An effective chamber of strange duality

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Abstract

We prove strange duality on the projective plane in an explicit chamber of positive ranks. We also determine the nef and ample thresholds of determinant line bundles on the full moduli spaces of one-dimensional sheaves on del Pezzo surfaces of degree one.

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1 Introduction

Strange duality compares spaces of generalized theta functions on moduli spaces of sheaves. Le Potier's exposition [16] starts with the curve dualities of Beauville and Beauville–Narasimhan–Ramanan, and develops a surface construction using the determinant of cohomology. Danila [10] traces the plane conjecture to Le Potier's December 1996 lecture at Luminy. On surfaces, positive-rank sheaves, Hilbert schemes of points and pure one-dimensional sheaves lead to different numerical ranges of the problem. We study two positive-rank classes on the projective plane and determinant positivity on del Pezzo surfaces of degree one.

1.1 The determinant map and earlier results

Let X be a smooth complex projective surface with an ample divisor H. Write $K(X)$ for the Grothendieck group of coherent sheaves and $M_H(v)$ for the coarse moduli space of H-Gieseker-semistable sheaves of numerical class v. Positive-rank sheaves in this notation are torsion-free; sheaves of rank zero and dimension one are pure. The Euler characteristic and the Ext pairing are

$$\chi(E) = \sum_i (-1)^i \dim H^i(X, E), \qquad \chi(E, F) = \sum_i (-1)^i \dim \operatorname{Ext}^i_X(E, F).$$

Strange-duality orthogonality means $\chi(v \otimes w) = 0$, with the product taken in $K(X)$. For a family $\mathcal{E}$ on $X \times S$, let p_S and p_X be the projections to S and X. The associated inverse determinant of cohomology is

$$\lambda_w(\mathcal{E}) = \left(\det Rp_{S*}(\mathcal{E} \otimes^{\mathbf{L}} p_X^* w)\right)^{-1}.$$

When this line descends, we denote it by Θ_w on $M_H(v)$. Suppose the inverse determinant on the product of the two moduli spaces descends as $\Theta_w \boxtimes \Theta_v$ and the tensor-product cohomology has amplitude $[0, 1]$. Orthogonality makes a local representative a map of vector bundles of equal rank. Its determinant gives a section $\vartheta_{v,w}$ and hence

$$\mathrm{SD}_{v,w} : H^0(M_H(v), \Theta_w)^\vee \longrightarrow H^0(M_H(w), \Theta_v). \tag{1.1}$$

This is Le Potier's construction in [16, §1.3]. The problem is to determine when this canonical map is an isomorphism.

Numerical classes are written as (rk, c_1, χ). On the plane $H = c_1(\mathcal{O}_{\mathbf{P}^2}(1))$. Drézet and Le Potier [11] established the existence theory of stable plane bundles and the irreducibility of the complete semistable moduli spaces. Exceptional bundles determine the boundary of this existence region. This structure underlies both the use of resolutions by exceptional bundles and the passage from a dense presentation locus to the whole moduli space.

Early effective cases of the plane duality were obtained by Danila. In [9], she proved duality between rank-two sheaves with $c_1 = 0$, $2 \le c_2 = n \le 19$ and pure one-dimensional sheaves of degree one and Euler characteristic zero. The latter moduli space is the dual projective plane: its points represent $\mathcal{O}_\ell(-1)$ for lines $\ell \subset \mathbf{P}^2$. The determinant section records jumping lines, linking duality to the Barth morphism. Danila's subsequent work [10] proves duality for $2 \le n \le 5$ and support degrees $d = 1, 2, 3$, using calculations on Hilbert schemes of points. These results combine explicit theta sections with computations of their spaces of global sections.

Abe [1] proved strange duality when one plane moduli space has height zero, the boundary determined by exceptional bundles. Yuan studied determinant bundles on moduli of pure sheaves on rational surfaces in [25], including their support morphisms, ample twists and spaces of sections in low-genus situations. His higher-rank results [26] include the pairing of rank-two sheaves with $c_1 = 0$, $c_2 \ge 3$ and pure sheaves of class $(0, 4H, 0)$ on the plane. Quiver methods give the broader statement of [27]: for the pair

$$(r, 0, r-n), \qquad (0, dH, 0), \qquad n \ge r > 0, \quad d > 0,$$

strange duality is injective, and it is an isomorphism if $n = r$, $n = r + 1$, or $d \le 3$. Here the dimension-one moduli space remains one of the two factors throughout the stated range.

A different method uses Quot schemes. Bertram–Goller–Johnson [5] relate strange duality on del Pezzo surfaces to Quot schemes and multiple-point formulas. Kernels and quotients provide pairs of sheaves at which the determinant tensor can be evaluated, making finite Quot schemes a source of rank estimates. Goller–Lin [13, Theorem 1.6] use Gaeta resolutions to prove injectivity when one class is $(1, 0, 1-\ell)$ and the other is $(r, L, r\ell)$, with $\ell \ge 1$, $r \ge 2$ and L satisfying their explicit sufficient positivity conditions. The version cited here treats the plane and specified two-step blowups of Hirzebruch surfaces with their prescribed polarizations. Its rank-one factor is the Hilbert scheme of ℓ points.

The linear-algebra approach begins with Beilinson's description of coherent sheaves on projective space [4]. Schofield–Van den Bergh [20] prove that determinantal semi-invariants span the semi-invariant spaces of an acyclic quiver in characteristic zero. Maiorana [17] identifies suitable Gieseker-semistable sheaf stacks with semistable quiver stacks. For strange duality these descriptions lead to three further tasks: compare determinant coefficients after imposing the relations, identify global sections on the complete moduli spaces, and identify the resulting pairing with the determinant of tensor-product cohomology. The proof below carries out these tasks for the following positive-rank chamber.

1.2 The plane theorem

Theorem 1.1. *Let n, s, r, d be integers satisfying*

$$n > s \ge 1, \qquad r \ge \max\{2, s\}, \qquad \frac{rn}{s} \in \mathbb{Z}, \qquad d > \frac{r}{s}(n-s)n. \tag{1.2}$$

Set

$$\rho = (s, 0, s-n), \qquad \sigma = \left(r, dH, \frac{rn}{s}\right), \qquad M_\rho = M_H(\rho), \quad M_\sigma = M_H(\sigma).$$

The determinant lines descend to the complete Gieseker moduli spaces, and the canonical map

$$\mathrm{SD}_{\rho,\sigma} : H^0(M_\rho, \Theta_\sigma)^\vee \longrightarrow H^0(M_\sigma, \Theta_\rho) \tag{1.3}$$

is an isomorphism between nonzero spaces.

Both ranks may exceed one, and the statement includes nonprimitive classes and strictly semistable points. For example,

$$\rho = (2, 0, -1), \qquad \sigma = (2, 4H, 3)$$

have second Chern classes 3 and 13. The determinant for this pair has an integral specialization of value 2, before and after the change of exceptional collection; the matrices and the determinant calculations are given in Section 4.6 and Appendix A.4.

The numerical location relative to the height-zero case can be made explicit. For a positive-rank plane class put $\mu = c_1 H/\operatorname{rk}$ and $\Delta = \mu^2/2 - \operatorname{ch}_2/\operatorname{rk}$. The Drézet–Le Potier boundary is

$$\delta(\mu) = \sup_{\substack{\alpha \text{ exceptional slope} \\ |\mu-\alpha|<3}} \left\{ 1 - \frac{3}{2}|\mu-\alpha| + \frac{1}{2}|\mu-\alpha|^2 - \Delta_\alpha \right\},$$

where Δ_α is the discriminant of the exceptional bundle of slope α; height zero means $\Delta = \delta(\mu)$ [8, §2.3]. For $0 \le t \le 3$ one has $1 - 3t/2 + t^2/2 \le 1$, and $\Delta_\alpha \ge 0$, so $\delta(\mu) \le 1$. In Theorem 1.1, $\Delta(\rho) = n/s > 1$ and $\Delta(\sigma) > 2$, the latter estimate being proved in Section 4. Both classes lie above this boundary.

We first establish a symmetric determinant comparison and prove the subrange $d > (r/s)(n-s)(n+1)$ in Theorem 3.1. The first exceptional mutation then gives a rectangular comparison and the stronger inequality (1.2). The relation algebras are treated with their generated ideals. A restriction theorem for derived Θ-strata identifies their character spaces with sections on M_ρ. A boundary estimate for complete prioritary families gives an injective pullback from M_σ to the Gaeta presentation space. The universal determinant identifies the canonical tensor under both maps, proving (1.3).

1.3 Determinant positivity

Positivity of determinant bundles supplies another part of the surface problem. The Bayer–Macrì positivity lemma [3] produces nef numerical classes from families of Bridgeland-semistable objects. Bertram–Martinez [6] relate changes of polarization to wall crossing and give a uniform large-volume description for pure one-dimensional sheaves. For the plane, Woolf [24] determines nef cones and many effective cones of moduli spaces of torsion sheaves. The support-determinant identity and the ample twists used below are those of [25, Propositions 3.0.3–3.0.4]. We combine these tools with an integral wall bound and explicit families on a del Pezzo surface of degree one.

Let X be such a surface, put $A = -K_X$, and for an integer $k \ge 1$ set

$$N_k = M_A(0, kA, 0), \qquad h : N_k \longrightarrow |kA|, \qquad P = h^*\mathcal{O}_{|kA|}(1).$$

The morphism h records the support with its multiplicities. Let $\mathcal{F}$ be the universal family on the sheaf stack, and let p be projection from its product with X to that stack. The inverse cohomology determinant $\Theta = (\det Rp_*\mathcal{F})^{-1}$ descends to N_k.

Theorem 1.2. *For $k \ge 1$, put $\nu_k = 1 + \binom{\lceil k/2 \rceil}{2}$. For every integer $r > 0$ and every integer n, on N_k one has*

$$\Theta^r \otimes P^n \text{ nef} \iff n \ge r\nu_k, \qquad \Theta^r \otimes P^n \text{ ample} \iff n > r\nu_k.$$

The proof includes reducible and nonreduced supports. The stability inequality bounds each wall by $\lceil k/2 \rceil - 1/2$. Two families, according to the parity of k, give curves on which $\deg P = 1$ and $\deg \Theta = -\nu_k$, proving sharpness. Section 5 also gives a sufficient discriminant bound for determinant positivity on any smooth complex projective surface, followed by the sharper degree-one calculation.

Recent work on K3 surfaces illustrates a further role for positivity. Zhang–Zhao [28] give counterexamples to the Marian–Oprea zero-or-isomorphism formulation and propose a refinement requiring both theta bundles to be big and nef; they also discuss ampleness as a sufficient positivity hypothesis for that refinement. Their examples concern K3 moduli spaces. In the present paper, Theorem 1.1 proves a canonical duality on the plane, while Theorem 1.2 computes the exact positivity range for the displayed determinant bundles in degree one.

1.4 Conventions

All schemes and stacks are over $\mathbb{C}$. For a sheaf F of positive rank on the plane, $p_F(t) = \chi(F(t))/\operatorname{rk}(F)$; polynomial inequalities mean inequalities for all sufficiently large real t. For a Harder–Narasimhan filtration, $p_{\min}$ and $p_{\max}$ denote the smallest and largest reduced polynomials of its positive-rank factors. Torsion factors are placed first. The symbol ch_2 denotes the degree of the second Chern character in numerical calculations on a surface. An extension closure $\langle \cdot \rangle_{\mathrm{ext}}$ is the smallest full subcategory containing the specified objects and closed under extensions.

For an algebraic group acting on an affine scheme, character spaces use covariant functions $f(gx) = \chi_\theta(g) f(x)$. Every relation or determinantal ideal is generated by the displayed entries or minors. Derived zero loci are represented by their entire Koszul complexes. We state the forms of the determinant spanning, Borel–Weil–Bott, moduli-stack and derived-restriction theorems used below, together with their hypotheses in each application.

2 The universal determinant

This section proves the algebraic comparison used in Theorem 3.1. It is useful to keep its hypotheses slightly more general. Let

$$n, k, a, b, c \in \mathbb{Z}_{>0}, \qquad an = bn + ck. \tag{2.1}$$

Choose vector spaces A, B, C of dimensions a, b, c, and a three-dimensional space V with basis x_0, x_1, x_2. Put

$$\mathcal{P} = \operatorname{Hom}(A, B) \otimes V \ \oplus \ \operatorname{Hom}(A, C) \otimes \operatorname{Sym}^2 V.$$

Write a point as (M, Q), with coefficients M_i and Q_{ij} for $i \le j$. The group

$$G = (\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)) \ltimes U, \qquad U = \operatorname{Hom}(B, C) \otimes V,$$

acts by changes of bases and by $(M, Q) \mapsto (M, Q + uM)$. Throughout this section a character space uses the covariance convention $f(gz) = \chi(g) f(z)$. Thus the ordinary representation on functions is the inverse character. Define

$$\mathscr{G} = \left\{ f \in \mathbb{C}[\mathcal{P}]^U : f(g \cdot (M, Q)) = (\det g_A)^{-n} (\det g_B)^{n} (\det g_C)^{k} f(M, Q) \right\}. \tag{2.2}$$

In this display the equation for g refers to the reductive factor. The central scalars give degrees bn and ck in M and Q, respectively, so this vector space is finite-dimensional.

Let E_0, E_1, E_2 have dimensions n, n, k. Introduce maps $X_j : E_0 \to E_1$ and $Z_{ij} = Z_{ji} : E_0 \to E_2$. The structured matrix is

$$\Gamma(X, Z) = \begin{pmatrix} X_0 & X_1 & X_2 \\ Z_{00} & Z_{01} & Z_{02} \\ Z_{01} & Z_{11} & Z_{12} \\ Z_{02} & Z_{12} & Z_{22} \end{pmatrix} : E_0^{\oplus 3} \longrightarrow E_1 \oplus E_2^{\oplus 3}.$$

The two rings needed here are

$$\begin{aligned} R_\Gamma &= \mathbb{C}[X_j, Z_{ij} = Z_{ji}]/I_{n+1}(\Gamma), \\ \mathcal{B} &= \mathbb{C}[X_0, X_1, X_2, Y_0, Y_1, Y_2]/(Y_i X_j - Y_j X_i \mid i < j), \end{aligned}$$

where $Y_i : E_1 \to E_2$. Matrix ideals mean the ideals generated by all the specified entries or minors. Both rings carry the natural action of $H = \mathrm{GL}(E_0) \times \mathrm{GL}(E_1) \times \mathrm{GL}(E_2)$. Set $\theta = (-a, b, c)$ and write R_θ for the corresponding covariance space. Equation (2.1) is exactly the condition that the common scalar subgroup acts trivially.

There is a ring map

$$\varphi : R_\Gamma \longrightarrow \mathcal{B}, \qquad X_j \longmapsto X_j, \quad Z_{ij} \longmapsto Y_i X_j. \tag{2.3}$$

It is well-defined: the relations make the lower blocks symmetric, and the resulting matrix factors through E_1. The universal evaluation matrix is

$$\mathcal{D}(M, Q; X, Z) = \begin{pmatrix} \sum_j M_j \otimes X_j \\ \sum_{i \le j} Q_{ij} \otimes Z_{ij} \end{pmatrix} : A \otimes E_0 \longrightarrow (B \otimes E_1) \oplus (C \otimes E_2). \tag{2.4}$$

It is square by (2.1). The coefficients of the quadratic monomials are chosen so that the second row is their ordinary evaluation; equivalently one can use the natural contraction of $\mathrm{Sym}^2 V$ and its dual.

Proposition 2.1. *For the positive integers in* (2.1)*, the polynomial* $\det \mathcal{D}$ *induces a perfect pairing, and the substitution* (2.3) *induces an isomorphism:*

$$\mathscr{G} \simeq (R_\Gamma)_\theta^\vee, \qquad \varphi_\theta : (R_\Gamma)_\theta \xrightarrow{\sim} \mathcal{B}_\theta.$$

The first isomorphism is the coefficient map of the displayed determinant. The character spaces are taken in the displayed quotient rings.

2.1 A free determinant pairing

We first replace $\mathrm{Sym}^2 V$ by an arbitrary finite-dimensional arrow-label space W and omit U. On the other side use the dual label spaces V^*, W^*. Thus the two affine spaces are

$$\begin{aligned} P_{\text{free}} &= (\mathrm{Hom}(A, B) \otimes V) \oplus (\mathrm{Hom}(A, C) \otimes W), \\ T_{\text{free}} &= (\mathrm{Hom}(E_0, E_1) \otimes V^*) \oplus (\mathrm{Hom}(E_0, E_2) \otimes W^*). \end{aligned}$$

Contraction of the labels gives the square matrix

$$\begin{pmatrix} \sum_j M_j \otimes X_j \\ \sum_\alpha Q_\alpha \otimes Z_\alpha \end{pmatrix}.$$

Its determinant has characters $(-n, n, k)$ and $(-a, b, c)$ on the two sets of vertices. Let F and S be the corresponding finite-dimensional spaces of semi-invariant functions.

Lemma 2.2. *This determinant belongs to $F \otimes S$ and its two coefficient maps $F^{\vee} \to S$ and $S^{\vee} \to F$ are isomorphisms. The pairing is natural under linear maps of the arrow-label spaces.*

Proof. The free star quiver has arrows from vertex 0 to vertices 1 and 2. The determinantal spanning theorem of Schofield–Van den Bergh [20, Theorem 2.3] says that all its semi-invariants are spanned by determinants of matrices of paths. We spell out why precisely the displayed matrix suffices for the present weight.

Let such a path matrix have p_i source copies and q_i target copies at vertex i. Its covariance on the first affine space is $\prod_i \det(g_i)^{q_i - p_i}$. The required character gives

$$p_0 = q_0 + n, \qquad q_1 = p_1 + n, \qquad q_2 = p_2 + k.$$

The diagonal blocks consist of scalar multiples of the identity path. On the dense set on which each of these scalar matrices has maximal rank, scalar changes of rows and columns cancel q_0, p_1, p_2 matching copies. What remains has n source copies at vertex 0, n target copies at vertex 1 and k target copies at vertex 2. The only remaining paths are arrows: this quiver has no paths of length two. Cancellation therefore leaves exactly the displayed type of evaluation matrix, up to a nonzero scalar factor. In particular its determinant lies in the span of evaluations of the universal tensor in the second variables.

