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Separated-Weights Interpolation on Higher-Genus CM Abelian Varieties and Irrationality Measure Two for Beta Values

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Abstract

We extend separated-weights jet surjectivity from one-dimensional group varieties to every absolutely simple complex abelian variety $A/\mathbb{C}$ of arbitrary dimension $g \geq 1$ with any effective ample Cartier divisor L. Combining this interpolation theorem with canonical Galois descent of second-kind algebraic 1-forms, covariant translated Appell–Humbert theta gauges, and a multivariable determinant argument over Galois composita, we prove $\mu(\mathcal{T}) = 2$ for every nonzero real period or quasi-period $\mathcal{T}$ of a $\mathbb{Q}$-rational CM eigenclass on an absolutely simple CM abelian variety over $\mathbb{Q}$.

Beyond the genus-1 first-kind rational CM elliptic case treated concurrently by G. V. and D. V. Chudnovsky, this yields higher-genus and second-kind bounds $\mu(B(a/b, a/b)) = \mu(\pi/B(a/b, a/b)) = 2$ and Beta Flint–Hills convergence $\sum_{n\geq 1} n^{-s} \sin^{-2}(n\, B(a/b, a/b)) < \infty$ $(s > 2)$ for all reduced $a/b \in (0, 1/2)$ with $b = 4$ or $4 \nmid b$; for example, in genus 2, $\mu(B(1/5, 1/5)) = \mu(\pi/B(1/5, 1/5)) = 2$ with $B(1/5, 1/5) = \Gamma(1/5)^2/\Gamma(2/5)$. Using Fermat quotients $y^b = x^r(1-x)^s$ $(b \geq 5)$ we obtain $\mu = 2$ for asymmetric and $4 \mid b$ Beta values on all primitive CM factors (existing iff $b \notin \{8, 12\}$), and at $b \in \{8, 12\}$ we classify all 32 reduced pairs (4 with $\mu = 2$, 28 with $\mu \leq 2d$), showing that every level $b \geq 3$ admits a reduced Beta value with $\mu = 2$.

Nine five-kernel-verified Lean 4.33.1 modules (147 targets) verify the determinant engine (cond. Thm. 3.5, Prop. 5.5), Galois/Cauchy bridge, $g \geq 1$ Jacobi/Riemann theta gauges, genus-2 $\mathbb{C}^2/\Lambda_{\tau_5}$, Sec. 2 Hensel/jet-drop/Stage 2–3 bookkeeping, and Cor. 3.9; algebraic geometry and curve-period proofs remain in prose without human peer review.

1 Introduction, main results, and verification scope

A rapid series of unrefereed autumn 2026 preprints introduced and developed the separated-weights interpolation method for irrationality exponents of 1-periods: first on $\mathbb{G}_m \times \mathbb{G}_a^m$ for $\mu(\pi) = 2$ [30], then for $\mu(\log r) = 2$ $(r \in \mathbb{Q}_{>0} \setminus \{1\})$ by J. Liu, K. Jiang, and P. Zhang [22] (with Lean 4 formalization) and for logarithmic values, ratios, and linear forms in logarithms by G. V. Chudnovsky [7, 8], and in the elliptic setting for the lemniscate period $\Omega = 2\int_0^1 (1-t^4)^{-1/2}\, dt = \frac{1}{2}B(1/4, 1/2)$ with a Lean 4 formalization by G. V. Chudnovsky [6] and for real first-kind periods of CM elliptic curves over $\mathbb{Q}$ with a rational invariant differential by G. V. and D. V. Chudnovsky [9]. Classical transcendence theory—beginning with Schneider [32] on Euler Beta values $B(\alpha, \beta)$ $(\alpha, \beta, \alpha+\beta \in \mathbb{Q} \setminus \mathbb{Z})$ and culminating in Wolfart–Wüstholz

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[35], Wüstholz's analytic subgroup theorem [36], and Huber–Wüstholz [16, Corollary 13.13]—establishes transcendence of every nonzero complete period of the first or second kind on a curve over $\overline{\mathbb{Q}}$ (hence $\mu \geq 2$). Effective irrationality measures for related Gamma values remain very large: Bruiltet [4] proved $\mu(\Gamma(1/4)) \leq 10^{143}$ and $\mu(\Gamma(1/3)) \leq 10^{151}$, recently sharpened to $\mu(\Gamma(1/4)) \leq 3.24 \cdot 10^{26}$ and $\mu(\Gamma(1/3)) \leq 1.74 \cdot 10^{26}$ by Niedbala Giraudin [28].

While the genus-1 first-kind case of rational CM elliptic periods is already covered by [6, 9], the new contributions of the present paper are *arbitrary higher genus* $g \geq 2$ and *second-kind quasi-periods*, yielding sharp (ineffective) irrationality exponent $\mu = 2$ for infinite families of symmetric and asymmetric Euler Beta values $B(a/b, c/b)$ and reciprocals $\pi/B(a/b, a/b)$, together with Beta Flint–Hills convergence. Transferring separated-weights interpolation beyond genus $g = 1$ encounters four coupled obstructions:

(1) *Intermediate-dimensional persistent subvarieties ($g \geq 2$):* Inside $A \times \mathbb{A}^m$, a persistent component of a derivative zero locus can project onto a singular proper subvariety $Y \subsetneq A$ of intermediate dimension $2 \leq \dim Y \leq g - 1$, where neither smooth curve normalization nor the surface Hodge index theorem applies.

(2) *Non-principal polarizations and splitting new factors:* Beyond odd primes $p = 2g+1$, the full Jacobian $J(y^2 = 4x^b \pm 1)$ splits off old subvarieties; the new factor $A_b = J_b^{\mathrm{new}}$ of dimension $g_b = \varphi(b)/2$ carries a non-principal polarization $L = (\Theta - Q_0)|_{A_b}$ ($\chi(L) > 1$), and its CM type Φ_b loses primitivity whenever $4 \mid b$ with $b \geq 8$ [11, Lemma 3.2.1].

(3) *Second-kind algebraic forms and translated theta growth:* Explicit Baker/Ônishi–Nakayashiki σ-function formulas [5, 29, 27] are tied to odd-degree hyperelliptic models and fixed basepoints, whereas composite, even-degree, and Fermat quotient curves require canonical Galois descent of second-kind representatives in $H^0(J, \Omega^1_J(2\Theta_O))_{\mathrm{cl}}$ and uniform translated theta gauges whose Hermitian-normalized modulus is lattice-periodic over arbitrary finite Galois basepoint fields $F/\mathbb{Q}$ [2].

(4) *Asymmetric Beta values and levels $4 \mid b$:* Hyperelliptic curves $y^2 = x^b - 1$ produce only symmetric Beta values $B(a/b, a/b)$ and fail to yield a simple new factor whenever $4 \mid b$, $b \geq 8$. Passing to Fermat quotient curves $C_{b;r,s} : y^b = x^r(1-x)^s$ [18, 13, 1] (with p-adic Gamma and Gauss-sum analogues in [12]) overcomes both barriers for all $b \geq 5$ except the Klein-four levels $b \in \{8, 12\}$, which require an exhaustive exact classification of all 32 reduced numerator pairs (a, c) satisfying $1 \leq a \leq c$, $a + c < b$, and $\gcd(a, c, b) = 1$.

We resolve all four obstructions inside a single self-contained framework. In Sections 2–5 we develop:

- the arbitrary-genus non-principally polarized interpolation theorem (Theorem 2.2) and quantitative multivariate Hensel denominator bound (Lemma 2.9);
- canonical second-kind descent, diagonal Hermiticity, and covariant translated theta gauges (Propositions 3.1, 3.3 and Lemmas 3.2, 3.4);
- the zero-penalty multivariable determinant theorem (Theorem 3.5), its quantified extension to algebraic classes (Proposition 3.6), a self-contained proof for rank-two imaginary quadratic CM orders on elliptic curves over $\mathbb{Q}$ (Proposition 3.7, covering $\mathbb{Z}[\sqrt{-2}]$ directly without relying on [9]), and Beta Flint–Hills convergence transfer (Corollary 3.9);
- an elementary self-contained proof (and Lean 4 formalization) of the hyperelliptic primitivity dichotomy of [11, Lemma 3.2.1] (Theorem 4.1), the symmetric Beta theorem for $b = 4$ or $4 \nmid b$ (Theorem 4.4, with the $b = 4$ model stated as Proposition 4.3), and in Appendix A explicit genus-2 and genus-3 Baker–Kleinian realizations with certified global theta enclosures (not used in any Diophantine proof);

- the Fermat quotient theorem for asymmetric and $4 \mid b$ levels $b \notin \{8, 12\}$ (Subsection 5.1, Theorems 5.1–5.3); and
- the exhaustive 32/32 exact classification of all reduced numerator pairs $1 \le a \le c$, $a + c < b$, $\gcd(a, c, b) = 1$ at $b \in \{8, 12\}$ (Theorem 5.4), the single-place upper bound $\mu \le 2d$ (Proposition 5.5), and the all-levels $b \ge 3$ existence theorem (Corollary 5.6).

Table 1 summarizes our main results together with the companion formal and computational artifacts.

1.1 Verification scope and cited external results

To keep the boundary completely transparent between our human-readable mathematical proofs, our companion Lean 4 formalizations, and our symbolic/numerical checks:

- *Main theorems (prose proofs, not yet human-refereed).* All main theorems of this paper—arbitrary-genus jet surjectivity (Theorem 2.2), canonical second-kind descent and the covariant translated theta gauge (Propositions 3.1, 3.3), the quantitative Hensel denominator bound (Lemma 2.9), the zero-penalty determinant theorem and its rank-two and algebraic-class variants (Theorem 3.5, Propositions 3.6–3.7), the hyperelliptic and Fermat-quotient period and homology-cycle constructions (Proposition 4.2, Theorem 5.1), and the level-8/12 classification and single-place bound (Theorem 5.4, Proposition 5.5)—are proved in prose in Sections 2–5 and have *not* yet been verified by human peer review. We cite without reproving the following published results: Kollár's simultaneous normalization [20] together with openness of geometric normality in characteristic zero [14, Théorème 12.2.4], Fulton's intersection positivity [10, Proposition 7.1(b), Theorem 12.2, and Example 12.2.7(a)], the Swanson–Huneke multiplicity formula [34, Theorem 11.2.7], the Lefschetz–Koizumi normal generation theorem [3, 19], Kleiman's ampleness criterion, relative Proj/Rees pushdown and Serre vanishing [15], [33, Tag 0AG7], the Appell–Humbert theta transformation law [3] and reduced theta normalization [2, Theorem 1.7, Definition 1.8, and §1.3], Ônishi's division identities [29, Definition 9.2 and Theorem 9.3], the CM simplicity criterion [24, Proposition 3.13 and Corollary 3.14], Aoki's Fermat-Jacobian classification for $b > 180$ [1, Theorem 0.2], and Wüstholz's analytic subgroup theorem together with the Huber–Wüstholz transcendence theorem [36] and [16, Corollary 13.13].
- *Companion Lean 4.33.1 formalizations (nine modules, 147 audited targets).* Following the conditional formalization pattern introduced in genus 1 by [6, 9] and extending it here to arbitrary genus $g \ge 1$ and second-kind quasi-periods, four new modules—`ConditionalDeterminantEngine.lean` (24 public targets), `GeometricBridge.lean` (19 public targets), `ConcreteCMThetaModels.lean` (20 public targets), and `HigherGenusThetaAndZeroEstimate.lean` (18 public targets)—formally prove in Lean 4.33.1: (a) the multivariable determinant engine of Theorem 3.5 and Proposition 3.7 (`theorem_3_5_master_determinant` and `theorem_3_5_from_geometric_inputs`: $\mu(\mathcal{T}) = 2$ and $\mu(r\mathcal{T}) = 2$ for all $r \in \mathbb{Q}^\times$, conditional on typed data); (b) the single-distinguished-place bound of Proposition 5.5 (`proposition_5_5_single_distinguished_place` and `proposition_5_5_from_geometric_inputs`: $\mu(\mathcal{T}) \le 2d$); (c) the totally complex Galois product formula over Mathlib's `NumberField/InfinitePlace` applied to one single non-singular minor Δ_{raw} and its actual Galois conjugates, the arbitrary-dimension g-polydisk Cauchy inequality via iterated torus integrals, the cleared-monomial E_L bound, and the $4^{H/w_*}$ binomial bound; (d) the genus-1 classical Jacobi theta-gauge and CM-field layer on $\Lambda_{\Omega,\kappa} = \Omega(\mathbb{Z} + i\kappa\mathbb{Z})$ (including a transparent non-singular 2×2, $E_L = 3$, $\det = -2$ witness with injective centre-index map `rowCen`, nonzero column degree $\beta_{\mathrm{col}} = (0, 1)$, meromorphic theta-ratio section $f_{\Omega/2}/f_0$, and publicly exposed exact infinite Taylor series in `C_mat`, plus

Result	Mathematical content	Companion artifacts	Verification status
Arbitrary-genus interpolation (Thm. 2.2, Lem. 2.9)	Jet surjectivity on $(A \setminus \mathrm{Supp}(L)) \times \mathbb{A}^m$ for every simple abelian g-fold A and ample L ($D = 3L$); multivariate Hensel denominators.	`HigherGenusThetaAndZeroEstimate` (Hensel u^{2n-1}, `MvPowerSeries` jet drop, Stage 2–3 bookkeeping); scalar lemmas in `ArbitraryGenusCM`, `CompositeBetaCM`, `Level8And12BetaCM`.	Prose alg. geom.; Lean Hensel, jet drop & Stage 2–3 bookkeeping
Descent, covariant gauge, determinant engine (Props. 3.1–3.7, Thm. 3.5, Cor. 3.9)	Canonical $H^0(J_F, \Omega^1_{J_F}(2\Theta_O))_{\mathrm{cl}} \xrightarrow{\sim} H^1_{\mathrm{dR}}(J/F)$, Λ_A-periodic Hermitian modulus $g_0^{(v)}(z) = e^{-Q(z)}\lvert f_v^{\mathrm{norm}}(z)\rvert$, zero-penalty $\mathbb{L} = K_b F$ determinant, rank-2 CM orders, Flint–Hills transfer.	`ConditionalDeterminantEngine`, `GeometricBridge`, `Concrete CMThetaModels`, `HigherGenusThetaAndZeroEstimate` (cond. Thm. 3.5, Galois/Cauchy, $g \geq 1$ Riemann & Jacobi theta gauges, direct Cor. 3.9).	Prose geom.; cond. Lean Thm. 3.5; Lean $g \geq 1$ gauges & Cor. 3.9
Hyperelliptic symmetric Beta, $b = 4$ or $4 \nmid b$ (Thms. 4.1, 4.4, Prop. 4.3)	Self-contained proof of [11, Lem. 3.2.1]; $\mu(B(a/b, a/b)) = \mu(\pi/B(a/b, a/b)) = 2$ and $\sum n^{-s} \sin^{-2}(nB(a/b, a/b)) < \infty$ ($\forall s > 2$); explicit $g = 2, 3$ enclosures in Appendix A.	`CompositeBetaCM` (stabiliser dichotomy); `HigherGenusThetaAndZeroEstimate` ($g = 2$ τ_5, diag. Herm., cond. C_5^-); `B4`, `D`, `E` Arb certs.	Prose curve-period; Lean dichotomy & $g = 2$ surface; Arb checks
Fermat quotients: asymmetric & $4 \mid b$ (Thms. 5.1–5.3)	Doubly cleared Pochhammer periods; $\mu = 2$ for 1st- and 2nd-kind Beta values on all primitive $C_{b;r,s}$ (existing iff $b \notin \{8, 12\}$); non-isogeny.	`FermatQuotient BetaCM` (infinite primitivity & $b = 8, 12$ in Lean); $5 \leq b \leq 180$ census + 568-bit Arb.	Prose proof; Lean families; Arb checks
Levels $b \in \{8, 12\}$ & all-levels $b \geq 3$ (Thm. 5.4, Prop. 5.5, Cor. 5.6)	Exhaustive 32/32 census of reduced pairs $1 \leq a \leq c$, $a + c < b$, $\gcd(a, c, b) = 1$: 4 rational CM periods ($\mu = 2$), 28 irrational twists ($\mu \leq 2d$); $\exists$ Beta with $\mu = 2$ at every $b \geq 3$.	`ConditionalDeterminantEngine` & `GeometricBridge` (cond. Prop. 5.5); `BETA_EXACT_CLASSIFICATION.json`.	Prose geom.; cond. Lean Prop. 5.5; FLINT/Arb

Table 1: Summary of main results and supporting formal and computational artifacts. The multivariable determinant reductions and Galois/Cauchy bridge (Theorem 3.5, Proposition 5.5) are conditionally formally verified in Lean 4.33.1, while the classical genus-1 Jacobi and arbitrary-genus ($g \geq 1$) Riemann theta-gauge bounds, the genus-2 principally polarized CM surface $\mathbb{C}^2/\Lambda_{\tau_5}$, the Section 2 Hensel/jet-drop lemmas and Stage 2–3 inequality bookkeeping, the Flint–Hills transfer (Corollary 3.9), and the combinatorial CM-type primitivity families are directly formally verified; all algebraic-geometric, jet-surjectivity, descent, large-H non-emptiness, and curve-period constructions remain proved in prose.

conditional instantiations `lemniscate_e4_mu_eq_two_of_algebraic_data` and `curve_e8_mu_eq_two_of_algebraic_data`); (e) the *arbitrary-genus ($g \geq 1$) Riemann/Siegel theta-gauge layer* on $\mathbb{C}^g \times \mathbb{H}_g$ (absolute summability, continuity, g-dimensional quasi-periodicity, unconditional rational-torsion nonvanishing, norm Appell–Humbert eq. (1), Proposition 3.3(i)–(iii) with explicit $c_0 = 1/2$, and standalone second-kind directional log-derivative for ϑ_O and its v-translate shifted by $-\pi H_\tau(e, v)$), the *concrete genus-2 principally polarized CM surface* $\mathbb{C}^2/\Lambda_{\tau_5}$ with CM by $\mathbb{Z}[\zeta_5]$ (exact $\tau_5 \in \mathbb{H}_2$ with $\Omega_1\tau_5 = \Omega_2$, exact $\operatorname{Im}\tau_5$ and $Y^{-1}\operatorname{Im}\tau_5 = I_2$, explicit surjective $\mathbb{Z}^4$-parametrization of Λ_{τ_5} matching the Minkowski embedding of $\mathbb{Z}[\zeta_5]$, unimodular symplectic pairing $\operatorname{Im} H_{\tau_5}(a + \tau_5 b, a' + \tau_5 b') = b^\top a' - a^\top b'$, exact genus-2 Lemma 3.2 diagonal Hermiticity, transparent genus-2 consistency witness, and conditional `genus2_c5_mu_eq_two_of_algebraic_data`), and formal Section 2 algebraic/inequality lemmas (univariate single-index Hensel u^{2n-1} denominator/uniqueness theorem underlying Lemma 2.9, commuting `MvPowerSeries` partial derivations and space/time separated-weights jet-order drop lower bounds underlying Lemma 2.5, and Theorem 2.2 Stage 2–3 real-inequality/combinatorics bookkeeping with $v_0 = 2N_{\mathrm{box}}^2\theta^\ell w_0$, uniform rational slack δ_{sl}, dilation closing bound, pole-vs-zero contradiction, and pigeonhole stabilization); and (f) the direct geometry-free Flint–Hills convergence transfer of Corollary 3.9. In addition, the five previously archived modules (66 targets) formalize the complete hyperelliptic stabiliser dichotomy (`CompositeBetaCM`, five theorems) and the infinite Fermat primitivity families, Klein-four obstructions at $b = 8, 12$, and $b = 11$ orbit separation (`FermatQuotientBetaCM`, eight theorems), plus 53 elementary scalar checks. Crucially, *no scheme, Chow variety, Kollár/Fulton/Swanson–Huneke algebraic geometry, curve-to-period identification, or transcendence theorem is constructed inside Lean 4*: irrationality of Ω, Section 2 jet surjectivity, Lemma 2.9 matrix integrality, Weierstrass/Kleinian Taylor matching, large-H non-emptiness of the algebraic-data structures (where all Diophantine content of the conditional $\mu = 2$ theorems resides), identification of $\mathbb{C}^2/\Lambda_{\tau_5}$ with $J(C_5^-)$ (class-number-1 uniqueness), and curve-period identities remain proved in prose in Sections 2–5; see Subsection 6.1 for exact scope and caveats.

- *Symbolic and interval computations.* Six companion Python scripts built on `SymPy` 1.14.0 and `python-flint` 0.9.0 / Arb [17] verify exact symbolic identities, finite CM-type stabiliser censuses ($3 \leq b \leq 100$ and $5 \leq b \leq 180$), exact FLINT irreducibility of all 32 level-8/12 twist polynomials, high-precision Gamma/Beta arithmetic relations, and outward-rounded global theta enclosures over $\mathbb{C}^2/\Lambda_{J_2^-}$ and $\mathbb{C}^3/\Lambda_{J_3^-}$. Homology-cycle integrality and period realizations are proved in prose, and *no main Diophantine theorem depends on the numerical enclosures of Appendix A.*

1.2 Principal notation conventions

To prevent symbol collisions across sections, we fix the following notation throughout:

- *Levels, dimensions, and centres:* $b \geq 3$ denotes the cyclotomic level (with odd half-level $m_0 := b/2$ when $b \equiv 2 \pmod 4$); $\varphi(b)$ is Euler's totient; $g = \dim A \geq 1$ is the dimension of the simple abelian variety A inside an ambient Jacobian J of genus g_{tot}; $m = g\ell \geq 1$ is the additive dimension (number of rational approximants); $K_{\mathrm{cen}} \geq 1$ denotes the number of interpolation centres (distinct from CM fields K, K_b).

- *Geometry and interpolation:* L denotes an effective ample Cartier divisor ($D = 3L$); $P_0 \in A^\circ := A \setminus \operatorname{Supp}(L)$ denotes the base centre on A (specialized to the origin 0_A in Section 3); $Q_0 \in J(\overline{\mathbb{Q}}) \setminus \Theta_O$ denotes the ambient translation point on J with conjugates $Q_\gamma = \gamma(Q_0) + T_\gamma$; $s_0 = 1, s_1, \ldots, s_{N_s}$ is a basis of $H^0(A, D)$; $\varpi_Y : \widetilde{Y} \to Y$ denotes normalization; ϕ_{cl} denotes a pole-clearing section; η_{target} denotes the closed algebraic target

differential on A°; $h_{\mathbf{u}}$ denotes a cleared higher-order differential operator; h_P denotes the normalized local jet order along a curve branch P.

- *Diophantine and determinant parameters:* $\mathcal{T}$ denotes the real target period or quasi-period; $\mu(\cdot)$ denotes the irrationality exponent; $T_i := \lfloor F_0 w_i/v_0 \rfloor + 1$ denotes the Taylor truncation order; $B_{\mathrm{den}} \in \mathbb{Z}_{>0}$ is the Hensel denominator base; $\theta, A_{\mathrm{cut}}, \beta_*, C, \varrho \in \mathbb{Q}_{>0}$ denote the determinant-budget parameters (with ϱ denoting the collision-row fraction, never η); $\delta_{\mathrm{sl}} \in \mathbb{Q}_{>0}$ and $q_{\mathrm{sl}} := (1+2\delta_{\mathrm{sl}})/(1+\delta_{\mathrm{sl}})$ denote the interpolation slack; $E_L := 3\lfloor H/w_0 \rfloor + \lfloor H/w_* \rfloor$ denotes the common theta-clearing exponent; ξ_j denote CM box multipliers (never eigenvalues); ζ_j denote automorphism eigenvalues on $\mathrm{Lie}(J)$.

- *CM types, fields, and Beta values:* $K_b = \mathbb{Q}(\zeta_b)$ is the b-th cyclotomic field and $\mathbb{L} = K_b F$ is the finite Galois compositum; $\Phi_b, \Phi_{b;r,s}$ denote CM types (called *primitive* strictly when their stabiliser is trivial) while $\Phi_n^{\mathrm{cyc}}(z)$ denotes the n-th cyclotomic polynomial; $B(\alpha,\beta)$ denotes the Euler Beta function; unordered pairs (a,c) with $1 \le a \le c$, $a+c<b$, $\gcd(a,c,b)=1$ are called *reduced numerator pairs*; $M_{b;r,s} := \mathrm{rad}(b)\Phi_{e_0}^{\mathrm{cyc}}(1)\Phi_{e_1}^{\mathrm{cyc}}(1)$ denotes the Fermat clearing integer; $\sigma(u)$ denotes the Kleinian sigma function and ϑ_O denotes a reduced Appell–Humbert theta function; σ_k denotes a CM embedding, $\tau \in \mathrm{Gal}(K_b/\mathbb{Q})$ and $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$ denote Galois automorphisms, and $\iota : \mathbb{L} \hookrightarrow \mathbb{C}$ denotes a complex embedding. In Appendix A only, $(S,T) = (x_1+x_2, x_1x_2)$ denote the classical symmetric-square coordinates on $\mathrm{Sym}^2(C_5^-)$.

2 Arbitrary-genus separated-weights interpolation on simple abelian varieties

Definition 2.1 (Ambient space and weighted jets)**.** Let $A/\mathbb{C}$ be an absolutely simple abelian variety of dimension $g \ge 1$. Let L be an effective ample Cartier divisor, $D = 3L$, $A^\circ = A \setminus \mathrm{Supp}(L)$, and $P_0 \in A^\circ$. Write s_0 for the canonical global section of $\mathcal{O}_A(D)$ with $\mathrm{div}(s_0) = D$. Division by s_0^k identifies $H^0(A,kD)$ with a finite-dimensional subspace of $\mathbb{C}[A^\circ]$. By the Lefschetz and Koizumi theorems [3, Theorem 4.5.1 and Theorem 7.3.1], [19], $D = 3L$ is very ample and normally generated on A (any ample line bundle on A is a translate of a symmetric ample line bundle, by divisibility of $\mathrm{Pic}^0(A)$ and surjectivity of the polarization isogeny $\phi_L : A \to \widehat{A}$). Kodaira–Mumford vanishing for ample line bundles and Riemann–Roch [3, Theorem 3.6.3] give $\chi(L) = h^0(A,L) > 0$ and $D^g = 3^g g!\,\chi(L)$.

Fix $m \ge 1$, $w_0, v_0 \in \mathbb{Q}_{>0}$, a rational number $0 < \theta < 1$, integers $w_1, \dots, w_m \in \mathbb{Z}_{>0}$, and $v_i = w_i/\theta$ for $1 \le i \le m$. For any integral projective subvariety $Y \subseteq A$ containing P_0, write $\varpi_Y : \widetilde{Y} \to Y$ for its normalization and $E = \varpi_Y^*(D|_Y)$. For every real $N \ge 0$, define the weighted section space

$$V_N^{(\widetilde{Y})} = \bigoplus_{\substack{\mathbf{b}\in\mathbb{N}^m \\ w\cdot\mathbf{b}\le N}} H^0\Big(\widetilde{Y}, \Big\lfloor \frac{N - w\cdot\mathbf{b}}{w_0} \Big\rfloor E\Big) X^{\mathbf{b}}, \qquad V_H = V_H^{(A)},$$

where $\varpi_Y^* s_0$ trivializes E on $\widetilde{Y}^\circ := \widetilde{Y} \setminus \mathrm{Supp}\, E$. If $s_0 = 1, s_1, \dots, s_{N_s}$ is a trivialized basis of $H^0(A,D)$, normal generation of D implies that V_H is spanned by monomials $s^{\mathbf{k}} X^{\mathbf{b}}$ of weighted degree $w_0|\mathbf{k}| + w\cdot\mathbf{b} \le H$.

Let η_{target} be a closed algebraic 1-form on A° whose poles along $\mathrm{Supp}(L)$ are bounded by $2L$ (i.e., $f^2\eta_{\mathrm{target}}$ is regular for every local defining equation f of L). Fix regular local parameters $\mathbf{e} = (e_1, \dots, e_g)$ of A at P_0, and for each $1 \le i \le m$ and integer $T_i \ge 1$ let $G_i \in \mathbb{C}[\mathbf{e}]$ be the unique polynomial of total degree $< T_i$ with $G_i(0) = 0$ matching the formal primitive of η_{target} at P_0 modulo $(\mathbf{e})^{T_i}$. Given $K_{\mathrm{cen}} \ge 1$ centres $(P_0, c_j) \in A^\circ \times \mathbb{A}^m$ ($1 \le j \le K_{\mathrm{cen}}$, $c_j = (c_{j1}, \dots, c_{jm}) \in \mathbb{C}^m$), set $U'_{ji} := X_i - c_{ji} - G_i(\mathbf{e})$ and define the weighted jet ideal $\mathcal{J}_{j,\ge H} \subset \mathcal{O}_{A^\circ\times\mathbb{A}^m,(P_0,c_j)}$ to be generated by all monomials $\mathbf{e}^{\mathbf{n}}(U'_j)^{\mathbf{b}}$ of jet weight $v_0|\mathbf{n}| + v\cdot\mathbf{b} \ge H$.

For any integral curve $C \subset A^\circ \times \mathbb{A}^m$, let C^ν be its smooth complete normalization. With $\mathrm{ord}_Q(0) = +\infty$ and $\mathrm{ord}_Q^-(f) = \max(0, -\mathrm{ord}_Q(f))$, define the weighted degree of C and the normalized local jet order at any branch $P \in C^\nu$ mapping to a centre (P_0, c_j) by

$$\deg_W(C) = \sum_{Q \in C^\nu} \max\left\{ \max_{0 \le r \le N_s} \frac{\mathrm{ord}_Q^-(s_r)}{w_0}, \max_{1 \le i \le m} \frac{\mathrm{ord}_Q^-(X_i)}{w_i} \right\},$$

$$h_P = \min\left\{ \min_{1 \le a \le g} \frac{\mathrm{ord}_P(e_a)}{v_0}, \min_{1 \le i \le m} \frac{\mathrm{ord}_P(U'_{ji})}{v_i} \right\} \qquad (P \mapsto (P_0, c_j)).$$

Since D is globally generated, $\max_{0 \le r \le N_s} \mathrm{ord}_Q^-(s_r)$ equals the pullback pole order of D at Q and is independent of the chosen basis; likewise $\min_a \mathrm{ord}_P(e_a)$ equals the order of vanishing of the maximal ideal $\mathfrak{m}_{A,P_0}$ along P. In any horizontal commuting frame (D_{0a}, ∂_{X_i}) (Lemma 2.5), a mixed derivative of horizontal multi-index $\mathbf{u}$ and additive multi-index $\mathbf{b}$ has jet cost $v_0|\mathbf{u}| + v \cdot \mathbf{b}$.

Theorem 2.2 (Non-principally polarized arbitrary-genus jet surjectivity)**.** *Let $A/\mathbb{C}$, L, $D = 3L$, $P_0 \in A^\circ$, and η_{target} be as in Definition 2.1. Suppose that the period map of η_{target} on $H_1(A^\circ, \mathbb{Z})$ factors through the surjection $H_1(A^\circ, \mathbb{Z}) \twoheadrightarrow H_1(A, \mathbb{Z})$ and that there exists a cycle $\lambda_{\mathrm{nz}} \in H_1(A, \mathbb{Z})$ with $\int_{\lambda_{\mathrm{nz}}} \eta_{\mathrm{target}} \neq 0$. Fix $m \ge 1$, $K_{\mathrm{cen}} \ge 1$, $w_0, v_0, F_0 \in \mathbb{Q}_{>0}$, and rational $\theta \in (0,1)$ satisfying $F_0\theta \ge 2$ and*

$$a_g = \frac{K_{\mathrm{cen}}\theta^m (w_0/v_0)^g}{3^g g!\, \chi(L)} < 1,$$

$$a_d = K_{\mathrm{cen}}\theta^m (w_0/v_0)^d < 1 \qquad (0 \le d < g).$$

Choose $\delta_{\mathrm{sl}} \in \mathbb{Q}_{>0}$ so that $(1+3\delta_{\mathrm{sl}})^{m+d} a_d < 1$ for every $0 \le d \le g$ and $q_{\mathrm{sl}}\theta < 1$, where $q_{\mathrm{sl}} := (1+2\delta_{\mathrm{sl}})/(1+\delta_{\mathrm{sl}})$. Then there exist positive constants $W_1, \dots, W_m$, where W_i depends only on $(A, L, D, \eta_{\mathrm{target}}, m, K_{\mathrm{cen}}, w_0, v_0, \theta, \delta_{\mathrm{sl}})$ and on the earlier weights $w_1, \dots, w_{i-1}$—and is completely independent of the centres (P_0, c_j)—such that for every tuple of positive integers $w_1, \dots, w_m$ with $w_i > W_i$ $(1 \le i \le m)$ and every set of K_{cen} centres (P_0, c_j) satisfying $c_{ji} \neq c_{j'i}$ whenever $j \neq j'$ for each $1 \le i \le m$, setting $T_i := \lfloor F_0 w_i / v_0 \rfloor + 1$, there exist positive integers R_{wt}, n_0 with $R_{\mathrm{wt}}/w_a, R_{\mathrm{wt}}/v_a \in \mathbb{Z}_{>0}$ $(0 \le a \le m)$ such that for every $H = nR_{\mathrm{wt}}$ with $n \ge n_0$ the jet evaluation map

$$V_H \longrightarrow \bigoplus_{j=1}^{K_{\mathrm{cen}}} \mathcal{O}_{A^\circ \times \mathbb{A}^m, (P_0, c_j)} / \mathcal{J}_{j, \ge H}$$

is surjective. If $A, L, P_0, \eta_{\mathrm{target}}$, the regular parameters $\mathbf{e}$, and the centres c_j are defined over a subfield $F \subset \mathbb{C}$, the same evaluation map is surjective over F.

Remark 2.3 (Relation to classical zero estimates and interpolation determinants). Theorem 2.2 plays the role of a multiplicity estimate on the product $A \times \mathbb{A}^m$ in the separated-weights regime, where $w_1 \ll w_2 \ll \cdots \ll w_m$ grow recursively without fixed ratios. Classical zero estimates on commutative algebraic groups and products due to Philippon [31] and Nakamaye [26] treat general positive multidegree tuples on products of projective or group varieties, yielding algebraic subgroup obstructions in the fixed-multidegree regime, while Laurent's interpolation determinant method [21] pairs such estimates with determinant expansions at a fixed number of points. In the separated-weights architecture initiated in [30, 22, 6, 9], allowing w_i to exceed an explicit threshold depending on $w_1, \dots, w_{i-1}$ forces every frame normal basis of any persistent component Z' to contain the largest non-tangent additive direction ∂_{X_i}, collapsing the obstruction to an exact differential $dX_i = \varpi_Y^* \eta_{\mathrm{target}}$ on Z'. The new geometric input of Theorem 2.2 is executing this recursive weight-separation induction on $A \times \mathbb{A}^m$ using a meromorphic horizontal commuting frame (cleared via Lemma 2.5), simultaneous normalization of singular proper intermediate projections $Y \subsetneq A$ of arbitrary dimension $2 \le d \le g-1$, and arbitrary non-principal ample

polarizations L on simple abelian varieties A. Because the thresholds W_i rely on Chow-variety boundedness (Lemma 2.4), they are ineffective; combined with the arithmetic determinant hypotheses of Theorem 3.5, Theorem 2.2 therefore yields sharp asymptotic irrationality exponents $\mu = 2$ rather than effective lower bounds on $|\mathcal{T} - p/q|$ (over $\mathbb{C}$ alone, Theorem 2.2 carries no arithmetic constraint on the periods of η_{target}; cf. Remark 3.8).

Theorem 2.2 is established through three lemmas (Lemmas 2.4, 2.5, and 2.7) followed by the induction on $\dim Y$ and the passage from curves to jets. Throughout, $d_{g,\max} := D^g = 3^g g!\,\chi(L)$ and $d_{0,g} := 1$ (since $D = 3L$ is very ample on A).

Lemma 2.4 (Bounded normalizations, uniform clearing ideals, and bad-locus degrees)**.** *Fix $1 \le d < g$ and an integer degree bound $d_{\max} \ge 1$. There exist uniform positive integers $d_{0,d}$, $l_{d,\max}$, and $D^{(d)}_{\text{bad}}$, depending only on $(A, D, d, d_{\max})$, such that for every integral projective subvariety $Y \subseteq A$ of dimension d and D-degree $\deg_D(Y) \le d_{\max}$ (with normalization $\varpi_Y : \widetilde{Y} \to Y$ and $E := \varpi_Y^*(D|_Y)$):*

(i) $d_{0,d}E$ is very ample on $\widetilde{Y}$;

(ii) there exist d global invariant 1-forms $\Omega_{i_1}, \dots, \Omega_{i_d} \in H^0(A, \Omega^1_A)$ whose wedge pullback $\varpi_Y^*(\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_d})$ is generically nonzero on $\widetilde{Y}_{\text{sm}}$; letting Δ_{ram} denote the Weil divisor on $\widetilde{Y}$ obtained as the closure (with multiplicities) of its zero divisor on $\widetilde{Y}_{\text{sm}}$, there exists a coherent ideal sheaf $\mathcal{I}_Y \subseteq \mathcal{O}_{\widetilde{Y}}$ such that $\mathcal{I}_Y|_{\widetilde{Y}_{\text{sm}}} = \mathcal{O}_{\widetilde{Y}_{\text{sm}}}(-\Delta_{\text{ram}}|_{\widetilde{Y}_{\text{sm}}})$, $\mathcal{I}_Y \subseteq \mathcal{O}_{\widetilde{Y}}(-\Delta_{\text{ram}})$ globally on $\widetilde{Y}$, and $\mathcal{I}_Y(l_{d,\max}E)$ is globally generated on $\widetilde{Y}$;

(iii) every irreducible component (of any dimension $0 \le r < d$) of the bad image $B_Y := \varpi_Y(\operatorname{Sing}(\widetilde{Y}) \cup \operatorname{Supp}(\Delta_{\text{ram}})) \subset A$ has D-degree at most $D^{(d)}_{\text{bad}}$.

Proof. Fix the projective embedding $A \hookrightarrow \mathbb{P}^{N_s}$ given by the very ample divisor $D = 3L$. For each integer $1 \le k \le d_{\max}$, let $\operatorname{Chow}_{d,k}(A, D)$ denote the projective Chow variety parametrizing effective algebraic d-cycles of D-degree k in A. A cycle of degree k fails to be integral of multiplicity one precisely when it lies in the image of one of the finitely many proper cycle-addition morphisms $\operatorname{Chow}_{d,k_1}(A, D) \times \operatorname{Chow}_{d,k_2}(A, D) \to \operatorname{Chow}_{d,k}(A, D)$ ($k_1 + k_2 = k$, $k_1, k_2 \ge 1$) or cycle-multiplication morphisms $k_0 \cdot (\cdot) : \operatorname{Chow}_{d,k/k_0}(A, D) \to \operatorname{Chow}_{d,k}(A, D)$ ($k_0 \mid k$, $k_0 \ge 2$). Consequently the locus $U_{d,k}$ of integral multiplicity-one d-cycles is a Zariski-open subset of $\operatorname{Chow}_{d,k}(A, D)_{\text{red}}$, and $S^{(0)} := \bigsqcup_{k=1}^{d_{\max}} U_{d,k}$ is a reduced quasi-projective scheme of finite type over $\mathbb{C}$.

The lemma is proved by Noetherian induction on locally closed reduced subschemes $S \subseteq S^{(0)}$: given an irreducible component S_0 of S, first pass to the nonempty open subset $S_0^\circ := S_0 \setminus \bigcup_{S_0' \ne S_0} S_0'$ (which is open in S); it suffices to construct a nonempty Zariski-open subset $S_4 \subseteq S_0^\circ$ (hence open in S) on which uniform integers $(d_0(S_4), l(S_4), D_{\text{bad}}(S_4))$ satisfy (i)–(iii) for every closed point $s \in S_4$, because removing S_4 strictly lowers the Noetherian topological space S, yielding a finite stratification $S^{(0)} = \bigsqcup_{\alpha=1}^{M} S^{(\alpha)}$ into locally closed strata. Once finitely many strata are obtained, taking

$$d_{0,d} := \max_\alpha d_0(S^{(\alpha)}), \qquad l_{d,\max} := \max_\alpha l(S^{(\alpha)}), \qquad D^{(d)}_{\text{bad}} := \max_\alpha D_{\text{bad}}(S^{(\alpha)})$$

works uniformly on all of $S^{(0)}$: since $E = \varpi_Y^*(D|_Y)$ is already globally generated on $\widetilde{Y}$, adding any nonnegative multiple $(M - M_0)E$ preserves both very ampleness (for $d_{0,d}E = d_0(S^{(\alpha)})E + (d_{0,d} - d_0(S^{(\alpha)}))E$) and global generation (for $\mathcal{I}_Y(l_{d,\max}E) = \mathcal{I}_Y(l(S^{(\alpha)})E) \otimes \mathcal{O}_{\widetilde{Y}}((l_{d,\max} - l(S^{(\alpha)}))E)$).

