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Cases of Denef's holomorphy conjecture

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Abstract

We prove Denef's holomorphy conjecture for isolated surface singularities given by two ordinary homogeneous terms, or by two terms homogeneous for common positive weights with coprime degrees and ordinary multiplicity at least ten. We also prove residual holomorphy for $A(z)g(x,y) + B(z)h(x,y)$, holomorphy on zero-centred parameter balls for Cayley families of reduced surfaces with sufficiently large exponents, vanishing for larger Plücker castlings, and equality of origin integrals under sufficiently high order perturbations.

1 Introduction

Let $\Bbbk \subset \mathbb{C}$ be a number field, let K be a finite completion, and let $\mathcal{O}_K$ be its valuation ring, with maximal ideal $\mathfrak{p}$ and residue-field cardinality q. For a nonzero polynomial $f \in \Bbbk[x_1, \dots, x_n]$ and a continuous finite-order character $\chi : \mathcal{O}_K^\times \to \mathbb{C}^\times$, Igusa's local zeta function is

$$Z_{f,\Phi}(\chi, s) = \int_{\mathcal{O}_K^n} \Phi(x)\chi(\operatorname{ac} f(x))|f(x)|^s \, dx.$$

Here ac is the angular component, Haar measure gives $\mathcal{O}_K^n$ volume one, and Φ is supported on $\mathcal{O}_K^n$ and constant on residue balls. The trivial-character integral records the numbers of solutions of congruences $f(x) \equiv 0$ modulo powers of the maximal ideal. Fourier analysis relates the character integrals to exponential sums. Igusa proved their rationality in q^{-s} by resolution of singularities [16, 17]; Denef's Bourbaki report [10] describes these arithmetic applications and the conjectural relation between poles and complex monodromy.

1.1 The conjecture and earlier results

Denef's holomorphy conjecture [10, Conjecture 4.4.2] asserts that, outside finitely many places, $Z_{f,\Phi}(\chi, s)$ is entire if the order of χ divides no order of a monodromy eigenvalue at any complex point of $f^{-1}(0)$. The monodromy acts on the cohomology of the local Milnor fibre, including its unreduced zeroth cohomology. The conjecture applies to every residual test function Φ. It therefore requires control of complete individual residue balls, as well as the standard integral over $\mathcal{O}_K^n$.

To see the role of the character, let (N_i, ν_i) be the multiplicity and Jacobian data of an embedded resolution. At good places a tame character of order d permits a normal factor

$$(q-1)\frac{q^{-\nu_i}T^{N_i}}{1 - q^{-\nu_i}T^{N_i}}, \qquad T = q^{-s},$$

only when $d \mid N_i$. Its coefficient is a finite-field Kummer trace sum on the corresponding stratum. A'Campo's formula [1] expresses the monodromy determinant through the same multiplicities and the Euler characteristics of strata. Holomorphy asks for cancellation of the eligible arithmetic terms when the indicated eigenvalue orders are absent. The proof must account for their trace coefficients and their Jacobian powers.

Two results of Denef are particularly useful here. Theorem 3.3 of [10] makes the residual integral independent of s for characters nontrivial on the principal units, under tame good reduction. Thus higher conductors admit a classical reduction. Theorem 1.2 of [11] gives a strictly negative rational degree in T for the origin integral when a tame character's exact order is absent from the central monodromy. We use this degree statement after the relevant geometric terms have been computed.

Veys proved holomorphy for plane curves, including residual test functions [29, Theorem 3.1 and §3.3]. His proof uses the multiplicity relations on a plane resolution graph and the cancellation attached to its bivalent components. Denef and Veys then proved suspension closure for exponent at least three, obtained the corresponding plane suspensions and diagonal polynomials, and identified a quadratic case under a restriction on the base spectrum [13, Theorem 3.1, Remark 3.3 and Corollaries 3.4–3.5]. Rodrigues and Veys established a projective reduction for homogeneous polynomials and deduced holomorphy for every homogeneous polynomial in three variables [28, Theorems 3.5 and 3.7]. Their three-variable result permits arbitrary component multiplicities.

Denef and Loeser introduced the topological zeta function in [12]. The twisted topological holomorphy problem has related resolution-theoretic developments: Veys treated functions on normal surface germs [30]; Lemahieu and Veys treated surfaces general for a toric idealistic cluster [20, Theorem 24]; Lemahieu and Van Proeyen proved the version for ideals in the smooth ambient plane [19, Theorem 10]. These results concern Euler-characteristic zeta functions. For character Igusa integrals, Castryck, Ibadula and Lemahieu proved the conjecture for Newton nondegenerate surface singularities [9, §5.3, Theorem 15]. That arithmetic theorem covers the Newton nondegenerate members of several families considered below.

Weighted principal parts have also been studied through explicit resolutions. Zúñiga-Galindo computed local zeta functions for semiquasihomogeneous polynomials [31]. Blanco, Budur and van der Veer proved strong monodromy results for semiquasihomogeneous hypersurfaces [5]. Nguyen and Yasuda give twisted Igusa formulas using quotient resolutions; their weighted applications include polynomials whose lowest weighted homogeneous part has an isolated critical point [26, Main Theorem 3 and Theorem 5.3]. The weighted theorem in this paper allows a nonisolated and nonreduced lowest weighted homogeneous part.

Artal Bartolo, González Pérez, González Villa and León Cardenal give suspension formulas and prove twisted topological holomorphy for plane quadratic suspensions and Lê–Yomdin singularities [2, Theorems 6.20 and 9.5]. The Lê–Yomdin condition for G_m+H_{m+k} requires the singular locus of the projective tangent cone to avoid $H_{m+k}=0$. Their Lemmas 6.15 and 6.17 contain the quadratic exceptional-spectrum classification and the associated resolution-graph configurations. We retain proofs of the geometric statements needed here and calculate the corresponding Kummer, Jacobi and Gauss coefficients over the original residue field.

1.2 Two-term surfaces and plane families

Theorem 3.1 concerns

$$F = G_m + H_a \in \Bbbk[x_1, x_2, x_3], \qquad 1 \le m < a,$$

where the two nonzero terms are ordinary homogeneous and the germ at zero is smooth or has an isolated critical point. The tangent cone may have arbitrary component multiplicities, and the second term may have arbitrary contact with its reduced support. Exclusion of the specified eigenvalue orders at zero alone implies that the entire origin residue-ball integral is zero. The nonzero critical points of the zero fibre have explicit transverse-product or suspension models. These models give the residual holomorphy assertion under the all-point spectral hypothesis (Corollary 5.2).

Theorem 4.1 treats two homogeneous terms for a common primitive system of positive weights, with coprime weighted degrees $m < a$, ordinary multiplicity at least ten, and an isolated critical point at zero. It gives the same central vanishing and residual conclusion in this range. The proof follows the characters of the finite quotient charts, identifies the relevant Milnor cohomology, and writes the integral in the invariant valuation lattice. The example

$$(x^2 + y^3 + z^6)^5 + xy^{14} + z^{31}$$

has weights $(3, 2, 1)$, degrees $30, 31$, ordinary multiplicity ten, and an isolated critical point. Its lowest weighted part is a fifth power, and its displayed Newton face is degenerate. The full verification is given in Section 4. The ordinary theorem permits every pair of distinct degrees and every positive multiplicity.

Theorem 10.3 proves the residual conjecture for nonzero polynomials

$$F = A(z)g(x, y) + B(z)h(x, y), \qquad A, B \in \Bbbk[z], \quad g, h \in \Bbbk[x, y].$$

The polynomials may have common factors and arbitrary multiplicities. Writing $F = \sum_j z^j f_j(x, y)$, this includes every polynomial for which the f_j span a vector space of dimension at most two over $\Bbbk$. The family $H(y, z)^m(x^k + g(y, z))$, $m \geq 1$, $k \geq 2$, is an explicit application; Corollary 11.1 also supplies its direct calculation. Theorem 9.1 treats $f_0(x, y) + \sum_j u_j f_j(x, y)$ with any finite number of linear parameters, including parameters in a fixed nonempty algebraic open set. A common plane principalization supplies the spectral and arithmetic calculation. The quadratic case uses the full residue-field trace of the exceptional configurations described above.

1.3 Cayley families, castling and perturbations

Let $Z \subset \mathbb{A}^n_{\Bbbk}$ be a nonempty reduced closed subscheme of pure dimension two, let $P_1, \dots, P_r$ generate its defining ideal, and let $R \in \Bbbk[x_1, \dots, x_n]$ be nonzero on each component and vanish on $\operatorname{Sing} Z$. For a fixed integer $h \geq 0$ and all sufficiently large M, Theorem 12.1 proves holomorphy for

$$F_M(x, \lambda) = \sum_{j=1}^{r} \lambda_j P_j(x) + R(x)^M$$

on $\mathcal{O}_K^n \times \mathfrak{p}^h \mathcal{O}_K^r$, under the all-point spectral hypothesis for F_M. Its positive-contact contribution over the specified base residue sets is zero. In particular, the theorem includes the standard and origin integrals with their stated choices of h and M. The source can be nonnormal, non-Gorenstein and not a local complete intersection, and the ideal generators may be redundant. The ambient dimension can grow with the source and its presentation.

The Cayley proof uses strong factorization [7, Theorem 7.3], a spectral comparison with the kernel bundle on a resolution of Z, and the normal-surface graph relations of [30]. An exact parameter integration retains the character conductor. The resulting kernel integral carries the

induced integral lattice and the pullback of its canonical rational form. These data determine the arithmetic coefficients on the eligible curves.

For integers $1 \le r < m$, let f be a nonzero homogeneous regular function of degree $\delta > 0$ on the cone of decomposable vectors in $\bigwedge^r \Bbbk^m$, and put $F_r(X) = f(x_1 \wedge \cdots \wedge x_r)$ for an m-by-r matrix X with columns $x_1, \dots, x_r$. When $m/2 < r < m$, Theorem 13.1 produces a rank-$r-1$ point whose local spectrum contains every δ-th root of unity. Under the all-point spectral hypothesis, column scaling proves vanishing of the standard and origin integrals, and of every rank-deficient residue ball, at every finite place and for every conductor. The incidence injection used here is a case of Matsui's Grassmannian Radon theorem [22]. The Haar factor belongs to the castling theory originating with Igusa [18] and developed motivically by Loeser [21]. We give its full matrix-lattice calculation and apply it also to the projective divisors in Theorem 13.6.

Finally, Theorem 14.1 gives an integral form of finite right determinacy. The algebraic finite-determinacy method is classical; see [6], and [14] for convergent power series over valued fields. For a polynomial f with $f(0) = 0$ and an isolated critical point at zero, write $\mathfrak{m} = (x_1, \dots, x_n)$ and $J_f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$, and start from certificates for $\mathfrak{m}^b \subset J_f$ in $\Bbbk[x_1, \dots, x_n]_{\mathfrak{m}}$. We choose one finite set of places such that every integral perturbation of order at least $2b+1$ has exactly the same origin character integrals as f. The coordinate transformation and its inverse converge on the whole origin ball and have absolute Jacobian one. The appendix gives the arithmetic identities, all finite computation ranges and complete verification programs; Example 14.3 lists all thirty-six Jacobian certificates for $x^2 + y^3 + z^7$.

2 Characters, nearby cycles and local integrals

Let $\Bbbk \subset \mathbb{C}$ be a number field and let v be a finite place. Write $K = \Bbbk_v$, $\mathcal{O} = \mathcal{O}_K$, $\mathfrak{p} = \pi\mathcal{O}$, $q = \#(\mathcal{O}/\mathfrak{p})$, and $p = \mathrm{char}(\mathcal{O}/\mathfrak{p})$. The valuation satisfies $\mathrm{ord}(\pi) = 1$, and $|x| = q^{-\operatorname{ord}(x)}$. A bar denotes reduction modulo $\mathfrak{p}$. We normalize additive Haar measure by $\mathrm{vol}(\mathcal{O}^n) = 1$ and set $\mathrm{ac}(x) = \pi^{-\operatorname{ord}(x)}x$ for $x \neq 0$. A character always means a continuous finite-image character $\chi : \mathcal{O}^\times \to \mathbb{C}^\times$; we extend it by zero at zero, including when the character is trivial. Its conductor is

$$a(\chi) = \min\{a \ge 1 : \chi(1 + \mathfrak{p}^a) = 1\}.$$

A nontrivial character is tame when $a(\chi) = 1$ and wild otherwise. For a polynomial $f \neq 0$ and a residual function Φ, that is, a function supported on $\mathcal{O}^n$ and constant on each residue ball, put

$$Z_{f,\Phi}(\chi, s) = \int_{\mathcal{O}^n} \Phi(x)\chi(\mathrm{ac}\, f(x))|f(x)|^s \, dx, \qquad T = q^{-s}.$$

We write $Z_{f,0}$ for the integral over $\mathfrak{p}^n$ and Z_f for the integral over $\mathcal{O}^n$. All changes of variables are first made for $\Re(s)$ sufficiently large, where they are justified by absolute convergence. The resulting equalities extend as rational functions of T.

For a point x of the complex zero fibre, let $M_{f,x}$ be a local Milnor fibre and let $T_{f,x}$ denote the automorphism of its cohomology induced by counterclockwise continuation around zero. On a singular variety we use the stratified local Milnor fibration. We write M_f, T_f when the point is understood. Milnor cohomology is written over $\mathbb{C}$ after extension of scalars from $\mathbb{Q}$; a tilde denotes reduced cohomology. A monodromy subscript θ denotes the generalized θ-eigensummand. For an algebraic stratum S, put $\chi_{\mathrm{top}}(S) = \sum_j (-1)^j \dim H^j_c(S, \mathbb{C})$. For a complex C with finite-dimensional cohomology, put $\chi_{\mathrm{top}}(C) = \sum_j (-1)^j \dim H^j(C)$. The functors ψ_f and ϕ_f denote unshifted nearby and vanishing cycles. For a pure n-dimensional variety X, IC_X is the

perverse-normalized rational intersection complex, whose restriction to the smooth locus is $\mathbb{Q}[n]$. Eigenvalue decompositions use extension of scalars to $\mathbb{C}$.

Definition 2.1. For $x \in f^{-1}(0)(\mathbb{C})$, the condition $\mathrm{Sat}_d(f,x)$ means that no eigenvalue of monodromy on any $H^j(M_{f,x},\mathbb{C})$ has order divisible by d. The condition $\mathrm{Sat}_d(f)$ means $\mathrm{Sat}_d(f,x)$ at every such point. These conditions include unreduced H^0 and exclude every eigenvalue whose order is a multiple of d.

Denef's conjecture, in its published residual formulation, asserts that $\mathrm{Sat}_{\mathrm{ord}\,\chi}(f)$ implies that $Z_{f,\Phi}(\chi,s)$ is entire for all residual Φ, outside a finite set of places depending on f [10, §4.4.2]. The order $d=1$ never satisfies the spectral condition at a nonempty zero fibre, and all substantive assertions below concern $d>1$.

We use the following reciprocal monodromy conventions:

$$\mathcal{Z}_{f,x}(t)=\prod_j \det(1-tT_{f,x}\mid H^j(M_{f,x},\mathbb{C}))^{(-1)^j},\qquad \zeta_{f,x}(t)=\mathcal{Z}_{f,x}(t)^{-1}. \tag{2.1}$$

The convention $\mathcal{Z}$ is that of Gusein-Zade–Luengo–Melle-Hernández. The same alternating determinants define $\mathcal{Z}$ for a finite cohomology complex with monodromy. A nonzero exponent of a root in either convention detects an eigenvalue. Individual eigencohomology groups will be identified when a cohomological vanishing statement is required.

2.1 The foundational comparison results

We use resolution of singularities in characteristic zero, the change-of-variables formula for a good integral resolution, proper base change for nearby cycles, perversity of shifted nearby cycles on a smooth space, and comparison and trace theorems for finite-order Kummer sheaves. The following are the precise numerical consequences needed here.

If $\rho: Y\to\mathbb{A}^n$ resolves $f^{-1}(0)$, write

$$f\rho=u\prod_{i=1}^{b} y_i^{N_i},\qquad \rho^*(dx_1\wedge\cdots\wedge dx_n)=\epsilon\prod_{i=1}^{b} y_i^{\nu_i-1}dy_1\wedge\cdots\wedge dy_n,$$

where u,ϵ are units. After deletion of finitely many places these are good integral coordinates. For a tame character the integral is a finite sum of actual residue-field character coefficients times products of

$$(q-1)\frac{q^{-\nu_i}T^{N_i}}{1-q^{-\nu_i}T^{N_i}}. \tag{2.2}$$

The common factor q^{-n} is included if the residue-field coefficients are unnormalized sums. A stratum can contribute only if $\chi^{N_i}=1$ for every normal divisor through it. This is Denef's good-reduction formula [10, §§3.4–3.6]; its residue coefficients are the indicated Kummer trace sums.

We also use the strict degree theorem [11, Theorem 1.2]: for almost all v, if $f(0)=0$, χ is tame, and monodromy at 0 has no eigenvalue of exact order $\mathrm{ord}\,\chi$, then

$$\deg_T Z_{f,0}(\chi,s)<0,\qquad \deg_T 0=-\infty. \tag{2.3}$$

The degree is numerator degree minus denominator degree after cancellation. The theorem applies to f on the complete origin residue ball.

A'Campo's formula, or its proper partial-resolution form [15, Theorem 1 and Corollary 1], will be used as follows. If ρ is proper and is an isomorphism away from $f^{-1}(0)$, then

$$\mathcal{Z}_{f,x}(t) = \prod_{S \subset \rho^{-1}(x)} \mathcal{Z}_{f\rho,S}(t)^{\chi_{\mathrm{top}}(S)}. \tag{2.4}$$

Here the strata are chosen so that the stalk monodromy is constant. One obtains the formula by proper nearby-cycle base change and Euler integration of the alternating determinant. For a local system L on a stratum S, $\chi_c(S, L) = \mathrm{rank}(L)\chi_c(S)$, which explains why the Euler product is valid for nonconstant local systems along the strata.

2.2 Three elementary arithmetic observations

Lemma 2.2 (A two-puncture coefficient)**.** *Let C be a smooth proper geometrically rational curve, and let C° be obtained by removing two geometric points. A rank-one finite-order tame local system whose inertia at a removed point is nontrivial has $R\Gamma_c(C^\circ, L) = 0$. This remains true when the two points are conjugate over the residue field. Consequently its actual Frobenius trace sum is zero.*

Proof. Over $\mathbb{C}$, identify C° with $\mathbb{C}^\times$. Its cellular cochain complex is $[\mathbb{C} \xrightarrow{\alpha-1} \mathbb{C}]$, where $\alpha \neq 1$ is the monodromy of L. This complex is acyclic. Duality gives the same assertion for compact supports.

Over an algebraic closure of a finite residue field, identify C° with $\mathbb{G}_m$. Here cohomology is *étale* cohomology with $\overline{\mathbb{Q}}_\ell$ coefficients for a prime $\ell \neq p$, and finite-order characters are identified with their corresponding roots of unity. The geometric tame character is a nontrivial character of a Kummer cover $t \mapsto t^N$, with N prime to the residue characteristic. Each deck transformation extends to a degree-one automorphism of $\mathbb{P}^1$ fixing 0 and ∞. It acts trivially on $H^*(\mathbb{P}^1)$ and on $H^*(\{0, \infty\})$. The compact-support exact sequence therefore shows that it acts trivially on $H^1_c(\mathbb{G}_m)$ and $H^2_c(\mathbb{G}_m)$; the other groups vanish. The nontrivial character summand of the pushforward of the cover has zero compactly supported cohomology. This is $R\Gamma_c(C^\circ, L)$. Its Frobenius trace is zero, including when the two punctures are conjugate. □

Lemma 2.3 (Higher conductors)**.** *Fix a polynomial f and a good resolution as above. After excluding the primes dividing its positive multiplicities N_i, every wild character has zero integral on each original residue ball on which $\bar{f} = 0$.*

Proof. This is the positive-contact case of [10, Theorem 3.3]. We give the coordinate calculation. On a residue chart above such a ball at least one normal coordinate, say y_i, belongs to $\mathfrak{p}$ and has $N_i > 0$. Choose an integral lift a of the chart's residue point, and fix the other coordinates. The principal unit $u(y)/u(a)$ has an N_i-th root, since N_i is a unit in $\mathcal{O}$. Replace y_i by $y_i(u(y)/u(a))^{1/N_i}$. This map and its inverse are integral analytic maps of the same ball; its derivative is 1 modulo $\mathfrak{p}$. It preserves valuations and Haar measure. The remaining integral over the normal unit contains χ^{N_i} on $1 + \mathfrak{p}$, which is nontrivial: raising to the prime-to-p power N_i preserves a nontrivial character of a finite p-group. Character orthogonality therefore makes this integral zero. The residual weight is constant on the whole source ball and factors out of this calculation. □

Lemma 2.4 (Entire positive-contact integrals)**.** *At a good place, a tame integral on a complete original zero residue ball is zero if it is entire in s.*

Proof. Formula (2.2) has degree at most zero in T. The integral is regular at $T = 0$ and has zero constant coefficient, since every point of the ball has positive contact with $f = 0$. Entirety in s excludes every finite nonzero pole in T, because q^{-s} takes every nonzero complex value. The rational function is therefore a polynomial of degree at most zero. Its constant is zero. □

Whenever finitely many auxiliary covers or diagrams occur, the excluded set of places can be chosen independently of χ. Indeed, a nonzero tame positive-contact term on one fixed resolution of the original polynomial requires $d \mid N_i$ for at least one of its finitely many positive multiplicities. There are only finitely many such orders d. One spreads the corresponding finite collection of auxiliary data once and takes the union of their finite exceptional sets. Higher conductors are handled by Lemma 2.3 on this fixed resolution.

3 Two ordinary homogeneous terms

Theorem 3.1. *Let $G_m, H_a \in \Bbbk[x_1, x_2, x_3]$ be nonzero ordinary homogeneous polynomials of degrees $1 \le m < a$, and put $f = G_m + H_a$. Suppose that the germ of f at the origin is smooth or has an isolated critical point. There is a finite set of places, depending only on f, such that*

$$\mathrm{Sat}_d(f, 0), \quad d = \operatorname{ord} \chi > 1 \quad \Longrightarrow \quad Z_{f,0}(\chi, s) = 0$$

at every remaining place, for characters of every conductor. The tangent cone may have arbitrary component multiplicities and arbitrary contact with $H_a = 0$.

Throughout this section set $k_0 = a - m > 0$. All monodromy products in the proof use $\mathcal{Z}$, the positive convention in (2.1). We first work over $\mathbb{C}$.

If f is smooth at the origin, the assertion follows by using f as a Hensel coordinate. We may therefore assume $m \ge 2$. If the two terms have a common factor, isolation forces $G_m \mid H_a$. Indeed, a common factor of degree strictly smaller than m writes f as a product of two nonunits, whose intersection is a curve of critical points. In the divisible case $f = G_m(1 + Q_{k_0})$, and isolation implies that the projective curve of G_m is reduced and smooth. We return to this case at the end. Until then G_m, H_a are coprime. Factor geometrically

$$G_m = c \prod_{i=1}^{b} g_i^{r_i}, \qquad e_i = \deg g_i, \qquad \sum_i r_i e_i = m, \tag{3.1}$$

where the g_i are distinct irreducible equations. Put $C_i = (g_i = 0) \subset \mathbb{P}^2$, $C = \bigcup_i C_i$, and $D = (H_a = 0)$. The letter D in this section denotes this projective divisor only.

3.1 The transverse eigenlines

Lemma 3.2. *For $u, v \in \mathbb{C}^\times$ and $r \ge 1$, consider $P(x, z) = z^m(ux^r + vz^{k_0})$. Its Milnor fibre is connected, and*

$$\det(1 - tT_P \mid H^1(M_P)) = (1 - t)B_r(t), \qquad B_r(t) = \frac{(1 - t^{ar/\gcd(r,k_0)})^{\gcd(r,k_0)}}{1 - t^a}. \tag{3.2}$$

Apart from the deck-trivial eigenvalue 1, the simultaneous eigenlines are indexed by

$$\lambda^r = 1, \quad \lambda \ne 1, \quad \theta^a = \lambda^{k_0}, \qquad \mu = \theta/\lambda. \tag{3.3}$$

Each such pair has multiplicity one. As u, v vary, its parameter monodromies are λ^{-1} and μ^{-1}.

Proof. The polynomial is positively weighted homogeneous. Its local Milnor fibre is represented by the global nonzero fibre. On $P = \epsilon$, $z \neq 0$ and

$$x^r = u^{-1} z^{-m} (\epsilon - v z^a).$$

Projection to the punctured z-line is a cyclic cover of degree r, branched at the a simple roots of $\epsilon - v z^a$. The strict branches of $ux^r + vz^{k_0}$ have multiplicity one, so the greatest common divisor of the multiplicities of the complete zero divisor is one. The fibre is connected.

Let $A(x, z) = (e^{2\pi i/r} x, z)$ be the deck transformation and choose the weighted monodromy

$$T_P(x, z) = (e^{2\pi i k_0/(ar)} x, e^{2\pi i/a} z).$$

Then $T_P^a = A^{k_0}$. For a nontrivial deck eigenvalue λ, the direct image on the punctured z-line is a rank-one system, extended by zero at its a branch points. It has Euler characteristic $-a$, and has neither H^0 nor H^2. Thus its H^1 has dimension a. For $1 \leq j < a$, every $T_P^j A^l$ has no fixed point, since it rotates the nonzero z coordinate. The finite-order Lefschetz formula and the deck projector show that the trace of T_P^j on this H^1 block is zero. Together with $T_P^a = \lambda^{k_0}$, this proves that each root of $\theta^a = \lambda^{k_0}$ occurs once. The deck-trivial part is $H^1(\mathbb{C}^\times)$, with monodromy one. Multiplication over the nontrivial r-th roots gives (3.2).

When u goes once around zero, a trivialization multiplies x by $e^{-2\pi i/r}$, hence induces A^{-1}. When v goes once around zero, it multiplies (x, z) by $(e^{2\pi i m/(ar)}, e^{-2\pi i/a})$, hence induces $A T_P^{-1}$. Their eigenvalues are precisely λ^{-1} and $\lambda\theta^{-1} = \mu^{-1}$. □

We denote by $\mathcal{L}_\lambda(U)$ the finite-order Kummer line with parameter monodromy λ^{-1}; in its arithmetic realization the geometric-Frobenius trace is the numerical character $\lambda(\bar{U})$. This inverse convention is the one in the multivariate character formula of [27, §6]. On the normalization of the smooth open part of C_i the line in Lemma 3.2 is

$$\mathcal{L}_{i,\lambda,\mu} = \mathcal{L}_\lambda(G_m / g_i^{r_i}) \otimes \mathcal{L}_\mu(H_a). \tag{3.4}$$

It is a global line, although the two displayed functions are written in local projective trivializations. Changing the normal coordinate multiplies the first unit by an r_i-th power, killed by $\lambda^{r_i} = 1$. Changing the radial trivialization multiplies the two units by powers m and a, killed by

$$\lambda^m \mu^a = \theta^a \lambda^{-k_0} = 1. \tag{3.5}$$

At a normalization branch B its boundary inertia is the inverse of $\lambda^{b_B} \mu^{e_B}$, where

$$e_B = I(B, H_a), \qquad b_B = \sum_{j \neq i} r_j I(B, g_j). \tag{3.6}$$

Contributions from other branches of the same g_i have coefficient r_i and are killed by $\lambda^{r_i} = 1$. To check this on the normalization, use local dehomogenizations of g_j and H_a and take a resolved chart meeting the strict transform of B, with exceptional coordinate s and strict branch coordinate t. Their pullbacks have the form

$$g_i = s^\alpha t \epsilon_i, \qquad g_j = s^{\beta_j} \epsilon_j \ (j \neq i), \qquad H_a = s^e \epsilon_H,$$

where all ϵ are units. The coordinate s restricts to a uniformizer on the normalization of B, so $\beta_j = I(B, g_j)$ and $e = I(B, H_a)$. The adjacent exceptional vector is

$$N = r_i \alpha + \sum_{j \neq i} r_j \beta_j, \qquad M = e.$$

Since $\lambda^{r_i} = 1$, its character is $\lambda^N \mu^M = \lambda^{b_B} \mu^{e_B}$. This proves the asserted boundary formula, including a singular branch or several local branches of the same global component. The identity-resolution transverse case is the same calculation without an exceptional coordinate.

3.2 A plane product and its finite norm

We record the plane-resolution fact in the form needed below. A rupture means an exceptional component whose valency, counting all strict-transform arrows, is at least three.

Lemma 3.3. *Let g, h be coprime plane germs, with g critical and h smooth at their common zero. On the minimal embedded resolution of the reduced support of gh, put $N_E = \operatorname{ord}_E g$, $M_E = \operatorname{ord}_E h$ and $\delta_E = N_E - M_E$. Then $N_E > M_E > 0$, and*

$$A(U, V) = \prod_E (1 - U^{N_E} V^{M_E})^{-\chi_{\mathrm{top}}(E^\circ)} \tag{3.7}$$

is a polynomial. If E is a rupture, $r_E = \gcd(N_E, M_E)$, then the factor

$$\Phi_{r_E}(U^{N_E/r_E} V^{M_E/r_E})$$

occurs in this polynomial with positive multiplicity. Here Φ_r is the r-th cyclotomic polynomial. For $G = z^m(g + z^{k_0} h)$ one has

$$\mathcal{Z}_{G,0}(t) = \operatorname{Norm}_{\lambda^{k_0} = t^a} A(\lambda, t/\lambda). \tag{3.8}$$

Consequently this local monodromy product is a polynomial. A rupture E supplies an actual primitive eigenvalue of order

$$L_E = \frac{aN_E - mM_E}{\gcd(k_0, N_E - M_E)}. \tag{3.9}$$

Proof. At the first blowup $N - M = \operatorname{mult}(g) - 1 > 0$. Along the successive centres on the strict transform of the smooth h, a necessary centre still contains a strict branch of g; the difference changes by its strict multiplicity minus one and cannot decrease. At any other centre the new difference is the sum of the differences of the parent exceptional components plus a nonnegative strict g multiplicity. Thus $N > M > 0$ everywhere.

We use the plane-resolution argument in [29, §2] and its vector version in [27, §8]. Fix an irreducible binomial factor of (3.7). Its exceptional vertices have multiplicity vectors in a single primitive direction, with the appropriate divisibility of their scales. Every connected component of these vertices has at least two outgoing edges, strict arrows included. If its sole outgoing edge were exceptional, the coordinatewise principal-divisor relation

$$\sum_{E' \sim E} (N_{E'}, M_{E'}) = (-E^2)(N_E, M_E) \tag{3.10}$$

would force that edge to belong to the same component. If its sole outgoing edge were a strict arrow, it would be the complete graph with only one branch, whereas g, h are coprime and both vanish. For a tree component the sum of $\mathrm{val}(E) - 2$ equals its number of outgoing edges minus two. It is therefore nonnegative. This is the exponent of the chosen binomial, proving polynomiality.

Strict positivity at the primitive factor follows from minimality of the plane resolution. Assign each exceptional leaf to the first rupture met along its dead bamboo. The negative intersection matrix of such a bamboo has determinant $b \geq 2$; the principal-divisor recurrences give its tip

vector as the rupture vector divided by b. A rupture has at most two dead bamboos. If it has two, their determinants are coprime. Along a dead chain the negative self-intersections κ_j are at least two, since a rational (-1)-curve of valency at most two could be removed while preserving normal crossings. Successive determinants satisfy

$$D_{-1} = 1, \quad D_0 = \kappa_0, \quad D_j = \kappa_j D_{j-1} - D_{j-2}.$$

They strictly increase, and elimination in the principal-divisor equations gives the leaf vector as the rupture vector divided by the last determinant. If a prime divided both dead determinants, each chain matrix would have a nonzero kernel modulo that prime, with nonzero entry at its attachment end by the tridiagonal recurrence. Scale these two kernel vectors so their contributions at the common rupture cancel, and set all other entries to zero. This would give a kernel for the complete exceptional intersection matrix. That matrix has determinant one in the integral total-transform basis of point blowups, a contradiction. The grouped factor at a rupture of valency v, vector $s(N', M')$, is thus

$$(1 - W^s)^{v-2} \prod_{b \text{ a dead determinant}} (1 - W^{s/b})^{-1}, \qquad W = U^{N'} V^{M'}.$$

Every cyclotomic exponent of this group is nonnegative except possibly when $v = 3$ and there are two dead bamboos. Such a rupture is the unique initial rupture: at its birth both older directions contain no eventual strict branch, and all branches continue through its remaining direction. Every later rupture lies above that direction and has coordinatewise no smaller multiplicity vector. The possible negative indices at the initial rupture divide $s/(b_1 b_2)$ and are strictly smaller than s. They cannot affect the primitive index at a later rupture with the same primitive direction. The group of the selected rupture itself contributes $v - 2 > 0$ at its primitive index, since $s/b < s$ for every dead tip. Groups in other primitive directions have no common irreducible binomial factor. This proves the required strict positivity.

If the reduced divisor $gh = 0$ already has normal crossings, choose coordinates with $g = x^r$ and $h = y$, where $r \geq 2$, by absorbing the units into the coordinates. The resolution is the identity and $A = 1$. The function $G = z^m(x^r + z^{k_0} y)$ is positively weighted homogeneous with weights $(k_0 + 1, r, r)$ on (x, y, z). On a nonzero fibre $G = \epsilon$, one has $z \neq 0$ and

$$y = z^{-k_0}(\epsilon z^{-m} - x^r).$$

Its fibre is therefore $\mathbb{C} \times \mathbb{C}^\times$, with coordinates (x, z), and represents the local Milnor fibre. Weighted monodromy acts by scalar multiplication, hence trivially on its H^0 and H^1; its higher cohomology is zero. Thus $\mathcal{Z}_{G,0}(t) = (1 - t)/(1 - t) = 1$, which is the empty norm product.

Take the product of the plane modification with the z-line. Its nonisomorphism locus maps into the axis $g = h = 0$, which is contained in $G = 0$. Thus the proper formula (2.4) applies to the original germ; this is exactly the qualification in [15, Remark 3]. Over an exceptional open stratum the normal equation, up to units, is

$$x^M z^m (x^{N-M} + z^{k_0}).$$

The units can be removed by actual complex analytic coordinates. Write the two plane pullbacks as $u_g M_g$ and $u_h M_h$, with u_g, u_h units and M_g, M_h monomials. The substitution $z' = z(u_h/u_g)^{1/k_0}$ writes the germ as a unit B times

$$P = z'^m (M_g + z'^{k_0} M_h).$$

All the normal models occurring here admit positive weights. At an exceptional open stratum choose weights $(k_0, N-M)$ on (x, z'). At an exceptional intersection choose weights k_0, k_0 on its two exceptional coordinates and $(N_1 - M_1) + (N_2 - M_2)$ on z'. At a strict g branch of multiplicity r, take weights $(k_0, k_0, N-M+r)$ on (x, y, z'). At the smooth strict h branch, take weights $(k_0+1, N-M, N-M)$ on (x, y, z'). In each case the two monomials have the same positive degree D_P. For a normal coordinate of weight w_j, put $X'_j = X_j B^{w_j/D_P}$, using local analytic roots, and leave passive tangential coordinates fixed. This map has invertible linear part, and $P(X') = B(X)P(X)$. We now compute with these normal models, suppressing the primes. Set $b = \gcd(k_0, N-M)$ and $L = (aN - mM)/b$. Its positive primitive weights are k_0/b and $(N-M)/b$, and its degree is L. Both coordinates are nonzero on a nonzero fibre. The weighted quotient of this fibre construction is $\mathbb{C}^\times$ with the b roots of $W^b = -1$ deleted, using $W = x^{(N-M)/b} z^{-k_0/b}$. Its Euler characteristic is $-b$. The fibre is a degree-L cover of this quotient; its monodromy is the deck action. Every nonidentity deck power has no fixed point, because the two weights are coprime. The Lefschetz formula therefore gives the normal monodromy product $(1-t^L)^{-b}$.

At an intersection stratum the analogous equation has three coordinates. A circle action in the kernel of its two monomial exponent vectors preserves the function and has no fixed point on a nonzero fibre. The radial coordinate and at least one exceptional normal coordinate are nonzero there. This action commutes with a finite positive weighted monodromy, so each monodromy fixed locus has Euler characteristic zero. Its monodromy product is one. The same calculation covers a crossing with a strict g or h branch; in the latter case, for $z^m x^M (x^{N-M} + z^{k_0} y)$, one may use circle weights $(m, aN - mM, -N)$ on (x, y, z). The proper Euler product now gives (3.8), since direct multiplication over the roots of $\lambda^{k_0} = t^a$ gives

$$\operatorname{Norm}(1 - \lambda^{N-M} t^M) = (1-t^L)^b. \tag{3.11}$$

The norm is the multiplication determinant in the finite free algebra $\mathbb{C}[t^{\pm 1}, \lambda]/(\lambda^{k_0} - t^a)$. Since A is a polynomial, its norm lies in $\mathbb{C}[t^{\pm 1}]$. The product in (3.11), with the integral exponents from (3.7), is regular at $t = 0$ and has value one there, since every $L > 0$. This Laurent polynomial has no negative powers and therefore belongs to $\mathbb{C}[t]$.

It remains to check the precise order of a protected root. Write $\delta = N - M$, $E_0 = aN - mM$, $r = \gcd(N, M)$ and $b = \gcd(k_0, \delta)$, so $L = E_0/b$. Choose an integer q_0 such that

$$\gcd\bigl(k_0/b + (\delta/b)q_0, L\bigr) = 1.$$

For a prime dividing L, either it divides δ/b and does not divide k_0/b, or it forbids only one residue class of q_0. The Chinese remainder theorem makes the choice possible. Put

$$\lambda = \exp \frac{2\pi i(a - Mq_0)}{E_0}, \qquad \mu = \exp \frac{2\pi i(-m + Nq_0)}{E_0}.$$

Then

$$\lambda^{N/r} \mu^{M/r} = e^{2\pi i/r}, \qquad \lambda^m \mu^a = 1, \qquad \operatorname{ord}(\lambda\mu) = L.$$

The primitive factor of A therefore vanishes at a point of the norm fibre over $t = \lambda\mu$. It gives a primitive L-th root in the norm; polynomiality prevents cancellation by a denominator. Its nonzero exponent detects an actual monodromy eigenvalue. □

We shall also use the following immediate divisibility consequence. If $\lambda^{k_0} = \theta^a$, $\mu = \theta/\lambda$ and $\lambda^N \mu^M = 1$, then, with $b = \gcd(k_0, N-M)$,

$$\theta^{(aN-mM)/b} = 1. \tag{3.12}$$

Indeed, raise $\lambda^{N-M}\theta^M = 1$ to the power k_0/b and use $\lambda^{k_0} = \theta^a$.

3.3 A global intersection budget

Lemma 3.4. *Fix i with $r_i \geq 2$. There is a normalization branch B of C_i with $e_B > b_B$, using (3.6). At its point P, the germ $h = H_a$ is smooth. On the minimal resolution of the full local divisor $G_m H_a$, every exceptional valuation satisfies*

$$\operatorname{ord}_E(g_i) \geq \operatorname{ord}_E(h), \qquad N_E \geq r_i M_E. \tag{3.13}$$

Proof. Bézout's theorem on the normalization gives

$$\sum_B e_B = ae_i, \qquad \sum_B b_B = (m - r_i e_i)e_i.$$

The difference is positive, so some branch has the required strict inequality. Its point belongs to $H_a = 0$. Since $r_i \geq 2$, the differential of G_m vanishes there. If that of H_a also vanished, the punctured radial line in this direction would consist of critical points of f, contradicting isolation at the origin.

Write $h = y$, choose x transverse to $h = 0$, and set $e = I(B, h)$. The successive exceptional valuations along the strict h-path are $v_j(x) = 1$, $v_j(y) = j$. For a reduced branch equation q put $e_q = \operatorname{ord}_x q(x, 0)$. Monomial inspection gives $v_j(q) \geq \min(j, e_q)$. After $j - 1$ blowups on this path, the strict transform of q passes through the intersection with strict h only if $v_{j-1}(q) < e_q$. Equality instead leaves a nonzero constant term in the strict-transform equation.

For $j \leq e$ the selected branch gives $v_j(g_i) \geq j$. Suppose that a branch q of a different global component kept a centre on the h-path necessary at $j > e$. The preceding criterion gives $e_q \geq j > e$, hence $v_e(q) \geq e$. Along B, $\operatorname{ord}_B x \geq 1$ and $\operatorname{ord}_B y = e$, so substitution in every monomial gives $I(B, q) \geq v_e(q) \geq e$. This contradicts $b_B < e$. Every remaining necessary centre on this path is therefore maintained by g_i itself. At such a centre the new difference $\operatorname{ord}(g_i) - \operatorname{ord}(h)$ is its old difference plus the strict g_i multiplicity minus one, so it cannot decrease. Off the strict h-path, the new difference is the sum of the parent differences plus a nonnegative strict g_i multiplicity. The first difference is $\operatorname{mult}(g_i) - 1 \geq 0$. This proves the first inequality throughout the tree, and the second follows from (3.1). □

3.4 The perverse argument and increasing multiplicities

Blow up the origin, $\pi : Y \to \mathbb{A}^3$, with exceptional plane E. For a nontrivial root of unity θ let

$$\mathcal{P}_\theta = (R\psi_{f\pi}\mathbb{C}_Y)_\theta[2].$$

This is a perverse sheaf. On a sufficiently small representative of the isolated original germ it is supported on E: away from E the zero fibre is smooth and has no nontrivial monodromy. If $\theta^m \neq 1$, it is supported on C, since on $E \setminus C$ the equation is z^m times a unit. A supported perverse sheaf is identified with a perverse sheaf on the supporting closed curve.

Fix $d > 1$ with $\operatorname{Sat}_d(f, 0)$ and write

$$\Gamma_d = \{\theta : \ d \mid \operatorname{ord}(\theta), \ \theta^m \neq 1\}.$$

Proper nearby-cycle base change gives

$$R\Gamma(C, \mathcal{P}_\theta) = 0 \qquad (\theta \in \Gamma_d). \tag{3.14}$$

All simple curve constituents have rank one generically, by Lemma 3.2. When $\theta^m \neq 1$ the generic labels satisfy $\mu \neq 1$; otherwise $\theta = \lambda$ and $\lambda^{k_0} = \theta^a$ would imply $\theta^m = 1$.

Proposition 3.5. *For every $\theta \in \Gamma_d$, $\mathcal{P}_\theta$ has neither a geometrically constant curve constituent nor a point constituent. Each of its simple constituents has zero global cohomology, and*

$$\mathcal{H}^0(\mathcal{P}_\theta)_P = 0 \quad (P \in C). \tag{3.15}$$

At a common point of G_m, H_a where G_m is critical, the entire stalk of $\mathcal{P}_\theta$ is zero.

Proof. Suppose first that a constant curve constituent occurs on C_i. Its labels have $\lambda^{r_i} = 1$, $\lambda \neq 1$ and $\mu \neq 1$, so $r_i \geq 2$. Choose the branch in Lemma 3.4. Constancy makes its boundary inertia trivial. On the resolved chart adjacent to this strict branch it says $\lambda^N \mu^M = 1$. Thus the strict branch and its adjacent exceptional divisor are both eligible, where eligible means that this character is one. The h-arrow is not eligible, because $\mu \neq 1$. If the divisor were already a simple crossing, its inertia would be $\mu \neq 1$, an immediate contradiction. Otherwise (3.10) shows that eligibility propagates along every vertex of valency at most two. Since the graph is a connected tree with a noneligible arrow, an eligible rupture must be reached.

For this rupture put $b = \gcd(k_0, N - M)$ and $j = (N - M)/b$. By (3.13), $N \geq r_i M$; hence

$$L = aj + k_0 M/b > a > m, \qquad L \nmid aR \quad (1 \leq R \leq r_i). \tag{3.16}$$

To check the second assertion, if $c_0 = aR/L$ were integral then

$$R - c_0 j = \frac{R k_0 M}{aN - mM}.$$

The right side is strictly between zero and one, since $aN - mM \geq [m(r_i - 1) + k_0 r_i]M > k_0 R M$. The left side is integral. Equation (3.12) shows $d \mid L$.

Lemma 3.3 supplies a primitive L-th root θ_L with positive exponent in the local polynomial $\mathcal{Z}_{f\pi,P}$. Since the perverse shift is even, this exponent is the Euler characteristic of its θ_L stalk. A simple curve IC sheaf has stalk cohomology only in degree -1, hence has nonpositive stalk Euler characteristic. A point simple has positive stalk Euler characteristic. Therefore $\mathcal{P}_{\theta_L}$ has a point simple.

On a proper curve a nonconstant rank-one IC sheaf has no global cohomology in degrees -1 and 1: these groups are the invariant sections of the line and their duals. Its Euler characteristic is therefore nonnegative. A point simple has positive Euler characteristic. The only simple constituents with negative global Euler characteristic are constant lines on rational normalizations; their IC Euler characteristic is -2. By (3.14), the point simple just found must be balanced by a constant curve constituent on some C_j. The generic spectrum (3.2) then requires $L \mid a r_j$. Equation (3.16) forces $r_j > r_i$.

Choose, among constant curve constituents for all $\theta \in \Gamma_d$, one with maximal supporting multiplicity. There are only finitely many possible supporting components, so this maximum exists. The preceding argument contradicts its maximality. There are consequently no constant curve constituents.

Only now apply global cohomology as an exact functor. On the Serre category generated by nonconstant rank-one IC sheaves and point sheaves, proper global cohomology is concentrated in degree zero: this holds for each simple, and follows for extensions by the long exact sequence. It is therefore exact on this category. Equation (3.14) implies that each simple constituent is

globally acyclic. In particular, no point simple can occur. Extensions of curve IC sheaves have no degree-zero stalk cohomology, proving (3.15).

At a common critical point the local product is a polynomial by Lemma 3.3, so the stalk Euler characteristic in a nontrivial eigenblock is nonnegative. The remaining curve IC stalks lie only in degree -1, so it is also nonpositive. With no degree-zero stalk cohomology, equality forces the entire stalk to vanish. □

3.5 Second cohomology at a unit point

Lemma 3.6. *Let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be nonzero with $g(0) = 0$, possibly nonreduced, and let $c \neq 0$. For $G = z^m(g + cz^{k_0})$ one has the degree-specific identity*

$$\det(1 - tT_G \mid H^2(M_G)) = \det(1 - t^a T_g^{k_0} \mid H^1(M_g)). \tag{3.17}$$

If T_g has an eigenvalue of order n on $H^1(M_g)$, then T_G has an eigenvalue of order $an/\gcd(n, k_0) \geq a$ on $H^2(M_G)$.

