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# ROBIN-TYPE CRITERIA FOR DIRICHLET, DEDEKIND, AND AUTOMORPHIC RIEMANN HYPOTHESES WITH FORMAL PROOFS IN LEAN

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Abstract. We prove Robin-type arithmetic equivalences for generalized Riemann hypotheses and verify their proofs in Lean at the specified analytic interfaces. For every number field $K$, Dedekind RH is equivalent to an eventual bound on the normalized ideal divisor sum. The main term is $e^{\gamma}\kappa_K \log\log N\mathfrak{a}$, with correction $e^{\gamma}\kappa_K(12+3\log|D_K|)/\sqrt{\log N\mathfrak{a}}$. Here $\kappa_K$ is the residue of $\zeta_K$ at 1. Lean checks this equivalence and the cyclotomic factorization that yields a criterion for all Dirichlet $L$-functions. For a fixed field we determine the sharp eventual ideal coefficient as a completed logarithmic derivative minus $2\sqrt{2}-2$.

For standard $L$-functions of unitary cuspidal representations of $\mathrm{GL}_n$, a one-sided square-root bound on a triangular prime-power sum characterizes critical-line location. Its converse uses Landau's theorem and the sign of a possible real pole. The formal proof isolates the analytic data used by this argument. Lean constructs those data for the actual rank-one Hecke family over every number field, including Dedekind norm twists. For higher ranks the Lean equivalence takes the Euler, continuation, growth, and functional-equation conditions as hypotheses. The paper supplies them for actual standard functions using cited analytic theorems. We express the sharp eventual triangular coefficient by a critical-line zero sum. Both sharp constants are proved in the manuscript and lie outside the Lean verification. The same coefficients give conditional tests for simplicity and prescribed multiplicity. The published verification compares and replays six location and four multiplicity equivalences. We give their exact statement types, pinned dependencies and recorded checker outputs, and describe an external replay that combines three of the arithmetic results with the Alpöge–Claude zero-location proofs.

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## 1. Introduction

This paper gives new arithmetic criteria for critical-line location and zero orders, together with a Lean proof development for the criteria. The formalization covers the full fieldwise ideal-divisor equivalence, its Dirichlet and Dedekind consequences, and the Mellin implication that connects the triangular prime-power bound to zero location. This makes it possible to see precisely which analytic inputs are needed in order to apply the same argument to another family of $L$-functions.

Robin's criterion detects the Riemann hypothesis through the sizes of divisor sums [46]. We extend the divisor-sum problem to integral ideals of a number field $K$, using the absolute ideal norm and the residue of $\zeta_K$ at 1. A logarithmic inequality for each local divisor factor allows us to charge the cost of a prime power against a single norm budget. This cancels the boundary term in the number-field Mertens formula. In the converse direction, an ideal formed from all prime powers up to a given norm recovers the same weighted prime-ideal error. The resulting

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fieldwise equivalence gives the Dirichlet GRH criterion on cyclotomic fields and the Dedekind ERH criterion on all number fields.

For the standard $L$-functions of unitary cuspidal automorphic representations, conjugation makes the paired logarithmic coefficients real, and their signs can vary. We use these coefficients in a triangular Riesz sum over prime powers. The same one-sided Mellin argument applies to its remainder and to the divisor-sum error. The key real poles have a positive residue before subtraction from a nonnegative comparison function. This sign also rules out a real exceptional zero in the relevant half-plane. Section 8 compares the fixed-weight equivalence with earlier smoothed prime-counting bounds and zero-location criteria.

1.1. **Notation and proof roadmap.** All ideal sums use nonzero integral ideals of $\mathcal{O}_K$. A prime-power sum is indexed by $(\mathfrak{p}, j)$, with $\mathfrak{p}$ a nonzero prime ideal and $j \geq 1$. Write $q = N\mathfrak{p}$ when working at one prime. All logarithms are natural.

| Notation | Meaning |
|---|---|
| $N\mathfrak{a}$, $\sigma_K(\mathfrak{a})$ | Absolute norm and sum of the norms of the integral divisors of $\mathfrak{a}$. |
| $D_K$, $\kappa_K$, $\Xi_K$ | Field discriminant, residue of $\zeta_K$ at 1, and pole-cleared Dedekind completion. |
| $\psi_K(x)$, $E_K(x)$ | $\psi_K(x) = \sum_{q^j \leq x} \log q$ over prime-power indices and $E_K(x) = \psi_K(x) - x$. |
| $H_0(x)$, $H(x)$ | Negative tails of $E_K(t)/t^2$ and of $E_K(t)(1+\log t)/(t^2 \log^2 t)$, respectively, from $x$ to infinity. The field is fixed in these formulas. |
| $\mathcal{U}_n(K)$, $\mathcal{U}_n^*(K)$ | Unitary cuspidal $\mathrm{GL}_n$ family and the family used for pairing. Only the pure norm characters in rank one are removed. |
| $\mathcal{T}_K$ | Tests $Z$, consisting of $\zeta_K$ and the products $L(\cdot,\pi)L(\cdot,\widetilde{\pi})$ for $\pi \in \mathcal{U}_n^*(K)$. |
| $m_Z$, $b_{\mathfrak{p},j}(Z)$ | Pole order of $Z$ at 1 and coefficient of $q^{-js}$ in the prime-power expansion of $-Z'/Z$. |
| $R_Z(x)$ | $m_Z x/2$ minus the sum of $b_{\mathfrak{p},j}(Z)(1 - q^j/x)$ over $q^j \leq x$. |
| $F_Z$, $\Lambda_\pi$ | Ordinary factor $(s-1)^{m_Z} Z(s)$ and pole-cleared standard completion of the individual representation. |
| $\mathcal{M}$, $m_\gamma$ | Mellin transform and the order at $1/2 + i\gamma$, respectively. In an ordinate subscript $\gamma$ denotes a frequency. |
| $\sigma(n)$, $h$, $\mathfrak{r}(n)$, $H_n$ | Integer divisor sum, classical log height $h = \log n$, logarithmic Robin margin, and $n$-th harmonic number. |

Section 2 gives the positive-transform argument. Section 3 applies it to the standard prime-power data and states the analytic inputs separately. Section 4 specializes the prime-power error to Dedekind zeta and transfers it to ideal divisor sums. Section 5 then uses cyclotomic factorization for Dirichlet GRH and the fieldwise quantifier for Dedekind ERH. Section 6 relates the rational case to the exact classical criteria, and Section 7 uses the individual coefficients to test zero orders. Sections 3.4 and 4.4 prove sharp asymptotic coefficients for the triangular and ideal Robin bounds. Section 9 records the verification evidence and exact reproduction inputs.

1.2. **The formal proof contribution.** The main formal results establish equivalences whose left sides quantify over the relevant zeros and whose right sides state the arithmetic inequalities. In the Dirichlet case the public theorem starts with every complex Dirichlet character of every positive modulus. The proof passes from those $L$-functions to the cyclotomic fields using an internal factorization theorem. The all-field result applies the Dedekind proof to each field, with

its own cutoff. The classical Robin, colossally abundant Robin, and Lagarias equivalences give three further public location theorems.

For the standard automorphic family the Lean interface separates the data of the Euler products and completions from the theorem about their zeros. The proof uses the local finiteness and summability of the prime-power data to obtain a Mellin transform. The growth bounds and functional equation control the zero expansion and reflection. The sign of the residue controls the real-pole case in the converse. The library instantiates this argument over every number field for unitary Hecke characters. The construction obtains the local coefficients from ideles, constructs the completed functions by Tate integrals and adelic Poisson summation, and treats the Dedekind norm twists by translation. For the actual higher-rank standard family, Section 3 supplies the mathematical argument using cited analytic results. The corresponding Lean equivalences take those analytic conditions as hypotheses.

The four multiplicity theorems assume the appropriate location condition and then characterize simplicity or prescribed zero orders by prime-power inequalities. The proof treats individual functions for their orders and uses contragredient pairs for the real-valued location remainder. The latter pairing can add orders at a common zero, which is why the two coefficient constructions are kept separate in the formal proof.

The `Challenge.lean` file states the ten headline equivalences over definitions built from Mathlib with proof placeholders. The `Solution.lean` gives the proofs. The published verification run compares the statement types and their fixed definitions, checks dependencies for allowed axioms, and replays the proofs using Lean's kernel and the independent NanoDa checker. Section 9 maps the ten results to the exposition, and Appendix B prints the exact types. The sections that follow link the principal mathematical steps to the proofs that use them. In a separate verification, Andrew Sutherland composed the exact RH and Dirichlet GRH proofs with an earlier revision of this development to derive three unconditional arithmetic bounds. His public record and its scope are described in Section 9.3.

The argument uses number-field Mertens theorems [17] and Landau's theorem for nonnegative transforms [39].

1.3. **The triangular sum in the zeta case.** Start with $K = \mathbb{Q}$ and $Z = \zeta$. A prime power $p^j$ enters the Chebyshev count $\psi(t)$ as a step of height $\log p$ at $t = p^j$. Its contribution to the average of $\psi$ up to $x$ is $\log p\,(x - p^j)/x$ when $p^j \le x$. For instance, at $x = 6$ the prime powers are $2, 3, 4, 5$, so the average and the remaining error are

$$\frac{1}{6}\int_0^6 \psi(t)\,dt = \frac{2}{3}\log 2 + \frac{1}{2}\log 3 + \frac{1}{3}\log 2 + \frac{1}{6}\log 5, \qquad R_\zeta(6) = 3 - \frac{1}{6}\int_0^6 \psi(t)\,dt.$$

The two copies of $\log 2$ come from different powers and have different weights. The weight of an entering power starts at zero, so the smoothed sum is continuous even though $\psi$ jumps. If $\psi(t)$ were exactly $t$, the average would be $x/2$. Thus $R_\zeta(x)$ measures the shortfall of cumulative prime-power weight relative to that model. Bounding that shortfall from above is the one-sided requirement.

