COUNTEREXAMPLES TO KAC'S CONJECTURE ON DENSE SIMPLE SUBALGEBRAS

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Abstract. We give counterexamples in the four polynomial Cartan series to Kac's 1974 conjecture on dense simple subalgebras.

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§ 1. Introduction

The four polynomial Cartan series come from the classical geometry of infinite transformation groups. Cartan's 1909 paper [1] identifies four simple transitive types: arbitrary transformations, transformations preserving volume, transformations preserving a nondegenerate alternating two-form, and contact transformations. Their infinitesimal versions give the series W, S, H, and K. Guillemin and Sternberg [2] developed an algebraic formulation using Taylor expansions of vector fields at a point. Formal vector fields carry a filtration by order of vanishing, and closed transitive subalgebras describe the corresponding formal local geometry.

Kac studied filtered Lie algebras with associated graded algebras of Cartan type in [3]. Let a subring of a formal power-series ring contain the coordinates and be stable under their partial derivatives. Over an algebraically closed field of characteristic zero, his Theorem 3 realizes every infinite-dimensional primitive transitive subalgebra of its vector-field algebra as a dense subalgebra of a Cartan algebra. For polynomial coefficients, density means that every vector field can be matched to any finite Taylor order at the distinguished origin. At the end of §9, Kac proposed the following conjecture [3, p. 834].

Conjecture (Kac, 1974). *A simple subalgebra dense in a simple Cartan algebra over a polynomial ring coincides with the ambient algebra.*

For W_n, Kac also gives the dense subalgebra of fields vanishing at a point $\xi \neq 0$. Choose a linear function ℓ with $\ell(0) = 0$ and $\ell(\xi) = 1$. For every $N \geq 1$ and $D \in W_n$, the field $(1 - \ell^N)D$ vanishes at ξ and agrees with D through Taylor degree $N - 1$ at the origin. The fields vanishing at ξ to order at least two form a proper nonzero ideal: bracketing with a field vanishing to first order preserves this condition. Simplicity is therefore the additional restriction in the conjecture.

Theorem 1.1. *Over $\mathbb{C}$, each of the Lie algebras*

$$W_n \quad (n \geq 2), \qquad S_n \quad (n \geq 2), \qquad H_{2m} \quad (m \geq 1), \qquad K_{2m+1} \quad (m \geq 1)$$

contains an explicitly defined proper, infinite-dimensional, simple Lie subalgebra dense in its natural filtration. Consequently, Kac's conjecture is false in all four standard simple Cartan series in these dimensions.

We use the convention $S_2 = H_2$. Kac's special series begins in dimension three because the two-dimensional case coincides with the Hamiltonian series. In dimension one, every

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infinite-dimensional simple subalgebra of the polynomial Witt algebra W_1 equals W_1, by Bell and Buzaglo [4, Proposition 4.3]. Thus dimension two is the first possible Witt case.

Kac also related the Witt problem to polynomial maps with constant nonzero Jacobian, now called Keller maps. The three-dimensional map recorded in [5, Eq. (2)] has determinant -2 and identifies distinct points. Its pullback on vector fields, its volume-preserving restriction, and its cotangent and contact lifts give proper dense simple subalgebras of W_n, S_n for $n \geq 3$ and of H_{2m}, K_{2m+1} for $m \geq 3$. These constructions are detailed in Appendix B. The constructions below use a single affine quadric throughout the ranges of Theorem 1.1, including W_2, H_2, H_4, K_3, and K_5.

The common coefficient algebra is

$$B = \mathbb{C}[a,b,c]/(ab - c(c-1)) \hookrightarrow \mathbb{C}[t,y], \qquad (a,b,c) \longmapsto (t + t^2 y, y, 1 + ty). \tag{1}$$

The quadric $\operatorname{Spec} B$ is smooth, and the displayed map identifies the affine plane with an open subset of it. The element t does not belong to B, although elements of B represent every finite Taylor expansion at the plane's origin. Restriction of global vector fields from the quadric consequently combines a global extension condition with arbitrary local jets. Polynomial products give the Witt examples in higher dimension. Hamiltonian fields, exact volume-preserving fields, and a contact form on the corresponding products give the other three series.

The simplicity of vector fields on smooth affine varieties has a separate history. Jordan [6, 7] studied ideals in derivation Lie algebras, and Siebert [8] related their ideal structure to affine geometry. Billig and Futorny [9, Theorem 1.1] give a direct proof of the resulting criterion: over an algebraically closed field of characteristic zero, the vector-field Lie algebra of a positive-dimensional irreducible affine variety is simple exactly when the variety is smooth. The Witt construction uses this setting on the quadric and its products.

The Hamiltonian algebra of the quadratic surface is also an established simple Lie algebra. Gotay, Grabowski, and Kaneshige [10] study polynomial Poisson algebras on coadjoint orbits of real semisimple Lie groups. They obtain the decomposition into constants and the derived algebra and prove that, for semisimple orbits, every Lie ideal is central or contains the derived algebra. For smooth Danielewski surfaces, Leuenberger and Regeta [11] prove simplicity of the Lie algebra generated by locally nilpotent vector fields. Dubouloz, Kunyavskii, and Regeta [12, Proposition 7 and Corollary 4] identify this algebra with the derived Hamiltonian algebra and compute its codimension. Their result gives simplicity of the full Hamiltonian algebra when the defining polynomial has degree two, as in (1).

For the special series, contraction with the volume form identifies volume-preserving fields with closed forms and exact fields with exact forms. Donzelli [13] studies the resulting quotient through algebraic de Rham cohomology. On the product of the quadric with an affine line, this distinction determines the simple subalgebra used in dimension three. In higher dimensions, every global volume-preserving field on the products considered here is exact. Sections 2 and 3 give the polynomial and geometric preliminaries; Sections 4–7 prove the four constructions.

§ 2. Polynomial Cartan algebras and their topology

All algebras and derivations are over $\mathbb{C}$. For a commutative algebra R, write $\operatorname{Der}(R) = \operatorname{Der}_{\mathbb{C}}(R)$, with bracket $[D,Z] = D \circ Z - Z \circ D$. In any Lie algebra, $\operatorname{ad}_u(v) = [u,v]$. A Lie algebra is simple if it is nonabelian and its only Lie ideals are zero and the whole algebra.

Let Ω^1_R be the module generated by the symbols df, $f \in R$, subject to linearity, $d\lambda = 0$ for $\lambda \in \mathbb{C}$, and $d(fg) = f\,dg + g\,df$. Put $\Omega^0_R = R$ and $\Omega^q_R = \bigwedge^q_R \Omega^1_R$. The exterior derivative d satisfies $d^2 = 0$ and $d(\beta \wedge \gamma) = d\beta \wedge \gamma + (-1)^q \beta \wedge d\gamma$ for $\beta \in \Omega^q_R$. Contraction ι_D is the degree -1 antiderivation determined by $\iota_D(f) = 0$ and $\iota_D(df) = D(f)$. The Lie derivative is $\mathcal{L}_D = d\iota_D + \iota_D d$. For a generator Ω of a free module of top-degree forms, define $\operatorname{div}_\Omega D$

by $\mathcal{L}_D\Omega = (\operatorname{div}_\Omega D)\Omega$. Since $df \wedge \iota_D\Omega = D(f)\Omega$,

$$\operatorname{div}_\Omega(fD) = D(f) + f \operatorname{div}_\Omega D. \tag{2}$$

Write $A_n = \mathbb{C}[x_1, \dots, x_n]$ and $W_n = \operatorname{Der}(A_n) = \bigoplus_{i=1}^n A_n\partial_{x_i}$. Subscripts on functions denote partial derivatives. In each ambient polynomial ring, $\mathfrak{m}$ denotes the ideal generated by its displayed coordinate variables, so it is the ideal of the origin. For a multi-index $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n_{\geq 0}$, write $|\nu| = \sum_i \nu_i$, $\nu! = \prod_i \nu_i!$, $x^\nu = \prod_i x_i^{\nu_i}$, and $\partial^\nu = \prod_i \partial_{x_i}^{\nu_i}$.

For the standard volume form $\Omega_0 = dx_1 \wedge \cdots \wedge dx_n$, put

$$S_n = \left\{ \sum_i f_i\partial_{x_i} : \sum_i \partial_{x_i} f_i = 0 \right\}. \tag{3}$$

Equivalently, $D \in S_n$ if and only if $d(\iota_D\Omega_0) = 0$. For coordinates $(q_1, p_1, \dots, q_m, p_m)$, set

$$\omega_0 = \sum_i dq_i \wedge dp_i, \qquad \{f, g\} = \sum_i (f_{q_i}g_{p_i} - f_{p_i}g_{q_i}).$$

Set $H_{2m} = \{D \in W_{2m} : \mathcal{L}_D\omega_0 = 0\}$. The Hamiltonian field $X_f = \{f, -\} = \sum_i (f_{q_i}\partial_{p_i} - f_{p_i}\partial_{q_i})$ satisfies $\iota_{X_f}\omega_0 = -df$. Its commutator is determined by

$$[X_f, X_g](q_i) = -\{f, g_{p_i}\} + \{g, f_{p_i}\} = -\partial_{p_i}\{f, g\},$$
$$[X_f, X_g](p_i) = \{f, g_{q_i}\} - \{g, f_{q_i}\} = \partial_{q_i}\{f, g\}.$$

Hence $[X_f, X_g] = X_{\{f,g\}}$; applying this equality to a function gives the Jacobi identity for $\{-, -\}$. The kernel of $f \mapsto X_f$ consists of the constants.

Lemma 2.1 (Polynomial homotopy)**.** *Let β be a closed polynomial r-form on $\mathbb{A}^n$, with $r > 0$. Write $\beta = \sum_{d\geq 0} \beta_d$, where the coefficients of β_d are homogeneous of degree d, and let $\mathcal{R} = \sum_i x_i\partial_{x_i}$. Then*

$$\gamma = \sum_{d\geq 0} \frac{\iota_{\mathcal{R}}\beta_d}{d + r} \quad \textit{satisfies} \quad d\gamma = \beta. \tag{4}$$

Proof. Each β_d is closed, because the components of $d\beta_d$ have distinct coefficient degrees. On each monomial form,

$$\mathcal{L}_{\mathcal{R}}(x^\nu\, dx_{i_1} \wedge \cdots \wedge dx_{i_r}) = (|\nu| + r)x^\nu\, dx_{i_1} \wedge \cdots \wedge dx_{i_r},$$

because $\mathcal{R}(x^\nu) = |\nu|x^\nu$ and $\mathcal{L}_{\mathcal{R}}dx_i = d(\mathcal{R}x_i) = dx_i$. Thus it acts on β_d as multiplication by $d + r$. Cartan's formula gives $d\iota_{\mathcal{R}}\beta_d = (d + r)\beta_d$. The sums are finite and the displayed denominators are nonzero. □

If $D \in H_{2m}$, then $\iota_D\omega_0$ is closed. The lemma gives $\iota_D\omega_0 = -df$ for a polynomial f. The nondegeneracy of ω_0 implies $D = X_f$. Therefore

$$H_{2m} \cong (A_{2m}/\mathbb{C}, \{-, -\}). \tag{5}$$

For contact coordinates $(t, p_1, q_1, \dots, p_m, q_m)$ put

$$\alpha_0 = dt + \sum_i p_i\, dq_i, \qquad K_{2m+1} = \{D \in W_{2m+1} : \mathcal{L}_D\alpha_0 \in A_{2m+1}\alpha_0\}. \tag{6}$$

For a polynomial f, define

$$Y_f = \left(f - \sum_i p_i f_{p_i}\right)\partial_t + \sum_i f_{p_i}\partial_{q_i} + \sum_i (p_i f_t - f_{q_i})\partial_{p_i}. \tag{7}$$

The equality $\alpha_0(Y_f) = f$ follows by cancellation of $\sum_i p_i f_{p_i}$. Moreover,

$$\iota_{Y_f}d\alpha_0 = \sum_i ((p_i f_t - f_{q_i})dq_i - f_{p_i}dp_i),$$

$$\mathcal{L}_{Y_f}\alpha_0 = df + \iota_{Y_f}d\alpha_0 = f_t\left(dt + \sum_i p_i\, dq_i\right) = f_t\alpha_0.$$

Conversely, if D is a contact field, write $f = \alpha_0(D)$. The equation $df + \iota_D d\alpha_0 = \lambda\alpha_0$ gives $\lambda = f_t$, $D(q_i) = f_{p_i}$, and $D(p_i) = p_i f_t - f_{q_i}$; its remaining coefficient is then determined by $\alpha_0(D) = f$. Hence $f \mapsto Y_f$ is a linear bijection onto K_{2m+1}.