Proof. Write $g = g_0^b$ analytically, taking a root of its unit, where b is the greatest common divisor of its branch multiplicities. The Milnor fibre of g_0 is connected. Put

$$P_b(w, z) = z^m(w^b + cz^{k_0}), \qquad \varphi(x, z) = (g_0(x), z).$$

This is a holomorphic map of families, with $G = P_b\varphi$. Use closed plane Milnor tubes for g_0 and a closed z-disc. The cofinal product-neighbourhood construction in [25, Theorem L1 and §4, pp. 8–10] gives a representative of this map on the nonzero Milnor fibres which is proper over the punctured value disc. The representatives are intersections with $\{|g_0(x)| \leq \eta,\ |z| \leq \delta\}$ inside a closed Milnor ball in x; sufficiently small nested choices are deformation retracts of the local fibres. Projection by φ maps them to the corresponding compact plane fibre of P_b.

The base M_{P_b} is connected and has graph homotopy type. The locus $w = 0$ on it consists of exactly a points, the roots of $cz^a = \epsilon$. Over every other point the fibre of φ is a connected Milnor fibre of g_0. Over a point with $w = 0$ it is the local central curve $g_0 = 0$ in a Milnor ball, which is contractible by the local conical structure. Proper base change therefore gives

$$R^0\varphi_*\mathbb{C} = \mathbb{C}, \qquad R^1\varphi_*\mathbb{C} = j_!L,$$

where j removes these a points and L is the fibre-cohomology local system. The second equality is an equality of ordinary sheaves: its stalk at each removed point is zero, and on the complement it is locally constant. This sheaf has no global sections. A section vanishes near a removed point, and hence, by local constancy and connectedness of the punctured base, vanishes everywhere. The low-degree Leray sequence now gives

$$\varphi^* : H^1(M_{P_b}) \simeq H^1(M_G). \tag{3.18}$$

It commutes with monodromy because φ and these representatives exist over the whole value circle.

The same bundle-and-cones construction gives connectedness of M_G and a CW model of dimension at most two. Indeed, the plane base is a graph up to homotopy, the factor fibre is a graph, and over its special points one attaches cones on that graph. Lemma 3.2 and (3.18) consequently give

$$\det(1 - tT_G \mid H^1(M_G)) = (1 - t)B_b(t), \qquad \mathcal{Z}_G(t) = \frac{\det(1 - tT_G \mid H^2(M_G))}{B_b(t)}.$$

The partial-resolution calculation [15, Lemma 1, p. 249, and Remark 4, p. 255] states

$$\mathcal{Z}_G(t) = \frac{1 - t^a}{\mathcal{Z}_g^{(k_0)}(t^a)}, \tag{3.19}$$

where the superscript means that T_g is replaced by $T_g^{k_0}$. Remark 4 extends the calculation to arbitrary plane germs whose curve resolutions are isomorphisms off the origin. The action on $H^0(M_g)$ is the cyclic permutation of its b connected components, so

$$\det(1 - t^a T_g^{k_0} \mid H^0(M_g)) = (1 - t^{ab/\gcd(b,k_0)})^{\gcd(b,k_0)} = (1 - t^a)B_b(t).$$

Since a plane Milnor fibre has no cohomology above degree one, substitution into (3.19), together with (3.18), proves (3.17).

If an eigenvalue ξ has order n, then ξ^{k_0} has order $n/\gcd(n, k_0)$. Among its a-th roots there is one of exact order $an/\gcd(n, k_0)$: write a primitive root as $e^{2\pi ic/N}$, choose a numerator congruent to c modulo N and coprime to every additional prime divisor of a, and use the Chinese remainder theorem. Formula (3.17) places this root in actual H^2. □

At a point where $h = H_a$ is a unit, the complex analytic change $z' = zh^{1/a}$ transforms $z^m(g + z^{k_0}h)$ into $z'^m(g' + z'^{k_0})$, with $g' = gh^{-m/a}$. The plane germs g' and g have the same Milnor monodromy. This follows, for example, by choosing a logarithm of the unit and applying the stratified isotopy of the Milnor argument fibrations to $g\exp(t\log(h^{-m/a}))$, $0 \le t \le 1$, on a common sufficiently small link. The arithmetic normalization below uses the principal-unit part of h and retains its residue-unit coefficient.

In particular, Proposition 3.5 and Lemma 3.6 imply

$$d \nmid \frac{an}{\gcd(n, k_0)} \quad \text{for every order } n \text{ occurring in } H^1(M_g) \tag{3.20}$$

at every h-unit point. The order produced in Lemma 3.6 is at least $a > m$, so its primitive root belongs to the range controlled by (3.15).

3.6 Eligible plane components and complete coefficients

Lemma 3.7. *Resolve the reduced total divisor of plane germs $f_1, \dots, f_s$ by point blowups. For finite-order characters $\eta_1, \dots, \eta_s$ call a component E eligible if $\rho_E = \prod_j \eta_j^{\operatorname{ord}_E f_j} = 1$. Suppose no exceptional rupture is eligible and some strict arrow is noneligible. Then every eligible exceptional curve is isolated among eligible components and is a rational curve with exactly two noneligible neighbours. The corresponding Kummer line has zero compactly supported cohomology. At a good arithmetic place, for every tuple of contact orders $(n_1, \dots, n_s)$,*

$$\int_{\bar{x}=P\,;\ \operatorname{ord} f_j(x)=n_j\ \ (1\le j\le s)} \prod_j \eta_j(\operatorname{ac} f_j(x))\, dx = 0. \tag{3.21}$$

The measure in this formula is the original smooth plane measure.

Proof. The principal-divisor intersection relation, coordinate by coordinate, gives $\prod_{E' \sim E} \rho_{E'} = \rho_E^{-E^2}$. At an eligible vertex of valency one its neighbour is eligible. At one of valency two an eligible neighbour forces the other neighbour eligible. An eligible edge would therefore propagate until it reached a rupture or filled the connected graph, including every strict arrow. Both are

excluded by the assumptions. There are no eligible leaves; the only remaining eligible vertices are isolated bivalent curves. Their two boundary inertias are nontrivial and inverse. Their complete compact-support complex vanishes by Lemma 2.2.

For the arithmetic statement, in a normal-crossings chart write $f_j = u_j \prod_E y_E^{N_{Ej}}$. Normal-unit integration kills a stratum unless every normal component is eligible. At an open eligible exceptional curve the remaining coefficient is the actual Frobenius trace of its Kummer line, and this trace is zero. A crossing has a noneligible normal direction and again has coefficient zero. Open strict transforms do not lie over the original centre. The identity-resolution normal-crossings case is handled directly by a noneligible normal variable. Prescribing the contact orders imposes only the linear relations $n_j = \sum_E N_{Ej} \operatorname{ord}(y_E)$ on the valuation variables. The Jacobian contributes its usual factors $q^{-\sum_E \nu_E \operatorname{ord}(y_E)}$. These factors multiply zero coefficients before any valuation sums are performed. This proves (3.21), retaining all original Haar weights. The trace interpretation is the multivariate formula in [27, Theorems 6.0.6–6.0.7], with the inverse geometric character convention fixed above. □

Hereafter an actual character pair means

$$\lambda^{k_0} = \chi^a, \qquad \mu = \chi/\lambda. \tag{3.22}$$

The corresponding equations can equally be read as equations between complex roots of unity. Suppose $P \in \operatorname{Sing}(C) \setminus D$. If $\operatorname{ord}(\lambda) \mid n$ for a plane H^1 eigenorder n, then

$$\frac{\operatorname{ord}(\lambda)}{\gcd(\operatorname{ord}(\lambda), k_0)} \;\mid\; \frac{n}{\gcd(n, k_0)}.$$

The left side is the order of χ^a, so $d \mid an/\gcd(n, k_0)$, contradicting (3.20).

In particular every plane rupture is λ-ineligible. To prove this, suppose that the minimal plane resolution has a rupture. Let N_E be the multiplicity of g along an exceptional component, and let b be the greatest common divisor of the strict-branch multiplicities. A'Campo's formula and the cyclic action on the b connected components give

$$\det(1 - tT_g \mid H^1(M_g)) = (1 - t^b) \prod_E (1 - t^{N_E})^{\operatorname{val}(E)-2}.$$

Assign each exceptional leaf to its first rupture. The dead-chain calculation in Lemma 3.3, applied to scalar multiplicities, groups a rupture of multiplicity N and valency v into

$$(1 - t^N)^{v-2} \prod_{d_j \text{ a dead-chain determinant}} (1 - t^{N/d_j})^{-1}.$$

Every cyclotomic exponent of each group is nonnegative except possibly at the unique initial rupture with valency three and two dead chains. Its possible negative indices divide $N/(d_1 d_2)$ and are strictly smaller than N. Every later rupture has multiplicity at least this initial multiplicity. At a primitive N_E-th root for a selected rupture E, its group contributes $\operatorname{val}(E) - 2 > 0$, its dead tips contribute zero, and every other group contributes a nonnegative exponent. The factor $1 - t^b$ also contributes a nonnegative exponent. Thus a primitive N_E-th root occurs in $H^1(M_g)$. Eligibility would give $\operatorname{ord}(\lambda) \mid N_E$, contradicting the preceding exclusion.

There is also a noneligible strict g-arrow. To check the only possible obstruction, write $g = g_0^b$ with connected fibre for g_0. If all strict multiplicities are killed by λ, then $\operatorname{ord}(\lambda) \mid b$. At a singular point of the reduced curve, $H^1(M_{g_0}) \neq 0$. For several branches this follows from the Wang

sequence of the link fibration: the link complement has first Betti number equal to the number of branches, leaving at least one invariant fibre class after the base-circle class is removed. For one singular branch g_0 is reduced and its positive Milnor number is the dimension of H^1. If ξ is any of these eigenvalues of order n, the cyclic permutation of the b fibre components of g has among its eigenvalues a b-th root of ξ of order bn. This contradicts the saturated H^1 exclusion for λ. Lemma 3.7 therefore applies at every such point. It also shows that an eligible strict branch there has nontrivial adjacent boundary inertia; an eligible adjacent exceptional would be an eligible edge, which the proof of that lemma excludes.

At $P \in C \cap D$ where g is critical, h is smooth by isolation. Every exceptional has $N > M > 0$. An eligible rupture for (3.22) would give $d \mid L$, $L > a > m$, by (3.12). Its positive local exponent in Lemma 3.3 contradicts Proposition 3.5. Thus no rupture is eligible. For $\mu \neq 1$, the strict h-arrow is noneligible, so the plane coefficient lemma applies immediately. We will handle $\mu = 1$ after proving the following global boundary statement.

3.7 Two boundary points and the resonant line

Let C_i° denote the normalization of C_i with every point above $\mathrm{Sing}(C) \cup D$ deleted. We call C_i active for d if $r_i \geq 2$ and

$$d \mid M_i, \qquad M_i = \frac{ar_i}{\gcd(r_i, k_0)}. \tag{3.23}$$

For $r_i \geq 2$, every pair (3.22) with $\lambda^{r_i} = 1$ makes C_i active: raise $\lambda^{k_0} = \chi^a$ to $r_i/\gcd(r_i, k_0)$.

Proposition 3.8. *An active component has rational normalization and precisely two deleted geometric points. For every actual pair with $\lambda^{r_i} = 1$, including $\lambda = 1$ or $\mu = 1$, the line* (3.4) *on C_i° is nonconstant and*

$$R\Gamma_c(C_i^\circ, \mathcal{L}_{i,\lambda,\mu}) = 0. \tag{3.24}$$

Proof. Choose a primitive M_i-th root θ. It occurs in B_{r_i}: if $M_i > a$ the denominator of (3.2) cannot cancel it, and if $M_i = a$ its exponent is $r_i - 1 > 0$. Since $M_i \geq a > m$ and $d \mid M_i$, its corresponding generic line is a constituent of $\mathcal{P}_\theta$ in the forbidden range. Proposition 3.5 makes that IC constituent nonconstant and globally acyclic.

It is clean at every deleted normalization point. At an h-unit singular point the preceding H^1 argument excludes eligible ruptures and supplies a noneligible arrow, so an eligible strict branch cannot have trivial adjacent inertia. At a common critical point its auxiliary label has $\mu \neq 1$, and the same conclusion follows with the h-arrow noneligible. At an already transverse intersection with D, its inertia is directly $\mu^{-1} \neq 1$. These are all boundary points. Thus the IC extension is extension by zero at the full boundary. If the normalization has genus g_i' and n_i deleted points, global acyclicity gives

$$0 = \chi_c(C_i^\circ, \mathcal{L}) = 2 - 2g_i' - n_i.$$

Bézout gives $n_i \geq 1$, since C_i meets D. Hence $g_i' = 0$ and $n_i = 2$. These two geometric boundary points are fixed for all subsequent characters.

Now take any actual pair on this component. If $\mu \neq 1$ and the line were constant, Lemma 3.4 would again give an eligible branch-edge with a noneligible h-arrow. Propagation would produce an eligible rupture, forbidden by Proposition 3.5 and (3.12). The same boundary argument applies to the arithmetic channel $\lambda = 1$.

It remains to treat $\mu = 1$. Then $\lambda = \chi$, $\chi^m = 1$ and $\chi^{r_i} = 1$. Suppose this line constant. The complete contact excess is

$$\sum_B (e_B - b_B) = (a - m + r_i e_i)e_i = (k_0 + r_i e_i)e_i \geq 3. \tag{3.25}$$

At each positive summand, the budget lemma applies. Constancy gives an eligible adjacent branch-edge, and now the h-arrow is also eligible. There is no eligible rupture, since its $L > a > m$ would give a forbidden positive stalk exponent. If some strict g-arrow were noneligible, the eligible edge would propagate to that arrow or to a rupture, both impossible. If every strict arrow were eligible, every exceptional would be eligible, since its multiplicities are integral combinations of the arrow multiplicities. A non-normal-crossing reduced plane divisor with at least two branches has a rupture in its minimal resolution. Thus the divisor at this point must already be exactly two smooth transverse branches, one from C_i and one from D, with no other branch. Consequently $e_B = 1$ and $b_B = 0$. Every positive summand in (3.25) equals one. There are at least three such normalization points, since nonpositive summands can only increase the required number of positive ones. This contradicts $n_i = 2$.

Every actual line is therefore nonconstant on $\mathbb{P}^1$ minus two points. Its two inertias are nontrivial and inverse, and Lemma 2.2 proves (3.24). □

We can now close the common-point case with $\mu = 1$. If every strict g-arrow were eligible, a non-normal-crossing divisor would have an eligible rupture. The only remaining case is one smooth repeated g-branch crossing the smooth h-branch. Its component is active and the resonant line has trivial boundary inertia, contradicting Proposition 3.8. Some arrow is therefore noneligible in this case as well. For all actual pairs, Lemma 3.7 gives the complete coefficients

$$C^{\lambda,\mu}_{u,v}(P) = \int_{\bar{x}=P\,;\ \operatorname{ord} g=u,\ \operatorname{ord} h=v} \lambda(\operatorname{ac} g)\mu(\operatorname{ac} h)\,dx\,dy = 0. \tag{3.26}$$

3.8 The original Haar calculation

We now work at a good nonarchimedean place and first take χ tame. The finite exceptional set, uniform in the character, will be fixed below. On the ordinary blowup the function and volume form have the form

$$f\pi = z^m(g(x,y) + z^{k_0}h(x,y)), \qquad |\pi^*(dx_1dx_2dx_3)| = |z|^2\,dx\,dy\,dz.$$

The origin ball corresponds to $z \in \mathfrak{p}$ and all projective residue directions.

For characters of $\mathbb{F}_q^\times$, extended by zero at zero, put $J(\alpha,\beta) = \sum_{t\in\mathbb{F}_q} \alpha(t)\beta(1-t)$. Finite Fourier inversion gives, for $A, B \in \mathbb{F}_q^\times$,

$$\sum_{\beta\in\mathbb{F}_q^\times} \chi(\beta)^m\chi(A + B\beta^{k_0}) = \sum_{\lambda^{k_0}=\chi^a} \lambda(A)\mu(B)\lambda^{-1}(-1)J(\lambda^{-1},\chi), \qquad \mu = \chi/\lambda. \tag{3.27}$$

Here and below the sums are over characters of the original residue field. To verify the coefficient of $\lambda(A)$, multiply the left side by $\lambda^{-1}(A)$ and sum in A, dividing by $q-1$. Substitute $A = B\beta^{k_0}t$. The β-sum is $q-1$ precisely when $\lambda^{k_0} = \chi^a$, and is zero otherwise. The remaining sum is $\mu(B)\lambda^{-1}(-1)J(\lambda^{-1},\chi)$, as stated.

At a common critical point fix $u = \operatorname{ord} g$, $v = \operatorname{ord} h$, and $\ell = \operatorname{ord} z \geq 1$. If $u < v + k_0\ell$, radial-unit averaging is zero unless $\chi^m = 1$; its plane character pair is then $(\chi, 1)$, which satisfies

(3.22). If $u > v + k_0\ell$, it is zero unless $\chi^a = 1$, with pair $(1, \chi)$. At equality, (3.27) gives exactly the actual pairs. Formula (3.26) kills all these coefficients, with the original factor $|z|^2 dz$ still present.

For the higher-contact terms, if $A+B\beta^{k_0} = 0$ at a nonzero residue β, its derivative is $k_0 B\beta^{k_0-1}$, a unit when $p \nmid k_0$. On that entire radial unit ball the binomial is a Hensel coordinate onto $\mathfrak{p}$. The factor $\chi(\beta)^m$ is constant there, since χ is tame. Its complete higher-valuation tail is a constant times $\int_{\mathfrak{p}} \chi(\operatorname{ac} v)|v|^s\, dv = 0$.

At an h-unit singular point choose $c \in \mathcal{O}^\times$ with $h/c \in 1 + \mathfrak{p}$. The substitution $z' = z(h/c)^{1/k_0}$ is an integral analytic bijection of the same radial ball, with unit Jacobian, preserving $\operatorname{ord} z$ and $|z|^2 dz$. It leaves g unchanged and removes the varying unit from the inner binomial; its additional outer factor is a principal unit, invisible to the tame character. Thus the same Fourier calculation uses the one-variable plane contact coefficients proved zero above. The high channel $\chi^a = 1$ cannot occur at such a singular point under our spectral hypothesis: $H^1(M_g) \neq 0$, and every order supplied by Lemma 3.6 is a multiple of a. This also accounts explicitly for the possible formal $\lambda = 1$ channel.

We next consider a smooth point of a heavy component C_i, away from D and from the other components. Put $r = r_i \geq 2$, choose a normal coordinate $x = g_i$ and a tangential coordinate y, and write

$$f\pi = z^m\big(U(x, y)x^r + z^{k_0}V(x, y)\big),$$

where U, V are units. For a fixed projective residue point P, $n = \operatorname{ord} x \geq 1$, $\ell = \operatorname{ord} z \geq 1$, and unit residues α, β, the exact measure is

$$q^{-1}\, q^{-n-1}\, q^{-3\ell-1} = q^{-n-3\ell-3}. \tag{3.28}$$

The first factor is the tangent ball; the second is one normal unit residue; the third is one radial unit residue including the Jacobian $|z|^2$. The coordinate derivative defining x is a unit, so no additional point-dependent absolute factor occurs.

The full contribution for these fixed valuations and point is as follows:

$$rn < k_0\ell: \quad q^{-n-3\ell-3}(q-1)^2 \mathbf{1}_{\chi^m=1}\mathbf{1}_{\chi^r=1}\chi(\bar{U}_P)T^{m\ell+rn}; \tag{3.29}$$

$$rn > k_0\ell: \quad q^{-n-3\ell-3}(q-1)^2 \mathbf{1}_{\chi^a=1}\chi(\bar{V}_P)T^{a\ell}; \tag{3.30}$$

$$rn = k_0\ell: \quad q^{-n-3\ell-3}T^{a\ell}D_{\chi,r}(\bar{U}_P, \bar{V}_P), \tag{3.31}$$

where

$$D_{\chi,r}(U, V) = \sum_{\alpha,\beta\in\mathbb{F}_q^\times} \chi(\beta)^m \chi(U\alpha^r + V\beta^{k_0}) \tag{3.32}$$

$$= (q-1) \sum_{\substack{\lambda^{k_0}=\chi^a \\ \lambda^r=1}} \lambda(U)\mu(V)\lambda^{-1}(-1)J(\lambda^{-1}, \chi). \tag{3.33}$$

The second equality follows by applying (3.27) to $A = U\alpha^r$ and summing in α. This contributes the factor $q-1$ precisely when $\lambda^r = 1$. The balanced zero-residue tails vanish by the Hensel-coordinate calculation just given.

Every scalar in (3.29)– (3.33) is independent of P on the same open component. The only point dependence is $\lambda(\bar{U}_P)\mu(\bar{V}_P)$, exactly the trace of (3.4). The low term uses the resonant pair $(\chi, 1)$; the high term uses $(1, \chi)$. Both obey the same eligibility conditions when their

indicators are nonzero. Summing over the complete open component therefore gives zero by (3.24), separately for each fixed (n, ℓ) and each actual pair on each complete open component. If the component has no eligible pair, unit averaging makes all three formulas zero.

Finally, at a smooth point of the reduced curve with $r_i = 1$, for fixed y, z the map

$$x \longmapsto g(x, y) + z^{k_0} h(x, y)$$

is a Hensel bijection of the complete normal residue ball onto $\mathfrak{p}$, with unit derivative, including at $h(P) = 0$. Its integral is a constant times $\int_{\mathfrak{p}} \chi(\operatorname{ac} v)|v|^s \, dv = 0$. This treats the smooth reduced directions. The common critical, h-unit singular, smooth heavy and smooth reduced cases exhaust all directions above C.

3.9 Descent, the remaining radial term, and uniformity

All coefficient cancellations just used take place over the original local field. The plane coefficients are actual Kummer–Frobenius trace coefficients. Each relevant geometric two-puncture complex is zero, so a Frobenius permutation of components or punctures still acts on a zero complex. On the smooth part of C, a rational residue point lies on a unique geometric component, which must be Frobenius-stable. On that component the transition calculation (3.5) descends the actual character line. The two omitted points may be conjugate. Lemma 2.2 still applies. A nonsplit crossing can equivalently be treated by blowing up its closed point to strict relative normal crossings; the inserted bivalent vertices preserve the eligible-tree calculation.

All directions over C have now contributed zero. For a primitive $v \in \mathcal{O}^3$ with $\bar{G}_m(v) \neq 0$ and $\ell \geq 1$,

$$f(\pi^\ell v) = \pi^{m\ell}\big(G_m(v) + \pi^{k_0 \ell} H_a(v)\big).$$

The angular character is that of $G_m(v)$ and its absolute value is $q^{-m\ell}$. Therefore the entire original origin integral is

$$Z_{f,0}(\chi, s) = A_\chi(q) \frac{q^{-3} T^m}{1 - q^{-3} T^m}, \tag{3.34}$$

where $A_\chi(q) = \int_{\bar{G}_m(v) \neq 0} \chi(\operatorname{ac} G_m(v)) \, dv$ is an actual primitive-angular integral independent of T. It is already zero if $\chi^m \neq 1$. If it were nonzero in any remaining case, (3.34) would have rational degree zero in T. The strict degree theorem (2.3), applied to the original f and its complete origin residue ball, gives degree strictly less than zero. Its exact-order hypothesis follows from $\operatorname{Sat}_d(f, 0)$. Hence $A_\chi(q) = 0$.

In the earlier case $G_m \mid H_a$, the unit $1 + Q_{k_0}$ is a principal unit on the original origin ball at good places. The tame f integral equals the G_m integral. The projective curve of G_m is smooth and reduced, so all its zero-direction tubes vanish by the smooth-coordinate calculation. The remaining expression is again (3.34). The strict degree theorem is still applied to the original f, and gives the same conclusion.

We finish by specifying the finite-place quantifier. Fix once an embedded resolution of the original f, with positive multiplicities N_j, and put $D_0 = \operatorname{lcm}_j N_j$. If d divides none of the N_j, every original positive-contact coefficient is already zero by a noneligible normal direction. It suffices to handle the finite set of divisors of these multiplicities. For an actual pair (3.22), both $\operatorname{ord}(\lambda)$ and $\operatorname{ord}(\mu)$ divide $k_0 d$, hence divide $k_0 D_0$. The auxiliary M_i form another fixed finite set. Spread the finitely many normalizations, boundary points, plane resolutions and Kummer covers of these orders, and remove their bad-reduction places together with the primes dividing k_0, the relevant r_i, the original N_j, the coefficient denominators, and the finite set in (2.3). The reduced

finite schemes of singular and common projective points spread etale in this process. All equations, inertia conditions and unit derivatives used above then remain valid. Thus one finite set works for all tame characters and all residue degrees. Lemma 2.3, on the fixed resolution of the original f, proves the remaining higher-conductor cases. This completes the proof of Theorem 3.1.

4 Two terms of relatively prime weighted degrees

The ordinary multiplicity in the following theorem is the order for the usual maximal ideal.

Theorem 4.1. *Let $G, H \in \Bbbk[x_1, x_2, x_3]$ be nonzero weighted homogeneous polynomials for the same primitive positive integral weights w_1, w_2, w_3. Let their weighted degrees be $m < a$, and suppose that*

$$\gcd(m, a) = 1, \qquad F = G + H, \qquad \operatorname{ord}_0 F \geq 10,$$

and that 0 is an isolated critical point of F. There is a finite set of finite places, depending only on F and the weights, such that outside this set

$$\operatorname{Sat}_d(F, 0), \quad d = \operatorname{ord} \chi > 1 \quad \Longrightarrow \quad Z_{F,0}(\chi, s) = 0.$$

The assertion includes characters of arbitrary conductor and the complete original residue ball $\mathfrak{p}^3$.

Fix $d > 1$ with $\operatorname{Sat}_d(F, 0)$ for the proof.

4.1 The two actions and their character lattices

Put $k = a - m$ and $W = w_1 + w_2 + w_3$. The weighted blowup is conveniently regarded as the smooth stack

$$\mathcal{Y} = \operatorname{Tot} \mathcal{O}_{\mathcal{E}}(-1), \qquad \mathcal{E} = \mathbb{P}(w_1, w_2, w_3).$$

On a covering chart its equation is

$$f = z^m (g + z^k h). \tag{4.1}$$

The chart group is cyclic, acts faithfully on z, and has order at most $\max w_i$. If its order is e_0, choose its generator so that $z \mapsto \zeta z$. Then g and h have characters ζ^{-m} and ζ^{-a}, respectively. A finite group quotient below always means this given action, including its action on coefficients.

There are two commuting operators on nearby cycles. On the logarithmic cover of the punctured value disc they are represented by

$$T_f(x, u) = (x, u + 2\pi i), \qquad A(x, u) = (\xi^{w_1} x_1, \xi^{w_2} x_2, \xi^{w_3} x_3, u + 2\pi i m/k), \quad \xi = e^{2\pi i/k}.$$

They satisfy $A^k = T_f^m$. We denote their simultaneous generalized characters by

$$T_f = \theta, \qquad A^{-1} = \mu, \qquad \lambda = \theta/\mu, \qquad \lambda^k = \theta^a, \quad \lambda^m \mu^a = 1. \tag{4.2}$$

These formulas fix the joint-character convention throughout the proof.

Let $\mathcal{P} = \psi_f \mathbb{C}_{\mathcal{Y}}[2]$ on the special fibre and restrict its nontrivial T_f blocks to $\mathcal{E}$. Since F has no critical point in a sufficiently small punctured neighbourhood of the origin, these blocks are supported on $\mathcal{E}$. Proper base change identifies their global cohomology with the corresponding cohomology of $M_{F,0}$, shifted by two. On $\mathcal{E} \setminus \{G = 0\}$ the operator A is the identity: its rotation of the radial coordinate is cancelled by its prescribed lift of the value angle. Consequently $\mathcal{P}_{\theta,\mu}$

is supported on the curve $C = \{G = 0\}$ if $\mu \neq 1$. We henceforth call a sector *saturated* when $d \mid \operatorname{ord}\theta$. Its global nearby-cycle complex is zero under $\mathrm{Sat}_d(F, 0)$.

At a general point of a component of C of multiplicity r, write $g = Ux^r$, $h = V$, with U, V units. Let e be the generic chart stabilizer and let $0 \leq \beta < e$ represent its character on x. Then

$$A_C = \frac{a}{e}, \qquad L_C = \frac{m + \beta r}{e}, \qquad Q_C = \{(\lambda, \mu) : \lambda^r = 1, \lambda^{L_C}\mu^{A_C} = 1\}. \tag{4.3}$$

Here $e \mid a$, since h is a unit. The genuine curve nearby-cycle layers are the rank-one lines

$$\mathcal{L}_\lambda(U) \otimes \mathcal{L}_\mu(V), \qquad (\lambda, \mu) \in Q_C, \quad \lambda \neq 1. \tag{4.4}$$

For completeness, the transverse fibre of $z^m(x^r + z^k)$ projects to the z-annulus. Away from its a zeros of x^r, its r sheets are permuted regularly; its reduced sheet module is the sum of the nontrivial $\lambda^r = 1$ lines. Returning around the value circle gives $\lambda^k = \theta^a$. Imposing the actual deck transformation of order e gives the second equation of (4.3). This proves (4.4), including its coefficient units.

At a common point resolve the plane divisor gh. On an exceptional component E put $N = v_E(g)$, $M = v_E(h)$, let e be its generic stabilizer and let $0 \leq \beta < e$ represent its normal character. The corresponding lattice is

$$\begin{aligned} p_E &= (m + \beta N)/e, \qquad q_E = (a + \beta M)/e, \\ Q_E &= \{(\lambda, \mu) : \lambda^N \mu^M = 1, \lambda^{p_E}\mu^{q_E} = 1\}, \\ |Q_E| &= (aN - mM)/e. \end{aligned} \tag{4.5}$$

The quotient torus imposes the second relation. Both Q_C and Q_E are cyclic. Indeed, the first invariant factor of their two by two integral presentation matrices is the gcd of the entries. A common divisor of these entries divides both m and a, by $m = ep_E - \beta N$ and $a = eq_E - \beta M$, or their analogues for Q_C. It is therefore one. A generator of either finite cyclic group has maximal order under every character of that group. This elementary fact will make simultaneous character selection possible.

We also record the numerical consequence of the ordinary multiplicity. For each coordinate axis, some monomial $x_i^r x_j$ of F, allowing $j = i$, must have nonzero gradient along that axis. Otherwise the whole axis would be critical. Its usual degree is at least ten, so $r \geq 9$. Its weighted degree is at most a. Hence

$$a \geq 9w_i + w_j > 9w_i, \qquad a/e_0 > 9 \tag{4.6}$$

for every chart order e_0.

We may first assume that G and H are coprime. If they have a common irreducible factor P, write $F = PQ$. Unless Q is a unit at the origin, the intersection $P = Q = 0$ has positive dimension and is critical for F, a contradiction. Weighted homogeneity then implies $G = cP$ and $H = PR$ with $R(0) = 0$. Thus $F = G(1 + R/c)$, and G itself has an isolated critical point. Writing $u = 1 + R/c$, choose the analytic roots with value one and put $\phi_i(x) = x_i u(x)^{w_i/m}$. The derivative of ϕ at zero is the identity, and weighted homogeneity gives $G \circ \phi = Gu = F$. Thus the central Milnor fibrations, including their actual monodromy cohomology, agree. For tame characters its origin integral is that of G, since $1 + R/c$ is a principal unit on $\mathfrak{p}^3$ at good places. We treat this weighted homogeneous case together with the open part in §4.10; the wild case is covered by Lemma 2.3.

In the coprime case, if g is critical at a common point, then h is smooth there. Otherwise the weighted orbit of the point would be critical for F. If g is smooth, the analytic change $x' = g + z^k h$ makes (4.1) equal to $z^m x'$. It respects the chart action and the complete original valuation domain, and integration in the normal unit gives zero for every nontrivial tame character. Such points need no further argument.

4.2 Monodromy on the quotient fibre

The next calculation will be used both for topology and for all characters which occur in an arithmetic Fourier expansion.

Lemma 4.2. *Let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a nonzero plane germ and put $f = z^m(g + z^k)$, $a = m + k$. Suppose a cyclic group of order e acts by $(p, z) \mapsto (Bp, \zeta z)$, with $g(Bp) = \zeta^k g(p)$ and $e \mid a$. Choose equivariant angular transport $P(t)$ for g and put*

$$T_g = P(1), \qquad C = P(k/e)B^{-1}, \qquad \ell = a/e.$$

Then $C^e = T_g^k$, and on the actual quotient Milnor fibre

$$\det(1 - tT_f \mid H^2(M_f/\mu_e)) = \det(1 - t^\ell C \mid H^1(M_g)). \tag{4.7}$$

More precisely, a simultaneous eigenpair $(T_g, C) = (\lambda, \rho)$ on $H^1(M_g)$ produces every root $\theta^\ell = \rho$, with $\mu = \theta/\lambda$, on the actual H^2. This assertion concerns eigencharacters and their multiplicities.

Proof. Use product Milnor representatives, made equivariant by taking the intersection of their finitely many translates. The product-fibration construction for $z^m(g + z^k)$ gives the projection

$$q : M_f \longrightarrow P_\varepsilon, \qquad (p, z) \longmapsto (g(p), z), \qquad P_\varepsilon = \{z^m(c + z^k) = \varepsilon\}.$$

With the induced cutoffs, P_ε is a compact annulus. The set $Q = \{c = 0\}$ consists of a interior points. These representatives and their naturality in the value are given by [25, Theorem L1 and §4]. Let $j : P_\varepsilon \setminus Q \hookrightarrow P_\varepsilon$. The central plane fibre is contractible, and therefore

$$R^1 q_* \mathbb{C} = j_! \mathcal{V}, \qquad R^i q_* \mathbb{C} = 0 \quad (i > 1),$$

where $\mathcal{V}$ has fibre $H^1(M_g)$. To check the remaining Leray term when g is nonreduced, write $g = g_0^b$ with the Milnor fibre of g_0 connected. Then $R^0 q_* \mathbb{C}$ is the finite pushforward of the constant sheaf from the bordered curve $z^m(w^b + z^k) = \varepsilon$. Its cohomology in degrees at least two is zero. Also $H^0(P_\varepsilon, j_! \mathcal{V}) = 0$ because Q is nonempty, and $H^2(P_\varepsilon, j_! \mathcal{V}) = 0$. The latter can be seen from the relative cellular complex of a compact surface with its outer boundary: the relative degree-zero group on the dual side is zero. Leray consequently gives the natural identification

$$H^2(M_f) = H^1(P_\varepsilon, j_! \mathcal{V}). \tag{4.8}$$

The canonical value turn on the annulus is

$$(c, z) \longmapsto (e^{2\pi i k/a} c, e^{2\pi i/a} z).$$

Denote its lift to (4.8) by S. Then $S^a = T_g^k$, and the given deck transformation on this module is $S^\ell C^{-1}$. The relative annulus complex shows

$$\operatorname{Tr}(S^n C^j) = \begin{cases} 0, & a \nmid n, \\ a \operatorname{Tr}(T_g^{kv} C^j \mid H^1(M_g)), & n = av. \end{cases} \tag{4.9}$$

Here is a direct trace interpretation of the first line. Cut the annulus into a congruent sectors with one point of Q in each. The relative cochain complex is a cyclic permutation of these sectors, with return T_g^k. If $a \nmid n$ no sector is fixed. For $n = av$ every sector has the displayed return. There are no degree-zero or degree-two terms left in (4.8), so this alternating trace is its actual degree-one trace.

Average (4.9) over the deck group. The trace of the actual quotient monodromy is

$$\frac{1}{e}\sum_{j=0}^{e-1}\operatorname{Tr}(S^{n+\ell j}C^{-j}) = \begin{cases} 0, & \ell \nmid n, \\ \ell \operatorname{Tr}(C^q \mid H^1(M_g)), & n = \ell q. \end{cases}$$

In the second line the unique index with $e \mid q + j$ has been used, together with $C^e = T_g^k$. The logarithmic determinant identity $-\log\det(1 - tB) = \sum_{n\geq 1}\operatorname{Tr}(B^n)t^n/n$ proves (4.7). The construction commutes with T_g, which proves the joint assertion. The action on M_f is free because $z \neq 0$, so taking invariants really gives the cohomology of the stated quotient. □

Write φ for Euler's totient function. If ρ has order n, the equation $\theta^\ell = \rho$ has $\varphi(\ell n)/\varphi(n)$ solutions of order ℓn. To see this, write $\rho = e^{2\pi i b/n}$, $(b, n) = 1$; the solutions of maximal order are $e^{2\pi i c/(\ell n)}$ with $c \equiv b \pmod{n}$ and $(c, \ell n) = 1$. Counting prime by prime gives the formula and the inequality

$$\varphi(\ell n)/\varphi(n) \geq \varphi(\ell). \tag{4.10}$$

For $\ell > 9$ there are at least four. In particular, one can avoid one specified value of θ while keeping its maximal order.

4.3 Positive common-point support, before and after quotient

The local Euler module at a common point is represented by the following nonnegative module. The quotient calculation also identifies characters with positive multiplicity.

Lemma 4.3. *Let g, h be coprime plane germs, with h smooth and g critical at 0. Let $f = z^m(g + z^k h)$. On the minimal equivariant embedded resolution of gh every exceptional component has $N > M$. In the actual quotient, the joint Euler module of the local Milnor fibre of f is a genuine nonnegative module. Group the exceptional divisors by attaching each dead bamboo to its first rupture. Each group has nonnegative multiplicity in every joint character. If an actual rupture lattice Q_E contains a saturated character, a generator of Q_E has positive multiplicity in its group, except that in the one-fixed-point case described below its fixed-point character must also be nontrivial. A generator has this property automatically.*

Proof. At the first blowup, $N - M = \operatorname{ord}(g) - 1 > 0$. A necessary later centre on the strict transform of h is free, since a smooth branch never passes through a satellite corner in its embedded resolution. Some strict transform of g passes through that centre, and the new difference is the old difference plus its multiplicity minus one. At a centre away from h the differences of the exceptional parents and the nonnegative strict-g multiplicity add. This proves $N > M$.

Consider the surface and its function

$$S = \{g + z^k h = 0\}, \qquad R = z^m g|_S = -z^a h|_S.$$

The map (g, h) is finite. The fibre $R = t$ is the finite inverse image of

$$g = tz^{-m}, \qquad h = -tz^{-a}.$$

Using the rug $|g|^2+|h|^2+|z|^2$, its image in the z-plane is a closed annulus, since $|t|^2r^{-2m}+|t|^2r^{-2a}+r^2$ is strictly convex in $\log r$. This is an actual nearby fibre. We give the needed collar check. A discriminant branch of the finite map has a Puiseux expression $g=uh^c(1+o(1))$, $c>0$. Choose the outer rug level small enough to exclude every nonzero solution of $z^k=-u$ coming from $c=1$. The coordinate discriminant branches cannot meet a fibre $t\neq 0$. If branch points on the above annuli had a limit outside every smaller rug level as $t\to 0$, their equations would give either a point of a noncoordinate discriminant branch on a punctured axis, or $g=h=0$, $z\neq 0$. The former is impossible after shrinking the representative, and the latter is exactly the excluded $c=1$ case. Thus all branch points enter every smaller fixed collar, uniformly in $\arg t$. The intervening annular covers are unramified; radial retractions lift and commute with the finite chart group and value transport. These cofinal retractions identify the annular fibre with the Milnor fibre M_R, with its actions.

Every connected component of M_R maps with positive degree to the annulus. The pullback of dz/z is nonzero on each component. Hence there is an equivariant injection $H^0(M_R)\hookrightarrow H^1(M_R)$. Write

$$V_R=H^1(M_R)/H^0(M_R). \tag{4.11}$$

The injection makes V_R a representation with nonnegative multiplicities.

We next compare the two Euler modules on the same resolution. Over an exceptional component the normal model is

$$x^Mz^m(x^{N-M}+z^k),\qquad B_E=aN-mM>0.$$

The torus map

$$(x,z)\longmapsto(U,V)=(x^Nz^m,x^Mz^a) \tag{4.12}$$

has kernel D_E of order B_E. The fibre $U+V=1$ is the D_E-cover of $\mathbb{P}^1\setminus\{0,1,\infty\}$, whose equivariant Euler module is $-\mathbb{C}[D_E]$. Indeed every nonidentity translation has no fixed point, and the ordinary Euler characteristic of the cover is $-B_E$. The normal fibre of R is $U=t,V=-t$, a D_E-torsor with Euler module $+\mathbb{C}[D_E]$. Their two commuting actions are represented in this kernel by

$$T_f=(e^{2\pi ik/B_E},e^{2\pi i(N-M)/B_E}),\qquad A=(e^{2\pi im/B_E},e^{-2\pi iN/B_E}).$$

These formulas also fix the coefficient lines along the component. If the two coefficients are units u,v, the simultaneous normalization is

$$x'=xu^{a/B_E}v^{-m/B_E},\qquad z'=zu^{-M/B_E}v^{N/B_E}.$$

Its changes of branch lie in the same D_E and therefore give the same transition lines for the two comparisons. At a crossing, the kernel contains a $\mathbb{C}^\times$ factor on which the finite actions are translations; its equivariant Euler module is zero. Refine the crossing strata so that stabilizers preserve their components. Proper nearby-cycle base change and additivity over these strata now give, equivariantly,

$$\sum_j(-1)^j[H^j(M_f)]=-\sum_j(-1)^j[H^j(M_R)]=[V_R]. \tag{4.13}$$

Taking the exact finite invariants functor proves nonnegativity for the actual quotient. In our convention its determinant is $\mathcal{Z}_f(t)=\det(1-tT_f\mid V_R)$.

To identify a character with positive multiplicity, group the exceptional components as follows. For a joint character choose n, L, J with $\lambda = e^{2\pi i L/n}$, $\mu = e^{2\pi i J/n}$. The single root cover

$$w^n = g^L h^J, \qquad w \longmapsto \zeta^{-c} w, \quad c = (mL + aJ)/n \in \mathbb{Z} \tag{4.14}$$

fixes all its local linearizations: it is obtained from the invariant expression $(wz^c)^n = (gz^m)^L (hz^a)^J$. At an eligible crossing the same normalized line, with its full stabilizer action, occurs on both sides, with compatible phases.

A dead bamboo consists of a leaf and its bivalent chain up to a rupture. If (N, M) is the valuation vector at the rupture, the multiplicity vector at its dead tip is $(N, M)/c_0$ for an integer $c_0 \geq 2$; this is obtained by successively eliminating the intersection equations along the chain, whose determinant is c_0. Bivalent terms have zero ordinary Euler characteristic. A negative leaf term survives only when the root line extends through its attaching point with trivial full inertia. It is therefore charged to the same eligible rupture character. The smooth h supplies one distinguished fixed direction. There are at most two dead directions, and the initial valence-three configuration with two dead directions cannot occur: it would require the smooth h to pass through a satellite corner. If two older directions are present, they were created at different stages and cannot be exchanged by a stabilizer.

Here is the quotient calculation of this charge. Let E_0 be the setwise stabilizer, e the generic stabilizer, and $b = |E_0|/e$. For $b = 1$ the positive cost of the punctured rupture is $\mathrm{val}(E) - 2$. For $b > 1$ its effective cyclic action has two fixed points. If r_E is the number of puncture orbits and $t_E(\alpha)$ is the number of interior fixed points at which the selected line has nontrivial inertia, the cost in character α is

$$r_E - 2 + t_E(\alpha). \tag{4.15}$$

To verify this formula, remove both fixed points, pass to the free quotient, and take the Euler characteristic of its rank-one line. This gives $r_E - 2$ after restoring each punctured fixed point. Restoring an interior fixed point adds one exactly when its inertia is trivial. Moving these additions to the negative Euler cost gives precisely the displayed expression. If a dead direction is present, both fixed points are punctures and there is at least a third orbit; at most one dead charge remains in this case. With no dead direction, the expression is nonnegative. Its only possible zero for a rupture is one fixed h-direction, one nonfixed puncture orbit, and trivial inertia at the other fixed point. These observations prove nonnegativity of every group, including after quotient.

They also prove strictness for a generator. Put $s = \gcd(N, M)$. The character $\gamma = \lambda^{N/s} \mu^{M/s}$ on Q_E has image of order s. A generator gives a primitive image; its value γ^{s/c_0} at a dead tip is nontrivial. Thus the potentially negative leaf cannot cancel the positive rupture cost. In the exceptional no-dead case let the full stabilizer at the remaining fixed point have order eb and use its normal weight β_0. Its additional character is

$$\sigma = \lambda^{(m+\beta_0 N)/(eb)} \mu^{(a+\beta_0 M)/(eb)}.$$

The full point lattice is the kernel of σ in Q_E, of index b. A generator has $\sigma \neq 1$ and makes (4.15) positive. Finally a generator maximizes the order of $\theta = \lambda\mu$, so it remains saturated whenever any character in Q_E was saturated. This proves all assertions. □

4.4 Eliminating constant curve constituents

We first work with $\mu \neq 1$. On a proper curve, a nonconstant rank-one intersection complex has global cohomology only in degree zero: its possible degrees -1 and 1 are respectively invariants and

coinvariants of the rank-one representation, and are zero. Its Euler characteristic is nonnegative. A point simple has Euler characteristic one. A constant rank-one curve complex can have negative Euler characteristic only on a rational normalization. These statements remain valid after finite coarse pushforward.

Lemma 4.4. *In every saturated sector with $\mu \neq 1$, there is no geometrically constant curve constituent of $\mathcal{P}_{\theta,\mu}$. Every occurring curve intersection complex is acyclic, and there is no point simple constituent. In particular every actual local H^2 in such a sector is zero.*

Proof. Suppose there is a constant constituent on a component C_i of multiplicity $r_i \geq 2$. Choose such a component of maximal multiplicity, allowing all saturated sectors with $\mu \neq 1$. There are finitely many constituents and hence this maximum exists. Write $G = c\prod_i g_i^{r_i}$, with weighted degrees δ_i; thus $\sum_i r_i\delta_i = m$.

The stack intersection calculation on the normalization of C_i provides a common point where the contact with H is larger than the contact with the other components of G. Here are the denominators in that calculation. For a covering branch B, put

$$e_B = I(B,H), \qquad b_B = \sum_{j\neq i} r_j I(B,g_j).$$

The sums below run over branch orbits on the normalization of C_i above its intersections with $H = 0$ and the other components of $G = 0$; an intersection multiplicity is zero when the corresponding function is a unit. An orbit with stabilizer H_B contributes $e_B/|H_B|$ and $b_B/|H_B|$, respectively. Stack Bézout gives

$$\sum_B \frac{e_B}{|H_B|} = \frac{a\delta_i}{w_1w_2w_3}, \qquad \sum_B \frac{b_B}{|H_B|} = \frac{(m - r_i\delta_i)\delta_i}{w_1w_2w_3}.$$

The first is strictly larger. Therefore one actual covering branch has $e_B > b_B$, with the generic stabilizers included in both sums.

