COUNTEREXAMPLES TO TWO CONJECTURES OF KAC

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Abstract. We give counterexamples to Kac's 1974 conjecture on dense simple subalgebras and his 1980 conjecture on prehomogeneous quiver representations.

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Part I. Dense simple subalgebras

§ 1. Introduction

The four polynomial Cartan series come from the classical geometry of infinite transformation groups. Cartan's 1909 paper [1] identifies four simple transitive types: arbitrary transformations, transformations preserving volume, transformations preserving a nondegenerate alternating two-form, and contact transformations. Their infinitesimal versions give the series W, S, H, and K. Guillemin and Sternberg [2] developed an algebraic formulation using Taylor expansions of vector fields at a point. Formal vector fields carry a filtration by order of vanishing, and closed transitive subalgebras describe the corresponding formal local geometry.

Kac studied filtered Lie algebras with associated graded algebras of Cartan type in [3]. Let a subring of a formal power-series ring contain the coordinates and be stable under their partial derivatives. Over an algebraically closed field of characteristic zero, his Theorem 3 realizes every infinite-dimensional primitive transitive subalgebra of its vector-field algebra as a dense subalgebra of a Cartan algebra. For polynomial coefficients, density means that every vector field can be matched to any finite Taylor order at the distinguished origin. At the end of §9, Kac proposed the following conjecture [3, p. 834].

Conjecture (Kac, 1974). *A simple subalgebra dense in a simple Cartan algebra over a polynomial ring coincides with the ambient algebra.*

For W_n, Kac also gives the dense subalgebra of fields vanishing at a point $\xi \neq 0$. Choose a linear function ℓ with $\ell(0) = 0$ and $\ell(\xi) = 1$. For every $N \geq 1$ and $D \in W_n$, the field $(1 - \ell^N)D$ vanishes at ξ and agrees with D through Taylor degree $N - 1$ at the origin. The fields vanishing at ξ to order at least two form a proper nonzero ideal: bracketing with a field vanishing to first order preserves this condition. Simplicity is therefore the additional restriction in the conjecture.

Theorem 1.1. *Over $\mathbb{C}$, each of the Lie algebras*

$$W_n \quad (n \geq 2), \qquad S_n \quad (n \geq 2), \qquad H_{2m} \quad (m \geq 1), \qquad K_{2m+1} \quad (m \geq 1)$$

contains an explicitly defined proper, infinite-dimensional, simple Lie subalgebra dense in its natural filtration. Consequently, Kac's conjecture is false in all four standard simple Cartan series in these dimensions.

We use the convention $S_2 = H_2$. Kac's special series begins in dimension three because the two-dimensional case coincides with the Hamiltonian series. In dimension one, every

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infinite-dimensional simple subalgebra of the polynomial Witt algebra W_1 equals W_1, by Bell and Buzaglo [4, Proposition 4.3]. Thus dimension two is the first possible Witt case.

Kac also related the Witt problem to polynomial maps with constant nonzero Jacobian, now called Keller maps. The three-dimensional map recorded in [5, Eq. (2)] has determinant -2 and identifies distinct points. Its pullback on vector fields, its volume-preserving restriction, and its cotangent and contact lifts give proper dense simple subalgebras of W_n, S_n for $n \geq 3$ and of H_{2m}, K_{2m+1} for $m \geq 3$. These constructions are detailed in Section 9. The constructions below use a single affine quadric throughout the ranges of Theorem 1.1, including W_2, H_2, H_4, K_3, and K_5.

The common coefficient algebra is

$$B = \mathbb{C}[a,b,c]/(ab - c(c-1)) \hookrightarrow \mathbb{C}[t,y], \qquad (a,b,c) \longmapsto (t + t^2 y, y, 1 + ty). \tag{1}$$

The quadric $\operatorname{Spec} B$ is smooth, and the displayed map identifies the affine plane with an open subset of it. The element t does not belong to B, although elements of B represent every finite Taylor expansion at the plane's origin. Restriction of global vector fields from the quadric consequently combines a global extension condition with arbitrary local jets. Polynomial products give the Witt examples in higher dimension. Hamiltonian fields, exact volume-preserving fields, and a contact form on the corresponding products give the other three series.

The simplicity of vector fields on smooth affine varieties has a separate history. Jordan [6, 7] studied ideals in derivation Lie algebras, and Siebert [8] related their ideal structure to affine geometry. Billig and Futorny [9, Theorem 1.1] give a direct proof of the resulting criterion: over an algebraically closed field of characteristic zero, the vector-field Lie algebra of a positive-dimensional irreducible affine variety is simple exactly when the variety is smooth. The Witt construction uses this setting on the quadric and its products.

The Hamiltonian algebra of the quadratic surface is also an established simple Lie algebra. Gotay, Grabowski, and Kaneshige [10] study polynomial Poisson algebras on coadjoint orbits of real semisimple Lie groups. They obtain the decomposition into constants and the derived algebra and prove that, for semisimple orbits, every Lie ideal is central or contains the derived algebra. For smooth Danielewski surfaces, Leuenberger and Regeta [11] prove simplicity of the Lie algebra generated by locally nilpotent vector fields. Dubouloz, Kunyavskii, and Regeta [12, Proposition 7 and Corollary 4] identify this algebra with the derived Hamiltonian algebra and compute its codimension. Their result gives simplicity of the full Hamiltonian algebra when the defining polynomial has degree two, as in (1).

For the special series, contraction with the volume form identifies volume-preserving fields with closed forms and exact fields with exact forms. Donzelli [13] studies the resulting quotient through algebraic de Rham cohomology. On the product of the quadric with an affine line, this distinction determines the simple subalgebra used in dimension three. In higher dimensions, every global volume-preserving field on the products considered here is exact. Sections 2 and 3 give the polynomial and geometric preliminaries; Sections 4–7 prove the four constructions.

§ 2. Polynomial Cartan algebras and their topology

Throughout Part I, all algebras and derivations are over $\mathbb{C}$. For a commutative algebra R, write $\operatorname{Der}(R) = \operatorname{Der}_{\mathbb{C}}(R)$, with bracket $[D,Z] = D \circ Z - Z \circ D$. In any Lie algebra, $\operatorname{ad}_u(v) = [u,v]$. A Lie algebra is simple if it is nonabelian and its only Lie ideals are zero and the whole algebra.

Let Ω^1_R be the module generated by the symbols df, $f \in R$, subject to linearity, $d\lambda = 0$ for $\lambda \in \mathbb{C}$, and $d(fg) = f\,dg + g\,df$. Put $\Omega^0_R = R$ and $\Omega^q_R = \bigwedge^q_R \Omega^1_R$. The exterior derivative d satisfies $d^2 = 0$ and $d(\beta \wedge \gamma) = d\beta \wedge \gamma + (-1)^q \beta \wedge d\gamma$ for $\beta \in \Omega^q_R$. Contraction ι_D is the degree -1 antiderivation determined by $\iota_D(f) = 0$ and $\iota_D(df) = D(f)$. The Lie derivative

is $\mathcal{L}_D = d\iota_D + \iota_D d$. For a generator Ω of a free module of top-degree forms, define $\operatorname{div}_\Omega D$ by $\mathcal{L}_D\Omega = (\operatorname{div}_\Omega D)\Omega$. Since $df \wedge \iota_D\Omega = D(f)\Omega$,

$$\operatorname{div}_\Omega(fD) = D(f) + f \operatorname{div}_\Omega D. \tag{2}$$

Write $A_n = \mathbb{C}[x_1, \dots, x_n]$ and $W_n = \operatorname{Der}(A_n) = \bigoplus_{i=1}^n A_n\partial_{x_i}$. Subscripts on functions denote partial derivatives. In each ambient polynomial ring, $\mathfrak{m}$ denotes the ideal generated by its displayed coordinate variables, so it is the ideal of the origin. For a multi-index $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n_{\geq 0}$, write $|\nu| = \sum_i \nu_i$, $\nu! = \prod_i \nu_i!$, $x^\nu = \prod_i x_i^{\nu_i}$, and $\partial^\nu = \prod_i \partial_{x_i}^{\nu_i}$.

For the standard volume form $\Omega_0 = dx_1 \wedge \cdots \wedge dx_n$, put

$$S_n = \left\{ \sum_i f_i\partial_{x_i} : \sum_i \partial_{x_i} f_i = 0 \right\}. \tag{3}$$

Equivalently, $D \in S_n$ if and only if $d(\iota_D\Omega_0) = 0$. For coordinates $(q_1, p_1, \dots, q_m, p_m)$, set

$$\omega_0 = \sum_i dq_i \wedge dp_i, \qquad \{f, g\} = \sum_i (f_{q_i}g_{p_i} - f_{p_i}g_{q_i}).$$

Set $H_{2m} = \{D \in W_{2m} : \mathcal{L}_D\omega_0 = 0\}$. The Hamiltonian field $X_f = \{f, -\} = \sum_i (f_{q_i}\partial_{p_i} - f_{p_i}\partial_{q_i})$ satisfies $\iota_{X_f}\omega_0 = -df$. Its commutator is determined by

$$\begin{aligned} [X_f, X_g](q_i) &= -\{f, g_{p_i}\} + \{g, f_{p_i}\} = -\partial_{p_i}\{f, g\}, \\ [X_f, X_g](p_i) &= \{f, g_{q_i}\} - \{g, f_{q_i}\} = \partial_{q_i}\{f, g\}. \end{aligned}$$

Hence $[X_f, X_g] = X_{\{f,g\}}$; applying this equality to a function gives the Jacobi identity for $\{-, -\}$. The kernel of $f \mapsto X_f$ consists of the constants.

Lemma 2.1 (Polynomial homotopy)**.** *Let β be a closed polynomial r-form on $\mathbb{A}^n$, with $r > 0$. Write $\beta = \sum_{d\geq 0} \beta_d$, where the coefficients of β_d are homogeneous of degree d, and let $\mathcal{R} = \sum_i x_i\partial_{x_i}$. Then*

$$\gamma = \sum_{d\geq 0} \frac{\iota_{\mathcal{R}}\beta_d}{d + r} \quad \textit{satisfies} \quad d\gamma = \beta. \tag{4}$$

Proof. Each β_d is closed, because the components of $d\beta_d$ have distinct coefficient degrees. On each monomial form,

$$\mathcal{L}_{\mathcal{R}}(x^\nu\, dx_{i_1} \wedge \cdots \wedge dx_{i_r}) = (|\nu| + r)x^\nu\, dx_{i_1} \wedge \cdots \wedge dx_{i_r},$$

because $\mathcal{R}(x^\nu) = |\nu|x^\nu$ and $\mathcal{L}_{\mathcal{R}}dx_i = d(\mathcal{R}x_i) = dx_i$. Thus it acts on β_d as multiplication by $d + r$. Cartan's formula gives $d\iota_{\mathcal{R}}\beta_d = (d + r)\beta_d$. The sums are finite and the displayed denominators are nonzero. □

If $D \in H_{2m}$, then $\iota_D\omega_0$ is closed. The lemma gives $\iota_D\omega_0 = -df$ for a polynomial f. The nondegeneracy of ω_0 implies $D = X_f$. Therefore

$$H_{2m} \cong (A_{2m}/\mathbb{C}, \{-, -\}). \tag{5}$$

For contact coordinates $(t, p_1, q_1, \dots, p_m, q_m)$ put

$$\alpha_0 = dt + \sum_i p_i\, dq_i, \qquad K_{2m+1} = \{D \in W_{2m+1} : \mathcal{L}_D\alpha_0 \in A_{2m+1}\alpha_0\}. \tag{6}$$

For a polynomial f, define

$$Y_f = \left(f - \sum_i p_i f_{p_i}\right)\partial_t + \sum_i f_{p_i}\partial_{q_i} + \sum_i (p_i f_t - f_{q_i})\partial_{p_i}. \tag{7}$$

The equality $\alpha_0(Y_f) = f$ follows by cancellation of $\sum_i p_i f_{p_i}$. Moreover,

$$\iota_{Y_f} d\alpha_0 = \sum_i ((p_i f_t - f_{q_i}) dq_i - f_{p_i} dp_i),$$

$$\mathcal{L}_{Y_f} \alpha_0 = df + \iota_{Y_f} d\alpha_0 = f_t \left(dt + \sum_i p_i \, dq_i \right) = f_t \alpha_0.$$

Conversely, if D is a contact field, write $f = \alpha_0(D)$. The equation $df + \iota_D d\alpha_0 = \lambda \alpha_0$ gives $\lambda = f_t$, $D(q_i) = f_{p_i}$, and $D(p_i) = p_i f_t - f_{q_i}$; its remaining coefficient is then determined by $\alpha_0(D) = f$. Hence $f \mapsto Y_f$ is a linear bijection onto K_{2m+1}.

The corresponding bracket on functions is

$$\langle f, g \rangle = Y_f(g) - g f_t, \qquad [Y_f, Y_g] = Y_{\langle f,g \rangle}. \tag{8}$$

The identity $[\mathcal{L}_D, \mathcal{L}_Z] = \mathcal{L}_{[D,Z]}$ holds on functions and their differentials, and hence on all forms. Therefore

$$\mathcal{L}_{[Y_f,Y_g]} \alpha_0 = [\mathcal{L}_{Y_f}, \mathcal{L}_{Y_g}] \alpha_0 = (Y_f(g_t) - Y_g(f_t)) \alpha_0,$$

so $[Y_f, Y_g]$ is contact. Moreover, $\alpha_0([Y_f, Y_g]) = Y_f(g) - (\mathcal{L}_{Y_f} \alpha_0)(Y_g)$. We reserve $\{-,-\}$ for the Poisson bracket and $\langle -,- \rangle$ for this contact bracket.

Lemma 2.2. *Let L be a Lie subalgebra of W_n whose evaluation at the origin is surjective. Set $L_0 = \{D \in L : D(0) = 0\}$, $L_{-1} = L$, and*

$$L_j = \{D \in L_{j-1} : [D, L] \subset L_{j-1}\} \quad (j \geq 1).$$

For $\mathfrak{m} = (x_1, \ldots, x_n)$ one has

$$L_j = L \cap \mathfrak{m}^{j+1} W_n \qquad (j \geq 0). \tag{9}$$

Consequently, a subalgebra $M \subset L$ is dense in this topology if for each $D \in L$ and each $r \geq 1$ it contains an element congruent to D modulo $\mathfrak{m}^r W_n$.

Proof. The assertion is the definition for $j = 0$. Suppose it is known for $j - 1$. Bracketing a field whose coefficients vanish to order $j + 1$ with a polynomial field gives coefficients vanishing to order at least j. This proves one inclusion. Conversely, let $D \in L_{j-1}$ have a nonzero homogeneous coefficient part of degree j. For a field $Z \in L$, the degree $j - 1$ part of $[D, Z]$ is minus the directional derivative of that part along $Z(0)$. Evaluation is surjective and a nonzero positive-degree polynomial in characteristic zero has a nonzero partial derivative. Thus some $[D, Z]$ does not lie in L_{j-1}. This proves the other inclusion and the density criterion. □

All four ambient algebras have surjective evaluation. In the contact case, $Y_1(0) = \partial_t$, $Y_{p_i}(0) = \partial_{q_i}$, and $Y_{q_i}(0) = -\partial_{p_i}$. Equation (9) is the natural-filtration topology used in [3, §§0,1,9]. The usual contact grading assigns weight two to t and weight one to p_i, q_i; its filtration gives the same topology, since a monomial of ordinary degree d has weighted degree between d and $2d$.