This reasoning also covers singular choices of the scalar blocks. For fixed copy numbers the determinant is polynomial in all path-matrix coefficients. The inverse image of the finite-dimensional linear span in question is a closed subset of the coefficient space and contains the dense set just considered. It is therefore the whole coefficient space. We conclude that the map $S^{\vee} \to F$ is surjective. Apply the same argument to the free star of dimensions (n, n, k) and character $(-a, b, c)$. Cancellation now leaves a, b, c copies, so $F^{\vee} \to S$ is also surjective. These two maps are the two coefficient maps of one tensor and have the same rank. They are thus both isomorphisms.

Finally, evaluation is contraction of dual label spaces. Applying a linear map to a label space and its transpose to the other side preserves the displayed matrix. This proves naturality, including for a quotient of one label space and the corresponding inclusion on the other side. □

2.2 The unipotent radical and the rank ideal

Temporarily let V_1, V_2 be independent three-dimensional spaces and use nine independent quadratic labels. Put

$$\widetilde{\mathcal{P}} = (\operatorname{Hom}(A, B) \otimes V_2) \oplus (\operatorname{Hom}(A, C) \otimes V_1 \otimes V_2).$$

Write $D = C \otimes V_1$ and $W_0 = A \otimes V_2^*$. Then

$$\widetilde{\mathcal{P}} = \operatorname{Hom}(W_0, B \oplus D), \qquad U = \operatorname{Hom}(B, D),$$

where U acts by the lower block shear and is the full unipotent radical of the indicated two-block parabolic. Let $\widetilde{\mathscr{G}}$ be its invariants of the original vertex character.

For a partition λ, write $|\lambda|$ for its size, $\ell(\lambda)$ for its length, λ' for its conjugate, and (p^q) for the rectangle with q parts equal to p. The notation $\mathbf{S}_\lambda E$ denotes the Schur functor of E. We use the Schur and highest-weight rules over $\mathbb{C}$ [12]. Cauchy decomposition gives

$$\mathbb{C}[\widetilde{\mathcal{P}}] = \bigoplus_{\lambda} \mathbf{S}_\lambda(W_0) \otimes \mathbf{S}_\lambda(B \oplus D)^*.$$

The U-invariant part of the second factor is

$$\mathbf{S}_{(\lambda_1,\dots,\lambda_b)}B^* \otimes \mathbf{S}_{(\lambda_{b+1},\dots)}D^*.$$

This branching assertion follows from the unique highest-weight line. The maximal unipotents of the two Levi blocks together with U generate the appropriate maximal unipotent of the full group. Taking its invariants in the irreducible representation gives one line. Since the Levi is reductive, its representation on the U-invariants is completely reducible; each irreducible summand would contribute a highest-weight line. There is exactly one summand, and its two highest weights are the restrictions of the full highest weight. This is the displayed expression, with the dual convention fixed by the action on functions.

Extracting $\det(B^*)^n$ forces

$$\lambda = (n^b, \tau), \qquad \tau_1 \le n.$$

Here $b \ge 1$ is used. Extraction of $\det(C^*)^k$ gives $|\tau| = ck$; extraction of $\det(A)^n$ is compatible because $|\lambda| = an$. For partitions α, β, γ of the same integer m, set

$$g(\alpha,\beta,\gamma) = \dim \operatorname{Hom}_{\mathfrak{S}_m}([\gamma], [\alpha]\otimes[\beta]),$$

where $[\lambda]$ is the irreducible symmetric-group representation indexed by λ. With fixed one-dimensional vertex determinant factors suppressed, we have

$$\begin{aligned}\widetilde{\mathscr{G}} \simeq \bigoplus_{\substack{|\tau|=ck\\ \tau_1\le n}} \left(\bigoplus_{\omega} g(\tau,(k^c),\omega)\,\mathbf{S}_\omega V_1^*\right) \\ \otimes \left(\bigoplus_{\nu} g((n^b,\tau),(n^a),\nu)\,\mathbf{S}_\nu V_2^*\right).\end{aligned} \tag{2.5}$$

As usual, a Schur functor of a vector space of insufficient dimension is zero.

On the auxiliary side allow all nine Z_{ij} to be independent. The matrix Γ is then a generic $(n+3k)\times 3n$ matrix. Let $\widetilde{S}$ be its polynomial coordinate ring, $I_{\mathrm{rk}} = I_{n+1}(\Gamma)$, and $\widetilde{R} = \widetilde{S}/I_{\mathrm{rk}}$. The determinantal straightening theorem gives its Cauchy decomposition with partitions of length at most n; in particular this generic determinantal ring is an integral domain [23, Chapter 6]. Write $c^{\lambda}_{\alpha,\beta}$ for the Littlewood–Richardson coefficient, the multiplicity of $\mathbf{S}_\lambda$ in $\mathbf{S}_\alpha \otimes \mathbf{S}_\beta$. In restriction to the upper n-dimensional block, extraction of the rectangle (b^n) gives

$$c^{\lambda}_{(b^n),\mu} \neq 0 \quad\Longleftrightarrow\quad \lambda = (b^n) + \mu, \qquad c^{(b^n)+\mu}_{(b^n),\mu} = 1.$$

Indeed, for partitions of length at most n the corresponding tensor product on an n-dimensional space is simply $\det{}^b \otimes \mathbf{S}_\mu$. Thus the character of the *dual* of the required auxiliary space is

$$\begin{aligned}(\widetilde{R}_\theta)^\vee \simeq \bigoplus_{\substack{|\mu|=ck\\ \ell(\mu)\le n}} \left(\bigoplus_{\omega} g(\mu,(c^k),\omega)\,\mathbf{S}_\omega V_1^*\right) \\ \otimes \left(\bigoplus_{\nu} g((b^n)+\mu,(a^n),\nu)\,\mathbf{S}_\nu V_2^*\right).\end{aligned} \tag{2.6}$$

The dual reflects the duality of the auxiliary and original arrow labels. Set $\mu = \tau'$. Then

$$(n^b,\tau)' = (b^n) + \tau', \qquad (k^c)' = (c^k), \qquad (n^a)' = (a^n).$$

The symmetric-group identity

$$g(\alpha', \beta', \gamma) = g(\alpha, \beta, \gamma)$$

follows by tensoring both of the first two factors by the sign representation. Consequently (2.5) and (2.6) agree.

The determinant gives the following coefficient map. On the nonempty dense open of the generic rank variety where $X = [X_0\ X_1\ X_2]$ has rank n, there are unique maps $Y_i : E_1 \to E_2$ such that $Z_{ij} = Y_i X_j$. Under a shear $Q_{ij} \mapsto Q_{ij} + u_i M_j$, the lower row of the universal evaluation matrix changes by

$$\sum_{i,j} (u_i M_j) \otimes (Y_i X_j) = \left(\sum_i u_i \otimes Y_i \right) \left(\sum_j M_j \otimes X_j \right).$$

This is an elementary row operation, so its determinant is unchanged. The generic rank variety is reduced and irreducible. The same polynomial identity therefore holds on all of it.

By Lemma 2.2, coefficients of the universal determinant span $\widetilde{S}_\theta$. Taking a character for the reductive group H is exact, so their images span $\widetilde{R}_\theta$ as well. The preceding row-operation identity puts the other coefficient space in $\widetilde{\mathscr{G}}$. We obtain an injection

$$(\widetilde{R}_\theta)^\vee \longrightarrow \widetilde{\mathscr{G}}.$$

The character computation makes the two dimensions equal. This injection, which is the coefficient map of the actual determinant, is an isomorphism.

2.3 The symmetric label quotient

Identify V_1 and V_2 with V. Multiplication gives the quotient $V \otimes V \to \operatorname{Sym}^2 V$. On the original affine spaces it induces $\widetilde{\mathcal{P}} \to \mathcal{P}$, compatible with the shear action. Thus functions pulled back from $\mathcal{P}$ are the functions independent of the antisymmetric directions. On the auxiliary side the transpose is the inclusion of symmetric labels, whose ideal is

$$I_{\mathrm{sk}} = (Z_{ij} - Z_{ji} \mid i < j) \subset \widetilde{S}.$$

Under the perfect free pairing of Lemma 2.2, the original symmetric function subspace is the annihilator of $(I_{\mathrm{sk}})_\theta$. Apply that lemma also with label space $\operatorname{Sym}^2 V$; naturality identifies its coefficient map with the inclusion of the symmetric function subspace and the transpose of the auxiliary restriction map. The previous subsection identifies the full shear-invariant subspace with the annihilator of $(I_{\mathrm{rk}})_\theta$. The required space $\mathscr{G}$ is the intersection of these two subspaces.

The dual of an intersection of annihilators in a finite-dimensional perfect pairing is the quotient by the sum of the two annihilated subspaces. Reductivity of H now gives

$$\mathscr{G}^\vee \simeq \frac{\widetilde{S}_\theta}{(I_{\mathrm{sk}})_\theta + (I_{\mathrm{rk}})_\theta} \simeq \left(\frac{\widetilde{S}}{I_{\mathrm{sk}} + I_{\mathrm{rk}}} \right)_\theta = (R_\Gamma)_\theta.$$

This proves the first assertion of Proposition 2.1.

2.4 A determinant calculation on the Stiefel open

We prove a cohomological lemma before applying it to the second comparison. Let Q_0 and E_1 have dimension n, let F have dimension m, and set $E = E_1 \oplus F$. Write a map $A : Q_0 \to E$ as (A_0, A'), and put

$$U_0 = \operatorname{Inj}(Q_0, E), \qquad P_0 = \operatorname{Hom}(Q_0, F), \qquad p : U_0 \to P_0.$$

The group $G_1 = \mathrm{GL}(E_1)$ acts on A_0 by left multiplication and acts trivially on P_0. Denote by C_b the exact functor extracting covariance $\det(g_1)^b$. In ordinary function-representation notation it extracts $\det(E_1^*)^b$.

Lemma 2.3. *For $b \in \mathbb{Z}_{\geq 0}$ and any bounded coherent complex N on P_0 with trivial G_1-action, the multiplication and pullback map is a quasi-isomorphism*

$$\det(A_0)^b N \xrightarrow{\sim} C_b \,\mathrm{R}\Gamma(U_0, p^* N).$$

It is compatible with additional group actions on Q_0, F, N and with polynomial spectator variables.

Proof. The map $U_0 \to \mathrm{Gr}(n, E)$ is the frame bundle of the tautological subbundle S. It is affine. For a vector bundle E' of rank ℓ and an integral dominant weight λ of length ℓ, rational Schur notation means

$$\mathbf{S}_\lambda E' = (\det E')^{-t} \otimes \mathbf{S}_{\lambda + (t^\ell)} E'$$

for any integer t making $\lambda + (t^\ell)$ a partition; determinant twisting makes the definition independent of t. Algebraic Peter–Weyl decomposes its direct image into rational Schur bundles $\mathbf{S}_\lambda S^*$, with finite-dimensional Q_0 multiplicity spaces. Here $\lambda_1 \geq \cdots \geq \lambda_n$ ranges over all integer dominant weights. This follows also from Cauchy decomposition of square-matrix functions followed by inversion of the determinant.

Let $N_E = n + m$. The Borel–Weil–Bott rule for these conventions sorts

$$(\lambda_1 + N_E - 1, \ldots, \lambda_n + m, m - 1, \ldots, 0), \tag{2.7}$$

then subtracts $(N_E - 1, \ldots, 0)$ [7, Proposition A.1]. Repeated entries mean that all cohomology vanishes. Otherwise the output is an irreducible representation of E^*, and the number of inversions is its cohomological degree.

Suppose $m > 0$ and the degree is positive. The last m entries in (2.7) occupy all integers $0, \ldots, m-1$. Each of its first n entries must consequently be at least m or at most -1. Let $q_0 > 0$ be the number of negative entries among them. In decreasing order the sequence consists of $n - q_0$ large entries, the m fixed tail entries, and the q_0 negative entries. If Λ denotes the resulting highest weight, its nth coordinate satisfies

$$\Lambda_n = -q_0 \quad (q_0 \leq m), \qquad \Lambda_n \leq -m - 1 \quad (q_0 > m). \tag{2.8}$$

Indeed, in the first case the nth sorted entry is $m - q_0$, from which one subtracts m; in the second it is negative.

The branching rule from GL_d to its standard GL_{d-1} block is $\Lambda_i \geq \mu_i \geq \Lambda_{i+1}$ [19, Theorem 2.1]. Iterating it down to $\mathrm{GL}(E_1^*)$ shows that every occurring highest weight μ satisfies $\mu_n \leq \Lambda_n$. The rule applies to rational integer weights by first twisting by a large determinant power and then untwisting. By (2.8), the nonnegative rectangle (b^n) cannot occur in positive cohomology. Cohomology on the projective Grassmannian commutes with direct sums of quasi-coherent sheaves, so the Peter–Weyl decomposition gives

$$C_b H^j(U_0, \mathcal{O}_{U_0}) = 0 \qquad (j > 0,\ b \geq 0).$$

For $m > 0$, the complement of U_0 in the matrix affine space has codimension $m + 1 \geq 2$. Hartogs extension on that normal affine space gives $H^0(U_0, \mathcal{O}) = \mathbb{C}[A_0, A']$. The covariance space of exponent $b \geq 0$ on the square block A_0 is the line generated by its bth determinant:

on invertible matrices this follows by left translation, and the resulting polynomial identity extends to every matrix. Hence

$$C_b H^0(U_0, \mathcal{O}) = \det(A_0)^b \mathbb{C}[P_0].$$

When $m = 0$, the open is a general linear group and is affine. Left translation gives the same assertion directly.

The map p is flat, being the restriction to an open of an affine-space projection. The affine base has a regular polynomial coordinate ring, so N has a bounded resolution by finite projective modules. Each such module is a direct summand of a finite free module. The preceding two cohomology assertions therefore apply term by term to this resolution, and multiplication by $\det(A_0)^b$ commutes with all differentials over the base. Exactness of C_b proves the stated quasi-isomorphism. This is an assertion about the displayed natural map. If further groups act, that map is equivariant, and its being a quasi-isomorphism can be checked after forgetting the extra action. The same proof includes spectators. □

2.5 Elimination with the whole Koszul complex

Let $W_0 = E_0^{\oplus 3}$ and $E = E_1 \oplus E_2^{\oplus 3}$. For a generic matrix $T : W_0 \to E$, put

$$D_n = \operatorname{Spec}\bigl(\mathbb{C}[T]/I_{n+1}(T)\bigr).$$

Its incidence resolution is

$$Y_n = \operatorname{Tot}\bigl(\operatorname{Hom}(W_0, S)\bigr) \longrightarrow D_n, \qquad S \subset E \otimes \mathcal{O}_{\mathrm{Gr}(n,E)}.$$

The coordinate pullback gives

$$\mathbb{C}[D_n] \xrightarrow{\sim} \mathrm{R}\Gamma(Y_n, \mathcal{O}_{Y_n}). \tag{2.9}$$

Here the right side is concentrated in degree zero. To recall the calculation, the projection to the Grassmannian has direct image

$$\operatorname{Sym}(W_0 \otimes S^*) = \bigoplus_{\ell(\lambda) \le n} \mathbf{S}_\lambda W_0 \otimes \mathbf{S}_\lambda S^*.$$

Nonnegative Borel–Weil–Bott gives $\mathbf{S}_\lambda E^*$ in degree zero and no higher cohomology. This is exactly the Cauchy decomposition of the generic determinantal ring. Coordinate pullback is nonzero on each summand, as is seen on a product of minors of the required highest weight; the determinantal straightening theorem identifies its kernel with $I_{n+1}(T)$. This proves (2.9) for coordinate pullback.

The resolution can also be written

$$Y_n = [\, U_0 \times \operatorname{Hom}(W_0, Q_0) \,/\, \mathrm{GL}(Q_0) \,], \qquad T = AB,$$

where $U_0 = \operatorname{Inj}(Q_0, E)$ and $\dim Q_0 = n$. This action is free, so the displayed quotient is the incidence scheme itself. Split $A = (A_0, A_0', A_1', A_2')$ according to the four target blocks and write $B = (B_0, B_1, B_2)$ according to the source blocks.

Let K_D be the Koszul complex of the linear equations identifying the three pairs of off-diagonal lower blocks of T. We put it in nonpositive cohomological degrees. It is a finite complex of free $\mathbb{C}[D_n]$-modules and

$$H^0(K_D) = R_\Gamma.$$

We retain all degrees of this Koszul complex. Pulling it back and using (2.9) term by term gives a quasi-isomorphism

$$K_D \xrightarrow{\sim} \mathrm{R}\Gamma(Y_n, K_D|_{Y_n}).$$

On the frame space its equations are exactly

$$A_i' B_j - A_j' B_i = 0 \qquad (i < j).$$

In particular they are independent of the square block A_0. Write K_{rel} for their entire Koszul complex on the affine space of (A', B). Lemma 2.3, with B as spectator variables, gives

$$C_b \, \mathrm{R}\Gamma(U_0 \times \mathrm{Hom}(W_0, Q_0), K_{\mathrm{rel}}) \simeq \det(A_0)^b K_{\mathrm{rel}} \qquad (b \geq 0).$$

All the subsequent character and invariant functors are for general linear groups in characteristic zero and are exact. They commute with taking H^0 of these complexes. Taking degree-zero cohomology of the derived comparison gives the two displayed quotient rings.

We check the characters and the map. The actions are

$$A_0 \mapsto g_1 A_0 h^{-1}, \qquad A_i' \mapsto g_2 A_i' h^{-1}, \qquad B_j \mapsto h B_j g_0^{-1}, \quad h \in \mathrm{GL}(Q_0).$$

After extracting the G_1-covariance, a term is $\det(A_0)^b f(A', B)$. The first factor has internal covariance $\det(h)^{-b}$. To take $\mathrm{GL}(Q_0)$-invariants, the second factor must therefore have covariance $\det(h)^b$. The remaining external characters are $\det(g_0)^{-a}$ and $\det(g_2)^c$. We obtain

$$(R_\Gamma)_{(-a,b,c)} \simeq \Big(\mathbb{C}[A', B]/(A_i' B_j - A_j' B_i)\Big)_{(-a,b,c)}.$$

On the open where A_0 is invertible, change the internal frame to set $A_0 = I$. Then $X_j = B_j$ and $Z_{ij} = A_i' B_j$. Thus this is the actual map (2.3). More formally, the natural multiplication description in Lemma 2.3 identifies its inverse on the entire complex, and evaluation at $A_0 = I$ gives this substitution. Since the coefficient module is polynomial in the independent square block A_0, multiplication by its determinant is injective on that module. The second assertion of Proposition 2.1 is proved.

2.6 Nonvanishing in the effective chamber

Proposition 2.4. *Assume the positive integer balance* (2.1)*, and suppose*

$$0 < k < n, \qquad a \geq (a - b)(n - 1).$$

Then $\mathcal{B}_\theta$ and the determinant pairing of Proposition 2.1 are nonzero.