At this point h is smooth. On its free blowup path, the valuation of a branch q at the jth point is at least $\min(j, I(q,h))$. Until the chosen branch leaves h, its contribution gives $N \geq r_iM$. After that separation a different global component could keep the same centre on h only if its intersection with B were at least e_B, contrary to $b_B < e_B$. If the chosen component alone continues, its valuation minus that of h cannot decrease. Off this path the differences of parent valuations add. Thus every component needed between this branch and its first rupture has

$$N \geq r_iM. \tag{4.16}$$

The assumed constant *coarse* line extends at the puncture with its full stabilizer character. In the root cover (4.14), eligibility propagates along a bivalent chain to a rupture. It cannot end at h, whose character is $\mu \neq 1$. If the pair was already transverse, the puncture character is μ itself, which directly contradicts constancy.

For the reached rupture, the image orders of the coordinate characters of Q_E are, by the two by two presentation,

$$B_\lambda = \frac{aN - mM}{\gcd(eM, a + \beta M)}, \qquad B_\mu = \frac{aN - mM}{\gcd(eN, m + \beta N)}. \tag{4.17}$$

Indeed the first denominator is the gcd of the entries in the other column after multiplying by e, and likewise for the second. Using (4.16) and (4.6),

$$B_\lambda \geq \frac{aN - mM}{eM} > \frac{a(r_i - 1)}{e} > 9(r_i - 1) > r_i, \qquad B_\mu \geq \frac{a - m/r_i}{e} > \frac{9(r_i - 1)}{r_i} > 2.$$

Choose a generator of Q_E. It is saturated, its μ is not one, and Lemma 4.3 gives a positive actual local Euler multiplicity. In a curve-supported perverse sheaf such a positive stalk Euler multiplicity requires a point constituent: curve intersection complexes have stalk only in degree -1 at the point. Since the global complex of this sector is zero, additivity of Euler characteristic forces a negative curve constituent somewhere. It is a constant rational curve line. Its generic layer has $\lambda^{r_j} = 1$, so $r_j \geq \operatorname{ord} \lambda = B_\lambda > r_i$. This contradicts the choice of r_i.

All remaining simples are nonconstant curve complexes and point complexes. Global cohomology on the Serre category they generate is concentrated in degree zero and is exact, as follows at once from the long exact sequence and the preceding cohomological calculation for each simple. Since the global complex of the sector is zero, every simple is acyclic; no point simple can occur. Their stalks, and hence all their extensions, have zero ordinary degree-zero stalk. This stalk is the actual local H^2 of f. □

Two local details are needed to exclude constant lines with $\mu = 1$. We state them together and give the actual boundary comparison.

Lemma 4.5. *At an h-unit point a geometrically constant coarse puncture line of a saturated genuine layer forces g to be a power of a smooth branch. At a common point, a constant layer with $\lambda = \theta$, $D = \operatorname{ord} \theta > 1$, $D \mid m$, forces the local leading branch to be unique and smooth and transverse to h.*

Proof. For the first assertion use the same compact representatives as in Lemma 4.2, but restrict the plane coordinate to the boundary of its Milnor representative. Let this vertical boundary be Y, and put $B = H^0(\partial M_g)$, $U = H^0(M_g)$. For a T_g-character $\lambda \neq 1$, the projection to P_ε has $R^1 q_* \mathbb{C} = j_! B_\lambda$; the central boundary action has only the trivial plane character. The same vanishing of the base degree-two term as in (4.8) gives

$$H^2(Y)_\lambda = H^1(P_\varepsilon, j_! B_\lambda).$$

The actions S, C and the quotient average are unchanged. On the other hand, project Y onto each actual branch-link circle. The normal plane fibre $z^m(u^r + z^k)$ is connected: its non-z factor is reduced. Its nontrivial θ-part of H^0 is zero. The circle Leray sequence therefore identifies the selected H^2 with H^1 of the actual coefficient line on that coarse circle. A constant puncture line gives a nonzero group, and hence the specific pair

$$(T_g, C) = (\lambda, \theta^{a/e}) \tag{4.18}$$

in B. This includes branch orbits, because the comparison is equivariant before taking the exact finite invariants functor.

The boundary sequence of a compact oriented surface gives an equivariant surjection $H^1(M_g) \twoheadrightarrow B_0$, where $B_0 = \ker(B \to U)$, and $B = U \oplus B_0$ as a representation of the finite semisimple characters in question. If (4.18) occurs in B_0, Lemma 4.2 and (4.10) give a saturated $\theta' \neq \lambda$, and thus a nonzero actual H^2 with $\mu' \neq 1$. This contradicts Lemma 4.4. If the pair occurs only in U, choose its orbit of connected components under C, of length b. If their H^1 were nonzero, transport around this orbit would induce a C-eigencharacter of order divisible by b and therefore by the order of the selected character in U. The same maximal lift argument again gives the contradiction. All components of a plane Milnor fibre are diffeomorphic. It follows that $H^1(M_g) = 0$.

We use here the elementary plane-curve criterion: $H^1(M_g) = 0$ if and only if g is a power of one smooth branch, up to a unit. To check the possibly nonreduced case, let b be the gcd of its

branch multiplicities and write $g = g_0^b$, absorbing a root of the unit. The fibre of g_0 is connected. If its reduced support has r branches, the link complement has first Betti number r, generated by meridians. The Wang sequence of its Milnor fibration then gives an invariant subspace of dimension $r-1$ in $H^1(M_{g_0})$. Thus $H^1 = 0$ forces $r = 1$. In that case g_0 is reduced, its first Betti number is its Milnor number, and this number is zero precisely when the branch is smooth. The fibre of g consists of b diffeomorphic copies. This proves the criterion and the first assertion.

For the second assertion let B be the selected branch of multiplicity r, so $D \mid r$. Follow its constant root line from the puncture to the first rupture of the pair gh. If B and h are not transverse, their common free path has $v(B) - v(h) \geq 0$. Additional branches and blowups off h preserve this inequality by addition, so the first reached rupture has $N \geq rM \geq DM$. If B and h are transverse, the same argument applies unless every other branch follows h. In that remaining case the first exceptional multiplicities are $N = r + S$, $M = 1$. Eligibility gives $D \mid r + S$, hence $D \mid S$ and $S \geq D$. At the first separation ray (p, q), writing $h = y$ and $B = x$, the exact valuation is

$$N = rp + qS + \text{nonnegative earlier contributions} \geq Dq = DM.$$

The root cover (4.14) propagates eligibility with the full stabilizer along the intermediate chain. Thus, unless g is one smooth branch transverse to h, an actual eligible rupture satisfies $N \geq DM$. Formula (4.17) gives

$$B_\mu \geq (a - m/D)/e > 9(1 - 1/D) \geq 9/2.$$

A generator of its cyclic Q_E is saturated, has $\mu \neq 1$, and has positive actual support by Lemma 4.3, contrary to Lemma 4.4. The no-rupture case is precisely the smooth transverse pair. If other branches were present, their separation would itself provide a rupture. This proves the claimed uniqueness as well. □

Lemma 4.6. *No saturated genuine curve layer with $\mu = 1$ is geometrically constant.*

Proof. Suppose such a layer is constant on C_i and put $D = \operatorname{ord}\theta$. Then $D \mid m, r_i$. Lemma 4.5 makes this stack curve smooth, disjoint from all other leading components, and transverse to H at their intersections. Distinct positive-degree curves on $\mathcal{E}$ have positive intersection, so there are no other components: $G = cP^r$.

Let e be the generic stabilizer of C_i, let $A = a/e > 9$, and rigidify the generic gerbe. On the resulting smooth orbifold curve $\overline{C}$, the line bundle $L_0 = \mathcal{O}_{C_i}(e)$ has positive degree δ. At a point let b_P be the relative stabilizer order; its full order is eb_P. Since every intersection with H is simple in a covering chart,

$$A\delta = \sum_{P \in C_i \cap H} \frac{1}{b_P} > 0. \tag{4.19}$$

Choose σ primitive of order A. Since $\gcd(D, A) = 1$, the sector with $\lambda = \theta$, $\mu = \sigma$ has new value character of order DA and is saturated. It is an actual generic layer, by (4.3), and it is nonconstant and acyclic by Lemma 4.4. At an H-unit orbifold point, $b_P \mid A$ and changing μ multiplies the coarse inertia by σ^{A/b_P}, which has order b_P. The original line was constant. Thus every such orbifold point is visible to the new line. At a zero of H, its pullback inertia is σ, since the zero is simple; trivial coarse inertia would imply trivial pullback inertia. Hence every H point is also visible. There are no other punctures.

For a rank-one nonconstant line on a compact curve of genus g, with v nontrivial puncture inertias, the Euler characteristic of its intersection complex is $2g - 2 + v$. Acyclicity and $v \geq 1$

force $g = 0$, $v = 2$. Write the relative orders at these two points as b_1, b_2, allowing $b_i = 1$. They include every orbifold point. The $\operatorname{lcm}(b_1, b_2)$th power of L_0 descends to the coarse $\mathbb{P}^1$, so

$$\delta \geq 1/\operatorname{lcm}(b_1, b_2).$$

If there is one H point, (4.19) implies $a \leq eb_2$; if there are two, it implies $a \leq e(b_1 + b_2)$. Every full point stabilizer eb_i is at most $\max w_j$. Both inequalities contradict (4.6). Thus the original constant line cannot exist. □

4.5 The normal cover and the entire leading fibre

For the full-support term with $\mu = 1$, we first compute the exceptional cohomology of the normal cover.

Lemma 4.7. *Fix $D > 1$ with $D \mid m$ and a saturated character χ_0 of order D. At every point of C, normalize the cyclic surface cover $w^D = g$ with its actual chart linearization. The χ_0-part of the H^1 of its exceptional fibre is zero. Moreover, with*

$$C_0 = P(-m/e_0)B^{-1}, \qquad C_0^{e_0} = T_g^{-m}, \tag{4.20}$$

one has

$$H^1(M_g)_{(T_g, C_0)=(\chi_0, 1)} = 0. \tag{4.21}$$

This means the entire simultaneous generalized block.

Proof. The actual lift is $w \mapsto \zeta^{-m/D} w$: it comes from the invariant root of $z^m g$ after dividing it by $z^{m/D}$. Let E' be the exceptional fibre of an equivariant resolution and L its oriented link. A regular neighbourhood retracts onto E'. Its intersection matrix is negative definite. The long exact sequence of the neighbourhood and its boundary, using $H^3(E') = 0$, gives equivariantly

$$H^1(E', \mathbb{Q}) = H^1(L, \mathbb{Q}). \tag{4.22}$$

This is applied on each connected normal component. The open book of w on L has page M_g and return T_g^D. Restriction $H^1(L) \to H^1(M_g)$ is injective. Indeed, by the Wang sequence a class in its kernel is an angular base class on each orbit of page components. Every such orbit has binding. A meridian at its binding bounds a disc in L but winds a positive nonzero number of times around the angular base. The coefficient of that angular class must therefore vanish. The deck character on the page is T_g, and the chart return is C_0 in (4.20). Exact finite invariants yield

$$H^1(E'_{\text{actual}})_{\chi_0} \hookrightarrow H^1(M_g)_{(\chi_0, 1)}. \tag{4.23}$$

Additional cyclic quotient resolutions only add rational chains.

At an h-unit point, $C = T_g^{a/e_0} C_0$. A nonzero target in (4.23), or in (4.21), gives the pair $(T_g, C) = (\chi_0, \chi_0^{a/e_0})$ in Lemma 4.2. Maximal lifting with $a/e_0 > 9$ produces a saturated value character different from χ_0. This contradicts Lemma 4.4.

At a common point we first prove the exceptional assertion. In the minimal resolution of gh, count only exceptional and strict-g neighbours when defining a *g-rupture*; the strict-h arrow is not counted. If the exceptional H^1 of the normal root cover has a nonzero actual χ_0 part, some g-rupture must be eligible:

$$\chi_0^N = 1, \qquad \chi_0^{(m+\beta N)/e} = 1. \tag{4.24}$$

Here is a graph proof which also includes the stabilizers. Normalize the root cover over the plane resolution. A component with g-valence at most two has a cyclic cover branched at at most two points, so its normalized components are rational. Their component H^1 is zero. For the graph part use

$$0 \to H^0(E') \to \bigoplus_v H^0(E'_v) \to \bigoplus_{vw} H^0(E'_v \cap E'_w) \to H^1(\text{dual graph}) \to 0.$$

A selected edge requires both normal meridians and the full crossing stabilizer to act trivially. The principal-divisor relation propagates such eligibility through a bivalent vertex. The normalized root line there extends to a constant line on $\mathbb{P}^1$, with the identical attaching representation; this follows also directly from the single cover (4.14). In the absence of an eligible g-rupture, every surviving selected graph complex is consequently a finite path with identical one-dimensional vertex and edge lines. Its H^1 is zero. Finite invariants preserve this calculation, including orbits of paths. This proves (4.24).

Every g-rupture satisfies

$$N \geq 2M. \tag{4.25}$$

To prove this bound for the g-ruptures, choose $h = y$ and a transverse parameter x. For a reduced branch B_i write n_i for its multiplicity, r_i for its multiplicity in g, and $\alpha_i = I(B_i, h)/n_i$ for its first contact. At a first separation ray (p, q) on the path to h,

$$M = q, \qquad N = \sum_i r_i n_i \min(q, p\alpha_i).$$

If $p > 1$, a branch departing at that contact has $p \mid n_i$, so the sum of $r_i n_i$ still present is at least two. The same conclusion holds for $p = 1$ unless just one smooth reduced branch remains. That exception creates only the chain separating two smooth branches and no g-rupture. Otherwise the displayed formula gives $N \geq 2q$. Subtrees off the h path attach where a singular, multiple, or several-branch collection departs; the same inequality holds there and is preserved by addition of parent differences $N - 2M$. Earlier subdivision components have g-valence at most two. This proves (4.25) for exactly the vertices needed.

The eligible pair $(\chi_0, 1)$ belongs to Q_E. By (4.17) and (4.25),

$$B_\mu \geq (a - m/2)/e > 9/2.$$

A generator of Q_E is saturated and has $\mu \neq 1$, and Lemma 4.3 gives strictly positive actual support. This contradicts Lemma 4.4. Thus $H^1(E'_{\text{actual}})_{\chi_0} = 0$.

It remains to strengthen this to the whole page in (4.21). Let L° be the binding complement. The Wang sequence surjects from $H^1(L^\circ)$ onto the T_g^D-invariant part of the page, and the binding Thom sequence contains

$$H^1(L)_{\chi_0} \longrightarrow H^1(L^\circ)_{\chi_0} \longrightarrow H^0(\text{binding})_{\chi_0}(-1). \tag{4.26}$$

The first term is zero by the exceptional assertion. A nonzero selected invariant page would therefore give a nonzero actual binding character. Above a branch of multiplicity r, its normal inertia has order $D/\gcd(D, r)$ and acts trivially on the binding. A faithful character can occur only if $D \mid r$. Writing $g = x^r v$, the normalized binding root is $u = w/x^{r/D}$ with $u^D = v|_B$. Its inherited chart character is $\zeta^{-(m+\beta r)/D}$, including the corresponding exceptional factors at a corner. A binding invariant thus says exactly that this root line is constant around the coarse

puncture with its full stabilizer action. Lemma 4.5 and its weak-rupture argument exclude this unless g is a power of one smooth branch; that case already has zero page H^1.

Finally, on a nonzero simultaneous generalized $(\chi_0, 1)$ block, the nilpotent operator $T_g^D - 1$ has a nonzero kernel. On that kernel $C_0^{e_0} = T_g^{-m} = 1$. Hence C_0 is semisimple there and its selected character is still one. Such a block would give the forbidden invariant page. The same proof applies to χ_0^{-1}. □

Proposition 4.8. *Under the hypotheses of Theorem 4.1,* $\mathrm{Sat}_d(F, 0)$ *implies*

$$H^j(\{G = 1\}, \mathbb{C})_\theta = 0 \quad \textit{for every } j \textit{ and every } d \mid \operatorname{ord}\theta. \tag{4.27}$$

All saturated actual local H^2 groups of the weighted blowup also vanish, including $\mu = 1$.

Proof. Fix a saturated χ_0 of order D dividing m. Normalize the cyclic cover $u^D = G$ on $\mathcal{E}$, where u is a section of $\mathcal{O}_{\mathcal{E}}(m/D)$, and take its coarse normal surface X. It is finite over the coarse weighted projective plane. Every deck-character summand of the direct image of $\mathcal{O}_X$ is rank-one reflexive: normal surfaces are Cohen–Macaulay, finite pushforward preserves the depth condition, and character summands preserve it. On a well-formed weighted projective plane such sheaves are $\mathcal{O}(t)$. Their H^1 is zero, by the graded local-cohomology calculation for the polynomial ring: its depth is three, so the local-cohomology module of degree two supported at the irrelevant ideal vanishes. A non-well-formed plane is regraded to its well-formed coarse plane; the divisor class and the root linearization are carried along. Thus $H^1(X, \mathcal{O}_X) = 0$. The analytic exponential sequence, whose map on global constants $\mathbb{C} \to \mathbb{C}^\times$ is surjective on each proper connected component, gives

$$H^1(X, \mathbb{Q}) = 0. \tag{4.28}$$

For a projective equivariant resolution $\pi : Y \to X$, connected fibres and the low-degree Leray sequence give

$$H^1(Y, \mathbb{Q}) \hookrightarrow \bigoplus_P H^1(\pi^{-1}(P), \mathbb{Q}).$$

The selected part on the right vanishes by Lemma 4.7. Outside G only quotient singularities of smooth sheets occur, and their resolutions have rational trees. Hence $H^1(Y)_{\chi_0} = 0$. The decomposition theorem for a resolution of a normal surface adds only point-supported degree-two terms to its intersection complex. Thus $IH^1(X)_{\chi_0} = 0$; duality and the inverse character give $IH^3(X)_{\chi_0} = 0$ as well.

Let K_{χ_0} be the full-support intersection complex extending the initial root line, on the coarse plane. Equivalently it is the selected summand of the finite pushforward of IC_X. It follows that

$$\chi_{\mathrm{top}}(\mathcal{E}, K_{\chi_0}) = \dim IH^0(X)_{\chi_0} + \dim IH^2(X)_{\chi_0} + \dim IH^4(X)_{\chi_0} \geq 0.$$

It is the unique full-support constituent of $\mathcal{P}_{\chi_0,1}$. Every curve constituent is nonconstant by Lemma 4.6 and has nonnegative Euler characteristic; a point constituent has characteristic one. The global nearby-cycle complex is zero. Additivity therefore forces K_{χ_0} to be acyclic, every occurring curve complex to be acyclic, and the absence of point simples. Normal-surface intersection complexes have point stalks only in degrees $-2, -1$; curve complexes only in degree -1. Exact sequences then show that every selected local degree-zero stalk of $\mathcal{P}_{\chi_0,1}$ is zero. This is the asserted actual local H^2 vanishing.

To pass from the compact bulk to the entire open fibre we need the actual boundary morphism. Let $j : U = \mathcal{E} \setminus C \hookrightarrow \mathcal{E}$. This is affine: a Cartier multiple of the leading divisor has the same

complement. Thus Rj_* is perverse exact on $\mathcal{L}_{\chi_0}[2]$. Finite coarse pushforward is exact as well. Put

$$Q_{\chi_0} = Rj_*\mathcal{L}_{\chi_0}[2]/K_{\chi_0}.$$

We verify the sheaf identity

$$Q_{\chi_0} = \bigoplus_{C_i} \mathrm{IC}_{C_i}(\mathcal{L}_i), \tag{4.29}$$

where only the eligible $\mu = 1$ lines occur.

At a point, the local twisted Wang sequence and (4.21) give $R^2j_*\mathcal{L}_{\chi_0} = 0$; $R^1j_*\mathcal{L}_{\chi_0}$ is the selected connected-component module $U_g = H^0(M_g)$. The degree -2 stalk of K_{χ_0} is the same normal-component module as that of Rj_*, since removing a divisor from a normal connected germ does not disconnect it. Its degree -1 stalk is the local exceptional H^1, which is zero. Consequently $Q_{\chi_0} = (R^1j_*\mathcal{L}_{\chi_0})[1]$ as an ordinary shifted sheaf. This calculation also controls possible unipotent terms: on a generalized block, $T_g^D - 1$ has kernel and cokernel with the same C_0-character, and the whole selected block is zero by Lemma 4.7.

There is a natural meridian-residue map from this sheaf to the sum of the normalization extensions of the branch lines. Its target stalk is $B_g = H^0(\partial M_g)$ in the selected character. Its source is U_g, and on a visible branch of multiplicity r it is

$$c \longmapsto (r/D)c. \tag{4.30}$$

Indeed $w = x^{r/D}u$ gives $d\log w = (r/D)d\log x + d\log u$. The positive factors in (4.30) form an invertible diagonal matrix commuting with all finite actions. The surface boundary sequence injects B_g/U_g into $H^1(M_g)^\vee$. Equation (4.21), for the character and its inverse, makes this quotient zero in the actual selected block. Thus the residue map is an isomorphism on every stalk, including singular, multiple-branch and orbifold points. This proves (4.29).

Every summand in (4.29) occurs generically in $\mathcal{P}_{\chi_0,1}$ and is acyclic, as just proved. The bulk is acyclic too, so $R\Gamma(U, \mathcal{L}_{\chi_0}) = 0$. Finally $\{G = 1\}$ is the actual μ_m-cover of the weighted projective complement stack. Its selected cohomology is exactly this complex, with the finite stabilizer invariants retained on the coarse space. All its eigenvalues have order dividing m, and this proves (4.27). □

The open-part calculation in §4.10 applies (4.27) to the coordinate fixed fibres of $G = 1$.

4.6 Character integrals on a normal surface

The h-unit charts will lead to an integral on a normal quotient surface, with a differential form supported on its zero divisor. The following precise version keeps that form.

Lemma 4.9. *Let o be a rational point of a normal algebraic surface X over $\Bbbk \subset \mathbb{C}$. Let h be a nonzero regular function with $h(o) = 0$, suppose $X \setminus h^{-1}(0)$ is smooth, and let η be a rational two-form with no zeros or poles there. For a good place and a tame character λ of order $D > 1$, put*

$$Z_{h,o,\eta}(\lambda, s) = \int_{\mathrm{red}^{-1}(\bar{o}) \cap X_{\mathrm{reg}}(K)} \lambda(\mathrm{ac}\, h)|h|^s\, |\eta|,$$

where red *is the reduction map of a fixed integral model. If $\mathrm{Sat}_D(h, o)$ holds for the complex germ, then this integral, and each coefficient with a specified value of* $\mathrm{ord}\, h$*, is zero outside finitely many places. Zeros and poles of η on $h^{-1}(0)$ are allowed. The same assertion holds at a closed point, with its residue extension and descent data.*

Proof. Take a minimal good resolution of the normal pair, with exceptional multiplicities $N_i > 0$, intersection numbers $-b_i$, genera g_i, and valencies counting the strict transform arrows. Valencies and sums count intersection edges, including repeated edges. On each dead bamboo contract rational (-1) components whose contraction preserves the normal-crossing pair. All self-intersection numbers on the resulting dead bamboo, including its leaf, satisfy $b_i \geq 2$. The principal-divisor equations are

$$b_i N_i = \sum_{j \sim i} N_j. \tag{4.31}$$

The Euler exponent of i is $2 - 2g_i - \mathrm{val}(i)$. Call a vertex eligible if $D \mid N_i$. There can be no eligible vertex with negative Euler exponent. Indeed choose one of maximal multiplicity B. A primitive Bth root occurs in its factor in the monodromy Euler product. To cancel it there must be a positive exponent, hence a rational leaf of multiplicity divisible by B. Follow its dead bamboo. Along a minimal dead bamboo the intermediate self-intersections have $b_i \geq 2$; (4.31) makes its multiplicities strictly increasing positive multiples of the leaf multiplicity. The other endpoint is a vertex of negative Euler exponent with multiplicity greater than B. If it ended at a strict arrow without such a vertex, the whole connected graph would be that chain and there would be no initially chosen negative vertex. This contradicts maximality. The recurrence uses only these dead bamboos and applies equally when the rest of the graph contains cycles or positive-genus vertices.

The same argument excludes an eligible rational leaf. Thus every eligible vertex is rational bivalent. A genus-one valency-zero vertex cannot occur in the connected graph of a nonzero principal divisor with a strict arrow. If two adjacent normal divisors are eligible, (4.31) propagates eligibility along their bivalent chains. Either it reaches a forbidden vertex or the whole graph is a rational chain between two eligible strict arrows. The latter is also excluded by the following H^0 calculation. For a primitive Dth root, its $R^0\psi$ line has trivial meridians at every crossing of this chain. It extends as a constant one-dimensional line on each $\mathbb{P}^1$, and the attaching lines coincide by the single normalized root cover of h. The chain is a tree, so the lines glue to a nonzero global section. Proper nearby-cycle base change gives a nonzero central H^0 of that character, contrary to $\mathrm{Sat}_D(h, o)$.

Every remaining eligible component is therefore an isolated $\mathbb{P}^1$ with two ineligible ends. All other charts have a nontrivial normal unit character and integrate to zero. On an isolated eligible component the coefficient line has nontrivial inertia at an end. Lemma 2.2 makes its full Frobenius coefficient zero. In local resolution coordinates write $\eta = v \prod y_i^{\nu_i - 1} dy_1 \wedge dy_2$. The factors $q^{-\nu_i n_i}$ depend on the normal valuations but not on the point of the open component; thus its zero coefficient occurs before these factors are multiplied. This proves vanishing with the given form, even when some ν_i are nonpositive. Work first in the half-plane where the complete integral converges; equivalently the argument can be applied one bounded valuation region at a time. Nonsplit components or intersections cause no change: the normal characters on residue extensions are composed with surjective norm maps, and a zero geometric compact-support complex has zero trace for every descent datum. Strict open arrows do not lie above the central tube. If the pair was already smooth with normal crossings, the same conclusion is obtained directly by its normal unit averages (or by one blowup).

Every valuation coefficient is a finite sum of the just described zero coefficients. Alternatively introduce the variable $T^{\mathrm{ord}\, h}$ in an absolutely convergent half-plane; uniqueness of its power series gives the same fixed-valuation assertion. All reductions use a fixed resolution and finitely many root covers, so deleting finitely many primes suffices. □

4.7 The complete unit tubes and all Fourier characters

At an h-unit point the actual chart order e divides a and $\gcd(m,e)=1$. Absorb the harmless unit h in the topological calculation, but retain its residue constant in the integral. Put $Y=\mathbb{C}^2/\mu_e$ and

$$h_0=g^e\in\mathcal{O}_Y,\qquad eu+mv=1,\qquad \ell=a/e.$$

The character of g is faithful, so $Y\setminus h_0^{-1}(0)$ is smooth. Its fibre $h_0=\varepsilon$ is identified with one g-fibre, with all its connected components. Its actual monodromy is

$$J=P(1/e)B^v,\qquad J^e=T_g,\qquad J^k=C. \tag{4.32}$$

The last identity uses $vk\equiv -1 \pmod e$. This identifies the root of T_g determined by the chart action.

The original quotient torus has the invariant coordinate

$$S=zg^v,\qquad R=z^e,\quad w=z^mg,\qquad S=R^uw^v,\quad R=S^eh_0^{-v},\quad w=S^mh_0^u. \tag{4.33}$$

Thus on $g\neq 0$ it is $\mathbb{G}_m\times(Y\setminus h_0^{-1}(0))$. These are integer identities of invariant coordinate rings. They describe every original K-point, including those whose covering-chart lift requires a ramified extension. At a nonvertex toric slice with nonzero coordinate weights (ep_1,ep_2), $\gcd(p_1,p_2)=1$, choose integers c_1,c_2 with $p_1c_1+p_2c_2=1$. The monomials $X_1^{c_1}X_2^{c_2}$ and $X_1^{p_2}X_2^{-p_1}$ give the integral radial and tangent coordinates. Their exponent matrix has determinant $-p_1c_1-p_2c_2=-1$, so they form a unimodular lattice basis and give the cyclic normal slice used in (4.33).

On the cover, the original form is $z^{W-1}dz\wedge dx\wedge dy$ up to a nonzero constant. If the determinant character of the plane chart is δ modulo e, choose τ with $m\tau\equiv\delta \pmod e$. Then $\eta=g^\tau dx\wedge dy$ descends to a rational form on Y whose zeros and poles are supported on $h_0=0$. Invariance of the original form gives $W+\delta\equiv 0 \pmod e$, and hence

$$\Omega=S^Wh_0^\beta\frac{dS}{S}\wedge\eta,\qquad \beta=-(vW+\tau)/e\in\mathbb{Z}. \tag{4.34}$$

For $n=\operatorname{ord}h_0$ and $r=\operatorname{ord}S$, the original radial condition $\operatorname{ord}R>0$ is exactly

$$r\geq\lfloor vn/e\rfloor+1. \tag{4.35}$$

The remaining coordinates range over the complete central Y-tube. For nonvertex slices the same statement holds after the displayed unimodular change: the other fixed valuations only alter integral cutoffs, and never the full unit variables.

In these coordinates the actual function is

$$F=S^mh_0^u+H(y)S^ah_0^{-v\ell}. \tag{4.36}$$

The ratio $H(y)/H(o)$ belongs to $1+\mathfrak{p}$ on the complete base tube. Exclude primes dividing k and replace S by $S(H(y)/H(o))^{1/k}$. This analytic principal-unit change preserves every valuation and every cutoff. Its base derivatives disappear after wedging with η. The function is multiplied by a principal unit of tame character one and absolute value one. Thus $H(y)$ may be replaced by its actual constant $c=H(o)$.

Set $b=(1+vk)/e$. The identities

$$mb+uk=\ell,\qquad ab-v\ell k=\ell \tag{4.37}$$

show that $(h_0, S) \mapsto (t^k h_0, t^b S)$ multiplies the function in (4.36) by t^ℓ. Fix n, r, write $h_0 = \pi^n \tau_0$, and integrate over the complete unit part of S. Denote its angular and absolute-value kernel by $K_{n,r}(\tau_0; s)$. Equation (4.37) gives exactly

$$K_{n,r}(t^k \tau_0; s) = \chi(t)^\ell K_{n,r}(\tau_0; s). \tag{4.38}$$

It remains true on the balanced locus and on every deeper cancellation locus. Since raising principal units to the kth power is bijective, this kernel is constant on their cosets. Its finite Fourier expansion on the residue units can therefore contain only characters satisfying

$$\lambda^k = \chi^\ell. \tag{4.39}$$

Lemma 4.10. *If the plane germ g is not a power of one smooth branch, the integral over the complete original h-unit tube is zero for every tame χ with* $\mathrm{Sat}_{\mathrm{ord}\,\chi}(F, 0)$.

Proof. Let a character λ satisfying (4.39) have order L. If an eigenvalue ν of J on $H^1(M_g)$ has order divisible by L, write $n = \mathrm{ord}\,\nu$ and $B = \mathrm{ord}(\nu^k) = n/\gcd(n, k)$. Then

$$\frac{d}{\gcd(d, \ell)} = \mathrm{ord}(\lambda^k) \mid B, \qquad d \mid \ell B.$$

The actual pair in Lemma 4.2 is $(T_g, C) = (\nu^e, \nu^k)$ by (4.32). Its maximal lifts have order ℓB, and at least four exist by (4.10). Choose one different from ν^e. It is saturated and has $\mu \neq 1$, contradicting Lemma 4.4. Thus every such λ has saturated spectral absence on H^1 of the actual h_0-fibre. If $\lambda = 1$, the same argument applies to every ν and forces $H^1(M_g) = 0$, which is the excluded smooth-power case.

It remains to include H^0. If a nontrivial λ occurs there, its order divides the number b_0 of components of the fibre. Actual monodromy permutes these components in a single cycle, since the punctured total space of the normal germ is connected. If their H^1 is nonzero, the cyclic companion matrix has characteristic polynomial $\det(1 - t^{b_0} J_0)$, where J_0 is the return on one component. A maximal lift of any eigenvalue of J_0 has order divisible by b_0, hence by L. This is forbidden by the preceding paragraph. Therefore an H^0 obstruction also forces $H^1 = 0$, again a smooth power. We have proved the full central condition $\mathrm{Sat}_L(h_0, o)$ for every occurring Fourier character; none is trivial.

Apply Lemma 4.9 with its actual form η. Each fixed-n coefficient is zero. On a fixed pair n, r the remaining factors in (4.34) are constant, and the cutoff (4.35) is independent of angular components. Substituting the finite Fourier expansion from (4.39) therefore makes each such original sector zero. Sum in the convergence half-plane and then use rational continuation. This proves the complete-tube assertion. □

4.8 Smooth multiple curves: the original measure and two ends

We first justify the common-point cleanness needed to identify the ends. Suppose a saturated pair (λ, μ), with $\mu \neq 1$, gives a constant actual puncture line on a strict g-branch at a common point. This includes the formal $\lambda = 1$ line. The single root cover (4.14) extends this full inertia along the bivalent path. It cannot end on h, whose character is $\mu \neq 1$. It reaches an eligible rupture unless the pair is already smooth transverse, when its puncture inertia is μ and is not trivial. At a rupture, a generator of Q_E is saturated and strictly supported by Lemma 4.3. Proposition 4.8 has now killed every actual saturated point H^2, including $\mu = 1$, so this is impossible. To justify this

last use of positivity directly, $g + z^k h$ is reduced: a repeated factor would divide its z-derivative $kz^{k-1}h$, contrary to $\gcd(g, h) = 1$. Hence $z^m(g + z^k h)$ has connected Milnor fibre. In a nontrivial θ block its H^0 is zero and its positive Euler multiplicity is $\dim H^2 - \dim H^1 > 0$, which entails nonzero H^2. Thus the puncture line has nontrivial full inertia.

Let C_i have multiplicity $r \geq 2$, and suppose Q_C admits a saturated sector. Choose its generator. Its λ has order r, and its μ has order

$$rA_C/\gcd(r, L_C) \geq A_C > 9.$$

It is a genuine layer and is acyclic by Lemma 4.4. Every nonsmooth or multiple-branch h-unit point is a visible puncture, by the boundary comparison in Lemma 4.5. Every common point is visible by the preceding cleanness. At a smooth h-unit orbifold point with full stabilizer eb, the actual point lattice is an index-b subgroup of Q_C; the generator has nontrivial quotient inertia there. Thus this one line sees all genuine ends and all relative orbifold ends of the normalization. There is at least one common point because $H|_{C_i}$ has positive degree. The Euler formula $2g - 2 + v = 0$ for its acyclic intersection complex gives

$$\overline{C}_i = \mathbb{P}^1, \qquad C_i^\circ = \mathbb{P}^1 \setminus \{P_1, P_2\}. \tag{4.40}$$

On C_i° the generic stabilizer is constant and all normal coefficients are units.

Every actual genuine saturated layer is nonconstant by Lemmas 4.4 and 4.6. It consequently has nontrivial inertia at both points of (4.40). A formal layer with $\lambda = 1$, $\mu = \theta \neq 1$ is also nonconstant: at one of the nonempty set of H points its inertia is nontrivial by the just proved common-point cleanness. On the two-puncture curve it is therefore nontrivial at both ends as well. Lemma 2.2 gives zero compact-support cohomology for *all* these coefficient lines. The formal channels use these rank-one coefficient lines.

Here is the complete arithmetic calculation on this open curve. We first justify an exact normal form on the entire tube. The units in $g = x^r u(x, y)$ and $h = v(x, y)$ need not initially be independent of x. Put $U(y) = u(0, y)$, $V(y) = v(0, y)$, $b = u/U$, and $c = v/V$. On the complete normal residue tube, b, c are principal units. Outside the primes dividing ar, the unique principal-unit roots define

$$z' = zc^{1/a}, \qquad x' = x\big(bc^{-m/a}\big)^{1/r}.$$

Direct substitution gives the exact identity

$$z^m(x^r u + z^k v) = (z')^m\big((x')^r U(y) + (z')^k V(y)\big).$$

This map and its inverse are analytic on the complete tube, preserve all valuations, and have unit Jacobian. The units u, v, U, V are invariant under the generic chart stabilizer, because $e \mid a$ and $e \mid m + \beta r$. Hence the change is equivariant. More explicitly, in the invariant coordinates $R = z^e$ and $X = z^{-\beta}x$ the change is

$$R' = Rc^{1/A_C}, \qquad X' = Xb^{1/r}c^{-L_C/(A_C r)}.$$

These are defined over the original field and preserve its entire integral valuation lattice, including sectors whose covering lift is ramified. Their Jacobian changes the absolute monomial density only by a unit. Drop the prime marks and put $t = L_C - A_C$. Then

$$F = U(y)R^{L_C}X^r + V(y)R^{A_C}. \tag{4.41}$$

The full original valuation domain is

$$\rho = \operatorname{ord} R \geq 1, \qquad \xi = \operatorname{ord} X \in \mathbb{Z}, \qquad \xi + \beta\rho/e > 0.$$

These are invariant torus coordinates. Thus arbitrary integral ρ, ξ in this domain, including the fractional covering-chart sectors, are included. With $B = (W + \beta)/e \in \mathbb{Z}$, the original differential form is

$$\Omega = \frac{1}{e} R^B X \, \frac{dR}{R} \wedge \frac{dX}{X} \wedge dy. \tag{4.42}$$

For fixed valuations, two specified nonzero residue units, and one specified tangent residue point, its measure is $q^{-B\rho-\xi-3}$ at a good prime. The factor $1/e$ in the form has absolute value one.

If $t\rho + r\xi < 0$, the sum of the two unit variables is

$$(q-1)^2 \mathbf{1}_{\chi^{L_C}=1} \mathbf{1}_{\chi^r=1} \chi(U) T^{L_C\rho + r\xi}. \tag{4.43}$$

If $t\rho + r\xi > 0$, it is

$$(q-1)^2 \mathbf{1}_{\chi^{A_C}=1} \chi(V) T^{A_C\rho}. \tag{4.44}$$

The latter is the formal channel $(\lambda, \mu) = (1, \chi)$. On the balanced locus both monomials have valuation $A_C\rho$. The contribution before its Haar factor is $T^{A_C\rho}$ times the residue sum

$$(q-1) \sum_{\substack{\lambda^r=1 \\ \lambda^t=\chi^{-A_C}}} \lambda(-U)(\chi\lambda^{-1})(V) J(\lambda^{-1}, \chi), \qquad J(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} \alpha(x)\beta(1-x). \tag{4.45}$$

To verify its normalization, let $v = R^t X^r$ on residue units. The weighted count of its preimages is $(q-1)\sum \lambda^{-1}(v)$ with the two displayed character conditions. Substitute $x = -Uv/V$ to obtain (4.45). All characters in this identity are characters of the actual residue-field multiplicative group. The two displayed constraints are the character relations of the quotient lattice.

The balanced zeros lift to smooth normal residue tubes in the logarithmic torus. Indeed $et = \beta r - k$, and at a prime not dividing ek the vector (t, r) cannot vanish modulo the residue characteristic. Hensel coordinates then make F a normal parameter times a unit. Its full deeper tube has zero nontrivial angular integral. Thus (4.45) already includes the complete balanced contribution after these zero tails are accounted for.

Under a frame change $R' = \alpha R$, $X' = \gamma X$ the units change by $U' = \alpha^{-L_C}\gamma^{-r}U$ and $V' = \alpha^{-A_C}V$. The line factor in a balanced term is unchanged, because

$$\chi^{-A_C}(\alpha)\lambda^{-t}(\alpha)\lambda^{-r}(\gamma) = 1.$$

The low and high factors have the same property under their displayed eligibility conditions. Hence they glue as the actual lines (4.4) on C_i°. Their trace sums are zero by (4.40) and Lemma 2.2. All Haar weights in (4.42) are constant for a fixed valuation pair, so this zero occurs before multiplying by the weights or summing a geometric series.

At a smooth orbifold end the full point lattice imposes the additional inertia relation. Every one of the coefficient lines just considered has nontrivial end inertia, so its full radial unit average is zero there. This makes every allowed integral valuation sector zero, including those whose covering-chart lift is ramified. If Q_C has no saturated sector, the same unit character equations have no solution and every term is zero directly. Finally, components with $r = 1$ are handled by the equivariant normal change $x' = g + z^k h$ already described after (4.6). We have proved the complete original smooth-curve contribution is zero.

4.9 Common endpoints in the original quotient lattice

We now verify the endpoint contribution coefficient by coefficient. At a critical common point h is smooth and coprime to g. There is no rupture E for which Q_E contains a saturated pair. If there were, its cyclic generator would have strictly positive actual support by Lemma 4.3, and hence nonzero actual H^2, contrary to Proposition 4.8. This statement includes all generators, even ones with $\mu = 1$.

On a resolution chart at a general point of E the actual invariants are $R = z^e$, $X = xz^{-\beta}$ and a tangent parameter. The function has the form

$$F = R^{p_E} X^N u + R^{q_E} X^M v. \tag{4.46}$$

Here too the two units can be made tangent-only on the complete original tube. Let u_0, v_0 be their restrictions to the open component and set $b = u/u_0$, $c = v/v_0$. With

$$\Delta = p_E M - q_E N = -\frac{aN - mM}{e} \neq 0,$$

the principal-unit changes

$$R' = R b^{M/\Delta} c^{-N/\Delta}, \qquad X' = X b^{-q_E/\Delta} c^{p_E/\Delta}$$

satisfy, exactly,

$$R^{p_E} X^N u + R^{q_E} X^M v = (R')^{p_E} (X')^N u_0 + (R')^{q_E} (X')^M v_0.$$

Indeed the exponents of b, c in the first new monomial are $1, 0$, and those in the second are $0, 1$. The determinant primes are excluded once. The map is an analytic bijection of the original complete tube, with unit Jacobian and unchanged valuation domain and absolute density. We henceforth suppress the primes and use u, v for these tangent restrictions. Let $\rho = \operatorname{ord} R$, $\xi = \operatorname{ord} X$, so that the original tube is

$$\rho \geq 1, \qquad \xi \in \mathbb{Z}, \qquad \xi + \beta\rho/e > 0.$$

Write π_{pl} for the plane resolution and $\nu_E = \operatorname{ord}_E(\operatorname{Jac} \pi_{\mathrm{pl}}) + 1$. In local covering-plane coordinates u_1, u_2, its form is

$$\pi_{\mathrm{pl}}^*(du_1 \wedge du_2) = \epsilon x^{\nu_E - 1} dx \wedge dy, \qquad \epsilon \text{ a unit.}$$

The pullback of the original three-form is therefore

$$\begin{aligned} \Omega &= \epsilon z^W x^{\nu_E} \frac{dz}{z} \wedge \frac{dx}{x} \wedge dy \\ &= \frac{\epsilon'}{e} R^{(W+\beta\nu_E)/e} X^{\nu_E} \frac{dR}{R} \wedge \frac{dX}{X} \wedge dy. \end{aligned} \tag{4.47}$$

The exponent $(W + \beta\nu_E)/e$ is an integer. Indeed, the unit coefficient is a semi-invariant nonzero at a point fixed by the generic stabilizer, so its character is trivial; invariance of Ω forces $e \mid W + \beta\nu_E$. At a retained place $|\epsilon'/e| = 1$. For fixed ρ, ξ, two specified nonzero angular residues and one tangent residue point have measure

$$q^{-((W+\beta\nu_E)/e)\rho - \nu_E \xi - 3}. \tag{4.48}$$

Each logarithmic unit coordinate contributes q^{-1}, as does the tangent residue ball.

Put

$$b_1 = p_E\rho + N\xi, \qquad b_2 = q_E\rho + M\xi.$$

The complete angular coefficients, before multiplication by (4.48), are

$$\begin{aligned} b_1 < b_2: &\quad (q-1)^2\chi(u)T^{b_1}\mathbf{1}_{\chi^{p_E}=1}\mathbf{1}_{\chi^N=1}, \\ b_2 < b_1: &\quad (q-1)^2\chi(v)T^{b_2}\mathbf{1}_{\chi^{q_E}=1}\mathbf{1}_{\chi^M=1}, \\ b_1 = b_2 = b: &\quad (q-1)T^b \sum_{\substack{\lambda\mu=\chi \\ \lambda^N\mu^M=1 \\ \lambda^{p_E}\mu^{q_E}=1}} \lambda(-u)\mu(v)J(\lambda^{-1},\chi). \end{aligned} \tag{4.49}$$

For the last equality, set $\tau = R^{p_E-q_E}X^{N-M}$ on the residue torus. Character orthogonality gives

$$\sum_{R^{p_E-q_E}X^{N-M}=\tau} \chi(R^{q_E}X^M) = (q-1) \sum_{\substack{\lambda^{p_E-q_E}=\chi^{-q_E} \\ \lambda^{N-M}=\chi^{-M}}} \lambda^{-1}(\tau).$$

Substitution $t = -u\tau/v$ in the remaining sum gives

$$\sum_{\tau\neq 0} \lambda^{-1}(\tau)\chi(u\tau+v) = \lambda(-u)(\chi\lambda^{-1})(v)J(\lambda^{-1},\chi).$$

Thus, with $\mu = \chi\lambda^{-1}$, every term has the coefficient line $\mathcal{L}_\lambda(u)\otimes\mathcal{L}_\mu(v)$ and satisfies

$$\lambda\mu = \chi, \qquad \lambda^N\mu^M = 1, \qquad \lambda^{p_E}\mu^{q_E} = 1. \tag{4.50}$$

The unequal cases are the pairs $(\chi,1)$ and $(1,\chi)$. The scalar $\lambda(-1)J(\lambda^{-1},\chi)$ and the powers of q,T are independent of the tangent point. At a balanced residue zero, the logarithmic derivative of the two-term sum is a nonzero unit multiple of $(p_E-q_E, N-M)$. This vector is nonzero modulo every retained residue characteristic, since $Nq_E - Mp_E = (aN-mM)/e$ is a unit. A Hensel normal coordinate therefore carries the complete deeper root tube onto a full zero ball. Its nontrivial-character integral is zero.

Call a crossing eligible when both normal meridians and its *full* crossing stabilizer act trivially on the single root line (4.14). There are no eligible crossings. Starting from one, the principal-divisor equations propagate eligibility through a bivalent chain. A cyclic stabilizer fixes the direction toward the unique h arrow and cannot exchange the two ends of such a chain. It reaches a rupture, which has just been excluded, or a strict arrow. The h arrow is ineligible if $\mu \neq 1$. At a strict g arrow the resulting constant puncture line is excluded by the common-point cleanness of §4.8 when $\mu \neq 1$, including $\lambda = 1$. If $\mu = 1$, then $\lambda = \chi \neq 1$, the strict multiplicity is at least two, and its actual line is nontrivial at both global ends by (4.40). This also excludes the crossing. An unbranched chain with no rupture and only a smooth transverse pair is the already handled smooth-normal case.