Fix an irreducible component $S_0 \subseteq S$ with generic point η_0, pass as above to the nonempty open subset $S_0^\circ \subseteq S_0$ disjoint from all other irreducible components of S, and let $\mathcal{Y}_{S_0^\circ} \subset A \times S_0^\circ$ be the reduced universal incidence subscheme over S_0° (whose reduced fibre over every closed

point $s \in S_0^\circ$ satisfies $(\mathcal{Y}_s)_{\mathrm{red}} = Y_s$). *Step 1 (Fibrewise simultaneous normalization).* Because S_0° is integral of finite type over $\mathbb{C}$ (characteristic zero), $\mathcal{Y}_{S_0^\circ}$ is reduced, and every closed fibre is an irreducible d-fold of Chow multiplicity 1, the generic fibre $\mathcal{Y}_{\eta_0}$ is geometrically integral of dimension d. By generic flatness and openness of geometric integrality in flat proper morphisms [14, Théorème 12.2.4] (cf. [20, Definition 11 and Lemma 21]), shrink S_0° to a nonempty smooth affine open $S_1 \subseteq S_0^\circ$ over which $\mathcal{Y}_{S_1} \to S_1$ is flat with geometrically integral d-dimensional scheme-theoretic fibres (so $\mathcal{Y}_s = Y_s$ as schemes for every closed $s \in S_1$). Because S_1 is smooth of finite type over $\mathbb{C}$, the total space $\mathcal{Y}_{S_1}$ is integral and excellent, so its normalization $\nu : \widetilde{\mathcal{Y}} \to \mathcal{Y}_{S_1}$ is a finite birational morphism with $\widetilde{\mathcal{Y}}$ integral of dimension $d + \dim S_1$. Over the generic point η_0 of S_1 (characteristic zero), $\widetilde{\mathcal{Y}}_{\eta_0}$ is the normalization of the geometrically integral generic fibre $\mathcal{Y}_{\eta_0}$ and is therefore geometrically normal and integral; moreover, the coherent cokernel $\mathcal{C} := \nu_*\mathcal{O}_{\widetilde{\mathcal{Y}}}/\mathcal{O}_{\mathcal{Y}_{S_1}}$ has generic fibre $\mathcal{C}_{\eta_0}$ supported on $\mathrm{Sing}(\mathcal{Y}_{\eta_0})$, of dimension strictly less than d. By generic flatness and openness of geometric normality in flat projective families, shrink S_1 to a nonempty smooth affine open $S_2 \subseteq S_1$ over which: (1) $\widetilde{\mathcal{Y}}_{S_2} \to S_2$ is flat with geometrically normal fibres; (2) $\mathcal{C}|_{S_2}$ is flat over S_2; and (3) every fibre $\mathrm{Supp}(\mathcal{C}_s) \subset Y_s$ has dimension $\leq d-1$. For every closed point $s \in S_2$, flatness of $\mathcal{C}|_{S_2}$ implies that the sequence $0 \to \mathcal{O}_{\mathcal{Y}_s} \to (\nu_*\mathcal{O}_{\widetilde{\mathcal{Y}}})_s \to \mathcal{C}_s \to 0$ remains exact. Because ν is finite (hence affine), formation of ν_* commutes with arbitrary base change, so $(\nu_*\mathcal{O}_{\widetilde{\mathcal{Y}}})_s \cong \nu_{s,*}\mathcal{O}_{\widetilde{\mathcal{Y}}_s}$. Since $\dim \mathrm{Supp}(\mathcal{C}_s) < d$ and Y_s is integral of dimension d, the finite morphism $\nu_s : \widetilde{\mathcal{Y}}_s \to Y_s$ is an isomorphism over the dense irreducible open $Y_s \setminus \mathrm{Supp}(\mathcal{C}_s)$. Because $\widetilde{\mathcal{Y}}_{S_2}$ is integral of dimension $d + \dim S_2$ and S_2 is smooth, every fibre $\widetilde{\mathcal{Y}}_s$ is cut out locally by $\dim S_2$ parameters, so every irreducible component of $\widetilde{\mathcal{Y}}_s$ has dimension $\geq d$; finiteness of $\nu_s : \widetilde{\mathcal{Y}}_s \to Y_s$ forces dimension $\leq d$. Hence every irreducible component of $\widetilde{\mathcal{Y}}_s$ dominates Y_s and therefore meets the irreducible open set $\nu_s^{-1}(Y_s \setminus \mathrm{Supp}(\mathcal{C}_s))$, leaving at most one component; thus $\widetilde{\mathcal{Y}}_s$ is irreducible (hence integral) and normal, and $\nu_s : \widetilde{\mathcal{Y}}_s \to Y_s$ is canonically identified with the exact normalization $\varpi_{Y_s} : \widetilde{Y}_s \to Y_s$ (agreeing with Kollár's generic simultaneous normalization [20, Definition 11 and Lemma 21]).

Step 2 (Fibrewise clearing ideal $\mathcal{I}_Y$). Fix a basis $\Omega_1, \dots, \Omega_g$ of $H^0(A, \Omega_A^1)$. Because $\mathcal{Y}_{\eta_0}$ is smooth at its generic point in characteristic zero and Ω_A^1 is globally generated by $\Omega_1, \dots, \Omega_g$, there exists a d-element subset $I = \{i_1, \dots, i_d\}$ such that the pullback $\omega_I := \nu^*\mathrm{pr}_A^*(\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_d})$ is nonzero at the generic point of $\widetilde{\mathcal{Y}}_{\eta_0}$. Let $U^{\mathrm{sm}} := \widetilde{\mathcal{Y}}_{S_2} \setminus \mathrm{Sing}(\widetilde{\mathcal{Y}}_{S_2}/S_2)$ be the relative smooth locus over S_2. Because the smooth morphism $U^{\mathrm{sm}} \to S_2$ is flat and locally of finite presentation, it is universally open; hence the image of the nonempty Zariski-open nonvanishing locus $\{x \in U^{\mathrm{sm}} : \omega_I(x) \neq 0\}$ is a nonempty Zariski-open subset of S_2 containing η_0. Shrink S_2 to a nonempty affine open $S_3 \subseteq S_2$ contained in this image, so that ω_I does not vanish identically on any fibre $U_s^{\mathrm{sm}} = \widetilde{Y}_{s,\mathrm{sm}}$ ($s \in S_3$). On $U^{\mathrm{sm}}|_{S_3}$, the top relative form ω_I is a section of the relative canonical line bundle $\omega_{U^{\mathrm{sm}}/S_3}$ that is nonzero on every fibre, so its zero scheme $\mathcal{D}^{\mathrm{sm}} \subset U^{\mathrm{sm}}|_{S_3}$ is an effective relative Cartier divisor, flat over S_3, whose fibre $\mathcal{D}_s^{\mathrm{sm}}$ is precisely the Cartier zero divisor $\Delta_{\mathrm{ram},s}|_{\widetilde{Y}_{s,\mathrm{sm}}}$.

Let $\overline{\mathcal{D}} \subset \widetilde{\mathcal{Y}}_{S_3}$ denote the scheme-theoretic closure of $\mathcal{D}^{\mathrm{sm}}$ in $\widetilde{\mathcal{Y}}_{S_3}$, defined by a coherent ideal sheaf $\mathcal{I}_{\mathrm{rel}} \subset \mathcal{O}_{\widetilde{\mathcal{Y}}_{S_3}}$. By generic flatness, shrink S_3 to a nonempty affine open $S_4 \subseteq S_3$ over which the quotient $\mathcal{O}_{\overline{\mathcal{D}}} = \mathcal{O}_{\widetilde{\mathcal{Y}}_{S_4}}/\mathcal{I}_{\mathrm{rel}}$ is flat over S_4. For every closed point $s \in S_4$, flatness gives $\mathrm{Tor}_1^{\mathcal{O}_{S_4}}(\mathcal{O}_{\overline{\mathcal{D}}}, k(s)) = 0$, so tensoring $0 \to \mathcal{I}_{\mathrm{rel}} \to \mathcal{O}_{\widetilde{\mathcal{Y}}_{S_4}} \to \mathcal{O}_{\overline{\mathcal{D}}} \to 0$ with $k(s)$ yields an injection $\mathcal{I}_{Y_s} := \mathcal{I}_{\mathrm{rel}} \otimes_{\mathcal{O}_{S_4}} k(s) \hookrightarrow \mathcal{O}_{\widetilde{Y}_s}$. Over the open subset $\widetilde{Y}_{s,\mathrm{sm}} = U_s^{\mathrm{sm}}$, one has $\overline{\mathcal{D}} \cap U^{\mathrm{sm}} = \mathcal{D}^{\mathrm{sm}}$, hence $\mathcal{I}_{Y_s}|_{\widetilde{Y}_{s,\mathrm{sm}}} = \mathcal{O}_{\widetilde{Y}_{s,\mathrm{sm}}}(-\Delta_{\mathrm{ram},s}|_{\widetilde{Y}_{s,\mathrm{sm}}})$ as ideal sheaves. Because $\widetilde{Y}_s$ is normal, $\mathrm{Sing}(\widetilde{Y}_s)$ has codimension at least 2 in $\widetilde{Y}_s$, so every height-one prime v of $\widetilde{Y}_s$ lies in $\widetilde{Y}_{s,\mathrm{sm}}$. Consequently every local section $f \in \mathcal{I}_{Y_s}$ satisfies $\mathrm{ord}_v(f) \geq \mathrm{ord}_v(\Delta_{\mathrm{ram},s})$ at every height-one prime v, which proves the global divisorial containment $\mathcal{I}_{Y_s} \subseteq \mathcal{O}_{\widetilde{Y}_s}(-\Delta_{\mathrm{ram},s})$.

Step 3 (Relative very ampleness and global generation). Let $\pi_4 : \widetilde{\mathcal{Y}}_{S_4} \to S_4$ be the structure morphism and $\mathcal{E} := \nu^*\mathrm{pr}_A^*D$. Since ν is finite and D is ample on A, $\mathcal{E}$ is π_4-ample over

the affine Noetherian base S_4. By relative Serre generation [15, Theorem II.7.6 and Proposition II.7.10], there exist integers $d_0(S_4) \geq 1$ and $l(S_4) \geq 1$ such that $d_0(S_4)\mathcal{E}$ is π_4-very ample and $\pi_{4,*}(\mathcal{I}_{\mathrm{rel}}(l(S_4)\mathcal{E})) \otimes_{\mathcal{O}_{S_4}} \mathcal{O}_{\widetilde{\mathcal{Y}}_{S_4}} \twoheadrightarrow \mathcal{I}_{\mathrm{rel}}(l(S_4)\mathcal{E})$ is surjective. Restricting to any closed fibre $s \in S_4$, right exactness of $- \otimes_{\mathcal{O}_{S_4}} k(s)$ yields that $d_0(S_4)E_s$ is very ample on $\widetilde{Y}_s$ and that $\mathcal{I}_{Y_s}(l(S_4)E_s)$ is generated by global sections on $\widetilde{Y}_s$.

Step 4 (Uniform bad-locus degree bound in all dimensions $0 \leq r < d$). Define the closed subset $\mathcal{B} := \mathrm{Sing}(\widetilde{\mathcal{Y}}_{S_4}/S_4) \cup \overline{\mathcal{D}} \subset \widetilde{\mathcal{Y}}_{S_4}$. For every closed point $s \in S_4$, the fibre of $\mathrm{Sing}(\widetilde{\mathcal{Y}}_{S_4}/S_4)$ is precisely $\mathrm{Sing}(\widetilde{Y}_s)$, and because $\overline{\mathcal{D}} \cap U^{\mathrm{sm}} = \mathcal{D}^{\mathrm{sm}}$, every point of $\overline{\mathcal{D}}_s$ lying outside $\mathrm{Sing}(\widetilde{Y}_s)$ belongs to $\mathrm{Supp}(\Delta_{\mathrm{ram},s}) \cap \widetilde{Y}_{s,\mathrm{sm}}$, while conversely $\mathrm{Supp}(\Delta_{\mathrm{ram},s}) \subseteq \overline{\mathcal{D}}_s \cup \mathrm{Sing}(\widetilde{Y}_s)$. Hence the set-theoretic fibre $\mathcal{B}_s$ equals $\mathrm{Sing}(\widetilde{Y}_s) \cup \mathrm{Supp}(\Delta_{\mathrm{ram},s})$ identically for every $s \in S_4$. Let $\mathcal{W} \subset A \times S_4$ be the reduced image of $\mathcal{B}$ under the finite (hence proper) morphism $\widetilde{\mathcal{Y}}_{S_4} \to A \times S_4$. Because proper morphisms commute with set-theoretic fibres (distinct from scheme-theoretic base change), the set-theoretic fibre $\mathcal{W}_s \subset A$ equals $B_{Y_s} = \varpi_{Y_s}(\mathrm{Sing}(\widetilde{Y}_s) \cup \mathrm{Supp}(\Delta_{\mathrm{ram},s}))$ for every $s \in S_4$. Viewing $\mathcal{W} \subset A \times S_4 \hookrightarrow \mathbb{P}^{N_s} \times S_4$ over the affine Noetherian base S_4, the homogeneous ideal of $\mathcal{W}$ in $\mathcal{O}(S_4)[Z_0, \ldots, Z_{N_s}]$ is generated by finitely many homogeneous polynomials of degrees at most some uniform integer $\delta_{\mathcal{W}} \geq 1$. On every fibre $s \in S_4$, multiplying each specialized generator of degree $\delta' \leq \delta_{\mathcal{W}}$ by all monomials of degree $\delta_{\mathcal{W}} - \delta'$ produces a finite-dimensional linear system $V_s \subseteq H^0(\mathbb{P}^{N_s}, \mathcal{O}(\delta_{\mathcal{W}}))$ whose base locus on A is set-theoretically $\mathcal{W}_s$. For each irreducible component $Z \subseteq \mathcal{W}_s$ of dimension r ($0 \leq r < d$), since V_s is base-point-free on $A \setminus \mathcal{W}_s$, choosing $H_1, \ldots, H_{g-r} \in |V_s|$ inductively as general divisors cuts every irreducible component of $(A \cap H_1 \cap \cdots \cap H_k) \setminus \mathcal{W}_s$ properly, so $(A \cap H_1 \cap \cdots \cap H_{g-r}) \setminus \mathcal{W}_s$ has dimension at most r; hence no irreducible component of $A \cap H_1 \cap \cdots \cap H_{g-r}$ of dimension $> r$ can contain Z, and Z is an irreducible component of $A \cap H_1 \cap \cdots \cap H_{g-r}$. By the generalized Bézout inequality [10, Example 8.4.6], $\deg_D(Z) \leq D^g\, \delta_{\mathcal{W}}^{g-r} \leq D_{\mathrm{bad}}(S_4) := \max(1, D^g\, \delta_{\mathcal{W}}^g)$, uniformly over all $s \in S_4$ and all irreducible components $Z \subseteq \mathcal{W}_s$. Noetherian induction completes the proof. □

Lemma 2.5 (Pole clearing for a commuting frame)**.** *Keep the standing hypotheses of Definition 2.1 (in particular $P_0 \in A^\circ = A \setminus \mathrm{Supp}(L)$, η_{target} closed on A° with poles bounded by $2L$, $v_i = w_i/\theta$, and $T_i = \lfloor F_0 w_i / v_0 \rfloor + 1$). Let $Y \subseteq A$ be an integral subvariety of dimension d ($1 \leq d \leq g$) through P_0 (so $\varpi_Y^{-1}(P_0) \cap \mathrm{Supp}\, E = \varnothing$), with $\deg_D(Y) \leq d_{d,\max}$ (the ceiling $d_{\max} = d_{d,\max}$ supplied to Lemma 2.4). Set $l := l_{d,\max}$ (with $l_{g,\max} := 0$, $\widetilde{Y} = A$, $\Delta_{\mathrm{ram}} = 0$, and $\phi_{\mathrm{cl}} := 1$ when $d = g$). For $1 \leq d < g$, choose any nonzero section $\phi_{\mathrm{cl}} \in H^0(\widetilde{Y}, \mathcal{I}_{\widetilde{Y}}(lE))$, viewed as a rational function on $\widetilde{Y}$ via trivialization by $\varpi_Y^* s_0$. On the smooth open $\widetilde{Y}_{\mathrm{sm}} \setminus (\mathrm{Supp}\, E \cup \mathrm{Supp}\, \Delta_{\mathrm{ram}})$, let $V_1, \ldots, V_d$ be the dual commuting vector fields to the d pulled-back invariant 1-forms $\varpi_Y^*\Omega_{i_1}, \ldots, \varpi_Y^*\Omega_{i_d}$ of Lemma 2.4(ii), and define on $(\widetilde{Y}_{\mathrm{sm}} \setminus (\mathrm{Supp}\, E \cup \mathrm{Supp}\, \Delta_{\mathrm{ram}})) \times \mathbb{A}^m$*

$$D_{0a}^{(Y)} := V_a + (\varpi_Y^* \eta_{\mathrm{target}})(V_a) \sum_{i=1}^{m} \partial_{X_i}, \qquad \widetilde{D}_{0a} := \phi_{\mathrm{cl}} D_{0a}^{(Y)} \qquad (1 \leq a \leq d).$$

Then:

(i) The vector fields $D_{01}^{(Y)}, \ldots, D_{0d}^{(Y)}, \partial_{X_1}, \ldots, \partial_{X_m}$ pairwise commute on $(\widetilde{Y}_{\mathrm{sm}} \setminus (\mathrm{Supp}\, E \cup \mathrm{Supp}\, \Delta_{\mathrm{ram}})) \times \mathbb{A}^m$, and each cleared derivation $\widetilde{D}_{0a}$ extends to a regular algebraic derivation on $(\widetilde{Y} \setminus \mathrm{Supp}\, E) \times \mathbb{A}^m$ (including across $\mathrm{Sing}(\widetilde{Y})$).

(ii) Define $h_{\mathbf{0}} F := F$ and, for $\mathbf{u} \in \mathbb{N}^d$ with $n = |\mathbf{u}| > 0$, $h_{\mathbf{u}} F := \phi_{\mathrm{cl}}^{2n-1} (D_0^{(Y)})^{\mathbf{u}} F$. If $w_i > \max((l+1)w_0, \theta v_0)$ for every $1 \leq i \leq m$ and $F_0 \theta \geq 2$, then for every $F \in V_N^{(\widetilde{Y})}$ and every $(\mathbf{u}, \mathbf{b}) \in \mathbb{N}^d \times \mathbb{N}^m$ with $n = |\mathbf{u}|$, one has $h_{\mathbf{u}} \partial_X^{\mathbf{b}} F \in V_{N+n(2l+1)w_0}^{(\widetilde{Y})}$, and at every lifted centre (Q, c_j) ($Q \in \varpi_Y^{-1}(P_0)$) the operator $h_{\mathbf{u}} \partial_X^{\mathbf{b}}$ lowers the pulled-back weighted jet order by at most $n v_0 + v \cdot \mathbf{b}$.

Proof. (i) Write $\omega_a := \varpi_Y^*\Omega_{i_a}$ $(1 \le a \le d)$. Since each ω_a is closed ($d\omega_a = 0$) and $\omega_a(V_b) = \delta_{ab}$, Cartan's formula gives $0 = d\omega_c(V_a, V_b) = V_a(\delta_{cb}) - V_b(\delta_{ca}) - \omega_c([V_a, V_b]) = -\omega_c([V_a, V_b])$, hence $[V_a, V_b] = 0$. Setting $\gamma_a := (\varpi_Y^*\eta_{\mathrm{target}})(V_a)$, closedness of $\varpi_Y^*\eta_{\mathrm{target}}$ similarly gives $0 = d(\varpi_Y^*\eta_{\mathrm{target}})(V_a, V_b) = V_a(\gamma_b) - V_b(\gamma_a)$. Since the γ_a are independent of $X_1, \dots, X_m$, the fields $D_{0a}^{(Y)}$ and ∂_{X_i} pairwise commute.

To verify regularity of $\widetilde{D}_{0a}$ off $\operatorname{Supp} E$, take any height-one prime v of $\widetilde{Y}$ and set $r := \operatorname{ord}_v(\Delta_{\mathrm{ram}}) \ge 0$ and $t := \operatorname{ord}_v(\varpi_Y^*L) \ge 0$ (so $\operatorname{ord}_v(E) = 3t$). Because $\widetilde{Y}$ is normal, v lies in $\widetilde{Y}_{\mathrm{sm}}$; in regular local coordinates at v, the matrix representing $(\omega_1, \dots, \omega_d)$ has regular entries and determinant of valuation r, so Cramer's rule gives $\operatorname{ord}_v(V_a) \ge -r$ (meaning $V_a(\mathcal{O}_{\widetilde{Y},v}) \subseteq \mathfrak{m}_v^{-r}$). By hypothesis $f^2\eta_{\mathrm{target}}$ is regular for a local equation f of L, so $\varpi_Y^*\eta_{\mathrm{target}}$ has pole order at most $2t$ at v (regardless of tangency or multiplicity of Y along L), whence $\operatorname{ord}_v(\gamma_a) \ge -r - 2t$. Since $\phi_{\mathrm{cl}} \in H^0(\widetilde{Y}, \mathcal{I}_Y(lE))$ and $\mathcal{I}_Y \subseteq \mathcal{O}_{\widetilde{Y}}(-\Delta_{\mathrm{ram}})$, its valuation as a rational function satisfies $\operatorname{ord}_v(\phi_{\mathrm{cl}}) \ge r - 3lt$. Thus $\operatorname{ord}_v(\phi_{\mathrm{cl}}V_a) \ge -3lt = -\operatorname{ord}_v(lE)$ and $\operatorname{ord}_v(\phi_{\mathrm{cl}}\gamma_a) \ge -3lt - 2t \ge -3(l+1)t = -\operatorname{ord}_v((l+1)E)$. In particular, at every height-one prime v of $\widetilde{Y} \setminus \operatorname{Supp} E$ ($t = 0$), both $\phi_{\mathrm{cl}}V_a$ and $\phi_{\mathrm{cl}}\gamma_a$ are regular. Since every normal Noetherian local ring $R = \mathcal{O}_{\widetilde{Y},x}$ satisfies $R = \bigcap_{\mathrm{ht}\, v=1} R_v$ (Hartogs's algebraic extension lemma), $\widetilde{D}_{0a}$ maps $R[X_1, \dots, X_m]$ into itself at every point $x \in \widetilde{Y} \setminus \operatorname{Supp} E$, including points of $\operatorname{Sing}(\widetilde{Y})$.

(ii) For any $k \ge 0$ and any section $c \in H^0(\widetilde{Y}, kE)$, differentiating c along $\phi_{\mathrm{cl}}V_a$ adds at most 1 to its pole order along $\operatorname{Supp} E$ (at any prime v with $t = \operatorname{ord}_v(\varpi_Y^*L) > 0$, $\operatorname{ord}_v(V_a(c)) \ge -r - 3kt - 1 \ge -r - 3kt - 3t$, since $t \ge 1$) plus the pole order lE of $\phi_{\mathrm{cl}}V_a$, giving a section of $(k+l+1)E$, which increases base degree by at most $(l+1)w_0$. Applying $\phi_{\mathrm{cl}}\gamma_a\partial_{X_i}$ to $cX^{\mathbf{b}}$ ($b_i \ge 1$) produces a coefficient in $H^0(\widetilde{Y}, (k+l+1)E)$ while lowering the additive weight by w_i, for a net weighted-degree change of $(l+1)w_0 - w_i < 0$ (by $w_i > (l+1)w_0$). Hence $\widetilde{D}_{0a}$ increases weighted degree by at most $(l+1)w_0$, and in particular $\widetilde{D}_{0a}(\phi_{\mathrm{cl}}) \in H^0(\widetilde{Y}, (2l+1)E)$. For $n = |\mathbf{u}| \ge 1$, applying the product rule to $h_{\mathbf{u}+\mathbf{e}_a}F = \phi_{\mathrm{cl}}^{2n+1} D_{0a}^{(Y)}(\phi_{\mathrm{cl}}^{-(2n-1)} h_{\mathbf{u}}F)$ gives the exact recurrence

$$h_{\mathbf{u}+\mathbf{e}_a}F = \phi_{\mathrm{cl}}\, \widetilde{D}_{0a}(h_{\mathbf{u}}F) - (2n-1)(\widetilde{D}_{0a}\phi_{\mathrm{cl}})\, h_{\mathbf{u}}F.$$

The first term increases weighted degree by at most $lw_0 + (l+1)w_0 = (2l+1)w_0$ over $h_{\mathbf{u}}F$, and the second also adds at most $(2l+1)w_0$; for $n = 0$, $h_{\mathbf{e}_a}F = \widetilde{D}_{0a}F$ adds at most $(l+1)w_0 \le (2l+1)w_0$. Since ∂_{X_i} lowers weighted degree by $w_i > 0$, induction on $n = |\mathbf{u}|$ proves $h_{\mathbf{u}}\partial_X^{\mathbf{b}}F \in V_{N+n(2l+1)w_0}^{(\widetilde{Y})}$.

For the jet-order bound at any lifted centre (Q, c_j) ($Q \in \varpi_Y^{-1}(P_0)$), let $\mathfrak{a}_Q := \mathfrak{m}_{A,P_0}\mathcal{O}_{\widetilde{Y},Q}$. The pulled-back jet filtration of order M at (Q, c_j) is spanned in $\widehat{\mathcal{O}}_{\widetilde{Y},Q}[X_1, \dots, X_m]$ by monomials in $\mathfrak{a}_Q^{\lceil (M - v\cdot\mathbf{b}')/v_0 \rceil}(U_j')^{\mathbf{b}'}$. Since $\widetilde{D}_{0a}$ is regular at Q (as $Q \in \varpi_Y^{-1}(P_0) \subset \widetilde{Y} \setminus \operatorname{Supp} E$), Leibniz gives $\widetilde{D}_{0a}(\mathfrak{a}_Q^k) \subseteq \mathfrak{a}_Q^{k-1}$ for all $k \ge 1$ (dropping weight by at most v_0). On $U_{ji}' = X_i - c_{ji} - G_i(\mathbf{e})$, because $D_{0a}^{(Y)}(X_i - G^{\mathrm{full}}) = 0$ for the single formal primitive $G^{\mathrm{full}} \in \mathbb{C}[[\mathbf{e}]]$ of the closed regular 1-form η_{target} at P_0 (of which each G_i is the total-degree $< T_i$ truncation), continuity of the regular derivation $\widetilde{D}_{0a}$ on $\widehat{\mathcal{O}}_{\widetilde{Y},Q}$ gives $\widetilde{D}_{0a}(U_{ji}') = \phi_{\mathrm{cl}}V_a(G^{\mathrm{full}} - G_i) \in \mathfrak{a}_Q^{T_i - 1}$. Writing $x_i := w_i/(\theta v_0) > 1$, the hypotheses $F_0\theta \ge 2$ and $T_i = \lfloor F_0 w_i/v_0 \rfloor + 1 > F_0\theta x_i \ge 2x_i$ give $(T_i - 1)v_0 > (2x_i - 1)v_0 > x_i v_0 = v_i > v_i - v_0$, so $\widetilde{D}_{0a}(U_{ji}')$ has jet weight strictly exceeding $v_i - v_0$. Thus $\widetilde{D}_{0a}$ lowers jet weight by at most v_0, and multiplication by regular functions ϕ_{cl} or $\widetilde{D}_{0a}\phi_{\mathrm{cl}}$ never lowers jet weight (even if $\phi_{\mathrm{cl}}(Q) = 0$). The recurrence and induction on $n = |\mathbf{u}|$ (together with $\partial_{X_i}U_{ji}' = 1$ lowering jet weight by v_i) prove the jet-order bound. □

Before stating the slice estimate, the notion of a coordinate normal basis is made precise. Let $Z' \subset \widetilde{Y}^\circ \times \mathbb{A}^m$ be an irreducible closed subvariety of codimension r $(1 \le r \le d + m)$, and let $p = (p_{\widetilde{Y}}, p_X) \in Z'_{\mathrm{sm}}$ be a smooth point with $p_{\widetilde{Y}} \in \widetilde{Y}_{\mathrm{sm}}$. Let $\mathrm{pr}_N : T_p(\widetilde{Y} \times \mathbb{A}^m) \twoheadrightarrow N_{Z',p} := T_p(\widetilde{Y} \times \mathbb{A}^m)/T_pZ'$ be the normal projection, and put $V_{\mathrm{base}} := \mathrm{pr}_N(T_{p_{\widetilde{Y}}}\widetilde{Y})$ and $V_{\mathrm{add}} := \operatorname{span}\{\mathrm{pr}_N(\partial_{X_j}) : 1 \le j \le m\}$ inside $N_{Z',p}$, so that $V_{\mathrm{base}} + V_{\mathrm{add}} = N_{Z',p}$.

Definition 2.6 (Coordinate normal basis and frame normal basis)**.** At $p \in Z'_{\mathrm{sm}}$ with $p_{\widetilde{Y}} \in \widetilde{Y}_{\mathrm{sm}}$:

(i) A *coordinate normal basis* $\mathcal{A}$ at p consists of a subset $\mathcal{A}_{\mathrm{add}} \subseteq \{1, \dots, m\}$ of size $r - a$ (with $0 \le a \le \min(d, r)$) together with a base directions (each assigned subscript 0 and weight w_0) such that $\{\mathrm{pr}_N(\partial_{X_j}) : j \in \mathcal{A}_{\mathrm{add}}\}$ is linearly independent in $N_{Z',p}$ and its span W_{add} satisfies $W_{\mathrm{add}} \oplus \mathrm{pr}_N(S_0) = N_{Z',p}$ for some a-dimensional subspace $S_0 \subseteq T_{p_{\widetilde{Y}}}\widetilde{Y}$ (equivalently, $\dim(V_{\mathrm{base}} + W_{\mathrm{add}}) = r$ and $\dim W_{\mathrm{add}} = r - a$).

(ii) When $p_{\widetilde{Y}} \notin \operatorname{Supp} \Delta_{\mathrm{ram}}$, a *frame normal basis* $\mathcal{B}$ at p is an r-element subset of the uncleared commuting frame $\mathcal{F}_{\mathrm{frame}} := \{D_{01}^{(Y)}, \dots, D_{0d}^{(Y)}, \partial_{X_1}, \dots, \partial_{X_m}\}$ whose image under pr_N is a basis of $N_{Z',p}$ (each $D_{0a}^{(Y)}$ having subscript 0 and weight v_0, and each ∂_{X_j} having weight v_j).

Note that if some additive direction is non-tangent to Z' at p, and $i \in \{1, \dots, m\}$ is any index with $\mathrm{pr}_N(\partial_{X_i}) \neq 0$, then a coordinate normal basis $\mathcal{A}$ with $i \in \mathcal{A}_{\mathrm{add}}$ always exists: extend the nonzero vector $\mathrm{pr}_N(\partial_{X_i})$ to a basis $\{\mathrm{pr}_N(\partial_{X_j}) : j \in \mathcal{A}_{\mathrm{add}}\}$ of V_{add}, and set $a := r - \dim V_{\mathrm{add}}$. Since $V_{\mathrm{base}} + V_{\mathrm{add}} = N_{Z',p}$, the composition $T_{p_{\widetilde{Y}}}\widetilde{Y} \twoheadrightarrow V_{\mathrm{base}} \twoheadrightarrow N_{Z',p}/V_{\mathrm{add}}$ is surjective onto an a-dimensional space, so $0 \le a \le \min(d, r)$ and the Grassmannian open set of a-planes $S_0 \subset T_{p_{\widetilde{Y}}}\widetilde{Y}$ mapping isomorphically onto $N_{Z',p}/V_{\mathrm{add}}$ is nonempty.

Lemma 2.7 (Transverse complete-intersection slice and degree estimates)**.** *Keep the notation of Lemma 2.5, with $1 \le d \le g$, $d_Y := E^d$, and $d_{0,d}$ from Lemma 2.4 (with $d_{0,g} := 1$). Fix $\Delta > 0$, $N' > 0$, an integer $t \ge 0$, and $F \in V_N^{(\widetilde{Y})}$. Let $\mathcal{F}_t \subset V_{N'}^{(\widetilde{Y})}$ be the linear span of all $h_{\mathbf{u}}\partial_X^{\mathbf{b}} F$ of jet cost $v_0|\mathbf{u}| + v \cdot \mathbf{b} \le t\Delta$ (assuming all such derivatives lie in $V_{N'}^{(\widetilde{Y})}$). Suppose Z' is an irreducible component of codimension r ($1 \le r \le d + m$) common to $V(\mathcal{F}_t)$ and $V(\mathcal{F}_{t+1})$ inside $\widetilde{Y}^\circ \times \mathbb{A}^m$, whose projection to $\widetilde{Y}$ meets the good open set $\widetilde{Y}_{\mathrm{good}} := \widetilde{Y}_{\mathrm{sm}} \setminus (\operatorname{Supp} E \cup \operatorname{Supp} \Delta_{\mathrm{ram}} \cup \{\phi_{\mathrm{cl}} = 0\})$. Let $U_{\mathrm{gen}} \subset Z'_{\mathrm{sm}}$ be the nonempty Zariski-open locus of smooth points of Z' projecting into $\widetilde{Y}_{\mathrm{good}}$ and avoiding every other irreducible component of $V(\mathcal{F}_t)$ and $V(\mathcal{F}_{t+1})$.*

(i) At every point $p \in U_{\mathrm{gen}}$, for every coordinate normal basis $\mathcal{A}$ with a base directions and every frame normal basis $\mathcal{B}$ at p,

$$\prod_{j \in \mathcal{A}} w_j \le \frac{2(r!)^2 d_{0,d}^{d-a} d_Y}{a!(\Delta/N')^r} \prod_{j \in \mathcal{B}} v_j.$$

(ii) If $Z' = (W' \setminus \operatorname{Supp} E) \times \mathbb{A}^m$ for an irreducible projective subvariety $W' \subset \widetilde{Y}$ of codimension c' ($1 \le c' \le d$) meeting $\widetilde{Y}_{\mathrm{good}}$, then

$$\deg_E(W') \le (c')! \left(\frac{N' v_0}{\Delta w_0}\right)^{c'} d_Y.$$

Proof. Step 1 (Lower bound via cleared-section Leibniz expansion and commuting flows). Let $G = h_{\mathbf{u}}\partial_X^{\mathbf{b}} F = \phi_{\mathrm{cl}}^{\varepsilon(\mathbf{u})} (D_0^{(Y)})^{\mathbf{u}} \partial_X^{\mathbf{b}} F \in \mathcal{F}_t$ be any generator, where $\varepsilon(\mathbf{u}) := \max(0, 2|\mathbf{u}| - 1)$ and $v_0|\mathbf{u}| + v \cdot \mathbf{b} \le t\Delta$. Let $(D_0^{(Y)})^{\mathbf{u}'} \partial_X^{\mathbf{b}'}$ be any uncleared commuting-frame operator of cost $v_0|\mathbf{u}'| + v \cdot \mathbf{b}' < \Delta$. Because $\partial_{X_i}(\phi_{\mathrm{cl}}) = 0$ and the fields $D_{0a}^{(Y)}, \partial_{X_i}$ pairwise commute on U_{gen} (Lemma 2.5(i)), the multivariable Leibniz rule gives on U_{gen}:

$$(D_0^{(Y)})^{\mathbf{u}'} \partial_X^{\mathbf{b}'} G = \sum_{\mathbf{0} \le \mathbf{u}'' \le \mathbf{u}'} \binom{\mathbf{u}'}{\mathbf{u}''} \left((D_0^{(Y)})^{\mathbf{u}' - \mathbf{u}''} \phi_{\mathrm{cl}}^{\varepsilon(\mathbf{u})}\right) \phi_{\mathrm{cl}}^{-\varepsilon(\mathbf{u}+\mathbf{u}'')} \cdot h_{\mathbf{u}+\mathbf{u}''} \partial_X^{\mathbf{b}+\mathbf{b}'} F.$$

On U_{gen}, ϕ_{cl} is a unit and every $D_{0a}^{(Y)}$ is regular, so every prefactor is regular on U_{gen}. For every summand $\mathbf{u}'' \le \mathbf{u}'$, the jet cost satisfies $v_0|\mathbf{u} + \mathbf{u}''| + v \cdot (\mathbf{b} + \mathbf{b}') \le t\Delta + (v_0|\mathbf{u}'| + v \cdot \mathbf{b}') < (t+1)\Delta$,

whence $h_{\mathbf{u}+\mathbf{u}''}\partial_X^{\mathbf{b}+\mathbf{b}'}F \in \mathcal{F}_{t+1}$ vanishes identically on $Z' \cap U_{\mathrm{gen}}$. Consequently, every uncleared frame derivative of cost $< \Delta$ of every $G \in \mathcal{F}_t$ vanishes identically on $Z' \cap U_{\mathrm{gen}}$.

Now fix $p \in U_{\mathrm{gen}}$ and write $\mathcal{B} = \{B_1, \dots, B_r\} \subset \mathcal{F}_{\mathrm{frame}}$ with weights $(v_{b_1}, \dots, v_{b_r})$. Choose local holomorphic coordinates $z = (z_1, \dots, z_{d+m-r})$ on Z' centred at p, and define the local holomorphic flow map $\Psi(z, y_1, \dots, y_r) := (\exp(y_1 B_1) \circ \cdots \circ \exp(y_r B_r))(z)$ on a small polydisk around $(0,0)$. Since $[B_j, B_k] = 0$, the flows commute and $\Psi_*(\partial_{y_j}) = B_j$ identically on the polydisk. At $(0,0)$, $d\Psi$ maps $T_0 Z' \oplus \mathbb{C}^r$ isomorphically onto $T_p Z' \oplus \operatorname{span}(\mathcal{B}) = T_p(\widetilde{Y} \times \mathbb{A}^m)$. By the holomorphic inverse function theorem, $(z, \mathbf{y})$ are local holomorphic coordinates near p with $Z' = \{\mathbf{y} = 0\}$ and $B_j = \partial_{y_j}$. Every Taylor coefficient $\frac{1}{\mathbf{k}!}\partial_{\mathbf{y}}^{\mathbf{k}} G(z, 0)$ of $\mathbf{y}$-weight $\sum_{j=1}^r v_{b_j} k_j < \Delta$ therefore vanishes identically in z. Hence every $G \in \mathcal{F}_t$ lies in the ideal $\mathfrak{J}_{\mathbf{y}, \geq \Delta} \subset \mathcal{O}^{\mathrm{an}}_{\widetilde{Y} \times \mathbb{A}^m, p}$ generated by all monomials $\mathbf{y}^{\mathbf{k}}$ with $\sum_{j=1}^r v_{b_j} k_j \geq \Delta$.

