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AN ELEMENTARY CARRY-FREE PROOF OF $h(n!) = O\big((\log n)^3 \log\log n\big)$ FOR ERDŐS PROBLEM #18

VINNOR

Abstract. A positive integer m is called *practical* if every integer $1 \le t \le m$ can be written as a sum of distinct positive divisors of m. For practical m, let $h(m)$ denote the smallest integer such that every $1 \le t \le m$ is a sum of at most $h(m)$ distinct positive divisors of m. Answering all three parts of Erdős Problem #18 in one elementary theorem, we prove unconditionally that

$$h(n!) = O\big((\log n)^3 \log\log(n+16)\big),$$

improving the November 2026 frontier $O((\log n)^9)$ of Howe–Huffman–Kohlmeyer–Kruer (`hexagon:2611.00748v1`) by nearly six powers of $\log n$ and approaching the information-theoretic lower bound $\Omega((\log n)^2)$ up to a factor of $\log n \cdot \log\log n$.

Our proof abandons 2-adic covering modulo 2^k and its associated carry chains altogether, working instead in a carry-free Chinese Remainder product of prime fields $\mathbb{Z}/M'\mathbb{Z} \cong \prod_{q\in Q'} \mathbb{F}_q$. Three elementary ingredients drive the bound: (i) a pointwise Kullback–Leibler contraction inequality on $\mathbb{F}_q$ with absolute constant $c_1 = 1/484$ yielding a universal dilation entropy-gain bound *linear* in $H_{\mathrm{nat}}(V)$ (eliminating quadratic and convex f_4 variance penalties); (ii) an elementary 2-factor smooth `Finset` pigeonhole dilator $c_1c_2 \mid m!$ hitting every residue in $\mathbb{F}_q^*$ with probability at least $\frac{1/40}{q-1}$ using zero character sums and zero Pólya–Vinogradov bounds; and (iii) adaptive greedy CRT coordinate matching paired with an 11-prime strip tail recursion $m \mapsto m/2$. Zero analytic number theory is used beyond Erdős's 1932 central binomial coefficient bounds. The entire proof is formally verified end-to-end in Lean 4.36.0 + Mathlib4 (13,586 lines, zero `sorry`, depending solely on `propext`, `Classical.choice`, and `Quot.sound`).

1. Introduction

A positive integer m is *practical* if every integer $1 \le t \le m$ can be expressed as a sum of distinct positive divisors of m. For practical m, let $h(m)$ denote the least integer r such that every $1 \le t \le m$ is a sum of at most r distinct positive divisors of m. Every factorial $n!$ is practical, and Erdős [2, 3, 4] posed three linked questions cataloged as Erdős Problem #18 [1]:

(a) Are there infinitely many practical integers m with $h(m) < (\log\log m)^{O(1)}$?
(b) Is $h(n!) < n^{o(1)}$?
(c) Is $h(n!) < (\log n)^{O(1)}$?

Since $\log(n!) = (1+o(1))n\log n$ while $\log\tau(n!) = \Theta(n/\log n)$, pigeonhole forces the universal lower bound

$$h(n!) \ge \frac{\log(n!)}{\log(\tau(n!)+1)} = \Omega\big((\log n)^2\big).$$

Historically, Erdős [2] gave $h(n!) < n$, Vose [8] constructed infinitely many practical m with $h(m) \ll (\log m)^{1/2}$, Tenenbaum–Yokota [7] and Yokota [9] obtained $h(n!) \ll n/(\log n)^{1/2-\varepsilon}$, and Hughes [6] sharpened this to $h(n!) \le (2\log 2 + o(1))\, n/\log n$. In 2026, formal verification drove rapid progress: Part (a) was addressed with $h(m) \ll (\log\log m)^2$ by Price and van Doorn, Part (b) ($h(n!) < n^{o(1)}$) was formalized via L^2 flattening on `conjectures.io`, and Part (c) was established with $h(n!) = O((\log n)^9)$ by Howe, Huffman, Kohlmeyer, and Kruer [5] using harmonic dyadic shifts and finite Shannon entropy modulo 2^k. Table 1 summarizes this progression.

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Formal verification repository: https://github.com/vinnor/erdos-18. Human-directed, AI-assisted mathematical research and Lean 4 formalization; see Section 5 for full provenance.

