Runs Off The Floor:
Exponential fibre growth for a vincular stack-sorting map

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Abstract

Ever dealt the same cards two ways and got the same result? Let M_n be the maximum fibre size of the greedy stack-sorting map on permutations of $[n]$ whose stack, read from top to bottom, avoids the vincular pattern $1\underline{23}$. We construct large fibres by opening new slots in the stack while keeping the same output across different deals. Two slots already give $M_n \geq n2^{n-5}$ for $n \geq 5$, so $M_n > 2^{n-2}$ for every $n \geq 9$, contrary to Zhao's conjecture. Allowing the number of slots to vary gives

$$M_n \geq \frac{\lambda^{n-2}}{\lfloor (n+1)/2 \rfloor} \qquad (n \geq 2),$$

where $\lambda \approx 2.247$ is the unique root of $t^3 - 2t^2 - t + 1 = 0$ in $(2, 9/4)$. Thus $\liminf_{n\to\infty} M_n^{1/n} \geq \lambda > 2$. Value complementation gives the same maximum fibre sizes for the map avoiding $3\underline{21}$.

Keywords: stack-sorting, vincular pattern, fibre, permutation, lower bound
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1. The floor

Deal $9, 6, 7$, with 9 at the bottom. Here comes 5. The adjacent pair $6, 9$, read down the stack, blocks the push because $5 < 6 < 9$. Off goes 7, which was minding its own business. A blocked push sends the top card to the output, and 5 has to try again. The pair is still there. Off goes 6. In goes 5. Deal $6, 7, 5$ instead, with 9 still in your hand. This time 5 walks straight in. A floor can do more than hold the cards up.

Write $[n] = \{1, \ldots, n\}$ and let $\mathfrak{S}_n$ be the set of permutations of $[n]$. Throughout, $[a, b] = \{j \in \mathbb{Z} : a \leq j \leq b\}$, so this interval is empty when $a > b$. Juxtaposition denotes concatenation of entries and words. Right-greedy stacks with classical pattern restrictions appear in the work of Cerbai, Claesson and Ferrari [3]. We use the selective adjacency introduced by Babson and Steingrímsson [1], as described by Claesson [4]. Following Zhao [8], a stack contains $1\underline{23}$ if, when read from top to bottom, it has entries $a < b < c$ in that order, with b, c in adjacent stack positions. The entry a need not be adjacent to b.

The map $F_n = \mathrm{SC}_{1\underline{23}}$ processes its input from left to right. It pushes the next entry if the resulting stack avoids $1\underline{23}$. Otherwise, it pops the top entry to the output and retries the same input entry. After the input is exhausted, it empties the stack to the output. Adjacency is tested in the current stack. Set

$$M_n = \max_{\pi \in \mathfrak{S}_n} |F_n^{-1}(\pi)|.$$

In the introduction to [8], Zhao conjectures $M_n = 2^{n-2}$ and the same formula for the complementary pattern $3\underline{21}$.

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For consecutive patterns, where all three positions must be adjacent, Defant and Zheng [5] determined the maximum fibre size exactly for four patterns of length three, with the value 2^{n-2} for $\underline{213}$ and $\underline{231}$, and asked for nontrivial estimates in the case of $\underline{123}$. Here we fix one adjacent pair and ask how much freedom this leaves at the input. The movement-sequence argument of Berlow [2] gives the Catalan upper bound C_{n-1} for the maximum fibre size, as discussed for vincular stacks by Zhao [8, arXiv version, Section 4.1].

We start with the known single-floor construction. A small card clears the current block and stays on the floor. A high card follows it and opens the next slot. For a fixed number of slots, we can change where we cut the blocks and how many cards we deal into each slot while keeping the output fixed. Two slots already beat Zhao's count for every $n \geq 9$. Comparing the family sizes across different slot counts gives the exponential lower bound in Theorem 1.

Theorem 1. *For every integer $n \geq 5$,*

$$M_n \geq n2^{n-5}. \tag{1}$$

Let λ be the unique root in $(2, 9/4)$ of $t^3 - 2t^2 - t + 1$. For every integer $n \geq 2$,

$$M_n \geq \frac{\lambda^{n-2}}{\lfloor (n+1)/2 \rfloor}. \tag{2}$$

Consequently the conjectured value fails for every $n \geq 9$, and $\liminf_{n\to\infty} M_n^{1/n} \geq \lambda > 2$. The maximum fibre sizes for $\mathrm{SC}_{\underline{321}}$ are equal to M_n.