§ 3. The common affine quadric

Inside $A_2 = \mathbb{C}[t, y]$ set

$$a = t + t^2 y, \qquad b = y, \qquad c = 1 + ty, \qquad h = 2c - 1. \tag{10}$$

We identify the algebra in (1) with its image in A_2. To justify the presentation, the polynomial $ab - c(c-1)$ is primitive and linear in a over the unique factorization domain $\mathbb{C}[b, c]$; b and $c(c-1)$ are relatively prime. It is therefore irreducible. After inverting b, its quotient is $\mathbb{C}[b, b^{-1}, c]$. The substitution $b = y$, $c = 1 + ty$ gives an isomorphism $\mathbb{C}[b, b^{-1}, c] \to \mathbb{C}[y, y^{-1}, t]$, with inverse $y \mapsto b$, $t \mapsto (c-1)/b$. The quotient is a domain and

injects into its localization, which proves the claimed embedding. Also $t = (c-1)/b$ in the fraction field, so

$$\operatorname{Frac}(B) = \mathbb{C}(t, y), \qquad h^2 - 4ab = 1. \tag{11}$$

The quadric $Q = \operatorname{Spec} B$ is smooth: the partial derivatives $b, a, -h$ of its equation cannot vanish simultaneously, by (11). The map $\mathbb{A}^2 \to Q$ identifies $\mathbb{A}^2$ with $Q \setminus V(b, c)$, where $V(b, c)$ is the common zero set of b, c. Write $D(u)$ for the locus where a function u is nonzero. The inverse map is $y = b$ and $t = (c-1)/b$ on $D(b)$, and $t = a/c$ on $D(c)$. The relation $ab = c(c-1)$ makes these formulas agree on $D(bc)$. On $D(b)$ one obtains $1 + ty = c$ and $tc = a$. On $D(c)$ one obtains $ty = c - 1$ and $tc = a$. Thus both recover (10). Conversely, $b = y$ and $c = 1 + ty$ never vanish simultaneously on $\mathbb{A}^2$, so the domains cover exactly the image.

Lemma 3.1. *One has $t \notin B$. For every integer $N \geq 1$, the polynomial*

$$t_N = a \sum_{j=0}^{N-1} (1-c)^j \in B \quad \textit{satisfies} \quad t_N = t\big(1 - (-ty)^N\big). \tag{12}$$

In particular,

$$t_N - t = (-1)^{N+1} t^{N+1} y^N \in (t, y)^{2N+1}, \tag{13}$$

and $B \to A_2/(t, y)^r$ is surjective for every $r \geq 1$.

Proof. If t belonged to B, the identity $bt = c-1$ would hold there. Its image in $B/(b, c) = \mathbb{C}[a]$ would read $0 = -1$, which is impossible. For the second assertion, use $a = tc$ and the finite geometric-series identity $c \sum_{j=0}^{N-1} (1-c)^j = 1 - (1-c)^N$. For $P(t, y) = \sum_{i \geq 0} p_i(y) t^i$, one has

$$P(t_N, y) - P(t, y) = (t_N - t) \sum_{i \geq 1} p_i(y) \sum_{\ell=0}^{i-1} t_N^{i-1-\ell} t^\ell.$$

Choosing $2N + 1 \geq r$ proves surjectivity. The same argument works after adjoining any number of independent polynomial variables. Differentiation also gives the exact identity

$$dt_N - dt = (-1)^{N+1}\big((N+1) t^N y^N \, dt + N t^{N+1} y^{N-1} \, dy\big). \tag{14}$$

□

Put $g = 3t^2 + 2t^3 y$ and define

$$E = \partial_t, \qquad T = g\partial_t + \partial_y, \qquad \eta = h \, da - 4a \, dc. \tag{15}$$

Their values on the generators are

	a	b	c
E	h	0	b
T	$4a^2$	1	ah

(16) $\qquad [E, T] = 6aE.$

Indeed, $E(a) = 1 + 2ty = h$, $E(b) = 0$, and $E(c) = y = b$, while

$$\begin{aligned} T(a) &= (3t^2 + 2t^3 y)(1 + 2ty) + t^2 = 4t^2 + 8t^3 y + 4t^4 y^2 = 4a^2, \\ T(b) &= 1, \qquad T(c) = 3t^2 y + 2t^3 y^2 + t = (t + t^2 y)(1 + 2ty) = ah, \\ [E, T] &= (\partial_t g) E = (6t + 6t^2 y) E = 6aE. \end{aligned}$$

These values show that E and T preserve B.

Differentiating the relation gives $b \, da + a \, db = h \, dc$. Together with (11), this yields the two exact identities

$$da = h\eta + 4a^2 db, \qquad dc = b\eta + ah \, db. \tag{17}$$

For the first identity, use $h\eta + 4a^2db = h^2da - 4a(b\,da + a\,db) + 4a^2db = da$. For the second, use $b\eta + ah\,db = h(b\,da + a\,db) - 4ab\,dc = dc$. The evaluations are

$$\eta(E) = h^2 - 4ab = 1, \qquad \eta(T) = 4a^2h - 4a^2h = 0, \qquad db(E) = 0, \quad db(T) = 1.$$

Thus

$$\Omega^1_B = B\eta \oplus B\,db, \qquad \mathrm{Der}(B) = BE \oplus BT. \tag{18}$$

If $D(a) = u$, $D(b) = v$, and $D(c) = w$, the relation $bu + av = hw$ gives the explicit expression

$$D = (hu - 4aw)E + vT. \tag{19}$$

Its values on a, b, c are u, v, w: the first is $h^2u - 4ahw + 4a^2v = (h^2 - 4ab)u$, and the third is $h(bu + av) - 4abw = (h^2 - 4ab)w$.

The volume form and its chart expressions are

$$\omega = \eta \wedge db = dt \wedge dy, \qquad \eta = dt - g\,dy, \qquad d\eta = -6a\,\omega. \tag{20}$$

Indeed, $da = h\,dt + t^2dy$ and $dc = y\,dt + t\,dy$, so

$$\eta = (h^2 - 4ay)dt + (ht^2 - 4at)dy = dt - g\,dy.$$

Moreover, (17) gives $da \wedge dc = a(h^2 - 4ab)\eta \wedge db = a\omega$, and hence

$$d\eta = 2dc \wedge da - 4da \wedge dc = -6a\omega.$$

The form ω generates Ω^2_B.

For every $r \geq 0$ put

$$R_r = B[z_1, \ldots, z_r], \qquad \Omega_r = \omega \wedge dz_1 \wedge \cdots \wedge dz_r. \tag{21}$$

The forms $\eta, db, dz_1, \ldots, dz_r$ and the derivations $E, T, \partial_{z_1}, \ldots, \partial_{z_r}$ are dual bases over R_r. Here E, T annihilate the z_i. Since $\iota_E\omega = db$ and $\iota_T\omega = -\eta$, (20) gives $\mathrm{div}_{\Omega_r} E = 0$, $\mathrm{div}_{\Omega_r} T = 6a$, and $\mathrm{div}_{\Omega_r} \partial_{z_i} = 0$. Thus (2) yields

$$\mathrm{div}_{\Omega_r}\left(AE + CT + \sum_i U_i\partial_{z_i}\right) = E(A) + T(C) + 6aC + \sum_i \partial_{z_i}U_i. \tag{22}$$

Lemma 3.2. *The ring B has no nonzero proper associative ideal stable under all its derivations. The same holds for $B[z_1, \ldots, z_r]$ for every $r \geq 0$.*

Proof. Besides E, consider the global derivation

$$F = hT - 4a^2E = -t^2\partial_t + h\partial_y. \tag{23}$$

It acts on (a, b, c) as $(0, h, a)$. Let I be a nonzero associative ideal of B stable under E, F. For $0 \neq u \in I$, applying E exactly $\deg_t u$ times gives a nonzero $P(y) \in I$. On $\mathbb{C}[y]$, $EF = 2y\partial_y$. If $P = \sum_{j\in S} P_j y^j$ with each $P_j \neq 0$, choose $k \in S$. The operator

$$\prod_{j\in S\setminus\{k\}} \frac{\frac{1}{2}EF - j}{k - j}$$

sends P to $P_k y^k$. Consequently $y^k \in I$ for some $k \geq 0$. For $k > 0$, $F(y^k) = ky^{k-1}h$ lies in I. Multiplication by h and $h^2 = 1 + 4ay$ give

$$y^{k-1} = \frac{1}{k}hF(y^k) - 4ay^k \in I.$$

Repeated descent gives $1 \in I$.

For $B[z_1, \ldots, z_r]$, a nonzero ideal stable under all derivations can be differentiated successively to its highest current degree in each z_i. At each step the result is nonzero in characteristic zero, and the final result lies in B. The intersection with B is stable under E, F, so the preceding argument applies. □

§ 4. The Witt series

We use Jordan's derivation criterion [6, 7], with the bracket identities in [9, Lemmas 2.2–2.5].

Lemma 4.1. *Let R be a commutative $\mathbb{C}$-algebra that is a domain, and let $0 \neq L \subset \mathrm{Der}_{\mathbb{C}}(R)$ be both a Lie subalgebra and an R-submodule. If R has no nonzero proper L-stable associative ideal, then L is a simple Lie algebra.*

Proof. Let $0 \neq J \lhd L$ and choose $0 \neq \mu \in J$. For any $f, u \in R$,

$$\mu(f)\mu \in J, \qquad \mu(f)\mu(u)\mu \in J, \qquad f\mu^2(u)\mu \in J. \tag{24}$$

The first assertion follows from $[\mu, f\mu] = \mu(f)\mu$. For the second, expand

$$[\mu, f\mu(u)\mu] - [f\mu, \mu(u)\mu] = 2\mu(f)\mu(u)\mu.$$

The first bracket has its first entry in J; the second has its second entry in J by the first assertion. Subtracting $\mu(f)\mu(u)\mu$ from $\mu(f\mu(u))\mu$ proves the third assertion.

Choose f with $\mu(f) \neq 0$. There exists u with $\mu^2(u) \neq 0$: take $u = f$ if possible, and otherwise take $u = f^2$, since then $\mu^2(f^2) = 2\mu(f)^2$. Put $q = \mu^2(u)$. Equation (24) gives $Rq\mu \subset J$. For $\tau \in L$ the two brackets are

$$[q\mu, f\tau] = q\mu(f)\tau + fq[\mu, \tau] - f\tau(q)\mu,$$
$$[fq\mu, \tau] = fq[\mu, \tau] - \tau(fq)\mu.$$

Since $\tau(fq) = \tau(f)q + f\tau(q)$, subtraction gives

$$[q\mu, f\tau] - [fq\mu, \tau] - \tau(f)q\mu = \mu(f)q\tau. \tag{25}$$

Each left-hand term lies in J. Consequently,

$$I = \{r \in R : rL \subset J\}$$

contains the nonzero element $\mu(f)q$. This set is an associative ideal: if $r \in I$, then $(sr)\tau = r(s\tau) \in J$ for every $s \in R$. It is also L-stable, because

$$\delta(r)\tau = [\delta, r\tau] - r[\delta, \tau] \in J \qquad (\delta, \tau \in L).$$

Hence $I = R$ and $J = L$. Finally, L is nonabelian, since $[\mu, f\mu] = \mu(f)\mu \neq 0$. □

For $n \geq 2$, regard R_{n-2} from (21) as a subring of $\mathbb{C}[t, y, z_1, \ldots, z_{n-2}]$ and define

$$\mathfrak{w}_n = R_{n-2}E \oplus R_{n-2}T \oplus \bigoplus_{i=1}^{n-2} R_{n-2}\partial_{z_i} \subset W_n. \tag{26}$$

By (18), this is the restriction of $\mathrm{Der}(R_{n-2})$. It is closed under brackets, either by this identification or by (16) and $[fD, uZ] = fu[D, Z] + fD(u)Z - uZ(f)D$. Lemmas 3.2 and 4.1 prove that it is simple. The elements b^jE, $j \geq 0$, are linearly independent, so it is infinite-dimensional.

All elements of $\mathfrak{w}_n$ preserve R_{n-2}, whereas

$$\partial_y(c) = t \notin R_{n-2}. \tag{27}$$

The nonmembership follows from Lemma 3.1 by setting all z_i equal to zero. Thus $\partial_y \notin \mathfrak{w}_n$ and $\mathfrak{w}_n \subsetneq W_n$.

Here is an explicit approximation to the missing field. Put

$$g_N = 3t_N^2 + 2t_N^3 b, \qquad D_N = T - g_N E \in \mathfrak{w}_n.$$

The identity

$$g_N - g = (t_N - t)\big(3(t_N + t) + 2y(t_N^2 + t_N t + t^2)\big) \tag{28}$$

gives $D_N - \partial_y \in \mathfrak{m}^{2N+2}W_n$, where $\mathfrak{m} = (t, y, z_1, \dots, z_{n-2})$. For $z = (z_1, \dots, z_{n-2})$ and $V = P\partial_t + Q\partial_y + \sum_i U_i\partial_{z_i} \in W_n$, put

$$V_N = P(t_N, b, z)E + Q(t_N, b, z)D_N + \sum_i U_i(t_N, b, z)\partial_{z_i} \in \mathfrak{w}_n.$$

The difference is

$$\begin{aligned} V_N - V = {} & (P(t_N, b, z) - P + Q(t_N, b, z)(g - g_N))\partial_t \\ & + (Q(t_N, b, z) - Q)\partial_y + \sum_i (U_i(t_N, b, z) - U_i)\partial_{z_i}. \end{aligned}$$

Every coefficient lies in $\mathfrak{m}^{2N+1}$ by (13) and (28). As N is arbitrary, Lemma 2.2 proves density.

§ 5. The Hamiltonian series

The standard bracket $\{u, v\} = u_t v_y - u_y v_t$ preserves B, since

$$\{a, b\} = a_t = h, \qquad \{a, c\} = ht - t^2 y = a, \qquad \{c, b\} = c_t = b. \tag{29}$$

The coordinates $e = b$, $f = -a$, and $h = 2c - 1$ put these relations in $\mathfrak{sl}_2$ form:

$$\{h, e\} = 2e, \qquad \{h, f\} = -2f, \qquad \{e, f\} = h, \qquad h^2 + 4ef = 1. \tag{30}$$

The inverse formulas $a = -f$, $b = e$, $c = (h + 1)/2$ identify B with $\mathbb{C}[e, f, h]/(h^2 + 4ef - 1)$.

Lemma 5.1. *For $d \geq 0$, let $B_{\leq d}$ be the image of the polynomials of total degree at most d in e, f, h. For $k, j \geq 0$ set*

$$v_{k,j} = (\operatorname{ad} f)^j(e^k), \qquad V_k = \operatorname{span}_{\mathbb{C}}\{v_{k,j} : 0 \leq j \leq 2k\},$$

where ad *refers to the Poisson bracket. Then V_k is an irreducible module for the three operators* $\operatorname{ad} e, \operatorname{ad} f, \operatorname{ad} h$, $\dim V_k = 2k + 1$, *and*

$$B_{\leq d} = \bigoplus_{k=0}^{d} V_k, \qquad B = \bigoplus_{k \geq 0} V_k, \qquad V_0 = \mathbb{C}1. \tag{31}$$

Proof. The relation $h^2 = 1 - 4ef$ is monic in h. Division by this relation gives the basis $e^i f^j h^\epsilon$ with $i, j \geq 0$ and $\epsilon \in \{0, 1\}$; reduction does not increase total degree. Hence

$$\dim B_{\leq d} = \frac{(d + 1)(d + 2)}{2} + \frac{d(d + 1)}{2} = (d + 1)^2.$$

The three adjoint operators preserve this filtration. Write them as $\mathcal{E}, \mathcal{F}, \mathcal{H}$, respectively. The Jacobi identity and (30) give

$$[\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, \qquad [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}, \qquad [\mathcal{E}, \mathcal{F}] = \mathcal{H}.$$

Starting from $\mathcal{H}(e^k) = 2ke^k$, the identity $\mathcal{H}\mathcal{F}v = \mathcal{F}\mathcal{H}v - 2\mathcal{F}v$ gives the first formula in

$$\mathcal{H}v_{k,j} = (2k - 2j)v_{k,j}, \qquad \mathcal{E}v_{k,j} = j(2k - j + 1)v_{k,j-1} \quad (j \geq 1). \tag{32}$$

For the second formula, $\mathcal{E}(e^k) = 0$ and $\mathcal{E}\mathcal{F}(e^k) = \mathcal{H}(e^k) = 2ke^k$. If it holds at j, then $\mathcal{E}\mathcal{F}v_{k,j} = \mathcal{F}\mathcal{E}v_{k,j} + \mathcal{H}v_{k,j}$, whose coefficient is $j(2k - j + 1) + 2k - 2j = (j + 1)(2k - j)$.

