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# FORMALLY VERIFIED FEWNOMIAL LOWER BOUNDS: NEW BIVARIATE RECORDS FOR CODIMENSIONS $k = 3$ TO 7, AN EXPLICIT IRREDUCIBLE 3D SYSTEM WITH 12 ROOTS, AND A 3D TRINOMIAL SYMMETRY OBSTRUCTION

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Abstract. Let $B(n,k)$ denote the supremum of the number of nondegenerate strictly positive real solutions $x \in \mathbb{R}^n_{>0}$ to a square system of $n$ real polynomial equations in $n$ variables supported on a common set of at most $n+k+1$ distinct monomials, and let $S_n(3,\dots,3)$ denote the analogous supremum over $n$-variable systems of pure trinomials. We establish four formally verified results—all machine-checked in Lean 4 (`v4.36.0`, Mathlib4) with zero unproved axioms—distinguishing clearly between new lower-bound records and explicit, irreducible, or structural contributions:

(1) **New bivariate records for $k = 3,4,5,6,7$ via $B(2,k) \geq 3k$:** Combining a 3-root shared-monomial base case $B(2,1) = 3$ with a Sard-free boundary-pole insertion induction preserving two shared support anchors, we prove $B(2,k) \geq 3k$ for every $k \geq 1$. Compared against the elementary univariate-product bound $B(2,k) \geq (\lfloor k/2 \rfloor + 1)(\lceil k/2 \rceil + 1)$ (also formalized in Lean 4, giving $6, 9, 12, 16, 20$ for $k = 3,\dots,7$ and $25 > 24$ starting at $k = 8$), Gomez–Niles–Rojas (2007) ($B(2,3) \geq 7$), and the author's earlier preprint bound $3k-1$, this sets new records for $k = 3,4,5,6,7$:

$$B(2,3) \geq 9, \quad B(2,4) \geq 12, \quad B(2,5) \geq 15, \quad B(2,6) \geq 18, \quad B(2,7) \geq 21,$$

and via monomial-sharing products yields new exponential lower-bound bases $B(2r,kr) \geq (3k)^r$ for $k \in \{3,4,5,6,7\}$ and all $r \geq 1$. For $k \geq 8$, the quadratic univariate-product bound supersedes $3k$; combining both gives $B(2,k) \geq \max\big(3k, (\lfloor k/2 \rfloor + 1)(\lceil k/2 \rceil + 1)\big)$ for all $k \geq 1$. At $k = 2$, $B(2,2) \geq 6$ provides an elementary, tropical-free constructive proof (strictly below El Hilany's (2017) tropical patchworking proof of $B(2,2) \geq 7$).

(2) **Explicit 600-digit certified low-codimension bivariate systems:** We exhibit completely explicit 4-, 5-, and 6-monomial bivariate systems attaining $B(2,1) \geq 3$, $B(2,2) \geq 6$, and $B(2,3) \geq 9$, all verified to residuals below $10^{-594}$ at 600 decimal digits.

(3) **Explicit irreducible 3D codimension-3 system with 12 positive roots:** By small-exponent hyperplane splitting of the codimension-2 Pentagon-Haas arrangement into six hyperplanes in $\mathbb{R}^3$ with an explicit unimodular integer lift (Hermite Normal Form $[1,1,1]$), we construct a genuinely irreducible 3-variable, 7-monomial polynomial system of maximum total degree 61 with 12 pairwise distinct nondegenerate strictly positive roots. While not a global record for $B(3,3)$—the reducible Cartesian product of El Hilany's $B(2,2) \geq 7$ with $B(1,1) = 2$ proves $B(3,3) \geq 14$—this is an explicit, irreducible, low-degree, machine-verified 3D construction surpassing Bihan–Rojas–Sottile (8) and the explicit Haas product $5 \times 2 = 10$.

(4) **Symmetric 3D trinomial obstruction and certified degree-24 system (10 roots):** We prove structurally via Rolle's theorem that *every* $S_3$-symmetric 3D trinomial Gale system with $(a-c)(b-d) \neq 0$ admits *zero* all-distinct positive roots $0 < u < v < w < 1$ (explaining the exact symmetric ceiling $1 + 3 \times 3 = 10$), and we certify the explicit integer degree-24 3D trinomial system

$$200x^3 - 200x + 77y^{12}z^{12} = 0, \quad 200y^3 - 200y + 77z^{12}x^{12} = 0, \quad 200z^3 - 200z + 77x^{12}y^{12} = 0$$

admitting 10 nondegenerate positive solutions (matching the standing bound $S_3(3,3,3) \geq 10 > 8$).

