Quadrics and strong maximal rank

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Abstract. We prove the quadratic Strong Maximal Rank Conjecture on the injective side of its original numerical range, including the dimension of every failure component. The proof combines a bound for the projective incidence of quadratic relations with a positive Chern intersection. We also obtain uniform quadratic-kernel bounds and rank estimates for higher multiplication cokernels.

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1. Introduction

We work over $\mathbf{C}$. Let C be a smooth projective connected curve of genus g, and let L be a line bundle of degree d. Put $V = H^0(C, L)$, $c = \dim V$, and, for an integer $m \geq 2$,

$$\mu_{m,L} \colon \operatorname{Sym}^m V \longrightarrow H^0(C, L^m).$$

For a very ample L, its kernel is the space of degree-m equations of the embedded curve, and its cokernel measures failure of degree-m normal generation. We write

$$Q_m(L) = H^0(C, L^m)/\operatorname{im}\mu_{m,L}, \qquad b_m(L) = \dim Q_m(L).$$

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The parameter spaces are

$$W^r_d(C) = \{L \in \mathrm{Pic}^d(C) : h^0(C,L) \geq r+1\},$$
$$G^r_d(C) = \{(L,V) : L \in \mathrm{Pic}^d(C),\ V \subseteq H^0(C,L),\ \dim V = r+1\}.$$

Brill–Noether theory determines their dimensions on a general curve, and the Gieseker–Petri theorem gives smoothness of $G^r_d(C)$ [**2, 14**]. Multiplication imposes further determinantal conditions on these spaces. We use the numerical notation

$$r = c-1, \qquad s = g-d+r, \qquad \rho = \rho(g,r,d) = g-(r+1)(g-d+r).$$

The classical maximal-rank problem asks whether a general embedded curve imposes independent conditions on hypersurfaces of each degree. Its history includes Severi's work of 1915 and Harris's formulation in 1982; see the historical discussion in [**15**]. The quadratic case was proved by Ballico [**5**]. Jensen–Payne [**15**] gave a tropical proof using independence of piecewise-linear functions on chains of loops. Liu–Osserman–Teixidor i Bigas–Zhang [**18**] developed a proof through limit linear series on chains of elliptic curves. Larson [**16**] proved the Maximal Rank Theorem in all degrees. These statements concern a general linear series on a general curve.

Uniform results for every line bundle have a different history. Teixidor i Bigas [**25**] proved in 2003 that $\mu_{2,L}$ is injective for every L of degree at most $g+1$ on a general curve, by degeneration to an elliptic chain. Ballico [**4**] treated corresponding injectivity statements for nonfree rank-one sheaves on nodal and cuspidal curves with general normalization. Farkas [**10**, Remark 5.6] recorded the bound $\dim\ker\mu_{2,L} \leq a(a+1)$ when $d \leq g+a+1$, for $a \geq 0$. On the surjective side, Ballico–Fontanari [**6**] obtained normal-generation criteria for every line bundle in explicit ranges of degree and speciality, using vector-bundle extensions and Clifford-type estimates.

Aprodu and Farkas proposed the Strong Maximal Rank Conjecture in their 2008 preprint, published in 2011 as [**1**]. Their formulation arose from the study of syzygetic loci and effective divisors on moduli spaces of curves. Its original range is

$$s \geq 0, \qquad 0 \leq \rho < r-2, \qquad m \geq 2. \tag{1}$$

For a general C, the conjecture predicts that the locus in $W^r_d(C)$ where $\mu_{m,L}$ fails to have maximal rank has dimension

$$\rho - 1 - \left|\binom{r+m}{m} - (md+1-g)\right|, \tag{2}$$

with a negative value meaning that the locus is empty. The numerical restrictions make every series complete and very ample. Petri injectivity gives $H^1(L^m) = 0$ for $m \geq 2$, so the two ranks in this formula are the actual source and target dimensions. The determinantal lower bound applies to each irreducible component; an upper bound of the displayed size therefore determines every component. When $\rho = 0$, monodromy reduces the strong statement to ordinary maximal rank. Aprodu–Farkas also recorded the nonnegative-dimensional quadratic case $(g,r,d) = (14,5,17)$ in [**1**, Remark 5.6].

The later strong results illustrate the role of these dimension questions in moduli theory. Liu–Osserman–Teixidor i Bigas–Zhang [**19**] and Farkas–Jensen–Payne [**13**] proved quadratic injectivity for every $L \in W^6_{g+3}(C)$ when $g = 22, 23$. Farkas–Jensen–Payne [**12**, Theorem 1.5] proved quadratic surjectivity for every $L \in W^5_{16}(C)$ on a general genus-thirteen curve. Their tropical constructions also control degeneracy loci over moduli divisors. The resulting divisor calculations establish general type for $\overline{\mathcal{M}}_{22}$, $\overline{\mathcal{M}}_{23}$, and the Prym moduli space $\overline{\mathcal{R}}_{13}$ [**13, 12**]. Here $\mathcal{M}_g$ parametrizes smooth curves of genus g, $\overline{\mathcal{M}}_g$ is its stable-curve compactification, and $\overline{\mathcal{R}}_g$ parametrizes stable Prym curves.

Intersection theory supplies a complementary source of results. Cotterill–Alonso Gonzalo–Zhang [**8**] computed virtual strong-maximal-rank classes, proved positivity in several ranges, and related these calculations to canonical-determinant Brill–Noether theory. Their Section 5.2.2

gives the Schur sum for the injective-side class. Lian's 2026 preprint [**17**] constructs unobstructed maps from general curves to smooth quadrics. Its Corollary 4.4.2 gives an expected-dimensional failure component throughout the nonnegative-dimensional injective band considered below; Proposition 4.4.1 gives a one-dimensional kernel at the constructed point. In Section 10 we express the injective-side Schur sum as a centered Haar moment and prove its positivity. The upper bound below controls the whole relation incidence and consequently every failure component.

We retain both the line bundle and its projective quadratic equation.

THEOREM 1.1. *Let s, t, ρ be integers with $s \geq 2$, $t \geq 3$, let $c = s + t$, and set*

$$g = cs + \rho, \qquad d = g + t - 1, \qquad 0 \leq \rho \leq c - 4.$$

Put

$$K = \binom{t-1}{2}, \qquad m_0 = \binom{s}{2}, \qquad \Delta = K - m_0.$$

For a sufficiently general smooth curve C of genus g, the projective relation incidence

$$\mathcal{R}_d^{c-1}(C) = \{(L, [q]) : L \in W_d^{c-1}(C),\ 0 \neq q \in \ker \mu_{2,L}\}$$

satisfies

$$\dim \mathcal{R}_d^{c-1}(C) \leq \Delta - 1. \tag{3}$$

A negative upper bound means that the incidence is empty. For every integer $\kappa \geq 1$,

$$\dim\{L \in W_d^{c-1}(C) : \dim \ker \mu_{2,L} = \kappa\} \leq \Delta - \kappa. \tag{4}$$

If $\Delta \geq 0$, then $b_2(L) \leq \rho$ for every such L, and $\mu_{m,L}$ is surjective for every $m \geq 3$. If $\Delta \leq 0$, every $\mu_{2,L}$ is injective.

The proof combines a cuspidal diagonal filtration with an all-order elliptic residue calculation. It includes nonfree torsion-free limits and all collisions of the auxiliary divisors.

COROLLARY 1.2. *Let $g \geq 2$, r, d satisfy the original range* (1)*, and suppose $\binom{r+2}{2} \leq 2d+1-g$. For a sufficiently general curve C, the quadratic failure locus in $W_d^r(C)$ is empty if the number in* (2)*, with $m = 2$, is negative. Otherwise every irreducible component has that dimension, and the general point of every component has a one-dimensional kernel. The nonnegative case consists exactly of*

$$r = 2s + 1, \qquad g = (2s + 2)s + \rho, \qquad d = g + s + 1, \qquad s \leq \rho \leq 2s - 2, \quad s \geq 2.$$

Its failure locus is nonempty of pure dimension $s - 1$, and every $\mu_{m,L}$, $m \geq 3$, is surjective.

The proof is given after the intersection calculation in Section 10. On the strictly surjective side, Theorem 1.1 instead gives the bounds in Corollary 9.3, including $\dim\{L : b_2(L) \geq e\} \leq \rho - e$ for $e \geq 1$.

Our second result relates the cokernels for successive powers of one line bundle. For an integer $m \geq 3$ and $\epsilon \in Q_m(L)^*$, multiplication defines

$$\delta_\epsilon \colon V \longrightarrow Q_{m-1}(L)^*, \qquad \delta_\epsilon(v)([a]) = \epsilon(va). \tag{5}$$

It is well defined because ϵ annihilates products of m sections.

THEOREM 1.3. *Fix $g \geq 2, d, c$, with $c \geq 5$ and*

$$0 \leq \rho = g - c(g - d + c - 1) \leq c - 2.$$

On a sufficiently general smooth curve C of genus g, let L be any very ample line bundle of degree d with $h^0(L) = c$. Every nonzero $\epsilon \in Q_3(L)^*$ satisfies $\operatorname{rank} \delta_\epsilon \geq c - 2$, and every nonzero

$\epsilon \in Q_m(L)^*$, $m \geq 4$, *satisfies* $\operatorname{rank} \delta_\epsilon \geq c-1$. *Consequently*

$$b_3 \leq 3\max\{0, b_2 - c + 3\}, \tag{6}$$

$$b_m \leq 2\max\{0, b_{m-1} - c + 2\} \quad (m \geq 4). \tag{7}$$

In particular:

(i) *if* $b_2 \leq c-3$, *then* $\mu_{m,L}$ *is surjective for all* $m \geq 3$;
(ii) *if* $b_2 \leq c-2$, *then* $b_3 \leq 3$ *and* $\mu_{m,L}$ *is surjective for all* $m \geq 4$.

The theorem applies to every line bundle in the original range (1) with $c \geq 5$: in that range very ampleness and completeness follow from Brill–Noether theory and the secant theorem of Farkas [**11**]. At the two additional values $\rho = c-3, c-2$, very ampleness remains an explicit hypothesis. The proof uses finite Frobenius quotients to handle arbitrary nonreduced divisors. Rank-two extensions and the ordinary Petri map then convert a large contraction kernel into a forbidden divisor.

For an integer h, set

$$\mathsf{B}(h) = \sum_{j \geq 1} \max\{0, 2h+1-4j\} = \begin{cases} 0, & h \leq 1, \\ \binom{h}{2}, & h \geq 2. \end{cases} \tag{8}$$

The sum is finite.

Theorem 1.4. *For fixed* $g \geq 2, d \geq 0$, *a sufficiently general smooth curve* C *of genus* g *has the following property. For every degree-*d *line bundle* L *and every subspace* $V \subseteq H^0(C, L)$,

$$\dim \ker(\operatorname{Sym}^2 V \longrightarrow H^0(C, L^2)) \leq \mathsf{B}(d-g). \tag{9}$$

In particular all these maps are injective when $d \leq g+1$.

The proof keeps torsion-free limits of line bundles. Evaluation at smooth points specializes relations without requiring symmetric powers of a nonfree sheaf to commute with specialization. The central estimate is an explicit diagonal filtration of symmetric biforms on the normalization of a rational cuspidal curve.

Corollary 1.5. *Let* $s \geq 2$, $t \geq 3$, $c = s+t$, *and*

$$g = cs + \rho, \qquad d = g + t - 1, \qquad 0 \leq \rho \leq c-4.$$

For a sufficiently general C, *every* $L \in W^{c-1}_d(C)$ *has* $h^0(L) = c$ *and satisfies, with* $K = \binom{t-1}{2}$,

$$\max\{0, K - \tbinom{s}{2} - \rho\} \leq \dim \ker \mu_{2,L} \leq K, \tag{10}$$

$$\max\{0, \tbinom{s}{2} + \rho - K\} \leq b_2(L) \leq \tbinom{s}{2} + \rho. \tag{11}$$

If $\binom{s}{2} + \rho \leq c-3$, *all* $\mu_{m,L}$, $m \geq 3$, *are surjective. If* $\binom{s}{2} + \rho \leq c-2$, *then* $b_3(L) \leq 3$ *and all* $\mu_{m,L}$, $m \geq 4$, *are surjective.*

The first alternative covers every original parameter with $s = 2$ and $t \geq 3$. It also covers $(g, r, d) = (18, 5, 20)$: the quadratic target has dimension 23, whereas the quadratic source has dimension 21, so quadratic surjectivity is impossible. The conclusion concerns surjectivity from degree three onward. For $s = 2$ and $r \geq 9$, normal generation in the original range is already supplied by [**6**, Theorem 3]. The contraction estimates also apply when the quadratic cokernel is positive, as in the genus-eighteen example.

The later sections describe subline bundles of canonical extensions associated with $Q_2(L)^*$, and compute the rank-three incidence, retaining the fixed divisor of its three-dimensional support. The underlying square-pencil description is classical; compare Park [**22**]. Retaining the support divisor corrects the odd-degree emptiness assertion in [**23**, v1, Theorem 3.2]. An appendix determines the scheme structure of an explicit Robin rank-drop locus, including its exceptional support lines.

2. General-curve inputs and notation

The notation $\mathbf{P}(E)$ parametrizes lines in E, and $\mathrm{Gr}(k,E)$ parametrizes its k-dimensional subspaces. For $V \subseteq H^0(C,L)$, write $\mu_{m,V} : \mathrm{Sym}^m V \to H^0(C,L^m)$ for multiplication on that subspace. An effective divisor may be nonreduced. For $D \geq 0$, put

$$q_L(D) = \mathrm{rank}(H^0(L) \longrightarrow H^0(L|_D)), \qquad \ell(D) = \deg D.$$

For fixed numerical data, *general curve* means a curve in a nonempty Zariski open subset of $\mathcal{M}_g$. Only finitely many divisor lengths and ranks occur in the secant arguments, so a single open subset suffices for every L, D considered.

Write K_C for the canonical bundle. We use the Brill–Noether and Gieseker–Petri theorems [**2, 14**]. Thus $G^r_d(C)$, when nonempty on a general curve, is smooth of dimension $\rho(g,r,d)$, and for every A,

$$H^0(A) \otimes H^0(K_C A^{-1}) \longrightarrow H^0(K_C) \tag{12}$$

is injective. The third input is the following form of [**11**, Theorem 0.1 and the incidence estimate preceding Corollary 0.2].

Theorem 2.1 (Farkas's secant estimate). *Let $c = r+1$, $1 \leq q \leq r$, and $\ell > q$. For a general C, the incidence of $(D,(L,V)) \in C^{(\ell)} \times G^r_d(C)$ with $\mathrm{rank}(V \to H^0(L|_D)) \leq q$ has dimension at most*

$$\rho - (\ell - q)(c - q) + \ell. \tag{13}$$

If this number is negative, the incidence is empty.

The substitution in the cited theorem is $e = \ell$, $f = \ell - q$. Its condition $r - e + f \geq 0$ becomes $q \leq r$. This upper-bound theorem has no restriction $\ell \leq g$. The full symmetric product includes all multiplicity partitions. Write

$$F_c(\ell, q) = (\ell - q)(c - q) - \ell = \ell(c - q - 1) - q(c - q). \tag{14}$$

Lemma 2.2. *On a Petri curve, if $h^0(L) \geq 2$, then $H^0(K_C L^{-2}) = 0$ and $H^1(L^a) = 0$ for every $a \geq 2$. More generally, if $h^0(L) = c \geq 2$ and $D \geq 0$ has $\deg D \leq c - 2$, then*

$$H^1(L^2(-D)) = 0. \tag{15}$$

In particular L^2 separates every subscheme of length at most $c-2$; for $c \geq 3$ it is $(c-3)$-very ample.

Proof. If $u \in H^0(K_C L^{-2})$ is nonzero and $v, w \in H^0(L)$ are independent, then $v \otimes uw - w \otimes uv$ is a nonzero Petri-kernel tensor. Its nonzeroness follows because multiplication by a nonzero section on an integral curve is injective. This contradicts (12). For $a > 2$, multiplication by a nonzero section of L^{a-2} injects $H^0(K_C L^{-a})$ into $H^0(K_C L^{-2})$. Apply Serre duality. For the stronger assertion, $h^0(L(-D)) \geq c - \deg D \geq 2$. If $0 \neq u \in H^0(K_C L^{-2}(D))$, choose independent $v, w \in H^0(L(-D))$. Their zeros cancel the allowed poles of u, so $uv, uw \in H^0(K_C L^{-1})$, and the same nonzero Petri tensor $v \otimes uw - w \otimes uv$ gives a contradiction. Serre duality proves (15). The exact sequence for restriction to D then proves surjectivity of $H^0(L^2) \to H^0(L^2|_D)$. □

For a line bundle L, put $V = H^0(L)$. For $p \geq 0$, the Koszul group $K_{p,3}(C,L)$ is the middle cohomology of

$$\bigwedge^{p+1} V \otimes H^0(L^2) \longrightarrow \bigwedge^{p} V \otimes H^0(L^3) \longrightarrow \bigwedge^{p-1} V \otimes H^0(L^4),$$

where the last term is zero for $p = 0$. The arrows are given by

$$(v_1 \wedge \cdots \wedge v_a) \otimes s \longmapsto \sum_{i=1}^{a} (-1)^{i-1} (v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_a) \otimes (v_i s).$$

PROPOSITION 2.3 (The Petri tail of the section ring). *Let L be a globally generated line bundle on a Petri curve, and put $V = H^0(L)$, $c = \dim V \geq 3$. For the complete section ring $R(L) = \bigoplus_{a\geq 0} H^0(L^a)$, one has*

$$K_{p,3}(C, L) = 0 \qquad (0 \leq p \leq c-3), \qquad \dim K_{c-2,3}(C, L) = h^1(L).$$

In particular, multiplication

$$V \otimes H^0(L^{a-1}) \longrightarrow H^0(L^a)$$

is surjective for every $a \geq 3$. Thus the defect module $\bigoplus_{a\geq 2} Q_a(L)$, with its natural $\operatorname{Sym} V$-module structure, is generated by $Q_2(L)$.

PROOF. Let

$$0 \longrightarrow M_L \longrightarrow V \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0$$

be the evaluation sequence. The second Koszul differential has kernel $H^0(\bigwedge^p M_L \otimes L^3)$, by a local splitting of this sequence. Taking exterior powers and tensoring gives

$$0 \longrightarrow \bigwedge^{p+1} M_L \otimes L^2 \longrightarrow \bigwedge^{p+1} V \otimes L^2 \longrightarrow \bigwedge^{p} M_L \otimes L^3 \longrightarrow 0.$$

Since $H^1(L^2) = 0$ by Lemma 2.2,

$$K_{p,3}(C, L) \simeq H^1(\bigwedge^{p+1} M_L \otimes L^2).$$

The bundle M_L has rank $c-1$ and determinant $(\det V) \otimes L^{-1}$. Serre duality therefore identifies the dual of this group, up to the constant one-dimensional factor $(\det V)^*$, with

$$H^0(\bigwedge^{c-2-p} M_L \otimes K_C L^{-1}).$$

Put $k = c-2-p$. If $k \geq 1$, this is the kernel of

$$\bigwedge^{k} V \otimes H^0(K_C L^{-1}) \longrightarrow \bigwedge^{k-1} V \otimes H^0(K_C).$$

The arrow factors through exterior comultiplication $\bigwedge^k V \to \bigwedge^{k-1} V \otimes V$, followed by the Petri map. Exterior multiplication composed with the first map is k times the identity, up to a fixed sign; hence that map is injective in characteristic zero. The Petri map is injective as well, so the displayed kernel vanishes. If $k = 0$, the space is $H^0(K_C L^{-1})$. This proves both Koszul assertions.

The assertion for $a = 3$ is exactly $K_{0,3} = 0$. Choose a two-dimensional generating subspace $V_0 \subseteq V$: two general sections have no common zero on a curve. For $a \geq 4$, its evaluation sequence, tensored with L^{a-1}, is

$$0 \longrightarrow L^{a-2} \longrightarrow V_0 \otimes L^{a-1} \longrightarrow L^a \longrightarrow 0.$$

Lemma 2.2 gives $H^1(L^{a-2}) = 0$, which proves surjectivity. Passing to the quotients by products of degree-one sections yields $VQ_{a-1} = Q_a$. □

REMARK 2.4. The complete section ring and the homogeneous coordinate ring have different degree-two pieces when $Q_2(L) \neq 0$. The action $V \otimes Q_2(L) \to Q_3(L)$ is surjective by Proposition 2.3. The contraction estimates below bound its target. The Koszul vanishing is a direct consequence of the classical Petri theorem.

LEMMA 2.5. *Suppose $0 \leq \rho = g - c(g-d+c-1) \leq c-2$. On a general curve, every $L \in W^{c-1}_d(C)$ has exactly c sections. If $\rho \leq c-4$, every such L is very ample.*

PROOF. Put $s_0 = g-d+c-1$. If $s_0 < 0$, then $\rho = g - cs_0 \geq g + c > c-2$, a contradiction. A line bundle with at least $c+1$ sections would belong to $W^c_d(C)$, but $\rho(g, c, d) = \rho - c - s_0 - 1 < 0$. The Brill–Noether theorem excludes it. Very ampleness follows from [**11**, Corollary 0.4] with divisor length two, since $\rho + 2 - r = \rho - c + 3 < 0$. □

3. Finite algebras and interpolation on divisors

DEFINITION 3.1. A functional $\lambda\colon A \to \mathbf{C}$ on a nonzero finite-dimensional commutative unital algebra is *faithful* if $(a,b) \mapsto \lambda(ab)$ is nondegenerate. For $1 \in W \subseteq A$, W^m denotes the linear span of all products of m elements of W.

LEMMA 3.2. *For a nonzero functional λ on a finite-dimensional commutative unital algebra R, the subspace*

$$I_\lambda = \{a \in R : \lambda(ab) = 0 \text{ for all } b \in R\}$$

is an ideal, and λ induces a faithful functional on R/I_λ. If $R = H^0(\mathcal{O}_D)$ for a divisor on a smooth curve, then $R/I_\lambda = H^0(\mathcal{O}_{D'})$ for a unique nonempty subdivisor $D' \le D$.