The corresponding bracket on functions is

$$\langle f, g\rangle = Y_f(g) - g f_t, \qquad [Y_f, Y_g] = Y_{\langle f,g\rangle}. \tag{8}$$

The identity $[\mathcal{L}_D, \mathcal{L}_Z] = \mathcal{L}_{[D,Z]}$ holds on functions and their differentials, and hence on all forms. Therefore

$$\mathcal{L}_{[Y_f,Y_g]}\alpha_0 = [\mathcal{L}_{Y_f}, \mathcal{L}_{Y_g}]\alpha_0 = (Y_f(g_t) - Y_g(f_t))\alpha_0,$$

so $[Y_f, Y_g]$ is contact. Moreover, $\alpha_0([Y_f, Y_g]) = Y_f(g) - (\mathcal{L}_{Y_f}\alpha_0)(Y_g)$. We reserve $\{-,-\}$ for the Poisson bracket and $\langle -,-\rangle$ for this contact bracket.

Lemma 2.2. *Let L be a Lie subalgebra of W_n whose evaluation at the origin is surjective. Set $L_0 = \{D \in L : D(0) = 0\}$, $L_{-1} = L$, and*

$$L_j = \{D \in L_{j-1} : [D, L] \subset L_{j-1}\} \quad (j \geq 1).$$

For $\mathfrak{m} = (x_1, \ldots, x_n)$ one has

$$L_j = L \cap \mathfrak{m}^{j+1} W_n \qquad (j \geq 0). \tag{9}$$

Consequently, a subalgebra $M \subset L$ is dense in this topology if for each $D \in L$ and each $r \geq 1$ it contains an element congruent to D modulo $\mathfrak{m}^r W_n$.

Proof. The assertion is the definition for $j = 0$. Suppose it is known for $j-1$. Bracketing a field whose coefficients vanish to order $j+1$ with a polynomial field gives coefficients vanishing to order at least j. This proves one inclusion. Conversely, let $D \in L_{j-1}$ have a nonzero homogeneous coefficient part of degree j. For a field $Z \in L$, the degree $j-1$ part of $[D, Z]$ is minus the directional derivative of that part along $Z(0)$. Evaluation is surjective and a nonzero positive-degree polynomial in characteristic zero has a nonzero partial derivative. Thus some $[D, Z]$ does not lie in L_{j-1}. This proves the other inclusion and the density criterion. □

All four ambient algebras have surjective evaluation. In the contact case, $Y_1(0) = \partial_t$, $Y_{p_i}(0) = \partial_{q_i}$, and $Y_{q_i}(0) = -\partial_{p_i}$. Equation (9) is the natural-filtration topology used in [3, §§0,1,9]. The usual contact grading assigns weight two to t and weight one to p_i, q_i; its filtration gives the same topology, since a monomial of ordinary degree d has weighted degree between d and $2d$.

§ 3. The common affine quadric

Inside $A_2 = \mathbb{C}[t, y]$ set

$$a = t + t^2 y, \qquad b = y, \qquad c = 1 + ty, \qquad h = 2c - 1. \tag{10}$$

We identify the algebra in (1) with its image in A_2. To justify the presentation, the polynomial $ab - c(c-1)$ is primitive and linear in a over the unique factorization domain $\mathbb{C}[b, c]$; b and $c(c-1)$ are relatively prime. It is therefore irreducible. After inverting b, its quotient is $\mathbb{C}[b, b^{-1}, c]$. The substitution $b = y$, $c = 1 + ty$ gives an isomorphism $\mathbb{C}[b, b^{-1}, c] \to \mathbb{C}[y, y^{-1}, t]$, with inverse $y \mapsto b$, $t \mapsto (c-1)/b$. The quotient is a domain and injects into its localization, which proves the claimed embedding. Also $t = (c-1)/b$ in the fraction field, so

$$\mathrm{Frac}(B) = \mathbb{C}(t, y), \qquad h^2 - 4ab = 1. \tag{11}$$

The quadric $Q = \operatorname{Spec} B$ is smooth: the partial derivatives $b, a, -h$ of its equation cannot vanish simultaneously, by (11). The map $\mathbb{A}^2 \to Q$ identifies $\mathbb{A}^2$ with $Q \setminus V(b, c)$, where $V(b, c)$ is the common zero set of b, c. Write $D(u)$ for the locus where a function u is nonzero.

The inverse map is $y = b$ and $t = (c-1)/b$ on $D(b)$, and $t = a/c$ on $D(c)$. The relation $ab = c(c-1)$ makes these formulas agree on $D(bc)$. On $D(b)$ one obtains $1 + ty = c$ and $tc = a$. On $D(c)$ one obtains $ty = c - 1$ and $tc = a$. Thus both recover (10). Conversely, $b = y$ and $c = 1 + ty$ never vanish simultaneously on $\mathbb{A}^2$, so the domains cover exactly the image.

Lemma 3.1. *One has $t \notin B$. For every integer $N \geq 1$, the polynomial*

$$t_N = a \sum_{j=0}^{N-1} (1-c)^j \in B \quad \textit{satisfies} \quad t_N = t(1 - (-ty)^N). \tag{12}$$

In particular,

$$t_N - t = (-1)^{N+1} t^{N+1} y^N \in (t, y)^{2N+1}, \tag{13}$$

and $B \to A_2/(t,y)^r$ is surjective for every $r \geq 1$.

Proof. If t belonged to B, the identity $bt = c-1$ would hold there. Its image in $B/(b,c) = \mathbb{C}[a]$ would read $0 = -1$, which is impossible. For the second assertion, use $a = tc$ and the finite geometric-series identity $c\sum_{j=0}^{N-1}(1-c)^j = 1 - (1-c)^N$. For $P(t,y) = \sum_{i\geq 0} p_i(y) t^i$, one has

$$P(t_N, y) - P(t, y) = (t_N - t) \sum_{i \geq 1} p_i(y) \sum_{\ell=0}^{i-1} t_N^{i-1-\ell} t^\ell.$$

Choosing $2N + 1 \geq r$ proves surjectivity. The same argument works after adjoining any number of independent polynomial variables. Differentiation also gives the exact identity

$$dt_N - dt = (-1)^{N+1}((N+1)t^N y^N \, dt + N t^{N+1} y^{N-1} \, dy). \tag{14}$$

□

Put $g = 3t^2 + 2t^3 y$ and define

$$E = \partial_t, \qquad T = g\partial_t + \partial_y, \qquad \eta = h \, da - 4a \, dc. \tag{15}$$

Their values on the generators are

	a	b	c
E	h	0	b
T	$4a^2$	1	ah

(16) $\qquad [E, T] = 6aE.$

Indeed, $E(a) = 1 + 2ty = h$, $E(b) = 0$, and $E(c) = y = b$, while

$$\begin{aligned} T(a) &= (3t^2 + 2t^3 y)(1 + 2ty) + t^2 = 4t^2 + 8t^3 y + 4t^4 y^2 = 4a^2, \\ T(b) &= 1, \qquad T(c) = 3t^2 y + 2t^3 y^2 + t = (t + t^2 y)(1 + 2ty) = ah, \\ [E, T] &= (\partial_t g)E = (6t + 6t^2 y)E = 6aE. \end{aligned}$$

These values show that E and T preserve B.

Differentiating the relation gives $b \, da + a \, db = h \, dc$. Together with (11), this yields the two exact identities

$$da = h\eta + 4a^2 db, \qquad dc = b\eta + ah \, db. \tag{17}$$

For the first identity, use $h\eta + 4a^2 db = h^2 da - 4a(b \, da + a \, db) + 4a^2 db = da$. For the second, use $b\eta + ah \, db = h(b \, da + a \, db) - 4ab \, dc = dc$. The evaluations are

$$\eta(E) = h^2 - 4ab = 1, \qquad \eta(T) = 4a^2 h - 4a^2 h = 0, \qquad db(E) = 0, \quad db(T) = 1.$$

Thus

$$\Omega^1_B = B\eta \oplus B \, db, \qquad \mathrm{Der}(B) = BE \oplus BT. \tag{18}$$

If $D(a) = u$, $D(b) = v$, and $D(c) = w$, the relation $bu + av = hw$ gives the explicit expression

$$D = (hu - 4aw)E + vT. \tag{19}$$

Its values on a, b, c are u, v, w: the first is $h^2u - 4ahw + 4a^2v = (h^2 - 4ab)u$, and the third is $h(bu + av) - 4abw = (h^2 - 4ab)w$.

The volume form and its chart expressions are

$$\omega = \eta \wedge db = dt \wedge dy, \qquad \eta = dt - g\,dy, \qquad d\eta = -6a\,\omega. \tag{20}$$

Indeed, $da = h\,dt + t^2 dy$ and $dc = y\,dt + t\,dy$, so

$$\eta = (h^2 - 4ay)dt + (ht^2 - 4at)dy = dt - g\,dy.$$

Moreover, (17) gives $da \wedge dc = a(h^2 - 4ab)\eta \wedge db = a\omega$, and hence

$$d\eta = 2dc \wedge da - 4da \wedge dc = -6a\omega.$$

The form ω generates Ω^2_B.

For every $r \geq 0$ put

$$R_r = B[z_1, \ldots, z_r], \qquad \Omega_r = \omega \wedge dz_1 \wedge \cdots \wedge dz_r. \tag{21}$$

The forms $\eta, db, dz_1, \ldots, dz_r$ and the derivations $E, T, \partial_{z_1}, \ldots, \partial_{z_r}$ are dual bases over R_r. Here E, T annihilate the z_i. Since $\iota_E\omega = db$ and $\iota_T\omega = -\eta$, (20) gives $\operatorname{div}_{\Omega_r} E = 0$, $\operatorname{div}_{\Omega_r} T = 6a$, and $\operatorname{div}_{\Omega_r} \partial_{z_i} = 0$. Thus (2) yields

$$\operatorname{div}_{\Omega_r}\left(AE + CT + \sum_i U_i\partial_{z_i}\right) = E(A) + T(C) + 6aC + \sum_i \partial_{z_i}U_i. \tag{22}$$

Lemma 3.2. *The ring B has no nonzero proper associative ideal stable under all its derivations. The same holds for $B[z_1, \ldots, z_r]$ for every $r \geq 0$.*

Proof. Besides E, consider the global derivation

$$F = hT - 4a^2E = -t^2\partial_t + h\partial_y. \tag{23}$$

It acts on (a, b, c) as $(0, h, a)$. Let I be a nonzero associative ideal of B stable under E, F. For $0 \neq u \in I$, applying E exactly $\deg_t u$ times gives a nonzero $P(y) \in I$. On $\mathbb{C}[y]$, $EF = 2y\partial_y$. If $P = \sum_{j\in S} P_j y^j$ with each $P_j \neq 0$, choose $k \in S$. The operator

$$\prod_{j\in S\setminus\{k\}} \frac{\frac{1}{2}EF - j}{k - j}$$

sends P to $P_k y^k$. Consequently $y^k \in I$ for some $k \geq 0$. For $k > 0$, $F(y^k) = ky^{k-1}h$ lies in I. Multiplication by h and $h^2 = 1 + 4ay$ give

$$y^{k-1} = \frac{1}{k}hF(y^k) - 4ay^k \in I.$$

Repeated descent gives $1 \in I$.