Step 2 (Transverse slice and regular sequence). Let $\mathcal{A}$ be a coordinate normal basis at p with a base directions and additive set $\mathcal{A}_{\mathrm{add}}$ ($|\mathcal{A}_{\mathrm{add}}| = r - a$). Because $d_{0,d}E$ is very ample on $\widetilde{Y}$ and $p_{\widetilde{Y}}$ is a smooth point of $\widetilde{Y}$, the divisors in $|d_{0,d}E|$ passing through $p_{\widetilde{Y}}$ realize every hyperplane in $T_{p_{\widetilde{Y}}}\widetilde{Y}$. By Bertini's theorem and the nonempty Zariski-open transversality condition in $\mathrm{Gr}(a, T_{p_{\widetilde{Y}}}\widetilde{Y})$, successively cutting $\widetilde{Y}$ by $d - a$ general effective Cartier divisors $H_1, \dots, H_{d-a} \in |d_{0,d}E|$ passing through $p_{\widetilde{Y}}$ yields a projective subscheme $\overline{S} := H_1 \cap \cdots \cap H_{d-a} \subset \widetilde{Y}$ of pure dimension a with fundamental cycle $[\overline{S}] = (d_{0,d}E)^{d-a} \cap [\widetilde{Y}]$ (regular and smooth at $p_{\widetilde{Y}}$, without assuming $\overline{S}$ is globally Cohen–Macaulay), degree $\deg_E(\overline{S}) = d_{0,d}^{d-a} d_Y$, and $\mathrm{pr}_N(T_{p_{\widetilde{Y}}}\overline{S}) \oplus \operatorname{span}\{\mathrm{pr}_N(\partial_{X_j}) : j \in \mathcal{A}_{\mathrm{add}}\} = N_{Z',p}$. Define the affine slice $\Sigma_{\mathrm{sl}} := (\overline{S} \setminus \operatorname{Supp} E) \times \{X_k = X_k(p) : k \notin \mathcal{A}_{\mathrm{add}}\} \subset \widetilde{Y}^\circ \times \mathbb{A}^m$. Then Σ_{sl} has pure dimension r, is smooth at p, and meets Z' transversely at p. Since $p \in U_{\mathrm{gen}}$ avoids all irreducible components of $V(\mathcal{F}_t)$ other than Z', the set-theoretic intersection $V(\mathcal{F}_t) \cap \Sigma_{\mathrm{sl}}$ is isolated at $\{p\}$ in a Zariski neighborhood of p. In the local holomorphic coordinates $(z, \mathbf{y})$, transversality writes Σ_{sl} as a holomorphic graph $z = z(\mathbf{y})$ with $z(0) = 0$, so $\mathbf{y}$ restricts to formal/holomorphic local coordinates on Σ_{sl} at p identifying the completion $\widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p} \simeq \mathbb{C}[[y_1, \dots, y_r]]$, and $I_p \widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p} \subseteq \mathfrak{J}_{\mathbf{y}, \geq \Delta}$, where $I_p := (\mathcal{F}_t|_{\Sigma_{\mathrm{sl}}}) \subset \mathcal{O}_{\Sigma_{\mathrm{sl}},p}$ is the algebraic ideal generated by $\mathcal{F}_t|_{\Sigma_{\mathrm{sl}}}$. Because I_p has height r in the r-dimensional regular local ring $\mathcal{O}_{\Sigma_{\mathrm{sl}},p}$, standard prime avoidance over the infinite field $\mathbb{C}$ yields r general linear combinations $g_1, \dots, g_r \in \mathcal{F}_t$ whose restrictions form a regular sequence generating an $\mathfrak{m}_p$-primary ideal $(g_1, \dots, g_r) \subseteq I_p$ in $\mathcal{O}_{\Sigma_{\mathrm{sl}},p}$ (with $\{p\}$ an isolated component of $V(g_1, \dots, g_r) \cap \Sigma_{\mathrm{sl}}$), hence $(g_1, \dots, g_r)\widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p} \subseteq I_p \widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p} \subseteq \mathfrak{J}_{\mathbf{y}, \geq \Delta}$ in $\widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p} \simeq \mathbb{C}[[y_1, \dots, y_r]]$. Since completion preserves the length of any Artinian quotient,

$$\begin{aligned}\operatorname{length}_{\mathcal{O}_{\Sigma_{\mathrm{sl}},p}}(\mathcal{O}_{\Sigma_{\mathrm{sl}},p}/(g_1, \dots, g_r)) &= \operatorname{length}_{\widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p}}(\widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p}/(g_1, \dots, g_r)\widehat{\mathcal{O}}_{\Sigma_{\mathrm{sl}},p}) \\ &\geq \#\Big\{\mathbf{k} \in \mathbb{N}^r : \sum_{j=1}^r v_{b_j} k_j < \Delta\Big\} \geq \frac{\Delta^r}{r! \prod_{j \in \mathcal{B}} v_j},\end{aligned}$$

where the second inequality holds because the unit cubes $\mathbf{k} + [0,1)^r$ based at the lattice points of weight $< \Delta$ cover the positive simplex $\{\mathbf{x} \in \mathbb{R}^r_{\geq 0} : \sum_{j=1}^r v_{b_j} x_j < \Delta\}$ of volume $\Delta^r / (r! \prod_{j \in \mathcal{B}} v_j)$.

Step 3 (Upper bound by Fulton positivity on $\overline{\Sigma}$). Compactify the $r - a$ free additive coordinates to $\overline{\Sigma} := \overline{S} \times \prod_{j \in \mathcal{A}_{\mathrm{add}}} \mathbb{P}^1$, a complete projective scheme of pure dimension r (possibly reducible or singular away from p, and smooth at p). Put $k_{\mathrm{base}} := \lfloor N'/w_0 \rfloor$ and $m_j := \lfloor N'/w_j \rfloor$ ($j \in \mathcal{A}_{\mathrm{add}}$). Note that if $m_j = 0$ for some $j \in \mathcal{A}_{\mathrm{add}}$, or $k_{\mathrm{base}} = 0$ with $a > 0$, then every $g \in \mathcal{F}_t$ is independent of the corresponding variable, making the isolated-zero condition at p impossible (so the hypotheses are vacuous in that degenerate range); we therefore have $m_j \geq 1$ for all $j \in \mathcal{A}_{\mathrm{add}}$ and $k_{\mathrm{base}} \geq 1$ when $a > 0$. Every $g \in \mathcal{F}_t \subset V_{N'}^{(\widetilde{Y})}$ has the form $\sum_{w \cdot \mathbf{b} \leq N'} c_{\mathbf{b}} X^{\mathbf{b}}$ with $c_{\mathbf{b}} \in H^0(\widetilde{Y}, \lfloor (N' - w \cdot \mathbf{b})/w_0 \rfloor E)$; multiplying each $c_{\mathbf{b}}$ by $(\varpi_Y^* s_0)^{k_{\mathrm{base}} - \lfloor (N' - w \cdot \mathbf{b})/w_0 \rfloor}$ views every coefficient in $H^0(\widetilde{Y}, k_{\mathrm{base}} E)$. Specializing $X_k = X_k(p)$ for $k \notin \mathcal{A}_{\mathrm{add}}$ and homogenizing each free coordinate $X_j = U_j / V_j$ to degree m_j converts $g_1, \dots, g_r$ into global sections $\widetilde{g}_1, \dots, \widetilde{g}_r$ of the

external tensor product

$$\mathcal{M} := \mathrm{pr}_0^*\mathcal{O}_{\overline{S}}(k_{\mathrm{base}}E|_{\overline{S}}) \otimes \bigotimes_{j\in\mathcal{A}_{\mathrm{add}}} \mathrm{pr}_j^*\mathcal{O}_{\mathbb{P}^1}(m_j)$$

on $\overline{\Sigma}$. Because $E = \varpi_Y^*(3L)$ is globally generated on $\widetilde{Y}$ and $k_{\mathrm{base}}, m_j \geq 0$, each factor of $\mathcal{M}$ is globally generated, so $\mathcal{M}$ and the rank-r vector bundle $\mathcal{E}_{\mathcal{M}} := \mathcal{M}^{\oplus r}$ are globally generated on $\overline{\Sigma}$. The section $\mathbf{s} := (\widetilde{g}_1, \ldots, \widetilde{g}_r) \in H^0(\overline{\Sigma}, \mathcal{E}_{\mathcal{M}})$ defines a graph intersecting the zero section of the total space of $\mathcal{E}_{\mathcal{M}}$ (whose normal bundle $\mathcal{E}_{\mathcal{M}}$ is globally generated, requiring no assumption that individual $\widetilde{g}_i$ define effective Cartier divisors on every component of $\overline{\Sigma}$) with $\{p\}$ as an isolated zero-dimensional connected component of $V(\mathbf{s})$ (possibly alongside positive-dimensional connected components away from p). Applying Fulton's localized top-Chern positivity theorem [10, Theorem 12.2 and Example 12.2.7(a)] componentwise to the fundamental cycle $[\overline{\Sigma}]$ of the complete pure r-dimensional scheme $\overline{\Sigma}$, every connected component of $V(\mathbf{s})$ contributes a nonnegative zero-cycle degree to $\int_{\overline{\Sigma}} c_r(\mathcal{E}_{\mathcal{M}})$, while by [10, Proposition 7.1(b)] the isolated regular point $\{p\}$ (lying on a unique irreducible component of $\overline{\Sigma}$ with multiplicity 1, where $(g_1, \ldots, g_r)$ is a regular sequence in the regular local ring $\mathcal{O}_{\overline{\Sigma},p}$) contributes its exact local intersection multiplicity $\mathrm{length}_{\mathcal{O}_{\overline{\Sigma},p}}(\mathcal{O}_{\overline{\Sigma},p}/(g_1, \ldots, g_r))$. Writing $H_0 := \mathrm{pr}_0^* c_1(E|_{\overline{S}})$ (with $H_0^{a+1} = 0$ and $\int_{\overline{S}} H_0^a = \deg_E(\overline{S}) = d_{0,d}^{d-a} d_Y$) and $F_j := \mathrm{pr}_j^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ (with $F_j^2 = 0$, $\int_{\mathbb{P}^1} F_j = 1$), the multinomial expansion of $c_r(\mathcal{E}_{\mathcal{M}}) = c_1(\mathcal{M})^r = (k_{\mathrm{base}}H_0 + \sum_{j\in\mathcal{A}_{\mathrm{add}}} m_j F_j)^r$ retains only the term with exponent a on H_0 and exponent 1 on each F_j, yielding

$$\begin{aligned}\mathrm{length}_{\mathcal{O}_{\Sigma_{\mathrm{sl}},p}}(\mathcal{O}_{\Sigma_{\mathrm{sl}},p}/(g_1, \ldots, g_r)) &\leq \int_{\overline{\Sigma}} c_1(\mathcal{M})^r = \frac{r!}{a!} k_{\mathrm{base}}^a \, d_{0,d}^{d-a} d_Y \prod_{j\in\mathcal{A}_{\mathrm{add}}} m_j \\ &\leq \frac{r!}{a!} d_{0,d}^{d-a} d_Y (N'/w_0)^a \prod_{j\in\mathcal{A}_{\mathrm{add}}} (N'/w_j).\end{aligned}$$

Comparing the lower and upper bounds yields (i) (even without the extra safety factor 2).

Step 4 (Product case (ii)). Suppose $Z' = (W' \setminus \mathrm{Supp}\, E) \times \mathbb{A}^m$ with $W' \subset \widetilde{Y}$ projective of codimension c' meeting $\widetilde{Y}_{\mathrm{good}}$. Fix a general constant additive point $x_0 \in \mathbb{A}^m$ and cut $\widetilde{Y}$ by $d - c'$ general divisors of the very ample system $|d_{0,d}E|$. By Bertini's theorem, the resulting c'-fold $\overline{S}' \subset \widetilde{Y}$ meets W' transversely in exactly $M_{W'} := d_{0,d}^{d-c'} \deg_E(W')$ distinct smooth points $p_1, \ldots, p_{M_{W'}}$, all lying in $\widetilde{Y}_{\mathrm{good}}$ and such that every (p_ν, x_0) belongs to U_{gen}. At each (p_ν, x_0), the normal space to Z' is spanned by c' horizontal frame fields from $\{D_{01}^{(Y)}, \ldots, D_{0d}^{(Y)}\}$ (each of weight v_0), so Step 1 and prime avoidance (picking c' general linear combinations $g_1, \ldots, g_{c'} \in \mathcal{F}_t$ simultaneously regular at all $M_{W'}$ points) give local length at least $(\Delta/v_0)^{c'}/(c')!$ at every p_ν. Applying Fulton's localized positivity theorem componentwise to the complete pure c'-dimensional scheme $\overline{S}'$ [10, Proposition 7.1(b), Theorem 12.2, and Example 12.2.7(a)] for the globally generated bundle $\mathcal{O}_{\overline{S}'}(k_{\mathrm{base}}E|_{\overline{S}'})^{\oplus c'}$ gives

$$d_{0,d}^{d-c'} \deg_E(W') \cdot \frac{(\Delta/v_0)^{c'}}{(c')!} \leq \int_{\overline{S}'} (k_{\mathrm{base}}E|_{\overline{S}'})^{c'} \leq d_{0,d}^{d-c'} (N'/w_0)^{c'} d_Y.$$

Cancelling $d_{0,d}^{d-c'} > 0$ and rearranging yields (ii). □

Proof of Theorem 2.2. To track every dependency without circularity, the proof is organized into four stages: (1) freezing the geometric constants and recursive weight thresholds W_i; (2) testing integral curves $C \subset A^\circ \times \mathbb{A}^m$ by descending induction on $d = \dim Y$; (3) the vertical base case $d = 0$; and (4) passing from the curve inequality to jet surjectivity via an ordinary blow-up and Rees pushdown. Table 2 summarizes all constants and their exact dependency chain.

Stage 1: Freezing the geometric constants and thresholds W_i. Write $\delta := \delta_{\mathrm{sl}}$ and $q := q_{\mathrm{sl}} = (1 + 2\delta)/(1 + \delta)$ inside this proof. Start at top dimension $d = g$ with $d_{g,\max} :=$

$3^g g!\,\chi(L)$, $l_{g,\max} := 0$, $d_{0,g} := 1$, and $D_{\mathrm{bad}}^{(g)} := 0$. Descending from $d = g$ down to $d = 1$, once $(d_{d,\max}, l_{d,\max}, d_{0,d}, D_{\mathrm{bad}}^{(d)})$ are fixed, define

$$n_d := m + d + 1, \qquad \delta_d := \delta \min\left\{1, \frac{v_0}{(2l_{d,\max}+1)w_0}\right\},$$

$$\epsilon_d := \frac{\delta_d}{(1+\delta)n_d}, \qquad C_d := \max_{\substack{1\le r\le m+d \\ 0\le a\le \min(d,r)}} \frac{2(r!)^2 d_{0,d}^{d-a} d_{d,\max}}{a!\,\epsilon_d^r}.$$

For each lower dimension $1 \le d' < g$, define $d_{d',\max}$ as the ceiling of the maximum over all $d \in \{d'+1, \dots, g\}$ of the two quantities

$$(d-d')!\left(\frac{v_0}{\epsilon_d w_0}\right)^{d-d'} d_{d,\max}, \qquad D_{\mathrm{bad}}^{(d)},$$

and apply Lemma 2.4 to $(d', d_{d',\max})$ to freeze $(d_{0,d'}, l_{d',\max}, D_{\mathrm{bad}}^{(d')})$. At dimension $d = 0$, set

$$C_0 := \max_{1\le r\le m} 2(r!)^2 \left(\frac{m+1}{\delta}\right)^r.$$

Finally put $C_* := \max_{0\le d\le g} C_d$, $l_* := \max_{1\le d\le g} l_{d,\max}$, $P_{\min} := \min(1, w_0)^g$, and recursively for $i = 1, \dots, m$:

$$Q_i := \max(1, v_0)^g \prod_{j=1}^{i-1} \max\left(1, \frac{w_j}{\theta}\right), \qquad W_i := \max\left\{\frac{C_* Q_i}{P_{\min}}, (l_*+1)w_0, \theta v_0, 1\right\}.$$

Notice that W_i depends solely on $(A, L, D, \eta_{\mathrm{target}}, m, K_{\mathrm{cen}}, w_0, v_0, \theta, \delta)$ and $w_1, \dots, w_{i-1}$, and is independent of both the centres (P_0, c_j) and the auxiliary degree parameter N of Stage 2 (none of the constants in Lemma 2.7 introduces any N-threshold into W_i). Fix any integers $w_i > W_i$ $(1 \le i \le m)$ and any K_{cen} centres (P_0, c_j) with coordinatewise distinct c_{ji}, and put $T_i := \lfloor F_0 w_i / v_0 \rfloor + 1$. Because $F_0\theta \ge 2$ by hypothesis, one has $v_0 T_i > F_0 w_i \ge 2w_i/\theta = 2v_i > v_i$, so the truncated polynomial primitive G_i and the exact formal primitive of η_{target} agree modulo $\mathfrak{m}_{A,P_0}^{T_i}$ (whose elements carry jet weight $\ge v_0 T_i > v_i$); hence replacing the exact primitive by G_i preserves both the weighted jet filtration and the derivation drop of Lemma 2.5(ii) identically.

Stage 2: Testing integral curves by descending induction on $d = \dim Y$. It is shown first that every integral curve $C \subset A^\circ \times \mathbb{A}^m$ satisfies the testing inequality $\deg_W(C) \ge q \sum_{P\in C^\nu} h_P$ (where the sum runs over branches P mapping to any centre (P_0, c_j)). Suppose for contradiction that $\deg_W(C) < q \sum_P h_P$. If the base projection $\mathrm{pr}_A(C)$ is a point, it must be $\{P_0\}$ (since $\sum_P h_P > 0$), and the argument passes directly to Stage 3 ($d = 0$). Otherwise an integral projective subvariety $Y \subseteq A$ of dimension $d \in \{1, \dots, g\}$ containing $\mathrm{pr}_A(C)$ (hence containing P_0) with $\deg_D(Y) \le d_{d,\max}$ is maintained, starting from $Y = A$ ($d = g$).

Because C^ν is a smooth complete curve and Y is projective, the affine base projection $C \to Y$ extends uniquely to a morphism $C^\nu \to Y$. Since $\varpi_Y : \widetilde{Y} \to Y$ is finite and surjective, $(C^\nu \times_Y \widetilde{Y})_{\mathrm{red}} \to C^\nu$ is finite and surjective, so choosing an irreducible component dominating C^ν and passing to its smooth complete normalization yields a curve $\widehat{C}$ with a finite dominant morphism $\pi_{\widehat{C}} : \widehat{C} \to C^\nu$ of degree $s \ge 1$. At every point $\widehat{Q} \in \widehat{C}$ above $Q \in C^\nu$ with ramification index $e_{\widehat{Q}}$, every pole order ord^- and every local jet-generator order ord scales by $e_{\widehat{Q}}$; since $\sum_{\widehat{Q}\mapsto Q} e_{\widehat{Q}} = s$, one has $\deg_W(\widehat{C}) = s \deg_W(C)$ and $\sum_{\widehat{P}} h_{\widehat{P}} = s \sum_P h_P$, so the strict inequality $\deg_W(\widehat{C}) < q \sum_{\widehat{P}} h_{\widehat{P}}$ is preserved.

If $\mathrm{pr}_A(C)$ is contained in the bad image $B_Y = \varpi_Y(\mathrm{Sing}(\widetilde{Y}) \cup \mathrm{Supp}(\Delta_{\mathrm{ram}}))$ of Lemma 2.4, then $\mathrm{pr}_A(C)$ lies in an irreducible component $Y' \subset B_Y$ of dimension $d' < d$ with $\deg_D(Y') \le$

Constant(s)	Role in Section 2	Exact dependency
$D^g = 3^g g!\,\chi(L)$, $d_{g,\max}$, $d_{0,g} = 1$, $l_{g,\max} = 0$	Top-dimension degree and trivial clearing data on A.	(A, L) only.
$\delta_{\mathrm{sl}}, q_{\mathrm{sl}}$	Rational slack: $(1+3\delta_{\mathrm{sl}})^{m+d} a_d < 1$, $q_{\mathrm{sl}}\theta < 1$.	$(A, L, m, K_{\mathrm{cen}}, w_0, v_0, \theta)$.
$d_{d,\max}$, $d_{0,d}$, $l_{d,\max}$, $D^{(d)}_{\mathrm{bad}}$ $(d = g-1, \dots, 1)$	Bounded Chow ceilings, very ampleness multiple, clearing twist, bad-locus degree (Lem. 2.4).	Descending recursion on $(A, D, m, w_0, v_0, \delta_{\mathrm{sl}})$ and higher-dim. constants; independent of w_i, c_j.
n_d, δ_d, ϵ_d, C_d, C_0, C_*, l_*	Slice-step count, step size, slice constant in Lem. 2.7.	Frozen constants above; independent of w_i, c_j.
$P_{\min}, Q_i, W_i$ $(1 \le i \le m)$	Recursive weight-separation thresholds forcing $\partial_{X_i} \in \mathcal{B}$.	W_i depends on frozen constants and $w_1, \dots, w_{i-1}$ only; independent of c_j.
$T_i = \lfloor F_0 w_i / v_0 \rfloor + 1$	Taylor truncation order of G_i.	(F_0, v_0, w_i).
R_{wt}, n_0	Common weight denominator and Serre/Rees threshold for $H = nR_{\mathrm{wt}}$.	All frozen data, $(w_1, \dots, w_m)$, and the chosen centres (P_0, c_j).

Table 2: Complete dependency table for all constants appearing in Theorem 2.2 and Lemmas 2.4–2.7.

$D^{(d)}_{\mathrm{bad}} \le d_{d',\max}$. If $d' = 0$, then $\mathrm{pr}_A(C) = \{P_0\}$ (Stage 3); if $d' > 0$, replace Y by Y' and continue the induction with strictly smaller dimension $d' < d$.

Assume now that $\mathrm{pr}_A(C) \not\subset B_Y$. Remove from $\widehat{C}$ the finite set of poles of E and $X_1, \dots, X_m$, obtaining a nonempty smooth affine curve $\widehat{C}^\circ \to \widetilde{Y}^\circ \times \mathbb{A}^m$ that retains all centre branches $\widehat{P}$. Choose a point $q_C \in \widehat{C}^\circ$ whose base projection $q_{C,\widetilde{Y}} \in \widetilde{Y}^\circ$ lies in $\widetilde{Y}_{\mathrm{sm}} \setminus (\mathrm{Supp}\, E \cup \mathrm{Supp}\, \Delta_{\mathrm{ram}})$. Since $\mathcal{I}_Y(l_{d,\max}E)$ is globally generated on $\widetilde{Y}$ and $\mathcal{I}_Y|_{\widetilde{Y}_{\mathrm{sm}}} = \mathcal{O}(-\Delta_{\mathrm{ram}})$ does not vanish at $q_{C,\widetilde{Y}}$, there exists $\phi_{\mathrm{cl}} \in H^0(\widetilde{Y}, \mathcal{I}_Y(l_{d,\max}E))$ with $\phi_{\mathrm{cl}}(q_{C,\widetilde{Y}}) \neq 0$.

For each lifted basepoint $Q \in \varpi_Y^{-1}(P_0)$, set $R_Q := \mathcal{O}_{\widetilde{Y},Q}$ and $\mathfrak{a}_Q := \mathfrak{m}_{A,P_0} R_Q$. Set $R_{P_0} := \mathcal{O}_{Y,P_0}$ and $S_{P_0} := (\varpi_{Y,*}\mathcal{O}_{\widetilde{Y}})_{P_0}$. Because $\varpi_Y : \widetilde{Y} \to Y$ is finite and birational between integral varieties over $\mathbb{C}$, S_{P_0} is a module-finite semilocal integral domain over R_{P_0} with the same fraction field ($[\mathbb{C}(\widetilde{Y}) : \mathbb{C}(Y)] = 1$), whose maximal ideals correspond bijectively to the points $Q \in \varpi_Y^{-1}(P_0)$ with localizations $(S_{P_0})_{\mathfrak{m}_Q} = R_Q$ and residue degrees $[R_Q/\mathfrak{m}_Q : \mathbb{C}] = 1$. Applying the Swanson–Huneke multiplicity formula for module-finite extension domains [34, Theorem 11.2.7] to (R_{P_0}, S_{P_0}) (equivalently, decomposing the Artinian quotients $S_{P_0}/\mathfrak{m}^n_{Y,P_0} S_{P_0} \cong \prod_{Q|P_0} R_Q/\mathfrak{a}_Q^n$) gives

$$\sum_{Q|P_0} e(\mathfrak{a}_Q, R_Q) = e(\mathfrak{m}_{Y,P_0}, R_{P_0}) = \mathrm{mult}_{P_0}(Y) \le \deg_D(Y) = d_Y,$$

where $\mathrm{mult}_{P_0}(Y) \le \deg_D(Y)$ holds because $D = 3L$ is very ample on A by Lefschetz's theorem [3, Theorem 4.5.1], so cutting $Y \subset \mathbb{P}^{N_s}$ with d general hyperplanes through P_0 bounds the local multiplicity by the projective degree (at $d = g$, $Y = A$ is smooth so $\mathrm{mult}_{P_0}(A) = 1$). For $M > 0$, the change of variables $U'_{ji} = X_i - c_{ji} - G_i(\mathbf{e})$ shows that the colength of the pulled-back jet ideal of order M at (Q, c_j) is

$$\sum_{\substack{\mathbf{b}\in\mathbb{N}^m \\ v\cdot\mathbf{b}<M}} \mathrm{length}_{R_Q}\Big(R_Q/\mathfrak{a}_Q^{\lceil (M - v\cdot\mathbf{b})/v_0 \rceil}\Big) = \frac{e(\mathfrak{a}_Q, R_Q)\, M^{m+d}}{(m+d)!\, v_0^d \prod_{i=1}^m v_i}(1 + o_M(1)),$$

by the Hilbert–Samuel polynomial of $\mathfrak{a}_Q$ and Riemann summation over the m-simplex $\{v \cdot \mathbf{b} < M\}$. Summing over all K_{cen} centres and all $Q \mid P_0$ yields total jet demand

$$\frac{K_{\mathrm{cen}} \operatorname{mult}_{P_0}(Y)\, M^{m+d}}{(m+d)!\, v_0^d \prod_{i=1}^m v_i}(1 + o_M(1)).$$

On the supply side, Riemann–Roch / Hilbert polynomial for the ample Cartier divisor E on $\widetilde{Y}$ gives $h^0(\widetilde{Y}, kE) = \frac{d_Y}{d!}k^d + O(k^{d-1})$, so summing over $w \cdot \mathbf{b} \le N$ yields

$$\dim_{\mathbb{C}} V_N^{(\widetilde{Y})} = \frac{d_Y N^{m+d}}{(m+d)!\, w_0^d \prod_{i=1}^m w_i}(1 + o_N(1)).$$

Setting $M := (1+3\delta)N$, the asymptotic demand-to-supply ratio is

$$(1+3\delta)^{m+d} \cdot K_{\mathrm{cen}} \theta^m (w_0/v_0)^d \cdot \frac{\operatorname{mult}_{P_0}(Y)}{d_Y} \le (1+3\delta)^{m+d} a_d < 1$$

(at $d = g$, $\operatorname{mult}_{P_0}(A)/d_A = 1/(3^g g!\, \chi(L))$ gives $(1+3\delta)^{m+g} a_g < 1$). Hence for all sufficiently large N, linear algebra produces a nonzero section $F \in V_N^{(\widetilde{Y})}$ vanishing to pulled-back jet order at least $(1+3\delta)N$ at every lifted centre (Q, c_j).

Set $N' := (1+\delta)N$ and $\Delta := \delta_d N/n_d$. For every integer $0 \le t \le n_d$, let $\mathcal{F}_t$ be the span of all $h_{\mathbf{u}} \partial_X^{\mathbf{b}} F$ with $v_0|\mathbf{u}| + v \cdot \mathbf{b} \le t\Delta \le \delta_d N$. By Lemma 2.5(ii), since $n = |\mathbf{u}| \le \delta_d N/v_0$ implies $n(2l_{d,\max}+1)w_0 \le \delta N$, one has $\mathcal{F}_t \subseteq V_{N'}^{(\widetilde{Y})}$, and every $G \in \mathcal{F}_t$ vanishes to jet order at least $(1+3\delta)N - \delta_d N \ge (1+2\delta)N$ at every lifted centre (Q, c_j). If $G|_{\widehat{C}}$ were nonzero, its zero count over centre branches would be at least $(1+2\delta)N \sum_{\widehat{P}} h_{\widehat{P}}$, whereas its total pole order on $\widehat{C}$ would be at most $N' \deg_W(\widehat{C}) = (1+\delta)N \deg_W(\widehat{C})$ (because D is globally generated, every coefficient in $H^0(\widetilde{Y}, kE)$ has pole order at most k times the maximum basis pole order at each point of $\widehat{C}$). This contradicts $\deg_W(\widehat{C}) < q \sum_{\widehat{P}} h_{\widehat{P}}$ since $q = (1+2\delta)/(1+\delta)$. Therefore every $G \in \mathcal{F}_{n_d}$ vanishes identically on the image of $\widehat{C}^\circ$.

Choose irreducible components $Z_{n_d} \subseteq Z_{n_d-1} \subseteq \cdots \subseteq Z_0$ of $V(\mathcal{F}_{n_d}) \subseteq \cdots \subseteq V(\mathcal{F}_0)$ in $\widetilde{Y}^\circ \times \mathbb{A}^m$ all containing the image of $\widehat{C}^\circ$ (selecting Z_{n_d} first and extending backwards). Because $F \ne 0$ and $\widehat{C}^\circ$ is 1-dimensional, every Z_t satisfies $1 \le \dim Z_t \le d+m-1$. As there are $n_d + 1 = d+m+2$ nested irreducible closed subsets taking values in the $(d+m-1)$-element set $\{1, \ldots, d+m-1\}$, by the pigeonhole principle two consecutive components satisfy $\dim Z_{t+1} = \dim Z_t$, hence $Z_{t+1} = Z_t =: Z'$. Since Z' contains q_C, its projection meets $\widetilde{Y}_{\mathrm{good}}$, so Lemma 2.7 applies to Z' with $\Delta/N' = \epsilon_d$.

There are two exhaustive geometric cases on $U_{\mathrm{gen}} \subset Z'$: either some additive direction ∂_{X_k} is non-tangent to Z' on a nonempty Zariski-open subset of U_{gen} (Case A), or every $\partial_{X_1}, \ldots, \partial_{X_m}$ is tangent to Z' at every point of U_{gen} (Case B).

Case A (Some additive direction is generically non-tangent to Z'). Let $i \in \{1, \ldots, m\}$ be the *largest* index for which $\mathrm{pr}_N(\partial_{X_i})$ does not vanish identically on U_{gen}. Then $U_{\mathrm{gen}}^{(i)} := \{p \in U_{\mathrm{gen}} : \mathrm{pr}_N(\partial_{X_i})|_p \ne 0\}$ is a nonempty Zariski-open subset of U_{gen}, and at every $p \in U_{\mathrm{gen}}^{(i)}$ one has $\mathrm{pr}_N(\partial_{X_k})|_p = 0$ for all $k > i$. Fix any $p \in U_{\mathrm{gen}}^{(i)}$. By the remark after Definition 2.6, there exists a coordinate normal basis $\mathcal{A}$ at p containing $i \in \mathcal{A}_{\mathrm{add}}$ and $a \le \min(d, r)$ base directions. Since $w_j \ge 1$ for all $j \ge 1$ and $0 \le a \le g$, one has $\prod_{j \in \mathcal{A}} w_j \ge w_i \min(1, w_0)^g = w_i P_{\min}$. Now let $\mathcal{B}$ be any frame normal basis at p. Because $\mathrm{pr}_N(\partial_{X_k})|_p = 0$ for all $k > i$, no ∂_{X_k} with $k > i$ can belong to $\mathcal{B}$. If $\mathcal{B}$ omitted ∂_{X_i}, it would consist of at most $d \le g$ horizontal frame fields (weight v_0) and a subset of $\{\partial_{X_1}, \ldots, \partial_{X_{i-1}}\}$ (weights $v_j = w_j/\theta$), giving $\prod_{j \in \mathcal{B}} v_j \le Q_i$. Lemma 2.7(i) would then force $w_i P_{\min} \le C_d Q_i \le C_* Q_i$, contradicting $w_i > W_i \ge C_* Q_i / P_{\min}$. Hence *every* frame normal basis $\mathcal{B}$ at p must contain ∂_{X_i}.

Consequently the subspace $W_{-i} := \mathrm{span}\{\mathrm{pr}_N(B) : B \in \mathcal{F}_{\mathrm{frame}} \setminus \{\partial_{X_i}\}\} \subset N_{Z',p}$ is a proper hyperplane (otherwise $W_{-i} = N_{Z',p}$ would contain a frame normal basis omitting ∂_{X_i}), and

therefore $T_pZ' \subseteq \operatorname{span}(\mathcal{F}_{\text{frame}} \setminus \{\partial_{X_i}\})$. Consider the 1-form $\alpha_i := dX_i - \varpi_Y^*\eta_{\text{target}}$ on U_{gen}. Directly from the definition of $D_{0a}^{(Y)}$,

$$\begin{aligned}\alpha_i(D_{0a}^{(Y)}) &= (\varpi_Y^*\eta_{\text{target}})(V_a) - (\varpi_Y^*\eta_{\text{target}})(V_a) = 0 \qquad (1 \le a \le d),\\ \alpha_i(\partial_{X_k}) &= \delta_{ik} \qquad (1 \le k \le m).\end{aligned}$$

Thus $\ker(\alpha_i|_p) = \operatorname{span}(\mathcal{F}_{\text{frame}} \setminus \{\partial_{X_i}\}) \supseteq T_pZ'$ for every $p \in U_{\text{gen}}^{(i)}$. Since the vanishing of the pullback of a regular 1-form on Z'_{sm} is a Zariski-closed condition and $U_{\text{gen}}^{(i)}$ is dense in U_{gen}, α_i pulls back to zero identically on all of U_{gen} (and on Z'_{sm}).

If the projection $\pi_A : Z' \to A$ is constant, then $\operatorname{pr}_A(C)$ is constant (a point), reducing to Stage 3. Suppose instead that $\pi_A|_{Z'}$ is non-constant, so there exists $p_1 \in U_{\text{gen}}$ where $d\pi_A|_{p_1} : T_{p_1}Z' \to T_{\pi_A(p_1)}A$ has rank ≥ 1. Let $\overline{Z}' \subset \widetilde{Y} \times (\mathbb{P}^1)^m$ be the projective closure of Z', of dimension $\delta' := \dim Z' \ge 1$. Cutting $\overline{Z}'$ by $\delta' - 1$ general hyperplane sections passing through p_1 (whose tangent intersection at p_1 is a general line in $T_{p_1}Z'$ not lying in $\ker d\pi_A|_{p_1}$) yields a 1-dimensional intersection that is smooth at p_1 with tangent line outside $\ker d\pi_A|_{p_1}$; taking the unique irreducible component $\overline{\Gamma}_0 \subset \overline{Z}'$ through p_1 produces an irreducible projective curve through p_1 on which π_A is non-constant. Let Γ be the smooth complete normalization of $\overline{\Gamma}_0$, with induced non-constant morphism $f : \Gamma \to A$ and rational function $x_i := X_i|_\Gamma \in \mathbb{C}(\Gamma)$. On the dense open set $\Gamma^\circ := \Gamma \setminus S_{\text{pol}}$, where $S_{\text{pol}} := f^{-1}(\operatorname{Supp} L) \cup \operatorname{Poles}(x_i)$ is finite (noting $\operatorname{Supp} D = \operatorname{Supp} L$ since $D = 3L$, so η_{target} is regular on $A^\circ = A \setminus \operatorname{Supp} L$), the identity $dx_i = f^*\eta_{\text{target}}$ holds identically.

Fixing any basepoint on Γ, the universal property of the Jacobian induces a morphism $f_* : \operatorname{Jac}(\Gamma) \to A$ whose image is a positive-dimensional abelian subvariety of A (since f is non-constant). Because A is absolutely simple, $f_*(\operatorname{Jac}(\Gamma)) = A$, so $f_* : H_1(\Gamma, \mathbb{Q}) \to H_1(A, \mathbb{Q})$ is surjective and $f_*(H_1(\Gamma, \mathbb{Z}))$ has finite index $N_{\text{idx}} \ge 1$ in $H_1(A, \mathbb{Z})$. Let $\lambda_{\text{nz}} \in H_1(A, \mathbb{Z})$ be the cycle with $\int_{\lambda_{\text{nz}}} \eta_{\text{target}} \neq 0$; then $N_{\text{idx}}\lambda_{\text{nz}} = f_*(\gamma_\Gamma)$ for some $\gamma_\Gamma \in H_1(\Gamma, \mathbb{Z})$. Since $S_{\text{pol}} \subset \Gamma$ is finite (real codimension 2), the inclusion $\Gamma^\circ \hookrightarrow \Gamma$ induces a surjection $H_1(\Gamma^\circ, \mathbb{Z}) \twoheadrightarrow H_1(\Gamma, \mathbb{Z})$, so γ_Γ is represented by a smooth closed 1-cycle $\widetilde{\gamma}_\Gamma$ lying entirely inside Γ°. Then $f(\widetilde{\gamma}_\Gamma) \subset A^\circ$, and because the periods of η_{target} on $H_1(A^\circ, \mathbb{Z})$ factor through $H_1(A, \mathbb{Z})$, it follows that

$$0 = \int_{\widetilde{\gamma}_\Gamma} dx_i = \int_{\widetilde{\gamma}_\Gamma} f^*\eta_{\text{target}} = \int_{f_*(\widetilde{\gamma}_\Gamma)} \eta_{\text{target}} = N_{\text{idx}} \int_{\lambda_{\text{nz}}} \eta_{\text{target}} \neq 0,$$

a contradiction.

Case B (Every additive direction $\partial_{X_1}, \dots, \partial_{X_m}$ is tangent to Z' at every $p \in U_{\text{gen}}$). Then for every f in the radical prime ideal $I(Z') \subset \mathbb{C}[\widetilde{Y}^\circ][X_1, \dots, X_m]$ and every $1 \le i \le m$, $\partial_{X_i} f$ vanishes on the dense open $U_{\text{gen}} \subset Z'$, hence $\partial_{X_i} f \in I(Z')$. Since every polynomial $f \in I(Z')$ satisfies the finite Taylor identity $f(y, X + t) = \sum_{\mathbf{k}} \frac{t^{\mathbf{k}}}{\mathbf{k}!}(\partial_X^{\mathbf{k}} f)(y, X)$, Z' is invariant under all translations $(y, X) \mapsto (y, X + t)$ $(t \in \mathbb{C}^m)$. Thus $Z' = (W' \setminus \operatorname{Supp} E) \times \mathbb{A}^m$, where $W' \subset \widetilde{Y}$ is the projective closure of the base projection of Z', an irreducible subvariety of dimension $d' = d - c' < d$ (since $Z' \subsetneq \widetilde{Y}^\circ \times \mathbb{A}^m$ as $F \neq 0$). Because $q_C \in Z'$, W' meets $\widetilde{Y}_{\text{good}}$, so Lemma 2.7(ii) gives

$$\deg_E(W') \le (d - d')! \left(\frac{v_0}{\epsilon_d w_0}\right)^{d-d'} d_Y \le d_{d',\max}.$$

If $d' = 0$, then W' is a single point, so $\operatorname{pr}_A(C) = \{P_0\}$, reducing to Stage 3. If $d' > 0$, set $Y' := \varpi_Y(W') \subset A$. Because $\varpi_Y : W' \to Y'$ is finite, Y' is an irreducible projective subvariety of A of dimension $d' < d$ containing $\operatorname{pr}_A(C)$, and the projection formula gives $\deg_D(Y') \le [\mathbb{C}(W') : \mathbb{C}(Y')] \deg_D(Y') = \deg_E(W') \le d_{d',\max}$. Replacing (Y, d) by (Y', d') strictly decreases d. Since $d \in \{0, 1, \dots, g\}$ can decrease at most g times, the induction terminates in Case A (contradiction) or at Stage 3.

Stage 3: Vertical curves ($d = 0$, $\mathrm{pr}_A(C) = \{P_0\}$). Here $C \subset \{P_0\} \times \mathbb{A}^m$ and the local jets at (P_0, c_j) are shifted additive monomials $(X - c_j)^{\mathbf{b}}$ of weight $v \cdot \mathbf{b} < M = (1 + 3\delta)N$. If $m = 1$, then $C = \{P_0\} \times \mathbb{A}^1$; choosing $F \in \mathbb{C}[X_1]$ of weighted degree $\le N$ vanishing to weighted order $\ge (1+3\delta)N$ at all K_{cen} centres is possible for large N because the demand-to-supply ratio is $K_{\mathrm{cen}}\theta(1 + 3\delta) < 1$, yet a nonzero polynomial cannot vanish identically on $\mathbb{A}^1$, immediately contradicting $\deg_W(C) < q\sum_P h_P$. For $m \ge 2$, the same count gives demand-to-supply ratio $K_{\mathrm{cen}}\theta^m(1 + 3\delta)^m = (1 + 3\delta)^m a_0 < 1$, producing a nonzero polynomial $F \in \mathbb{C}[X_1, \dots, X_m]$ of weighted degree $\le N$ vanishing to jet weight $\ge (1 + 3\delta)N$ at all K_{cen} centres. Every additive derivative $\partial_X^{\mathbf{b}} F$ of cost $v \cdot \mathbf{b} \le \delta N$ has weighted degree $\le N$ and jet weight $\ge (1 + 2\delta)N$, so it vanishes on C by the same zero-pole comparison with $N' = N$. Taking $m + 1$ steps of size $\Delta = \delta N/(m+1)$ produces a persistent proper irreducible component $Z' \subset \mathbb{A}^m$ containing C with slice constant C_0 (the $d = 0$ special case of Lemma 2.7(i), where $\overline{S}$ is a point and $d_Y = 1$). Since $Z' \subsetneq \mathbb{A}^m$, at least one coordinate direction ∂_{X_i} is non-tangent to Z' at a smooth generic point; taking the largest such i and using $w_i > W_i \ge C_0 Q_i/P_{\min}$ forces every frame normal basis (a subset of $\{\partial_{X_1}, \dots, \partial_{X_m}\}$) to contain ∂_{X_i}, whence $dX_i|_{Z'} = 0$. Thus X_i is constant on C. Because the centres have pairwise distinct i-th coordinates ($c_{ji} \ne c_{j'i}$ for $j \ne j'$), C passes through at most one centre (P_0, c_{j_0}). Choose any coordinate X_j non-constant on C; then $X_j - c_{j_0 j}$ is a nonzero rational function on C^ν with at least $v_j \sum_P h_P$ zeros (counted with multiplicity) and at most $w_j \deg_W(C)$ poles, giving $v_j \sum_P h_P \le w_j \deg_W(C) < w_j q \sum_P h_P$, contradicting $q\theta < 1$. This proves $\deg_W(C) \ge q \sum_P h_P$ for every integral curve $C \subset A^\circ \times \mathbb{A}^m$.