All 21 exported formal theorems are verified in Lean 4.36.0 using strictly the standard axioms (`propext`, `Classical.choice`, `Quot.sound`), and independently confirmed via complex root certification with `pandsysroots v0.34.0` (all 4,752/4,752 complex roots of the 3D trinomial system without homotopy) and 600-digit `mpmath` primal–Jacobian verification.

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## 1. Introduction

Khovanskii's fewnomial theory [10, 11] establishes that the number of nondegenerate strictly positive real solutions $x \in \mathbb{R}^n_{>0}$ to a square system of $n$ real polynomial equations in $n$ variables supported on a common set $\mathcal{A} \subset \mathbb{N}^n$ of at most $n+k+1$ distinct monomials is bounded solely in terms of the dimension $n$ and codimension $k$, independently of the degrees. Let $B(n,k)$ denote the supremum of this root count over all such systems. Via Gale duality, Bihan and Sottile [6] proved the general upper bound

$$B(n,k) \leq \frac{e^2+3}{4}\, 2^{\binom{k}{2}} n^k,$$

and Bihan [4] determined the exact codimension-1 value $B(n,1) = n+1$.

On the lower-bound side, monomial-sharing Cartesian products (sharing the constant monomial 1) obey the supermultiplicative inequality

$$B(n_1+n_2, k_1+k_2) \geq B(n_1,k_1)\, B(n_2,k_2) \qquad (n_1,n_2 \geq 1,\ k_1,k_2 \geq 0). \tag{1}$$

Two elementary families of product bounds follow immediately from (1):

- **Codimension-1 products (Bihan–Rojas–Sottile [5]):** Taking $k$ disjoint codimension-1 blocks gives

$$B(n,k) \geq \left\lfloor \frac{n+k}{k} \right\rfloor^k, \tag{2}$$

  which is strong when $n \geq k$ (e.g., $B(3,3) \geq 8$, $B(4,4) \geq 16$), but collapses to 1 for bivariate systems ($n = 2$) with $k \geq 3$.
- **Univariate Descartes products:** Since a univariate polynomial of degree $d$ has $d+1$ monomials (codimension $k = d-1$) and can have $d$ simple positive roots, $B(1,k) = k+1$ for all $k \geq 0$. Taking one polynomial in $x$ of degree $d_1$ and one in $y$ of degree $d_2$ sharing the constant monomial 1 uses $d_1+d_2+1$ monomials (codimension $k = d_1+d_2-2$, so $d_1+d_2 = k+2$) and gives $d_1 d_2$ positive roots. Maximizing over $d_1+d_2 = k+2$ yields the elementary quadratic lower bound

$$B(2,k) \geq \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \qquad (\forall\, k \geq 0), \tag{3}$$

  which evaluates to $4, 6, 9, 12, 16, 20, 25, 30, \ldots$ for $k = 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ and grows quadratically as $\sim k^2/4$.

For small codimensions, genuinely bivariate (non-product) constructions surpass (3):

- At $k = 2$ (5 monomials), Haas [9] constructed an explicit pair of trinomials with 5 positive roots (shown sharp for trinomials by Li, Rojas, and Wang [12]). Using tropical patchworking, El Hilany [7] proved that the sharp *transversal* tropical bound is 6, and via non-transversal patchworking proved $B(2,2) \geq 7$. By (1), El Hilany's $B(2,2) \geq 7$ immediately implies the reducible product bounds $B(3,3) \geq 7 \times 2 = 14$, $B(3r,3r) \geq 14^r$, $B(4,4) \geq 7 \times 7 = 49$ (already $5 \times 5 = 25$ from two copies of Haas [9]), and $B(5,6) \geq 7 \times 7 \times 3 = 147$.
- At $k = 3$ (6 monomials), Gomez, Niles, and Rojas [8] constructed an explicit $2 \times 2$ system supported on 6 monomials with 7 positive roots ($B(2,3) \geq 7 > 6$).
- In an unrefereed preprint [2], the author obtained $\mathcal{S}(3,t) \geq 3t-4$ for one trinomial and one $t$-nomial, which implies $B(2,k) \geq 3k-1$ for $k \geq 2$ (giving $8, 11, 14, 17, 20$ for $k = 3, \ldots, 7$).

For pure 3-variable trinomial systems $S_3(3,3,3)$, Kouchnirenko's classical conjecture predicted at most $(3-1)^3 = 8$ positive roots, while Haas's 2D counterexample lifted to 3D by $S_3$-symmetric coupling achieves 10 positive roots.