For $B[z_1, \ldots, z_r]$, a nonzero ideal stable under all derivations can be differentiated successively to its highest degree in each z_i. At each step the result is nonzero in characteristic zero, and the final result lies in B. The intersection with B is stable under E, F, so the preceding argument applies. □

§ 4. The Witt series

We use Jordan's derivation criterion [6, 7], with the bracket identities in [9, Lemmas 2.2–2.5].

Lemma 4.1. *Let R be a commutative $\mathbb{C}$-algebra that is a domain, and let $0 \neq L \subset \operatorname{Der}_{\mathbb{C}}(R)$ be both a Lie subalgebra and an R-submodule. If R has no nonzero proper L-stable associative ideal, then L is a simple Lie algebra.*

Proof. Let $0 \neq J \lhd L$ and choose $0 \neq \mu \in J$. For any $f, u \in R$,

$$\mu(f)\mu \in J, \qquad \mu(f)\mu(u)\mu \in J, \qquad f\mu^2(u)\mu \in J. \tag{24}$$

The first assertion follows from $[\mu, f\mu] = \mu(f)\mu$. For the second, expand

$$[\mu, f\mu(u)\mu] - [f\mu, \mu(u)\mu] = 2\mu(f)\mu(u)\mu.$$

The first bracket has its first entry in J; the second has its second entry in J by the first assertion. Subtracting $\mu(f)\mu(u)\mu$ from $\mu(f\mu(u))\mu$ proves the third assertion.

Choose f with $\mu(f) \neq 0$. There exists u with $\mu^2(u) \neq 0$: take $u = f$ if possible, and otherwise take $u = f^2$, since then $\mu^2(f^2) = 2\mu(f)^2$. Put $q = \mu^2(u)$. Equation (24) gives $Rq\mu \subset J$. For $\tau \in L$ the two brackets are

$$[q\mu, f\tau] = q\mu(f)\tau + fq[\mu, \tau] - f\tau(q)\mu,$$
$$[fq\mu, \tau] = fq[\mu, \tau] - \tau(fq)\mu.$$

Since $\tau(fq) = \tau(f)q + f\tau(q)$, subtraction gives

$$[q\mu, f\tau] - [fq\mu, \tau] - \tau(f)q\mu = \mu(f)q\tau. \tag{25}$$

Each left-hand term lies in J. Consequently,

$$I = \{r \in R : rL \subset J\}$$

contains the nonzero element $\mu(f)q$. This set is an associative ideal: if $r \in I$, then $(sr)\tau = r(s\tau) \in J$ for every $s \in R$. It is also L-stable, because

$$\delta(r)\tau = [\delta, r\tau] - r[\delta, \tau] \in J \qquad (\delta, \tau \in L).$$

Hence $I = R$ and $J = L$. Finally, L is nonabelian, since $[\mu, f\mu] = \mu(f)\mu \neq 0$. □

For $n \geq 2$, regard R_{n-2} from (21) as a subring of $\mathbb{C}[t, y, z_1, \ldots, z_{n-2}]$ and define

$$\mathfrak{w}_n = R_{n-2}E \oplus R_{n-2}T \oplus \bigoplus_{i=1}^{n-2} R_{n-2}\partial_{z_i} \subset W_n. \tag{26}$$

By (18), this is the restriction of $\mathrm{Der}(R_{n-2})$. It is closed under brackets, either by this identification or by (16) and $[fD, uZ] = fu[D, Z] + fD(u)Z - uZ(f)D$. Lemmas 3.2 and 4.1 prove that it is simple. The elements b^jE, $j \geq 0$, are linearly independent, so it is infinite-dimensional.

All elements of $\mathfrak{w}_n$ preserve R_{n-2}, whereas

$$\partial_y(c) = t \notin R_{n-2}. \tag{27}$$

The nonmembership follows from Lemma 3.1 by setting all z_i equal to zero. Thus $\partial_y \notin \mathfrak{w}_n$ and $\mathfrak{w}_n \subsetneq W_n$.

Here is an explicit approximation to the missing field. Put

$$g_N = 3t_N^2 + 2t_N^3 b, \qquad D_N = T - g_N E \in \mathfrak{w}_n.$$

The identity

$$g_N - g = (t_N - t)\big(3(t_N + t) + 2y(t_N^2 + t_N t + t^2)\big) \tag{28}$$

gives $D_N - \partial_y \in \mathfrak{m}^{2N+2}W_n$, where $\mathfrak{m} = (t, y, z_1, \ldots, z_{n-2})$. For $z = (z_1, \ldots, z_{n-2})$ and $V = P\partial_t + Q\partial_y + \sum_i U_i\partial_{z_i} \in W_n$, put

$$V_N = P(t_N, b, z)E + Q(t_N, b, z)D_N + \sum_i U_i(t_N, b, z)\partial_{z_i} \in \mathfrak{w}_n.$$

The difference is

$$V_N - V = \big(P(t_N, b, z) - P + Q(t_N, b, z)(g - g_N)\big)\partial_t$$
$$+ \big(Q(t_N, b, z) - Q\big)\partial_y + \sum_i \big(U_i(t_N, b, z) - U_i\big)\partial_{z_i}.$$

Every coefficient lies in $\mathfrak{m}^{2N+1}$ by (13) and (28). As N is arbitrary, Lemma 2.2 proves density.

§ 5. The Hamiltonian series

The standard bracket $\{u, v\} = u_t v_y - u_y v_t$ preserves B, since

$$\{a, b\} = a_t = h, \qquad \{a, c\} = ht - t^2 y = a, \qquad \{c, b\} = c_t = b. \tag{29}$$

The coordinates $e = b$, $f = -a$, and $h = 2c - 1$ put these relations in $\mathfrak{sl}_2$ form:

$$\{h, e\} = 2e, \qquad \{h, f\} = -2f, \qquad \{e, f\} = h, \qquad h^2 + 4ef = 1. \tag{30}$$

The inverse formulas $a = -f$, $b = e$, $c = (h + 1)/2$ identify B with $\mathbb{C}[e, f, h]/(h^2 + 4ef - 1)$.

Lemma 5.1. *For $d \geq 0$, let $B_{\leq d}$ be the image of the polynomials of total degree at most d in e, f, h. For $k, j \geq 0$ set*

$$v_{k,j} = (\operatorname{ad} f)^j(e^k), \qquad V_k = \operatorname{span}_{\mathbb{C}}\{v_{k,j} : 0 \leq j \leq 2k\},$$

where ad *refers to the Poisson bracket. Then V_k is an irreducible module for the three operators* $\operatorname{ad} e, \operatorname{ad} f, \operatorname{ad} h$, $\dim V_k = 2k + 1$, *and*

$$B_{\leq d} = \bigoplus_{k=0}^{d} V_k, \qquad B = \bigoplus_{k \geq 0} V_k, \qquad V_0 = \mathbb{C}1. \tag{31}$$

Proof. The relation $h^2 = 1 - 4ef$ is monic in h. Division by this relation gives the basis $e^i f^j h^\epsilon$ with $i, j \geq 0$ and $\epsilon \in \{0, 1\}$; reduction does not increase total degree. Hence

$$\dim B_{\leq d} = \frac{(d+1)(d+2)}{2} + \frac{d(d+1)}{2} = (d+1)^2.$$

The three adjoint operators preserve this filtration. Write them as $\mathcal{E}, \mathcal{F}, \mathcal{H}$, respectively. The Jacobi identity and (30) give

$$[\mathcal{H}, \mathcal{E}] = 2\mathcal{E}, \qquad [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}, \qquad [\mathcal{E}, \mathcal{F}] = \mathcal{H}.$$

Starting from $\mathcal{H}(e^k) = 2ke^k$, the identity $\mathcal{H}\mathcal{F}v = \mathcal{F}\mathcal{H}v - 2\mathcal{F}v$ gives the first formula in

$$\mathcal{H}v_{k,j} = (2k - 2j)v_{k,j}, \qquad \mathcal{E}v_{k,j} = j(2k - j + 1)v_{k,j-1} \quad (j \geq 1). \tag{32}$$

For the second formula, $\mathcal{E}(e^k) = 0$ and $\mathcal{E}\mathcal{F}(e^k) = \mathcal{H}(e^k) = 2ke^k$. If it holds at j, then $\mathcal{E}\mathcal{F}v_{k,j} = \mathcal{F}\mathcal{E}v_{k,j} + \mathcal{H}v_{k,j}$, whose coefficient is $j(2k - j + 1) + 2k - 2j = (j + 1)(2k - j)$.

Since $\mathcal{F}(e) = -h$, $\mathcal{F}(h) = 2f$, and $\mathcal{F}(f) = 0$, for an indeterminate λ and $k \geq 1$ one has

$$\begin{aligned} \partial_\lambda(e - \lambda h - \lambda^2 f)^k &= k(e - \lambda h - \lambda^2 f)^{k-1}(-h - 2\lambda f) \\ &= \mathcal{F}\big((e - \lambda h - \lambda^2 f)^k\big). \end{aligned}$$

For $k = 0$, both operators kill 1. The value at $\lambda = 0$ is e^k. Comparing coefficients and expanding by the multinomial formula therefore gives, for every $k, j \geq 0$,

$$v_{k,j} = j! \sum_{\substack{\rho + 2\sigma = j \\ \rho, \sigma \geq 0 \\ \rho + \sigma \leq k}} \frac{(-1)^{\rho+\sigma} k!}{(k - \rho - \sigma)!\, \rho!\, \sigma!} e^{k-\rho-\sigma} h^\rho f^\sigma. \tag{33}$$

For $j = 2k$ the only summand has $\rho = 0$, $\sigma = k$; for $j > 2k$ the sum is empty. Thus

$$v_{k,2k} = (-1)^k (2k)! f^k \neq 0, \qquad v_{k,2k+1} = 0. \tag{34}$$

All $v_{k,j}$ with $0 \le j \le 2k$ are nonzero, since a zero iterate would force $v_{k,2k} = 0$. Their distinct $\mathcal{H}$-weights make them linearly independent. Equations (32) and (34) prove stability of V_k. For a nonzero vector $\sum_{j=0}^{2k} a_j v_{k,j}$, choose $a_j \neq 0$. The operator

$$\prod_{\substack{0\le \ell\le 2k\\ \ell\neq j}} \frac{\mathcal{H}-(2k-2\ell)}{2(\ell-j)}$$

projects it onto $a_j v_{k,j}$. Thus every nonzero stable subspace contains a weight vector. Repeated application of $\mathcal{E}$, using the nonzero coefficients in (32), gives $v_{k,0}$; application of $\mathcal{F}$ then gives every $v_{k,j}$. Hence V_k is irreducible.

