\vartheta_{25}$, the monic degree-66 squarefree residue polynomial over $\mathbb{F}_p$ has ascending coefficients starting $602473, 690131, 745416, 406876, 624504$ and ending $648681, 791705, 720648, 803753, 1$. The SHA-256 digest of the comma-separated decimal integers (without spaces or trailing newline) is `f3f0 9b758d572706874d91f79d627d89f60c0a6529ff50257326201b0e9e7aeb`.

Smooth-cover ramification rows used in Theorem 4.2(b). The companion table `companion_tables/special_places.json` records 131 exceptional closed places (with 13,024 generic simple branch points of index 2 and unramified affine nodes). The following exhaustive geometric counts of different coefficients $f_x = \sum_{P|x}(e_P - 1)$ are invoked in Theorem 4.2(b) (exponents indicate multiplicities; index-1 sheets omitted):

f_x	Geometric count	Base values	Ramified indices
5	1	$R = 5$	$3, 3, 2$
6	1	$R = -5$	$3, 3, 2, 2$
7	12	includes 381924, 348525 ($v_R(D_p) = 141$)	$4, 4, 2$
120	4	$R = -3 \pm 4i,\ 3 \pm 4i$	$6^2, 4^9, 3^{25}, 2^{33}$

Here $2(6-1)+9(4-1)+25(3-1)+33(2-1)=120$, and $v_R(D_p)=141$ is odd at both 381924 and 348525.

Equation of the septic C_3. Writing $f(x,y)=\sum_{j=0}^{7}a_j(x)y^j$, the curve C_3 of Proposition 5.1 (`NEW_CURVES_CERT.json`) is defined by

$$\begin{aligned} a_0 &= 3x^7-6x^6+3x^5, & a_1 &= -5x^5+4x^4+x^3,\\ a_2 &= 2x^5-x^4+x^3+x^2, & a_3 &= -3x^3+x^2-2x,\\ a_4 &= 3x^2-2x, & a_5 &= 3x^2+3x-70,\\ a_6 &= 3, & a_7 &= -3. \end{aligned}$$

B Notes on the conditional 125-place programme

This appendix records auxiliary symbolic identities computed for the conditional analysis of the 125 remaining special base places (none of which is used in Propositions 3.4, 3.5, Theorem 3.6's upper bound, Lemma 4.1, or Theorem 4.2).

- *Euclidean norms at $P(z)=0$ and $Q(z)=0$.* Dividing $F_*=\zeta P^2+(a_{3,P}z+a_{2,P})P+r_1z+r_0$ gives $a_{3,P}=f_3-12\zeta$, $a_{2,P}=f_2-6f_3-14\zeta$, and norm $\mathcal{R}_P=25a_{3,P}^2-6a_{3,P}a_{2,P}+a_{2,P}^2$ (with signs reversed at 6 for Q). Assuming $P\mid F_*$ (i.e., $r_0=r_1=0$), one has $F_*'(r)=(a_{3,P}r+a_{2,P})P'(r)$ at each simple root r of P ($\mathrm{Disc}(P)=-64$), so vanishing of $\mathcal{R}_P$ detects when F_* has a multiple root at a root of P.

- *Regular cluster eliminations.* At a regular cluster of m roots with differences $x_j=y_i-y_j$ and $D=\prod_{j\neq i}x_j$, (1) gives leading term $D^2[-11(\sum x_j^{-1})^2-9\sum x_j^{-2}]$ in $24B(y_i)D^3h^\sharp(y_i)$. For $m=3$, the matching polynomial $\Pi(s)=10s^5-31s^4+31s^3+31s^2-31s+10$ has resultants 1,073,217,600, 317,520,000, and 1,323 against $10s^2+11s+10$, $31s^2-31s+10$, and s^2-s+1 (all units at p).

- *Chart ($R^2\pm 8R-25=0, b=\infty$).* In the coordinate $\beta=b/t$ near 0 at a root r of $R^2\pm 8R-25=0$ with $r'=-25/r$ and $u=R-r$, the linear perturbation coefficient of β in $\mathrm{Disc}(q_2)$ for the quadratic factor q_2 near r' is a unit multiple of $r'P'(r')P(r)-rP'(r)P(r')$, taking values $\pm 800\sqrt{41}$ with exact norm $-2^{10}\cdot 5^4\cdot 41=-26{,}240{,}000\equiv 760{,}081 \pmod p$ (certified in `verify_r8_c_beta_unit.py` and `verify_paper4_new_checks.py`), excluding $0<v(\beta)<2v(u)$ and producing the candidate operator β/u^2.

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