The failure at $n = 9$ was found earlier by computer in a public record [6], which reports 144 preimages for each of 765432819 and 765432918, checks 129 preimages of 765432819 in Lean, and lists, without proof, the fibre sizes $n2^{n-5}$ of $(n-2)(n-3)\cdots 2\,(n-1)\,1\,n$ for $5 \leq n \leq 10$.

Section 6 gives two further families in which some floor cards leave early. With fourteen cards, they bring the count for one output to at least 7424, above the largest standard construction count of 7168.

2. When a push clears a block

The test at 5 works on any legal stack.

Lemma 2. *If the current stack avoids $1\underline{23}$, pushing x is forbidden exactly when an adjacent pair b, c in the old stack, read from top to bottom, satisfies $x < b < c$.*

Proof. A new forbidden occurrence must use the newly pushed entry: the order and adjacency of the old entries are unchanged. Since x is above every old entry, it must be the first member of that occurrence. Thus a forbidden push requires an adjacent pair b, c with $x < b < c$. Conversely, any such pair forms a forbidden occurrence together with x. Popping the top leaves a suffix of the old stack, so it cannot introduce a forbidden occurrence; the criterion remains applicable after each pop. □

For the rest of the construction we write stacks from bottom to top. The blocking pair b, c in Lemma 2 then appears as c, b.

Suppose an increasing block with minimum b has been pushed onto a floor topped by a high card $H > b$. The pair H, b stays adjacent until b itself is popped. An incoming card $x < b$ therefore keeps asking for another pop until the whole block is off. Once the floor is exposed, x can go in if none of the floor's descending adjacent pairs has a smaller endpoint greater than x.

Figure 1 follows one deal from the first attempt to push 5 to the final output.

Require all three pattern positions to be consecutive and 5 goes straight in above 7. Remove the adjacency requirement and, later in the deal, 4 can use 8 and 9 across 1 to pop 8. Under our rule 5 waits and 8 stays.

Work backwards from 765. Cut it as 7 | 65 or 76 | 5, reverse the pieces and deal them above 9. Let 1 come next. Either cut leaves 765 on the screen.

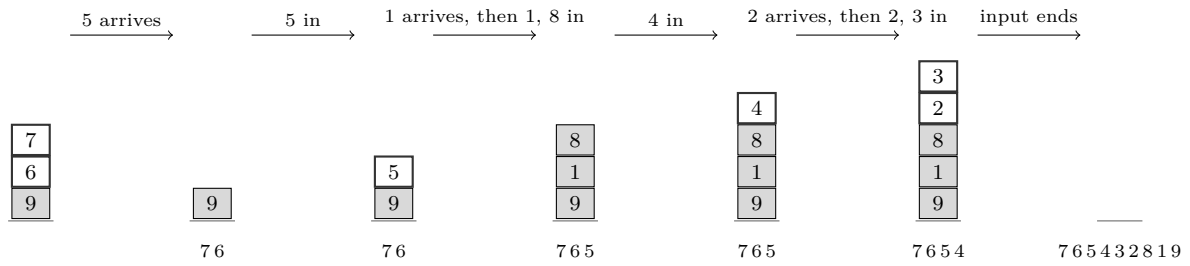

Figure 1: The run of F_9 on input 9 6 7 5 1 8 4 2 3, with the output so far beneath each stack. Shaded cards form the floor and stay until the final emptying. A card smaller than the current block clears it before going in. When the input ends, the remaining stack empties to the output. The TikZ code for this illustration was generated using Claude (Anthropic, Opus 5.5).

For a consecutive interval I of q integers, let $\mathcal{L}(I)$ consist of the following words. Partition I into nonempty consecutive intervals, called the blocks, put the blocks in decreasing order of their values, and write the entries of each block in increasing order. For $I = \varnothing$, there is one empty word. If $a_q = |\mathcal{L}(I)|$, then

$$a_0 = 1, \qquad a_q = 2^{q-1} \ (q \geq 1), \qquad A(x) := \sum_{q \geq 0} a_q x^q = \frac{1-x}{1-2x}. \tag{3}$$

Indeed, there are $q-1$ possible cuts for $q \geq 1$, and the maximal increasing runs of the resulting word recover the blocks. On standardising the values, these are the reverse-layered permutations, the class $\mathrm{Av}(132, 213)$ counted by Simion and Schmidt [7] and described in terms of runs by Defant and Zheng [5, Section 4].