TABLE 1. Progression of upper bounds on $h(n!)$ for Erdős Problem #18.

Reference	Upper Bound on $h(n!)$	Method / Verification
Erdős (1974) [2]	$< n$	Elementary induction
Tenenbaum–Yokota / Yokota [7, 9]	$O(n/(\log n)^{1/2-\varepsilon})$	Decreasing greedy
Hughes (Sep 2026) [6]	$(2\log 2 + o(1))\, n/\log n$	Reciprocal divisor gaps
`conjectures.io` (Sep 2026)	$n^{o(1)}$	Radix + L^2 flattening (claimed, Lean 4)
Howe et al. (Nov 20, 2026) [5]	$O((\log n)^9)$	Harmonic shifts + entropy (Lean 4)
This repo (2-adic milestone)	$O((\log n)^3(\log\log n)^2)$	Convex f_4 + s-prime blocks (Lean 4)
This work (Theorem 1.1)	$\boldsymbol{O((\log n)^3 \log\log n)}$	**Carry-free $\prod \mathbb{F}_q$ + 2-factor smooth pigeonhole (Lean 4)**
Counting lower bound	$\Omega((\log n)^2)$	Information-theoretic floor

Theorem 1.1. *There exists an absolute constant $C > 0$ such that for every integer $n \geq 2$ and every integer $1 \leq t \leq n!$, there exists a finite set S of positive divisors of $n!$ with $\sum_{d\in S} d = t$ and*

$$|S| \leq C\,(\ln n)^3 \ln\ln(n+16).$$

Corollary 1.2. *Unconditionally, $h(n!) = O((\ln n)^3 \ln\ln(n+16)) = O((\ln n)^4)$, hence $h(n!) < n^{o(1)}$ and $h(n!) < (\ln n)^{O(1)}$. Moreover, since $n!$ is practical and $\ln\ln(n!) = \ln n + O(\ln\ln n)$, there are infinitely many practical integers m (namely $m = n!$) satisfying $h(m) = O((\ln\ln m)^3 \ln\ln\ln m) < (\ln\ln m)^4$.*

Notation. Throughout, log and ln both denote the natural logarithm (and $\log_2$ denotes the base-2 logarithm); the shift $+16$ in $\ln\ln(n+16)$ ensures $\ln\ln(n+16) > 1$ for all $n \geq 2$. Write $y\# := \prod_{p\leq y} p$ for the primorial. All entropies are measured in nats: for a distribution p on $\mathbb{Z}/q\mathbb{Z}$, $H_{\mathrm{nat}}(p) := -\sum_{x\in\mathbb{Z}/q\mathbb{Z}} p(x)\ln p(x)$ and $\Delta_q^{\mathrm{nat}}(p) := \ln q - H_{\mathrm{nat}}(p) = D_{\mathrm{KL}}^{\mathrm{nat}}(p \,\|\, u_q)$, where u_q is the uniform distribution on $\mathbb{Z}/q\mathbb{Z}$.

2. Pointwise KL Contraction and Universal $\mathbb{F}_q$ Dilation

Working in $\mathbb{Z}/2^k\mathbb{Z}$ forces binary carry propagation and convex bitwise variance bounds $f_4(u) = u/\ln(4/u)$, losing an extra $\ln(4/g) = \Theta(\log\log n)$ factor. Over a prime field $\mathbb{F}_q$, averaging over multiplicative dilations $c \in \mathbb{F}_q^*$ gives an entropy gain strictly linear in $H_{\mathrm{nat}}(V)$ (`Erdos18/PrimeFieldEntropy.lean`).

Lemma 2.1 (Pointwise KL contraction)**.** *Let $q \geq 3$ be an integer, p any probability distribution on $\mathbb{Z}/q\mathbb{Z}$, and $r \in (0, 1/2]$. Define $T_r(p)(x) := r\,p(x) + (1-r)\frac{1-p(x)}{q-1}$. Then with $c_1 := 1/484$,*

$$D_{\mathrm{KL}}^{\mathrm{nat}}\big(p \,\|\, T_r(p)\big) \;\geq\; \frac{1}{484} \cdot \frac{\min\big(\ln(1/r),\, \ln q\big)}{\ln q} \cdot \Delta_q^{\mathrm{nat}}(p).$$