PROOF. Associativity gives $\lambda((xa)b) = \lambda(a(xb)) = 0$ for $a \in I_\lambda$, $x, b \in R$. Thus it is an ideal. Taking $b = 1$ shows $I_\lambda \subseteq \ker\lambda$; the induced pairing has zero radical by construction. The quotient is nonzero. Locally the ring of a divisor of length e is $\mathbf{C}[z]/(z^e)$; every quotient by an ideal is $\mathbf{C}[z]/(z^a)$, $0 \le a \le e$. Taking the product over the support proves the last assertion. □

LEMMA 3.3 (Length bound). *Let A be a nonzero finite-dimensional commutative unital algebra with a faithful functional λ, and let $W \subseteq A$ contain 1, generate A, and have dimension q. If $m \ge 2$ is an integer and $\lambda(W^m) = 0$, then $q \ge 2$ and*

$$\dim A \ge 2q + m - 2. \tag{16}$$

PROOF. Since $1 \in W$, $W^i \subseteq W^{i+1}$. If equality holds for some $1 \le i < m$, the space W^i is stable under W, hence under A. It contains 1, so equals A. Then $W^m = A$, contradicting $\lambda \ne 0$. All these inclusions are strict, and $\dim W^{m-1} \ge q + m - 2$. The spaces W and W^{m-1} are orthogonal for the perfect pairing, giving (16). If $q = 1$, generation implies $A = W = \mathbf{C}1$, where $\lambda(1) = 0$ is incompatible with faithfulness. □

EXAMPLE 3.4. For every $m \ge 2$, equality occurs for

$$A = \mathbf{C}[z]/(z^{m+2}), \quad W = \langle 1, z\rangle, \quad \lambda\Big(\sum_{i=0}^{m+1} a_i z^i\Big) = a_{m+1}.$$

The pairing matrix in $1, z, \dots, z^{m+1}$ has entry 1 exactly when the indices sum to $m+1$, and entry 0 otherwise. Its determinant is $(-1)^{(m+2)(m+1)/2}$: the only nonzero term is the reversal permutation. Thus λ is faithful. Products in W^m have degree at most m, so $\lambda(W^m) = 0$.

LEMMA 3.5. *If L is very ample, choose $v_0 \in H^0(L)$ nonvanishing on the support of an effective divisor D. After trivializing $L|_D$ by v_0, the image of $H^0(L)$ contains 1 and generates $H^0(\mathcal{O}_D)$. The assertion also holds on every subdivisor.*

PROOF. The embedding defined by L, on the affine patch $v_0 \ne 0$, has coordinate ring generated by the ratios v/v_0. The ring of the closed finite subscheme D is a quotient of this ring. Restriction gives the required generators. □

LEMMA 3.6 (Hopf's inequality). *Let U, W be nonzero finite-dimensional vector spaces. If a bilinear map $U \times W \to T$ has no zero product of two nonzero vectors, its image spans a space of dimension at least $\dim U + \dim W - 1$.*

PROOF. Replace T by the linear span of the image. If the kernel of $U \otimes W \to T$ is zero, the assertion is immediate. Otherwise its projectivization is disjoint from the Segre variety in $\mathbf{P}(U \otimes W)$. Writing $a = \dim U, b = \dim W$, the projective dimension theorem gives

$$(ab - \dim T - 1) + (a + b - 2) < ab - 1,$$

which is equivalent to the claimed integer inequality. □

PROPOSITION 3.7 (Cubic interpolation). *Under the hypotheses of Theorem 1.3, if $D \geq 0$ and $h^0(L(-D)) \geq 3$, then $\operatorname{Sym}^3 H^0(L) \to H^0(L^3|_D)$ is surjective.*

PROOF. Suppose otherwise. Trivialize by a section nonvanishing on the support. A nonzero functional annihilating products gives a faithful quotient $A = H^0(\mathcal{O}_{D'})$, $\varnothing \neq D' \leq D$. Let W be the image of $H^0(L)$, $q = \dim W$, and $\ell = \deg D'$. The preceding lemmas give

$$2 \leq q \leq c-3, \qquad \ell \geq 2q+1.$$

The penalty $F_c(\ell, q)$ is increasing in ℓ, and hence

$$F_c(\ell, q) \geq q(c-q-2) + (c-q-1).$$

This concave quadratic has endpoint values $3c-11$ and $c-1$ on $[2, c-3]$. The former exceeds the latter by $2(c-5) \geq 0$. Thus $F_c(\ell, q) \geq c-1 > \rho$, contradicting the secant theorem. Its conditions hold: $\ell > q$, $q \leq r$, and $f = \ell - q < \ell$. All these divisor lengths are at most d, because $L(-D)$ has a nonzero section. □

PROPOSITION 3.8 (Quartic interpolation). *Under the same hypotheses, if $D \geq 0$ and $h^0(L(-D)) \geq 2$, then $\operatorname{Sym}^m H^0(L) \to H^0(L^m|_D)$ is surjective for every $m \geq 4$.*

PROOF. Suppose quartic interpolation fails and take a faithful quotient (A, W, λ) with $\lambda(W^4) = 0$. If $q = \dim W \leq c-3$, cubic interpolation gives $W^3 = A$, hence $W^4 = A$, a contradiction. Thus $q = c-2 \geq 3$.

If $v, w \in W$ are nonzero and $vw = 0$, faithfulness makes $\lambda_v(a) = \lambda(va)$ nonzero. Its faithful quotient kills w, so the image of W has dimension at most $c-3$. But $\lambda_v(W^3) = \lambda(vW^3) = 0$, contradicting cubic interpolation on the associated subdivisor. Multiplication on $W \times W$ therefore has no nonzero zero product. Hopf's inequality gives $\dim W^2 \geq 2q-1$. As $\lambda(W^4) = 0$, W^2 is totally isotropic, and $\ell = \dim A \geq 4q-2$. Since $c - q = 2$,

$$F_c(\ell, q) = \ell - 2q \geq 2q - 2 > q \geq \rho,$$

again impossible. The strict inequality uses $q \geq 3$. Multiplication by a section nonvanishing on D propagates quartic interpolation to all $m \geq 4$. □

4. Contraction ranks and higher cokernels

LEMMA 4.1 (Support of an extension class). *Let $m \geq 3$ and $0 \neq \epsilon \in Q_m(L)^*$. By Serre duality it defines an extension*

$$0 \longrightarrow K_C L^{1-m} \longrightarrow E_\epsilon \longrightarrow L \longrightarrow 0. \tag{17}$$

Suppose a line subbundle $A \subseteq E_\epsilon$ maps nontrivially to L, with image $A = L(-D)$. If $\operatorname{Sym}^m H^0(L) \to H^0(L^m|_D)$ is surjective, then $\epsilon = 0$, a contradiction.

PROOF. The pullback of (17) to A splits, because its inclusion into E_ϵ lifts $A \to L$. The image of ϵ in $H^1(K_C L^{-m}(D))$ is therefore zero. By Serre duality, ϵ annihilates $H^0(L^m(-D))$. It defines a functional on the image of restriction to $H^0(L^m|_D)$. Extend that functional to the whole finite-dimensional target. Its value on all restricted products is zero. Surjectivity forces this functional, and hence ϵ, to be zero. □

PROPOSITION 4.2. *Under the hypotheses of Theorem 1.3, every nonzero $\epsilon \in Q_m(L)^*$ satisfies*

$$\operatorname{rank} \delta_\epsilon \geq \begin{cases} c-2, & m = 3, \\ c-1, & m \geq 4. \end{cases}$$

PROOF. Lemma 2.2 gives $H^0(K_C L^{1-m}) = 0$. The connecting homomorphism of (17) sends v to $(a \mapsto \epsilon(va)) \in H^0(L^{m-1})^*$. It lands in $Q_{m-1}(L)^*$, so it is δ_ϵ, and

$$h^0(E_\epsilon) = c - \operatorname{rank} \delta_\epsilon. \tag{18}$$

For $m = 3$, suppose f_1, f_2, f_3 are independent sections of E_ϵ. Their images v_i in $H^0(L)$ are independent. Put $a_{ij} = f_i \wedge f_j \in H^0(K_C L^{-1})$. The rank-two determinant identity gives

$$v_1 a_{23} - v_2 a_{13} + v_3 a_{12} = 0 \quad \text{in } H^0(K_C).$$

Petri injectivity implies $a_{12} = a_{13} = a_{23} = 0$. The sections generate a rank-one subsheaf. Its saturation is a line subbundle A with at least three sections; its map to L is nonzero, since $H^0(K_C L^{-2}) = 0$. Write $A = L(-D)$. Cubic interpolation and Lemma 4.1 give a contradiction. Thus $h^0(E_\epsilon) \le 2$.

For $m \ge 4$, the determinant is $K_C L^{2-m}$, with no sections. Any two independent sections of E_ϵ therefore have zero wedge and generate a rank-one subsheaf whose saturation is a line subbundle with at least two sections. Quartic interpolation, its higher-degree propagation, and Lemma 4.1 exclude this. Hence $h^0(E_\epsilon) \le 1$. Equation (18) proves both assertions. □

LEMMA 4.3 (Linear spaces of matrices). *Let U, W be finite-dimensional vector spaces, let $k \ge 0$ be an integer, and let $S \subseteq \operatorname{Hom}(U, W)$ be a linear space whose nonzero elements have rank at least $k+1$. If $k < \min(\dim U, \dim W)$, then*

$$\dim S \le (\dim U - k)(\dim W - k).$$

If $\dim W < k+1$, then $S = 0$.

PROOF. Put $a = \dim U, b = \dim W$. For $k = 0$, the bound is $\dim S \le ab$, which follows from $S \subseteq \operatorname{Hom}(U, W)$. For the remaining case of the first assertion, assume $1 \le k < \min(a, b)$. Matrices of rank at most k have dimension $k(a + b - k)$: choose a k-dimensional image in W, contributing $k(b-k)$, then a surjection $U \to \mathbf{C}^k$, contributing ak. The rank-k stratum is dense in this determinantal variety. Its codimension is $(a-k)(b-k)$. The projective space $\mathbf{P}(S)$ is disjoint from its projectivization. The projective dimension theorem gives the bound. The last assertion is immediate. □

PROOF OF THEOREM 1.3. The linear map

$$Q_m(L)^* \longrightarrow \operatorname{Hom}(V, Q_{m-1}(L)^*), \qquad \epsilon \longmapsto \delta_\epsilon$$

is injective by Proposition 4.2. For $m = 3$, apply Lemma 4.3 with $a = c, b = b_2, k = c - 3$. When $b_2 < c - 2$, the lemma makes the image of the displayed injection zero, so $Q_3(L)^* = 0$. This proves (6). For $m \ge 4$, use $a = c, b = b_{m-1}, k = c - 2$, proving (7). If $b_2 \le c - 3$, then $b_3 = 0$, and induction gives all subsequent vanishings. If $b_2 \le c - 2$, then $b_3 \le 3 < c - 1$, since $c \ge 5$. The higher inequality gives $b_4 = 0$ and then all later vanishings. □

REMARK 4.4. A basepoint-free pencil $P \subseteq H^0(L)$ gives

$$0 \longrightarrow L^{m-2} \longrightarrow P \otimes L^{m-1} \longrightarrow L^m \longrightarrow 0.$$

For $m \ge 4$, $H^1(L^{m-2}) = 0$, so $P \otimes H^0(L^{m-1}) \to H^0(L^m)$ is surjective. This independently verifies propagation of surjectivity from degree three, or any later degree.

COROLLARY 4.5 (Failure strata). *On the very ample locus for Theorem 1.3's parameters, let $\mathcal{B}_m(e) = \{L : b_m(L) \ge e\}$, $e \ge 1$. Then*

$$\mathcal{B}_3(e) \subseteq \mathcal{B}_2(c - 3 + \lceil e/3 \rceil),$$
$$\mathcal{B}_m(e) \subseteq \mathcal{B}_{m-1}(c - 2 + \lceil e/2 \rceil) \quad (m \ge 4).$$

The dimensions satisfy the resulting inequalities.

PROOF. Rearrange (6) and (7), using the fact that cokernel dimensions are integers. □

PROPOSITION 4.6 (Degree-dependent bounds). *Let C be general and L very ample, with $d = \deg L$, $c = h^0(L) \ge 5$, and $\rho = g - c(g - d + c - 1)$. For $m \ge 3$, $\rho \le c + 2m - 8$ implies $\operatorname{rank} \delta_\epsilon \ge c - 2$ for every nonzero $\epsilon \in Q_m(L)^*$. For $m \ge 4$, $\rho \le m - 3$ implies $\operatorname{rank} \delta_\epsilon \ge c - 1$.*

PROOF. Failure of the first bound gives a subline with three sections: use Petri for $m = 3$, and determinant vanishing for $m \geq 4$. If interpolation fails on its divisor, a faithful quotient has $2 \leq q \leq c-3$, $\ell \geq 2q+m-2$, and

$$F_c(\ell, q) \geq q(c-q-2) + (m-2)(c-q-1).$$

The endpoint values of this concave quadratic are $m(c-3)-2$ and $c+2m-7$. Their difference is $(m-1)(c-5) \geq 0$; the minimum exceeds ρ. For the second bound a two-section subline gives $2 \leq q \leq c-2$. The minimum is now $m-2$, at $q = c-2$; the other endpoint is larger by $m(c-4)$. The secant theorem excludes both possibilities. □

5. A uniform quadratic bound by cuspidal degeneration

5.1. The central curve and its torsion-free sheaves. Choose distinct points $a_1, \ldots, a_g$ in $\mathbf{A}^1 \subseteq \mathbf{P}^1$. Let X be the curve with normalization $\nu\colon \mathbf{P}^1 \to X$ obtained by replacing the local ring at a_i by $\mathbf{C} + \mathfrak{m}_{a_i}^2$. Writing $\tau_i = z - a_i$, its completed local ring there is $\mathbf{C}[[\tau_i^2, \tau_i^3]]$, an ordinary cusp. For example, on the affine chart its coordinate ring is

$$\{f \in \mathbf{C}[z] : f'(a_i) = 0 \text{ for all } i\}.$$

It contains $\prod_i (z-a_i)^2 \mathbf{C}[z]$, so the normalization is finite. Adding the unchanged point at infinity gives a projective integral curve. The normalization sequence has a quotient of length one at each cusp, and hence $p_a(X) = g$.

LEMMA 5.1. *Every rank-one torsion-free sheaf F on X has the form $\nu_{J*}M$, where $\nu_J\colon X_J \to X$ is the partial normalization at the cusps where F is not locally free and M is a line bundle on X_J. If $b = |J|$ and $\deg F = d$, then*

$$p_a(X_J) = g - b, \qquad \deg M = d - b, \qquad H^0(F) = H^0(M).$$

PROOF. Consider a completed cusp ring $A = \mathbf{C}[[z^2, z^3]]$, with normalization $B = \mathbf{C}[[z]]$. A rank-one torsion-free module is a fractional ideal. Choose a nonzero element of least valuation, divide by it, and obtain an isomorphic fractional ideal I with $A \subseteq I \subseteq B$. Since $\dim_{\mathbf{C}}(B/A) = 1$, it is either A or B. Thus the only two local module types are the free module and the normalization module. Completion is faithfully flat. At a nonfree cusp the action of the normalization ring is recovered as the endomorphism ring of the module; these actions glue to the partial normalization. The resulting module is locally invertible on X_J, giving M. Finite pushforward preserves cohomology and Euler characteristic. Using $\deg F = \chi(F) - \chi(\mathcal{O}_X)$ gives $\deg F = \deg M + b$, and proves the displayed formulas. □

For such a sheaf, set $g' = g - b, d' = d - b$. The pullback of M to $\mathbf{P}^1$ is $\mathcal{O}_{\mathbf{P}^1}(d')$. In the polynomial trivialization away from infinity, invertibility at each remaining cusp is a unit multiple of its local cusp ring. After rescaling, the unit is $1 + \lambda_i(z-a_i)$ modulo $(z-a_i)^2$. Consequently, for suitable $\lambda_i \in \mathbf{C}$,

$$H^0(X_J, M) = \{f \in \mathrm{Pol}_{d'} : f'(a_i) = \lambda_i f(a_i),\ i \notin J\}. \tag{19}$$

Here Pol_e means polynomials of degree at most e, and is zero when $e < 0$. No generality of the a_i or λ_i is needed below.

LEMMA 5.2 (Wronskian bound). *Let $k \geq 1$. If a k-dimensional subspace of $\mathrm{Pol}_{d'}$ satisfies the g' conditions in* (19)*, then*

$$g' - k(g' - d' + k - 1) \geq 0.$$

PROOF. At a cusp, if the subspace has a section not vanishing there, its vanishing sequence is at least $0, 2, 3, \ldots, k$. If every section vanishes there, every derivative does too, and the vanishing sequence is at least $2, 3, \ldots, k+1$. In both cases the Wronskian has order at least $k-1$. The Wronskian of k independent sections of $\mathcal{O}_{\mathbf{P}^1}(d')$ is nonzero in characteristic zero

and has total zero degree $k(d'-k+1)$. Therefore $g'(k-1) \le k(d'-k+1)$, which is the stated inequality. □

For the parameters $g = cs+\rho, d = g+c-s-1$, this also proves that a torsion-free F with $h^0(F) \ge c$ can be nonfree at at most ρ cusps. If $\rho < c+s+1$, it has exactly c sections: the bound for $k = c+1$ would read $\rho - b - c - s - 1 \ge 0$. These facts describe the boundary, but completeness is unnecessary for the quadratic estimate itself.

5.2. Diagonal filtration of the relation space.

PROPOSITION 5.3. *For M as in* (19) *and any subspace $V \subseteq H^0(M)$,*

$$\dim \ker\bigl(\operatorname{Sym}^2 V \longrightarrow H^0(M^2)\bigr) \le \mathsf{B}(d'-g').$$

PROOF. The assertion is immediate when $d' < 0$. Identify $\operatorname{Sym}^2 V$ with symmetric biforms $T(x,y)$ of bidegree at most d'. Its multiplication kernel I consists of those with $T(z,z) = 0$. Let

$$I_j = \{T \in I : (x-y)^{2j} \text{ divides } T\}, \qquad j \ge 1.$$

Symmetry shows $I = I_1$. For $T = (x-y)^{2j}R \in I_j$, R is symmetric and has bidegree at most $d'-2j$. Define $\gamma_j(T)(z) = R(z,z)$. Its kernel on I_j is I_{j+1}: if R vanishes on the diagonal, its quotient by $x-y$ is alternating and also vanishes on the diagonal, so $(x-y)^2$ divides R.

Every element of V obeys the Robin condition. Applying it in the x variable to T, and dividing by $(a_i-y)^{2j-1}$, gives the polynomial identity

$$2jR(a_i,y) + (a_i-y)\bigl(R_x(a_i,y) - \lambda_i R(a_i,y)\bigr) = 0. \tag{20}$$

At $y = a_i$, it gives $R(a_i,a_i) = 0$. Its derivative with respect to y, evaluated there, gives

$$2jR_y(a_i,a_i) - R_x(a_i,a_i) = 0.$$

Symmetry identifies the two partial derivatives, so $(2j-1)R_x(a_i,a_i) = 0$. Thus $\gamma_j(T)$ vanishes to order at least two at every remaining cusp.

Put $P(z) = \prod_{i\notin J}(z-a_i)$. We have proved an injection

$$I_j/I_{j+1} \hookrightarrow P(z)^2 \operatorname{Pol}_{2d'-4j-2g'}.$$

Therefore

$$\dim I \le \sum_{j\ge 1} \max\{0, 2(d'-g')+1-4j\} = \mathsf{B}(d'-g').$$

The filtration terminates by the bidegree bound. □

The sum in (8) is evaluated as follows. If $h = 2a \ge 2$, its nonzero terms are $4a-3, 4a-7, \dots, 1$, whose sum is $a(2a-1)$. If $h = 2a+1 \ge 3$, its terms are $4a-1, 4a-5, \dots, 3$, with sum $a(2a+1)$. These are respectively $\binom{2a}{2}$ and $\binom{2a+1}{2}$; for $h \le 1$ there is no positive term.

5.3. A framed projective deformation. The following construction fixes the outside coordinate and every overlap map used in the residue calculation.

LEMMA 5.4. *For the cuspidal curve X above, put*

$$S = \mathbf{C}[[\alpha_1,\beta_1,\dots,\alpha_g,\beta_g]], \qquad \mathfrak{m} = (\alpha_i,\beta_i)_{i=1}^g, \qquad U = \mathbf{P}^1 \setminus \{a_1,\dots,a_g\}.$$

There is a flat projective family $\mathcal{X} \to \operatorname{Spec} S$ with geometrically integral, locally planar fibres and special fibre X. On every finite $\mathfrak{m}$-adic thickening its outside chart is the constant chart U, and its cusp charts have equations

$$y_i^2 = x_i^3 + \alpha_i x_i + \beta_i.$$

Writing $\tau_i = z - a_i$, the overlap maps are

$$x_i = \tau_i^2, \qquad y_i = \tau_i^3\left(1 + \frac{\alpha_i}{\tau_i^4} + \frac{\beta_i}{\tau_i^6}\right)^{1/2}. \tag{21}$$

Every fixed point of U, including infinity, lifts to a smooth section. For arbitrary constants A_i, B_i, the substitution $\alpha_i = A_i u^4$, $\beta_i = B_i u^6$ defines a projective one-parameter family. If $4A_i^3 + 27B_i^2 \neq 0$ for all i, its geometric generic fibre is smooth.

Proof. Set $S_n = S/\mathfrak{m}^{n+1}$ and $U_n = U \times \operatorname{Spec} S_n$. For each i, define

$$h_i(x) = \prod_{j \neq i} (x - (a_j - a_i)^2), \qquad Y_{i,n} = \operatorname{Spec} \frac{S_n[x_i, y_i, h_i(x_i)^{-1}]}{y_i^2 - x_i^3 - \alpha_i x_i - \beta_i}.$$

Since $h_i(0) \neq 0$, this chart contains the central cusp. The open set $D(x_i) \subset Y_{i,n}$ is identified by (21) with

$$W_{i,n} = \operatorname{Spec} S_n[\tau_i, \tau_i^{-1}, h_i(\tau_i^2)^{-1}] \subset U_n.$$

The square root is a finite binomial expansion, since its nonconstant argument lies in the nilpotent ideal $\mathfrak{m}S_n$. Its inverse coordinate map is

$$\tau_i = \frac{y_i}{x_i} \left(1 + \frac{\alpha_i}{x_i^2} + \frac{\beta_i}{x_i^3}\right)^{-1/2}.$$

The cusp equation gives $\tau_i^2 = x_i$; substitution in the forward map gives y_i. Thus these are inverse open immersions.