Proof. Put $\tau = a - b = ck/n$. This is a positive integer and $n\tau = kc$. On the open where X_0 is invertible set

$$P = X_1 X_0^{-1}, \qquad Y = Y_0, \qquad \mathcal{K}(P, Y) = \begin{pmatrix} Y \\ YP \\ \vdots \\ YP^{n-1} \end{pmatrix}.$$

Let L be the finite-dimensional span of products of τ maximal $n \times n$ minors of $\mathcal{K}$. Under

$$(P, Y) \longmapsto (gPg^{-1}, hYg^{-1}), \quad g \in \mathrm{GL}_n, \ h \in \mathrm{GL}_k,$$

every maximal minor has g-covariance $\det(g)^{-1}$, and its span is GL_k-stable. Hence all of L has g-covariance $\det(g)^{-\tau}$.

Fix a diagonal P_0 with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. For each i let

$$f_i(z) = \prod_{j \neq i} \frac{z - \lambda_j}{\lambda_i - \lambda_j}.$$

Its degree is $n-1$ and $f_i(P_0)$ projects onto the ith coordinate. For arbitrary covectors $h_i \in (\mathbb{C}^k)^*$, multilinearity of the determinant gives

$$\det \begin{pmatrix} h_1 Y f_1(P_0) \\ \vdots \\ h_n Y f_n(P_0) \end{pmatrix} = \prod_{i=1}^{n} h_i(Y_{\bullet i}).$$

The determinant on the left is a linear combination of maximal Krylov minors. Conversely each such minor has degree one in each column of Y at this diagonal specialization. Products show that evaluation at P_0 is a surjective GL_k-equivariant map from L onto the space of polynomials of multidegree $(\tau, \dots, \tau)$ in those columns.

There is an explicit nonzero polynomial of the desired GL_k-covariance in that image. Number the columns modulo n, and put

$$J_j = (jk, jk+1, \dots, jk+k-1) \pmod{n} \quad (0 \leq j < c), \qquad H(Y) = \prod_{j=0}^{c-1} \det Y_{J_j}.$$

Here Y_{J_j} is the submatrix with the indicated ordered columns. Their order fixes the sign of each determinant. As $k < n$, its entries are distinct. The complete list of entries is $0, 1, \dots, ck-1$ modulo n. Since $ck = n\tau$, each column appears exactly τ times. The polynomial H is a product of nonzero maximal minor polynomials, hence is nonzero, and $H(hY) = \det(h)^c H(Y)$. Exactness of reductive character extraction lifts H to $f(P, Y) \in L$ satisfying

$$f(gPg^{-1}, hYg^{-1}) = \det(g)^{-\tau} \det(h)^c f(P, Y), \qquad f(P_0, Y) = H(Y).$$

We must bound the denominator after replacing P by $X_1 X_0^{-1}$. Let $\delta = \det X_0$, and localize the ambient polynomial ring at its prime divisor (δ). Complete this discrete valuation ring, calling it R. The matrix

$$T = \delta P = X_1 \operatorname{adj}(X_0)$$

is integral over R. Modulo δ it has rank one and nonzero trace at the generic point. Nonvanishing of the trace can be checked, for example, at $X_0 = \operatorname{diag}(0, 1, \dots, 1)$ and $X_1 = I$. Thus its characteristic polynomial modulo δ is $z^{n-1}(z-u)$ with $u \neq 0$.

Hensel factorization lifts these two coprime factors to monic coprime factors over R. Bezout polynomials in T give complementary idempotents, and hence an invariant decomposition $R^n = L_1 \oplus L_0$ of ranks 1 and $n-1$. On L_0, the reduction of T is zero, so $P = T/\delta$ is integral. On L_1 it has a pole of order one. In an integral invertible basis adapted to this decomposition, every row YP^j is integral in the last $n-1$ columns, and its first entry has pole order at most j. An $n \times n$ determinant uses the first column exactly once. Its expansion along that column therefore shows that every maximal minor of $\mathcal{K}$ has pole order at most $n-1$. An integral invertible change of basis changes such a determinant only by a unit. Products of τ minors have pole order at most $\tau(n-1)$, as does their linear combination f.

It follows that

$$F(X, Y) = (\det X_0)^a f(X_1 X_0^{-1}, Y_0)$$

is an ambient polynomial because $a \geq \tau(n-1)$. Its only possible denominator was a power of the irreducible polynomial δ, and its valuation along that divisor is now nonnegative. Under the three vertex groups, its covariance is

$$\det(g_0)^{-a} \det(g_1)^{a-\tau} \det(g_2)^{c} = \det(g_0)^{-a} \det(g_1)^{b} \det(g_2)^{c}.$$

It remains nonzero in the raw quotient $\mathcal{B}$. To check this, take

$$X_0 = I, \quad X_1 = P_0, \quad X_2 = 0, \qquad Y_0 = Y, \quad Y_1 = YP_0, \quad Y_2 = 0.$$

All three matrix relations hold. Choose Y with $H(Y) \neq 0$; then $F = H(Y) \neq 0$. We have constructed a nonzero element of $\mathcal{B}_\theta$. Proposition 2.1 identifies its dual with $\mathscr{G}$ by the universal tensor, so that tensor is nonzero as well. □

3 The symmetric Beilinson presentation

Theorem 3.1 (Symmetric presentation)**.** *Let n, s, r, d be integers satisfying*

$$n > s \geq 1, \qquad r \geq \max\{2, s\}, \qquad \frac{rn}{s} \in \mathbb{Z}, \qquad d > \frac{r}{s}(n-s)(n+1). \tag{3.1}$$

For $\rho = (s, 0, s-n)$ and $\sigma = (r, dH, rn/s)$, the determinant lines descend to M_ρ and M_σ, and the canonical map (1.3) *is an isomorphism between nonzero spaces.*

We retain the integers of Theorem 3.1 and put

$$k = n - s, \qquad q = \frac{r}{s}, \qquad c = qn, \qquad \tau = qk = c - r, \qquad b = d - 2\tau, \qquad a = b + \tau.$$

Thus $q \geq 1$, $k \geq 1$, $\tau \in \mathbf{Z}$, and the strict bound in the theorem is equivalent to

$$b > qk(n-1). \tag{3.2}$$

In particular a, b, c are positive integers and $an = bn + ck$. Write $\alpha = (n, n, k)$ and $\theta = (-a, b, c)$. Throughout this section, semistability of a representation means that every subrepresentation of dimension β satisfies $\theta(\beta) \geq 0$. Its character line has fiber character $\prod_i \det(g_i)^{\theta_i}$; its sections on an affine representation scheme are functions satisfying $f(gx) = \prod_i \det(g_i)^{\theta_i} f(x)$.

3.1 A numerical inequality for every subdimension

The Euler form of the Beilinson algebra in the order of vertices used here is

$$\chi_Q(u, v) = \sum_{i=0}^{2} u_i v_i - 3u_0 v_1 - 3u_1 v_2 + 3u_0 v_2. \tag{3.3}$$

The last term comes from the three commutativity relations from vertex 0 to vertex 2.

Lemma 3.2. *Suppose* (3.2) *holds. For every integral vector $\beta = (x, y, z)$ with $0 \leq x, y \leq n$, $0 \leq z \leq k$ and $\beta \neq 0, \alpha$, one has*

$$\theta(\beta) < 0 \quad \Longrightarrow \quad \theta(\beta) + \chi_Q(\beta, \alpha - \beta) < 0. \tag{3.4}$$

Moreover, the signs of $\theta(\beta)$ agree, including zero, with those of the polynomial

$$(-st + k)x + (st - 2k)y + nz$$

for sufficiently large real t.

Proof. The character and the Euler expression are

$$\theta(\beta) = b(y-x) - qkx + qnz, \tag{3.5}$$

$$\chi_Q(\beta, \alpha-\beta) = x(n-x) + y(n-y) + z(k-z) - 3x(n-y) - 3y(k-z) + 3x(k-z). \tag{3.6}$$

If $y > x$, then $x \le n-1$, and $\theta(\beta) \ge b - qk(n-1) > 0$. This case cannot occur in the premise of (3.4).

If $x = y$ and $\theta(\beta) < 0$, put $D = kx - nz > 0$. Expanding (3.6) gives

$$\chi_Q(\beta, \alpha-\beta) - D = -(x-z)(n+k-x-z). \tag{3.7}$$

Since $n > k$ and $kx > nz$, one has $x > z$. The second factor is positive: it could vanish in the indicated box only at $x = n, z = k$, where $D = 0$. As $q \ge 1$, we obtain $\theta(\beta) + \chi_Q = -qD + \chi_Q \le -D + \chi_Q < 0$.

It remains to consider $x > y$. Put

$$h = x - y \ge 1, \qquad \varepsilon = b - qk(n-1) > 0, \qquad D = k(n-1)h + kx - nz.$$

The inequalities $x \ge h$ and $z \le k$ give

$$D \ge k(n-1)h + kh - nk = kn(h-1) \ge 0, \qquad \theta(\beta) = -\varepsilon h - qD < 0. \tag{3.8}$$

On substituting $y = x - h$, set $F(x, h, z) = \chi_Q(\beta, \alpha-\beta) - D$. Then

$$F(x,h,z) = x^2 - (h+k+n)x - h^2 - hkn + 4hk - hn - 3hz + (k+n)z - z^2. \tag{3.9}$$

This is convex as a function of $x \in [h, n]$. Direct subtraction gives

$$F(n,h,z) - F(h,h,z) = k(h-n) \le 0, \tag{3.10}$$

$$F(h,h,z) - F(1,1,z) = -(h-1)\{h + k(n-3) + 2n + 3z + 1\} \le 0. \tag{3.11}$$

The brace in the second line is positive. For $n \ge 3$ this is immediate; for $n = 2$, necessarily $k = 1$, and it is again positive. Finally, for $t = k - z \in \mathbf{Z}_{\ge 0}$ and $s = n - k \ge 1$,

$$F(1,1,z) = -2n - 1 + t(3 - s - t) \le -2n. \tag{3.12}$$

Indeed $t(3-s-t) \le t(2-t) \le 1$ for an integer $t \ge 0$. Equations (3.10)–(3.12) show that $F \le -2n$. Therefore

$$\theta(\beta) + \chi_Q(\beta, \alpha-\beta) = -\varepsilon h - (q-1)D + F < 0.$$

These same three cases prove the assertion about signs. The leading coefficient of the polynomial character is $s(y-x)$. For $x = y$ its constant term is $nz - kx$, and (3.5) is q times that constant term. □

3.2 Restriction from the derived relation space

Let V_0, V_1, V_2 have dimensions n, n, k, let $G_\alpha = \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$, and let $\mathfrak{X}$ be the quotient by G_α of the derived zero locus of

$$Y_i X_j - Y_j X_i = 0 \qquad (0 \le i < j \le 2).$$

Its classical coordinate ring is $\mathcal{B}$. We denote its character line by L_θ.

We first specify the derived structure. In the path algebra, add three arrows of degree -1 from 0 to 2, with differential the three commutativity relations. There are no arrows entering 0 or leaving 2; hence a new arrow has no nontrivial composable path on either side. The three relations are linearly independent in the nine-dimensional space of paths of length two. The resulting relative semi-free differential graded algebra has no negative cohomology, and its degree-zero cohomology is the Beilinson algebra. Applying the representation functor therefore gives exactly the finite Koszul algebra on the $3nk$ scalar relation entries. In particular this construction retains the ordinary relation ideal in degree zero.

Thus $\mathfrak{X}$ is an algebraic derived stack of finite presentation, with affine diagonal and bounded structure sheaf. Its cotangent complex is perfect of amplitude $[-1, 1]$; equivalently it is quasi-smooth. The deformation complex at a representation is

$$\bigoplus_{i=0}^{2} \operatorname{End}(V_i) \longrightarrow \operatorname{Hom}(V_0, V_1)^{\oplus 3} \oplus \operatorname{Hom}(V_1, V_2)^{\oplus 3} \longrightarrow \operatorname{Hom}(V_0, V_2)^{\oplus 3}. \tag{3.13}$$

Its terms are in cohomological degrees $0, 1, 2$. For $\xi = (\xi_0, \xi_1, \xi_2)$ and arrow variations $(\dot{X}_i, \dot{Y}_i)$, the differentials are

$$\begin{aligned} d_0(\xi)_i &= (\xi_1 X_i - X_i \xi_0,\ \xi_2 Y_i - Y_i \xi_1), \\ d_1(\dot{X}, \dot{Y})_{ij} &= \dot{Y}_i X_j + Y_i \dot{X}_j - \dot{Y}_j X_i - Y_j \dot{X}_i \quad (i < j). \end{aligned}$$

Their composite is $\xi_2(Y_i X_j - Y_j X_i) - (Y_i X_j - Y_j X_i)\xi_0 = 0$. This complex computes the Ext groups over the Beilinson algebra; its shift by 1 is the tangent complex.

Proposition 3.3. *Under* (3.2)*, restriction induces an equivalence*

$$\mathrm{R}\Gamma(\mathfrak{X}, L_\theta) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathfrak{X}^{\theta\text{-}ss}, L_\theta). \tag{3.14}$$

Proof. Use the norm

$$\|\lambda\|^2 = \sum_{v=0}^{2} \sum_{j=1}^{\alpha_v} \lambda_{v,j}^2$$

on one-parameter subgroups. By [15, Theorem 3.8 and Corollary 3.9], the Harder–Narasimhan and Hesselink stratifications for the bound quiver agree, including their quotient stacks. There are finitely many types. The classical strata have unique derived lifts [14, Lemma 1.2.3]. Removing strata in the order of the stratification, we may work with a single closed derived Θ-stratum $i : \mathfrak{S} \hookrightarrow \mathfrak{X}'$ in the open that remains. Its center will be denoted by $\mathfrak{Z}$.

Here is the local-cohomology criterion we use. For an algebraic derived stack locally almost finitely presented with affine diagonal over a locally noetherian base, [14, Proposition 2.1.3] identifies $\mathrm{RHom}(F, G)$ with its restriction off a closed Θ-stratum if i^*F has baric weights at least w and $G \in \mathrm{QC}(\mathfrak{X}')^{<w}$. Here QC denotes the derived category of quasi-coherent complexes. When the structure sheaf is bounded, the latter condition is equivalent to $i^{\mathrm{QC},!}G \in \mathrm{QC}(\mathfrak{S})^{<w}$ [14, §1.7.1]. These hypotheses hold for every open $\mathfrak{X}'$ here.

We calculate this last weight for our perfect line. Both $\mathfrak{X}'$ and $\mathfrak{S}$ are quasi-smooth [14, Lemma 3.1.4], so their dualizing complexes are shifted invertible complexes. Coherent duality for a closed immersion [14, Appendix B, pp. 88–89] gives

$$i^{\mathrm{QC},!}\mathcal{O}_{\mathfrak{X}'} \simeq \omega_{\mathfrak{S}} \otimes i^* \omega_{\mathfrak{X}'}^{-1}, \qquad i^{\mathrm{QC},!}L_\theta \simeq i^* L_\theta \otimes i^{\mathrm{QC},!}\mathcal{O}_{\mathfrak{X}'}. \tag{3.15}$$

On a closed affine chart the quasi-coherent right adjoint is derived Hom along the corresponding algebra map. On a bounded coherent input it agrees with coherent shriek. Writing $\mathcal{O}_{\mathfrak{X}'} =$

$\omega_{\mathfrak{X}'} \otimes \omega_{\mathfrak{X}'}^{-1}$ and using the projection formula for the perfect inverse line gives the first identity; the second uses that L_θ is dualizable. These identities descend from the charts. The graded fiber weights of the dualizing complexes are the determinant weights of their cotangent complexes [14, Proposition 3.1.5 and Corollary 3.1.5.1].

Let the associated graded representation have dimensions $d_1, \dots, d_\ell$ and weights $w_1 > \cdots > w_\ell$. The adapted weights are proportional to $-\theta(d_i)/|d_i|$, where $|d_i| = d_{i,0} + d_{i,1} + d_{i,2}$. Put $\beta_t = d_1 + \cdots + d_t$. The total slope is zero, and every proper HN prefix has strictly positive slope $-\theta(\beta_t)/|\beta_t|$; in particular $\theta(\beta_t) < 0$.

Here a superscript > 0 denotes the direct sum of positive-weight parts on the center. We compute the normal weight including the relations. For $i < j$, a tangent map from the ith graded factor to the jth has weight $w_j - w_i < 0$. The dual cotangent map has positive weight. The vertex, arrow and relation terms in (3.13) have signs $-, +, -$ in the cotangent determinant. Consequently

$$\eta := \operatorname{wt} \det(\mathbb{L}_{\mathfrak{X}'}^{>0}|_{\mathfrak{Z}}) = -\sum_{i<j} (w_i - w_j)\chi_Q(d_i, d_j). \tag{3.16}$$

The relative dualizing line in (3.15) has weight $-\eta$. The character line has weight $\sum_i w_i\theta(d_i)$. Bilinearity gives

$$\chi_Q(\beta_t, \alpha - \beta_t) = \sum_{i \le t < j} \chi_Q(d_i, d_j).$$

The coefficient of $\chi_Q(d_i, d_j)$ after summing over t is $\sum_{t=i}^{j-1}(w_t - w_{t+1}) = w_i - w_j$. Also

$$\sum_i w_i\theta(d_i) = \sum_{t=1}^{\ell-1}(w_t - w_{t+1})\theta(\beta_t) + w_\ell\theta(\alpha) = \sum_{t=1}^{\ell-1}(w_t - w_{t+1})\theta(\beta_t).$$

Consequently

$$\begin{aligned} \operatorname{wt}(i^{\mathrm{QC},!}L_\theta) &= \sum_i w_i\theta(d_i) + \sum_{i<j}(w_i - w_j)\chi_Q(d_i, d_j) \\ &= \sum_{t=1}^{\ell-1}(w_t - w_{t+1})\{\theta(\beta_t) + \chi_Q(\beta_t, \alpha - \beta_t)\} < 0. \end{aligned} \tag{3.17}$$

The strict inequality follows from Lemma 3.2. Since $i^{\mathrm{QC},!}L_\theta$ is a shifted line, this fiber calculation is the required strict baric bound by [14, Lemma 1.5.4].

