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One-runner extension and phase stability of a 57-runner shifted configuration

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Abstract

Fix the 57-runner configuration credited to `tomdif` in the public shifted lonely runner certificate collection. Over every additional distinct positive integer speed and arbitrary real phase, the optimal covering radius is 1359451/79000000. A single speed, 528344, attains it, with a closed interval of optimal phases. The exact radius and a perturbation barrier are Lean-verified. An independent weighted residual certificate shows that seven extra runners, even with repeated speeds, cannot cover the time circle at radius 1/59. Any 58-runner cover at that radius shares at most 50 speed–phase pairs with the fixed configuration. We also give restricted exclusion computations for coordinated phase changes and for several deletion-and-insertion problems.

1 Setup

Let $\|x\| = \min_{k\in\mathbb{Z}} |x-k|$. For a family C of speed–phase pairs (v,s) with $v \in \mathbb{Z}_{>0}$ and $s \in \mathbb{R}/\mathbb{Z}$, define

$$d_C(t) = \min_{(v,s)\in C} \|vt+s\|, \qquad \gamma(C) = \max_{t\in\mathbb{R}/\mathbb{Z}} d_C(t).$$

The observer at the origin contributes no moving runner. A shifted lonely runner counterexample with n distinct speeds satisfies $\gamma(C) < 1/(n+1)$. Blanco, Criado and Santos give counterexamples for $5 \le n \le 17$ [1, Theorem 1.8], and Poliakova gives them for $n \ge 95$ [2, Theorem 2]. The public collection [4] gives exact rational certificates at each $n = 18, \dots, 57$, since verified in Lean [5]. Zhang's Question 8.5 asks both about all sizes $n \ge 5$ and about global minimizers [3]. Here the primary results hold for one specified seed, and the later tests use specified speed and phase restrictions.

Let B be the 57-runner configuration from `certificates.json` in the public collection at commit `75a0b27a`. Its coordinates are reproduced in the companion `known_slr_57.json`. The exact value $\gamma(B) = 43021/2500000$ comes from that certificate. The public verifier checks all 40 covers in rational arithmetic and recomputes the maximum at $n = 57$ by its full event enumeration. We use $d_0 = 1/59$ for the 58-runner target.

For time intervals at radius $d \in (0, 1/2)$, the bands of (v,s) are $[(k-s-d)/v, (k-s+d)/v]$ with $k \in \mathbb{Z}$. For strict coverage take the bands open. We work on $[0,1]$ with 0 and 1 identified. The v bands modulo one for a single runner are disjoint, apart from the representation at this seam.

2 Exact optimum with one extra runner

Theorem 1. *Among configurations obtained from B by adding one runner with a distinct positive integer speed and arbitrary real phase, the least covering radius is*

$$d_* = \frac{1359451}{79000000}.$$

The speed is uniquely 528344*. At that speed the full set of optimal phases modulo one is the closed interval with left endpoint* 4737613/31600000 *and length* 178783/158000000*.*

Proof. For weak coverage at a specified d, sweep and merge the closed bands of B. The complement comprises the open gaps where $d_B(t) > d$. A new runner covers an open gap if and only if it covers its closure, by continuity. For the seed here 0 is covered. At $d = d_*$ there are eleven positive-length gaps. The longest has length 53/1264000000. Since the phase image of a time interval $[a, b]$ is connected, covering it requires $v(b - a) \le 2d$. In particular any possible speed at d_* is at most 820800.

For each candidate speed v, order the closed gaps by decreasing length and write the first as $[a_0, b_0]$. The gap $[a, b]$ restricts the phase to the circular window

$$[-va - d,\ -vb + d] \pmod 1, \tag{1}$$

empty when the endpoints are reversed. With $u = s + va_0 + d \bmod 1$, the first window sets $0 \le u \le 2d - v(b_0 - a_0)$. Every other window starts at $v(a_0 - a) \bmod 1$ and has length $2d - v(b - a)$. Choose each start in $[-1/2, 1/2]$, and discard negative lengths. Here $2d < 1/4$, so the range of u can intersect only this representative, without a translate by a nonzero integer. Intersect the closed intervals in this coordinate.