Since $\mathcal{F}(e) = -h$, $\mathcal{F}(h) = 2f$, and $\mathcal{F}(f) = 0$, for an indeterminate λ and $k \geq 1$ one has

$$\begin{aligned} \partial_\lambda(e - \lambda h - \lambda^2 f)^k &= k(e - \lambda h - \lambda^2 f)^{k-1}(-h - 2\lambda f) \\ &= \mathcal{F}\big((e - \lambda h - \lambda^2 f)^k\big). \end{aligned}$$

For $k = 0$, both operators kill 1. The value at $\lambda = 0$ is e^k. Comparing coefficients and expanding by the multinomial formula therefore gives, for every $k, j \geq 0$,

$$v_{k,j} = j! \sum_{\substack{\rho+2\sigma=j \\ \rho,\sigma\geq 0 \\ \rho+\sigma\leq k}} \frac{(-1)^{\rho+\sigma} k!}{(k-\rho-\sigma)!\,\rho!\,\sigma!} e^{k-\rho-\sigma} h^\rho f^\sigma. \tag{33}$$

For $j = 2k$ the only summand has $\rho = 0$, $\sigma = k$; for $j > 2k$ the sum is empty. Thus

$$v_{k,2k} = (-1)^k (2k)! f^k \neq 0, \qquad v_{k,2k+1} = 0. \tag{34}$$

All $v_{k,j}$ with $0 \leq j \leq 2k$ are nonzero, since a zero iterate would force $v_{k,2k} = 0$. Their distinct $\mathcal{H}$-weights make them linearly independent. Equations (32) and (34) prove stability of V_k. For a nonzero vector $\sum_{j=0}^{2k} a_j v_{k,j}$, choose $a_j \neq 0$. The operator

$$\prod_{\substack{0\leq \ell\leq 2k \\ \ell\neq j}} \frac{\mathcal{H} - (2k - 2\ell)}{2(\ell - j)}$$

projects it onto $a_j v_{k,j}$. Thus every nonzero stable subspace contains a weight vector. Repeated application of $\mathcal{E}$, using the nonzero coefficients in (32), gives $v_{k,0}$; application of $\mathcal{F}$ then gives every $v_{k,j}$. Hence V_k is irreducible.

Set

$$\mathcal{C} = \mathcal{H}^2 + 2\mathcal{E}\mathcal{F} + 2\mathcal{F}\mathcal{E}.$$

Its three commutators are

$$\begin{aligned} [\mathcal{E}, \mathcal{C}] &= -2\mathcal{E}\mathcal{H} - 2\mathcal{H}\mathcal{E} + 2\mathcal{E}\mathcal{H} + 2\mathcal{H}\mathcal{E} = 0, \\ [\mathcal{F}, \mathcal{C}] &= 2\mathcal{F}\mathcal{H} + 2\mathcal{H}\mathcal{F} - 2\mathcal{H}\mathcal{F} - 2\mathcal{F}\mathcal{H} = 0, \\ [\mathcal{H}, \mathcal{C}] &= 2(2\mathcal{E}\mathcal{F} - 2\mathcal{E}\mathcal{F}) + 2(-2\mathcal{F}\mathcal{E} + 2\mathcal{F}\mathcal{E}) = 0. \end{aligned}$$

Its value on e^k is $4k^2 e^k + 4k e^k = 4k(k+1)e^k$. Since it commutes with $\mathcal{F}$, it has the scalar $4k(k+1)$ on V_k. These scalars are distinct, so the sum of $V_0, \ldots, V_d$ is direct. Its dimension is $\sum_{k=0}^{d}(2k+1) = (d+1)^2$. Since it lies in $B_{\leq d}$, the dimension count proves (31). □

Proposition 5.2. *The Lie algebra $B/\mathbb{C}$ is simple.*

Proof. Let $0 \neq J \lhd B/\mathbb{C}$. For a nonzero element of J choose d with representative in $B_{\leq d}$. Polynomial projection in $\mathcal{C}$ isolates a nonzero component in some V_k, $k \geq 1$; the projection onto V_k in this finite sum is

$$\prod_{\substack{0\leq j\leq d \\ j\neq k}} \frac{\mathcal{C} - 4j(j+1)}{4k(k+1) - 4j(j+1)}.$$

The ideal is stable under these operators. Irreducibility and (32) put the class of e^k in J.

The identity

$$\{e^k, eh\} = -2k e^{k+1} \tag{35}$$

puts $e^{k+1} \bmod \mathbb{C}$ in J. Applying $\operatorname{ad} f$ twice and using $h^2 = 1 - 4ef$ gives

$$\begin{aligned} A_k := (\operatorname{ad} f)^2(e^{k+1}) &= (k+1)k e^{k-1} h^2 - 2(k+1)e^k f \\ &= (k+1)e^{k-1}(k - (4k+2)ef), \end{aligned} \tag{36}$$

$$C_k := \{e^k, fh\} = k e^{k-1} h^2 - 2k e^k f = k e^{k-1}(1 - 6ef). \tag{37}$$

Both classes lie in J. The ef coefficients cancel because $-6(k+1)(4k+2)+6(k+1)(4k+2) = 0$, and the constant coefficient is $6k(k+1) - (k+1)(4k+2) = 2(k+1)(k-1)$. Hence

$$6A_k - \frac{(k+1)(4k+2)}{k} C_k = 2(k+1)(k-1)e^{k-1}. \tag{38}$$

For $k > 1$ the scalar is nonzero. Descent gives $e \bmod \mathbb{C} \in J$. For every $j \geq 1$, $\{e, he^{j-1}\} = -2e^j$ then gives $e^j \bmod \mathbb{C} \in J$. Applying $\operatorname{ad} f$ and (31) proves $J = B/\mathbb{C}$. The bracket $\{h, e\} = 2e$ shows nonabelianity. □

Let $B' = \operatorname{span}_{\mathbb{C}}\{\{u, v\} : u, v \in B\}$ be the derived Lie algebra of B.

Lemma 5.3. *There is a linear functional $\tau : B \to \mathbb{C}$ with $\tau(1) = 1$, $\tau(\{B, B\}) = 0$, and*

$$B = \mathbb{C}1 \oplus B', \qquad B' \cong B/\mathbb{C}, \qquad d\Omega^1_B = B'\omega. \tag{39}$$

On the normal basis it is given by

$$\tau(e^i f^j) = \begin{cases} (i!)^2/(2i+1)!, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \tau(e^i f^j h) = 0. \tag{40}$$

Proof. For $i, j \geq 0$, omit terms with zero coefficient. The first four generator actions are

$$\{h, e^i f^j\} = 2(i-j)e^i f^j, \qquad \{h, e^i f^j h\} = 2(i-j)e^i f^j h,$$
$$\{e, e^i f^j\} = j e^i f^{j-1} h, \qquad \{f, e^i f^j\} = -i e^{i-1} f^j h.$$

The remaining two actions, reduced by $h^2 = 1 - 4ef$, are

$$\begin{aligned} \{e, e^i f^j h\} &= j e^i f^{j-1} h^2 - 2e^{i+1} f^j \\ &= j e^i f^{j-1} - (4j+2) e^{i+1} f^j, \\ \{f, e^i f^j h\} &= -i e^{i-1} f^j h^2 + 2e^i f^{j+1} \\ &= -i e^{i-1} f^j + (4i+2) e^i f^{j+1}. \end{aligned}$$

Let $c_j = (j!)^2/(2j+1)!$. For $j \geq 1$,

$$\frac{c_j}{c_{j-1}} = \frac{j^2}{(2j)(2j+1)} = \frac{j}{2(2j+1)}, \qquad j c_{j-1} = (4j+2) c_j.$$

The first four displayed actions have trace zero: a term containing h has trace zero, and $2(i-j)$ vanishes when $i = j$. In the fifth action, only $i = j - 1$ can contribute, and its trace is $j c_{j-1} - (4j+2)c_j = 0$. In the sixth, only $j = i - 1$ can contribute, and its trace is $-i c_{i-1} + (4i+2)c_i = 0$. Thus τ is invariant under $\operatorname{ad} e, \operatorname{ad} f, \operatorname{ad} h$.

For a monomial $v = q_1 \cdots q_r$, where $r \geq 1$ and $q_i \in \{e, f, h\}$, put $Q_i = \prod_{\ell \neq i} q_\ell$; empty products are 1. For any $w \in B$, invariance applied to wQ_i and the Leibniz rule give

$$\begin{aligned} \tau(\{v, w\}) &= \sum_i \tau(Q_i \{q_i, w\}) \\ &= -\sum_i \tau(w\{q_i, Q_i\}) \\ &= -\sum_{i \neq j} \tau\left(w \prod_{\ell \neq i,j} q_\ell \, \{q_i, q_j\} \right) \\ &= -\sum_{i<j} \tau\left(w \prod_{\ell \neq i,j} q_\ell (\{q_i, q_j\} + \{q_j, q_i\}) \right) = 0. \end{aligned}$$

The bracket with a constant is zero, so linearity proves this for every $v, w \in B$. Thus $B' \subset \ker \tau$ and $B' \cap \mathbb{C} = 0$. For $k \geq 1$, $e^k = (2k)^{-1}\{h, e^k\}$ belongs to B'. The derived algebra is an ideal by the Jacobi identity, so it contains every V_k with $k \geq 1$. Equation (31) and $\tau(1) = 1$ therefore give $B' = \bigoplus_{k \geq 1} V_k = \ker \tau$, and the quotient map identifies B' with $B/\mathbb{C}$.

Every algebraic one-form is a sum of forms $u\,dv$. On the affine chart,

$$d(u\,dv) = du \wedge dv = (u_t v_y - u_y v_t)\, dt \wedge dy = \{u, v\}\omega.$$

The free module of global forms injects into its restriction to this chart, so the equality holds globally and proves the last assertion of (39). □

For $m \geq 1$ set

$$\begin{aligned} P_m &= B[u_2, v_2, \ldots, u_m, v_m], \\ \{u_i, v_j\} &= \delta_{ij}, \qquad \{u_i, u_j\} = \{v_i, v_j\} = 0, \end{aligned} \tag{41}$$

where δ_{ij} is 1 for $i = j$ and 0 otherwise. Use the bracket of (29) on B and set all mixed brackets with B equal to zero. The associated polynomial Hamiltonian fields are taken with respect to $dt \wedge dy + \sum_{i=2}^{m} du_i \wedge dv_i$. Define

$$\mathfrak{h}_{2m} = \{X_F : F \in P_m\} \subset H_{2m}.$$

Proposition 5.4. *For every $m \geq 1$, $P_m/\mathbb{C}$ is a simple Lie algebra.*

Proof. For $m = 1$ this is Proposition 5.2. Suppose $m \geq 2$ and $0 \neq J \lhd P_m/\mathbb{C}$. If $J \cap (B/\mathbb{C}) \neq 0$, this intersection is an ideal of $B/\mathbb{C}$, so $B/\mathbb{C} \subset J$. In this proof membership in J is understood modulo constants. For every polynomial G in the additional variables,

$$\{e, fG\} = hG, \qquad \{e, hG\} = -2eG, \qquad \{f, hG\} = 2fG.$$

Thus eG, fG, hG all belong to J. Fix an added pair u, v. Writing $G = \sum_{j \geq 0} G_j v^j$ with G_j independent of v, put $Q = \sum_{j \geq 0} G_j v^{j+1}/(j+1)$. Then $\{u, Q\} = \partial_v Q = G$ and

$$\{eu, fQ\} - huQ = efG, \qquad \{hu, hQ\} = h^2 G$$

belong to J. The first entries eu, hu lie in J, and huQ already does. The identity $h^2 + 4ef = 1$ gives $G \in J$. In particular $u \in J$. The same finite integration formula makes $\{u, -\} = \partial_v$ surjective on P_m, and hence $J = P_m/\mathbb{C}$.

Suppose now that $J \cap (B/\mathbb{C}) = 0$. Choose a nonzero class in J of least total degree d in the added variables. One has $d > 0$. Bracketing with their linear Hamiltonians gives all partial derivatives in those variables. If $d > 1$, some derivative has nonzero positive degree $d - 1$, contrary to minimality. Hence $d = 1$, and a representative has the form

$$F = b_0 + \sum_{j=1}^{2m-2} b_j w_j, \qquad b_j \in B,$$

where $w_1, \ldots, w_{2m-2}$ list the added variables. Each derivative b_j, $j > 0$, has class in $J \cap (B/\mathbb{C})$, so every such b_j is a scalar. For $b \in B$, $\{b, F\} = \{b, b_0\}$ also has zero class. Thus the class of b_0 is central in $B/\mathbb{C}$. The center is an ideal, and simplicity and nonabelianity force it to be zero. Subtracting the constant b_0 leaves a nonzero scalar linear form ℓ in the added variables, with $\ell \in J$.

Write $\{\ell, -\} = \delta = \sum_j c_j \partial_{w_j} \neq 0$. Choose i with $c_i \neq 0$ and set

$$z = w_i/c_i, \qquad z_j = w_j - c_j z \quad (j \neq i).$$

These are linear coordinates: their inverse is $w_i = c_i z$ and $w_j = z_j + c_j z$. They satisfy $\delta z = 1$, $\delta z_j = 0$, and $\delta|_B = 0$. For $F = \sum_{\nu \geq 0} F_\nu z^\nu$ with $F_\nu \in B[z_j : j \neq i]$,

$$\delta \left(\sum_{\nu \geq 0} \frac{F_\nu z^{\nu+1}}{\nu + 1} \right) = F. \tag{42}$$

Thus $\{\ell, -\}$ is surjective on P_m, and $J = P_m/\mathbb{C}$. The bracket $\{h, e\} = 2e$ proves nonabelianity. □

The Hamiltonian map has kernel $\mathbb{C}$, so the proposition proves simplicity of $\mathfrak{h}_{2m}$. The independent classes of b^j, $j \geq 1$, prove infinite-dimensionality. Comparing coefficients in the added variables gives $P_m \cap \mathbb{C}[t, y] = B$. If $X_t = X_F$ for $F \in P_m$, then $t - F$ is a scalar, contradicting $t \notin B$. Thus $\mathfrak{h}_{2m}$ is proper. Substituting t_N, b, u_i, v_i into any polynomial

Hamiltonian gives a difference in $\mathfrak{m}^{2N+1}$; its Hamiltonian field differs by an element of $\mathfrak{m}^{2N}W_{2m}$. This proves density for every m.

§ 6. The special series

Fix $n \geq 2$ and put $r = n-2$. For $D = AE + CT + \sum_i U_i \partial_{z_i} \in \mathrm{Der}(R_r)$, contraction with Ω_r is given by

$$\begin{aligned}\iota_D \Omega_r &= A\, db \wedge dz_1 \wedge \cdots \wedge dz_r - C\, \eta \wedge dz_1 \wedge \cdots \wedge dz_r \\ &\quad + \sum_{i=1}^{r} (-1)^{i+1} U_i\, \omega \wedge dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_r.\end{aligned}$$

Here a hat means that the indicated factor is omitted, and an empty exterior product is 1. The displayed forms constitute a basis of $\Omega^{n-1}_{R_r}$, so contraction is an isomorphism from $\mathrm{Der}(R_r)$ onto $\Omega^{n-1}_{R_r}$. Define

$$\mathfrak{s}_n = \{D \in \mathrm{Der}(R_r) : \iota_D \Omega_r \in d\Omega^{n-2}_{R_r}\}, \tag{43}$$

and identify its elements with their restrictions to the affine chart. They preserve Ω_r, since $d^2 = 0$, and therefore lie in S_n.

The identity $[\mathcal{L}_U, \iota_V] = \iota_{[U,V]}$ is zero on functions; on df it reads $U(V(f)) - V(U(f)) = [U,V](f)$. Both sides are graded derivations, so it holds on all differential forms. If U, V preserve Ω_r, then $d\iota_V \Omega_r = 0$ and this identity gives

$$\iota_{[U,V]}\Omega_r = \mathcal{L}_U(\iota_V \Omega_r) = d(\iota_U \iota_V \Omega_r). \tag{44}$$

Thus $\mathfrak{s}_n$ is a Lie subalgebra. When $r = 0$, its defining condition is $\iota_D \omega \in dB$, so $\mathfrak{s}_2 = \mathfrak{h}_2$.