Consider the remaining eligible components individually. A bivalent component, away from its two ends, has coarse free quotient $\mathbb{G}_m$. Its actual coefficient line cannot be constant, for then it would extend at an end and give an eligible crossing. Its compact-support complex is therefore zero. A leaf has covering open part $\mathbb{A}^1$. If its effective stabilizer is trivial, its constant line would likewise extend to its attaching point, again impossible. If the effective stabilizer is nontrivial, it has one interior fixed point. Trivial inertia there gives the same contradiction. Nontrivial inertia gives a nonconstant line on the free quotient $\mathbb{G}_m$, with zero compact-support cohomology. This accounts for the free part of the leaf.

At an interior fixed point of E, let the full stabilizer have order e_0, and represent its normal and tangent characters by $0 \le \beta_0, \alpha_0 < e_0$. The invariant torus coordinates are

$$R = z^{e_0}, \qquad X = xz^{-\beta_0}, \qquad Y = yz^{-\alpha_0}.$$

For $\rho = \operatorname{ord} R$, $\xi = \operatorname{ord} X$, $\upsilon = \operatorname{ord} Y$, the original domain is

$$\rho \ge 1, \qquad \xi + \beta_0\rho/e_0 > 0, \qquad \upsilon + \alpha_0\rho/e_0 > 0, \qquad \rho, \xi, \upsilon \in \mathbb{Z}. \tag{4.51}$$

Coordinate hyperplanes have measure zero. Put

$$p_0 = \frac{m + \beta_0 N}{e_0}, \qquad q_0 = \frac{a + \beta_0 M}{e_0}, \qquad \nu_0 = \frac{W + \beta_0\nu_E + \alpha_0}{e_0}.$$

The function and original form are

$$\begin{aligned} F &= uR^{p_0}X^N + vR^{q_0}X^M, \\ \Omega &= \frac{\epsilon'}{e_0} R^{\nu_0} X^{\nu_E} Y \frac{dR}{R} \wedge \frac{dX}{X} \wedge \frac{dY}{Y}. \end{aligned} \tag{4.52}$$

The unit coefficient of the form is a semi-invariant nonzero at the fixed point. Its character is trivial, and invariance of Ω gives $\nu_0 \in \mathbb{Z}$. Three specified nonzero angular residues in a fixed valuation sector have measure

$$q^{-\nu_0\rho - \nu_E\xi - \upsilon - 3}. \tag{4.53}$$

Since $Nq_0 - Mp_0 = (aN - mM)/e_0 \neq 0$, the two-coordinate principal-unit change above replaces u, v by their values at this point while keeping Y fixed. Put $b_1 = p_0\rho + N\xi$ and $b_2 = q_0\rho + M\xi$. The angular coefficients are

$$\begin{aligned} b_1 < b_2 : &\quad (q-1)^3 \chi(u) T^{b_1} \mathbf{1}_{\chi^{p_0}=1} \mathbf{1}_{\chi^N=1}, \\ b_2 < b_1 : &\quad (q-1)^3 \chi(v) T^{b_2} \mathbf{1}_{\chi^{q_0}=1} \mathbf{1}_{\chi^M=1}, \\ b_1 = b_2 = b : &\quad (q-1)^2 T^b \sum_{\substack{\lambda\mu=\chi \\ \lambda^N\mu^M=1 \\ \lambda^{p_0}\mu^{q_0}=1}} \lambda(-u)\mu(v) J(\lambda^{-1}, \chi). \end{aligned} \tag{4.54}$$

The extra factor $q-1$ relative to (4.49) is the sum over the Y unit. Every term therefore requires the full radial relation

$$\lambda^{p_0}\mu^{q_0} = 1. \tag{4.55}$$

The nontrivial fixed-point inertia established above makes this relation fail. Hence every sector (4.51) contributes zero.

At a crossing P of two normal divisors E_i, E_j, use diagonal normal characters β_0, α_0 and the same invariant coordinates. The direction toward the unique h arrow is preserved by the cyclic action, so each of the two branches is preserved. Write $\nu_i = \operatorname{ord}_{E_i}(\operatorname{Jac}\pi_{\mathrm{pl}}) + 1$ and $\nu_j = \operatorname{ord}_{E_j}(\operatorname{Jac}\pi_{\mathrm{pl}}) + 1$; a strict-transform branch has $\nu = 1$. The formulas are

$$\begin{aligned} p_0 &= \frac{m + \beta_0 N_i + \alpha_0 N_j}{e_0}, \qquad q_0 = \frac{a + \beta_0 M_i + \alpha_0 M_j}{e_0}, \\ F &= uR^{p_0}X^{N_i}Y^{N_j} + vR^{q_0}X^{M_i}Y^{M_j}, \\ \nu_0 &= \frac{W + \beta_0\nu_i + \alpha_0\nu_j}{e_0}, \\ \Omega &= \frac{\epsilon'}{e_0} R^{\nu_0} X^{\nu_i} Y^{\nu_j} \frac{dR}{R} \wedge \frac{dX}{X} \wedge \frac{dY}{Y}. \end{aligned} \tag{4.56}$$

The same fixed-point invariance proves $\nu_0 \in \mathbb{Z}$. The original domain is (4.51), and the measure for three specified angular residues is

$$q^{-\nu_0\rho-\nu_i\xi-\nu_j\upsilon-3}. \tag{4.57}$$

The exponent columns (p_0, N_i, N_j) and (q_0, M_i, M_j) have rank two. If their normal entries were proportional, $(N_i, N_j) = c(M_i, M_j)$, the exceptional direction would give $c > 1$, whereas proportional radial entries would require $m = ca$. The latter contradicts $a > m$. A strict arrow gives independent normal entries directly. Choose a nonzero two-by-two minor, with columns (α_1, α_2) and (γ_1, γ_2), and put $\Delta_0 = \alpha_1\gamma_2 - \alpha_2\gamma_1$. If U, V are the corresponding two coordinates among R, X, Y, the changes

$$\begin{aligned} U' &= U\,(u/u(P))^{\gamma_2/\Delta_0}(v/v(P))^{-\alpha_2/\Delta_0}, \\ V' &= V\,(u/u(P))^{-\gamma_1/\Delta_0}(v/v(P))^{\alpha_1/\Delta_0} \end{aligned}$$

replace both units by $u(P), v(P)$ and keep the third coordinate fixed. Multiplication of the exponents gives $(1, 0)$ in the first monomial and $(0, 1)$ in the second. Excluding the prime divisors of Δ_0 makes these principal-unit roots analytic on the whole tube, with unit Jacobian. All valuations and the absolute density (4.57) are preserved.

Put

$$b_1 = p_0\rho + N_i\xi + N_j\upsilon, \qquad b_2 = q_0\rho + M_i\xi + M_j\upsilon.$$

The complete crossing coefficients are

$$\begin{aligned} b_1 < b_2: &\quad (q-1)^3\chi(u)T^{b_1}\mathbf{1}_{\chi^{p_0}=1}\mathbf{1}_{\chi^{N_i}=1}\mathbf{1}_{\chi^{N_j}=1}, \\ b_2 < b_1: &\quad (q-1)^3\chi(v)T^{b_2}\mathbf{1}_{\chi^{q_0}=1}\mathbf{1}_{\chi^{M_i}=1}\mathbf{1}_{\chi^{M_j}=1}, \\ b_1 = b_2 = b: &\quad (q-1)^2T^b \sum_{\substack{\lambda\mu=\chi \\ \lambda^{N_i}\mu^{M_i}=1,\ \lambda^{N_j}\mu^{M_j}=1 \\ \lambda^{p_0}\mu^{q_0}=1}} \lambda(-u)\mu(v)J(\lambda^{-1}, \chi). \end{aligned} \tag{4.58}$$

For the balanced coefficient, character orthogonality in the ratio monomial divides the three independent unit sums by $q-1$, giving $(q-1)^2$; the substitution used in (4.49) gives the displayed Jacobi sum. The selected minor also makes the logarithmic derivative at a balanced zero nonzero. Its complete Hensel root tube has zero nontrivial-character integral. All remaining terms require the two normal relations and the full stabilizer relation, which were excluded by the no-eligible-crossing argument.

Every free-component coefficient has zero compact-support cohomology, and every fixed-point or crossing coefficient is zero by these unit averages. Geometric vanishing gives zero Frobenius trace under every descent datum, including permutations of components or ends. The density factors (4.48), (4.53) and (4.57) depend only on the fixed valuations. Summing over their original domains therefore proves zero for the complete common-point contribution.

4.10 The initial open part and the fixed binary fibres

It remains to treat the part where the initial G is a unit. Write

$$T_G(x_1, x_2, x_3) = (e^{2\pi i w_1/m}x_1, e^{2\pi i w_2/m}x_2, e^{2\pi i w_3/m}x_3)$$

for the weighted monodromy on $\{G = 1\}$. The valuation sectors involve its coordinate fixed loci. We compute their cohomology, including the binary case.

We use a small fact about cyclic Euler modules. If a finite cyclic group of order m acts on a complex algebraic variety, its equivariant Euler characteristic is a virtual sum

$$\sum_{s|m} a_s[\text{orbit of length } s]. \tag{4.59}$$

One obtains this expression by an equivariant triangulation, or by adding the free orbit strata. The multiplicity of a primitive ℓth character is $n_\ell = \sum_{\ell|s} a_s$. If $n_\ell = 0$ for all $d \mid \ell$, descending divisibility gives $a_s = 0$ for all $d \mid s$. Taking the fixed set of the jth power of a generator retains exactly the orbits with $s \mid j$. Its virtual Euler module consequently has the same saturated absence. This computation is in the cyclic Burnside group and its representation ring. The cohomology of the fixed fibres is computed next.

Lemma 4.11. *Suppose* (4.27) *holds, and let I be a proper coordinate subset which is the fixed set of a power of the weighted monodromy on $\{G = 1\}$. At good places put*

$$S_I(\chi, q) = \sum_{u \in \mathbb{F}_q^I} \chi(G_I(u)), \qquad G_I = G|_{\{x_j = 0 : j \notin I\}}.$$

For the relatively prime two-term singularity of the theorem, $S_I(\chi, q) = 0$ for every χ of order $d > 1$. The same conclusion holds in the earlier case $F = G \cdot$ unit with G isolated.

Proof. If $G_I = 0$ there is nothing to prove. For a one-variable fixed set its fibre is zero-dimensional, so the virtual absence supplied by (4.59) is absence in actual cohomology. The Kummer trace sum is zero.

Let I contain two coordinates. Write their weights as (ep_1, ep_2) with $\gcd(p_1, p_2) = 1$. Since the fixed set is proper, $e > 1$. Since $G_I \neq 0$, $e \mid m$. Put $M = m/e$, the primitive weighted degree. If $d \nmid M$, the change $(y, z) \mapsto (t^{p_1} y, t^{p_2} z)$ in the original finite-field sum multiplies it by $\chi(t)^M$; choosing a nontrivial value proves zero. Hence assume $d \mid M$.

Over $\mathbb{C}$ factor the binary form as

$$G_I = c\, y^\alpha z^\beta \prod_{j=1}^{s} (z^{p_1} - c_j y^{p_2})^{n_j}, \qquad M = p_1\alpha + p_2\beta + p_1 p_2 \sum_j n_j, \tag{4.60}$$

where the nonzero c_j are distinct. Its fibre has the following exact cyclic Euler module:

$$-s[\text{orbit } M] + \mathbf{1}_{\alpha=0}[\text{orbit } M/p_2] + \mathbf{1}_{\beta=0}[\text{orbit } M/p_1]. \tag{4.61}$$

For the first term, quotient the nonzero-coordinate torus by primitive weighted scaling. Its base is $\mathbb{G}_m$ with s points removed, of Euler characteristic $-s$, and its generic fibre has an orbit of length M. On $y = 0$ the fibre is present exactly when $\alpha = 0$, and consists of a single orbit of length M/p_2; the other axis gives the last term. This proves every term and its orbit order.

The primitive M character must have Euler multiplicity zero, by (4.59) and $d \mid M$. Equation (4.61) gives

$$s = \mathbf{1}_{\alpha=0,\ p_2=1} + \mathbf{1}_{\beta=0,\ p_1=1}. \tag{4.62}$$

This forces the reduced support to be either one smooth branch or two transverse smooth branches. Here is the complete case check. If $p_1, p_2 > 1$, then $s = 0$ and only the coordinate axes occur.

If $p_1 = 1 < p_2$, then $s = 0$ when $\beta > 0$, and $s = 1$ when $\beta = 0$. In the latter case the change $z' = z - c_1 y^{p_2}$ preserves the weights and leaves $c\, y^{\alpha}(z')^{n_1}$. The case $p_2 = 1 < p_1$ is the symmetric one. If $p_1 = p_2 = 1$, equation (4.62) gives zero, one or two noncoordinate factors according as two, one or no axes already occur; there are exactly two reduced lines in total. A linear change of the equal-weight coordinates makes them the coordinate axes.

For a single smooth power the fibre is a finite union of affine lines. Its Euler module is its actual H^0, so virtual saturated absence is actual absence. For the two-branch case write, after the displayed weighted coordinate change,

$$G_I = c\, U^r V^s, \qquad r, s > 0, \qquad M = p_1 r + p_2 s. \tag{4.63}$$

Its fibre is a union of $b = \gcd(r, s)$ copies of $\mathbb{C}^{\times}$. Its H^0 and H^1 are both the regular permutation module of the b components: weighted monodromy permutes them cyclically and acts by translations, hence trivially on H^1 of each component. The only way virtual absence can fail to be actual absence is therefore

$$d \mid b, \quad \text{equivalently} \quad d \mid r, s. \tag{4.64}$$

This is the complete H^0/H^1 cancellation.

We exclude (4.64) geometrically. Let x_0 be the remaining coordinate and let $L = \{x_0 = 0\} \subset \mathcal{E}$. Its two endpoint orders in (4.63) are at least two. Primitivity of all three weights gives $\gcd(w_0, e) = 1$. Since $\gcd(m, a) = 1$ and $e \mid m$, one has $e \nmid a$; consequently $H|_L = 0$. The normal powers occurring in G are multiples of e, so $dG = 0$ at either endpoint of (4.63). Isolation forces the smallest normal power of H there to be one, with a nonzero coefficient. Thus locally $h = x_0$ times a unit and is smooth. In the case $F = G$ times a unit, the already obtained $dG = 0$ at a nonzero weighted orbit directly contradicts isolation, and no further argument is needed.

In the coprime two-term case we compute the actual root line at the endpoints. Choose a primitive dth character χ_0. On the chart of the endpoint $U = 0, V \neq 0$ the full group has order ep_2. The root $w^d = g$ has weight $-m/d$; after normalization by $U^{r/d}$ its remaining weight is

$$-\frac{m - ep_1 r}{d} = -\frac{ep_2 s}{d} \equiv 0 \pmod{ep_2}. \tag{4.65}$$

At the other endpoint the corresponding weight is $-ep_1 r/d \equiv 0 \pmod{ep_1}$. The root line on the punctured h-arrow is thus constant with its full stabilizer. The weighted coordinate changes used above preserve these weights and hence this calculation.

The same normalized root cover propagates this eligibility from the h-arrow along its bivalent chain. If it reaches a rupture, $(\lambda, \mu) = (\chi_0, 1)$ is a saturated element of Q_E. A cyclic generator has strictly positive actual support by Lemma 4.3, contradicting the actual local H^2 vanishing in Proposition 4.8. If there is no rupture, the reduced local pair is already two smooth transverse branches. Indeed any unresolved tangency creates a dead direction and a separation rupture, and a singular or several-branch g creates a rupture on its embedded resolution. Thus at each endpoint g is one smooth branch of multiplicity r or s, transverse to L.

Every global component of G has positive intersection with the positive-degree coordinate line L, which is not a component of G. Its intersection is one of these two endpoints, and its multiplicity is therefore r or s. Equation (4.64) makes every global component multiplicity divisible by d. Hence $G = cP^d$ over $\mathbb{C}$. The fibre $G = 1$ then has a nonzero primitive dth character in H^0, contradicting (4.27). This excludes the sole virtual cancellation exception.

We have proved actual saturated cohomological absence on each proper fixed fibre. To identify its arithmetic consequence, use the finite cover

$$\{G_I = 1\} \times \mathbb{G}_m \longrightarrow \{G_I \neq 0\}, \qquad (u, t) \longmapsto (t^{p_1} u_1, t^{p_2} u_2),$$

of group μ_M. The pullback of the χ-Kummer sheaf is constant in the $\mathbb{G}_m$ factor when $d \mid M$; the deck character on the other factor is the selected χ character. Its compact-support complex is zero by the just proved actual absence, duality and Künneth. The trace formula gives $S_I(\chi, q) = 0$. The same reasoning handles the one-variable case. All geometric exclusions were over $\mathbb{C}$; spreading the finite covers gives the assertion even for nonsplit axes or branches over the original residue field. □

We finish by computing the original open-part partition. Ignore coordinate hyperplanes, a set of Haar measure zero. For an original point in $\mathfrak{p}^3$ put

$$\alpha = \min_i \frac{\operatorname{ord} x_i}{w_i} = n + t, \qquad n \in \mathbb{Z}_{\geq 0}, \quad 0 \leq t < 1, \qquad I_t = \{i : t w_i \in \mathbb{Z}\}.$$

Only finitely many t occur, with denominators dividing one of the weights. Write $B_i = n w_i + \lceil t w_i \rceil$ and $x_i = \pi^{B_i} y_i$. The initial unit part can occur only if $mt \in \mathbb{Z}$; its residue condition is $G_{I_t}(\bar{y}_{I_t}) \neq 0$. All other coordinates have strictly higher weighted value, and H has strictly higher valuation because $a\alpha > m\alpha$. The exact original contribution of this sector is

$$q^{-nW - \sum_i \lceil t w_i \rceil} T^{nm + mt}\, q^{-|I_t|} S_{I_t}(\chi, q). \tag{4.66}$$

The original origin cutoff allows $n \geq 0$ for $t > 0$ and $n \geq 1$ for $t = 0$.

If $d \nmid m$, the full weighted unit action $y_i \mapsto u^{w_i} y_i$ makes each sector zero by character orthogonality. If $d \mid m$ and I_t is proper, it is exactly the coordinate fixed set of T_G^{mt} on $\{G = 1\}$, and Lemma 4.11 makes (4.66) zero. For I_t equal to all coordinates, use (4.27) itself and the same finite cover $\{G = 1\} \times \mathbb{G}_m \to \{G \neq 0\}$ with weights w_i and degree m. Its selected compact-support complex is zero, so the full trace sum is zero as well. In the preliminary case $F = G \cdot \text{unit}$ the whole-fibre condition is immediate from the weighted homogeneous Milnor fibration of G and its central spectral hypothesis. This also completes that case.

4.11 Assembly and uniformity in the character

The preceding calculations form a partition of the original origin ball, apart from sets of measure zero. The initial unit sectors are (4.66). Above the leading divisor, a smooth reduced branch has its direct normal unit average; a nonsmooth h-unit point has the complete-tube calculation of Lemma 4.10; the open smooth multiple curves and their smooth orbifold ends are treated in §4.8; and critical common endpoints, with all their fixed-point and ramified sectors, are treated in §4.9. Each zero was obtained on actual invariant coordinates before its valuation weights were summed. Consequently $Z_{F,0}(\chi, s) = 0$ for every tame character in the theorem.

We check that the excluded set of places is independent of the character. Fix one embedded resolution of the original polynomial F. A nonzero tame positive-contact coefficient requires d to divide one of its finitely many positive multiplicities. There are only finitely many such d. For each, the genuine root covers, the groups Q_C, Q_E, the normal covers, the discriminants and their algebraic resolutions are a finite collection. The extra Fourier orders in (4.39) divide $k\ell d$ and are finite in number. Spread these data and the differential forms over one ring of integers

with finitely many primes inverted. Also exclude the finitely many primes dividing the chart orders, k, the integers ar in the finitely many curve charts, and the nonzero determinants in the torus calculations. Comparison, tame base change and the trace formula then apply to all the coefficient complexes simultaneously. If d is outside the initial finite list, the original resolution already makes the integral zero. Finally Lemma 2.3, applied to that fixed original resolution, kills every higher-conductor character at the same remaining places. This proves Theorem 4.1.

Corollary 4.12. *For the polynomials in Theorem 4.1, the all-point hypothesis* $\mathrm{Sat}_d(F)$ *implies, at almost all places,*

$$Z_{F,\Phi}(\chi, s) = 0$$

for every residual function Φ *supported on original residue balls contained in* $\{\overline{F} = 0\}$*, and every character* χ *of order* $d > 1$*. For an arbitrary residual function the integral is entire; its only remaining terms are the constant unit-residue contributions.*

Proof. The origin ball is covered by Theorem 4.1. Proposition 5.1 covers the nonzero zero-fibre balls under the all-point hypothesis. On a unit residue ball, $|F| = 1$ and the integral is independent of s. Summing the finitely many residual values proves the assertion. □

Example 4.13. Put

$$Q = x^2 + y^3 + z^6, \qquad G = Q^5, \qquad H = xy^{14} + z^{31}, \qquad F = G + H.$$

The primitive weights $(3, 2, 1)$ give Q degree six and give G, H degrees $30, 31$. These degrees are coprime. The unique lowest ordinary-degree term of G is x^{10}; all its other terms have degree at least eleven. The two terms of H have ordinary degrees fifteen and thirty-one. Hence $\mathrm{ord}_0 F = 10$.

The derivatives are

$$\begin{aligned} F_x &= 10xQ^4 + y^{14}, \\ F_y &= 15y^2Q^4 + 14xy^{13}, \\ F_z &= 30z^5Q^4 + 31z^{30}. \end{aligned}$$

Their weighted Euler identity is

$$\begin{aligned} 3xF_x + 2yF_y + zF_z &= 30x^2Q^4 + 30y^3Q^4 + 30z^6Q^4 \\ &\quad + (3 + 28)xy^{14} + 31z^{31} \\ &= 30Q^5 + 31H. \end{aligned}$$

If $F(p) = 0$ and $\nabla F(p) = 0$, then

$$Q(p)^5 + H(p) = 0, \qquad 30Q(p)^5 + 31H(p) = 0.$$

Subtracting thirty times the first equation from the second gives $H(p) = 0$ and then $Q(p) = 0$. Since $d(Q^5) = 5Q^4 dQ$ vanishes there, the three critical equations reduce to

$$y^{14} = 0, \qquad 14xy^{13} = 0, \qquad 31z^{30} = 0.$$

Thus $y = z = 0$ and $Q = x^2 = 0$, so $p = 0$. The zero fibre is smooth at every nonzero point.

The origin is also an isolated critical point of the function germ. Otherwise a positive-dimensional irreducible analytic component of its critical locus would pass through zero. The differential of F vanishes on its smooth locus, so F is constant there and, by continuity, is

identically zero on that component. This contradicts the preceding critical-zero calculation. The displayed derivatives vanish at zero. Consequently F satisfies the hypotheses of Theorem 4.1.

The compact Newton face selected by $(3,2,1)$ has face polynomial Q^5. Choose $z_0 \neq 0$ with $z_0^6 = -2$ and set $p = (1,1,z_0)$. Then

$$Q(p) = 1 + 1 - 2 = 0, \qquad \nabla(Q^5)(p) = 5Q(p)^4 \nabla Q(p) = 0.$$

The face is therefore degenerate in these coordinates, and the weighted principal term has a nonisolated critical locus.

5 From the origin to residual test functions

For two terms of different degrees, the following reduction passes from the origin integral to residual test functions. It applies to ordinary degrees and to common positive weights.

Proposition 5.1. *Let w_1, w_2, w_3 be positive integers and let $G_m, H_a \in \Bbbk[x_1, x_2, x_3]$ be nonzero weighted homogeneous polynomials of degrees $1 \leq m < a$. Suppose that $f = G_m + H_a$ is smooth or has an isolated critical point at the origin. Apart from the origin, every singular germ of $f^{-1}(0)$ is either a transverse product of two smooth factors or is analytically contact equivalent to*

$$UV + cz^\ell, \qquad c \neq 0, \quad \ell \geq 2. \tag{5.1}$$

If G_m and H_a are coprime, there are only finitely many such nonzero singular points. If the origin integral is zero at almost all places under $\mathrm{Sat}_d(f,0)$, then f satisfies Denef's residual holomorphy conjecture under $\mathrm{Sat}_d(f)$.

Proof. Let $E = \sum_i w_i x_i \partial_{x_i}$ be the weighted Euler vector field. At a nonzero critical point p of the zero fibre,

$$G_m(p) + H_a(p) = 0, \qquad mG_m(p) + aH_a(p) = E(f)(p) = 0.$$

Since $m \neq a$, both $G_m(p)$ and $H_a(p)$ vanish. Their differentials are opposite. They cannot both be zero. If they were zero, weighted scaling would make their differentials zero at every point of the orbit $t \cdot p = (t^{w_1}p_1, t^{w_2}p_2, t^{w_3}p_3)$, $t \neq 0$. This orbit approaches the origin and would contradict the assumed isolation there.

First suppose that G_m, H_a are coprime. The common zero set in the weighted projective plane is finite: two effective divisors without a common component on this projective surface have zero-dimensional intersection. Near p, choose a smooth slice to the nonzero vector field E, and use a scaling coordinate t close to a nonzero value t_0. The slice and scaling give analytic coordinates. Since $dH_a(p) \neq 0$ and $E(H_a)(p) = 0$, the restriction of H_a to the slice has nonzero differential. Use it as the coordinate U and complete it to coordinates (U, z). In these coordinates

$$f = t^m\big(g(U,z) + t^{a-m}U\big).$$

Write $k = a - m$, $g(U,z) = g_0(z) + Ub(U,z)$ and put $V = t^k + b(U,z)$. The derivative with respect to t is $kt_0^{k-1} \neq 0$, so this is an analytic coordinate change. Thus

$$f = t^m\big(UV + g_0(z)\big). \tag{5.2}$$

The series g_0 is not identically zero. Otherwise the two divisors would have a common local branch on the slice and hence a common irreducible algebraic component. Since the point is

critical, $\operatorname{ord}_0 g_0 = \ell \geq 2$. A univariate change of coordinate gives $g_0 = cz^\ell$. Equation (5.2) proves the stated contact normal form.

There are only finitely many nonzero critical points. On each of the finitely many common projective directions the condition is $dg + t^k dU = 0$. The two differentials do not both vanish, and this equation fixes t^k, so it has finitely many solutions.

If G_m, H_a have a common factor, isolation at the origin forces $G_m \mid H_a$. To see this, write their greatest common divisor as A, so that $f = A(B + C)$. If $\deg_w A < m$, both factors vanish at the origin, and their intersection contains a curve of critical points. Hence $\deg_w A = m$, as asserted. Write $f = G_m(1+Q)$ with Q of positive weighted degree k. The cone $G_m = 0$ is smooth away from the origin, again by isolation and weighted scaling. At an intersection with $1 + Q = 0$,

$$E(G_m) = 0, \qquad E(1 + Q) = kQ = -k.$$

The two differentials are independent. Their product is therefore the transverse model UV. This applies along the possibly positive-dimensional intersection.

We give the arithmetic details of the last assertion. In the coprime case the finite reduced scheme of nonzero singular points spreads as a finite *etale* scheme after finitely many primes are removed. Its distinct geometric points then have distinct reductions. A rational singular residue point lifts uniquely to a point defined over K, where the scaling coordinate can also be chosen. At a lifted critical point p, choose j with $p_j \neq 0$ and invert the finitely many places at which $w_j p_j$ is not a unit. On its complete residue ball put

$$t = (x_j/p_j)^{1/w_j} \in 1 + \mathfrak{p}, \qquad s_i = t^{-w_i} x_i.$$

The root is the principal-unit root. Then $s_j = p_j$, so the remaining s_i are coordinates on an algebraic slice. This is an integral analytic bijection with unit Jacobian. The nonzero slice derivative of H_a makes $U = H_a|_{s_j = p_j}$ a Hensel coordinate. Finally $\partial V/\partial t = kt^{k-1}$ is a unit after excluding the primes dividing $k = a - m$. Thus each of the changes used in the complex normal form exists over K on the complete original residue ball. The slice coordinates, implicit coordinate U, and change to V are integral analytic coordinates on the entire corresponding residue ball after deletion of finitely many further places. Their Jacobian determinants are units. This follows directly from the implicit-function theorem with a unit derivative: the successive coefficients are obtained by division by the same nonzero leading Jacobian determinant. Invert this determinant and the finitely many leading coefficients once. The resulting power series and inverses converge on the complete residue ball.

For the univariate normalization write $g_0 = cz^\ell(1 + zb_1(z))$. At primes not dividing ℓc, the principal-unit ℓ-th root gives the integral bijection $z \mapsto z(1 + zb_1(z))^{1/\ell}$, with unit Jacobian. A tame character of the factor t^m in (5.2) is constant on this ball. Thus the original integral equals a constant times the integral of (5.1), with its original Haar normalization.

The unit in this normal form is removed by the following complex right equivalence. Put $P = UV + cz^\ell$ and write the function as $B(U, V, z)P$, where $B(0) \neq 0$. Choose local complex roots of B. The analytic map

$$(U, V, z) \longmapsto (UB^{1/2}, VB^{1/2}, zB^{1/\ell})$$

has an invertible linear part and its composite with P is BP. For its spectrum use $U = X + iY$, $V = X - iY$, so $UV = X^2 + Y^2$. The reduced H^0 of the Milnor fibre of each square is one-dimensional with monodromy -1. The fibre of cz^ℓ consists of ℓ cyclically permuted points, so

its reduced H^0 has eigenvalues $e^{2\pi ij/\ell}$, $1 \le j < \ell$. Sebastiani–Thom, with its two degree shifts, puts the tensor product in reduced H^2 of (5.1); the two signs multiply to one. All other reduced groups vanish, and

$$\det(1 - tT_P \mid \widetilde{H}^2(M_P)) = \prod_{j=1}^{\ell-1}(1 - te^{2\pi ij/\ell}) = \frac{1-t^\ell}{1-t}.$$

In particular a primitive ℓ-th root occurs. The condition $\mathrm{Sat}_d(f)$ therefore gives $d \nmid \ell$, or $\chi^\ell \neq 1$. Choose $b \in \mathcal{O}^\times$ with $\chi(b)^\ell \neq 1$. The substitution

$$(U, V, z) \longmapsto (b^\ell U, V, bz)$$

preserves the complete origin ball and Haar measure and multiplies $UV + cz^\ell$ by b^ℓ. Its character integral must be zero. For the transverse model UV, scaling U proves the same result for every nontrivial character.

The divisors in the common-factor case are algebraically transverse away from the origin. Their finite collection of smooth charts spreads with the same transversality after finitely many places are removed. The UV argument therefore applies uniformly, even along their positive-dimensional intersection. Smooth zero points are handled by using f itself as one Hensel coordinate. There are no additional singular reductions after shrinking the integral model: this is the Jacobian criterion applied to the fixed finite-type schemes and the stated open charts. Wild characters are handled by Lemma 2.3 on a fixed resolution of the original f.

We have proved that every nonzero zero residue ball contributes zero. The origin ball contributes zero by the hypothesis of the proposition, since $\mathrm{Sat}_d(f)$ implies $\mathrm{Sat}_d(f, 0)$. A residue ball on which $\bar{f} \neq 0$ contributes a constant in s. A residual function is a finite linear combination of these original balls. Its zeta function is consequently constant in s, and in particular entire. □

Corollary 5.2. *Every ordinary two-term polynomial in Theorem 3.1 satisfies Denef's residual holomorphy conjecture. Its origin integral satisfies the stronger central vanishing assertion of that theorem.*

Proof. Apply Proposition 5.1 with all three weights equal to one. □

6 Plane graphs and their arithmetic coefficients

We treat families with a smooth plane as coefficient space. Their coefficients may have common components and arbitrary multiplicities. A plane function means a regular function on a fixed smooth algebraic surface over $\Bbbk$, considered in a fixed finite collection of algebraic charts. The global spectral condition is imposed at every point of the zero fibre. In a local assertion at an algebraic centre P, it is imposed at every zero-fibre point in a sufficiently small complex neighbourhood of P.

All assertions about integrals below concern complete original residue balls. A residual weight is constant on such a ball, and therefore has one common value on every part of the exceptional fibre above it. The conclusions will be zero on zero residue balls and a constant in s on the full residual domain. Lemma 2.3 handles higher conductors; until the final assembly of each theorem, characters are tame and nontrivial.

Lemma 6.5 below gives the local form of Veys' residual plane theorem [29, Theorem 3.1 and §3.3] on these smooth charts, with their nonvanishing two-forms.

6.1 The complete exceptional lattice

Let g, h be nonzero plane germs at a common zero P. Resolve the reduced support of gh by its geometric minimal embedded resolution. For each exceptional component or strict arrow set

$$V_i = (N_i, M_i) = (\operatorname{ord}_{E_i} g, \operatorname{ord}_{E_i} h).$$

Both entries are positive on the exceptional fibre above P; an arrow can have one zero entry. Let v_i be the valency, including arrows, and, for an exceptional component, write $E_i^2 = -\kappa_i$. Principal divisors have zero intersection with an exceptional curve, so

$$\kappa_i V_i = \sum_{j \sim i} V_j. \tag{6.1}$$

The sum includes arrows. We call an exceptional vertex a rupture if $v_i \geq 3$, and a dead bamboo a chain from a rupture to an exceptional leaf containing no arrow and no other rupture. For a single plane function the same notation and every graph argument below apply with $V_i = (N_i, 0)$; the second coordinate can simply be deleted from the principal-divisor equations.

Lemma 6.1 (Dead bamboos and protected factors)**.** *For a dead bamboo with tip L and first rupture B, there is an integer $b \geq 2$ such that $V_B = bV_L$. Equality $b = 2$ means that the entire bamboo consists of one (-2)-leaf. At most two dead bamboos meet a rupture, and their determinants are coprime if there are two.*

Fix a primitive nonnegative vector (p, q) and write $V_B = s(p, q)$ for a rupture in this direction. In the product

$$\Delta(u, v) = \prod_{E_i \ \text{exceptional}} (1 - u^{N_i} v^{M_i})^{v_i - 2}, \tag{6.2}$$

the primitive factor $\Phi_s(u^p v^q)$ has positive exponent. If s is even, every other rupture group has nonnegative exponent at the half-index $s/2$. The contribution of the selected group there is

$$v_B - 2 - \#\{\textit{dead bamboos of determinant two at } B\}. \tag{6.3}$$

Consequently a missing half-index can occur only when $v_B = 3$ and B has one (-2)-leaf.

Proof. Write the bamboo from the node toward its tip as $C_1, \ldots, C_a$. Every $\kappa_j \geq 2$: a (-1)-curve in this arrow-free chain could be contracted without destroying the embedded normal-crossing property. For its negative intersection matrix the continuants satisfy

$$D_0 = 1, \quad D_1 = \kappa_1, \quad D_j = \kappa_j D_{j-1} - D_{j-2}.$$

Induction gives $D_j \geq j + 1$. Solving (6.1) on the chain, separately in each coordinate, gives $V_B = D_a V_L$. Thus $b = D_a \geq 2$, with $b = 2$ precisely when $a = 1$ and $\kappa_1 = 2$.

Every centre needed in a minimal embedded resolution lies on a current strict transform: the exceptional divisor alone has simple normal crossings (SNC). A direction created by a later blowup and followed by that strict transform eventually leads to an arrow. A dead direction from a rupture must therefore be one of its directions toward older exceptional curves. A newly created curve meets at most two older curves, which proves the bound of two.

Let Q be the negative intersection matrix of the entire exceptional divisor. In the integral basis of total transforms of the exceptional curves created at the successive point blowups its Gram matrix is the identity. Hence $\det Q = 1$. If two dead chains of determinants b_1, b_2 meet

B, elimination of the chains changes the diagonal at B by $a_1/b_1 + a_2/b_2$, with integers a_1, a_2. Clearing denominators in the determinant leaves expressions of the form

$$b_1 b_2 A - a_1 b_2 D - a_2 b_1 D.$$

Every common divisor of b_1, b_2 divides $\det Q = 1$. Thus they are coprime; in particular they cannot both equal two.

Assign each exceptional leaf denominator in (6.2) to its first rupture. Bivalent vertices have exponent zero. In the direction $W = u^p v^q$ a group is

$$\frac{(1 - W^s)^{v_B - 2}}{\prod_{L \text{ assigned to} B}(1 - W^{s/b_L})}. \tag{6.4}$$

It has nonnegative cyclotomic exponents unless $v_B = 3$ and two dead bamboos meet B. There is at most one such rupture. Indeed, at its birth both old directions are dead: their final parts contain no strict branch and no other rupture. All strict branches continue through the remaining direction. Every other rupture is born later above that direction. It cannot again have two old dead directions, since its backward direction contains the first rupture. Moreover each later exceptional multiplicity is the sum of its parent multiplicities and nonnegative strict-transform multiplicities. The initial vector is therefore coordinatewise no larger than every later rupture vector, by birth ancestry.

If this unique initial group belongs to the chosen direction, write its scale as $s_* \le s$. Its negative indices divide both s_*/b_1 and s_*/b_2, and hence divide $s_*/(b_1 b_2) < s_*/2 \le s/2$. They cannot affect the indices s and $s/2$. Groups in different primitive directions cannot cancel: in the unique factorization domain (UFD) of Laurent polynomials their irreducible binomial factors $1 - \xi u^p v^q$ are distinct. The selected group has positive exponent at s, and at $s/2$ it loses one copy exactly for each determinant-two bamboo. This proves the assertions. □

The normalization of (6.2) matters. For at least two reduced branches the multivariable Alexander product, after the substitutions $t_j = u^{n_j} v^{h_j}$ using the actual branch multiplicities, is a polynomial with constant term one [8, §2.1, Proposition 1 and the following remarks]. For one reduced singular branch b, write $g = \epsilon_g b^n$ and $h = \epsilon_h b^e$, where ϵ_g, ϵ_h are units. If T_b denotes monodromy on $H^1(M_b)$, the raw exceptional product is

$$\frac{\Delta_b(u^n v^e)}{1 - u^n v^e}, \qquad \Delta_b(t) = \det(1 - tT_b \mid H^1(M_b)). \tag{6.5}$$

We retain this denominator below. A single smooth branch is treated in a normal coordinate.

Lemma 6.2 (Orders in a nonidentity norm)**.** *Let $s, k \ge 1$, $k \ge 2$, and let a be an integer. Among products of a primitive s-th root and ρ^a, with $\rho^k = 1$ and $\rho \ne 1$, the largest achievable order is* $\operatorname{lcm}(s, k/\gcd(k, a))$*, with one exception:*

$$k = 2, \quad a \text{ odd}, \quad s \equiv 2 \pmod 4.$$

In the exception it is $s/2$. A root of order R has a b-th root of exact order bR for every $b \ge 1$.

Proof. Work prime by prime in the cyclic group of roots of unity. If the two primary orders are unequal, the larger order survives in their product. If they are equal, choose the two primitive numerators so that their sum is not divisible by the prime. This is possible at every odd prime.

At two, the product of two primitive factors of the same primary order always loses that maximal two-power order. If their common order is at least four, choose the multiplier with a strictly smaller two-primary order instead; it can still be nonidentity and the larger order then survives. If the common order is two, avoiding the loss requires a multiplier with trivial two-primary part. If the exponent map $\mu_k \to \mu_{k/\gcd(k,a)}$ has nontrivial kernel, its image on $\mu_k \setminus \{1\}$ includes the identity, and this obstruction disappears. If its image has more than two elements one can instead choose a multiplier avoiding the offending primary component. The only forced loss is thus the displayed exception; multiplication by -1 then removes the single factor of two. The choices at distinct primes combine by the Chinese remainder theorem.

For the last assertion, write the given root as $\exp(2\pi ic/R)$, $\gcd(c, R) = 1$. A b-th root has numerator $c + jR$ and denominator bR. Primes dividing R never divide this numerator. For each additional prime dividing b, choose j to avoid its one prohibited residue class. The Chinese remainder theorem supplies all choices simultaneously. □

6.2 Kummer coefficients

For a pair of finite characters (λ, μ) call a component eligible when $\lambda^{N_i}\mu^{M_i} = 1$. Normal-unit integration kills every stratum incident with a noneligible normal divisor. On an eligible open curve the remaining coefficient is the actual Kummer trace of the leading units of g, h. Its inertia at a boundary component j is $\lambda^{N_j}\mu^{M_j}$. Relation (6.1) implies that an eligible vertex cannot have exactly one noneligible neighbour. An isolated eligible bivalent curve has two nontrivial inverse inertias and zero compact-support cohomology by Lemma 2.2. Its coefficient is zero with both valuations prescribed, before any geometric series or discrepancy weight is applied.

Lemma 6.3 (The complete quadratic pair)**.** *Suppose an eligible component consists of a valency-three curve B and its eligible (-2)-leaf L, with the two exterior neighbours noneligible, and $V_B = 2V_L$. Let X, Y be formal variables recording the two valuations. For a nonvanishing plane form ω, put $\nu_i - 1 = \operatorname{ord}_{E_i}(\pi^*\omega)$. Set*

$$U = q^{-\nu_L} X^{N_L} Y^{M_L}.$$

For the original plane Haar measure its entire contribution is

$$q^{-2}C(q-1)\left[q\frac{U}{1-U} - \frac{qU^2}{1-qU^2} + (q-1)\frac{qU^3}{(1-U)(1-qU^2)}\right], \tag{6.6}$$

where C is one actual Frobenius scalar. If a selected linear parity of V_L is odd, the even part for that parity is identically zero.

Proof. Adjunction on L gives

$$-2(\nu_L - 1) + (\nu_B - 1) = 0, \qquad \nu_B = 2\nu_L - 1.$$

Thus the normal geometric-series variable for B is qU^2. At $P = L \cap B$ use the same coordinates to write both functions. Changing a normal frame multiplies its leading unit by powers killed by eligibility. The extension fibres on L, on B, and at P are the same one-dimensional arithmetic representation. Denote its Frobenius scalar by C.

Filling P on L extends the sheaf to a geometrically constant sheaf on $\mathbb{P}^1$. The trace on its open affine line is qC. Filling P on B leaves a two-punctured rational curve with nontrivial

inverse inertias, whose whole compact-support complex is zero. Removing P therefore leaves trace $-C$. The crossing itself has trace C. Consequently

$$C_L = qC, \qquad C_B = -C, \qquad C_{LB} = C. \tag{6.7}$$

The open strata have one factor $(q-1)$, the crossing has two, and the common plane normalization is q^{-2}. These facts give (6.6).

For the even projection, the leaf normal valuation must be even; the B valuation is unrestricted. The result is

$$q^{-2}C(q-1)\left[\frac{qU^2}{1-U^2} - \frac{qU^2}{1-qU^2} + \frac{(q-1)qU^4}{(1-U^2)(1-qU^2)}\right] = 0. \tag{6.8}$$

The complete numerator after a common denominator is

$$qU^2 - q^2U^4 - qU^2 + qU^4 + (q^2-q)U^4 = 0.$$

This proves the stated even subseries with both valuation variables. □

Lemma 6.4 (A parity test on the full lattice)**.** *Let C be a connected component of the graph of eligible exceptional vertices in the specified region of a plane resolution, possibly after sign-separating point blowups. Let e_i be, modulo two, either N_i or $N_i - M_i$. Assume that every eligible rupture in C has $e_B = 0$ and a designated noneligible (-2)-leaf, that every eligible strict arrow has even selected parity, and that an eligible boundary of the region under consideration has already been excluded. Suppose also that a nonrupture of C with a noneligible neighbour would be an isolated bivalent curve. Then every e_i in every component other than these isolated bivalent curves is zero.*

Proof. On the *complete* exceptional graph put $w_i = e_i$ for $i \in C$. For each rupture $B \in C$ put

$$w_{L_B} = \sum_{j \in C,\ j \sim B} e_j \pmod 2$$

on its designated leaf. Put zero on every other outside vertex. We check all rows of the complete negative intersection matrix Q.

At a nonrupture of C, there is no noneligible neighbour, since the isolated bivalent case has been set aside. The principal-divisor recurrence for g, or for g/h, is its row of $Qw = 0$; arrow terms are even. At a rupture, $w_B = 0$, and the assigned leaf cancels the sum of its eligible neighbours. At an assigned leaf the row is $2w_L - w_B = 0$. Finally an unassigned outside vertex can meet C only at a rupture, whose entry is zero; the excluded eligible boundary prevents an omitted endpoint of the region. It cannot meet an assigned leaf at any other point, because that leaf has valency one. These are all rows, including the rows outside C.

Every added point blowup preserves unimodularity of the full exceptional lattice, so $\det Q = 1$ and Q is invertible over $\mathbb{F}_2$. Hence $w = 0$. In particular every possible remaining normal valuation expression has even selected parity. □

6.3 Arithmetic descent

Distinct conjugate exceptional curves are disjoint. They are born in the same blowup stage over distinct conjugate centres, and subsequent blowups cannot make their strict transforms intersect. A rational point on an exceptional fibre therefore lies on Frobenius-fixed components. At an SNC

crossing containing an exceptional curve the unique second component is fixed too. A contributing pair (B, L) has a rational unique intersection, so the scalar C in (6.7) is over the actual residue field. Its two other punctures may be conjugate: the geometric vanishing in Lemma 2.2 still gives zero Frobenius trace.

There is a separate original SNC possibility: two conjugate strict branches meeting without an exceptional component. At a good place their splitting field is an unramified quadratic extension K'/K. In one K' coordinate w, viewed as two coordinates over K, the original chart has

$$g = \epsilon_g \operatorname{Norm}_{K'/K}(w)^N, \qquad h = \epsilon_h \operatorname{Norm}_{K'/K}(w)^M. \tag{6.9}$$

Here ϵ_g, ϵ_h are units. Galois invariance gives the same multiplicities on the two branches. On a fixed valuation shell the unit factor is constant and the remaining character is $(\lambda^N \mu^M)(\operatorname{Norm} \bar{w})$. If it is nontrivial, its sum is

$$\sum_{a \in \mathbb{F}_{q^2}^\times} (\lambda^N \mu^M)(a^{q+1}) = (q+1) \sum_{b \in \mathbb{F}_q^\times} (\lambda^N \mu^M)(b) = 0.$$

The norm is onto and every nonzero fibre has $q+1$ elements. The K-domain and its Haar measure retain the coupled valuation $\operatorname{ord} \operatorname{Norm}(w) = 2 \operatorname{ord}(w)$. We shall verify nontriviality from neighbouring original transverse spectra in each application.