Proof. Set $\theta := \frac{qr-1}{q-1}$ and $u_x := qp(x) \in [0, q]$, so $\frac{1}{q}\sum_x u_x = 1$ and $qT_r(p)(x) = 1 + \theta(u_x - 1)$. If $r \leq 1/q$, then $\theta \leq 0$ and $\ln(1+z) \leq z$ gives

$$D_{\mathrm{KL}}^{\mathrm{nat}}(p \,\|\, T_r(p)) - \Delta_q^{\mathrm{nat}}(p) = -\sum_x p(x)\ln\big(1+\theta(u_x-1)\big) \geq -\theta\Big(q\sum_x p(x)^2 - 1\Big) \geq 0$$

by Cauchy–Schwarz. If $r \in (1/q, 1/2]$, then $\theta \in (0, 1/2)$ and $\theta < r$. Writing $B(u) := u\ln u - (u-1)$ and $g_\theta(u) := u\ln\frac{u}{\theta u+1-\theta} - (1-\theta)(u-1)$, we have $\Delta_q^{\mathrm{nat}}(p) = \frac{1}{q}\sum_x B(u_x)$ and $D_{\mathrm{KL}}^{\mathrm{nat}}(p \,\|\, T_r(p)) = \frac{1}{q}\sum_x g_\theta(u_x)$. For $u \in [0, 10/\theta]$, $B(1) = B'(1) = g_\theta(1) = g_\theta'(1) = 0$ and $g_\theta''(u) = \frac{(1-\theta)^2}{u(\theta u+1-\theta)^2} \geq \frac{1/4}{121u} = \frac{1}{484}B''(u)$, integrating twice to $g_\theta(u) \geq \frac{1}{484}B(u) \geq \frac{1}{484}\frac{\ln(1/r)}{\ln q}B(u)$ since $B(u) \geq 0$ and $\ln(1/r) < \ln q$. For $u \in (10/\theta, q]$, $u > 20 > e$, $\theta u + 1 - \theta < \frac{11}{10}\theta u$, and $\ln(1/\theta) - (1-\theta) - \ln(11/10) > 0.14\ln(1/\theta) > 0.14\ln(1/r)$, whence $g_\theta(u) > 0.14\,u\ln(1/r) \geq \frac{1}{484}\frac{\ln(1/r)}{\ln q}B(u)$. □

Lemma 2.2 (Universal $\mathbb{F}_q$ dilation entropy gain)**.** *Let $q \geq 3$ be prime, let U, V be independent random variables on $\mathbb{F}_q$, and let W be a probability distribution on $\mathbb{F}_q^*$ satisfying $W(c) \geq c_{\mathrm{dil}}/(q-1)$ for all $c \in \mathbb{F}_q^*$ ($c_{\mathrm{dil}} \geq 0$). Then*

$$\sum_{c \in \mathbb{F}_q^*} W(c)\big(H_{\mathrm{nat}}(U + cV) - H_{\mathrm{nat}}(U)\big) \;\geq\; \frac{c_{\mathrm{dil}}}{2420 \ln q}\left(H_{\mathrm{nat}}(V) - \frac{5 \ln q}{q}\right)\Delta_q^{\mathrm{nat}}(U).$$

Proof. Since $H_{\mathrm{nat}}(U + cV) - H_{\mathrm{nat}}(U) \geq 0$ for every $c \in \mathbb{F}_q^*$, it suffices to treat uniform $W(c) = 1/(q-1)$. Let $p(x) := \mathbb{P}(U = x)$ and $r(y) := \mathbb{P}(V = y)$. If $\max_y r(y) = 1$, then $H_{\mathrm{nat}}(V) = 0$ and the right-hand side is $\leq 0 \leq$ LHS. Otherwise $1 - r(y) > 0$ for all $y \in \mathbb{F}_q$; by the exact Jensen-gap identity for $-t \ln t$ and the substitution $x' = x - cy$,

$$H_{\mathrm{nat}}(U + cV) - H_{\mathrm{nat}}(U) = \sum_{y \in \mathbb{F}_q} r(y) \sum_{x' \in \mathbb{F}_q} p(x') \ln \frac{p(x')}{r(y)p(x') + (1 - r(y))\, t_c(x', y)},$$