Set

$$\mathcal{C} = \mathcal{H}^2 + 2\mathcal{E}\mathcal{F} + 2\mathcal{F}\mathcal{E}.$$

Its three commutators are

$$\begin{aligned}
[\mathcal{E},\mathcal{C}] &= -2\mathcal{E}\mathcal{H} - 2\mathcal{H}\mathcal{E} + 2\mathcal{E}\mathcal{H} + 2\mathcal{H}\mathcal{E} = 0,\\
[\mathcal{F},\mathcal{C}] &= 2\mathcal{F}\mathcal{H} + 2\mathcal{H}\mathcal{F} - 2\mathcal{H}\mathcal{F} - 2\mathcal{F}\mathcal{H} = 0,\\
[\mathcal{H},\mathcal{C}] &= 2(2\mathcal{E}\mathcal{F} - 2\mathcal{E}\mathcal{F}) + 2(-2\mathcal{F}\mathcal{E} + 2\mathcal{F}\mathcal{E}) = 0.
\end{aligned}$$

Its value on e^k is $4k^2e^k + 4ke^k = 4k(k+1)e^k$. Since it commutes with $\mathcal{F}$, it has the scalar $4k(k+1)$ on V_k. These scalars are distinct, so the sum of $V_0, \dots, V_d$ is direct. Its dimension is $\sum_{k=0}^{d}(2k+1) = (d+1)^2$. Since it lies in $B_{\le d}$, the dimension count proves (31). □

Proposition 5.2. *The Lie algebra $B/\mathbb{C}$ is simple.*

Proof. Let $0 \neq J \lhd B/\mathbb{C}$. For a nonzero element of J choose d with representative in $B_{\le d}$. Polynomial projection in $\mathcal{C}$ isolates a nonzero component in some V_k, $k \ge 1$; the projection onto V_k in this finite sum is

$$\prod_{\substack{0\le j\le d\\ j\neq k}} \frac{\mathcal{C}-4j(j+1)}{4k(k+1)-4j(j+1)}.$$

The ideal is stable under these operators. Irreducibility and (32) put the class of e^k in J.

The identity

$$\{e^k, eh\} = -2ke^{k+1} \tag{35}$$

puts e^{k+1} mod $\mathbb{C}$ in J. Applying $\operatorname{ad} f$ twice and using $h^2 = 1 - 4ef$ gives

$$\begin{aligned}
A_k := (\operatorname{ad} f)^2(e^{k+1}) &= (k+1)ke^{k-1}h^2 - 2(k+1)e^k f\\
&= (k+1)e^{k-1}(k-(4k+2)ef), \qquad (36)
\end{aligned}$$

$$C_k := \{e^k, fh\} = ke^{k-1}h^2 - 2ke^k f = ke^{k-1}(1-6ef). \tag{37}$$

Both classes lie in J. The ef coefficients cancel because $-6(k+1)(4k+2)+6(k+1)(4k+2) = 0$, and the constant coefficient is $6k(k+1) - (k+1)(4k+2) = 2(k+1)(k-1)$. Hence

$$6A_k - \frac{(k+1)(4k+2)}{k}C_k = 2(k+1)(k-1)e^{k-1}. \tag{38}$$

For $k > 1$ the scalar is nonzero. Descent gives e mod $\mathbb{C} \in J$. For every $j \ge 1$, $\{e, he^{j-1}\} = -2e^j$ then gives e^j mod $\mathbb{C} \in J$. Applying $\operatorname{ad} f$ and (31) proves $J = B/\mathbb{C}$. The bracket $\{h, e\} = 2e$ shows nonabelianity. □

Let $B' = \operatorname{span}_{\mathbb{C}}\{\{u, v\} : u, v \in B\}$ be the derived Lie algebra of B.

Lemma 5.3. *There is a linear functional $\tau : B \to \mathbb{C}$ with $\tau(1) = 1$, $\tau(\{B, B\}) = 0$, and*

$$B = \mathbb{C}1 \oplus B', \qquad B' \cong B/\mathbb{C}, \qquad d\Omega^1_B = B'\omega. \tag{39}$$

On the normal basis it is given by

$$\tau(e^i f^j) = \begin{cases} (i!)^2/(2i+1)!, & i = j, \\ 0, & i \neq j, \end{cases} \qquad \tau(e^i f^j h) = 0. \tag{40}$$

Proof. For $i, j \geq 0$, omit terms with zero coefficient. The first four generator actions are

$$\{h, e^i f^j\} = 2(i-j)e^i f^j, \qquad \{h, e^i f^j h\} = 2(i-j)e^i f^j h,$$
$$\{e, e^i f^j\} = j e^i f^{j-1} h, \qquad \{f, e^i f^j\} = -i e^{i-1} f^j h.$$

The remaining two actions, reduced by $h^2 = 1 - 4ef$, are

$$\begin{aligned} \{e, e^i f^j h\} &= j e^i f^{j-1} h^2 - 2e^{i+1} f^j \\ &= j e^i f^{j-1} - (4j+2) e^{i+1} f^j, \\ \{f, e^i f^j h\} &= -i e^{i-1} f^j h^2 + 2e^i f^{j+1} \\ &= -i e^{i-1} f^j + (4i+2) e^i f^{j+1}. \end{aligned}$$

Let $c_j = (j!)^2/(2j+1)!$. For $j \geq 1$,

$$\frac{c_j}{c_{j-1}} = \frac{j^2}{(2j)(2j+1)} = \frac{j}{2(2j+1)}, \qquad j c_{j-1} = (4j+2) c_j.$$

The first four displayed actions have trace zero: a term containing h has trace zero, and $2(i-j)$ vanishes when $i = j$. In the fifth action, only $i = j-1$ can contribute, and its trace is $j c_{j-1} - (4j+2)c_j = 0$. In the sixth, only $j = i-1$ can contribute, and its trace is $-i c_{i-1} + (4i+2)c_i = 0$. Thus τ is invariant under $\operatorname{ad} e, \operatorname{ad} f, \operatorname{ad} h$.

For a monomial $v = q_1 \cdots q_r$, where $r \geq 1$ and $q_i \in \{e, f, h\}$, put $Q_i = \prod_{\ell \neq i} q_\ell$; empty products are 1. For any $w \in B$, invariance applied to wQ_i and the Leibniz rule give

$$\begin{aligned} \tau(\{v, w\}) &= \sum_i \tau(Q_i \{q_i, w\}) \\ &= -\sum_i \tau(w \{q_i, Q_i\}) \\ &= -\sum_{i \neq j} \tau\left(w \prod_{\ell \neq i,j} q_\ell \, \{q_i, q_j\} \right) \\ &= -\sum_{i<j} \tau\left(w \prod_{\ell \neq i,j} q_\ell (\{q_i, q_j\} + \{q_j, q_i\}) \right) = 0. \end{aligned}$$

The bracket with a constant is zero, so linearity proves this for every $v, w \in B$. Thus $B' \subset \ker \tau$ and $B' \cap \mathbb{C} = 0$. For $k \geq 1$, $e^k = (2k)^{-1}\{h, e^k\}$ belongs to B'. The derived algebra is an ideal by the Jacobi identity, so it contains every V_k with $k \geq 1$. Equation (31) and $\tau(1) = 1$ therefore give $B' = \bigoplus_{k \geq 1} V_k = \ker \tau$, and the quotient map identifies B' with $B/\mathbb{C}$.

Every algebraic one-form is a sum of forms $u\,dv$. On the affine chart,

$$d(u\,dv) = du \wedge dv = (u_t v_y - u_y v_t)\, dt \wedge dy = \{u, v\}\omega.$$

The free module of global forms injects into its restriction to this chart, so the equality holds globally and proves the last assertion of (39). □

For $m \geq 1$ set

$$\begin{gathered} P_m = B[u_2, v_2, \ldots, u_m, v_m], \\ \{u_i, v_j\} = \delta_{ij}, \qquad \{u_i, u_j\} = \{v_i, v_j\} = 0, \end{gathered} \tag{41}$$

where δ_{ij} is 1 for $i = j$ and 0 otherwise. Use the bracket of (29) on B and set all mixed brackets with B equal to zero. The associated polynomial Hamiltonian fields are taken with respect to $dt \wedge dy + \sum_{i=2}^{m} du_i \wedge dv_i$. Define

$$\mathfrak{h}_{2m} = \{X_F : F \in P_m\} \subset H_{2m}.$$

Proposition 5.4. *For every $m \geq 1$, $P_m/\mathbb{C}$ is a simple Lie algebra.*

Proof. For $m = 1$ this is Proposition 5.2. Suppose $m \geq 2$ and $0 \neq J \lhd P_m/\mathbb{C}$. If $J \cap (B/\mathbb{C}) \neq 0$, this intersection is an ideal of $B/\mathbb{C}$, so $B/\mathbb{C} \subset J$. In this proof membership in J is understood modulo constants. For every polynomial G in the additional variables,

$$\{e, fG\} = hG, \qquad \{e, hG\} = -2eG, \qquad \{f, hG\} = 2fG.$$

Thus eG, fG, hG all belong to J. Fix an added pair u, v. Writing $G = \sum_{j\geq 0} G_j v^j$ with G_j independent of v, put $Q = \sum_{j\geq 0} G_j v^{j+1}/(j+1)$. Then $\{u, Q\} = \partial_v Q = G$ and

$$\{eu, fQ\} - huQ = efG, \qquad \{hu, hQ\} = h^2 G$$

belong to J. The first entries eu, hu lie in J, and huQ already does. The identity $h^2 + 4ef = 1$ gives $G \in J$. In particular $u \in J$. The same finite integration formula makes $\{u, -\} = \partial_v$ surjective on P_m, and hence $J = P_m/\mathbb{C}$.

Suppose now that $J \cap (B/\mathbb{C}) = 0$. Choose a nonzero class in J of least total degree d in the added variables. One has $d > 0$. Bracketing with their linear Hamiltonians gives all partial derivatives in those variables. If $d > 1$, some derivative has nonzero positive degree $d - 1$, contrary to minimality. Hence $d = 1$, and a representative has the form

$$F = b_0 + \sum_{j=1}^{2m-2} b_j w_j, \qquad b_j \in B,$$

where $w_1, \ldots, w_{2m-2}$ list the added variables. Each derivative b_j, $j > 0$, has class in $J \cap (B/\mathbb{C})$, so every such b_j is a scalar. For $b \in B$, $\{b, F\} = \{b, b_0\}$ also has zero class. Thus the class of b_0 is central in $B/\mathbb{C}$. The center is an ideal, and simplicity and nonabelianity force it to be zero. Subtracting the constant b_0 leaves a nonzero scalar linear form ℓ in the added variables, with $\ell \in J$.

