A sharp phase neighbourhood for consecutive lonely runners

Chengyi Fang

Revised 27 November 2026

Abstract

For the speeds $1, \dots, p-1$, with $p \geq 7$ prime, we determine the loneliness as a function of all initial phases on the largest centred cube supporting the local formula. It equals $1/p$ plus the maximum of the absolute values of $(p-1)/2$ explicit linear forms. The first competing peaks arise at Farey denominator sum $p+1$. An obstruction on doubling orbits yields a larger tube in which loneliness cannot fall below $1/p$ whenever doubling modulo sign has odd order.

hexagon:2611.01186v1

1 The phase function

Write $\|x\| = \min_{m \in \mathbb{Z}} |x - m|$. For phases $u = (u_1, \dots, u_{p-1}) \in \mathbb{R}^{p-1}$ put

$$F(u) = \max_{t \in \mathbb{R}/\mathbb{Z}} \min_{1 \leq j < p} \|jt + u_j\|, \qquad M(u) = \max_{1 \leq a < p/2} \left| \frac{(p-a)u_a - a u_{p-a}}{p} \right|.$$

The maximum exists by compactness. Blanco, Criado and Santos determine the minimum over all phases for several consecutive speed sets, including $9/67 < 1/7$ for $1, \dots, 6$ [1, Prop. 5.12(2)]. Zhang discusses the global minimum over both velocities and phases and asks for its minimizers [3, Question 8.5]. Poliakova constructs configurations with small gaps for large runner counts [2]. Beck and Everett give necessary short integer relations among the velocities when a shifted gap is small [4, Prop. 4 and Thm. 5].

Here the velocities are fixed and the question is how the gap depends on the phases near the common start. In the discussion of Zhang's problem, `aadrianvv` identified the Farey tight points at zero phases and the complementary-speed peaks with opposite orientations. Perturbing a complementary pair raises one exposed peak as it lowers the other [5]. We determine the full local formula, its largest centred cube, and a separate bound on a wider tube.

Theorem 1. *Let $p \geq 7$ be prime. Take $Q = (p-1)/2$ if $p \equiv 3 \pmod 4$ and $Q = (p-3)/2$ otherwise. Set $r_0 = 1/(2Q(p+1))$. On $\max_j |u_j| \leq r_0$ one has $F(u) = 1/p + M(u)$. Every larger centred cube contains a point where $F(u) > 1/p + M(u)$.*

In particular the common start is a local minimum. Within this cube its level set is the linear section $u_a/a = u_{p-a}/(p-a)$ for each complementary pair. This allows a different common time shift for each pair. Here M uses the indicated real lifts of the phases. Both F and the linear forms are unchanged by replacing u_j by $u_j + j\tau$ for a common τ.

Proof. Write $d = 1/p$, $r = \max_j |u_j| \leq r_0$. Choose a attaining M, put $b = p - a$, and take k with $ka \equiv 1 \pmod p$. At $t = \pm k/p - (u_a + u_b)/p$, the two phase corrections relative to $\pm k/p$ are $X = (bu_a - au_b)/p$ and $-X$. Take the sign making the two distances $d + |X|$ (here $d + r < 1/2$). All the remaining speeds had distance at least $2d$ before the correction and change by at most $(1 + 2(p-1)/p)r$. Since $4r_0 \leq d$, they still have distance at least $d + r$. This proves the lower bound.

Decompose $[0,1]$ using the reduced fractions of denominator at most $p-1$. Consecutive terms $P/q < R/h$ satisfy $qR - hP = 1$ and $q + h \geq p$, by the standard Farey neighbour identities [6, Ch. III]. For t in this interval put $A = qt - P + u_q$ and $B = R - ht - u_h$. If either is negative its absolute value is at most r. Otherwise the distance to the nearer of these two runners is at most

$$\min(A, B) \leq \frac{hA + qB}{q + h} = \frac{1 + hu_q - qu_h}{q + h}.$$

When $q+h = p$ the last quantity is at most $d+M$; when $q+h \geq p+2$ it is at most $1/(p+2)+r_0 \leq d$.

Suppose $q+h = p+1$, and now denote the smaller denominator by q. Regardless of endpoint order the numerator of the phase contribution is bounded by $h|u_q| + qr$. One has $q \leq Q$. Indeed the two denominators are coprime, and if $p = 4m+1$ the candidate $2m$ is even. Using $M \geq ((p-q)|u_q| - qr)/p$, we obtain

$$\frac{h|u_q| + qr}{p+1} - M \leq \frac{q|u_q|}{p(p+1)} + qr\left(\frac{1}{p} + \frac{1}{p+1}\right) \leq \frac{2qr}{p}.$$

Thus the peak lies below or at $1/(p+1) + 2Qr_0/p + M = d + M$.