One added variable. Suppose $r = 1$ and write $z = z_1$. Extend each $v \in \mathrm{Der}(B)$ to $B[z]$ by $v(z) = 0$, and put $\delta_v = \mathrm{div}_\omega(v)$. Formula (22) gives $\delta_{AE+CT} = E(A) + T(C) + 6aC$. For $k \geq 0$ define

$$T_k(v) = z^k v - \frac{z^{k+1}}{k+1} \delta_v \partial_z. \tag{45}$$

Every global divergence-free field on $B[z]$ has a unique finite expression

$$q\partial_z + \sum_{k \geq 0} T_k(v_k), \qquad q \in B, \quad v_k \in \mathrm{Der}(B). \tag{46}$$

Indeed, write its horizontal part as $\sum_k z^k v_k$. The equation for its vertical coefficient is $\partial_z D(z) = -\sum_k z^k \delta_{v_k}$, whose polynomial solutions are $D(z) = q - \sum_k z^{k+1}\delta_{v_k}/(k+1)$.

Each $T_k(v)$ is exact. Since $d\iota_v \omega = \delta_v \omega$,

$$\begin{aligned}-d\left(\frac{z^{k+1}}{k+1}\iota_v\omega\right) &= -z^k dz \wedge \iota_v\omega - \frac{z^{k+1}}{k+1}\delta_v\omega \\ &= z^k \iota_v\omega \wedge dz - \frac{z^{k+1}}{k+1}\delta_v\omega = \iota_{T_k(v)}(\omega \wedge dz).\end{aligned} \tag{47}$$

Moreover, $\iota_{q\partial_z}(\omega \wedge dz) = q\omega$. If $q\omega = d\lambda$ over B, the same λ is a primitive over $B[z]$. Conversely, pulling a primitive over $B[z]$ back along $z = 0$ gives a primitive over B. Lemma 5.3 therefore yields

$$\mathfrak{s}_3 = B'\partial_z \;\oplus\; \bigoplus_{k \geq 0} T_k(\mathrm{Der}(B)). \tag{48}$$

Thus $\partial_z \notin \mathfrak{s}_3$, and $\mathfrak{s}_3$ has codimension one in the global divergence-free fields on $Q \times \mathbb{A}^1$.

For $q \in B$, the product rule for divergence gives $\operatorname{div}_\omega(qv) = q\delta_v + v(q)$. Hence

$$[q\partial_z, T_{k+1}(v)] = (k+1)z^k qv - z^{k+1}(q\delta_v + v(q))\partial_z$$

$$= (k+1)T_k(qv) \qquad (k \geq 0), \tag{49}$$

$$[q\partial_z, T_0(v)] = -(q\delta_v + v(q))\partial_z = -\operatorname{div}_\omega(qv)\partial_z. \tag{50}$$

The fields in $B\partial_z$ commute. Induction on j using these two formulas gives, for every $k, j \geq 0$,

$$(\operatorname{ad}(e\partial_z))^j T_k(v) = \begin{cases} \dfrac{k!}{(k-j)!}\, T_{k-j}(e^j v), & 0 \leq j \leq k, \\ -k!\, \operatorname{div}_\omega(e^{k+1}v)\partial_z, & j = k+1, \\ 0, & j \geq k+2. \end{cases} \tag{51}$$

For $j < k$, the next bracket multiplies the coefficient by $k - j$; at $j = k$ equation (50) gives the second line, and the next bracket is zero.

Proposition 6.1. *The Lie algebra $\mathfrak{s}_3$ is simple.*

Proof. Let $0 \neq J \lhd \mathfrak{s}_3$. We first show $J \cap B'\partial_z \neq 0$. Choose a nonzero element of J and write it as in (48). If every v_k is zero, the assertion follows. Otherwise, let N be the largest index with $v_N \neq 0$ and apply $(\operatorname{ad}(e\partial_z))^N$. Since $e \in B'$, the result belongs to J. Equation (51) shows that it has the form $T_0(v) + q_0\partial_z$, where

$$v = N!e^N v_N \in \operatorname{Der}(B), \qquad q_0 \in B'.$$

The vector field v is nonzero because B is a domain and $\operatorname{Der}(B)$ is free over B.

At least one $q \in \{e, f, h\}$ satisfies $\operatorname{div}_\omega(qv) \neq 0$. Indeed, if all three divergences vanished, the product rule would give

$$v(e) = -\delta_v e, \qquad v(f) = -\delta_v f, \qquad v(h) = -\delta_v h.$$

Applying v to $h^2 + 4ef = 1$ would then yield

$$0 = 2h\, v(h) + 4f\, v(e) + 4e\, v(f) = -2\delta_v(h^2 + 4ef) = -2\delta_v.$$

It would follow that v vanishes on the generators e, f, h, contrary to $v \neq 0$. For the resulting q, the bracket $[q\partial_z, T_0(v) + q_0\partial_z]$ is a nonzero element of $J \cap B'\partial_z$ by (50). Its coefficient lies in B' because $(\operatorname{div}_\omega(qv))\omega = d(\iota_{qv}\omega)$.

The space $M = \{q \in B' : q\partial_z \in J\}$ is therefore nonzero. For $u \in B'$, the Hamiltonian field X_u has zero divergence, so $T_0(X_u) = X_u \in \mathfrak{s}_3$. Since

$$[X_u, q\partial_z] = \{u, q\}\partial_z,$$

M is a Lie ideal of B'. Lemma 5.3 and Proposition 5.2 imply $M = B'$. Equation (49) now gives $T_k(qv) \in J$ for every $q \in B'$, $v \in \operatorname{Der}(B)$, and $k \geq 0$. Using $e, h \in B'$ and $h^2 + 4ef = 1$, we obtain

$$T_k(v) = T_k\big(h(hv)\big) + 4T_k\big(e(fv)\big) \in J.$$

Together with $B'\partial_z \subset J$, this proves $J = \mathfrak{s}_3$. The bracket $[X_h, e\partial_z] = 2e\partial_z$ proves non-abelianity. □

At least two added variables. Suppose $r \geq 2$, and let $\mathcal{S}_r$ denote the Lie algebra of global Ω_r-divergence-free fields on R_r. For $v \in \operatorname{Der}(B)$, write $\delta_v = \operatorname{div}_\omega(v)$.

Proposition 6.2. *For every $r \geq 2$, the algebra $\mathcal{S}_r$ is simple and equals $\mathfrak{s}_{r+2}$.*

Proof. Let $0 \neq J \lhd \mathcal{S}_r$. All ∂_{z_i} belong to $\mathcal{S}_r$. Express a nonzero field of J in the free frame E, T, ∂_{z_i}, and choose a nonzero coefficient monomial z^α of maximal total z-degree. Apply ∂_z^α to its coefficients by successive brackets with ∂_{z_i}. Every lower-degree monomial is killed.

A different monomial of the same degree has some exponent smaller than the corresponding α_i and is also killed. The resulting field is nonzero and has the form

$$D_0 = v + \sum_i q_i \partial_{z_i}, \qquad v \in \mathrm{Der}(B), \quad q_i \in B, \quad \delta_v = 0.$$

If $v \neq 0$, choose $q \in B$ with $v(q) \neq 0$. Since $q\partial_{z_j} \in \mathcal{S}_r$,

$$[D_0, q\partial_{z_j}] = v(q)\partial_{z_j} \in J.$$

If $v = 0$, choose $q_i \neq 0$ and $j \neq i$. The field $z_i\partial_{z_j}$ belongs to $\mathcal{S}_r$, and $[D_0, z_i\partial_{z_j}] = q_i\partial_{z_j} \in J$. After renumbering the added coordinates, we have $0 \neq q\partial_{z_1} \in J$.

Set $M = \{q \in B : q\partial_{z_1} \in J\}$. For $q \in M$ and $u \in B$, the following brackets use divergence-free fields:

$$\begin{aligned} [q\partial_{z_1}, z_1 u\partial_{z_2}] &= qu\partial_{z_2}, \\ [z_2\partial_{z_1}, qu\partial_{z_2}] &= -qu\partial_{z_1}. \end{aligned} \tag{52}$$

Thus M is an associative ideal of B. For any $v \in \mathrm{Der}(B)$, extend v trivially in the added variables and put $\widetilde{v} = v - z_2\delta_v\partial_{z_2}$. Then

$$\mathrm{div}_{\Omega_r} \widetilde{v} = \delta_v - \partial_{z_2}(z_2\delta_v) = 0, \qquad [\widetilde{v}, q\partial_{z_1}] = v(q)\partial_{z_1}.$$

Hence M is stable under every derivation of B. Lemma 3.2 gives $M = B$, so $\partial_{z_1} \in J$.

For a polynomial $\sum_{j\geq 0} a_j z_i^j$ with coefficients independent of z_i, define

$$\mathcal{I}_i\left(\sum_{j\geq 0} a_j z_i^j\right) = \sum_{j\geq 0} \frac{a_j z_i^{j+1}}{j+1}. \tag{53}$$

Given $D \in \mathcal{S}_r$, write

$$\begin{aligned} D &= AE + CT + \sum_{i=1}^{r} U_i \partial_{z_i}, \\ Y_0 &= \mathcal{I}_1(A)E + \mathcal{I}_1(C)T + \sum_{i=1}^{r} \mathcal{I}_1(U_i)\partial_{z_i}. \end{aligned}$$

Then $[\partial_{z_1}, Y_0] = D$. Formula (22) gives

$$\begin{aligned} \mathrm{div}_{\Omega_r} Y_0 &= \mathcal{I}_1\left(E(A) + T(C) + 6aC + \sum_{i=2}^{r} \partial_{z_i} U_i\right) + U_1 \\ &= -\mathcal{I}_1(\partial_{z_1} U_1) + U_1 = U_1\big|_{z_1=0}. \end{aligned}$$

Put $Q = \mathcal{I}_2(U_1|_{z_1=0})$ and $Y = Y_0 - Q\partial_{z_2}$. The polynomial Q is independent of z_1, and

$$\mathrm{div}_{\Omega_r} Y = U_1|_{z_1=0} - \partial_{z_2} Q = 0, \qquad [\partial_{z_1}, Y] = D.$$

It follows that $D \in J$, so $J = \mathcal{S}_r$. The bracket $[\partial_{z_1}, z_1\partial_{z_2}] = \partial_{z_2}$ proves nonabelianity.

The same construction and (44) give

$$\iota_D \Omega_r = d(\iota_{\partial_{z_1}} \iota_Y \Omega_r).$$

Thus every $D \in \mathcal{S}_r$ belongs to $\mathfrak{s}_{r+2}$. The defining exactness condition gives the reverse inclusion. □

Properness and density in every dimension. Every field in $\mathfrak{s}_n$ preserves R_{n-2}, whereas the ambient divergence-free field ∂_y does not, by (27). Hence $\mathfrak{s}_n \subsetneq S_n$.

Take $D \in S_n$ on the affine chart and put $\beta = \iota_D \Omega_0$, where $\Omega_0 = dt \wedge dy \wedge dz_1 \wedge \cdots \wedge dz_r$. Since $d\beta = 0$, Lemma 2.1 supplies a polynomial $(n-2)$-form γ with $d\gamma = \beta$. Define the algebra map

$$\psi_N^* : \mathbb{C}[t, y, z_1, \ldots, z_r] \longrightarrow R_r, \qquad t \longmapsto t_N, \quad y \longmapsto b, \quad z_i \longmapsto z_i.$$

Extend it to forms by $\psi_N^*(df) = d(\psi_N^* f)$ and $\psi_N^*(\lambda \wedge \mu) = \psi_N^* \lambda \wedge \psi_N^* \mu$. The Leibniz rule and $d^2 = 0$ show that this extension commutes with d. Contraction determines a unique global derivation D_N satisfying

$$\iota_{D_N} \Omega_r = d(\psi_N^* \gamma) = \psi_N^* \beta. \tag{54}$$

Consequently $D_N \in \mathfrak{s}_n$.

Write $z = (z_1, \ldots, z_r)$ and $D = P\partial_t + Q\partial_y + \sum_i U_i \partial_{z_i}$. Set $P_N = P(t_N, b, z)$, with $Q_N, U_{i,N}$ defined in the same way. On the chart, $\Omega_r = \Omega_0$ and $dt_N = (\partial_t t_N) dt + (\partial_y t_N) dy$. Substitution in the contraction formula gives

$$\begin{aligned} D_N = {} & (P_N - Q_N \partial_y t_N)\partial_t + Q_N (\partial_t t_N)\partial_y \\ & + \sum_i U_{i,N} (\partial_t t_N) \partial_{z_i}. \end{aligned} \tag{55}$$

Each substituted coefficient differs from the original coefficient by an element of $\mathfrak{m}^{2N+1}$, by (13). Equation (14) gives $\partial_t t_N - 1, \partial_y t_N \in \mathfrak{m}^{2N}$. Subtracting D from (55), its coefficients are

$$P_N - P - Q_N \partial_y t_N, \qquad Q_N - Q + Q_N(\partial_t t_N - 1),$$
$$U_{i,N} - U_i + U_{i,N}(\partial_t t_N - 1).$$

They lie in $\mathfrak{m}^{2N}$, so

$$D_N - D \in \mathfrak{m}^{2N} W_n. \tag{56}$$

Lemma 2.2 proves density. For example, in dimension three,

$$E(t_N)\partial_z = \big(1 - (N+1)(-ty)^N\big)\partial_z \in \mathfrak{s}_3, \qquad E(t_N)\omega = d(t_N\, db).$$

The difference from ∂_z lies in $\mathfrak{m}^{2N} W_3$. The fields X_{b^j} for $n = 2$ and $j \geq 1$, $b^j \partial_z$ for $n = 3$ and $j \geq 1$, and $b^j \partial_{z_1}$ for $n \geq 4$ and $j \geq 0$ give linearly independent families in the respective algebras.

§ 7. The contact series

For $m \geq 1$ use coordinates $(t, y, p, u_2, v_2, \ldots, u_m, v_m)$ and the standard contact form

$$\alpha_0 = dt + p\, dy + \sum_{i=2}^{m} u_i\, dv_i.$$

Thus $(p_1, q_1) = (p, y)$ and $(p_i, q_i) = (u_i, v_i)$ for $i \geq 2$ in (7). Set

$$s = p + g = p + 3t^2 + 2t^3 y, \qquad D_m = B[s, u_2, v_2, \ldots, u_m, v_m] \subset \mathbb{C}[t, y, p, u_2, v_2, \ldots, u_m, v_m]. \tag{57}$$

The element s is algebraically independent over B, since it is monic of degree one in the independent variable p. Define

$$\mathfrak{k}_{2m+1} = \{Y_F : F \in D_m\}. \tag{58}$$

The contact bracket. The Reeb derivation is $R = \partial_t$, with p, y, u_i, v_i held fixed: $\alpha_0(R) = 1$ and $\iota_R d\alpha_0 = 0$. Its values are

$$Ra = h, \quad Rb = 0, \quad Rc = b, \quad Rs = 6a, \quad Ru_i = Rv_i = 0. \tag{59}$$

Thus R preserves D_m. Each element is an ambient polynomial of finite t-degree, so it is killed by a sufficiently high power of R. For $U, V \in B$, (7) gives

$$\langle U, V\rangle = UE(V) - VE(U), \qquad \langle U, s\rangle = 6aU - T(U). \tag{60}$$

For the second formula, $Y_U(p) = pE(U) - \partial_y U$ and $Y_U(g) = 6aU$. Subtracting $sE(U) = (p+g)E(U)$ leaves $6aU - \partial_y U - gE(U)$. The six brackets of the base generators are

$$\begin{aligned}
\langle a, b\rangle &= aE(b) - bE(a) = -bh,\\
\langle a, c\rangle &= ab - ch = c(c-1) - c(2c-1) = -c^2,\\
\langle a, s\rangle &= 6a^2 - T(a) = 6a^2 - 4a^2 = 2a^2,\\
\langle b, c\rangle &= bE(c) - cE(b) = b^2,\\
\langle b, s\rangle &= 6ab - T(b) = 6ab - 1,\\
\langle c, s\rangle &= 6ac - T(c) = 6ac - ah = a(4c+1).
\end{aligned} \tag{61}$$

For $F \in B[s]$ and $2 \le i, j \le m$, the added generators satisfy

$$\langle u_i, v_j\rangle = \delta_{ij}, \quad \langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0, \quad \langle F, u_i\rangle = 0, \quad \langle F, v_i\rangle = -v_i RF. \tag{62}$$

Indeed, $Y_{u_i} = \partial_{v_i}$ and $Y_{v_i} = v_i\partial_t - \partial_{u_i}$; also $Y_F(u_i) = u_i RF$ and $Y_F(v_i) = 0$.