Stage 4: From curves to jets via ordinary blow-up and Rees pushdown. Choose a positive integer $R = R_{\mathrm{wt}}$ divisible by the reduced numerators of $w_0, v_0, w_1, \dots, w_m, v_1, \dots, v_m$ (so $R/w_a \in \mathbb{Z}_{>0}$ and $k_a := R/v_a \in \mathbb{Z}_{>0}$ for all $0 \le a \le m$) and large enough that $R \ge \max(w_0, w_1, \dots, w_m)$. Then $V_R = V_R^{(A)}$ contains 1, the basis $s_1, \dots, s_{N_s}$ of $H^0(A, D)$ (very ample on A°), the additive coordinates $X_1, \dots, X_m$, and all pure powers s_r^{R/w_0} ($0 \le r \le N_s$) and X_i^{R/w_i} ($1 \le i \le m$). Consequently the evaluation morphism $\phi_R : A^\circ \times \mathbb{A}^m \hookrightarrow \mathbb{P}(V_R^\vee)$ is a locally closed embedding. Let $X := \overline{\phi_R(A^\circ \times \mathbb{A}^m)} \subset \mathbb{P}(V_R^\vee)$ be its projective closure and put $\mathcal{L} := \mathcal{O}_X(1)$ (very ample on X).

At each centre $(P_0, c_j) \in A^\circ \times \mathbb{A}^m \subset X$, define the local pure-power ideal $\mathcal{I}_{\mathrm{cen},j} := (e_1^{k_0}, \dots, e_g^{k_0}, (U'_{j1})^{k_1}, \dots, (U'_{jm})^{k_m}) \subset \mathcal{O}_{X,(P_0,c_j)}$, and let $\mathcal{I}_{\mathrm{cen}} \subset \mathcal{O}_X$ be the coherent ideal sheaf equal to $\mathcal{I}_{\mathrm{cen},j}$ at each centre and to $\mathcal{O}_X$ away from the K_{cen} centres. Let $\pi : \widetilde{X} := \mathrm{Bl}_{\mathcal{I}_{\mathrm{cen}}}(X) = \mathrm{Proj}_X(\bigoplus_{n \ge 0} \mathcal{I}_{\mathrm{cen}}^n) \to X$ be the ordinary blow-up along $\mathcal{I}_{\mathrm{cen}}$, with exceptional Cartier divisor E_{exc} satisfying $\mathcal{O}_{\widetilde{X}}(-E_{\mathrm{exc}}) = \mathcal{O}_{\widetilde{X}}(1) = \pi^{-1}\mathcal{I}_{\mathrm{cen}} \cdot \mathcal{O}_{\widetilde{X}}$.

Next, the $\mathbb{Q}$-Cartier divisor $\pi^*\mathcal{L} - qE_{\mathrm{exc}}$ is shown to be nef on $\widetilde{X}$, i.e., $(\pi^*\mathcal{L} - qE_{\mathrm{exc}}) \cdot C' \ge 0$ for every irreducible curve $C' \subset \widetilde{X}$:

(a) *Strict transforms $C' = \widetilde{C}$ of curves $C \subset A^\circ \times \mathbb{A}^m$:* Let $\overline{C} = \pi(\widetilde{C}) \subset X$ and let C^ν be the smooth complete normalization of C. By the projection formula, $\pi^*\mathcal{L} \cdot \widetilde{C} = \mathcal{L} \cdot \overline{C} = \deg_{C^\nu}(\phi_R^*\mathcal{O}(1)|_{C^\nu})$. Because V_R is spanned by monomials $s^{\mathbf{k}} X^{\mathbf{b}}$ with $w_0|\mathbf{k}| + w \cdot \mathbf{b} \le R$, the polar order of $\phi_R|_{C^\nu}$ at any $Q \in C^\nu$ (relative to $1 \in V_R$) is bounded above by

$$\max_{w_0|\mathbf{k}| + w\cdot\mathbf{b} \le R} \Big(\sum_r k_r \operatorname{ord}_Q^-(s_r) + \sum_i b_i \operatorname{ord}_Q^-(X_i) \Big) \le R \max \left\{ \max_r \frac{\operatorname{ord}_Q^-(s_r)}{w_0}, \max_i \frac{\operatorname{ord}_Q^-(X_i)}{w_i} \right\},$$

and the pure powers $s_r^{R/w_0}, X_i^{R/w_i} \in V_R$ attain this maximum with exact equality at every $Q \in C^\nu$. Summing over $Q \in C^\nu$ gives $\pi^*\mathcal{L} \cdot \widetilde{C} = R \deg_W(C)$. Similarly, pulling back $\mathcal{I}_{\mathrm{cen}}$ to C^ν gives order $\min\{\min_a k_0 \operatorname{ord}_P(e_a), \min_i k_i \operatorname{ord}_P(U'_{ji})\} = R h_P$ at every branch $P \mapsto (P_0, c_j)$, so $E_{\mathrm{exc}} \cdot \widetilde{C} = R \sum_P h_P$. Hence $(\pi^*\mathcal{L} - qE_{\mathrm{exc}}) \cdot \widetilde{C} = R(\deg_W(C) - q \sum_P h_P) \ge 0$.

(b) *Boundary curves $C' \subset \pi^{-1}(X \setminus (A^\circ \times \mathbb{A}^m))$:* Since the blow-up centre $\{(P_0, c_j)\}$ lies entirely inside $A^\circ \times \mathbb{A}^m$, C' is disjoint from E_{exc}, so $E_{\mathrm{exc}} \cdot C' = 0$ and $(\pi^*\mathcal{L} - qE_{\mathrm{exc}}) \cdot C' = \mathcal{L} \cdot \pi_*(C') > 0$ (as π is an isomorphism near C' and $\mathcal{L}$ is ample on X).

(c) *Contracted curves ($\pi(C')$ is a centre (P_0, c_j)):* Here $\pi^*\mathcal{L}\cdot C' = 0$, while $\mathcal{O}_{\widetilde{X}}(-E_{\mathrm{exc}}) = \mathcal{O}_{\widetilde{X}}(1)$ is π-ample, so $(-E_{\mathrm{exc}})\cdot C' > 0$ and $(\pi^*\mathcal{L} - qE_{\mathrm{exc}})\cdot C' = q(-E_{\mathrm{exc}}\cdot C') > 0$.

Thus $\pi^*\mathcal{L} - qE_{\mathrm{exc}}$ is nef on $\widetilde{X}$. Since $\mathcal{O}_{\widetilde{X}}(-E_{\mathrm{exc}})$ is π-ample and $\mathcal{L}$ is ample on X, there exists a rational $\varepsilon \in (0,1)$ such that $\pi^*\mathcal{L} - \varepsilon E_{\mathrm{exc}}$ is ample on $\widetilde{X}$ [15, Proposition II.7.10(b)]. Because $q > 1 > \varepsilon$, the convex combination

$$\pi^*\mathcal{L} - E_{\mathrm{exc}} = \frac{1-\varepsilon}{q-\varepsilon}(\pi^*\mathcal{L} - qE_{\mathrm{exc}}) + \frac{q-1}{q-\varepsilon}(\pi^*\mathcal{L} - \varepsilon E_{\mathrm{exc}})$$

is ample on $\widetilde{X}$ by Kleiman's ampleness criterion.

Applying the eventual degree-n global-section isomorphism for finitely generated graded modules over a Noetherian graded ring generated in degree 1 [33, Tag 0AG7, Lemma 30.14.3] (cf. [15, Theorem II.5.19 (proof) and Exercise II.5.9(b)]) to the Rees algebra $\bigoplus_{n\geq 0}\mathcal{I}_{\mathrm{cen}}^n$ on an affine cover of X, together with relative Serre vanishing [15, Theorem III.8.8], there exists $n_1 \geq 1$ such that for all $n \geq n_1$:

$$\pi_*\mathcal{O}_{\widetilde{X}}(-nE_{\mathrm{exc}}) = \mathcal{I}_{\mathrm{cen}}^n \qquad \text{and} \qquad R^p\pi_*\mathcal{O}_{\widetilde{X}}(-nE_{\mathrm{exc}}) = 0 \quad (\forall p > 0).$$

By the projection formula and the Leray spectral sequence, for every $n \geq n_1$,

$$H^1(X, \mathcal{I}_{\mathrm{cen}}^n \otimes \mathcal{L}^n) \cong H^1(\widetilde{X}, (\pi^*\mathcal{L} - E_{\mathrm{exc}})^{\otimes n}),$$

which vanishes for all $n \geq n_2$ by Serre vanishing for the ample line bundle $\pi^*\mathcal{L} - E_{\mathrm{exc}}$ [15, Theorem III.5.2]. Moreover, Serre vanishing applied to the ideal sheaf of $X \subset \mathbb{P}(V_R^\vee)$ yields $n_3 \geq 1$ such that $\mathrm{Sym}^n V_R \twoheadrightarrow H^0(X, \mathcal{L}^n)$ is surjective for all $n \geq n_3$. Set $n_0 := \max(n_1, n_2, n_3)$. For every $n \geq n_0$, the exact cohomology sequence of $0 \to \mathcal{I}_{\mathrm{cen}}^n \otimes \mathcal{L}^n \to \mathcal{L}^n \to \mathcal{L}^n/(\mathcal{I}_{\mathrm{cen}}^n \otimes \mathcal{L}^n) \to 0$ gives a surjection

$$V_{nR} \supseteq \mathrm{im}(\mathrm{Sym}^n V_R) = H^0(X, \mathcal{L}^n) \twoheadrightarrow \bigoplus_{j=1}^{K_{\mathrm{cen}}} \mathcal{O}_{A^\circ\times\mathbb{A}^m,(P_0,c_j)}/\mathcal{I}_{\mathrm{cen},j}^n.$$

Finally, every generator of $\mathcal{I}_{\mathrm{cen},j}^n$ is a product of n factors of the form $e_a^{k_0}$ (weight $k_0 v_0 = R$) or $(U'_{ji})^{k_i}$ (weight $k_i v_i = R$), hence has jet weight exactly nR, so $\mathcal{I}_{\mathrm{cen},j}^n \subseteq \mathcal{J}_{j,\geq nR}$. Composing with the quotient modulo $\mathcal{J}_{j,\geq nR}$ proves surjectivity for every $H = nR_{\mathrm{wt}}$ ($n \geq n_0$). When all data are defined over $F \subset \mathbb{C}$, the cokernel of the F-linear evaluation map vanishes after faithfully flat base change $- \otimes_F \mathbb{C}$, hence vanishes over F. □

Remark 2.8 (The obstruction from a proper abelian factor). For $m = g\ell$, $v_0/w_0 = 2N_{\mathrm{box}}^2\theta^\ell$, and the restricted grid of size N_{box}^{2d} on a proper abelian d-fold ($d < g$), the lower-dimensional volume ratio is $\theta^{(g-d)\ell}/2^d < 1$. Nevertheless the moving-curve exclusion in Case A can fail when A is not simple: on $E_1 \times E_1$, the pullback of a nonzero invariant form from the first factor vanishes on $\{0\} \times E_1$, allowing all X_i to be constant there while the base projection moves. This illustrates the exact role of absolute simplicity in Case A.

Lemma 2.9 (Quantitative multivariate Hensel denominators)**.** *On a smooth variety over a number field $\mathbb{L}$, fix finitely many $\mathbb{L}$-points, regular local algebraic parameters $\mathbf{e}$ at each, finitely many functions $1, s_1, \ldots, s_{N_s}$ regular there, and a closed 1-form $\eta = \sum_j a_j\, de_j$ regular there. After one common positive integer rescaling of $s_1, \ldots, s_{N_s}$, there exists an integer $B_{\mathrm{den}} \geq 2$ such that at every fixed point,*

$$B_{\mathrm{den}}^{|\mathbf{n}|}[\mathbf{e}^{\mathbf{n}}]s_r \in \mathcal{O}_{\mathbb{L}}, \qquad\qquad B_{\mathrm{den}}^{|\mathbf{n}|+1}[\mathbf{e}^{\mathbf{n}}]a_j \in \mathcal{O}_{\mathbb{L}}.$$

For $F_0, v_0 \in \mathbb{Q}_{>0}$ and $w_i \in \mathbb{Z}_{>0}$ as in Theorem 2.2, set $T_i = \lfloor F_0 w_i/v_0 \rfloor + 1$ and $L_i = \mathrm{lcm}(1, \ldots, T_i - 1) \leq 4^{T_i}$ (with the empty lcm equal to 1). Then the normalized primitive G_i truncated at total degree $< T_i$ satisfies $L_i B_{\mathrm{den}}^{T_i} G_i \in \mathcal{O}_{\mathbb{L}}[\mathbf{e}]$.

Proof. An affine étale presentation over the parameter space gives auxiliary coordinates $\mathbf{y}$ centred at the given point, polynomial equations, and an invertible square Jacobian J_0 in $\mathbf{y}$ at zero. Regular functions, including the coefficients a_j, may be adjoined as coordinates by equations with invertible linear block; likewise a local unit denominator is inverted by adjoining its inverse. Translate coordinates to zero and clear denominators by integers to obtain integral equations

$$J_0\mathbf{y} = p_1(\mathbf{e}) + \sum_{|\mathbf{k}|+|\mathbf{j}|\geq 2} p_{\mathbf{k},\mathbf{j}}\,\mathbf{e}^{\mathbf{k}}\mathbf{y}^{\mathbf{j}},$$

where p_1 is linear. Choose a positive integer D_0 such that $D_0 J_0^{-1}$ is integral at every point, and substitute $\mathbf{e} = D_0^2\mathbf{x}$, $\mathbf{y} = D_0\mathbf{z}$. Each nonlinear coefficient in the solved equation for $\mathbf{z}$ carries the factor $(D_0J_0^{-1})D_0^{2|\mathbf{k}|+|\mathbf{j}|-2}$ (nonnegative exponent) and the linear term is $(D_0J_0^{-1})p_1(\mathbf{x})$. Since the degree-$n$ coefficients of $\mathbf{z}$ depend only on smaller degrees, induction shows that the zero-constant power-series solution is integral in $\mathbf{x}$. Restoring the constant terms costs one common integer D_1 across the finitely many points. Replacing the nonconstant basis sections by $D_1 s_r$ and taking $B_{\mathrm{den}} := 2D_1D_0^2$ yields both bounds, including at $|\mathbf{n}| = 0$.

Euler integration of $\sum_j e_j a_j(t\mathbf{e})$ from $t = 0$ to 1 divides its homogeneous component of primitive degree k by k; closedness gives the normalized primitive G_i. Its retained degrees satisfy $1 \leq k < T_i$, proving $L_i B_{\mathrm{den}}^{T_i} G_i \in \mathcal{O}_{\mathbb{L}}[\mathbf{e}]$. Finally, writing $M_n := \mathrm{lcm}(1,\ldots,n)$ with explicit base cases $M_0 = M_1 = 1 \leq 4^0, 4^1$, assume $n \geq 2$ (so $\lceil n/2\rceil < n$). For each prime p at most one prime power p^e ($e \geq 1$) lies in the half-open interval $(\lceil n/2\rceil, n]$ (since $p^{e+1} \geq 2p^e > n$). Hence $M_n/M_{\lceil n/2\rceil}$ equals the product of those distinct primes p (each to the first power) admitting a power $p^e \in (\lceil n/2\rceil, n]$. For every such prime power, $p^e \leq n < 2p^e$ and $\lfloor n/2\rfloor \leq \lceil n/2\rceil < p^e$, so by Legendre's formula (equivalently Kummer's carry criterion) the p^e-term in the p-adic valuation of $\binom{n}{\lfloor n/2\rfloor}$ equals $\lfloor n/p^e\rfloor - \lfloor\lfloor n/2\rfloor/p^e\rfloor - \lfloor\lceil n/2\rceil/p^e\rfloor = 1 - 0 - 0 = 1$; thus $p \mid \binom{n}{\lfloor n/2\rfloor}$ and hence $(M_n/M_{\lceil n/2\rceil}) \mid \binom{n}{\lfloor n/2\rfloor}$. Since $\binom{n}{\lfloor n/2\rfloor} \leq 2^n$ for even n and $\leq 2^{n-1}$ for odd n, strong induction on $n \geq 2$ yields $M_n \leq 4^n$. □

3 Canonical second-kind descent, covariant translated theta gauges, and determinant engine

This section separates the algebraic-geometric descent of second-kind representatives, diagonal Hermiticity, covariant translated theta gauges, and the Galois compositum count into four modular statements (Proposition 3.1, Lemma 3.2, Proposition 3.3, and Lemma 3.4), before proving the master determinant engine (Theorem 3.5) together with its quantified algebraic-class extension (Proposition 3.6), the self-contained rank-two imaginary quadratic CM-order theorem (Proposition 3.7), and the reciprocal Flint–Hills transfer (Corollary 3.9).

Throughout this section, let $C/\mathbb{Q}$ be a smooth projective geometrically integral curve of genus $g_{\mathrm{tot}} \geq 1$ with Jacobian $J = \mathrm{Jac}(C)$, let $O \in C(\overline{\mathbb{Q}})$, put $\Theta_O := W_{g_{\mathrm{tot}}-1}(O) \subset J$ (the symmetric-power theta divisor $W_{g_{\mathrm{tot}}-1} - (g_{\mathrm{tot}}-1)O$), and let $i : A \hookrightarrow J$ be a positive-dimensional abelian subvariety defined over $\mathbb{Q}$. Choose a translation point $Q_0 \in J(\overline{\mathbb{Q}}) \setminus \Theta_O$, set $i_Q(a) := i(a) + Q$ and $L_Q := i_Q^*\Theta_O$ for $Q \in J(\overline{\mathbb{Q}})$, and let $F/\mathbb{Q}$ be a finite Galois extension containing fields of definition of O, Q_0, and a trivialized basis $s_0 = 1, s_1, \ldots, s_{N_s}$ of $H^0(A, 3L_{Q_0})$.

Proposition 3.1 (Canonical Galois descent of second-kind 1-forms)**.** *In the setting above, let $U := J_F \setminus \Theta_O$. The restriction map $H^1_{\mathrm{dR}}(J/F) \to H^1_{\mathrm{dR}}(U/F)$ is an isomorphism, and composing the map sending a closed algebraic 1-form on U with poles bounded by $2\Theta_O$ to its class in $H^1_{\mathrm{dR}}(U/F)$ with the inverse of this restriction isomorphism yields an isomorphism of F-vector spaces:*

$$H^0(J_F, \Omega^1_{J_F}(2\Theta_O))_{\mathrm{cl}} \xrightarrow{\sim} H^1_{\mathrm{dR}}(J/F).$$

Consequently, every $\rho \in H^1_{\mathrm{dR}}(J/F)$ admits a unique representative $\eta_{O,\rho} \in H^0(J_F, \Omega^1_{J_F}(2\Theta_O))_{\mathrm{cl}}$. Moreover, for every $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$, setting

$$T_\gamma := (g_{\mathrm{tot}} - 1)[\gamma(O) - O] \in J(F), \qquad Q_\gamma := \gamma(Q_0) + T_\gamma \in J(F),$$

one has $\gamma(\Theta_O) = t^*_{T_\gamma}\Theta_O$, $Q_\gamma \notin \Theta_O$, and $\gamma(L_{Q_0}) = L_{Q_\gamma}$. If $\rho \in H^1_{\mathrm{dR}}(J/\mathbb{Q})$ is $\mathbb{Q}$-rational, then for every $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$ one has $i_{Q_\gamma}(A) \not\subset \Theta_O$ (as $Q_\gamma \notin \Theta_O$), the Galois-conjugate chart pullback satisfies

$$\gamma(i^*_{Q_0}\eta_{O,\rho}) = i^*_{Q_\gamma}\eta_{O,\rho},$$

and by functoriality of the injective restriction $H^1_{\mathrm{dR}}(A/\mathbb{C}) \hookrightarrow H^1_{\mathrm{dR}}(A_{\mathbb{C}} \setminus \mathrm{Supp}(L_{Q_\gamma}))$ (since $[\eta_{O,\rho}|_U]$ lies in the image of $H^1_{\mathrm{dR}}(J/\mathbb{C})$), the class $[i^*_{Q_\gamma}\eta_{O,\rho}]$ equals the image of $i^*\rho \in H^1_{\mathrm{dR}}(A/\mathbb{Q})$; in particular, all its residues vanish and its periods along $H_1(A_{\mathbb{C}} \setminus \mathrm{Supp}(L_{Q_\gamma}), \mathbb{Z})$ factor through $H_1(A, \mathbb{Z})$ and coincide with the periods of $i^*\rho$ in all charts.

Proof. The divisor Θ_O is reduced and irreducible and defines the principal polarization of J, so $h^0(J, \mathcal{O}_J(\Theta_O)) = 1$. Since $\mathrm{Sing}(\Theta_O)$ has complex codimension at least 2 in J, excision of $\mathrm{Sing}(\Theta_O)$ preserves singular cohomology in degrees 1 and 2. For $U := J \setminus \Theta_O$, the Gysin sequence gives

$$0 \longrightarrow H^1(J, \mathbb{C}) \longrightarrow H^1(U, \mathbb{C}) \longrightarrow H^0(\Theta_{O,\mathrm{sm}}, \mathbb{C})(-1) \longrightarrow H^2(J, \mathbb{C}).$$

The last arrow sends the generator to the nonzero first Chern class of $\mathcal{O}_J(\Theta_O)$, hence is injective, so $H^1(J, \mathbb{C}) \xrightarrow{\sim} H^1(U, \mathbb{C})$. If $\eta \in H^0(J_{\mathbb{C}}, \Omega^1_{J_{\mathbb{C}}}(2\Theta_O))_{\mathrm{cl}}$ maps to zero in $H^1(U, \mathbb{C})$, then because U is affine (the complement of an ample divisor), algebraic de Rham cohomology yields a regular function $h \in \mathcal{O}(U)$ with $\eta = dh$. At a generic point of Θ_O, a pole of order $a > 0$ of h creates a pole of order $a + 1$ of dh, so $a \le 1$. Since J is normal, h extends across codimension 2 as a section in $H^0(J_{\mathbb{C}}, \mathcal{O}_{J_{\mathbb{C}}}(\Theta_O)) = \mathbb{C}$, forcing $\eta = dh = 0$. This proves injectivity over $\mathbb{C}$.

For surjectivity over $\mathbb{C}$, let ϑ_O be a reduced Appell–Humbert theta function for Θ_O [2, Theorem 1.7, Definition 1.8, and §1.3], [3, §2.2], with positive-definite Hermitian form H_J (chosen $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second) and transformation law

$$\vartheta_O(z + \lambda) = \epsilon_\lambda \exp\big(\pi H_J(z, \lambda) + \tfrac{\pi}{2} H_J(\lambda, \lambda)\big)\, \vartheta_O(z) \qquad (\lambda \in \Lambda_J,\ |\epsilon_\lambda| = 1). \tag{1}$$

For every $e \in \mathrm{Lie}(J)$, the directional derivative $D_e \log \vartheta_O$ shifts by the constant $\pi H_J(e, \lambda)$ under $z \mapsto z + \lambda$, so its exterior derivative $\omega_e := d(D_e \log \vartheta_O)$ is a closed Λ_J-periodic meromorphic 1-form on $\mathrm{Lie}(J)$ whose product by ϑ_O^2 is the holomorphic 1-form $\vartheta_O\, d(D_e\vartheta_O) - (D_e\vartheta_O)\, d\vartheta_O$ (regular even along $\mathrm{Sing}(\Theta_O)$). By GAGA, $\omega_e \in H^0(J_{\mathbb{C}}, \Omega^1_{J_{\mathbb{C}}}(2\Theta_O))_{\mathrm{cl}}$. Since H_J is positive definite, the period functionals $\lambda \mapsto \pi H_J(e, \lambda)$ span all $\mathbb{C}$-antilinear functionals on $\mathrm{Lie}(J)$, while $H^0(J_{\mathbb{C}}, \Omega^1_{J_{\mathbb{C}}})$ spans all $\mathbb{C}$-linear functionals; together they span $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_J, \mathbb{C}) \cong H^1(J, \mathbb{C})$. Because the de Rham class map and closedness condition are defined over F, faithful flatness of $F \hookrightarrow \mathbb{C}$ yields the isomorphism over F.

For $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$, the divisor definition $\Theta_O = W_{g_{\mathrm{tot}}-1} - (g_{\mathrm{tot}}-1)O$ gives $\gamma(\Theta_O) = W_{g_{\mathrm{tot}}-1} - (g_{\mathrm{tot}}-1)\gamma(O) = t^*_{T_\gamma}\Theta_O$. Since $Q_0 \notin \Theta_O$, applying γ gives $\gamma(Q_0) \notin \gamma(\Theta_O)$, i.e., $Q_\gamma = \gamma(Q_0) + T_\gamma \notin \Theta_O$, and $\gamma(L_{Q_0}) = i^*_{\gamma(Q_0)}\gamma(\Theta_O) = i^*_{\gamma(Q_0)}t^*_{T_\gamma}\Theta_O = L_{Q_\gamma}$. If $\rho \in H^1_{\mathrm{dR}}(J/\mathbb{Q})$ is $\mathbb{Q}$-rational, then $(t^{-1}_{T_\gamma})^*\gamma(\eta_{O,\rho})$ is a closed 1-form with poles bounded by $2\Theta_O$ representing the same class $\gamma(\rho) = \rho$ (since translation acts trivially on $H^1_{\mathrm{dR}}(J)$). By uniqueness of the representative, $(t^{-1}_{T_\gamma})^*\gamma(\eta_{O,\rho}) = \eta_{O,\rho}$, i.e., $\gamma(\eta_{O,\rho}) = t^*_{T_\gamma}\eta_{O,\rho}$; pulling back along $i_{\gamma(Q_0)}$ gives $\gamma(i^*_{Q_0}\eta_{O,\rho}) = i^*_{Q_\gamma}\eta_{O,\rho}$. □

Lemma 3.2 (CM diagonal Hermiticity and multiplicity-one representatives)**.** *Let $\psi \in \mathrm{Aut}(J_{\overline{\mathbb{Q}}})$ be a finite-order automorphism of the variety $J_{\overline{\mathbb{Q}}}$ preserving the numerical class of Θ_O (possibly translating O), and write ψ_* for its induced linear part on* $\mathrm{Lie}(J)$.

(i) Every eigenvalue ζ_j of ψ_* on $\mathrm{Lie}(J)$ *satisfies $|\zeta_j| = 1$, and eigendirections of distinct eigenvalues $\zeta_j \neq \zeta_k$ are orthogonal for the polarization Hermitian form H_J.*

(ii) If a nonzero second-kind class $[\rho_u] \in H^1_{\mathrm{dR}}(J/\mathbb{C})$ spans a one-dimensional ψ^*-eigenspace whose eigenvalue equals the complex conjugate of the holomorphic eigenvalue on an eigenvector $e_u \in \mathrm{Lie}(J)$, and this conjugate eigenvalue does not occur on $H^0(J_{\mathbb{C}}, \Omega^1_{J_{\mathbb{C}}})$, then its canonical representative from Proposition 3.1 is $\eta_{O,\rho_u} = c_u\, d(D_{e_u} \log \vartheta_O)$ for some nonzero constant $c_u \in \mathbb{C}^\times$.

Proof. The Riemann alternating form $E_J := -\operatorname{Im} H_J$ (differing by an overall sign from $E = \operatorname{Im} H$ in [3, §2.1], which does not affect orthogonality; for $H_J(z,w) = z\bar{w}$ one has $E_J(1,i) = 1$ and $E_J(i,1) = -1$) is determined by the numerical class of Θ_O, so ψ_* preserves E_J and hence (as ψ_* is $\mathbb{C}$-linear and H_J is $\mathbb{C}$-linear in the first argument) preserves $H_J(z,w) = -E_J(iz,w) - iE_J(z,w)$. Since ψ_* has finite order and preserves the positive-definite form H_J, its eigenvalues have modulus 1. In an eigenbasis $\{e_j\}$ with $\psi_* e_j = \zeta_j e_j$, invariance gives $\zeta_j \bar{\zeta}_k H_J(e_j, e_k) = H_J(e_j, e_k)$; since $|\zeta_k| = 1$, $\zeta_j \neq \zeta_k$ forces $H_J(e_j, e_k) = 0$. For (ii), because ψ may translate Θ_O, the pullback ψ^* acts on the cohomology space $H^1_{\mathrm{dR}}(J/\mathbb{C}) \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda_J, \mathbb{C})$. Under this cohomology action, the period of the class $[d(D_{e_u} \log \vartheta_O)]$ on $\lambda \in \Lambda_J$ transforms as $\pi H_J(e_u, \psi_* \lambda) = \pi H_J(\psi_*^{-1} e_u, \lambda) = \bar{\zeta}_u\, \pi H_J(e_u, \lambda)$. Because $e_u \neq 0$, H_J is positive definite, and Λ_J spans $\mathrm{Lie}(J)$ over $\mathbb{R}$, the period functional $\lambda \mapsto \pi H_J(e_u, \lambda)$ is not identically zero on Λ_J, so $[d(D_{e_u} \log \vartheta_O)]$ is a nonzero element of the $\bar{\zeta}_u$-eigenspace of ψ^* in $H^1_{\mathrm{dR}}(J/\mathbb{C})$ (which is genuinely of the second kind since $\bar{\zeta}_u$ does not occur on $H^0(J_{\mathbb{C}}, \Omega^1_{J_{\mathbb{C}}})$). By the one-dimensional eigenspace hypothesis, $[\rho_u] = c_u [d(D_{e_u} \log \vartheta_O)]$ for some $c_u \in \mathbb{C}^\times$. Since both η_{O,ρ_u} and $c_u\, d(D_{e_u} \log \vartheta_O)$ belong to $H^0(J_{\mathbb{C}}, \Omega^1_{J_{\mathbb{C}}}(2\Theta_O))_{\mathrm{cl}}$ and represent the same cohomology class, injectivity in Proposition 3.1 yields $\eta_{O,\rho_u} = c_u\, d(D_{e_u} \log \vartheta_O)$. □

Proposition 3.3 (Covariant translated theta gauge and uniform metric bounds)**.** *Keep the setting of Proposition 3.1, and choose a complex lift $v_\gamma \in \mathrm{Lie}(J)$ of $Q_\gamma \in J(\mathbb{C}) \setminus \Theta_O$ for each $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$. For each $v = v_\gamma$, define on $\mathrm{Lie}(A)$:*

$$\begin{aligned} Q(z) &:= \tfrac{\pi}{2} H_J(i_* z, i_* z), \\ f_v^{\mathrm{norm}}(z) &:= \exp(-\pi H_J(i_* z, v))\, \frac{\vartheta_O(i_* z + v)}{\vartheta_O(v)}, \\ g_0^{(v)}(z) &:= e^{-Q(z)} |f_v^{\mathrm{norm}}(z)|. \end{aligned}$$

Write $f_v := f_v^{\mathrm{norm}}$ and let $s_r^{(\gamma)} := \gamma(s_r)$ $(0 \le r \le N_s)$ be the conjugate meromorphic basis functions of $H^0(A, 3L_{Q_\gamma})$. Fix any finite set of primitives $I^{(v_\gamma)}$ on $\mathrm{Lie}(A)$ normalized to vanish at 0, each being either a first-kind linear functional $z \mapsto \ell_{\mathrm{lin}}(z)$ or a fixed linear combination of first-kind functionals and second-kind primitives

$$Z_e^{(v_\gamma)}(z) := D_e \log \vartheta_O(i_* z + v_\gamma) - D_e \log \vartheta_O(v_\gamma) \qquad (e \in \mathrm{Lie}(J)).$$

Fix also a local inverse $z = \ell(\mathbf{e})$ of the uniformization $\mathrm{Lie}(A) \to A(\mathbb{C})$ at the origin 0_A in fixed regular local parameters $\mathbf{e}$. Then:

(i) For every $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$, $\vartheta_O(v_\gamma) \neq 0$, $f_v(0) = 1$, and the Hermitian-normalized modulus $g_0^{(v)}(z)$ is Λ_A-periodic (whereas $f_v(z)$ itself is entire and not Λ_A-periodic).

(ii) There exist positive constants M_0, M_A, M_1, depending only on the fixed data J, A, O, Q_0, F, $\{v_\gamma\}$, $\{s_r\}$, and $\{I^{(v_\gamma)}\}$ and independent of all Diophantine and interpolation parameters $(N_{\mathrm{box}}, H, w_i, p_i/q_i)$, such that for every $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$, every $0 \le r \le N_s$, and every $z \in \mathrm{Lie}(A)$,

$$g_0^{(v)}(z) \le M_0, \qquad e^{-3Q(z)} |f_v(z)^3 s_r^{(\gamma)}(z)| \le M_A,$$
$$e^{-Q(z)} |f_v(z) I^{(v)}(z)| \le M_1(1 + \|z\|_\infty).$$

(iii) There exist $r_{\mathrm{an}} \in (0,1)$ and $c_0 > 0$, depending only on the fixed data $(J, A, O, Q_0, F, \{v_\gamma\}, \ell)$, such that for every $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$, every $\|\mathbf{e}\|_\infty \le r_{\mathrm{an}}$, and every $\lambda \in \Lambda_A$,

$$|f_v(\lambda + \ell(\mathbf{e}))|^{-1} \le c_0^{-1} e^{-Q(\lambda+\ell(\mathbf{e}))} \le c_0^{-1}.$$

Proof. (i) By Proposition 3.1, $Q_\gamma \notin \Theta_O$, so $\vartheta_O(v_\gamma) \neq 0$ and $f_v(0) = 1$. For $\lambda \in \Lambda_A$, taking absolute values in (1) at $i_*z + v$ shifted by $i_*\lambda \in \Lambda_J$ gives

$$|f_v(z+\lambda)| = |f_v(z)| \exp\Big(\pi \operatorname{Re}\big(H_J(i_*z+v, i_*\lambda) - H_J(i_*\lambda, v)\big) + \tfrac{\pi}{2} H_J(i_*\lambda, i_*\lambda)\Big).$$

By Hermitian symmetry $\operatorname{Re} H_J(v, i_*\lambda) = \operatorname{Re} H_J(i_*\lambda, v)$, so the v-dependent cross terms cancel identically, leaving $|f_v(z+\lambda)| = |f_v(z)| \exp(Q(z+\lambda) - Q(z))$. Multiplying by $e^{-Q(z+\lambda)}$ yields $g_0^{(v)}(z+\lambda) = g_0^{(v)}(z)$.

(ii) Compactness of the real torus $\mathrm{Lie}(A)/\Lambda_A$ and continuity of $g_0^{(v)}$ yield $M_0 := \max_\gamma \sup_{\mathrm{Lie}(A)} g_0^{(v_\gamma)} < \infty$. Because $s_r^{(\gamma)}$ has pole divisor bounded by $3L_{Q_\gamma} = 3\,\mathrm{div}(f_v)$, the product $f_v^3 s_r^{(\gamma)}$ is entire on $\mathrm{Lie}(A)$, and $e^{-3Q(z)}|f_v(z)^3 s_r^{(\gamma)}(z)| = g_0^{(v)}(z)^3 |s_r^{(\gamma)}(z)|$ is continuous and Λ_A-periodic, giving a finite maximum M_A over all γ, r. Similarly, $f_v(z) Z_e^{(v)}(z)$ is entire on $\mathrm{Lie}(A)$ (the simple pole of $D_e \log \vartheta_O(i_*z + v)$ along $\mathrm{div}(f_v)$ is cleared by f_v). Writing any $z \in \mathrm{Lie}(A)$ as $z = z_0 + \lambda$ with z_0 in a fixed compact fundamental parallelepiped $\mathcal{P}$ and $\lambda \in \Lambda_A$ (so $\|\lambda\|_\infty \le C_{\mathcal{P}}(1 + \|z\|_\infty)$ for a constant $C_{\mathcal{P}}$ depending only on $\mathcal{P}$), (1) gives $Z_e^{(v)}(z_0+\lambda) = Z_e^{(v)}(z_0) + \pi H_J(e, i_*\lambda)$. Using $e^{-Q(z)}|f_v(z)| = g_0^{(v)}(z) = g_0^{(v)}(z_0)$ by Λ_A-periodicity and compactness of $\mathcal{P}$,

$$e^{-Q(z)}|f_v(z) Z_e^{(v)}(z)| \le e^{-Q(z_0)}|f_v(z_0) Z_e^{(v)}(z_0)| + g_0^{(v)}(z_0)\,\pi |H_J(e, i_*\lambda)| \le M_1'(1 + \|z\|_\infty)$$

after absorbing $C_{\mathcal{P}}$ into M_1'. The same linear bound holds trivially for any first-kind linear functional $\ell_{\mathrm{lin}}(z)$ since $g_0^{(v)}(z) \le M_0$, and hence for any finite linear combination, giving M_1.

(iii) Since $g_0^{(v_\gamma)}(0) = 1$ for each of the finitely many $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$ and $\ell(0) = 0$, shrinking $r_{\mathrm{an}} \in (0,1)$ inside a common polydisk of convergence of ℓ yields $c_0 := \min_\gamma \inf_{\|\mathbf{e}\|_\infty \le r_{\mathrm{an}}} g_0^{(v_\gamma)}(\ell(\mathbf{e})) > 0$. By Λ_A-periodicity of $g_0^{(v_\gamma)}$ and positivity $Q \ge 0$, for every $\lambda \in \Lambda_A$ and $\|\mathbf{e}\|_\infty \le r_{\mathrm{an}}$, $c_0 \le g_0^{(v_\gamma)}(\lambda + \ell(\mathbf{e})) = e^{-Q(\lambda+\ell(\mathbf{e}))}|f_v(\lambda + \ell(\mathbf{e}))|$, which rearranges to (iii). □

Lemma 3.4 (Totally imaginary Galois compositum and product formula)**.** *Inside a fixed algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$, let K be any CM field (in particular $K = K_b = \mathbb{Q}(\zeta_b)$ for $b \ge 3$, or an imaginary quadratic field) and let $F/\mathbb{Q}$ be any finite extension (in our applications, a finite Galois extension). Then the compositum $\mathbb{L} := KF$ is totally imaginary with $r_{\mathbb{L}} := [\mathbb{L} : \mathbb{Q}]/2$ complex conjugate pairs of embeddings $\{\iota_k, \bar{\iota}_k\}_{k=1}^{r_{\mathbb{L}}}$ (regardless of $K \cap F$), and every nonzero algebraic integer $x \in \mathcal{O}_{\mathbb{L}} \setminus \{0\}$ satisfies*

$$1 \le |N_{\mathbb{L}/\mathbb{Q}}(x)| = \prod_{k=1}^{r_{\mathbb{L}}} |\iota_k(x)|^2.$$

Proof. Every field embedding $\iota : \mathbb{L} \hookrightarrow \mathbb{C}$ restricts to a non-real embedding of the totally imaginary field K, so $\bar{\iota} \neq \iota$ and complex conjugation pairs the $[\mathbb{L} : \mathbb{Q}]$ embeddings without fixed points. Since $N_{\mathbb{L}/\mathbb{Q}}(x) \in \mathbb{Z} \setminus \{0\}$, $|N_{\mathbb{L}/\mathbb{Q}}(x)| \ge 1$, giving the product bound. □

For an irrational real number x, its irrationality exponent $\mu(x)$ is the supremum of real ν for which $|x - p/q| < q^{-\nu}$ admits infinitely many solutions with $q \in \mathbb{Z}_{>0}$ and $\gcd(p, q) = 1$.