The completed zeta function supplies the reflection used in the proof. The factor $\pi^{-s/2}\Gamma(s/2)$ adds the archimedean local factor. The product with $\zeta(s)$ has simple poles at 0 and 1. Multiplying by $s(s-1)$ removes those poles and gives the entire function $\Xi_{\mathbb{Q}}(s)$ in Section 4. Its zeros are the nontrivial zeta zeros, and $\Xi_{\mathbb{Q}}(1-s) = \Xi_{\mathbb{Q}}(s)$. Once the Mellin argument excludes zeros to the right of $1/2$, this identity excludes zeros to the left.

The common starting point for a general test is the logarithmic derivative of an Euler product. Its coefficients count prime powers with their local weights. Giving the term of norm $v$ the weight $1 - v/x$, for $v \le x$, has a precise analytic effect. Its Mellin transform contributes the factor $v^{-s}/(s(s+1))$. For the zeta functions the pole at 1 also contributes the main term $x/2$.

Subtracting the triangular sum from this term produces $R_Z(x)$ in (3.2). Formula (3.3) relates its transform directly to $Z'/Z$. The denominator $s(s+1)$ makes the contributions of the nontrivial zeros absolutely summable after inversion.

If the zeros lie on the critical line, the term from a zero $\rho$ is proportional to $x^\rho/(\rho(\rho+1))$. Its dependence on $x$ has magnitude $\sqrt{x}$. Summing with multiplicity gives the square-root estimate. Conversely, an eventual bound $R_Z(x) \le C\sqrt{x}$ makes $C\sqrt{x} - R_Z(x)$ nonnegative beyond a cutoff. A zero to the right of the line would create a pole of its transform. Landau's theorem would then force a singularity on the real axis at the abscissa of convergence. Every possible real pole there has negative residue for this nonnegative comparison function, because $Z'/Z$ has positive residue at a real zero and $R_Z$ is subtracted. The transform would approach $-\infty$ along the real axis, contradicting nonnegativity. The completed functional equation excludes zeros to the left by reflection.

Transfer to ideal divisor sums. For $Z = \zeta_K$, each local parameter of the trivial Hecke character is 1. The coefficient at $\mathfrak{p}^j$ is therefore $\log N\mathfrak{p}$, and the pole order is one. In this specialization, put $E_K(u) = \psi_K(u) - u$, where $\psi_K(u)$ sums $\log N\mathfrak{p}$ over all prime ideals $\mathfrak{p}$ and powers $j \ge 1$ with $(N\mathfrak{p})^j \le u$. For $x \ge 1$ the exact identity is

$$R_{\zeta_K}(x) = \frac{x}{2} - \frac{1}{x}\int_1^x \psi_K(u)\,du = \frac{1}{2x} - \frac{1}{x}\int_1^x E_K(u)\,du.$$

Thus the triangular test bounds the primitive of the very same prime-ideal error that appears in the ideal Mertens formula.

The divisor-sum criterion expresses that zero condition through a different arithmetic quantity. Write the normalized ideal divisor sum as a product of one geometric sum for each prime ideal dividing the ideal. Taking logarithms, the local inequalities bound this product using the single budget $\log N\mathfrak{a}$. In the Mertens formula (4.2), the budget absorbs the boundary term $E_K(x)/(x\log x)$. The remaining integral of $E_K$, together with the norm-budget comparison, yields the correction in (4.1). For the converse, take the integral ideal with one extra factor $\mathfrak{p}$ for each prime power $(N\mathfrak{p})^j \le x$. Its logarithmic norm is $\psi_K(x)$. The ideal bound on this particular family gives a one-sided bound on the weighted prime-ideal error, so the same Mellin and residue argument recovers Dedekind RH. The triangular bound and the ideal bound are therefore equivalent for each fixed field.

Taking every number field gives Dedekind ERH. Restricting the fields to $\mathbb{Q}(\zeta_m)$ gives Dirichlet GRH because the zeta function of each cyclotomic field factors into the primitive Dirichlet $L$-functions belonging to its characters. Over $\mathbb{Q}$, the integral ideal $(n)$ has norm $n$ and its ideal divisor sum is $\sigma(n)$. Here the residue and the absolute discriminant are both 1, so the field criterion reads

$$\frac{\sigma(n)}{n} \le e^\gamma \left(\log\log n + \frac{12}{\sqrt{\log n}}\right) \qquad \text{for all sufficiently large } n.$$

This rational specialization is equivalent to RH. Robin's theorem [46] identifies the same condition with the strict uncorrected inequality for every $n > 5040$. Section 6 states the exact classical criteria and points to their proofs.

| Proof transition | Mathematical step and its Lean counterpart |
|---|---|
| Prime powers $\longleftrightarrow$ zero location | Transform the triangular weights and apply the zero-pole test (Sections 2 and 3). Lean proves the kernel and the converse from supplied analytic data. |
| Dedekind zeros $\longleftrightarrow$ ideal divisor sums | Use the Mertens boundary term, a norm budget and least common multiples of ideals (Section 4). Lean proves the fixed-field equivalence. |
| Cyclotomic and all-field tests | Use cyclotomic factorization or quantify the fixed-field equivalence over all number fields (Section 5). Both transfers are formalized. |
| Location assumed $\longrightarrow$ order criterion | Change from pairs to individual factors and recover the zero orders by Fourier averaging (Section 7). Lean verifies the four order equivalences listed in Section 9. |

## 2. A one-sided Mellin criterion

For a real, locally integrable function $U$ on $[1,\infty)$, write

$$\mathcal{M}U(s)=\int_1^\infty U(x)x^{-s-1}\,dx$$

where the integral converges absolutely.

**Lemma 2.1** (Landau's theorem and the sign of a residue)**.** *Suppose $\mathcal{M}U$ converges absolutely in some right half-plane and extends meromorphically to $\Re s>\alpha$, with every real pole there simple and of positive residue. If $U(x)\le Cx^\alpha$ for all sufficiently large $x$, then $\mathcal{M}U$ has no pole in $\Re s>\alpha$.*

*Proof.* Choose $X>1$ beyond the cutoff and consider

$$G(x)=\mathbf{1}_{[X,\infty)}(x)(Cx^\alpha-U(x))\ge 0.$$

Its transform continues meromorphically to $\Re s>\alpha$ as

$$\mathcal{M}G(s)=\frac{CX^{\alpha-s}}{s-\alpha}-\mathcal{M}U(s)+\int_1^X U(x)x^{-s-1}\,dx.$$

The integral over the compact interval is entire. If a pole of $\mathcal{M}U$ lay to the right of $\alpha$, the abscissa of convergence $\sigma_c$ of the transform of $G$ would be at least the real part of that pole. Indeed, the convergent transform is holomorphic to the right of $\sigma_c$, and agrees there with the meromorphic continuation by uniqueness. Now $\sigma_c>\alpha$. Landau's theorem [39], applied to the Laplace transform after $x=e^u$, forces a singularity at the real point $\sigma_c$. The continuation can only have a simple pole there and its residue is negative. As $s\downarrow\sigma_c$ on the real axis the transform would tend to $-\infty$, contradicting its nonnegative integral. □

For the inversions below, it suffices to compare absolutely convergent transforms on $\Re s=2$. Under $x=e^u$ those transforms are Fourier transforms of $L^1$ functions on $u\ge 0$. Uniqueness gives equality almost everywhere, which gives pointwise equality whenever the two expressions are continuous.

## 3. The triangular criterion for the standard automorphic family

Grand RH is the assertion that every nontrivial zero of every automorphic $L$-function lies on $\Re s=1/2$. The equivalence proved in this section concerns the standard cuspidal family. We use the unitary normalization of the standard $L$-function $L(s,\pi)$ of a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$, with $n\ge 1$ and $K$ any number field. Its nontrivial zeros are the

zeros of the pole-cleared completed standard function. The completed functional equation is centered at $1/2$. Write $\widetilde{\pi}$ for the contragredient. A pure imaginary norm character $|\cdot|_{\mathbb{A}_K}^{it}$ of $\mathrm{GL}_1$ has $L(s,\pi)=\zeta_K(s+it)$; its zero condition is the same as that of $\zeta_K$ by vertical translation.

For a fixed number field $K$, let $\mathcal{U}_n(K)$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_K)$, up to isomorphism. Put $\mathcal{U}_n^*(K)=\mathcal{U}_n(K)$ for $n\geq 2$ and

$$\mathcal{U}_1^*(K)=\mathcal{U}_1(K)\setminus\{|\cdot|_{\mathbb{A}_K}^{it} : t\in\mathbb{R}\}.$$

Define the test family over $K$ by

$$\mathcal{T}_K=\{\zeta_K\}\ \cup\ \{L(\cdot,\pi)L(\cdot,\widetilde{\pi}) : n\geq 1,\ \pi\in\mathcal{U}_n^*(K)\}.$$

The family $\mathcal{T}_K$ consists of the Euler products to which we apply a real inequality. The contragredient conjugates the local parameters, so the product adds the conjugate logarithmic coefficients and gives a real sum. A zero of either entire completed factor is a zero of the pair. Thus the pairing makes the arithmetic test real while retaining the zeros that the test is meant to detect. The superscript $*$ changes only the tests. The global pole classification explains the exclusion. In rank one, a character trivial on the norm-one idele classes descends along the idelic norm to a continuous unitary character of $\mathbb{R}_{>0}$. These characters are $r\mapsto r^{it}$, $t\in\mathbb{R}$. For them $L(s,\chi_t)=\zeta_K(s+it)$, and clearing the translated poles gives the completion $\Xi_K(s+it)$. Vertical translation preserves real parts, so the single test $\zeta_K$ covers their entire zero-location assertion. Tate's theory gives an entire completion for a character nontrivial on norm-one classes [13]. This includes characters ramified at a finite prime as well as characters unramified at all finite primes that are nontrivial on norm-one classes. All these pole-free characters enter through the pair with their inverse. For $n\geq 2$, the unitary cuspidal standard completion is entire by Godement–Jacquet, so every representation enters through its contragredient pair.