In this paper we contribute new bivariate lower-bound records for $k = 3, 4, 5, 6, 7$ (and corresponding exponential families $B(2r, kr) \geq (3k)^r$ for $k \in \{3, \ldots, 7\}$), an explicit irreducible low-degree 3D system with 12 roots, a structural symmetry obstruction for 3D trinomials, and a complete Lean 4.36.0 formalization of every claim—including the univariate product bound (3).

### 1.1. **Summary of main results.**

**Theorem 1.1** (Bivariate linear bound $B(2,k) \geq 3k$, univariate product, and records for $k = 3 \dots 7$)**.**

(a) *For every integer codimension $k \geq 1$, there exists a genuine bivariate polynomial system $f_1, f_2 \in \mathbb{R}[x,y]$ supported on a common set of at most $k+3$ distinct monomials admitting at least $3k$ pairwise distinct, strictly positive, nondegenerate real solutions in $\mathbb{R}^2_{>0}$; that is,*

$$B(2,k) \geq 3k \qquad (\forall\, k \geq 1).$$

*Combined with the elementary univariate-product bound* (3) *(formalized in* `UnivariateProduct.lean`*), we obtain for every $k \geq 1$:*

$$B(2,k) \geq \max\left(3k,\ \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{k}{2} \right\rceil + 1\right)\right).$$

(b) ***Exact record window** ($k = 3,4,5,6,7$)**:** The linear bound $3k$ strictly beats* (3) *(as well as all prior published bounds and the author's earlier preprint bound $3k-1$) precisely for $k \in \{3,4,5,6,7\}$, setting five new bivariate records:*

$$B(2,3) \geq 9, \quad B(2,4) \geq 12, \quad B(2,5) \geq 15, \quad B(2,6) \geq 18, \quad B(2,7) \geq 21.$$

*For all $k \geq 8$, the quadratic univariate-product bound* (3) *strictly exceeds $3k$ (beginning with $25 > 24$ at $k = 8$ and $30 > 27$ at $k = 9$).*

(c) ***Exponential product families:** Via* (1)*, $B(2r,kr) \geq (3k)^r$ for all $r,k \geq 1$. For $k \in \{3,4,5,6,7\}$, this yields new exponential lower-bound bases $9^r, 12^r, 15^r, 18^r, 21^r$: indeed, combining El Hilany's $B(2,2) \geq 7$ with univariate factors $B(1,j) = j+1$ over $2r$ variables and total codimension $kr$ yields asymptotic base at most $\sup_{x \in [0,1)} 7^x \left(\frac{k+2-4x}{2(1-x)}\right)^{2(1-x)} \approx 8.660, 10.714, 13.255, 16.398, 20.287$ for $k = 3,4,5,6,7$, all strictly below $3k$. For $k \geq 8$, the univariate product bound $\big((\lfloor k/2 \rfloor + 1)(\lceil k/2 \rceil + 1)\big)^r$ again dominates $(3k)^r$.*

**Theorem 1.2** (Explicit 600-digit certified bivariate systems)**.** *The following explicit bivariate systems realize $B(2,1) \geq 3$, $B(2,2) \geq 6$, and $B(2,3) \geq 9$ with nondegenerate strictly positive roots verified to 600 decimal digits:*

(1) *(4 shared monomials, 3 roots) $x^{32} + y^{32} - 1 = 0$, $20xy^{64} + 18x^{32} - 16 = 0$.*
(2) *(5 shared monomials, 6 roots) $x^{65} + xy^{64} - x = 0$, $x^9 y^{128} + \frac{3}{5}x^{65} - \frac{11}{20}x + \frac{17121}{50000}y^{64000} = 0$.*
(3) *(6 shared monomials, 9 roots) $u + (1-u) = 1$ coupled with*

$$-\frac{5345401633}{10^{11}}u^{-9/512}(1-u)^{5\cdot 10^{11}} + \frac{17121}{50000}u^{-1/64}(1-u)^{1000} - \frac{11}{20} + u^{1/8}(1-u)^2 + \frac{3}{5}u = 0,$$

*which shares both $(0,0)$ and $(1,0)$ with $u + (1-u) = 1$ and clears to a 6-monomial polynomial system with 9 nondegenerate positive roots.*

**Theorem 1.3** (Explicit irreducible 3D codimension-3 Pentagon-Haas lift (12 roots))**.** *There exists an explicit, genuinely irreducible 3-variable polynomial system $f_1, f_2, f_3 \in \mathbb{R}[x,y,z]$ supported on 7 distinct monomials of maximum total degree 61 (unimodular Hermite Normal Form $[1,1,1]$) admitting 12 pairwise distinct, strictly positive, nondegenerate real solutions in $\mathbb{R}^3_{>0}$.*

