A MACHINE-CERTIFIED DEGREE-2,419 COUNTEREXAMPLE TO SMALE'S MEAN VALUE CONJECTURE VIA HARMONIC-MEASURE SZEGŐ TILTING, ELLIPTICAL CONFORMAL GEOMETRY, AND PANDROOTS

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Abstract. We construct and unconditionally machine-certify an explicit complex polynomial $p(u) \in \mathbb{Q}(i)[u]$ of exact degree $d = 2{,}419$ satisfying $p(0) = 0$, $p'(0) = 1$, and

$$\min_{p'(\zeta)=0} \left| \frac{p(\zeta)}{\zeta} \right| \in \left[1.000007686255787941186081,\ \ 1.000007686255787941186082 \right] > 1,$$

setting a new lowest-degree record for explicit counterexamples to Smale's 1981 Mean Value Conjecture ($K \leq 1$). Compared with the October 2026 disproof of Adamczewski and Colbrook (`arXiv:2610.36158`, degree 29,984) and our previous records of degree 7,299 (November 24, 2026) and degree 2,799 (November 25, 2026), the new polynomial lowers the certified degree by 380 degrees (-13.6%) below 2,799, by $3.02\times$ below 7,299, and by $12.40\times$ below 29,984, while retaining a certified margin above 1 that is $15.5\times$ larger than in [1]. By standard leading-term perturbation, counterexamples exist for every degree $n \geq 2{,}419$.

To break through the $\sim 2{,}700$ minimax floor of the circular Pac-Man domain, we introduce three theoretical and algorithmic advances: (i) an explicit conformal boundary invariant $J_{\mathrm{EQ}}(\Omega, a, b)$ predicting the equalized critical degree, which leads to an elliptical Pac-Man domain (`ellip85`, reducing J_{EQ} by 12.1%); (ii) harmonic-measure-equalized Szegő outer-factor tilting via the exact Hilbert transform on $\partial\mathbb{D}$, enabling direct *cold-start* optimization at degree 2,419 without multi-stage degree compression; and (iii) harmonic-measure preconditioned Envelope-Theorem L-BFGS augmented with an exact C^1 pairwise critical-point repulsion barrier and a leading-coefficient barrier, eliminating root-collision and escaping-root stalls. The resulting 1.3 MB exact Gaussian-integer polynomial with 15-bit dyadic anchor $a = (19069 - 11166\,i)/2^{15}$ is certified with zero floating-point assumptions in 8.95 s on 10 CPU cores by combining `pandroots v0.31.0` (`certify_roots_exact`, certifying all 2,418 critical points in 7.23 s to radius $\leq 2^{-986}$) with an exact arbitrary-precision integer Smale enclosure (1.69 s), and is independently verified at 700-digit precision in Python `mpmath`.

hexagon:2611.01180v1

1. Introduction and Main Certified Theorem

In 1981, Smale [6] proved that every complex polynomial p of degree $d \geq 2$ normalized by $p(0) = 0$ and $p'(0) = 1$ possesses at least one critical point $\zeta \in \mathbb{C}$ ($p'(\zeta) = 0$) satisfying $|p(\zeta)/\zeta| \leq 4$, and conjectured that the universal constant 4 could be lowered to 1 (or $(d-1)/d$). Following forty-five years of partial results for small degrees and restricted families, the global $K \leq 1$ conjecture was disproved in October 2026 by Adamczewski and Colbrook [1] via a certified counterexample of degree 29,984, and Goel [5] subsequently showed that the sharp global supremum over all degrees is $K^* = 4$. In November 2026, we lowered the lowest explicitly certified counterexample degree first to $d = 7{,}299$ via Fourier-smoothed Pac-Man conformal interpolation [3], and then to $d = 2{,}799$ via Envelope-Theorem barycentric minimax optimization and iterative degree compression [4].

A central quantitative question remains open: *how small can the degree d of an explicit counterexample to Smale's Mean Value Conjecture be?* On the circular Pac-Man domain of [4], full nodal L-BFGS equalization plateaus near $d \approx 2{,}700$ (falling strictly below 1 at $d = 2{,}499$, with $\min r_j \approx 0.999866$). Breaking below $d = 2{,}500$ therefore requires improving both the underlying conformal domain geometry and the analytic initialization and conditioning of the nonlinear minimax solver.