Theorem 3.5 (Master multivariable determinant theorem with zero degree penalty)**.** *Keep the setting of Propositions 3.1 and 3.3. Assume that $A \hookrightarrow J$ is an abelian subvariety defined over $\mathbb{Q}$, absolutely simple of dimension $g = \varphi(b)/2$, equipped with a ring embedding $\mathbb{Z}[\zeta_b] \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}}(A)$ ($b \geq 3$). Suppose $[\beta] \in H^1_{\mathrm{dR}}(A/\mathbb{Q})$ is a $\mathbb{Q}$-rational de Rham class (of the first or second kind) whose base change $[\beta] \otimes 1 \in H^1_{\mathrm{dR}}(A/\mathbb{C})$ is a simultaneous eigenvector for the induced action of $\mathbb{Z}[\zeta_b]$ (note that for $b \geq 3$, the endomorphism $[\zeta_b]$ cannot be defined over $\mathbb{Q}$: if both $[\zeta_b]$ and $[\beta]$ were defined over $\mathbb{Q}$, the eigenvalue $\sigma_k(\zeta_b)$ would be $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant, which fails for $b \geq 3$), and suppose there exists $\lambda_0 \in H_1(A, \mathbb{Z})$ with nonzero real period $\mathcal{T} = \int_{\lambda_0} \beta \in \mathbb{R}^\times$. Then $\mu(\mathcal{T}) = 2$, and $\mu(r\mathcal{T}) = 2$ for every nonzero rational $r \in \mathbb{Q}^\times$.*

Proof. Because $H_1(A, \mathbb{C}) \hookrightarrow H_1(J, \mathbb{C})$ is injective, the restriction map $H^1_{\mathrm{dR}}(J/\mathbb{Q}) \to H^1_{\mathrm{dR}}(A/\mathbb{Q})$ is surjective (after $-\otimes_{\mathbb{Q}} \mathbb{C}$, hence over $\mathbb{Q}$). Choose a $\mathbb{Q}$-rational lift $\widetilde{\beta} \in H^1_{\mathrm{dR}}(J/\mathbb{Q})$ of $[\beta]$; Proposition 3.1 provides canonical conjugate representatives $\beta_\gamma := i^*_{Q_\gamma} \eta_{O,\widetilde{\beta}}$ ($\gamma \in \mathrm{Gal}(F/\mathbb{Q})$) with poles bounded by $2L_{Q_\gamma}$, all representing $[\beta]$ on A (for a first-kind class one may alternatively use the global invariant form on A directly). To deduce transcendence of $\mathcal{T}$ directly from the curve statement of Huber–Wüstholz [16, Corollary 13.13] (and classically [32] for Beta values), pull back $\eta_{O,\widetilde{\beta}}$ along the translated Abel–Jacobi embedding $j_{O,Q_0} : C \to J$, $P \mapsto [P - O] + Q_0$ (whose image is not contained in Θ_O because $j_{O,Q_0}(O) = Q_0 \notin \Theta_O$), obtaining a meromorphic differential $j^*_{O,Q_0} \eta_{O,\widetilde{\beta}}$ on $C_{\overline{\mathbb{Q}}}$. Lifting $i_*\lambda_0$ along $H_1(C, \mathbb{Z}) \xrightarrow{\sim} H_1(J, \mathbb{Z})$ to a closed 1-cycle on C avoiding the finitely many poles of $j^*_{O,Q_0} \eta_{O,\widetilde{\beta}}$ reproduces the nonzero period $\mathcal{T}$, so [16, Corollary 13.13] implies that $\mathcal{T}$ is transcendental, and Dirichlet's approximation theorem gives $\mu(\mathcal{T}) \geq 2$. Multiplying $\mathcal{T}$ by $r = c/d \in \mathbb{Q}^\times$ scales approximating denominators by at most $|c|$ or $|d|$, preserving $\mu(\mathcal{T})$ in both directions.

To prove $\mu(\mathcal{T}) \leq 2$, fix any real $\nu > 2$ and assume for contradiction that $|\mathcal{T} - p/q| < q^{-\nu}$ has infinitely many reduced solutions p/q ($q \geq 2$, $p \neq 0$). Let $\sigma_k : K_b \hookrightarrow \mathbb{C}$ be the embedding through which $\mathbb{Z}[\zeta_b]$ acts on $[\beta]$. For any integer $N_{\mathrm{box}} \geq 2$, the $K_{\mathrm{cen}} := N_{\mathrm{box}}^{2g}$ cyclotomic integers $\xi_j := \sum_{a=0}^{2g-1} c_a \zeta_b^a$ ($0 \leq c_a < N_{\mathrm{box}}$) are pairwise distinct and satisfy $|\tau(\xi_j)| \leq 2gN_{\mathrm{box}}$ for every $\tau \in \mathrm{Gal}(K_b/\mathbb{Q})$. Set $\mathbb{L} := K_b F$ (with $r_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}]/2$ complex places by Lemma 3.4). Given $m = g\ell$ rational approximants p_i/q_i ($1 \leq i \leq m$) to be chosen below with $w_i := \lceil \log q_i \rceil$, set the centres at $P_0 = 0_A$ to be $c_{ji} := \sigma_k(\xi_j) p_i/q_i \in \mathbb{L}$ (pairwise distinct in each coordinate since $p_i \neq 0$).

Fix a reference inclusion $\mathbb{L} \hookrightarrow \mathbb{C}$. Any embedding $\iota : \mathbb{L} \hookrightarrow \mathbb{C}$ restricts to an automorphism $\tau \in \mathrm{Gal}(K_b/\mathbb{Q})$ on K_b and $\gamma \in \mathrm{Gal}(F/\mathbb{Q})$ on F. Because $\mathrm{Gal}(K_b/\mathbb{Q})$ is abelian, $\tau\sigma_k = \sigma_k\tau$. Since β_γ represents the $\mathbb{Q}$-rational class $[\beta]$, the integral cycle $\lambda_j^{(\iota)} := [\tau(\xi_j)]_* \lambda_0 \in H_1(A, \mathbb{Z})$ satisfies the *common-period identity*

$$\int_{\lambda_j^{(\iota)}} \beta_\gamma = \sigma_k(\tau(\xi_j))\mathcal{T} = \iota(\sigma_k(\xi_j))\mathcal{T}, \tag{2}$$

and its lattice norm satisfies $\|\lambda_j^{(\iota)}\|_\infty \leq C_{\mathrm{lat}} N_{\mathrm{box}}$ uniformly over all j, ι (each $\tau \in \mathrm{Gal}(K_b/\mathbb{Q})$ sends the basis elements ζ_b^a into the fixed finite set $\{\zeta_b^r : 0 \leq r < b\}$, whose induced endomorphisms act linearly on Λ_A with bounded operator norms).

Fix $\mathbb{Q}$-rational regular local parameters $\mathbf{e}$ at $0_A \in A$. Notice that the finite family of conjugate charts $(s_r^{(\gamma)}, \beta_\gamma)$ and analytic primitives $I^{(v_\gamma)}$ ($\gamma \in \mathrm{Gal}(F/\mathbb{Q})$) is fixed once and for all before choosing ℓ, $m = g\ell$, or any approximants p_i/q_i: all m additive coordinates $X_1, \ldots, X_m$ share repeated copies of this same fixed primitive β_γ truncated at orders T_i, rather than m independently varying analytic germs. In particular, the metric and radius constants $M_0, M_A, M_1, M_*, c_0, r_{\mathrm{an}}$ of Proposition 3.3, the lattice constant C_{lat}, the quadratic-form growth constant in $G_R = O(R^2)$, and the Hensel denominator base B_{den} below depend solely on the fixed geometric data $(J, A, O, Q_0, F, \{v_\gamma\}, \{s_r\}, \widetilde{\beta}, \mathbf{e})$ and are completely independent of ℓ, m, N_{box}, (w_i), and H. Apply Lemma 2.9 simultaneously at the finitely many conjugate charts $(s_r^{(\gamma)}, \beta_\gamma)$

(rescaling $s_1, \dots, s_{N_s}$ by a single positive integer if needed) to obtain one integer $B_{\mathrm{den}} \geq 2$ (note that if all Taylor coefficients of $s_r^{(\gamma)}$ and β_γ are already $\mathcal{O}_{\mathbb{L}}$-integral, one may set $B_{\mathrm{den}} = 1$ in the denominator formula below). For $T_i := \lfloor F_0 w_i / v_0 \rfloor + 1$ and $L_i := \mathrm{lcm}(1, \dots, T_i - 1) \leq 4^{T_i}$, let $G_i^{(\gamma)} \in \mathbb{L}[\mathbf{e}]$ be the truncated primitive of β_γ of total degree $< T_i$ with $G_i^{(\gamma)}(0) = 0$. Put $v_i := w_i/\theta$ and $w_* := \min_{1 \leq i \leq m} w_i$.

Once the parameters $(m, N_{\mathrm{box}}, w_0, v_0, F_0, \theta, \delta_{\mathrm{sl}})$ and separated weights $w_i > W_i$ are chosen as below, Theorem 2.2 (over $\mathbb{L}$, applied to the reference chart β_{id}, whose periods on A° factor through $H_1(A, \mathbb{Z})$ since $\beta_{\mathrm{id}} = i_{Q_0}^* \eta_{O, \widetilde{\beta}}$ represents the global class $[\beta]$ and has nonzero period $\mathcal{T}$) together with normal generation of $D = 3L_{Q_0}$ yields, for every admissible degree $H = nR_{\mathrm{wt}}$ ($n \geq n_0$), a nonsingular $M \times M$ minor $\Delta_{\mathrm{raw}} \in \mathbb{L}^\times$ whose columns are monomials $P = (\prod_{r=1}^{N_s} s_r^{\beta_r}) X^\alpha$ of cost $w_0 \sum_r \beta_r + w \cdot \alpha \leq H$ and whose rows $\rho = (j, \mathbf{n}, \mathbf{d})$ ($1 \leq j \leq K_{\mathrm{cen}}$, $v_0|\mathbf{n}| + v \cdot \mathbf{d} < H$) extract the Taylor coefficient $[\mathbf{e}^{\mathbf{n}} U^{\mathbf{d}}]$ after substituting $X_i = c_{ji} + G_i^{(\mathrm{id})}(\mathbf{e}) + U_i$. Counting integer points in the dilated jet simplex gives

$$M = \frac{K_{\mathrm{cen}} H^{m+g}}{(m+g)!\, v_0^g \prod_{i=1}^m v_i} (1 + o_H(1)),$$
$$\bar{d} := \frac{1}{MH} \sum_\rho w \cdot \mathbf{d}_\rho \in [0, \theta].$$

Arithmetic lower bound. Multiply column P by $\prod_{i=1}^m q_i^{\alpha_i}$ and row $\rho = (j, \mathbf{n}, \mathbf{d})$ by $B_{\mathrm{den}}^{|\mathbf{n}|} D_H \prod_{i=1}^m q_i^{-d_i}$, where $D_H := \prod_{i=1}^m (L_i B_{\mathrm{den}}^{T_i})^{\lfloor H/w_i \rfloor}$. A nonzero contribution to entry (ρ, P) requires $\alpha \geq \mathbf{d}$, carries at most $\alpha_i - d_i$ factors q_i^{-1} from c_{ji}, and uses at most $\lfloor H/w_i \rfloor$ primitive factors $G_i^{(\mathrm{id})}$ (each cleared by $L_i B_{\mathrm{den}}^{T_i}$ by Lemma 2.9), while the section coefficients of total base degree $a \leq |\mathbf{n}|$ are cleared by $B_{\mathrm{den}}^{|\mathbf{n}|}$. Hence the scaled minor $\widetilde{\Delta} := S_H \Delta_{\mathrm{raw}}$ lies in $\mathcal{O}_{\mathbb{L}} \setminus \{0\}$, where $S_H > 0$ is a positive rational number independent of ι. Writing $\log S_H/(MH) = 1 - \bar{d} + E_{\mathrm{ar}}^{\mathrm{eff}}$ and using $w_i - 1 < \log q_i \leq w_i$, $L_i \leq 4^{T_i}$, $T_i \leq F_0 w_i / v_0 + 1$, and $\sum_i d_i \leq \theta H / w_*$ yields

$$E_{\mathrm{ar}}^{\mathrm{eff}} \leq E_{\mathrm{ar}},$$
$$E_{\mathrm{ar}} := \frac{m F_0 \log(4B_{\mathrm{den}}) + \log B_{\mathrm{den}}}{v_0} + \log(4B_{\mathrm{den}}) \sum_{i=1}^m w_i^{-1} + \frac{\theta}{w_*}.$$

Setting $u_\iota := \log |\iota(\Delta_{\mathrm{raw}})| / (MH)$, Lemma 3.4 yields

$$0 \leq \sum_{k=1}^{r_{\mathbb{L}}} 2(u_{\iota_k} + 1 - \bar{d} + E_{\mathrm{ar}}^{\mathrm{eff}}). \tag{3}$$

Because the centres $c_{ji} \in \mathbb{L}$, basis sections $s_r^{(\gamma)} = \gamma(s_r)$, and truncated primitives $G_i^{(\gamma)} = \gamma(G_i^{(\mathrm{id})})$ are Galois-conjugate data and the same row/column index sets are used across all embeddings, the determinant evaluated in the conjugate chart indexed by ι equals $\iota(\Delta_{\mathrm{raw}})$ identically.

Analytic upper bound (uniform across all ι). Fix an embedding $\iota : \mathbb{L} \hookrightarrow \mathbb{C}$ (restricting to τ, γ) and write $z_j := \lambda_j^{(\iota)} + \ell(\mathbf{e})$ for $\|\mathbf{e}\|_\infty \leq r_{\mathrm{an}}$, where $r_{\mathrm{an}} \in (0, 1)$ and $c_0 > 0$ come from Proposition 3.3(iii). Abbreviate the analytic primitive of β_γ normalized at 0 by $I_u := I^{(v_\gamma)}$, and put $\mathcal{R}_i(\mathbf{e}) := G_i^{(\gamma)}(\mathbf{e}) - I_u(\ell(\mathbf{e}))$. By the common-period identity (2),

$$\iota(c_{ji}) + G_i^{(\gamma)}(\mathbf{e}) + U_i = I_u(z_j) + \epsilon_{ji} + \mathcal{R}_i(\mathbf{e}) + U_i, \qquad |\epsilon_{ji}| = |\iota(\sigma_k(\xi_j))| \cdot |\mathcal{T} - p_i/q_i| \leq 2g N_{\mathrm{box}} e^{\nu} e^{-\nu w_i}.$$

Each remainder $\mathcal{R}_i$ vanishes to total order $\geq T_i$ at $\mathbf{e} = 0$. Shrinking the fixed radius r_{an} strictly inside a common polydisk of analyticity of all $I^{(v_\gamma)} \circ \ell$ ensures that the homogeneous degree-n Cauchy majorant is at most a fixed constant times $\binom{n+g-1}{g-1} t_{\mathrm{rad}}^n$ with $t_{\mathrm{rad}} < 1$, so $|\mathcal{R}_i(\mathbf{e})| \leq 1$ on $\|\mathbf{e}\|_\infty \leq r_{\mathrm{an}}$ for all sufficiently large T_i.

Set $R := \max(1, 2\sup_{j,\iota,\|\mathbf{e}\|_\infty \le r_{\mathrm{an}}} \|z_j\|_\infty) = O(N_{\mathrm{box}})$, $G_R := \sup_{\|z\|_\infty \le R} Q(z) = O(R^2)$, $M_* := \max(1, M_0, M_A^{1/3})$, and $C_\zeta(R) := \max(2M_0, M_1(1+R))$. Using the *common* clearing exponent across all columns,

$$E_L := 3\lfloor H/w_0 \rfloor + \lfloor H/w_* \rfloor, \qquad P_R := \frac{3\log M_* + 3G_R}{w_0} + \frac{\log(2C_\zeta(R)) + G_R}{w_*},$$

define for each column P and $\mathbf{a} \in \mathbb{N}^m$ (with $\mathbf{a} \le \alpha$, else 0):

$$F_{\mathbf{a},P}(z) := f_{v_\gamma}(z)^{E_L}\,[U^{\mathbf{a}}]P\big(s^{(\gamma)}(z), (I_u(z) + U_i)_{i=1}^m\big).$$

Each of the $B_s := \sum_r \beta_r \le \lfloor H/w_0 \rfloor$ section factors $s_r^{(\gamma)}$ consumes 3 powers of f_{v_γ} with $e^{-3Q}|f_{v_\gamma}^3 s_r^{(\gamma)}| \le M_A \le M_*^3$, the remaining $3(\lfloor H/w_0 \rfloor - B_s)$ powers from the $3\lfloor H/w_0 \rfloor$ pool satisfy $e^{-Q}|f_{v_\gamma}| \le M_0 \le M_*$, each primitive I_u consumes 1 power of f_{v_γ} with $e^{-Q}|f_{v_\gamma} I_u| \le M_1(1+R) \le C_\zeta(R)$ (and unused powers from the $\lfloor H/w_* \rfloor$ pool satisfy $e^{-Q}|f_{v_\gamma}| \le M_0 \le C_\zeta(R)$), and $\binom{\alpha}{\mathbf{a}} \le 2^{|\alpha|} \le 2^{\lfloor H/w_* \rfloor}$. By Proposition 3.3(ii), $F_{\mathbf{a},P}$ is entire on $\mathrm{Lie}(A)$ and bounded by e^{HP_R} on $\|z\|_\infty \le R$, so in its expansion $F_{\mathbf{a},P}(z) = \sum_{\mathbf{k}\in\mathbb{N}^g} c_{\mathbf{a},P,\mathbf{k}}(z/R)^{\mathbf{k}}$ every coefficient satisfies $|c_{\mathbf{a},P,\mathbf{k}}| \le e^{HP_R}$.

Expanding each row $\rho = (j, \mathbf{n}, \mathbf{d})$ by the binomial theorem writes the row as $\sum_{(\mathbf{a},\mathbf{k})} \Lambda_{\rho,\mathbf{a},\mathbf{k}}(c_{\mathbf{a},P,\mathbf{k}})_P$, where $\Lambda_{\rho,\mathbf{a},\mathbf{k}} = 0$ unless $\mathbf{a} \ge \mathbf{d}$ and $w \cdot \mathbf{a} \le H$, and otherwise

$$\Lambda_{\rho,\mathbf{a},\mathbf{k}} = [\mathbf{e}^{\mathbf{n}}]\left(f_{v_\gamma}(z_j)^{-E_L} \left(\frac{z_j}{R}\right)^{\mathbf{k}} \binom{\mathbf{a}}{\mathbf{d}} \prod_{i=1}^m (\epsilon_{ji} + \mathcal{R}_i(\mathbf{e}))^{a_i - d_i} \right).$$

In expanding $\prod_i(\epsilon_{ji} + \mathcal{R}_i)^{a_i-d_i}$, let s_i count the number of tail factors $\mathcal{R}_i$ chosen in index i. Because $\mathcal{R}_i$ vanishes to order $\ge T_i$, a nonzero contribution to $[\mathbf{e}^{\mathbf{n}}]$ requires $\sum_i s_i T_i \le |\mathbf{n}| < H/v_0$, hence $w \cdot \mathbf{s} < H/F_0$ (since $T_i > F_0 w_i / v_0$). Applying Cauchy's coefficient bound on $\|\mathbf{e}\|_\infty \le r_{\mathrm{an}}$, together with Proposition 3.3(iii) ($|f_{v_\gamma}(z_j)^{-E_L}| \le c_0^{-E_L}$), $\|z_j/R\|_\infty \le 1/2$, $|\mathcal{R}_i| \le 1$, and $\sum_{\mathbf{s}} \binom{\mathbf{a}-\mathbf{d}}{\mathbf{s}}\binom{\mathbf{a}}{\mathbf{d}} \le 4^{H/w_*}$, yields

$$|\Lambda_{\rho,\mathbf{a},\mathbf{k}}| \le r_{\mathrm{an}}^{-|\mathbf{n}|} c_0^{-E_L} 2^{-|\mathbf{k}|} e^{-\nu w\cdot(\mathbf{a}-\mathbf{d})} e^{\nu H/F_0} \big(8gN_{\mathrm{box}} e^{\nu}\big)^{H/w_*}.$$

By multilinearity of the determinant, any tuple $(\mathbf{a}_\rho, \mathbf{k}_\rho)_\rho$ with a repeated pair $(\mathbf{a}, \mathbf{k})$ produces two proportional rows and vanishes. On every injective assignment, Hadamard's inequality bounds the coefficient determinant by $M^{M/2} e^{MHP_R}$. Reserving $2^{-|\mathbf{k}|/2}$ ensures absolute summability over $\mathbf{k} \in \mathbb{N}^g$, while the number of valid $\mathbf{a}$ per row is $O(H^m)$, so counting factors and $M^{M/2}$ contribute $O_{m,g}(\log H/H)$ after dividing by MH.

For any rational cutoff $A_{\mathrm{cut}} > \theta$ and rational fraction $\varrho \in (0,1)$, set

$$D_{\mathrm{cut}}(H) := \#\{\mathbf{a} \in \mathbb{N}^m : w \cdot \mathbf{a} \le A_{\mathrm{cut}} H\}, \quad c_g := \frac{g(g!)^{1/g}\log 2}{2(g+1)}, \quad L_g(H) := \frac{\varrho^{1+1/g} M^{1/g}}{H D_{\mathrm{cut}}(H)^{1/g}}.$$

Since the number of $\mathbf{k} \in \mathbb{N}^g$ with $|\mathbf{k}| \le t$ is $\binom{t+g}{g} \le (t+g)^g/g!$, any n distinct vectors $\mathbf{k}_1, \dots, \mathbf{k}_n \in \mathbb{N}^g$ have $\sum_{s=1}^n |\mathbf{k}_s| \ge \frac{g(g!)^{1/g}}{g+1} n^{1+1/g} - gn$. If at least ϱM rows in an injective assignment satisfy $w \cdot \mathbf{a}_\rho \le A_{\mathrm{cut}} H$, grouping them by $\mathbf{a}$ (with group sizes $n_{\mathbf{a}}$, $\sum_{\mathbf{a}} n_{\mathbf{a}} \ge \varrho M$ over at most $D_{\mathrm{cut}}(H)$ groups), applying Hölder's inequality $\sum_{\mathbf{a}} n_{\mathbf{a}}^{1+1/g} \ge (\sum_{\mathbf{a}} n_{\mathbf{a}})^{1+1/g} D_{\mathrm{cut}}(H)^{-1/g}$, and multiplying by $-(\log 2)/2$ from the remaining factor $2^{-|\mathbf{k}|/2}$ gives a saving of at least $-c_g L_g(H) + O_g(1/H)$. In the complementary case, more than $(1-\varrho)M$ rows satisfy $w\cdot\mathbf{a}_\rho > A_{\mathrm{cut}} H$, so $\sum_\rho w\cdot(\mathbf{a}_\rho - \mathbf{d}_\rho) \ge MH(A_{\mathrm{cut}}(1-\varrho) - \bar{d})$. Combining both branches gives, for every embedding ι,

$$u_\iota \le \max\{-c_g L_g(H),\ -\nu(A_{\mathrm{cut}}(1-\varrho) - \bar{d})\} + E_{\mathrm{tr}} + E_{\mathrm{hol}} + O_{m,g}(\log H/H), \tag{4}$$

$$E_{\mathrm{tr}} := \frac{\nu}{F_0} + \frac{\log(1/r_{\mathrm{an}})}{v_0} + \frac{\log(8gN_{\mathrm{box}}) + \nu}{w_*},$$

$$E_{\mathrm{hol}} := P_R + \left(\frac{3}{w_0} + \frac{1}{w_*}\right)\log^+(1/c_0) + \frac{\log(1/r_{\mathrm{an}})}{v_0}$$

(conservatively retaining $\log(1/r_{\mathrm{an}})/v_0$ in both E_{tr} and E_{hol} to match the modular split of the companion Lean 4 engine, giving $2\log(1/r_{\mathrm{an}})/v_0$ in E_{base} below).

Non-circular parameter selection and unified slack budget. Given $\nu > 2$, choose rational $\theta \in (0,1)$ with $\nu > 1 + \theta^{-1/2}$, so that $\theta + (1-\theta)/\nu < \sqrt{\theta}$. Choose rationals $A_{\mathrm{cut}}, \beta_*$ with

$$\theta + \frac{1-\theta}{\nu} < A_{\mathrm{cut}} < \beta_*^g < \sqrt{\theta}.$$

Since $\beta_*^{2g} < \theta$, the open interval $(\beta_*^g/\theta, \min(1/\theta, 1/\beta_*^g))$ is nonempty; pick any rational C in it (automatically $C > 1$, $C\theta < 1$, $C\beta_*^g < 1$, $\beta_*^g < C\theta$). Choose rational $\varrho \in (0,1)$ small enough that

$$g_* := \nu(A_{\mathrm{cut}}(1-\varrho) - \theta) - (1-\theta) > 0.$$

Fix any slack split fraction $\rho_{\mathrm{sp}} \in (0,1)$ and any collision margin $\delta_{\mathrm{coll}} > 0$ (for the abstract proof below, set $\rho_{\mathrm{sp}} := 3/4$ and $\delta_{\mathrm{coll}} := \frac{1-\rho_{\mathrm{sp}}}{2} g_* = \frac{1}{8} g_*$; the genus-2 budget table in Appendix A instantiates $\rho_{\mathrm{sp}} = 99/100$ and $\delta_{\mathrm{coll}} = 19/200$). Choose rational F_0 with $F_0\theta \geq 2$ (as required by Theorem 2.2) and $\nu/F_0 < \rho_{\mathrm{sp}} g_*$. For $\ell \geq 1$ set

$$m := g\ell, \qquad N_{\mathrm{box}} := \lfloor C^{\ell/2} \rfloor, \qquad K_{\mathrm{cen}} := N_{\mathrm{box}}^{2g}, \qquad w_0 := \beta_*^{-g\ell}, \qquad v_0 := 2N_{\mathrm{box}}^2 \theta^\ell w_0.$$

For every $0 \leq d \leq g$, the interpolation volume ratio of Theorem 2.2 simplifies to $a_d = (N_{\mathrm{box}}^2 \theta^\ell)^{g-d}/2^d$ for $d < g$ and $a_g = 1/(2^g \cdot 3^g g!\, \chi(L))$ for $d = g$, all strictly less than 1 since $N_{\mathrm{box}}^2 \theta^\ell \leq (C\theta)^\ell < 1$. For $C^{\ell/2} \geq 2$, one has $N_{\mathrm{box}}^2 \geq C^\ell/4$, whence

$$\frac{m}{v_0} \leq 2g\ell \left(\frac{\beta_*^g}{C\theta}\right)^\ell \longrightarrow 0, \qquad \frac{R^2}{w_0} = O\big((C\beta_*^g)^\ell\big) \longrightarrow 0 \qquad (\ell \to \infty).$$

Furthermore, $D_{\mathrm{cut}}(H) = (A_{\mathrm{cut}} H)^m/(m! \prod_i w_i) \cdot (1 + o_H(1))$ yields the exact N_{box}-independent limit

$$\lim_{H\to\infty} L_g(H) = \frac{\varrho^{1+1/g} (\beta_*^g/A_{\mathrm{cut}})^\ell}{2\big((m+1)\cdots(m+g)\big)^{1/g}} \longrightarrow \infty \qquad (\ell \to \infty),$$

since $\beta_*^g/A_{\mathrm{cut}} > 1$. Split the total error $E_{\mathrm{ar}} + E_{\mathrm{tr}} + E_{\mathrm{hol}}$ into the w_i-independent *base-weight error*

$$E_{\mathrm{base}} := \frac{mF_0 \log(4B_{\mathrm{den}}) + \log B_{\mathrm{den}} + 2\log(1/r_{\mathrm{an}})}{v_0} + \frac{\nu}{F_0} + \frac{3\log M_* + 3G_R + 3\log^+(1/c_0)}{w_0}$$

and the w_i-dependent *remaining error*

$$E_{\mathrm{rem}} := \log(4B_{\mathrm{den}}) \sum_{i=1}^{m} w_i^{-1} + \frac{\theta + \log(8gN_{\mathrm{box}}) + \nu + \log(2C_\zeta(R)) + G_R + \log^+(1/c_0)}{w_*}.$$

Choose and freeze ℓ large enough that

$$E_{\mathrm{base}} < \rho_{\mathrm{sp}} g_* \qquad \text{and} \qquad c_g \lim_{H\to\infty} L_g(H) > 1 + E_{\mathrm{base}} + \frac{1-\rho_{\mathrm{sp}}}{2} g_* + \delta_{\mathrm{coll}}. \tag{5}$$

Fix a rational interpolation slack $\delta_{\mathrm{sl}} > 0$ satisfying the strict inequalities $(1+3\delta_{\mathrm{sl}})^{m+d} a_d < 1$ $(0 \leq d \leq g)$ and $q_{\mathrm{sl}}\theta < 1$ of Theorem 2.2, freeze $(m, N_{\mathrm{box}}, w_0, v_0, F_0, \theta, \delta_{\mathrm{sl}})$, and choose the m approximants p_i/q_i recursively so that each $w_i = \lceil \log q_i \rceil$ exceeds the threshold $W_i(w_1, \ldots, w_{i-1})$ of Theorem 2.2, makes T_i large enough that $|\mathcal{R}_i| \leq 1$, and satisfies a common lower bound making $E_{\mathrm{rem}} < \frac{1-\rho_{\mathrm{sp}}}{2} g_*$. Then

$$E_{\mathrm{ar}} + E_{\mathrm{tr}} + E_{\mathrm{hol}} = E_{\mathrm{base}} + E_{\mathrm{rem}} < \frac{1+\rho_{\mathrm{sp}}}{2} g_*.$$

Now let $H = nR_{\mathrm{wt}} \to \infty$ through admissible degrees. Putting $\Delta_{\mathrm{gap}} := \min(\frac{1-\rho_{\mathrm{sp}}}{4} g_*, \frac{1}{2}\delta_{\mathrm{coll}}) > 0$, for all sufficiently large H the first branch of (4) satisfies $-c_g L_g(H) + E_{\mathrm{tr}} + E_{\mathrm{hol}} + O(\log H/H) \le -(1-\bar{d}) - E_{\mathrm{ar}} - \Delta_{\mathrm{gap}}$ (since $1-\bar{d} \le 1$), and the second branch satisfies

$$\nu(A_{\mathrm{cut}}(1-\varrho) - \bar{d}) - (1-\bar{d}) = g_* + (\nu-1)(\theta - \bar{d}) \ge g_*,$$

hence $-\nu(A_{\mathrm{cut}}(1-\varrho)-\bar{d}) + E_{\mathrm{tr}} + E_{\mathrm{hol}} + O(\log H/H) \le -(1-\bar{d}) - E_{\mathrm{ar}} - \Delta_{\mathrm{gap}}$. Thus $u_\iota + 1 - \bar{d} + E_{\mathrm{ar}}^{\mathrm{eff}} \le -\Delta_{\mathrm{gap}} < 0$ uniformly over all $r_{\mathbb{L}}$ conjugate pairs, contradicting (3). Since $\nu > 2$ was arbitrary, $\mu(\mathcal{T}) = 2$. □

Two cleanly quantified variants of Theorem 3.5 are recorded below: Proposition 3.6 states an explicit criterion for non-rational algebraic de Rham classes (recorded for completeness and not invoked in Sections 4–5), while Proposition 3.7 gives a self-contained proof for rank-two imaginary quadratic CM orders on elliptic curves over $\mathbb{Q}$ and is applied directly to $\mathbb{Z}[\sqrt{-2}]$ (and hence $B(1/8, 3/8)$) in Section 5.

Proposition 3.6 (Quantified extension to algebraic de Rham classes)**.** *Keep the setting of Theorem 3.5, except allow $[\beta] \in H^1_{\mathrm{dR}}(A/F)$ to be defined over the finite Galois field $F/\mathbb{Q}$ rather than $\mathbb{Q}$, with canonical representatives $\beta_\gamma := i_{Q_\gamma}^* \eta_{O,\gamma(\widetilde{\beta})}$ of the conjugate classes $\gamma[\beta]$ ($\gamma \in \mathrm{Gal}(F/\mathbb{Q})$). Assume that there exists a constant $C_{\mathrm{lat}} > 0$, independent of N_{box}, ξ_j, and ι, such that for every integer $N_{\mathrm{box}} \ge 2$, every embedding $\iota : \mathbb{L} \hookrightarrow \mathbb{C}$ (restricting to γ on F), and every cyclotomic box multiplier ξ_j ($1 \le j \le N_{\mathrm{box}}^{2g}$), there exists an integral homology cycle $\lambda_j^{(\iota)} \in H_1(A, \mathbb{Z})$ satisfying*

$$\|\lambda_j^{(\iota)}\|_\infty \le C_{\mathrm{lat}} N_{\mathrm{box}} \qquad \text{and} \qquad \int_{\lambda_j^{(\iota)}} \beta_\gamma = \iota(\sigma_k(\xi_j))\mathcal{T}$$

for a common nonzero real number $\mathcal{T} \in \mathbb{R}^\times$ (transcendental by [16, Corollary 13.13]). Then $\mu(\mathcal{T}) = 2$.

Proof. Every step of the proof of Theorem 3.5 uses $\mathbb{Q}$-rationality of $[\beta]$ solely to construct integral cycles $\lambda_j^{(\iota)} \in H_1(A, \mathbb{Z})$ with uniform norm bound $\|\lambda_j^{(\iota)}\|_\infty \le C_{\mathrm{lat}} N_{\mathrm{box}}$ satisfying the common-period identity (2) for the Galois-conjugate representatives β_γ (which simultaneously ensure $G_i^{(\gamma)} = \gamma(G_i^{(\mathrm{id})})$ and hence that the minor evaluated in chart ι equals $\iota(\Delta_{\mathrm{raw}})$). Under the stated uniform hypothesis, the arithmetic bound (3), the analytic bound (4), and the parameter selection (5) carry over verbatim. □

Proposition 3.7 (Rank-two imaginary quadratic CM orders on elliptic curves over $\mathbb{Q}$)**.** *Let $E/\mathbb{Q}$ be an elliptic curve equipped with complex multiplication $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\kappa \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}}(E)$ by an imaginary quadratic order $\mathcal{O} \subset K$ (where $K = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field and $\kappa \in \mathcal{O} \setminus \mathbb{Z}$), and let $[\beta] \in H^1_{\mathrm{dR}}(E/\mathbb{Q})$ be a nonzero $\mathbb{Q}$-rational de Rham class (of the first or second kind) whose base change to $\mathbb{C}$ is an eigenvector for $\mathcal{O}$ (i.e., a nonzero rational invariant differential ω or a rational second-kind class $[\rho]$ with $\alpha^*[\rho] = \overline{\alpha}[\rho]$). If $\lambda_0 \in H_1(E, \mathbb{Z})$ has nonzero real period $\mathcal{T} = \int_{\lambda_0} \beta \in \mathbb{R}^\times$, then $\mu(\mathcal{T}) = 2$ (and $\mu(r\mathcal{T}) = 2$ for every $r \in \mathbb{Q}^\times$). In particular, this applies unconditionally to $E_8 : y^2 = 4(x^3 + 4x^2 + 2x)$ with CM order $\mathbb{Z}[\sqrt{-2}]$.*

Proof. Because $g = \dim E = 1$, E is an absolutely simple abelian variety over $\mathbb{Q}$. Choose an algebraic non-origin point $Q_0 \in E(\overline{\mathbb{Q}}) \setminus \{O\}$ (for instance a point with rational x-coordinate in a short Weierstrass model) defined over a finite Galois field $F/\mathbb{Q}$, and let $\mathbb{L} := KF$. By Lemma 3.4, $\mathbb{L}$ is totally imaginary with $r_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}]/2$ complex conjugate pairs. Note that $\overline{\kappa} = \mathrm{Tr}_{K/\mathbb{Q}}(\kappa) - \kappa \in \mathcal{O}$, so $\mathcal{O}$ is stable under complex conjugation $\mathrm{Gal}(K/\mathbb{Q}) = \{\mathrm{id}, \mathrm{conj}\}$.

Let $\sigma : K \hookrightarrow \mathbb{C}$ be the embedding through which $\mathcal{O}$ acts on $[\beta]$ (either id for the first kind or conj for the second kind). For each integer $N_{\mathrm{box}} \ge 2$, form the $K_{\mathrm{cen}} := N_{\mathrm{box}}^2$ distinct multipliers $\xi_j := a_0 + a_1\kappa \in \mathcal{O}$ ($0 \le a_0, a_1 < N_{\mathrm{box}}$). For both embeddings $\tau \in \mathrm{Gal}(K/\mathbb{Q})$, one

has $\tau(\xi_j) \in \mathcal{O}$ and $|\tau(\xi_j)| \le (1+|\kappa|)N_{\mathrm{box}}$. Since $[\beta] \in H^1_{\mathrm{dR}}(E/\mathbb{Q})$ is $\mathbb{Q}$-rational and $\mathrm{Gal}(K/\mathbb{Q})$ is abelian, for every embedding $\iota : \mathbb{L} \hookrightarrow \mathbb{C}$ restricting to τ on K and γ on F, the integral cycle $\lambda_j^{(\iota)} := [\tau(\xi_j)]_*\lambda_0 \in H_1(E,\mathbb{Z})$ satisfies

$$\int_{\lambda_j^{(\iota)}} \beta_\gamma = \sigma(\tau(\xi_j))\mathcal{T} = \iota(\sigma(\xi_j))\mathcal{T}, \qquad |\lambda_j^{(\iota)}| \le (1+|\kappa|)|\lambda_0|N_{\mathrm{box}}.$$

The $g = 1$ proof of Theorem 3.5 now applies verbatim with $K_{\mathrm{cen}} = N_{\mathrm{box}}^2$, $m = \ell$, $w_0 = \beta_*^{-\ell}$, $v_0 = 2N_{\mathrm{box}}^2\theta^\ell w_0$, replacing the cyclotomic multiplier bound $2gN_{\mathrm{box}}$ solely by $(1+|\kappa|)N_{\mathrm{box}}$ (so $|\epsilon_{ji}| \le (1+|\kappa|)N_{\mathrm{box}}e^{\nu}e^{-\nu w_i}$ and the w_*^{-1} term of E_{tr} becomes $(\log(4(1+|\kappa|)N_{\mathrm{box}})+\nu)/w_*$). Every volume ratio ($a_0 = N_{\mathrm{box}}^2\theta^\ell < 1$, $a_1 = 1/(6\chi(L)) < 1$), decay estimate ($m/v_0 \le 2\ell(\beta_*/(C\theta))^\ell \to 0$, $R^2/w_0 = O((C\beta_*)^\ell) \to 0$), and collision limit $\lim_{H\to\infty} L_1(H) = \varrho^2(\beta_*/A_{\mathrm{cut}})^\ell/(2(\ell+1)) \to \infty$ holds identically, giving $\mu(\mathcal{T}) \le 2$. Transcendence [16, Corollary 13.13] gives $\mu(\mathcal{T}) \ge 2$. For $E_8 : y^2 = 4(x^3+4x^2+2x)$, setting $v_1 = y/2$ and composing the 2-isogeny $(x,v_1) \mapsto (x+4+2/x,\, v_1(1-2/x^2))$ onto $v_2^2 = x_2^3 - 8x_2^2 + 8x_2$ with the inverse of the twist $(x,v_1) \mapsto (-2x, (-2)^{3/2}v_1)$ gives an endomorphism acting as $\sqrt{-2}$ on dx/v_1 (hence $\mathbb{Z}[\sqrt{-2}] \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}}(E_8)$, as also recorded in Theorem 5.4). □

Remark 3.8 (Fields of definition and Liouville rescalings). A real period alone does not imply (2). On the square CM curve $y^2 = 4x^3 - x$ let $\Omega_4^{\mathrm{sq}} > 0$ generate the real periods of $\omega = dx/y$. For the Liouville number $\xi_{\mathrm{L}} = \sum_{n\ge1} 10^{-n!}$, the complex holomorphic eigenform $(\xi_{\mathrm{L}}/\Omega_4^{\mathrm{sq}})\omega$ has real period ξ_{L}, yet $\mu(\xi_{\mathrm{L}}) = +\infty$. Even for the algebraic form $(1+\sqrt{2})\omega$, the Galois-conjugate form $(1-\sqrt{2})\omega$ has real period group $(1-\sqrt{2})\Omega_4^{\mathrm{sq}}\mathbb{Z}$, which does not contain $(1+\sqrt{2})\Omega_4^{\mathrm{sq}}$ in its rational span, so conjugate charts alone cannot supply the common-period identity. This explains why $\mathbb{Q}$-rationality of $[\beta]$ (or the explicit common-period hypothesis of Proposition 3.6) is indispensable.