*Remark* 1.4 (Scope of Theorem 1.3 relative to El Hilany (2017) and product bounds). El Hilany [7] proved $B(2,2) \geq 7$ via non-transversal tropical patchworking. By (1), that theorem immediately yields reducible product lower bounds strictly exceeding the raw root counts of our 3D lift and its products:

$$B(3,3) \geq 7 \times 2 = 14, \qquad B(3r,3r) \geq 14^r,$$
$$B(4,4) \geq 7 \times 7 = 49 \ \text{ (and } 5 \times 5 = 25 \text{ from Haas)}, \qquad B(5,6) \geq 7 \times 7 \times 3 = 147.$$

(Combining our record $B(2,3) \geq 9$ with El Hilany's 7 also gives $B(5,6) \geq 9 \times 7 \times 2 = 126$, still below 147.) Consequently, Theorem 1.3 is *not* a global record for $B(3,3)$, nor are the internal product corollaries $B(3r,3r) \geq 12^r$, $B(4,4) \geq 24$, and $B(5,6) \geq 108$ records. The contribution

of Theorem 1.3 is rather an *explicit, irreducible (non-product), low-degree (*deg $\leq 61$*), and formally machine-verified* 3D codimension-3 system achieving 12 positive roots—surpassing Bihan–Rojas–Sottile (8) and the explicit Haas product $5 \times 2 = 10$. Similarly, at $k = 2$, $B(2, 2) \geq 6$ lies below El Hilany's 7 and serves as an elementary, explicit, tropical-free construction surpassing Haas (5).

**Theorem 1.5** (Symmetric 3D trinomial obstruction and certified $S_3(3, 3, 3) \geq 10$)**.**

(a) ***Zero all-distinct orbits:*** *For any real exponents $a, b, c, d \in \mathbb{R}$ with $(a - c)(b - d) \neq 0$ and any constant $K \in \mathbb{R}$, the $S_3$-symmetric 3D trinomial Gale system*

$$a \log u_i + b \log(1 - u_i) + c \log(u_j u_k) + d \log((1 - u_j)(1 - u_k)) = K \qquad (\{i, j, k\} = \{1, 2, 3\})$$

*admits* no *strictly ordered solution $0 < u_1 < u_2 < u_3 < 1$.*

(b) ***Explicit degree-24 pure trinomial system (10 roots):*** *The explicit integer 3-variable trinomial polynomial system*

$$200x^3 - 200x + 77y^{12}z^{12} = 0, \qquad 200y^3 - 200y + 77z^{12}x^{12} = 0, \qquad 200z^3 - 200z + 77x^{12}y^{12} = 0$$

*admits 10 pairwise distinct, strictly positive, nondegenerate real solutions in $\mathbb{R}^3_{>0}$; hence $S_3(3, 3, 3) \geq 10 > 8$ (matching the standing bound from $S_3$-coupled Haas trinomials).*

TABLE 1. Comparison of all formally verified bounds in this paper against the univariate Descartes product (3), the best published prior bounds (BRS [5], Haas [9], GNR [8], and El Hilany [7] under (1)), and the author's unrefereed preprint [2].

| Invariant | Univ. prod. | Best published | Preprint | **This paper (Lean 4)** | Status |
|---|---|---|---|---|---|
| $B(2,2)$ | 4 | 7 (EH) / 5 (Haas) | 5 | 6 (explicit) | Not a record; explicit $> 5$ |
| $B(2,3)$ | 6 | 7 (GNR) | 8 | **9** (explicit) | **New record** |
| $B(2,4)$ | 9 | 9 (univ. prod.) | 11 | **12** | **New record** |
| $B(2,5)$ | 12 | 12 (univ. prod.) | 14 | **15** | **New record** |
| $B(2,6)$ | 16 | 16 (univ. prod.) | 17 | **18** | **New record** |
| $B(2,7)$ | 20 | 20 (univ. prod.) | 20 | **21** | **New record** |
| $B(2,8)$ | 25 | 25 (univ. prod.) | 23 | 24 / 25 (comb.) | Not a record ($25 > 24$) |
| $B(2,k)$ ($k \geq 8$) | $(\lfloor \frac{k}{2} \rfloor + 1)(\lceil \frac{k}{2} \rceil + 1)$ | univ. prod. ($\sim k^2/4$) | $3k-1$ | $\max(3k, \text{univ.})$ | Univ. prod. wins ($k \geq 8$) |
| $B(2r,kr)$ ($k \in [3,7]$) | $\leq 20^r$ | $< (3k)^r$ (EH+Desc.) | $(3k-1)^r$ | $(\mathbf{3k})^{\mathbf{r}}$ ($\forall r \geq 1$) | **New exp. base ($k = 3 \ldots 7$)** |
| $B(3,3)$ | 8 | 14 (EH×2) / 10 (Haas×2) | 10 | 12 (irred., deg 61) | Not a record; explicit irred. |
| $B(3r,3r)$ | $8^r$ | $14^r$ (EH×2) / $10^r$ (Haas×2) | $10^r$ | $12^r$ (irred. prod.) | Not a record ($12 < 14$) |
| $B(4,4)$ | 16 | 49 ($7^2$, EH) / 25 ($5^2$, Haas) | 20 | 24 (internal prod.) | Not a record ($24 < 25, 49$) |
| $B(5,6)$ | 72 | 147 ($7^2 \times 3$, EH) | — | 108 (internal prod.) | Not a record ($108 < 147$) |
| $S_3(3,3,3)$ | 8 (Kouch.) | 10 (Haas-3D) | — | 10 (deg 24, Lean+Rust) | Ties prior; obstruction + cert. |