For each nonzero prime ideal $\mathfrak{p}$ put $q=N\mathfrak{p}$. Use the actual standard local factor

$$L_{\mathfrak{p}}(s,\pi)=P_{\pi,\mathfrak{p}}(q^{-s})^{-1},\qquad P_{\pi,\mathfrak{p}}(X)=\prod_{k=1}^{d_{\pi,\mathfrak{p}}}(1-\alpha_{\pi,\mathfrak{p},k}X).$$

The polynomial has constant coefficient 1 and degree $d_{\pi,\mathfrak{p}}\leq n$. Its parameters are the Satake parameters when $\pi_{\mathfrak{p}}$ is unramified. At a ramified prime the polynomial is the standard local polynomial of $\pi_{\mathfrak{p}}$. In the Weil–Deligne description it is the determinant of $1-X\,\mathrm{Fr}_{\mathfrak{p}}$ on the inertia invariants in the kernel of monodromy, with Frobenius normalized to give the local Euler factor [32, § 1]. Padding the list to length $n$ by zeros preserves the polynomial. A ramified Hecke character has $P_{\pi,\mathfrak{p}}=1$, but ramified representations in higher rank can contribute nonzero parameters.

Define $a_{\pi,\mathfrak{p},j}=\sum_{k=1}^{d_{\pi,\mathfrak{p}}}\alpha_{\pi,\mathfrak{p},k}^{j}$ for every $j\geq 1$, at ramified and unramified primes alike. The series for $-\log P_{\pi,\mathfrak{p}}(X)$ at $X=0$ has coefficient $a_{\pi,\mathfrak{p},j}/j$. With $X=q^{-s}$, negative logarithmic differentiation of $L_{\mathfrak{p}}$ gives the coefficient $(\log q)a_{\pi,\mathfrak{p},j}$. Unitarity identifies the local contragredient with the complex conjugate representation. The two standard polynomials therefore have conjugate coefficients. For $Z\in\mathcal{T}_K$, its pole order at 1 and the logarithmic coefficients of its local factors are

$$(m_Z, b_{\mathfrak{p},j}(Z))=\begin{cases}(1,\log q), & Z=\zeta_K,\\ (0,2(\log q)\Re a_{\pi,\mathfrak{p},j}), & Z=L(\cdot,\pi)L(\cdot,\widetilde{\pi}).\end{cases}$$

In the second line $2\Re a_{\pi,\mathfrak{p},j}=a_{\pi,\mathfrak{p},j}+a_{\widetilde{\pi},\mathfrak{p},j}$. The proof below justifies the absolutely convergent identity

$$-\frac{Z'}{Z}(s)=\sum_{\mathfrak{p}}\sum_{j\geq 1}b_{\mathfrak{p},j}(Z)(N\mathfrak{p})^{-js},\qquad \Re s>1. \tag{3.1}$$

The index is a pair $(\mathfrak{p}, j)$, so two pairs with equal norm both contribute.
What happens at a ramified Hecke prime. In degree one let $\chi_{\mathfrak{p}}$ be the local Hecke character. If it is trivial on the local units, its value $\alpha = \chi_{\mathfrak{p}}(\varpi)$ on a uniformizer is independent of the uniformizer and the Euler factor is $(1 - \alpha q^{-s})^{-1}$. For a unitary character, writing $\alpha = e^{i\theta}$ gives $2(\log q)\cos(j\theta)$ for the paired coefficient. Local reciprocity identifies the local units with the image of inertia in the abelianized local Weil group. If the character is nontrivial on the units, its one-dimensional parameter has no nonzero inertia-fixed vector. The determinant defining the local polynomial is then on a zero-dimensional space. Thus $P_{\chi,\mathfrak{p}} = 1$, so the logarithmic derivative of this local factor is zero at every prime power. The inverse character is ramified there as well, giving $b_{\mathfrak{p},j} = 0$ for the pair at all $j \geq 1$. The character remains part of the test family, and its ramification still enters the conductor in the completion. This is Tate's local factor [13], implemented by the zero branch of `heckeLocalParameter`. The paired coefficient always counts both factors, including when a representation is self-dual.
A signed test over $\mathbb{Q}$. The nontrivial character $\chi_4$ modulo 4 is the quadratic Hecke character with $\chi_4(3) = -1$, $\chi_4(5) = 1$, and ramification at 2. It equals its inverse, so its test is $Z = L(\cdot, \chi_4)^2$. The logarithmic derivative of the square doubles the individual coefficients. At $x = 6$ only the primes 3 and 5 contribute. The powers of 2 have zero coefficient, and $m_Z = 0$ because the factor is pole-free. Thus the signed remainder at this point is

$$R_Z(6) = -\left(-2\log 3\left(1 - \frac{3}{6}\right) + 2\log 5\left(1 - \frac{5}{6}\right)\right) = \log 3 - \frac{1}{3}\log 5.$$

This example uses both signs in the same test. The conductor 4 and the odd archimedean character enter the individual completion through $4^{s/2}\pi^{-(s+1)/2}\Gamma((s+1)/2)$. The completion is entire and gives the reflection of the zeros. Squaring preserves their locations and doubles their orders, which is why the location test can use the pair while the multiplicity test uses individual factors.

**Theorem 3.1** (Triangular criterion for standard automorphic $L$-functions)**.** *Write $\Lambda_\pi$ for the pole-cleared completed standard function. With $K$ ranging over number fields, the following conditions are equivalent.*

(i) *For every $K$, every $n \geq 1$, every $\pi \in \mathcal{U}_n(K)$, and every $s \in \mathbb{C}$, $\Lambda_\pi(s) = 0$ implies $\Re s = 1/2$.*
(ii) *For every $K$ and every $Z \in \mathcal{T}_K$, there exist $X_{K,Z} \in [1, \infty)$ and $C_{K,Z} \in \mathbb{R}$ such that for every real $x \geq X_{K,Z}$,*

$$R_Z(x) := \frac{m_Z x}{2} - \sum_{\substack{\mathfrak{p},\ j \geq 1 \\ (N\mathfrak{p})^j \leq x}} b_{\mathfrak{p},j}(Z)\left(1 - \frac{(N\mathfrak{p})^j}{x}\right) \leq C_{K,Z}\sqrt{x}. \tag{3.2}$$

A term at equality $(N\mathfrak{p})^j = x$ vanishes. The strict cutoff in the Lean sum consequently gives the same value. The formal predicate `for a single test` quantifies over a real cutoff at least 1, a real constant, and all $x$ above the cutoff. Both witnesses are inside the quantifier over the family index, allowing dependence on $K$ and on the tested representation or on the Dedekind test.

3.1. **Analytic input for the standard family.** We use the standard continuation and functional equation of Godement–Jacquet [8, Theorem 13.8] and Tate in degree one [13], with the standard unitary normalization and growth stated in [33, § 2.1]. Jacquet–Shalika [11, Part I, Theorem 5.3] gives absolute convergence on $\Re s > 1$ for the partial Rankin–Selberg product omitting ramified and infinite places. On the set $I$ of unramified prime-power indices $(\mathfrak{p}, j)$,

abbreviate $a_{\pi,\mathfrak{p},j}$ as $a_{\mathfrak{p},j}$. The corresponding self Rankin–Selberg coefficients are $|a_{\mathfrak{p},j}|^2$, so at every real $\sigma > 1$,

$$\sum_I \frac{|a_{\mathfrak{p},j}|}{jq^{j\sigma}} \le \left(\sum_I \frac{|a_{\mathfrak{p},j}|^2}{jq^{j\sigma}}\right)^{1/2} \left(\sum_I \frac{1}{jq^{j\sigma}}\right)^{1/2} < \infty, \qquad q = N\mathfrak{p}.$$

For the first factor the local logarithm at real $\sigma$ has nonnegative coefficients $|a_{\mathfrak{p},j}|^2/j$. The local parameter bound [33, (2.2)] is strictly below $q^{1/2}$, so the local Taylor series converges even for products of two parameters. The sum of these nonnegative local logarithms is the logarithm of the convergent partial Euler product. The second factor is bounded by $(\log \zeta_K(\sigma))^{1/2}$. To pass to the differentiated series on $\Re s \ge \sigma > \sigma_0 > 1$, multiply the absolute log-series majorant at $\sigma_0$ by $ve^{-(\sigma-\sigma_0)v}$ with $v = \log(q^j)$. This multiplier is bounded by $1/(e(\sigma - \sigma_0))$. At a ramified prime put $r_{\mathfrak{p}} = q^{-\sigma} \max_k |\alpha_{\pi,\mathfrak{p},k}|$, with maximum zero for an empty list. The parameter estimate in [33, (2.2)] gives $r_{\mathfrak{p}} < 1$. The local absolute logarithm is bounded by $n \sum_{j\ge1} r_{\mathfrak{p}}^j/j$, and its derivative series by $n(\log q)r_{\mathfrak{p}}/(1 - r_{\mathfrak{p}})$. There are only finitely many ramified primes. Thus the logarithm and logarithmic derivative converge absolutely, locally uniformly on $\Re s > 1$, and exponentiation gives nonvanishing there.

In the same notation, the archimedean factors are products of $\Gamma_{\mathbb{R}}(s + \mu)$ and $\Gamma_{\mathbb{C}}(s + \mu)$, with $\Re\mu > -1/2$ by [33, (2.2)]. The conductor factor is $Q^{s/2}$ with $Q > 0$. The pole-cleared completion has order at most one and satisfies the functional equation with a nonzero root number, as recorded in [33, § 2.1]. For a pair, multiply the completions of the two individual factors. For the separate test $Z = \zeta_K$, use $\Xi_K$. This gives a pole-cleared completed function

$$s^m Q_Z^{s/2} G_Z(s) F_Z(s), \qquad F_Z(s) = (s-1)^m Z(s),$$

where $G_Z$ is a product of the indicated Gamma factors. The cleared ordinary function $F_Z$ is entire of order at most one. Indeed, the completed function and the reciprocal Gamma and conductor factors have order at most one. Division by $s^m$ has this same growth bound outside the unit disk, and the singularity at 0 is removable for $F_Z$. On $\Re s > 1/2$, every completion factor in this display is finite and nonzero. The functional equation sends a completed zero at $s$ to a completed zero of the dual at $1 - s$. Thus the standard-family critical-line condition is equivalent to nonvanishing of each $F_Z$ on $\Re s > 1/2$. In the paired case, a zero of either factor gives a zero of their product, since both factors are entire.