For $d = d_*$ the finite enumeration leaves exactly one new speed and the stated interval of phases. The arithmetic can be reproduced without floating-point operations. Multiply d and the gap endpoints by their least common denominator D. For each $1 \le v \le 820800$ absent from the seed, initialize

$$(\ell, h) = (0,\ 2dD - vD(b_0 - a_0)).$$

For every other gap set z to $vD(a_0 - a)$ reduced modulo D into $[-D/2, D/2]$. Update

$$\ell \leftarrow \max(\ell, z), \qquad h \leftarrow \min(h, z + 2dD - vD(b - a)).$$

Drop the speed as soon as $\ell > h$. The single surviving interval, after dividing by D and shifting by $-va_0 - d$, is as stated. At its left endpoint the eleven closed gaps also pass a direct test of (1).

For the lower bound use

$$t_* = \frac{38586773}{79000000}.$$

At this point all seed distances are at least d_*, with equality for speeds 44 and 114. These inequalities are checked by substituting the rational seed. The distance of the only feasible new speed at its leftmost phase is 66324111/158000000 there. Changing the phase anywhere inside the feasible interval can decrease the distance by at most 178783/158000000, so the new distance is still strictly greater than d_*. Thus no weak cover at d_* can have smaller radius. The isolated point t_* belongs to the complement of the *open* seed bands at d_*; the lower-bound argument uses it even though the complement of the closed bands has only positive-length gaps. Source and certificate are in `optimal_extension.py` and `optimal_extension.json`. □

An independent Lean proof of the radius uses six pairs of rational times where all seed distances are at least d_*. For every $1 \le v \le 820800$ at least one pair (a, b) satisfies $\|v(a - b)\| \ge 2d_*$. The Lean

kernel checks the finite modular inequality. If the new runner had distance strictly less than d_* at both times, the triangle inequality on $\mathbb{R}/\mathbb{Z}$ would contradict it. For larger v the proof uses two times separated by $2d_*/v$ inside the longest closed gap of the seed coverage. The upper bound in Lean uses a chain of rational closed bands for the attaining configuration. This proves the radius with no bound on speeds and no assumption of rational phases. The uniqueness and phase interval in Theorem 1 use the Python enumeration.

Corollary 2. *Keep the 57 speeds of B and move each of its phases by at most ϵ in circular distance. With any additional runner the covering radius is at least $d_* - \epsilon$. At $\epsilon = 1207609/4661000000$ this is $1/59$.*

Proof. For fixed time and speed the distance to the nearest integer is 1-Lipschitz in the phase on $\mathbb{R}/\mathbb{Z}$. The same Lipschitz bound, in the maximum displacement over runners, holds after taking a minimum over runners and a maximum over time. Compare to the old phases with the extra runner unchanged. The value $d_* - 1/59 = 1207609/4661000000$ is exact. A common change of time origin replaces all phases by $s_i + v_i t_0$ without changing the maximum. The barrier holds in any box about such a time translate of B. □

3 A weighted obstruction to seven additions

Theorem 3. *If all runners of B are retained, appending any seven runners of positive integer speed and arbitrary real phase cannot achieve radius at most $1/59$. Repeated speeds among the extra runners or between the extra and retained runners are allowed. Consequently a 58-runner configuration of radius at most $1/59$ can contain at most 50 of the original 57 speed–phase pairs.*

Proof. Use $Q = 59000000$ and the coordinate $x = Qt$. The open gaps (a, b) in the closed bands of B at d_0 number 207. In each take the integer interval $[\lceil Qa \rceil, \lfloor Qb \rfloor]$ and use its interior. The companion file `residual_weights.json` gives the nonnegative integer density w_j on each, in increasing order of a. The 121 positive-length intervals of positive weight have total weight $W = \sum_j w_j = 303345$ and total weighted length $T = \sum_j w_j(b_j - a_j) = 153120431$, where a_j, b_j are the integer endpoints. Normalize to probability mass by dividing the densities by T. For each supported interval, each old runner's phase on it is between $k + d_0$ and $k + 1 - d_0$ for one integer k, as a rational endpoint check confirms. Its interior lies outside all the old bands, and the seed covers measure zero for this probability measure.