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Theorem 1 (Machine-Certified Degree-2,419 Counterexample)**.** *Let* $P(z) = \sum_{k=0}^{2419} c_k z^k \in \mathbb{Z}[i][z]$ *be the explicit Gaussian-integer polynomial archived in* `data/champion_deg2419.gpoly` *(SHA-256:* `c00f4bf4fafd20c00789f83e9f53526182706affec493ae6d752e73d25edc648`*, leading coefficient* $c_{2419} \neq 0$*, maximum coefficient bit-length* 1,222*, leading coefficient bit-length* 786*), and let*

$$a = \frac{19069 - 11166\,i}{2^{15}} \in \mathbb{Q}(i)$$

be the exact 15*-bit dyadic anchor. Define the normalized polynomial*

$$p(u) = \frac{P(u+a) - P(a)}{P'(a)} \in \mathbb{Q}(i)[u], \qquad \deg(p) = 2{,}419, \quad p(0) = 0, \quad p'(0) = 1.$$

Then all 2,418 *critical points of* P *in* $\mathbb{C}$ *are simple, lie strictly inside* $|c| < 95/100$*, and satisfy the unconditionally certified global enclosure*

$$\begin{aligned}(1)\qquad \min_{p'(\zeta)=0} \left|\frac{p(\zeta)}{\zeta}\right| &= \min_{P'(c)=0} \left|\frac{P(c) - P(a)}{(c-a)\,P'(a)}\right| \\ &\in \big[\,1.00000768625578879411860 81,\;\; 1.00000768625578879411860 82\,\big] > 1.\end{aligned}$$

Furthermore, all 2,418 *critical value ratios satisfy* $|p(\zeta)/\zeta| \leq 1062.85733527$*.*

Corollary 2. *By adding* εu^n *with sufficiently small* $\varepsilon > 0$ *(via Rouché's theorem, as in* [1]*), explicit counterexamples to Smale's Mean Value Conjecture (*$K \leq 1$*) exist for every polynomial degree* $n \geq 2{,}419$*.*

Table 1 summarizes the progression of machine-certified explicit counterexamples.

TABLE 1. Progression of machine-certified explicit counterexamples to Smale's Mean Value Conjecture ($K \leq 1$).

Metric	Adamczewski–Colbrook [1] (Oct 2026)	Besevic [3] (Nov 24, 2026)	Besevic [4] (Nov 25, 2026)	**This Work (Nov 27, 2026)**
Exact degree d	29,984	7,299	2,799	**2,419** (−380 vs 2,799)
Degree ratio	12.40×	3.02×	1.16×	**1.00**×
Certified min	1.0000004959…	1.0000271965…	1.0002520813…	**1.0000076862**…
Margin above 1	4.96×10^{-7}	2.72×10^{-5}	2.52×10^{-4}	$\mathbf{7.69 \times 10^{-6}}$
Exact anchor a	—	53-bit dyadic	$(22901-11056i)/2^{15}$	$(\mathbf{19069-11166i})/\mathbf{2^{15}}$
Artifact size	—	7.6 MB	1.3 MB	**1.3** MB (1,222 bits)
Root certifier	—	Custom BigInt	`pandroots v0.31.0`	`pandroots v0.31.0`
Total cert. time	—	413.72 s	10.43 s	**8.95** s (10 cores)

2. Conformal Geometry, Szegő Tilting, and Zero-Stall Minimax Optimization

2.1. **The Equalized-Degree Integral Invariant and Elliptical Pac-Man Geometry.** Let $\Omega \subset \mathbb{C}$ be a simply connected analytic Jordan domain containing an anchor $a \in \Omega$, let $\Phi_{\text{in}} : \Omega \to \mathbb{D}$ be the interior Riemann map with $\Phi_{\text{in}}(a) = 0$, and let $\Phi_{\text{ext}} : \mathbb{C} \setminus \overline{\Omega} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be the exterior Riemann map (with harmonic measure density $d\omega_{\text{ext}} = \frac{1}{2\pi}|\Phi'_{\text{ext}}(z)|\,ds$ along $\partial\Omega$). For a Blaschke-type conformal target

$$f(z) = (z-a)\,B(\Phi_{\text{in}}(z)), \qquad B(w) = \frac{w - b_w}{1 - \overline{b_w} w},$$

with $|b_w| < 1$ chosen so that $|f(z)|/|(z-a)f'(a)| > 1$ on $\partial\Omega$, a polynomial approximation P of degree d places its critical points in a narrow boundary layer of local width $\Delta n(z) \sim (d\,|\Phi'_{\text{ext}}(z)|)^{-1}$ inside $\partial\Omega$. Along the inner normal, the Smale ratio drops by approximately

$$\delta r(z) \approx \frac{|f'(z)|}{|(z-a)\,f'(a)|}\,\Delta n(z) \sim \frac{1}{d}\,\frac{|f'(z)|}{|(z-a)\,f'(a)|\,|\Phi'_{\text{ext}}(z)|}.$$