## 2. Proof of Theorem 1.1: Bivariate Bound $B(2, k) \geq 3k$ and Univariate Product

In our earlier preprint proving $\mathcal{S}(3, t) \geq 3t - 4$ [2], we started from a 2-term rational-exponent function $F_2(u) = -1 + 2u^{1/32}(1-u)^{1/32}$ with 2 simple roots on $(0, 1)$, and inserted one boundary-pole term $-c_{\text{new}} u^{-\delta}(1 - u)^M$ per step to add 3 simple roots per new monomial. Because $F_2(u)$ shared only *one* monomial (the constant 1) with the curve $u + (1 - u) = 1$, a $(k + 2)$-term equation $g$ yielded $k + 4$ total monomials instead of $k + 3$, giving $B(2, k) \geq 3k - 1$.

To gain the extra $+1$ root at every codimension $k \geq 1$, we replace the 2-term base case by a 3-term base case that shares *two* monomials—both $(0, 0)$ and $(1, 0)$—with $u + (1-u) - 1 = 0$ and already possesses 3 simple roots on $(0, 1)$ while satisfying the exact same inductive gap condition $0 < a_2 - a_1 < 1/16$.

### 2.1. The $k = 1$ shared-support base case.

Consider the 3-term rational-exponent function on $(0, 1)$:

$$F_1(u) = -\frac{4}{5} + u^{1/32}(1 - u)^2 + \frac{9}{10}u,$$

with leading exponents $a_1 = 0 < a_2 = 1/32$, gap $a_2 - a_1 = 1/32 < 1/16$, opposite-sign leading coefficients $(-4/5) \cdot 1 < 0$, and shared-support anchors $(s, 0) = (0, 0)$ and $(s + 1, 0) =$

$(1,0)$. Evaluating $F_1(t^{32}) = -\frac{4}{5} + t(1-t^{32})^2 + \frac{9}{10}t^{32}$ at the exact rational endpoints $t \in \{\frac{3}{4}, \frac{7}{8}, \frac{47}{50}, \frac{24}{25}, \frac{124}{125}, \frac{997}{1000}\}$ produces three strict sign changes on the pairwise disjoint intervals

$$I_1 = \big[(\tfrac{3}{4})^{32}, (\tfrac{7}{8})^{32}\big], \qquad I_2 = \big[(\tfrac{47}{50})^{32}, (\tfrac{24}{25})^{32}\big], \qquad I_3 = \big[(\tfrac{124}{125})^{32}, (\tfrac{997}{1000})^{32}\big].$$

On each $I_j = [t_a^{32}, t_b^{32}]$, trivial monotonicity of $u^{1/32}$, $u^{-31/32}$, and $1-u$ yields uniform strict derivative bounds:

$$F_1'(u) \geq +1.057 > 0 \text{ on } I_1, \qquad F_1'(u) \leq -0.313 < 0 \text{ on } I_2, \qquad F_1'(u) \geq +0.448 > 0 \text{ on } I_3.$$

By the Intermediate Value Theorem, $F_1$ has three strictly increasing simple roots $0 < r_0 < r_1 < r_2 < 1$. Clearing the denominator $D = 32$ via $u = x^{32}, 1-u = y^{32}$ gives the explicit 4-monomial base system $x^{32} + y^{32} - 1 = 0$, $20xy^{64} + 18x^{32} - 16 = 0$ of Theorem 1.2(1).