3. From one floor to many slots

Begin with n as the only floor card. It stays at the bottom until the final emptying, since a stack holding one card accepts every push. Feed it a word from $\mathcal{L}([1, n-1])$. Each block leaves in decreasing order and the blocks themselves come in decreasing order of value. The output is $(n-1)(n-2)\cdots 1\, n$. This is the single-floor case of the preimage problem studied by Zhao [8, arXiv version, Lemma 4.15]. For $n \geq 2$, there are exactly 2^{n-2} preimages of this output. Indeed, n must enter first to leave last. Above n, pushing a smaller card onto a larger one would put the two out of order in the decreasing output. Thus the middle cards accumulate in increasing blocks, and the blocking-pair test clears a whole block whenever a push is forbidden. The decreasing output forces these blocks to be consecutive intervals in decreasing order of value.

The floor is already doing useful work, can we make it do more? Could we clear one lot of cards and still have somewhere to put the next? Reserve 1 and $n-1$ and deal them in that order between two slots. The 1 clears what is left of the first slot and settles on n. Leave 1 on top and it is too small to make the blocking pair under the next block. Put $n-1$ above it and there is a high card under that block again. The new floor has only one descending adjacent pair, $n, 1$, read from bottom to top. Its smaller endpoint is 1, so none of the remaining middle cards is blocked by the floor. This is the move we will repeat.

With nine cards, try 67 | 5 above 9, then 567. Two increasing runs in that slot, then one. In the first deal, 5 sends 7 and 6 off the floor. In the second, 5 goes in first and leaves the clearing to 1. Just after 1 goes in, both stacks are $9, 1$, both output prefixes are 765, and both have $8, 4, 2, 3$ to come. We can forget where the first slot was cut. Every remaining move is the same.

Could we put 8 in before 1 and get away with it? When 1 arrives, it sends 8 out along with the cards still in the first slot. The stacks still meet at $9, 1$, but the output prefixes are now 7685 and 8765. With 8 gone, 2 can go in without popping 4. The rest of the deal cannot undo the difference between the prefixes. Table 1 shows all four runs.

One more run. Move 4 from the second slot to the first, giving 9 6 7 | 5 | 4 1 8 2 3. Now 1 sends 4 out where 2 did before, and the final output is still 765432819. Can we deal the blocks in any order, then? Keep the

Input	Stack just after 1 enters	Output so far	Final output
967518423	9, 1	765	765432819
956718423	9, 1	765	765432819
967581423	9, 1	7685	768532419
956781423	9, 1	8765	876532419

Table 1: Four runs, with the positions of 1 and 8 exchanged in the last two. Stacks are written bottom to top.

last five cards $1, 8, 4, 2, 3$ as in the original runs. Deal the three one-card blocks above 9 as $7 \mid 6 \mid 5$, then as $6 \mid 5 \mid 7$. The first output starts 765 and the second starts 675. The order of the blocks matters. We can leave a slot empty, too. The floor is already there. Deal $1, 8$ right after 9 and the first slot closes without a pop. There is one way to leave it empty, so $a_0 = 1$.

Moving 4 changed the slot lengths from $3 + 3$ to $4 + 2$. Either split still leaves cuts to choose. Fix $k \geq 1$ and $n \geq 2k - 1$. Set $m = n - 2k + 1$. Split the middle interval $[k, n - k]$ into k consecutive intervals $I_1, \ldots, I_k$, the slots, in decreasing order of values, with arbitrary sizes $m_1, \ldots, m_k \geq 0$ summing to m. A slot is empty if its size is zero. Choose $w_i \in \mathcal{L}(I_i)$ and form the input

$$\tau = n\, w_1\, 1\, (n-1)\, w_2\, 2\, (n-2)\, w_3 \cdots (k-1)\, (n-k+1)\, w_k. \tag{4}$$

Thus the pair $i, (n - i)$ lies between w_i and w_{i+1} for $1 \leq i < k$; for $k = 1$ the input is simply $n\, w_1$. The low separators $1, \ldots, k - 1$, middle entries $k, \ldots, n - k$, and high entries $n - k + 1, \ldots, n$ are disjoint and exhaust $[n]$.