Apply the quoted restriction theorem with $F = \mathcal{O}_{\mathfrak{X}'}$, $G = L_\theta$ and $w = 0$. The pullback of F has weight zero, whereas (3.17) gives the required negative shriek weight for G. It follows that derived global sections are unchanged on removing $\mathfrak{S}$. Induction over the finite stratification proves (3.14). □

3.3 The semistable locus and its determinant line

Proposition 3.4. *Restriction of polynomial covariants gives an isomorphism*

$$\mathcal{B}_\theta \xrightarrow{\sim} H^0(M_\rho, \Theta_\sigma). \tag{3.18}$$

Proof. We use the second Beilinson equivalence of [17, §5.2], namely

$$\Psi(F) = \mathrm{RHom}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), F)[1],$$

with vertices in the order $(\mathcal{O}(2), \mathcal{O}(1), \mathcal{O})$. For a class (R, DH, χ) its dimension vector and polynomial character are given in [17, equations (5.2)–(5.3)]. For $\rho = (s, 0, -k)$ they are

$$\alpha = (n, n, k), \qquad \theta_G(t) = (-st + k, st - 2k, n). \tag{3.19}$$

The admissible region of [17, Theorem 5.4] is $-tR < tD + \chi < R$, $D \neq -R$, with the inequalities interpreted polynomially. It contains ρ. That theorem is obtained from [17, Corollary 4.15], which identifies the moduli stacks of semistable objects before identifying their coarse moduli spaces.

Our finite character is $q\theta_G(d/r)$, and Lemma 3.2 shows that it has exactly the same signs on all subdimension vectors as the polynomial character. Thus its entire semistable stack is the stack of Gieseker-semistable sheaves of class ρ.

For any such sheaf F, slope semistability and $K_{\mathbf{P}^2} = -3H$ give

$$\operatorname{Ext}^2(F,F) = \operatorname{Hom}(F, F(-3))^\vee = 0.$$

The complex (3.13), under the derived equivalence, computes these same Ext groups. Its last cohomology is therefore zero at every semistable point. The derivative of the relation map is surjective there. The semistable derived locus is consequently the ordinary smooth semistable stack.

Every semistable F has

$$H^0(F(-2)) = H^0(F(-1)) = H^0(F) = 0, \qquad H^2(F(-2)) = H^2(F(-1)) = H^2(F) = 0. \tag{3.20}$$

For the first two H^0 groups this follows from slope semistability. At twist zero a nonzero map $\mathcal{O} \to F$ would contradict Gieseker semistability, because $\chi(F)/s = -k/s < 0 < \chi(\mathcal{O})$. For H^2, Serre duality reduces to the vanishing of maps from F to $\mathcal{O}(-1), \mathcal{O}(-2), \mathcal{O}(-3)$, respectively. Riemann–Roch gives

$$\dim H^1(F(-2)) = n, \qquad \dim H^1(F(-1)) = n, \qquad \dim H^1(F) = k.$$

Multiplication by the three linear forms gives the arrows X_i, Y_i; their compositions satisfy the commutativity relations. The character line L_θ is the determinant line Θ_σ: the Gaeta class of σ is $[\mathcal{O}(-1)^b \oplus \mathcal{O}^c] - [\mathcal{O}(-2)^a]$, and (3.20) gives the determinant exponents $(-a, b, c)$.

For descent, write a closed polystable point as $F = \bigoplus_j F_j \otimes \mathbf{C}^{m_j}$. Each stable factor has rank $s_j > 0$, first Chern class zero, and $\chi(F_j)/s_j = -k/s$. Its dimension vector is $(s_j/s)\alpha$. The factor GL_{m_j} of the stabilizer acts on L_θ by $\det^{\theta((s_j/s)\alpha)} = 1$. The line therefore descends by [2, Theorem 10.3]. Exact pushforward and the structure-sheaf identity for the good moduli morphism identify its sections on the stack with those on M_ρ.

On the affine derived quotient, reductivity of G_α makes extraction of a character summand exact. Hence $H^0\,\mathrm{R}\Gamma(\mathfrak{X}, L_\theta) = \mathcal{B}_\theta$. Now (3.18) follows from Proposition 3.3 and the classical description of its semistable endpoint. □

The space M_ρ is nonempty by a direct construction. Choose n distinct points p_i and covectors $\ell_i \in (\mathbf{C}^s)^\vee$ such that any s are independent, and take the kernel

$$0 \longrightarrow F \longrightarrow \mathcal{O}^s \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{p_i} \longrightarrow 0.$$

It has class ρ and is slope-semistable. A saturated subsheaf of $\mathcal{O}^s$ of rank h and first Chern class zero is $W \otimes \mathcal{O}$ for a constant h-plane W. Indeed its Plücker coordinates, after taking the reflexive hull, are constant sections of its trivial determinant; saturation then gives the asserted subsheaf. At most $s - h$ of the covectors vanish on W. Thus any rank-h subsheaf of F of slope zero, for $0 < h < s$, has Euler characteristic at most $h - (n - s + h) = -k$. Its reduced constant term is at most $-k/h < -k/s$. Subsheaves of negative slope cannot destabilize. This proves Gieseker stability; for $s = 1$ the construction is the usual ideal sheaf.

3.4 A boundary estimate for complete prioritary families

For a torsion-free sheaf of rank R on the plane, write

$$\mu(E) = \frac{c_1(E)H}{R}, \qquad \Delta(E) = \frac{\mu(E)^2}{2} - \frac{\mathrm{ch}_2(E)}{R}.$$

Call E line-prioritary if $\mathrm{Ext}^2(E, E(-1)) = 0$.

Lemma 3.5. *Let S be a smooth finite type parameter scheme carrying a flat family of torsion-free line-prioritary sheaves on $\mathbf{P}^2$, of fixed rank $R \geq 2$ and discriminant Δ. Suppose its classifying morphism to the stack of line-prioritary sheaves is smooth. If*

$$\Delta \geq 1 + \frac{1}{R}, \tag{3.21}$$

then the complement of its slope-stable locus has codimension at least two. In particular a nonempty such family has a nonempty stable open.

Proof. Suppose an irreducible component Z of the nonstable locus had codimension at most one. At its geometric generic point take the saturated slope-Harder–Narasimhan filtration and refine its semistable factors into slope-stable factors. Denote the resulting factors by G_i, their ranks by $R_i > 0$, their slopes by $\mu_1 \geq \cdots \geq \mu_\ell$, and their discriminants by $\Delta_i \geq 0$. There are at least two factors, including when the original sheaf is strictly slope-semistable. A same-slope refinement can be obtained by choosing a same-slope subobject of minimum positive rank, saturating it, and proceeding with the torsion-free quotient. Bogomolov's inequality gives $\Delta_i \geq 0$ for all these factors.

The flag descends to a finite extension of the generic field and spreads over a dense open of the corresponding cover. Choose a fixed line avoiding the finite singularity sets of all factors and prefixes at the generic point. On a neighborhood of that point the restrictions are bundles and the prefixes restrict as subbundles. The restriction morphism to the stack of bundles on the line is smooth by [22, Lemma 4] and the classifying hypothesis.

A bundle $V = \bigoplus_{j=1}^{R} \mathcal{O}(e_j)$ on $\mathbf{P}^1$, with $e_1 \geq \cdots \geq e_R$, has splitting stratum of codimension

$$h^1(\mathrm{End}\, V) = \sum_{i<j} \max(e_i - e_j - 1, 0). \tag{3.22}$$

Indeed the bundle stack has dimension $-R^2$, the splitting stratum has dimension $-h^0(\mathrm{End}\, V)$, and their difference is the displayed number. Codimension at most one allows only a balanced type or

$$\mathcal{O}(t+1) \oplus \mathcal{O}(t)^{R-2} \oplus \mathcal{O}(t-1). \tag{3.23}$$

The complement of these types is closed of codimension at least two [22, Lemma 3], so its smooth inverse image cannot contain the generic point of Z.

For a rank-A subbundle of a balanced type, the largest possible degree excess over $A\deg(V)/R$ is $\min(A, e) - Ae/R \leq A(R-A)/R$, where e is the number of upper summands. For (3.23), the excess is at most one. Since $A(R-A) \geq R-1$, every prefix of rank $A_j = R_1 + \cdots + R_j$ therefore satisfies

$$S_j := \sum_{i \leq j} R_i(\mu_i - \mu(E)) \leq \frac{A_j(R - A_j) + 1}{R}. \tag{3.24}$$

Set $w_{ij} = R_i R_j$, $d_{ij} = \mu_i - \mu_j \geq 0$, and

$$U = \sum_{i<j} w_{ij} d_{ij}, \quad V_2 = \sum_{i<j} w_{ij} d_{ij}^2, \quad W = \sum_{i<j} w_{ij}, \quad L = \mu_1 - \mu_\ell.$$

Write $\delta_j = \mu_j - \mu_{j+1}$ and $\mu = \mu(E)$. Reordering the finite sums gives

$$U = \sum_{i<j} R_i R_j \sum_{t=i}^{j-1} \delta_t = \sum_t \delta_t \Big(\sum_{i\le t} R_i\Big)\Big(\sum_{j>t} R_j\Big) = \sum_t A_t(R-A_t)\delta_t,$$
$$V_2 = \frac{1}{2}\sum_{i,j} R_i R_j(\mu_i-\mu_j)^2 = R\sum_i R_i\mu_i^2 - \Big(\sum_i R_i\mu_i\Big)^2 = R\sum_i R_i(\mu_i-\mu)^2.$$

Since $\sum_i R_i(\mu_i-\mu)=0$, we also have

$$\begin{aligned}\sum_{j=1}^{\ell-1}\delta_j S_j &= \sum_{i=1}^{\ell} R_i(\mu_i-\mu)\sum_{j=i}^{\ell-1}\delta_j \\ &= \sum_i R_i(\mu_i-\mu)(\mu_i-\mu_\ell) = \sum_i R_i(\mu_i-\mu)^2.\end{aligned}$$

The empty inner sum for $i=\ell$ is zero. Combining these identities with (3.24) and $A_j(R-A_j)\ge R-1$ yields

$$V_2 = R\sum_{j=1}^{\ell-1}(\mu_j-\mu_{j+1})S_j \le U+L, \tag{3.25}$$
$$U = \sum_{j=1}^{\ell-1} A_j(R-A_j)(\mu_j-\mu_{j+1}) \ge (R-1)L. \tag{3.26}$$

All coefficients in (3.25) are nonnegative, which is why the one-sided prefix estimate suffices.

Put

$$m_0 = \min\{R - \max_i R_i, R/2\} \ge 1, \qquad \gamma = 1 - m_0/R.$$

Additivity of the Chern character gives

$$\mathrm{ch}_2(E) = \sum_i R_i\left(\frac{\mu_i^2}{2} - \Delta_i\right), \qquad \Delta(E) = \frac{\sum_i R_i\Delta_i}{R} - \frac{V_2}{2R^2}.$$

Together with Riemann–Roch on the plane, this yields

$$\chi(G_i,G_j) = w_{ij}\left(1 + \frac{d_{ij}^2}{2} - \frac{3d_{ij}}{2} - \Delta_i - \Delta_j\right), \tag{3.27}$$
$$\sum_i R_i\Delta_i = R\Delta(E) + \frac{V_2}{2R}. \tag{3.28}$$

Since $R - R_i \ge m_0$ and $\Delta_i \ge 0$, these formulas imply

$$Q := -\sum_{i<j}\chi(G_i,G_j) \ge m_0 R\Delta(E) - \sum_{i<j} w_{ij}\left(1 + \frac{\gamma}{2}d_{ij}^2 - \frac{3}{2}d_{ij}\right). \tag{3.29}$$

The subtracted sum is at most

$$\begin{aligned}W + \frac{\gamma-3}{2}U + \frac{\gamma}{2}L &\le W + \frac{(\gamma-3)(R-1)+\gamma}{2}L \\ &= W + \frac{3-2R-m_0}{2}L \le W.\end{aligned} \tag{3.30}$$

Here we used both (3.25) and (3.26), with the negative sign of $\gamma-3$.

We also need the integer-rank estimate

$$W \le m_0(R-1). \tag{3.31}$$

If $m = R - \max R_i \le R/2$, the remaining positive integer ranks have square sum at least m, so

$$W = \frac{R^2 - \sum_i R_i^2}{2} \le \frac{R^2 - (R-m)^2 - m}{2} = m\left(R - \frac{m+1}{2}\right) \le m(R-1).$$

Otherwise $m_0 = R/2$, and $\sum_i R_i^2 \ge R$ gives $W \le R(R-1)/2$. Combining (3.29)–(3.31), we obtain

$$Q \ge m_0\{R\Delta(E) - (R-1)\} \ge 2. \tag{3.32}$$

We relate this number to the boundary. For $i \ge j$,

$$\operatorname{Ext}^2(G_i, G_j) = \operatorname{Hom}(G_j, G_i(-3))^\vee = 0$$

by the order of slopes. The factor stacks are therefore smooth of dimension $-\chi(G_i, G_i)$, and the stack of extensions $0 \to E_{j-1} \to E_j \to G_j \to 0$ has relative dimension $-\chi(G_j, E_{j-1})$. This is the difference $\dim \operatorname{Ext}^1(G_j, E_{j-1}) - \dim \operatorname{Hom}(G_j, E_{j-1})$; the Hom automorphisms are part of the calculation. Consequently the stack of the fixed numerical flags has dimension

$$-\sum_i \chi(G_i, G_i) - \sum_{i>j} \chi(G_i, G_j). \tag{3.33}$$

These dimension statements remain valid when Hom dimensions jump: locally the extension stack is the quotient of the two terms of a perfect complex of amplitude $[0, 1]$.

Forgetting the flag is representable by a relative Quot flag scheme, so its image has no larger dimension. The prioritary stack itself is smooth of dimension $-\chi(E, E)$. To see the vanishing needed here, multiplication by an equation of the line injects $\operatorname{Hom}(E, E(-3))$ into $\operatorname{Hom}(E, E(-2)) = 0$. Subtracting (3.33) from the ambient dimension gives precisely Q. Smooth pullback to S adds the same relative dimension to both sides. Since the chosen flag exists at the geometric generic point of Z and spreads after a finite extension, its image closure contains Z. Thus $\operatorname{codim}_S Z \ge Q \ge 2$, contrary to its choice. This also excludes a codimension-zero nonstable component and proves the final assertion. □

3.5 Gaeta presentations and Hartogs extension

Proposition 3.6. *The stable torsion-free cokernels in*

$$\mathcal{P} = \operatorname{Hom}(\mathcal{O}(-2)^a, \mathcal{O}(-1)^b \oplus \mathcal{O}^c)$$

form a nonempty open with complement of codimension at least two. Their image is dense in M_σ. Pullback and extension give an injection

$$j : H^0(M_\sigma, \Theta_\rho) \hookrightarrow \mathscr{G}, \tag{3.34}$$

where $\mathscr{G}$ is the full presentation semi-invariant space of Proposition 2.1.

Proof. Write $K = \mathcal{O}(-2)^a$ and $T = \mathcal{O}(-1)^b \oplus \mathcal{O}^c$. At any fixed $p \in \mathbf{P}^2$, evaluation from $\mathcal{P}$ to $\operatorname{Hom}(K_p, T_p)$ is surjective, since $\mathcal{O}(1)$ and $\mathcal{O}(2)$ are globally generated. The rank-defect condition in this matrix space has codimension $r+1$. Its incidence with $\mathbf{P}^2$ therefore has dimension $\dim \mathcal{P} - r + 1$. Properness of the projection and upper semicontinuity of fiber dimension show that the parameters with a positive-dimensional defect locus form a closed set of dimension at most $\dim \mathcal{P} - r$.

Outside this set the map $K \to T$ is a sheaf injection, its cokernel E is locally free in codimension one, and its local projective dimension is at most one. At a closed point the depth lemma gives $\operatorname{depth} E \geq 1$, excluding zero-dimensional torsion. Local freeness in codimension one excludes one-dimensional torsion. Thus E is torsion-free. Fiberwise injectivity gives a flat cokernel family. The torsion-free open $\mathcal{P}_{\mathrm{tf}}$ consequently has complement of codimension at least $r \geq 2$.

The resolution gives, for every member,

$$H^1(E) = H^1(E(1)) = 0, \qquad H^0(E(-1)) = H^0(E(-2)) = 0.$$

The first pair implies $\operatorname{Ext}^1(T, E) = 0$. Given a deformation of E over an Artin small extension, its quotient from T lifts and remains surjective. Its kernel is a flat deformation of K, locally free because its special fiber is locally free. Since $\operatorname{Ext}^1(K, K) = 0$, a chosen trivialization of this kernel lifts. This proves that the classifying morphism of the family is smooth. The second pair, by precomposition with $T \twoheadrightarrow E$, gives $\operatorname{Hom}(E, E(-2)) = 0$, hence $\operatorname{Ext}^2(E, E(-1)) = 0$. The family is line-prioritary.

Its class is σ: the rank is $b + c - a = r$, the first Chern class is $(2a - b)H = dH$, and the Euler characteristic is c. Put $\mu = d/r$ and $u = k/s > 0$. Riemann–Roch yields

$$\Delta(\sigma) = \frac{\mu^2 + 3\mu}{2} - u. \tag{3.35}$$

The hypothesis gives $\mu > k + (k+1)u$. The right side of (3.35) is increasing for $\mu > 0$, and its value at the lower endpoint is

$$\frac{k(k+3)}{2} + \frac{2k^2 + 5k + 1}{2}u + \frac{(k+1)^2}{2}u^2 > 2. \tag{3.36}$$

In particular $\Delta(\sigma) > 1 + 1/r$. Lemma 3.5 now shows that the slope-stable locus is nonempty and has complement of codimension at least two in $\mathcal{P}_{\mathrm{tf}}$, and hence in $\mathcal{P}$. Slope stability implies Gieseker stability, so the same assertion holds for the stable torsion-free presentation open $\mathcal{P}^s$.

Its smooth classifying image is open in the stable stack. The full plane moduli space M_σ is irreducible by [11, Theorem D]; the stable image is therefore dense. The semistable stack is smooth because $\operatorname{Ext}^2(E, E) = \operatorname{Hom}(E, E(-3))^\vee = 0$, and its characteristic-zero good quotient is reduced (in fact normal). A section vanishing after pullback to $\mathcal{P}^s$ thus vanishes on M_σ.