2.2. **Anchor-preserving +3 boundary-pole insertion and Laurent lifting.** Define the inductive invariant GenInv($k$) ($k \geq 1$) to consist of $k+2$ nonzero rational-exponent terms $(c_j, a_j, b_j) \in \mathbb{Q}^\times \times \mathbb{Q}^2$ with pairwise distinct exponents, leading terms $a_1 < a_2$ with $a_2 - a_1 < 1/16$ and $c_1 c_2 < 0$, $3k$ strictly increasing simple roots in $(0,1)$, and a shift anchor $s \in \mathbb{Q}$ such that both $(s,0)$ and $(s+1,0)$ appear among the exponents $(a_j, b_j)$.

Given $I \in \text{GenInv}(k)$, the Sard-free boundary-pole insertion step [2] normalizes the leading sign and shifts all $u$-exponents by $-a_1$, then prepends a single new term $(-c_{\text{new}}, -\delta, M)$ with sufficiently large $M \in \mathbb{N}$ and sufficiently small $\delta \in (0, a_2 - a_1) \cap \mathbb{Q}_{>0}$ to create 3 new strictly monotone simple roots near $u = 0^+$ while preserving all $3k$ existing simple roots by $C^1$-stability. Crucially, shifting all $u$-exponents by $-a_1$ maps the anchor pair $(s,0), (s+1,0)$ to $(s',0), (s'+1,0)$ with $s' = s - a_1$, so GenInv($k$) $\implies$ GenInv($k+1$) for every $k \geq 1$.

To lift $I \in \text{GenInv}(k)$ to a genuine bivariate polynomial system, shift all $u$-exponents by $-I.s$ so that both $(0,0)$ and $(1,0)$ appear in the shifted list $L$ of length $k+2$, and set $D = \text{expDenom}(L) \in \mathbb{N}_{>0}$. Under the bijection $(x,y) = (u^{1/D}, (1-u)^{1/D})$, the Laurent system

$$f_0(x,y) = x^D + y^D - 1 = 0, \qquad f_1(x,y) = \sum_{(c,a,b) \in L} c\, x^{Da} y^{Db} = 0$$

has support $S = \{(0,D)\} \cup \text{supp}(f_1)$ satisfying $(0,0) \in S$ and $|S| \leq (k+2)+1 = k+3 = 2+k+1$. Its logarithmic Jacobian at $(x,y) = (u^{1/D}, (1-u)^{1/D})$ satisfies the exact algebraic identity: if $Duv_0 + D(1-u)v_1 = 0$, then $v_1 = -\frac{u}{1-u} v_0$ and

$$\text{logDeriv}_0(f_1)\, v_0 + \text{logDeriv}_1(f_1)\, v_1 = Du\, F_L'(u)\, v_0,$$

which vanishes if and only if $v_0 = v_1 = 0$ since $D > 0$, $u > 0$, and $F_L'(u) \neq 0$. Clearing negative exponents uniformly yields a genuine polynomial system proving $B(2,k) \geq 3k$, and taking $r$-fold disjoint monomial-sharing products yields $B(2r, kr) \geq (3k)^r$. This proof is formalized in `Codim3Records/BivariateCodimK.lean`.

2.3. **Formalized univariate Descartes product bound ($k \geq 8$ regime).** For every $d \geq 1$, the monic polynomial $P_d(x) = \prod_{j=1}^{d}(x-j)$ has support inside $\{0, 1, \dots, d\}$ (size $d+1 = 1 + (d-1) + 1$, containing the constant monomial 0) and $d$ simple positive roots $x = 1, \dots, d$ with $P_d'(r) = \prod_{i \neq r}(r-i) \neq 0$. Applying the monomial-sharing product (1) to $P_{d_1}(x)$ and $P_{d_2}(y)$ with $d_1 = \lfloor k/2 \rfloor + 1$ and $d_2 = \lceil k/2 \rceil + 1$ yields $B(2,k) \geq (\lfloor k/2 \rfloor + 1)(\lceil k/2 \rceil + 1)$ for all $k \geq 0$ (formalized as `univariate_product_B2k` and `bivariate_combined_bound` in `Codim3Records/UnivariateProduct.lean`). Comparing $3k$ with $(\lfloor k/2 \rfloor + 1)(\lceil k/2 \rceil + 1)$ directly confirms that $3k$ strictly wins on $k \in \{3,4,5,6,7\}$ and loses for all $k \geq 8$.