Write $\mathrm{rev}(w)$ for the reversal of a word w. For $1 \leq i \leq k$, let B_i be the base word

$$B_i = n, 1, n-1, 2, n-2, \ldots, i-1, n-i+1. \tag{5}$$

Here $B_1 = (n)$; the word is obtained by starting with n and appending $j, n - j$ for $1 \leq j < i$. Define

$$\pi_{n,k} = (n-k)(n-k-1)\cdots k \ \mathrm{rev}(B_k),$$

where the decreasing prefix is empty if $m = 0$.

Lemma 3. *Every input in* (4) *has output* $\pi_{n,k}$. *For fixed* n, k, *distinct choices of slot lengths and block cuts give distinct inputs, and therefore*

$$|F_n^{-1}(\pi_{n,k})| \geq L_{n,k} := [x^{n-2k+1}]\, A(x)^k. \tag{6}$$

Proof. We prove by induction on i that the stack is B_i immediately when slot i is reached. This holds for $i = 1$, after the initial n is pushed. The descending adjacent pairs in B_i have smaller endpoints exactly $1, \ldots, i - 1$. Every middle entry is at least $k > i - 1$, so it can be pushed onto B_i. The top entry of B_i is $H = n - i + 1$, which exceeds every middle entry.

Consider an increasing block $b, b+1, \ldots, c$ of w_i. The first entry b is pushed onto B_i. Among the adjacent pairs created while this block is pushed, the only descending one is H, b. Every later entry of the block exceeds b and is therefore pushed without a pop. If the next block starts with x, then $x < b$ because the blocks are arranged in decreasing value order. While any of the preceding block remains, its minimum b is still directly above H. The pair H, b therefore forces another pop. The whole block is popped in decreasing order. Once b is popped, only B_i remains. Each descending adjacent pair in B_i has its smaller endpoint at most $i - 1 < x$. Popping stops with H still at the top, and x is then pushed. The same argument applies to each later block of w_i.

If $i < k$, the next separator i similarly pops any remaining middle block, since its entries are at least $k > i$. The separator i is then pushed onto B_i, since every descending adjacent pair in B_i has its smaller endpoint less than i. If w_i is empty, no pop is needed. The next high entry $n - i$ exceeds every smaller endpoint of a descending adjacent pair, now among $1, \ldots, i$, because $n - i \geq k > i$. It is pushed legally, establishing B_{i+1}. No entry of the base has been popped.

At input exhaustion, the last middle block, if present, is popped in decreasing order, followed by $\mathrm{rev}(B_k)$. All middle blocks, both within and between slots, are in decreasing value order. Their combined output is therefore exactly $(n-k)(n-k-1)\cdots k$. This proves the output formula. It also covers empty slots, $k=1$, and $m=0$.

The fixed separator and high-entry values delimit the words w_i in the input, including empty words. The input thus recovers their lengths m_i and their middle intervals. Within each w_i, its maximal increasing runs recover all the chosen cuts. Hence the construction is injective. For given lengths there are $\prod_i a_{m_i}$ choices. Summing over weak compositions of m and using (3) gives (6). □

Corollary 4. *For $n \geq 5$, the fibre of $(n-2)(n-3)\cdots 2\,(n-1)\,1\,n$ contains at least $n2^{n-5}$ inputs.*

Proof. Take $k=2$, so $m=n-3\geq 2$. In the sum $L_{n,2}=\sum_{j=0}^{m} a_j a_{m-j}$, the two endpoint terms are each 2^{m-1}, and the $m-1$ interior terms are each 2^{m-2}. Thus

$$L_{n,2} = 2\cdot 2^{m-1} + (m-1)2^{m-2} = (m+3)2^{m-2} = n2^{n-5}.$$

□

With nine cards and two nonempty slots there are five ways to split the six middle cards by length, and each split gives 16 inputs. Allowing a slot to be empty adds 32 at either end, for a total of 144 inputs giving 765432819. For general n, the ratio of the two-slot bound to 2^{n-2} is $n/8$, so the construction exceeds Zhao's conjectured value at every $n \geq 9$.