where $t_c(x', y) := \frac{1}{1-r(y)} \sum_{z \neq y} r(z)p(x' + c(y - z))$. Because $y - z \in \mathbb{F}_q^*$ for $z \neq y$, averaging over $c \in \mathbb{F}_q^*$ gives $\mathbb{E}_c[t_c(x', y)] = \frac{1-p(x')}{q-1}$, so convexity of $t \mapsto a \ln \frac{a}{ra+(1-r)t}$ yields

$$\mathbb{E}_{c \in \mathbb{F}_q^*}\big[H_{\mathrm{nat}}(U + cV) - H_{\mathrm{nat}}(U)\big] \;\geq\; \sum_{y \in \mathbb{F}_q} r(y)\, D_{\mathrm{KL}}^{\mathrm{nat}}\big(p \,\|\, T_{r(y)}(p)\big).$$

If $r_{\max} := \max_y r(y) \leq 1/2$, applying Lemma 2.1 to all y and separating $\{r(y) < q^{-2}\}$ (at most q terms contributing $\leq \frac{2 \ln q}{q}$) yields the bound. If $r_{\max} = 1 - \epsilon > 1/2$, dropping $y_{\max}$ and applying Lemma 2.1 to $y \neq y_{\max}$ uses $H_{\mathrm{nat}}(V) \leq (1 + 1/\ln 2)S_{\neq} < 2.5 S_{\neq}$ to finish with factor $\frac{1}{5}c_1 = \frac{1}{2420}$. □

3. Elementary 2-Factor Smooth Pigeonhole Dilator

Instead of Dirichlet characters or Pólya–Vinogradov bounds, we construct a nearly uniform multiplicative dilator modulo every large prime $q \in (0.56m, m]$ by elementary pigeonhole on smooth numbers (`Erdos18/PrimeFieldDilation.lean`).

Lemma 3.1 (2-factor smooth pigeonhole dilator)**.** *For $m \geq 1$, define the $0.28m$-smooth set*

$$A(m) := \big\{1 \leq a \leq \lfloor 14m/25 \rfloor : p \mid a \text{ prime} \implies p \leq \lfloor 7m/25 \rfloor\big\}.$$

Then:

1. *For all $m \geq 16$ and $c_1, c_2 \in A(m)$, we have $c_1 c_2 < m^2$ and $c_1 c_2 \mid m!$.*
2. *For all $m \geq 4 \cdot 4^{45}$, $|A(m)| \geq \frac{44}{45}\lfloor 14m/25 \rfloor$.*
3. *For all $m \geq 4 \cdot 4^{45}$, every prime q with $\lfloor 14m/25 \rfloor < q \leq m$, and every $x \in \mathbb{F}_q^*$,*

$$\mathbb{P}_{c_1, c_2 \sim \mathrm{Unif}(A(m))}\big(c_1 c_2 \equiv x \pmod q\big) \;\geq\; \frac{1/40}{q-1}.$$

Proof. For (1), $c_1, c_2 < m$ is immediate, and for any prime $p \leq 7m/25$, $v_p(c_1 c_2) \leq 2\lfloor \log_p(14m/25) \rfloor \leq \lfloor m/p \rfloor \leq v_p(m!)$ (for $p > \sqrt{14m/25}$, $v_p(c_1 c_2) \leq 2 < 3 \leq \lfloor m/p \rfloor$).

For (2), since $2(\lfloor 7m/25 \rfloor + 1) > \lfloor 14m/25 \rfloor$, the complement of $A(m)$ in $[1, \lfloor 14m/25 \rfloor]$ consists solely of the primes in $(7m/25, 14m/25]$. By Erdős's 1932 primorial bound $\prod_{p \leq y} p \leq 4^y$ (`Chebyshev.lean`), the number of such primes is at most $\frac{1}{45}\lfloor 14m/25 \rfloor$ for $m \geq 4 \cdot 4^{45}$.

For (3), $A(m)$ injects into $\mathbb{F}_q^*$ because $14m/25 < q$. For any $x \in \mathbb{F}_q^*$, the sets $A(m)$ and $xA(m)^{-1}$ inside $\mathbb{F}_q^*$ satisfy $|A(m) \cap xA(m)^{-1}| \geq 2|A(m)| - (q-1) \geq q/20$, whence dividing by $|A(m)|^2 \leq (14m/25)^2$ yields $\geq \frac{1/40}{q-1}$. □

4. Carry-Free CRT Covering and Global Assembly

We assemble Theorem 1.1 in `Erdos18/PrimeFieldCovering.lean` and `Erdos18/PrimeFieldMain.lean`.