## 3. Proof of Theorem 1.3: Explicit Irreducible 3D Lift (12 Roots)

To construct an *explicit, irreducible, low-degree* 3D codimension-3 system surpassing both Bihan–Rojas–Sottile (8) and the explicit Haas product $5 \times 2 = 10$, we lift our codimension-2 Pentagon-Haas arrangement [3] (which has 5 lines in $\mathbb{R}^2$ and 8 roots) to 6 hyperplanes in $\mathbb{R}^3$ by splitting the line $1 - v$ into two planes $L_3 = w$ and $L_4 = 1 - v + w$:

$$L_0 = u, \quad L_1 = 1 - u, \quad L_2 = v, \quad L_3 = w, \quad L_4 = 1 - v + w, \quad L_5 = \tfrac{13}{10}u + v - \tfrac{7}{25}.$$

Setting $s = w/(1-v) > 0$, the first two Gale equations become the Pentagon-Haas equations with constants shifted by $(1+s)^{-1}$, tracing 8 smooth curves in $(u, v, w)$-space passing near $s_0 = 1/4$ ($C_0 = \frac{383}{160}$, $C_1 = 19600$). Adding a third master equation with integer exponent vector $A_2 = (1, -1, -5, 2, -11, -9)$ and exact rational constant $C_2 = \frac{1}{894}$ slices these 8 space curves transversely at 12 **pairwise distinct strictly positive points** inside the open 3-polytope $\{L_i > 0\}$:

$$A = \begin{pmatrix} 2 & 4 & -14 & 0 & -1 & -2 \\ 8 & -1 & 6 & 0 & -1 & -28 \\ 1 & -1 & -5 & 2 & -11 & -9 \end{pmatrix}, \qquad C = \Big(\frac{383}{160},\ 19600,\ \frac{1}{894}\Big).$$

The $3 \times 6$ integer matrix $A$ is *unimodular*: its Hermite Normal Form diagonal is $[1, 1, 1]$, with reduced kernel basis $B \in \mathbb{Z}^{6\times 3}$ and exact right inverse $R \in \mathbb{Z}^{6\times 3}$ ($AB = 0$, $AR = I_3$):

$$B = \begin{pmatrix} 8 & -7 & 8 \\ 6 & 6 & 14 \\ 3 & 1 & 5 \\ -24 & 0 & 7 \\ -8 & 0 & -4 \\ 3 & -2 & 3 \end{pmatrix}, \qquad R = \begin{pmatrix} -2 & -6 & 0 \\ -9 & -1 & 4 \\ -3 & -1 & 1 \\ 1 & 2 & 16 \\ 3 & 2 & 2 \\ -1 & -2 & 0 \end{pmatrix}.$$

Shifting the rows of $B$ by $(24, 7, 4)$ clears all negative exponents to 7 distinct nonnegative monomials in $\mathbb{N}^3$ of maximum total degree 61 (for row 1: $(30, 13, 18)$ has degree 61). The three linear relations among the hyperplanes ($L_0 + L_1 = 1$, $L_2 - L_3 + L_4 = 1$, $\frac{13}{10}L_0 + L_2 - L_5 = \frac{7}{25}$) lift directly via `gale3_lift_to_laurent` to a genuine irreducible 3-variable, 7-monomial polynomial system.

In `Codim3Records/PentagonLiftB33.lean`, all 12 Gale roots are certified inside pairwise disjoint 3D $\ell_\infty$-boxes by our general 3D Newton–Banach contraction theorem (`gale3_banach_box_root` in `BanachTools3D.lean`) using normalized base-10 mantissa logarithm bounds, proving `pentagon_lift_B33_ge_12` (as well as the internal product corollaries `exponential_B3r_3r`, `product_B44_ge_24`, and `product_B56_ge_108`, which as noted in Table 1 are not global records). At 600-digit precision, the 12 primal roots achieve maximum relative residual $4.26 \times 10^{-584}$, minimum primal Jacobian determinant $8.00 \times 10^{62}$, and minimum $\ell_\infty$-separation 0.365.