3.2. **Analytic hypotheses of the Lean theorem.** The quantified number-field and representation assertion of Theorem 3.1 uses the actual standard functions. The independently replayed Lean declaration `for a supplied family` has a different input interface. Fix any index type, with countably many prime-power slots for each index $i$. An index can carry the field and a representation. Supply completed functions $\Lambda_i$, dual indices $i^\vee$, and data $N_{i,k}$, $b_{i,k}$, $m_i$, and $F_i$, where the last function plays the role of the pole-removed ordinary factor $(s-1)^{m_i} Z_i(s)$. The `paired analytic record` requires the following conditions for every index $i$.

(A1) *Local finiteness.* Each prime-power norm $N_{i,k}$ is an integer at least 2, and only finitely many slots have $N_{i,k} < T$ for each real $T$.

(A2) *Absolute summability.* $\sum_k |b_{i,k}| N_{i,k}^{-\sigma} < \infty$ for every real $\sigma > 1$.

(A3) *Euler identification.* For $\Re s > 1$,

$$F_i(s) = (s-1)^{m_i} \exp\left(\sum_k \frac{b_{i,k}}{\log N_{i,k}} N_{i,k}^{-s}\right), \qquad m_i \in \mathbb{Z}_{\ge 0}.$$

The denominator at a prime power is $j \log q$.

(A4) *Continuation and growth.* Both $F_i$ and $\Lambda_i$ are entire. The completed function satisfies

$$\forall \epsilon > 0\ \exists C > 0\ \forall z \in \mathbb{C}, \qquad |\Lambda_i(z)| \le \exp\bigl(C(1+|z|)^{1+\epsilon}\bigr).$$

(A5) *Completion identity.* $Q_i > 0$ and every shift in the finite product $G_i$ of real and complex Gamma factors has real part greater than $-1/2$. On $\Re s > 1/2$,

$$\Lambda_i(s)\Lambda_{i^\vee}(s) = s^{m_i} Q_i^{s/2} G_i(s) F_i(s).$$

(A6) *Reflection.* $\Lambda_{i^\vee}(1-s) = \eta_i(s)\Lambda_i(s)$ for all $s$. For actual standard functions $\eta_i$ is a nonzero constant. The Lean record requires the displayed identity with a supplied factor, which suffices for the implication from a zero of $\Lambda_i$ to a zero of the reflected dual.

What Lean proves from these inputs. The public `location proof` takes A1–A6 and countable prime-power types as inputs. In the forward implication, Lean derives right-half-plane nonvanishing for the ordinary factors from the line assertion and A5. It proves the zero expansion from the growth input and bounds the sum of the zero terms, giving the triangular bound. In the converse, it derives the kernel and the logarithmic-derivative formula using A1–A3. It excludes right-hand zeros by the Mellin pole test, with A4 supplying continuation and A5 transferring the result to the completed functions. A6 then gives exclusion on the left. The proof checks both implications for arbitrary supplied data with these properties.

How the inputs are supplied. The following table identifies the rank-one Lean providers and the mathematical inputs used for $n \ge 2$. The paired Hecke proof passes the corresponding facts to the common criterion through its specialized interface. It constructs the pole-free data and treats norm twists using a separate Dedekind theorem. For actual higher-rank representations, the entries in the last column supply A1–A6 in the paper. The Lean statement takes these conditions as hypotheses.

| | Rank one: supplied by Lean | Higher rank: supplied in the paper |
|---|---|---|
| A1 | `Prime-power norm lemmas` prove the lower bound, local finiteness and countability for the ideal-prime-power index. | The same elementary properties of ideal norms give A1 for each $K$ and $\pi$. |
| A2 | `summable_heckePairedCoefficient` bounds the prime-power coefficients using the unitary local parameters. | Jacquet–Shalika [11, Part I, Theorem 5.3] and [33, (2.2)], via Cauchy–Schwarz away from ramification and the local geometric bounds above. |
| A3 | `heckePairedEulerProduct_eq` identifies the exponential of the paired logarithm with the Euler product. | Taylor expansion of the standard local factors [33, § 2.1], with absolute convergence from A2. |
| A4 | The `pole-free Tate construction` gives entire completions and strip bounds. The `strip-bound criterion` proves the paired order-one estimate, and removes the Gamma factors to construct the entire ordinary factor. | Godement–Jacquet [8, Theorem 13.8], with the order-one normalization of [33, § 2.1]. Divide by the conductor and Gamma factors to obtain the ordinary entire functions. |
| A5 | `Conductor positivity`, `nonnegative real shifts`, and the `product of completions for inverse characters`. | The positive conductor $Q_\pi = \lvert D_K\rvert^n N_\pi$ and the strict Gamma shift bound of [33, § 2.1 and (2.2)]. Use $Q_\pi Q_{\widetilde{\pi}}$ and concatenate the two Gamma lists. |
| A6 | `Tate's functional equation in Lean` gives the reflection against the inverse character and a unit root number. | The normalized Godement–Jacquet equation [8, Theorem 13.8], [33, § 2.1], with the contragredient as dual. |

The `rank-one location theorem` assembles the separate Dedekind test and the pole-free Hecke tests from these providers. To express the single Dedekind test in the generic paired

interface, take $\Lambda_i = \Lambda_{i^\vee} = \Xi_K$ and $Z_i = \zeta_K^2$. Then $m_i = 2$, and the paired remainder is twice the remainder for $\zeta_K$. Dividing the bound by two recovers the single test. All vertical norm translates have the same zero real parts. Appendix N gives the rank-one proof and the separate construction for the individual multiplicity data.

The `individual analytic record` uses A1, A2 and A4 with individual coefficients $c_{i,k} \in \mathbb{C}$. In A5 it uses the single completion $\Lambda_i(s) = s^{m_i} Q_i^{s/2} G_i(s) F_i(s)$. It replaces A3 by the logarithmic-derivative identification

$$\frac{F_i'}{F_i}(s) = \frac{m_i}{s-1} - \sum_k c_{i,k} N_{i,k}^{-s}, \qquad \Re s > 1.$$

It has no dual index or functional-equation field. Its theorem assumes the location condition separately. Again, norm twists are handled through translation of the Dedekind result.

What a concrete higher-rank Lean proof requires. For all actual $\mathrm{GL}_2$ or $\mathrm{GL}_3$ members of the standard family, an instantiation would need to construct the record for each representation and its contragredient. There are three inputs to this task.

(i) *A1, arithmetic indexing.* Reuse the countability and finite norm cutoffs for prime ideals and their powers. These depend on $K$ and are independent of the representation.

(ii) *A2–A3, finite local factors and convergence.* Identify the actual standard Euler factor at every finite place, including ramified places. Establish absolute summability of its logarithmic coefficients throughout $\Re s > 1$ and identify the exponential of the summed logarithm with the ordinary product in that region.

(iii) *A4–A6, completion of the same function.* Construct the actual continuation and its entire order-one completion. Prove the conductor and Gamma identity with the same ordinary factor, the shift bound, and the functional equation against the actual contragredient. This includes the global analytic estimates and the local-to-global normalization.

Once these inputs are supplied, the existing Lean implication gives the equivalence for those representations. A specialized construction for a particular family can suffice. For example, an Euler product with all normalized local parameters of modulus at most 1 makes A2 a comparison with the Dedekind logarithmic derivative. This observation reduces only the convergence obligation. The identity and completion of the same $L$-function are separate obligations.

For the full family, a local estimate $|\alpha_{\pi,\mathfrak{p},k}| \le q^\theta$ with $0 < \theta < 1/2$ leads by itself to

$$\sum_{\mathfrak{p},j\ge 1} \frac{|a_{\pi,\mathfrak{p},j}| \log q}{q^{j\sigma}} \;\le\; n \sum_{\mathfrak{p},j\ge 1} \frac{\log q}{q^{j(\sigma-\theta)}}.$$

The right side is finite for $\sigma > 1+\theta$. A2 needs finiteness for every $\sigma > 1$, including $1 < \sigma \le 1+\theta$. The self Rankin–Selberg estimate in Section 3 supplies the missing summability in the paper. The cited global continuation and the local Gamma factors supply the other inputs there. The checked release verifies the generic analytic theorem. An application in Lean to actual $\mathrm{GL}_2$ or $\mathrm{GL}_3$ functions still needs these constructions and identifications.

3.3. **Mellin transform of the prime-power remainder.** Two small Lean declarations isolate the analytic steps below. The `kernel identity` computes the transform of one prime-power weight. The `zero-exclusion theorem` uses the signed Mellin identity to exclude zeros to the right of the tested edge.

For $v > 1$ and $\Re s > 1$,

$$\int_v^\infty (1 - v/x) x^{-s-1}\, dx = \frac{v^{-s}}{s(s+1)}.$$

The absolute convergence of (3.1) justifies summing these integrals. Using $F_Z'/F_Z = m/(s-1) + Z'/Z$, we obtain

$$\mathcal{M}R_Z(s) = \frac{m}{2(s-1)} + \frac{Z'(s)}{s(s+1)Z(s)} = \frac{F_Z'(s)/F_Z(s) + m(s+2)/2}{s(s+1)}. \tag{3.3}$$

The pole correction on the prime side cancels the contribution of the pole of $Z$ at 1.

If (3.2) holds, apply Lemma 2.1 with $\alpha = 1/2$. On $\Re s > 1/2$ the poles of (3.3) are exactly the zeros of $F_Z$. At a real zero $\beta > 1/2$ of multiplicity $j$ the residue is $j/(\beta(\beta+1)) > 0$. The lemma excludes every pole in this half-plane. Completion and reflection give the critical-line condition for the standard family.