At speed v, change coordinates again to $y = vx$. Modulo Q, the covered y-coordinates lie in a sliding window of length $U = 2d_0Q = 2000000$. The contribution from $[a_j, b_j]$ consists of $\lfloor v(b_j - a_j)/Q \rfloor$ whole periods and one remainder arc. Each period contributes U to the intersection length. The remainder contributes the arc's intersection length with the sliding window. The sum of these lengths, with weights w_j, divided by vT, is the covered mass. For fixed v the length is continuous and piecewise affine in the window start. Breakpoints occur where one of the window endpoints meets a remainder endpoint, which are integral coordinates modulo Q.

For $1 \le v \le 59940$ we enumerate those breakpoints. A verifier forms the piecewise constant density on the residue circle by adding the weights of the remainder arcs, integrates it by a prefix sum of integer rectangle areas, and subtracts the values at the two window endpoints. Wrapped arcs are split at zero or handled by periodic extension of the integral. For each v we add the independent whole-period term and compare exact rational masses. The largest over the finite range is

$$\beta = \frac{20159589}{153120431},$$

attained for speed 160 and window start 9192800. The second verifier computes the same maximum by accumulating slope changes of the window intersection function. For every $v \ge 59941$, the whole

periods contribute at most U/Q to the normalized mass, and the remainders contribute at most $UW/(vT)$. Thus the mass is bounded by

$$\frac{U}{Q}+\frac{UW}{vT} \;\le\; \frac{7735870501306}{77359044788527} \;<\; \beta.$$

Seven new runners cover mass at most $7\beta = 141117123/153120431 < 1$. This proves the obstruction even with weak coverage. If a 58-runner cover kept at least 51 of the old pairs, restoring every missing old pair would leave all 57 old pairs and at most seven additional pairs, a contradiction. □

Linear programming proposed the weights and two exact interval checks reproduce the bound. With all weights equal the precursor calculation `residual_mass.py` gives a weaker mass cap 12371/51040 and hence at least five extra runners. An earlier direct two-runner calculation uses just 20 of the seed gaps.

On an interval of length L, a runner of speed v covers measure at most $2d_0L + 2d_0/v$, by splitting into full periods and one remainder. For two runners with speeds $v \le w$, the longest gap, of width 26269/295000000, forces $v \le 4d_0/((1-4d_0)L)$, hence $v \le 816$. Divide the slower phase into 32 closed boxes of half-width 1/64. A gap with midpoint t and width l is missed throughout a box centred at s if $\|vt+s\| > d_0 + 1/64 + vl/2$. For each of the 26112 speed-box pairs among the 20 widest gaps at least eight are missed. The fast speed has finite upper bound $2d_0/l$ from the widest such gap, and for every two missed midpoints t_i, t_j it must satisfy $\|w(t_i - t_j)\| \le 2d_0$. All candidate sets are empty by rational arithmetic. A separate single-runner check eliminates the unused speeds through 380 at d_0 using the four widest gap constraints. The two-runner calculation permits both speeds to coincide.

4 Deletion and insertion with fixed surviving phases

We also tested reducing each of two fixed configurations by one runner. The first seed is the zero-phase speed set $1, \dots, 59$ and the second is Poliakova's 95-runner construction, built by applying rule A to $2, 4, \dots, 192$ with its phases evaluated at 1/97 [2, Theorem 7]. The target radii are 1/59 and 1/95, respectively. This section uses strict coverage. Each new speed must be distinct from the surviving speeds, and the two new speeds, when present, distinct from each other. A deleted speed may be re-used with a different phase.

Deleting two and inserting one

Sweep the endpoints of all original open bands. Record each endpoint and open cell with its exact set of covering runners whenever that set has size at most two. At an endpoint first remove arcs ending there, record the endpoint, and then insert arcs starting there. Deleting a specified pair exposes exactly the pieces whose covering sets lie in the pair. The set so exposed is closed, so we can merge the closures of its cells and include every exposed singleton. If the widest residual component has positive length L, the new speed is at most $\lfloor 2d/L \rfloor$ and its allowable phase windows are the open form of (1), including the constraints of singletons. If the residual has only rational singletons, a multiple of their denominators hits all of them at phase zero. An unused such speed exists. The empty residual requires no phase constraint. The resulting complete finite computations give the following.

Target	Deletion pairs	Largest speed bound	Feasible
58	1711	170	0
94	4465	580	0

See `single_replacement.py` and its exact results.