For arbitrary polynomial functions F, G, formula (7) gives

$$Y_1 = R, \qquad \langle 1, F\rangle = RF, \qquad Y_{FG} = FY_G + GY_F - FG\,R. \tag{63}$$

For the last identity, the ∂_t coefficient is

$$\begin{aligned}
FG - \sum_i p_i(F_{p_i}G + FG_{p_i}) = F\left(G - \sum_i p_i G_{p_i}\right)\\
+ G\left(F - \sum_i p_i F_{p_i}\right) - FG.
\end{aligned}$$

The ∂_{q_i} coefficient is $(FG)_{p_i} = FG_{p_i} + GF_{p_i}$, and the ∂_{p_i} coefficient is

$$p_i(FG)_t - (FG)_{q_i} = F(p_iG_t - G_{q_i}) + G(p_iF_t - F_{q_i}).$$

Consequently,

$$\begin{aligned}
\langle F, GH\rangle &= G\,Y_F(H) + H\,Y_F(G) - GH\,RF\\
&= G\langle F, H\rangle + H\langle F, G\rangle + GH\,RF,
\end{aligned} \tag{64}$$

$$\begin{aligned}
\langle FG, H\rangle &= F\,Y_G(H) + G\,Y_F(H) - FG\,RH - HR(FG)\\
&= F\langle G, H\rangle + G\langle F, H\rangle - FG\,RH.
\end{aligned} \tag{65}$$

Equations (59)–(62), (63), and these multiplication rules prove by induction on products of generators that D_m is closed under the contact bracket. Thus (58) is a Lie subalgebra, identified with $(D_m, \langle -, -\rangle)$ by $F \mapsto Y_F$.

On $Q \times \mathbb{A}^1_s \times \mathbb{A}^{2m-2}$, the global form

$$\alpha = \eta + s\,db + \sum_{i=2}^{m} u_i\,dv_i$$

restricts to α_0 by (20). Its exterior derivative is

$$d\alpha = (ds - 6a\eta) \wedge db + \sum_{i=2}^{m} du_i \wedge dv_i.$$

Terms with a repeated db, dv_i, or η vanish in the top exterior power. The global coframe gives the nowhere-vanishing top form

$$\begin{aligned}\alpha \wedge (d\alpha)^m &= m!\,\eta \wedge ds \wedge db \wedge \bigwedge_{i=2}^{m} (du_i \wedge dv_i) \\ &= -m!\,\omega \wedge ds \wedge \bigwedge_{i=2}^{m} (du_i \wedge dv_i).\end{aligned}$$

Simplicity. The function

$$r = ah - bs = t - py \in D_m \quad \text{satisfies} \quad Rr = h^2 + 2ab - 6ab = 1. \tag{66}$$

Proposition 7.1. *The Lie algebra $(D_m, \langle -, - \rangle)$ is simple for every $m \geq 1$.*

Proof. Let $0 \neq J \lhd D_m$, choose $0 \neq F \in J$, and let $d = \deg_t F$ in the ambient polynomial ring. Since $\langle 1, F \rangle = RF = \partial_t F$, the element $(\operatorname{ad} 1)^d F \in J$ is nonzero and independent of t. On such elements,

$$\operatorname{ad} b = b\partial_t - \partial_p = -\partial_p, \qquad \operatorname{ad} u_i = \partial_{v_i}, \qquad \operatorname{ad} v_i = v_i \partial_t - \partial_{u_i} = -\partial_{u_i}.$$

Successively differentiate to the highest current degree in each of p, u_i, v_i. Each step is a bracket with an element of D_m, preserves independence of t, and leaves a nonzero result. This gives a nonzero polynomial $P(b) \in J$.

Formula (7) applied to $r = t - py$ gives $Y_r(b) = -b$ and $Rr = 1$, hence

$$\langle r, P(b) \rangle = -bP'(b) - P(b). \tag{67}$$

Write $P(b) = \sum_{j \in S} P_j b^j$, where S is its finite set of nonzero coefficient indices, and choose $k \in S$. The eigenvalues of $\operatorname{ad} r$ on b^j are $-(j+1)$. Therefore the operator

$$\prod_{j \in S \setminus \{k\}} \frac{\operatorname{ad} r + (j+1)}{j - k}$$

sends P to $P_k b^k$, so $b^k \in J$.

For $k > 0$, $Y_s(b) = 1$ and $Rs = 6a$ give

$$U_k = \langle s, b^k \rangle = kb^{k-1} - 6ab^k \in J. \tag{68}$$

For $F \in B$, (60) gives $\langle a, F \rangle = aE(F) - hF$. Thus

$$\langle a, b^{k-1} \rangle = -hb^{k-1}, \qquad \langle a, ab^k \rangle = a(hb^k) - hab^k = 0,$$

and $\langle a, U_k \rangle = -khb^{k-1} \in J$. Consequently $hb^{k-1} \in J$, and

$$\begin{aligned} V_k := \langle a, hb^{k-1} \rangle &= 2ab^k - h^2 b^{k-1} \\ &= -b^{k-1} - 2ab^k \in J. \end{aligned} \tag{69}$$

The combination

$$\begin{aligned} U_k - 3V_k &= kb^{k-1} - 6ab^k + 3b^{k-1} + 6ab^k \\ &= (k+3)b^{k-1} \end{aligned} \tag{70}$$

lowers the exponent. Since $k + 3 \neq 0$, descent gives $1 \in J$.

For $F \in D_m$, choose $M \geq 0$ with $R^{M+1} F = 0$ and put

$$I(F) = \sum_{j=0}^{M} \frac{(-1)^j r^{j+1} R^j F}{(j+1)!} \in D_m. \tag{71}$$

Using $Rr = 1$ and the Leibniz rule gives

$$\begin{aligned} RI(F) &= \sum_{j=0}^{M} \frac{(-1)^j r^j R^j F}{j!} + \sum_{j=0}^{M} \frac{(-1)^j r^{j+1} R^{j+1} F}{(j+1)!} \\ &= F + \sum_{j=1}^{M} \frac{((-1)^j + (-1)^{j-1}) r^j R^j F}{j!} + \frac{(-1)^M r^{M+1} R^{M+1} F}{(M+1)!} = F. \end{aligned}$$

Hence $F = \langle 1, I(F) \rangle \in J$ and $J = D_m$. The bracket $\langle 1, r \rangle = 1$ proves nonabelianity. □

Properness and density. The equality $s = p + g$ is triangular and monic in p. A nonzero coefficient of highest s-degree remains the nonzero coefficient of the same p-degree. Thus an element of D_m independent of p has s-degree zero. Comparing coefficients of the independent u_i, v_i then gives

$$D_m \cap \mathbb{C}[t, y] = B. \tag{72}$$

In particular $t \notin D_m$. Since $F \mapsto Y_F$ is injective, $Y_t \notin \mathfrak{k}_{2m+1}$, proving properness.

Set

$$p_N = s - 3t_N^2 - 2t_N^3 b \in D_m.$$

Equations (12) and (28) give $t_N - t \in \mathfrak{m}^{2N+1}$ and $p_N - p \in \mathfrak{m}^{2N+2}$. For any polynomial contact Hamiltonian F, substituting t_N, b, p_N, u_i, v_i for its coordinates gives $F_N \in D_m$ with $F_N - F \in \mathfrak{m}^{2N+1}$. Formula (7) then gives $Y_{F_N} - Y_F \in \mathfrak{m}^{2N} W_{2m+1}$, proving density in the natural filtration. The independent functions b^j, $j \geq 0$, and injectivity of the Hamiltonian correspondence prove infinite-dimensionality.

The four constructions prove Theorem 1.1.

§ 8. Simplicity of the ambient algebras

The polynomial Cartan algebras in Theorem 1.1 are simple. For W_n, let I be a nonzero associative ideal of A_n stable under all derivations. Choose $P = \sum_\beta c_\beta x^\beta \in I$ and a nonzero coefficient c_α of maximal total degree. Then

$$\partial^\alpha P = \alpha! c_\alpha \neq 0.$$

Indeed, every other exponent β of degree at most $|\alpha|$ has $\beta_i < \alpha_i$ for some i, so its monomial is killed. Thus $I = A_n$, and Lemma 4.1 proves simplicity of W_n.

For S_n, $n \geq 2$, let I be a nonzero Lie ideal and choose $0 \neq D = \sum_{i,\beta} c_{i,\beta} x^\beta \partial_{x_i} \in I$. Choose α of maximal total degree among its nonzero coefficients. Brackets with the constant fields ∂_{x_i} give

$$\prod_{j=1}^{n} (\operatorname{ad} \partial_{x_j})^{\alpha_j} D = \alpha! \sum_i c_{i,\alpha} \partial_{x_i} \neq 0.$$

Hence I contains a nonzero constant field $v = \sum_i c_i \partial_{x_i}$. Choose i with $c_i \neq 0$. For every $j \neq i$, the field $x_i \partial_{x_j}$ belongs to S_n, and

$$[v, x_i \partial_{x_j}] = c_i \partial_{x_j}.$$

Using any such j also gives $[\partial_{x_j}, x_j \partial_{x_i}] = \partial_{x_i}$. Thus every constant field belongs to I.

For $i < j$, put

$$D_{ij}(F) = F_{x_j} \partial_{x_i} - F_{x_i} \partial_{x_j}, \qquad \theta_{ij} = \iota_{\partial_{x_j}} \iota_{\partial_{x_i}} \Omega_0.$$

The constant forms θ_{ij} form a basis of the polynomial $(n-2)$-forms. Their signs satisfy, for every k,

$$dx_k \wedge \theta_{ij} = \delta_{kj} \iota_{\partial_{x_i}} \Omega_0 - \delta_{ki} \iota_{\partial_{x_j}} \Omega_0.$$

Consequently,

$$\begin{aligned} d(F\theta_{ij}) &= \sum_k F_{x_k}\, dx_k \wedge \theta_{ij} \\ &= F_{x_j}\iota_{\partial_{x_i}}\Omega_0 - F_{x_i}\iota_{\partial_{x_j}}\Omega_0 = \iota_{D_{ij}(F)}\Omega_0. \end{aligned}$$

For any $D \in S_n$, apply Lemma 2.1 to $\iota_D\Omega_0$ and write its primitive as $\sum_{i<j} F_{ij}\theta_{ij}$. The displayed identity and injectivity of contraction give $D = \sum_{i<j} D_{ij}(F_{ij})$.

Fix an index r. If $F = \sum_{\ell\geq 0} F_\ell x_r^\ell$ with $\partial_{x_r} F_\ell = 0$, set $G = \sum_{\ell\geq 0} F_\ell x_r^{\ell+1}/(\ell+1)$. Then $\partial_{x_r} G = F$ and

$$[\partial_{x_r}, D_{ij}(G)] = D_{ij}(\partial_{x_r} G) = D_{ij}(F) \in I.$$

It follows that $I = S_n$. The bracket $[\partial_{x_1}, x_1\partial_{x_2}] = \partial_{x_2}$ proves nonabelianity.

For $H_{2m} = A_{2m}/\mathbb{C}$, let I be a nonzero Lie ideal and choose a representative F of degree $d \geq 1$ of a nonzero class in I. Brackets with the linear Hamiltonians give all coordinate partial derivatives. Choose a monomial $c_\alpha x^\alpha$ of degree d with $c_\alpha \neq 0$, and choose j with $\alpha_j > 0$. Writing ε_j for the jth unit multi-index, the polynomial $\partial^{\alpha-\varepsilon_j} F$ has degree at most one, and its x_j-coefficient is $c_\alpha \alpha! \neq 0$. Thus I contains the class of a nonzero linear Hamiltonian ℓ. Its Hamiltonian derivation is a nonzero constant-coefficient derivative. The linear coordinates and polynomial antiderivative constructed in Proposition 5.4 show that this derivative is surjective on A_{2m}. Hence $I = A_{2m}/\mathbb{C}$. The bracket $\{q_1^2, p_1\} = 2q_1$ proves nonabelianity.

For the contact bracket on the full polynomial ring,

$$\langle 1, F\rangle = \partial_t F, \qquad \langle p_i, F\rangle = \partial_{q_i} F, \qquad \langle q_i, F\rangle = q_i\partial_t F - \partial_{p_i} F.$$

Given a nonzero element F of a Lie ideal, set $d = \deg_t F$. Then $(\operatorname{ad} 1)^d F$ is nonzero and independent of t. On such polynomials, $\operatorname{ad} p_i = \partial_{q_i}$ and $\operatorname{ad} q_i = -\partial_{p_i}$. Successively differentiating to the highest current degree in each q_i, p_i gives a nonzero constant. After scalar multiplication, the ideal therefore contains 1. For $F = \sum_{j\geq 0} F_j t^j$, with the F_j independent of t,

$$\langle 1, \sum_{j\geq 0} \frac{F_j t^{j+1}}{j+1}\rangle = F.$$

The ideal is the whole algebra. Finally, $\langle 1, t\rangle = 1$ proves nonabelianity and hence simplicity of K_{2m+1}.

§ 9. Keller maps and related examples

A Keller map is a polynomial map with constant nonzero Jacobian determinant. The map recorded in [5, Eq. (2)] is

$$\begin{aligned} F_1(x,y,z) &= (1+xy)^3 z + y^2(1+xy)(4+3xy), \\ F_2(x,y,z) &= y + 3x(1+xy)^2 z + 3xy^2(4+3xy), \\ F_3(x,y,z) &= 2x - 3x^2 y - x^3 z. \end{aligned} \tag{73}$$

Here and below J_F denotes the matrix of partial derivatives of the components of F. On $x \neq 0$, use the coordinates

$$\xi = 1 + xy, \qquad \zeta = 2 - 3xy - x^2 z.$$

The identities $xy = \xi - 1$ and $x^2 z = 5 - 3\xi - \zeta$ give

$$\begin{aligned} x^2 F_1 &= (5-3\xi-\zeta)\xi^3 + (\xi-1)^2\xi(3\xi+1) = \xi(\xi+1) - \xi^3\zeta, \\ xF_2 &= \xi - 1 + 3\xi^2(5-3\xi-\zeta) + 3(\xi-1)^2(3\xi+1) = 4\xi + 2 - 3\xi^2\zeta, \\ F_3 &= x\zeta. \end{aligned}$$

Put $P = \xi(\xi+1) - \xi^3\zeta$ and $Q = 4\xi + 2 - 3\xi^2\zeta$. The coordinate Jacobian is

$$\det \frac{\partial(x,\xi,\zeta)}{\partial(x,y,z)} = \det \begin{pmatrix} 1 & 0 & 0 \\ y & x & 0 \\ -3y - 2xz & -3x & -x^2 \end{pmatrix} = -x^3,$$

whereas

$$\det \frac{\partial(P/x^2, Q/x, x\zeta)}{\partial(x,\xi,\zeta)} = x^{-3} \det \begin{pmatrix} -2P & P_\xi & P_\zeta \\ -Q & Q_\xi & Q_\zeta \\ \zeta & 0 & 1 \end{pmatrix}.$$

Expanding the latter determinant gives $P_\xi(Q + \zeta Q_\zeta) - Q_\xi(2P + \zeta P_\zeta)$. Since

$$P_\xi = 2\xi + 1 - 3\xi^2\zeta, \quad P_\zeta = -\xi^3, \quad Q_\xi = 4 - 6\xi\zeta, \quad Q_\zeta = -3\xi^2,$$
$$Q + \zeta Q_\zeta = 2P_\xi, \qquad 2P + \zeta P_\zeta = 2\xi^2 + 2\xi - 3\xi^3\zeta,$$

this determinant is

$$\begin{aligned} &2(2\xi + 1 - 3\xi^2\zeta)^2 - (4 - 6\xi\zeta)(2\xi^2 + 2\xi - 3\xi^3\zeta) \\ &\quad = 8\xi^2 + 8\xi + 2 - 24\xi^3\zeta - 12\xi^2\zeta + 18\xi^4\zeta^2 \\ &\qquad - \left(8\xi^2 + 8\xi - 24\xi^3\zeta - 12\xi^2\zeta + 18\xi^4\zeta^2\right) = 2. \end{aligned}$$

The chain rule therefore gives $\det J_F = -2$ on $x \neq 0$, and hence everywhere, since the determinant is a polynomial.