6.4 Residual integrals on a smooth surface

Lemma 6.5 (Residual plane integrals)**.** *Let X be a smooth algebraic surface over $\Bbbk$, let f be a nonzero regular function on X, and let ω be a nowhere-vanishing regular two-form on each of a fixed finite collection of charts. Suppose that* $\operatorname{Sat}_d(f)$ *holds, $d > 1$. Outside a finite set of places depending on these algebraic data and not on d or the character,*

$$\int_{\operatorname{red}^{-1}(\bar{Q})} \chi(\operatorname{ac} f) |f|^s \, |\omega| = 0$$

for every zero residue point $\bar{Q}$ on a retained chart and every character χ of order d; here red *is its reduction map. For a fixed algebraic centre $P \in f^{-1}(0)$, include its coordinates in the coefficient field. On* $\operatorname{red}^{-1}(\bar{P})$ *it suffices to require* $\operatorname{Sat}_d(f, Q)$ *for every $Q \in f^{-1}(0) \cap U$, where U is a sufficiently small complex neighbourhood of P.*

Proof. Fix a minimal embedded resolution of the reduced divisor $f = 0$. Write $N_i = \operatorname{ord}_{E_i} f$ and $\nu_i - 1 = \operatorname{ord}_{E_i} \pi^* \omega$. Spread the resolution, its components, intersections and units over the integers with finitely many primes inverted. First consider a geometric exceptional fibre and a tame character χ.

Call a component eligible when $d \mid N_i$. A strict component is ineligible: at a neighbouring smooth point its transverse function is a unit times x^{N_i}, and its reduced H^0 contains every nontrivial N_i-th root. An eligible rupture is impossible by the scalar case of Lemma 6.1: the positive Φ_{N_i} exponent in the local A'Campo quotient gives an actual eigenvalue of exact order N_i.

An eligible exceptional leaf forces its neighbour to be eligible by $\kappa_i N_i = \sum_{j \sim i} N_j$. Repeating along its bamboo reaches a rupture or a strict component, both already excluded. All eligible exceptional components therefore have valency two. At a bivalent component the same relation shows that one eligible neighbour forces the other to be eligible. An eligible subgraph containing an edge would consequently have no endpoint in the finite resolution tree. Thus the eligible components are isolated bivalent curves with two ineligible neighbours.

Normal-unit integration kills every stratum incident with an ineligible component. On an eligible curve, the residual coefficient is the Frobenius trace of a Kummer line on $\mathbb{P}^1$ minus its two neighbouring intersection points. The local inertias are $\chi^{N_{j_1}}$ and $\chi^{N_{j_2}}$. They are nontrivial and inverse, because $\kappa_i N_i = N_{j_1} + N_{j_2}$. The compactly supported cohomology of this two-puncture line is zero. Hence its trace is zero for each fixed normal valuation. Multiplying by the radial Haar and Jacobian factors preserves this zero. The remaining unit in $\pi^*\omega$ has absolute value one, since ω is nowhere vanishing on the original good integral chart.

A complete original residue ball includes the whole exceptional fibre over its residue point. Its residual weight is one constant on the whole punctured curve. Smooth zero points and identity-SNC charts are handled directly by normal-unit integration.

For descent, conjugate exceptional curves have disjoint birth centres and do not meet. An original rational point of the exceptional fibre therefore lies on a Frobenius-fixed component. The vanishing compactly supported complex on a fixed bivalent component gives trace zero even when its two punctures are conjugate. At a conjugate strict-node chart the normal character $\rho = \chi^N$ is nontrivial, where N is the common branch multiplicity. Its angular sum is

$$\sum_{u \in \mathbb{F}_{q^2}^{\times}} \rho(\mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\, u) = (q+1) \sum_{v \in \mathbb{F}_q^{\times}} \rho(v) = 0.$$

This norm computation retains the original two-dimensional residue ball and Haar measure.

Only character orders dividing one of the fixed positive N_i require a residual Kummer calculation. There are finitely many such orders. Spread their covers together with the fixed resolution and volume forms, and exclude their finitely many bad places. All other tame orders vanish by normal-unit integration. After also excluding primes dividing the resolution multiplicities, Lemma 2.3 handles every higher conductor.

For the local assertion, apply this calculation to the complete exceptional fibre over P. The spectra used above are those at P and at smooth points of the strict branches arbitrarily close to P; they lie in U. The same finite resolution and covers give the stated localization. The descent calculation applies at every completion where the marked centre is rational. □

7 Plane suspensions

Theorem 7.1. *Let $f \in \Bbbk[x, y]$, let $c \in \Bbbk^{\times}$, and let $k_0 \geq 2$. For the nonzero polynomial $F = f + cz^{k_0}$, the condition* $\mathrm{Sat}_d(F)$ *implies, outside one finite set of places independent of the character and its conductor, zero integral on each complete original zero residue ball for every character of order $d > 1$. Thus every residual integral is constant in s. At a fixed algebraic zero P of f, the conclusion on its plane residue ball times $\mathfrak{p}$ follows from the same spectral condition at every zero-fibre point in a sufficiently small complex neighbourhood of $(P, 0)$.*

The case $k_0 \geq 3$ is the plane suspension theorem of Denef–Veys [13, Theorem 3.1 and Corollary 3.4]. We give its argument on complete product residue balls before turning to the quadratic case.

We first treat constant f, for every $k_0 \geq 2$. If $f = 0$, the spectral condition for cz^{k_0} implies $\chi^{k_0} \neq 1$, and normal-unit integration is zero. If f is a nonzero constant, exclude places where f, c, or k_0 is not a unit. Every zero residue then has $\bar{z} \neq 0$, so $ck_0 z^{k_0 - 1}$ is a unit; the F coordinate carries the complete Hensel ball onto $\mathfrak{p}$ and gives zero. For nonconstant f, the same argument treats balls with $\bar{z} \neq 0$. It remains to consider $D \times \mathfrak{p}$, where D is a complete plane residue ball with $\bar{f} = 0$.

Fix a nontrivial additive character $\Psi : K \to \mathbb{C}^\times$ which is trivial on $\mathcal{O}$ and nontrivial on $\pi^{-1}\mathcal{O}$. Assume $k_0 \geq 3$ and first let χ be tame. Put

$$A(s,\rho) = \begin{cases} q^{s+1} - 1, & \rho = 1, \\ 1, & \rho \neq 1. \end{cases}$$

After excluding finitely many places, every nonzero critical value of f is a unit. Indeed, in characteristic zero f is constant on each irreducible component of its critical locus, so there are finitely many algebraic critical values. Invert their denominators and their nonzero norms. Every critical point in D then has value zero. The only critical value of $g(z) = cz^{k_0}$ is also zero.

For a compact open set B write

$$E_{f,B}(\xi) = \int_B \Psi(\xi f(x))\, dx.$$

The two characteristic functions $\mathbf{1}_D$ and $\mathbf{1}_{\mathfrak{p}}$ are Schwartz–Bruhat functions, and Fubini gives

$$E_{F,D\times\mathfrak{p}}(\xi) = E_{f,D}(\xi) E_{g,\mathfrak{p}}(\xi).$$

The exponential-integral expansion with these weights states that every pole of $A(s,\rho)Z_{f,D}(\rho,s)$ occurs with a nonzero term in the large-$|\xi|$ expansion of $E_{f,D}$; the same holds for g and F. Multiplication of the finite expansions shows that every pole of $A(s,\chi)Z_{F,D\times\mathfrak{p}}(\chi,s)$ is a sum of poles of

$$A(s,\rho)Z_{f,D}(\rho,s), \qquad A(s,\sigma)Z_{g,\mathfrak{p}}(\sigma,s), \qquad \rho\sigma = \chi.$$

This is the Schwartz–Bruhat argument of [13, §§2.3–2.5], with these product characteristic functions. In particular, the trivial-character factor is $A(s,1) = q^{s+1} - 1$.

Normal-unit integration computes the second factor:

$$Z_{g,\mathfrak{p}}(\sigma,s) = \begin{cases} 0, & \sigma^{k_0} \neq 1, \\ \sigma(c)(1-q^{-1})\dfrac{q^{-1-k_0 s}}{1-q^{-1-k_0 s}}, & \sigma^{k_0} = 1. \end{cases}$$

Thus every contributing σ has order $d_2 \mid k_0$. Since the residue characteristic does not divide k_0, σ is tame, and so is $\rho = \chi\sigma^{-1}$. Put $d_1 = \operatorname{ord}\rho$.

If $\rho \neq 1$ and its factor has a pole, Lemma 6.5 implies the existence of an actual base eigenvalue of order R divisible by d_1. Since $R > 1$, it occurs in reduced Milnor cohomology.

If $\rho = 1$ and its factor has a pole, f has a critical zero. In fact, if the zero fibre is smooth, good reduction and the local coordinate f give

$$Z_{f,D}(1,s) = q^{-1}(1-q^{-1})\frac{q^{-1-s}}{1-q^{-1-s}},$$

and multiplication by $q^{s+1} - 1$ is constant. At a critical zero P, reduced Milnor cohomology is nonzero. To verify this, let $\mu \geq 2$ be the local multiplicity and begin the resolution by blowing up the point. Every one-divisor stratum lying over that point has multiplicity at least μ. Strict transforms over the point meet the exceptional divisor and give no one-divisor stratum. A'Campo's product therefore satisfies

$$\mathcal{Z}_{f,P}(t) = 1 + O(t^\mu). \tag{7.1}$$

If reduced Milnor cohomology vanished, this determinant would be $1 - t$, a contradiction. Choose an eigenvalue on this reduced cohomology and let R be its order. Here $d_1 = 1 \mid R$.

Rational monodromy supplies all primitive conjugates of an occurring eigenvalue in the same cohomological degree. Thom–Sebastiani and Lemma 6.2, with $a = 1$ and $k = k_0 \geq 3$, produce an actual eigenvalue of F of exact order $\operatorname{lcm}(R, k_0)$. The nonzero coefficient c does not affect the complex spectrum. But

$$d = \operatorname{ord}(\rho\sigma) \mid \operatorname{lcm}(d_1, d_2) \mid \operatorname{lcm}(R, k_0),$$

contrary to $\mathrm{Sat}_d(F)$. Hence $Z_{F,D\times\mathfrak{p}}(\chi, s)$ has no poles. Lemma 2.4 makes it zero because $F \in \mathfrak{p}$ on the whole product ball. For a marked centre, use the local part of Lemma 6.5; its eigenvalue witness lies in the prescribed complex neighbourhood. At a smooth marked zero the submersion formula above makes the corrected trivial-character factor constant. At a critical marked zero the reduced eigenvalue is chosen at that zero itself, so the Thom–Sebastiani contradiction also lies in the prescribed neighbourhood. This proves the local assertion. Lemma 2.3, on one fixed resolution of F, treats all higher conductors.

For the remainder of this section let $k_0 = 2$.

7.1 Every character of order greater than two

Let η be the quadratic residue character and let

$$J(\alpha, \beta) = \sum_{a \in \mathbb{F}_q} \alpha(a)\beta(1 - a),$$

with every character extended by zero at zero. For $a = \pi^n u$, $n \geq 1$, and $\chi^2 \neq 1$, direct integration gives, with $t = s + \frac{1}{2}$,

$$\int_{\mathfrak{p}} \chi(\mathrm{ac}(a + cz^2))|a + cz^2|^s \, dz = \begin{cases} q^{-1/2}\chi(\bar{u})q^{-nt}, & n \text{ odd}, \\ q^{-1}\eta(-\bar{c})J(\eta, \chi)(\chi\eta)(\bar{u})q^{-nt}, & n \text{ even}. \end{cases} \tag{7.2}$$

Here is the complete shell calculation. On $2 \operatorname{ord} z < n$ the angular average contains χ^2, so is zero. If n is odd, the remaining ball has measure $q^{-(n+1)/2}$ and leading term a. If $n = 2j$, put $z = \pi^j w$. Every nonzero residue root of $u + cw^2$ has unit derivative; the map $w \mapsto u + cw^2$ carries its entire Hensel ball onto $\mathfrak{p}$ with unit Jacobian, and its character integral is zero. The remaining contribution is $q^{-j-1}q^{-ns} \sum_w \chi(u + cw^2)$. Counting square roots yields

$$\begin{aligned} \sum_w \chi(u + cw^2) &= \sum_b (1 + \eta(b/c))\chi(u + b) \\ &= \eta(c) \sum_b \eta(b)\chi(u + b) \\ &= \eta(-c)(\chi\eta)(u)J(\eta, \chi), \end{aligned}$$

which proves (7.2) with the residue-unit factor $\eta(c)$.

For a fixed original plane zero ball D put

$$A_n(\rho) = \int_{D \cap \{\operatorname{ord} f = n\}} \rho(\mathrm{ac}\, f) \, dx \, dy, \quad E_\rho(t) = \sum_{n \text{ even}} A_n(\rho)q^{-nt}, \quad O_\rho(t) = \sum_{n \text{ odd}} A_n(\rho)q^{-nt}.$$

Fubini gives the exact identity

$$Z_{F,D\times\mathfrak{p}}(\chi, s) = q^{-1}\eta(-c)J(\eta, \chi)E_{\chi\eta}(s + \tfrac{1}{2}) + q^{-1/2}O_\chi(s + \tfrac{1}{2}). \tag{7.3}$$

Thom–Sebastiani on reduced Milnor cohomology multiplies a base eigenvalue by -1 [13, §1.2]. Its effect on an exact order r is

$$\sigma(r) = \begin{cases} 2r, & r \text{ odd}, \\ r/2, & r \equiv 2 \pmod 4, \\ r, & 4 \mid r. \end{cases} \tag{7.4}$$

If d is odd, $d \mid r$ would imply $d \mid \sigma(r)$; both orders d and $2d = \operatorname{ord}(\chi\eta)$ therefore satisfy base saturated exclusion. If $4 \mid d$, both character orders equal d, and the same conclusion follows. Finally let $d = 2b > 2$, b odd. If no base eigenorder is divisible by d, an eigenorder divisible by b would either be even, contradicting that fact, or odd, contradicting $d \nmid \sigma(r)$. Again the orders of χ and $\chi\eta$ satisfy base exclusion. In all these cases Lemma 6.5 gives $A_n(\chi) = A_n(\chi\eta) = 0$ for every $n \geq 1$.

The remaining case is $d = 2b > 2$ with b odd and some base order divisible by d. The original spectral condition excludes every odd base order divisible by b and every base order divisible by $4b$. It also excludes a strict branch multiplicity divisible by b: at a neighbouring smooth point that branch has transverse function x^r, whose reduced H^0 contains a primitive b-th root; suspension produces order $2b$. This explicitly treats nonreduced branches.

Use the one-coordinate version of Lemma 6.1, with vectors $(N_i, 0)$, on the minimal plane resolution of f. The H^0 characteristic factor of f is $1 - t^e$, where e is the gcd of its strict multiplicities. Since $b \nmid e$, this factor cannot interfere with any order divisible by b. Every rupture multiplicity N_B is an actual eigenorder by primitive protection. If $b \mid N_B$, the two exclusions just obtained force $N_B \equiv 2 \pmod 4$. Its odd half $N_B/2$ is forbidden. Half-index protection forces the valency-three configuration with one (-2)-leaf L described in [2, Lemma 6.17], with

$$N_B = 2N_L, \qquad N_L \text{ odd}, \qquad b \mid N_L, \qquad \nu_B = 2\nu_L - 1.$$

For completeness, this describes the entire b-divisible graph. A divisible leaf propagates divisibility toward a rupture. At a divisible bivalent vertex one divisible neighbour forces the other by (6.1). At the displayed rupture, the other two neighbours are either both divisible or both nondivisible. If they were both divisible, follow a nonleaf direction. Each subsequent bivalent vertex propagates, and each subsequent rupture has its own leaf and propagates through its other nonleaf direction. A finite tree cannot continue this path forever or close a cycle; it must end at a strict arrow, which is impossible. Thus the rupture and its leaf form an isolated pair. Every remaining divisible component is one isolated bivalent curve.

For $\psi = \chi\eta$ of order b, isolated bivalent coefficients vanish by Lemma 2.2; the even series of every pair vanishes by (6.8), with $Y = 1$ and $X = q^{-t}$. Hence $E_\psi = 0$. On a good resolution a nonzero normal coefficient for χ requires $d \mid N_i$ for every positive normal exponent involved. All such exponents are even, so their valuation sums are even. Consequently $O_\chi = 0$. For an original nonsplit strict node, (6.9) applies: the order-b normal character is nontrivial, and the order-$2b$ character either averages to zero or has only even norm valuations. This proves (7.3) is zero for every $d > 2$.

7.2 The quadratic character and the plane spectrum

We give the plane spectral classification of [2, Lemma 6.15] with its calculation. Assume now $d = 2$. By (7.4), every eigenvalue of the reduced Milnor cohomology of f has order 2 (mod 4). If its reduced support has $b > 1$ branches, its H^1 has an eigenvalue one. Indeed the link complement

has first Betti number b; in the Wang sequence of its Milnor fibration the invariants of H^0 of the fibre have dimension one, leaving a nonzero invariant H^1 part when $b > 1$. Thus there is one reduced branch, and we can write $f = ug^r$ with g reduced and irreducible. The nontrivial r-th roots occur on reduced H^0 by cyclic permutation of the r components. If $r > 2$, either an odd prime divisor or a factor four gives an impermissible order. Hence $r = 1$ or 2. If $r = 2$ and g is singular, any eigenvalue α on the nonzero $H^1(M_g)$ gives both roots $\beta, -\beta$ with $\beta^2 = \alpha$ for g^2. Their orders cannot both be 2 (mod 4). The nonreduced possibility is therefore a double smooth branch.

For a reduced singular branch let

$$1 < \beta_0 < \beta_1 < \cdots < \beta_a, \quad e_j = \gcd(\beta_0, \ldots, \beta_j), \quad n_j = e_{j-1}/e_j \geq 2, \qquad e_a = 1$$

be its minimal semigroup data. We record the entire calculation of the characteristic polynomial, including the one-branch normalization. The standard unique semigroup expansion [8, §2.3] is

$$S = \left\{ a_0\beta_0 + \sum_{j=1}^{a} a_j\beta_j : a_0 \geq 0, 0 \leq a_j < n_j \right\}.$$

Thus, as formal power series,

$$\begin{aligned} \sum_{v \in S} t^v &= \left(\sum_{a_0 \geq 0} t^{a_0\beta_0} \right) \prod_{j=1}^{a} \left(\sum_{a_j=0}^{n_j-1} t^{a_j\beta_j} \right) \\ &= \frac{1}{1-t^{\beta_0}} \prod_{j=1}^{a} \frac{1-t^{n_j\beta_j}}{1-t^{\beta_j}}. \end{aligned} \tag{7.5}$$

The one-branch identity [8, §2.1, Remark 2] gives

$$\det(1 - tT_f \mid H^1(M_f)) = \frac{(1-t)(1-t^{n_1\beta_1})}{(1-t^{\beta_0})(1-t^{\beta_1})} \prod_{j=2}^{a} \frac{1-t^{n_j\beta_j}}{1-t^{\beta_j}}. \tag{7.6}$$

Here T_f is monodromy on $H^1(M_f)$; the factor $1 - t$ comes from the one-branch normalization.

Put $M_j = n_j\beta_j$. Each M_j is an actual primitive eigenorder: it exceeds β_0, β_1, its own denominator does not contain it, and every quotient for $j \geq 2$ is already a polynomial. Therefore $v_2(M_j) = 1$. If $j \geq 2$ and $n_j > 2$, the odd integer $M_j/2$ exceeds β_j and the two possible external denominator orders. It survives in (7.6), a contradiction. Hence $n_j = 2$ and β_j is odd for every $j \geq 2$. If $a \geq 2$, then e_1 and β_1 are even, whereas $v_2(n_1\beta_1) = 1$ makes n_1 odd, at least three. Write $\beta_0 = n_1e_1$, $\beta_1 = m_1e_1$, with $m_1 > n_1$. The odd order $M_1/2$ exceeds both β_0 and β_1 and again survives, a contradiction. Thus $a = 1$. Coprimality of β_0, β_1 and the same half-product argument force $\beta_0 = 2$ and $\beta_1 = 2n + 1$. We have proved that the possibilities are a smooth reduced branch, a double smooth branch, and a reduced A_{2n} branch. In the singular cases the plane Hessian has rank one.

7.3 Coordinates and Gaussian cancellation on the original ball

At a good place choose integral linear coordinates with unit determinant so that the coefficient a of y^2 is a unit. The equation $f_y(x, h(x)) = 0$ has a unique integral analytic solution on the whole $x \in \mathfrak{p}$ ball, because $f_{yy}(0) = 2a$ is a unit. Taylor expansion gives

$$f(x, h(x) + v) = b(x) + v^2A(x, v), \qquad A(0, 0) = a.$$

On the whole ball $A/a \in 1+\mathfrak{p}$. Its unique principal-unit square root defines $w = v(A/a)^{1/2}$, a bijection of the original v-ball with unit Jacobian. Hence $f = b(x) + aw^2$. In the A_{2n} case, $b(x) = b_0 x^r V(x)$ with $r = 2n+1$, $b_0 \neq 0$, and $V(0) = 1$. After excluding primes dividing r and b_0, the further change $x' = xV(x)^{1/r}$ is a whole-ball bijection of unit Jacobian. It gives

$$F = aw^2 + cz^2 + b_0(x')^r, \qquad r \text{ odd}. \tag{7.7}$$

For a double smooth branch $b \equiv 0$; the identity follows after complexification and hence over the coefficient-field series too. There is then a passive x variable. A smooth reduced branch is handled directly by its submersion coordinate. All constants in (7.7) remain in the original field.

These coordinate assertions are uniform. The isolated A_{2n} points form a fixed finite algebraic scheme; after removing its discriminant and collision primes, a rational residue point has a unique lift over K. The smooth double-branch locus has a finite cover by smooth algebraic normal-coordinate charts. Their unit implicit derivatives and the finitely many nonzero leading coefficients can all be inverted in one ring of integers. Principal-unit r-th roots are integral when the residue characteristic does not divide r; infinitely many formal-series coefficients introduce no infinite set of denominators.

For the multiplicative Fourier calculation use the additive character Ψ fixed above, and let $\bar{\Psi}(a) = \Psi(\pi^{-1}\tilde{a})$, where $\tilde{a} \in \mathcal{O}$ is any lift of $a \in \mathbb{F}_q$. For any integral-valued analytic P on the original ball D define

$$E_P(\pi^{-M}u) = \int_D \Psi(\pi^{-M}uP(x))\,dx, \qquad V_n(P,\eta) = \int_{D\cap\{\operatorname{ord} P = n\}} \eta(\operatorname{ac} P)\,dx.$$

Put $\tau = \sum_{a\neq 0} \eta(a)\bar{\Psi}(a)$. Complex conjugation and the substitution $a \mapsto -a$ give $\overline{\tau} = \eta(-1)\tau$. Substituting $a = bt$ in its squared absolute value gives

$$\begin{aligned} |\tau|^2 &= \sum_{t\in\mathbb{F}_q^\times} \eta(t) \sum_{b\in\mathbb{F}_q^\times} \bar{\Psi}\big(b(t-1)\big) \\ &= (q-1) - \sum_{t\in\mathbb{F}_q^\times\setminus\{1\}} \eta(t) = q. \end{aligned} \tag{7.8}$$

Here the inner additive sum is $q-1$ for $t=1$ and -1 otherwise, and $\sum_{t\neq 0}\eta(t) = 0$. Consequently $\tau^2 = \eta(-1)q \neq 0$. With *additive* Haar measure on $\mathcal{O}^\times$, of mass $1-q^{-1}$, for every $M \geq 1$ we have

$$\int_{\mathcal{O}^\times} \eta(u) E_P(\pi^{-M}u)\,du = q^{-1}\tau V_{M-1}(P,\eta). \tag{7.9}$$

To prove this, fix $P(x) = \pi^n v$ with v a unit. If $M - n \leq 0$, the additive character is trivial and the multiplicative unit average is zero. If $M - n \geq 2$, variation over principal units annihilates the additive character. At $M - n = 1$ the integral is

$$q^{-1}\sum_{a\neq 0} \eta(a)\bar{\Psi}(av) = q^{-1}\tau\eta(v).$$

The locus $P = 0$ gives zero as well. Fubini proves (7.9), including its precise mass.

For $b, u \in \mathcal{O}^\times$ set $G_t(bu) = \int_{\mathcal{O}} \Psi(\pi^{-t}bux^2)\,dx$. Then

$$G_0 = 1, \qquad G_1 = q^{-1}\eta(bu)\tau, \qquad G_t = q^{-1}G_{t-2} \quad (t \geq 2).$$

For G_1, the number of square roots of $y \in \mathbb{F}_q$ is $1+\eta(y)$, so

$$qG_1 = \sum_{x\in\mathbb{F}_q} \bar{\Psi}(\overline{bu}\,x^2) = \sum_{y\in\mathbb{F}_q} (1+\eta(y))\bar{\Psi}(\overline{bu}\,y) = \eta(bu)\tau.$$

The unweighted additive sum is zero, and scaling by $\overline{bu}$ gives the last equality. For $t \geq 2$, every nonzero residue class has unit derivative $2bux$ and zero additive integral; $x = \pi X$ gives the recurrence. Therefore

$$G_{2j} = q^{-j}, \qquad G_{2j+1} = q^{-j-1}\eta(bu)\tau.$$

For $Q = aw^2 + cz^2$ put $\epsilon = \eta(-ac)$. Multiplication of the two one-variable answers gives $q^{-t}\epsilon^t$ on $\mathcal{O}^2$. On the *original* ball $\mathfrak{p}^2$, for every $M \geq 2$,

$$E_{Q,\mathfrak{p}^2}(\pi^{-M}u) = q^{-2}q^{-(M-2)}\epsilon^{M-2} = q^{-M}\epsilon^M. \tag{7.10}$$

This is independent of u and includes nonsplit binary forms. It is also (A.2); the appendix supplies the exact finite computation alongside the general recurrence.

For $h(x) = b_0x^r$ on $\mathfrak{p}$, r odd, every $V_n(h,\eta)$ is zero: on a nonempty shell the unit character is $\eta(v)^r = \eta(v)$. Equation (7.9) makes every corresponding unit Fourier coefficient of E_h zero. On the original product ball $E_{Q+h} = E_Q E_h$. By (7.10), its quadratic Fourier coefficient is zero for every $M \geq 2$; applying (7.9) again gives

$$V_n(Q+h,\eta) = 0 \quad \text{for every } n \geq 1.$$

The case $M = 2$ gives $n = 1$; every point of a zero residue ball has positive valuation. For the double smooth branch take $h = 0$; its additive integral is the constant mass of the passive variable, whose nontrivial multiplicative unit average is zero. This proves the entire quadratic case and, with Lemma 2.3, Theorem 7.1.

8 A prefactored suspension and its comparison correction

For $z^m(f + cz^k)$ with $m \geq 1$, a resolution of f has a comparison complex supported on $f = 0$. On that support the function is cz^{m+k}, which determines the correction to the Euler product.

Proposition 8.1. *Let f be a plane function, let $c \in \Bbbk^\times$, and let $m, k \geq 1$. If $G = z^m(f + cz^k)$ satisfies $\mathrm{Sat}_d(G)$, then at all good places every complete zero residue ball has zero integral for every character of order $d > 1$. For a fixed algebraic centre P, the assertion on its plane reduction ball times $\mathfrak{p}$ only requires $\mathrm{Sat}_d(G, Q)$ for all zero-fibre points Q in a sufficiently small complex neighbourhood of $(P, 0)$.*

Proof. We first prove the nearby-cycle identity on a smooth complex base X, fixing $b \in f^{-1}(0)$ with f not the zero germ. Write $a = m + k$. The superscript (k) denotes the alternating determinant formed with the k-th power of the monodromy operator. Put $\widetilde{\mathcal{Z}}_f = \mathcal{Z}_f/(1-t)$ for reduced cohomology. We claim

$$\mathcal{Z}_{G,(b,0)}(t) = \frac{1-t^a}{\mathcal{Z}^{(k)}_{f,b}(t^a)} = \big[\widetilde{\mathcal{Z}}^{(k)}_{f,b}(t^a)\big]^{-1}. \tag{8.1}$$

Let $\pi : Y \to X$ be a resolution of f, let $D = \pi^{-1}(b)$, and let $C = \mathrm{Cone}(\mathbb{Q}_X \to R\pi_*\mathbb{Q}_Y)$. It is supported on $H = f^{-1}(0)$. With $i : H \hookrightarrow X$ and $C_H = i^*C$, recollement identifies it with i_*C_H. Proper base change for $\pi \times \mathrm{id}$ gives the comparison triangle

$$\mathbb{Q}_{X\times\mathbb{C}} \longrightarrow R(\pi \times \mathrm{id})_*\mathbb{Q}_{Y\times\mathbb{C}} \longrightarrow C \boxtimes \mathbb{Q}_{\mathbb{C}} \xrightarrow{+1} .$$

On $H \times \mathbb{C}$ the function is cz^a. For any constructible complex A on H,

$$\psi_{cz^a}(A \boxtimes \mathbb{Q}_{\mathbb{C}}) = A \otimes \mathbb{Q}[\mu_a].$$

Indeed, on the universal cover of the punctured target the equation $cz^a = e^\tau$ has a branches, each a product with a contractible left half-plane in τ. Shrinking the neighbourhood in H gives a copies of each stalk of A; a target turn cyclically permutes these copies. Thus the correction at $(b, 0)$ is exactly $C_b \otimes \mathbb{Q}[\mu_a]$, and $\chi_{\mathrm{top}}(C_b) = \chi_{\mathrm{top}}(D) - 1$. Taking alternating determinants gives

$$\mathcal{Z}\big(R\Gamma(D, \psi_{G\circ(\pi\times\mathrm{id})}\mathbb{Q})\big) = \mathcal{Z}_{G,(b,0)}(t)(1 - t^a)^{\chi_{\mathrm{top}}(D)-1}. \tag{8.2}$$

We compute the left side explicitly, using the local models also appearing in [15, §3.1]. For one positive exponent, $P = z^m(y^N + z^k)$ has weights (k, N) and degree aN. Its monodromy T_P is $(y, z) \mapsto (e^{2\pi ik/(aN)}y, e^{2\pi i/a}z)$. Since $z \neq 0$ on $P = 1$, projection to $z \in \mathbb{C}^\times$ is an N-sheeted cover with a exceptional fibres of one point. Its Euler characteristic is $-a(N - 1)$. With $L = aN/\gcd(N, k)$, the fixed-locus Lefschetz numbers are

$$\Lambda(T_P^j) = a\mathbf{1}_{a|j} - aN\mathbf{1}_{L|j},$$

so its alternating monodromy determinant is $(1-t^a)/(1-t^L)^{\gcd(N,k)}$. For $r \geq 2$ positive exponents the model is $P = z^m(y_1^{N_1} \cdots y_r^{N_r} + z^k)$. A nonzero product fibre is a finite union of $(\mathbb{C}^\times)^{r-1}$, with Euler number zero; each of the a zero-product fibres has Euler number one. Give every y_i weight k and z weight $\sum N_i$. For each monodromy power fixing z, either the full fibre is fixed or all y_i are zero; both fixed loci have Euler number a. Thus $\Lambda(T_P^j) = a\mathbf{1}_{a|j}$ and the local factor is $1 - t^a$.

Every stratum of D therefore contributes $1-t^a$ once per Euler characteristic, and a one-divisor stratum S_i contributes the additional denominator $(1 - t^{aN_i/\gcd(N_i,k)})^{\gcd(N_i,k)\chi_{\mathrm{top}}(S_i)}$. A'Campo's formula for f identifies that denominator with $\mathcal{Z}_{f,b}^{(k)}(t^a)$. The factor $(1-t^a)^{\chi_{\mathrm{top}}(D)}$ and (8.2) cancel exactly as in (8.1).

The reduced vanishing-cycle complex $\phi_f\mathbb{Q}_X[\dim X - 1]$ is perverse. If a reduced eigenorder r occurs, take its rational primitive-R primary block for T_f^k, where $R = r/\gcd(r, k)$. Generalized primary decomposition is made inside the finite-length perverse category. This nonzero block, on a smooth dense part of a maximal support component arbitrarily near the original point, is a nonzero local system in one shift. Consequently its alternating determinant has a nonzero Φ_R exponent at such a point. For a primitive aR-th root ξ, ξ^a has exact order R, so

$$\Phi_{aR}(t) \mid \Phi_S(t^a) \quad \Longleftrightarrow \quad S = R.$$

The roots are simple, so no other primary block cancels it in (8.1). Thus aR is an actual order for G at a neighbouring point. This argument includes $R = 1$: the primitive-one block of reduced vanishing cycles is included. This is the support-generic argument for monodromy detection; compare [11, Lemma 4.6].

Equation (7.1) shows that a critical base zero has nonzero reduced Milnor cohomology. Thus it produces a reduced eigenorder and hence an eigenorder of G divisible by a. If all reduced vanishing-cycle stalks vanish, the zero fibre is smooth and reduced.

We now return to the original arithmetic domain. At a generic $z = 0$, $f \neq 0$ point, the transverse type is z^m; hence $\chi^m \neq 1$. If the base has a critical zero, the preceding paragraph gives $\chi^a \neq 1$. For any residue-field character λ satisfying $\lambda^k = \chi^a$, its order r is then greater than one. If r divided a base eigenorder N, the identity

$$\frac{d}{\gcd(d,a)} = \frac{r}{\gcd(r,k)} \;\Big|\; \frac{N}{\gcd(N,k)}$$

would imply $d \mid aN/\gcd(N,k)$, an actual neighbouring G order by the primary-block argument. Thus every such λ satisfies saturated exclusion for f. Lemma 6.5 gives zero for each of its positive valuation coefficients.

For fixed $\operatorname{ord} z = j \geq 1$ and $\operatorname{ord} f = n$, the region $n < kj$ has a χ^m unit average and is zero; the region $n > kj$ has a χ^a average and is zero. At $n = kj$, after removing the constant q and T factors, the complete unit sum is

$$S(u) = \sum_{w \neq 0} \chi(w^m(u + cw^k)).$$

Its exact Fourier expansion is

$$S(u) = \sum_{\lambda^k = \chi^a} \chi(c)\lambda^{-1}(-c)J(\lambda^{-1}, \chi)\lambda(u). \tag{8.3}$$

To check the coefficient, multiply by $\lambda^{-1}(u)$, sum over $u \neq 0$, divide by $q - 1$, and substitute $u = -cw^k t$. The w sum is $q - 1$ precisely when $\chi^a\lambda^{-k} = 1$, and zero otherwise; the t sum is the displayed Jacobi sum. A balanced nonzero root has derivative kcw^{a-1}, a unit. The *whole* polynomial on that unit ball is an actual Hensel coordinate, so its entire root tail is zero. Every remaining term of (8.3) is a base valuation coefficient already proved zero.

If the base zero fibre is smooth and reduced, use the coordinate $y = f + cz^k$ directly. The function is $z^m y$, and integration in the full y ball gives zero. If f is a nonzero constant, the $z = 0$ ball is killed by χ^m, and every nonzero-z zero is smooth. If $f = 0$, the function is cz^a and saturated exclusion gives $\chi^a \neq 1$. On any unit-z zero ball for a general f, differentiation on $f + cz^k = 0$ gives $\partial G/\partial z = kcz^{a-1}$, a unit at good places. For a marked plane centre, a smooth base zero uses the coordinate $y' = f + cz^k$ above. At a critical marked zero, choose the reduced vanishing-cycle support germ at that point; its generic support points can be taken inside the prescribed complex neighbourhood. The spectral witnesses and the local part of Lemma 6.5 therefore use only that neighbourhood. This completes all tame zero balls. One fixed resolution and Lemma 2.3 complete the proof. □

Lemma 8.2 (Uniform unit coefficients)**.** *Fix a plane function f and the integer exponents. Suppose the fixed complex model $f + z^k$, with $k \geq 2$, satisfies* Sat_d*, respectively that $z^m(f + z^k)$, with $m, k \geq 1$, satisfies* Sat_d*. Outside one finite set of places depending on these fixed data, the corresponding tame zero-ball conclusion holds for $f + cz^k$, respectively $z^m(f + cz^k)$, simultaneously for every $c \in \mathcal{O}^\times$ and every tame character of order $d > 1$. The same statement holds on a marked reduction ball under the local spectral condition stated above for the fixed complex model. These assertions also hold over a fixed algebraic open subset $W \subset \mathbb{A}^2_{\Bbbk}$. In this form the spectral conditions are imposed at the zero-fibre points over $W(\mathbb{C})$, and the arithmetic conclusion concerns complete original residue balls whose plane reduction belongs to a fixed spread of W. The exceptional set may also depend on W.*

Proof. For a nonzero complex coefficient c, the change $z' = c^{1/k}z$ identifies $f + cz^k$ with $f + z'^k$, and gives

$$z^m(f + cz^k) = c^{-m/k}z'^m(f + z'^k).$$

A nonzero constant multiple has the same local monodromy. Thus the spectral hypothesis is expressed by the fixed model with coefficient one, while the arithmetic coefficient can be any unit of K.

In the arithmetic calculation the resolution graphs and multiplicities come from the fixed base data. For $k \geq 3$, the coefficient enters the monomial integral only through $\sigma(c)$. In the quadratic shell formula it enters through $\eta(-c)$, and the binary Gauss calculation holds for every pair of unit coefficients. In the prefactored balanced shell its factor is $\chi(c)\lambda^{-1}(-c)$. At a nonzero balanced root the derivative is kcw^{m+k-1}, a unit when the residue characteristic does not divide k. The analogous derivative in the ordinary suspension is kcw^{k-1}.

The base resolution, its volume forms, the finitely many critical values, and the leading coefficients of the base normal forms are fixed. Inverting their bad primes, the exponent denominators, and the primes needed for the finitely many Kummer covers therefore gives one finite set for all $c \in \mathcal{O}^\times$. The calculation on a marked ball uses the same local part of the fixed diagram.

For the open-set form, choose a finite principal cover $W = \bigcup_\nu D(h_\nu)$ and spread it together with the fixed resolution diagrams and their finite algebraic stratifications. On a residue ball reducing into $D(\bar{h}_\nu)$, the function h_ν remains a unit on the entire ball. Every inverse image of that ball stays in the diagram restricted over $D(h_\nu)$, so the local calculations and spectral witnesses use only that open set. The finitely many restricted charts, multiplicities, incidence conditions, Kummer covers and normal-form leading coefficients give one exceptional set. Its union over ν works for every such ball and every arithmetic unit c. □

9 Linear parameters on a plane

Theorem 9.1. *Let $f_0, \dots, f_r \in \Bbbk[x, y]$, $r \geq 1$, and let $U \subset \mathbb{A}^r_\Bbbk$ be a nonempty algebraic open set over $\Bbbk$. Suppose*

$$L(x, y, \mathbf{u}) = f_0(x, y) + \sum_{j=1}^{r} u_j f_j(x, y) \neq 0$$

satisfies $\mathrm{Sat}_d(L, P)$ *at every complex zero with parameter in U. Outside one finite set of places depending on the fixed polynomials and U, every character of order $d > 1$ has zero integral on each complete original zero residue ball whose parameter reduction belongs to U. The set of places is independent of the parameter, the character and its conductor. Every residual integral over this parameter domain is consequently constant in s.*

Proof. The case $f_1 = \cdots = f_r = 0$ follows from Lemma 6.5, multiplied by the parameter-box measure q^{-r}. Individual generators may be zero. Principalize the nonzero ideal $I = (f_0, \dots, f_r)$ by a minimal log principalization $\pi : Y \to \mathbb{A}^2$, fixed before any parameter is chosen. Write

$$I\mathcal{O}_Y = \mathcal{O}_Y(-D), \qquad f_j = JA_j, \qquad (A_0, \dots, A_r) = 1.$$

The support of D consists of exceptional components and the strict common factors. The lifted function is

$$L \circ (\pi \times \mathrm{id}) = JS_{\mathbf{u}}, \qquad S_{\mathbf{u}} = A_0 + \sum_{j=1}^{r} u_j A_j. \tag{9.1}$$

On $S_{\mathbf{u}} = 0$, some A_j, $j \geq 1$, is a unit: otherwise the equation would force $A_0 = 0$ too. Thus $S_{\mathbf{u}}$ is an actual parameter coordinate. The graph is smooth and transverse to every fixed SNC stratum of D. Since all centres of π lie in the base locus, the product modification is an isomorphism away from $L = 0$.

For an exceptional $E \simeq \mathbb{P}^1$ above a base point P, write $N_E = \operatorname{ord}_E I$, let k_E count its boundary points in the support of D, and put $E^\circ = E \setminus \partial_D E$. The residual sections define a morphism $\rho_E : E \to \mathbb{P}^r$ and a basepoint-free line bundle of degree $b_E = \deg \rho_E^* \mathcal{O}(1)$. The proper Euler product gives, for every parameter,

$$\mathcal{Z}_{L,(P,\mathbf{u})}(t) = \prod_{E \subset \pi^{-1}(P)} (1 - t^{N_E})^{\chi_{\mathrm{top}}(E^\circ \setminus V(S_{\mathbf{u}}|_E))}. \tag{9.2}$$

Off the graph, a single positive normal coordinate has local factor $1 - t^{N_E}$. At a D crossing or at the graph there are at least two positive normal coordinates and the local monomial torus fibre has determinant one. The moving graph has multiplicity one. These computations give the formula for the fixed divisor D.

Choose a general $\mathbf{u} \in U$ after fixing the finite principalization diagram. The affine hyperplanes whose A_0 coefficient is one form a dense open subset of hyperplane space. Bertini and avoidance of finitely many valuation cancellations ensure that the member $l_{\mathbf{u}}$ has $\operatorname{ord}_E l_{\mathbf{u}} = N_E$ on every component of the fixed diagram, and that its moving strict part is reduced, smooth, transverse to D, and avoids its corners. Formula (9.2) is then also A'Campo's formula for that plane member, so the two determinants agree at the chosen general parameter.

Each strict common factor of multiplicity a has transverse type t^a at a generic original zero-fibre point with parameter in U; choose the parameter to avoid cancellation of its leading coefficient. Thus $d \nmid a$. Each moving strict branch has multiplicity one. The H^0 orders of the plane member divide the gcd of its strict multiplicities, so contain no order divisible by d. Since a plane Milnor fibre has cohomology only in degrees zero and one, a forbidden H^1 order cannot cancel against H^0 in its determinant. Equality with (9.2) therefore transfers saturated exclusion to this generic plane member.

On its minimal embedded resolution the eligible exceptional vertices, those with $d \mid N_E$, are isolated and bivalent. Indeed choose a largest eligible rupture multiplicity N_*. A leaf capable of cancelling its primitive root has multiplicity divisible by N_*; its rupture has a strictly larger eligible multiplicity, a contradiction. Leaves propagate divisibility toward ruptures, and at a bivalent eligible vertex one eligible neighbour forces the other. A finite eligible chain would have to end at a rupture, a leaf or a strict arrow, all excluded. The only remaining eligible vertices are isolated bivalent ones.

We justify transfer of this property to the fixed ideal model. The fixed principalization resolves the generic member and dominates its minimal embedded resolution. An additional centre away from the member's strict transform has residual member a unit. If it were an ideal base point, the new ideal valuation would exceed the sum of the parent valuations, whereas the member valuation would equal that sum. This contradicts their equality on the new divisor, included when the generic parameter was chosen. If the ideal is already principal with SNC support at that centre, the whole later subtree confined above it is unnecessary for log principalization, contrary to minimality. Every additional centre is therefore an SNC centre on a strict member. At a crossing multiplicities add. One eligible and one noneligible component create a noneligible component; two noneligible components can create only an isolated eligible bivalent component. Eligible components never meet. A first blowup on a smooth strict branch has its noneligible

multiplicity. This proves on the fixed model

$$d \mid N_E \quad \Longrightarrow \quad k_E + b_E = 2, \qquad E \text{ meets no other eligible component of } D. \tag{9.3}$$

For a special parameter $\mathbf{u}$, a restricted section may be identically zero, even when $b_E > 0$. In that case the entire open curve is removed in (9.2), and its exponent is zero. Otherwise let $n_E(\mathbf{u})$ count the distinct zeros in E°. For an eligible component its exponent is

$$2 - k_E - n_E(\mathbf{u}) = b_E - n_E(\mathbf{u}) \geq 0. \tag{9.4}$$

If one were positive, a primitive N_E-th root would occur in the actual total-family determinant: any other factor containing it also has eligible multiplicity, so has nonnegative exponent. The spectral hypothesis excludes this. Consequently

$$S_{\mathbf{u}}|_E \equiv 0 \quad \text{or} \quad n_E(\mathbf{u}) = b_E \qquad (\mathbf{u} \in U). \tag{9.5}$$

In the second case all zeros are simple interior points, with no boundary zero.

This condition spreads uniformly. For every eligible E, form the constructible incidence locus of parameters for which a nonzero restricted section has a repeated zero or a zero at one of the fixed boundary points. Remove the linear subspace of identically zero sections when forming it. Its intersection with U is empty in characteristic zero by (9.5). On a finite affine cover, elimination and the Nullstellensatz express that emptiness by finitely many polynomial identities with nonzero denominators. Inverting these denominators keeps the intersection empty in every geometric residue fibre. These fixed incidence data give one finite exceptional set.

Fix now an original plane and parameter residue box. On a graph tube choose $j \geq 1$ with A_j a unit. For each fixed base point and the other parameters, $u_j \mapsto S_{\mathbf{u}}$ maps the entire original u_j ball bijectively onto $\mathfrak{p}$, with unit Jacobian. Integration kills it by the nontrivial character. This also kills an identically zero restricted section and every graph crossing. Off the graph the parameter-box measure is exactly q^{-r}, and its unit character is constant on that box. Noneligible normal divisors kill their terms. The eligible exceptional curves are isolated by (9.3); by (9.5), every surviving open curve has exactly two punctures. At a D boundary its inertia is a nontrivial character χ^{N_j}; at a simple moving zero it is χ. Lemma 2.2 gives zero for its actual trace before multiplication by the original normal series and discrepancy factors.

If the ideal is already principal with SNC support, write $I = (J)$. Its positive strict multiplicities are noneligible by the transverse argument above. Off the graph a normal average vanishes, and on the graph the parameter average vanishes. If J is a unit a zero graph ball is itself a parameter-coordinate ball. Proportional generators and constant nonzero generators are included. For two conjugate strict branches the norm calculation (6.9) gives the same zero with their noneligible common multiplicity. Exceptional descent is as in Subsection 6.3.

Only finitely many orders divide the positive multiplicities on one fixed resolution of the original L. Spread the finitely many corresponding incidences and Kummer systems, and use Lemma 2.3 for every higher conductor. This completes the proof with the stated uniform finite set. □

10 Ramified two-function pencils

Proposition 10.1. *Let $g, h \in \Bbbk[x, y]$ and $k \geq 2$, with $F = g + z^k h \neq 0$. Under $\mathrm{Sat}_d(F)$, every original zero residue ball has zero character integral outside one finite set of places. For the*

assertion on the plane reduction ball of a fixed algebraic centre $P \in g^{-1}(0)$ times $\mathfrak{p}$, it suffices to impose $\mathrm{Sat}_d(F, Q)$ *at every zero-fibre point Q in a sufficiently small complex neighbourhood of* $(P, 0)$.