Write $\{\ell, -\} = \delta = \sum_j c_j \partial_{w_j} \neq 0$. Choose i with $c_i \neq 0$ and set

$$z = w_i/c_i, \qquad z_j = w_j - c_j z \quad (j \neq i).$$

These are linear coordinates: their inverse is $w_i = c_i z$ and $w_j = z_j + c_j z$. They satisfy $\delta z = 1$, $\delta z_j = 0$, and $\delta|_B = 0$. For $F = \sum_{\nu\geq 0} F_\nu z^\nu$ with $F_\nu \in B[z_j : j \neq i]$,

$$\delta \left(\sum_{\nu \geq 0} \frac{F_\nu z^{\nu+1}}{\nu + 1} \right) = F. \tag{42}$$

Thus $\{\ell, -\}$ is surjective on P_m, and $J = P_m/\mathbb{C}$. □

The Hamiltonian map has kernel $\mathbb{C}$, so the proposition proves simplicity of $\mathfrak{h}_{2m}$. The independent classes of b^j, $j \geq 1$, prove infinite-dimensionality. Comparing coefficients in the added variables gives $P_m \cap \mathbb{C}[t, y] = B$. If $X_t = X_F$ for $F \in P_m$, then $t - F$ is a scalar, contradicting $t \notin B$. Thus $\mathfrak{h}_{2m}$ is proper. Substituting t_N, b, u_i, v_i into any polynomial Hamiltonian gives a difference in $\mathfrak{m}^{2N+1}$; its Hamiltonian field differs by an element of $\mathfrak{m}^{2N} W_{2m}$. This proves density for every m.

§ 6. The special series

Fix $n \geq 2$ and put $r = n-2$. For $D = AE + CT + \sum_i U_i \partial_{z_i} \in \mathrm{Der}(R_r)$, contraction with Ω_r is given by

$$\begin{aligned}\iota_D \Omega_r &= A\, db \wedge dz_1 \wedge \cdots \wedge dz_r - C\, \eta \wedge dz_1 \wedge \cdots \wedge dz_r \\ &\quad + \sum_{i=1}^{r} (-1)^{i+1} U_i\, \omega \wedge dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_r.\end{aligned}$$

Here a hat means that the indicated factor is omitted, and an empty exterior product is 1. The displayed forms constitute a basis of $\Omega^{n-1}_{R_r}$, so contraction is an isomorphism from $\mathrm{Der}(R_r)$ onto $\Omega^{n-1}_{R_r}$. Define

$$\mathfrak{s}_n = \{D \in \mathrm{Der}(R_r) : \iota_D \Omega_r \in d\Omega^{n-2}_{R_r}\}, \tag{43}$$

and identify its elements with their restrictions to the affine chart. They preserve Ω_r, since $d^2 = 0$, and therefore lie in S_n.

The identity $[\mathcal{L}_U, \iota_V] = \iota_{[U,V]}$ is zero on functions; on df it reads $U(V(f)) - V(U(f)) = [U,V](f)$. Both sides are graded derivations, so it holds on all differential forms. If U, V preserve Ω_r, then $d\iota_V \Omega_r = 0$ and this identity gives

$$\iota_{[U,V]} \Omega_r = \mathcal{L}_U(\iota_V \Omega_r) = d(\iota_U \iota_V \Omega_r). \tag{44}$$

Thus $\mathfrak{s}_n$ is a Lie subalgebra. When $r = 0$, its defining condition is $\iota_D \omega \in dB$, so $\mathfrak{s}_2 = \mathfrak{h}_2$.

One added variable. Suppose $r = 1$ and write $z = z_1$. Extend each $v \in \mathrm{Der}(B)$ to $B[z]$ by $v(z) = 0$, and put $\delta_v = \mathrm{div}_\omega(v)$. Formula (22) gives $\delta_{AE+CT} = E(A) + T(C) + 6aC$. For $k \geq 0$ define

$$T_k(v) = z^k v - \frac{z^{k+1}}{k+1} \delta_v \partial_z. \tag{45}$$

Every global divergence-free field on $B[z]$ has a unique finite expression

$$q\partial_z + \sum_{k \geq 0} T_k(v_k), \qquad q \in B, \quad v_k \in \mathrm{Der}(B). \tag{46}$$

Indeed, write its horizontal part as $\sum_k z^k v_k$. The equation for its vertical coefficient is $\partial_z D(z) = -\sum_k z^k \delta_{v_k}$, whose polynomial solutions are $D(z) = q - \sum_k z^{k+1} \delta_{v_k}/(k+1)$.

Each $T_k(v)$ is exact. Since $d\iota_v \omega = \delta_v \omega$,

$$\begin{aligned}-d\left(\frac{z^{k+1}}{k+1} \iota_v \omega\right) &= -z^k dz \wedge \iota_v \omega - \frac{z^{k+1}}{k+1} \delta_v \omega \\ &= z^k \iota_v \omega \wedge dz - \frac{z^{k+1}}{k+1} \delta_v \omega = \iota_{T_k(v)}(\omega \wedge dz).\end{aligned} \tag{47}$$

Moreover, $\iota_{q\partial_z}(\omega \wedge dz) = q\omega$. If $q\omega = d\lambda$ over B, the same λ is a primitive over $B[z]$. Conversely, pulling a primitive over $B[z]$ back along $z = 0$ gives a primitive over B. Lemma 5.3 therefore yields

$$\mathfrak{s}_3 = B'\partial_z \ \oplus \ \bigoplus_{k \geq 0} T_k(\mathrm{Der}(B)). \tag{48}$$

Thus $\partial_z \notin \mathfrak{s}_3$, and $\mathfrak{s}_3$ has codimension one in the global divergence-free fields on $Q \times \mathbb{A}^1$.

For $q \in B$, the product rule for divergence gives $\mathrm{div}_\omega(qv) = q\delta_v + v(q)$. Hence

$$\begin{aligned}[q\partial_z, T_{k+1}(v)] &= (k+1) z^k qv - z^{k+1}(q\delta_v + v(q))\partial_z \\ &= (k+1) T_k(qv) \qquad (k \geq 0),\end{aligned} \tag{49}$$

$$[q\partial_z, T_0(v)] = -(q\delta_v + v(q))\partial_z = -\,\mathrm{div}_\omega(qv)\partial_z. \tag{50}$$

The fields in $B\partial_z$ commute. Induction on j using these two formulas gives, for every $k, j \geq 0$,

$$(\mathrm{ad}(e\partial_z))^j T_k(v) = \begin{cases} \dfrac{k!}{(k-j)!}\, T_{k-j}(e^j v), & 0 \leq j \leq k, \\ -k!\ \mathrm{div}_\omega(e^{k+1}v)\partial_z, & j = k+1, \\ 0, & j \geq k+2. \end{cases} \tag{51}$$

For $j < k$, the next bracket multiplies the coefficient by $k - j$; at $j = k$ equation (50) gives the second line, and the next bracket is zero.

Proposition 6.1. *The Lie algebra $\mathfrak{s}_3$ is simple.*

Proof. Let $0 \neq J \lhd \mathfrak{s}_3$. We first show $J \cap B'\partial_z \neq 0$. Choose a nonzero element of J and write it as in (48). If every v_k is zero, the assertion follows. Otherwise, let N be the largest index with $v_N \neq 0$ and apply $(\mathrm{ad}(e\partial_z))^N$. Since $e \in B'$, the result belongs to J. Equation (51) shows that it has the form $T_0(v) + q_0\partial_z$, where

$$v = N!e^N v_N \in \mathrm{Der}(B), \qquad q_0 \in B'.$$

The vector field v is nonzero because B is a domain and $\mathrm{Der}(B)$ is free over B.

At least one $q \in \{e, f, h\}$ satisfies $\mathrm{div}_\omega(qv) \neq 0$. Indeed, if all three divergences vanished, the product rule would give

$$v(e) = -\delta_v e, \qquad v(f) = -\delta_v f, \qquad v(h) = -\delta_v h.$$

Applying v to $h^2 + 4ef = 1$ would then yield

$$0 = 2h\, v(h) + 4f\, v(e) + 4e\, v(f) = -2\delta_v(h^2 + 4ef) = -2\delta_v.$$

It would follow that v vanishes on the generators e, f, h, contrary to $v \neq 0$. For the resulting q, the bracket $[q\partial_z, T_0(v) + q_0\partial_z]$ is a nonzero element of $J \cap B'\partial_z$ by (50). Its coefficient lies in B' because $(\mathrm{div}_\omega(qv))\omega = d(\iota_{qv}\omega)$.

The space $M = \{q \in B' : q\partial_z \in J\}$ is therefore nonzero. For $u \in B'$, the Hamiltonian field X_u has zero divergence, so $T_0(X_u) = X_u \in \mathfrak{s}_3$. Since

$$[X_u, q\partial_z] = \{u, q\}\partial_z,$$

M is a Lie ideal of B'. Lemma 5.3 and Proposition 5.2 imply $M = B'$. Equation (49) now gives $T_k(qv) \in J$ for every $q \in B'$, $v \in \mathrm{Der}(B)$, and $k \geq 0$. Using $e, h \in B'$ and $h^2 + 4ef = 1$, we obtain

$$T_k(v) = T_k(h(hv)) + 4T_k(e(fv)) \in J.$$

Together with $B'\partial_z \subset J$, this proves $J = \mathfrak{s}_3$. The bracket $[X_h, e\partial_z] = 2e\partial_z$ proves non-abelianity. □

At least two added variables. Suppose $r \geq 2$, and let $\mathcal{S}_r$ denote the Lie algebra of global Ω_r-divergence-free fields on R_r.

Proposition 6.2. *For every $r \geq 2$, the algebra $\mathcal{S}_r$ is simple and equals $\mathfrak{s}_{r+2}$.*

Proof. Let $0 \neq J \lhd \mathcal{S}_r$. All ∂_{z_i} belong to $\mathcal{S}_r$. Express a nonzero field of J in the free frame E, T, ∂_{z_i}, and choose a nonzero coefficient monomial z^α of maximal total z-degree. Apply ∂_z^α to its coefficients by successive brackets with ∂_{z_i}. Every lower-degree monomial is killed. A different monomial of the same degree has some exponent smaller than the corresponding α_i and is also killed. The resulting field is nonzero and has the form

$$D_0 = v + \sum_i q_i\partial_{z_i}, \qquad v \in \mathrm{Der}(B), \quad q_i \in B, \quad \delta_v = 0.$$

If $v \neq 0$, choose $q \in B$ with $v(q) \neq 0$. Since $q\partial_{z_j} \in \mathcal{S}_r$,

$$[D_0, q\partial_{z_j}] = v(q)\partial_{z_j} \in J.$$

If $v = 0$, choose $q_i \neq 0$ and $j \neq i$. The field $z_i\partial_{z_j}$ belongs to $\mathcal{S}_r$, and $[D_0, z_i\partial_{z_j}] = q_i\partial_{z_j} \in J$. After renumbering the added coordinates, we have $0 \neq q\partial_{z_1} \in J$.

Set $M = \{q \in B : q\partial_{z_1} \in J\}$. For $q \in M$ and $u \in B$, the following brackets use divergence-free fields:

$$\begin{aligned}[q\partial_{z_1}, z_1 u\partial_{z_2}] &= qu\partial_{z_2},\\ [z_2\partial_{z_1}, qu\partial_{z_2}] &= -qu\partial_{z_1}.\end{aligned} \tag{52}$$

Thus M is an associative ideal of B. For any $v \in \mathrm{Der}(B)$, extend v trivially in the added variables and put $\widetilde{v} = v - z_2\delta_v\partial_{z_2}$. Then

$$\mathrm{div}_{\Omega_r}\widetilde{v} = \delta_v - \partial_{z_2}(z_2\delta_v) = 0, \qquad [\widetilde{v}, q\partial_{z_1}] = v(q)\partial_{z_1}.$$

Hence M is stable under every derivation of B. Lemma 3.2 gives $M = B$, so $\partial_{z_1} \in J$.