Set $P_0 = (0, 0, -1/4)$ and $P_\varepsilon = (\varepsilon, -3\varepsilon/2, 13/2)$ for $\varepsilon \in \{1, -1\}$. Substitution gives $F(P_0) = (-1/4, 0, 0)$. At P_ε, one has $xy = -3/2$, and the three components are

$$\begin{aligned} F_1(P_\varepsilon) &= -13/16 + 9/16 = -1/4, \\ F_2(P_\varepsilon) &= \varepsilon(-3/2 + 39/8 - 27/8) = 0, \\ F_3(P_\varepsilon) &= \varepsilon(2 + 9/2 - 13/2) = 0. \end{aligned} \tag{74}$$

Thus $F(P_0) = F(P_1) = F(P_{-1})$, while x takes the values $0, 1, -1$.

Proposition 9.1. *The map* (73)*, its products with identity maps, and its cotangent and contact lifts give proper dense simple subalgebras of*

$$W_n, \ S_n \quad (n \geq 3), \qquad H_{2m}, \ K_{2m+1} \quad (m \geq 3).$$

Each subalgebra is isomorphic to its ambient algebra.

Proof. For $n \geq 3$, write $\mathbf{x} = (x, y, z, x_4, \ldots, x_n)$ and extend F by the components $x_4, \ldots, x_n$. Then $F(0) = 0$ and $\det J_F = -2$. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be independent target coordinates, and put $R_F = \mathbb{C}[F_1, \ldots, F_n] \subset \mathbb{C}[\mathbf{x}]$. The components are algebraically independent. Indeed, a nonzero relation $A(F) = 0$ of minimal degree would give

$$(A_{u_1}(F), \ldots, A_{u_n}(F)) J_F = 0$$

by differentiation. Invertibility of J_F makes every entry zero. Some derivative of the nonconstant polynomial A is nonzero in characteristic zero, producing a relation of smaller degree.

Let $\mathbf{x}_j$ denote the jth coordinate in the displayed tuple. The inverse matrix J_F^{-1} is polynomial, by the adjugate formula. Define

$$\delta_i = \sum_j (J_F^{-1})_{ji} \partial_{\mathbf{x}_j}, \qquad \delta_i(F_j) = \delta_{ij}. \tag{75}$$

These fields form a basis of W_n over $\mathbb{C}[\mathbf{x}]$. The commutator $[\delta_i, \delta_j]$ annihilates every F_k; multiplication of its coefficient vector by J_F is therefore zero, so $[\delta_i, \delta_j] = 0$. For polynomials A_i, B_i in $\mathbf{u}$,

$$\left[\sum_i A_i(F)\delta_i, \sum_j B_j(F)\delta_j\right] = \sum_j \left(\sum_i (A_i \partial_{u_i} B_j - B_i \partial_{u_i} A_j)\right)(F)\delta_j.$$

Consequently the map

$$\Phi : \sum_i A_i(\mathbf{u})\partial_{u_i} \longmapsto \sum_i A_i(F)\delta_i$$

is an injective Lie homomorphism, with image $\bigoplus_i R_F\delta_i$. The points in (74), with all added coordinates zero, show $x \notin R_F$. Thus $x\delta_1$ is absent from the image, which is proper.

The substitution $\mathbf{u} = F(\mathbf{x})$ has a formal inverse at zero. Set $H_1(\mathbf{u}) = J_F(0)^{-1}\mathbf{u}$. If $F(H_{d-1}) - \mathbf{u}$ vanishes through degree $d-1$, let E_d be its homogeneous part of degree d and set

$$H_d = H_{d-1} - J_F(0)^{-1}E_d \qquad (d \geq 2).$$

The added term changes the linear part of F by $-E_d$ and its higher-degree parts only in degrees at least $d+1$. Thus $F(H_d) - \mathbf{u}$ vanishes through degree d. The limit H satisfies $F(H(\mathbf{u})) = \mathbf{u}$. If two formal tuples with zero constant term first differ in degree $d \geq 1$, their images under F first differ by $J_F(0)$ times that degree-d difference. It follows that $H(F(\mathbf{x})) = \mathbf{x}$. For a polynomial $A(\mathbf{x})$, truncate $A(H(\mathbf{u}))$ below degree N to a polynomial A_N. Then

$$A_N(F(\mathbf{x})) - A(\mathbf{x}) \in \mathfrak{m}^N.$$

Expressing any vector field in the basis (75) and applying this approximation to its coefficients proves density of $\Phi(W_n)$.

For the special series, put $\Omega_{\mathbf{x}} = d\mathbf{x}_1 \wedge \cdots \wedge d\mathbf{x}_n$ and $\Omega_{\mathbf{u}} = du_1 \wedge \cdots \wedge du_n$. Then $F^*\Omega_{\mathbf{u}} = -2\Omega_{\mathbf{x}}$. Applying $\mathcal{L}_{\delta_i}$ to the left side differentiates each dF_j to $d(\delta_i F_j) = 0$. Thus every δ_i has zero divergence, and (2) gives

$$\operatorname{div}_{\Omega_{\mathbf{x}}}\left(\sum_i A_i(F)\delta_i\right) = \left(\sum_i \partial_{u_i}A_i\right)(F).$$

Hence Φ restricts to an embedding of S_n into itself. The coefficient of δ_1 in ∂_z is $\partial_z F_1 = (1+xy)^3$. Its values at P_0 and P_1 are 1 and $-1/8$, so it does not lie in R_F. Therefore $\partial_z \notin \Phi(S_n)$.

For $D \in S_n$, its coefficients in formal $\mathbf{u}$ coordinates are $\widehat{A}_i(\mathbf{u}) = D(F_i)(H(\mathbf{u}))$. The preceding divergence formula, applied to formal series, gives $\sum_i \partial_{u_i}\widehat{A}_i = 0$. Writing $\widehat{A}_i = \sum_{d\geq 0} A_{i,d}$ in homogeneous parts gives $\sum_i \partial_{u_i}A_{i,d} = 0$ for each d, because the resulting degrees $d-1$ are distinct. Truncating all coefficients below degree N produces a polynomial field in S_n whose image differs from D by an element of $\mathfrak{m}^N W_n$. This proves density.

For the Hamiltonian series, use $n = m$ and introduce $\mathbf{p} = (p_1, \ldots, p_m)^T$. The polynomial map

$$G(\mathbf{x}, \mathbf{p}) = \big(F(\mathbf{x}), J_F(\mathbf{x})^{-T}\mathbf{p}\big) = (\mathbf{u}, \mathbf{v}) \tag{76}$$

has block-triangular Jacobian with determinant $\det J_F \det J_F^{-T} = 1$. For $\lambda_{\mathbf{x}} = \sum_i p_i\, d\mathbf{x}_i$ and $\lambda_{\mathbf{u}} = \sum_i v_i\, du_i$,

$$G^*\lambda_{\mathbf{u}} = (J_F^{-T}\mathbf{p})^T J_F\, d\mathbf{x} = \mathbf{p}^T\, d\mathbf{x} = \lambda_{\mathbf{x}}.$$

Taking $-d$ shows that G preserves the standard symplectic forms. For $\mathbb{J} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, with I_m the identity matrix, this is $J_G^T\mathbb{J}J_G = \mathbb{J}$, equivalently $J_G\mathbb{J}J_G^T = \mathbb{J}$. For $f, g \in \mathbb{C}[\mathbf{u}, \mathbf{v}]$, write ∇ for the column gradient. The chain rule gives

$$\{f \circ G, g \circ G\} = (\nabla f \circ G)^T J_G \mathbb{J} J_G^T (\nabla g \circ G) = \{f, g\} \circ G.$$

The same minimal-degree argument proves algebraic independence of the components of G. Composition therefore embeds the polynomial Poisson algebra, and modulo constants gives an embedding of H_{2m}. The collision points with $\mathbf{p} = 0$ have the same image under G, while x separates them. Hence $x - f \circ G$ cannot be constant, and X_x is absent from the image. Since $G(0) = 0$ and $J_G(0)$ is invertible, the formal construction above approximates every Hamiltonian function to any prescribed order. An error in $\mathfrak{m}^{N+1}$ gives a Hamiltonian-field error in $\mathfrak{m}^N W_{2m}$, proving density.

For the contact series, add a coordinate t and set $C(t, \mathbf{x}, \mathbf{p}) = (t, G(\mathbf{x}, \mathbf{p}))$. The forms $\alpha_{\mathbf{x}} = dt + \lambda_{\mathbf{x}}$ and $\alpha_{\mathbf{u}} = dt + \lambda_{\mathbf{u}}$ satisfy $C^*\alpha_{\mathbf{u}} = \alpha_{\mathbf{x}}$, and J_C^{-1} is polynomial. For target polynomials $f, g \in \mathbb{C}[t, \mathbf{u}, \mathbf{v}]$, let $Z = J_C^{-1}(Y_f \circ C)$. The chain rule gives

$$\alpha_{\mathbf{x}}(Z) = f \circ C, \qquad \mathcal{L}_Z \alpha_{\mathbf{x}} = C^*(\mathcal{L}_{Y_f} \alpha_{\mathbf{u}}) = (f_t \circ C)\alpha_{\mathbf{x}}.$$

The uniqueness established in (7) implies $Z = Y_{f \circ C}$. Since C fixes t and its other components are independent of t, one obtains

$$\langle f \circ C, g \circ C \rangle = Y_{f \circ C}(g \circ C) - (g \circ C)\partial_t(f \circ C) = \langle f, g \rangle \circ C.$$

The components of C are algebraically independent by the same minimal-degree argument. Thus composition gives an embedding of K_{2m+1}. At $t = 0$ and $\mathbf{p} = 0$, the collision points exclude x from the image on functions, so Y_x is absent from the image on fields. The formal inverse of C gives arbitrary jets of contact Hamiltonians; formula (7) transfers an error in $\mathfrak{m}^{N+1}$ to an error in $\mathfrak{m}^N W_{2m+1}$. All four images are simple by their isomorphisms with the ambient algebras, whose simplicity was proved in Section 8. $\square$

Part II. Prehomogeneous quiver representations

§ 10. Introduction

A representation of a quiver is a collection of vector spaces and linear maps indexed by its vertices and arrows. Fixing the dimensions gives an affine space of matrices, on which the product of the vertex general linear groups acts by change of bases. The existence of a dense orbit means that a nonempty open set of these matrix collections has a single isomorphism class. Dimension vectors with this property are called prehomogeneous.

Reflection functors connect this orbit problem with the root system of the underlying graph. Bernstein, Gel'fand and Ponomarev [14] constructed reflection functors at sinks and sources by taking kernels and cokernels of the incident maps. These constructions realize the corresponding simple reflections on dimension vectors in their natural domains. Kac [17] developed the relation between representations of arbitrary graphs, infinite root systems and invariant theory. In [17, §2.8(b), p. 87], he proposed a reduction criterion for the existence of a dense orbit using modified sink and source reflections.

The limitation of that criterion was recorded by Kac in [18, §3(d), p. 326]. He attributed to Ringel the example

$$1 \rightrightarrows 2 \longrightarrow 3, \qquad d = (3, 6, 1).$$

This dimension vector is prehomogeneous and cannot be reduced to zero using those reflections alone. Kac therefore also allowed reversal of an individual arrow when the stabilizer of general deleted-arrow data has equal maximal orbit dimensions on the arrow space and its dual. He denoted by D the set obtained using both operations, and by D_1 the set obtained using reflections alone. In the same remark he observed that Ringel's vector belongs to D. The revised conjecture, stated in [18, §3(d), p. 325], asserts that the prehomogeneous dimension vectors are precisely the elements of D.

The distinction between the two operations can be seen directly in Ringel's example. Reversing its single arrow between vertices 2 and 3 makes the middle vertex a sink. The subsequent reflections can then reduce the dimensions to zero. Section 16 gives the stabilizer calculation and the complete reduction. Reflection constructions for hereditary algebras were also studied by Ringel in [20]; the reduction example above is cited here from Kac's explicit account.

For acyclic quivers, the canonical decomposition records the dimensions of the indecomposable summands of a general representation. Kac [19, §1.21, Proposition (c)] characterized prehomogeneity by the occurrence of only real roots in this decomposition. Schofield [21] gave a recursive procedure for determining the decomposition using the Euler form and generic dimensions of spaces of homomorphisms and extensions. Derksen and Weyman [15] gave an algorithm in terms of exceptional sequences and the Euler form. Their later treatment [16, §2.4] formulates the dense-orbit criterion using real Schur roots: these are dimension vectors of representations with scalar endomorphisms and no self-extensions. Kac's reduction conjecture specifies the individual operations by which a dimension vector should reach zero.

In this part we use a three-vertex chain with at least two parallel arrows in each of its two links. For integers $r, s \geq 2$, $c \geq 1$ and $a \geq rsc$, the dimension vector (a, ra, c) has an explicit open orbit. Every sequence of sink and source reflections keeps all three coordinates positive. At each such state, reversing an individual arrow creates a directed two-cycle. The trace around that cycle precludes a dense orbit, and hence precludes the reversal allowed by Kac. This proves that (a, ra, c) lies outside D. The example with $r = s = 2$ and $(a, ra, c) = (4, 8, 1)$ has scalar endomorphism ring. We give its matrices, an integer inverse for an 80×80 submatrix of the orbit differential, and the complete sparse matrix and verification code in Sections 18 and 19.

§ 11. The reduction operations and the main theorem

Let k be an algebraically closed field, of arbitrary characteristic. Let $Q = (Q_0, Q_1)$ be a finite quiver. Parallel arrows are separate elements of Q_1. For $d = (d_i)_{i \in Q_0} \in \mathbb{Z}_{\geq 0}^{Q_0}$, set

$$\mathrm{Rep}(Q, d) = \bigoplus_{b : u \to v \in Q_1} \mathrm{Hom}_k(k^{d_u}, k^{d_v}), \qquad G_d = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(k).$$

Write I_n for the $n \times n$ identity matrix. Here $\mathrm{GL}_0(k)$ is the trivial group. The action is $(g \cdot M)_b = g_v M_b g_u^{-1}$. We use the full representation space of Q, with no path relations. A pair (Q, d) is *prehomogeneous* if G_d has a dense orbit in $\mathrm{Rep}(Q, d)$. Such an orbit is open: an orbit is constructible, so a dense orbit contains a nonempty open subset, and its translates cover the orbit.

For a representation M, its endomorphism space $\mathrm{End}_Q(M)$ consists of the tuples $(T_i)_{i \in Q_0}$ with $T_i \in \mathrm{End}_k(k^{d_i})$ and $T_v M_b = M_b T_u$ for each arrow $b : u \to v$. These are linear equations in the entries of the T_i. The stabilizer of M is the open subset where every T_i is invertible. It contains the identity, so it is smooth and has the dimension of this linear space in every characteristic.

A vertex is a *sink* if no arrow starts there and a *source* if no arrow ends there. An isolated vertex is both. The two operations of [18, §3(d)] have the following form.

(a) Modified reflection. Choose a sink or source i. Reverse all arrows incident with i, leave the other coordinates unchanged, and replace d_i by

$$d_i' = \max\left\{0, \sum_{b \text{ incident with } i} d_{\text{other endpoint of } b} - d_i\right\}. \tag{77}$$

Each parallel arrow contributes one term to the sum.

(b) Admissible arrow reversal. Choose one arrow $b : u \to v$ and put

$$U = \mathrm{Rep}(Q \setminus \{b\}, d), \qquad W = \mathrm{Hom}_k(k^{d_u}, k^{d_v}).$$

Let H be the stabilizer in G_d of a general point of U. The arrow b may be reversed, with d unchanged, if

$$\max_{w \in W} \dim H \cdot w = \max_{\lambda \in W^*} \dim H \cdot \lambda. \tag{78}$$

The pairing $(X, Y) \mapsto \mathrm{tr}(YX)$ identifies the dual representation W^* with $\mathrm{Hom}_k(k^{d_v}, k^{d_u})$. Along the prehomogeneous states used below, U has an open orbit. Its point stabilizers are conjugate, which specifies H in (78).

Definition 11.1. The set $D(Q)$ consists of the dimension vectors that can be transformed to zero by a finite sequence of operations (a) and (b). The orientation is updated after every operation. The subset $D_1(Q)$ is defined by using only (a).

Kac's 1980 conjecture [18, §3(d)] asserts

$$(Q, d) \text{ is prehomogeneous} \quad \Longleftrightarrow \quad d \in D(Q).$$

For positive integers r, s, let $Q_{r,s}$ have three vertices, r arrows from 1 to 2, s arrows from 2 to 3, and no other arrows. Thus

$$Q_{r,s}: \qquad 1 \xrightarrow{r} 2 \xrightarrow{s} 3,$$

where each label is an arrow multiplicity.