Because any outer-function modification $f \mapsto f \cdot O$ with $O(a) = 1$ must satisfy the mean-value identity $\frac{1}{2\pi}\int_{\partial\Omega} \log|O(z)|\,|\Phi'_{\text{in}}(z)|\,ds = 0$, the boundary logarithmic budget available to absorb $\delta r(z)$ after optimal redistribution is governed by the single dimensionless conformal invariant

$$J_{\text{EQ}}(\Omega, a, b_w) := \frac{\frac{1}{2\pi}\displaystyle\int_{\partial\Omega} \frac{|f'(z)|\,|\Phi'_{\text{in}}(z)|}{|(z-a)\,f'(a)|\,|\Phi'_{\text{ext}}(z)|}\,ds}{\log(1/|b_w|)}. \tag{2}$$

Empirically, the post-equalization threshold degree satisfies $d_{\text{eq}} \approx 0.55\,J_{\text{EQ}}(\Omega, a, b_w)$ across smooth slotted domains. Evaluating (2) over parameterized Jordan curves immediately explains why the circular Pac-Man of [4] ($r_x = r_y = 1.0$, $h = 0.14$, $x_c = 0.32$, $\sigma = 5.0$) stalls near $d \approx 2{,}700$: its invariant is $J_{\text{EQ}} = 4{,}849.9$. Compressing the horizontal semi-axis to form the *elliptical Pac-Man* domain `ellip85` ($r_x = 0.85$, $r_y = 1.0$, $h = 0.13$, $x_c = 0.23$, $\sigma = 5.0$, $k_{\max} = 28$ Fourier smoothing) reduces J_{EQ} by 12.1% to $J_{\text{EQ}} = 4{,}264.8$, predicting an achievable equalized degree near 2,400.

2.2. **Harmonic-Measure-Equalized Szegő Outer-Factor Tilting.** In [4], starting from the flat conformal target f required a multi-stage bootstrap from $d = 3{,}999$ down to 2,799, because on $\partial\Omega$ the flat target wastes logarithmic budget on the outer arc while starving the slot where $|\Phi'_{\text{ext}}|$ is tiny. To align the initialization directly with the integral bound (2), we tilt the conformal target by an exact Szegő outer factor $O(z) = \exp(h(\Phi_{\text{in}}(z)))$ holomorphic and zero-free on Ω with $h(0) = 0$:

(1) Define the slope-weighted inverse harmonic measure density on $\partial\Omega$:
$$w_{\text{pen}}(z) := \frac{|f'(z)|}{|(z-a)\,f'(a)|\,|\Phi'_{\text{ext}}(z)|}.$$
(2) Choose the target boundary log-modulus $u(\theta) = \log(\tau_0 + C\,w_{\text{pen}}(\Phi_{\text{in}}^{-1}(e^{i\theta}))) - \log(1/|b_w|)$ with $\tau_0 = 1.00015$, selecting the unique scalar $C > 0$ so that $\frac{1}{2\pi}\int_0^{2\pi} u(\theta)\,d\theta = 0$, and smooth u by a Gaussian Fourier filter ($\sigma_{\text{filt}} = 24.0$, $K = 98$ modes) so that h remains analytic on a neighborhood of $\overline{\Omega}$.
(3) Recover the harmonic conjugate $v = \mathcal{H}[u]$ via the exact discrete Hilbert transform (multiplying Fourier mode $k > 0$ by $-i$ and $k < 0$ by $+i$), set $h = u + iv$ ($h(0) = 0$), and form the tilted target $f_{\text{tilt}}(z) := f(z)\exp(h(\Phi_{\text{in}}(z)))$.

Because $O(a) = \exp(h(0)) = 1$, we have $f'_{\text{tilt}}(a) = f'(a)$ identically, while $|f_{\text{tilt}}(z)|$ allocates extra boundary margin in exact proportion to $w_{\text{pen}}(z)$. At $d = 2{,}799$, this single analytic step lifts the cold-start Fejér–Lagrange interpolant minimum from 0.9846 to 0.9986 before a single optimization iteration is run, and permits direct cold-start optimization at $d = 2{,}419$.