4. How many floors?

With nine cards the construction counts for $k=1,\ldots,5$ are $128, 144, 66, 14, 1$. Each extra slot costs two floor cards. Giving 1 and 8 to the floor leaves fewer cards in the slots and still gives more deals. At $k=5$ the floor has eaten the whole deck. There is one deal. A fixed choice of k gives $L_{n,k}=\Theta_k(n^{k-1}2^n)$ as $n\to\infty$. Indeed, $A(x)=\frac{1}{2}(1+(1-2x)^{-1})$, so the highest pole order of $A(x)^k$ is k with positive leading coefficient. To beat base 2, we let the number of slots vary with the size of the deck.

For $n\geq 1$, sum the construction counts over all permissible k:

$$s_n = \sum_{k=1}^{\lfloor (n+1)/2\rfloor} L_{n,k}.$$

The auxiliary case $n=1$ is the one-letter input, so $s_1=1$. For $n\geq 2$, Lemma 3 gives $L_{n,k}\leq |F_n^{-1}(\pi_{n,k})|\leq M_n$ for each permissible k. There are $\lfloor (n+1)/2\rfloor$ such summands, giving

$$M_n \geq \frac{s_n}{\lfloor (n+1)/2\rfloor}. \tag{7}$$

At least one choice of k has a count at least as large as the average. When the middle prefix is nonempty, its first card $n-k$ tells us the slot count. The only coincidence is for $n=2r-1$, $r\geq 2$. With $r-1$ slots, the middle cards come out as $r, r-1$. With r slots, these same two cards come off first in the final emptying. The rest of the output agrees.

We start with one floor card and one slot, giving $xA(x)$. Each time we raise the floor for another slot, two cards go onto the floor and one more slot opens. This multiplies the term by $x^2A(x)$. Only finitely many k can contribute to any fixed coefficient. Hence (6) yields

$$\sum_{n\geq 1} s_n x^n = \sum_{k\geq 1} x^{2k-1}A(x)^k = \frac{xA(x)}{1-x^2A(x)} = \frac{x(1-x)}{1-2x-x^2+x^3}. \tag{8}$$

Comparing coefficients gives

$$s_1=1,\quad s_2=1,\quad s_3=3,\quad s_4=6,\qquad s_n = 2s_{n-1}+s_{n-2}-s_{n-3}\quad (n\geq 4). \tag{9}$$

The negative coefficient can be removed by keeping differences. Put $d_n = s_n - s_{n-1}$ for $n \geq 2$, and set

$$v_n = (s_n, d_n, d_{n-1})^{\mathsf{T}} \quad (n \geq 3), \qquad T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Equation (9) gives

$$v_{n+1} = T v_n \quad (n \geq 3), \qquad v_4 = (6, 3, 2)^{\mathsf{T}}. \tag{10}$$

Its first two coordinates follow from $s_{n+1} = 2s_n + d_{n-1}$ and $d_{n+1} = s_n + d_{n-1}$, and the third is the shift d_n.

Let $f(t) = t^3 - 2t^2 - t + 1$. Since $f(2) = -1$, $f(9/4) = 1/64$, and $f'(t) > 0$ for $t \geq 2$, there is a unique root $\lambda \in (2, 9/4)$. The defining equation implies

$$\lambda^2 - 2\lambda = 1 - \lambda^{-1}, \qquad \lambda^2(\lambda - 2) = \lambda - 1.$$

The positive vector

$$u = (1, 1 - \lambda^{-1}, \lambda - 2)^{\mathsf{T}}$$

satisfies $Tu = \lambda u$. All three coordinates of v_4 dominate those of $\lambda^2 u$:

$$\lambda^2 < \frac{81}{16} < 6, \qquad \lambda^2 - \lambda < \frac{45}{16} < 3, \qquad \lambda^2(\lambda - 2) = \lambda - 1 < \frac{5}{4} < 2.$$

Since T has nonnegative entries, so does T^{n-4} for $n \geq 4$. Applying this matrix to the componentwise inequality $v_4 \geq \lambda^2 u$, and using (10) and $Tu = \lambda u$, gives

$$v_n = T^{n-4} v_4 \geq \lambda^2 T^{n-4} u = \lambda^{n-2} u \qquad (n \geq 4).$$

In particular $s_n \geq \lambda^{n-2}$. The remaining cases are $s_2 = 1 = \lambda^0$ and $s_3 = 3 > \lambda$. Combining this with (7) proves (2) for every $n \geq 2$.