Theorem 11.2. *Let $r, s \geq 2$, $c \geq 1$ and $a \geq rsc$ be integers, and set $d = (a, ra, c)$. Then $\mathrm{Rep}(Q_{r,s}, d)$ has an open G_d-orbit and $d \notin D(Q_{r,s})$.*

In particular, the quiver $1 \rightrightarrows 2 \rightrightarrows 3$ with dimension vector $(4, 8, 1)$ is a counterexample to the conjecture.

§ 12. Preservation of the dense-orbit property

Lemma 12.1. *The pairs before and after operation* (a) *are prehomogeneous simultaneously.*

Proof. An open orbit meets every nonempty invariant open subset of the irreducible representation space, and is then contained in that subset. We may therefore work on the full-rank loci below. Suppose first that i is a sink. Let $V = k^{d_i}$ and $E = \bigoplus_{b:j\to i} k^{d_j}$, and put $n = \dim V$, $m = \dim E$. Let U be the direct sum of the spaces for the other arrows, and let $G' = \prod_{j\neq i} \mathrm{GL}_{d_j}(k)$. The incident maps combine into one map $f : E \to V$, so the representation space is $U \times \mathrm{Hom}(E, V)$. A repeated vertex in E is acted on by the same element of its vertex group on each occurrence.

Assume $n \leq m$. The surjections $E \to V$ form a nonempty open set. The map

$$(u, f) \longmapsto (u, \ker f)$$

takes this set onto $U \times \mathrm{Gr}(m-n, E)$, where $\mathrm{Gr}(t, E)$ denotes the variety of t-dimensional subspaces of E. Its fibre over (u, L) consists of the isomorphisms $E/L \to V$, and is a single $\mathrm{GL}(V)$-orbit. The map is locally a product with $\mathrm{GL}(V)$: on a standard Grassmannian chart a fixed complementary subspace trivializes the quotient E/L.

After reflection, the dimension at i is $m - n$ and the incident maps combine into a map $h : k^{m-n} \to E$. The injections form a nonempty open set. The map $(u, h) \mapsto (u, \mathrm{im}\, h)$ has the same target $U \times \mathrm{Gr}(m-n, E)$ and is locally a product with $\mathrm{GL}_{m-n}(k)$. Both maps are G'-equivariant. In each case, the inverse image of a G'-orbit in the target is a single orbit of the full vertex group. Since the maps are surjective and open, that inverse image is open exactly when the base orbit is open. Thus the two representation spaces have open orbits simultaneously. This also covers $n = m$ and $n = 0$.

If $n > m$, use the open set of injections $E \to V$. The group $\mathrm{GL}(V)$ acts transitively on these injections. For a fixed injection f and any $g' \in G'$, write g'_E for the induced automorphism of E. The automorphism $f g'_E f^{-1}$ of $f(E)$ extends, using a complementary subspace, to an automorphism of V. Hence the inverse image of each G'-orbit in U is one $G' \times \mathrm{GL}(V)$-orbit in the injection locus. Projection to U is open, so an open orbit exists there exactly when one exists in U. Formula (77) gives dimension zero at i; the reflected representation space is U, with group G'.

For a source, dualizing every vertex space and every arrow map interchanges sources and sinks. Transpose inverse on each vertex group makes this an equivariant isomorphism of the corresponding orbit problems. The sink argument therefore proves the source case. □

Lemma 12.2. *A legal operation* (b) *applied to a prehomogeneous pair produces a prehomogeneous pair.*

Proof. Use U, W as in Section 11, and write $G = G_d$. The projection of a dense G-orbit in $U \oplus W$ is a dense G-orbit in U, hence is open. Choose a point u on this orbit and put $H = G_u$. For every $w \in W$, the stabilizer of (u, w) in G is $H_w = \{h \in H : h \cdot w = w\}$. The dimension formula for orbits gives

$$\begin{aligned} \dim G \cdot (u, w) &= \dim G - \dim H_w \\ &= (\dim G - \dim H) + (\dim H - \dim H_w) \\ &= \dim U + \dim H \cdot w. \end{aligned}$$

The dense orbit in $U \oplus W$ therefore gives an H-orbit of dimension $\dim W$. Equality (78) gives an H-orbit of dimension $\dim W^* = \dim W$ in W^*. Applying the same formula to $U \oplus W^*$ gives a G-orbit whose dimension equals that of the whole space. Its closure is the whole irreducible affine space, so the orbit is open. □

§ 13. The open orbit

Write a representation of $Q_{r,s}$ of dimension (a, ra, c) as

$$A_i : k^a \longrightarrow k^{ra} \quad (1 \le i \le r), \qquad B_j : k^{ra} \longrightarrow k^c \quad (1 \le j \le s).$$

Set

$$A = [A_1 \ \cdots \ A_r], \qquad C = \begin{bmatrix} B_1A_1 \\ \vdots \\ B_1A_r \\ B_2A_1 \\ \vdots \\ B_sA_r \end{bmatrix}. \tag{79}$$

Thus A is an $ra \times ra$ matrix and C is an $rsc \times a$ matrix; the blocks of C are ordered first by j and then by i.

Proposition 13.1. *If $a \ge rsc$, the conditions*

$$\det A \neq 0, \qquad \operatorname{rank} C = rsc \tag{80}$$

define a nonempty open $G_{(a,ra,c)}$-orbit.

Proof. Choose the A_i as the standard inclusions of the r consecutive a-dimensional blocks, so that $A = I_{ra}$. Choose $C = [I_{rsc}\ 0]$. Its consecutive $c \times a$ blocks uniquely specify the blocks of the matrices B_j, giving (80). Thus this polynomial open condition is nonempty.

Under $(g_1, g_2, g_3) \in G_{(a,ra,c)}$ the matrices transform as

$$A \longmapsto g_2 A \operatorname{diag}(g_1^{-1}, \dots, g_1^{-1}), \qquad C \longmapsto \operatorname{diag}(g_3, \dots, g_3) C g_1^{-1},$$

with r copies in the first diagonal matrix and rs in the second. Hence the conditions are invariant. For any representation satisfying them, the choice $g_1 = I_a$, $g_2 = A^{-1}$ and $g_3 = I_c$ normalizes A to I_{ra}.

The transformations preserving this normalization are exactly

$$(g, \operatorname{diag}(g, \dots, g), h), \qquad g \in \mathrm{GL}_a(k), \quad h \in \mathrm{GL}_c(k).$$

They act on C by $C \mapsto \operatorname{diag}(h, \dots, h) C g^{-1}$. The subgroup with $h = I_c$ is transitive on the surjections $k^a \to k^{rsc}$. Indeed, for two surjections C, C', choose sections σ, σ' of them and bases of their kernels. Define g by mapping a chosen basis of $\ker C$ to the chosen basis of $\ker C'$ and by $g(\sigma(v)) = \sigma'(v)$ for every $v \in k^{rsc}$. Then $C'g = C$, so $C' = Cg^{-1}$. Thus every point of (80) belongs to the same orbit. □

The endomorphism dimension of this orbit can also be read from the normal form. Put $p = rsc$ and $t = a - p$, and let M be the representation with $A = I_{ra}$ and $C = [I_p\ 0]$. An endomorphism is a triple (P, R, H) of matrices of sizes $a \times a$, $ra \times ra$ and $c \times c$, respectively, satisfying $R = \operatorname{diag}(P, \dots, P)$ and

$$CP = \operatorname{diag}(H, \dots, H) C.$$

The first p rows of P must therefore be $[\operatorname{diag}(H, \dots, H)\ 0]$, while its last t rows are arbitrary. It follows that

$$\dim \operatorname{End}_{Q_{r,s}}(M) = c^2 + ta = a^2 + c^2 - rsac. \tag{81}$$

The stabilizer has the same dimension. Subtracting (81) from $\dim G_{(a,ra,c)} = a^2 + r^2a^2 + c^2$ gives $r^2a^2 + rsac = \dim \operatorname{Rep}(Q_{r,s}, (a, ra, c))$.

§ 14. All admissible reflection sequences

Lemma 14.1. *Let $r, s \geq 2$. Start with the orientation $Q_{r,s}$ and positive coordinates x, y, z satisfying $y > \max\{x, z\}$. Every finite sequence of operations* (a) *keeps all three coordinates strictly positive.*

Proof. First consider the ordinary linear reflections

$$E_1(x, y, z) = (ry - x, y, z),$$
$$E_3(x, y, z) = (x, y, sy - z),$$
$$F(x, y, z) = (x, rx + sz - y, z).$$

They satisfy

$$E_1^2 = E_3^2 = F^2 = 1, \qquad E_1E_3 = E_3E_1. \tag{82}$$

These identities concern linear maps on $\mathbb{R}^3$.

Record each bundle direction by a bit: 1 means left to right, and 0 means right to left. Initially the bits are $(1, 1)$. The endpoints are always sources or sinks. Reflection at endpoint 1 toggles the first bit, and reflection at endpoint 3 toggles the second. The middle vertex is a source or sink exactly when the bits differ; its reflection toggles both bits, which remain different.

Consider a word whose reflections are admissible for these orientations. Using (82), reduce each consecutive block of endpoint reflections to one element of

$$K = \{1, E_1, E_3, E_1E_3\}.$$

If an identity endpoint block lies between two F's, cancel the two F's and combine the neighbouring endpoint blocks. Repeating this procedure terminates, because each cancellation reduces the number of F's by two. Endpoint commutations preserve admissibility since both endpoints are always available. Each cancellation restores the same orientation, so the suffix remains admissible. The final linear map is unchanged by all these steps.

If the reduced word contains an F, the block before its first F must toggle exactly one of the two initially equal bits. It is therefore E_1 or E_3. Between two consecutive remaining F's, the nonidentity endpoint block must preserve the condition that the bits differ. It is therefore E_1E_3. The terminal endpoint block is arbitrary. Consequently every reduced word with an F has the following form, written in chronological order:

$$E_i,\ F,\ \underbrace{(E_1E_3, F), \ldots, (E_1E_3, F)}_{n \text{ blocks}},\ E, \qquad i \in \{1, 3\},\ n \geq 0,\ E \in K. \tag{83}$$

A word with no F reduces to one element of K.

Suppose $y > \max\{x, z\} > 0$. Put $X = ry - x$ and $Z = sy - z$. Since $r, s \geq 2$, we have $X > y > 0$ and $Z > y > 0$. Thus every endpoint block in K gives positive coordinates, including both intermediate steps of E_1E_3.

For the first pair E_1, F in (83), the new middle coordinate is $Y = rX + sz - y$, and

$$Y - X = (r - 1)X + sz - y > 0, \qquad Y - z = rX + (s - 1)z - y > 0.$$

For the first pair E_3, F, it is $Y = rx + sZ - y$, and

$$Y - x = (r - 1)x + sZ - y > 0, \qquad Y - Z = rx + (s - 1)Z - y > 0.$$

In either case the new middle coordinate strictly exceeds both new endpoints. For each later block E_1E_3, F, the new endpoints are $X, Z > y$ and the new middle coordinate is $Y = rX + sZ - y$. Then

$$Y - X = (r - 1)X + sZ - y > 0, \qquad Y - Z = rX + (s - 1)Z - y > 0.$$

Induction through (83), followed by its last endpoint block, proves that the final value of every orientation-admissible ordinary word is positive.

Every prefix of an orientation-admissible word is itself admissible. Its ordinary linear value is therefore positive. Now suppose that a sequence of the modified reflections (77) first produces a zero coordinate. Before that step it agrees with the ordinary linear sequence. At that step the ordinary new coordinate would be nonpositive. This contradicts positivity for the corresponding prefix. Thus all modified reflections agree with the ordinary ones along the sequence, and all its coordinates stay positive. □

§ 15. Excluding arrow reversals

Lemma 15.1. *If a quiver contains opposite arrows between two vertices of positive dimensions, its full representation space has no dense base-change orbit.*

Proof. Write the two maps as $X : k^u \to k^v$ and $Y : k^v \to k^u$, where $u, v > 0$. A change of bases conjugates YX, so $\operatorname{tr}(YX)$ is an invariant polynomial. Choose first basis vectors e_1, f_1 at these vertices. Let $X(e_1) = f_1$, $Y(f_1) = te_1$, and let both maps vanish on the other basis vectors. Then $\operatorname{tr}(YX) = t$, so the invariant is nonconstant over k. An invariant polynomial constant on a dense orbit is constant on the whole space, a contradiction. □

Proof of Theorem 11.2. Proposition 13.1 gives the open orbit. The inequalities $a \geq rsc$, $r, s \geq 2$ and $c \geq 1$ imply $ra > \max\{a, c\} > 0$. Lemma 14.1 therefore applies to every sequence consisting of operations (a).

Suppose that a finite reduction from $(Q_{r,s}, (a, ra, c))$ to zero contains an operation (b), and consider its first occurrence. All preceding operations are of type (a). At the resulting state, all vertex dimensions are positive by Lemma 14.1, and there is an open orbit by Lemma 12.1. Each parallel bundle is still uniformly oriented, because every previous reflection reversed all incident arrows.

Reversing an individual arrow leaves an arrow in the opposite direction in the same bundle, since $r, s \geq 2$. Lemma 15.1 shows that the new state has no dense orbit. Lemma 12.2 shows that a legal operation (b) would give a dense orbit. This contradiction excludes the first operation (b).

The reduction would consequently consist entirely of operations (a). By Lemma 14.1, every coordinate would remain positive, so the reduction could not reach zero. Thus $d \notin D(Q_{r,s})$. □

§ 16. Ringel's earlier example

For comparison with the change from D_1 to D in [18, §3(d)], consider $1 \rightrightarrows 2 \to 3$ with dimension vector $(3, 6, 1)$. Denote the arrows by $A_1, A_2 : k^3 \to k^6$ and $B : k^6 \to k$. After deleting B, the locus where $[A_1\ A_2]$ is invertible is one open orbit. Normalize A_1, A_2 to the two standard inclusions. Its stabilizer is

$$H = \{(g, \operatorname{diag}(g, g), \lambda) : g \in \mathrm{GL}_3(k),\ \lambda \in k^*\}.$$

On the arrow space $\operatorname{Hom}(k^6, k)$, write $B = (b_1, b_2)$ as two rows of length three. The condition that these rows are independent defines an open orbit under H, because the right action of $\mathrm{GL}_3(k)$ is transitive on matrices of row rank two. On the dual arrow space $\operatorname{Hom}(k, k^6)$, a map is a pair of columns of length three. The independent pairs form an open orbit under the left action of $\mathrm{GL}_3(k)$. Both maximal orbit dimensions are six. Thus the single arrow $2 \to 3$ can be reversed by (b).

After that reversal, the following table gives a complete reduction. An orientation entry displays the directions of the double link and the single link, respectively.

Operation just performed	Orientation	Dimension vector
(b), at $2 \to 3$	$1 \rightrightarrows 2 \leftarrow 3$	$(3, 6, 1)$
(a), at 2	$1 \leftleftarrows 2 \to 3$	$(3, 1, 1)$
(a), at 1	$1 \rightrightarrows 2 \to 3$	$(0, 1, 1)$
(a), at 3	$1 \rightrightarrows 2 \leftarrow 3$	$(0, 1, 0)$
(a), at 2	$1 \leftleftarrows 2 \to 3$	$(0, 0, 0)$

The four changed coordinates are $2 \cdot 3 + 1 - 6 = 1$, $\max\{0, 2 \cdot 1 - 3\} = 0$, $1 - 1 = 0$, and $\max\{0, 2 \cdot 0 + 0 - 1\} = 0$. This verifies $(3, 6, 1) \in D$ directly.

§ 17. An integral certificate for $(4, 8, 1)$

For $r = s = 2$, take the four integer matrices

$$A_1 = \begin{bmatrix} I_4 \\ 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 \\ I_4 \end{bmatrix}, \qquad \begin{aligned} B_1 &= (1, 0, 0, 0, 0, 1, 0, 0), \\ B_2 &= (0, 0, 1, 0, 0, 0, 0, 1). \end{aligned} \tag{84}$$

The rows $B_1A_1, B_1A_2, B_2A_1, B_2A_2$ are, in that order, the four standard coordinate rows. An endomorphism (P, R, t) satisfies $RA_i = A_iP$, whence $R = \operatorname{diag}(P, P)$. The equalities $tB_j = B_jR$ then give, successively, the four rows of $P = tI_4$. Hence the endomorphism ring is k and the stabilizer is the common scalar group k^*.