For a polynomial $\sum_{j\geq 0} a_j z_i^j$ with coefficients independent of z_i, define

$$\mathcal{I}_i\left(\sum_{j\geq 0} a_j z_i^j\right) = \sum_{j\geq 0}\frac{a_j z_i^{j+1}}{j+1}. \tag{53}$$

Given $D \in \mathcal{S}_r$, write

$$\begin{aligned}D &= AE + CT + \sum_{i=1}^{r} U_i\partial_{z_i},\\ Y_0 &= \mathcal{I}_1(A)E + \mathcal{I}_1(C)T + \sum_{i=1}^{r}\mathcal{I}_1(U_i)\partial_{z_i}.\end{aligned}$$

Then $[\partial_{z_1}, Y_0] = D$. Formula (22) gives

$$\begin{aligned}\mathrm{div}_{\Omega_r} Y_0 &= \mathcal{I}_1\left(E(A) + T(C) + 6aC + \sum_{i=2}^{r}\partial_{z_i}U_i\right) + U_1\\ &= -\mathcal{I}_1(\partial_{z_1}U_1) + U_1 = U_1\big|_{z_1=0}.\end{aligned}$$

Put $Q = \mathcal{I}_2(U_1|_{z_1=0})$ and $Y = Y_0 - Q\partial_{z_2}$. The polynomial Q is independent of z_1, and

$$\mathrm{div}_{\Omega_r} Y = U_1|_{z_1=0} - \partial_{z_2}Q = 0, \qquad [\partial_{z_1}, Y] = D.$$

It follows that $D \in J$, so $J = \mathcal{S}_r$. The bracket $[\partial_{z_1}, z_1\partial_{z_2}] = \partial_{z_2}$ proves nonabelianity.

The same construction and (44) give

$$\iota_D\Omega_r = d(\iota_{\partial_{z_1}}\iota_Y\Omega_r).$$

Thus every $D \in \mathcal{S}_r$ belongs to $\mathfrak{s}_{r+2}$. The defining exactness condition gives the reverse inclusion. □

Properness and density in every dimension. Every field in $\mathfrak{s}_n$ preserves R_{n-2}, whereas the ambient divergence-free field ∂_y does not, by (27). Hence $\mathfrak{s}_n \subsetneq S_n$.

Take $D \in S_n$ on the affine chart and put $\beta = \iota_D\Omega_0$, where $\Omega_0 = dt \wedge dy \wedge dz_1 \wedge \cdots \wedge dz_r$. Since $d\beta = 0$, Lemma 2.1 supplies a polynomial $(n-2)$-form γ with $d\gamma = \beta$. Define the algebra map

$$\psi_N^* : \mathbb{C}[t, y, z_1, \ldots, z_r] \longrightarrow R_r, \qquad t \longmapsto t_N, \quad y \longmapsto b, \quad z_i \longmapsto z_i.$$

Extend it to forms by $\psi_N^*(df) = d(\psi_N^* f)$ and $\psi_N^*(\lambda \wedge \mu) = \psi_N^*\lambda \wedge \psi_N^*\mu$. The Leibniz rule and $d^2 = 0$ show that this extension commutes with d. Contraction determines a unique global derivation D_N satisfying

$$\iota_{D_N}\Omega_r = d(\psi_N^*\gamma) = \psi_N^*\beta. \tag{54}$$

Consequently $D_N \in \mathfrak{s}_n$.

Write $z = (z_1, \dots, z_r)$ and $D = P\partial_t + Q\partial_y + \sum_i U_i \partial_{z_i}$. Set $P_N = P(t_N, b, z)$, with $Q_N, U_{i,N}$ defined in the same way. On the chart, $\Omega_r = \Omega_0$ and $dt_N = (\partial_t t_N)dt + (\partial_y t_N)dy$. Substitution in the contraction formula gives

$$\begin{aligned} D_N = (P_N - Q_N \partial_y t_N)\partial_t + Q_N(\partial_t t_N)\partial_y \\ + \sum_i U_{i,N}(\partial_t t_N)\partial_{z_i}. \end{aligned} \tag{55}$$

Each substituted coefficient differs from the original coefficient by an element of $\mathfrak{m}^{2N+1}$, by (13). Equation (14) gives $\partial_t t_N - 1, \partial_y t_N \in \mathfrak{m}^{2N}$. Subtracting D from (55), its coefficients are

$$P_N - P - Q_N \partial_y t_N, \qquad Q_N - Q + Q_N(\partial_t t_N - 1),$$
$$U_{i,N} - U_i + U_{i,N}(\partial_t t_N - 1).$$

They lie in $\mathfrak{m}^{2N}$, so

$$D_N - D \in \mathfrak{m}^{2N} W_n. \tag{56}$$

Lemma 2.2 proves density. For example, in dimension three,

$$E(t_N)\partial_z = \big(1 - (N+1)(-ty)^N\big)\partial_z \in \mathfrak{s}_3, \qquad E(t_N)\omega = d(t_N\, db).$$

The difference from ∂_z lies in $\mathfrak{m}^{2N} W_3$. The fields X_{b^j} for $n = 2$ and $j \geq 1$, $b^j \partial_z$ for $n = 3$ and $j \geq 1$, and $b^j \partial_{z_1}$ for $n \geq 4$ and $j \geq 0$ give linearly independent families in the respective algebras.

§ 7. The contact series

For $m \geq 1$ use coordinates $(t, y, p, u_2, v_2, \dots, u_m, v_m)$ and the standard contact form

$$\alpha_0 = dt + p\, dy + \sum_{i=2}^{m} u_i\, dv_i.$$

Thus $(p_1, q_1) = (p, y)$ and $(p_i, q_i) = (u_i, v_i)$ for $i \geq 2$ in (7). Set

$$s = p + g = p + 3t^2 + 2t^3 y, \qquad D_m = B[s, u_2, v_2, \dots, u_m, v_m] \subset \mathbb{C}[t, y, p, u_2, v_2, \dots, u_m, v_m]. \tag{57}$$

The element s is algebraically independent over B, since it is monic of degree one in the independent variable p. Define

$$\mathfrak{k}_{2m+1} = \{Y_F : F \in D_m\}. \tag{58}$$

The contact bracket. The Reeb derivation is $R = \partial_t$, with p, y, u_i, v_i held fixed: $\alpha_0(R) = 1$ and $\iota_R d\alpha_0 = 0$. Its values are

$$Ra = h, \quad Rb = 0, \quad Rc = b, \quad Rs = 6a, \quad Ru_i = Rv_i = 0. \tag{59}$$

Thus R preserves D_m. Each element is an ambient polynomial of finite t-degree, so it is killed by a sufficiently high power of R. For $U, V \in B$, (7) gives

$$\langle U, V \rangle = UE(V) - VE(U), \qquad \langle U, s \rangle = 6aU - T(U). \tag{60}$$

For the second formula, $Y_U(p) = pE(U) - \partial_y U$ and $Y_U(g) = 6aU$. Subtracting $sE(U) = (p+g)E(U)$ leaves $6aU - \partial_y U - gE(U)$. The six brackets of the base generators are

$$\begin{aligned}
\langle a, b\rangle &= aE(b) - bE(a) = -bh,\\
\langle a, c\rangle &= ab - ch = c(c-1) - c(2c-1) = -c^2,\\
\langle a, s\rangle &= 6a^2 - T(a) = 6a^2 - 4a^2 = 2a^2,\\
\langle b, c\rangle &= bE(c) - cE(b) = b^2,\\
\langle b, s\rangle &= 6ab - T(b) = 6ab - 1,\\
\langle c, s\rangle &= 6ac - T(c) = 6ac - ah = a(4c+1).
\end{aligned} \tag{61}$$

For $F \in B[s]$ and $2 \le i, j \le m$, the added generators satisfy

$$\langle u_i, v_j\rangle = \delta_{ij}, \quad \langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0, \quad \langle F, u_i\rangle = 0, \quad \langle F, v_i\rangle = -v_i RF. \tag{62}$$

Indeed, $Y_{u_i} = \partial_{v_i}$ and $Y_{v_i} = v_i\partial_t - \partial_{u_i}$; also $Y_F(u_i) = u_i RF$ and $Y_F(v_i) = 0$.

For arbitrary polynomial functions F, G, formula (7) gives

$$Y_1 = R, \qquad \langle 1, F\rangle = RF, \qquad Y_{FG} = FY_G + GY_F - FG\,R. \tag{63}$$

For the last identity, the ∂_t coefficient is

$$\begin{aligned}
FG - \sum_i p_i(F_{p_i}G + FG_{p_i}) = F\left(G - \sum_i p_i G_{p_i}\right)\\
+ G\left(F - \sum_i p_i F_{p_i}\right) - FG.
\end{aligned}$$

The ∂_{q_i} coefficient is $(FG)_{p_i} = FG_{p_i} + GF_{p_i}$, and the ∂_{p_i} coefficient is

$$p_i(FG)_t - (FG)_{q_i} = F(p_iG_t - G_{q_i}) + G(p_iF_t - F_{q_i}).$$

Consequently,

$$\begin{aligned}
\langle F, GH\rangle &= G\,Y_F(H) + H\,Y_F(G) - GH\,RF\\
&= G\langle F, H\rangle + H\langle F, G\rangle + GH\,RF,
\end{aligned} \tag{64}$$

$$\begin{aligned}
\langle FG, H\rangle &= F\,Y_G(H) + G\,Y_F(H) - FG\,RH - HR(FG)\\
&= F\langle G, H\rangle + G\langle F, H\rangle - FG\,RH.
\end{aligned} \tag{65}$$

Equations (59)–(62), (63), and these multiplication rules prove by induction on products of generators that D_m is closed under the contact bracket. Thus (58) is a Lie subalgebra, identified with $(D_m, \langle -, -\rangle)$ by $F \mapsto Y_F$.

On $Q \times \mathbb{A}^1_s \times \mathbb{A}^{2m-2}$, the global form

$$\alpha = \eta + s\,db + \sum_{i=2}^{m} u_i\,dv_i$$

restricts to α_0 by (20). Its exterior derivative is

$$d\alpha = (ds - 6a\eta) \wedge db + \sum_{i=2}^{m} du_i \wedge dv_i.$$

Terms with a repeated db, dv_i, or η vanish in the top exterior power. The global coframe gives the nowhere-vanishing top form

$$\begin{aligned}
\alpha \wedge (d\alpha)^m &= m!\,\eta \wedge ds \wedge db \wedge \bigwedge_{i=2}^{m}(du_i \wedge dv_i)\\
&= -m!\,\omega \wedge ds \wedge \bigwedge_{i=2}^{m}(du_i \wedge dv_i).
\end{aligned}$$

Simplicity. The function

$$r = ah - bs = t - py \in D_m \quad \text{satisfies} \quad Rr = h^2 + 2ab - 6ab = 1. \tag{66}$$

Proposition 7.1. *The Lie algebra $(D_m, \langle -, - \rangle)$ is simple for every $m \geq 1$.*

Proof. Let $0 \neq J \lhd D_m$, choose $0 \neq F \in J$, and let $d = \deg_t F$ in the ambient polynomial ring. Since $\langle 1, F \rangle = RF = \partial_t F$, the element $(\operatorname{ad} 1)^d F \in J$ is nonzero and independent of t. On such elements,

$$\operatorname{ad} b = b\partial_t - \partial_p = -\partial_p, \qquad \operatorname{ad} u_i = \partial_{v_i}, \qquad \operatorname{ad} v_i = v_i \partial_t - \partial_{u_i} = -\partial_{u_i}.$$

Successively differentiate to the highest degree in each of p, u_i, v_i. Each step is a bracket with an element of D_m, preserves independence of t, and leaves a nonzero result. This gives a nonzero polynomial $P(b) \in J$.