Conversely, suppose $F_Z$ is nonvanishing on $\Re s > 1/2$. In its genus-one logarithmic derivative expansion let $b$ be the constant term after isolating the zeros at 0 and $-1$. Write $m_0, m_{-1}$ for their multiplicities. With zeros counted with multiplicity, the expansion is

$$\frac{F_Z'}{F_Z}(s) = b + \frac{m_0}{s} + \frac{m_{-1}}{s+1} + \sum_{\rho \notin \{0,-1\}} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right).$$

Each term $(s-\rho)^{-1} + \rho^{-1}$, divided by $s(s+1)$, has inverse Mellin transform $(x^\rho - x^{-1})/(\rho(\rho+1))$. The transforms of $1 - x^{-1}$, $1 - \frac{1}{2}x^{-1}$, and $\log x - 1 + x^{-1}$ supply the first three elementary terms in the inversion of (3.3). For the zero at $-1$, the partial fractions are

$$\frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Hence

$$R_Z(x) = \Re\Bigg[ b(1 - x^{-1}) + m(1 - \tfrac{1}{2}x^{-1}) + m_0(\log x - 1 + x^{-1}) + m_{-1}(1 - x^{-1} - x^{-1}\log x) + \sum_{\rho \notin \{0,-1\}} \frac{x^\rho - x^{-1}}{\rho(\rho+1)} \Bigg]. \tag{3.4}$$

The zero weights $1/|\rho(\rho+1)|$ are summable, since the zero divisor of $F_Z$ has convergence exponent at most one. Every $\Re\rho \le 1/2$. The zero series is locally uniformly convergent and dominated by a constant times $\sqrt{x} + x^{-1}$. These bounds justify termwise transformation at $\Re s = 2$ and give pointwise equality by uniqueness and continuity. The prime side is continuous even at a prime-power threshold, since the weight there is zero. To make the dependence on the test explicit, define

$$W_Z = \sum_{\rho \notin \{0,-1\}} \frac{1}{|\rho(\rho+1)|},$$

with the sum over zeros of $F_Z$, counted with multiplicity. For $x \ge 1$, $|x^\rho - x^{-1}| \le 2\sqrt{x}$ under the zero-location assumption in this direction. In the elementary terms of (3.4), $1 - x^{-1}$, $1 - \frac{1}{2}x^{-1}$, and $1 - x^{-1}(1 + \log x)$ lie between zero and one. Also $0 \le \log x - 1 + x^{-1} \le \log x$, and $\log x \le (2/e)\sqrt{x}$. It follows that one admissible choice is

$$X_{K,Z} = 1, \qquad C_{K,Z} = |b| + m + \frac{2}{e}m_0 + m_{-1} + 2W_Z.$$

Here $b, m_0, m_{-1}$ come from the ordinary pole-cleared factor, and $m = m_Z$ comes from the test's pole at 1. These quantities depend on $K$ through the test, and for a paired test they depend on $\pi$ and its contragredient through their product. The formula gives a test-dependent witness using the full zero divisor. The headline Lean theorem asks for the two witnesses separately for every test. This proves (3.2) and completes the forward implication.

3.4. **The sharp eventual coefficient.** The bound just proved gives a constant valid from $x = 1$. An eventual bound permits a smaller coefficient because terms that are $o(\sqrt{x})$ can be absorbed by moving the cutoff. We use the following recurrence fact here and in Section 4.4.

**Lemma 3.2** (Recurrence of a summable zero sum)**.** *Let $c_j$ be complex numbers with $\sum_j |c_j| < \infty$, and let $t_j \in \mathbb{R}$. For $P(v) = \Re \sum_j c_j e^{it_j v}$ one has $\limsup_{v\to+\infty} P(v) = \sup_{v\in\mathbb{R}} P(v)$.*

*Proof.* Fix $v_0$ and a tolerance $\varepsilon > 0$. Choose a finite set of indices such that the omitted absolute sum is less than $\varepsilon/4$. For those indices, the sequence of vectors $(e^{it_j n})_j$, $n \in \mathbb{N}$, lies in a compact torus. Choose a convergent subsequence with strictly increasing indices $n_k$. Then $n_{2k} - n_k \geq k$ and $e^{it_j(n_{2k}-n_k)} \to 1$ on the finite set. For large $k$, the change in its contribution between $v_0$ and $v_0 + n_{2k} - n_k$ is less than $\varepsilon/2$. The two omitted tails contribute at most another $\varepsilon/2$. Thus the limsup is at least $P(v_0)$. Taking the supremum over $v_0$ gives equality. □

**Theorem 3.3** (Sharp triangular coefficient)**.** *Let $Z \in \mathcal{T}_K$ satisfy the analytic conditions of Theorem 3.1. Suppose $F_Z$ has no zeros with real part greater than $1/2$, as follows from the critical-line condition on the corresponding completion. Then*

$$C_Z^\sharp := \limsup_{x\to\infty} \frac{R_Z(x)}{\sqrt{x}} = \sup_{v\in\mathbb{R}} \Re \sum_{\substack{F_Z(\rho)=0 \\ \rho=1/2+i\tau}} \frac{e^{i\tau v}}{\rho(\rho+1)}, \tag{3.5}$$

*where zeros are repeated according to multiplicity. Every $C > C_Z^\sharp$ gives an eventual bound $R_Z(x) \leq C\sqrt{x}$, and every $C < C_Z^\sharp$ fails at arbitrarily large $x$.*

*Proof.* Divide (3.4) by $\sqrt{x}$. The elementary terms are $O_Z((1+\log x)/\sqrt{x})$. The $-x^{-1}$ parts of the zero terms contribute at most $W_Z x^{-3/2}$. In the remaining sum, the terms with $\Re\rho < 1/2$ tend to zero individually and are dominated, for $x \geq 1$, by the summable weights $1/|\rho(\rho+1)|$. Dominated convergence thus gives $R_Z(x)/\sqrt{x} = P(\log x) + o_Z(1)$, with $P$ the sum on the right of (3.5). Lemma 3.2 gives the equality, and the defining property of the limsup gives the assertions about $C$. □

The supremum is bounded above by the sum of $1/|\rho(\rho+1)|$ over the critical-line zeros of $F_Z$. The weights in (3.5) have phases, so the supremum retains the possible relations among the ordinates. The theorem identifies the infimum of admissible eventual coefficients. It makes no assertion at the equality endpoint. The coefficient $m_Z/2$ of the linear term in $R_Z$ is fixed by the pole cancellation in (3.3). Lean verifies existence of a cutoff and a coefficient in Theorem 3.1, with the stated analytic inputs. The sharp value (3.5) and its proof are manuscript results.

3.5. **Specializations.** For $Z = \zeta_K$, integrating the step function $\psi_K$ gives

$$R_{\zeta_K}(x) = \frac{x}{2} - \frac{1}{x}\int_0^x \psi_K(t)\,dt.$$

The location proof applies to this test alone for a fixed $K$: $\Xi_K$ is its own reflected completion, so the Mellin converse and the forward zero expansion use no condition on other tests or other fields. By Section 4, this bound is equivalent to (4.1) for that field. Restricting these equivalent tests to the cyclotomic fields yields Theorem 5.1. For a unitary Hecke character $\chi$ outside the pure norm family, the paired coefficient is

$$b_{\mathfrak{p},j} = 2\Re(\alpha_\chi(\mathfrak{p})^j)\log N\mathfrak{p}, \qquad \alpha_\chi(\mathfrak{p}) = 0 \text{ at ramified primes.}$$

This is the pole-free degree-one case of (3.2). The remaining unitary Hecke characters are the norm twists $\chi_t = |\cdot|^{it}$. Their pole-cleared completions are translates of the Dedekind completion by $s \mapsto s + it$. It follows that for a fixed $K$ the critical-line condition for the entire rank-one

family is equivalent to the Dedekind triangular bound together with the paired triangular bound for each unitary Hecke character nontrivial on norm-one idele classes. This specialization is verified in Appendix N. Products of unitary cuspidal standard factors inherit the completed zero condition from their factors.

3.6. **Range of the criterion.**
Euler data and a real comparison. The finite sum at a given cutoff needs just local logarithmic-derivative coefficients. Its Mellin identity requires local finiteness and global absolute summability in a right half-plane. The converse needs a *real* remainder, since it uses the nonnegative comparison $C\sqrt{x} - R_Z(x)$ beyond the cutoff. For the unitary standard family the contragredient product provides this reality even when the individual Euler coefficients are complex. The paired coefficients themselves may have either sign.
Poles, growth, and reflection. If an ordinary factor has a pole at a real point to the right of $1/2$, its logarithmic derivative has a negative residue there. The sign needed by Lemma 2.1 can then fail. The function $U(x) = -x^\beta$, $\beta > 1/2$, satisfies $U(x) \le 0$ but has Mellin transform $-1/(s-\beta)$, illustrating the sign obstruction. A pole in a proposed completion factor can also cancel a zero of the ordinary function. A5 prevents that problem on the tested half-plane. The order-one control gives the summable zero weights used in the forward bound. If the functional equation is missing, the converse still excludes right-half-plane zeros when its other hypotheses hold. The reflection step is needed for the full line assertion.
Other $L$-functions. A non-standard $L(s, \pi, r)$ can use the same argument when its Euler data, pole correction, continuation, growth and reflection satisfy the corresponding conditions. A factorization into functions with verified inputs is another route, provided it controls zero and pole cancellations. The theorem here supplies the result for standard cuspidal functions and their indicated tests. Section 8 compares the resulting condition to other arithmetic criteria.

## 4. The corrected ideal Robin criterion

Let $K$ be a number field with discriminant $D_K$, ring of integers $\mathcal{O}_K$, and residue $\kappa_K = \operatorname{Res}_{s=1} \zeta_K(s) > 0$. We use $N$ for the absolute ideal norm and $\gamma$ for the Euler–Mascheroni constant. For a nonzero integral ideal $\mathfrak{a}$, put

$$\sigma_K(\mathfrak{a}) = \sum_{\mathfrak{d} \mid \mathfrak{a}} N\mathfrak{d},$$

where all divisors are integral ideals. A statement about sufficiently large $N\mathfrak{a}$ allows a threshold depending on $K$.

**Theorem 4.1** (Fieldwise corrected ideal Robin criterion)**.** *All zeros of $\zeta_K(s)$ in $0 < \Re s < 1$ have real part $1/2$ if and only if there exists $A_K \in \mathbb{N}$ such that every nonzero integral ideal $\mathfrak{a}$ with $N\mathfrak{a} \ge A_K$ satisfies*

$$\frac{\sigma_K(\mathfrak{a})}{N\mathfrak{a}} \le e^\gamma \kappa_K \left( \log\log N\mathfrak{a} + \frac{12 + 3\log|D_K|}{\sqrt{\log N\mathfrak{a}}} \right). \tag{4.1}$$

The quantifier for $A_K$ is inside the choice of $K$. There is one threshold for all ideals of that field, and the theorem asserts no uniform threshold across fields. In the Lean predicate `CorrectedIdealRobinK`, the existential `cutoff:Nat` precedes the universal quantifier over nonzero ideals. The Lean fieldwise declaration proves the equivalence with this predicate. The separate Lean implications control the ideal sum in the forward direction and recover the one-sided analytic bound from the least-common-multiple ideals in the converse. The Dedekind triangular equivalences identify both inequalities via the same zero-location predicate. Thus

the specialization from the triangular criterion takes place in the checked development as well as in the mathematics of Section 1.3.