The orbit dimension is therefore

$$(4^2 + 8^2 + 1) - 1 = 80 = 2 \cdot 4 \cdot 8 + 2 \cdot 8 = \dim \operatorname{Rep}(Q_{2,2}, (4, 8, 1)).$$

The orbit differential is the integral linear map

$$\begin{aligned} \partial &: \operatorname{Mat}_{4\times4}(\mathbb{Z}) \oplus \operatorname{Mat}_{8\times8}(\mathbb{Z}) \oplus \mathbb{Z} \longrightarrow \operatorname{Mat}_{8\times4}(\mathbb{Z})^2 \oplus \operatorname{Mat}_{1\times8}(\mathbb{Z})^2, \\ &(P, R, t) \longmapsto (RA_1 - A_1P, RA_2 - A_2P, tB_1 - B_1R, tB_2 - B_2R). \end{aligned} \tag{85}$$

Here $\operatorname{Mat}_{m\times n}(\mathbb{Z})$ denotes integer matrices of that size. Use row-major order within each matrix, the domain order (P, R, t), and the four displayed output blocks. The resulting matrix J has size 80×81. Let J_0 be its first eighty columns, corresponding to the restriction $t = 0$.

Proposition 17.1. *The map with matrix J_0 is an isomorphism of free abelian groups. In the specified coordinate order,* $\det J_0 = 1$.

Proof. We first give its inverse with integer coefficients. For arbitrary output data $U_1, U_2 \in \operatorname{Mat}_{8\times4}(\mathbb{Z})$ and $v_1, v_2 \in \operatorname{Mat}_{1\times8}(\mathbb{Z})$, put

$$U = [U_1\ U_2], \qquad w_j = -v_j - B_jU \quad (j = 1, 2).$$

Write $w_{j,\nu}$ for the νth coordinate of w_j and define

$$P = \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{1,4} \\ w_{1,5} & w_{1,6} & w_{1,7} & w_{1,8} \\ w_{2,1} & w_{2,2} & w_{2,3} & w_{2,4} \\ w_{2,5} & w_{2,6} & w_{2,7} & w_{2,8} \end{bmatrix}, \qquad R = U + \operatorname{diag}(P, P). \tag{86}$$

The first two blocks of (85) are U_1, U_2. Moreover $B_1 \operatorname{diag}(P, P) = w_1$ and $B_2 \operatorname{diag}(P, P) = w_2$, by the positions of the nonzero entries of B_1, B_2. Thus $-B_jR = -B_jU - w_j = v_j$. This proves surjectivity over $\mathbb{Z}$. The same equations force (86), so the inverse is unique. Consequently $\det J_0 = \pm 1$.

For the sign, change the domain coordinates from (P, R) to (P, Z), where $Z = R - \operatorname{diag}(P, P)$. This change has determinant one. The first sixty-four output coordinates are the entries of the first four columns of Z followed by those of its last four columns, each in row-major order. Reordering these into the row-major order of Z has $16\binom{8}{2} = 448$ inversions, hence positive sign. Add B_jZ to each last output row. These row operations

have determinant one and leave the last sixteen coordinates equal to the negative entries of P, in row-major order. Finally moving the sixteen P coordinates past the sixty-four Z coordinates has sign $(-1)^{16\cdot 64} = 1$, and negating sixteen coordinates has sign $(-1)^{16} = 1$. Thus the total determinant is 1. $\square$

Reduction of the inverse (86) modulo any prime shows directly that the differential has rank eighty in every characteristic. The vector with blocks $(I_4, I_8, 1)$ spans its one-dimensional kernel.

§ 18. The complete matrix certificate

The column numbers of J are

$$P_{ij} : 4(i-1)+j \quad (1 \le i,j \le 4), \qquad R_{ij} : 16+8(i-1)+j \quad (1 \le i,j \le 8), \qquad t : 81.$$

The rows for the first arrow are $4(i-1)+j$, and those for the second are $32+4(i-1)+j$, with $1 \le i \le 8$, $1 \le j \le 4$. Rows $64+j$ and $72+j$, for $1 \le j \le 8$, correspond to the third and fourth arrows. In the table below, $+c$ means that column c has entry 1, and $-c$ means that it has entry -1. All unlisted entries are zero. The table gives all eighty rows and all 132 nonzero entries of J.

Row	Signed columns	Row	Signed columns
1	−1, +17	41	+37
2	−2, +18	42	+38
3	−3, +19	43	+39
4	−4, +20	44	+40
5	−5, +25	45	+45
6	−6, +26	46	+46
7	−7, +27	47	+47
8	−8, +28	48	+48
9	−9, +33	49	−1, +53
10	−10, +34	50	−2, +54
11	−11, +35	51	−3, +55
12	−12, +36	52	−4, +56
13	−13, +41	53	−5, +61
14	−14, +42	54	−6, +62
15	−15, +43	55	−7, +63
16	−16, +44	56	−8, +64
17	+49	57	−9, +69
18	+50	58	−10, +70
19	+51	59	−11, +71
20	+52	60	−12, +72
21	+57	61	−13, +77
22	+58	62	−14, +78
23	+59	63	−15, +79
24	+60	64	−16, +80
25	+65	65	−17, −57, +81
26	+66	66	−18, −58
27	+67	67	−19, −59
28	+68	68	−20, −60
29	+73	69	−21, −61
30	+74	70	−22, −62, +81
31	+75	71	−23, −63
32	+76	72	−24, −64
33	+21	73	−33, −73
34	+22	74	−34, −74
35	+23	75	−35, −75, +81
36	+24	76	−36, −76
37	+29	77	−37, −77
38	+30	78	−38, −78
39	+31	79	−39, −79
40	+32	80	−40, −80, +81

The integral inverse. Let $S = J_0^{-1}$. Its rows are indexed by the entries of (P, R); its columns are indexed by the entries of (U_1, U_2, v_1, v_2), in the row-major orders of (85). The following table gives all eighty rows and all 176 nonzero entries, using the same signed-column convention. Every unlisted entry is zero.

Row	Signed columns	Row	Signed columns
1	$-1,\ -21,\ -65$	41	$+13,\ -41,\ -61,\ -77$
2	$-2,\ -22,\ -66$	42	$+14,\ -42,\ -62,\ -78$
3	$-3,\ -23,\ -67$	43	$+15,\ -43,\ -63,\ -79$
4	$-4,\ -24,\ -68$	44	$+16,\ -44,\ -64,\ -80$
5	$-33,\ -53,\ -69$	45	$+45$
6	$-34,\ -54,\ -70$	46	$+46$
7	$-35,\ -55,\ -71$	47	$+47$
8	$-36,\ -56,\ -72$	48	$+48$
9	$-9,\ -29,\ -73$	49	$+17$
10	$-10,\ -30,\ -74$	50	$+18$
11	$-11,\ -31,\ -75$	51	$+19$
12	$-12,\ -32,\ -76$	52	$+20$
13	$-41,\ -61,\ -77$	53	$-1,\ -21,\ +49,\ -65$
14	$-42,\ -62,\ -78$	54	$-2,\ -22,\ +50,\ -66$
15	$-43,\ -63,\ -79$	55	$-3,\ -23,\ +51,\ -67$
16	$-44,\ -64,\ -80$	56	$-4,\ -24,\ +52,\ -68$
17	$-21,\ -65$	57	$+21$
18	$-22,\ -66$	58	$+22$
19	$-23,\ -67$	59	$+23$
20	$-24,\ -68$	60	$+24$
21	$+33$	61	$-33,\ -69$
22	$+34$	62	$-34,\ -70$
23	$+35$	63	$-35,\ -71$
24	$+36$	64	$-36,\ -72$
25	$+5,\ -33,\ -53,\ -69$	65	$+25$
26	$+6,\ -34,\ -54,\ -70$	66	$+26$
27	$+7,\ -35,\ -55,\ -71$	67	$+27$
28	$+8,\ -36,\ -56,\ -72$	68	$+28$
29	$+37$	69	$-9,\ -29,\ +57,\ -73$
30	$+38$	70	$-10,\ -30,\ +58,\ -74$
31	$+39$	71	$-11,\ -31,\ +59,\ -75$
32	$+40$	72	$-12,\ -32,\ +60,\ -76$
33	$-29,\ -73$	73	$+29$
34	$-30,\ -74$	74	$+30$
35	$-31,\ -75$	75	$+31$
36	$-32,\ -76$	76	$+32$
37	$+41$	77	$-41,\ -77$
38	$+42$	78	$-42,\ -78$
39	$+43$	79	$-43,\ -79$
40	$+44$	80	$-44,\ -80$

The elimination record. For step $\ell = 1, \ldots, 79$ of the fraction-free elimination in Section 19, let ρ_ℓ be the position of the chosen pivot row in the current matrix before the row swap, π_ℓ the pivot after that swap, and ϵ_ℓ the cumulative row-swap sign through step ℓ. The divisor is $\delta_1 = 1$ and $\delta_\ell = \pi_{\ell-1}$ for $\ell \geq 2$. The complete pivot record is

ℓ	ρ_ℓ	π_ℓ	δ_ℓ	ϵ_ℓ	ℓ	ρ_ℓ	π_ℓ	δ_ℓ	ϵ_ℓ
1	1	-1	1	1	41	61	-1	1	-1
2	2	1	-1	1	42	62	1	-1	1
3	3	-1	1	1	43	63	-1	1	-1
4	4	1	-1	1	44	64	1	-1	1
5	5	-1	1	1	45	45	1	1	1
6	6	1	-1	1	46	46	1	1	1
7	7	-1	1	1	47	47	1	1	1
8	8	1	-1	1	48	48	1	1	1
9	9	-1	1	1	49	49	1	1	1
10	10	1	-1	1	50	50	1	1	1
11	11	-1	1	1	51	51	1	1	1
12	12	1	-1	1	52	52	1	1	1
13	13	-1	1	1	53	65	-1	1	-1
14	14	1	-1	1	54	66	1	-1	1
15	15	-1	1	1	55	67	-1	1	-1
16	16	1	-1	1	56	68	1	-1	1
17	49	-1	1	-1	57	57	1	1	1
18	50	1	-1	1	58	58	1	1	1
19	51	-1	1	-1	59	59	1	1	1
20	52	1	-1	1	60	60	1	1	1
21	33	1	1	-1	61	69	-1	1	-1
22	34	1	1	1	62	70	1	-1	1
23	35	1	1	-1	63	71	-1	1	-1
24	36	1	1	1	64	72	1	-1	1
25	53	-1	1	-1	65	65	1	1	1
26	54	1	-1	1	66	66	1	1	1
27	55	-1	1	-1	67	67	1	1	1
28	56	1	-1	1	68	68	1	1	1
29	37	1	1	-1	69	73	-1	1	-1
30	38	1	1	1	70	74	1	-1	1
31	39	1	1	-1	71	75	-1	1	-1
32	40	1	1	1	72	76	1	-1	1
33	57	-1	1	-1	73	73	1	1	1
34	58	1	-1	1	74	74	1	1	1
35	59	-1	1	-1	75	75	1	1	1
36	60	1	-1	1	76	76	1	1	1
37	41	1	1	-1	77	77	-1	1	1
38	42	1	1	1	78	78	1	-1	1
39	43	1	1	-1	79	79	-1	1	1
40	44	1	1	1					

The final diagonal entry and cumulative row-swap sign are both 1, so the determinant is 1. All 167480 divisions have zero remainder.

§ 19. Exact verification code

The following Python 3 program uses integer arithmetic and the standard library. It constructs all columns of (85), checks the scalar kernel, computes $\det J_0$ by fraction-free elimination, and verifies both products with the integer inverse (86). Each division in elimination is checked for zero remainder. A successful run prints the matrix size, the number of nonzero entries, the determinant and the two inverse identities. Section 18 prints the full matrix, its inverse and the elimination record.

```
def flatten(matrix):
    return [value for row in matrix for value in row]

def reshape(values, rows, cols):
    assert len(values) == rows * cols
    return [list(values[i * cols:(i + 1) * cols]) for i in range(rows)]

def multiply(left, right):
    assert len(left[0]) == len(right)
    columns = list(zip(*right))
    return [[sum(x * y for x, y in zip(row, col)) for col in columns]
            for row in left]

def difference(left, right):
    return [[x - y for x, y in zip(a, b)] for a, b in zip(left, right)]

A1 = [[int(i == j) for j in range(4)] for i in range(8)]

A2 = [[int(i == j + 4) for j in range(4)] for i in range(8)]

B1 = [[1, 0, 0, 0, 0, 1, 0, 0]]

B2 = [[0, 0, 1, 0, 0, 0, 0, 1]]

def differential(vector):
    assert len(vector) == 81
    p = reshape(vector[:16], 4, 4)
    r = reshape(vector[16:80], 8, 8)
    t = vector[80]
    parts = [difference(multiply(r, a), multiply(a, p)) for a in (A1, A2)]
    parts += [difference([[t * x for x in b[0]]], multiply(b, r))
              for b in (B1, B2)]
    return [value for part in parts for value in flatten(part)]

def inverse_at_t_zero(vector):
    """Recover (P,R) integrally from (U1,U2,v1,v2) when t=0."""
    assert len(vector) == 80
    u1 = reshape(vector[:32], 8, 4)
    u2 = reshape(vector[32:64], 8, 4)
    v1, v2 = vector[64:72], vector[72:80]
    u = [left + right for left, right in zip(u1, u2)]
    w1 = [-v - z for v, z in zip(v1, multiply(B1, u)[0])]
    w2 = [-v - z for v, z in zip(v2, multiply(B2, u)[0])]
    p = [w1[:4], w1[4:], w2[:4], w2[4:]]
    r = [row[:] for row in u]
    for block in (0, 4):
        for i in range(4):
            for j in range(4):
                r[block + i][block + j] += p[i][j]
    return flatten(p) + flatten(r)
```

```
def matrix_of_linear_map(function, domain_dimension):
    columns = []
    for j in range(domain_dimension):
        vector = [0] * domain_dimension
        vector[j] = 1
        columns.append(function(vector))
    return [list(row) for row in zip(*columns)]

def bareiss(matrix):
    """Fraction-free elimination; every exact division is asserted."""
    a = [row[:] for row in matrix]
    n = len(a)
    assert all(len(row) == n for row in a)
    previous = sign = 1
    divisions = 0
    ledger = []
    for k in range(n - 1):
        row = next((i for i in range(k, n) if a[i][k] != 0), None)
        if row is None:
            return 0, ledger, divisions
        if row != k:
            a[k], a[row] = a[row], a[k]
            sign = -sign
        pivot = a[k][k]
        ledger.append({"step": k + 1, "pivot_row_before_swap": row + 1,
                       "pivot": pivot, "divisor": previous,
                       "cumulative_row_swap_sign": sign})
        for i in range(k + 1, n):
            coefficient = a[i][k]
            for j in range(k + 1, n):
                numerator = pivot * a[i][j] - coefficient * a[k][j]
                quotient, remainder = divmod(numerator, previous)
                assert remainder == 0
                a[i][j] = quotient
                divisions += 1
            a[i][k] = 0
        previous = pivot
    determinant = sign * a[-1][-1]
    ledger.append({"final_diagonal": a[-1][-1], "row_swap_sign": sign,
                   "determinant": determinant})
    return determinant, ledger, divisions

J = matrix_of_linear_map(differential, 81)
J0 = [row[:80] for row in J]
S = matrix_of_linear_map(inverse_at_t_zero, 80)
I = [[int(i == j) for j in range(80)] for i in range(80)]
assert multiply(J0, S) == I
assert multiply(S, J0) == I
det, ledger, divisions = bareiss(J0)
assert det == 1
scalar = [int(i == j) for size in (4, 8, 1)
          for i in range(size) for j in range(size)]
assert differential(scalar) == [0] * 80
assert len(J) == 80 and all(len(row) == 81 for row in J)
assert sum(x != 0 for row in J for x in row) == 132
print({"shape": (80, 81), "nonzero_entries": 132,
       "determinant": det, "left_inverse": True,
       "right_inverse": True, "exact_divisions": divisions})
```

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