For sharpness fix $q = Q$ and $h = p + 1 - q$. We have $\gcd(q, p+1) = 1$: the odd number q differs from $(p+1)/2$ by one or two. Give indices $q, p-q$ phase r, index h phase $-r$, and index $p-h$ phase $-(p-h)r/h$. The four indices are distinct; set all other phases to zero. Then $M = (p-2q)r/p$. At $t = k/(p+1)$, $kq \equiv 1 \pmod{p+1}$, the two distances for q, h are $1/(p+1) + r$, while all others are at least $2/(p+1) - r$. For $r_0 < r \leq 1/(2(p+1))$ the former is strictly greater than $d + M$ and the latter at least as large as the former. Such points occur arbitrarily close to the boundary. □

2 An obstruction on a wider tube

Let $G = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ and $R(u) = \min_{\tau \in \mathbb{R}/\mathbb{Z}} \max_j \|u_j + j\tau\|$. The order of 2 in G is odd for every $p \equiv 3 \pmod 4$ since $|G| = (p-1)/2$ is then odd.

Theorem 2. *Let $p \geq 7$ be prime. If the order of 2 in G is odd, then*

$$R(u) \leq \frac{2(p+1)}{3p-1}\,\frac{1}{p} \quad \Longrightarrow \quad F(u) \geq 1/p.$$

Proof. Translate time, take real phase representatives, and put $r = \max_j |u_j|$ equal to the attained minimum $R(u)$. Suppose $F(u) < d$. Even with lifts of magnitude at most d, at the two times $\pm k/p$, $ka \equiv 1 \pmod p$, only the speeds $a, p-a$ can have distance strictly below d. At positive k/p this requires $u_a < 0$ or $u_{p-a} > 0$; at negative k/p the signs reverse. Consequently both phases in the pair have the same strict sign. For any $|z| \leq (d-r)/(p-1)$ the phases $u_j + jz$ still have magnitude at most d. Hence no such phase vanishes, which implies

$$|u_j| > j(d-r)/(p-1). \tag{1}$$

Take $1 \leq a < p/2$, $h = p - a$. At $t = \pm k/p - u_h/h$ the runner h is at distance d, and at one sign runner a is at distance at least d (its correction has size at most $2r < 2d$ and $3d < 1/2$). Every correction $u_j - ju_h/h$ has magnitude at most $(1 + (p-1)/h)r \leq 2d$, by $h \geq (p+1)/2$ and the assumed radius. A runner attaining distance less than d must therefore have had nominal distance exactly $2d$, so its class in G is $[2a]$ and its correction has magnitude greater than d. If its phase had the same sign as u_h, one signed difference would be at most $|u_j| < d$ and the other, by (1), would satisfy

$$j|u_h|/h - |u_j| < \frac{j}{p-1}[(1 + (p-1)/h)r - d] \leq d.$$

Thus the signs attached to the classes $[a]$ and $[2a]$ must be opposite. This is impossible around an odd cycle of doubling. □

The order condition is equivalent to the existence of an odd positive integer n with $2^n \equiv 1$ or $-1 \pmod{p}$. The order divides any such return exponent, and an odd order itself is such an exponent.

For $p = 59$ the exact formula holds on the cube of radius $1/3480$, and the obstruction holds on a tube of radius $15/(22 \cdot 59)$. For general p, determining the distance from the common-start orbit to the first strict improvement remains open here. For composite moduli the nonunit speeds can introduce further active constraints, and the argument does not give the same formula.

Verification. The companion Lean development gives the exact closed-cube identity, its sharpness, and the closed-tube bound. It checks the correspondence between the supremum and infimum over real time and the attained extrema on one turn. It also checks the equality between the all-index and half-index maxima for the complementary-pair forms. The distance to the integers is represented by $|x - \mathrm{round}(x)|$. The formal proof establishes the Farey interval property by finite mediant refinement from $0/1$ and $1/1$. The accompanying Python file `check.py` independently evaluates the complete piecewise-linear envelope for rational examples using integer arithmetic. At a maximum, opposite active slopes occur, including a cusp of a single runner. Consequently $(i + j)t + u_i + u_j$ is integral for some i, j (possibly equal), and enumeration on one period suffices. These tests are supplementary.

Contributions. Chengyi Fang initiated and funded MathLab, set its research goals, supervised the work, and arranged dissemination. OpenAI Codex coordinated the research, proof engineering, and verification. GPT-6.2 Sol subagents assisted with mathematical auditing and publication work. GPT-6.2 Astra supplied the $p = 59$ precursor and a formalization draft for the wider tube, and carried out independent model review. AI systems produced the mathematics, programs, and manuscript. Chengyi Fang has not independently checked any part of the proofs. The original proof and exact examples accompany MathLab v1.0.0.

References

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