The determinant trivialization along a presentation has character

$$\det_a^{-n} \det_b^{n} \det_c^{k}.$$

The entire change-of-presentation group acts, including $\operatorname{Hom}(\mathbf{C}^b, \mathbf{C}^c) \otimes H^0(\mathcal{O}(1))$ and its shears on the quadratic block. The pullback of a moduli section is a regular covariant on $\mathcal{P}^s$. Hartogs extension on the affine space $\mathcal{P}$ extends it uniquely to a polynomial, and uniqueness preserves covariance under the entire group. This gives (3.34). □

3.6 The canonical determinant tensor

We verify the construction on the complete product of the two moduli spaces. If E, F are torsion-free on a smooth surface, then $\operatorname{Tor}_i(E, F) = 0$ for all $i > 0$. Indeed E has a locally free resolution of length one; its first Tor with F is a generically zero subsheaf of a direct

sum of copies of the torsion-free sheaf F. It is therefore zero. For our two semistable classes, $F^\vee$ is a slope-semistable bundle of slope zero, so

$$H^2(E \otimes F) = \mathrm{Hom}(E, F^\vee(-3))^\vee = 0 \tag{3.37}$$

because $\mu(E) = d/r > 0$. On the plane the tensor Euler characteristic is

$$\chi(E \otimes F) = r\chi(F) + s\chi(E) - rs + c_1(E)c_1(F) = -rk + sc - rs = 0.$$

Relative perfectness, the fiberwise Tor vanishing and (3.37) imply that the derived pushforward in a family has amplitude $[0, 1]$ and Euler rank zero. Its inverse determinant therefore has its canonical determinant section, with the convention of [16].

For a stable Jordan–Hölder factor G of a σ-sheaf, $\chi(G)/\operatorname{rk}(G) = n/s$, so

$$\chi(G \otimes \rho) = s\chi(G) - n \operatorname{rk}(G) = 0. \tag{3.38}$$

For a stable factor F_j of a ρ-sheaf, $c_1(F_j) = 0$ and $\chi(F_j)/\operatorname{rk}(F_j) = -k/s$, so

$$\chi(\sigma \otimes F_j) = r\chi(F_j) + (c - r)\operatorname{rk}(F_j) = 0. \tag{3.39}$$

The multiplicity groups at all closed polystable points consequently act trivially on the determinant lines. By [2, Theorem 10.3] the two lines, and their product section, descend to the complete good moduli spaces.

The determinant line on the product is $\Theta_\sigma \boxtimes \Theta_\rho$, up to a fixed one-dimensional factor. We give the K-theoretic verification to make the separation of the two factors explicit. The resolution of the diagonal is

$$0 \longrightarrow \mathcal{O}(-2) \boxtimes \Omega^2(2) \longrightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \longrightarrow \mathcal{O} \boxtimes \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

For a family E on $\mathbf{P}^2 \times S$, put $e_i = [\mathcal{O}(-i)]$ and $V_i = Rp_{S*}(E \otimes \Omega^i(i))[i]$, for $0 \le i \le 2$. These are perfect coefficient complexes, and the diagonal resolution gives

$$[E] = \sum_{i=0}^{2} e_i \boxtimes [V_i].$$

For a family F on $\mathbf{P}^2 \times T$, define W_j in the same way. Their virtual ranks v_i, w_j are constant and depend only on σ, ρ. On $S \times T$ one therefore has

$$[Rp_*(E \otimes^{\mathbf{L}} F)] = \sum_{i,j=0}^{2} C_{ij}[V_i] \boxtimes [W_j], \qquad (C_{ij}) = (\chi(e_i e_j)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

For perfect complexes A, B, $\det(A \otimes B) = (\det A)^{\operatorname{rk} B} \otimes (\det B)^{\operatorname{rk} A}$. Thus the inverse determinant above is

$$\left(\bigotimes_i (\det V_i)^{-\sum_j C_{ij} w_j} \right) \boxtimes \left(\bigotimes_j (\det W_j)^{-\sum_i C_{ij} v_i} \right). \tag{3.40}$$

Fixing either family at a point identifies the corresponding variable factor with its defining determinant line: on S it is Θ_ρ, and on T it is Θ_σ. This construction is functorial on the sheaf stacks and their atlases, so it applies without universal families on the coarse spaces and descends by (3.38)–(3.39). Reordering the two factors gives the asserted external product on $M_\rho \times M_\sigma$. A normalization of the fixed one-dimensional factor changes the strange-duality map by one nonzero scalar.

Proof of Theorem 3.1. For F of class ρ, tensor its Gaeta partner's resolution with F and use (3.20). The derived pushforward is represented in degrees $0, 1$ by

$$\mathbf{C}^a \otimes H^1(F(-2)) \longrightarrow (\mathbf{C}^b \otimes H^1(F(-1))) \oplus (\mathbf{C}^c \otimes H^1(F)). \tag{3.41}$$

Both terms have dimension $an = bn + ck$. The differential is the universal block evaluation matrix of Proposition 2.1: its linear entries evaluate on X_i, and its quadratic entries evaluate on Y_iX_j. Its determinant is exactly the pulled-back cohomological theta section. This identification holds in families; changes of bases and unipotent presentation shears give the same determinant transition functions on both sides.

Let

$$u : (R_\Gamma)_\theta \xrightarrow{\sim} \mathcal{B}_\theta \xrightarrow{\sim} H^0(M_\rho, \Theta_\sigma)$$

be the actual substitution from Proposition 2.1 followed by Proposition 3.4, and let $\lambda : (R_\Gamma)_\theta^\vee \xrightarrow{\sim} \mathscr{G}$ be the pairing supplied by that universal determinant. With j as in (3.34), the identification (3.41) proves the identity of linear maps

$$j \circ \mathrm{SD}_{\rho,\sigma} = \lambda \circ u^\vee. \tag{3.42}$$

The right side is an isomorphism and j is injective. It follows first that j is surjective and then that $\mathrm{SD}_{\rho,\sigma}$ is an isomorphism. Nonvanishing follows from Proposition 2.4 and (3.18). □

4 The first exceptional mutation

We now prove Theorem 1.1. Throughout this section put

$$\begin{aligned} k = n - s, \qquad q = r/s, \qquad \tau = qk, \qquad c = qn, \\ a = d - \tau, \qquad b = d - 2\tau, \qquad e = 3a - b = 2d - \tau. \end{aligned}$$

Thus $q \geq 1$, τ, c are positive integers and

$$d > qkn, \qquad a, b, e > 0, \qquad en = 2an + ck. \tag{4.1}$$

The assertion about b follows from $n \geq 2$: $d > \tau n$ implies $b = d - 2\tau \geq 1$. The relation algebra in this section has six relation labels.

4.1 Changing the presentation in families

Fix a three-dimensional vector space W and an Euler sequence

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{i} \mathcal{O}(-1) \otimes W \xrightarrow{p} \Omega^1(1) \longrightarrow 0. \tag{4.2}$$

Here a fixed volume form identifies $T_{\mathbf{P}^2}(-2)$ with $\Omega^1(1)$. It fixes the subsequent identifications of arrow-label spaces and changes determinants only by a fixed nonzero scalar.

Let A, B, C, D be coefficient spaces of dimensions a, b, c, e. The two presentation spaces are

$$\begin{aligned} \mathcal{P}_{\mathrm{old}} &= \mathrm{Hom}(\mathcal{O}(-2) \otimes A, \mathcal{O}(-1) \otimes B \oplus \mathcal{O} \otimes C), \\ \mathcal{P}_{\mathrm{new}} &= \mathrm{Hom}(\mathcal{O}(-1) \otimes D, \Omega^1(1) \otimes A \oplus \mathcal{O} \otimes C). \end{aligned}$$

Write $\mathscr{G}_{\mathrm{old}}$ for the covariance space (2.2), and $\mathscr{G}_{\mathrm{new}}$ for the full presentation-group covariance space of character $\det_D^{-n} \det_A^{2n} \det_C^{k}$. Both presentation groups include the target shears.

Lemma 4.1. *There is a natural isomorphism $\mathscr{G}_{\mathrm{old}} \simeq \mathscr{G}_{\mathrm{new}}$. On the respective full-rank coefficient opens it is induced by an equivalence of presentation quotient stacks in families. This equivalence preserves the actual cohomological theta tensor, up to the fixed scalar in* (4.2).

Proof. Write $i_A = i \otimes 1_A$ and $p_A = p \otimes 1_A$, with the coefficient factors ordered as $A \otimes W$. For an old map (M, Q), the linear coefficient map is $m : A \otimes W \to B$, with $M = mi_A$. On the open where m is surjective its kernel is a vector bundle D. Locally on any base, the quadratic component Q lifts to $L : \mathcal{O}(-1) \otimes A \otimes W \to \mathcal{O} \otimes C$. Indeed, applying $\mathrm{Hom}(-, \mathcal{O})$ to (4.2) gives an exact sequence

$$0 \to \mathrm{Hom}(\Omega^1(1), \mathcal{O}) \to \mathrm{Hom}(\mathcal{O}(-1) \otimes W, \mathcal{O}) \to \mathrm{Hom}(\mathcal{O}(-2), \mathcal{O}) \to 0,$$

since $\mathrm{Ext}^1(\Omega^1(1), \mathcal{O}) = H^1(T(-1)) = 0$. The new map is $(p_A|_D, -L|_D)$. Conversely, the component of a new map into $\Omega^1(1) \otimes A$ is a coefficient map $u : D \to A \otimes W$. On its subbundle-inclusion open take $B = \operatorname{coker} u$ and extend the negative of its $\mathcal{O} \otimes C$ component locally to a map L. Then $Q = Li_A$ recovers the old presentation.

To account for every choice and stabilizer, use the common stack of exact coefficient sequences

$$0 \to D \xrightarrow{u} A \otimes W \xrightarrow{m} B \to 0$$

together with L. Besides $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C) \times \mathrm{GL}(D)$, it has two commuting translation groups, acting by

$$L \longmapsto L + hp_A, \qquad L \longmapsto L + vm,$$

where $h : \Omega^1(1) \otimes A \to \mathcal{O} \otimes C$ and $v : \mathcal{O}(-1) \otimes B \to \mathcal{O} \otimes C$. Quotienting by the first translation replaces L by Q; quotienting also by $\mathrm{GL}(D)$ forgets the frame of $\ker m$. The remaining action is exactly the full old presentation group. In the reverse order the second translation replaces L by $L|_D$, and quotienting by $\mathrm{GL}(B)$ forgets the frame of $\operatorname{coker} u$, giving the full new presentation group. The maps forgetting a lift L are torsors under the indicated vector bundles; forgetting coefficient frames gives the indicated general linear torsors. Exact sequences of coefficient bundles split locally even over nonreduced bases, so these constructions descend for arbitrary families. The common quotient retains the stabilizers of the combined translation action.

The two discarded rank loci have determinantal codimensions $3a - b + 1 = e + 1$ and $3a - e + 1 = b + 1$. Both are at least two. Hartogs on each entire affine presentation space therefore extends its regular covariants uniquely. Equivariance of the extension follows by equality on the dense coefficient open. The character identity is

$$(\det D)^{-n} (\det A)^{2n} (\det C)^k = (\det A)^{-n} (\det B)^n (\det C)^k,$$

because $\det D = (\det A)^3 (\det B)^{-1}$ up to a constant line. This proves the claimed isomorphism.

For the comparison of complexes put $T = \mathcal{O}(-1) \otimes A \otimes W$, $U = \Omega^1(1) \otimes A$, $B' = \mathcal{O}(-1) \otimes B$ and $C' = \mathcal{O} \otimes C$. Put $D' = \mathcal{O}(-1) \otimes D$ and write

$$\mathcal{C} = [\, T \xrightarrow{(p_A, m, -L)} U \oplus B' \oplus C' \,].$$

Denote the old and new presentation complexes by $\mathcal{C}_{\mathrm{old}}$ and $\mathcal{C}_{\mathrm{new}}$. All three complexes have terms in degrees $-1, 0$. There are canonical short exact sequences of complexes

$$0 \longrightarrow \mathcal{C}_{\mathrm{old}} \longrightarrow \mathcal{C} \longrightarrow [U \xrightarrow{1} U] \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{C}_{\mathrm{new}} \longrightarrow \mathcal{C} \longrightarrow [B' \xrightarrow{1} B'] \longrightarrow 0.$$

The first inclusion is i_A on sources and $(b', c') \mapsto (0, b', -c')$ on targets. The quotient on sources is $T/i_A(\mathcal{O}(-2) \otimes A) = U$, and its differential is the identity. The second inclusion is $D' \hookrightarrow T$ on sources and $(u', c') \mapsto (u', 0, c')$ on targets. Here the source quotient is $T/D' = B'$ via m, and the quotient differential is again the identity. Thus both inclusions are quasi-isomorphisms. They commute with base change and the group actions on the common stack. Let S be a common parameter base, $\mathcal{F}$ a family of the other class on $X \times S$, and $\pi : X \times S \to S$ the projection. Tensoring the two exact sequences with $\mathcal{F}$, applying derived pushforward, and taking determinants identifies both theta sections with the section of $\big(\det R\pi_*(\mathcal{C} \otimes^{\mathbf{L}} \mathcal{F})\big)^{-1}$. The identity complexes have their canonical acyclicity trivializations. The sign on the old C' summand is fixed, so the comparison multiplies the theta tensor by a fixed nonzero scalar. □

4.2 The rectangular determinant and the skew cut

Let E_0, E_1, E_2 have dimensions $n, 2n, k$. Write $X_j : E_0 \to E_1$ and $Z_{ij} : E_0 \to E_2$, with $Z_{ji} = -Z_{ij}$ and $Z_{ii} = 0$, and set

$$\Gamma' = \begin{pmatrix} X_0 & X_1 & X_2 \\ 0 & Z_{01} & Z_{02} \\ -Z_{01} & 0 & Z_{12} \\ -Z_{02} & -Z_{12} & 0 \end{pmatrix}, \qquad R' = \mathbb{C}[X_j, Z_{ij}\ (i < j)]/I_{2n+1}(\Gamma').$$

The composition of the two three-dimensional arrow-label spaces for $\mathcal{O}(-1), \Omega^1(1), \mathcal{O}$ is the alternating quotient. In corresponding bases a new presentation has coefficients $M_j : D \to A$ and $N_{ij} : D \to C$ for $i < j$. Its universal evaluation matrix is

$$\mathcal{D}' = \begin{pmatrix} \sum_j M_j \otimes X_j \\ \sum_{i<j} N_{ij} \otimes Z_{ij} \end{pmatrix} : D \otimes E_0 \longrightarrow (A \otimes E_1) \oplus (C \otimes E_2). \tag{4.3}$$

It is square by (4.1). Put $\theta' = (-e, a, c)$. The second raw ring is

$$\begin{aligned} \mathcal{B}' = \mathbb{C}[X_0, X_1, X_2, Y_0, Y_1, Y_2]/(&Y_0X_0, Y_1X_1, Y_2X_2, \\ &Y_0X_1 + Y_1X_0,\ Y_0X_2 + Y_2X_0,\ Y_1X_2 + Y_2X_1), \end{aligned} \tag{4.4}$$

where $Y_i : E_1 \to E_2$ and matrix equations mean all their entries.

Proposition 4.2. *The coefficient map of* $\det \mathcal{D}'$ *and the actual substitution* $Z_{ij} \mapsto Y_iX_j$ *give isomorphisms*

$$\mathcal{G}_{\mathrm{new}} \simeq (R'_{\theta'})^\vee, \qquad R'_{\theta'} \xrightarrow{\sim} \mathcal{B}'_{\theta'}.$$

Both assertions concern the displayed raw ideals.

Proof. We give the rectangular character calculation. Before the alternating quotient replace its labels by $V_1 \otimes V_2$, with $\dim V_1 = \dim V_2 = 3$, and take all nine Z_{ij} independent. The full radical is $\mathrm{Hom}(A, C \otimes V_1)$ acting on $\mathrm{Hom}(D \otimes V_2^*, A \oplus C \otimes V_1)$. The proof of Lemma 2.2 applies to auxiliary dimensions $(n, 2n, k)$: path cancellation leaves respectively $n, 2n, k$ copies, and the reverse cancellation leaves e, a, c copies. Surjectivity in both directions is therefore again for the coefficient maps of the same tensor. Its characters are $(-n, 2n, k)$ and $(-e, a, c)$.

In the parabolic Cauchy decomposition extraction of $\det(A^*)^{2n}$ forces the partition $((2n)^a, \tau_0)$ with $|\tau_0| = ck$ and $(\tau_0)_1 \le 2n$. Suppressing the one-dimensional vertex factors, the

character of its radical invariants is

$$\bigoplus_{\substack{|\tau_0|=ck\\(\tau_0)_1\le 2n}} \left(\bigoplus_{\omega} g(\tau_0,(k^c),\omega)\mathbf{S}_\omega V_1^*\right) \otimes \left(\bigoplus_{\nu} g(((2n)^a,\tau_0),(n^e),\nu)\mathbf{S}_\nu V_2^*\right). \tag{4.5}$$

For the generic matrix of size $(2n+3k)\times 3n$ and rank at most $2n$, the character of the *dual* auxiliary covariance space is

$$\bigoplus_{\substack{|\mu_0|=ck\\\ell(\mu_0)\le 2n}} \left(\bigoplus_{\omega} g(\mu_0,(c^k),\omega)\mathbf{S}_\omega V_1^*\right) \otimes \left(\bigoplus_{\nu} g((a^{2n})+\mu_0,(e^n),\nu)\mathbf{S}_\nu V_2^*\right). \tag{4.6}$$

Indeed the rank bound restricts partitions to length at most $2n$, and extracting (a^{2n}) from the upper block adds this rectangle. Putting $\mu_0=\tau_0'$ identifies the two formulas, since

$$((2n)^a,\tau_0)' = (a^{2n})+\tau_0', \quad (n^e)'=(e^n), \quad (k^c)'=(c^k),$$

and simultaneous conjugation of two arguments preserves a Kronecker coefficient. The dual in (4.6) accounts for the dual arrow labels.

On the dense open where $[X_0\ X_1\ X_2]$ has rank $2n$, write $Z_{ij}=Y_iX_j$. A full shear adds $(\sum_i u_i\otimes Y_i)(\sum_j M_j\otimes X_j)$ to the lower evaluation block. Its determinant is unchanged. The generic determinantal ring is integral, so this identity holds throughout it. Exact reductive character extraction and the free determinant spanning argument give an injective coefficient map from the dual rank-ring space to the radical invariants. Formulas (4.5)–(4.6) make it an isomorphism of the actual determinant coefficients.

Now take $V_1=V_2=V$ and the quotient $V\otimes V\to\bigwedge^2 V$. Its dual is the skew-label inclusion, with ideal $I_{\mathrm{sym}}=(Z_{ii},Z_{ij}+Z_{ji})$. Naturality of the free pairing identifies functions pulled back from this quotient with the annihilator of $(I_{\mathrm{sym}})_{\theta'}$. Full radical invariance is the annihilator of the raw rank ideal. Their intersection is dual to the quotient by the *sum* of these two ideals. This proves the first assertion for the displayed generated ideals. In these bases N_{ij} changes by $u_iM_j-u_jM_i$; the skew evaluation convention pairs each coefficient once with its dual skew label.