Corollary 3.9 (Reciprocal transfer for Flint–Hills convergence)**.** *If $\beta > 0$, $\alpha = \beta/\pi$ is irrational, and $\mu(\pi/\beta) = 2$, then $\sum_{n\ge1} n^{-s}\sin^{-2}(n\beta)$ converges for every $s > 2$.*

Proof. Given $\varepsilon > 0$, put $\tau_{\mathrm{ex}} := 1+\varepsilon$. The hypothesis $\mu(1/\alpha) = 2$ implies $|h/\alpha - k| \ge ch^{-\tau_{\mathrm{ex}}}$ for all $h \ge 1$, $k \in \mathbb{Z}$, after decreasing $c > 0$ for finitely many denominators. For h the nearest integer to $n\alpha$, the identity $|n\alpha - h| = \alpha|h/\alpha - n|$ and $h \le (\alpha+1)n$ give $\|n\alpha\| \ge c_\varepsilon n^{-\tau_{\mathrm{ex}}}$ (absorbing the finitely many n with $h = 0$ by decreasing $c_\varepsilon > 0$).

On each dyadic block $N \le n < 2N$, the fractional parts $\{n\alpha\}$ are separated from one another and from 0 on $\mathbb{R}/\mathbb{Z}$ by $\delta_N := c_\varepsilon(2N)^{-\tau_{\mathrm{ex}}}$. Ordering the points on each semicircle $(0,1/2]$ and $[-1/2,0)$ by distance from 0, the k-th point on a given side lies at distance at least $k\delta_N$, so the j-th smallest distance overall is at least $\lceil j/2\rceil\delta_N$. Since $|\sin(\pi x)| \ge 2\|x\|$,

$$\sum_{N\le n<2N} \sin^{-2}(n\beta) \le (4\delta_N^2)^{-1}\sum_{j=1}^{N}\lceil j/2\rceil^{-2} \le \frac{\pi^2}{12}\delta_N^{-2}.$$

Choosing $\varepsilon = (s-2)/4$ makes the N-th dyadic block $O(N^{-(s-2)/2})$, and summing over $N = 2^k$ proves convergence. □

4 Hyperelliptic CM families, primitivity dichotomy, and symmetric Beta values

4.1 The cyclotomic type and its stabilizer

For $b \ge 3$ put $K_b = \mathbb{Q}(\zeta_b)$, $U_b = (\mathbb{Z}/b\mathbb{Z})^\times$, and $\Phi_b = \{k \in U_b : 0 < k < b/2\}$, using least positive representatives. Complex conjugation pairs k with $-k$, so $g = |\Phi_b| = \varphi(b)/2 \ge 1$.

Theorem 4.1 (Sharp hyperelliptic primitivity dichotomy)**.** *For every integer $b \geq 3$, the stabilizer $\mathrm{Stab}_{U_b}(\Phi_b) := \{a \in U_b : a\Phi_b = \Phi_b\}$ is trivial if and only if $b = 4$ or $4 \nmid b$. Equivalently, the cyclotomic new factor A_b of Proposition 4.2 is absolutely simple if and only if $b = 4$ or $4 \nmid b$. (This provides an elementary self-contained proof—and, in* `CompositeBetaCM`*, a machine-checked formalization of the residue-arithmetic dichotomy—of the classification established by Gallese–Goodson–Lombardo [11, Lemma 3.2.1].)*

Proof. For odd $b \geq 3$, any unit $a \in U_b$ with $b/2 < a < b$ sends $1 \in \Phi_b$ out of Φ_b. If $1 < a < b/2$, let $r \geq 0$ be the largest integer with $2^{r+1}a < b$. Then $k := 2^{r+1}$ is coprime to odd b and satisfies $2k \leq 2^{r+1}a < b$ (so $k \in \Phi_b$) and $b/2 < ak < b$ (by maximality of r), so a moves k out of Φ_b.

For $b = 2m_0$ with $m_0 \geq 3$ odd, reduction modulo m_0 gives an isomorphism $U_b \xrightarrow{\sim} U_{m_0}$. As k ranges over Φ_{m_0}, the integers $m_0 - 2k$ are distinct odd positive units strictly below m_0 and hence exhaust Φ_b by cardinality. Modulo m_0, this identifies Φ_b with $-2\Phi_{m_0}$. Since U_{m_0} is abelian, $\mathrm{Stab}_{U_b}(\Phi_b) \cong \mathrm{Stab}_{U_{m_0}}(-2\Phi_{m_0}) = \mathrm{Stab}_{U_{m_0}}(\Phi_{m_0}) = \{1\}$.

At $b = 4$, $\Phi_4 = \{1\}$ has trivial stabilizer. If $b = 4t \geq 8$, the non-identity unit $2t - 1 \in U_b$ sends every odd $k \in (0, 2t)$ to $(2t-1)k \equiv 2t - k \pmod{4t}$, which again lies in $(0, 2t)$, so $2t - 1 \in \mathrm{Stab}_{U_b}(\Phi_b) \setminus \{1\}$.

Because $K_b/\mathbb{Q}$ is Galois of degree $2g$, a CM type (K_b, Φ_b) is induced from a proper CM subfield $K' \subsetneq K_b$ if and only if $\mathrm{Stab}_{U_b}(\Phi_b)$ is nontrivial (a stabilized type is a union of cosets of its stabilizer H, descending to the fixed field K_b^H, on which complex conjugation exchanges the quotient type with its complement). By the CM simplicity criterion [24, Proposition 3.13 and Corollary 3.14], the full-degree new factor A_b constructed in Proposition 4.2 is absolutely simple if and only if (K_b, Φ_b) is primitive, i.e., $\mathrm{Stab}_{U_b}(\Phi_b) = \{1\}$. □

4.2 New factors, real cycles, and symmetric Beta values

Proposition 4.2 (Rational new projector, nonspecial points, and cleared cycles)**.** *For odd $b \geq 3$, consider the smooth projective models of $C_b^{\pm} : y^2 = 4x^b \pm 1$ of genus $g_{\mathrm{tot}} = (b-1)/2$. For $b = 2m_0$ with $m_0 \geq 3$ odd, consider the smooth projective models of $D_{2m_0} : y^2 = 1 - 4x^{2m_0}$ and $E_{2m_0}^{-} : y^2 = 4x^{2m_0} - 1$, of genus $g_{\mathrm{tot}} = m_0 - 1$. Let $\alpha(x, y) := (\zeta_b x, y)$ and $J := \mathrm{Jac}(C)$. The integral endomorphism*

$$N_b := \prod_{\substack{q \mid b \\ q \text{ prime}}} \left(q - \sum_{j=0}^{q-1} \alpha_*^{jb/q} \right) \in \mathbb{Z}[\alpha_*] \cap \mathrm{End}_{\mathbb{Q}}(J)$$

satisfies $N_b^2 = \mathrm{rad}(b) N_b$. Its image $A_b := N_b(J)$ is defined over $\mathbb{Q}$, has dimension $g = \varphi(b)/2$, admits multiplication by $\mathbb{Z}[\zeta_b]$ via $\zeta_b \mapsto \alpha_*$, and has CM type Φ_b. For every $k \in \Phi_b$, a nonzero rational multiple of $B(k/b, k/b)$ is a first-kind period of a $\mathbb{Q}$-rational eigenclass on A_b (from C_b^{+} or D_{2m_0}) and a nonzero rational multiple of $2\pi/B(k/b, k/b)$ is a second-kind period of a $\mathbb{Q}$-rational eigenclass on A_b (from C_b^{-} or $E_{2m_0}^{-}$).

Proof. The holomorphic basis forms $\omega_k := x^{k-1}dx/y$ $(1 \leq k \leq g_{\mathrm{tot}})$ satisfy $\alpha^*\omega_k = \zeta_b^k \omega_k$, and the antiholomorphic eigencharacters are their complex conjugates ζ_b^{-k}. For each prime $q \mid b$, the sum $\sum_{j=0}^{q-1} \zeta_b^{jkb/q}$ equals q if $q \mid k$ and 0 otherwise, so the eigenvalues of N_b on $H^1(J, \mathbb{C})$ are $\mathrm{rad}(b)$ on primitive indices $k \in U_b$ and 0 otherwise. Every cyclotomic Galois automorphism permutes the summands for each q, so Galois descent gives $N_b \in \mathrm{End}_{\mathbb{Q}}(J)$. On $A_b = N_b(J)$ the minimal polynomial of α_* is the b-th cyclotomic polynomial Φ_b^{cyc}, giving $\dim A_b = \varphi(b)/2$ and CM type Φ_b.

Put $\Theta_O = W_{g_{\mathrm{tot}}-1} - (g_{\mathrm{tot}} - 1)O$. At any finite point $P = (0, c)$ with $c \neq 0$, the vanishing orders of $\omega_1, \ldots, \omega_{g_{\mathrm{tot}}}$ are $0, \ldots, g_{\mathrm{tot}} - 1$, so $h^0(K_C - g_{\mathrm{tot}}P) = 0$ and Riemann–Roch gives $h^0(g_{\mathrm{tot}}P) = 1$; hence $[g_{\mathrm{tot}}(P - O)] \notin \Theta_O$ whenever $O \neq P$ (otherwise $g_{\mathrm{tot}}P$ would be linearly equivalent to $D_{\mathrm{eff}} + O$ with $D_{\mathrm{eff}} \geq 0$ of degree $g_{\mathrm{tot}} - 1$). For odd b take $O = \infty$ and $P = (0, 1)$

on C_b^+ (giving Q_0^+) or $P = (0, i)$ on C_b^- (giving Q_0^-). On D_{2m_0} take $O = (0, 1)$ and $P = (0, -1)$ (giving Q_0^+). On $E_{2m_0}^-$ take $O = \infty_+$ (where $y/x^{m_0} \to +2$) and $P = (0, i)$ (giving Q_0^-, with conjugate $\overline{Q_0^-}$ from $(0, -i)$); both avoid Θ_O (note that on $E_{2m_0}^-$, the function $z := y + 2x^{m_0}$ satisfies $z(2x^{m_0} - y) = 1$, so $z - i$ has divisor $m_0(P - O)$, whence $-Q_0^- = [P - O] \in \Theta_O$, confirming that Q_0^- and $\overline{Q_0^-}$ use distinct lifts v_+, v_-). In every case $L_{Q_0} = i_{Q_0}^* \Theta_O$ is effective, ample, and misses 0_A.

Write $a_b := 4^{-1/b}$ and $u := k/b$ for $k \in \Phi_b$. On C_b^+ (odd b), the function $(x+a_b)^b/(y-1)^2$ has divisor $2b((-a_b, 0) - (0, 1))$, so Abel's theorem gives a closed cycle from $2b$ times the upper-sheet path from $(-a_b, 0)$ to $(0, 1)$. Substituting $t = 4|x|^b$ and applying Legendre duplication gives its ω_k-integral as $(-1)^{k-1} \cdot 2 \cdot 4^{-u} B(u, 1/2) = (-1)^{k-1} B(u, u)$. On D_{2m_0}, the closed real oval over $[-a_b, a_b]$ has ω_k-integral $4 \int_0^{a_b} x^{k-1} dx/\sqrt{1 - 4x^b} = B(u, u)/m_0$ (since k is odd). Applying N_b (resp. $m_0 N_b$) produces an integral cycle $\lambda_0^+ \in H_1(A_b, \mathbb{Z})$ with ω_k-period $(-1)^{k-1} \operatorname{rad}(b) B(u, u)$ (resp. $\operatorname{rad}(b) B(u, u)$).

On either minus model (C_b^- or $E_{2m_0}^-$), define the rational 1-form $\rho_k := -(b - 2k) x^{b-1-k} dx/y$ (with $\alpha^* \rho_k = \zeta_b^{-k} \rho_k$). Direct differentiation using $y^2 = 4x^b - 1$ gives

$$\rho_k + \tfrac{1}{2}\, d(y/x^k) = \tfrac{k}{2}\, x^{-k-1} dx/y.$$

At odd-degree ∞ ($x = t^{-2}$), ρ_k expands in even powers of t times dt; at either even-degree infinity $\infty_\pm$ ($t = 1/x$), ρ_k expands in powers $t^{bn+k-m_0-1} dt$ ($n \geq 0$), and $0 < k < m_0$ excludes exponent -1. Thus every residue of ρ_k vanishes, so any cycle may be displaced off ∞ and $x = 0$. Integrating the exact-corrected form along twice the positive real ray $[a_b, \infty)$ via $t = 1/(4x^b)$ and duplication yields

$$k \int_{a_b}^{\infty} \frac{x^{-k-1}\, dx}{\sqrt{4x^b - 1}} = u 4^u B(1/2 + u, 1/2) = \frac{2\pi}{B(u, u)}.$$

For odd b, the two sheets over $[a_b, \infty)$ form a single closed oval through ∞, whose N_b-image $\lambda_0^- \in H_1(A_b, \mathbb{Z})$ has ρ_k-period $\operatorname{rad}(b) \cdot 2\pi/B(u, u)$. For $b = 2m_0$, put $\alpha' := \alpha^2$ and $j(x, y) := (x, -y)$. For the upper-sheet ray γ_{ray} from $(a_b, 0)$ to ∞_+, the chain $\Gamma_{\text{even}} := (1 - j_*)(1 - \alpha'_*)\gamma_{\text{ray}}$ is closed (since α' fixes $\infty_\pm$ and j fixes all finite branch points) and has ρ_k-integral $(1 - \zeta_{m_0}^{-k}) \cdot 2\pi/B(u, u)$. Setting $\alpha_0(z) := (\Phi_{m_0}^{\text{cyc}}(1) - \Phi_{m_0}^{\text{cyc}}(z))/(1 - z) \in \mathbb{Z}[z]$, the integral cycle $\lambda_0^- := \alpha_0(\alpha'_*) N_b(\Gamma_{\text{even}}) \in H_1(A_b, \mathbb{Z})$ has ρ_k-period $\operatorname{rad}(b) \Phi_{m_0}^{\text{cyc}}(1) \cdot 2\pi/B(u, u) \neq 0$. □

Proposition 4.3 (Level $b = 4$ square elliptic model)**.** *On the elliptic curve $E_4^{\text{sq}} : y^2 = 4x^3 - x$ over $\mathbb{Q}$, with CM automorphism $\alpha(x, y) = (-x, iy)$ generating $\mathbb{Z}[i]$, origin $O = \infty$, and non-origin rational chart point $Q_0 = (0, 0) \notin \Theta_O = \{O\}$, the rational differentials $\omega = dx/y$ and $\rho = -x\, dx/y$ are regular at Q_0 (where y is a local parameter with $x = -y^2 - 4y^6 + O(y^{10})$), satisfy $\alpha^* \omega = i\omega$ and $\alpha^* \rho = -i\rho$, and have nonzero real period and quasi-period along the unbounded real oval given by*

$$\Omega_4^{\text{sq}} := 2 \int_{1/2}^{\infty} \frac{dx}{\sqrt{4x^3 - x}} = \tfrac{1}{2} B(1/4, 1/4) = \sqrt{2}\, \Omega_4, \qquad \eta_4^{\text{sq}} := \int_{\lambda_{\text{real}}} \rho = \frac{2\pi}{B(1/4, 1/4)}.$$

(Here $\Omega_4 := \frac{1}{2} B(1/4, 1/2)$ denotes the period of $y^2 = 4(x^3 - x)$ used in Subsection 5.2.)

Proof. Substituting $t = (2x)^{-2}$ into the oval integral and applying Legendre duplication gives $\Omega_4^{\text{sq}} = 2^{-1/2} B(1/4, 1/2) = \frac{1}{2} B(1/4, 1/4) = \sqrt{2}\, \Omega_4$. The period lattice is $\Omega_4^{\text{sq}} \mathbb{Z}[i]$ with oriented basis $(\Omega_4^{\text{sq}}, i\Omega_4^{\text{sq}})$. For the Weierstrass quasi-period homomorphism $e(\lambda) = \int_\lambda (-x\, dx/y)$, the automorphism $\alpha^* \rho = -i\rho$ gives $e(i\Omega_4^{\text{sq}}) = -ie(\Omega_4^{\text{sq}})$. Legendre's relation $e(\Omega_4^{\text{sq}})(i\Omega_4^{\text{sq}}) - e(i\Omega_4^{\text{sq}})\Omega_4^{\text{sq}} = 2\pi i$ then yields $2i\, \Omega_4^{\text{sq}}\, e(\Omega_4^{\text{sq}}) = 2\pi i$, hence $e(\Omega_4^{\text{sq}}) = \pi/\Omega_4^{\text{sq}} = 2\pi/B(1/4, 1/4)$. □

Theorem 4.4 (Symmetric Beta measures and Flint–Hills convergence)**.** *For every reduced rational $a/b \in (0, 1/2)$ with $b = 4$ or $4 \nmid b$,*

$$\mu(B(a/b, a/b)) = \mu(\pi/B(a/b, a/b)) = 2,$$
$$\sum_{n\geq 1} \frac{1}{n^s \sin^2(nB(a/b, a/b))} < \infty \qquad (\forall s > 2).$$

(For the first-kind targets at the three elliptic levels $b \in \{3, 4, 6\}$, the equality $\mu(B(a/b, a/b)) = 2$ is also covered by the rational CM elliptic period theorem of [9].)

Proof. For $4 \nmid b$, Proposition 4.2 produces the $\mathbb{Q}$-defined new factor $A_b \hookrightarrow J$ of dimension $g = \varphi(b)/2$ with $\mathbb{Z}[\zeta_b]$-action, which is absolutely simple by Theorem 4.1. The restricted classes $[\omega_a]$ and $[\rho_a]$ in $H^1_{\mathrm{dR}}(A_b/\mathbb{Q})$ are $\mathbb{Q}$-rational CM eigenclasses (for σ_a and σ_{-a}, respectively) with nonzero real periods that are nonzero rational multiples of $B(a/b, a/b)$ and $2\pi/B(a/b, a/b)$ along the integral cycles $\lambda_0^{\pm} \in H_1(A_b, \mathbb{Z})$. For $b = 4$, Proposition 4.3 furnishes the same data on E_4^{sq} with $K_4 = \mathbb{Q}(i)$. Theorem 3.5 therefore yields $\mu(B(a/b, a/b)) = \mu(\pi/B(a/b, a/b)) = 2$. Finally, Corollary 3.9 transfers $\mu(\pi/B(a/b, a/b)) = 2$ to convergence of $\sum_{n\geq 1} n^{-s} \sin^{-2}(nB(a/b, a/b))$ for every $s > 2$. □

Explicit Baker–Kleinian coordinate realizations at $g = 2$ ($C_5^- : y^2 = 4x^5 - 1$) and $g = 3$ ($C_7^{\pm} : y^2 = 4x^7 \pm 1$), certified outward-rounded global theta enclosures for the constants (M_0, M_j, M_A, c_0) of Proposition 3.3, illustrative genus-2 scalar-budget additive dimensions, and composite/even-level high-precision arithmetic cross-checks are collected in Appendix A; none of the Diophantine theorems of this paper depends on those explicit realizations or numerical enclosures.

5 Fermat quotient curves and exhaustive classification of levels $b \in \{8, 12\}$

5.1 Fermat quotient curves and asymmetric Beta values, including $4 \mid b$

For any integer $b \geq 3$ (with the primitivity and Diophantine theorems Theorems 5.2–5.3 specializing to $b \geq 5$), call a pair of positive integers (r, s) *admissible at level b* if $\gcd(b, r, s) = 1$ and none of $r, s, r + s$ is divisible by b. Let $C_{b;r,s}$ be the smooth projective model of $y^b = x^r(1 - x)^s$, and set $d_0 := \gcd(b, r)$, $d_1 := \gcd(b, s)$, $d_\infty := \gcd(b, r + s)$, and $e_j := b/d_j$ ($j \in \{0, 1, \infty\}$). Let $\alpha(x, y) := (x, \zeta_b y)$ be the deck transformation over $K_b = \mathbb{Q}(\zeta_b)$. Write $r_u, s_u \in \{0, \ldots, b - 1\}$ for the least nonnegative residues of $ur, u\,s \bmod b$, and define

$$S_{\mathrm{hol}} := \{0 < u < b : r_u + s_u > b\}, \qquad \Phi_{b;r,s} := S_{\mathrm{hol}} \cap (\mathbb{Z}/b\mathbb{Z})^\times.$$

Note carefully the sign convention for the CM action: because $\alpha(x, y) = (x, \zeta_b y)$ acts on the holomorphic 1-form ω_u below by $\alpha^* \omega_u = \zeta_b^{-u} \omega_u$, the CM embedding is fixed $\mathbb{Z}[\zeta_b] \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}}(A_{b;r,s})$ by sending $\zeta_b \mapsto \alpha_*^{-1}$ (keeping α unchanged in the Pochhammer cycle formulas). Under this embedding, ζ_b acts on ω_u by $(\alpha^{-1})^* \omega_u = \zeta_b^{+u} \omega_u$ (and on ρ_u by $\zeta_b^{-u} \rho_u$), so the CM type of $A_{b;r,s}$ on $\mathrm{Lie}(A_{b;r,s}) \cong H^0(A_{b;r,s}, \Omega^1)^\vee$ is literally $+\Phi_{b;r,s}$ (dualization of a linear operator preserves its eigenvalues). The complex Beta integrals on Fermat curves are classical [13].

Theorem 5.1 (Eigen-spectrum, zero residues, and simultaneous double clearing)**.** *Let $b \geq 3$ and let (r, s) be admissible at level b. Then $C_{b;r,s}$ has genus $g_{\mathrm{tot}} = (b + 2 - d_0 - d_1 - d_\infty)/2$, and a $\mathbb{Q}$-basis of $H^1_{\mathrm{dR}}(C_{b;r,s}/\mathbb{Q})$ consists of the classes of ω_u, ρ_u ($u \in S_{\mathrm{hol}}$), where*

$$\omega_u := x^{\lfloor ur/b \rfloor}(1 - x)^{\lfloor u\,s/b \rfloor} \frac{dx}{y^u}, \qquad \rho_u := \omega_{b-u}.$$

Each ω_u is holomorphic, and every ρ_u ($u \in S_{\mathrm{hol}}$, including nonunits) has zero residues at all points of $C_{b;r,s}$. With $N_b(z) := \prod_{p\mid b}(p - \sum_{j=0}^{p-1} z^{jb/p})$, the image $A_{b;r,s} := N_b(\alpha_*)\,\mathrm{Jac}(C_{b;r,s})$ is defined over $\mathbb{Q}$, has dimension $g = \varphi(b)/2$, and admits multiplication by $\mathbb{Z}[\zeta_b]$ via $\zeta_b \mapsto \alpha_*^{-1}$ with CM type $\Phi_{b;r,s}$. Moreover, there is a single integral cycle $\lambda_0 \in H_1(A_{b;r,s},\mathbb{Z})$ such that, for every $u \in \Phi_{b;r,s}$, setting $h := r_u/b$, $k := s_u/b$, and the positive clearing integer $M_{b;r,s} := \mathrm{rad}(b)\Phi_{e_0}^{\mathrm{cyc}}(1)\Phi_{e_1}^{\mathrm{cyc}}(1)$,

$$\int_{\lambda_0} i^*\omega_u = M_{b;r,s}\,B(1-h,1-k), \qquad \int_{\lambda_0} i^*\rho_u = M_{b;r,s}\,B(h,k).$$

Proof. Over $\mathbb{C}(x)$, the radicand $x^r(1-x)^s$ has divisor valuations $(r,s,-r-s)$ at $(0,1,\infty)$; since $\gcd(b,r,s)=1$, for every prime $p \mid b$ at least one valuation is coprime to p, so Kummer theory gives an irreducible degree-b cover with d_j points of ramification index e_j above $j \in \{0,1,\infty\}$. Riemann–Hurwitz yields $2g_{\mathrm{tot}} - 2 = -2b + \sum_{j\in\{0,1,\infty\}} d_j(e_j-1) = b - d_0 - d_1 - d_\infty$. Direct local valuation computation gives the orders of ω_u at the points above $(0,1,\infty)$ as

$$\frac{b-r_u}{d_0} - 1, \qquad \frac{b-s_u}{d_1} - 1, \qquad \frac{r_u+s_u-b}{d_\infty} - 1.$$

Among $0 < u < b$, the subsets where $b \mid ur$, $b \mid u\,s$, or $b \mid u(r+s)$ have sizes d_0-1, d_1-1, $d_\infty-1$ and are pairwise disjoint by $\gcd(b,r,s)=1$. On each remaining pair $\{u, b-u\}$, $r_u + s_u + r_{b-u} + s_{b-u} = 2b$ and $b \nmid (r_u+s_u)$, so exactly one index lies in S_{hol}, giving g_{tot} holomorphic forms ω_u with distinct eigencharacters ζ_b^{-u} under α^*.

For $\rho_u = \omega_v$ ($v := b-u$, $u \in S_{\mathrm{hol}}$), the only poles lie above ∞. Choosing a local parameter t at a point above ∞ with $x^{-1} = t^{e_\infty}$, expanding $(1-x)^{-s_v/b} = (-1)^{-s_v/b} t^{s_v/d_\infty}(1-t^{e_\infty})^{-s_v/b}$ gives up to a nonzero constant the local expansion

$$\rho_u = \sum_{j\ge 0} (-1)^j \binom{-s_v/b}{j} t^{(r_v+s_v)/d_\infty + (j-1)e_\infty - 1}\,dt.$$

Since $v \notin S_{\mathrm{hol}}$ and $b \nmid (r_v+s_v)$, one has $0 < (r_v+s_v)/d_\infty < e_\infty$, so the exponent of t is never congruent to $-1 \pmod{e_\infty}$ and in particular never equals -1. Thus all residues of ρ_u vanish. The nonvanishing integrals below show no ρ_u is exact, and since their α^*-eigencharacters ζ_b^u are pairwise distinct and disjoint from $\{\zeta_b^{-w} : w \in S_{\mathrm{hol}}\}$, the $2g_{\mathrm{tot}}$ classes form a $\mathbb{Q}$-basis of $H^1_{\mathrm{dR}}(C_{b;r,s}/\mathbb{Q})$.

Let ℓ_0 be the positive real lift of $(0,1)$ to $C_{b;r,s}$. Since α^{d_j} fixes each point above $j \in \{0,1\}$, the Pochhammer chain $\Gamma := (1-\alpha_*^{d_0})(1-\alpha_*^{d_1})\ell_0$ is a closed 1-cycle. Along ℓ_0, $\int_{\ell_0}\omega_u = B(1-r_u/b, 1-s_u/b)$ and $\int_{\ell_0}\rho_u = B(r_u/b, s_u/b)$ (with ρ_u integrated over a cycle displaced off ∞ where its residue vanishes, matching analytic continuation of the Beta integral), so integrating over Γ multiplies these values by $(1-\zeta_b^{-ud_0})(1-\zeta_b^{-ud_1})$ and $(1-\zeta_b^{+ud_0})(1-\zeta_b^{+ud_1})$, respectively (both nonzero since $b \nmid ur, u\,s$).

As in Proposition 4.2, $N_b(\alpha_*) \in \mathrm{End}_{\mathbb{Q}}(\mathrm{Jac}(C_{b;r,s}))$ has eigenvalue $\mathrm{rad}(b)$ on unit indices $u \in U_b$ and 0 on nonunits, so its image $A_{b;r,s}$ is defined over $\mathbb{Q}$, has dimension $\varphi(b)/2$, and carries $\mathbb{Z}[\zeta_b]$ via $\zeta_b \mapsto \alpha_*^{-1}$ with CM type $\Phi_{b;r,s}$. Factor $N_b(\alpha_*) = i \circ q$ through the inclusion $i : A_{b;r,s} \hookrightarrow \mathrm{Jac}(C_{b;r,s})$, put $a_n(z) := (\Phi_n^{\mathrm{cyc}}(1) - \Phi_n^{\mathrm{cyc}}(z))/(1-z) \in \mathbb{Z}[z]$, and define

$$\lambda_0 := a_{e_0}(\alpha_*^{d_0})\,a_{e_1}(\alpha_*^{d_1})\,q_*\Gamma \;\in\; H_1(A_{b;r,s},\mathbb{Z}).$$

In pairing λ_0 with $i^*\omega_u$ or $i^*\rho_u$ ($u \in \Phi_{b;r,s}$), the operator N_b acts once with eigenvalue $\mathrm{rad}(b)$, and since both $\zeta_b^{-ud_j}$ and $\zeta_b^{+ud_j}$ are primitive e_j-th roots of unity, $(1-z)a_{e_j}(z) = \Phi_{e_j}^{\mathrm{cyc}}(1)$ at both roots simultaneously, giving the displayed period identities. □

Theorem 5.2 (Primitivity classification and non-isogeny)**.** *Let $b \ge 5$.*

(i) The CM type $\Phi_{b;1,2}$ is primitive ($\mathrm{Stab}_{U_b}(\Phi_{b;1,2}) = \{1\}$) if and only if $b \notin \{7, 8, 12, 15, 20\}$ (with the finite range $5 \le b \le 180$ verified by exact integer enumeration and $b > 180$ deduced from Aoki's classification [1, Theorem 0.2]).

(ii) An admissible pair (r, s) at level $b \ge 5$ with primitive CM type $\mathrm{Stab}_{U_b}(\Phi_{b;r,s}) = \{1\}$ exists if and only if $b \notin \{8, 12\}$.

(iii) The simple new factors $A_{11;1,2}$, $A_{13;1,2}$, and every simple new factor $A_{b;r,s}$ with $4 \mid b$, $b \ge 16$, are non-isogenous over $\mathbb{C}$ to the entire new factor A_n of any hyperelliptic curve $y^2 = x^n - 1$.

Proof. By [24, Proposition 3.13 and Corollary 3.14] and [18], $A_{b;r,s}$ is absolutely simple if and only if $\mathrm{Stab}_{U_b}(\Phi_{b;r,s}) = \{1\}$. For $(r, s) = (1, 2)$, a unit $u \in U_b$ belongs to $\Phi_{b;1,2}$ iff $u \in (b/3, b/2) \cup (2b/3, b)$; its complement is $D_- := U_b \cap \big((0, b/3) \cup (b/2, 2b/3)\big)$, which has the same stabilizer. Any $a \in \mathrm{Stab}(D_-)$ satisfies $a = a \cdot 1 \in D_-$.

Infinite elementary families (also formalized in `FermatQuotientBetaCM`). For $b = 2^k \ge 16$, write $o(t)$ for the smallest odd integer strictly larger than t. If $3 \le a < b/6$, then $u := o(2b/(3a)) \in D_-$ satisfies $u < b/3$ and $2b/3 < au < b$ ($au \notin D_-$). If $b/6 < a < b/3$, $u := b/2 + 1 \in D_-$ gives $au \equiv b/2 + a \in (2b/3, b)$. If $a = b/2 + t$ with odd $0 < t < b/6$, then $t = 1$ uses $u := b/4 - 1 \in D_-$ (image $3b/4 - 1 > 2b/3$), while $t \ge 3$ uses $u := o(b/(6t)) \in D_-$ ($u < b/3$, image $b/2 + tu \in (2b/3, b)$). For prime $b = p \ge 11$: $a = 2$ uses $u = \lfloor p/4 \rfloor \in D_-$ (image $2u \in (p/3, p/2)$); $3 \le a < p/3$ uses $u = \lfloor 2p/(3a) \rfloor + 1$ (image $au \in (2p/3, p)$); $5p/9 < a < 2p/3$ uses $u = 3$ (image $3a - p \in (2p/3, p)$); $a = (p + t)/2$ with $0 < t \le p/9$ has $t = 1 \implies a^{-1} = 2$ (already excluded) and $t \ge 3$ uses $u = 2(\lfloor p/(3t) \rfloor + 1) \in D_-$ (image $au \equiv tk \in (p/3, p/2)$). At $b = 5$, $\Phi_{5;1,2} = \{2, 4\}$ has trivial stabilizer.

Finite range $5 \le b \le 180$ and levels $b \in \{8, 12\}$. Direct integer computation over U_b for all $5 \le b \le 180$ (recorded in `mise_F_fermat_asymmetric_cert.json`) shows that the only non-trivial stabilizers for $(r, s) = (1, 2)$ in $[5, 180]$ are

$$(7; \{1, 2, 4\}),\ (8; \{1, 5\}),\ (12; \{1, 7\}),\ (15; \{1, 2, 4, 8\}),\ (20; \{1, 11\}).$$

At $b \in \{7, 15, 20\}$, the admissible pairs $(7; 1, 3)$, $(15; 1, 7)$, and $(20; 1, 4)$ each have stabilizer $\{1\}$. At $b \in \{8, 12\}$, $U_b \cong (\mathbb{Z}/2\mathbb{Z})^2$ is the Klein four group, so any CM type Φ has size 2 and its unique non-identity ratio $u_2 u_1^{-1}$ exchanges $u_1 \leftrightarrow u_2$ and stabilizes Φ; hence no primitive CM type exists at $b = 8, 12$.

All levels $b > 180$ via Aoki's classification [1, Theorem 0.2]. Aoki's theorem is quoted explicitly. Let $\mathfrak{A}_b^1 := \{(a_0, a_1, a_2) \in (\mathbb{Z}/b\mathbb{Z} \setminus \{0\})^3 : a_0 + a_1 + a_2 = 0\}$, call two triples equivalent ($\sim$) if they agree up to permutation and common scaling by a unit $t \in U_b$, and define the finite exceptional set of 33 levels (maximum 180)

$$\begin{aligned}\mathcal{E} := \{&2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 30,\\ &36, 39, 40, 42, 48, 54, 60, 66, 72, 78, 84, 90, 120, 156, 180\}.\end{aligned}$$

For $\gamma \in \mathfrak{A}_b^1$ with $\gcd(\gamma, b) = 1$, Aoki [1, §0, Theorem 0.2] defines four non-primitive types:

- **Type II-1:** $\gamma \sim (1, w, -1 - w)$ with $w^2 \equiv 1 \pmod b$, $w \not\equiv \pm 1$, and $w \not\equiv b/2 + 1$ if $\mathrm{ord}_2(b) \ge 3$ (stabilizer $\{1, w\}$);
- **Type II-2:** $\gamma \sim (1, 1, -2)$ with $\mathrm{ord}_2(b) \ge 2$ (stabilizer $\{1, b/2 - 1\}$);
- **Type II-3:** $\gamma \sim (1, b/2+1, b/2-2)$ with $\mathrm{ord}_2(b) \ge 3$ (stabilizer $\{1, b/4-1, b/2+1, 3b/4-1\}$);
- **Type III:** $\gamma \sim (1, w, w^2)$ with $1 + w + w^2 \equiv 0 \pmod b$ (stabilizer $\{1, w, w^2\}$);

and **Type I** for all remaining primitive triples. Aoki's Theorem 0.2 states that for every $b \notin \mathcal{E}$ (in particular every $b > 180$), the stabilizer W_γ is trivial (equivalently $A_{[\gamma]}$ is simple) if and only if γ is of Type I.

Checking $\gamma = (1, 2, -3) \in \mathfrak{A}_b^1$ against each non-primitive type for $b > 180$ gives:

- *Not Type II-2:* Equivalence to $(1, 1, -2)$ requires two entries to be congruent modulo b. The pairwise differences of $(1, 2, -3)$ are $\pm 1, \pm 4, \pm 5$, none divisible by $b > 180$.
- *Not Type II-1 or Type II-3:* Both types contain a pair of entries whose squares are congruent modulo b ($1^2 \equiv w^2$ in Type II-1; $(b/2+1)^2 = b^2/4 + b + 1 \equiv 1^2 \pmod b$ when $8 \mid b$ in Type II-3), a property invariant under permutation and unit scaling ($x^2 - y^2 \mapsto t^2(x^2 - y^2)$). The nonzero pairwise square differences of $(1, 2, -3)$ have absolute values $2^2 - 1^2 = 3$, $(-3)^2 - 2^2 = 5$, and $(-3)^2 - 1^2 = 8$, none divisible by $b > 180$.
- *Not Type III:* Equivalence to $(1, w, w^2)$ with $1 + w + w^2 \equiv 0 \pmod b$ forces $x^2 + xy + y^2 \equiv 0 \pmod b$ on a pair of entries (invariant under permutation and unit scaling). On every pair from $(1, 2, -3)$, $x^2 + xy + y^2 = 7$, not divisible by $b > 180$.

Hence $(1, 2, -3)$ is of Type I for every $b > 180$, proving (i) and (ii).

Proof of (iii) (Non-isogeny to hyperelliptic new factors). Two simple isogenous CM abelian varieties have isomorphic rational endomorphism fields (which coincide inside $\mathbb{C}$ when Galois) and equivalent CM types [24, Proposition 3.13]. The conductor of K_n is n if $n \not\equiv 2 \pmod 4$ and $n/2$ if $n \equiv 2 \pmod 4$. For $4 \mid b$ ($b \geq 16$), the only hyperelliptic new factor with CM field K_b has level b, which is not simple by Theorem 4.1 and thus cannot be isogenous to simple $A_{b;r,s}$. For $b \in \{11, 13\}$, the only hyperelliptic levels with CM field K_b are b and $2b$, whose CM types are $D_b := U_b \cap (0, b/2)$ and $-2D_b$ (same U_b-orbit). If $aD_b = \Phi_{b;1,2}$ then $a \in \Phi_{b;1,2}$; for the elements $a \in \Phi_{b;1,2}$ in increasing order, explicit witnesses $h \in D_b$ with $(ah \bmod b) \notin \Phi_{b;1,2}$ are $(3, 5, 3, 2, 4)$ at $b = 11$ and $(3, 5, 3, 2, 3, 5)$ at $b = 13$. □

Theorem 5.3 (Asymmetric and $4 \mid b$ Beta irrationality measures)**.** *Let $b \geq 5$ and let (r, s) be an admissible pair with $\mathrm{Stab}_{U_b}(\Phi_{b;r,s}) = \{1\}$. For every $u \in \Phi_{b;r,s}$, setting $h := r_u/b$ and $k := s_u/b$,*

$$\mu(B(1-h, 1-k)) = \mu(B(h, k)) = 2.$$

Proof. By Theorem 5.1 and Theorem 5.2, $A_{b;r,s} \hookrightarrow \mathrm{Jac}(C_{b;r,s})$ is an absolutely simple $\mathbb{Q}$-abelian variety of dimension $g = \varphi(b)/2$ with $\mathbb{Z}[\zeta_b]$-action, carrying the $\mathbb{Q}$-rational eigenclasses $[i^*\omega_u]$ and $[i^*\rho_u]$ whose periods along the single integral cycle $\lambda_0 \in H_1(A_{b;r,s}, \mathbb{Z})$ are $M_{b;r,s}B(1-h, 1-k)$ and $M_{b;r,s}B(h, k)$. Because $\mathrm{Jac}(C_{b;r,s}) \setminus \Theta_O$ is nonempty and defined over a number field, an algebraic translation point $Q_0 \in \mathrm{Jac}(C_{b;r,s})(\overline{\mathbb{Q}}) \setminus \Theta_O$ exists, and Propositions 3.1 and 3.3 supply the conjugate charts and covariant theta gauges. Theorem 3.5 (with the positive integer multiplier $M_{b;r,s}$) gives $\mu(B(1-h, 1-k)) = \mu(B(h, k)) = 2$. Table 3 records 568-bit Arb arithmetic cross-checks for eight flagship curves. □

5.2 Exhaustive classification of levels 8 and 12 and existence at every level

For a short model $y^2 = 4f(x)$ over $\mathbb{R}$, let $\Omega := \int_{e_{\max}}^{\infty} dx/\sqrt{f(x)}$ denote the positive real period of dx/y around the unbounded real oval ($e_{\max}$ the largest real root of f). Set

$$f_8(x) := x^3 + 4x^2 + 2x, \qquad f_4(x) := x^3 - x, \qquad f_3(x) := x^3 - 1,$$

with corresponding base periods $\Omega_8, \Omega_4, \Omega_3$, and write $\Omega_{4,t}$ for $x^3 - tx$ and $\Omega_{3,t}$ for $x^3 - t$ ($t > 0$). (Thus $\Omega_4^{\mathrm{sq}} = \sqrt{2}\,\Omega_4$ in Proposition 4.3.)

Table 3: Eight flagship Fermat quotients. The last two columns strictly exceed the recorded upper endpoints for the integral-versus-Gamma comparison and the cleared-period identity residual in the 568-bit certificate `mise_F_fermat_asymmetric_cert.json`. Sample arguments are numerators over b.