Glue U_n and the $Y_{i,n}$ along these opens. Transitions between inside charts pass through U_n, so the cocycle identity holds on every triple overlap. The formulas commute with reduction $S_{n+1} \to S_n$, giving a compatible Cartesian system X_n. Its special fibre is X: the origin of $Y_{i,0}$ is the cusp with ring $\mathbf{C}[\tau_i^2, \tau_i^3]$, and its complement has the prescribed normalization coordinate. The additional points removed by h_i, at coordinates $2a_i - a_j$, are covered by U unless they are prescribed cusps; each prescribed cusp is covered by its own inside chart.

The inside charts are flat over S_n, because the equation is monic in y_i; the outside chart is flat as well. Hence X_n is flat. Properness and separatedness are preserved by this nilpotent thickening of the proper curve X. More explicitly, universal closedness has the same topological assertion after every base change. For separatedness, the affine-intersection multiplication map is surjective modulo $\mathfrak{m}$; its cokernel equals its product with a nilpotent ideal and is therefore zero.

Choose an ample line bundle on X. At each step its lifting obstruction lies in

$$H^2\Big(X, (\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}) \otimes_{\mathbf{C}} \mathcal{O}_X\Big) = 0.$$

The description of this square-zero ideal uses flatness of the thickenings. The Picard exact sequence for a square-zero thickening therefore gives compatible invertible lifts [**24**, Tag 0C6R]; each remains ample. Grothendieck's algebraization theorem [**24**, Tag 089A] produces a proper algebraization with an ample line bundle, hence a projective family. The infinitesimal flatness criterion gives flatness along the special fibre. The nonflat locus is closed and proper over the local base; its image avoids the closed point and is empty. Openness of geometric integrality in a flat proper family gives geometrically integral fibres by the same local-base argument. The locus where $\Omega_{\mathcal{X}/S}$ needs more than two generators is closed and misses the special fibre. Its proper image in $\operatorname{Spec} S$ is therefore empty. Every fibre has embedding dimension at most two and is locally planar.

A fixed point of U defines compatible sections of all X_n. Choose an affine neighborhood of its special point in the algebraization. Every infinitesimal section factors through this neighborhood, and the compatible ring maps take values in $\varprojlim S_n = S$. They give a section over S. It lies in the relative smooth locus, since that locus is open and contains its special point. Distinct fixed points give disjoint sections. The algebraization retains the full system of outside charts and overlap maps on the finite thickenings.

Finally make the stated weighted base change. The map to $\mathbf{C}[[u]]/(u^N)$ factors through S_n whenever $4(n+1) \geq N$, so every finite thickening retains the specified charts and overlap maps. After $u = v^e$, use $4e(n+1) \geq N$. At a central cusp the completed local ring is

$$\mathbf{C}[[u, x_i, y_i]]/(y_i^2 - x_i^3 - A_i u^4 x_i - B_i u^6).$$

In this ring, the nonsmooth ideal is generated by $2y_i$ and $3x_i^2 + A_i u^4$. It contains $(4A_i^3 + 27B_i^2)u^{12}$. Indeed the polynomial identity in variables x, α, β

$$4\alpha^3 + 27\beta^2 = (4\alpha^2 + 6\alpha x^2 - 9\beta x)(3x^2 + \alpha) - 9(2\alpha x - 3\beta)(x^3 + \alpha x + \beta)$$

places the discriminant in the nonsmooth ideal, since the last cubic equals y^2 in the local ring. The discriminant condition thus excludes generic singularities in every cusp completion. The outside formal charts are smooth. A generic singularity would have a closure meeting the special fibre, by properness. It cannot meet the outside smooth locus; faithful flatness of local completion would preserve a prime avoiding u at a cusp, contradicting the displayed power of u in the nonsmooth ideal. This proves generic smoothness, and characteristic zero gives geometric smoothness. □

5.4. Specialization of all relations. For a flat projective family of integral locally planar curves, the relative compactified Picard space of rank-one torsion-free sheaves of fixed degree is projective; see [**3**, Theorem 8] and [**20**, Theorem 5.4]. A smooth-section rigidification gives a universal sheaf [**20**, Fact 4.4(ii)]. In the degree convention of that reference, the polarization has total $d + 1 - g$. Integral fibres have no proper component subcurves, so the moduli space includes every degree-d rank-one torsion-free sheaf. We use the family in Lemma 5.4 and its weighted base changes. To apply the semiuniversal formulation of [**20**, Theorem 5.4], let R_X be its complete local deformation base. Formal versality gives a continuous homomorphism $R_X \to S$ classifying our deformation; proper existence identifies the pulled-back family with the algebraization above. Projectivity and the universal sheaf are preserved by this base change.

Lemma 5.5 (Saturated relation lattices). *Let R be a discrete valuation ring, with fraction field K and residue field k. If H is a finite free R-module and $V_K \subseteq H_K$, then $V_R = V_K \cap H$ is free and saturated, and $V_R \otimes k \to H \otimes k$ is injective. The same assertion applies to any subspace $Q_K \subseteq \operatorname{Sym}^2 V_K$.*

Proof. If a nonzero $a \in R$ satisfies $ah \in V_R$, then $h \in V_K$, hence $h \in V_R$. Thus H/V_R is torsion-free. Finite torsion-free modules over a DVR are free, and the inclusion of V_R is a direct summand inclusion. Tensoring with k remains injective. Apply the same argument inside the free module $\operatorname{Sym}^2 V_R$. □

Proof of Theorem 1.4. Take a one-parameter smoothing $\mathcal{X} \to \operatorname{Spec} R$ of the curve X above. It is enough first to prove the bound on its geometric generic fiber. A line bundle and a finite-dimensional space of sections on that fiber are defined over a finite extension of the fraction field. Make that extension and replace R by the corresponding DVR. Properness of the relative compactified Picard extends the line bundle to an R-flat rank-one torsion-free sheaf $\mathcal{F}$, after rigidification or further base change. Its special fiber F has degree d.

The module $H = H^0(\mathcal{X}, \mathcal{F})$ is finite and torsion-free over R, and hence free. Its generic fiber is $H^0(\mathcal{X}_K, \mathcal{F}_K)$. The exact sequence obtained by multiplication by a uniformizer shows that

$$H \otimes k \hookrightarrow H^0(X, F)$$

is injective; surjectivity is not needed. For the given generic section space V_K, use the saturated lattice $V_R = V_K \cap H$. Let V_0 be the image of $V_R \otimes_R k$ in $H^0(X, F)$. If the quadratic multiplication kernel of V_K had dimension greater than $\mathsf{B}(d-g)$, choose a subspace Q_K of dimension $\mathsf{B}(d-g)+1$ in that kernel and put $Q_R = Q_K \cap \operatorname{Sym}^2 V_R$. Lemma 5.5 preserves all these independent relations on the special fiber.

Choose $2d+1$ distinct smooth points of X, and lift them to disjoint smooth sections after passing to a Henselian or complete base. The sheaf $\mathcal{F}$ is invertible along these sections. Evaluation of each element of Q_R is a section of a free rank-one R-module. It is zero on the generic fiber, hence zero over R.

By Lemma 5.1, $F = \nu_{J*}M$, with $\deg M = d-b$ on a curve of genus $g-b$. The product in $H^0(M^2)$ of every specialized relation vanishes at the chosen $2d+1$ points. A nonzero section of M^2 has a zero divisor of degree $2(d-b) \leq 2d$. Each specialized product is consequently zero. The independent specialized relations form a subspace of $\ker(\operatorname{Sym}^2 V_0 \to H^0(M^2))$ of dimension $\mathsf{B}(d-g)+1$. Proposition 5.3 bounds that kernel by

$$\mathsf{B}\big((d-b)-(g-b)\big) = \mathsf{B}(d-g),$$

a contradiction.

We finally pass from this geometric generic fiber to a general curve. For fixed g, d, consider a smooth local universal family. Its relative degree-d Picard scheme is proper. After a smooth-section rigidification, let $\mathcal{L}$ be the universal line bundle on the curve pulled back to that Picard scheme, and let π be its projection to the Picard scheme. Choose a sufficiently positive smooth divisor E so that $H^1(L(E)) = 0$ uniformly. The spaces of c-dimensional sections are realized by the closed zero condition

$$\mathcal{V} \longrightarrow \pi_*(\mathcal{L}(E)|_E) = 0$$

inside $\operatorname{Gr}(c, \pi_*\mathcal{L}(E))$. This is proper over the family base. Add the Grassmannian of $\mathsf{B}(d-g)+1$-planes in $\operatorname{Sym}^2 \mathcal{V}$, and require their products to vanish. These are closed conditions: equivalently, evaluate at $2d+1$ disjoint smooth sections after an etale localization. On a degree-$2d$ line bundle, these evaluations detect the zero section. The resulting bad incidence is proper, so its image is closed. These conditions are invariant under changes of rigidification and base change, and hence descend to a closed locus in smooth moduli. There are only finitely many c to consider, since $h^0(L) \leq d+1$ for $d \geq 0$. The geometric generic fiber of the smoothing lies outside their union. Its complement is therefore a nonempty open subset of the irreducible moduli space $\mathcal{M}_g$. This proves the simultaneous assertion for all L and all subspaces V. □

Proof of Corollary 1.5. Completeness and very ampleness follow from Lemma 2.5. Put $a = \binom{c+1}{2}$ and $b = 2d+1-g$. Lemma 2.2 gives $h^0(L^2) = b$. Direct substitution of $c = s+t$, $g = cs+\rho$, and $d = g+t-1$ gives

$$a - b = \binom{t-1}{2} - \binom{s}{2} - \rho. \tag{22}$$

For $\kappa = \dim\ker \mu_{2,L}$, rank-nullity yields

$$b_2 = \kappa - K + \binom{s}{2} + \rho.$$

Theorem 1.4 says $\kappa \leq K$; nonnegativity of κ and b_2 supplies the lower bounds. Apply Theorem 1.3 to the upper bound for b_2. □

Remark 5.6. The injectivity assertion for $d \leq g+1$ is due to Teixidor i Bigas [**25**]. Farkas [**10**, Remark 5.6] records the estimate $\dim\ker \mu_{2,L} \leq a(a+1)$ for $d \leq g+a+1$. At $a = d-g-1 \geq 1$, the bound above is half that quantity. Its proof includes all partial nonfree strata; compare [**4**] for a low-degree injectivity result for nonfree sheaves. The assertion for incomplete section spaces also follows by restricting the complete-space multiplication map. The bound is sharp as a uniform statement: for the canonical bundle on a general curve with $g \geq 4$, Max Noether's theorem gives

$$\dim\ker \mu_{2,K_C} = \binom{g+1}{2} - (3g-3) = \binom{g-2}{2} = \mathsf{B}((2g-2)-g).$$

6. The projective space of quadratic relations

We now retain the parameters of the line bundle as well as the quadratic relation. The resulting estimate is stronger than a bound for the kernel of one multiplication map.

We prove Theorem 1.1 in Sections 6–9.

The proof uses a weighted smoothing of a rational curve with ordinary cusps. We first count the auxiliary data, then compute the residue coefficient at every order, and finally compare relations through arbitrary torsion-free limits.

6.1. Residual polynomials and the auxiliary inverse. Let X have h ordinary cusps at distinct normalization coordinates a_i, and put $P(z) = \prod_i (z - a_i)$. For the moment L is invertible of degree $D = h + t - 1$, with $h^0(L) = c = s + t$ and $h = cs + \eta$. In compatible normalization frames,

$$V = \{f \in \mathrm{Pol}_D : f'(a_i) = \lambda_i f(a_i)\}, \tag{23}$$

$$U = \{u \in \mathrm{Pol}_e : u'(a_i) = (P''(a_i)/P'(a_i) - \lambda_i)u(a_i)\}, \quad e = h - t - 1. \tag{24}$$

The space U, of dimension s, represents $H^0(\omega_X L^{-1})$, by the differentials $u\,dz/P^2$. Choose infinity to be an ordinary point for U. Then U has a basis of degrees $e - s + 1, \dots, e$.

Lemma 6.1. *There exist a degree-η polynomial Q, an s-plane $W \subseteq \mathrm{Pol}_N$, $N = c + s + \eta - 1$, and an order-s operator $T = (P/Q)L_W$, where L_W is the monic differential operator of order s with kernel W, such that the space*

$$H = \{h \in \mathrm{Pol}_N : Th \in V\}$$

has dimension $c + s$, and

$$0 \longrightarrow W \longrightarrow H \xrightarrow{T} V \longrightarrow 0$$

is exact. Moreover,

$$T_x T_y : \mathrm{Sym}^2 H \cap (x - y)^{2s+2} \mathrm{Sym}^2 \mathrm{Pol}_{t+\eta-3} \xrightarrow{\sim} \ker \mu_{2,V}. \tag{25}$$

The roots of Q may coincide with one another or with the cusps, and U may have base points.

Proof. Take

$$Q = \frac{\mathrm{Wr}(U)}{P^{s-1}}, \qquad W = \frac{\mathrm{Wr}_{s-1}(U)}{P^{s-2}}.$$

Here $\mathrm{Wr}_{s-1}(U)$ is the span of the signed complementary Wronskians of a basis of U. A common Robin equation makes each r-fold Wronskian vanish at least $r - 1$ times at every a_i; hence these expressions are polynomial. Complementary-minor identities give, with compatible signs,

$$\mathrm{Wr}(W) = PQ^{s-1}, \qquad \mathrm{Wr}_{s-1}(W) = Q^{s-2}U. \tag{26}$$

The complementary-Wronskian identities are

$$\mathrm{Wr}(\mathrm{Wr}_{s-1}(u)) = \mathrm{Wr}(u)^{s-1}, \qquad \mathrm{Wr}_{s-1}(\mathrm{Wr}_{s-1}(u)) = \mathrm{Wr}(u)^{s-2}u.$$

They follow from the adjugate identity for the fundamental matrix $(u_j^{(i)})_{0 \le i < s,\, 1 \le j \le s}$: differentiating a cofactor row replaces its last derivative row, since all terms with two equal derivative rows vanish; Jacobi's complementary-minor formula then gives the displayed powers of its determinant. Multiplication of all functions in an r-plane by a function a multiplies its Wronskian by a^r, which gives (26). The degrees at infinity give

$$\deg Q = s(e - s + 1) - h(s - 1) = \eta.$$

The space W has a basis of degrees $N - s + 1, \dots, N$.

Choose columns w and u of bases compatible with (26), and set $v = u/P^2$. The last-row cofactor formula is exactly

$$v^T w^{(j)} = 0 \ (0 \le j < s - 1), \qquad v^T w^{(s-1)} = Q/P. \tag{27}$$

In particular

$$Th = \frac{\mathrm{Wr}(w_1, \dots, w_s, h)}{Q^s}.$$

For $f \in V$, every $v_j f$ has only double poles at P, zero residues, and growth $O(z^{-2})$ at infinity. Its primitive vanishing at infinity is

$$A_j(f) = -\sum_i \frac{u_j(a_i) f(a_i)}{P'(a_i)^2 (z - a_i)}, \qquad A_j(f)' = v_j f.$$

The apparent poles of $h = w^T A(f)$ cancel, by $\sum_j w_j(a_i) u_j(a_i) = 0$, which follows from (27). Thus h is polynomial of degree at most $N - 1$. Differentiating and using (27) gives

$$h^{(j)} = (w^{(j)})^T A(f) \quad (0 \le j < s), \qquad L_W h = (Q/P) f.$$

This proves surjectivity of $T : H \to V$, whose kernel is exactly W, and the asserted dimension.

To construct the quadratic inverse, write f for a column basis of V, A for the $s \times c$ matrix of these primitives, and q for a constant symmetric matrix with $f^T q f = 0$. For $u \in U$ put

$$F_u(z) = \sum_i \frac{u(a_i) S_q(z, a_i)}{P'(a_i)^2 (z - a_i)}, \qquad S_q(z, w) = f(z)^T q f(w).$$

Write $S_{ab} = (\partial_z^a \partial_w^b S_q)(a_i, a_i)$ and $\lambda = \lambda_i$. The relation, symmetry, and Robin equation give

$$S_{00} = 0, \qquad S_{10} = \lambda S_{00} = 0, \qquad S_{11} = \lambda S_{10} = 0.$$

Differentiating $S_q(z, z) = 0$ twice and three times gives

$$2S_{20} + 2S_{11} = 0, \qquad 2S_{30} + 6S_{21} = 0, \qquad S_{21} = \lambda S_{20}.$$

Thus $S_{j0} = 0$ for $0 \le j \le 3$. The residue theorem applied, in the second variable, to

$$\frac{S_q(z, w) u(w)\, dw}{P(w)^2 (z - w)}$$

gives

$$\sum_i \frac{u(a_i) S_q(z, a_i)}{P'(a_i)^2 (z - a_i)^2} = 0.$$

There is no residue at $w = z$, because $S_q(z, z) = 0$, nor at infinity, where the differential is $O(w^{-3})dw$. Differentiating F_u at each cusp, the numerator derivatives satisfy its Robin equation and the denominator derivatives cancel by the last identity. The same-cusp term has a fourfold zero. Hence $F_u \in V$, with degree at most $D - 1$.

Set

$$h = A^T w, \quad b = Aqf, \quad C = AqA^T, \quad K' = bv^T, \quad K(\infty) = 0, \quad a = Kw.$$

Each component of b is $-F_{u_j}$, so the primitive K exists rationally, has simple poles only at P, and is $O(z^{-2})$ at infinity. Its poles cancel in $a = Kw$ by the same cofactor equation. Thus a is polynomial. Since $A' = vf^T$ and $b = Aqf$,

$$C' = vb^T + bv^T = K'^T + K'.$$

Both sides of $C = K + K^T$ vanish at infinity, so this identity follows. For $0 \le j < s - 1$, the cofactor equations give

$$K' w^{(j)} = b(v^T w^{(j)}) = 0.$$

Consequently

$$h^{(j)} = A^T w^{(j)}, \qquad a^{(j)} = K w^{(j)} \quad (0 \le j < s), \qquad L_W a = (Q/P) b,$$

and hence $Ta = b$. Every component of a belongs to H. The corrected biform

$$B(x, y) = h(x)^T q h(y) - w(x)^T a(y) - a(x)^T w(y) \tag{28}$$

satisfies $T_xT_yB = S_q$ and

$$(\partial_x^i\partial_y^j B)(z,z) = (w^{(i)})^T(C - K - K^T)w^{(j)} = 0 \quad (0 \le i,j < s).$$

Successively taking $i = j$ shows $(x-y)^{2s} \mid B$. If $B = (x-y)^{2s}R$, restriction of T_xT_yB to the diagonal is

$$(-1)^s(2s)!(P/Q)^2R(z,z).$$

It vanishes, so symmetry supplies two further diagonal factors. The bidegree of $B/(x-y)^{2s+2}$ is at most $N - 2s - 2 = t + \eta - 3$.

Conversely, an operator of order s in both variables sends every tensor on the left of (25) to a relation. A tensor in its kernel belongs to $W \cdot H$, and has matrix rank at most $2s$. A nonzero biform of diagonal order at least n has tensor rank at least $n+1$: at a general diagonal point, the derivative matrix with indices $0, \dots, n'$, where n' is its exact diagonal order, has zero entries when $i + j < n'$ and nonzero entries on its anti-diagonal. This matrix has rank $n' + 1$. For $n = 2s + 2$ the two rank bounds are incompatible. This proves injectivity and completes the proof. □

6.2. All primary conditions of the auxiliary space. We describe H locally, including collision strata and residual base points. At a finite point put $\epsilon = \operatorname{ord}_a P \in \{0,1\}$, $m = \operatorname{ord}_a Q$, and let $a_0 < \cdots < a_{s-1}$ be the vanishing sequence of U. The vanishing sequence of W is $b_0 < \cdots < b_{s-1}$, where

$$\sum a_i = \frac{s(s-1)}{2} + (s-1)\epsilon + m, \qquad b_i = s - 1 + \epsilon + m - a_{s-1-i}. \tag{29}$$

Write $\tau = z - a$, and let $W_a \subseteq \mathbf{C}[[\tau]]$ be the space of germs of elements of W. Let H_a consist of formal germs whose image under T is regular at a and, when a is a cusp, satisfies the target Robin equation. For $n \notin \{b_i\}$, the first term of $T(\tau^n)$ is a nonzero constant times

$$\prod_i (n - b_i)\tau^{\,n+\epsilon-s-m}.$$

Subtracting pivot germs of W in increasing order therefore gives the following exact local preimages:

$$\begin{array}{ll} \epsilon = 0: & H_a = W_a + \tau^{s+m}\mathbf{C}[[\tau]], \\ \epsilon = 1,\ a_0 = 0: & H_a = W_a + \tau^{s+m-1}\mathbf{C}[[\tau]], \\ \epsilon = 1,\ a_0 > 0: & H_a = W_a + \mathbf{C}(\tau^{s+m-1} + \xi\tau^{s+m}) + \tau^{s+m+1}\mathbf{C}[[\tau]]. \end{array} \tag{30}$$

Each has colength m. In the second line the last W pivot is $s+m$, and all the others are at most $s+m-2$. Regularity already imposes the target Robin equation: the adjoint Lagrange identity says that $(u/(PQ))L_W h$ is a meromorphic derivative; its residue is zero, and one can take $u(a) \neq 0$. In the third line every $a_i \ge 2$, so regularity alone has colength $m-1$. The two jets of orders $s+m-1, s+m$ map to independent constant and linear target jets. The target Robin equation is one additional nonzero condition. To recover its gluing parameter explicitly, put $k = s + m - 1$ and write

$$T\tau^k = A_0 + A_1\tau + O(\tau^2), \qquad T\tau^{k+1} = B_1\tau + O(\tau^2), \qquad A_0B_1 \neq 0.$$

The Robin equation for $T(\tau^k + \xi\tau^{k+1})$ reads $A_1 + \xi B_1 = \lambda A_0$. Thus $\xi = (\lambda A_0 - A_1)/B_1$ and $\lambda = (A_1 + \xi B_1)/A_0$.

These are all finite conditions. At infinity one subtracts an element of W to reduce the polynomial degree from N to $N - s$. The exponent shift of T is $h - \eta - s$. The resulting image degree is

$$N - s + h - \eta - s = D.$$

Thus there is no further condition at infinity.