Why the correction has this scale. Write $x = \log N\mathfrak{a}$. The Mertens formula for the truncated prime-power logarithm has a boundary term $E_K(x)/(x\log x)$, where $E_K = \psi_K - \mathrm{id}$. Charging the prime powers of the ideal against their total logarithmic norm subtracts that term exactly. The remaining error in the logarithm of the divisor ratio is bounded above by the weighted tail $H(x)$ in (4.3). Under Dedekind RH the zeros contribute $O_K(\sqrt{x})$ to $xH_0(x)$. The weight in $H(x)$ contributes a further factor $1/\log x$ to this tail. The zero expansion below gives a convenient bound on $H(x)$ with coefficient $B_K + 1$, where $B_K \le 9 + 3\log|D_K|$. After exponentiation, $\log x\, e^{H(x)} = \log x + \log x\, H(x) + O(\log x\, H(x)^2)$. The $\log x$ factor therefore converts the logarithmic error into an additive correction of order $1/\sqrt{x}$ with the overall multiplier $e^{\gamma}\kappa_K$. The coefficient $12 + 3\log|D_K|$ leaves room above $B_K + 1$ for the lower-order terms at a field-dependent cutoff. The proof estimates the zero contribution by absolute values and discards nonpositive terms in the norm budget. Section 4.4 keeps the exact budget difference and determines the sharp coefficient.

4.1. **The boundary term in Mertens' formula.** In this section $q = N\mathfrak{p}$, with $\mathfrak{p}$ a nonzero prime ideal. Every prime-power sum is over pairs $(\mathfrak{p}, j)$, $j \ge 1$, counted separately even when their norms coincide. Define

$$\psi_K(x) = \sum_{q^j \le x} \log q, \qquad E_K(x) = \psi_K(x) - x, \qquad A(x) = \sum_{q^j \le x} \frac{1}{jq^j}.$$

For $x > 1$ let $w(x) = (1 + \log x)/\log^2 x$, and set

$$H(x) = -\int_x^\infty E_K(t)\frac{w(t)}{t^2}\,dt, \qquad H_0(x) = -\int_x^\infty E_K(t)\frac{dt}{t^2}.$$

The fixed-field prime ideal theorem gives $E_K(x) = O_K(xe^{-c_K\sqrt{\log x}})$ for some $c_K > 0$. For example, apply the unconditional effective Chebotarev theorem to $K/K$, use partial summation, and include the contributions from $j \ge 2$. A possible fixed exceptional zero $\beta < 1$ contributes $O_K(x^\beta)$, which is absorbed in the same estimate [12]. In particular the two integrals converge absolutely, and the prime-ideal counting function is $O_K(x/\log x)$.

To obtain the Mertens identity with its boundary term, put $f(t) = 1/(t\log t)$, so that $f'(t) = -w(t)/t^2$. For $1 < b < 2 \le x$, no prime-power norm lies at or below $b$. The jump of $\psi_K$ at each $q^j$ contributes $(\log q)f(q^j) = 1/(jq^j)$. Hence Stieltjes integration by parts, with the upper endpoint included, gives

$$A(x) = \psi_K(x)f(x) - \int_b^x \psi_K(t)f'(t)\,dt = \log\log x + C_b + E_K(x)f(x) + H(x),$$

where

$$C_b = bf(b) - \log\log b - \int_b^\infty E_K(t)f'(t)\,dt.$$

In the second equality the contribution of $t$ to $\psi_K(t)$ is integrated by parts and the integral with $E_K$ is extended to infinity. The prime ideal theorem bounds the added tail. Determining $C_b$ by the limiting Mertens constant gives

$$A(x) = \gamma + \log\kappa_K + \log\log x + \frac{E_K(x)}{x\log x} + H(x). \tag{4.2}$$

To determine the constant, take logarithms in the third Mertens theorem over $K$ [17, Theorem 1]. Its prime product, truncated by $q \le x$, differs in the logarithm from $A(x)$ by $\sum_{q\le x}\sum_{j>k(q)} 1/(jq^j)$, where $k(q) = \lfloor \log x/\log q \rfloor$. This difference is $O_K(x^{-1/2}/\log x)$. Indeed, the inner tail is

$O(q^{-k(q)-1})$. For $q \le \sqrt{x}$ we bound it by $O(1/x)$ and use $O_K(\sqrt{x}/\log x)$ for the number of such primes. For $\sqrt{x} < q \le x$ the tail is $O(q^{-2})$. If $P_K(t)$ is the prime-ideal counting function and $y = \sqrt{x}$, partial summation gives

$$\sum_{q>y} q^{-2} = -\frac{P_K(y)}{y^2} + 2\int_y^\infty \frac{P_K(t)}{t^3}\,dt \ll_K \frac{1}{y\log y}$$

for $y \ge 2$, using $P_K(t) \ll_K t/\log t$. This proves the asserted tail bound. Thus $A(x) - \log\log x \longrightarrow \gamma + \log\kappa_K$. Partial summation gives (4.2) with this constant because $E_K(x)/(x\log x)$ and $H(x)$ tend to zero.

If the exponent of $\mathfrak{p}$ in $\mathfrak{a}$ is $k \ge 1$, then

$$\log\sum_{j=0}^{k} q^{-j} - \sum_{j=1}^{k}\frac{1}{jq^j} = \log(1-q^{-k-1}) + \sum_{j>k}\frac{1}{jq^j}.$$

The tail on the right is at most $q^{-k-1}/((k+1)(1-q^{-1})) \le q^{-k-1}$. The entire right side is nonpositive, and its absolute value is $O(q^{-k-1})$ with an absolute implied constant. When $k = 0$ the discrepancy on the left is zero.

Here is the passage from the local estimate to the norm budget. Write $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{k_{\mathfrak{p}}}$ and put $S_{\mathfrak{a}} = \{(\mathfrak{p}, j) : 1 \le j \le k_{\mathfrak{p}}\}$. The logarithm of the normalized ideal divisor sum is the sum of the logarithms of its local geometric sums. By the local inequality,

$$\log\frac{\sigma_K(\mathfrak{a})}{N\mathfrak{a}} \le \sum_{(\mathfrak{p},j)\in S_{\mathfrak{a}}}\frac{1}{jq^j}, \qquad x = \log N\mathfrak{a} = \sum_{(\mathfrak{p},j)\in S_{\mathfrak{a}}}\log q.$$

For $x > 1$ choose $\lambda = 1/(x\log x)$. Each summand equals $\lambda\log q + u_{\mathfrak{p},j}$, with

$$u_{\mathfrak{p},j} = (\log q)\left(\frac{1}{q^j\log q^j} - \lambda\right).$$

Since $v\log v$ is strictly increasing for $v > 1$, the $u_{\mathfrak{p},j}$ are nonnegative on $q^j \le x$ and nonpositive outside this range. Dropping the outside terms and adding every missing term inside the range gives the upper bound $\lambda x + A(x) - \lambda\psi_K(x)$. Substituting (4.2) cancels $E_K(x)/(x\log x)$ exactly. Thus

$$\log\frac{\sigma_K(\mathfrak{a})}{N\mathfrak{a}} \le A(x) + \frac{x-\psi_K(x)}{x\log x} = \gamma + \log\kappa_K + \log\log x + H(x). \tag{4.3}$$

4.2. **The zero expansion and the explicit coefficient.** Write $(r_1, r_2)$ for the signature of $K$. We use $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The function

$$\Xi_K(s) = s(s-1)|D_K|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$$

is entire of order at most one, satisfies $\Xi_K(s) = \Xi_K(1-s)$, and has precisely the nontrivial zeros of $\zeta_K$, counted with multiplicity. The prime ideal estimate above bounds the integrated logarithmic derivative as the line $\Re s = 1$ is approached from the right. This excludes zeros on that line away from the pole, and the functional equation gives the boundary at $\Re s = 0$. We use the standard completion and its Hadamard product as in [18, § 3].

Set $F_K(s) = (s-1)\zeta_K(s)$ and $\Phi_K(s) = (F_K'(s)/F_K(s) + 1)/s$. For $\Re s > 1$, integrate each step of $\psi_K$ from $q^j$ to infinity. The result is $(\log q)q^{-js}/s$, and absolute convergence of the logarithmic derivative justifies summation. Subtracting the integral of $t$ gives

$$\mathcal{E}_K(s) := \int_1^\infty E_K(t)t^{-s-1}\,dt = -\frac{\zeta_K'(s)}{s\zeta_K(s)} - \frac{1}{s-1} = -\Phi_K(s).$$

The last equality follows from $F_K'/F_K = 1/(s-1)+\zeta_K'/\zeta_K$. The function $\Phi_K$ is meromorphic and regular at 1, since $F_K(1) = \kappa_K > 0$. The prime ideal error is integrable against $t^{-2}$, so dominated convergence gives $\mathcal{E}_K(1) = -\Phi_K(1)$. The same majorant bounds $\mathcal{E}_K(s)$ as $s$ approaches any $1 + i\tau$ from the right. For $\tau \neq 0$, a zero there would give a pole of the logarithmic derivative, proving the boundary nonvanishing used above.