If the exposed compact set is symmetric about time zero and $d < 1/4$, a strict single-runner cover can be chosen at phase 0 or $1/2$. Indeed for each exposed pair $t, -t$, both $s + vt$ and $s - vt$ are within d of integers k, l, respectively. The average of the signed errors is $s - (k + l)/2$ and half their difference is $vt - (k - l)/2$. Modulo one there is a unique phase in $\{0, 1/2\}$ within d of s, independent of t. It is the residue of both half-integers $(k + l)/2, (k - l)/2$. Thus the corresponding phase gives distance less than d at every exposed time. The computation above treats the continuous phases directly in both seeds.

Deleting three and inserting two

We selected twelve triples per seed by the smallest number of exposed positive-length cells before merging, breaking ties by seed order. Sweep all open bands as above, now keeping covering sets of size up to three. For each triple let H be the union of closed residual components. We reduce the search on H as follows.

Let $v < w$ be the new speeds and let $S = 2d/v,\ F = 2d/w$ be their band lengths. For $d < 1/4$ any complete gap between two slow bands is longer than a fast band. Thus each covered component of H can meet at most one slow band. Its two portions outside that band each have length strictly less than F because each is contained in one open fast band. If it misses the slow bands entirely its length is less than F. Therefore the maximum length L of a residual component requires

$$L < S + 2F < 6d/v.$$

This caps v, and caps w whenever $L > S$. When a component $[a, b]$ has length at least F, it must meet a slow band, whose endpoints must be strictly below $a + F$ and strictly above $b - F$, respectively. This requires the slow phase to lie in

$$(-d - va - vF,\ d - vb + vF) \pmod 1.$$

Intersect these necessary conditions from all qualifying components. Failure persists as w increases, since F decreases. Two more finite filters can be used. If $g = \gcd(v, w)$ and $gL \geq 1$, then one common period $1/g$ inside H cannot be covered by the two runners, whose combined covered fraction is at most $4d < 1$. If $L \geq S + F$, both wings of a longest component require distinct fast bands. If their centers differ by m/w, the component's endpoints force $L < m/w + F$, while both fast bands meeting the respective slow-band boundaries forces $m/w - F < S$.

For each remaining fixed speed pair, split the slow and fast phase axes at the finitely many values where a missed time band starts or stops meeting a component of H. For a speed v and phase s its closed missed band with index k is $[(k + d - s)/v, (k + 1 - d - s)/v]$. The set of relevant indices is constant on the open phase cells. For a fast phase b_0, a missed time in one residual component exists precisely when that component and one closed gap from each runner meet pairwise. Three closed real intervals meet all together if they meet pairwise. The two gaps with indices k, j meet if and only if

$$w(k + d) - v(j + 1 - d) \ \leq\ ws - vb_0 \ \leq\ w(k + 1 - d) - v(j + d).$$

Thus on each open phase rectangle feasibility is the existence of a point in its open interval of values $ws - vb_0$ outside a finite union of closed intervals. For fixed k, consecutive j's translate these forbidden blocks by v. Their common length $(w + v)(1 - 2d)$ exceeds v, so they can be merged before the interval-union test. A cover strict on the compact set H has a positive margin and persists under small phase changes. It is sufficient to test only open phase rectangles, with every time endpoint still included in H.

For the 24 selected triples the exact solver proved infeasibility. It used a fallback speed cap of 2500, but for each allowed slower speed either a finite analytic bound covered the entire fast range or the monotone phase condition excluded the entire remaining tail. The solver distinguishes these global exclusions from cap-limited negative outcomes. The exact solver is `two_speed_search.py`. The driver `triple_replacement.py` selects the triples and `triple_replacement.json` records each verdict. The following are the triples, listed as speeds.