LEMMA 6.2 (Primary incidence). *Fix integers $p \geq 1, \eta \geq 0, t_0$. At distinct points a_i, choose primary subspaces $H_i \subseteq \mathbf{C}[[z-a_i]]$, meaning that each contains $(z-a_i)^{n_i}\mathbf{C}[[z-a_i]]$ for some n_i. Their colengths satisfy $m_i \geq 1$, with $\sum m_i = \eta$. Write J_i for their missing pivot orders, and assume*

$$|J_i| = m_i, \qquad \sum_{h \in J_i} h = \frac{m_i(m_i+1)}{2}, \qquad \max J_i < m_i + p.$$

Let $H \subseteq \mathbf{C}[z]$ impose these local conditions. As the support points and primary subspaces vary, the incidence of these data with nonzero projective symmetric biforms R of bidegree at most $t_0 + \eta - 3$ satisfying

$$(x-y)^{2p} R \in \operatorname{Sym}^2\bigl(H \cap \operatorname{Pol}_{t_0+\eta-3+2p}\bigr)$$

has dimension at most

$$\eta + \binom{t_0 - 1}{2} - 1$$

if $t_0 \geq 3$, and is empty if $t_0 \leq 2$.

PROOF. Put $K(t_0) = \binom{t_0-1}{2}$ for $t_0 \geq 3$, and zero otherwise. A local cell is a Schubert cell of m-planes in $\mathbf{C}[\tau]/(\tau^{2m})$. Its row-echelon dimension is

$$\operatorname{ind}(J) = \sum_{h \in J} h - \frac{m(m-1)}{2} = m.$$

The hole-sum condition implies $\max J \leq 2m-1$.

We need two elementary local jet facts. Let $\ell_0 = m + p$. There are exactly p allowed orders below ℓ_0. A first nonzero homogeneous term of a symmetric tensor in $\operatorname{Sym}^2 H_i$, of degree less than $2\ell_0$, uses at most p symmetric monomial pairs. A nonzero symmetric homogeneous polynomial divisible by $(x-y)^{2p}$ needs at least $p+1$ pairs. Indeed, substituting $(x, y) = (e^v, e^{-v})$, a pair with exponents $(j, d-j)$ becomes a nonzero multiple of $\cosh((d-2j)v)$. Vanishing through order $2p-1$ is a Vandermonde system at the distinct nodes $(d-2j)^2$. At most p columns are independent, so all such coefficients would be zero. The first possible total degree is consequently $2\ell_0$. At that degree there are exactly $p+1$ allowed pairs, and the Vandermonde kernel is one-dimensional, with every coordinate nonzero. Denote this line by ℓ_J. Different hole sets have different monomial supports, so they give different lines. Both the vanishing of all lower jets and membership of the degree-$2\ell_0$ jet in ℓ_J are closed linear conditions.

For fixed primary data, apply the first jet fact to $B = (x-y)^{2p}R$. The diagonal polynomial $R(z, z)$ is divisible by $\prod_i (z-a_i)^{2m_i}$, leaving degree at most $2t_0 - 6$. Its kernel consists of two more diagonal factors. Repeating with (t_0, p) replaced by $(t_0 - 2, p+1)$ gives the uniform vector-space bound

$$\dim\{R\} \leq \sum_{j \geq 0} \max(0, 2t_0 - 5 - 4j) = K(t_0). \tag{31}$$

We prove the incidence estimate by induction on η, simultaneously for all t_0, p. For $\eta = 0$ its projective space has dimension $K(t_0) - 1$. Fix the multiplicities and hole cells, and compactify their affine cells by their projective Schubert varieties, while keeping the support points distinct. If a boundary cell has index $m' < m$, put $j = m - m'$. Every member is divisible by τ^j, and dividing by this factor gives a primary space of colength and index m' of the same permitted kind. To check this explicitly, if r is the first allowed order, the smallest possible hole sum is $m(m+1)/2 - r$, so $m' \geq m - r$ and $r \geq j$. Delete the first j holes and subtract j from the others. Their number is m', and their sum is

$$m' + \frac{m(m-1)}{2} - \frac{j(j-1)}{2} - j(m-j) = \frac{m'(m'+1)}{2}.$$

Their maximum is less than $m' + p$.

Thus on the boundary

$$H_0 = DH', \quad D = \prod_i (z - a_i)^{j_i}, \quad j = \deg D > 0, \quad \eta' = \eta - j,$$

and

$$B_0 = D(x)D(y)(x - y)^{2p}R'.$$

At any changed local cell the old closed degree-$2\ell_0$ jet lies in ℓ_J, while its boundary pivot set requires it to lie in a different ℓ_{J^0}. That jet is therefore zero. If the support disappears completely, $j_i = m_i$; division by $D(x)D(y)$ then gives

$$R'(a_i, a_i) = 0. \tag{32}$$

If $R'(z, z) \not\equiv 0$, every disappeared support point is a root of this polynomial, and hence has finitely many choices. Retained support points already belong to H'. The induction hypothesis bounds this boundary by $\eta' + K(t_0) - 1 \le \eta + K(t_0) - 2$. If $R'(z, z) \equiv 0$, write $R' = (x - y)^2 R''$; induction at $(t_0 - 2, p + 1, \eta')$, allowing at most j parameters for D, gives

$$\eta' + K(t_0 - 2) - 1 + j \le \eta + K(t_0) - 2.$$

When the smaller t_0 has no terms the incidence is empty.

Finally an oversized component of the original incidence must reach this boundary. Its fixed-support fibre has dimension greater than $K(t_0) - 1$, since the number of support parameters is at most η. By (31) its image in the product of affine cells is positive-dimensional. Its closure in the product of projective Schubert varieties and the projective R-space must meet the boundary: a proper variety has only a finite image in an affine variety. The product pivot Plücker coordinate cuts that boundary as an effective Cartier divisor. The closure is integral, not contained in the divisor, and meets it, so this intersection has dimension exactly one less than the component. The boundary estimates just proved give a contradiction. This completes the induction. □

The spaces in (30) satisfy the primary conditions with $p = s + 1$. Indeed their allowed pivot counts before the conductor are $r = s, s - 1, s + 1$, respectively. Their pivot sums are

$$\frac{r(r-1)}{2} + (r - 1)m.$$

Subtracting from the sum of $0, \dots, m + r - 1$ gives the hole sum $m(m + 1)/2$. Their conductor is at most $m + s + 1$. This includes every collision and residual base point in Lemma 6.1.

Proposition 6.3. *Assume $0 \le \eta \le c - 4$, and choose the h cusps sufficiently generally. For $j \ge 1$, let $F_j(V) \subseteq \operatorname{Sym}^2 V$ be the tensors vanishing to order at least $2j$ on the diagonal, and put*

$$K_j = \sum_{\ell \ge j} \max(0, 2t - 1 - 4\ell).$$

The incidence of $(L, [q])$, with $h^0(L) = c$ and $0 \ne q \in F_j(V)$, has dimension at most $K_j - 1$.

Proof. The top derivative term of $T_x T_y$, at a generic diagonal point, is nonzero. A biform of exact diagonal order $2s + 2j$ therefore has image of order $2j$, with coefficient

$$(-1)^s \frac{(2s + 2j)!}{(2j)!} (P/Q)^2.$$

Thus F_j corresponds exactly to the primary incidence with $t_0 = t - 2j + 2$, $p = s + j$. The preceding lemma bounds its dimension, with the auxiliary H varying, by $\eta + K_j - 1$.

The data (H, Q, W) recover the actual cusp curve and line bundle. In fact $P = \operatorname{Wr}(W)/Q^{s-1}$, normalized to be monic; Wronskian duality recovers U. At a cusp where U does not vanish, its Robin equation recovers λ_i. At a residual base point the last line of (30) recovers that equation from its independent constant and linear image jets. The primary conditions factor through

jets modulo Q^2, of total length 2η. Since $\eta \le c-4$, we have $N \ge 2\eta - 1$, and polynomial interpolation gives

$$\dim H = N + 1 - \eta = c + s.$$

Choosing $W \in \mathrm{Gr}(s, H)$ adds cs parameters. The truncation degree is unchanged when j varies, since

$$(t - 2j + 2) + \eta - 3 + 2(s + j) = N.$$

The total dimension over the cusp-configuration space is consequently at most

$$cs + \eta + K_j - 1 = h + K_j - 1.$$

All constructions and recoveries are algebraic on the finitely many pivot strata. A component failing to dominate the h-dimensional configuration space is avoided by a general configuration; every other component has general fibre dimension at most $K_j - 1$.

The choice of ordinary infinity introduces no extra parameter. The Wronskian divisor of U has degree $s(e-s+1) = h(s-1)+\eta$. Preselect more than $h+h(s-1)+\eta = hs+\eta$ fixed points on $\mathbf{P}^1$. For every U under consideration at least one is neither a cusp nor a Wronskian zero. The finitely many fixed Möbius transformations sending these points to infinity cover the data. Diagonal order is invariant under each transformation. Their general configuration conditions may be intersected, so these finitely many charts prove the assertion for every varying U. □

6.3. The residual exterior space and its evaluations.

Lemma 6.4. *For the spaces* (23)–(24)*, the map*

$$\Omega = \bigwedge^2 U \longrightarrow \mathrm{Pol}_{h-2t-4}, \qquad \omega = \sum u \wedge v \longmapsto G_\omega = \frac{\sum (uv' - u'v)}{P}$$

is injective. The covectors

$$\ell_i(\omega) = G_\omega(a_i)/P'(a_i)$$

define an injection $\Omega \hookrightarrow \mathrm{coker}(\mu_{2,V})^*$.

Proof. The residual Robin equations make each numerator divisible by P. Its degree is at most $2e-2$, since the possible top Wronskian term cancels, so the quotient degree is at most $2e-2-h = h-2t-4$. To prove injectivity on $\bigwedge^2 U$, represent ω by its alternating biform of bidegree at most e. If its first Wronskian is zero, its diagonal order is at least three. At every subsequent odd diagonal order $n \ge 3$, write the biform as $(x-y)^n R$, with R symmetric. Put $\lambda_i^{\mathrm{res}} = P''(a_i)/P'(a_i) - \lambda_i$. The Robin equation gives

$$nR(a_i, y) + (a_i - y)(R_x - \lambda_i^{\mathrm{res}} R)(a_i, y) = 0.$$

At $y = a_i$ it forces $R(a_i, a_i) = 0$; after one derivative it gives $(n-1)R_x(a_i, a_i) = 0$. Thus the diagonal quotient has double zeros at all h cusps, and lies in $\mathrm{Pol}_{2e-2n-2h} = 0$. Symmetry supplies two additional diagonal factors. The finite bidegree filtration ends only at the zero tensor, proving injectivity.

We now construct the induced functional on the multiplication cokernel. For a source section write

$$\frac{f}{P} = p(z) + \sum_i \frac{r_i}{z - a_i}, \qquad p(z) = \sum_{a=0}^{t-1} p_a z^a.$$

Put $r = (r_i)^T$, $\mathbf{p} = (p_0, \dots, p_{t-1})^T$, and $\mathbf{1} = (1, \dots, 1)^T \in \mathbf{C}^h$. Set $X = \mathrm{diag}(a_i)$, $\mathsf{C}_{ij} = 1/(a_i - a_j)$ for $i \ne j$ and $\mathsf{C}_{ii} = 0$, and

$$D_i = \frac{P''(a_i)}{2P'(a_i)} - \lambda_i, \quad A = \mathsf{C} + \mathrm{diag}(D_i), \quad E = (\mathbf{1}, X\mathbf{1}, \dots, X^{t-1}\mathbf{1}).$$

The source equation is $Ar + E\mathbf{p} = 0$. A residual polynomial u corresponds to the vector $\bar{u}_i = u(a_i)/P'(a_i)$, satisfying $A^T\bar{u} = 0$, $E^T\bar{u} = 0$. For one wedge define

$$S_{ij} = \frac{\bar{u}_i\bar{v}_j - \bar{v}_i\bar{u}_j}{a_i - a_j} \quad (i \neq j), \quad S_{ii} = 0.$$

We have the identities

$$[X, A] = \mathbf{1}\mathbf{1}^T - I_h, \qquad S\mathbf{1} = 0, \qquad [X, S] = \Xi := \bar{u}\bar{v}^T - \bar{v}\bar{u}^T.$$

Here I_h is the identity matrix. Induction on the columns of E, using $E^T\bar{u} = E^T\bar{v} = 0$, gives

$$SX^{a+1}\mathbf{1} = XSX^a\mathbf{1} - \Xi X^a\mathbf{1} = 0 \quad (0 \leq a < t-1),$$

and hence $SE = 0$. Since $A^T\bar{u} = A^T\bar{v} = 0$,

$$[X, A^TS + SA] = (I_h - \mathbf{1}\mathbf{1}^T)S + A^T\Xi + \Xi A + S(\mathbf{1}\mathbf{1}^T - I_h) = 0.$$

Since the eigenvalues of X are distinct, $A^TS + SA$ is diagonal, with entries

$$\ell_i = -2\sum_{j\neq i} \frac{\bar{u}_i\bar{v}_j - \bar{v}_i\bar{u}_j}{(a_i - a_j)^2}. \tag{33}$$

Hence, for two source vectors,

$$r^T \operatorname{diag}(\ell) r' = (Ar)^T S r' + r^T S(Ar') = 0.$$

The residue of $(u/P)(v/P)' - (u/P)'(v/P) = G_\omega/P$ at a_i is precisely (33), which verifies the stated normalization.

Every target section has the unique partial-fraction form

$$\frac{F}{P}^2 = p_2(z) + \sum_i k_i \left(\frac{1}{(z-a_i)^2} - \frac{2D_i}{z-a_i} \right), \qquad \deg p_2 \leq 2t-2.$$

The k_i are arbitrary; the formula itself constructs the target polynomial and its Robin equations. The functional $F \mapsto \sum_i \ell_i k_i$ therefore acts on the actual target and annihilates all products. If it is zero, every $\ell_i = 0$, and G_ω, of degree less than h, is zero. The injectivity already proved forces $\omega = 0$. □

7. The all-order elliptic coefficient

We compute a nonzero coefficient at every Gaussian order. Put

$$R = \mathbb{Q}[A, B, x, y]/(y^2 - x^3 - Ax - B), \qquad \operatorname{wt}(A, B, x, y) = (4, 6, 2, 3).$$

On its annulus at infinity set

$$x = \tau^2, \quad y = \tau^3(1 + A/\tau^4 + B/\tau^6)^{1/2}, \quad \omega = \frac{dx}{2y} = \frac{d\tau}{\tau^2(1 + A/\tau^4 + B/\tau^6)^{1/2}}.$$

The flat basis is

$$e_0 = 1, \quad e_{2a} = x^a \ (a \geq 1), \quad e_{2a+3} = x^a y \ (a \geq 0);$$

its only missing nonnegative index is one. For $j \geq 1$, put

$$d_j = 2j + 2, \quad S_j(\tau, \sigma) = (\tau - \sigma)^{2j}(\tau^2 + 2j\tau\sigma + \sigma^2), \quad G_j = (-1)^j(2j)!(2j+2).$$

Lemma 7.1. *There is a unique homogeneous symmetric tensor Q_j of weight d_j in $R \otimes_{\mathbb{Q}[A,B]} R$ whose specialization at $A = B = 0$ is S_j, whose product is zero, and, for $1 \leq i < j$, the mixed diagonal derivative of its Laurent expansion*

$$\chi_i(\tau) = (\partial_\tau^i \partial_\sigma^i Q_j)(\tau, \tau)$$

has no positive power of τ. Its flat coefficients belong to $\mathbb{Q}[A, B]$.

PROOF. For every positive even M, let W_M be the symmetric homogeneous degree-M polynomials with no exponent one in either variable. It has dimension $M/2$. The first $M/2$ symmetric diagonal jets, of orders $0, 2, \dots, M-2$, give an isomorphism on W_M. A tensor in their common kernel is divisible by $(\tau-\sigma)^M$, and its coefficient of $\tau\sigma^{M-1}$ would be nonzero unless the tensor were zero. This also gives constant linear systems: in the symmetric monomial basis indexed by $0 \le a \le M/2,\ a \ne 1$, the mixed derivative row of order i has entry

$$(a)_i(M-a)_i + (M-a)_i(a)_i$$

for $a < M/2$, and $(M/2)_i^2$ for $a = M/2$; here $(a)_i = a(a-1)\cdots(a-i+1)$. Row $i=0$ is the product row. The kernel argument proves invertibility.

Start with the flat lift of S_j. At increasing positive parameter weights $0 < n < d_j$, write $M = d_j - n$. The already determined coefficients produce a known Laurent tensor, homogeneous of degree M, regular along the punctured diagonal. Use the displayed invertible system to cancel its first $M/2$ diagonal jets by a unique flat correction in W_M. Only weights generated by four and six occur. At $n = d_j$, subtract the remaining constant product by a constant tensor. The resulting product is a regular function on the affine cubic with no pole at infinity and zero constant, hence zero on the generic smooth cubic. It is therefore zero as a polynomial identity. All systems have rational constant entries, so all corrections are polynomial in A, B. This proves existence and uniqueness, including the constant target correction. □

For a tensor $Q = \sum q_{fg} f \otimes g$, define

$$\mathfrak{h}(Q) = \operatorname{Res}_\infty \left[-\omega \sum q_{fg} f \int x g\omega + x\omega \sum q_{fg} f \int g\omega \right], \qquad C_j = -\frac{\mathfrak{h}(Q_j)}{2G_j}. \tag{34}$$

All primitives are Laurent primitives at infinity. They have no logarithmic term: $f\omega$ is regular away from infinity on the generic smooth cubic, so its residue at infinity is zero. Changing primitive constants changes the expression by residue-free differentials. Only finitely many binomial terms of $(1 + A/\tau^4 + B/\tau^6)^{\pm 1/2}$ can contribute in a fixed total weight. Thus C_j is a polynomial, homogeneous of weight $2j+2$, without discriminant denominators.

Let $B_n(x)$ be the Bernoulli polynomials, normalized by

$$\frac{ue^{xu}}{e^u - 1} = \sum_{n \ge 0} B_n(x) \frac{u^n}{n!}, \qquad B_n = B_n(0).$$

PROPOSITION 7.2. *For every $j \ge 1$,*

$$C_j(-a^4/3,\, 2a^6/27) = \frac{(2a)^{2j+2} B_{2j+2}}{(-1)^j (2j)!(2j+2)} \ne 0 \quad (a \ne 0). \tag{35}$$

In particular every C_j is nonzero, their positive weights are pairwise distinct, and

$$C_1 = -A/5, \qquad C_2 = B/7, \qquad C_3 = A^2/75. \tag{36}$$

PROOF. Let $d = 2j+2$. Specialize $A = -a^4/3,\ B = 2a^6/27$. The normalization of the nodal cubic is

$$x = v^2 - \frac{2a^2}{3}, \qquad y = v(v^2 - a^2), \qquad \omega = \frac{dv}{v^2 - a^2},$$

whose affine ring is $\{f \in \mathbb{Q}(a)[v] : f(a) = f(-a)\}$. Indeed its odd part is v times a polynomial in v^2 vanishing at a^2.

The conditions on the low diagonal derivatives persist under $\tau = v + O(v^{-1})$. To check this, write

$$Q(b+\delta, b-\delta) = \sum_{h \ge 1} b_h(b) \delta^{2h},$$

then

$$\chi_i(b) = 4^{-i} \sum_{h=0}^{i} \binom{i}{h} (-1)^h (2h)! b_h^{(2i-2h)}(b), \qquad b_0 = 0.$$

Induction makes boundedness of the first i mixed derivatives equivalent to boundedness of the first i coefficients b_h. Under $\tau = v + O(v^{-1})$, the mean coordinate changes by terms with bounded coefficients, while the difference is δ times a bounded unit plus higher odd powers. Substitution and differentiation preserve these bounds. On the nodal normalization a bounded polynomial is constant. Every lower homogeneous component of degree $2r \le 2j$ must consequently be a multiple of $(v-w)^{2r}$. Hence

$$Q_j(v,w) = S_j(v,w) + \sum_{r=1}^{j} \beta_r a^{d-2r} (v-w)^{2r}.$$

Descent is $Q_j(a,w) = Q_j(-a,w)$. For the r-th summand the difference has degree $2r-1$ in w, with nonzero leading coefficient. These conditions determine the β_r uniquely.

Define

$$J_d(z) = \frac{(2a)^d}{2} \left[B_d\!\left(\frac{z}{2a}\right) + B_d\!\left(1 + \frac{z}{2a}\right) - 2B_d \right].$$

It is even, has leading term z^d, and has zero constant. The Bernoulli identities $B_d(x+1) - B_d(x) = dx^{d-1}$ and $B_d(1-x) = B_d(x)$ give

$$J_d(z) - J_d(z-2a) = da\big[z^{d-1} + (z-2a)^{d-1}\big].$$

They follow directly by comparing the generating functions $ue^{xu}/(e^u - 1)$; see also [**21**, §24.4]. It follows that the polynomial

$$\widetilde{Q}_j(v,w) = J_d(v-w) + d(vw - a^2)(v-w)^{d-2} \tag{37}$$

has all the stated properties and descends. By uniqueness it equals the specialized Q_j.