Since $\lfloor (n+1)/2 \rfloor \leq n$, the result also gives $M_n \geq \lambda^{-2} \lambda^n / n$. Taking nth roots and using $(\lambda^2 n)^{1/n} \to 1$ proves the asserted lower limit. Thus $M_n = \Omega(\lambda^n / n)$.

5. Turn the values over

Defant and Zheng [5, Lemma 1.2] give the complement identity for consecutive patterns. Zhao [8, arXiv version, Lemma 2.1] records it for vincular patterns. Here is how it acts on our two maps.

What happens if we turn the values over and keep the $1\underline{23}$ rule? On the stack $9, 6, 7$, written bottom to top, 5 is blocked. The change $x \mapsto 10 - x$ sends the stack to $1, 4, 3$ and leaves 5 itself fixed. Now 5 can go straight in. Turn the pattern over too, to $3\underline{21}$. The pair $4, 1$ keeps 5 out again, and off come 3 and then 4, just as 7 and 6 did before.

Let comp_n replace each entry x by $n + 1 - x$, and let $G_n = \mathrm{SC}_{3\underline{21}}$. Complementation reverses inequalities while preserving all positions and adjacencies. A proposed push in an F_n stack is therefore legal precisely when the corresponding push in the complementary G_n stack is legal. Induction over pushes, forced pops, retries and final emptying gives

$$G_n(\mathrm{comp}_n(\tau)) = \mathrm{comp}_n(F_n(\tau)), \qquad G_n = \mathrm{comp}_n \circ F_n \circ \mathrm{comp}_n .$$

Since comp_n^2 is the identity, for every $\tau, \pi \in \mathfrak{S}_n$ we have

$$F_n(\tau) = \pi \quad \Longleftrightarrow \quad G_n(\mathrm{comp}_n(\tau)) = \mathrm{comp}_n(\pi).$$

Thus comp_n is a bijection between these two fibres. It also permutes the possible outputs, so the maximum fibre sizes are equal, completing the proof of Theorem 1.

6. What did we save?

For each $2 \le n \le 12$, a C++ program enumerated all $n!$ input permutations, applied the stack rule, and counted the occurrences of each output. I used OpenAI Codex to develop and run the enumeration. A separate C implementation reproduced the maximum fibre sizes. Table 2 compares the resulting M_n with $\max_k L_{n,k}$. In the computed range $9 \le n \le 12$, the two maximizing outputs are $\pi_{n,2}$ and $(n-2)(n-3)\cdots 2\,n\,1\,(n-1)$. At $n = 9$ these are 765432819 and 765432918, with 144 preimages each. Try swapping n and $n-1$ in the two-slot construction. A forbidden triple could change only if it contained both high cards as its adjacent pair, but 1 keeps them apart on the stack until final emptying. The swap gives a second family of $n2^{n-5}$ deals for $n \ge 5$, with the last three output cards now $n, 1, n-1$. Zhao [8, arXiv version, Lemma 4.16] uses the same observation about swapping consecutive values.

n	2	3	4	5	6	7	8	9	10	11	12
M_n	1	2	4	8	16	32	64	144	320	704	1536
$\max_k L_{n,k}$	1	2	4	8	16	32	64	144	320	704	1536

Table 2: Maximum fibre sizes of F_n by exhaustive computation, and the largest construction count.

The floor does not always have to stay until the very end. Can some cards come off the floor early and still leave the same output? For any $n \ge 5$, split the middle cards $[3, n-3]$ into as many consecutive intervals as there are words in a row, in decreasing order of values. Empty intervals are allowed. Each of u, v, w is chosen from $\mathcal{L}$ for its interval.