Target 58	Target 94
$(1, 2, 6)$	$(2, 3, 9)$
$(1, 2, 59)$	$(3, 4, 9)$
$(1, 3, 4)$	$(2, 4, 9)$
$(1, 3, 59)$	$(4, 6, 9)$
$(1, 4, 6)$	$(2, 3, 4)$
$(1, 4, 59)$	$(2, 3, 6)$
$(1, 6, 59)$	$(2, 6, 9)$
$(2, 3, 59)$	$(3, 4, 6)$
$(2, 6, 59)$	$(3, 6, 9)$
$(1, 2, 3)$	$(2, 3, 16)$
$(1, 2, 4)$	$(2, 9, 16)$
$(2, 3, 4)$	$(3, 9, 16)$

5 Tests in phase boxes

For fixed distinct positive speeds and $0 < d < 1/2$, the family of open runner bands covers $\mathbb{R}/\mathbb{Z}$ if and only if every right endpoint of every band lies in another band. For necessity the band that ends there is open and no other band of the same runner contains that endpoint. For sufficiency, a proper finite nonempty union of open arcs has a boundary point at a right endpoint of one arc, including the case of a singleton complement next to it. By compactness pointwise strict coverage is equivalent to $\gamma < d$.

Let the nominal phases be s_i and vary them to $s_i - v_i y_i$ with $|v_i y_i| \le \epsilon$. For each original right endpoint $e = (k + d - s_i)/v_i$, the exact condition is that at least one choice of $j \ne i$ and $l \in \mathbb{Z}$ satisfies

$$\frac{l - s_j - v_j e - d}{v_j} < y_i - y_j < \frac{l - s_j - v_j e + d}{v_j}.$$

The bounds on the delays exclude all but finitely many l. After picking a disjunct for each endpoint, this is a system of strict and weak difference inequalities with a zero reference for the delay box.

For rational nominal phases, radius and half-width let D be the product of their common denominator and the lcm of the speeds. All bounds after scaling delays by D are integral. A real feasible selection has nonnegative weight on every directed cycle and strictly positive weight on each cycle using a strict edge. Every simple cycle has at most $n + 1$ edges, so tightening each strict bound by $1/(n + 1)$ leaves all cycle weights nonnegative. Multiply again by $n + 1$. Integer shortest-path potentials then give a point satisfying the tightened constraints. Subtract the potential of the reference from every coordinate to reset it to zero. Conversely any point on this grid satisfying those tightened constraints solves the original strict ones. This reduction is exact for continuous phases and time. The interval and difference-constraint approach is related to [1, Section 5].

At $d = 1/59$ the exact integer-difference encoding in Z3 returned infeasible in the following two boxes.

Fixed speeds and nominal phases	Half-width	Endpoints
$1, \ldots, 58$, zero phases	$3/125$	1711
Stored 58-runner candidate	$1/32$	3217

A separate real-arithmetic Z3 encoding independently excludes the standard-speed case at half-width $1/50$ using 203 endpoints. The two larger results use solver verdicts from the integer encoding. The real encoding timed out on those larger boxes. The nominal phases for the second row are the exact rationals in `chain58_box.json`, obtained from the stored search output. A common time translation of all phases preserves the conclusion for each box.

An additional finite-menu SAT calculation used the full time circle at $64/(64K+1)$, for $K = 59$, with at most $K-1$ runners chosen from finite speed and phase options. The union of closed coverage arcs covers the circle precisely when it covers every open cell cut out by their rational endpoints. A clause requires at least one active option per cell not covered by a fixed runner. Further constraints allow at most one phase per speed and enforce the runner budget. Seven menus at $K = 59$ were infeasible at that fixed radius. The exact parameters and verdicts are in `direct_rule_search.log` and `direct_rule_frontier.log`. The radius is strictly smaller than $1/K$, so even within a searched menu the verdict is scoped to that fixed radius. A global 58-runner question remains.

Data, formalization and attribution

The exact data and replay commands are in the versioned MathLab repository. The `MathLab.Frozen` namespace contains `frozen_optimum`, its attaining cover, and `perturbation_barrier`. The formalization concerns extending and perturbing the already published seed. The seed and its exact radius are credited to `tomdif`; `aadrianvv` provided the earlier certificates and the upstream public verifier. The upstream Lean cover checker, which was reused under its MIT license, has its separate attribution in the formal project. The finite phase and replacement checks use rational or integer arithmetic as described above.

Chengyi Fang initiated and funded the project, set its objectives, supervised its execution and directed dissemination. OpenAI Codex led proof development, computation, formalization and integration. GPT-6.2 Sol subagents assisted with mathematical auditing and publication. GPT-6.2 Astra developed the exact two-runner phase solver and assisted with formal proofs and independent model review through the Responses API. AI systems produced the mathematics, software and manuscript. Chengyi Fang has not independently checked any part of the proofs.

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References

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