Proposition 4.1 (Adaptive CRT contraction and mod-M' covering)**.** *For $m \geq 4096$, define the base prime pool $P_{\mathrm{pool}} := \{p \text{ prime} : 2 \leq p \leq m/8\}$ (with $N := |P_{\mathrm{pool}}|$) and the core divisor*

$$M_{\mathrm{core}}(m) := 2^{\lfloor m/4 \rfloor}\, m\#\, \lfloor m/3 \rfloor\#^2\, \lfloor \sqrt{m} \rfloor\#^{2\lfloor \log_2 m \rfloor} \;\Big|\; m!,$$

which satisfies $M_{\mathrm{core}}(m) \leq 2^{6m}$ and is practical for all $m \geq 20$. Dividing $(m/2, m]$ into 16 equal subintervals and choosing the densest subinterval via quantitative Bertrand (`Chebyshev.lean`) produces an upper endpoint $m_{\mathrm{dil}} \in (m/2, m]$ and two additional prime subsets $Q_{\mathrm{tag}}, Q_{\mathrm{strip}} \subset (\lfloor 14 m_{\mathrm{dil}}/25 \rfloor, m_{\mathrm{dil}}]$ pairwise disjoint from P_{pool} and from each other, with $12 \leq |Q_{\mathrm{strip}}| \leq \lfloor N/2 \rfloor$ and $M_{\mathrm{strip}} := \prod_{q \in Q_{\mathrm{strip}}} q$. For any nonempty subset $Q' \subseteq Q_{\mathrm{strip}}$ with $M' := \prod_{q \in Q'} q$:

(1) (Adaptive single-step contraction) *For every distribution X on $\mathbb{Z}/M'\mathbb{Z}$, ordering Q' by decreasing conditional coordinate deficit via the CRT chain rule and choosing the subset size $r \in \{1, \ldots, |Q'|\}$ (with top-r coordinate subset $Q_S \subseteq Q'$) greedily yields a divisor law $Y = p_{\mathrm{tag}} c_1 c_2 M_{\mathrm{comp}} Z_r \mid M_{\mathrm{core}}(m)$—where $c_1, c_2 \in A(m_{\mathrm{dil}})$, $M_{\mathrm{comp}} := \prod_{q \in Q' \setminus Q_S} q$ zeros out all coordinates outside Q_S, and Z_r is uniform over squarefree r-prime products from P_{pool}—satisfying*

$$H_{\mathrm{nat}}(X + Y) - H_{\mathrm{nat}}(X) \;\geq\; \eta\, \Delta^{\mathrm{nat}}_{M'}(X), \qquad \eta \geq \frac{1}{193600\, \ln m\, (\ln\ln(m+16) + 4)}.$$

Here the $\ln\ln(m{+}16)$ factor arises solely from the injective harmonic sum $\sum_{r=1}^{\lfloor N/2 \rfloor} \frac{1}{r \ln(N/r)} = \ln\ln N + O(1)$.

(2) (Mod-M' covering) *Iterating (1) across $2R(m) = O((\ln m)^2 \ln\ln(m+16))$ draws with pairwise distinct tag primes $p_{\mathrm{tag}} \in Q_{\mathrm{tag}}$ (using $\Delta^{\mathrm{nat}}_{M'}(X_0) \leq \ln M_{\mathrm{core}}(m) \leq 6m \ln 2$) covers every residue class modulo M' as a sum of at most $2R(m)$ strictly distinct positive divisors of $M_{\mathrm{core}}(m)$, each $< m^3 M'$ and not divisible by M'.*

(3) (Exact carry-free digit representation) *Every $r \in [1, M_{\mathrm{strip}}/m^5)$ is a sum of at most $2R(m) + 6\log_2 m + 8$ strictly distinct positive divisors of $M_{\mathrm{core}}(m)$: if $r < 2m^6$, write r directly in binary using powers of 2 dividing $M_{\mathrm{core}}(m)$; if $2m^6 \leq r < M_{\mathrm{strip}}/m^5$, greedily pick a nonempty subset $Q' \subseteq Q_{\mathrm{strip}}$ whose product $M' := \prod_{q \in Q'} q$ satisfies $M' \leq \lfloor r/(2m^5) \rfloor < mM'$, subtract a mod-$M'$ cover S_1 from (2) (with total sum $< m^5 M' \leq r/2 < r$), and write the nonnegative quotient $(r - \sum S_1)/M' < 2m^6$ directly in binary as a sum of terms $2^b M'$.*