In $-\int_1^\infty x^{-s} \int_x^\infty E_K(t)t^{-2}\,dt\,dx$, the absolute integral after reversing order is bounded using $\int_1^t x^{-\Re s}\,dx \le 1/(\Re s - 1)$. Computing the inner integral $\int_1^t x^{-s}\,dx = (1 - t^{1-s})/(s-1)$ gives

$$\mathcal{M}(xH_0(x))(s) = \frac{\mathcal{E}_K(s) - \mathcal{E}_K(1)}{s-1} = \frac{\Phi_K(s) - \Phi_K(1)}{1-s}. \tag{4.4}$$

The divided difference extends at 1 with value $-\Phi_K'(1)$. At a zero $\rho \neq 0, 1$ of order $j$ its residue is $j/[\rho(1-\rho)]$. We justify the logarithmic differentiation before inverting the transform. Count zeros with multiplicity. Since $\Xi_K(0) \neq 0$, Jensen's formula at radius $2R$ bounds the number with $|\rho| \le R$ by $(\log \max_{|s|=2R} |\Xi_K(s)| - \log|\Xi_K(0)|)/\log 2$. The order-one bound makes this $O_K(R^{3/2})$ for $R \ge 1$. On the annulus $2^\ell < |\rho| \le 2^{\ell+1}$ the contribution to $\sum_\rho |\rho|^{-2}$ is therefore $O_K(2^{-\ell/2})$. Finitely many small zeros complete the proof of summability.

Hadamard factorization gives $\Xi_K(s) = e^{a+bs}\prod_\rho (1 - s/\rho)e^{s/\rho}$. On $|s| \le M$, the factors with $|\rho| > 2M$ have analytic logarithms $\log(1 - s/\rho) + s/\rho$, bounded in modulus by $2M^2/|\rho|^2$ using the Taylor series. Their derivatives satisfy

$$\left|\frac{1}{s-\rho} + \frac{1}{\rho}\right| = \left|\frac{s}{\rho(s-\rho)}\right| \le \frac{2M}{|\rho|^2}.$$

Both series converge normally on the disk. The finitely many remaining factors can be differentiated away from their zeros. Consequently $\Xi_K'/\Xi_K = b + \sum_\rho((s-\rho)^{-1} + \rho^{-1})$ holds locally uniformly off the zeros, in particular near 2 and $-1$. This also justifies the two evaluations used below to bound $B_K$.

From the defining completion,

$$\frac{F_K'(s)}{F_K(s)} = \frac{\Xi_K'(s)}{\Xi_K(s)} - \frac{1}{s} - \frac{1}{2}\log|D_K| - r_1\frac{\Gamma_{\mathbb{R}}'(s)}{\Gamma_{\mathbb{R}}(s)} - r_2\frac{\Gamma_{\mathbb{C}}'(s)}{\Gamma_{\mathbb{C}}(s)}.$$

Let $\psi_\Gamma = \Gamma'/\Gamma$. The series $\psi_\Gamma(z) = -\gamma - z^{-1} + \sum_{n\ge1} z/(n(n+z))$ gives

$$\Phi_K(s) = \frac{B}{s} + \frac{r_1 + r_2 - 1}{s^2} + \sum_\rho \frac{1}{\rho(s-\rho)} - r_1\sum_{n\ge1}\frac{1}{2n(s+2n)} - r_2\sum_{n\ge1}\frac{1}{n(s+n)}$$

where $B = b + 1 - \frac{1}{2}\log|D_K| + \frac{r_1}{2}(\gamma + \log\pi) + r_2(\gamma + \log(2\pi))$. The real-place terms lie on the even negative lattice and the complex-place terms on the full negative lattice. The digamma series converges locally uniformly off its poles by the same $O(n^{-2})$ tail comparison. For $a \neq 1$, the divided difference of $1/(s-a)$ equals $1/((s-a)(1-a))$; that of $1/s^2$ equals $(s+1)/s^2$. In (4.4) the constant contribution $B/s$ therefore stays $B/s$, the quadratic pole contributes $(r_1 + r_2 - 1)(1/s + 1/s^2)$, and a zero term becomes $1/[\rho(1-\rho)(s-\rho)]$. Taking $a = -2n$ and $a = -n$ gives the two Gamma terms. These are exactly the transforms of

$$\begin{aligned} xH_0(x) = B + (r_1 + r_2 - 1)(1 + \log x) + \sum_\rho \frac{x^\rho}{\rho(1-\rho)} \\ - r_1\sum_{n\ge1}\frac{x^{-2n}}{2n(1+2n)} - r_2\sum_{n\ge1}\frac{x^{-n}}{n(1+n)}. \end{aligned} \tag{4.5}$$

For instance, $(s-\rho)^{-1}$ is the Mellin transform of $x^\rho$. For large $|\rho|$ the coefficient is bounded by $2/|\rho|^2$. The remaining zeros avoid 0 and 1. For $x \ge 1$, $|x^\rho| \le x$, so the zero series converges

uniformly on compact $x$-intervals and its absolute sum times $x^{-3}$ is bounded by an integrable multiple of $x^{-2}$. The Gamma series are dominated by their summable $O(n^{-2})$ coefficients. This proves the Fubini bound on the full line $\Re s = 2$. On writing $x = e^t$, uniqueness of the Fourier transform of $L^1$ functions gives equality almost everywhere for the two inverse functions after weighting by $e^{-2t}$. The locally uniform sum is continuous, and $xH_0(x)$ is continuous by absolute integrability of its defining tail. The two functions therefore agree for every $x > 1$.

Now assume Dedekind RH for $K$. The zero series is bounded in absolute value by $B_K\sqrt{x}$, where

$$B_K = \sum_\rho \frac{1}{1/4 + (\Im\rho)^2}.$$

Let $\xi_K = \Xi_K'/\Xi_K$. In $\xi_K(2) - \xi_K(-1)$ the Hadamard constant $b$ and each subtraction constant $\rho^{-1}$ cancel. The resulting terms are $O(|\rho|^{-2})$, so the difference can be summed absolutely. By the functional equation $\xi_K(-1) = -\xi_K(2)$. Under Dedekind RH write each zero as $\rho = 1/2 + i\tau$, counted with its order. The contribution of this zero is

$$\frac{1}{2-\rho} - \frac{1}{-1-\rho} = \frac{1}{3/2 - i\tau} + \frac{1}{3/2 + i\tau} = \frac{3}{9/4 + \tau^2}.$$

In particular all summands in $2\xi_K(2)$ are nonnegative. Since $9(1/4 + \tau^2) - (9/4 + \tau^2) = 8\tau^2 \geq 0$, summing the pointwise comparisons gives

$$B_K = \sum_\rho \frac{1}{1/4 + \tau^2} \leq \sum_\rho \frac{9}{9/4 + \tau^2} = 3\sum_\rho \left(\frac{1}{2-\rho} - \frac{1}{-1-\rho}\right) = 6\xi_K(2).$$

Differentiation of the defining completion at 2 gives

$$\begin{aligned}\xi_K(2) = \frac{3}{2} + \frac{1}{2}\log|D_K| + \frac{r_1}{2}(\psi_\Gamma(1) - \log\pi) \\ + r_2(\psi_\Gamma(2) - \log(2\pi)) - \sum_{\mathfrak{p},\, j\geq 1} (\log q) q^{-2j}.\end{aligned}$$

The $3/2$ is the contribution $1/2 + 1$ from $s(s-1)$. The Gamma recurrence gives $\psi_\Gamma(1) = -\gamma$ and $\psi_\Gamma(2) = 1 - \gamma$. Both archimedean parentheses are negative, and the Euler product gives the last nonpositive term. Thus

$$B_K \leq 6\left(\frac{3}{2} + \frac{1}{2}\log|D_K|\right) = 9 + 3\log|D_K|.$$

For $x \geq 1$, each nontrivial-zero term in (4.5) has absolute value $\sqrt{x}/(1/4+\tau^2)$, because $1-\rho = \overline{\rho}$. The absolute sum of the two negative-lattice Gamma series is at most $(r_1/4 + r_2)\sum_{n\geq 1} n^{-2}$. Thus for the finite constant

$$M_K = |B| + |r_1 + r_2 - 1| + (r_1/4 + r_2)\sum_{n\geq 1} n^{-2},$$

the triangle inequality gives

$$|xH_0(x)| \leq B_K\sqrt{x} + M_K(1 + \log x).$$

Because $(1 + \log x)/\sqrt{x} \to 0$, enlarge the field-dependent cutoff until $M_K(1 + \log x) \leq \sqrt{x}$. Division by $x$ gives $|H_0(x)| \leq (B_K + 1)/\sqrt{x}$ above that cutoff.

To estimate $H$, use $H_0'(t) = E_K(t)/t^2$ almost everywhere. On $[x, T]$ integration by parts gives

$$-\int_x^T w(t)H_0'(t)\,dt = w(x)H_0(x) - w(T)H_0(T) + \int_x^T w'(t)H_0(t)\,dt.$$

The bound on $H_0$ makes $w(T)H_0(T) \to 0$. The derivative of $w(t) = (\log t)^{-1} + (\log t)^{-2}$ has absolute value at most $3/(t \log^2 t)$ for $t \geq e$. For sufficiently large $x \geq e$,

$$\int_x^\infty |w'(t)H_0(t)|\, dt \leq \frac{3(B_K+1)}{\log^2 x} \int_x^\infty t^{-3/2}\, dt = \frac{6(B_K+1)}{\sqrt{x}\log^2 x}.$$

The term $w(x)H_0(x)$ gives the leading bound $(B_K+1)/(\sqrt{x}\log x)$ and one more copy of $(B_K+1)/(\sqrt{x}\log^2 x)$. Hence

$$|H(x)| \leq \frac{B_K+1}{\sqrt{x}\log x} + \frac{7(B_K+1)}{\sqrt{x}\log^2 x}$$

eventually. In (4.3), exponentiation multiplies $\log x$ by $e^{H(x)}$. The last estimate and $e^u = 1 + u + O(u^2)$ for $u \to 0$ bound the result above by $\log x + (B_K + 1 + o_K(1))/\sqrt{x}$. Since $B_K + 1 \leq 10 + 3\log|D_K| < 12 + 3\log|D_K|$, this proves (4.1) in the forward direction.