$(b;r,s)$	(g_{tot},g)	$M_{b;r,s}$	Sample	Integral	Cleared
$(11;1,2)$	$(5,5)$	1331	$(1,2)$	2.20×10^{-167}	1.20×10^{-163}
$(13;1,2)$	$(6,6)$	2197	$(1,2)$	2.19×10^{-167}	2.38×10^{-163}
$(16;1,2)$	$(7,4)$	8	$(1,2)$	2.18×10^{-167}	3.93×10^{-166}
$(16;1,3)$	$(6,4)$	8	$(1,3)$	2.14×10^{-167}	1.26×10^{-165}
$(16;1,4)$	$(6,4)$	8	$(1,4)$	1.83×10^{-167}	2.34×10^{-166}
$(20;1,4)$	$(6,4)$	50	$(1,4)$	3.60×10^{-167}	2.21×10^{-165}
$(24;1,2)$	$(10,4)$	6	$(1,2)$	4.33×10^{-167}	6.31×10^{-166}
$(24;1,3)$	$(9,4)$	12	$(1,3)$	3.61×10^{-167}	1.44×10^{-165}

Theorem 5.4 (Exact 32/32 classification at levels 8 and 12)**.** *Among all reduced unordered numerator pairs (a,c) satisfying $1\le a\le c$, $a+c<b$, and $\gcd(a,c,b)=1$, there are exactly 10 pairs at $b=8$ and 22 pairs at $b=12$ (32 pairs in total). Precisely 4 of these 32 Beta values are rational multiples of real periods of elliptic curves over $\mathbb{Q}$ with a nonzero $\mathbb{Q}$-rational invariant differential:*

$$B(1/8,3/8)=4\Omega_8, \qquad B(1/8,5/8)=2^{7/4}\Omega_4=4\Omega_{4,2},$$
$$B(1/12,5/12)=4\cdot3^{1/4}\Omega_4=12\Omega_{4,27}, \qquad B(1/12,7/12)=3\cdot2^{5/6}\Omega_3=6\Omega_{3,2}.$$

All four satisfy $\mu=2$ unconditionally (proved within this paper by Proposition 3.7 for E_8 with CM by $\mathbb{Z}[\sqrt{-2}]$ and by Theorem 3.5 / Proposition 3.7 for the $\mathbb{Z}[i]$ and $\mathbb{Z}[\zeta_3]$ models; also covered independently by [9]). Each of the remaining 28 pairs is a real period on a CM elliptic twist defined over a real number field of exact degree $d\in\{2,3,4,6\}$ listed in Table 4 and satisfies $\mu(B(a/b,c/b))\le 2d$ (leaving whether $\mu=2$ holds for those 28 values open).

Proof. In $B(1/8,3/8)=\int_0^\infty u^{-7/8}(1+u)^{-1/2}du$, setting $u=x^4$ gives $B(1/8,3/8)/4=\int_0^\infty dx/\sqrt{x(x^4+1)}$. Splitting at $x=1$ and inverting on $(0,1)$ gives $\int_1^\infty(1+x)\,dx/\sqrt{x(x^4+1)}$. Substituting $X:=x+x^{-1}-2\in(0,\infty)$ and $w:=\sqrt{x(x^4+1)}(x-1)/x^2$ gives $w^2=X^3+4X^2+2X=f_8(X)$ and $dX/w=(1+x)\,dx/\sqrt{x(x^4+1)}$, proving $B(1/8,3/8)=4\Omega_8$. To verify CM by $\mathbb{Z}[\sqrt{-2}]$ on $v_1^2=x^3+4x^2+2x$ ($v_1=y/2$), the 2-isogeny with kernel $\{O,(0,0)\}$ is $(x,v_1)\mapsto(x+4+2/x,\,v_1(1-2/x^2))$ onto $v_2^2=x_2^3-8x_2^2+8x_2$ (pulling dx_2/v_2 back to dx/v_1). The twist $(x,v_1)\mapsto(-2x,(-2)^{3/2}v_1)$ maps $v_1^2=x^3+4x^2+2x$ isomorphically onto $v_2^2=x_2^3-8x_2^2+8x_2$; composing with its inverse produces an endomorphism of E_8 acting as $\sqrt{-2}$ on dx/v_1 ($c_4=160$, $\Delta=512$, $j=8000$). Proposition 3.7 therefore yields $\mu(B(1/8,3/8))=2$ unconditionally.

For the remaining three $d=1$ rows, $t=x^{-2}$ and $t=x^{-3}$ yield $\Omega_4=B(1/4,1/2)/2$ and $\Omega_3=B(1/6,1/2)/3$. Gamma multiplication and reflection give

$$\Gamma(1/8)\Gamma(5/8)=2^{3/4}\sqrt{\pi}\,\Gamma(1/4),$$
$$\Gamma(1/12)\Gamma(5/12)\Gamma(3/4)=2\pi\cdot3^{1/4}\Gamma(1/4),$$
$$\Gamma(1/12)\Gamma(7/12)=2^{5/6}\sqrt{\pi}\,\Gamma(1/6).$$

Dividing by $\Gamma(3/4)$, $\Gamma(1/2)$, and $\Gamma(2/3)$ respectively, and using $\Omega_{4,t}=t^{-1/4}\Omega_4$ and $\Omega_{3,t}=t^{-1/6}\Omega_3$ ($t>0$), produces the three rational elliptic realizations (CM by $\mathbb{Z}[i]$ or $\mathbb{Z}[\zeta_3]$), so Theorem 3.5 (and Proposition 3.7) gives $\mu=2$.

For the full 32-pair enumeration, write $B(a/b,c/b)=f\,\Omega_j$ ($f>0$), with $(k_8,k_4,k_3)=(2,4,6)$ and $d:=[\mathbb{Q}(f^{k_j}):\mathbb{Q}]$. Rescaling the base model by $(x,y)\mapsto(f^{-2}x,f^{-3}y)$ produces a twist

over $\mathbb{Q}(f^{k_j}) \subset \mathbb{R}$ whose invariant differential has real period $f\,\Omega_j$. At $b = 8$, permuting $(a, c, 8-a-c)$ scales f by $\sin(\pi u/8)/\sin(\pi u_0/8)$ via reflection; combining the seeds $(1,3,4)$, $(1,2,5)$ with duplication $B(x,x) = 2^{1-2x}B(x,1/2)$ yields $B(1/8,1/8) = 4\sqrt{1+\sqrt{2}}\,\Omega_8$ and $B(3/8,3/8) = 4\sqrt{(\sqrt{2}-1)/2}\,\Omega_8$, and exact quadratic polynomials for scaled f^{k_j} with coefficient triples $(1,-2,-1)$, $(1,-6,1)$, $(1,-4,2)$, $(1,0,-2)$, $(1,2,-1)$.

At $b = 12$, put $\varsigma := \sin(5\pi/12) = (\sqrt{6}+\sqrt{2})/4$, $a_4 := \Gamma(1/4)$, $a_3 := \Gamma(1/3)$, and $G_j := \Gamma(j/12)$. Duplication and triplication at $1/12$ and $1/6$, together with reflection $G_j = \pi/(G_{12-j}\sin(\pi j/12))$ for $j > 6$, yield

$$G_1 = \sqrt{2\cdot 3^{3/4}\varsigma}\,a_4a_3/\sqrt{\pi}, \qquad G_5 = \sqrt{3^{-1/4}/\varsigma}\,a_4\sqrt{\pi}/a_3,$$
$$G_2 = 2^{-1/3}\sqrt{3}\,a_3^2/\sqrt{\pi}, \qquad (G_3,G_4,G_6) = (a_4,a_3,\sqrt{\pi}).$$

Dividing G_aG_c/G_{a+c} by $\Omega_4 = a_4^2/(2\sqrt{2\pi})$ or $\Omega_3 = a_3^3/(2^{4/3}\pi)$ gives f explicitly for all 22 pairs. After rational scaling, the quadratic minimal polynomials are $z^2 \pm 6z - 3$, $z^2 - 52z + 1$, and $z^2 - 14z + 1$ (nonsquare discriminants); the cubic pairs $(1,10),(2,5)$ (scales $384, 48$) have $z^3 - 2, z^3 - 4$; the six quartic pairs have $z^4 + a'z^2 + c'$ with

(a,c)	$(1,2)$	$(1,9)$	$(2,3)$	$(2,7)$	$(2,9)$	$(3,7)$
scale	15552	1944	216	1728	1944	216
a'	-90	-90	-270	270	90	270
c'	-3	-3	-27	-27	-3	-27

(irreducible because the quadratic in z^2 has nonsquare discriminant and negative norm $c' < 0$, so z^2 is not a square in that quadratic field); and the two sextic pairs $(1,1),(5,5)$ (scales $384, 48$) satisfy $(z^3)^2 - 5404z^3 + 4 = 0$ and $(z^3)^2 - 10808z^3 + 16 = 0$ (nonsquare discriminants and noncube norms $4, 16$, so adjoining z has relative degree 3).

Finally, no alternative elliptic curve $E'/\mathbb{Q}$ with rational invariant differential can realize any of the 28 remaining values as a rational multiple of its period. Indeed, if it did, the product $E' \times E_j$ would have a period vector on an algebraic line; Wüstholz's analytic subgroup theorem [36] forces a $\overline{\mathbb{Q}}$-isogeny $E' \to E_j$, so E' and E_j share the same imaginary quadratic CM field K. If $\mathrm{End}(E')$ has conductor n in $\mathcal{O}_K$, the kernel of $n\mathcal{O}_K$ is Galois-stable and yields a $\mathbb{Q}$-isogenous curve $E''/\mathbb{Q}$ with maximal CM order $\mathcal{O}_K$ whose real period differs by a nonzero rational factor. Because $\mathbb{Z}[\sqrt{-2}]$, $\mathbb{Z}[i]$, and $\mathbb{Z}[\zeta_3]$ each have class number 1 (norm Euclidean), E'' is a $\overline{\mathbb{Q}}$-twist of E_j. The $\overline{\mathbb{Q}}$-isomorphism scales differentials by $v \in \overline{\mathbb{Q}}^\times$ whose Galois conjugates differ by an automorphism of E_j (of order dividing k_j), forcing $v^{k_j} \in \mathbb{Q}^\times$. Since all period ratios on E_j lie in K, $f^{k_j} \in K \cap \mathbb{R} = \mathbb{Q}$, i.e., $d = 1$. Hence none of the 28 pairs with $d \geq 2$ admits a $\mathbb{Q}$-rational CM elliptic realization. Proposition 5.5 gives $\mu(B(a/b,c/b)) \leq 2d$ for all 28 pairs. □

Proposition 5.5 (Single distinguished conjugate pair)**.** *Let $F \subset \mathbb{R}$ be a number field of degree $d = [F:\mathbb{Q}]$ and let E/F be an elliptic curve with complex multiplication and a nonzero F-rational invariant differential ω. Every nonzero real period $\mathcal{T}$ of ω (at the given real embedding $F \hookrightarrow \mathbb{R}$) satisfies $\mu(\mathcal{T}) \leq 2d$.*

Proof. Let K be the imaginary quadratic CM field of E. Since the distinguished embedding of F is real, $F \cap K = \mathbb{Q}$, so $\mathbb{L} := FK$ has degree $2d$ over $\mathbb{Q}$ and every embedding of $\mathbb{L}$ into $\mathbb{C}$ is nonreal; hence $\mathbb{L}$ has exactly d complex conjugate pairs of embeddings (normality and total reality of F are unnecessary, and these d complex places of FK are distinct from the archimedean places of F whenever F has complex places). Fix the distinguished pair $\iota_0, \bar{\iota}_0$ above $F \hookrightarrow \mathbb{R}$. Because every automorphism fixing F and K fixes the tangent action of $\mathrm{End}_{\overline{F}}(E)$ on ω, the CM endomorphisms descend to FK. Take a $\mathbb{Z}$-basis $(1,\kappa)$ of $\mathrm{End}_{\overline{F}}(E)$ and form the $K_{\mathrm{cen}} := N_{\mathrm{box}}^2$ multipliers $\xi_j = a_0 + a_1\kappa$ $(0 \leq a_0, a_1 < N_{\mathrm{box}})$, satisfying $|\iota(\xi_j)| \ll_E N_{\mathrm{box}}$ at all embeddings.

Table 4: Exhaustive 32/32 classification of all reduced unordered numerator pairs (a, c) satisfying $1 \le a \le c$, $a + c < b$, $\gcd(a, c, b) = 1$ at levels $b \in \{8, 12\}$ (10 at $b = 8$, 22 at $b = 12$). The last column records the proven bound for $\mu(B(a/b, c/b))$.

b	Base	d	Reduced numerator pairs (a, c)	Exponent
8	f_8	1	$(1,3)$	$= 2$
8	f_8	2	$(1,1),(1,4),(1,6),(2,3),(3,3),(3,4)$	≤ 4
8	f_4	1	$(1,5)$	$= 2$
8	f_4	2	$(1,2),(2,5)$	≤ 4
12	f_4	1	$(1,5)$	$= 2$
12	f_4	2	$(1,3),(1,6),(1,8),(3,4)$ $(3,5),(3,8),(4,5),(5,6)$	≤ 4
12	f_4	3	$(1,10),(2,5)$	≤ 6
12	f_4	6	$(1,1),(5,5)$	≤ 12
12	f_3	1	$(1,7)$	$= 2$
12	f_3	2	$(1,4),(4,7)$	≤ 4
12	f_3	4	$(1,2),(1,9),(2,3),(2,7),(2,9),(3,7)$	≤ 8

Identifying $\mathrm{End}_{\overline{F}}(E)$ with a subring of $\mathbb{C}$ via its action on ω, a cycle of period $\mathcal{T}$ yields multiples of periods $\xi_j\mathcal{T}$ at the distinguished place.

Take $P_0 = 0_E$ as interpolation centre and choose a finite affine x-coordinate $a \in F$ in a short Weierstrass model of E/F, defining the effective ample degree-2 divisor $L := x^{-1}(a) = P + (-P)$ avoiding 0_E ($\chi(L) = \deg(L) = 2$). Fix $\nu > 2d$, assume infinitely many reduced approximants $|p_i/q_i - \mathcal{T}| < q_i^{-\nu}$ ($p_i \neq 0$), and set $w_i := \lceil \log q_i \rceil$, $v_i := w_i/\theta$, $w_* := \min_i w_i$, and $c_{ji} := \xi_j p_i/q_i$. Theorem 2.2 (with $g = 1$, L of degree 2) over $\mathbb{L} = FK$ produces, for every admissible degree $H = nR_{\mathrm{wt}}$ ($n \ge n_0$), a nonsingular $M \times M$ minor $\Delta_{\mathrm{raw}} \in \mathbb{L}^\times$ whose columns are monomials $P = (\prod_r s_r^{\beta_r})X^\alpha$ in $H^0(E, 3L)$ and X_i with $w_0 \sum_r \beta_r + w \cdot \alpha \le H$ (in particular $\sum_r \beta_r \le \lfloor H/w_0 \rfloor$ and $|\alpha| \le \lfloor H/w_* \rfloor$) and whose rows $\rho = (j, n, \mathbf{d})$ satisfy $v_0 n + v \cdot \mathbf{d} < H$ (in particular $n < H/v_0$ and $\bar{d} := \frac{1}{MH} \sum_\rho w \cdot \mathbf{d}_\rho \in [0, \theta]$), with row count $M = \frac{N_{\mathrm{box}}^2 \theta^m}{v_0 (m+1)! \prod_i w_i} H^{m+1}(1 + o_H(1))$. Applying Lemma 2.9 across all conjugate charts gives a single B_{den} independent of $(m, N_{\mathrm{box}}, w_i, H)$, and the exact row/column scaling $S_H > 0$ of Theorem 3.5 produces a nonzero algebraic integer $S_H \Delta_{\mathrm{raw}} \in \mathcal{O}_{\mathbb{L}} \setminus \{0\}$ (with the same error E_{ar} as in Theorem 3.5), yielding the norm lower bound

$$\sum_{\iota \bmod \mathrm{conj}} \frac{\log |\iota(\Delta_{\mathrm{raw}})|}{MH} \;\ge\; -d(1 - \bar{d} + E_{\mathrm{ar}}).$$

At each of the $d - 1$ non-distinguished conjugate pairs, fix a common convergence disk $|e| \le r < 1$ strictly inside a common larger disk $|e| \le r' < 1$ on which the fixed conjugate analytic elliptic logarithms $\ell^{(\iota)}$ and basis sections $s_r^{(\iota)}$ are holomorphic and bounded. Because every $G_i^{(\iota)}$ is a Taylor partial sum of the single fixed analytic germ $\ell^{(\iota)}$ on $|e| \le r'$, Cauchy's coefficient bound on $|e| = r'$ and the geometric series $\sum_{k \ge 0} (r/r')^k = (1 - r/r')^{-1}$ yield a constant C_0 independent of T_i and m with $|G_i^{(\iota)}(e)| \le C_0$ on $|e| \le r$, while $|s_r^{(\iota)}(e)| \le C_1$ ($C_1 \ge 1$) and $|\iota(c_{ji})| \le C_2 N_{\mathrm{box}}$ (since $|p_i/q_i| \le |\mathcal{T}| + 1$). Extracting coefficients on $|e| = r$, $|U_i'| = 1$ via Cauchy's inequality—using $n < H/v_0$, $\sum_r \beta_r \le H/w_0$, $|\alpha| \le H/w_*$, polynomial growth $M = O_m(H^{m+1})$, and no period lattice or Diophantine approximation at the $d - 1$ non-distinguished pairs—gives

$$\frac{\log |\iota(\Delta_{\mathrm{raw}})|}{MH} \le E_{\mathrm{bad}} + o_H(1), \qquad E_{\mathrm{bad}} := \frac{\log(1/r)}{v_0} + \frac{\log C_1}{w_0} + \frac{\log(1 + C_0 + C_2 N_{\mathrm{box}})}{w_*}.$$

At the distinguished pair, by contrast, $c_{ji} = \xi_j p_i/q_i$ approximates the lattice translates $\xi_j\mathcal{T} \in \Lambda_E$. To match the exact degree-2 divisor $L = P + (-P)$, choose a complex lift $u \in \mathbb{C}$ of P and let ϑ be a reduced principal theta function for $\{0_E\}$ with quadratic form $Q_{\mathrm{princ}}(z)$ satisfying

(1). Define the degree-2 normalized theta section

$$f_L(z) := \frac{\vartheta(z+u)\vartheta(z-u)}{\vartheta(u)\vartheta(-u)}, \qquad Q_L(z) := 2Q_{\mathrm{princ}}(z).$$

Because $0_E \notin L$, one has $\pm u \notin \Lambda_E$ and hence $\vartheta(u)\vartheta(-u) \neq 0$ (including when $P = -P$ is 2-torsion, where $\mathrm{div}(f_L) = 2[P] = L$ with multiplicity 2). Directly from (1), the cross terms $\pi H(z,\lambda)$ and $\pm\pi H(u,\lambda)$ combine so that $\pi H(u,\lambda) + \pi H(-u,\lambda) = 0$, whence $f_L(0) = 1$, the Hermitian-normalized modulus $g_L(z) := e^{-Q_L(z)}|f_L(z)|$ is Λ_E-periodic with global maximum $M_0 < \infty$, $f_L^3 s_r$ is entire on $\mathbb{C}$ for every $s_r \in H^0(E, 3L)$ with periodic metric bound $e^{-3Q_L(z)}|f_L(z)^3 s_r(z)| \le M_A$, and $g_L(\ell(e)) \ge c_0 > 0$ on a small convergence disk $|e| \le r_{\mathrm{an}}$ of the local elliptic logarithm ℓ (shrunk so that every tail $\mathcal{R}_i(e) := G_i^{(\iota_0)}(e) - \ell(e)$ satisfies $|\mathcal{R}_i(e)| \le 1$ on $|e| \le r_{\mathrm{an}}$ once T_i is sufficiently large). By Λ_E-periodicity of g_L and $Q_L \ge 0$, every $\lambda \in \Lambda_E$ and $|e| \le r_{\mathrm{an}}$ satisfy $|f_L(\lambda + \ell(e))|^{-1} \le c_0^{-1} e^{-Q_L(\lambda+\ell(e))} \le c_0^{-1}$.

Set $C_{\mathrm{mul}} := 1 + |\kappa|$ (so $|\xi_j| \le C_{\mathrm{mul}} N_{\mathrm{box}}$), write $z_j := \xi_j \mathcal{T} + \ell(e)$, and observe that $\iota_0(c_{ji}) + G_i^{(\iota_0)}(e) + U_i = z_j + \epsilon_{ji} + \mathcal{R}_i(e) + U_i$ with $|\epsilon_{ji}| = |\xi_j|\,|\mathcal{T} - p_i/q_i| \le C_{\mathrm{mul}} N_{\mathrm{box}} e^{\nu} e^{-\nu w_i}$. Put $R := \max(1, 2\sup_{j,|e|\le r_{\mathrm{an}}} |z_j|) = O_E(N_{\mathrm{box}})$, $G_R := \sup_{|z|\le R} Q_L(z) = O_E(N_{\mathrm{box}}^2)$, $M_* := \max(1, M_0, M_A^{1/3})$, $C_\zeta(R) := \max(2M_0, M_0(1+R))$,

$$E_L := 3\lfloor H/w_0 \rfloor + \lfloor H/w_* \rfloor, \qquad P_R := \frac{3\log M_* + 3G_R}{w_0} + \frac{\log(2C_\zeta(R)) + G_R}{w_*}.$$

For each column monomial $P = (\prod_r s_r^{\beta_r}) X^\alpha$ (of cost $w_0 \sum_r \beta_r + w \cdot \alpha \le H$) and each $\mathbf{a} \in \mathbb{N}^m$ ($\mathbf{a} \le \alpha$), define the entire function

$$F_{\mathbf{a},P}(z) := f_L(z)^{E_L}\,[U^{\mathbf{a}}] P\big(s(z), (z+U_i)_{i=1}^m\big) = \sum_{k\ge 0} c_{\mathbf{a},P,k} (z/R)^k.$$

On $|z| \le R$, each of the $\sum_r \beta_r \le \lfloor H/w_0 \rfloor$ section factors uses 3 powers of f_L with $|f_L^3 s_r| \le M_A e^{3G_R} \le M_*^3 e^{3G_R}$, unused powers from $3\lfloor H/w_0 \rfloor$ satisfy $|f_L| \le M_* e^{G_R}$, each additive factor z uses 1 power of f_L with $|f_L(z) z| \le M_0 R e^{G_R} \le C_\zeta(R) e^{G_R}$, and $\binom{\alpha}{\mathbf{a}} \le 2^{\lfloor H/w_* \rfloor}$, so $|F_{\mathbf{a},P}(z)| \le e^{HP_R}$ and Cauchy's inequality gives $|c_{\mathbf{a},P,k}| \le e^{HP_R}$.

Expanding each row $\rho = (j, n, \mathbf{d})$ by the binomial theorem gives $\sum_{(\mathbf{a},k)} \Lambda_{\rho,\mathbf{a},k} (c_{\mathbf{a},P,k})_P$, where $\Lambda_{\rho,\mathbf{a},k} = 0$ unless $\mathbf{a} \ge \mathbf{d}$ and $w \cdot \mathbf{a} \le H$, and otherwise

$$\Lambda_{\rho,\mathbf{a},k} = [e^n] \left(f_L(z_j)^{-E_L} \left(\frac{z_j}{R}\right)^k \binom{\mathbf{a}}{\mathbf{d}} \prod_{i=1}^m (\epsilon_{ji} + \mathcal{R}_i(e))^{a_i - d_i} \right).$$

Any term choosing s_i factors of $\mathcal{R}_i$ (which vanishes to order $\ge T_i > F_0 w_i / v_0$) contributes to $[e^n]$ only if $\sum_i s_i T_i \le n < H/v_0$, whence $w \cdot \mathbf{s} < H/F_0$. Applying Cauchy's bound on $|e| = r_{\mathrm{an}}$ with $|f_L(z_j)^{-E_L}| \le c_0^{-E_L}$, $|z_j/R| \le 1/2$, $|\mathcal{R}_i| \le 1$, and $\sum_{\mathbf{s}} \binom{\mathbf{a}-\mathbf{d}}{\mathbf{s}} \binom{\mathbf{a}}{\mathbf{d}} \le 4^{H/w_*}$ gives

$$|\Lambda_{\rho,\mathbf{a},k}| \le r_{\mathrm{an}}^{-n} c_0^{-E_L} 2^{-k} e^{-\nu w \cdot (\mathbf{a}-\mathbf{d})} e^{\nu H/F_0} \big(4 C_{\mathrm{mul}} N_{\mathrm{box}} e^{\nu}\big)^{H/w_*}.$$

By multilinearity and absolute convergence of the row expansions (for fixed H, finitely many $\mathbf{a}$ satisfy $w \cdot \mathbf{a} \le H$, while $|\Lambda_{\rho,\mathbf{a},k}|$ decays geometrically as 2^{-k} and $|c_{\mathbf{a},P,k}| \le e^{HP_R}$), we may expand $\iota_0(\Delta_{\mathrm{raw}})$ termwise over tuples $(\mathbf{a}_\rho, k_\rho)_\rho$; only injective assignments contribute, and Hadamard's inequality bounds the coefficient minor by $M^{M/2} e^{MHP_R}$. Reserving $2^{-k/2}$ for summability over $k \ge 0$, let $D_{\mathrm{cut}}(H) := \#\{\mathbf{a} \in \mathbb{N}^m : w \cdot \mathbf{a} \le A_{\mathrm{cut}} H\}$, $c_1 := (\log 2)/4$, and $L_1(H) := \varrho^2 M / (H D_{\mathrm{cut}}(H))$ for rational $A_{\mathrm{cut}} > \theta$ and $\varrho \in (0,1)$. In dimension 1, any $n_{\mathbf{a}}$ distinct nonnegative integers k have sum at least $n_{\mathbf{a}}(n_{\mathbf{a}} - 1)/2$. Hence if at least ϱM rows satisfy $w \cdot \mathbf{a}_\rho \le A_{\mathrm{cut}} H$, grouping by $\mathbf{a}$ over at most $D_{\mathrm{cut}}(H)$ groups and applying Cauchy–Schwarz $\sum_{\mathbf{a}} n_{\mathbf{a}}^2 \ge (\varrho M)^2 / D_{\mathrm{cut}}(H)$ shows that the remaining factor $2^{-k/2}$ contributes at most

$-c_1 L_1(H) + O(1/H)$ after dividing by MH. Otherwise more than $(1-\varrho)M$ rows satisfy $w\cdot\mathbf{a}_\rho > A_{\mathrm{cut}}H$, giving $\sum_\rho w\cdot(\mathbf{a}_\rho-\mathbf{d}_\rho) \geq MH(A_{\mathrm{cut}}(1-\varrho)-\bar{d})$. Combining both branches yields

$$\frac{\log|\iota_0(\Delta_{\mathrm{raw}})|}{MH} \leq \max\{-c_1L_1(H),\, -\nu(A_{\mathrm{cut}}(1-\varrho)-\bar{d})\} + E_{\mathrm{tr}} + E_{\mathrm{hol}} + O_m(\log H/H),$$

where

$$E_{\mathrm{tr}} := \frac{\nu}{F_0} + \frac{\log(1/r_{\mathrm{an}})}{v_0} + \frac{\log(4C_{\mathrm{mul}}N_{\mathrm{box}})+\nu}{w_*},$$
$$E_{\mathrm{hol}} := P_R + \left(\frac{3}{w_0}+\frac{1}{w_*}\right)\log^+(1/c_0) + \frac{\log(1/r_{\mathrm{an}})}{v_0}.$$

Finally, since $\nu/d > 2$, choose rational $\theta \in (0,1)$ with $\nu/d > 1+\theta^{-1/2}$, rationals $A_{\mathrm{cut}}, \beta_*$ satisfying $\theta+(1-\theta)/(\nu/d) < A_{\mathrm{cut}} < \beta_* < \sqrt{\theta}$, rational $C \in (\beta_*/\theta, \min(1/\theta, 1/\beta_*))$, and rational $\varrho \in (0,1)$ so that $\delta_0 := \nu(A_{\mathrm{cut}}(1-\varrho)-\theta) - d(1-\theta) > 0$. Notice that for every $\bar{d} \in [0,\theta]$,

$$\nu(A_{\mathrm{cut}}(1-\varrho)-\bar{d}) - d(1-\bar{d}) = \delta_0 + (\nu-d)(\theta-\bar{d}) \geq \delta_0.$$

Choose rational F_0 with $F_0\theta \geq 2$ and $\nu/F_0 < \min(\delta_0,1)/8$, and for $m \geq 1$ set $N_{\mathrm{box}} := \lfloor C^{m/2} \rfloor$, $K_{\mathrm{cen}} := N_{\mathrm{box}}^2$, $w_0 := \beta_*^{-m}$, and $v_0 := 2N_{\mathrm{box}}^2\theta^m w_0$. Then the interpolation volume ratios satisfy $a_0 = N_{\mathrm{box}}^2\theta^m \leq (C\theta)^m < 1$ and $a_1 = 1/(2\cdot 3\cdot\chi(L)) = 1/12 < 1$, while as $m\to\infty$ one has $m/v_0 \leq 2m(\beta_*/(C\theta))^m \to 0$, $R^2/w_0 = O_E((C\beta_*)^m) \to 0$, and $\lim_{H\to\infty} L_1(H) = \varrho^2(\beta_*/A_{\mathrm{cut}})^m/(2(m+1)) \to \infty$. Freeze m large enough that $c_1\lim_{H\to\infty}L_1(H) > d+2$ and all w_i-independent terms of $dE_{\mathrm{ar}}+E_{\mathrm{tr}}+E_{\mathrm{hol}}+(d-1)E_{\mathrm{bad}}$ sum to less than $\min(\delta_0,1)/4$. Then choose p_i/q_i recursively so that $w_i > W_i$, $|\mathcal{R}_i| \leq 1$ on $|e| \leq r_{\mathrm{an}}$, and all w_i-dependent terms of $dE_{\mathrm{ar}}+E_{\mathrm{tr}}+E_{\mathrm{hol}}+(d-1)E_{\mathrm{bad}}$ sum to less than $\min(\delta_0,1)/4$. Summing the distinguished upper bound with $(d-1)(E_{\mathrm{bad}}+o_H(1))$ and comparing with the norm lower bound $-d(1-\bar{d}+E_{\mathrm{ar}})$ yields, as admissible $H\to\infty$,

$$0 \leq \max\{d(1-\bar{d}) - c_1L_1(H),\, -\delta_0\} + \tfrac{1}{2}\min(\delta_0,1) + o_H(1) \leq -\tfrac{1}{2}\min(\delta_0,1) + o_H(1),$$

a contradiction (using $d(1-\bar{d}) \leq d$ in the first branch). Since $\nu > 2d$ was arbitrary, $\mu(\mathcal{T}) \leq 2d$. □

Corollary 5.6 (All-levels existence theorem)**.** *For every integer $b \geq 3$, there exists a reduced numerator pair (a,c) satisfying $1 \leq a \leq c$, $a+c<b$, $\gcd(a,c,b)=1$, and $\mu(B(a/b,c/b)) = 2$.*

Proof. For $b \in \{3,4\}$, take $(a,c)=(1,1)$ in Theorem 4.4. For $b \in \{8,12\}$, take $(a,c)=(1,3)$ and $(1,5)$, respectively, in Theorem 5.4. For every other $b \geq 5$, Theorem 5.2(ii) furnishes an admissible pair (r,s) with $\mathrm{Stab}_{U_b}(\Phi_{b;r,s}) = \{1\}$; for any $u \in \Phi_{b;r,s}$, setting $a' := b - r_u$ and $c' := b - s_u$ (ordered as $a \leq c$) gives $a+c<b$, $\gcd(a,c,b) = \gcd(ur, u\,s, b) = 1$, and $\mu(B(a/b,c/b)) = 2$ by Theorem 5.3. □

6 Companion artifacts, reproducibility, and AI provenance

6.1 Companion Lean 4 formalizations and computational checks

Scope of the Lean 4 companion archive. The companion repository ships nine Lean 4.33.1 modules [25, 23] (14,273 total lines, 147 public theorem targets, zero `sorry`, zero warnings, `autoImplicit = false`, strictly standard axioms {`propext`, `Classical.choice`, `Quot.sound`}). Its main mathematical formal results fall into five complementary parts:

- `ConditionalDeterminantEngine.lean` (4,145 lines, 24 public targets): machine-checked conditional formalization of the multivariable determinant engine in arbitrary dimension $g \geq 1$ (matching and generalizing the genus-1 conditional Lean 4 methodology of [6, 9]), together with the direct geometry-free Flint–Hills convergence transfer. Specifically, it proves

in Lean 4: (i) standard `irrationalityMeasure` via Mathlib's `LiouvilleWith` (under which $\mu(\mathbb{Q}) = 1$, $\mu(x) \geq 2$ for every irrational x by Dirichlet's theorem, and $\mu(rx) = \mu(x)$ for $r \in \mathbb{Q}^\times$); (ii) Chebyshev's LCM bound $\operatorname{lcm}(1, \ldots, n) \leq 4^n$; (iii) Hadamard's determinant inequality over $\mathbb{C}$ (via Gram–Schmidt) and an infinite-series (`tsum`) multilinear determinant bound killing repeated rows; (iv) exact d-dimensional weighted simplex counting bounds, g-dimensional lattice shell packing, Hölder grouping, the two-branch analytic exponent bound (4), the arithmetic scaling bound $\log S_H/(MH) \leq 1 - \bar{d} + E_{\mathrm{ar}}$, and $H \to \infty$ asymptotics for $M(H)$, $D_{\mathrm{cut}}(H)$, $L_g(H)$, and all $O_{m,g}(\log H/H)$ errors; (v) non-circular parameter selection (`noncircular_parameter_selection`) strictly matching the prose quantifier order (fixed geometric constants first, then rationals $(\theta, A_{\mathrm{cut}}, \beta_*, C, \varrho, F_0)$, then ℓ, with all w_*-dependent terms absorbed recursively via `choose_separated_approximants`); (vi) **conditional Theorem 3.5 and Proposition 3.7** (`theorem_3_5_master_determinant`: `GeometricAnalyticHypotheses g T` $\implies$ $\mu(\mathcal{T}) = 2$ and $\mu(r\mathcal{T}) = 2$ for all $r \in \mathbb{Q}^\times$); (vii) **conditional Proposition 5.5** (`proposition_5_5_single_distinguished_place`: `SinglePlaceHypotheses d T` $\implies$ $\mu(\mathcal{T}) \leq 2d$); and (viii) **direct geometry-free Corollary 3.9** (`corollary_3_9_flint_hills_convergence`: $\beta > 0$, $\beta/\pi \notin \mathbb{Q}$, and $\mu(\pi/\beta) = 2$ $\implies$ Mathlib `Summable` of $\sum_{n \geq 1} n^{-s} \sin^{-2}(n\beta)$ for every $s > 2$, with no unformalized geometric hypotheses).

- `GeometricBridge.lean` (1,980 lines, 19 public targets; imports `ConditionalDeterminantEngine`): narrows the gap between abstract determinant bounds and algebraic-analytic inputs by proving inside Lean 4: (i) Lemma 3.4 over Mathlib's `NumberField`, `IsTotallyComplex`, and `InfinitePlace` (`lemma_3_4_totally_imaginary_product_formula`: $[\mathbb{L} : \mathbb{Q}] = 2r_{\mathbb{L}}$ and $1 \leq \prod_w \|w(x)\|^2$ for nonzero $x \in \mathcal{O}_{\mathbb{L}}$); (ii) the Galois product-formula lower bound for *one single* nonsingular minor $\Delta_{\mathrm{raw}} \in \mathbb{L}^{M \times M}$ and its actual Galois-conjugate matrices `pl.embedding.mapMatrix` Δ_{raw} (`galois_product_formula_of_nonsingular_minor` and its single-place counterpart), closing the earlier gap of unlinked complex matrices and an assumed product formula; (iii) minor extraction from a spanning family and the Euler-integrated single-coefficient denominator-clearing step; (iv) the arbitrary-dimension g-polydisk Cauchy inequality by genuine induction on g via iterated torus integrals (`multivar_polydisk_cauchy_bound`), together with the covariant gauge inverse bound, the five-factor cleared-monomial bound with $E_L = 3\lfloor H/w_0 \rfloor + \lfloor H/w_* \rfloor$, and the $4^{H/w_*}$ binomial bound; and (v) assembly of `GoodPlaceData` and `BadPlaceData` yielding the bridged conditional theorems `theorem_3_5_from_geometric_inputs` and `proposition_5_5_from_geometric_inputs` (the latter specialized to $g = 1$, matching Proposition 5.5; all 19 verbatim signatures are archived in `LEAN_GEOMETRIC_BRIDGE_EXACT_TYPES.md`).

- `ConcreteCMThetaModels.lean` (2,221 lines, 20 public targets; imports `GeometricBridge`): discharges the complex-analytic theta-gauge and CM-field layer of Section 3 on rectangular genus-1 CM lattices $\Lambda_{\Omega,\kappa} = \Omega(\mathbb{Z} + i\kappa\mathbb{Z})$ ($\kappa > 0$, normalized with $\operatorname{Im} H(\Omega, i\kappa\Omega) = -1$) by proving inside Lean 4: (i) total complexity and exact place counts $\varphi(b)/2$ for every Mathlib cyclotomic field $\mathbb{Q}(\zeta_b)$ ($b \geq 3$, fields only) and for `FieldE8` := AdjoinRoot$(X^2 + 2) \cong \mathbb{Q}(\sqrt{-2})$ (1 infinite place), together with explicit injective, norm-bounded CM boxes `gaussianBox` ($C_{\mathrm{mul}} = 2$, embedding into $\Lambda_{\Omega,1}$) and `e8CMBox` ($C_{\mathrm{mul}} = 3$, embedding into $\Lambda_{\Omega,1/\sqrt{2}}$, matching $\Lambda(E_8) = \Omega_8(\mathbb{Z} + (i/\sqrt{2})\mathbb{Z})$); (ii) using Mathlib's actual classical two-variable Jacobi theta function `jacobiTheta`$_2$ and derivative `jacobiTheta`$_2'$, nonvanishing at the origin ($1 < \operatorname{Re}\theta_3(0; i\kappa)$), the norm form of the Appell–Humbert transformation law (1), and Proposition 3.3(i)–(iii) for `cmCovariantGauge` (exact $\Lambda_{\Omega,\kappa}$-periodicity of $e^{-Q(z)}|f_v(z)|$, non-effective compactness bound $M_0 \geq 1$, and uniform positive local lower bound $c_0 > 0$ on $|e| \leq r_{\mathrm{an}}$); (iii) a standalone second-kind (theta log-derivative) lemma proving the quasi-period shift

$\pi H(1,\lambda)$ and global linear growth $e^{-Q}|f_v Z_v| \le M_1(1+|z|)$ (not used in the first-kind instantiations below; identification of `cmReducedThetaDeriv` with an analytic derivative rests on Mathlib's `jacobiTheta₂'`), plus a standalone lemniscate quartic Lemma 2.9 clearing check ($\mathrm{lcm}(1,\dots,T-1)\cdot 4^T$); (iv) two unconditional consistency witnesses inhabiting all 29 fields of `ThetaGaugePlaceData` over $\mathbb{Q}(i)$ with the actual classical Jacobi theta gauge: a degenerate $H = 1/2$ ($E_L = 0$, 1×1) witness (`lemniscate_theta_gauge_place_data_nonempty`) and a *transparent* 2×2, $E_L = 3$, $\det = -2$ *witness* (`lemniscate_theta_gauge_nontrivial_EL_witness`) at $H = 1$, $w_0 = 1$, $w_* = 2$, returning an explicit witness D of type `ThetaGaugePlaceData 1 1 2 1 (N_box^2) ...` with injective centre-index map `rowCen` $=![0,1]$ (indexing 0 and ζ_4 in `gaussianBox`), nonzero column degree $\beta_{\mathrm{col}} = (0,1)$, evaluation matrix `evalMat` $=!![1,1;\,1,-1]$ ($\det = -2$), meromorphic theta-ratio section $s_{\mathrm{sec},0}(z) = f_{\Omega/2}(z)/f_0(z)$, and publicly exposed exact infinite Taylor series of f_0^3 and $f_0^2 f_{\Omega/2}$ in `C_mat` (with $\alpha_{\mathrm{col}} = d_{\mathrm{row}} = 0$, $R_{\mathrm{rem}} \equiv 0$, and $\ell = \mathrm{id}$, so the additive/rational-approximant branch of `row_binomial_expansion` is vacuous); and (v) conditional instantiations: the generic CM elliptic reduction `concrete_cm_elliptic_mu_eq_two_of_algebraic_data` and its concrete specializations `lemniscate_e4_mu_eq_two_of_algebraic_data` ($\mathbb{Q}(i)$, $\kappa = 1$, wired directly to `gaussianBox`) and `curve_e8_mu_eq_two_of_algebraic_data` ($\mathbb{Q}(\sqrt{-2})$, $\kappa = 1/\sqrt{2}$, wired directly to `e8CMBox`, with $\mu(4\Omega_8) = 2$ noted as the $r = 4$ corollary), plugging the proved Jacobi theta gauge and CM box directly into `CMThetaGeometricInputs` (all 20 verbatim signatures with binder types shown, plus full non-elided `#print` of `ThetaGaugePlaceData` and `ConcreteEllipticCurveAlgebraicData`, are archived in `LEAN_CONCRETE_CM_MODELS_EXACT_TYPES.md`).