Formula (7) applied to $r = t - py$ gives $Y_r(b) = -b$ and $Rr = 1$, hence

$$\langle r, P(b) \rangle = -bP'(b) - P(b). \tag{67}$$

Write $P(b) = \sum_{j \in S} P_j b^j$, where S is its finite set of nonzero coefficient indices, and choose $k \in S$. The eigenvalues of $\operatorname{ad} r$ on b^j are $-(j+1)$. Therefore the operator

$$\prod_{j \in S \setminus \{k\}} \frac{\operatorname{ad} r + (j+1)}{j - k}$$

sends P to $P_k b^k$, so $b^k \in J$.

For $k > 0$, $Y_s(b) = 1$ and $Rs = 6a$ give

$$U_k = \langle s, b^k \rangle = kb^{k-1} - 6ab^k \in J. \tag{68}$$

For $F \in B$, (60) gives $\langle a, F \rangle = aE(F) - hF$. Thus

$$\langle a, b^{k-1} \rangle = -hb^{k-1}, \qquad \langle a, ab^k \rangle = a(hb^k) - hab^k = 0,$$

and $\langle a, U_k \rangle = -khb^{k-1} \in J$. Consequently $hb^{k-1} \in J$, and

$$\begin{aligned} V_k := \langle a, hb^{k-1} \rangle &= 2ab^k - h^2 b^{k-1} \\ &= -b^{k-1} - 2ab^k \in J. \end{aligned} \tag{69}$$

The combination

$$\begin{aligned} U_k - 3V_k &= kb^{k-1} - 6ab^k + 3b^{k-1} + 6ab^k \\ &= (k+3)b^{k-1} \end{aligned} \tag{70}$$

lowers the exponent. Since $k + 3 \neq 0$, descent gives $1 \in J$.

For $F \in D_m$, choose $M \geq 0$ with $R^{M+1} F = 0$ and put

$$I(F) = \sum_{j=0}^{M} \frac{(-1)^j r^{j+1} R^j F}{(j+1)!} \in D_m. \tag{71}$$

Using $Rr = 1$ and the Leibniz rule gives

$$\begin{aligned} RI(F) &= \sum_{j=0}^{M} \frac{(-1)^j r^j R^j F}{j!} + \sum_{j=0}^{M} \frac{(-1)^j r^{j+1} R^{j+1} F}{(j+1)!} \\ &= F + \sum_{j=1}^{M} \frac{((-1)^j + (-1)^{j-1}) r^j R^j F}{j!} + \frac{(-1)^M r^{M+1} R^{M+1} F}{(M+1)!} = F. \end{aligned}$$

Hence $F = \langle 1, I(F) \rangle \in J$ and $J = D_m$. The bracket $\langle 1, r \rangle = 1$ proves nonabelianity. $\square$

Properness and density. The equality $s = p + g$ is triangular and monic in p. A nonzero coefficient of highest s-degree remains the nonzero coefficient of the same p-degree. Thus an element of D_m independent of p has s-degree zero. Comparing coefficients of the independent u_i, v_i then gives

$$D_m \cap \mathbb{C}[t, y] = B. \tag{72}$$

In particular $t \notin D_m$. Since $F \mapsto Y_F$ is injective, $Y_t \notin \mathfrak{k}_{2m+1}$, proving properness.

Set

$$p_N = s - 3t_N^2 - 2t_N^3 b \in D_m.$$

Equations (12) and (28) give $t_N - t \in \mathfrak{m}^{2N+1}$ and $p_N - p \in \mathfrak{m}^{2N+2}$. For any polynomial contact Hamiltonian F, substituting t_N, b, p_N, u_i, v_i for its coordinates gives $F_N \in D_m$ with $F_N - F \in \mathfrak{m}^{2N+1}$. Formula (7) then gives $Y_{F_N} - Y_F \in \mathfrak{m}^{2N} W_{2m+1}$, proving density in the natural filtration. The independent functions b^j, $j \geq 0$, and injectivity of the Hamiltonian correspondence prove infinite-dimensionality.

The four constructions prove Theorem 1.1.

Appendix A. Simplicity of the ambient algebras

The polynomial Cartan algebras in Theorem 1.1 are simple. For W_n, let I be a nonzero associative ideal of A_n stable under all derivations. Choose $P = \sum_\beta c_\beta x^\beta \in I$ and a nonzero coefficient c_α of maximal total degree. Then

$$\partial^\alpha P = \alpha! c_\alpha \neq 0.$$

Indeed, every other exponent β of degree at most $|\alpha|$ has $\beta_i < \alpha_i$ for some i, so its monomial is killed. Thus $I = A_n$, and Lemma 4.1 proves simplicity of W_n.

For S_n, $n \geq 2$, let I be a nonzero Lie ideal and choose $0 \neq D = \sum_{i,\beta} c_{i,\beta} x^\beta \partial_{x_i} \in I$. Choose α of maximal total degree among its nonzero coefficients. Brackets with the constant fields ∂_{x_i} give

$$\prod_{j=1}^{n} (\operatorname{ad} \partial_{x_j})^{\alpha_j} D = \alpha! \sum_i c_{i,\alpha} \partial_{x_i} \neq 0.$$

Hence I contains a nonzero constant field $v = \sum_i c_i \partial_{x_i}$. Choose i with $c_i \neq 0$. For every $j \neq i$, the field $x_i \partial_{x_j}$ belongs to S_n, and

$$[v, x_i \partial_{x_j}] = c_i \partial_{x_j}.$$

Using any such j also gives $[\partial_{x_j}, x_j \partial_{x_i}] = \partial_{x_i}$. Thus every constant field belongs to I.

For $i < j$, put

$$D_{ij}(F) = F_{x_j} \partial_{x_i} - F_{x_i} \partial_{x_j}, \qquad \theta_{ij} = \iota_{\partial_{x_j}} \iota_{\partial_{x_i}} \Omega_0.$$

The constant forms θ_{ij} form a basis of the polynomial $(n-2)$-forms. Their signs satisfy, for every k,

$$dx_k \wedge \theta_{ij} = \delta_{kj} \iota_{\partial_{x_i}} \Omega_0 - \delta_{ki} \iota_{\partial_{x_j}} \Omega_0.$$

Consequently,

$$\begin{aligned} d(F\theta_{ij}) &= \sum_k F_{x_k}\, dx_k \wedge \theta_{ij} \\ &= F_{x_j} \iota_{\partial_{x_i}} \Omega_0 - F_{x_i} \iota_{\partial_{x_j}} \Omega_0 = \iota_{D_{ij}(F)} \Omega_0. \end{aligned}$$

For any $D \in S_n$, apply Lemma 2.1 to $\iota_D \Omega_0$ and write its primitive as $\sum_{i<j} F_{ij} \theta_{ij}$. The displayed identity and injectivity of contraction give $D = \sum_{i<j} D_{ij}(F_{ij})$.

Fix an index r. If $F = \sum_{\ell \geq 0} F_\ell x_r^\ell$ with $\partial_{x_r} F_\ell = 0$, set $G = \sum_{\ell \geq 0} F_\ell x_r^{\ell+1}/(\ell+1)$. Then $\partial_{x_r} G = F$ and

$$[\partial_{x_r}, D_{ij}(G)] = D_{ij}(\partial_{x_r} G) = D_{ij}(F) \in I.$$

It follows that $I = S_n$. The bracket $[\partial_{x_1}, x_1 \partial_{x_2}] = \partial_{x_2}$ proves nonabelianity.

For $H_{2m} = A_{2m}/\mathbb{C}$, let I be a nonzero Lie ideal and choose a representative F of degree $d \geq 1$ of a nonzero class in I. Brackets with the linear Hamiltonians give all coordinate partial derivatives. Choose a monomial $c_\alpha x^\alpha$ of degree d with $c_\alpha \neq 0$, and choose j with $\alpha_j > 0$. Writing ε_j for the jth unit multi-index, the polynomial $\partial^{\alpha-\varepsilon_j} F$ has degree at most one, and its x_j-coefficient is $c_\alpha \alpha! \neq 0$. Thus I contains the class of a nonzero linear Hamiltonian ℓ. Its Hamiltonian derivation is a nonzero constant-coefficient derivative. The linear coordinates and polynomial antiderivative constructed in Proposition 5.4 show that this derivative is surjective on A_{2m}. Hence $I = A_{2m}/\mathbb{C}$. The bracket $\{q_1^2, p_1\} = 2q_1$ proves nonabelianity.

For the contact bracket on the full polynomial ring,

$$\langle 1, F\rangle = \partial_t F, \qquad \langle p_i, F\rangle = \partial_{q_i} F, \qquad \langle q_i, F\rangle = q_i \partial_t F - \partial_{p_i} F.$$

Given a nonzero element F of a Lie ideal, set $d = \deg_t F$. Then $(\operatorname{ad} 1)^d F$ is nonzero and independent of t. On such polynomials, $\operatorname{ad} p_i = \partial_{q_i}$ and $\operatorname{ad} q_i = -\partial_{p_i}$. Successively differentiating to the highest degree in each q_i, p_i gives a nonzero constant. After scalar multiplication, the ideal therefore contains 1. For $F = \sum_{j\geq 0} F_j t^j$, with the F_j independent of t,

$$\langle 1, \sum_{j\geq 0} \frac{F_j t^{j+1}}{j+1}\rangle = F.$$

The ideal is the whole algebra. Finally, $\langle 1, t\rangle = 1$ proves nonabelianity and hence simplicity of K_{2m+1}.

Appendix B. Keller maps and related examples

A Keller map is a polynomial map with constant nonzero Jacobian determinant. The map recorded in [5, Eq. (2)] is

$$\begin{aligned} F_1(x,y,z) &= (1+xy)^3 z + y^2(1+xy)(4+3xy),\\ F_2(x,y,z) &= y + 3x(1+xy)^2 z + 3xy^2(4+3xy),\\ F_3(x,y,z) &= 2x - 3x^2 y - x^3 z. \end{aligned} \tag{73}$$

Here and below J_F denotes the matrix of partial derivatives of the components of F. On $x \neq 0$, use the coordinates

$$\xi = 1 + xy, \qquad \zeta = 2 - 3xy - x^2 z.$$

The identities $xy = \xi - 1$ and $x^2 z = 5 - 3\xi - \zeta$ give

$$\begin{aligned} x^2 F_1 &= (5 - 3\xi - \zeta)\xi^3 + (\xi-1)^2\xi(3\xi+1) = \xi(\xi+1) - \xi^3\zeta,\\ xF_2 &= \xi - 1 + 3\xi^2(5 - 3\xi - \zeta) + 3(\xi-1)^2(3\xi+1) = 4\xi + 2 - 3\xi^2\zeta,\\ F_3 &= x\zeta. \end{aligned}$$

Put $P = \xi(\xi+1) - \xi^3\zeta$ and $Q = 4\xi + 2 - 3\xi^2\zeta$. The coordinate Jacobian is

$$\det \frac{\partial(x,\xi,\zeta)}{\partial(x,y,z)} = \det \begin{pmatrix} 1 & 0 & 0\\ y & x & 0\\ -3y - 2xz & -3x & -x^2 \end{pmatrix} = -x^3,$$

whereas

$$\det \frac{\partial(P/x^2, Q/x, x\zeta)}{\partial(x,\xi,\zeta)} = x^{-3} \det \begin{pmatrix} -2P & P_\xi & P_\zeta\\ -Q & Q_\xi & Q_\zeta\\ \zeta & 0 & 1 \end{pmatrix}.$$

Expanding the latter determinant gives $P_\xi(Q+\zeta Q_\zeta) - Q_\xi(2P+\zeta P_\zeta)$. Since

$$P_\xi = 2\xi + 1 - 3\xi^2\zeta, \quad P_\zeta = -\xi^3, \quad Q_\xi = 4 - 6\xi\zeta, \quad Q_\zeta = -3\xi^2,$$
$$Q + \zeta Q_\zeta = 2P_\xi, \qquad 2P + \zeta P_\zeta = 2\xi^2 + 2\xi - 3\xi^3\zeta,$$

this determinant is

$$\begin{aligned} &2(2\xi + 1 - 3\xi^2\zeta)^2 - (4 - 6\xi\zeta)(2\xi^2 + 2\xi - 3\xi^3\zeta) \\ &\quad = 8\xi^2 + 8\xi + 2 - 24\xi^3\zeta - 12\xi^2\zeta + 18\xi^4\zeta^2 \\ &\qquad - \left(8\xi^2 + 8\xi - 24\xi^3\zeta - 12\xi^2\zeta + 18\xi^4\zeta^2\right) = 2. \end{aligned}$$

The chain rule therefore gives $\det J_F = -2$ on $x \neq 0$, and hence everywhere, since the determinant is a polynomial.