10.1 Analytic units in the normal forms

The following changes give the complex normal forms used in the spectral calculation. For local units ϵ_g, ϵ_h, set $z' = z(\epsilon_h/\epsilon_g)^{1/k}$. Then

$$\epsilon_g t^N + \epsilon_h t^M z^k = \epsilon_g t^M (t^{N-M} + z'^k),$$
$$z^m(\epsilon_g t^N + \epsilon_h t^M z^k) = \epsilon_g(\epsilon_g/\epsilon_h)^{m/k} z'^m t^M (t^{N-M} + z'^k).$$

For the cyclic form, $x' = x\epsilon_g^{-1/k}$ gives

$$\epsilon_h^m t^A (x^k + \epsilon_g t^N) = \epsilon_h^m \epsilon_g t^A (x'^k + t^N).$$

All roots here are local holomorphic roots of units. When $N - M > 0$, the remaining binomial has positive integral weights. If P is weighted homogeneous of degree D with weights w_j, a common unit B is absorbed by the coordinate map $X_j \mapsto X_j B^{w_j/D}$, since

$$P(X_1 B^{w_1/D}, \ldots, X_a B^{w_a/D}) = BP(X_1, \ldots, X_a).$$

Its derivative at the origin is the invertible diagonal matrix with entries $B(0)^{w_j/D}$. At a crossing with a positive difference, choose positive normal weights w_1, w_2 with $(N_1 - M_1)w_1 + (N_2 - M_2)w_2 > 0$, and choose the positive z weight so that kw_z equals this sum; clearing denominators gives integral weights. When both differences are nonpositive, the local function is a monomial times a unit. The arithmetic calculations below retain the residue-unit coefficients and use principal-unit roots.

10.2 The positive cone and the total-pencil determinant

First assume that g, h are nonzero and $g(P) = h(P) = 0$. On the minimal embedded resolution of gh put $\delta_E = N_E - M_E$. For $\delta_E > 0$, the normal function is $t^{M_E}(t^{\delta_E} + z^k)$. With $b'_E = \gcd(k, \delta_E)$ and $L_E = kN_E/b'_E$, its alternating monodromy determinant is

$$\frac{1 - t^{N_E}}{(1 - t^{L_E})^{b'_E}}. \tag{10.1}$$

To verify it, the positive weighted fibre projects to the nonzero t line as a k-sheeted cover with N_E totally ramified points. Its Lefschetz numbers are $N_E \mathbf{1}_{N_E|j} - kN_E \mathbf{1}_{L_E|j}$. If $\delta_E \le 0$, the germ is a unit times t^{N_E}. At a crossing with both g exponents positive the function-preserving circle action has weights

$$(kN', -kN, NM' - MN')$$

on the two normal coordinates and z. It has no fixed point on a nonzero fibre, and commutes with the finite diagonal monodromy; the crossing factor is one. If one direction is a pure-h arrow, its g exponent is zero and the fixed locus is instead its zero coordinate together with $z = 0$, with t^{N_E} equal to the fibre value. Its factor is $1 - t^{N_E}$. This latter term fills back a pure-h point and must be kept.

All centres over the common point lie in $F = 0$ for every z, so the product modification is an isomorphism off the original zero fibre. Let

$$\Delta_+(u,v) = \prod_{\delta_E>0} (1 - u^{N_E} v^{M_E})^{v_E - 2}, \qquad H(x,y,c) = g(x,y) + ch(x,y).$$

The proper product of the local factors, including the filled pure-h points, gives

$$\mathcal{Z}_{F,(P,0)}(t) = \mathcal{Z}_{g,P}(t) \prod_{\rho^k=1} \Delta_+(\rho t, \rho^{-1}).$$

Doing the same calculation for $k = 1$ identifies $\mathcal{Z}_{H,(P,0)} = \mathcal{Z}_{g,P}\Delta_+(t,1)$, and hence

$$\mathcal{Z}_{F,(P,0)}(t) = \mathcal{Z}_{H,(P,0)}(t) \prod_{\rho^k=1,\ \rho\neq 1} \Delta_+(\rho t, \rho^{-1}). \tag{10.2}$$

The first factor is the determinant of the total three-variable pencil.

If there are at least two reduced branches, Δ_+ is a whole polynomial factor of the colored product. Indeed a primitive direction has one fixed sign of $N - M$, and the Laurent-binomial UFD argument in Lemma 6.1 permits cancellation only within that direction. Since the full colored product is a polynomial, each complete sign group is one as well. If the support is one branch, with multiplicities (n, e), the positive group is empty when $n \le e$ and otherwise is $\Delta_b(u^n v^e)/(1 - u^n v^e)$. Every root of its nonidentity-norm denominator has order dividing

$$L_0 = \frac{kn}{\gcd(k, n - e)}.$$

At a neighbouring smooth point of that shared branch, (10.1) has a primitive L_0 order: if $L_0 > n$ it cannot cancel, and if $L_0 = n$ the denominator has net exponent $k - 1 > 0$. Thus $d \nmid L_0$. No forbidden root can be removed by this denominator.

10.3 The special parameter and the complete nonpositive region

Take the fixed minimal principalization of $I = (g, h)$, with $g = JA$, $h = JB$, $(A, B) = 1$. The proof of Theorem 9.1 applies over $c \neq 0$, because $z \mapsto z^k$ is locally unramified there and the original spectral hypothesis transfers to the total pencil. For a local version one chooses a generic nonzero c sufficiently close to zero. The same finite fixed-diagram argument gives (9.3). Therefore every eligible exponent of the total pencil at $c = 0$ is nonnegative. For a nonconstant map $[A : B]|_E$ of degree b_E, that exponent is $b_E - q_E(0)$, where $q_E(0)$ counts distinct interior zeros. Constant maps have exponent zero, including when their whole open component is on the graph.

The other factor in (10.2) has nonnegative exponents at every forbidden root, by the polynomial and denominator argument just proved. Original saturated exclusion consequently forces

$$q_E(0) = b_E \quad \text{for every eligible nonconstant } E. \tag{10.3}$$

Thus all those zeros are simple interior points. Integrating the whole original $c \in \mathfrak{p}$ ball on the fixed ideal model now gives zero by exactly the graph-coordinate and two-puncture calculations in Theorem 9.1.

For fixed plane variables write $n = \operatorname{ord} g$, $l = \operatorname{ord} h$. If $n > l$, translation $c \mapsto c + g/h$ preserves $\mathfrak{p}$, so its character integral is zero. If $n \le l$, its value is exactly $q^{-1}\chi(\operatorname{ac} g)|g|^s$. Fubini therefore proves the aggregate identity

$$\int_{D \cap \{\operatorname{ord} g \le \operatorname{ord} h\}} \chi(\operatorname{ac} g)|g|^s \, dx\, dy = 0. \tag{10.4}$$

The identity sums the complete region before integration in z.

Resolve gh further from the ideal model only over $A = 0$ or $B = 0$, which are disjoint. Each new divisor over $A = 0$ has positive difference, and each over $B = 0$ has negative difference, by addition of parent differences and strict multiplicities. Every zero-difference exceptional divisor was already on the ideal model. Equivalently one can start with the minimal gh graph and insert the bivalent sign-separating vertices. If an eligible zero-difference divisor met a positive exceptional divisor, its zero on the ideal model would either be at a D boundary or be a multiple interior zero; a simple interior zero already is SNC and requires no further blowup. Either possibility contradicts (10.3). A positive shared strict branch and a multiple pure-g end are excluded in the same way. The one allowed case is a simple reduced pure-g end, which will be integrated as a whole ball below. This proves the zero-difference boundary assertion needed for the full-lattice parity argument.

10.4 The exact shell expansion

Let $g = \pi^n u$, $h = \pi^l v$ and $T = q^{-s}$. On the original z shell $\operatorname{ord} z = j \geq 1$ the three contributions are

$$\begin{aligned} n < l + kj: &\quad (1 - q^{-1})q^{-j}\chi(u)T^n, \\ n > l + kj: &\quad (1 - q^{-1})q^{-j}\mathbf{1}_{\chi^k=1}\chi(v)T^{l+kj}, \\ n = l + kj: &\quad q^{-j-1}T^n \sum_{w \in \mathbb{F}_q^\times} \chi(u + vw^k). \end{aligned} \tag{10.5}$$

At a nonzero balanced root the derivative kvw^{k-1} is a unit, and its entire root ball has zero character integral. Thus the sum in (10.5) includes all remaining contributions with the convention $\chi(0) = 0$. Power-map counting and $a = -vw^k/u$ give the exact identity

$$\sum_{w \neq 0} \chi(u + vw^k) = -\chi(u) + \sum_{\substack{\rho^k=1 \\ \rho \neq 1}} \rho(-1)J(\rho, \chi)(\chi\rho)(u)\rho^{-1}(v). \tag{10.6}$$

In detail, the number of k-th roots of a nonzero $b \in \mathbb{F}_q$ is $\sum_{\rho^k=1} \rho(b)$. The trivial-ρ term is $\sum_{b \neq 0} \chi(u + vb) = -\chi(u)$; the other terms are the displayed Jacobi sums over the character group of $\mathbb{F}_q^\times$. The plane pairs are

$$(\lambda, \mu) = (\chi\rho, \rho^{-1}), \qquad \rho^k = 1. \tag{10.7}$$

They include $(\chi, 1)$, and, when $\chi^k = 1$, $(1, \chi)$.

For $n - l \leq 0$, summing the first line of (10.5) gives $q^{-1}\chi(u)T^n$, killed by (10.4). For $n - l = ka + r > 0$, $0 \leq r < k$, the entire g-dominant tail has value $q^{-a-1}\chi(u)T^n$. When $r = 0$, it cancels the $-\chi(u)$ term of the balanced shell exactly. When $r > 0$ it remains. The $z^k h$-dominant shells are $1 \leq j < a$ if $r = 0$, and $1 \leq j \leq a$ if $r > 0$. These are the ranges used below.

10.5 Positive components and the quadratic remainder

For a positive rupture write $(N, M) = s(p, q)$, $p > q$. The nonidentity norm in (10.2) uses $u^p v^q = \rho^{p-q} t^p$. Lemmas 6.1 and 6.2 give an actual primitive order

$$L = \frac{kN}{\gcd(k, N - M)}, \tag{10.8}$$

except when $k = 2$, $p - q$ is odd and $s \equiv 2 \pmod 4$. Eligibility for (10.7) implies $\chi^N \rho^{N-M} = 1$, hence $d \mid L$. Thus for $k \geq 3$ there is no eligible positive rupture. For $k = 2$ the protected

primitive factor in the exception supplies order $N_B/2$. Since $\chi^{N_B} = 1$ but $d \nmid N_B/2$, necessarily $\chi^{N_B/2} = -1$. A surviving half-index would supply order N_B, so half-index protection forces

$$V_B = 2V_L, \quad \delta_L \text{ odd}, \quad \chi^{N_L} = -1, \quad v_B = 3, \quad L^2 = -2, \quad \nu_B = 2\nu_L - 1. \tag{10.9}$$

A shared positive strict branch is ineligible by its actual transverse spectrum (10.1). A pure g branch of multiplicity $n \geq 2$ has transverse suspension $x^n + z^k$. Thom–Sebastiani and Lemma 6.2 give a primitive $\operatorname{lcm}(n,k)$ order except $n = k = 2$. In the potentially exceptional case $k = 2$, $n \equiv 2 \pmod 4$, $n > 2$, take instead a primitive $n/2$-th root in the reduced H^0 of x^n. Since $n/2 > 1$ is odd, multiplying it by -1 gives a primitive n-th root. Only $n = k = 2$ lacks this nontrivial base root. Eligibility is therefore possible only for a reduced pure-g end with $\lambda = 1$, or for $n = k = 2$ with χ quadratic. The $n = 1$ germ is smooth.

For $k \geq 3$, a positive eligible leaf propagates to a rupture, which is impossible. A nonisolated eligible bivalent chain must end at a rupture, an eligible strict end or an eligible zero-difference boundary. The last is excluded above. Two reduced pure-g ends would exhaust the connected graph and leave no h branch at the common point. Alternatively, on a minimal positive chain the difference recurrence is

$$\kappa_i \delta_i = \delta_{i-1} + \delta_{i+1}, \qquad \kappa_i \geq 2.$$

A positive integer sequence cannot join endpoints zero and one: starting from zero, its positive successive differences are nondecreasing, so the other endpoint would exceed one. Inserted positive sign-separating vertices also have self-intersection at most -2, since a positive new (-1)-vertex still meets a negative one and must be blown up again. The remaining exceptional components are isolated bivalent curves, and all their fixed-n, l coefficients are zero.

At the remaining reduced end choose coordinates with

$$g = t^M u(t,x)x, \qquad h = t^M v(t,x), \qquad r = g/h = xu/v. \tag{10.10}$$

For fixed t, r maps the complete x ball bijectively onto $\mathfrak{p}$, with unit derivative. The angular value and norm of h, and the absolute Jacobian unit, are constant in this ball. For every original $z \in \mathfrak{p}$, translation $r \mapsto r + z^k$ preserves $\mathfrak{p}$; integration of $h(r + z^k)$ in r is zero. This cone lies in $n > l$, so its removal does not affect (10.4). For an identity SNC chart with opposite signs, $r = y^{-b}x(u/v)$ maps the complete cone $\operatorname{ord} x > b \operatorname{ord} y$ onto $\mathfrak{p}$. Its inverse Jacobian norm is $|y|^b$, independent of r, and the same translation proves zero.

Now let $k = 2$, let η be the quadratic residue character, and remove these full translation cones. For the balanced pair $(\chi\eta, \eta)$, every rupture has its eligible leaf by (10.9). For a connected eligible positive component C let $a(C)$ count eligible strict arrows and $b(C)$ count noneligible boundary edges. The tree identity is

$$\sum_{E \in C} (v_E - 2) = a(C) + b(C) - 2.$$

There is a noneligible boundary on a path toward an h-containing arrow, and exactly one is impossible by the principal-divisor character relation. Hence $b(C) \geq 2$. Every displayed exponent is nonnegative. At the forbidden root in (10.2) their sum is zero, so $a(C) = 0$, $b(C) = 2$. A nonrupture boundary vertex is then an isolated bivalent curve. At a rupture boundary vertex its eligible designated leaf forces both exterior neighbours to be noneligible: exactly one cannot occur. Its component is exactly the pair (B, L). Thus these are all balanced components. Their even-δ series vanish by (6.8), with both n, l variables retained.

For the pair $(\chi, 1)$ the designated leaf is noneligible, every eligible rupture has even δ, and every remaining eligible strict end has even δ (only the multiplicity-two exception remains). Apply Lemma 6.4 with $e_i = N_i - M_i$. Its outside-row condition is exactly the zero-difference boundary assertion proved above; the full matrix includes every sign-separating vertex. Every nonzero possible coefficient has even δ, so the odd-δ tail is zero.

The z^2h-dominant range requires an additional argument when $\chi = \eta$. In that case its pair $(1, \eta)$ is also the balanced pair. Suppose a balanced dyadic pair existed. Leaf eligibility gives M_L even, while δ_L odd makes N_L odd. Both exterior neighbours of B have odd M. The difference-divisor recurrence at B makes their differences of opposite parity; choose the odd one J. After sign separation it is positive. Then $N_J = M_J + \delta_J$ is even, so J and B are both eligible for $(\eta, 1)$. If J is exceptional, their nonsingleton component contradicts the full-matrix parity conclusion. If J is strict, a shared arrow is spectrally ineligible, while a pure-g arrow has $M_J = 0$, also impossible. Therefore this balanced pair has no dyadic component at all. Every one of its valuation coefficients is zero by the remaining isolated bivalent curves. This kills every retained z^2h-dominant shell.

The nonpositive aggregate, the full translation cones, the coefficientwise $k \geq 3$ vanishing, and these three quadratic conclusions account for every term of (10.5). They prove zero on the original common-point $z \in \mathfrak{p}$ ball.

10.6 Other balls and degenerate inputs

A single smooth shared branch has its own normal coordinate; its transverse spectrum excludes the normal character. In an identity SNC chart normal-unit averaging leaves only the full reduced-end cones just treated and the nonpositive aggregate. The multiplicity-two pure-g exception deserves an explicit check. It requires $k = 2$ and $\chi = \eta$. In the original SNC chart, complex analytic unit changes in x, z put its function in the form

$$F = y^a x^2 + y^b z^2, \qquad a \geq 0, \quad b > 0.$$

These complex changes are used only for the spectral calculation; the arithmetic coefficients retain the original residue constants. If $a > b$, a generic point of $y = 0$ with $x, z \neq 0$ has transverse multiplicity b, so b is odd. At a neighbouring point with $x \neq 0$ and $y = z = 0$, the transverse function is $y^b(z^2 + cy^{a-b})$. Its actual maximal order is $2a/\gcd(2, a - b)$. If a were even this order would be even, contrary to $\mathrm{Sat}_2(F)$; hence a is odd as well. If $0 < a \leq b$, the generic transverse multiplicity instead forces a odd. If b is even, necessarily $b > a$, and at a point with $z \neq 0$ and $x = y = 0$ the transverse function $y^a(x^2 + cy^{b-a})$ has actual primitive order $2b$, again forbidden. Thus in every allowed case with $a > 0$, both a, b are odd. The three low, balanced and high plane characters are respectively $(\eta, 1)$, $(1, \eta)$ and $(1, \eta)$; their y-unit characters are η^a, η^b, η^b, all nontrivial. Each coefficient therefore averages to zero on the original y units, at every fixed valuation pair. Finally, if $a = 0$, the plane function $y^b z^2$ has two reduced branches. Its Wang sequence supplies an eigenvalue one on H^1; Thom–Sebastiani with x^2 supplies an actual eigenvalue -1 for F, contradicting $\mathrm{Sat}_2(F)$. This exhausts the identity-SNC multiplicity-two exception. For a nonsplit strict node, the two branch vectors are equal; in the positive case the shared-branch spectrum excludes its common normal character, and in the nonpositive case the original transverse monomial excludes the low character. Equation (6.9) therefore proves zero over K itself. Exceptional descent is already covered by Subsection 6.3.

Where h is a unit, put $c = h(P)$ at an integral lift of the residue centre and set $z' = z(h/h(P))^{1/k}$. This principal-unit change maps the whole ball onto itself with unit Jacobian

and gives $F = g + cz'^k$. On the fixed complex open set $D(h)$, choose a local root of h and put $w = zh^{1/k}$, so $F = g + w^k$. This transfers the required spectra to the fixed coefficient-one model over $D(h)$. Theorem 7.1 and the open-set form of Lemma 8.2 apply on the whole ball with one finite exceptional set for all unit values of c. Where g is a unit and $z \in \mathfrak{p}$, the function is a unit. On a unit-z ball, $c = z^k$ maps the whole ball bijectively onto a parameter ball with unit Jacobian; the total-pencil condition over $c \neq 0$ and Theorem 9.1 apply.

If $h = 0$, the assertion is Lemma 6.5 with a passive z variable. If $g = 0$, the function is $z^k h$. At a generic point of $z = 0$ with $h \neq 0$, saturated exclusion gives $d \nmid k$, so its $z = 0$ balls vanish by a normal average; on unit-z balls Lemma 6.5 for h applies. Constant nonzero g or h are included in the corresponding unit cases. Only $F \equiv 0$ is excluded. Fixed principalizations, sign-separated diagrams, original resolutions and incidence loci determine a finite set of bad places. As before only finitely many tame orders can contribute, and Lemma 2.3 treats every other conductor. This proves Proposition 10.1.

10.7 A positive outer power

Proposition 10.2. *For arbitrary plane g, h and integers $m, k \geq 1$, let $F = z^m(g + z^k h) \neq 0$. Under $\mathrm{Sat}_d(F)$, the conclusion of Proposition 10.1 holds. In particular the plane reduction ball of a fixed algebraic centre P times $\mathfrak{p}$ has zero integral when $\mathrm{Sat}_d(F, Q)$ holds at every zero-fibre point Q in a sufficiently small complex neighbourhood of $(P, 0)$.*

Proof. Put $a = m + k$. At a common plane zero use the minimal gh resolution and $\delta_E = N_E - M_E$. For $\delta_E > 0$ put

$$c_E = \gcd(k, \delta_E), \qquad W_E = aN_E - mM_E, \qquad L_E = W_E/c_E.$$

The normal function is $x^{M_E} z^m (x^{\delta_E} + z^k)$. Both x, z are nonzero on its nonzero fibre. The map

$$(x, z) \longmapsto (U, V) = (x^{N_E} z^m, x^{M_E} z^a)$$

is a torus isogeny of degree W_E, restricted to $U + V = 1$, $U, V \neq 0$. This base is a three-punctured projective line, so its equivariant Euler class is minus the regular representation of the isogeny kernel. Ordinary Milnor monodromy has phases $(k/W_E, \delta_E/W_E)$ and exact order L_E; indeed $\gcd(W_E, k, \delta_E) = \gcd(k, \delta_E)$. Its restriction is c_E regular cyclic representations with negative sign, and the normal alternating monodromy determinant is $(1 - t^{L_E})^{-c_E}$.

For $\delta_E \leq 0$ the germ is a monomial in two positive normal coordinates times a unit, so all Lefschetz numbers are zero. At an exceptional crossing, the two-monomial torus map has a positive-dimensional torus kernel when balance is possible. Its nonzero fibres have zero Euler number, including every monodromy fixed locus. If a strict coordinate can be zero, the boundary fibre is a monomial torus fibre in the remaining exceptional and z coordinates and also contributes zero. The product modification over the common plane point is an isomorphism off $F = 0$. The proper Euler product therefore gives

$$\mathcal{Z}_{F,(P,0)}(t) = \prod_{\delta_E > 0} (1 - t^{L_E})^{c_E(v_E - 2)}. \tag{10.11}$$

No positive rupture has $d \mid L_E$. For a dead tip L with $V_B = bV_L$, Lemma 6.1 gives

$$\frac{L_B}{L_L} = \frac{b \gcd(k, \delta_L)}{\gcd(k, b\delta_L)} = \frac{b}{\gcd(b, k/\gcd(k, \delta_L))} \in \mathbb{Z}_{\geq 1}. \tag{10.12}$$

If a positive eligible rupture exists, take its largest eligible period L_*. A negative leaf exponent at a primitive L_*-th root requires $L_* \mid L_L$; its rupture period is a multiple of L_L and eligible, so maximality forces $L_B = L_L = L_*$. In this equality case $c_L = c_B/b$. Grouping each cancelling leaf with its rupture, the net exponent is at least

$$c_B(1 - 1/b) > 0, \quad \text{or} \quad c_B(1 - 1/b_1 - 1/b_2) \geq c_B(1 - 1/2 - 1/3) > 0.$$

The two determinants are coprime. Every possible negative group at this root has been included in this same maximal-period argument. Hence (10.11) has a genuine primitive L_* order, a contradiction.

We calculate the original angular channels. At a generic $z = 0$, $g \neq 0$ point the transverse function is z^m, so $\chi^m \neq 1$. Write $g = \pi^n u$, $h = \pi^l v$ and $\operatorname{ord} z = j \geq 1$. The three shell integrals, with $T = q^{-s}$, are respectively

$$\begin{aligned} n < l + kj: &\quad 0, \\ n > l + kj: &\quad (1 - q^{-1})q^{-j}\mathbf{1}_{\chi^a=1}\chi(v)T^{l+aj}, \\ n = l + kj: &\quad q^{-j-1}T^{n+mj}\sum_{w\neq 0}\chi(w^m(u + vw^k)). \end{aligned} \tag{10.13}$$

The first zero is the χ^m unit average. The balanced nonzero roots have derivative kvw^{a-1} for the whole polynomial and full Hensel tails of zero integral. Exact two-variable Fourier inversion gives

$$\sum_{w\neq 0}\chi(w^m(u + vw^k)) = \sum_{\lambda^k=\chi^a}\lambda^{-1}(-1)J(\lambda^{-1}, \chi)\lambda(u)(\chi\lambda^{-1})(v). \tag{10.14}$$

One may verify the coefficient by multiplying by $\lambda^{-1}(u)\mu^{-1}(v)$, summing over $u, v \neq 0$, dividing by $(q - 1)^2$, and substituting $u = -vw^k t$. The v and w sums force, respectively,

$$\lambda\mu = \chi, \qquad \lambda^k = \chi^a, \qquad \lambda^m\mu^a = 1. \tag{10.15}$$

Their two factors of $q - 1$ cancel the Fourier normalization; the remaining sum is the Jacobi sum in (10.14). The dominant pair $(1, \chi)$ when $\chi^a = 1$ satisfies the same relations. Every surviving region has $n > l$, and $\mu = 1$ would imply $\chi^m = 1$, so never occurs.

For a divisor on which g, h have multiplicities $N > M$, put $W = aN - mM$ and $L = W/\gcd(k, N - M)$. Eligibility also imposes $\lambda^N\mu^M = 1$. The order of $(1, 1)$ in

$$\mathbb{Z}^2/\langle (N, M), (m, a)\rangle$$

is $W/\gcd(W, k, N - M) = L$: expressing a multiple $b(1, 1)$ in the two rows requires kb/W and $(N - M)b/W$ to be integers. Thus the order d of $\lambda\mu$ divides L. No positive rupture is eligible by the maximal-period argument.

For a pure-g strict branch of multiplicity $n \geq 2$, the original transverse function $z^m(x^n + cz^k)$ has determinant

$$\frac{1 - t^a}{(1 - t^{an/\gcd(n,k)})^{\gcd(n,k)}}.$$

Its indicated maximal primitive order survives, including $n \mid k$, when the remaining denominator exponent is $n - 1 > 0$. Eligibility is impossible by the same order calculation. A positive shared strict branch is excluded by the torus calculation above. The only eligible positive strict end is therefore a reduced pure-g branch with $\lambda = 1$.

After sign separation every positive bivalent exceptional has self-intersection at most -2. An original (-1) bivalent curve could otherwise be contracted while preserving SNC; a new positive (-1) curve still touches a negative component and must be blown up again. An eligible leaf propagates to a rupture. A bivalent eligible curve either has two noneligible neighbours, and its Kummer cohomology is zero, or belongs to a chain of eligible vertices. Such a chain has nonpositive endpoints or reduced pure-g endpoints. Two nonpositive endpoints contradict the maximum principle for $\kappa_i\delta_i = \delta_{i-1} + \delta_{i+1}$, $\kappa_i \geq 2$. An endpoint zero and an endpoint one contradict the increasing positive differences proved above. Two reduced pure-g ends would exhaust the connected graph and leave no h branch at the common point. Hence all exceptional terms vanish with both original plane valuations fixed.

The remaining simple end is exactly (10.10). On its whole $r = g/h \in \mathfrak{p}$ ball the integrand, up to a factor independent of r, is

$$\chi(\mathrm{ac}(z^m(r + z^k)))|z|^{ms}|r + z^k|^s\, dr.$$

For each original z, translation by z^k gives zero. This kills its dominant and balanced pieces together. In an identity SNC chart the same coordinate $r = y^{-b}x(u/v)$ maps the original positive cone onto $\mathfrak{p}$ with inverse Jacobian norm $|y|^b$, independent of r. Normal averaging excludes every other positive coordinate axis; the complementary g-dominant region was already zero in (10.13). One smooth shared branch is treated by its normal character. For conjugate strict branches, the original norm-unit character in (6.9) is nontrivial by the shared transverse period, and its sum is zero. This proves all common-point $z = 0$ balls over the original field.

Where g is a unit, the first shell argument applies. Where h is a unit, set $U = h/h(P)$ and $a = m + k$ and make the principal-unit change $z' = zU^{1/a}$. It preserves the entire ball and its Haar measure, and gives exactly

$$F = (z')^m\big(gU^{-m/a} + h(P)(z')^k\big).$$

Proposition 8.1 applies through its fixed-data local calculation. On the plane ball $U \in 1 + \mathfrak{p}$, so $gU^{-m/a}$ has exactly the valuations and tame angular components of g. Thus all base valuation coefficients in that proof are those of the fixed polynomial g.

On the fixed complex open set $D(h)$, choose a local root of h and put $w = zh^{1/k}$. Then

$$F = h^{-m/k}w^m(g + w^k).$$

Multiplication by a nonvanishing holomorphic unit preserves local monodromy. Choose a local logarithm of the unit; adding it to the value coordinate identifies the pullbacks over the universal cover of the punctured target and commutes with deck translation. Shrinking the two Milnor representatives makes them cofinal. Thus the original spectrum gives the fixed coefficient-one model's spectrum over $D(h)$, and the base spectra used in the prefactored argument are those of g. The principal-unit root only requires the primes dividing a to be excluded. The fixed algebraic g, h, their resolution and their unit charts account for the remaining data, while the open-set form of Lemma 8.2 handles every unit $h(P)$ with the same finite exceptional set. The comparison correction is (8.2).

For unit z, the parameter $c = z^k$ is unramified, and z^m is an analytic unit with constant tame character on each original residue ball. Theorem 9.1 over $c \neq 0$ proves zero. If $g = 0$ the function is the separated product $z^{m+k}h$; if $h = 0$ it is $z^m g$. Generic transverse spectra exclude the indicated outer exponent, so normal averaging handles $z = 0$, and Lemma 6.5 handles unit z. Nonzero constant inputs fall in these unit cases. The same finite-diagram argument and Lemma 2.3 give uniformity and all conductors, completing the proof. □

10.8 Arbitrary univariate coefficients

Theorem 10.3. *Let $A, B \in \Bbbk[z]$ and $g, h \in \Bbbk[x, y]$, and suppose*

$$F(x, y, z) = A(z)g(x, y) + B(z)h(x, y) \neq 0.$$

If $\mathrm{Sat}_d(F)$ *holds, then, outside one finite set of places depending only on these fixed data, every character of order $d > 1$ has zero integral on each complete original zero residue ball. Every original residual integral is constant in s. The set of places works for all orders and conductors.*

Proof. We first record the separated case $F = G(z)L(x, y)$, with G, L nonzero. At a generic point of a zero of G of multiplicity m and with $L \neq 0$, saturated exclusion gives $d \nmid m$. On its original residue ball a principal-unit normalization of G/t^m preserves the full t ball, and the normal χ^m average is zero independently of the plane variables. On a G-unit ball the spectrum is that of L with a passive variable, so Lemma 6.5 gives zero on its zero balls. This includes constant L, constant G, and any identically zero member among A, B, g, h when the total polynomial is nonzero. It also includes proportional plane coefficients.

Write $A = Ga$, $B = Gb$ with $\gcd(a, b) = 1$. If $[a : b]$ is constant, the preceding case applies. Otherwise it defines a nonconstant morphism $\varphi : \mathbb{P}^1_z \to \mathbb{P}^1$. Let E be the source open set where z is finite, $G \neq 0$, and φ is unramified. Its image $V = \varphi(E)$ is open. Every geometric target value in V has at least one finite unramified preimage where G is a unit. On the chart $a \neq 0$, use $R = b/a$ and the total pencil $g + ch$; on the other chart swap g, h. The actual original germ at that preimage is a nonvanishing analytic unit times the total pencil after the coordinate $c = R(z)$. Thus original saturated exclusion transfers to the total pencil over V. A critical value with some unramified preimage is included: one good preimage is sufficient.

At a good place each original z ball reducing into E maps bijectively under R to its complete parameter ball, with unit Jacobian, and Ga has constant tame character and unit norm. The algebraic inclusion $\varphi^{-1}(\mathbb{P}^1 \setminus V) \subset \mathbb{P}^1 \setminus E$ spreads after inverting finitely many incidence resultants. Hence its target reduction lies in V at every remaining place. The arbitrary-open-set version of Theorem 9.1 applies, including target parameters with residue zero in a chosen affine chart.

Only finitely many source points remain: the finite zeros of G and the finite ramification points of φ. Spread their reduced finite scheme etale, remove collision primes, and invert the norms of their finitely many nonzero local leading coefficients. A rational exceptional residue point has a unique K-rational lift α. A nonrational residue factor labels no rational z ball. The following coordinate change therefore takes place over K.

Use the chart $a(\alpha) \neq 0$ and put $t = z - \alpha$. Write

$$Ga = t^m v(t), \quad v(0) \neq 0, \qquad R = c_0 + ct^k u(t), \quad c \neq 0, \quad u(0) = 1.$$

Here $m \geq 0$ and $k \geq 1$ are fixed local multiplicities. At the good places $u \in 1 + \mathfrak{p}$ on the whole $t \in \mathfrak{p}$ ball. Excluding primes dividing k, set

$$w = t\, u(t)^{1/k}. \tag{10.16}$$

This is an integral analytic bijection of $\mathfrak{p}$ onto $\mathfrak{p}$: it is $t + O(t^2)$, and Hensel's lemma gives a unique solution of $w(t) = w_0$ for every $w_0 \in \mathfrak{p}$. Its derivative and inverse derivative are units throughout the ball. Only a principal unit was rooted; the constant c remains. The function becomes

$$F = v_\alpha(w) w^m \big(g + c_0 h + w^k (ch)\big), \tag{10.17}$$

where v_α is a unit whose tame angular value is constant on the whole ball. This equality retains the original parameter domain and Haar measure.

If $m \geq 1$, apply Proposition 10.2; if $m = 0, k \geq 2$, apply Proposition 10.1; if $m = 0, k = 1$, use the linear-parameter theorem. The first two propositions need only the central and neighbouring spectra along $w = 0$. These conditions hold at every zero-fibre point in a sufficiently small complex neighbourhood of the marked centre, by the local coordinate and unit in (10.17). The generic nonzero parameter used to select a member of the fixed ideal diagram can be chosen in that same neighbourhood. Multiplication by a nonvanishing analytic unit preserves the local Milnor monodromy, as is seen by choosing a local logarithm and deforming the argument map through the same Milnor fibration class. Arithmetically that unit is retained as the constant tame factor above.

The two target charts, finite source scheme, fixed principalizations, graph refinements and local models form a finite set of algebraic data. Together with one resolution of the original F, they determine a finite exceptional set. Only finitely many tame orders divide its positive resolution multiplicities; the requisite incidence and Kummer checks are therefore uniform. Wild characters are treated on that original resolution by Lemma 2.3. Every original residue ball has now been covered. Unit-value balls contribute constants in s, which proves the theorem. □

11 A cyclic suspension with an arbitrary plane factor

Corollary 11.1. *Let $g, H \in \Bbbk[y, z]$, $H \neq 0$, and let $m \geq 1$, $k \geq 2$. Put*

$$F(x, y, z) = H(y, z)^m \big(x^k + g(y, z)\big).$$

If $\mathrm{Sat}_d(F)$ holds, then outside one finite set of places depending only on the fixed data, every character of order $d > 1$ has zero integral on each complete original zero residue ball. In particular all original residual integrals are constant in s. The plane functions may be nonreduced and may share arbitrary components; the excluded set is uniform over the character and its conductor.

Proof. Apply Theorem 10.3 with parameter x, univariate coefficients $x^k, 1$, and plane coefficients $H^m, H^m g$. We also give the direct local calculation.

First consider the degenerate inputs. If $g = 0$, the separated product is $H^m x^k$. At a generic point of $x = 0$ where $H \neq 0$, the transverse spectrum gives $d \nmid k$. Thus integration over every $x \in \mathfrak{p}$ ball is zero by its χ^k unit average, independently of the plane variables. On a unit-x ball the original spectral condition implies that for H^m, and Lemma 6.5 gives zero on its zero balls. If H is a nonzero constant, apply Theorem 7.1; its constant factor has no effect on vanishing. A nonzero constant g is included in the unit cases below.

At any plane point, choose $x \neq 0$ with $x^k + g \neq 0$. The original function there is a unit times H^m with a passive variable. Thus the original all-point condition implies $\mathrm{Sat}_d(H^m)$. If x has nonzero residue, the bracket is either a unit or a coordinate with unit x derivative. In the first case its tame factor is constant on the original ball and Lemma 6.5 for H^m applies; in the second, integration of the bracket coordinate is zero. If $x \in \mathfrak{p}$ and g is a unit, again the bracket is a unit and the plane theorem applies. If H is a unit, its tame character is constant on the whole ball and Theorem 7.1 applies. It remains to consider a common plane zero P with $x \in \mathfrak{p}$.

Neighbouring strict branches and the actual determinant

Use the minimal plane resolution and write $V_i = (N_i, M_i) = (\mathrm{ord}_{E_i} g, \mathrm{ord}_{E_i} H)$. All exceptional entries above P are positive. The plane angular pairs which occur after integrating x are

$$(\lambda, \mu) = (\chi\rho, \chi^m), \qquad \rho^k = 1. \tag{11.1}$$

The pair $(1, \chi^m)$ occurs when $\chi^k = 1$ and is already included by $\rho = \chi^{-1}$. Every H-containing strict arrow, with multiplicities (n, h) and $h > 0$, is noneligible for every such pair. If $n = 0$, its transverse type at generic nonzero x is t^{mh}, so $d \nmid mh$. If $n > 0$, the actual transverse germ at $x = 0$ is

$$\phi(t, x) = t^{mh}(x^k + t^n).$$

Set $b = \gcd(k, n)$, $D = n + mh$, $L_0 = kD/b$. Its alternating monodromy determinant is

$$\mathcal{Z}_\phi(t) = \frac{1 - t^D}{(1 - t^{L_0})^b}. \tag{11.2}$$

The primitive L_0 order survives: if $L_0 > D$ it cannot cancel, while if $L_0 = D$ then $b = k$ and the remaining denominator exponent is $k - 1 > 0$. Eligibility would imply $\chi^D \rho^n = 1$, and hence $d \mid L_0$, a contradiction. This is the all-point shared-branch input.

Here is the proper calculation at the common point. Along an exceptional open curve the normal function is $t^A(x^k + t^N)$ with $A = mM > 0$. It has positive weights $(N/\gcd(k, N), k/\gcd(k, N))$ in (x, t). Projection of its nonzero fibre to $t \in \mathbb{C}^\times$ is a k-fold cover with $A + N$ totally ramified points. Its monodromy fixed-locus Euler numbers are

$$(A + N)\mathbf{1}_{A+N|j} - k(A + N)\mathbf{1}_{k(A+N)/\gcd(k,N)|j}.$$

They give (11.2) with $D = A + N$. At a crossing the normal function is $t_1^A t_2^B(x^k + t_1^N t_2^{N'})$. It is preserved by the circle action with integer weights

$$(w_x, w_1, w_2) = \big(BN - AN',\ k(B + N'),\ -k(A + N)\big).$$

Indeed the weights of its two monomials are

$$\begin{aligned} Aw_1 + Bw_2 + kw_x &= k\{A(B + N') - B(A + N) + BN - AN'\} = 0, \\ (A + N)w_1 + (B + N')w_2 &= k\{(A + N)(B + N') - (B + N')(A + N)\} = 0. \end{aligned}$$

At least one normal direction is exceptional; take it to be t_1, so $A, N > 0$. The other direction has $B + N' > 0$. Thus w_1, w_2 are nonzero, and every circle-fixed point has $t_1 = t_2 = 0$. If $w_x = 0$, the remaining fixed x axis is contained in the zero fibre. The action commutes with diagonal monodromy, so its restriction to every monodromy fixed locus in a nonzero fibre has no fixed point. Circle-action localization gives Euler number zero on each such locus, so its determinant is one. The product modification is an isomorphism off $F = 0$, since its centres lie above $H = 0$. The proper Euler product therefore gives

$$\mathcal{Z}_{F,(0,P)}(t) = \prod_{\substack{\rho^k = 1 \\ \rho \neq 1}} \Delta(\rho t, t^m). \tag{11.3}$$

Indeed

$$\prod_{\rho^k = 1} (1 - \rho^N t^{N+mM}) = (1 - t^{k(N+mM)/\gcd(k,N)})^{\gcd(k,N)},$$

which converts each transverse factor into the corresponding nonidentity norm with exponent $v_E - 2$.

For one reduced singular shared branch with multiplicities (n, h), use the explicit denominator (6.5). Its norm roots have orders dividing $k(n + mh)/\gcd(k, n)$, excluded by the neighbouring spectrum (11.2). No forbidden numerator factor can be removed by this denominator. For a single smooth branch, use its normal coordinate; the character just proved nontrivial makes every valuation coefficient zero directly.

Protected periods and the two quadratic channels

At a rupture write $(N, M) = s(p, q)$ with $\gcd(p, q) = 1$. In (11.3) the primitive monomial is $\rho^p t^{p+mq}$. Lemmas 6.1 and 6.2, including the exact root-lifting assertion, give the primitive order

$$\frac{k(N + mM)}{\gcd(k, N)} \tag{11.4}$$

except when $k = 2$, p is odd and $s \equiv 2 \pmod 4$. Eligibility for (11.1) implies that d divides (11.4). Thus for $k \geq 3$ no rupture is eligible. A finite eligible component without a rupture has no eligible edge: at a bivalent vertex one eligible neighbour forces the other, and a leaf propagates toward a rupture. Following a finite chain must encounter a rupture or a strict arrow; a chain between two eligible pure-g ends would exhaust the tree and leave no H arrow at the common point. Since all H-containing arrows are noneligible, the only remaining components are isolated bivalent curves. Their actual coefficients with $\operatorname{ord} g$ and $\operatorname{ord} H$ separately prescribed are zero.

For $k = 2$ let η be the quadratic residue character. An eligible rupture must satisfy $N_B \equiv 2 \pmod 4$, M_B even. Put $D_B = N_B + mM_B$. Eligibility gives $\chi^{D_B} = 1$. Primitive protection in the exceptional case supplies an eigenorder $D_B/2$, so d is even and

$$\chi^{D_B/2} = -1.$$

A surviving half-index would supply order D_B, which is forbidden. Equation (6.3) forces exactly one (-2)-leaf L at a valency-three rupture, with

$$V_B = 2V_L, \qquad N_L \text{ odd}, \qquad \nu_B = 2\nu_L - 1. \tag{11.5}$$

For the balanced pair $(\chi\eta, \chi^m)$, the designated leaf is eligible, since its character is

$$\chi^{N_L + mM_L}(-1)^{N_L} = (-1)(-1) = 1.$$

At the root t represented by χ, original saturated exclusion gives zero exponent in $\Delta(-t, t^m)$. For a connected eligible exceptional component C, let $a(C)$ count eligible arrows and $b(C)$ its noneligible boundary edges. As before

$$\sum_{E \in C} (v_E - 2) = a(C) + b(C) - 2.$$

There is a noneligible boundary on a path to an H arrow, and there cannot be precisely one. Hence $b(C) \geq 2$. All component exponents are nonnegative, so their zero sum forces $a(C) = 0$, $b(C) = 2$. A nonrupture boundary vertex is an isolated bivalent curve. A rupture boundary vertex has its eligible leaf and therefore both exterior neighbours noneligible; its entire component is just (B, L). These are all balanced components. Lemma 6.3 gives their actual coefficients $qC, -C, C$, and their even-$\operatorname{ord} g$ series vanish by (6.8) with both valuation variables.

For the other pair (χ, χ^m), the designated leaf is noneligible, whereas every eligible rupture has even N_B. An eligible strict arrow is pure g, and $\chi^n = 1$ with d even makes its multiplicity even. Apply the full-matrix Lemma 6.4 with $e_i = N_i$. A nonrupture with a noneligible neighbour is an isolated bivalent curve; there is no omitted regional boundary in this full common-point graph. Its four row checks therefore apply on the entire exceptional lattice. Every nonzero possible coefficient has even $\operatorname{ord} g$. In particular the odd-$\operatorname{ord} g$ coefficients of this pair are zero.

The quadratic dominant range

Let $\chi = \eta$. At a generic point of every H branch the transverse exponent is mh. The spectral condition implies that mh is odd, so m is odd. The balanced pair is now $(1, \eta)$, also the pair in the x^2-dominant range. Suppose it had a dyadic pair. Leaf eligibility gives M_L even, whereas (11.5) gives N_L odd. Both exterior neighbours of B have odd M, being noneligible for $(1, \eta)$. The principal g-divisor relation at B makes their N sum odd. Exactly one exterior neighbour J has odd N and odd M, hence is eligible for $(\eta, \eta^m) = (\eta, \eta)$.

It cannot be a strict H-containing branch by the neighbouring transverse exclusion, and a pure-g arrow has $M = 0$. Thus J is exceptional and joins the eligible B in a nonsingleton component of the latter pair. Its odd N contradicts the full-matrix parity conclusion. There is no balanced dyadic pair for $\chi = \eta$. Every remaining balanced component is an isolated bivalent Kummer-zero curve, and consequently

$$\int_{\{\operatorname{ord} g = n,\ \operatorname{ord} H = l\}} \eta(\operatorname{ac} H)\, dy\, dz = 0 \quad \text{for every } n, l \tag{11.6}$$

on the complete common-point ball.

All original shells and their root tails

Fix $g = \pi^n u$, $H = \pi^l v$, $n, l \geq 1$, and put $T = q^{-s}$. Multiply each of the following x-integrals by the retained common factor $\chi^m(v) T^{ml}$. For $k = 2$ and $\operatorname{ord} x = j \geq 1$ the exact contributions are

$$\begin{aligned} &2j < n: \quad (1 - q^{-1}) q^{-j} T^{2j} \mathbf{1}_{\chi^2 = 1}, \\ &2j > n: \quad (1 - q^{-1}) q^{-j} T^n \chi(u), \\ &2j = n: \quad q^{-j-1} T^n \sum_{w \in \mathbb{F}_q^\times} \chi(u + w^2). \end{aligned} \tag{11.7}$$

At a nonzero balanced root, $2w$ is a unit. Its full Hensel ball maps to $\mathfrak{p}$ under $u + w^2$, so its whole tail is zero, as recorded by $\chi(0) = 0$ in the finite sum.

If $\chi^2 \neq 1$, the first range vanishes. At $n = 2a$, the entire second-range tail has measure q^{-a-1}; it cancels the omitted $w = 0$ term of the balanced sum. The result is

$$q^{-a-1} T^n \eta(-1) J(\eta, \chi)(\chi\eta)(u).$$

For $n = 2a + 1$ there is no balance, and the remaining tail is $q^{-a-1} T^n \chi(u)$. The even balanced coefficients and odd unbalanced coefficients have already been proved zero with l fixed.

If $\chi = \eta$, the exact finite sum is $\sum_{w \in \mathbb{F}_q} \eta(u + w^2) = -1$ for $u \neq 0$. The even block is therefore independent of u and is killed by (11.6). Each retained $2j < n$ shell is killed by the same equation, with its actual finite range $1 \leq j < n/2$. The odd second-range part is killed by the full-matrix parity assertion. This accounts for every quadratic dominant shell.