- `HigherGenusThetaAndZeroEstimate.lean` (3,484 lines, 18 public targets; imports `GeometricBridge` and `ConcreteCMThetaModels`): lifts the formalization beyond genus 1 into the genuinely new arbitrary-genus ($g \ge 2$) and Section 2 regime by proving inside Lean 4: (i) *arbitrary-genus ($g \ge 1$) Riemann/Siegel theta functions* on $\mathbb{C}^g \times \mathbb{H}_g$ (`SiegelPeriodMatrix g`, `riemannTheta`): absolute summability, continuity, exact g-dimensional lattice quasi-periodicity under $z \mapsto z + a + \tau b$, unconditional existence of a rational torsion nonvanishing basepoint $v \in \mathbb{Q}^g$ via finite torsion-grid Fourier averaging (`riemannTheta_exists_nonvanishing`), the arbitrary-genus norm Appell–Humbert law (1), Proposition 3.3(i)–(iii) (exact Λ_τ-periodicity, global compactness bound $M_0 \ge 1$, uniform local lower bound with explicit $c_0 = 1/2$), and a standalone arbitrary-genus second-kind directional log-derivative theorem proving actual complex directional differentiability of ϑ_O, exact quasi-period shift $\pi H_\tau(e,\lambda)$ for $D_e \log \vartheta_O$ and its v-translate shifted by $-\pi H_\tau(e,v)$ (identifying this shifted expression with $D_e \log f_v$ remains in prose), and global linear growth $M_1(1+\|z\|)$ of `riemannSecondKindCleared` (not wired into the first-kind genus-2 theorem below); (ii) *concrete genus-2 principally polarized CM surface* $\mathbb{C}^2/\Lambda_{\tau_5}$ with CM by $\mathbb{Z}[\zeta_5]$ ($\mathbb{Q}(\zeta_5)$ = `CyclotomicField 5 ℚ`, 2 infinite places): injective norm-bounded 4D CM box `zeta5Box` (N_{box}^4 points, $C_{\mathrm{mul}} = 4$), exact symmetric period matrix $\tau_5 \in \mathbb{H}_2$ with $\Omega_1\tau_5 = \Omega_2$, exact closed-form $\mathrm{Im}\,\tau_5$ and matrix inverse $Y^{-1}\,\mathrm{Im}\,\tau_5 = I_2$, explicit surjective $\mathbb{Z}^4$-parametrization of Λ_{τ_5} matching the Minkowski embedding of $\mathbb{Z}[\zeta_5]$ (injectivity follows in prose from Minkowski independence, while the 4D box inclusion is proved in the auxiliary non-target lemma `zeta5Box_in_tau5_lattice`), exact unimodular symplectic pairing $\mathrm{Im}\,H_{\tau_5}(a+\tau_5 b, a'+\tau_5 b') = b^\top a' - a^\top b'$, exact genus-2 Lemma 3.2 diagonal Hermiticity $H_{\tau_5}(\Omega_1^{-1}u, \Omega_1^{-1}v) = \frac{4\sin(2\pi/5)}{5} u_0\bar{v}_0 + \frac{4\sin(4\pi/5)}{5} u_1\bar{v}_1$, a transparent genus-2 consistency witness returning an explicit D of type `ThetaGaugePlaceData 2 2 1 1 (N_box^4) ...` with publicly exposed Λ_{τ_5}, Q_{τ_5}, f_{v_5}, first-kind $I_u(z) = \Omega(\Omega_1 z)_0$, `zeta5Box`, and $C_{\mathrm{lat}} = 13$ (degenerate $H = 1/2$, $E_L = 0$, 1×1 regime), and conditional `genus2_c5_mu_eq_two_of_algebraic_data` wiring these proved genus-2 objects into `CMThetaGeometricInputs 2 (CyclotomicField 5 ℚ) Ω`; and (iii) *formal Sec-*

tion 2 algebraic and inequality lemmas: the general univariate single-index Hensel denominator ($u^{2n-1}c_n \in \mathbb{Z}$), Euler-integral clearing ($\mathrm{lcm}(1,\dots,T)\cdot u^{2T}$), and deterministic uniqueness theorem (`general_hensel_implicit_power_series_denominators`; the reduction from multivariate Lemma 2.9 remains in prose); commuting formal partial derivations, the exact Leibniz rule, and unweighted, weighted, and explicit space/time separated-weights jet-order drop lower bounds on Mathlib's `MvPowerSeries` (`multivar_commuting_derivation_jet_order_drop`, underlying Lemma 2.5); and the real-inequality/combinatorics bookkeeping of Theorem 2.2 Stages 2–3 (`higher_genus_stage2_stage3_zero_estimate_contradiction`; no geometric variety Y or degree $\deg W$ is constructed in Lean): exact volume ratios $a_d = 2^{-d}(N_{\mathrm{box}}^2\theta^\ell)^{g-d} < 1$ $(0 \le d < g)$ and $a_g = 1/(6^g g!\chi_L) < 1$ for $v_0 = 2N_{\mathrm{box}}^2\theta^\ell w_0$, existence of uniform rational slack $\delta_{\mathrm{sl}} \in \mathbb{Q}_{>0}$, abstract dilation closing bound ($v_0 = \kappa w_0$) and specialization $N_{\mathrm{box}}^{2a} \le C_{\mathrm{drop}}C_{\mathrm{int}}(2\theta^\ell)^{g-a}$, intermediate demand-to-supply ratio, pole-vs-zero inequality contradiction, and pigeonhole stabilization of monotone chains (all 18 verbatim signatures with binder types shown, plus `#print Genus2C5AlgebraicData`, are archived in `LEAN_HIGHER_GENUS_THETA_EXACT_TYPES.md`).

- `CompositeBetaCM.lean` (five core theorems, culminating in `exact_hyperelliptic_cm_primitivity_dichotomy`) and `FermatQuotientBetaCM.lean` (eight core theorems): complete residue-arithmetic proofs that $\mathrm{Stab}_{U_n}(\Phi_n) = \{1\} \iff n = 4$ or $4 \nmid n$ (Theorem 4.1), that $\Phi_{b;1,2}$ is primitive for all $b = 2^k \ge 16$, all primes $p \ge 11$, and $b = 5$, that no primitive CM type exists at $b \in \{8, 12\}$, and that $\Phi_{11;1,2}$ is not in the hyperelliptic U_{11}-orbit (Theorem 5.2).

Explicit boundaries of the conditional Lean formalization. To make the exact verification boundary completely transparent, the Lean kernel certifies the Diophantine determinant reduction, the Galois/Cauchy bridge, the genus-1 classical Jacobi and arbitrary-genus ($g \ge 1$) Riemann theta-gauge layers, the concrete genus-2 principally polarized CM surface $\mathbb{C}^2/\Lambda_{\tau_5}$, and the Section 2 Hensel/jet-drop/Stage 2–3 lemmas inside `ConditionalDeterminantEngine.lean`, `GeometricBridge.lean`, `ConcreteCMThetaModels.lean`, and `HigherGenusThetaAndZeroEstimate.lean`, *not* an unconditional formal proof of $\mu = 2$ for any concrete period. Specifically: (1) no algebraic-geometric scheme, Chow variety, Kollár simultaneous normalization, Fulton/Swanson–Huneke intersection bound on proper subvarieties $Y \subsetneq A$, contour period integral, or transcendence theorem (Huber–Wüstholz) is constructed in Lean 4; (2) no Diophantine exponent ν appears anywhere in `CMThetaGeometricInputs`, `SinglePlaceGeometricInputs`, `ConcreteEllipticCurveAlgebraicData`, or `Genus2C5AlgebraicData`, and the conjugate matrices are genuinely `pl.embedding.mapMatrix` Δ_{raw} linked through one single algebraic minor Δ_{raw} over a totally complex number field $\mathbb{L}$; (3) the transparent 2×2, $E_L = 3$, $\det = -2$ witness `lemniscate_theta_gauge_nontrivial_EL_witness` (and the degenerate $H = 1/2$, $E_L = 0$, 1×1 witnesses in genus 1 and genus 2) certify that all 29 fields of `ThetaGaugePlaceData` are jointly satisfiable with actual classical Jacobi and genus-2 Riemann theta gauges, an injective centre-index map, a meromorphic theta-ratio section, and exact infinite Taylor series at fixed weight $H = 1$ (or $H = 1/2$); however, all witnesses set $R_{\mathrm{rem}} \equiv 0$, $\ell = \mathrm{id}$, and $\alpha_{\mathrm{col}} = d_{\mathrm{row}} = 0$ (with $w = 2 > H = 1$ in the 2×2 witness, making the additive/rational-approximant branch of `row_binomial_expansion` vacuous), and they do *not* prove satisfiability as $H \to \infty$; (4) in the conditional $\mu(\Omega) = 2$ theorems (`lemniscate_e4_mu_eq_two_of_algebraic_data`, `curve_e8_mu_eq_two_of_algebraic_data`, and `genus2_c5_mu_eq_two_of_algebraic_data`), the real parameter $\Omega > 0$ is a free scalar and the hypothesis records `ConcreteEllipticCurveAlgebraicData` and `Genus2C5AlgebraicData` encapsulate `Irrational` Ω, Section 2 jet surjectivity (`Submodule.span` $= \top$), Lemma 2.9 entrywise matrix integrality, and Weierstrass/Kleinian Taylor matching to the proved theta gauge via `row_binomial_expansion`; as a Liouville-number sanity test shows (instantiating Ω with a Liouville number makes the theorem prove that the data structure is empty), the

entire Diophantine content of the conditional $\mu = 2$ theorems resides inside those hypothesis records; and (5) non-emptiness of `CMThetaGeometricInputs`, `ConcreteEllipticCurveAlgebraicData`, and `Genus2C5AlgebraicData` as $H \to \infty$, identification of the principally polarized CM surface $\mathbb{C}^2/\Lambda_{\tau_5}$ with $J(C_5^-)$ (class-number-1 uniqueness), identification of the free scalar Ω with the curve periods of E_4, E_8, or C_5^- (including $\kappa = 1/\sqrt{2}$ for E_8), and the Beta identity $B(1/8, 3/8) = 4\Omega_8$ are not proved in Lean 4 and rest completely on the prose proofs of Sections 2–5, which still require human peer review. The remaining 53 public lemmas across the five frozen modules (`Genus2QuasiPeriods`, `ArbitraryGenusCM`, `CompositeBetaCM`, `FermatQuotientBetaCM`, and `Level8And12BetaCM`, kept byte-identical to release `v1.3.0`) check elementary scalar and algebraic identities. Verbatim kernel-printed signatures, axiom sets, and digests for all 147 targets are archived in the downloadable inventories `LEAN_CONDITIONAL_ENGINE_EXACT_TYPES.md` (24 targets), `LEAN_GEOMETRIC_BRIDGE_EXACT_TYPES.md` (19 targets), `LEAN_CONCRETE_CM_MODELS_EXACT_TYPES.md` (20 targets), `LEAN_HIGHER_GENUS_THETA_EXACT_TYPES.md` (18 targets), and `LEAN_66_TARGETS_EXACT_TYPES.md` (66 targets).

All nine Lean modules are pinned to `leanprover/lean4:v4.33.1` (commit `819816b2`, Mathlib revision `0df444a3`) via `lean-toolchain`, `lakefile.toml`, and `lake-manifest.json`, and were verified by `lean`, `leanchecker`, and external kernel replay on `lean4export` (commit `15f6055e`, built under Lean 4.33.1) using `con-leche -verified`, `con-ron -verified`, and `nanoda_bin` 0.4.17 (`github.com/ammkrn/nanoda_lib`, external Rust kernel bundled in the `leanprover/lean4:v4.36.0` toolchain distribution alongside `con-leche`/`con-ron`, invoked via `nanoda_config.template.json`). The relocatable runner `scripts/reproduce_lean4.py` deterministically replays all nine modules and checks both `.olean` and `.export` SHA-256 digests against the nine archived JSON receipts.

Table 5 summarizes the six companion `SymPy` 1.14.0 and `python-flint` 0.9.0 (FLINT 3.6.0 / Arb [17]) scripts certifying our symbolic identities, finite censuses, polynomial irreducibilities, high-precision Gamma/Beta relations, and the illustrative genus-2/3 enclosures of Appendix A (exact internal Arb bit precisions: 40 dps = 136 bits, 50 dps = 169 bits, 120 dps = 402 bits, 512 bits in `beta_exact_classification.py`, and 170 dps = 568 bits).

6.2 Pure-AI provenance and author responsibility

This manuscript reports **pure AI-generated research** directed, scoped, and supervised by the human author (Robert Kadlec, Masaryk University, Brno, Czech Republic; ORCID 0009-0003-9245-8130) and forms part of a broader research project investigating the respective competencies of humans and large language models. No mathematical proof, algebraic identity, Lean 4 module, Python script, or paragraph of prose was manually written by the human author; all technical execution was carried out locally under a multi-AI workflow combining Antigravity/Gemini (problem decomposition, derivations, global box scripts, manuscript assembly, and packaging), OpenAI Codex and ChatGPT Web (section prototypes, level-8/12 classification, symbolic/numerical cross-checks, and multi-iteration adversarial review), and Anthropic Claude Code (independent read-only adversarial auditing of the proofs, citations, and companion archive).

The author takes full responsibility for the content of this paper. No AI output or internal AI audit is treated as self-certifying or as a substitute for human peer review, and human peer review has not yet been performed on this preprint. Full details of the multi-AI workflow are archived in `AI_DISCLOSURE_AND_PROVENANCE.md`.

6.3 Artifact availability and open problems

The companion release archive (CC BY 4.0 for the manuscript and documentation; Apache-2.0 for all Lean 4 sources and Python scripts) is permanently archived on Zenodo under the all-versions concept DOI `doi:10.5281/zenodo.24560090` (release `v2.2.0`, version DOI

Script	Certified symbolic and interval content	Cert. SHA-256
`certify_mise_B4_genus2_quasiperiods.py`	Genus-2 C_5^- exact Q ($\det = 1$), $\mathbb{Z}[i][[S,T]]$ jets (≤ 14), 6 odd characteristics, 568-bit Arb periods/Legendre/shifts, $r = 1/64$ polydisk $c_0 \geq 0.354345$, 8-embedding Cauchy+alias checks.	`f4ed42b0 ac240280`
`certify_mise_B4_4d_global_box.py`	402-bit zero-wrapping identities; 136-bit 512-box 2D global enclosure over $\mathbb{C}^2/\Lambda_{J_2^-}$ ($M_0 \leq 1.187637$, $M_J \leq 4.622797$); 169-bit outward-rounded literal eq. (5) witnesses ($\delta_{\mathrm{coll}} = 19/200$, Table 6).	`8cf91b6c 40a8b451`
`certify_mise_D_genus3_cm.py`	Exact Q_p, S_p ($p \in \{3,5,7,11,13\}$), all 6 $g = 3$ Baker $\wp_{ij}$, 568-bit periods/Legendre/6D zero-wrapping, 36 even theta-nulls, 136-bit Schur calibration $\lvert c_{\mathrm{hyp}}\rvert = 1/\lvert H_{13}\rvert$, polydisk $c_0 \geq 0.9314024$, 256-box bounds.	`70cbbbbf b49f5ac6`
`certify_mise_E_composite_even_beta.py`	Exact stabilizer census ($3 \leq b \leq 100$: 74 primitive, 24 split), N_b on 139 characters, $\alpha_0(z)$ division, zero residues at $\infty_\pm$, 568-bit Arb Gamma/Beta and Legendre checks ($b \in \{4,6,9,10,14,15\}$).	`2a1988d8 13a74a6e`
`certify_mise_F_fermat_asymmetric.py`	Exact census $5 \leq b \leq 180$ (171/176 primitive for $(1,2)$; exceptions $\{7,8,12,15,20\}$), Laurent exponents and N_b, 568-bit Arb + independent series-integral checks ($< 10^{-160}$) across 11 curves (8 in Table 3).	`8e9fb9e9 1cefddff`
`beta_exact_classification.py`	Exhaustive 32/32 level-8 (10) and level-12 (22) exact FLINT irreducible twist polynomials and 512-bit Arb checks (4 rational $d = 1$ periods, 28 irrational twists $d \in \{2,3,4,6\}$) plus C_5^- Baker checks.	`635835c4 ebde7db3`

Table 5: Companion `SymPy` 1.14.0 and `python-flint` 0.9.0 (Arb) certification scripts and archived JSON receipt digests (first 16 hex characters).

doi:10.5281/zenodo.24571119). It contains all nine staged `.lean` modules, their nine five-kernel verification receipts, pinned Lean/Mathlib manifests, the replay harness `scripts/reproduce_lean4.py`, all five exact-type inventories (`LEAN_*_EXACT_TYPES.md`, listed in Subsection 6.1), all six Python/`SymPy`/`python-flint` scripts and JSON certificates (total runtime ≈ 68 s under `requirements.txt`), the pure-ASCII LaTeX source and compiled PDF, and a `SHA256SUMS.txt` manifest.

Four natural open problems remain directly accessible from this framework: (i) sharpening the single-place upper bounds $\mu \le 2d$ to $\mu = 2$ for the 28 irrational-twist reduced numerator pairs at $b \in \{8, 12\}$ (notably $\mu(B(1/8, 1/8)) \le 4$ and $\mu(B(1/12, 1/12)) \le 12$); (ii) handling non-simple CM factors where individual differential forms avoid proper abelian subvarieties (bypassing the differential-annihilation obstruction of Remark 2.8); (iii) extending sharp irrationality measure 2 to individual Gamma values such as $\Gamma(1/4)$ and $\Gamma(1/3)$ or to non-CM elliptic periods; and (iv) constructing large-H inhabitants of `ConcreteEllipticCurveAlgebraicData`, `Genus2C5AlgebraicData`, and `CMThetaGeometricInputs` in Lean 4 (formalizing the algebraic geometry of Section 2 jet surjectivity, Lemma 2.9 matrix integrality, Weierstrass/Kleinian Taylor matching, and curve-to-period identifications including $B(1/8, 3/8) = 4\Omega_8$ and $B(1/5, 1/5)$) to upgrade the five-kernel-verified conditional reductions to an unconditional end-to-end formalization.

A Explicit genus-2 and genus-3 Baker–Kleinian realizations and certified global enclosures

Scope note: None of the Diophantine theorems of this paper (Theorems 4.4, 5.3, 5.4, or Corollaries 3.9, 5.6) depends on the numerical enclosures of this appendix; those theorems follow unconditionally from the prose proofs in Sections 2–5. Here we provide explicit Baker–Kleinian coordinate realizations at $g = 2, 3$, certify concrete global values of the theta constants (M_0, M_j, M_A, c_0) in Proposition 3.3 with rigorous outward-rounded Arb interval enclosures (explicit domain coverage and zero omitted tails), and exhibit illustrative scalar-budget additive dimensions at $g = 2$. All homology-cycle integrality claims below rest on the algebraic proofs in prose, while the companion Python/`SymPy`/`python-flint` scripts certify the symbolic identities, Gamma/Beta arithmetic checks, and interval bounds.

Genus two ($C_5^- : y^2 = 4x^5 - 1$). Put $S := x_1 + x_2$, $T := x_1 x_2$, and $v := (y_1 - y_2)/(x_1 - x_2)$ on $\mathrm{Sym}^2(C_5^-)$. In the Abel coordinates of $(dx/y, x\,dx/y)$, Baker's polar identities [5] give

$$\wp_{22} = S, \qquad \wp_{12} = -T, \qquad \wp_{11} = \frac{v^2}{4} - S(S^2 - T),$$

where $4(x_1 - x_2)^2 \wp_{11} = -2 + 4x_1^2 x_2^2 (x_1 + x_2) - 2y_1 y_2$. The nine basis sections of $H^0(J(C_5^-), 3\Theta)$ in the rational trivialization are

$$1,\ \wp_{22},\ \wp_{12},\ \wp_{11},\ \wp_{222},\ \wp_{122},\ \wp_{112},\ \wp_{111},\ \wp_{11}\wp_{22} - \wp_{12}^2.$$

Letting the second point tend to ∞ via $x_2 = t^{-2}$, $y_2 = 2t^{-5}(1 + O(t^{10}))$ shows that along Θ the quadratic principal coefficients are $(1, -x, x^2)$ and the cubic ones are $(2, -2x, 2x^2, -2x^3, -y)$, proving linear independence and spanning ($h^0(3\Theta) = 9$).

At $Q_0^- = [2(0, i) - 2\infty]$ (a 5-torsion point since $\mathrm{div}(y - i) = 5((0, i) - \infty)$), the symmetric coordinates (S, T) are $\mathbb{Q}(i)$-rational regular local parameters (with complex-conjugate local parameters at $\overline{Q_0^-}$, matching the Galois-conjugate charts Q_γ in Proposition 3.1). Putting $H_r := (x_1^{r+1} - x_2^{r+1})/(x_1 - x_2)$ ($H_{-1} := 0$), the identity $x_1^r dx_1 + x_2^r dx_2 = H_r\, dS - H_{r-1}\, dT$ together with $1/y = -i \sum_{j \ge 0} \binom{2j}{j} x^{5j}$ and $y = i(1 - 2\sum_{j \ge 1} \binom{2j-2}{j-1} x^{5j}/j)$ (Catalan integers) implies that the Abel Jacobian lies in $\mathrm{GL}_2(\mathbb{Z}[i][[S, T]])$ (constant term $\mathrm{diag}(-i, i)$) and that *every* basis

section s_r ($0 \le r \le 8$) and both second-kind differentials $d\zeta_k = -\sum_j \wp_{kj}\, du_j$ have $\mathbb{Z}[i]$-integral Taylor coefficients to all orders in (S,T) (and identically in the conjugate chart at $\overline{Q_0^-}$). In particular, applying Lemma 2.9 and Theorem 3.5 across the two conjugate charts allows $B_{\mathrm{den}} = 1$ on C_5^- around Q_0^- and $\overline{Q_0^-}$, so $\log B_{\mathrm{den}}/v_0 = 0$ and $\log(4B_{\mathrm{den}}) = 2\log 2$. The staged script `certify_mise_B4_genus2_quasiperiods.py` verifies these integral expansions symbolically to ordinary degree 12 (sections) and degree 14 (differentials/primitives).

For any odd level $b = 2g_{\mathrm{tot}} + 1$ on C_b^-, let $c_j := \alpha_*^j \gamma_{\mathrm{real}}$ ($0 \le j < 2g_{\mathrm{tot}}$). The finite rays meet only at ∞, where in local parameter $x^{g_{\mathrm{tot}}}/y$ their tangents rotate by $2\pi g_{\mathrm{tot}}/b$, giving the alternating intersection matrix $(Q_b)_{ij} = (-1)^{j-i}$ ($i < j$). Defining $a_k := c_{2k} + c_{2k+1}$ and $b_k := c_{2k+1} + \sum_{r=k+1}^{g_{\mathrm{tot}}-1} a_r$ ($0 \le k < g_{\mathrm{tot}}$) yields a unimodular integer change of basis S_b satisfying $S_b Q_b S_b^t = \left(\begin{smallmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$, certifying that (a_k, b_k) is an integral symplectic basis of $H_1(C_b^-, \mathbb{Z})$ (explicitly checked for $b \in \{3,5,7,9,11,13\}$).

For C_5^-, let W_j, e_j be the real ω_j- and ρ_j-periods, $\kappa_j := e_j/(2W_j) > 0$, $u = \Omega_A x + \Omega_B y$ ($x, y \in \mathbb{R}^2$), and $Q(u) := \sum_{j=1}^2 \kappa_j |u_j|^2$. Define the $\Lambda_{J_2^-}$-periodic covariant metrics $g_0 := e^{-Q}|\sigma|$ and $g_j := e^{-Q}|\partial_{u_j}\sigma - 2\kappa_j \bar{u}_j \sigma|$ ($j = 1,2$), with $M_0 := \sup g_0$, $M_j := \sup g_j$, and $M_J^3 := \max_{0 \le r \le 8} \sup(g_0^3 |s_r|)$. Writing $\tau = \Omega_A^{-1}\Omega_B = X + iY$, the script `certify_mise_B4_4d_global_box.py` certifies the exact zero-wrapping identities $\mathrm{Re}(\frac{1}{2} u^t M_{\mathrm{sym}} u) - \sum_j \kappa_j |u_j|^2 = -\pi y^t Y y$ and $M_{\mathrm{sym}} u - 2\,\mathrm{diag}(\kappa)\bar{u} = 2\pi i (\Omega_A^t)^{-1} y$ across the full 4D real basis at 120-digit (402-bit internal Arb precision in `python-flint` 0.9.0) to $< 2 \times 10^{-117}$. Absorbing $e^{-\pi y^t Y y}$ into every theta term eliminates exponential wrapping and, via the triangle inequality over unit-modulus x-phases, produces pure 2D majorants in (y_0, y_1) valid uniformly for all $x \in \mathbb{R}^2$. By parity $f(-u) = f(u)$, the half-torus $[0, 1/2] \times [-1/2, 1/2]$ (exact area $1/2$) covers 100% of $\mathbb{C}^2/\Lambda_{J_2^-}$. Covering $[0,1/2] \times [-1/2,1/2]$ with $16 \times 32 = 512$ exact Arb boxes of side $1/32$ at 40-digit (136-bit internal Arb precision), using theta cutoff $N = 6$ ($13^2 = 169$ terms) and adding explicit analytic geometric-series upper bounds for all omitted Gaussian tails ($t_0 \le 8.19 \times 10^{-22}$, $t_1^{\mathrm{cov}} \le 3.59 \times 10^{-20}$, $t_2 \le 1.39 \times 10^{-18}$, $t_3 \le 5.67 \times 10^{-17}$) with outward Arb `.upper()` rounding throughout, certifies globally over all of $\mathbb{C}^2$:

$$M_0 \le 1.187637, \qquad M_1 \le 1.255992,$$
$$M_2 \le 1.339228, \qquad M_J \le 4.622797.$$

In `certify_mise_B4_genus2_quasiperiods.py` (run at 170 decimal digits, i.e., ≥ 512 bits / 568 internal bits, recorded under `flint_512bit_checks` with `precision_bits_min: 512`), evaluating the odd-characteristic theta series on the polydisk $\max(|S|,|T|) \le r_{\mathrm{an}} := 1/64$ with outward `.lower()` rounding and explicit tails yields $c_0 \ge 0.354345$, and at the centre $|g_0(u_0) - 1| < 2.75 \times 10^{-167}$. The same script encloses all six odd-characteristic gradients (Baker gradient $(1.13746122339 \pm 4 \times 10^{-12}, < 8 \times 10^{-168})$; the remaining five have $|u_2\text{-comp}| = 1.00370502274 \pm 4 \times 10^{-12}$ and ratio $1.31950791077 \pm 3 \times 10^{-12}$) and all eight $\mathrm{Gal}(\mathbb{Q}(\zeta_{20})/\mathbb{Q})$ pointwise and 2D Cauchy-plus-alias coefficient checks at $(S,T) = (1/200, 1/400)$.

Genus-two scalar-budget additive dimensions (Table 6). To illustrate the concrete size of the dimension parameter $m = 2\ell$ needed to satisfy the unified scalar budget (5) of Theorem 3.5 for the genus-2 targets $\pi/B(k/5, k/5)$ ($k = 1,2$) at $\nu = 2 + \varepsilon$, inserting the certified global constants $(M_0, M_J, c_0, r_{\mathrm{an}}) = (1.187637, 4.622797, 0.354345, 1/64)$ into (5) with split parameters $\rho_{\mathrm{sp}} = 99/100$ and $\delta_{\mathrm{coll}} = 19/200$ tests the literal inequalities

$$E_{\mathrm{base}}^{\mathrm{cons}} < \frac{99}{100} g_* \qquad \text{and} \qquad c_2 \lim_{H \to \infty} L_2(H) > 1 + E_{\mathrm{base}}^{\mathrm{cons}} + \frac{1}{200} g_* + \frac{19}{200}$$

(leaving explicit positive slack $\frac{1-\rho_{\mathrm{sp}}}{2} g_* = \frac{1}{200} g_*$ for E_{rem} and contradiction margin $\Delta_{\mathrm{gap}} = \frac{1}{400} g_*$). Because the Taylor expansions on C_5^- are $\mathbb{Z}[i]$-integral ($B_{\mathrm{den}} = 1$, hence $\log B_{\mathrm{den}}/v_0 = 0$

and $\log(4B_{\mathrm{den}}) = 2\log 2$), the script `certify_mise_B4_4d_global_box.py` (running the scalar-budget phase at 50 decimal digits, i.e., 169 internal Arb bits, recorded dynamically in the JSON certificate) uses the strictly conservative upper bound

$$E_{\mathrm{base}}^{\mathrm{cons}} := \frac{4\log(2)\, mF_0 + 2\log(1/r_{\mathrm{an}})}{v_0^{\mathrm{lo}}} + \frac{\nu}{F_0} + \frac{3\log M_* + 3G_R^{\mathrm{hi}} + 3\log^+(1/c_0)}{w_0} \;\geq\; E_{\mathrm{base}},$$

where $v_0^{\mathrm{lo}} := 2(C^{\ell/2}-1)^2\theta^\ell w_0 \leq v_0$, $M_* = \max(1, M_0, M_A^{1/3}) = \max(1, M_0, M_J)$ (matching Theorem 3.5 since $M_A = M_J^3$), and $G_R^{\mathrm{hi}} := (\kappa_1+\kappa_2)((8W_{\max}+2)C^{\ell/2})^2 \geq G_R$: indeed, because J_2^- is already simple, taking the real oval $\lambda_0 = \gamma_{\mathrm{real}}$ directly gives $\|\lambda_j^{(\iota)}\|_\infty \leq 4(N_{\mathrm{box}}-1)W_{\max}$, while the script certifies in Arb that the Abel–Jacobi lift u_0 of Q_0^- satisfies $\|u_0\|_\infty \leq 2.735235$ and the local chart on $|S|, |T| \leq 1/64$ satisfies $\|\ell(S,T)\|_\infty \leq 0.015748$, whence $\|u_0\|_\infty + \|\ell(S,T)\|_\infty < 2.76 < 4W_{\max} \approx 10.405$ and therefore $R = 2\sup\|u_0 + \lambda_j^{(\iota)} + \ell(S,T)\|_\infty \leq 2(4(N_{\mathrm{box}}-1)W_{\max} + 2.76) < (8W_{\max}+2)C^{\ell/2}$ for every $N_{\mathrm{box}} \geq 1$. Table 6 lists the complete exact rational witness tuples $(\theta, A_{\mathrm{cut}}, \beta_*, C, \varrho, F_0, g_*)$, the minimal integers ℓ_{suff} satisfying literal (5) (with $\ell_{\mathrm{suff}}-1$ verified to fail), and strictly outward-rounded bounds from `mise_B4_4d_global_box_cert.json` (every decimal string and every IEEE-754 float—converted via exact binary ratio `as_integer_ratio()`—is verified in Arb both before and after JSON serialization so that every printed upper bound strictly exceeds the true value while staying below $\frac{99}{100}g_*$, and every printed lower bound lies strictly below the true value while exceeding the right-hand side of (5)). *Important caveat:* because the geometric weight thresholds W_i in Theorem 2.2 are ineffective (arising from Chow-variety boundedness in Lemma 2.4), Table 6 records only the scalar-budget dimension $m_{\mathrm{suff}}(\varepsilon)$ for (5) and does *not* yield an effective irrationality measure.

Table 6: Genus-two scalar-budget additive dimensions $m_{\mathrm{suff}} = 2\ell_{\mathrm{suff}}$ for $\pi/B(k/5, k/5)$ $(k = 1, 2)$ at $\nu = 2+\varepsilon$ with complete exact rational witnesses and strictly outward-rounded bounds from `mise_B4_4d_global_box_cert.json` ($\rho_{\mathrm{sp}} = 99/100$, $\delta_{\mathrm{coll}} = 19/200$, $B_{\mathrm{den}} = 1$ conservatively bounded via $4\log 2$, $r_{\mathrm{an}} = 1/64$, $c_0 \geq 0.354345$; each ℓ_{suff} is minimal for literal (5)). Because the thresholds W_i of Theorem 2.2 are ineffective, no effective irrationality measure follows.

Parameter / Bound	$\varepsilon = 0.5$ $(\nu = \frac{5}{2})$	$\varepsilon = 0.1$ $(\nu = \frac{21}{10})$	$\varepsilon = 0.01$ $(\nu = \frac{201}{100})$	$\varepsilon = 0.001$ $(\nu = \frac{2001}{1000})$
θ	25/36	441/484	40401/40804	4004001/4008004
A_{cut}	247/300	1469/1540	134669/135340	1906667/1907620
β_*	227/250	9767/10000	12469/12500	4998751/5000000
C	6/5	22/21	202/201	2002/2001
ϱ	1/618	1/11018	1/1010018	1/100100018
F_0	1 313	20 213	1 776 338	175 262 588
g_* (exact)	$\frac{989}{74160}$	$\frac{17629}{24239600}$	$\frac{1616029}{204023636000}$	$\frac{160160029}{2004002360360000}$
g_* (approx.)	1.33360×10^{-2}	7.27281×10^{-4}	7.92079×10^{-6}	7.99201×10^{-8}
$E_{\mathrm{base}}^{\mathrm{cons}}$ upper	1.904037×10^{-3}	1.038936×10^{-4}	7.841567×10^{-6}	$7.91208779 \times 10^{-8}$
$c_2 \lim L_2$ lower	1.097608	1.095140	413 834.424384	1 702.494156
Eq. (5) coll. margin $>$	6.3729×10^{-4}	3.327×10^{-5}	4.13833×10^{5}	1.70139×10^{3}
ℓ_{suff} (minimal)	15 970	609 150	12 962 365	1 553 749 758
$m_{\mathrm{suff}} = 2\ell_{\mathrm{suff}}$	**31 940**	**1 218 300**	**25 924 730**	**3 107 499 516**

Genus three ($C_7^\pm : y^2 = 4x^7 \pm 1$). Over $\mathbb{Q}(i)$, $(x,y) \mapsto (-x, iy)$ maps C_7^+ to C_7^-. Writing e_1, e_2, e_3 for the elementary symmetric polynomials in x_1, x_2, x_3, the six rational Baker functions on C_7^- are $\wp_{33} = e_1$, $\wp_{23} = -e_2$, $\wp_{13} = e_3$, and—setting for $\{i,j,k\} = \{1,2,3\}$

$$R_k := \frac{-2 + 4x_i^3x_j^3(x_i+x_j) - 2y_iy_j}{4(x_i-x_j)^2} - \wp_{13}(x_i^2+x_j^2) - \wp_{23}x_ix_j(x_i+x_j) - \wp_{33}x_i^2x_j^2$$

and defining $r_0 + r_1z + r_2z^2 := \sum_{k=1}^3 R_k \prod_{j\neq k}(z-x_j)/(x_k-x_j)$—yields $\wp_{22} = r_2$, $\wp_{12} = -r_1 - e_1r_2$, and $\wp_{11} = r_0 - e_1\wp_{12} - e_2r_2$ (by Baker's polar identity $R_k = \wp_{11} + (x_i+x_j)\wp_{12} + x_ix_j\wp_{22}$), each

with pole order at most 2 along Θ [5]. If $w_0'+w_1'z+w_2'z^2$ interpolates y_i at x_i, the exact primitive corrections to the symmetric sums of ρ_1, ρ_2, ρ_3 are $(w_1' + e_1 w_2'/2,\ w_2'/2,\ 0)$.

On the hyperelliptic genus-3 curve C_7^-, there is a unique effective even theta characteristic (the hyperelliptic pencil g_2^1, whose square is canonical). In the (Q_7, S_7) marking it is $\delta_{\mathrm{hyp}} = \frac{1}{2}(1,1,1;1,0,1)$: enclosing all 36 even theta constants with explicit tails ($< 3.65 \times 10^{-7}$) shows the other 35 satisfy $|\theta[\delta](0)| \geq 0.558596$. In Nakayashiki's Schur normalization [27], the leading weight-6 term is the Jacobi–Trudi polynomial for $(3,2,1)$ in (u_3, u_2, u_1), namely $u_1u_3 - u_2^2 - \frac{1}{3}u_2u_3^3 + \frac{1}{45}u_3^6$. Since the theta-null and gradient vanish at 0, multiplication by the Baker exponential does not alter the quadratic part, so the calibration scalar converting raw $\theta[\delta_{\mathrm{hyp}}]$ to Schur σ is $|c_{\mathrm{hyp}}| = 1/|H_{13}| \in 0.817670994878909 \pm 2.82 \times 10^{-16}$ (verified with Hessian cube $N = 5$ and tail $t_2 \leq 2.19 \times 10^{-16}$).

At $Q_0^- = [3(0, i/2) - 3\infty]$ in the monic model $y'^2 = x^7 - 1/4$, letting z lift $[(0, i/2) - \infty]$, one has $7z \in \Lambda$ and $u_0^- = 3z$. By Ônishi's division identities [29, Definition 9.2 and Theorem 9.3], $|\psi_n(z)| = |\sigma(nz)|/|\sigma_\sharp(z)|^{n^2}$ for $n = 3, 4$, where $\psi_3(z) = \pm(2y'(P))^3$ and $\psi_4(z) = \pm \det(D^r x^s)_{1\leq r,s\leq 3}/(1!2!3!)$ ($D = 2y'd/dx$), which is lower triangular at $x = 0$ with diagonal $r!(2y')^r$, giving $|\psi_3(z)| = |\psi_4(z)| = 1$. Setting $t := e^{-Q(z)}|\sigma_\sharp(z)|$, quadratic scaling $Q(nz) = n^2Q(z)$ gives $g_0(3z) = t^9$ and $g_0(4z) = t^{16}$; evenness, periodicity, and nonspeciality of $3P$ give $g_0(4z) = g_0(-3z) = g_0(3z) > 0$, forcing $t = 1$ and hence $g_0(u_0^-) = 1$ identically.

In `certify_mise_D_genus3_cm.py`, period matrices, Legendre relations, and 6D zero-wrapping identities ($< 1.83 \times 10^{-166}$) are computed at 170 digits (568 bits); all theta enclosures run at 40 digits (136 bits) with outward Arb rounding and zero omitted tails. Covering $[0, 1/2] \times [-1/2, 1/2]^2$ (exact y-volume 1/2, covering 100% of $\mathbb{C}^3/\Lambda_{J_3^-}$ by parity and x-phase majorization) with 256 cubes of side 1/8 (tails $t_0 \leq 9.81 \times 10^{-10}$, $t_1 \leq 6.34 \times 10^{-8}$) certifies global bounds for (g_0, g_1, g_2, g_3):

$$\begin{aligned} \text{raw } \theta[\delta_{\mathrm{hyp}}]: &\quad (2.4267,\ 3.0570,\ 4.0926,\ 3.5890), \\ \text{Schur-normalized } \sigma: &\quad (1.984216565,\ 2.499547713,\ 3.346382380,\ 2.934553580), \end{aligned}$$

with central value $g_0(u_0^-) \in 1 \pm 1.07 \times 10^{-15}$ and Schur polydisk lower bound $c_0 \geq 0.9314024$ on $\|\Delta u\|_\infty \leq 1/128$.

Composite and even-level computational cross-checks. The staged script `certify_mise_E_composite_even_beta.py` checks all 98 stabilizers for $3 \leq b \leq 100$ (74 trivial, 24 non-trivial), evaluates N_b on 139 characters across $b \in \{6, 9, 10, 14, 15, 18, 21, 25, 30\}$, verifies polynomial clearing at $b \in \{6, 10, 14, 18, 30\}$, and performs 568-bit Arb Gamma/Beta and Legendre arithmetic checks at $b \in \{4, 6, 9, 10, 14, 15\}$ (bounding the $b = 9$ normalized mixed Legendre residual by 1.64×10^{-168}, the $b = 9$ new quasi-period residual by 3.36×10^{-169}, and the $b = 15$ maximum scaled new residual by 2.24×10^{-167}, with second-kind clearing multipliers $18, 50, 98$ at $b = 6, 10, 14$).

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