For the second assertion apply the incidence calculation (2.9) with rank $2n$, source $E_0^{\oplus 3}$ and target $E_1\oplus E_2^{\oplus 3}$. Its proof by Cauchy and nonnegative Borel–Weil–Bott is unchanged for these dimensions. Pull back all degrees of the Koszul complex of the six skew-label equations. In a frame Q_0 of dimension $2n$, factor the matrix as AB, with upper square block $A_0:Q_0\to E_1$. The pulled-back equations are $A_i'B_i=0$ and $A_i'B_j+A_j'B_i=0$, independent of A_0. Lemma 2.3, with its n replaced by $2n$ and exponent $a\ge 0$, extracts exactly $\det(A_0)^a$ times this relation complex. The internal action $h\in\mathrm{GL}(Q_0)$ on this factor is $\det(h)^{-a}$, so invariants select covariance $\det(h)^a$ in the remaining variables. The other characters are $\det(g_0)^{-e}$ and $\det(g_2)^c$. Taking H^0 after these exact reductive operations gives $\mathcal{B}'_{\theta'}$, retaining possible negative Koszul cohomology until the comparison has been made. Evaluation at $A_0=I$ is exactly $X_j=B_j$, $Z_{ij}=A_i'B_j$. As in the proof of Proposition 2.1, the natural multiplication by the independent square-block determinant proves that this is the substitution on the entire character space. □

4.3 The new inequality on the entire dimension box

For $\alpha' = (n, 2n, k)$ the Euler form is

$$\chi'(x, y) = \sum_{i=0}^{2} x_i y_i - 3x_0 y_1 - 3x_1 y_2 + 6x_0 y_2.$$

The coefficient six is the number of relation labels in (4.4).

Lemma 4.3. *For every proper nonzero integral $\beta = (u, v, z)$ with $0 \le u \le n$, $0 \le v \le 2n$, $0 \le z \le k$, one has*

$$\theta'(\beta) < 0 \quad \Longrightarrow \quad \theta'(\beta) + \chi'(\beta, \alpha' - \beta) < 0.$$

The signs of $\theta'(\beta)$, including zero, agree with the polynomial character $st(v-2u)+k(u-v)+nz$.

Proof. Put $m = v - 2u$ and $\varepsilon = d - qkn > 0$. Then $\theta' = m(d - qk) - qku + qnz$. If $m > 0$, then $u \le n - 1$ and $\theta' \ge d - qk - qk(n - 1) > 0$. If $m = 0$, set $D_0 = ku - nz$. The premise is $D_0 > 0$, and direct expansion gives

$$\chi'(\beta, \alpha' - \beta) - D_0 = z(k - z) - u(n - u) - (ku - nz) = -(u - z)(n + k - u - z) < 0. \quad (4.7)$$

Here $n > k$ and $D_0 > 0$ give $u > z$; the second factor could vanish in the box only at $u = n, z = k$, where $D_0 = 0$. Since $q \ge 1$, $\theta' + \chi' \le -D_0 + \chi' < 0$.

Suppose $m = -h < 0$. Now $1 \le h \le 2n$ and $\lceil h/2 \rceil \le u \le n$. Set

$$D_0 = k(n - 1)h + ku - nz, \qquad F(u, h, z) = \chi'(\beta, \alpha' - \beta) - D_0.$$

One has

$$D_0 \ge k\{(n - 1)h + \lceil h/2 \rceil - n\} \ge 0, \qquad \theta' = -\varepsilon h - qD_0 < 0.$$

The second inequality is equality at $h = 1$ and its bracket increases with h. The complete Euler expansion is

$$\begin{aligned} \chi'(\beta, \alpha' - \beta) &= u(n - u) + (2u - h)(2n - 2u + h) + z(k - z) \\ &\quad - 3u(2n - 2u + h) - 3(2u - h)(k - z) + 6u(k - z) \\ &= u^2 - nu - h^2 + h(3k - 2n + u - 3z) + z(k - z). \end{aligned} \quad (4.8)$$

Subtracting D_0 gives

$$F = u^2 + (h - k - n)u - h^2 - hkn + 4hk - 2hn - 3hz + (k + n)z - z^2.$$

Thus F is convex in u and it suffices to treat its two endpoints. Write $t_0 = k - z$ and $s = n - k$. At the upper endpoint,

$$F(n, h, z) = -h^2 - hkn - hs + 3ht_0 - st_0 - t_0^2 < 0. \quad (4.9)$$

For $n \ge 3$ this follows from $t_0 \le k$, since $-hkn + 3ht_0 \le hk(3 - n) \le 0$. If $n = 2$, then $k = s = 1$; the two values $t_0 = 0, 1$ give $-h^2 - 3h$ and $-h^2 - 2$.

For the even lower endpoint put $E_j = F(j, 2j, z)$, $j \ge 1$. For the odd endpoint put $O_j = F(j + 1, 2j + 1, z)$, $j \ge 0$. Their polynomial expansions are

$$\begin{aligned} E_j &= -j^2 + j(-2kn + 7k - 5n - 6z) + (k + n)z - z^2, \\ O_j &= -j^2 + j(-2kn + 7k - 5n - 6z + 1) \\ &\quad - kn + 3k - 3n + 1 + (k + n - 3)z - z^2. \end{aligned}$$

Subtracting successive values and substituting $z = k - t_0$, $n = s + k$ at the initial values gives

$$E_{j+1} - E_j = -2j - 1 + k(7 - 2n) - 5n - 6z < 0, \tag{4.10}$$
$$E_1 = -kn - 4k - 5s - 1 + t_0(6 - s - t_0) < 0, \tag{4.11}$$
$$O_{j+1} - O_j = -2j + k(7 - 2n) - 5n - 6z < 0, \tag{4.12}$$
$$O_0 = -3n + 1 + t_0(3 - s - t_0) \le -3n + 2 < 0. \tag{4.13}$$

For the increments, $n \ge 4$ makes $7 - 2n < 0$; for $n = 2, 3$ use $1 \le k \le n - 1$ directly. Since $s \ge 1$ and t_0 is integral, $t_0(6-s-t_0) \le 6$ and $t_0(3-s-t_0) \le 1$, proving both base inequalities. The first has value at most $-kn - 4k - 5s + 5 < 0$. Consequently $F < 0$ throughout the box and

$$\theta' + \chi' = -\varepsilon h - (q - 1)D_0 + F < 0.$$

The same cases prove sign agreement: the leading coefficient of the polynomial character is sm, and when $m = 0$ its constant term is $nz - ku$, whose q-multiple is θ'. Equations (4.7)–(4.13) display all seven polynomial identities used here; their exact residuals are also checked in the accompanying calculation certificate. □

4.4 The complete first Beilinson heart

Proposition 4.4. *The restriction of covariant functions is an isomorphism*

$$\mathcal{B}'_{\theta'} \xrightarrow{\sim} H^0(M_\rho, \Theta_\sigma).$$

It includes strictly semistable points and the descended determinant line.

Proof. For the first collection in [17, §5.1], twisted by $\mathcal{O}(-1)$, the tilting bundles in vertex order are

$$T_0 = \mathcal{O}(1), \qquad T_1 = T_{\mathbf{P}^2}(-1), \qquad T_2 = \mathcal{O}.$$

The shifted tilting functor is $F \mapsto \mathrm{RHom}(T_0 \oplus T_1 \oplus T_2, F)[1]$. Its representation heart is

$$\mathcal{K} = \langle \mathcal{O}(-2)[1], \mathcal{O}(-1), \mathcal{O}[-1] \rangle_{\mathrm{ext}}.$$

The six symmetric relations are exactly (4.4). The inverse numerical class of (u, v, z) is $(-u + v - z, (2u - v)H, -z)$, so the polynomial weight for ρ is

$$\theta_G(u, v, z) = st(v - 2u) + k(u - v) + nz, \qquad \theta' = q\theta_G(d/r).$$

For a Gieseker-semistable F of class ρ, its three derived Hom coordinates are concentrated in degree one and have dimensions

$$h^1(F(-1)) = n, \qquad h^1(F \otimes \Omega^1(1)) = 2n, \qquad h^1(F) = k. \tag{4.14}$$

The H^0 vanishings follow from semistability against $\mathcal{O}(1), T(-1), \mathcal{O}$; for the last bundle use $\chi(F)/s = -k/s < 1$. The H^2 vanishings follow by Serre duality against their canonical twists, all of negative slope. Riemann–Roch then gives (4.14).

We apply the polynomial region $R^G \cap S$ of [17, equations (4.5)–(4.6)]. In the present notation,

$$R^G = \{v : \mathrm{rk}(v) > 0,\ \max_i p_{T_i(-3)} < p_v < \min_i p_{T_i}\},$$

where polynomials are compared for large positive t. The least p_{T_i} is $p_{\mathcal{O}}$, strictly greater than p_ρ; every $T_i(-3)$ has negative slope. Thus $\rho \in R^G$. The point conditions defining S require

positive polynomial weight for every nonzero subobject of the point complex, and vanishing sheaf H^{-1} for each quotient of leading weight zero.

For $x \in \mathbf{P}^2$, choose independent linear equations ℓ_0, ℓ_1 cutting out x. The point complex is

$$K_x = [\mathcal{O}(-2) \xrightarrow{(\ell_0,\ell_1)} \mathcal{O}(-1)^2 \xrightarrow{(\ell_1,-\ell_0)} \mathcal{O}] \simeq \mathcal{O}_x[-1],$$

with terms in degrees $-1, 0, 1$. Its proper nonzero subdimension vectors and their weights are

β	$(0,0,1)$	$(0,1,1)$	$(0,2,1)$
$\theta_G(\beta)$	n	$st-k+n$	$2st-2k+n.$

A nonzero source forces the whole middle space, and a nonzero constant middle subspace has nonzero image under $(\ell_1, -\ell_0)$, proving that the list is complete. All three polynomial weights are positive. The whole complex has weight $n-k=s>0$, as required by the condition on all nonzero subobjects. Its only nonzero quotients of leading weight zero are the whole complex and the quotient of dimension $(1,2,0)$. Their sheaf H^{-1} vanishes, in the latter case because (ℓ_0, ℓ_1) is injective. These verify both conditions defining S.

We now compare the two stability conditions. Let F be a sheaf in $\mathcal{K}$ which is not Gieseker-semistable, and let F_+ contain all its HN factors whose reduced polynomial is strictly greater than p_F, including torsion at the beginning if present. Set $F_- = F/F_+$. For their positive-rank factors one has

$$p_{\min}(F_+) > p_F \geq p_{\max}(F_-).$$

For every tilting summand T_i the inequalities defining R^G give

$$\operatorname{Ext}^2(T_i, F_+) = \operatorname{Hom}(F_+, T_i(-3))^* = 0, \qquad \operatorname{Hom}(T_i, F_-) = 0.$$

Maps from torsion to a vector bundle vanish as well. Since F lies in the heart, $\operatorname{Hom}(T_i, F) = \operatorname{Ext}^2(T_i, F) = 0$. The inclusion of F_+ and the long exact sequence now give $\operatorname{Hom}(T_i, F_+) = 0$ and $\operatorname{Ext}^2(T_i, F_-) = 0$, the latter also using $\operatorname{Ext}^3(T_i, F_+) = 0$ on a surface. Thus both terms lie in $\mathcal{K}$, and $0 \to F_+ \to F \to F_- \to 0$ is exact there. The strict polynomial inequality makes it destabilizing.

For the reverse implication, let $K = [K^{-1} \xrightarrow{\delta^{-1}} K^0 \xrightarrow{\delta^0} K^1]$ be an object of $\mathcal{K}$, represented by the corresponding complex of sums of $\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}$. Write $\delta_x^{-1}, \delta_x^0$ for its fiber differentials and ω_x for the fiber of $\omega_{\mathbf{P}^2}$ at x. Derived Hom to a point and Serre duality give

$$\begin{aligned} \operatorname{Hom}_{\mathcal{K}}(K, K_x) &\simeq \operatorname{coker}(\delta_x^0)^\vee, \\ \operatorname{Hom}_{\mathcal{K}}(K_x, K) &\simeq \ker(\delta_x^{-1}) \otimes \omega_x^{-1}. \end{aligned} \tag{4.15}$$

Indeed the first group is degree -1 cohomology of the dual fiber complex. For the second, Serre duality identifies it with $\operatorname{Hom}(K, \mathcal{O}_x[1] \otimes \omega_x)^\vee$; degree 1 cohomology of the dual fiber complex is $\ker(\delta_x^{-1})^\vee$. These are the point-map identifications used in [17, Lemmas 4.6–4.9 and 4.12].

For a sheaf $F \in \mathcal{K}$ choose a quotient Q whose polynomial weight $\theta_G(Q)$ is maximal, first in its leading coefficient and then in its constant term. Such a maximum exists because there are finitely many subdimension vectors. Its kernel A has minimal polynomial weight. If the last differential of A is not surjective at a point, it gives a nonzero map from A to that point object. Its image J has positive polynomial weight by the first point condition, so $\theta_G(\ker(A \to J)) < \theta_G(A)$, a contradiction. Thus $H^1(A) = 0$.

For Q, successively take cokernels of nonzero maps from point objects. Such a map exists if its first differential fails to be injective on a fiber. A point object has leading weight

zero and all its subobjects have nonnegative leading weight, so each image has nonpositive leading weight. A negative value would make the next quotient of F have greater leading weight, contradicting maximality. Thus every such image has leading weight zero, and the second point condition gives its vanishing sheaf H^{-1}. Each successive quotient still has maximal leading weight. The process terminates in the finite-length heart at a quotient with vanishing H^{-1}, since its first map is fiberwise injective or the quotient is zero. The cohomology sequences, read backwards, give $H^{-1}(Q) = 0$. The ordinary cohomology sequence of $0 \to A \to F \to Q \to 0$ then makes A, Q sheaves and this sequence an ordinary sheaf sequence. If F is unstable in the heart, its chosen quotient has positive polynomial weight and the resulting sheaf sequence destabilizes F. For a semistable heart object, apply the first test to the object itself, and the second to itself as a quotient of maximal leading weight zero. Both outer ordinary cohomology sheaves vanish, so it is a sheaf. An equal-polynomial Jordan–Hölder prefix obeys the same Hom and Ext^2 vanishings, so the comparison includes strictly semistable objects.

These are the comparisons of [17, Theorem 4.14 and Corollary 4.15]. They also hold in families: the two outer Ext vanishings make the shifted derived Hom coordinates locally free and compatible with base change; the relative perfect inverse of a representation family whose fibers are sheaves is flat by the fiberwise Tor-amplitude criterion. The inverse relative tilting functors therefore identify the entire polynomial-semistable representation and sheaf stacks. Lemma 4.3 identifies this locus with the finite-character semistable stack.

The derived representation scheme is the finite Koszul zero locus of the six groups of entries in (4.4). Indeed, adjoining six arrows of degree -1 from vertex 0 to vertex 2 gives the relative semi-free resolution: the six quadratic labels are independent, and no nontrivial paths compose with these new arrows. Thus the derived quotient is bounded and quasi-smooth, with the same three-term deformation complex as (3.13), except that the relation term has six copies. Its first differential is d_0 above, and its last differential has entries

$$\dot{Y}_i X_i + Y_i \dot{X}_i \quad (0 \le i \le 2), \qquad \dot{Y}_i X_j + Y_i \dot{X}_j + \dot{Y}_j X_i + Y_j \dot{X}_i \quad (i < j).$$

On the semistable locus $\mathrm{Ext}^2(F, F) = 0$, so it is the smooth classical sheaf stack.

Apply the stratification and shriek argument of Proposition 3.3 to this bound quiver. For a stratum with descending weights w_i and HN prefixes β_t, the identical determinant calculation now gives

$$\mathrm{wt}(i^{\mathrm{QC},!} L_{\theta'}) = \sum_t (w_t - w_{t+1})\{\theta'(\beta_t) + \chi'(\beta_t, \alpha' - \beta_t)\} < 0.$$

All prefixes have negative character, and the strict inequality is Lemma 4.3. The finiteness, boundedness, affine diagonal and quasi-smoothness hypotheses used there have just been verified here. The same restriction theorem consequently identifies derived global sections before and after removal of the unstable strata. In degree zero it identifies $\mathcal{B}'_{\theta'}$ with sections on the full semistable stack.

For a stable Jordan–Hölder factor of rank s_j, the dimension vector is $(s_j/s)(n, 2n, k)$. Its character is zero by (4.1). Therefore every multiplicity-group stabilizer acts trivially on the line, and [2, Theorem 10.3] gives descent to the complete coarse space. The character of inverse cohomology determinant is $(-e, a, c)$, as also follows directly from (4.14) and the new presentation. This is the claimed line Θ_σ. □

4.5 The canonical map in the larger chamber

Proof of Theorem 1.1. First check that the old Gaeta boundary argument remains applicable. For $u_0 = k/s > 0$ the new bound says $\mu = d/r > k(1 + u_0)$. The discriminant in (3.35),

evaluated at this lower endpoint, is

$$\frac{k(k+3)}{2}+\left(k^2+\frac{3k}{2}-1\right)u_0+\frac{k^2u_0^2}{2}>2.$$

Its constant term is at least two and the linear coefficient is at least $3/2$; the expression increases with $\mu>0$. Thus $\Delta(\sigma)>2>1+1/r$. All multiplicities a,b,c are positive and $r\geq 2$. The proof of Proposition 3.6 uses precisely these conditions for the torsion-free incidence estimate, the complete prioritary family and Lemma 3.5. Its density and determinant arguments therefore give the same injection

$$j_{\mathrm{old}}:H^0(M_\sigma,\Theta_\rho)\hookrightarrow\mathscr{G}_{\mathrm{old}}.$$

Compose with Lemma 4.1 to obtain j_{new}.

The product-amplitude and factorwise descent calculations (3.37), (3.38), (3.39) and (3.40) require the same two classes and $d>0$ and remain valid. They construct the actual canonical theta tensor on the complete product. On a new presentation, its two-term pushforward is

$$D\otimes H^1(F(-1))\longrightarrow(A\otimes H^1(F\otimes\Omega^1(1)))\oplus(C\otimes H^1(F)).$$

In the fixed Euler bases this is exactly (4.3). The alternating composition gives $Z_{ij}=Y_iX_j$, including every relation in (4.4). Lemma 4.1 also identifies this tensor with the old cohomological tensor on their common stack, so the comparison is compatible with j_{new}.