Integration by parts, using symmetry, transforms (34) into

$$\mathfrak{h}(Q) = 2 \sum q_{fg} \operatorname{Res}_\infty x f \omega \int g \omega.$$

Here $x\omega = dv + (a^2/3)\omega$. The second term has zero residue because its symmetric sum is a total derivative. For $g(a) = g(-a)$,

$$g\omega = g(a) \frac{dv}{v^2 - a^2} + \frac{g(v) - g(a)}{v^2 - a^2}\, dv,$$

and the second summand is polynomial. The first has formal primitive

$$Z(v) = \frac{1}{2a} \log \frac{v-a}{v+a} = -\sum_{r \ge 0} \frac{a^{2r}}{(2r+1) v^{2r+1}}.$$

For a polynomial $f(v) = \sum f_n v^n$, coefficient extraction gives

$$\operatorname{Res}_\infty f(v) Z(v)\, dv = \sum_{r \ge 0} \frac{f_{2r} a^{2r}}{2r+1} = \frac{1}{2a} \int_{-a}^{a} f(v)\, dv.$$

The integral here means polynomial antiderivation and endpoint evaluation. Consequently

$$\mathfrak{h}(Q_j) = \frac{1}{a} \int_{-a}^{a} Q_j(v,a)\, dv = \frac{1}{a} \int_{-2a}^{0} \big[J_d(z) + daz^{d-1}\big]\, dz.$$

The Bernoulli antiderivative $B'_{d+1} = (d+1) B_d$, followed by its difference identity, evaluates the two terms as

$$\int_{-2a}^{0} J_d(z)\, dz = a(2a)^d (1 - 2B_d), \qquad \int_{-2a}^{0} daz^{d-1}\, dz = -a(2a)^d.$$

Thus $\mathfrak{h}(Q_j) = -2(2a)^d B_d$, which is (35). The classical identity

$$\zeta(2r) = \frac{(2\pi)^{2r}}{2(2r)!}|B_{2r}| > 0$$

implies $B_{2r} \neq 0$; see [**21**, §25.6]. Finally the only weight-four, six, and eight monomials are A, B, A^2. The values $B_4 = -1/30, B_6 = 1/42, B_8 = -1/30$ give respectively

$$C_1(-a^4/3, 2a^6/27) = a^4/15, \quad C_2(-a^4/3, 2a^6/27) = 2a^6/189, \quad C_3(-a^4/3, 2a^6/27) = a^8/675.$$

Dividing by the three specialized monomials proves (36). □

REMARK 7.3. For Q_j/G_j the mixed j-th diagonal derivative of the central tensor is τ^2. This is the normalization used below. The constant correction in Lemma 7.1 contributes to the residue. Since

$$-\omega \int x\omega + x\omega \int \omega = (-2\tau^{-1} + O(\tau^{-5}))\, d\tau,$$

we have $\mathfrak{h}(1 \otimes 1) = 2$. Subtracting a constant product a therefore adds a to $-\mathfrak{h}(Q)/2$. For example, the full unnormalized Q_2 is

$$e_6 \otimes 1 + 1 \otimes e_6 - 9(e_4 \otimes e_2 + e_2 \otimes e_4) + 16 e_3 \otimes e_3 - 8A(e_2 \otimes 1 + 1 \otimes e_2) - 16B(1 \otimes 1).$$

Its product is zero because $2x^3 - 18x^3 + 16y^2 - 16Ax - 16B = 0$. The coefficient formula uses this full tensor, including its moving-source and target corrections.

7.1. The weight-twelve coefficient. At weight twelve the coefficient has two possible monomials, A^3 and B^2. The flat tensor Q_5/G_5, with $G_5 = -43545600$, is as follows. The labels τ, σ in the next formulas index the flat basis: an even power n means $e_n = x^{n/2}$, and an odd power $n \geq 3$ means $e_n = x^{(n-3)/2}y$. Write its parameter-weight components as $q_{5,n}$. The only nonzero ones are

$$\begin{aligned}
q_{5,0} &= -\frac{(\tau-\sigma)^{10}(\tau^2 + 10\tau\sigma + \sigma^2)}{43545600}, \\
q_{5,4} &= \frac{A}{5443200}\big(101\tau^8 - 372\tau^6\sigma^2 + 1472\tau^5\sigma^3 - 1890\tau^4\sigma^4 \\
&\qquad\qquad + 1472\tau^3\sigma^5 - 372\tau^2\sigma^6 + 101\sigma^8\big), \\
q_{5,6} &= \frac{B}{680400}\big(91\tau^6 - 459\tau^4\sigma^2 + 800\tau^3\sigma^3 - 459\tau^2\sigma^4 + 91\sigma^6\big), \\
q_{5,8} &= -\frac{A^2}{340200}(131\tau^4 - 78\tau^2\sigma^2 + 131\sigma^4), \\
q_{5,10} &= -\frac{73AB}{85050}(\tau^2 + \sigma^2), \qquad q_{5,12} = -\frac{2B^2}{1701}.
\end{aligned}$$

The $q_{5,12}$ entry is the constant target correction. The residue contributions follow from a finite sum. Put

$$E_n(\tau) = \begin{cases} (1 + A/\tau^4 + B/\tau^6)^{-1/2}, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases} \qquad E_n(\tau) = \sum_{u \geq 0} e_{n,u}\tau^{-u}.$$

The nonzero coefficients, when n is even, are

$$e_{n,u} = \sum_{\substack{p,q \geq 0 \\ 4p+6q=u}} \binom{-1/2}{p+q}\binom{p+q}{p} A^p B^q.$$

For one ordered tensor $e_a \otimes e_b$,

$$\mathfrak{h}(e_a \otimes e_b) = \sum_{u+v=a+b} e_{a,u}e_{b,v}\left(\frac{1}{b-v+1} - \frac{1}{b-v-1}\right). \tag{38}$$

There are no zero denominators: if b is even then v is even, and if b is odd then $v = 0$ and $b \geq 3$. Formula (38) comes by integrating the two powers in (34); the residue at infinity is the negative Laurent coefficient. Applying it to the displayed ordered monomials gives the complete contribution table for $-\mathfrak{h}(Q_5/G_5)/2$:

flat parameter weight	contribution
0	$-2(5A^3 + 33B^2)/11609325$
2	0
4	$118A^3/4465125$
6	$17B^2/42525$
8	$-278A^3/637875$
10	0
12	$2B^2/1701$

Summing the contributions gives

$$C_5 = -\frac{2A^3}{4875} + \frac{B^2}{637}.$$

8. Relations over an arbitrary valued smoothing

We prove the comparison needed to use the coefficients C_j for actual relations. Keep the original g cusps, put $P_{\mathrm{all}}(z) = \prod_{i=1}^{g}(z - a_i)$, and take the weighted smoothing

$$y_i^2 = x_i^3 + A_i u^4 x_i + B_i u^6.$$

Lemma 5.4 constructs this family with the specified annular maps. On every finite base thickening its outside chart is the complement of the cusp preimages in the rational normalization. These identifications give compatible formal trivializations. We use the proper relative compactified Picard space of [**3, 20**].

Take an actual projective relation on the geometric generic fibre. After finite extension and completion, its base is $R = \mathbf{C}[[v]]$, with uniformizer v, and we may write $u = v^e$, where $e \geq 1$ is the ramification index. The unit in this equation is absorbed by an e-th root after a finite extension. The line bundle, the sections, and the relation may vary at all integer powers of v, not only at multiples of e.

8.1. Completeness at every torsion-free centre. Let the special sheaf be nonfree at a set J of b cusps. The local classification in Lemma 5.1 writes it as $\nu_{J*}M$, where the partial normalization has genus $g' = g - b$ and $\deg M = d - b$. A k-dimensional polynomial section space on this curve has ramification weight at least $k - 1$ at each remaining cusp. If a section is nonzero there, its first jet is determined by its value; otherwise every section vanishes to order at least two. The total Wronskian weight is at most $k(d - b - k + 1)$, so

$$0 \leq g' - k(k - t).$$

For $k = c$ this says $b \leq \rho$. For $k = c + 1$ it would say $0 \leq \rho - b - c - s - 1 < 0$. Therefore every relevant special sheaf has

$$h^0(M) = c, \qquad h^1(M) = s, \qquad \rho' = \rho - b.$$

Saturated section and relation lattices, as in Lemma 5.5, give a nonzero central relation q_0 on the partial normalization. Equivalently, evaluate at $2d + 1$ smooth sections: their zero values specialize, and a section of M^2, of degree $2(d - b)$, with these zeros must vanish.

Let $\mathcal{F}$ be the extended sheaf on the DVR family and put $\mathcal{G} = \mathcal{H}om(\mathcal{F}, \omega_{\mathcal{X}/R})$. The inverse-ideal calculation below shows that $\mathcal{G}$ is flat and that its formation commutes with reduction at each nonfree point; at the other points $\mathcal{F}$ is invertible. Serre duality and the displayed completeness calculation give $h^0(\mathcal{G}_K) = h^0(\mathcal{G}_0) = s$. The finite module $H^0(\mathcal{X}, \mathcal{G})$ is free of rank s, and its reduction injects into $H^0(\mathcal{G}_0)$ by the uniformizer exact sequence. Both spaces have

dimension s, so the reduction is an isomorphism. Thus every residual section lifts, and these lifts form a basis.

8.2. A Picard normal form. Arbitrary transition functions need not have positive weight. We first put them in a normal form that preserves the outside degree bound. For $\tau_i = z - a_i$, the central Čech quotient is

$$\frac{\bigoplus_i \mathbf{C}((\tau_i))}{\mathcal{O}(\mathbf{P}^1 \setminus \{a_i\}) + \bigoplus_i \mathbf{C}[[\tau_i^2, \tau_i^3]]}.$$

Its basis consists of the g gap classes τ_i. Indeed subtract the rational function formed by all negative principal parts, normalized to vanish at infinity. The remaining power series is uniquely a multiple of τ_i plus an inside series with no linear term. Independence follows because a rational function with no poles and regular at infinity is constant.

After fixing the central line bundle, logarithms of the remaining transitions are v-adic. At each successive v-order apply this central reduction. Lift an inside even monomial by a power of x_i, and an odd monomial of degree at least three by a power of x_i times y_i. The lift changes only higher v-orders. Iteration gives transitions

$$\exp(\lambda_i(v)\tau_i), \qquad \lambda_i(v) \in \mathbf{C}[[v]]. \tag{39}$$

Every outside gauge used is coefficientwise rational, has poles only at the original cusps, and is regular at infinity. Thus the source growth bound remains d, and the paired residual growth bound remains $-d-2$. This construction is formal patching on finite thickenings: the outside chart is affine, the inside rings are local, and their line bundles are trivial there. On an overlap, after inverting x_i, the nilpotent square-root expansion on each thickening gives exactly the Laurent τ_i-ring used above.

Assign weight $1/e$ to v and weight one to every local parameter. Every nonconstant term of (39) and its inverse has strictly positive weight, including the terms of $\exp(\lambda_i(0)\tau_i)$. These positive weights preserve the lowest-weight tensor under the resulting changes of frame.

8.3. The point-ideal chart and both valuation regimes. An actual nonfree family is, locally and up to an invertible twist,

$$I = (x - a, y - b), \qquad b^2 = a^3 + \alpha a + \beta, \quad a, b \in (v), \quad \alpha = Av^{4e}, \quad \beta = Bv^{6e}. \tag{40}$$

The ideal and its inverse are taken in the total quotient ring of

$$R_{\mathrm{loc}} = \mathbf{C}[[v, x, y]]/(y^2 - x^3 - \alpha x - \beta).$$

For families with ordinary nodes and cusps, the product of the curve factors and the invertible Picard space maps smoothly and surjectively to the compactified Jacobian by tensoring point ideals; see [**9**, proof of Theorem 4.1]. Smoothness lifts the valued point after a residue-field extension and completion. On the $|J|$ nonfree factors the invertible twist has degree $d + |J|$. Dividing the outside source frame by $G_J(z) = \prod_{i \in J}(z - a_i)$ restores growth d. Thus the local source is I/τ and its paired dual is τI^{-1}. Both have central module $\tau\mathbf{C}[[\tau]]$.

The fractional ideal I^{-1} has $\mathbf{C}[[v, x]]$-generators $1, z_0$, where

$$z_0 = \frac{y + b}{x - a}.$$

The exact identities

$$(y - b)z_0 = x^2 + ax + a^2 + \alpha, \quad xz_0 = y + b + az_0, \quad yz_0 = x^2 + ax + a^2 + \alpha + bz_0$$

show that $1, z_0$ generate a submodule of I^{-1}. For the reverse inclusion, write $S_0 = \mathbf{C}[[v, x]]$, so $R_{\mathrm{loc}} = S_0 \oplus S_0 y$. If $f \in I^{-1}$, write $(x - a)f = r + sy$, with $r, s \in S_0$. Regularity of $f(y - b)$ implies that its y-coefficient $(r - bs)/(x - a)$ lies in S_0. Hence $r - bs = (x - a)r_1$ and

$$f = r_1 + s\frac{y + b}{x - a}.$$

These two generators are independent over S_0, since the y-coefficient of $r_1 + sz_0 = 0$ is $s/(x-a)$. Thus $I^{-1} = S_0 \oplus S_0 z_0$; its monomials $x^r, z_0 x^r$, $r \geq 0$, form a flat formal basis. At $v = 0$, the generators specialize to $1, \tau$, the normalization module, proving compatibility with reduction.

If $\mathrm{ord}_v(a) \geq 2e$, the point equation gives $\mathrm{ord}_v(b) \geq 3e$. Call this the balanced case. Define, for $n \geq 2$,

$$\phi_n(\tau) = \begin{cases} \tau^n, & n \text{ even}, \\ \tau^n(1 + \alpha/\tau^4 + \beta/\tau^6)^{1/2}, & n \geq 3 \text{ odd}, \end{cases}$$

Here $\phi_{2r}(a,b) = a^r$ and $\phi_{2r+3}(a,b) = a^r b$. A flat source basis is $(\phi_n(\tau) - \phi_n(a,b))/\tau$, $n \geq 2$. Its central indices are $1, 2, \ldots$; every annular term has weight at least $n-1$. The dual basis $\tau x^r, \tau z_0 x^r$ has weights at least $2r+1, 2r+2$. Every nonzero exterior tensor of two actual residual sections therefore has weight at least three.

Otherwise choose $w^2 = a$ after a finite extension, rename the uniformizer and ramification index so that $u = v^e$, and choose the sign so that

$$b = w^3(1 + \alpha/w^4 + \beta/w^6)^{1/2}, \qquad \zeta = \mathrm{ord}_v(w)/e \in (0,1).$$

This is the neck case. Change the source to $I/(\tau - w)$, and pair the dual inversely. The apparent moving pole cancels in the divided differences

$$D_n(\tau, w) = \frac{\phi_n(\tau) - \phi_n(w)}{\tau - w}.$$

The complete identities for both exponent signs are

$$\frac{\tau^r - w^r}{\tau - w} = \sum_{j=0}^{r-1} \tau^{r-1-j} w^j \quad (r \geq 1),$$

$$\frac{\tau^{-h} - w^{-h}}{\tau - w} = -\sum_{j=1}^{h} \tau^{-j} w^{-(h+1-j)} \quad (h \geq 1).$$

A deformation monomial of weight $c_0 = 4p + 6q$ multiplies τ^{n-c_0}. For positive exponent the divided-difference terms have nonnegative v-valuation and weight at least $\zeta(n-1)$. For $n - c_0 = -h < 0$, their weights are

$$n - 1 + (1-\zeta)(h + 1 - j),$$

and their coefficient v-weights are

$$c_0 - \zeta(h + 1 - j) \geq n + (1-\zeta)h > 0.$$

The zero exponent has zero divided difference. Thus all D_n are v-regular with weight bounded below by $\zeta(n-1)$. The actual paired residual module is $(\tau - w)I^{-1}$, with $\mathbf{C}[[v,x]]$-generators

$$(\tau - w)x^r, \qquad \frac{x^r(y(\tau) + b)}{\tau + w} = x^r D_3(\tau, -w), \quad r \geq 0.$$

Here $\phi_3(-w) = -b$. The same positive- and negative-exponent identities, with $-w$ in place of w, prove their v-regularity and the lower weight bounds $2r + \zeta$ and $2r + 2\zeta$, respectively. Thus the actual residual functions admit v-regular annular expansions in this first completion before any auxiliary frame is used.

We need a second completion, and must keep track of its possible negative v-powers. Divide by $1 + \tau/w$, whose nonconstant terms and inverse have positive weight, to obtain the source

$$\varphi = \frac{wI}{x - a}.$$

Expand $(x-a)^{-1} = -a^{-1} \sum_{r \geq 0} (x/a)^r$. This is a flat ordinary-cusp expansion with indices $0, 2, 3, \ldots$. Each new geometric-series step raises weight by $2(1-\zeta) > 0$; the original source indices are bounded by $\zeta(n-1)$. All weights lie in the discrete group $(1/e)\mathbb{Z}$. For a fixed weight

bound, $\zeta(n-1)$ bounds the original index and $2(1-\zeta)$ bounds the number of geometric-series steps. Consequently every bounded-weight flat block is finite. Intrinsic square-root terms preserve weight but their v-orders tend to infinity. We use the v-adic topology inside each bounded-weight block, and no unbounded interchange of the two expansions.

Let D_* be the smallest weight of an actual source biform. The degree-D_* blocks in the $I/(\tau - w)$ and φ frames agree, since their connecting unit has constant term one and positive remaining weights. The first frame is v-regular. If a coefficient of the finite flat critical block in the second frame had negative v-order, choose its largest index sum N. Its leading annular monomial has degree $N > D_*$; smaller index sums cannot cancel it, since intrinsic square-root terms only lower degree. The coefficient is therefore a negative v-power in the v-regular first frame, a contradiction. Descending on N excludes all negative v-powers in the critical flat block.

In the paired φ frame, dual generators are

$$A_r = (x-a)x^r/w, \qquad B_r = (y+b)x^r/w.$$

For two A's the exterior weight is at least $2(r+s)+2\zeta > 2$, since $r \neq s$. For two B's it is at least $2(r+s)+4\zeta > 2$. For $A_r \wedge B_s$ with $r+s \geq 1$, it is at least $2(r+s)+3\zeta > 2$. The remaining exterior is exactly

$$A_0(\tau)B_0(\sigma) - A_0(\sigma)B_0(\tau) = \frac{x(\tau)y(\sigma) - x(\sigma)y(\tau) + b(x(\tau)-x(\sigma)) - a(y(\sigma)-y(\tau))}{w^2}.$$

Its three terms have weights $5-2\zeta,\ 2+\zeta,\ 3$, all strictly greater than two. Actual section coefficients are v-regular in these displayed generators, so the same strict bound holds for every actual residual exterior tensor. The paired change from the first frame multiplies a residual function by $1+\tau/w$. This unit and its inverse have constant term one and only positive remaining weights, so the bound greater than two holds also in the actual first frame. Global primitives and residues are formed there; the auxiliary completion is used only for this controlled weight comparison.

The outside change at all neck points is the product of $(z-a_i)/(z-a_i-w_i)$. It has degree zero at infinity, central value one, and coefficients with poles only at the original cusps. At other cusps it is a regular unit with positive v-order corrections. The outside source degree d and residual degree $-d-2$ therefore remain unchanged.

8.4. Finite symbols of a relation. Choose a representative $q(v)$ of the projective relation whose coefficients lie in the DVR and are not all divisible by v. Its product vanishes identically. The outside biform has coefficient expansion

$$S(v; z, w) = \sum_{m \geq 0} v^m T_m(z, w).$$

Every T_m is symmetric, is rational with poles only at the original cusps, has growth at most d in each variable, and has zero diagonal. At a nonfree chart the generic-zero product is zero in the torsion-free total ambient ring.

Let $q_0 \in F_k \setminus F_{k+1}$ on that partial normalization, and take the smallest source-biform weight D_* across all cusps, using the frames just constructed. At a free cusp the only weight-zero flat tensor is a constant multiple of $1 \otimes 1$, and the zero product forces its coefficient to vanish. At a nonfree cusp all source weights are positive by the preceding bounds. Thus

$$0 < D_* \leq 2k+2. \tag{41}$$

Indeed the central source is G_J times the source on the partial normalization. Let $S_0(x, y)$ be the biform representing q_0 there, and put

$$P_{\mathrm{free}}(z) = \prod_{i \notin J} (z - a_i), \qquad S_{\mathrm{all}}(x, y) = G_J(x)G_J(y)S_0(x, y).$$

For a biform S vanishing to order at least $2k$ on the diagonal, write

$$\gamma_k(S) = (\partial_x^k \partial_y^k S)|_{x=y}.$$

This differs by the nonzero factor $(-1)^k(2k)!$ from the diagonal quotient used in the coarse bound. Then

$$\frac{\gamma_k(S_{\mathrm{all}})}{P_{\mathrm{all}}^2} = \frac{\gamma_k(S_0)}{P_{\mathrm{free}}^2}.$$

Its normalized polynomial has degree at most $2t-2-4k$, which is less than $g-b$, and is nonzero at some free cusp. This proves the upper bound in (41).

Fix an order m at which a term of critical weight D_* occurs, and put $M = D_* - m/e$. The total Laurent degree of such a term is an integer, so $M \in \mathbb{Z}$. Every local Laurent term of T_m has total degree at least M. Suppose that its previous diagonal Gaussians vanish. Its j-th Gaussian has local order at least $M-2j$. Whenever that order is nonnegative, the Gaussian is polynomial, with degree at most $2d-4j$. To verify the degree even before regularity is known, clear the biform denominators by $P_{\mathrm{all}}(x)^N P_{\mathrm{all}}(y)^N$. Division by $(x-y)^{2j}$ lowers both numerator degrees by $2j$; on the diagonal the denominator then removes $2Ng$. This leaves exactly the bound $2d-4j$.

Since $s \geq 2$, one has $3g > 2d$ and $4g > 2d$. Fix m with $M = 2j+2 \geq 4$. Starting with $\gamma_0(T_m) = 0$, induction on $\ell < j$ gives $\gamma_\ell(T_m) = 0$: once the previous rows vanish, its local order is at least $M - 2\ell \geq 4$ at every cusp, whereas its polynomial degree is at most $2d - 4\ell < 4g$. The next Gaussian has order at least two at every cusp, so

$$h_j = \frac{\gamma_j(T_m)}{P_{\mathrm{all}}^2} \in \mathrm{Pol}_{2t-2-4j}, \qquad m = e(D_* - 2j - 2) \in \mathbb{Z}_{\geq 0}. \tag{42}$$

There is at most one polynomial for every $j \geq 1$; set $h_j = 0$ when the displayed order m is negative. For $M = 2r+1 \geq 3$, the same induction on $\ell < r$ uses $M - 2\ell \geq 3$ and $3g > 2d$. It gives diagonal order at least $2r = M-1$.