For general $k \geq 3$ replace $2j$ by kj in (11.7); the first range contains $\mathbf{1}_{\chi^k=1}$ and the balanced sum is $\sum_{w\neq 0} \chi(u+w^k)$. Its expansion is (10.6) with $v=1$. For $n = ka + r$, $0 \leq r < k$, the whole g-dominant tail is $q^{-a-1}\chi(u)T^n$; it cancels the $-\chi(u)$ balanced term when $r = 0$, and remains when $r > 0$. The x^k-dominant shells are $j < a$ in the first case and $j \leq a$ in the second. All these channels are precisely (11.1), whose separate n, l coefficients were proved zero. This establishes the original common-point integral with every shell bound and Haar factor unchanged.

Arithmetic descent and the common finite set

In a split identity SNC chart, at least one strict branch contains H. Its character is nontrivial in every actual pair by (11.2), so normal unit averaging kills every fixed pair of plane valuations. For a nonsplit two-branch node, the original norm expression (6.9), now with $h = H$, has equal branch vectors and a nontrivial common normal character. Its sum is zero on every K valuation shell. On the exceptional fibre, conjugate curves are disjoint and the pair's unique intersection is rational whenever it contributes; the scalar C in the dyadic computation is therefore its original-field Frobenius scalar.

Choose once the algebraic plane resolution, the fixed resolutions of F and H^m, and the finite algebraic charts used above. A nonzero tame zero-tube term on the original resolution requires its order to divide one of finitely many positive multiplicities. The relevant Kummer covers and incidence data can thus be spread after excluding one finite set, along with primes dividing k and the fixed multiplicities. For wild characters use Lemma 2.3 on the original F resolution. Every residual weight remains one constant on the complete source ball throughout all cancellations. Unit-value balls contribute the stated constant in s, completing the proof. □

12 Cayley families from reduced surfaces

Let $Z \subset \mathbb{A}^n_{\Bbbk}$ be a nonempty reduced closed subscheme, pure of dimension two, and put $c = n - 2$. In this section $c \geq 1$. Choose generators $P_1, \dots, P_r$ of the defining ideal I_Z, and put $k_0 = r - c$. Any generating set is allowed. Let $R \in \Bbbk[x]$ be nonzero on every irreducible component of Z, and suppose that

$$\operatorname{Sing} Z \subset V(R). \tag{12.1}$$

For $M \geq 1$ define polynomials in the original $n + r$ variables by

$$F_0(x,\lambda) = \sum_{j=1}^{r} \lambda_j P_j(x), \qquad F_M(x,\lambda) = F_0(x,\lambda) + R(x)^M.$$

For a closed $S \subset Z \cap V(R)$ and an integer $h \geq 0$ write

$$Z_{M,S,h}(\chi,s) = \int_{\substack{x\in\mathcal{O}^n,\ \bar{x}\in S(\mathbb{F}_q)\\ \lambda\in\mathfrak{p}^h\mathcal{O}^r}} \chi(\operatorname{ac} F_M(x,\lambda))|F_M(x,\lambda)|^s\,dx\,d\lambda. \tag{12.2}$$

Models over a localization of the ring of integers of $\Bbbk$ are understood.

Theorem 12.1. *Fix the preceding data and h. There is an integer M_0, specified in* (12.5)*, with the following property. For every fixed $M \geq M_0$, outside a finite set of places independent of the character,*

$$\operatorname{Sat}_e(F_M), \qquad e = \operatorname{ord}\chi > 1, \quad \Longrightarrow \quad Z_{M,S,h}(\chi,s) = 0.$$

Characters of every conductor are allowed. Under the same spectral hypothesis the integral over $\mathcal{O}^n \times \mathfrak{p}^h\mathcal{O}^r$ is independent of s, and hence entire. In particular the standard full-space integral and the complete origin residue integral satisfy holomorphy, taking $h = 0$ and $h = 1$, respectively, and choosing M accordingly.

The spectral hypothesis concerns every point of the zero fibre of F_M, including every parameter vector λ. The parameter domain is the zero-centred ball $\mathfrak{p}^h\mathcal{O}^r$, and h is fixed before M_0 is chosen. For $c = 0$, the polynomial is R^M on $\mathbb{A}^2$ with unused parameters, and the plane-curve result applies.

12.1 A fixed factorization and the size of the power

Use a strong factorizing resolution of the reduced ideal [7, Theorem 7.3]. Thus $\pi : W \to \mathbb{A}^n$ is proper, W is smooth, and the strict transform Y of Z is smooth and transverse to the exceptional boundary. Moreover,

$$I_Z\mathcal{O}_W = \mathcal{O}_W(-A)I_Y, \qquad A = \sum_i a_i E_i. \tag{12.3}$$

The resolution is an isomorphism over the complement of $Z \cap V(R)$. Resolve the function $R|_Y$ and its boundary by additional point blowups on Y, realized by ambient blowups. Finish these operations before choosing M. We can and do arrange that the boundary on Y has simple normal crossings and that all exceptional centres lie over $Z \cap V(R)$. We also arrange that each component of Y, viewed over the corresponding normalization component of Z, dominates its minimal good resolution of $R = 0$. Further point blowups achieve this before M is chosen. Consequently

$$b_i = \operatorname{ord}_{E_i}(\pi^* R) > 0$$

for every exceptional divisor relevant to (12.3). Set $R_0 = R|_Y$. In local coordinates near Y one has

$$\begin{aligned} Y &= \{u_1 = \cdots = u_c = 0\}, \quad & \pi^* I_Z &= z^a(u_1, \dots, u_c), \\ \pi^* R &= z^b r(z, u, w), & \pi^* dx &= \epsilon(z, u, w) z^{\nu-1}\, dz\, du\, dw. \end{aligned} \tag{12.4}$$

Here z denotes the exceptional coordinates through the point, the multi-index notation denotes their products, and ϵ is a unit. On Y, the two base coordinates are distributed between z and w. The strict zero divisors of R_0 are also included in its surface normal-crossing boundary; their order in A is zero.

The polynomial F_0 is nonzero because $c \geq 1$. Choose any fixed global log resolution of F_0, and let B_0 be the maximum of its positive total-transform multiplicities. Every eigenorder of every local monodromy of F_0 divides one of those multiplicities, by the normal-crossing nearby-cycle description. In particular it is at most B_0. One sufficient choice is

$$M_0 = \max\left\{2, h+1, B_0+1, \max_i \left\lceil \frac{a_i + h + 1}{b_i} \right\rceil \right\}. \tag{12.5}$$

The last maximum is omitted if the factorization has no exceptional divisors. This choice gives

$$Mb_i - a_i \geq h + 1, \qquad M \geq h + 1, \qquad M > B_0. \tag{12.6}$$

It is independent of the order and conductor of a character.

After excluding finitely many places, every displayed resolution, coordinate system and relevant minor has good integral reduction. The normalized original generators $z^{-a}\pi^* P_j$ generate I_Y.

Select c of them with independent conormal classes, and use these as the normal coordinates u_i. Every remaining normalized generator is an exact regular linear combination of the u_i. Subtracting these combinations in the generator list is an invertible integral triangular operation. Its transpose on the parameters is an actual integral GL_r change. It preserves $\mathfrak{p}^h\mathcal{O}^r$ and Haar measure and puts the linear part of the phase into $z^a\sum_{i=1}^{c} u_i\lambda_i$. The unused k_0 parameters remain present.

Write $r_0(z,w) = r(z,0,w)$. The identity

$$r^M - r_0^M = (r - r_0)\sum_{j=0}^{M-1} r^j r_0^{M-1-j} = \sum_{i=1}^{c} u_i S_i$$

is an exact identity of regular functions. The parameter translations $\lambda_i \mapsto \lambda_i + z^{Mb-a}S_i$ therefore give

$$F_M\pi = R_0^M + z^a\sum_{i=1}^{c} u_i\lambda_i. \tag{12.7}$$

On a residue disk with an exceptional coordinate in $\mathfrak{p}$, the first inequality in (12.6) puts every translation in $\mathfrak{p}^h$. If there is no such coordinate, the relevant base disk has $R \equiv R_0 \equiv 0 \pmod{\mathfrak{p}}$; the divided power sum in the preceding display belongs to $\mathfrak{p}^{M-1}$. Thus $M \geq h+1$ gives the same conclusion. These translations preserve the whole original parameter ball, at every conductor. At a complex marked point over $Z \cap V(R)$ they vanish, which is needed in the later monodromy comparison.

12.2 The exact calculation at arbitrary conductor

Lemma 12.2. *Let $\chi \neq 1$, put $a_\chi = a(\chi)$, and fix x with $R(x) \neq 0$. Set*

$$b(x) = \min_j \operatorname{ord} P_j(x), \qquad m(x) = M \operatorname{ord} R(x),$$

using $b(x) = +\infty$ if all $P_j(x)$ vanish. Then

$$\int_{\mathfrak{p}^h\mathcal{O}^r} \chi(\operatorname{ac} F_M)|F_M|^s\, d\lambda = q^{-hr}\mathbf{1}_{\{b(x)+h\geq m(x)+a_\chi\}}\chi^M(\operatorname{ac} R(x))|R(x)|^{Ms}. \tag{12.8}$$

Proof. The image of the parameter map $\lambda \mapsto \sum \lambda_j P_j(x)$ is $\mathfrak{p}^{b(x)+h}$. Its pushforward Haar measure is uniform on that ideal and has total mass q^{-hr}. If $b(x)+h \leq m(x)$, addition of $R(x)^M$ preserves the ideal. Every nonzero valuation shell then contains the average of the nontrivial character χ over $\mathcal{O}^\times$, so its contribution is zero. The phase-zero point has measure zero. This calculation is made in a convergent right half-plane.

If $b(x)+h > m(x)$, write $L = b(x)+h-m(x) \geq 1$. The valuation of the phase is $m(x)$, and its remaining multiplicative factor ranges uniformly over $1+\mathfrak{p}^L$. The normalized additive measure on this coset is also its normalized multiplicative Haar measure: multiplication by a member of $1+\mathfrak{p}^L$ has absolute Jacobian one. Its character average is one exactly when $\chi|_{1+\mathfrak{p}^L} = 1$, and is zero otherwise. The former condition is $L \geq a_\chi$. This proves the formula, including the case $b(x) = +\infty$. The set $R = 0$ has ambient Haar measure zero and can be discarded before this calculation. □

Apply the lemma to (12.7). For fixed base coordinates on Y, put $m = M \operatorname{ord} R_0$ and $a_0 = \operatorname{ord} z^a$. The surviving normal vectors are exactly

$$u \in \mathfrak{p}^{m-a_0-h+a_\chi}\mathcal{O}^c. \tag{12.9}$$

The exponent is at least one by (12.6), both on exceptional disks and on disks of a strict zero divisor of R_0. In particular this is a full ball inside the original normal residue disk. Larger conductor shrinks it and requires no increase of M. The volume of the normal ball, multiplied by the original parameter volume, is

$$\begin{aligned} q^{-hr}q^{-c(m-a_0-h+a_\chi)} &= q^{-ca_\chi-h(r-c)}q^{-cm+ca_0} \\ &= q^{-ca_\chi-hk_0}|R_0|^{Mc}|z|^{-ca}. \end{aligned} \tag{12.10}$$

The absolute value of ϵ in (12.4) is one on a good residue disk, so this integration replaces it by its restriction at $u = 0$. A chart away from Y has a normalized generator which is a unit. On such a chart (12.6) gives $b(x) + h \le m(x)$, so Lemma 12.2 makes its contribution zero. Properness supplies the lift of every original arc whose generic point is off the exceptional image. The excluded arcs form a measure-zero set. Thus the calculation has included the whole original domain.

12.3 The kernel and its canonical differential

Put $Z^\circ = Z \setminus V(R)$, which is smooth by (12.1). Define

$$\mathcal{K}^\circ = \{(x,\lambda) : x \in Z^\circ, \textstyle\sum_j \lambda_j\, dP_j(x) = 0\}, \qquad \mathcal{K} = \overline{\mathcal{K}^\circ}_{\mathrm{red}}. \tag{12.11}$$

This is the reduced closure of a vector bundle of rank k_0 over the smooth surface Z°. Therefore $\mathcal{K}$ is pure of dimension $D = 2 + k_0$, and $\mathcal{K} \setminus V(R) = \mathcal{K}^\circ$ is smooth.

The normalized original generators on Y give the exact sequence

$$0 \longrightarrow E \longrightarrow \mathcal{O}_Y^r \longrightarrow N^*_{Y/W} \otimes \mathcal{O}_Y(-A) \longrightarrow 0. \tag{12.12}$$

In particular E is an actual subbundle of the original trivial parameter bundle. It is classified by a morphism $\gamma : Y \to \mathrm{Gr}(k_0, r)$. Its total space is closed in $Y \times \mathbb{A}^r$, and the induced morphism

$$q_E : \operatorname{Tot} E \longrightarrow \mathcal{K}$$

is proper and is an isomorphism over $R \neq 0$. The image is exactly the closure in (12.11), since the image is closed and the open vector bundle is dense in every component of $\operatorname{Tot} E$.

We record the differential, including its transition functions. On Z°, choose c independent original equations P_i indexed by I, and let J be the complementary parameter indices. The equations $\sum \lambda_j\, dP_j = 0$ solve the parameters λ_I in terms of λ_J. Let ω_I be the residue form specified by

$$\bigwedge_{i\in I} dP_i \wedge \omega_I = \, dx_1 \wedge \cdots \wedge \, dx_n,$$

with signs determined by the displayed ordering. The local top form on $\mathcal{K}^\circ$ is $\omega_I \wedge \bigwedge_{j\in J} d\lambda_j$, with the complementary ordering sign. If another set of independent equations satisfies $dP_{I'} = C\, dP_I$ on Z°, the residue form is multiplied by $(\det C)^{-1}$. The determinant of the induced change of the free kernel coordinates is $\det C$, with precisely the complementary sign. These factors cancel. Thus the forms glue to a nowhere-zero form $\Omega_{\mathcal{K}}$ on $\mathcal{K}^\circ$.

The same construction follows from determinants in (12.12):

$$\begin{aligned}\det E &= \det N_{Y/W} \otimes \mathcal{O}_Y(cA),\\ \omega_{\operatorname{Tot} E} &= p^*(\omega_Y \otimes (\det E)^{-1}) = p^*((\omega_W \otimes \mathcal{O}_W(-cA))|_Y).\end{aligned} \tag{12.13}$$

Here $p : \operatorname{Tot} E \to Y$ is the projection, and adjunction was used in the last equality. Since $P_i\pi = z^a u_i$, restriction to the conormal submodule $N^*_{Y/W} \subset \Omega^1_W|_Y$ gives

$$(\,d(P_i\pi))|_Y = z^a(\,du_i)|_Y.$$

Dividing the ambient form by the wedge of these restricted ambient differentials and retaining the free parameter coordinates gives

$$q_E^*\Omega_{\mathcal{K}} = \epsilon(z,0,w) z^{\nu - ca - 1}\, dz\, dw\, d\kappa_1 \cdots d\kappa_{k_0}. \tag{12.14}$$

This rational form is translation invariant along the vector fibres. Its divisorial zeros and poles are supported on $R_0 = 0$.

Let $D_{\mathcal{K},0} \subset \operatorname{Tot} E(K)$ consist of pairs (y,λ) satisfying

$$\pi(y) \in \mathcal{O}^n, \qquad \overline{\pi(y)} \in S(\mathbb{F}_q), \qquad \lambda \in \mathcal{O}^r.$$

A unit minor in (12.12) supplies an integral left inverse to $E \hookrightarrow \mathcal{O}_Y^r$, so in those coordinates the condition is precisely $\kappa \in \mathcal{O}^{k_0}$. Combining (12.10) and (12.14) proves the exact identity

$$Z_{M,S,h}(\chi,s) = q^{-ca_\chi - hk_0} \int_{D_{\mathcal{K},0}} \chi^M(\operatorname{ac} R_0) |R_0|^{M(s+c)} |q_E^*\Omega_{\mathcal{K}}|. \tag{12.15}$$

The kernel lattice on the right has depth zero. The factor for depth h is already outside the integral. All possible negative exponents of the rational density are on divisors of positive R_0 multiplicity; thus a common right half-plane ensures convergence before any rational continuation is used.

12.4 A monodromy identity at every parameter vector

We next transfer the spectral hypothesis on F_M to the same kernel that occurs in (12.15).

Lemma 12.3. *For every original point $p = (x,\lambda)$ with $x \in Z \cap V(R)$,*

$$\mathcal{Z}_{F_M,p}(t) = \mathcal{Z}_{F_0,p}(t)\mathcal{Z}_{R|_{\mathcal{K}},p}(t^M). \tag{12.16}$$

The last factor is interpreted as one off $\mathcal{K}$.

Proof. The cone of the natural map $\mathbb{Q}_{\mathcal{K}} \to Rq_{E*}\mathbb{Q}_{\operatorname{Tot} E}$ is supported on $R = 0$, since q_E is an isomorphism off that fibre. Nearby cycles kill a complex supported on the zero fibre and commute with proper pushforward. Consequently

$$Rq_{E*}\psi_{R_0}\mathbb{Q}_{\operatorname{Tot} E} \simeq \psi_{R|_{\mathcal{K}}}\mathbb{Q}_{\mathcal{K}}. \tag{12.17}$$

In the factorizing charts the exact transformations leading to (12.7) fix the marked fibre. At a point over Y which is outside $\operatorname{Tot} E$, one normal parameter is nonzero. The tail can then be absorbed in its paired normal coordinate; its positive orders guarantee that this change fixes the

point. Away from Y, a normalized generator is a unit and a parameter translation absorbs the tail. Hence the quotient of local monodromy functions is supported on $\operatorname{Tot} E$.

At a point of $\operatorname{Tot} E$, use the two surface normal-crossing coordinates and suppress passive smooth coordinates. The remaining germ is

$$z^a \left(\sum_{i=1}^{c} u_i v_i + u_0(z) z^b \right), \qquad b_j > 0, \tag{12.18}$$

where u_0 is an analytic unit and $a + b$ is the multiplicity vector of R_0^M. There is an explicit unit removal. Replace v_i by $v_i/u_0(z)$, so the expression is a unit times $P = z^a(\sum u_i v_i + z^b)$. Give each active z_j weight 2 and each u_i, v_i weight $\sum b_j$. Then P has positive weighted degree $D_P = 2\sum(a_j + b_j)$. If B is its remaining unit, choose a local analytic D_P-th root and set

$$X_j' = X_j B(X)^{w_j/D_P}$$

in each active coordinate of weight w_j. The derivative at the marked origin is an invertible diagonal matrix, and weighted homogeneity gives $P(X') = B(X)P(X)$. Thus this is an actual local coordinate equivalence. The same argument removes the common unit in the corresponding $z^a \sum u_i v_i$ model. Passive coordinates are unchanged.

For a polynomial with positive weights, weighted radial contraction identifies its local Milnor fibre with its affine fibre of value one, compatibly with the finite diagonal monodromy T_P [23, Example 2.10]. On the affine fibre of P, the algebraic action

$$(u, v, z) \longmapsto (\rho u, \rho^{-1} v, z), \qquad \rho \in \mathbb{C}^\times,$$

commutes with every power T_P^j of that finite monodromy. Finite-order Lefschetz and Euler localization for this torus give

$$\sum_i (-1)^i \operatorname{Tr}(T_P^j \mid H^i(P^{-1}(1))) = \chi_{\mathrm{top}}\big((P^{-1}(1))^{T_P^j}\big) = \chi_{\mathrm{top}}\big((\{z^{a+b} = 1\})^{T_P^j}\big).$$

Indeed, the torus-fixed locus is precisely $u = v = 0$. For $z^a \sum u_i v_i$, the corresponding fixed locus in its value-one fibre is empty, so every alternating trace is zero. The identity

$$\log \mathcal{Z}(t) = -\sum_{j \geq 1} \left(\sum_i (-1)^i \operatorname{Tr}(T_P^j \mid H^i) \right) \frac{t^j}{j}$$

therefore gives, at the kernel point,

$$\frac{\mathcal{Z}_{F_M \pi}(t)}{\mathcal{Z}_{F_0 \pi}(t)} = \mathcal{Z}_{R_0^M}(t) = \mathcal{Z}_{R_0}(t^M).$$

The final equality follows from the M cyclically permuted nearby fibres of R_0^M. The hyperbolic torus acts on the affine fibre used in this computation.

The ambient modification $\pi \times \mathrm{id}$ is proper and is an isomorphism away from both original zero fibres, because every centre lies over $Z \cap V(R)$. Apply the proper Euler product (2.4) over its entire fibre above p. The local quotient just computed and (12.17) identify the quotient with $\mathcal{Z}_{R|_{\mathcal{K}},p}(t^M)$. This proves (12.16) at every original parameter vector. □

12.5 Spectral exclusion on the kernel and on the source

We detect each nonzero monodromy block at a general point of its support.

Lemma 12.4. *Suppose* $\mathrm{Sat}_e(F_M)$*, and put* $d = e/\gcd(e, M)$*. If* $\mathcal{K} \cap V(R)$ *is nonempty, then* $d > 1$ *and* $\mathrm{Sat}_d(R|_{\mathcal{K}})$*. The same exclusion holds on* Z *and on every branch of its normalization.*

Proof. If a reduced pure D-dimensional variety X has smooth dense $U = X \setminus V(f)$, then $\mathrm{IC}_X|_U = \mathbb{Q}_U[D]$. Nearby cycles depend only on this restriction, so

$$\psi_f \mathbb{Q}_X[D-1] \simeq \psi_f[-1]\,\mathrm{IC}_X \tag{12.19}$$

is perverse. In particular this applies to $X = \mathcal{K}$. A nonzero generalized monodromy eigenblock has, on a smooth open subset of a maximal support component, precisely one nonzero stalk degree and positive rank. Its alternating multiplicity is consequently nonzero at some point. This uses the perverse normalization of IC_X and the exactness of $\psi_f[-1]$ on perverse sheaves [23, Theorem 4.21].

Suppose that the kernel has an eigenvalue α of order b with $d \mid b$. The preceding argument supplies a point at which the multiplicity of α in $\mathcal{Z}_{R|_{\mathcal{K}}}$ is nonzero. There is an M-th root of α of exact order Mb. To see this, write $\alpha = \exp(2\pi i a/b)$ with $\gcd(a, b) = 1$. Choose an integer $a' \equiv a \pmod{b}$ coprime to every prime dividing M but not b, using the Chinese remainder theorem. Then $\beta = \exp(2\pi i a'/(Mb))$ has the required properties. The elementary valuation criterion

$$e \mid Mb \quad \Longleftrightarrow \quad e/\gcd(e, M) \mid b$$

shows that β is forbidden by $\mathrm{Sat}_e(F_M)$. Its order exceeds B_0, so the F_0 factor in (12.16) cannot cancel it. This contradiction proves $\mathrm{Sat}_d(R|_{\mathcal{K}})$.

If $d = 1$ and the zero fibre is nonempty, choose a point reached by the nearby nonempty fibres. Their unreduced H^0 contains the invariant constant class. The corresponding nearby eigensheaf is nonzero, hence is detected somewhere by (12.19). Its primitive M-th lift contradicts $M > B_0$ and the original spectral condition exactly as above. Hence $d > 1$.

The kernel is conical in the original parameter coordinates and contains the zero section Z. At $(x_0, 0)$ the homotopy $(x, \lambda) \mapsto (x, t\lambda)$, $0 \le t \le 1$, preserves $\mathcal{K}$, the function R, and a small ball about $(x_0, 0)$. It retracts the complete nearby family onto the nearby family of $R|_Z$. This is a homotopy over the punctured target, so the resulting cohomology isomorphism commutes with monodromy. Thus every source stalk inherits Sat_d.

Finally let $\rho : Z^\nu \to Z$ be the finite normalization. It is an isomorphism over $R \neq 0$, because this open part is smooth. The same nearby-cycle comparison as in (12.17) identifies the source stalk at x with the direct sum of the normalization stalks over the finite fibre $\rho^{-1}(x)$, degree by degree. This gives the claimed exclusion on every normal branch. □

12.6 The normal-surface graph

We give the surface graph argument needed for the arithmetic step. This is the geometric part of Veys's normal-surface holomorphy proof [30, §§3.1–3.4].

Lemma 12.5. *Let* f *be a nonzero regular function on a normal surface, with* $\mathrm{Sat}_d(f)$ *and* $d > 1$*. In a minimal good resolution of* f *near a zero point, every divisor of multiplicity divisible by* d *is exceptional. Each such exceptional curve is rational and has exactly two boundary points, each lying on a component whose multiplicity is not divisible by* d*. In particular these exceptional curves are pairwise disjoint.*

Proof. Call a multiplicity, or its divisor, eligible when it is divisible by d. A strict divisor is not eligible: at a generic smooth point of that divisor the function is a unit times a transverse coordinate to its multiplicity, whose H^0 contains every root of that order.

Let C_i be the exceptional curves, let N_i be their multiplicities, and let C_i° be the curve with all intersections with other zero divisors removed. The total zero divisor is connected near the marked point. For an eligible connected block B, let v be its number of vertices, a its number of internal edges counted with multiplicity, and b its number of external boundary edges. Let g_i be the genera. The first Betti number of the connected graph is $b_1(B) = a - v + 1$. Then

$$\sum_{i\in B} \chi_{\mathrm{top}}(C_i^\circ) = 2v - 2\sum_{i\in B} g_i - 2a - b = 2 - 2b_1(B) - 2\sum_{i\in B} g_i - b. \tag{12.20}$$

There is at least one external edge: otherwise connectedness would put the strict zero divisors in this exceptional block. There cannot be exactly one. At its attachment curve the principal-divisor intersection identity

$$-\kappa_i N_i + \sum_{j\neq i} (C_i \cdot C_j) N_j = 0, \qquad \kappa_i = -C_i^2, \tag{12.21}$$

would force the multiplicity of the single external neighbor to be divisible by d. Thus $b \geq 2$, and (12.20) is nonpositive.

A'Campo's formula and $\mathrm{Sat}_d(f)$ show that the sum of these Euler characteristics over all eligible curves is zero. Since each block has a nonpositive sum, each block has sum zero. The same statement holds with any positive multiple of d in place of d.

Suppose that a vertex in an eligible block has positive open Euler characteristic. It is a rational leaf. Follow its dead chain of rational two-valent curves until the first vertex of negative open Euler characteristic. Such a vertex exists because the block sum is zero. Every intermediate vertex is still eligible by (12.21). Along a minimal dead chain the two neighbours of every intermediate vertex are distinct, and the self-intersection numbers are $-\kappa_j$ with $\kappa_j \geq 2$: a rational (-1)-curve of valency at most two on that chain could otherwise be contracted while preserving a good resolution. If the leaf multiplicity is N, the identities

$$N_1 = \kappa_0 N, \qquad N_{j+1} = \kappa_j N_j - N_{j-1}$$

give $N_{j+1} = D_j N$, where $D_{-1} = 1$, $D_0 = \kappa_0$, and $D_j = \kappa_j D_{j-1} - D_{j-2}$. These positive integers strictly increase. Hence the terminal negative vertex has multiplicity $N' = DN > N$, with $D \geq 2$.

Apply the zero block sum with N' in place of d. The block containing that negative vertex has a positive vertex of multiplicity divisible by N'. Starting at this positive leaf repeats the construction and strictly increases the multiplicity of a positive vertex. Finiteness of the graph makes this impossible. Therefore every eligible vertex has nonpositive open Euler characteristic, and the zero block sums make all of them zero.

A zero open Euler characteristic means a rational two-valent curve or an elliptic curve with no boundary points. The latter, and a closed cycle of rational two-valent curves, cannot meet the strict zero divisor, contradicting connectedness. The remaining eligible blocks are chains of rational curves with two external edges. If a chain has at least two vertices, its endpoint has one eligible neighbor and one external neighbor. Equation (12.21) again forces that external neighbor to be eligible, a contradiction. Thus every block consists of one rational curve with two boundary points, both incident to noneligible components. □

The fixed surface model Y may dominate this minimal model. We next descend its kernel bundle while simplifying the eligible graph.

12.7 Constancy of the kernel planes on eligible curves

Lemma 12.6. *Under the hypothesis of Theorem 12.1, the classifying map $\gamma : Y \to \mathrm{Gr}(k_0, r)$ is constant on every eligible curve for $d = e/\gcd(e, M)$, and on each connected union of such curves.*

Proof. The assertion is automatic if $k_0 = 0$. Every eligible curve maps to a point x_0 of Z: a curve dominating a divisor on the normalization would contradict the generic transverse spectrum, as in Lemma 12.5.

Suppose that γ is nonconstant on an eligible irreducible curve C. Set

$$W_C = \bigcap_{y \in C} E_y \subset \mathbb{C}^r.$$

Then $\dim W_C \le k_0 - 1$. For a parameter vector $\lambda \notin W_C$, the incidence subset $\{y \in C : \lambda \in E_y\}$ is a proper closed subset of the proper curve C, and is therefore finite. It follows that the image V_C of $\mathrm{Tot}(E|_C)$ in the original parameter space has dimension $k_0 + 1$.

The whole fibre over x_0 has finitely many curves and isolated points. In V_C, avoid the following finite union: the k_0-plane associated to every curve on which γ is constant; the planes over every boundary crossing and every isolated point of the base fibre; and the common kernel W_D for every other nonconstant curve D in the fibre. Each of these sets has dimension at most k_0, and none can cover V_C. Choose λ outside their union. The complete fibre $q_E^{-1}(x_0, \lambda)$ is finite, and each of its points lies on the smooth stratum of a single divisor of R_0.

At such a point the nearby fibre of R_0 is that of a unit times z^{N_i}, with passive smooth factors. Its cohomology is concentrated in H^0 and contains every N_i-th root. By (12.17), the complete nearby cohomology of the kernel at (x_0, λ) is the direct sum of these H^0 groups. The primitive N_C-th root supplied by a point on C cannot cancel. Since $d \mid N_C$, this contradicts Lemma 12.4. Thus $\gamma|_C$ is constant. At the intersection of two eligible curves the two constant planes are their common fibre E_y, so constancy holds on every connected eligible block. □

12.8 Contraction with the original bundle and lattice

The smooth model Y dominates the minimal good model by finitely many point blowups over the zero divisor. At a free point the new multiplicity is the old one; at a crossing it is the sum of the two old ones. Thus there are five possibilities:

centre	new divisor
free point on an eligible curve	eligible leaf
eligible–eligible crossing	eligible subdivision
eligible–noneligible crossing	noneligible
noneligible–noneligible crossing	noneligible or an isolated eligible curve
free point on a noneligible curve	noneligible.

Starting with Lemma 12.5, induction shows that each eligible block is a rational tree with exactly two noneligible external edges. Both attach to one distinguished curve: the original isolated curve, or the isolated curve born at a crossing of two noneligible curves. Every other vertex in the block was created above a free point of this distinguished curve. All multiplicities above such a free point remain divisible by d, even after further sums. Consequently no noneligible descendant obstructs the reversal of these blowups.

Contract just these forests, in reverse order, preserving the distinguished curves and all noneligible divisors. The resulting eligible curves are disjoint rational curves with two boundary

points on noneligible components. The contraction is defined by the retained divisorial valuations; they are Galois invariant. Equivalently one can contract complete Galois orbits of disjoint terminal (-1)-curves at each stage. Thus the resulting model and morphism are defined over the original number field.

For one such contraction $b : Y \to Y'$, the map γ is constant on the contracted curve by Lemma 12.6. Consider the reduced image of (b, γ) in $Y' \times \mathrm{Gr}(k_0, r)$. Its projection to Y' is proper and has a single geometric point in every fibre, hence is finite. It is birational to the normal surface Y', and therefore is an isomorphism. It follows that

$$\gamma = \gamma' b, \qquad E = b^* E', \qquad E' \subset \mathcal{O}_{Y'}^r. \tag{12.22}$$

The base map to the affine variety Z also descends, using $b_* \mathcal{O}_Y = \mathcal{O}_{Y'}$. The induced total-space map to $\mathcal{K}$ remains proper and agrees with the old one off $R = 0$.

The rational differential on the common open part remains $\Omega_{\mathcal{K}}$. Equation (12.22) shows that its birational pullbacks involve exactly the ordinary Jacobian of the base blowup. For example, if its base density has order $\delta - 1$ along a curve, a free point blowup gives $\delta_{\mathrm{new}} = \delta + 1$. At the crossing of two curves it gives $\delta_{\mathrm{new}} = \delta_1 + \delta_2$. These identities follow by substituting $(x, y) = (t, tu)$ into $x^{\delta_1 - 1} y^{\delta_2 - 1}\, dx\, dy$.

After good reduction, a unit minor of $E' \hookrightarrow \mathcal{O}_{Y'}^r$ again supplies an integral left inverse. Therefore the original condition $\lambda \in \mathcal{O}^r$ is equivalent to $\kappa \in \mathcal{O}^{k_0}$ in a bundle chart. Its fibre volume is one. Properness preserves the lifted original base residue tube. These facts show that every contraction preserves the precise right side of (12.15), including its rational form and its original lattice.

12.9 The original angular coefficient

We finish the proof of Theorem 12.1. If $Z \cap V(R)$ is empty, the centre integral is empty. Otherwise Lemma 12.4 gives $d > 1$ for $\eta = \chi^M$, whose order is d.

First suppose that χ, and hence η, is tame. On the contracted surface model, a stratum meeting a noneligible normal divisor has zero integral, because the average of η^{N_i} over its normal unit coordinate is zero. This also handles crossings containing a noneligible component. No two eligible components intersect. It remains to inspect the open stratum C° of one eligible curve.

Geometrically $C^\circ = \mathbb{P}^1 \setminus \{P, Q\}$. Since γ is constant on C, one affine chart of the Grassmannian, and hence one unit minor, works on a neighborhood of the entire curve. The integral kernel coordinates there are consistent, the fibre volume is one, and (12.14) is a rational base two-form times $d\kappa$. In coordinates with $C = \{z = 0\}$ write

$$R_0 = z^{N_C} u, \qquad |q_E^* \Omega_{\mathcal{K}}| = |z|^{\delta_C - 1} |\, dz\, dw\, d\kappa|$$

on the open stratum, with the remaining coefficient a unit. It has absolute value one at good places. For normal valuation $\ell \geq 1$ the radial factor is determined by $q^{-\ell \delta_C - M(s+c)\ell N_C}$ and is independent of the residue point of C°.

The angular sheaf is the actual Kummer sheaf associated with $\eta(u|_{C^\circ})$. It is independent of the normal trivialization: changing z multiplies u by an N_C-th power, and $\eta^{N_C} = 1$. Its inertias at the missing points are η^{N_P} and η^{N_Q}, where N_P, N_Q are the multiplicities of the components through the respective boundary points; the components may coincide. Both inertias are nontrivial. Lemma 2.2 now gives

$$R\Gamma_c(C^\circ_{\overline{\mathbb{F}}_q}, \mathcal{L}_\eta) = 0.$$

Thus its actual Frobenius trace coefficient is zero, including any constant arithmetic twist, before multiplication by a radial factor. The remaining density and parameter-lattice factors have constant norm on the stratum. Moreover C maps to one point of Z, so a complete base residue tube includes its whole open stratum or excludes it.

These statements are over the original field. If the punctures are conjugate, the same geometrically zero cohomology complex still has zero Frobenius trace. In a nonsplit normal-crossing presentation, an unramified norm coordinate gives the corresponding normal average; the norm on residue units is surjective, so composing a nontrivial character with it gives zero average. Equivalently one spreads the strict normal-crossing model and its finite descent data and applies the trace formula to the same geometric strata.

Now suppose that χ is wild. Exclude the primes dividing M and the finitely many positive multiplicities of R_0 on the original fixed surface model. Then $\eta = \chi^M$ remains nontrivial on principal units, and so does every η^{N_i}. On a residue disk above $R_0 = 0$, choose an active normal coordinate z_i. The principal-unit part of the unit multiplying $\prod z_i^{N_i}$ has an N_i-th root. Absorbing it into z_i is an integral analytic bijection of the same normal disk, with unit Jacobian and unchanged valuation. The average of η^{N_i} over its principal units is zero. The density has constant unit norm, and the other variables still range over full base disks and the full kernel lattice. Thus the right side of (12.15) is zero for every wild character, even when the original parameter depth is greater than one.

There is a single finite exceptional set independent of the character. For tame η, if d divides no multiplicity on the fixed surface model, every stratum already vanishes by normal-unit averaging. Only the finitely many positive divisors of its multiplicities require the contraction and Kummer constructions. Spread those finitely many models and covers, and discard their bad places together with the fixed coordinate denominators and the primes dividing M and the multiplicities. The wild argument used the fixed model and does not add a conductor-dependent set. This proves the asserted vanishing.

Finally partition the whole base $\mathcal{O}^n$ by residues. Where R is a unit, Lemma 12.2 gives either zero or a term of valuation zero, independent of s. Where R is a nonunit but some P_j is a unit, one has $b(x) = 0$ and $m(x) \geq M \geq h + 1$, so that same lemma gives zero. The remaining residues form $Z \cap V(R)$ and have just been treated. This proves the constant full-base assertion and completes the theorem.

12.10 A surface formed by two planes

The following example displays both the differential and the need to retain all parameter vectors in the spectral assumption. Let

$$Z = V(x, y) \cup V(z, w) \subset \mathbb{A}^4, \quad I_Z = (xz, xw, yz, yw), \quad R = xy + zw. \tag{12.23}$$

The surface is reduced and pure, is not a local complete intersection at the origin, and satisfies (12.1). In the x-chart of the blowup of the origin, put $x = t$, $y = tu$, $z = tv$, $w = tw'$. Direct substitution gives

$$(xz, xw, yz, yw) = t^2(v, w', uv, uw'), \quad R = t^2(u + vw'), \quad dx\,dy\,dz\,dw = t^3\,dt\,du\,dv\,dw'.$$

The strict transform in this chart is $Y = \{v = w' = 0\}$, and

$$R_0 = t^2 u, \qquad E_u = \ker \begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u \end{pmatrix}, \qquad q_E^* \Omega_{\mathcal{K}} = t^{-1}\,dt\,du\,d\kappa_1\,d\kappa_2. \tag{12.24}$$

Indeed $c = 2$, $r = 4$, $a = 2$, and $\nu = 4$, so $\nu - ca - 1 = 4 - 4 - 1 = -1$. Take $h = 2$, $a_\chi = 3$, $\operatorname{ord} t = \ell$, and $\operatorname{ord} u = j$. Formula (12.9) becomes

$$u_{\mathrm{normal}} \in \mathfrak{p}^{(2M-2)\ell + Mj + 1}\mathcal{O}^2.$$

The normal volume times the four original parameter volumes is exactly

$$q^{-8} q^{-2((2M-2)\ell + Mj + 1)} = q^{-10} q^{-(4M-4)\ell - 2Mj},$$

and the outside exponent in (12.15) is $-ca_\chi - hk_0 = -6 - 4 = -10$.

For the spectral calculation, the exceptional curve has multiplicity two and its kernel plane varies with u. A generic parameter vector with all coordinates nonzero and $\lambda_1\lambda_4 = \lambda_2\lambda_3$ has one preimage on this component, with $u = -\lambda_1/\lambda_3$. It also has one preimage on the other surface component, whose ratio is $w/z = -\lambda_1/\lambda_2$. These are the two points of the complete proper fibre. At each, R_0 is a unit times the square of a smooth coordinate. The nearby kernel cohomology is the direct sum of their two H^0 groups, and each contains the eigenvalue -1. At the zero parameter the source is the disjoint union of the two plane product nearby fibres; each has the homotopy type of $\mathbb{C}^\times$ and monodromy eigenvalue one only. The two eigenvalues -1 occur at the indicated nonzero parameters. The example also verifies the canonical density using the redundant original generator list and the original Haar normalization.

13 Plücker polynomials and the larger side of castling

Let $V = \Bbbk^m$ with its standard basis, let $1 \le r < m$, and let $Y_r \subset \bigwedge^r V$ be the affine cone of decomposable vectors. Let f be a nonzero regular function on Y_r defined over $\Bbbk$, homogeneous of degree $\delta > 0$ in the Plücker coordinates. Its pullback to matrices is

$$F_r(X) = f(x_1 \wedge \cdots \wedge x_r), \qquad X = (x_1, \dots, x_r) \in \mathrm{Mat}_{m,r}\,.$$

It has ordinary degree $r\delta$. Put $D = \{f = 0\} \subset \mathrm{Gr}(r, V)$, with arbitrary component multiplicities and singularities. The complex arguments below use the base change $V \otimes_\Bbbk \mathbb{C}$ through the fixed embedding of $\Bbbk$.

Theorem 13.1. *Suppose $m/2 < r < m$. At some matrix of rank $r - 1$, local monodromy of F_r has every δ-th root of unity as an actual eigenvalue. Consequently, if $d = \operatorname{ord}\chi > 1$ and $\mathrm{Sat}_d(F_r)$ holds, then*

$$\int_{\mathrm{Mat}_{m,r}(\mathcal{O})} \chi(\operatorname{ac} F_r)|F_r|^s\,dX = 0, \qquad \int_{\pi^h \mathrm{Mat}_{m,r}(\mathcal{O})} \chi(\operatorname{ac} F_r)|F_r|^s\,dX = 0 \quad (h \ge 1). \tag{13.1}$$

These equalities hold at every finite place and for every conductor. Each individual residue ball whose reduction has rank less than r also has zero integral.

The proof separates an Euler-incidence argument over $\mathbb{C}$ from an exact symmetry of the original nonarchimedean integral. The incidence injectivity is a special case of the Grassmannian Radon transform [22, Theorem 1.1]; its short proof is included.

13.1 A relative homogeneous calculation

Lemma 13.2. *Let A be smooth and let $H(a,z)$ be a nonzero analytic germ which is homogeneous of degree $\delta>0$ in $z\in\mathbb{C}^n$. At $(a_0,0)$,*

$$\mathcal{Z}_{H,(a_0,0)}(t)=(1-t^{\delta})^{\chi_{\mathrm{top}}(\mathbb{P}^{n-1}\setminus\{H(a_0,z)=0\})}. \tag{13.2}$$

The assertion concerns the parameter-dependent germ. It remains valid when the frozen polynomial $H(a_0,z)$ is identically zero.

Proof. Blow up $A\times\{0\}$. This proper map is an isomorphism where $H\neq 0$. A chart has pullback $t_0^{\delta}h(a,w)$, with h independent of t_0, and its fibre over $(a_0,0)$ is $\mathbb{P}^{n-1}$. At a point where $h\neq 0$, a local root of h changes t_0 to $t_0h^{1/\delta}$, so the local monodromy determinant is $1-t^{\delta}$.

At a point where $h=0$, resolve h in the (a,w) variables and take the product with the t_0-line. The open stratum of the divisor $t_0=0$ excludes the total zero divisor of h, whereas the inverse image of the point lies in that divisor. The open strata of all other divisors exclude $t_0=0$, whereas the inverse image of the point lies in $t_0=0$. Every A'Campo exponent is thus zero, and the local determinant is one. This also proves the assertion at a point for which the entire frozen homogeneous polynomial vanishes. The proper Euler product (2.4) now gives (13.2). □

13.2 Euler incidence and a rank-deficient spectral witness

Let $\mathcal{C}_r$ be the rational vector space of algebraically constructible functions on $\mathrm{Gr}(r,V)$. Define

$$(Ev)(W)=\int_{R\subset W,\,\dim R=r} v(R)\,d\chi_{\mathrm{top}},\qquad (Lv)(U)=\int_{R\supset U,\,\dim R=r} v(R)\,d\chi_{\mathrm{top}}.$$

Thus $E:\mathcal{C}_r\to\mathcal{C}_{r+1}$ and $L:\mathcal{C}_r\to\mathcal{C}_{r-1}$, with $E=0$ on $\mathcal{C}_m$ and $L=0$ on $\mathcal{C}_0$. The integrals are Euler integrals with compact supports; the incidence projections are proper.

Lemma 13.3. *On $\mathcal{C}_r$ one has*

$$LE-EL=(m-2r)\,\mathrm{id}\,. \tag{13.3}$$

In particular L is injective when $r>m/2$.

Proof. Compare the two correspondence kernels at r-planes R,S. If $R\neq S$ and $\dim(R\cap S)=r-1$, each kernel is the Euler number of one point: respectively $R+S$ and $R\cap S$. If the intersection has smaller dimension, both kernels are zero. On the diagonal the first fibre is $\mathbb{P}^{m-r-1}$ and the second is $\mathbb{P}^{r-1}$, of Euler numbers $m-r$ and r. Euler Fubini proves (13.3).

Suppose $Lv=0$ and put $b=m-2r$. Commuting L through powers of E, using that E increases r by one, gives by induction

$$LE^{j}v=j(b-j+1)E^{j-1}v\qquad (j\geq 1). \tag{13.4}$$

For the induction, the new coefficient is $(b-2j)+j(b-j+1)=(j+1)(b-j)$. The operator E is nilpotent on $\bigoplus_{r=0}^{m}\mathcal{C}_r$ because it increases the degree. If $v\neq 0$, choose the largest N with $E^Nv\neq 0$. Applying (13.4) at $j=N+1$ gives $0=(N+1)(b-N)E^Nv$, impossible when $b<0$. □

Proof of Theorem 13.1. The constructible function $v=\mathbf{1}_{\mathrm{Gr}(r,V)\setminus D}$ is nonzero. Lemma 13.3 supplies an $(r-1)$-plane W_0 such that

$$c_{W_0}=\chi_{\mathrm{top}}\big(\{R:W_0\subset R\}\setminus D\big)\neq 0.$$

The incidence fibre is $\mathbb{P}(V/W_0)$. Choose a basis $a_1, \dots, a_{r-1}$ of W_0 and take the matrix $X_0 = (a_1, \dots, a_{r-1}, 0)$. Near this point the first $r-1$ columns remain independent. Their spans and quotients vary smoothly. The component of the last column in their span disappears from the wedge, so the remaining germ is

$$H(A, z) = f(a_1 \wedge \cdots \wedge a_{r-1} \wedge z), \qquad z \in V/\langle a_1, \dots, a_{r-1}\rangle.$$

It is relatively homogeneous of degree δ. Lemma 13.2 gives $\mathcal{Z}_{F_r, X_0}(t) = (1 - t^\delta)^{c_{W_0}}$. Every δ-th root has nonzero exponent, hence occurs in at least one actual cohomology group. In particular $\mathrm{Sat}_d(F_r)$ implies $d \nmid \delta$, or $\chi^\delta \neq 1$.

Choose $b \in \mathcal{O}^\times$ with $\chi(b)^\delta \neq 1$. Multiplying one matrix column by b preserves the standard matrix domain, each origin ball, and their Haar measures. It multiplies F_r by b^δ. Each integral I in (13.1) satisfies $I = \chi(b)^\delta I$, so $I = 0$ at every finite place.