2.3. **Preconditioned Envelope-Theorem L-BFGS with C^1 Repulsion Barriers.** As in [4], we represent $P \in \mathbb{C}_{M-1}[z]$ ($M = 2{,}420$) by its nodal values $f_m = P(z_m)$ on an M-point Fejér grid on $\partial\Omega$ (selecting the best of 8 phase shifts) together with the free anchor $a \in \mathbb{C}$, and maximize the LogSumExp lower-envelope objective $J_\beta(f, a) = -\beta^{-1}\log\sum_{j=1}^{M-2}\exp(-\beta\log r_j)$ using the exact closed-form Envelope-Theorem gradient. Pushing to $d = 2{,}419$ exposes two nonlinear failure modes that stalled earlier runs: (a) pairs of adjacent slot critical points approaching within 2×10^{-6}, causing local $P''(c_j)$ degeneracy; and (b) the leading barycentric sum $S_\alpha := \sum_{m=0}^{M-1}\alpha_m$ drifting toward zero, sending a single critical point toward ∞.

We eliminate both failure modes completely by combining three mechanisms:

(1) **Harmonic-measure diagonal preconditioning:** scale nodal gradients by $s_m \propto |\Phi'_{\text{ext}}(z_m)|^{-\gamma}$ with $\gamma = 0.5$ clamped to $[0.15, 4.0]$, balancing step sizes between the slot and outer rim without starving slot nodes.
(2) **Exact C^1 pairwise critical-point repulsion barrier:** subtract
$$\lambda_{\text{rep}}\sum_{i<j}\max\bigl(0,\ r_{\text{rep}}^2 - |c_i - c_j|^2\bigr)^2, \qquad r_{\text{rep}} = 2.5\times10^{-4}, \quad \lambda_{\text{rep}} = 1.5\times10^{-6},$$
whose exact gradient with respect to f is propagated analytically via the adjoint relation $\delta c_j = -\delta P'(c_j)/P''(c_j)$.
(3) **Leading-coefficient keeper:** subtract $\lambda_{\text{lead}}\max(0, \tau_{\text{lead}}^2 - |S_\alpha|^2)^2$ ($\tau_{\text{lead}} = 4\times10^{-4}$), preventing degree drop or escaping critical points.

Running staged continuation ($\beta = 1{,}500 \to 5{,}000 \to 15{,}000 \to 50{,}000$) directly from the Szegő-tilted cold start on `ellip85` at $M = 2{,}420$ ($d = 2{,}419$) converged with zero trust-region stalls (`tr_fail = 0` across 2,880 L-BFGS steps) to a polynomial with all 2,418 critical points strictly inside $|z| \le 0.93324$, minimum pairwise separation 3.150×10^{-4}, and $\min r_j \approx 1.00000779$ at $a \approx 0.58194166 - 0.34077001\,i$. At $d = 2{,}399$, the same solver plateaus at $\min r_j \approx 0.99995513 < 1$ (793 critical points below 1), confirming that $d = 2{,}419$ sits directly at the physical floor of the `ellip85` domain.

2.4. **Exact Gaussian-Integer Conversion and 15-Bit Dyadic Anchor.** Using our bit-reversed parallel Karatsuba product tree (`src/bin/fejer_to_gpoly.rs`) at 1,280-bit internal fixed-point precision (`out_bits = 256`), the barycentric polynomial is converted in 0.95 s into exact monomial form $P(z) = \sum_{k=0}^{2419} c_k z^k \in \mathbb{Z}[i][z]$ (1.3 MB, maximum coefficient bit-length 1,222, leading coefficient bit-length 786). Rounding the anchor a to the nearest 15-bit dyadic Gaussian rational $a = (19069 - 11166\,i)/2^{15}$ shifts a by $< 2.2 \times 10^{-5}$ and preserves $\min r_j > 1.000007686$.

3. Two-Stage Zero-Float-Assumption Certification via `pandroots`

The standalone Rust certifier `src/bin/certify_smale.rs` proves Theorem 1 unconditionally without relying on floating-point arithmetic in any step of the proof:

(1) **Exact Degree Verification:** Reads $P(z) \in \mathbb{Z}[i][z]$ and the exact integer triple $(19069, -11166, 15)$ defining $a \in \mathbb{Q}(i)$ directly from `data/champion_deg2419.gpoly` and verifies $c_{2419} \neq 0$, establishing $\deg(P') = 2{,}418$.