Proof of Theorem 1.1. For n below the explicit threshold m_{pf}, apply the finite base case (or Milestone 3). For $n \geq m_{\mathrm{pf}}$:

(1) *Stage 1 (Mixed-radix reduction):* Greedily factor $n!/M_{\mathrm{core}}(n) = D_1 \cdots D_L$ with $D_j \leq M_{\mathrm{strip}}(n)/n^5$ and $L = O(\ln n)$. Each mixed-radix digit is represented inside disjoint value intervals by Proposition 4.1(3), costing $L \cdot O((\ln n)^2 \ln\ln n) = O((\ln n)^3 \ln\ln n)$ divisors.

(2) *Stage 2 (Self-similar tail recursion $m \mapsto m/2$):* Represent $x < M_{\mathrm{core}}(m)$ by strong induction on m across three exhaustive branches:

- *Branch A* ($x < M_{\mathrm{strip}}/m^5$): direct call to Proposition 4.1(3).
- *Branch B* ($M_{\mathrm{strip}}/m^5 \leq x < m^5 M_{\mathrm{strip}}$): strip 11 primes from Q_{strip} to form $M'' \mid M_{\mathrm{strip}}$ with $M_{\mathrm{strip}}/M'' > m^{10}$, cover x mod M'' via Proposition 4.1(2) (sum $< m^5 M'' < x$), and write the quotient $(x - S_1)/M'' < m^{16} < 2^{m/4}$ directly in binary powers $2^b M''$, returning immediately.
- *Branch C* ($x \geq m^5 M_{\mathrm{strip}}$): cover x mod M_{strip} via Proposition 4.1(2), divide by M_{strip}, factor the smooth excess $U_{\mathrm{extra}} = M_{\mathrm{core}}(m)/(M_{\mathrm{strip}}\, M_{\mathrm{core}}(m/2))$ into $O(1)$ digits handled by Proposition 4.1(3), and recurse on $M_{\mathrm{core}}(m/2)$.

All three divisor groups are pairwise disjoint across $O(\ln n)$ recursion steps, contributing $O((\ln n)^3 \ln\ln(n+16))$ divisors in total.

Combining Stages 1 and 2 proves Theorem 1.1, and Corollary 1.2 follows immediately. □

Remark 4.2 (Natural barrier at $O((\log n)^3 \log\log n)$). Only two factors separate Theorem 1.1 from the counting floor $\Omega((\log n)^2)$: the injective harmonic factor $\ln\ln N$ in Proposition 4.1(1) and the $\ln(2m)$ contraction tail in Proposition 4.1(2). Pushing below $O((\log n)^3 \log\log n)$ across 2-adic, prime-field, and Archimedean completions appears to require either super-polynomial modular non-collapse/equidistribution lying beyond GRH and RH, or genuinely non-Fourier multi-term additive energy bounds for divisor sumsets.

5. Formal Verification in Lean 4 and Provenance

The entire proof is formally verified in Lean 4.36.0 with Mathlib4 (13,586 lines across 13 files, including 5,732 lines across the four new `PrimeField*.lean` modules) in the public repository https://github.com/vinnor/erdos-18. The entry point `erdos18_lean/Verify.lean` locks all four top-level theorem statements (`check_alpha5`, `check_alpha4`, `check_alpha3_loglog2`, and `check_alpha3_loglog1`) against definition shadowing via explicit `_root_` references and confirms via `#print axioms` that every theorem depends solely on `[propext, Classical.choice, Quot.sound]` with zero `sorry`, zero `admit`, and zero `native_decide`. Independent GitHub Actions CI compiles the project from scratch and executes the accompanying Python verification and adversarial stress suites.

Provenance Statement. This work was produced through human-directed, AI-assisted mathematical research and Lean 4 formalization (using Google Gemini models under human problem selection, multi-wave research architecture design, independent adversarial auditing gates, and CI verification). The Lean 4 kernel is the sole formal arbiter of correctness, and this manuscript presents the self-contained mathematical proof independently of the software tooling.

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