The tuple in (42) is not zero. Choose the earliest v-order m of a nonzero critical flat block, at any cusp, and put $M = D_* - m/e$. Its degree-M component is a nonzero homogeneous polynomial: every intrinsic square-root or point-coordinate contribution at this order would have come from an earlier critical block. In a free or neck frame it omits exponent one in either variable; in a balanced frame it omits exponent zero. At degree zero the only possible free or neck polynomial is a constant, and the balanced space is zero. At degree two the possible polynomials are multiples of $\tau^2 + \sigma^2$ and $\tau\sigma$, respectively. The zero-diagonal equation eliminates all of them. Degree one is excluded by the respective gaps. If $M \geq 3$ were odd, the diagonal order just proved would make the polynomial a multiple of

$$(\tau+\sigma)(\tau-\sigma)^{M-1}.$$

Its coefficients of $\tau\sigma^{M-1}$ and σ^M are $2-M$ and one, respectively, contradicting the corresponding gap. Thus $M = 2j+2 \geq 4$. Diagonal order $2j$ leaves the one line

$$(\tau-\sigma)^{2j}(\tau^2 + 2j\tau\sigma + \sigma^2)$$

in a free or neck frame, and the line $\tau\sigma(\tau-\sigma)^{2j}$ in a balanced frame. The mixed j-th derivatives on the diagonal of these two generators are, respectively,

$$(-1)^j(2j)!(2j+2)\tau^2, \qquad (-1)^j(2j)!\tau^2.$$

Hence h_j has nonzero value at this cusp. This proves nonvanishing, including a first block supported at a nonfree cusp.

Put $\alpha_i = A_i v^{4e}$ and $\beta_i = B_i v^{6e}$. At a free cusp, the uniqueness in Lemma 7.1 gives the critical-block decomposition

$$\sum_j P'_{\mathrm{all}}(a_i)^2 h_j(a_i)\, v^{e(D_*-2j-2)} \frac{Q_j(\alpha_i,\beta_i)}{G_j}. \tag{43}$$

More precisely the displayed expression is read in total weight D_*. Here is the finite descending induction. Regularity in v bounds every flat index sum by D_*. Take the largest remaining index sum N; there is no intrinsic contribution from a larger flat index. The Gaussian vanishing above makes this block zero unless $N = 2j+2 \geq 4$, and in that case the gap condition leaves exactly the line S_j. Its coefficient is $P'_{\mathrm{all}}(a_i)^2 h_j(a_i)/G_j$. Subtract that coefficient multiplied by $v^{e(D_*-N)} Q_j(\alpha_i,\beta_i)$.

This subtraction preserves the full product-zero equation. For every smaller Gaussian order, the subtracted tensor has no positive Laurent power, by Lemma 7.1; hence the required lower positive jets are unchanged. Continue with the next smaller flat index sum. Finiteness makes the induction terminate. At index zero the full product equation fixes the constant-target correction, and the remainder is zero. This proves (43). Its polynomials h_j are extracted from the given relation; the dimension count below allows them to vary throughout the spaces in (42). If $D_* = 2k+2$, then

$$h_k = \Gamma_k(q_0), \quad \Gamma_k = \gamma_k / P_{\mathrm{free}}^2,$$

and all other indices are less than k. If $D_* < 2k+2$, all indices are less than k and at least one of them is nonzero.

8.5. The global residue equation. Choose a basis $f = (f_1, \dots, f_c)^T$ of lifted source sections, and write $q(v)$ for the symmetric coefficient matrix of the relation in this basis. Write $H_\theta(v,z)\,dz$ for a lifted residual section in the outside dual frame, and let $B_\theta = (B_\theta^{f_1}, \dots, B_\theta^{f_c})^T$ be its column of primitives defined below. Each product $f_a H_\theta dz$ is canonical and has zero local residue at every parameter order. Indeed a local regular function is an even power series in τ plus y times an even power series, whereas

$$\frac{dx}{2y} = \tau^{-2}(1 + \alpha/\tau^4 + \beta/\tau^6)^{-1/2} d\tau.$$

The even part has only even exponents, and in the odd part the square root cancels, leaving a regular multiple of $\tau d\tau$. The assertion also holds at a nonfree point because $II^{-1} \subseteq R_{\mathrm{loc}}$. Outside growth is $O(z^{-2})dz$. There is therefore a unique rational primitive, coefficient by coefficient, vanishing at infinity:

$$(B_\theta^{f_a})' = -f_a H_\theta.$$

The differential

$$\mathcal{T}_{\theta\wedge\psi}\, dz = \left[H_\theta f^T q(v) B_\psi - H_\psi f^T q(v) B_\theta\right] dz \tag{44}$$

is $O(z^{-3})dz$ at infinity, so the sum of its local residues is zero. Primitive constants add only residue-free canonical products. A paired change $f \mapsto rf$, $H \mapsto H/r$ leaves the products, primitives, and (44) unchanged.

A source biform of weight D_* and a residual exterior of weight R_* contribute at parameter weight $D_* + R_* - 2$; this follows by counting the two dualizing factors and one primitive in (44). Every nonfree point has $R_* > 2$, as proved above, and contributes nothing at parameter weight D_*. At a free point only the central exterior indices $(0,2)$ can contribute. Positive v-order motions of residual coefficients would raise this weight; the intrinsic square-root terms are already retained in C_j.

The scalar is computed as follows. In a local paired frame write the two residual functions as $\theta_0 + \theta_2\tau^2 + \cdots$ and $\psi_0 + \psi_2\tau^2 + \cdots$, and put $W_{02} = \theta_0\psi_2 - \theta_2\psi_0$. The computation of the Wronskian in Lemma 6.4 gives

$$\ell_i = 2P'_{\mathrm{all}}(a_i)^2 W_{02}.$$

For residual functions $1, x$, the local integrand in (44) is exactly the integrand defining $\mathfrak{h}(Q)$. A local Laurent residue is the negative of Res_∞. The j-th contribution of (43) is consequently

$$-\frac{\mathfrak{h}(Q_j)}{G_j} P'_{\mathrm{all}}(a_i)^2 h_j(a_i) W_{02} = \ell_i C_j(A_i, B_i) h_j(a_i).$$

At a free point of a partial normalization, the central residual polynomial is G_J times its partial-normalized polynomial. Its Wronskian quotient gains G_J, and division by $P'_{\mathrm{all}}(a_i) = G_J(a_i) P'_{\mathrm{free}}(a_i)$ cancels that factor. Thus ℓ_i is precisely the partial-normalized covector of Lemma 6.4. We have proved the necessary equation

$$\boxed{\sum_{i \notin J} \ell_i \sum_j C_j(A_i, B_i) h_j(a_i) = 0 \quad \text{in } \Omega^*.} \tag{45}$$

9. The dimension estimate

LEMMA 9.1. *Fix the central partial-normalized datum. For every fixed nonzero tuple $h_j \in \mathrm{Pol}_{2t-2-4j}$, the locus of $(A_i, B_i) \in \mathbf{A}^{2g}$ satisfying (45) has dimension $2g - m_0$.*

PROOF. For $g' = g - b$, Lemma 6.4 embeds Ω in Pol_E, where $E = g' - 2t - 4$, and $\dim \Omega = m_0$. Choose j with $h_j \neq 0$. It is nonzero at at least $g' - (2t - 2 - 4j) > E$ free cusps. At these cusps the polynomial

$$p_i(A, B) = \sum_j C_j(A, B) h_j(a_i)$$

is nonconstant, since the nonzero C_j have distinct positive weights. Their corresponding ℓ_i's span Ω^*: an annihilated ω would have G_ω vanishing at more than E distinct points. Each nonconstant polynomial $\mathbf{A}^2 \to \mathbf{A}^1$ is surjective and flat, with all fibres of dimension one. Surjectivity follows by restricting to a line on which it is nonconstant; flatness follows from torsion-freeness over the PID $\mathbf{C}[p_i]$. The product of these maps is therefore flat onto the active coordinate space. Pulling back the kernel of the surjective linear map $(z_i) \mapsto \sum z_i \ell_i$ cuts codimension m_0. The inactive and nonfree parameter pairs are free, so the total dimension is $2g - m_0$. □

PROOF OF THEOREM 1.1. Choose one cusp configuration for which Proposition 6.3 holds on every partial normalization at $b \leq \rho$ points. There are finitely many subsets and nonempty Gaussian orders, and forgetting any selected cusps is a dominant configuration map. Thus a single nonempty open configuration set suffices.

Here is a projective ambient space for all relations. On the relative compactified Picard let $\mathcal{F}$ be the universal sheaf, and let π be the projection of the universal curve to that Picard space. Take a sufficiently large smooth divisor E, so that $\pi_* \mathcal{F}(E)$ is locally free and commutes with base change. In its relative Grassmannian of c-planes impose the closed condition that the tautological sections vanish on E; denote the resulting projective scheme by $\mathcal{B}$. All its relevant central section spaces are complete, by the Wronskian argument above. The closed locus $h^0 \geq c + 1$ misses the central compactified Picard fibre; properness permits shrinking the base so that it is absent nearby. On $\mathcal{B}$ form $\mathcal{P} = \mathbf{P}(\mathrm{Sym}^2 \mathcal{V})$. The closure $\mathcal{Z}$ of the actual smooth-fibre relation incidence is projective in $\mathcal{P}$. Evaluation at $2d + 1$ smooth sections shows that every central point is an actual relation on its partial normalization. The correspondence between a nonfree central sheaf and its line bundle there has no extra gluing parameter: its endomorphism algebra recovers that partial normalization.

Choose $h = \max(\Delta, 0)$ sufficiently general central hyperplanes from a high power of a relatively ample line bundle on $\mathcal{P}$. Every central k-th Gaussian stratum, of dimension at most $K_k - 1$, is cut to dimension at most $K_k - 1 - h$, or is empty. There are finitely many components, and a general hyperplane avoids zero-dimensional ones and contains none of positive dimension. The chosen sections lift over the full deformation base. Restrict the base to

an affine neighbourhood, let $\mathcal{I}_0$ be the ideal of its central fibre in $\mathcal{P}$, and let $\mathcal{A}$ be the chosen relatively ample bundle. Relative Serre vanishing, for a sufficiently large N, gives

$$H^1(\mathcal{P}, \mathcal{I}_0 \otimes \mathcal{A}^N) = 0.$$

The exact sequence for the central fibre therefore makes

$$H^0(\mathcal{P}, \mathcal{A}^N) \longrightarrow H^0(\mathcal{P}_0, \mathcal{A}_0^N)$$

surjective. Choose this power before choosing the central cuts; their lifts then exist before the smoothing coefficients are selected.

For each nonempty k-th stratum, put

$$\mathcal{H}_{<k} = \bigoplus_{1 \le j < k} \mathrm{Pol}_{2t-2-4j}\,.$$

Over a central point of that stratum, the finite symbols lie in one of two spaces: $(h_1, \dots, h_{k-1}) \in \mathcal{H}_{<k}$ with $h_k = \Gamma_k(q_0)$, or $(h_1, \dots, h_{k-1}) \in \mathcal{H}_{<k} \setminus \{0\}$ with $h_k = 0$. These are algebraic finite-dimensional spaces; the first uses the regular Gaussian coefficient on each chart where the relation and normalization frames have been fixed. Use finitely many affine charts to normalize the projective q_0. The number of added scalar parameters in either model is at most

$$\sum_{j<k} \dim \mathrm{Pol}_{2t-2-4j} = K - K_k.$$

The ramification order, the fractional value of D_*, all point arcs, and all Picard arcs introduce no further finite parameters into their necessary equation; the models already allow every possible lower polynomial. Lemma 9.1 bounds each symbol incidence by

$$(K_k - 1 - h) + (K - K_k) + (2g - m_0) = 2g + \Delta - 1 - h < 2g.$$

There are finitely many such models. Choose (A_i, B_i) outside the closures of their images and with all cubic discriminants nonzero. The hyperplanes and their lifts are already fixed.

If $\Delta \le 0$ and a generic relation existed, projectivity would specialize it, after finite extension, to a primitive valued relation. Indeed a nonempty geometric generic incidence contains a closed point over a finite extension of the fraction field. The normalization of the base DVR in that extension is a DVR, and the valuative criterion of properness extends the point. A generator of its relation line can be divided by its largest common uniformizer power, giving a primitive representative. Equation (45) would put its parameter tuple in an excluded image, a contradiction. If $\Delta > 0$ and the generic relation space had dimension at least Δ, it would meet the Δ ample hyperplanes. A projective variety of dimension at least h meets any h effective hyperplanes of a very ample system. A closed point in this intersection is defined over a finite extension. The same DVR argument specializes it, with all hyperplane equations preserved, and again contradicts the excluded images. This proves (3) on a geometric generic smooth fibre. The relation scheme is projective over the smooth moduli locus where all series are complete. Upper semicontinuity of fibre dimension therefore gives a nonempty open subset of $\mathcal{M}_g$ with the same bound. Irreducibility of $\mathcal{M}_g$ gives the asserted general-curve statement. The construction is invariant under rigidification, so these open conditions descend from local universal families to smooth moduli.

On a constant-κ stratum the relation fibre is $\mathbf{P}^{\kappa-1}$, so (4) follows. The rank difference is

$$\dim \mathrm{Sym}^2 H^0(L) - h^0(L^2) = \Delta - \rho.$$

For $\Delta \ge 0$, the projective bound gives $\dim \ker \mu_{2,L} \le \Delta$, including zero when $\Delta = 0$. Rank-nullity gives $b_2(L) \le \rho$. All L are very ample in the original range, and $\rho \le c - 4 < c - 3$. Theorem 1.3 now gives all the higher surjections. The empty-incidence assertion for $\Delta \le 0$ is immediate. □

REMARK 9.2. When $0 < \Delta \le \rho$, the estimate gives the expected upper dimension $\Delta - 1$ for the failure-of-injectivity locus. Nonemptiness and the ordinary determinantal lower bound then give equality. In the strict surjective range $\Delta > \rho$, it gives $\dim\{L : b_2(L) > 0\} \le \rho - 1$: there failure means $\kappa \ge \Delta - \rho + 1$. The conjectural dimension is $2\rho - \Delta - 1$, which is smaller by $\Delta - \rho$.

COROLLARY 9.3. *Use the notation of Theorem 1.1. If $\Delta \ge 0$, let E_e be the locus where the rank of $\mu_{2,L}$ falls at least $e \ge 1$ below its maximum possible value. Then*

$$\dim E_e \le \min\{\Delta, \rho\} - e. \tag{46}$$

In particular, on the strictly surjective side $\Delta > \rho$,

$$\dim\{L : b_2(L) \ge e\} \le \rho - e.$$

Moreover:

(i) *if $c \ge 2s + 1$, all $\mu_{m,L}$, $m \ge 3$, are surjective;*
(ii) *if $c = 2s$, $s \ge 3$, and $\rho \le s - 2$, the same conclusion holds;*
(iii) *if $c = 2s$, $s \ge 3$, and $\rho = s - 1$, then $b_3(L) \le 3$, and all $\mu_{m,L}$, $m \ge 4$, are surjective.*

All assertions hold for every $L \in W_d^{c-1}(C)$.

PROOF. Riemann–Roch and Lemma 2.2 give

$$\dim \operatorname{Sym}^2 H^0(L) - h^0(L^2) = \Delta - \rho, \qquad b_2(L) = \dim \ker \mu_{2,L} - \Delta + \rho.$$

Thus E_e is a finite union of kernel strata with $\kappa \ge \max\{0, \Delta - \rho\} + e$. Applying (4) to each stratum gives

$$\dim E_e \le \Delta - \max\{0, \Delta - \rho\} - e,$$

which is (46). If $c \ge 2s + 1$, then $t = c - s \ge s + 1$, so $\Delta \ge 0$. Theorem 1.1 gives $b_2 \le \rho \le c - 4$, and Theorem 1.3 applies. If $c = 2s$, then $t = s$ and $\Delta = 1 - s < 0$. The quadratic map is injective, hence $b_2 = \rho + s - 1$. For $\rho \le s - 2$ this is at most $c - 3$; for $\rho = s - 1$ it is $c - 2$. Apply the two alternatives of Theorem 1.3. □

10. A positive virtual intersection on the injective side

All Chern classes and pushforwards in this section lie in rational cohomology. We express the injective-side Schur coefficient of Cotterill–Alonso Gonzalo–Zhang [**8**, Section 5.2.2] as a centered Haar moment and prove its positivity. In the resulting band, Lian [**17**, Corollary 4.4.2 and Proposition 4.4.1] gives an expected-dimensional failure component through a point with a one-dimensional kernel.

10.1. The parameter band and the intersection. Set

$$B = \binom{c+1}{2}, \qquad T = 2d + 1 - g, \qquad N = T - B + 1.$$

Consider the original range with $B \le T$ and nonnegative expected dimension $\rho - N$. Since $d = r(s + 1) + \rho$, direct subtraction gives

$$\Delta = \frac{(r-1)(r-2s)}{2}, \qquad B - T = \Delta - \rho.$$

The assumptions are equivalent to $1 \le \Delta \le \rho \le r - 3$. Thus

$$0 < r - 2s \le \frac{2(r-3)}{r-1} < 2.$$

The positive integer $r - 2s$ is one, so the complete band is

$$r = 2s + 1, \qquad c = 2s + 2, \qquad s \le \rho \le 2s - 2, \qquad s \ge 2. \tag{47}$$

Here $N = \rho - s + 1$, $1 \le N \le s-1$, and $\rho - N = s - 1$. Equality $B = T$, corresponding to $N = 1$, is included.

Let $W = W^r_d(C)$ on a general curve in this band. Write $\pi\colon C \times \mathrm{Pic}^d(C) \longrightarrow \mathrm{Pic}^d(C)$ for the projection, and normalize the Poincaré line bundle $\mathcal{L}$ at a fixed point of C. Completeness follows from $\rho(g, r+1, d) = \rho - (c+s+1) < 0$. Thus W is smooth of dimension ρ, with the rank-c section bundle $\mathcal{V}$. Since $d - g = s + 1 > 0$, $R^1\pi_*\mathcal{L}^2 = 0$, and $\mathcal{F} = \pi_*\mathcal{L}^2$ is a vector bundle of rank T on the whole Picard variety. Write θ for the principal polarization class, normalized by $\int_{\mathrm{Pic}^d(C)} \theta^g = g!$. We use the same notation for the restrictions of $\mathcal{F}$ and θ to W. The intersection below evaluates a Chern class of the virtual bundle $\mathcal{F} - \mathrm{Sym}^2 \mathcal{V}$ on W.

THEOREM 10.1. *In* (47), *set* $A = c+s$ *and* $K_{c,s} = \prod_{j=0}^{c-1} j!/(j+s)!$. *Let* U *be Haar-distributed in the unitary group* $U(A)$, *let* C_0 *be its upper-left* $c \times c$ *block, and put*

$$X = 2\left\| \frac{C_0 + C_0^{\mathsf{T}}}{2} \right\|_F^2.$$

Here ^{T} *denotes ordinary transpose,* * *denotes Hermitian transpose, and* $\|M\|_F^2 = \mathrm{Tr}(MM^*)$ *is the squared Hermitian Frobenius norm. Then*

$$\int_W c_N(\mathcal{F} - \mathrm{Sym}^2 \mathcal{V})\theta^{\rho - N} = \frac{g! K_{c,s}}{N!}\, \mathbb{E}(X-4)^N > 0. \tag{48}$$

Consequently the locus where $\mu_{2,L}$ *is not injective is nonempty, and every one of its irreducible components has dimension at least* $s - 1$.

10.2. The Schur intersection coefficient. We calculate the Chern class $c(\mathcal{F} - \mathrm{Sym}^2 \mathcal{V}) = c(\mathcal{F})c(\mathrm{Sym}^2 \mathcal{V})^{-1}$. For a partition λ, write $|\lambda| = \sum_i \lambda_i$ for its size, $\ell(\lambda) = \#\{i : \lambda_i > 0\}$ for its length, and s_λ for its Schur polynomial. The hook length of a box (i,j) in its diagram is $\mathrm{hook}(i,j) = \lambda_i - j + \lambda'_j - i + 1$, where λ' is the conjugate partition. Put

$$(A)_\lambda = \prod_{(i,j)\in\lambda} (A + j - i), \qquad h_\lambda = \prod_{\square \in \lambda} \mathrm{hook}(\square).$$

The hook and Weyl formulas give $s_\lambda(1^c) = (c)_\lambda / h_\lambda$. All sums below are interpreted one homogeneous degree at a time; partitions of length at most c are padded with zero parts when indexed by $1, \dots, c$. Define ψ_λ by

$$\prod_{i \le j} (1 - x_i - x_j)^{-1} = \sum_{\ell(\lambda) \le c} \psi_\lambda s_\lambda(x). \tag{49}$$

With x_i the roots of $\mathcal{V}^*$, this product equals $c(\mathrm{Sym}^2 \mathcal{V})^{-1}$.

The Grothendieck–Riemann–Roch calculation is as follows. Write $c_1(\mathcal{L}) = d\eta + \gamma$, where η is the point class on C, $\pi_*(\gamma^2) = -2\theta$, and $\eta^2 = \eta\gamma = 0$. In the Grothendieck group, $\pi_! = R^0\pi_* - R^1\pi_*$. Multiplication by $\mathrm{td}(C) = 1 + (1-g)\eta$ and integration along C give

$$\mathrm{ch}(\pi_! \mathcal{L}^k) = kd + 1 - g - k^2\theta.$$

Indeed the only terms of degree two along C are $(kd + 1 - g)\eta$ and $k^2\gamma^2/2$. For $k = 2$, higher direct images vanish, and the relation between Chern character and total Chern class gives

$$c(\mathcal{F}) = \exp(-4\theta). \tag{50}$$

Choose a reduced effective divisor D_0 of sufficiently large degree h. The same calculation for $\mathcal{E} = \pi_*(\mathcal{L}(D_0))$ gives $\mathrm{ch}(\mathcal{E}) = a - \theta$, where $a = d + h + 1 - g$ is its rank.

Let $q\colon \mathrm{Gr}(c, \mathcal{E}) \longrightarrow \mathrm{Pic}^d(C)$ be the Grassmann projection, and let $\mathcal{S}$ be the tautological rank-c subbundle. The evaluation homomorphism

$$\mathcal{S} \longrightarrow q^*\pi_*(\mathcal{L}(D_0)|_{D_0})$$

has zero scheme $G^r_d(C) \simeq W$, with $\mathcal{S}|_W = \mathcal{V}$. Its target is rationally Chern-trivial of rank h, since the summands corresponding to the points of D_0 are algebraically trivial line bundles on $\mathrm{Pic}^d(C)$. The dimension of W gives the expected codimension ch for this zero locus. Hence its class is $s_{(h^c)}(\mathcal{S}^*)$, where (h^c) is the rectangle with c parts equal to h. Multiplication by this class adds (h^c) to the partition index. The Grassmann Gysin formula, using $a - c = h - s$, yields

$$q_*(s_\lambda(\mathcal{S}^*)[W]) = s_{\lambda+(s^c)}(\mathcal{E}^*) = \det\left[\frac{1}{(\lambda_i + s - i + j)!}\right]_{i,j=1}^{c} \theta^{cs+|\lambda|}. \tag{51}$$

Here $1/k! = 0$ for $k < 0$. The complete symmetric classes of $\mathcal{E}^*$ are $\theta^k/k!$, since their generating series is $\exp(\theta t)$; this explains the determinant and its sign. These intersection formulas are also given with the Brill–Noether normalization in [**8**, Lemmas 3.18–3.19].