Deal	Count at $n = 14$
$n\,u\,1\,(n-1)\,v\,2\,(n-2)\,w$	5632
$n\,u\,2\,(n-2)\,v\,1\,(n-1)$	1536
$n\,w\,1\,(n-1)\,(n-2)\,2$	256

The first row is the usual three-slot deal. In the second, 2 clears what remains of u, then $n-2$ goes in. The fourteen-card example has u empty and $v = 3\,4\cdots 11$. When 1 arrives, it clears what remains of v. The pair $2, n$, read down the stack, still keeps 1 out, so $n-2$ and 2 go too. While 1 waits to get in, a little of the floor goes out. Then $1, n-1$ go in, and the remaining floor comes out at the end.

Try the last row. The 1 clears whatever is left of w. Both high cards go in without popping a card, since the only ascending adjacent pair read down the old stack before $n-2$ enters is $1, n$. Now 2 sees the pair $n-2, n-1$ and sends $n-2$ out. Once that pair is gone, 2 goes in. Final emptying gives $2, n-1, 1, n$. With either of these new deals the middle cards have come out in decreasing order before $n-2$, so the whole output is $\pi_{n,3}$.

The orders of the five floor cards distinguish the three rows. The positions of those cards and the increasing runs still recover the slot lengths and cuts. Thus

$$|F_n^{-1}(\pi_{n,3})| \ge [x^{n-5}]\big(A(x)^3 + A(x)^2 + A(x)\big) \qquad (n \ge 5).$$

For $n \ge 7$, the last two terms give $L_{n-2,2} + a_{n-5} = n2^{n-7}$ inputs in addition to the standard family.

With fourteen cards, the three constructions alone give

$$M_{14} \ge 5632 + 1536 + 256 = 7424 > 7168 = \max_k L_{14,k}.$$

A depth-first search for the fixed target $\pi_{14,3}$ returned exactly these 7424 inputs. The C program tries each unused card next and keeps a branch only while the emitted cards match the target prefix and the stack, read top to bottom, is a subsequence of the remaining target. Later pushes cannot change the relative pop order of cards already on the stack. The search checks the final emptying. A separate Python implementation checked every accepted input directly by the avoidance rule.

The generating function (8) has a simple dominant pole at $1/\lambda$, since the other characteristic roots lie in $(-1, 0)$ and $(0, 1)$. Thus s_n is asymptotic to a positive constant times λ^n. The inequalities $s_n/\lfloor (n+1)/2 \rfloor \le$

$\max_k L_{n,k} \le s_n$ then give $(\max_k L_{n,k})^{1/n} \to \lambda$. Does $M_n^{1/n}$ still tend to the same λ once we count deals like this one?

Could we tell from the final screen alone whether 7 and 6 came off when 5 arrived? Both of the first two runs in Table 1 left 765432819. Save one more thing and we can find out. Write `P` for each successful push and `X` for each pop, including the final emptying. The two runs have action words

967518423 `PPPXXPXPPPXPPXXXXX`,
956718423 `PPPPXXXPPPXPPXXXXX`.

The action word alone is also insufficient. For instance, 123 and 213 both have action word `PPPXXX`, with outputs 321 and 312 respectively.

Put the replay beside the output. Number the pushes in order and read the action word from the start. At each `P`, push the next position number. At each `X`, pop that number and fill the corresponding input position with the next card from the output. Every position is filled exactly once. Thus the pair consisting of the final output and the complete action word determines the input. This is the recovery step in Berlow's bound [2, Theorem 3.2]. The first card stays to the end, so deleting the initial `P` and final `X` gives a Dyck word of semilength $n-1$. There are C_{n-1} such words, yielding the stated upper bound for a fixed final output.

Keep the replay beside the screen and we can lay out the old deal, card for card. Or leave just the nine cards on the screen and put 5 down first above 9 next time. The screen won't tell on us.

Code and data availability

The enumeration programs, the fixed-target search and independent checker, the list of 7424 preimages of $\pi_{14,3}$, and reproduction instructions are archived at Zenodo, `https://doi.org/10.5281/zenodo.24584942`.

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I declare that I have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

I set the research aim, developed the concept, restated the whole problem in my own words and rebuilt the narrative. Codex (GPT-6 Astra, OpenAI) took part in the constructions, proofs, computations and drafts. Claude (Opus 5.5, Anthropic) took part in literature searches, reference checks, prose editing and the TikZ code for Figure 1. I reviewed and edited all of it and take sole responsibility for the paper.

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