## 4. Proof of Theorem 1.5: 3D Trinomial Obstruction and $S_3(3,3,3) \geq 10$

4.1. **Why $S_3$-symmetric 3D trinomial Haas systems have zero all-distinct orbits.** Consider any $S_3$-symmetric 3D trinomial Gale system on $(0,1)^3$:

$$a \log u_i + b \log(1-u_i) + c(\log u_j + \log u_k) + d(\log(1-u_j) + \log(1-u_k)) = K$$

for $\{i, j, k\} = \{1, 2, 3\}$, with $(a-c)(b-d) \neq 0$. Suppose for contradiction that a strictly ordered solution $0 < u < v < w < 1$ exists. Subtracting the equations pairwise shows that the single-variable function

$$\varphi(x) = (a-c)\log x + (b-d)\log(1-x)$$

satisfies $\varphi(u) = \varphi(v) = \varphi(w)$. By Rolle's theorem on $[u, v]$ and $[v, w]$, there exist $u < \xi_1 < v < \xi_2 < w$ such that $\varphi'(\xi_1) = \varphi'(\xi_2) = 0$. Since $\varphi'(x) = \frac{a-c}{x} - \frac{b-d}{1-x}$, vanishing at $\xi_i \in (0,1)$ gives

$$(a-c)(1-\xi_i) = (b-d)\xi_i \qquad (i = 1, 2).$$

Subtracting yields $((a-c)+(b-d))(\xi_2 - \xi_1) = 0$. Since $\xi_2 > \xi_1$, we get $(a-c)+(b-d) = 0$, which substituted back gives $a - c = 0$, contradicting $(a-c)(b-d) \neq 0$.

Thus every positive solution of an $S_3$-symmetric 3D trinomial Gale system has at least two equal coordinates (1 diagonal root + orbits of size 3 with $u = v \neq w$), structurally explaining why symmetric 3D Haas systems attain at most $1 + 3 \times 3 = 10$ roots and demonstrating that breaking $S_3(3,3,3) \geq 11$ necessarily requires breaking $S_3$-symmetry or leaving the diagonal-cube hyperplane arrangement.

4.2. **Explicit degree-24 pure integer trinomial system with 10 roots.** Choosing $a = 1, b = 2, c = -12, d = 0$ and exact-square parameter $K = (77/200)^2 = 5929/40000$ in the Gale equations on $(0,1)^3$ gives

$$\frac{40000}{5929}\,\frac{u_i(1-u_i)^2}{u_j^{12}u_k^{12}} = 1 \qquad (\{i,j,k\} = \{1,2,3\}).$$

Setting $x = \sqrt{u}, y = \sqrt{v}, z = \sqrt{w} \in (0,1)$ and taking positive square roots yields the pure integer degree-24 trinomial system

$$200x^3 - 200x + 77y^{12}z^{12} = 0, \qquad 200y^3 - 200y + 77z^{12}x^{12} = 0, \qquad 200z^3 - 200z + 77x^{12}y^{12} = 0.$$

Its primal Jacobian satisfies the exact determinant identity

$$\det J_{\text{primal}}(x,y,z) = -8{,}000{,}000\,uvw(1-u)(1-v)(1-w)\det J_{\text{gale}}(u,v,w).$$

In `Codim3Records/SymmetricTrinomial3D.lean`, we certify the 4 orbit representatives (one diagonal root $u = v = w \approx 0.89394$ and three two-equal roots with $w \in \{0.30265, 0.42615, 0.65915\}$) via `gale3_banach_box_root`, expand them by exact cyclic $C_3$-symmetry to 10 pairwise disjoint 3D boxes, and lift directly to `AttainsTrinomialBound 3 10`.

## 5. Multi-Pillar Independent Verification

Every result in this paper is backed by three independent verification engines:

(1) **Formal proof in Lean 4.36.0 / Mathlib4:** All 21 exported theorems (including the bivariate records for $k = 3 \dots 7$, the univariate Descartes product bound, the combined bivariate bound, the 3D irreducible lift, and the 3D trinomial obstruction and system) compile cleanly via `lake build Codim3Records` with zero `sorry`, zero `admit`, zero `native_decide`, and strictly the three standard foundational axioms `[propext, Classical.choice, Quot.sound]` verified by `Codim3Records/Audit.lean`.

(2) **Complex root certification in Rust (`pandsysroots v0.34.0`):** Using simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy, `cargo run -release` globally certifies:
- all 3/3 positive real roots of the $B(2,1) \geq 3$ base system (BKK = 2,080, 7.45 s, max residual $5.33 \times 10^{-14}$), and
- all 4,752/4,752 complex roots (100% of the BKK volume) and all 10/10 strictly positive real roots of the degree-24 3D trinomial system $S_3(3,3,3) \geq 10$ in 10.95 s (max ball radius $1.95 \times 10^{-13}$, max residual $9.95 \times 10^{-14}$, $\min|\det J| = 2.929 \times 10^6$).

(3) **600-digit `mpmath` primal–Jacobian verification (`verify_600digits.py`):** Confirms all 5 explicit systems ($3, 6, 9, 12, 10$ roots) to residuals $\leq 4.26 \times 10^{-584}$ with exact primal Jacobian determinants and strict pairwise root separation.

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