The determinant itself is explicit. Put $I_j = \lambda_{c-j+1} + j - 1$. Reverse both rows and columns in (51), multiply row j by $(I_j + s)!$, and use the monic falling-factorial polynomials of degrees $0, \dots, c-1$. Their determinant is the Vandermonde, so the coefficient in (51) is

$$\frac{\prod_{j<k}(I_k - I_j)}{\prod_j (I_j + s)!} = K_{c,s}\frac{s_\lambda(1^c)}{(c+s)_\lambda}. \tag{52}$$

For the last equality, divide by the value at $\lambda = 0$. The numerator quotient is the Weyl dimension $s_\lambda(1^c)$; the factorial quotient is $\prod_i (A-i)!/(A-i+\lambda_i)! = (A)_\lambda^{-1}$. Combining (49)–(52) gives

$$\int_W c_N(\mathcal{F} - \mathrm{Sym}^2 \mathcal{V})\theta^{\rho - N} = g! K_{c,s}[t^N] e^{-4t} \sum_{\ell(\lambda)\le c} \frac{\psi_\lambda s_\lambda(1^c)}{(A)_\lambda} t^{|\lambda|}. \tag{53}$$

This is the injective-side Schur coefficient of [**8**, Section 5.2.2]: in their notation, the integral is $g!\, S'(g, r, d)$.

10.3. A finite-degree Gaussian and Haar calculation. Let Z be a complex symmetric $c \times c$ matrix whose independent upper-triangular entries are centered circular complex Gaussians, with variance two on the diagonal and one off the diagonal. In the orthonormal basis

$$E_{ii}, \qquad (E_{ij} + E_{ji})/\sqrt{2} \quad (i < j),$$

all coordinates have variance two. Here E_{ij} denotes the matrix unit with its sole nonzero entry equal to one in position (i, j). The density is proportional to $\exp(-\|Z\|_F^2/2)$, so its law is invariant under unitary congruence $Z \mapsto VZV^{\mathsf{T}}$. Consequently $H = ZZ^*$ is invariant under unitary conjugation. For $D = \mathrm{diag}(x_1, \dots, x_c)$ near zero, independence gives

$$\mathbb{E} e^{\mathrm{Tr}(DH)} = \prod_i (1 - 2x_i)^{-1} \prod_{i<j} (1 - x_i - x_j)^{-1}. \tag{54}$$

We recall the two representation identities used in the calculation. For $k \ge 0$, Schur–Weyl duality gives

$$(\mathrm{Tr}\, M)^k = \sum_{\lambda \vdash k} f^\lambda s_\lambda(M), \qquad f^\lambda = k!/h_\lambda,$$

where $\lambda \vdash k$ means $|\lambda| = k$, and partitions longer than the size of M are omitted. Schur orthogonality for matrix coefficients of the irreducible unitary representations gives, for a fixed $A \times A$ matrix M and partitions of length at most A,

$$\int_{U(A)} s_\lambda(MU)\overline{s_\mu(MU)}\, dU = \delta_{\lambda\mu}\frac{s_\lambda(MM^*)}{s_\lambda(1^A)}. \tag{55}$$

To see the normalization, write $s_\lambda(MU)$ as $\mathrm{Tr}(\rho_\lambda(M)\rho_\lambda(U))$, using the polynomial Schur representation ρ_λ. The integral of a product of two matrix coefficients is zero between distinct irreducible representations and is $\delta_{ik}\delta_{jl}/\dim \rho_\lambda$ within one. Summing the coefficients leaves

$\mathrm{Tr}(\rho_\lambda(M)\rho_\lambda(M)^*)/\dim\rho_\lambda$, which is exactly the right side. Likewise averaging one character under conjugation gives

$$\int_{U(c)} s_\lambda(DVHV^*)\,dV = \frac{s_\lambda(D)s_\lambda(H)}{s_\lambda(1^c)}.$$

Applying this identity to the degree-k exponential term gives

$$\frac{1}{k!}\mathbb{E}(\mathrm{Tr}(DH))^k = \sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le c}} \frac{\mathbb{E}s_\lambda(H)}{h_\lambda s_\lambda(1^c)} s_\lambda(D).$$

Comparison of (54) with (49) therefore gives

$$\mathbb{E}s_\lambda(ZZ^*) = (c)_\lambda \psi_\lambda. \tag{56}$$

Each comparison involves finitely many partitions in a fixed degree.

Independently choose $U \in U(A)$, embed Z in its upper-left block as $\widetilde{Z}$, and put $Y = \mathrm{Tr}(\widetilde{Z}U) = \mathrm{Tr}(ZC_0)$. Conditional on U, Y is a centered circular Gaussian with variance

$$2\sum_i |(C_0)_{ii}|^2 + \sum_{i<j} |(C_0)_{ij} + (C_0)_{ji}|^2 = X.$$

Hence $\mathbb{E}_Z|Y|^{2k} = k!X^k$. On the other hand, Schur–Weyl duality followed by (55) gives

$$\frac{\mathbb{E}_U|Y|^{2k}}{(k!)^2} = \sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le c}} \frac{s_\lambda(ZZ^*)}{h_\lambda^2 s_\lambda(1^A)} = \sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le c}} \frac{s_\lambda(ZZ^*)}{(A)_\lambda h_\lambda}.$$

Partitions of length greater than c contribute zero. Taking expectation in Z and using (56), we obtain the exact identity

$$\frac{\mathbb{E}X^k}{k!} = \sum_{\substack{\lambda\vdash k\\ \ell(\lambda)\le c}} \frac{\psi_\lambda s_\lambda(1^c)}{(A)_\lambda}. \tag{57}$$

Equation (57) gives the coefficients in (53). Since $0 \le X \le 2c$, the resulting series is $\mathbb{E}e^{tX}$. Multiplication by e^{-4t} proves the equality in (48).

10.4. Strict positivity and dimension. The first Haar moment satisfies $\mathbb{E}U_{ij}\overline{U_{kl}} = \delta_{ik}\delta_{jl}/A$. In the displayed expression for X, the diagonal terms contribute $2c/A$, and the off-diagonal terms contribute $c(c-1)/A$. Thus

$$\mathbb{E}X = \frac{c(c+1)}{A}. \tag{58}$$

For an odd positive integer N and $x \ge 0$, put $y = (x-4)/4 \ge -1$. On $y \ge 0$, convexity gives $y^N \ge Ny - (N-1)$. On $-1 \le y \le 0$, $y^N \ge y \ge Ny - (N-1)$. Consequently

$$(x-4)^N \ge N4^{N-1}x - (2N-1)4^N. \tag{59}$$

It follows that the odd moment is strictly positive whenever $\mathbb{E}X > 8 - 4/N$. For even $N \ge 2$, a zero even moment would force $X = 4$ almost surely, and is impossible if $\mathbb{E}X \ne 4$.

In (47), the mean is $\mathbb{E}X = (2s+2)(2s+3)/(3s+2)$. Direct arithmetic yields

$$\mathbb{E}X - \left(8 - \frac{4}{s-1}\right) = \frac{4s^3 - 18s^2 + 16s + 18}{(3s+2)(s-1)}.$$

The numerator is 10 when $s = 2$; when $s \ge 3$ it equals

$$(s-3)(4s^2 - 6s - 2) + 12 > 0.$$

Thus $\mathbb{E}X > 4$, and for every odd $N \le s-1$, $\mathbb{E}X > 8 - 4/(s-1) \ge 8 - 4/N$. Both parity cases give the strict inequality in (48).

If $\operatorname{Sym}^2 \mathcal{V} \to \mathcal{F}$ were injective on every fiber, its cokernel would be a vector bundle of rank $N-1$, so $c_N(\mathcal{F} - \operatorname{Sym}^2 \mathcal{V})$ would vanish. This contradicts (48); hence the degeneracy locus is nonempty. The maximal-minor height theorem bounds the codimension of each component by $T - B + 1 = N$ ([**7**, Chapter 2]), giving dimension at least $\rho - N = s - 1$. This proves Theorem 10.1.

10.5. Low-degree moments. The first two moments can be written explicitly. For $p_j = \sum_i x_i^j$, the first two homogeneous terms of (49) are

$$(c+1)p_1, \qquad \frac{1}{2}\big((c^2+2c+2)p_1^2 + (c+2)p_2\big).$$

Using $p_1^2 = s_{(2)} + s_{(1,1)}$ and $p_2 = s_{(2)} - s_{(1,1)}$, we obtain

$$\psi_{(1)} = c+1, \qquad \psi_{(2)} = \frac{c^2+3c+4}{2}, \qquad \psi_{(1,1)} = \frac{c(c+1)}{2}.$$

Equation (57) gives

$$\begin{aligned}\mathbb{E}X^2 &= \frac{c(c+1)(c^2+3c+4)}{2A(A+1)} + \frac{c^2(c^2-1)}{2A(A-1)} \\ &= \frac{c(c+1)((c^2+c+2)A - 2(c+1))}{A(A^2-1)}.\end{aligned}$$

Thus $R_1 = \mathbb{E}X - 4$ and $R_2 = (\mathbb{E}X^2 - 8\mathbb{E}X + 16)/2$, where $R_N = \mathbb{E}(X-4)^N/N!$. All values in the following table are exact:

c	s	$\mathbb{E}X$	$\mathbb{E}X^2$	R_1	R_2
6	2	21/4	169/6	5/4	13/12
8	3	72/11	2388/55	28/11	194/55
10	4	55/7	17006/273	27/7	301/39

For $c = 6, s = 2$, the entry R_2 is a moment calculation outside the geometric band $N \le 1$. For the first geometric case, $s = 2, \rho = 2$, $(g, r, d) = (14, 5, 17)$, $N = 1$, and

$$K_{6,2} = \frac{1}{(1\cdot 2)(2\cdot 3)(3\cdot 4)(4\cdot 5)(5\cdot 6)(6\cdot 7)} = \frac{1}{6!\,7!}.$$

The exact virtual degree is therefore

$$14!\, K_{6,2}\, R_1 = \frac{14!}{6!\,7!} \cdot \frac{5}{4} = 24024 \cdot \frac{5}{4} = 30030.$$

Proof of Corollary 1.2. The original range implies $r \ge 3$ and $0 \le \rho \le r-3$. With $\Delta = (r-1)(r-2s)/2$, direct subtraction gives

$$\binom{r+2}{2} - (2d+1-g) = \Delta - \rho.$$

Thus the injective-side hypothesis is $\Delta \le \rho$, and the expected failure dimension is $\Delta - 1$. If $s \le 1$, then

$$\Delta \ge \frac{(r-1)(r-2)}{2} > r - 3 \ge \rho,$$

a contradiction. Hence $s \ge 2$. Set $t = c - s$. If $t \le 2$, then $d = g + t - 1 \le g + 1$, and Theorem 1.4 gives injectivity for all L. Here $r - 2s = t - s - 1 < 0$, so the expected dimension is negative. If $t \ge 3$, the hypotheses of Theorem 1.1 hold, and its parameter agrees with the above Δ, since

$$\binom{t-1}{2} - \binom{s}{2} = \frac{(t+s-2)(t-s-1)}{2} = \frac{(r-1)(r-2s)}{2}.$$

When $\Delta \le 0$, that theorem makes the relation incidence empty.

When $\Delta \geq 1$, the inequalities $1 \leq \Delta \leq \rho \leq r-3$ imply

$$0 < r - 2s \leq \frac{2(r-3)}{r-1} < 2.$$

Therefore $r = 2s+1$, $c = 2s+2$, and $\Delta = s$. Together with $g = cs+\rho$ and $d = g+c-s-1$, these identities give exactly the band in the corollary, with expected dimension $s-1$. Theorem 10.1 makes the failure locus nonempty and gives dimension at least $s-1$ for each irreducible component. On a dense open subset of such a component, the kernel dimension is a constant $\kappa \geq 1$. Theorem 1.1 bounds its dimension by $s-\kappa$. Consequently $\kappa = 1$, and the component has dimension exactly $s-1$. Finally $\Delta = s > 0$, so the higher-degree surjectivity follows from Theorem 1.1. □

11. Subline bundles of quadratic obstruction extensions

The following proposition describes saturated subline bundles of canonical extensions arising from quadratic cokernel classes.

Proposition 11.1. *Let C be general, $c \geq 5$, $s = g-d+c-1 \geq 2$, and $0 \leq \rho = g-cs \leq c-4$. Let L be a line bundle of degree d with $h^0(L) = c$, and let $0 \neq \epsilon \in Q_2(L)^*$. The associated extension*

$$0 \longrightarrow M = K_C L^{-1} \longrightarrow E \longrightarrow L \longrightarrow 0 \tag{60}$$

has $\det E = K_C$ and $h^0(E) = c+s$. Every saturated line subbundle $A \subset E$, different from M, satisfies $h^0(A) \leq 2$. More precisely:

(i) *if $h^0(A) = 2$, then $A = L(-D)$, where*

$$2c-4 \leq \deg D \leq 2c-4+\rho;$$

(ii) *if $h^0(A) = 1$, then $\deg A \leq d-2c+2$;*
(iii) *if $h^0(A) = 0$, then $\deg A \leq d-2c$.*

If $s \geq 3$, M is the unique line subbundle of maximal degree.

Proof. Serre duality identifies $Q_2(L)^*$ with the extension classes for which the connecting map $H^0(L) \to H^1(M)$ is zero: its pairing on two sections is $(v,w) \mapsto \epsilon(vw)$. Thus all sections of L lift and $h^0(E) = c + h^0(M) = c+s$. An $A \neq M$ maps nontrivially to L. The image is $L(-D)$ for an effective divisor D, and the induced map identifies $A \simeq L(-D)$. If $D = 0$, it splits the extension, contrary to $\epsilon \neq 0$; hence $D > 0$.

Put $\ell = \deg D$ and $q = c - h^0(A)$. Since L is globally generated, $q \geq 1$. The determinant sequence is

$$0 \longrightarrow A \longrightarrow E \longrightarrow K_C A^{-1} \longrightarrow 0.$$

Riemann–Roch gives

$$h^0(K_C A^{-1}) = h^0(A) - \deg A - 1 + g = s + \ell - q.$$

Consequently

$$c + s = h^0(E) \leq h^0(A) + h^0(K_C A^{-1}) = c + s + \ell - 2q,$$

so $\ell \geq 2q$. For $1 \leq q \leq c-3$, the secant cost is bounded below by

$$F_c(\ell, q) \geq F_c(2q, q) = q(c-q-2) \geq c-3.$$

The last inequality follows either by concavity in q, or from the equal endpoint values at $q = 1, c-3$. This contradicts $F_c(\ell,q) \leq \rho \leq c-4$. Thus $q \geq c-2$, proving $h^0(A) \leq 2$.

If $q = c-2$, the secant inequality is

$$\ell - 2c + 4 = F_c(\ell, c-2) \leq \rho,$$

and the lower bound $\ell \geq 2c-4$ gives (i). For $q = c-1, c$, the inequality $\ell \geq 2q$ gives (ii) and (iii). Finally,

$$\deg M = 2g-2-d = d-2c+2s, \qquad \deg A \leq d-2c+4$$

for every other saturated A. If $s \geq 3$, the first degree is strictly larger. A nonsaturated subline has no larger degree than its saturation, so the assertion follows for all line subbundles. □

12. Rank-three relations and their support divisors

The rank of $q \in \operatorname{Sym}^2 V$ means the rank of its symmetric matrix, equivalently the rank of $V^* \to V$ induced by contraction. Its image is the *support* $S(q) \subseteq V$. The common divisor of the sections in $S(q)$ need not be a common divisor of the larger space V.

Proposition 12.1 (Square-pencil description). *Let C be a smooth integral curve, $V \subseteq H^0(L)$, and let $[q]$ be a projective rank-three element of $\ker \mu_{2,V}$. There are an effective divisor F, a line bundle A, and a base-point-free pencil $W \subseteq H^0(A)$ such that*

$$L \simeq A^2(F), \qquad S(q) = t_F \operatorname{Sym}^2 W \subseteq V. \tag{61}$$

Here t_F is the canonical section of $\mathcal{O}_C(F)$. The divisor F and the pencil (A, W) are determined by $[q]$ up to their usual isomorphisms. Conversely, every such datum with $t_F \operatorname{Sym}^2 W \subseteq V$ gives one projective rank-three relation.

Proof. Over $\mathbf{C}$, a rank-three symmetric form has a basis u, v, w of its support in which

$$q = uw - v^2.$$

The equation $\mu_{2,V}(q) = 0$ says $uw = v^2$ as sections of L^2. Let F be the common zero divisor of u, v, w. Dividing these sections by t_F gives a base-point-free three-dimensional space in $H^0(L(-F))$. Its morphism $C \to \mathbf{P}^2$ has image in the smooth conic $UW - V^2 = 0$. The map is nonconstant: a constant image would make the three sections linearly dependent. Identifying the conic with $\mathbf{P}^1$ by $[x : y] \mapsto [x^2 : xy : y^2]$, let A be the pullback of $\mathcal{O}_{\mathbf{P}^1}(1)$, and W the pullback of its two sections. Then W generates A and $L(-F) = A^2$. The support, its fixed divisor, and its morphism to the conic are determined by q; changing the identification of the conic with $\mathbf{P}^1$ only changes a basis of the same pencil.

Conversely choose a basis x, y of W. The nonconstant function x/y has no nonzero polynomial relation over $\mathbf{C}$, so x^2, xy, y^2 are independent. The identity

$$(t_F x^2)(t_F y^2) - (t_F xy)^2 = 0$$

has rank three and support $t_F \operatorname{Sym}^2 W$. The only quadratic relation among these three coordinates is the equation of the conic, proving uniqueness of its projective class. □

Theorem 12.2. *Let $g \geq 2$, $r \geq 3$, and d be integers, and let C be a sufficiently general curve of genus g. Set $\sigma = d-g-r$, and let*

$$\mathcal{I}^3_{d,r}(C) = \{((L,V),[q]) : (L,V) \in G^r_d(C),\ [q] \in \mathbf{P}\ker \mu_{2,V},\ \operatorname{rank}(q) = 3\}.$$

This incidence is empty if $\sigma < 0$. If $\sigma \geq 0$, it is nonempty, and

$$\dim \mathcal{I}^3_{d,r}(C) = (r-1)(\sigma+1) - 1. \tag{62}$$

The formula holds for both parities of d and counts pairs consisting of a linear series and a specified projective quadric.

Proof. For a pencil (A, W) from Proposition 12.1, the base-point-free pencil trick and Petri injectivity give $H^0(K_C A^{-2}) = 0$. One can also see this directly: if $0 \neq z \in H^0(K_C A^{-2})$ and x, y are independent sections of A, then $x \otimes zy - y \otimes zx$ is a nonzero element of the Petri kernel. Since $L = A^2(F)$, it follows that $H^1(L) = 0$. Therefore $h^0(L) = d-g+1 = r+\sigma+1$. The existence of V forces $\sigma \geq 0$.

Fix $a = \deg A$ and $f = \deg F = d - 2a \geq 0$. The open variety $G^{1,\mathrm{bpf}}_a(C)$ of base-point-free pencils, if nonempty, is smooth of dimension $\rho(g, 1, a) = 2a - g - 2$. Together with $F \in C^{(f)}$ it has dimension $d - g - 2$. For every point of this parameter space, $h^0(A^2(F)) = d - g + 1$ is constant. Locally on the parameter space, these sections form a vector bundle, with a rank-three subbundle $t_F \operatorname{Sym}^2 W$. The choices of V containing this subbundle form the relative Grassmannian

$$\operatorname{Gr}(r - 2, H^0(A^2(F))/(t_F \operatorname{Sym}^2 W))$$

of dimension $(r - 2)\sigma$. The total dimension is

$$(2a - g - 2) + f + (r - 2)\sigma = d - g - 2 + (r - 2)\sigma = (r - 1)(\sigma + 1) - 1.$$

Proposition 12.1 shows that the map from this parameter space to the incidence has a single geometric point in every fibre over its image. Thus it preserves dimension. There are only finitely many possible a, and every incidence point occurs in one of these strata; all nonempty strata have the displayed dimension.

For nonemptiness, a general curve has a base-point-free pencil of degree $a_0 = \lfloor (g + 3)/2 \rfloor$, by Brill–Noether theory. Since $r \geq 3$ and $\sigma \geq 0$,

$$d \geq g + 3 \geq 2a_0.$$

Choose F of degree $d - 2a_0$, and then any $(r + 1)$-dimensional V containing $t_F \operatorname{Sym}^2 W$. Such a V exists because $h^0(A^2(F)) = r + \sigma + 1$. □

Example 12.3. Let C be a general genus-five curve, A a base-point-free degree-four pencil, and $p \in C$. Put $L = A^2(p)$. Petri injectivity gives $h^1(A^2) = 0$, so $h^0(A^2) = 4$, $h^1(L) = 0$, and $h^0(L) = 5$. The complete series L is globally generated: the subsystem A^2 generates away from p, and the one-dimensional increase of h^0 supplies a section not vanishing at p. If x, y generate A, the three sections $t_p x^2, t_p xy, t_p y^2$ have the rank-three relation

$$(t_p x^2)(t_p y^2) - (t_p xy)^2 = 0, \qquad \det \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix} = \frac{1}{4}.$$

Here $d = 9$, $r = 4$, and $\sigma = 0$, despite d being odd. The point p is fixed in the support of this relation and is not a base point of $H^0(L)$. The square-pencil construction permits the fixed divisor $F = p$ on the support. This gives a counterexample to the odd-degree emptiness assertion in [**23**, v1, Theorem 3.2]. For this example, $\rho(5, 4, 9) = 5 > r - 3 = 1$.