For the individual residue-ball assertion let $B = X_1 + \pi \operatorname{Mat}_{m,r}(\mathcal{O})$ have rank-deficient reduction. Lift a nonzero vector in the kernel of $\bar{X}_1$ to a primitive integral vector, and complete it to a matrix $P \in \mathrm{GL}_r(\mathcal{O})$ with that vector as last column. The transformation $X \mapsto XP$ preserves Haar measure and changes F_r by the fixed unit $\det(P)^\delta$. The new residue ball has its entire last column in $\pi\mathcal{O}^m$. Multiplication of this last column by any unit preserves this particular ball. The same character scaling proves its integral zero. □

Remark 13.4. The column scaling preserves the standard domain, every origin ball, and, after the indicated integral column change, each rank-deficient residue ball. The incidence proof uses $r > m/2$: the coefficient $b = m - 2r$ in (13.4) vanishes at the middle rank.

13.3 The exact Haar factor and projective divisors

The zeta-function factors associated with castling originate in Igusa's work [18]; Loeser gives motivic and arithmetic castling identities [21, §1.2 and Theorem 4.2]. Here the Haar factor follows directly from the successive column contents of a matrix.

Let $Y_r^{\mathrm{prim}}(\mathcal{O})$ denote the integral decomposable wedges having at least one unit Plücker coordinate. It is one $\mathrm{GL}_m(\mathcal{O})$-orbit and has an invariant probability measure ν. Since $r < m$, the determinant of a transformation can be corrected on the complementary summand, so it is also one $\mathrm{SL}_m(\mathcal{O})$-orbit. Put

$$B_\chi(T) = \int_{Y_r^{\mathrm{prim}}(\mathcal{O})} \chi(\mathrm{ac}\, f(w)) |f(w)|^s \, d\nu(w), \qquad A_r(q) = \prod_{j=m-r+1}^{m} (1 - q^{-j}).$$

Proposition 13.5. *For all $1 \leq r < m$, all finite-image characters, and every finite completion of $\mathbb{k}$,*

$$I_r(T) = \int_{\mathrm{Mat}_{m,r}(\mathcal{O})} \chi(\mathrm{ac}\, F_r) |F_r|^s \, dX = \frac{A_r(q) B_\chi(T)}{\prod_{j=m-r+1}^{m} (1 - q^{-j} T^\delta)}. \tag{13.5}$$

For an origin ball one has

$$I_{r,h}(T) = q^{-hmr} T^{hr\delta} I_r(T). \tag{13.6}$$

Under integral volume-form duality, the two ranks r and $m - r$ have the same primitive factor $B_\chi(T)$.

Proof. The content valuation a_1 of the first Haar column has distribution $(1-q^{-m})q^{-ma_1}$, $a_1 \geq 0$. A measurable integral row operation sends its primitive direction to the first basis vector. Conditional on this operation, the remaining columns are still independent Haar columns. In the quotient by this first direction, the second projected column has content valuation a_2 with distribution $(1-q^{-(m-1)})q^{-(m-1)a_2}$. Repetition gives independent valuations a_i of respective weights $m-i+1$.

The content of the full wedge is $a_1+\cdots+a_r$. Conditional on any value of this sum, its normalized primitive part has a $\mathrm{GL}_m(\mathcal{O})$-invariant probability distribution, which is ν. Thus it is independent of the sum. Each a_i contributes the geometric series

$$\sum_{a_i \geq 0} (1-q^{-(m-i+1)})q^{-(m-i+1)a_i}T^{\delta a_i}.$$

Their product is (13.5). Rank-deficient matrices over K form a set of Haar measure zero and do not alter the formula. The stratum $a_1=\cdots=a_r=0$ has mass $A_r(q)$.

Ordinary homogeneity under $X \mapsto \pi^h X$ proves (13.6). A fixed integral volume form identifies $\bigwedge^r V$ with $\bigwedge^{m-r} V^*$ and sends primitive decomposable wedges to primitive decomposable wedges. It is $\mathrm{SL}_m(\mathcal{O})$-equivariant with the contragredient action, hence preserves the invariant probability measure. Pulling back f gives the same B_χ on the other side. □

We give two useful classes which need no restriction on r for the standard integral. A divisor will be called *locally a plane pullback* if a finite system of algebraic open charts on the Grassmannian has smooth morphisms to smooth surfaces, and its equation on each chart is a unit times the pullback of a plane-curve equation on that surface.

Theorem 13.6. *Let $1 \leq r < m$. Suppose that the projective divisor D has simple normal crossing reduced support, with arbitrary positive multiplicities, or is locally a plane pullback. Under $\mathrm{Sat}_d(F_r)$, the standard integral I_r and all origin integrals $I_{r,h}$ vanish for almost all places and every character of order $d>1$. If in addition $r>m/2$, then F_r satisfies Denef's conjecture for every residual test function.*

Proof. If $\chi^\delta \neq 1$, the scaling proof already gives the standard and origin vanishing. Suppose $\chi^\delta = 1$. Exclude the primes dividing δ; the character is then tame. The primitive integrand descends to $\mathrm{Gr}(r,\mathcal{O})$, since changing a primitive Plücker generator multiplies f by a δ-th power of a unit. The invariant probability measure gives each rational residue point mass $1/\#\,\mathrm{Gr}(r,\mathbb{F}_q)$. On a graph chart with $D_0=r(m-r)$ affine coordinates its density is exactly $q^{D_0}/\#\,\mathrm{Gr}(r,\mathbb{F}_q)$ times additive Haar measure: translations of graph matrices preserve the measure, and each coordinate residue ball has additive volume q^{-D_0}.

Full-rank matrices smoothly cover the nonzero Plücker cone. In local coordinates their function is $u^\delta g(z)$, with u nonzero and g a local projective equation. The nonzero u-disc has a logarithm; trivialization over that disc identifies its local Milnor fibration with that of g, times a smooth factor. Thus $\mathrm{Sat}_d(F_r)$ implies the corresponding spectral condition for each projective equation. For normal crossing support, every component of multiplicity b supplies a primitive b-th root at its generic point, so $\chi^b \neq 1$. Normal-unit averaging kills every divisor residue tube. For a plane pullback write $g(z)=\epsilon(z)h(\sigma(z))$, where ϵ is a unit and σ is smooth onto its open image W in the surface. A local complex root gives $u'=u\epsilon(z)^{1/\delta}$ and $F_r=(u')^\delta h(\sigma(z))$. The unit-disc trivialization above and the local smooth-product coordinates for σ identify its Milnor fibration with that of h, times a smooth factor. Hence $\mathrm{Sat}_d(F_r)$ gives $\mathrm{Sat}_d(h)$ on W. Smooth Hensel coordinates reduce each original tube to a plane residue tube times a smooth-coordinate

ball. Lemma 6.5, which gives the local surface form of the plane-curve holomorphy theorem [29, Theorem 3.1 and §3.3], applies on the indicated surface chart restricted to W. The unit factor has constant tame character on each source tube. It follows that $B_\chi(T)$ is entire in s.

Resolve the fixed projective divisor and spread the line-bundle trivializations. Formula (2.2) shows that $\deg_T B_\chi \le 0$. It is regular at $T = 0$. Entirety therefore makes it a constant C. If $C \neq 0$, formulas (13.5)–(13.6) with $h = 1$ give

$$I_{r,1}(T) = \frac{q^{-mr} A_r(q) C T^{r\delta}}{\prod_{j=m-r+1}^{m} (1 - q^{-j} T^{\delta})},$$

which has rational degree zero. The origin case of $\mathrm{Sat}_d(F_r)$ excludes an eigenvalue of exact order d. The strict degree theorem (2.3) is a contradiction. Hence $C = 0$. There are only finitely many orders $d \mid \delta$ in this argument, so one finite set of places suffices. Higher conductors have already been treated by scaling or Lemma 2.3.

Finally suppose $r > m/2$. Theorem 13.1 gives $\chi^\delta \neq 1$ and zero on every rank-deficient residue ball. For tame χ, on a full-rank ball over D the same local projective argument gives an entire integral, hence zero by Lemma 2.4; the tame unit factor is constant on this individual ball. Higher conductors are treated by Lemma 2.3 on a fixed resolution of F_r. A full-rank ball off D contributes an entire unit integral. This proves the assertion for every residual function. □

The standard and origin conclusions of Theorem 13.6 hold for every $1 \le r < m$; its residual conclusion holds for $r > m/2$.

14 Integral finite determinacy

We give an integral form of finite right determinacy [6, 14] which preserves the character integrals. It extends the preceding isolated classes to high order perturbations.

Theorem 14.1. *Let $f \in \Bbbk[x_1, \dots, x_n] \cap \mathfrak{m}^2$, where $\mathfrak{m} = (x_1, \dots, x_n)$, have an isolated critical point at zero. Choose $b \ge 1$ such that*

$$\mathfrak{m}^b \subset J_f = (\partial_1 f, \dots, \partial_n f) \quad \text{in } \Bbbk[x]_{\mathfrak{m}}. \tag{14.1}$$

There is a finite set S of places such that, for every polynomial $h \in \mathfrak{m}^{2b+1}$ with coefficients integral outside S, and every $v \notin S$, an integral analytic automorphism Ψ of the entire ball $\mathfrak{p}^n$ satisfies

$$(f + h) \circ \Psi = f, \qquad \Psi(x) = x + O(\mathfrak{m}^2), \qquad |\det D\Psi(x)| = 1. \tag{14.2}$$

Consequently

$$Z_{f+h,0}(\chi, s) = Z_{f,0}(\chi, s) \tag{14.3}$$

for every continuous finite-image character, of every conductor. The germs $f + h$ and f also have the same complex local monodromy. For an individual polynomial h with arbitrary coefficients, enlarge S only by their denominator places.

Proof. For every monomial x^α of total degree b choose, once, rational functions regular at zero such that

$$x^\alpha = \sum_i a_{\alpha i}(x) \partial_i f(x). \tag{14.4}$$

There are finitely many monomials and certificates. Invert the denominators of their coefficients and the nonzero constant terms of their denominator polynomials, together with the coefficients needed for f. Expanding the denominators as formal geometric series gives $a_{\alpha i} \in R[[x]]$, where $R = \mathcal{O}_{\Bbbk,S}$ for one finite S. All subsequent operations take place in $R[[x]]$.

Fix a rule assigning to each monomial of degree at least b one of its degree-b divisors. For $e \in \mathfrak{m}^N R[[x]]$, $N \geq b$, group the monomials according to this rule. Then

$$e = \sum_{|\alpha|=b} x^\alpha e_\alpha, \quad e_\alpha \in \mathfrak{m}^{N-b} R[[x]], \quad e = \sum_i c_i \partial_i f, \quad c_i = \sum_\alpha e_\alpha a_{\alpha i} \in \mathfrak{m}^{N-b} R[[x]].$$

Every step uses only multiplication and addition of integral series.

Put $e_0 = h$, $N_0 = 2b+1$. If $e_j \in \mathfrak{m}^{N_j}$, choose the c_i just described and set $\psi_j(x) = x - c(x)$. Its difference from the identity has order at least $N_j - b \geq 2$. Expanding each monomial of f by the binomial theorem gives

$$f(x-c) = f(x) - \sum_i c_i \partial_i f(x) + R_j, \qquad \operatorname{ord} R_j \geq 2(N_j - b).$$

A term with at least two c factors has this order because $f \in \mathfrak{m}^2$. The binomial coefficients are integers. The same expansion for e_j gives

$$\operatorname{ord}(e_j(x-c) - e_j(x)) \geq (N_j - 1) + (N_j - b) \geq 2(N_j - b).$$

Thus

$$(f + e_j) \circ \psi_j = f + e_{j+1}, \qquad N_{j+1} = 2(N_j - b) = 2b + 2^{j+1}.$$

The error orders tend to infinity, and the orders of $\psi_j - \mathrm{id}$ also tend to infinity. The compositions $\Psi_j = \psi_0 \circ \cdots \circ \psi_{j-1}$ converge in the $\mathfrak{m}$-adic topology to an element of $R[[x]]^n$ tangent to the identity. Recursive formal inversion uses only addition and multiplication, so its inverse also has coefficients in R. Passing to the limit proves $(f+h) \circ \Psi = f$ formally.

At $v \notin S$, every such integral power series and its inverse converge on $\mathfrak{p}^n$: terms of total degree j have absolute value at most $|\pi|^j$. Formal composition is valid there by convergence, so these maps are inverse analytic bijections of the complete ball. Since $D\Psi = I + O(\mathfrak{p})$, its determinant has absolute value one everywhere. The exact change-of-variables formula proves (14.3) for every character.

For complex monodromy, apply analytic Artin approximation [3, Theorem 1.2] to the analytic equation $(f+h)(Y) = f(X)$ and its formal solution $Y = \Psi(X)$. It gives a convergent exact solution congruent to Ψ modulo $\mathfrak{m}^2$ (take approximation precision $c = 2$). Its linear part is the identity, so the inverse function theorem makes it a complex analytic local automorphism. Analytic right equivalence identifies the local Milnor fibrations and their monodromy. □

Corollary 14.2. *Any established central vanishing theorem for an isolated germ f extends to every $f + h$ in Theorem 14.1. In particular, the ordinary and weighted two-term classes, and any other isolated case for which the required central vanishing has been established, may be enlarged by arbitrary perturbations in $\mathfrak{m}^{2b+1}$ once* (14.1) *is certified. One may always take $b = \mu(f)$, the length of the local Jacobian algebra, obtaining the sufficient bound $\operatorname{ord} h \geq 2\mu(f) + 1$. If, in addition, the zero fibre of $f + h$ is smooth away from zero, then $f + h$ satisfies the full residual holomorphy conjecture.*

Proof. Complex right equivalence identifies the spectral hypotheses, and (14.3) identifies the original integrals. In an Artinian local algebra of length μ, the descending chain of powers of the maximal ideal reaches zero by its μ-th power: an equality before zero would imply zero by Nakayama's lemma, whereas each strict inclusion consumes at least one dimension. Thus $\mathfrak{m}^{\mu} \subset J_f$. Under the additional smoothness assumption, the Jacobian criterion spreads that smoothness away from the origin after finitely many further places, which may depend on the chosen h, are removed. Every nonzero zero residue ball then has $f + h$ as a Hensel coordinate and contributes zero for every nontrivial character. The origin ball has already been treated, and balls off the zero fibre contribute constants in s. This proves the last assertion. □

Example 14.3. For $f = x^2 + y^3 + z^7$ one can take $b = 7$, since $J_f = (2x, 3y^2, 7z^6)$. The following rule gives the complete set of degree-seven certificates. For $i + j + k = 7$, put

$$x^i y^j z^k = \begin{cases} \frac{1}{2} x^{i-1} y^j z^k \, \partial_x f, & i \geq 1, \\ \frac{1}{3} y^{j-2} z^k \, \partial_y f, & i = 0,\ j \geq 2, \\ \frac{1}{7} y^j z^{k-6} \, \partial_z f, & i = 0,\ j \leq 1. \end{cases}$$

In the last case $k \geq 6$, so all displayed exponents are nonnegative. These mutually exclusive cases exhaust all 36 degree-seven monomials: 28 use $\partial_x f$, six use $\partial_y f$, and two use $\partial_z f$. Here are all the individual identities.

No.	Identity	No.	Identity
1	$x^7 = \frac{x^6}{2} \, \partial_x f$	19	$x^2 y^2 z^3 = \frac{x y^2 z^3}{2} \, \partial_x f$
2	$x^6 y = \frac{x^5 y}{2} \, \partial_x f$	20	$x^2 y z^4 = \frac{x y z^4}{2} \, \partial_x f$
3	$x^6 z = \frac{x^5 z}{2} \, \partial_x f$	21	$x^2 z^5 = \frac{x z^5}{2} \, \partial_x f$
4	$x^5 y^2 = \frac{x^4 y^2}{2} \, \partial_x f$	22	$x y^6 = \frac{y^6}{2} \, \partial_x f$
5	$x^5 y z = \frac{x^4 y z}{2} \, \partial_x f$	23	$x y^5 z = \frac{y^5 z}{2} \, \partial_x f$
6	$x^5 z^2 = \frac{x^4 z^2}{2} \, \partial_x f$	24	$x y^4 z^2 = \frac{y^4 z^2}{2} \, \partial_x f$
7	$x^4 y^3 = \frac{x^3 y^3}{2} \, \partial_x f$	25	$x y^3 z^3 = \frac{y^3 z^3}{2} \, \partial_x f$
8	$x^4 y^2 z = \frac{x^3 y^2 z}{2} \, \partial_x f$	26	$x y^2 z^4 = \frac{y^2 z^4}{2} \, \partial_x f$
9	$x^4 y z^2 = \frac{x^3 y z^2}{2} \, \partial_x f$	27	$x y z^5 = \frac{y z^5}{2} \, \partial_x f$
10	$x^4 z^3 = \frac{x^3 z^3}{2} \, \partial_x f$	28	$x z^6 = \frac{z^6}{2} \, \partial_x f$
11	$x^3 y^4 = \frac{x^2 y^4}{2} \, \partial_x f$	29	$y^7 = \frac{y^5}{3} \, \partial_y f$
12	$x^3 y^3 z = \frac{x^2 y^3 z}{2} \, \partial_x f$	30	$y^6 z = \frac{y^4 z}{3} \, \partial_y f$
13	$x^3 y^2 z^2 = \frac{x^2 y^2 z^2}{2} \, \partial_x f$	31	$y^5 z^2 = \frac{y^3 z^2}{3} \, \partial_y f$
14	$x^3 y z^3 = \frac{x^2 y z^3}{2} \, \partial_x f$	32	$y^4 z^3 = \frac{y^2 z^3}{3} \, \partial_y f$
15	$x^3 z^4 = \frac{x^2 z^4}{2} \, \partial_x f$	33	$y^3 z^4 = \frac{y z^4}{3} \, \partial_y f$
16	$x^2 y^5 = \frac{x y^5}{2} \, \partial_x f$	34	$y^2 z^5 = \frac{z^5}{3} \, \partial_y f$
17	$x^2 y^4 z = \frac{x y^4 z}{2} \, \partial_x f$	35	$y z^6 = \frac{y}{7} \, \partial_z f$
18	$x^2 y^3 z^2 = \frac{x y^3 z^2}{2} \, \partial_x f$	36	$z^7 = \frac{z}{7} \, \partial_z f$

They invert only $2, 3, 7$. Every integral $h \in (x, y, z)^{15}$ therefore has exactly the same origin character integral as this diagonal polynomial at every prime other than $2, 3, 7$, for every conductor. The diagonal germ has the known holomorphy property [13, Corollary 3.5]. One can also see the required central vanishing directly. Thom–Sebastiani gives the actual eigenvalue with exponent $1/2 + 1/3 + 1/7 = 41/42$, of order 42. Thus $\mathrm{Sat}_d(f, 0)$ implies $\chi^{42} \neq 1$. Scaling with integral weights $(21, 14, 6)$ preserves the complete original ball and multiplies its integral by $\chi(t)^{42}$ for every unit t, which forces zero. For a concrete global illustration, take $f_N = x^2 + y^3 + z^7 + z^N$ with any integer $N \geq 15$. A nonzero critical point of its zero fibre would have $x = y = 0$, $z^{N-7} = -1$ and $Nz^{N-7} = -7$, which would force $N = 7$. Thus zero is the only singular point of the zero fibre, and the last assertion of Corollary 14.2 applies. The same integral identity applies to every mixed perturbation in $(x, y, z)^{15}$.

A Expanded arithmetic and incidence certificates

This appendix gives the exact arithmetic identities, their finite computation ranges, complete programs and output tables. Character values are computed in cyclotomic rings or rational character groups. The source package includes every individual conductor and Gaussian record, all thirty-six Jacobian identities, and the remaining exact outputs.

A.1 The Jacobi factor and a resonant channel

The identity checked is precisely

$$\sum_{\alpha,\beta \in \mathbb{F}_q^\times} \chi(\beta)^m \chi(U\alpha^r + V\beta^k) = (q-1) \sum_{\substack{\lambda^k = \chi^{m+k} \\ \lambda^r = 1}} \lambda(U)(\chi/\lambda)(V)\lambda^{-1}(-1)J(\lambda^{-1}, \chi).$$

The proof, including the Fourier normalization, is in (3.27)–(3.33). For each of $q = 5, 7, 11$, the program below runs over every $1 \leq m, k, r \leq 4$, every nontrivial character, and every nonzero U, V. If g generates $\mathbb{F}_q^\times$, the character with exponent j takes g^b to X^{jb} in $\mathbb{Z}[X]/(\Phi_{q-1})$, where Φ_N is the Nth cyclotomic polynomial. Reduction uses the following full coefficient lists, in increasing degree:

$$\Phi_4 = 1 + X^2, \qquad \Phi_6 = 1 - X + X^2, \qquad \Phi_{10} = 1 - X + X^2 - X^3 + X^4.$$

The outputs are

q	generator	identities	failures
5	2	3,072	0
7	3	11,520	0
11	2	57,600	0

Their sum is $72,192$. As a negative control, omit the factor $q - 1$. The first failure for each field has $m = k = r = U = V = 1$ and $j = (q-1)/2$. The reduced differences are respectively $(-3, 0)$, $(-5, 0)$ and $(-9, 0, 0, 0)$.

Here is the entire table for $q = 7$, $(m, k, r) = (4, 3, 2)$, and the quadratic character, with rows

$U = 1, \dots, 6$ and columns $V = 1, \dots, 6$:

$$\begin{pmatrix} -6 & -6 & -6 & -6 & -6 & -6 \\ -6 & -6 & -6 & -6 & -6 & -6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ -6 & -6 & -6 & -6 & -6 & -6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix}.$$

The resonant channel cancels upon taking the trace sum over the complete geometric curve.

A.2 The quadratic norm and the dyadic identity

For a nonsquare $c \in \mathbb{F}_q^\times$, the norm is $N(a + b\sqrt{c}) = a^2 - cb^2$. Its zero fibre has one element, and every nonzero fibre has $q + 1$ elements: the norm on $\mathbb{F}_{q^2}^\times$ is the power $q + 1$, has image of order $q - 1$, and has kernel of order $q + 1$. The complete computed fibre counts, in the order $0, 1, \dots, q - 1$, are

q	c	counts
3	2	$(1, 4, 4)$
5	2	$(1, 6, 6, 6, 6)$
7	3	$(1, 8, 8, 8, 8, 8, 8)$
11	2	$(1, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12)$.

In particular, for every nontrivial character η,

$$\sum_{(a,b)\in\mathbb{F}_q^2} \eta(a^2 - cb^2) = (q + 1) \sum_{u\in\mathbb{F}_q^\times} \eta(u) = 0.$$

The exceptional quadratic cancellation uses the three coefficients $qC, -C, C$. With U an indeterminate, factoring out C gives

$$\frac{qU^2}{1 - U^2} - \frac{qU^2}{1 - qU^2} + \frac{(q - 1)qU^4}{(1 - U^2)(1 - qU^2)} = 0. \tag{A.1}$$

Multiplication by the common denominator gives

$$qU^2 - q^2U^4 - qU^2 + qU^4 + (q^2 - q)U^4 = 0.$$

Lemma 6.3 identifies these three coefficients on the resolution graph, including their Frobenius descent.

A.3 A Gaussian check on the original residue ball

Let q be odd, let η be the quadratic character, and choose an additive character Ψ trivial on $\mathcal{O}$ and nontrivial on $\pi^{-1}\mathcal{O}$. Put $\overline{\Psi}(\bar{a}) = \Psi(\pi^{-1}\widetilde{a})$ for any lift $\widetilde{a} \in \mathcal{O}$, and write $\tau(\eta) = \sum_{a\in\mathbb{F}_q^\times} \eta(a)\overline{\Psi}(a)$. Equation (7.8) and $\overline{\tau(\eta)} = \eta(-1)\tau(\eta)$ give $\tau(\eta)^2 = \eta(-1)q$. For units b, u set

$$G_t(bu) = \int_{\mathcal{O}} \Psi(\pi^{-t}bux^2)\, dx.$$

The square-root count used above gives $qG_1 = \sum_{y\in\mathbb{F}_q}(1+\eta(y))\overline{\Psi}(\overline{bu}\,y) = \eta(bu)\tau(\eta)$. Thus the initial values and recursion are

$$G_0 = 1, \quad G_1 = q^{-1}\eta(bu)\tau(\eta), \quad G_t = q^{-1}G_{t-2} \quad (t \geq 2).$$

For the recursion, the integral on each nonzero residue ball is zero because the derivative $2bux$ is a unit, while $x = \pi y$ on the zero residue ball supplies $q^{-1}G_{t-2}$. Thus

$$G_{2j} = q^{-j}, \qquad G_{2j+1} = q^{-j-1}\eta(bu)\tau(\eta).$$

For units a, c, multiplication for $Q(x,y) = ax^2 + cy^2$ gives

$$\int_{\mathcal{O}^2} \Psi(\pi^{-t}uQ)\,dx\,dy = q^{-t}\epsilon^t, \qquad \epsilon = \eta(-ac).$$

Rescaling the ball $\mathfrak{p}^2$ gives, for every $M \geq 2$,

$$\int_{\mathfrak{p}^2} \Psi(\pi^{-M}uQ)\,dx\,dy = q^{-2}q^{-(M-2)}\epsilon^{M-2} = q^{-M}\epsilon^M. \tag{A.2}$$

This is independent of u and includes nonsplit binary forms.

The additional exact program checks this formula for $p = 3, 5, 7$, $2 \leq M \leq 5$, and every nonzero residue a, c, u. Put $t = M - 2$. It counts all p^{2t} pairs in the global finite sum and reduces the integer coefficient polynomial modulo

$$\Phi_{p^t}(X) = \sum_{j=0}^{p-1} X^{jp^{t-1}} \quad (t \geq 1).$$

The case $t = 0$ is evaluated directly. The complete aggregate output is $32,256,864$ checks for $p = 3, 5, 7$, respectively: 1152 checks, all passing. The program records every parameter tuple and its exact rational integral. At $M = 2$, omission of the original-ball factor p^{-2} gives the false value 1 in place of $1/9, 1/25, 1/49$, respectively.

A.4 The matrix-content measure

For an $m \times r$ matrix distributed with normalized Haar measure, let $c(X)$ be the minimum valuation of its maximal minors. The Haar decomposition proved in Proposition 13.5 gives independent contents with generating function

$$\mathbb{E}(z^{c(X)}) = \prod_{j=m-r+1}^{m} \frac{1-q^{-j}}{1-q^{-j}z}.$$

For $m = 3$, $r = 2$, $q = 2$, put $A = (1-2^{-2})(1-2^{-3}) = 21/32$. Then

$$\Pr(c=0) = \frac{21}{32}, \qquad \Pr(c=1) = \frac{21}{32}\left(\frac{1}{4}+\frac{1}{8}\right) = \frac{63}{256}, \qquad \Pr(c\geq 2) = \frac{25}{256}.$$

All $4^6 = 4096$ matrices modulo 4 were enumerated. Multiplying these probabilities by 4096 gives respectively $2688, 1008, 400$, exactly the counts obtained by testing all three maximal minors. The same enumeration was performed independently by the Wolfram Language program below.

A.5 Incidence operators

For the finite subset model, the matrix L_r has rows indexed by $(r-1)$-subsets and columns indexed by r-subsets of $\{1,\dots,m\}$, with entry one for containment and zero otherwise. The full check is

$$L_{r+1}L_{r+1}^{\mathsf{t}} - L_r^{\mathsf{t}}L_r = (m-2r)I_{\binom{m}{r}},$$

where the first term is zero for $r = m$. The following table gives all computed ranks, in increasing $r = 1,\dots,m$. To verify the displayed identity entry by entry, index both square matrices by r-subsets I, J. If $I \neq J$, each product has entry one exactly when $|I \cap J| = r-1$: the first counts their unique common $(r+1)$-superset, and the second their unique common $(r-1)$-subset. Both entries are zero otherwise. On the diagonal the first product has $m-r$ extensions and the second has r subsets. The difference is therefore $(m-2r)I$ in every entry. All thirty-six computed matrices satisfy this identity.

m	$\operatorname{rank} L_1,\dots,\operatorname{rank} L_m$	number of identities
1	1	1
2	1, 1	2
3	1, 3, 1	3
4	1, 4, 4, 1	4
5	1, 5, 10, 5, 1	5
6	1, 6, 15, 15, 6, 1	6
7	1, 7, 21, 35, 21, 7, 1	7
8	1, 8, 28, 56, 56, 28, 8, 1	8

There are 36 identities. In the equal middle dimension for $m = 2, 4, 6, 8$, the kernel dimensions are $1, 2, 5, 14$; thus the finite model records the boundary kernels. The corresponding constructible-function calculation is proved in Lemma 13.3.

A.6 The arbitrary-conductor cutoff

Let χ have conductor a_χ, let $b \geq 1$, and let $A \geq a_\chi$. Write $H_{b,A} = \operatorname{im}(1+\mathfrak{p}^b \longrightarrow (\mathcal{O}/\mathfrak{p}^A)^\times)$. The principal-unit average is

$$\frac{1}{\#H_{b,A}} \sum_{u \in H_{b,A}} \chi(u) = \begin{cases} 1, & b \geq a_\chi, \\ 0, & b < a_\chi. \end{cases}$$

For $b \geq A$ the image group is $\{1\}$; this includes the depth $b = A+1$ used in the finite check. This follows from orthogonality of a character of the finite group: if $\chi(u_0) \neq 1$, multiplying every summation argument by u_0 shows that the sum equals $\chi(u_0)$ times itself. The code checks every nontrivial character modulo p^A, with $p = 3, 5, 7$, $1 \leq A \leq 3$, and every depth $1 \leq b \leq A+1$. The complete counts of tested character–depth pairs are

	$A=1$	$A=2$	$A=3$	total
$p=3$	2	15	68	85
$p=5$	6	57	396	459
$p=7$	10	123	1172	1305
total	18	195	1636	1849.

All 1849 checks passed. With $m = M \operatorname{ord} R_0$ and $a_0 = \operatorname{ord} z^a$ as in (12.9), the power of q in the Cayley reduction is checked by the polynomial identity

$$\begin{aligned} -hr - c(m - a_0 - h + a_\chi) &= -hr - cm + ca_0 + ch - ca_\chi \\ &= -ca_\chi - h(r - c) - cm + ca_0. \end{aligned}$$

Thus the kernel lattice has depth zero, while $q^{-ca_\chi - h(r-c)}$ records the parameter ball and the conductor.

A.7 Weighted finite-group checks

The exact rational program checks the Bézout covariance identities and the two presentations of the same finite character group. Its ranges are $1 \le m \le 25$, $m < a \le 40$, $(m, a) = 1$, every divisor e of a, $2 \le r \le 19$, and every compatible residue $\beta \pmod e$. The primitive-lift check runs over $10 \le \ell \le 25$, $1 \le k \le 17$, $1 \le L \le 34$, with the fixed choices $\lambda = \exp(2\pi i/L)$ and $\nu = \exp(2\pi i/n)$, where $n = L, 2L, 3L$. It tests every compatible nontrivial χ satisfying $\lambda^k = \chi^\ell$, and all ℓ-th roots of ν^k. The complete output is

Bézout covariance	1577
coprime presentation	21082
character-group equivalence	2651104
primitive lifts	483024
failures	0.

The weighted chapter proves the identities for all integer parameters; these counts record the finite ranges of the exact implementation.

A.8 The thirty-six Jacobian identities

For $f = x^2 + y^3 + z^7$, Example 14.3 gives every degree-seven monomial as a polynomial multiple of one of $\partial_x f = 2x$, $\partial_y f = 3y^2$, $\partial_z f = 7z^6$. The last program enumerates the exponent triples in decreasing lexicographic order, records each multiplier, and expands the thirty-six residuals. Its complete output is `jacobian_36_expanded.json`; all residuals equal zero.

A.9 Complete programs

The first program uses only the Python standard library. It prints the Jacobi check counts and negative controls, the full resonant table, the matrix counts and the norm counts displayed above.

```
#!/usr/bin/env python3
"""Exact finite checks; the general proofs are in the article.

Only integer arithmetic is used for character values. A vector of counts
represents a polynomial at a primitive (q-1)-st root, reduced modulo its
cyclotomic polynomial. No floating approximation or random sample occurs.
"""
from collections import Counter
from itertools import product, combinations
from math import prod
import json
from pathlib import Path
```

```
# Coefficients in increasing degree, including the monic leading term.
PHI = {4: [1, 0, 1], 6: [1, -1, 1], 10: [1, -1, 1, -1, 1]}


def primitive_root(q):
    for g in range(2, q):
        if len({pow(g, a, q) for a in range(q - 1)}) == q - 1:
            return g
    raise AssertionError("no primitive root")


def reduce_cyclotomic(counts, n):
    a = list(counts)
    p = PHI[n]
    degree = len(p) - 1
    for j in range(len(a) - 1, degree - 1, -1):
        c = a[j]
        for k, b in enumerate(p):
            a[j - degree + k] -= c * b
    return a[:degree]


def character_checks(q):
    n = q - 1
    g = primitive_root(q)
    logs = {pow(g, a, q): a for a in range(n)}
    minus_one = logs[q - 1]
    checked = 0
    first_wrong_normalization = None
    for m, k, r in product(range(1, 5), repeat=3):
        a = m + k
        for j in range(1, n):
            eligible = [l for l in range(n)
                        if (l * k - j * a) % n == 0 and l * r % n == 0]
            jacobi = {}
            for l in eligible:
                v = [0] * n
                for x in range(1, q):
                    y = (1 - x) % q
                    if y:
                        v[(-l * logs[x] + j * logs[y]) % n] += 1
                jacobi[l] = v
            for u, v in product(range(1, q), repeat=2):
                left, right_without_factor = [0] * n, [0] * n
                for alpha, beta in product(range(1, q), repeat=2):
                    value = (u * pow(alpha, r, q) + v * pow(beta, k, q)) % q
                    if value:
                        e = j * (m * logs[beta] + logs[value])
                        left[e % n] += 1
                for l in eligible:
                    shift = l * logs[u] + (j - l) * logs[v] - l * minus_one
                    for e, count in enumerate(jacobi[l]):
                        right_without_factor[(e + shift) % n] += count
                delta = [x - n * y for x, y in zip(left, right_without_factor)]
                assert reduce_cyclotomic(delta, n) == [0] * (len(PHI[n]) - 1)
                checked += 1
                wrong = reduce_cyclotomic(
```

```
                    [x - y for x, y in zip(left, right_without_factor)], n)
                if any(wrong) and first_wrong_normalization is None:
                    first_wrong_normalization = {
                        "m": m, "k": k, "r": r, "character_exponent": j,
                        "U": u, "V": v, "difference_without_q_minus_one": wrong}
    return {"q": q, "generator": g, "identities": checked,
            "failures": 0, "negative_control": first_wrong_normalization}


def resonant_table():
    q = 7
    square = {1, 2, 4}
    def eta(x):
        x %= q
        return 0 if x == 0 else (1 if x in square else -1)
    return [[sum(eta(beta) ** 4 * eta(u * alpha ** 2 + v * beta ** 3)
                 for alpha in range(1, 7) for beta in range(1, 7))
             for v in range(1, 7)] for u in range(1, 7)]


def minors_mod_four():
    counts = Counter()
    for entries in product(range(4), repeat=6):
        rows = [entries[0:2], entries[2:4], entries[4:6]]
        minors = [(rows[i][0] * rows[j][1] - rows[i][1] * rows[j][0]) % 4
                  for i, j in combinations(range(3), 2)]
        content = 0 if any(x % 2 for x in minors) else (1 if 2 in minors else 2)
        counts[content] += 1
    assert counts == {0: 2688, 1: 1008, 2: 400}
    return {"matrices": 4 ** 6, "content_0": counts[0],
            "content_1": counts[1], "content_at_least_2": counts[2]}


def norm_counts(q):
    squares = {a * a % q for a in range(1, q)}
    c = next(a for a in range(1, q) if a not in squares)
    counts = Counter((a * a - c * b * b) % q
                     for a, b in product(range(q), repeat=2))
    assert counts[0] == 1
    assert all(counts[a] == q + 1 for a in range(1, q))
    return {"q": q, "nonsquare": c,
            "counts_for_values_0_to_q_minus_1": [counts[a] for a in range(q)]}


def run():
    result = {"jacobi": [character_checks(q) for q in (5, 7, 11)],
              "resonant_q7_table": resonant_table(),
              "matrix_minors": minors_mod_four(),
              "quadratic_norms": [norm_counts(q) for q in (3, 5, 7, 11)]}
    expected = [[-6] * 6, [-6] * 6, [6] * 6,
                [-6] * 6, [6] * 6, [6] * 6]
    assert result["resonant_q7_table"] == expected
    target = Path(__file__).with_name("verification_results.json")
    target.write_text(json.dumps(result, indent=2) + "\n")
    print(json.dumps(result, indent=2))


if __name__ == "__main__":
```

```
    run()
```

The second program uses exact rational arithmetic from the Python standard library.

```
from fractions import Fraction
from math import gcd
import json
from pathlib import Path

def lcm(a,b): return a//gcd(a,b)*b

def order(x): return Fraction(x%1).denominator

checks={'bezout_covariance':0,'coprime_presentation':0,'primitive_lifts':0,'
    jacobi_group_equivalence':0}
for m in range(1,26):
 for a in range(m+1,41):
  if gcd(m,a)!=1: continue
  for e in range(1,a+1):
   if a%e: continue
   v=pow(m,-1,e) if e>1 else 0
   u=(1-m*v)//e
   k=a-m; ell=a//e; b=(1+v*k)//e
   assert e*u+m*v==1
   assert m*b+u*k==ell
   assert a*b-v*ell*k==ell
   assert (v*k+1)%e==0
   checks['bezout_covariance']+=1
   for r in range(2,20):
    for beta in range(e):
     if (m+beta*r)%e: continue
     L0=(m+beta*r)//e
     assert gcd(gcd(r,L0),ell)==1
     checks['coprime_presentation']+=1
     t=L0-ell
     # Finite rational character parameters: (lambda, mu) satisfy generic Q_C.
     for i in range(r):
      lam=Fraction(i,r)
      for j in range(ell):
       mu=(-L0*lam+j)/ell
       theta=(lam+mu)%1
       assert (r*lam)%1==0
       assert (L0*lam+ell*mu)%1==0
       assert (t*lam+ell*theta)%1==0
       assert (k*lam-a*theta)%1==0
       checks['jacobi_group_equivalence']+=1
for ell in range(10,26):
 for k in range(1,18):
  for L in range(1,35):
   # Fix lambda=exp(2*pi*i/L); enumerate every compatible nontrivial chi.
   lam=Fraction(1,L)
   for j in range(ell):
    chi=(k*lam+j)/ell
    d=order(chi)
    if d<=1: continue
    for mult in (1,2,3):
     n=L*mult; nu=Fraction(1,n)
```

```
    B=order(k*nu)
    assert ell*B%d==0
    roots=[(k*nu+j)/ell for j in range(ell)]
    maximal=[x for x in roots if order(x)==ell*B]
    assert len(maximal)>=sum(gcd(i,ell)==1 for i in range(1,ell+1))>=4
    checks['primitive_lifts']+=1
out={'checks':checks,'failures':0,'meaning':'Finite independent exact-rational checks; the proof
    uses Bezout, Smith normal form, and primitive root lifting, not enumeration.'}
Path(__file__).with_name('weighted_identities_results.json').write_text(json.dumps(out,indent=2)+
    '\n')
print(json.dumps(out))
```

The third program uses SymPy for primitive roots and exact polynomial remainders. It writes all 1849 character–depth records and prints the aggregate check count.

```
"""Exact unit-coset checks for the Cayley parameter integral.

This verifies finite identities only. The uniform proof is in the Cayley section of the article.
No floating point arithmetic is used.
"""
from collections import Counter
from pathlib import Path
import json
from math import gcd
import sympy as s

X = s.Symbol('X')
records = []
for p in (3, 5, 7):
    for modulus_depth in (1, 2, 3):
        modulus = p ** modulus_depth
        phi = (p - 1) * p ** (modulus_depth - 1)
        generator = int(s.primitive_root(modulus))
        logs = {pow(generator, j, modulus): j for j in range(phi)}
        for character_power in range(1, phi):
            # Actual conductor of the character on (Z/p^a Z)^*.
            conductor = 1
            for j in range(1, modulus_depth):
                if character_power * logs[(1 + p ** j) % modulus] % phi:
                    conductor = j + 1
            order = phi // gcd(phi, character_power)
            k = character_power // gcd(phi, character_power)
            cyclotomic = s.Poly(s.cyclotomic_poly(order, X), X)
            for level in range(1, modulus_depth + 2):
                coset_size = p ** max(modulus_depth - level, 0)
                counts = Counter(
                    k * logs[(1 + p ** level * t) % modulus] % order
                    for t in range(coset_size)
                )
                polynomial = s.Poly(sum(n * X ** e for e, n in counts.items()), X)
                remainder = s.rem(polynomial, cyclotomic)
                expected = coset_size if level >= conductor else 0
                assert remainder == s.Poly(expected, X), (
                    p, modulus_depth, character_power, conductor, level, remainder
                )
                records.append({
                    'p': p, 'modulus_depth': modulus_depth,
```

```
                    'character_power': character_power,
                    'order': order, 'conductor': conductor,
                    'principal_unit_depth': level,
                    'coset_size': coset_size, 'character_sum': expected,
                })

# The exact exponent in the c normal directions and r original parameter directions.
c, r, h, m, a0, a_chi = s.symbols('c r h m a0 a_chi', integer=True)
assert s.expand(-h*r-c*(m-a0-h+a_chi)
                - (-c*a_chi-h*(r-c)-c*m+c*a0)) == 0
output = {'checks': len(records), 'failures': 0, 'records': records,
          'normal_volume_exponent_identity': 'verified symbolically'}
Path(__file__).with_name('owner_conductor_results.json').write_text(
    json.dumps(output, indent=2)
)
print(json.dumps({'checks': len(records), 'failures': 0,
                  'normal_volume_exponent_identity': 'verified symbolically'}))
```

The Gaussian check uses only integer counters, exact rational numbers, and the displayed prime-power cyclotomic reduction:

```
"""Exact binary Gaussian checks on the original maximal-ideal ball.

All a,c,u run over nonzero residues. For M=2,...,5 the finite sums
are evaluated in Z[X]/Phi_(p^(M-2)), not with approximate roots.
The proof in the article covers all units and every M >= 2.
"""
from collections import Counter
from fractions import Fraction
from itertools import product
from pathlib import Path
import json


def eta(a, p):
    a %= p
    if not a:
        return 0
    return 1 if pow(a, (p - 1) // 2, p) == 1 else -1


def prime_power_reduce(coefficients, p, t):
    if t == 0:
        return [sum(coefficients)]
    base = p ** (t - 1)
    # Phi_(p^t) = 1 + X^base + ... + X^((p-1)*base).
    return [coefficients[i] - coefficients[(p - 1) * base + i % base]
            for i in range((p - 1) * base)]


def binary_sum(p, t, a, c, u):
    n = p ** t
    first = Counter(a * u * x * x % n for x in range(n))
    second = Counter(c * u * y * y % n for y in range(n))
    coefficients = [0] * n
    for i, count_i in first.items():
        for j, count_j in second.items():
```

```
            coefficients[(i + j) % n] += count_i * count_j
    return coefficients


records = []
summary = []
for p in (3, 5, 7):
    count = 0
    for t in range(4):
        n, m = p ** t, t + 2
        for a, c, u in product(range(1, p), repeat=3):
            coefficients = binary_sum(p, t, a, c, u)
            epsilon = eta(-a * c, p)
            expected_sum = n * epsilon ** t
            coefficients[0] -= expected_sum
            reduced = prime_power_reduce(coefficients, p, t)
            assert all(value == 0 for value in reduced), (p, m, a, c, u)
            original_ball_value = Fraction(expected_sum, p ** 2 * n ** 2)
            assert original_ball_value == Fraction(epsilon ** m, p ** m)
            records.append([p, m, a, c, u, str(original_ball_value)])
            count += 1
    summary.append({"p": p, "M_range": [2, 5], "checks": count,
                    "failures": 0,
                    "missing_ball_factor_at_M2": {
                        "wrong": "1", "correct": str(Fraction(1, p ** 2))}})

output = {"summary": summary, "total": len(records),
          "record_columns": ["p", "M", "a", "c", "u", "integral"],
          "records": records}
Path(__file__).with_name("quadratic_gauss_results.json").write_text(
    json.dumps(output, indent=2) + "\n")
print(json.dumps({"summary": summary, "total": len(records)}, indent=2))
```

The independent Wolfram Language program is

```
ClearAll["Global`*"];
inc[m_, r_] := Module[{lo = Subsets[Range[m], {r - 1}], hi = Subsets[Range[m], {r}]},
 Table[Boole[SubsetQ[hi[[j]], lo[[i]]]], {i, Length[lo]}, {j, Length[hi]}]];
rows = Flatten[Table[
 Table[With[{f = inc[m, r]},
  {m, r, Binomial[m, r], MatrixRank[f],
   If[r < m, inc[m, r + 1].Transpose[inc[m, r + 1]], ConstantArray[0, {Binomial[m, r], Binomial[
    m, r]}]]
   - Transpose[f].f == (m - 2 r) IdentityMatrix[Binomial[m, r]]}],
 {r, 1, m}], {m, 1, 8}], 1];
counts = Counts[Map[Function[v, Module[{x = Partition[v, 2], a},
 a = Mod[Det /@ (x[[#]] & /@ Subsets[Range[3], {2}]), 4];
 If[AnyTrue[a, OddQ], 0, If[AnyTrue[a, # == 2 &], 1, 2]]]], Tuples[Range[0, 3], 6]]];
<|"incidenceRows" -> rows, "minorContentCountsMod4" -> counts|>
```

The Jacobian certificate program uses exact symbolic polynomials:

```
"""List and verify every degree-seven Jacobian certificate exactly."""
from pathlib import Path
import json
import sympy as s
```

```
x, y, z = s.symbols("x y z")
f = x**2 + y**3 + z**7
records = []
for i in range(7, -1, -1):
    for j in range(7 - i, -1, -1):
        k = 7 - i - j
        monomial = x**i * y**j * z**k
        if i >= 1:
            variable = x
            coefficient = s.Rational(1, 2) * x**(i-1) * y**j * z**k
        elif j >= 2:
            variable = y
            coefficient = s.Rational(1, 3) * y**(j-2) * z**k
        else:
            variable = z
            coefficient = s.Rational(1, 7) * y**j * z**(k-6)
        residual = s.expand(monomial - coefficient * s.diff(f, variable))
        assert residual == 0
        records.append({
            "exponents": [i, j, k], "monomial": str(monomial),
            "derivative": str(variable), "coefficient": str(coefficient),
            "monomial_tex": s.latex(monomial),
            "coefficient_tex": s.latex(coefficient), "residual": str(residual),
        })
assert len(records) == 36
assert len({tuple(row["exponents"]) for row in records}) == 36
result = {"polynomial": str(f), "degree": 7, "count": 36,
          "identities": records}
Path("jacobian_36_expanded.json").write_text(json.dumps(result, indent=2) + "\n")
print(json.dumps({"certificates": len(records), "nonzero_residuals": 0}))
```

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