Let $u_{\mathrm{new}}:R'_{\theta'}\xrightarrow{\sim}H^0(M_\rho,\Theta_\sigma)$ be the actual substitution and restriction of Propositions 4.2 and 4.4, and let $\lambda_{\mathrm{new}}:(R'_{\theta'})^\vee\xrightarrow{\sim}\mathscr{G}_{\mathrm{new}}$ be the coefficient pairing. Choose compatible trivializations of the fixed one-dimensional determinant factors. Then

$$j_{\mathrm{new}}\circ\mathrm{SD}_{\rho,\sigma}=\lambda_{\mathrm{new}}\circ u_{\mathrm{new}}^\vee.$$

Its right side is an isomorphism and j_{new} is injective. It follows that j_{new} and then $\mathrm{SD}_{\rho,\sigma}$ are isomorphisms.

Finally $a-b=\tau$, $n\tau=kc$, and $a=d-\tau>\tau(n-1)$. Proposition 2.4 gives a nonzero old covariant, and Proposition 2.1 makes $\mathscr{G}_{\mathrm{old}}$ nonzero. Lemma 4.1 and the new comparisons therefore prove that both moduli section spaces are nonzero. □

4.6 A certificate in the added range

For $(s,n,r,d)=(2,3,2,4)$ one has $(k,q,\tau,a,b,c)=(1,1,1,3,2,3)$ and $d=4>3=qkn$. Here $\sigma=(2,4H,3)$ has $c_2(\sigma)=13$. Use the old raw variables in (2.4), with

$$X_0=I_3,\quad X_1=\begin{pmatrix}0&0&0\\0&1&0\\0&0&2\end{pmatrix},\quad X_2=0,\qquad Y_0=(1\ 1\ 1),\quad Y_1=(0\ 1\ 2),\quad Y_2=0.$$

The three relation residuals vanish entry by entry: $Y_0X_1-Y_1X_0=(0\ 0\ 0)$, $Y_0X_2-Y_2X_0=(0\ 0\ 0)$ and $Y_1X_2-Y_2X_1=(0\ 0\ 0)$. The nonzero lower blocks are

$$Z_{00}=(1\ 1\ 1),\qquad Z_{01}=(0\ 1\ 2),\qquad Z_{11}=(0\ 1\ 4).$$

Choose the Gaeta coefficients

$$M_0=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\qquad M_1=M_2=0,$$

and

$$Q_{00} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_{01} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $Q_{02} = Q_{12} = Q_{22} = 0$. The quadratic coefficients refer to the monomials $x_i x_j$ for $i \le j$, each taken once. The complete evaluation matrix is

$$\begin{pmatrix} M_0 \otimes X_0 \\ Q_{00} \otimes Z_{00} + Q_{01} \otimes Z_{01} + Q_{11} \otimes Z_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Its determinant is $\det(I_6) \det \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{smallmatrix} \right) = 4 - 2 = 2$. Nonvanishing of this polynomial and density of the stable Gaeta open give a stable presentation where it remains nonzero. The section and tensor comparisons identify the resulting theta section. Appendix A.4 gives the corresponding 21×21 matrix after the Euler change of presentation.

5 Determinant positivity on a surface

We begin with a sufficient discriminant bound for determinant positivity. All numerical intersections are taken on the surface. If v has positive rank r, put

$$\Delta(v) = \frac{c_1(v)^2}{2r^2} - \frac{\mathrm{ch}_2(v)}{r}.$$

The determinant convention throughout this paper is the inverse determinant of cohomology.

Theorem 5.1. *Let X be a smooth complex projective surface, H an integral ample divisor, and u the numerical class of a pure one-dimensional sheaf, with $c_1(u) = D \neq 0$. Suppose that the Simpson moduli space $N = M_H(u)$ is nonempty. Let $v \in K(X)$ have rank $r > 0$, and suppose that*

$$\frac{c_1(v)}{r} = \mu H, \qquad \chi(v \otimes u) = 0.$$

The inverse determinant line λ_v descends to N. It is ample if

$$\Delta(v) > \chi(\mathcal{O}_X) - \frac{K_X^2}{8} + \frac{(HD)^2}{8H^2}, \tag{5.1}$$

and it is nef if equality holds. If $c_1(v) = 0$, $\chi(u) = 0$, and D is nef, the strict sufficient bound can be replaced by

$$\Delta(v) > \chi(\mathcal{O}_X) + \frac{D^2 - K_X^2}{8}; \tag{5.2}$$

equality in this last bound implies nefness.

Write $\mathfrak{N}$ for the stack of semistable sheaves of class u, $\mathcal{F}$ for its universal family, and $p : X \times \mathfrak{N} \to \mathfrak{N}$ for the projection. The determinant line on N is obtained by descent from $\mathfrak{N}$.

Proof. Write $e = \chi(u)$ and $h = HD > 0$. Orthogonality says $e + \mu h = 0$. For every stable Jordan–Hölder factor F_i of a semistable sheaf of class u, equality of reduced Hilbert polynomials gives $\chi(F_i)/(Hc_1(F_i)) = e/h$. Consequently

$$\chi(v \otimes F_i) = r\chi(F_i) + c_1(v)c_1(F_i) = 0.$$

Every multiplicity factor in the stabilizer therefore acts trivially on the determinant line. The GIT descent criterion gives λ_v on N.

Set

$$B = \frac{K_X}{2} - \mu H, \qquad Z_\alpha(E) = -\operatorname{ch}_2^B(E) + \frac{\alpha^2 H^2}{2}\operatorname{rk}(E) + i\alpha H \operatorname{ch}_1^B(E), \qquad \alpha > 0,$$

where $\operatorname{ch}^B = e^{-B}\operatorname{ch}$. For a torsion-free sheaf E, put $\mu_H(E) = Hc_1(E)/\operatorname{rk}(E)$. Let $\mathcal{T}_B$ be the extension-closed category generated by torsion sheaves and slope-semistable torsion-free sheaves with $\mu_H > HB$, and let $\mathcal{F}_B$ be the extension-closed category generated by slope-semistable torsion-free sheaves with $\mu_H \le HB$. The tilt heart is $\operatorname{Coh}^B(X) = \langle \mathcal{F}_B[1], \mathcal{T}_B \rangle_{\mathrm{ext}}$. We use the geometric stability condition with this heart and charge Z_α. Its existence and the inequality

$$(H \operatorname{ch}_1^B(E))^2 - 2H^2 \operatorname{rk}(E) \operatorname{ch}_2^B(E) \ge 0 \tag{5.3}$$

for semistable objects are the surface results [18, Theorems 6.10 and 6.13]. The tilt heart is independent of α. In particular $Z_\alpha(u) = i\alpha h$. The uniform large-volume identification with ordinary Simpson semistability is [6, Proposition 6.3]. Its assertion is for a whole fixed numerical class, including $e = 0$. Indeed, their vector is $(1, -B, -\tau)$ with $\tau = (\alpha^2 H^2 - B^2)/2$. Its charge has the same real part as Z_α and imaginary part $H \operatorname{ch}_1^B$; multiplication of that imaginary part by $\alpha > 0$ preserves all phase comparisons.

Consider a first changing wall for a Simpson-semistable sheaf F as α decreases from large volume. Choose a same-phase semistable subobject U and semistable quotient V. The cohomology sequence for $0 \to U \to F \to V \to 0$ in the tilt heart gives $H^{-1}(U) = 0$, so U is a sheaf. If $\operatorname{rk}(U) = 0$, the torsion-free sheaf $H^{-1}(V)$ injects into U and hence vanishes. Thus U is an ordinary subsheaf of F, is pure, and the comparison is precisely

$$\chi(U) + \mu H c_1(U) \le 0.$$

This comparison is independent of α. Thus a changing wall has $s = \operatorname{rk}(U) > 0$ and $\operatorname{rk}(V) = -s$. Put $x = H \operatorname{ch}_1^B(U)$. The two imaginary parts are positive, so $0 < x < h$. Both real parts vanish at the wall. Applying (5.3) to the two factors gives

$$x^2 \ge \alpha^2 (H^2)^2 s^2, \qquad (h - x)^2 \ge \alpha^2 (H^2)^2 s^2.$$

Hence every such wall satisfies

$$\alpha \le \frac{h}{2H^2 s} \le \frac{h}{2H^2} =: \alpha_0. \tag{5.4}$$

All Simpson-semistable sheaves of class u remain semistable for $\alpha > \alpha_0$.

Choose α by

$$\alpha^2 H^2 = 2\Delta(v) + \frac{K_X^2}{4} - 2\chi(\mathcal{O}_X). \tag{5.5}$$

For any class $E \in K(X)$, Riemann–Roch gives

$$\begin{aligned} \frac{\chi(v \otimes E)}{r} = \operatorname{ch}_2(E) &+ \left(\mu H - \frac{K_X}{2}\right) c_1(E) \\ &+ \operatorname{rk}(E) \left(\chi(\mathcal{O}_X) - \frac{\mu H K_X}{2} + \frac{\operatorname{ch}_2(v)}{r}\right). \end{aligned}$$

On the other hand,

$$\Re Z_\alpha(E) = -\operatorname{ch}_2(E) + Bc_1(E) + \frac{\alpha^2H^2 - B^2}{2}\operatorname{rk}(E),$$
$$\frac{\alpha^2H^2 - B^2}{2} = -\chi(\mathcal{O}_X) + \frac{\mu HK_X}{2} - \frac{\operatorname{ch}_2(v)}{r}.$$

Thus the exact identity is

$$\Re Z_\alpha(E) = -\frac{\chi(v \otimes E)}{r}. \tag{5.6}$$

For a family $\mathcal{F}$ on $X \times T$, with T a smooth projective curve, let $p_X : X \times T \to X$ and $p_T : X \times T \to T$ be the projections, and write $E = R(p_X)_*\mathcal{F}$. The virtual rank of $R(p_T)_*(\mathcal{F} \otimes p_X^* v)$ is zero. Riemann–Roch on T and the projection formula therefore give

$$\deg_T \lambda_v = -\chi(v \otimes E).$$

The Bayer–Macrì positivity lemma [3, Theorem 1.1 and Lemma 3.3] says that

$$\ell_\alpha(T) = \Im\frac{-Z_\alpha(E)}{Z_\alpha(u)} = \frac{\Re Z_\alpha(E)}{\alpha h} = \frac{\deg_T \lambda_v}{r\alpha h}$$

is nonnegative. It applies to the coarse moduli space as follows. After a finite extension of the function field of any integral coarse curve, its generic point has a representative family. Normalize the curve in that extension, spread the family, and use semistable reduction at the finitely many missing discrete valuation rings. The resulting coarse morphism agrees with the original cover by separatedness. Degrees multiply by the degree of the cover. Hence the same nonnegativity holds on every coarse curve. Taking limits at α_0 gives a nef boundary class.

For ampleness we combine the nef bound with the GIT polarization, as in [6, Section 4]. Put $p(j) = hj + e$. For $\ell \gg m \gg 0$ sufficiently divisible, Simpson's ample GIT line is the descent of a positive power of the scalar-neutral line

$$(\det Rp_*\mathcal{F}(\ell H))^{p(m)}(\det Rp_*\mathcal{F}(mH))^{-p(\ell)}. \tag{5.7}$$

See [21, Theorem 1.21] and the explicit formulation in [15, Section 4.2 and Theorem 4.4]. For $e = 0$ this construction is [25, Proposition 3.0.4]. It is the inverse determinant associated to

$$w = p(\ell)[\mathcal{O}(mH)] - p(m)[\mathcal{O}(\ell H)].$$

Expanding its character gives

$$\operatorname{rk}(w) = h(\ell - m), \quad c_1(w) = -e(\ell - m)H,$$
$$\operatorname{ch}_2(w) = -\frac{\ell - m}{2}\{hm\ell + e(m + \ell)\}H^2.$$

Thus $c_1(w)/\operatorname{rk}(w) = \mu H$, and

$$\Delta(w) = \frac{\mu^2H^2}{2} + \frac{H^2}{2}\left(m\ell + \frac{e}{h}(m + \ell)\right) \longrightarrow +\infty.$$

After division by the rank, the numerical determinant class depends affinely on Δ, all other components being fixed. Along this line we extend the determinant homomorphism linearly to real numerical K-classes; this also defines the nef boundary class when its discriminant is not represented by an integral K-class. A class strictly beyond the nef boundary is a positive convex combination of that boundary and one of these ample classes with larger Δ. The sum of an ample real class and a nef real class is ample. This proves the strict assertion.

Finally, if $c_1(v) = 0$ and $\chi(u) = 0$, the stability condition for a pure subsheaf is simply $\chi(G) \leq 0$, independent of the polarization. Let $H_\varepsilon = D + \varepsilon A$ for an ample A and positive rational ε, clearing denominators when necessary. The determinant class and the Simpson moduli problem remain the same, and

$$\lim_{\varepsilon \downarrow 0} \frac{(H_\varepsilon D)^2}{H_\varepsilon^2} = D^2.$$

For $D^2 = 0$, the quotient is $\varepsilon^2(AD)^2/(2\varepsilon AD + \varepsilon^2 A^2)$ and tends to zero; for $D^2 > 0$ the limit is immediate. Strictness in (5.2) therefore implies ampleness for a sufficiently small ε. To justify the equality case explicitly, choose a closed point $x \in X$ and replace v by the rational K-class $v_\delta = v - r\delta[\mathcal{O}_x]$ for $\delta > 0$ rational. It has the same rank and first Chern class, remains orthogonal to u, and satisfies $\Delta(v_\delta) = \Delta(v) + \delta$. Clearing denominators, the strict case shows that its numerical determinant class is ample. These classes converge to λ_v as $\delta \downarrow 0$, so λ_v is nef. □

6 The sharp threshold in degree one

Let X be a smooth del Pezzo surface with $A = -K_X$ and $A^2 = 1$. For $k \geq 1$ set

$$N_k = M_A(0, kA, \chi = 0), \qquad h : N_k \longrightarrow |kA|, \qquad P = h^*\mathcal{O}_{|kA|}(1), \qquad \Theta = (\det Rp_*\mathcal{F})^{-1}.$$

Here $\mathfrak{N}_k$ is the moduli stack, $\mathcal{F}$ its universal family, and $p : X \times \mathfrak{N}_k \to \mathfrak{N}_k$ the projection. The argument above shows that Θ descends to the entire coarse moduli space. We write line bundles additively when discussing their numerical classes.

We prove Theorem 1.2, including its statements on reducible and nonreduced supports.

6.1 An integral improvement of the wall bound

The Chern character of a sheaf in N_k is $(0, kA, -k/2)$. Take $B = -A/2$ and $\omega = \alpha A$. Thus

$$Z_\alpha(E) = -\operatorname{ch}_2^B(E) + \frac{\alpha^2}{2}\operatorname{rk}(E) + i\alpha A \operatorname{ch}_1^B(E), \qquad Z_\alpha(F) = i\alpha k. \tag{6.1}$$

The large-volume result used above applies to this ray. In the notation of [6, Proposition 6.3], its real parameter is $\tau = \alpha^2/2 - 1/8$. Rescaling the imaginary part by a positive number preserves the ordering of phases. Starting in its large-volume range $\tau > 0$, we follow the same ray through the positive values of α in (6.1).

At a changing wall the factors from the preceding proof have ranks s and $-s$, where $s \geq 1$. Set $x = A\operatorname{ch}_1^B(U)$. The two Bogomolov inequalities and the integral lattice give, respectively,

$$\alpha s \leq x, \qquad \alpha s \leq k - x, \qquad x = Ac_1(U) + \frac{s}{2} \in \mathbb{Z} + \frac{s}{2}. \tag{6.2}$$

The closest this lattice can come to $k/2$ has distance zero if $k - s$ is even, and $1/2$ if $k - s$ is odd. Therefore

$$\alpha \leq \frac{k - \epsilon_{k,s}}{2s}, \qquad \epsilon_{k,s} = \begin{cases} 0 & k - s \text{ even}, \\ 1 & k - s \text{ odd}. \end{cases}$$

For $s = 1$ the bound is $k/2$ for odd k, and $(k-1)/2$ for even k. For $s \geq 2$ it is at most $k/4$. If $k = 2a + 1$, then $k/4 \leq a + 1/2$; if $k = 2a$ with $a \geq 1$, then $k/4 = a/2 \leq a - 1/2$. Consequently every changing wall satisfies

$$\alpha \leq R_k := \left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}. \tag{6.3}$$

Every Simpson-semistable sheaf of this class remains semistable for $\alpha > R_k$. The nef class at the endpoint will be obtained by a limit.

6.2 The determinant class and ampleness

Let $\mathcal{F}$ be a family over a smooth projective curve T and $E = R(p_X)_*\mathcal{F}$. Then

$$\deg_T \Theta = -\chi(E), \qquad \deg_T P = \operatorname{rk}(E). \tag{6.4}$$

The support identity is [25, Proposition 3.0.3], which attributes it to Le Potier. We include the degree calculation. The first equality follows from the zero virtual rank of $R(p_T)_*\mathcal{F}$ and Riemann–Roch on T. To check the sign in the second, flatness over T and purity of the fibers make $\mathcal{F}$ a Cohen–Macaulay sheaf of codimension one on $X \times T$. It therefore has a local equal-rank resolution by vector bundles. The determinant of its differential cuts out the relative Fitting support. Restricting to $\{x\} \times T$ for a general point $x \in X$ shows that $\det R(p_T)_*(\mathcal{F} \otimes^{\mathbf{L}} p_X^*\mathcal{O}_x)$ has the degree of the support hyperplane. The complex on T has virtual rank zero, so its determinant degree equals its Euler characteristic. Derived base change identifies this Euler characteristic with that of $E \otimes^{\mathbf{L}} \mathcal{O}_x$, namely $\operatorname{rk}(E)$. Thus the inverse point determinant has numerical class $-P$.