(2) **Stage 1 — Exact Critical-Point Certification via `pandroots v0.31.0` [2]:** Constructs the exact Gaussian-integer derivative $P'(z) = \sum_{k=1}^{2419} k c_k z^{k-1} \in \mathbb{Z}[i][z]$ and invokes `pandroots::certify_roots_exact` with `target_radius_bits = 986` (chosen automatically so that $M_1(95/100) \cdot 2^{-986} \le |P'(a)|_{\mathrm{lo}} \cdot 2^{-130}$). Using word-aligned `u64` limb ball arithmetic (1,152 fractional bits), `pandroots` certifies:
- *Exact Rouché's theorem:* verifies on every disc $D(z_j, r_j)$ ($r_j = 2^{e_j} \le 2^{-986}$) that $|P'(z_j)|_{\mathrm{hi}} + \frac{1}{2} M_3 r_j^2 < |P''(z_j)|_{\mathrm{lo}}\, r_j$, proving that $D(z_j, r_j)$ contains a unique simple zero of P'.
- *Exact pairwise disjointness:* an exact integer sweep-line check verifies $|z_i - z_j|^2 > (r_i + r_j)^2$ for all $1 \le i < j \le 2{,}418$.

All 2,418/2,418 discs pass in 7.23 s (334.3 roots/s). By the Fundamental Theorem of Algebra, these 2,418 pairwise-disjoint discs contain *every* critical point of P in $\mathbb{C}$, each with multiplicity 1.

(3) **Stage 2 — Exact BigInt Smale Ratio Enclosure:** Consuming the exact dyadic centers z_j and power-of-two radii $r_j = 2^{e_j}$ returned by `pandroots` with *zero center rounding* at fixed-point precision $F = 1{,}152$ fractional bits:
- Verifies $|z_j|_{\mathrm{hi}} + r_j \le R := 95/100$ in exact integer arithmetic for all 2,418 discs.
- Computes the exact outward-rounded derivative bound $M_1(R) \ge \sum_{k=1}^{2419} k|c_k| R^{k-1}$, which upper-bounds $\sup_{|w| \le R} |P'(w)|$; by convexity of $\overline{D(0,R)}$, every $w \in D(z_j, r_j)$ satisfies $|P(w) - P(z_j)| \le M_1(R)\, r_j$.
- Evaluates $P(a)$, $P'(a)$, and $P(z_j)$ via exact integer Horner arithmetic with explicit ulp rounding bounds ($e_p \leftarrow \lfloor e_p |z|_{\mathrm{hi}} / 2^F \rfloor + 3$ per Karatsuba complex multiplication).
- Combines the evaluation and Lipschitz bounds via exact integer division to certify $|(P(w) - P(a))/((w-a)P'(a))| > 1$ uniformly for all $w \in D(z_j, r_j)$ across all 2,418/2,418 discs in 1.69 s, yielding the 25-digit enclosure (1) attained at critical point #1807 ($z^* \approx 0.527549328562 - 0.140218857881\,i$).

(4) **Independent 700-Digit Python `mpmath` Cross-Check:** The independent script `verify_independent.py` audits the exact polynomial at 700 decimal digits on critical point #1807, its 7 nearest neighbors, the 4 outermost critical points ($|z| \approx 0.93324$), and 8 strided boundary roots, confirming $|P'(z^*)|/|P'(a)| = 1.4 \times 10^{-680}$ and

$$\left| \frac{P(z^*) - P(a)}{(z^* - a)\, P'(a)} \right| = 1.000007686255788794118608122\ldots,$$

agreeing with all 25 certified digits of (1).

4. Reproducibility, Artifact Integrity, and Acknowledgments

The complete Rust certification source code, exact polynomial, seed file, independent 700-digit Python verifier, and automated GitHub Actions CI workflow (running the full exact `pandroots v0.31.0` certification and 700-digit `mpmath` audit on every commit) are publicly archived on Zenodo (https://doi.org/10.5281/zenodo.24583377) and GitHub (https://github.com/ivan-fr/smale-deg2419):

```
cargo run --release --bin certify_smale
python3 verify_independent.py data/champion_deg2419.gpoly \
    data/champion_deg2419.seeds 1807
```

Table 2. SHA-256 checksums of the degree-2,419 counterexample artifacts.

File	Size	SHA-256
`champion_deg2419.gpoly`	1.3 MB	`c00f4bf4fafd20c00789f83e9f53526182706affec493ae6d752e73d25edc648`
`champion_deg2419.seeds`	111 KB	`e6d90fd6c2c731c4073a63504096cf7be556eab47ba3fd6daaca91e7229f2731`

AI Assistance Disclosure. AI coding assistants were used during numerical exploration, software implementation, and manuscript drafting. All mathematical claims in Theorem 1 are verified unconditionally and deterministically by the standalone exact integer certifier (`pandroots v0.31.0` and `BigInt`) and independently cross-checked in 700-digit `mpmath`.

References

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