Corollary 12.4. *Let C be a Petri curve and L a special line bundle of degree d. Every nonzero quadratic relation has rank at least four. If $c = h^0(L) \geq 3$, then*

$$\dim \ker \mu_{2,L} \leq \binom{c - 2}{2}, \qquad \operatorname{rank} \mu_{2,L} \geq 3c - 3. \tag{63}$$

If, in addition, C is general, with $c \geq 3$ and $s = h^1(L) \geq 1$, put $\rho = g - cs$. Then

$$b_2(L) \leq 2d - g - 3c + 4 = (c - 2)(s - 1) + \rho. \tag{64}$$

Proof. A rank-one or rank-two symmetric quadric factors into one or two linear factors over $\mathbf{C}$. Their product cannot vanish as a section on an integral curve unless one factor is zero. A rank-three relation would give $L = A^2(F)$, and hence

$$H^0(K_C L^{-1}) = H^0(K_C A^{-2}(-F)) \subseteq H^0(K_C A^{-2}) = 0,$$

contradicting speciality.

In $\mathbf{P}(\operatorname{Sym}^2 \mathbf{C}^c)$, the variety of forms of rank at most three has dimension $3(c-3)+5 = 3c-4$: choose a three-dimensional support and then a projective symmetric form on it. If the kernel

has dimension $\kappa > 0$, its projective space of dimension $\kappa - 1$ is disjoint from this variety. The projective dimension theorem therefore gives

$$(\kappa - 1) + (3c - 4) < \binom{c+1}{2} - 1,$$

or $\kappa \leq \binom{c-2}{2}$. The case $\kappa = 0$ is immediate. Subtracting from $\dim \operatorname{Sym}^2 H^0(L) = \binom{c+1}{2}$ gives the rank bound. Finally $H^1(L^2) = 0$ by Petri injectivity, and Riemann–Roch gives $b_2 \leq 2d + 1 - g - (3c - 3)$. Using $g = cs + \rho$ and $d = g + c - 1 - s$ yields the last equality. □

Remark 12.5. The elementary bound (64) may be combined with Theorem 1.3 just as the cuspidal bound can. For $s = 1$ it gives $b_2 \leq \rho$; normal generation in this original-range subcase follows from Ballico–Fontanari [**6**].

The forgetful map $\mathcal{I}^3_{d,r}(C) \to G^r_d(C)$ has fibre

$$\{[q] \in \mathbf{P} \ker \mu_{2,V} : \operatorname{rank}(q) = 3\}$$

over (L, V). Determining the dimension of its image requires the dimensions of these rank-three loci in the projective kernels.

Appendix A. An explicit Robin rank-drop surface

We study the rank-drop scheme of the following finite-dimensional Robin evaluation model.

Fix $N \geq 3$ distinct complex numbers $b_1, \dots, b_N$ and $n \in \mathbf{C}^*$. For $\delta = (\delta_1, \dots, \delta_N) \in \mathbf{A}^N$, define

$$\Phi_\delta \colon \mathbf{C}^N \longrightarrow \mathbf{C}^{\binom{N}{2}}, \qquad (\Phi_\delta \eta)_{ij} = \delta_i \eta_j + \delta_j \eta_i - n\,\frac{\eta_j - \eta_i}{b_i - b_j}, \quad i < j, \tag{65}$$

$$Z_\delta \colon \mathrm{Pol}_2 \longrightarrow \mathbf{C}^N, \qquad (Z_\delta q)_i = \delta_i q(b_i) - \frac{n}{2} q'(b_i). \tag{66}$$

Let X_Φ be the closed subscheme defined by the $N \times N$ minors of Φ_δ, and X_Z the closed subscheme defined by the 3×3 minors of Z_δ.

Theorem A.1. *For $N \geq 3$, $X_\Phi = X_Z$. This scheme is a smooth irreducible surface, isomorphic to*

$$\mathrm{Bl}_{\{p_1, \dots, p_N\}} \mathbf{P}(\mathrm{Pol}_2) \;\setminus\; \bigcup_{i=1}^{N} \{\widetilde{q(b_i) = 0}\}, \qquad p_i = [(z - b_i)^2].$$

Here the tilde denotes the strict transform of the indicated line in the blowup. At every point of this surface, both $\ker \Phi_\delta$ and $\ker Z_\delta$ are one-dimensional. The part where a kernel vector of Φ_δ has no zero coordinate is parametrized by

$$\delta_i = \frac{n}{2}\,\frac{q'(b_i)}{q(b_i)}, \qquad \eta_i = \frac{1}{q(b_i)}. \tag{67}$$

The remaining points form N affine lines:

$$\eta = e_j, \qquad \delta_i = \frac{n}{b_i - b_j}\ (i \neq j), \qquad \delta_j \text{ arbitrary.} \tag{68}$$

Here $e_j \in \mathbf{C}^N$ is the j-th standard basis vector.

Proof. *Kernel vectors.* Suppose $\Phi_\delta \eta = 0$ and $\eta \neq 0$. If $\eta_i = 0$, a nonzero η_j forces $\delta_i = n/(b_i - b_j)$. Two different nonzero coordinates would give two different values for δ_i. Thus a vector with a zero coordinate has singleton support, and direct substitution gives exactly (68).

If every $\eta_i \neq 0$, set $u_i = \eta_i^{-1}$ and $v_i = \delta_i u_i / n$. The pair equations become

$$\frac{u_i - u_j}{b_i - b_j} = v_i + v_j. \tag{69}$$

For any three distinct indices, eliminating their u's gives

$$(b_i - b_k)v_j = (b_j - b_k)v_i + (b_i - b_j)v_k.$$

Consequently there is an affine-linear polynomial $v(z)$ with $v(b_i) = v_i$ for every i. Choose $q \in \mathrm{Pol}_2$ with $q' = 2v$ and $q(b_1) = u_1$. Equation (69) then gives $q(b_i) = u_i$ for every i, proving (67). The converse follows from the exact quadratic identity

$$\frac{q(x) - q(y)}{x - y} = \frac{q'(x) + q'(y)}{2}. \tag{70}$$

A nonzero $q \in \ker Z_\delta$ vanishing at b_j also has $q'(b_j) = 0$, so it is proportional to $(z - b_j)^2$. Its other equations are exactly (68). This proves equality of the two rank-drop sets. For two elements $q_1, q_2 \in \ker Z_\delta$, the Wronskian $q_1 q_2' - q_1' q_2$ has degree at most two and vanishes at all the $N \geq 3$ marked points. It is zero, so q_1, q_2 are proportional. Hence $\operatorname{rank} Z_\delta = 2$ on the rank-drop set.

Also $\dim \ker \Phi_\delta = 1$ there. If two independent kernel vectors existed, a nonzero linear combination would have a zero coordinate and hence singleton support. Two distinct singleton vectors are impossible, since at a third node they prescribe different values of δ. A singleton and a full-support vector would give a polynomial q for which

$$(z - b_j)q'(z) - 2q(z)$$

vanishes at all b_i, $i \neq j$. This polynomial has degree at most one. As there are at least two such nodes, it vanishes identically, so q is proportional to $(z - b_j)^2$, contrary to full support.

The graph and its blowup. Consider the rational map

$$\mathbf{P}(\mathrm{Pol}_2) \dashrightarrow (\mathbf{P}^1)^N, \qquad [q] \longmapsto \big([\,(n/2)q'(b_i) : q(b_i)\,]\big)_i.$$

The i-th base ideal is generated by the independent linear forms $q(b_i), q'(b_i)$, and defines the reduced point p_i. Near this point all the other maps are regular. The graph of the remaining pencil is the blowup of its two-generator ideal. Since the points p_i are distinct, the complete graph is the blowup at all of them. Restrict to the affine target charts where every δ_i is finite. The graph equations become

$$\delta_i q(b_i) = \frac{n}{2} q'(b_i).$$

Locally, writing $u = q(b_i)$, $v = q'(b_i)$, this is the blowup chart $(n/2)v = \delta_i u$. It omits the strict transform of $u = 0$ and retains an affine line of the exceptional divisor. Thus the graph G_Z is the displayed smooth irreducible surface.

On a chart with an invertible 2×2 minor of Z_δ, linear elimination recovers its kernel line regularly from δ, and the remaining equations are precisely the 3×3 minors. Since Z_δ has rank two at every rank-drop point, these charts cover. Therefore $G_Z \to X_Z$ is a scheme isomorphism.

Reducedness at the singleton boundary. Let

$$G_\Phi = \{(\delta, [\eta]) : \Phi_\delta \eta = 0\} \subseteq \mathbf{A}^N \times \mathbf{P}^{N-1}.$$

The formula

$$\eta_i = \prod_{\ell \neq i} q(b_\ell) \tag{71}$$

defines a regular map $G_Z \to G_\Phi$: a polynomial in $\ker Z_\delta$ has either no marked root or a double root at exactly one marked point, so these coordinates never all vanish. Off the singleton boundary, the inversion leading to (69) identifies G_Φ with the corresponding smooth open subset of G_Z.

At a singleton e_j, normalize $\eta_j = 1$. For $i \neq j$, put $a_i = b_i - b_j$ and $\varepsilon_i = d\eta_i$; the base values are $\delta_i = n/a_i$. For distinct $i, \ell \neq j$, linearizing the pair (i, ℓ) gives

$$\left(\frac{n}{a_\ell} + \frac{n}{a_i - a_\ell}\right)\varepsilon_i + \left(\frac{n}{a_i} - \frac{n}{a_i - a_\ell}\right)\varepsilon_\ell = 0,$$

or, after multiplying by $a_i a_\ell (a_i - a_\ell)/n$,

$$a_i^2 \varepsilon_i = a_\ell^2 \varepsilon_\ell.$$

These have $N-2$ independent conditions. Thus $\varepsilon_i = \tau/a_i^2$ for one scalar τ. The pair (i,j) gives

$$d\delta_i = -\left(\delta_j + \frac{n}{a_i}\right)\frac{\tau}{a_i^2}.$$

The two parameters $\tau, d\delta_j$ are free. The tangent space of G_Φ therefore has dimension exactly two. Its underlying space has local dimension two: the graph parametrization is bijective on geometric points, and the full-support surface is dense, also at each exceptional line. Thus each local ring at the boundary has dimension and embedding dimension two, and is regular. This proves that G_Φ is reduced everywhere.

Finally, on a chart with an invertible $(N-1)\times(N-1)$ minor of Φ_δ, eliminate the normalized kernel vector. The residual equations are its maximal minors, so $G_\Phi \to X_\Phi$ is an isomorphism on that chart. The corank-one statement shows that these charts cover. Hence X_Φ is reduced. The two reduced closed subschemes $X_\Phi, X_Z \subseteq \mathbf{A}^N$ have the same geometric support, and therefore the same defining radical ideal. They are equal, and all the assertions follow. □

Remark A.2 (The two-node exception). When $N = 2$, Φ_δ has one row and a kernel everywhere. That row is zero exactly at

$$\delta_1 = \frac{n}{b_1 - b_2}, \qquad \delta_2 = \frac{n}{b_2 - b_1}.$$

Here $\dim\ker\Phi_\delta = 2$ and $\dim\ker Z_\delta = 2$; elsewhere both dimensions are one. The map

$$q \longmapsto (q(b_2), q(b_1))$$

identifies these two kernels. Equation (70) proves that it maps into $\ker\Phi_\delta$; a polynomial in its kernel would have two distinct double roots and hence would be zero. The incidence consequently has a $\mathbf{P}^1$ fibre at the exceptional parameter, whereas its image is all of $\mathbf{A}^2$.

The three-node determinant. For $N = 3$, take $(b_1, b_2, b_3) = (0, 1, 2)$ and $(\delta_1, \delta_2, \delta_3) = (x, y, z)$. Using the ordered constant, linear, and quadratic monomials as a basis of Pol_2, and indexing the rows of Φ by $(1,2), (1,3), (2,3)$, the two matrices are

$$\Phi = \begin{pmatrix} y-n & x+n & 0 \\ z-n/2 & 0 & x+n/2 \\ 0 & z-n & y+n \end{pmatrix}, \qquad Z = \begin{pmatrix} x & -n/2 & 0 \\ y & y-n/2 & y-n \\ z & 2z-n/2 & 4z-2n \end{pmatrix}.$$

Expansion along the first row gives

$$\begin{aligned} \det\Phi &= -(y-n)(x+n/2)(z-n) - (x+n)(z-n/2)(y+n) \\ &= -2xyz + \frac{3}{2}nxy - \frac{3}{2}nyz - \frac{1}{2}n^2x + n^2y - \frac{1}{2}n^2z, \\ \det Z &= x\big((y-n/2)(4z-2n) - (y-n)(2z-n/2)\big) \\ &\qquad + \frac{n}{2}\big(y(4z-2n) - z(y-n)\big) \\ &= 2xyz - \frac{3}{2}nxy + \frac{3}{2}nyz + \frac{1}{2}n^2x - n^2y + \frac{1}{2}n^2z. \end{aligned}$$

Thus

$$P = 4xyz - 3nxy + 3nyz + n^2x - 2n^2y + n^2z, \qquad \det\Phi = -\frac{P}{2}, \qquad \det Z = \frac{P}{2}.$$

For this configuration, the common determinantal ideal is (P).

Appendix B. A ramification criterion for dimension bounds

The following intersection argument reduces a dimension bound to maximal rank on a fixed-point ramification locus.

LEMMA B.1. *Let C be a smooth projective curve, $p \in C$, and let $Z \subseteq \mathrm{Pic}^d(C)$ be a projective integral subvariety on which $h^0(C, L) = n \geq 1$ is constant. If $t \geq 1$ is an integer and $\dim Z \geq t$, then some $L \in Z$ has*

$$H^0\big(C, L(-(n+t-1)p)\big) \neq 0.$$

PROOF. Since $n \geq 1$, we have $d \geq 0$. If $d = 0$, every line bundle in Z with a section is trivial, contradicting $\dim Z \geq t \geq 1$. Thus $d \geq 1$. Let $\pi\colon C \times Z \to Z$ be the projection. Normalize a Poincaré bundle $\mathcal{P}$ on $C \times Z$ at p, and put $\mathcal{E} = \pi_*\mathcal{P}$. Constancy of h^0, together with Riemann–Roch, gives cohomology and base change, so $\mathcal{E}$ is locally free of rank n. Let $p_Y\colon Y = \mathbf{P}_{\text{lines}}(\mathcal{E}) \to Z$ be the projection, and put $\xi = c_1(\mathcal{O}_Y(1))$. A point of Y is a line bundle together with a nonzero section up to scalar. Thus Y identifies with the inverse image of Z under the Abel map $C^{(d)} \to \mathrm{Pic}^d(C)$. In particular $\dim Y = \dim Z + n - 1$.

The divisor $X_p = \{D \in C^{(d)} : p \in D\}$ is ample. Indeed under the finite surjection $C^d \to C^{(d)}$, its line bundle pulls back to $\bigotimes_{i=1}^d \mathrm{pr}_i^*\,\mathcal{O}_C(p)$, which is ample; ampleness descends under a finite surjection. The restriction of its line bundle to Y is $\mathcal{O}_Y(1)$. Indeed, writing $\mathrm{pr}_Y\colon C \times Y \to Y$ for the projection, the universal Abel divisor has line bundle

$$(\mathrm{id}_C \times p_Y)^*\mathcal{P} \otimes \mathrm{pr}_Y^*\,\mathcal{O}_Y(1).$$

Restriction to $p \times Y$, using the normalization of $\mathcal{P}$, gives the asserted identity. This identity of line bundles holds even when every section in a fiber vanishes at p. Consequently ξ is ample.

Put $N = n+t-1$. The jet bundle $\mathcal{J}_N = \pi_*(\mathcal{P}|_{Np\times Z})$ has rank N. Its filtration by successive powers of the ideal of p has trivial line-bundle quotients on Z; each is $\mathcal{P}|_{p\times Z}$ tensored with a fixed one-dimensional cotangent space. Thus $c(\mathcal{J}_N) = 1$. The tautological section line gives an evaluation section of $\mathcal{O}_Y(1) \otimes p_Y^*\mathcal{J}_N$. If the lemma were false, this section would be nowhere zero. Its quotient would then be a vector bundle of rank $N-1$, forcing

$$0 = c_N(\mathcal{O}_Y(1) \otimes p_Y^*\mathcal{J}_N) = \xi^N.$$

But $\dim Y = \dim Z + n - 1 \geq N$. Multiplication by $\xi^{\dim Y - N}$ would make the positive top self-intersection of an ample line bundle zero, a contradiction. □

COROLLARY B.2. *Let C be a smooth projective curve, and let $m \geq 2$ be an integer. Let $W \subseteq \mathrm{Pic}^d(C)$ be a projective locus, endowed with its reduced structure, on which $h^0(L) = n \geq 1$ and $h^0(L^m)$ are constant. Fix $p \in C$ and an integer $t \geq 1$. Suppose that every $L \in W$ satisfying*

$$H^0(L(-(n+t-1)p)) \neq 0$$

has a maximal-rank degree-m multiplication map. Then the locus of non-maximal-rank maps in W has dimension at most $t-1$.

PROOF. Constancy of the two section dimensions realizes multiplication as a morphism of vector bundles. Its non-maximal-rank locus is closed, hence projective. Apply Lemma B.1 to the reduced structure on any irreducible component of dimension at least t. The resulting ramified line bundle contradicts the hypothesis. □

REMARK B.3. The restriction $n \geq 1$ is necessary: a positive-dimensional subvariety of a negative-degree Picard variety has $n = 0$ and no section. The corollary reduces the dimension bound to maximal rank on the fixed-point ramification locus.

References

[1] M. Aprodu and G. Farkas, *Koszul cohomology and applications to moduli*, in Grassmannians, Moduli Spaces and Vector Bundles, Clay Mathematics Proceedings **14**, American Mathematical Society, Providence, RI, 2011, 25–50.

[2] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves, Vol. I*, Grundlehren der mathematischen Wissenschaften **267**, Springer-Verlag, New York, 1985.

[3] D. Arinkin, *Cohomology of line bundles on compactified Jacobians*, Mathematical Research Letters **18** (2011), no. 6, 1215–1226; arXiv:0705.0190v2.

[4] E. Ballico, *Rank one torsion free sheaves on curves: the injectivity of the symmetric multiplication map*, International Journal of Pure and Applied Mathematics **12** (2004), no. 4, 483–484, ijpam.eu/contents/2004-12-4/7/7.pdf.

[5] E. Ballico, *Embeddings of general curves in projective spaces: the range of the quadrics*, Lithuanian Mathematical Journal **52** (2012), no. 2, 134–137, doi:10.1007/s10986-012-9161-9.

[6] E. Ballico and C. Fontanari, *Normally generated line bundles on general curves*, Journal of Pure and Applied Algebra **214** (2010), 837–840, doi:10.1016/j.jpaa.2009.08.002.

[7] W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Mathematics **1327**, Springer-Verlag, Berlin, 1988.

[8] E. Cotterill, A. Alonso Gonzalo and N. Zhang, *The strong maximal rank conjecture and higher-rank Brill–Noether theory*, Journal of the London Mathematical Society **104** (2021), 169–205, doi:10.1112/jlms.12427.

[9] E. Esteves and S. Kleiman, *The compactified Picard scheme of the compactified Jacobian*, arXiv:math/0410537v1.

[10] G. Farkas, *Gaussian maps, Gieseker–Petri loci and large theta-characteristics*, Journal für die reine und angewandte Mathematik **581** (2005), 151–173.

[11] G. Farkas, *Higher ramification and varieties of secant divisors on the generic curve*, Journal of the London Mathematical Society **78** (2008), 418–440, doi:10.1112/jlms/jdn038.

[12] G. Farkas, D. Jensen and S. Payne, *The nonabelian Brill–Noether divisor on $\overline{\mathcal{M}}_{13}$ and the Kodaira dimension of $\overline{\mathcal{R}}_{13}$*, Geometry & Topology **28** (2024), no. 2, 803–866, doi:10.2140/gt.2024.28.803.

[13] G. Farkas, D. Jensen and S. Payne, *The Kodaira dimension of $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$*, Cambridge Journal of Mathematics **13** (2025), no. 3, 431–607, doi:10.4310/CJM.250715010717.

[14] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Inventiones Mathematicae **66** (1982), 251–275, doi:10.1007/BF01389394.

[15] D. Jensen and S. Payne, *Tropical independence II: The maximal rank conjecture for quadrics*, Algebra & Number Theory **10** (2016), no. 8, 1601–1640, doi:10.2140/ant.2016.10.1601.

[16] E. Larson, *The maximal rank conjecture*, arXiv:1711.04906.

[17] C. Lian, *General curves on smooth quadric hypersurfaces*, arXiv:2611.23141v1.

[18] F. Liu, B. Osserman, M. Teixidor i Bigas and N. Zhang, *Limit linear series and ranks of multiplication maps*, Transactions of the American Mathematical Society **374** (2021), no. 1, 367–405, doi:10.1090/tran/8230.

[19] F. Liu, B. Osserman, M. Teixidor i Bigas and N. Zhang, *The strong maximal rank conjecture and moduli spaces of curves*, Algebra & Number Theory **18** (2024), 1403–1464, msp.org/ant/2024/18-8/p01.xhtml.

[20] M. Melo, A. Rapagnetta and F. Viviani, *Fine compactified Jacobians of reduced curves*, Transactions of the American Mathematical Society **369** (2017), no. 8, 5341–5402. The numbering used here is that of arXiv:1406.2299v3.

[21] F. W. J. Olver et al., eds., *NIST Digital Library of Mathematical Functions*, Section 24.4 and Section 25.6, accessed November 27, 2026.

[22] E. Park, *On rank 3 quadratic equations of projective varieties*, Transactions of the American Mathematical Society **377** (2024), 2049–2064, doi:10.1090/tran/9083.

[23] V. Robu, *The stratified and the strong maximal rank conjecture in $\mathbf{P}^3$ and $\mathbf{P}^4$*, arXiv:2605.24612v1. All references to this preprint are specifically to version 1.

[24] The Stacks Project Authors, *The Stacks Project*, stacks.math.columbia.edu, Tags 0C6R and 089A, accessed November 27, 2026.

[25] M. Teixidor i Bigas, *Injectivity of the symmetric map for line bundles*, Manuscripta Mathematica **112** (2003), no. 4, 511–517, doi:10.1007/s00229-003-0424-y.