Since $\chi(\mathcal{O}_X) = 1$ and $K_X = -A$, Riemann–Roch and twisting give

$$\begin{aligned}
\chi(E) &= \operatorname{rk}(E) + \frac{1}{2}Ac_1(E) + \operatorname{ch}_2(E),\\
\operatorname{ch}_2^{-A/2}(E) &= \operatorname{ch}_2(E) + \frac{1}{2}Ac_1(E) + \frac{1}{8}\operatorname{rk}(E),\\
\Re Z_\alpha(E) &= -\chi(E) + \left(\frac{7}{8} + \frac{\alpha^2}{2}\right)\operatorname{rk}(E).
\end{aligned}$$

The positivity lemma therefore gives the nef class

$$\frac{1}{\alpha k}\left(\Theta + \left(\frac{7}{8} + \frac{\alpha^2}{2}\right)P\right) \quad (\alpha > R_k) \tag{6.5}$$

on the coarse space. The curve-lifting argument in the proof of Theorem 5.1 applies without a universal coarse family. Put $q = \lceil k/2 \rceil$. The endpoint computation is

$$\frac{7}{8} + \frac{1}{2}\left(q - \frac{1}{2}\right)^2 = \frac{7}{8} + \frac{q^2}{2} - \frac{q}{2} + \frac{1}{8} = 1 + \frac{q(q-1)}{2} = \nu_k. \tag{6.6}$$

Closedness of the nef cone proves that $\Theta + \nu_k P$ is nef.

For this ray, the GIT calculation of [25, Proposition 3.0.4] reads as follows. With $\ell \gg m \gg 0$ divisible by a fixed very ample multiple of A, the Hilbert polynomial is $p(j) = kj$. The line

$$(\det Rp_*\mathcal{F}(\ell A))^{km}(\det Rp_*\mathcal{F}(mA))^{-k\ell}$$

has ample descent. On the Quot presentation the second determinant is a constant determinant of the space of generators, trivial for its special linear group; the line is a positive power of the Grassmannian GIT line. It is the inverse determinant of ku, where

$$\begin{aligned}
u &= \ell[\mathcal{O}(mA)] - m[\mathcal{O}(\ell A)],\\
\operatorname{rk}(u) &= \ell - m, \qquad c_1(u) = \ell mA - m\ell A = 0,\\
\operatorname{ch}_2(u) &= \frac{\ell m^2 - m\ell^2}{2} = -\frac{m\ell(\ell - m)}{2}.
\end{aligned}$$

By (6.4), the inverse determinant associated to u has class

$$(\ell - m)\Theta + \frac{m\ell(\ell - m)}{2}P.$$

The displayed GIT line is its kth power. In particular $\Theta + MP$ is ample for arbitrarily large $M = m\ell/2$. For any real $t > \nu_k$, choose such an $M > t$ and write

$$\Theta + tP = \frac{t - \nu_k}{M - \nu_k}(\Theta + MP) + \frac{M - t}{M - \nu_k}(\Theta + \nu_k P). \tag{6.7}$$

This is an ample real class plus a nef real class, hence is ample. This proves sufficiency in both assertions of the theorem.

6.3 Subsheaves on arbitrary divisors

We give a lemma that keeps the sharpness argument valid on all supports.

Lemma 6.1. *Let C be an effective Cartier divisor on a smooth surface and L a line bundle. If $G \subset \mathcal{O}_C(L)$ is a nonzero pure subsheaf with generic support cycle $D \leq C$, then*

$$G \subset \mathcal{O}_D(L - (C - D)),$$

and the quotient is zero-dimensional.

Proof. At the generic point of a component of multiplicity m in C, the ambient local ring is a discrete valuation ring and $\mathcal{O}_C$ is its quotient by (t^m). A submodule of generic length d is the ideal $(t^{m-d})/(t^m)$. Its image in $\mathcal{O}_{C-D}$ is zero. Globally the image of G in $\mathcal{O}_{C-D}(L)$ is therefore zero-dimensional; that target is pure, so the image is zero. The exact divisor sequence identifies the kernel of restriction with $\mathcal{O}_D(L - (C - D))$. The generic lengths are equal, hence its quotient by G is zero-dimensional. □

On our degree-one del Pezzo surface,

$$\chi(\mathcal{O}_X(tA)) = 1 + \frac{t(t+1)}{2}, \qquad \chi(\mathcal{O}_D(L)) = LD - \frac{D^2 + K_X D}{2}. \tag{6.8}$$

If $C \in |kA|$, $0 < D < C$, and $x = AD$, then x is an integer in $[1, k-1]$. Hodge index gives $D^2 \leq x^2$, and hence

$$D(C - D) = kx - D^2 \geq x(k - x) > 0. \tag{6.9}$$

Substituting $L = aA - (C - D)$ into (6.8) gives the useful identity

$$\chi(\mathcal{O}_D(aA - (C - D))) = \frac{(2a + 1 - k)x - D(C - D)}{2}. \tag{6.10}$$

6.4 The two sharp test families

Suppose first that $k = 2a + 1$. For every $C \in |kA|$, put $F_C = \mathcal{O}_C(aA)$. Formula (6.8) gives $\chi(F_C) = 0$. If $G \subset F_C$ has proper generic support D, the lemma and (6.10) give

$$\chi(G) \leq -\frac{D(C - D)}{2} < 0.$$

If G has full generic support and is proper, its quotient has positive finite length, so again $\chi(G) < 0$. Every F_C is therefore stable. The universal divisor gives the flat family

$$0 \longrightarrow \mathcal{O}_X(-(a+1)A) \boxtimes \mathcal{O}(-1) \longrightarrow \mathcal{O}_X(aA) \boxtimes \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{6.11}$$

on $X \times |kA|$. Taking the inverse determinant in this sequence gives

$$\begin{aligned}\Theta|_{|kA|} &= \mathcal{O}(-\chi(\mathcal{O}_X(-(a+1)A))) \\ &= \mathcal{O}\left(-1 - \frac{a(a+1)}{2}\right) = \mathcal{O}(-\nu_k), \qquad P|_{|kA|} = \mathcal{O}(1).\end{aligned}$$

The linear system contains a line, including for $k = 1$, when it is the anticanonical pencil.

Now let $k = 2a$ with $a \geq 1$. Fix any zero-dimensional subscheme $Z \subset X$ of length a, and set $B_Z = |I_Z(kA)|$. For $C \in B_Z$ put $F_C = I_{Z,C}(aA)$. It is pure as a subsheaf of $\mathcal{O}_C(aA)$, and

$$\chi(F_C) = \chi(\mathcal{O}_C(aA)) - a = a - a = 0.$$

For a subsheaf with proper generic support D, the lemma gives

$$\chi(G) \leq \frac{x - D(C-D)}{2} \leq \frac{x(1-k+x)}{2} \leq 0.$$

For a proper subsheaf with full generic support, the finite nonzero quotient instead gives $\chi(G) < 0$. Thus all the F_C are semistable. The family has the presentation

$$0 \longrightarrow \mathcal{O}_X(-aA) \boxtimes \mathcal{O}_{B_Z}(-1) \longrightarrow I_Z(aA) \boxtimes \mathcal{O}_{B_Z} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{6.12}$$

The first map is injective on every fiber: it is multiplication by a nonzero equation of C into the torsion-free sheaf $I_Z(aA)$. Both left-hand terms are flat over B_Z, so the fiberwise injectivity criterion proves flatness of the quotient. Kodaira vanishing and (6.8) give

$$\dim B_Z \geq h^0(\mathcal{O}_X(kA)) - a - 1 = \frac{k(k+1)}{2} - a = 2a^2 \geq 2.$$

Hence B_Z contains a line. Taking determinants in (6.12) gives

$$\Theta|_{B_Z} = \mathcal{O}\left(-1 - \frac{a(a-1)}{2}\right) = \mathcal{O}(-\nu_k), \qquad P|_{B_Z} = \mathcal{O}(1).$$

In both parity cases the moduli map on a parameter line is nonconstant: its composition with h is the original inclusion into $|kA|$. The degree of $\Theta^r P^n$ on that line is exactly

$$n - r\nu_k. \tag{6.13}$$

A negative degree excludes nefness, and a zero degree on this nonconstant curve excludes ampleness. Together with (6.7), this proves Theorem 1.2. □

Remark 6.2. For $k = 1, 2, 3, 4, 5, 6, 7, 8$ the thresholds ν_k are respectively $1, 1, 2, 2, 4, 4, 7, 7$. For odd k the value is $1 + (k^2 - 1)/8$, agreeing with the intrinsic sufficient bound in Theorem 5.1. For even k the lattice condition improves it to $1 + k(k-2)/8$.

A Exact calculation certificates

All computations in this appendix use integer or rational arithmetic.

A.1 The window polynomial

For $\alpha = (n, n, k)$ and $\beta = (x, y, z)$ the Euler expression is

$$C = x(n-x) + y(n-y) + z(k-z) - 3x(n-y) - 3y(k-z) + 3x(k-z).$$

If $y = x$, direct expansion gives

$$\begin{aligned} C - (kx - nz) &= x^2 - z^2 - (n+k)x + (n+k)z \\ &= -(x-z)(n+k-x-z). \end{aligned}$$

If $y = x - h$, the complete polynomial is

$$C = -h^2 + 3hk - hn - hx - 3hz + kz - nx + x^2 - z^2.$$

Put $D = k(n-1)h + kx - nz$ and $F = C - D$. Then

$$\begin{aligned} F &= x^2 - (h+k+n)x - h^2 - hkn + 4hk - hn \\ &\qquad - 3hz + kz + nz - z^2, \\ F(h,h,z) &= -h^2 - hkn + 3hk - 2hn - 3hz + kz + nz - z^2, \\ F(n,h,z) &= -h^2 - hkn + 4hk - 2hn - 3hz - kn + kz + nz - z^2, \\ F(1,1,z) &= -kn + kz + 3k + nz - 2n - z^2 - 3z - 1. \end{aligned}$$

Subtracting these expressions gives

$$\begin{aligned} F(n,h,z) - F(h,h,z) &= k(h-n), \\ F(h,h,z) - F(1,1,z) &= -(h-1)\{h + k(n-3) + 2n + 3z + 1\}. \end{aligned}$$

Finally, $n = s + k$ and $z = k - t$ give

$$F(1,1,k-t) = -2n - 1 + t(3 - s - t).$$

For the discriminant computation put $u = k/s > 0$. The endpoint expansion is

$$\begin{aligned} &\frac{(k+(k+1)u)^2 + 3(k+(k+1)u)}{2} - u \\ &\quad = \frac{k(k+3)}{2} + \frac{2k^2+5k+1}{2}u + \frac{(k+1)^2}{2}u^2. \end{aligned}$$

The dimension balance is the identity

$$(b + qk)n - bn - (qn)k = 0.$$

These are polynomial identities in the indicated symbols. Their use as inequalities, including the integrality of t, is proved in the main text.

A.2 A nonzero example with two positive ranks

Take $(s, n, r, d) = (2, 3, 2, 5)$. Then

$$k = 1, \quad q = 1, \quad \tau = 1, \quad (a, b, c) = (4, 3, 3), \quad d = 5 > 4 = qk(n+1).$$

On the relation side choose

$$X_0 = I_3, \quad X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad X_2 = 0, \quad Y_0 = (1\ 1\ 1), \quad Y_1 = (0\ 1\ 2), \quad Y_2 = 0.$$

All three relations vanish: $Y_0X_1 = Y_1X_0 = (0\ 1\ 2)$, $Y_0X_2 = Y_2X_0 = 0$, and $Y_1X_2 = Y_2X_1 = 0$. The nonzero quadratic matrices are

$$Z_{00} = (1\ 1\ 1), \qquad Z_{01} = (0\ 1\ 2), \qquad Z_{11} = (0\ 1\ 4).$$

For the Gaeta coefficients take

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad M_1 = M_2 = 0,$$

and

$$Q_{00} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with the other Q_{ij} zero. The coefficients use the monomials x_ix_j for $i \le j$, each taken once. The universal square matrix is

$$\begin{pmatrix} M_0 \otimes X_0 \\ Q_{00} \otimes Z_{00} + Q_{01} \otimes Z_{01} + Q_{11} \otimes Z_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Its determinant is

$$\det(I_9) \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} = 1(4 - 2) = 2.$$

Nonvanishing of this determinant and density of the stable Gaeta open give a stable presentation where it is nonzero. The section comparison identifies the resulting theta section.

A.3 The degree-one parity calculation

Write $k = 2a + 1$ or $k = 2a$. Every quantity used for the boundary and its test curve is displayed in the following table:

k	R_k	$\frac{7}{8} + \frac{R_k^2}{2}$	χ(left surface term)
$2a+1$	$a + \frac{1}{2}$	$1 + \frac{a(a+1)}{2}$	$\chi(\mathcal{O}_X(-(a+1)A)) = 1 + \frac{a(a+1)}{2}$
$2a$	$a - \frac{1}{2}$	$1 + \frac{a(a-1)}{2}$	$\chi(\mathcal{O}_X(-aA)) = 1 + \frac{a(a-1)}{2}$

For the even family, $k(k+1)/2 - a = (2a)(2a+1)/2 - a = 2a^2$. For both families, $\deg \Theta = -\nu_k$, $\deg P = 1$, and hence $\deg(\Theta^r P^n) = n - r\nu_k$.

A.4 The Euler change of the determinant

Take $(s, n, r, d) = (2, 3, 2, 4)$ and put

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad Y = (1\ 1\ 1), \qquad YP = (0\ 1\ 2).$$

The old relation matrices are

$$(X_0, X_1, X_2) = (I, P, 0), \qquad (Y_0, Y_1, Y_2) = (Y, YP, 0).$$

The old presentation has $(a, b, c) = (3, 2, 3)$, with

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_1 = M_2 = 0, \qquad Q_{00} = E_{13}, \quad Q_{01} = E_{23}, \quad Q_{11} = E_{33},$$

and $Q_{02} = Q_{12} = Q_{22} = 0$. Here E_{ij} denotes the 3×3 matrix with entry 1 in position (i, j) and zero elsewhere. These are the coefficients of the 9×9 example in Section 4.6; they give

$$D_{\text{old}} = I_6 \oplus \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix}, \qquad \det D_{\text{old}} = 2.$$

For the Euler sequence use

$$i = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, \qquad q_0 = (0, -x_2, x_1), \quad q_1 = (x_2, 0, -x_0), \quad q_2 = (-x_1, x_0, 0).$$

Each $q_j i$ is zero, so q_j descends to a map $\Omega^1(1) \to \mathcal{O}$, denoted by the same symbol. The composition matrix is

$$(q_i p_j)_{0 \le i,j \le 2} = \begin{pmatrix} 0 & -x_2 & x_1 \\ x_2 & 0 & -x_0 \\ -x_1 & x_0 & 0 \end{pmatrix},$$

where p_j is the image of the jth standard summand under the Euler quotient map. On the three-dimensional old cohomology spaces the Euler inclusion and quotient are

$$i_{\text{coh}} = \begin{pmatrix} I \\ P \\ 0 \end{pmatrix}, \qquad \pi_{\text{coh}} = \begin{pmatrix} -P & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Their product is zero and their ranks are 3 and 6. The new source is the old $E_1 = H^1(F(-1))$; $X_0 = I$ identifies its chosen basis with that of the old E_0. The quotient coordinates are $(v_1 - Pv_0, v_2)$ for $(v_0, v_1, v_2) \in E_1^{\oplus 3}$. In the quotient basis determined by π_{coh}, the new matrices are

$$X_0' = \begin{pmatrix} -P \\ 0 \end{pmatrix}, \quad X_1' = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad X_2' = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

$$Y_0' = (0 \ \ YP), \quad Y_1' = (0 \ \ -Y), \quad Y_2' = (Y \ \ 0).$$

For example $Y_0'\pi_{\mathrm{coh}} = (0\ 0\ YP)$, $Y_1'\pi_{\mathrm{coh}} = (0\ 0\ -Y)$, and $Y_2'\pi_{\mathrm{coh}} = (-YP\ Y\ 0)$, which are the three maps induced by q_0, q_1, q_2. All six relations are

$$\begin{aligned} Y_0'X_0' &= 0, & Y_1'X_1' &= 0, & Y_2'X_2' &= 0,\\ Y_0'X_1' + Y_1'X_0' &= 0 + 0 = 0,\\ Y_0'X_2' + Y_2'X_0' &= YP - YP = 0,\\ Y_1'X_2' + Y_2'X_1' &= -Y + Y = 0. \end{aligned}$$

Consequently

$$Z_{01}' = 0, \qquad Z_{02}' = YP, \qquad Z_{12}' = -Y.$$

Let (a_1, a_2, a_3) be the old source basis and (w_0, w_1, w_2) the Euler coefficient basis. Order $A \otimes W$ by the three w-blocks, with a_1, a_2, a_3 inside each block. The coefficient map is $m = (M_0\ 0\ 0)$, and its kernel has ordered basis

$$(a_3w_0, a_1w_1, a_2w_1, a_3w_1, a_1w_2, a_2w_2, a_3w_2).$$

For the kernel inclusion the three 3×7 blocks are

$$M_0' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_1' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$M_2' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Choose the lift

$$L = (x_0Q_{00} + x_1Q_{01} \quad x_1Q_{11} \quad 0).$$

Set $i_A = \begin{pmatrix} x_0I\\ x_1I\\ x_2I \end{pmatrix}$. Then

$$Li_A = x_0^2Q_{00} + x_0x_1Q_{01} + x_1^2Q_{11} = Q.$$

Restricting $-L$ to the displayed kernel and using the three skew labels $(-x_2, x_1, -x_0)$ gives

$$N_{01}' = 0_{3\times 7}, \qquad N_{02}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \qquad N_{12}' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Order the source tensor basis first by the seven displayed kernel vectors and then by the old three-dimensional cohomology basis. Order the target first by a_1, a_2, a_3, each followed by the six quotient coordinates, and then by the three C coordinates. The actual tensor matrix is

$$D_{\mathrm{new}} = \begin{pmatrix} \sum_{j=0}^{2} M_j' \otimes X_j'\\ N_{01}' \otimes Z_{01}' + N_{02}' \otimes Z_{02}' + N_{12}' \otimes Z_{12}' \end{pmatrix}.$$

$$D_{\text{new}} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The seven column blocks have width 3. The first eighteen rows are the A-coordinates and the last three are the C-coordinates.

Permute the columns into the order

$$(4, 5, 6, 13, 14, 15, 7, 8, 9, 16, 17, 18, 10, 11, 12, 19, 20, 21, 1, 2, 3).$$

At the level of the seven width-3 blocks this is $(2, 5, 3, 6, 4, 7, 1)$, which has 9 inversions; hence the permutation of individual columns has $9 \cdot 3^2 = 81$ inversions and sign -1. Next perform $R_{21} \leftarrow R_{21} + R_{14} + 2R_{15}$. The upper-left block is I_{18}, the lower-left block is zero, and the lower-right block is

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & -1 & -4 \end{pmatrix},$$

whose determinant is $(-1)(4-2) = -2$. Column permutation changes the original determinant by -1, while the row operation preserves it. Therefore $\det D_{\text{new}} = 2 = \det D_{\text{old}}$.

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