Set $P_0 = (0, 0, -1/4)$ and $P_\varepsilon = (\varepsilon, -3\varepsilon/2, 13/2)$ for $\varepsilon \in \{1, -1\}$. Substitution gives $F(P_0) = (-1/4, 0, 0)$. At P_ε, one has $xy = -3/2$, and the three components are

$$\begin{aligned} F_1(P_\varepsilon) &= -13/16 + 9/16 = -1/4, \\ F_2(P_\varepsilon) &= \varepsilon(-3/2 + 39/8 - 27/8) = 0, \\ F_3(P_\varepsilon) &= \varepsilon(2 + 9/2 - 13/2) = 0. \end{aligned} \tag{74}$$

Thus $F(P_0) = F(P_1) = F(P_{-1})$, while x takes the values $0, 1, -1$.

Proposition B.1. *The map* (73)*, its products with identity maps, and its cotangent and contact lifts give proper dense simple subalgebras of*

$$W_n,\ S_n \quad (n \geq 3), \qquad H_{2m},\ K_{2m+1} \quad (m \geq 3).$$

Each subalgebra is isomorphic to its ambient algebra.

Proof. For $n \geq 3$, write $\mathbf{x} = (x, y, z, x_4, \ldots, x_n)$ and extend F by the components $x_4, \ldots, x_n$. Then $F(0) = 0$ and $\det J_F = -2$. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be independent target coordinates, and put $R_F = \mathbb{C}[F_1, \ldots, F_n] \subset \mathbb{C}[\mathbf{x}]$. The components are algebraically independent. Indeed, a nonzero relation $A(F) = 0$ of minimal degree would give

$$(A_{u_1}(F), \ldots, A_{u_n}(F))J_F = 0$$

by differentiation. Invertibility of J_F makes every entry zero. Some derivative of the nonconstant polynomial A is nonzero in characteristic zero, producing a relation of smaller degree.

Let $\mathbf{x}_j$ denote the jth coordinate in the displayed tuple. The inverse matrix J_F^{-1} is polynomial, by the adjugate formula. Define

$$\delta_i = \sum_j (J_F^{-1})_{ji}\partial_{\mathbf{x}_j}, \qquad \delta_i(F_j) = \delta_{ij}. \tag{75}$$

These fields form a basis of W_n over $\mathbb{C}[\mathbf{x}]$. The commutator $[\delta_i, \delta_j]$ annihilates every F_k; multiplication of its coefficient vector by J_F is therefore zero, so $[\delta_i, \delta_j] = 0$. For polynomials A_i, B_i in $\mathbf{u}$,

$$\left[\sum_i A_i(F)\delta_i, \sum_j B_j(F)\delta_j\right] = \sum_j \left(\sum_i (A_i\partial_{u_i}B_j - B_i\partial_{u_i}A_j)\right)(F)\delta_j.$$

Consequently the map

$$\Phi : \sum_i A_i(\mathbf{u})\partial_{u_i} \longmapsto \sum_i A_i(F)\delta_i$$

is an injective Lie homomorphism, with image $\bigoplus_i R_F\delta_i$. The points in (74), with all added coordinates zero, show $x \notin R_F$. Thus $x\delta_1$ is absent from the image, which is proper.

The substitution $\mathbf{u} = F(\mathbf{x})$ has a formal inverse at zero. Set $H_1(\mathbf{u}) = J_F(0)^{-1}\mathbf{u}$. If $F(H_{d-1}) - \mathbf{u}$ vanishes through degree $d-1$, let E_d be its homogeneous part of degree d and set

$$H_d = H_{d-1} - J_F(0)^{-1}E_d \qquad (d \geq 2).$$

The added term changes the linear part of F by $-E_d$ and its higher-degree parts only in degrees at least $d+1$. Thus $F(H_d) - \mathbf{u}$ vanishes through degree d. The limit H satisfies $F(H(\mathbf{u})) = \mathbf{u}$. If two formal tuples with zero constant term first differ in degree $d \geq 1$, their images under F first differ by $J_F(0)$ times that degree-d difference. It follows that $H(F(\mathbf{x})) = \mathbf{x}$. For a polynomial $A(\mathbf{x})$, truncate $A(H(\mathbf{u}))$ below degree N to a polynomial A_N. Then

$$A_N(F(\mathbf{x})) - A(\mathbf{x}) \in \mathfrak{m}^N.$$

Expressing any vector field in the basis (75) and applying this approximation to its coefficients proves density of $\Phi(W_n)$.

For the special series, put $\Omega_{\mathbf{x}} = d\mathbf{x}_1 \wedge \cdots \wedge d\mathbf{x}_n$ and $\Omega_{\mathbf{u}} = du_1 \wedge \cdots \wedge du_n$. Then $F^*\Omega_{\mathbf{u}} = -2\Omega_{\mathbf{x}}$. Applying $\mathcal{L}_{\delta_i}$ to the left side differentiates each dF_j to $d(\delta_i F_j) = 0$. Thus every δ_i has zero divergence, and (2) gives

$$\operatorname{div}_{\Omega_{\mathbf{x}}}\left(\sum_i A_i(F)\delta_i\right) = \left(\sum_i \partial_{u_i} A_i\right)(F).$$

Hence Φ restricts to an embedding of S_n into itself. The coefficient of δ_1 in ∂_z is $\partial_z F_1 = (1+xy)^3$. Its values at P_0 and P_1 are 1 and $-1/8$, so it does not lie in R_F. Therefore $\partial_z \notin \Phi(S_n)$.

For $D \in S_n$, its coefficients in formal $\mathbf{u}$ coordinates are $\widehat{A}_i(\mathbf{u}) = D(F_i)(H(\mathbf{u}))$. The preceding divergence formula, applied to formal series, gives $\sum_i \partial_{u_i}\widehat{A}_i = 0$. Writing $\widehat{A}_i = \sum_{d\geq 0} A_{i,d}$ in homogeneous parts gives $\sum_i \partial_{u_i} A_{i,d} = 0$ for each d, because the resulting degrees $d-1$ are distinct. Truncating all coefficients below degree N produces a polynomial field in S_n whose image differs from D by an element of $\mathfrak{m}^N W_n$. This proves density.

For the Hamiltonian series, use $n = m$ and introduce $\mathbf{p} = (p_1, \ldots, p_m)^T$. The polynomial map

$$G(\mathbf{x}, \mathbf{p}) = \left(F(\mathbf{x}), J_F(\mathbf{x})^{-T}\mathbf{p}\right) = (\mathbf{u}, \mathbf{v}) \tag{76}$$

has block-triangular Jacobian with determinant $\det J_F \det J_F^{-T} = 1$. For $\lambda_{\mathbf{x}} = \sum_i p_i\, d\mathbf{x}_i$ and $\lambda_{\mathbf{u}} = \sum_i v_i\, du_i$,

$$G^*\lambda_{\mathbf{u}} = (J_F^{-T}\mathbf{p})^T J_F\, d\mathbf{x} = \mathbf{p}^T\, d\mathbf{x} = \lambda_{\mathbf{x}}.$$

Taking $-d$ shows that G preserves the standard symplectic forms. For $\mathbb{J} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, with I_m the identity matrix, this is $J_G^T \mathbb{J} J_G = \mathbb{J}$, equivalently $J_G \mathbb{J} J_G^T = \mathbb{J}$. For $f, g \in \mathbb{C}[\mathbf{u}, \mathbf{v}]$, write ∇ for the column gradient. The chain rule gives

$$\{f \circ G, g \circ G\} = (\nabla f \circ G)^T J_G \mathbb{J} J_G^T (\nabla g \circ G) = \{f, g\} \circ G.$$

The same minimal-degree argument proves algebraic independence of the components of G. Composition therefore embeds the polynomial Poisson algebra, and modulo constants gives an embedding of H_{2m}. The collision points with $\mathbf{p} = 0$ have the same image under G, while x separates them. Hence $x - f \circ G$ cannot be constant, and X_x is absent from the image. Since $G(0) = 0$ and $J_G(0)$ is invertible, the formal construction above approximates every Hamiltonian function to any prescribed order. An error in $\mathfrak{m}^{N+1}$ gives a Hamiltonian-field error in $\mathfrak{m}^N W_{2m}$, proving density.

For the contact series, add a coordinate t and set $C(t, \mathbf{x}, \mathbf{p}) = (t, G(\mathbf{x}, \mathbf{p}))$. The forms $\alpha_{\mathbf{x}} = dt + \lambda_{\mathbf{x}}$ and $\alpha_{\mathbf{u}} = dt + \lambda_{\mathbf{u}}$ satisfy $C^*\alpha_{\mathbf{u}} = \alpha_{\mathbf{x}}$, and J_C^{-1} is polynomial. For target polynomials $f, g \in \mathbb{C}[t, \mathbf{u}, \mathbf{v}]$, let $Z = J_C^{-1}(Y_f \circ C)$. The chain rule gives

$$\alpha_{\mathbf{x}}(Z) = f \circ C, \qquad \mathcal{L}_Z \alpha_{\mathbf{x}} = C^*(\mathcal{L}_{Y_f}\alpha_{\mathbf{u}}) = (f_t \circ C)\alpha_{\mathbf{x}}.$$

The uniqueness established in (7) implies $Z = Y_{f\circ C}$. Since C fixes t and its other components are independent of t, one obtains

$$\langle f \circ C, g \circ C\rangle = Y_{f\circ C}(g \circ C) - (g \circ C)\partial_t(f \circ C) = \langle f, g\rangle \circ C.$$

The components of C are algebraically independent by the same minimal-degree argument. Thus composition gives an embedding of K_{2m+1}. At $t = 0$ and $\mathbf{p} = 0$, the collision points exclude x from the image on functions, so Y_x is absent from the image on fields. The formal inverse of C gives arbitrary jets of contact Hamiltonians; formula (7) transfers an error in $\mathfrak{m}^{N+1}$ to an error in $\mathfrak{m}^N W_{2m+1}$. All four images are simple by their isomorphisms with the ambient algebras, whose simplicity was proved in Appendix A. □

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