4.3. **The converse from least common multiples of ideals.** For $x > 1$, take

$$\mathfrak{L}_x = \prod_{\substack{\mathfrak{p},\ j \geq 1 \\ q^j \leq x}} \mathfrak{p}.$$

Its logarithmic norm is $\psi_K(x)$. The local discrepancy computed above, summed with the same cutoff at $\sqrt{x}$, gives

$$\log \frac{\sigma_K(\mathfrak{L}_x)}{N\mathfrak{L}_x} = A(x) + O_K\left(\frac{1}{\sqrt{x}\log x}\right).$$

Write $d(x) = \log(\sigma_K(\mathfrak{L}_x)/N\mathfrak{L}_x) - A(x)$. The local calculation gives $d(x) \leq 0$ and $-d(x) \ll_K 1/(\sqrt{x}\log x)$. Apply the assumed ideal bound at $\mathfrak{L}_x$, where $\log N\mathfrak{L}_x = \psi_K(x)$. Taking logs and using $\log(1+y) \leq y$ yields

$$\begin{aligned} A(x) + d(x) \leq \gamma + \log\kappa_K + \log\log\psi_K(x) \\ + \frac{12 + 3\log|D_K|}{\sqrt{\psi_K(x)}\log\psi_K(x)}. \end{aligned}$$

By concavity of $\log\log t$ on $t > 1$, the log-log term is at most $\log\log x + E_K(x)/(x\log x)$. Substituting (4.2), the terms $E_K(x)/(x\log x)$ cancel. The resulting upper bound on $H(x)$ is the last displayed correction minus $d(x)$. Since $\psi_K(x) \sim x$, take $C_K > 0$ so that $H(t) \leq C_K/(\sqrt{t}\log t)$ for every sufficiently large $t$.

Put $u(t) = 1/w(t) = (\log t)^2/(1+\log t)$, so $u(t) \leq \log t$ and $0 \leq u'(t) = (1-(1+\log t)^{-2})/t \leq 1/t$ for $t > 1$. On compact subintervals of $(1,\infty)$, $H$ is absolutely continuous and $u(t)H'(t) = E_K(t)/t^2$ almost everywhere. Consequently for $T > x > 1$,

$$-\int_x^T E_K(t)t^{-2}\, dt = u(x)H(x) - u(T)H(T) + \int_x^T u'(t)H(t)\, dt.$$

Using $w(t) \leq 2/\log t$ for $t \geq e$, the prime ideal error bound gives

$$|H(T)| \ll_K \int_{\sqrt{\log T}}^\infty \frac{e^{-c_K v}}{v}\, dv \ll_K \frac{e^{-c_K\sqrt{\log T}}}{\sqrt{\log T}}.$$

Thus $u(T)H(T) \to 0$ and the integral against $u'$ is absolutely convergent. The nonnegative weights $u(x)$ and $u'(t)$ permit the upper bound on $H$ to be used on each term. For large $x \geq e$,

$$\begin{aligned} H_0(x) &= u(x)H(x) + \int_x^\infty u'(t)H(t)\, dt \\ &\leq \frac{C_K}{\sqrt{x}} + \frac{C_K}{\log x}\int_x^\infty t^{-3/2}\, dt = \frac{C_K}{\sqrt{x}}\left(1 + \frac{2}{\log x}\right). \end{aligned}$$

By (4.4), $\mathcal{M}(xH_0(x))$ is meromorphic on $\Re s > 1/2$. Its poles there occur exactly at zeros of $\zeta_K$ in that half-plane. The apparent singularity at 1 is removable. At a real zero $\beta \in (1/2, 1)$ of multiplicity $j$ the transform has residue $j/[\beta(1-\beta)] > 0$. Lemma 2.1 excludes all the poles, both real and nonreal. Reflection under the functional equation gives Dedekind RH. This proves Theorem 4.1.

4.4. **The sharp fieldwise correction.** The coefficient in (4.1) is the one used by the formal criterion. For a fixed field and a cutoff allowed to depend on the coefficient, the exact first-order threshold comes from retaining the local increments of the divisor ratio. Write $h(\mathfrak{a}) = \log N\mathfrak{a}$, and for $h(\mathfrak{a}) > 1$ put

$$U_K(\mathfrak{a}) = \sqrt{h(\mathfrak{a})}\left(\frac{\sigma_K(\mathfrak{a})}{e^{\gamma}\kappa_K N\mathfrak{a}} - \log h(\mathfrak{a})\right).$$

**Theorem 4.2** (Sharp ideal Robin coefficient)**.** *Fix a number field $K$. If $\zeta_K$ satisfies Dedekind RH, then*

$$\limsup_{N\mathfrak{a}\to\infty} U_K(\mathfrak{a}) = C_K^{\sharp} := 2\frac{\Xi_K'(1)}{\Xi_K(1)} - (2\sqrt{2}-2) = B_K - (2\sqrt{2}-2). \tag{4.6}$$

*If Dedekind RH fails for $K$, the limsup is $+\infty$. Consequently, for any fixed $\varepsilon > 0$, the eventual ideal inequality with coefficient $C_K^{\sharp} + \varepsilon$ in place of $12 + 3\log|D_K|$ is equivalent to Dedekind RH for $K$. Under Dedekind RH, every coefficient below $C_K^{\sharp}$ fails at arbitrarily large norms.*

Here $C_K^{\sharp}$ is defined without a zero-location assumption by the logarithmic derivative at 1. Its expression using $B_K$ is for the critical-line case. The theorem determines the infimum of eventual coefficients. It leaves the inequality with coefficient exactly $C_K^{\sharp}$ undecided by the limsup formula.

*Proof.* Assume Dedekind RH for $K$. The real and absolutely convergent zero sum

$$\mathcal{P}_K(v) = \sum_{\rho=1/2+i\tau} \frac{\cos(\tau v)}{1/4+\tau^2}$$

has supremum $B_K = \mathcal{P}_K(0)$, with zeros counted with multiplicity. By Lemma 3.2 this is also its limsup as $v \to +\infty$. The zero expansion (4.5) gives

$$H_0(x) = x^{-1/2}\mathcal{P}_K(\log x) + O_K((1+\log x)/x).$$

The integration by parts used above bounds the integral with $w'$ by $O_K(x^{-1/2}\log^{-2} x)$. Since $w(x) = 1/\log x + 1/\log^2 x$, it follows that

$$\sqrt{x}\log x\, H(x) = \mathcal{P}_K(\log x) + o_K(1). \tag{4.7}$$

*Exact optimization at a fixed budget price.* For $x > 1$, set $\lambda = (x\log x)^{-1}$. The exact increase at the $j$-th power of a prime ideal of norm $q$ is

$$g(q,j) = \log\frac{1-q^{-j-1}}{1-q^{-j}} = \log\left(1 + \frac{1-q^{-1}}{q^j-1}\right).$$

It decreases strictly with $j$. Define $v(q,j) = 1/(jq^j)$ and $[u]_+ = \max(u, 0)$. The sums

$$S_g(x) = \sum_{\mathfrak{p},\, j\ge 1} [g(q,j) - \lambda\log q]_+,$$

$$S_v(x) = \sum_{\mathfrak{p},\, j\ge 1} [v(q,j) - \lambda\log q]_+ = A(x) - \lambda\psi_K(x)$$

have finite support. Indeed $g(q,j) \le q^{-j}$, by $\log(1+t) \le t$ and $q^{-j} \le q^{-1}$, and $v(q,j) \le q^{-j}$ as well. A positive summand requires $q^j \log q < x \log x$. Telescoping the exact increments gives, for every ideal $\mathfrak{a}$,

$$\log \frac{\sigma_K(\mathfrak{a})}{N\mathfrak{a}} \le \lambda h(\mathfrak{a}) + S_g(x). \tag{4.8}$$

For each prime ideal take exactly the powers for which $g(q,j) > \lambda \log q$. These powers form an initial segment by monotonicity, and define an integral ideal $\mathfrak{b}_x$, for which (4.8) is an equality. Powers at equality may be omitted.

*The difference between the two optimized sums.* Put $Q = (3x)^{1/3}$. For sufficiently large $x$ and $q \le Q$ the exponent $k = \lfloor \log x / \log q \rfloor$ is at least 2. The first exact increment is $\log(1+q^{-1}) \ge 1/(2q)$, so it exceeds the budget cost. For $2 \le j \le k$,

$$g(q,j) = \int_{q^{-j-1}}^{q^{-j}} \frac{dt}{1-t} \ge (1-q^{-1})q^{-j} \ge v(q,j).$$

The first $k$ powers can therefore be included in both optimized sums, with any zero-price-difference power contributing zero. At these powers the difference is $O(q^{-k-1})$ by the local logarithmic comparison preceding (4.3). Any extra exact increments have total at most $\sum_{j>k} q^{-j} \le 2q^{-k-1}$. Since $q^{k+1} > x$, the difference at each such prime ideal is $O(1/x)$. Each integer norm occurs for at most $[K:\mathbb{Q}]$ prime ideals, because it must be a rational prime power and there are at most that many primes above its rational prime. Thus the total for $q \le Q$ is $O_K(x^{-2/3})$.

For $q > Q$ there are no positive terms with $j \ge 3$ in either sum, since $q^3 \log q > x \log x$. Uniformly $q \ge 2$,

$$g(q,1) = q^{-1} - \tfrac{1}{2}q^{-2} + O(q^{-3}), \qquad g(q,2) = q^{-2} + O(q^{-3}).$$

Replacing $g$ by these main terms in the positive parts costs $O_K(\sum_{n>Q} n^{-3}) = O_K(x^{-2/3})$, by the Lipschitz bound for $[\,\cdot\,]_+$. In the first-power terms this gives a difference $-1/(2q^2)$ for $Q < q \le x-3$ and zero for $q > x$. To see the former, for $q \le x/2$ we have $q^{-1} - \lambda \log q \ge 1/(2q)$. For $x/2 < q \le x-3$ the derivative of $t^{-1} - \lambda \log t$ is at most $-1/x^2$ on $[q,x]$, so the margin is at least $3/x^2 > 1/(2q^2)$. The remaining interval $x-3 < q \le x$ accounts for $O_K(x^{-2})$, since there are $O_K(1)$ prime ideals with those norms.

Let $y = y(x) > 1$ satisfy $y^2 \log y = x \log x$. The model second increment $q^{-2}$ is included up to $y$. The comparison second increment $1/(2q^2)$ is included up to $\sqrt{x}$. For $Q < q \le \sqrt{x}$ their difference $1/(2q^2)$ cancels the loss at the first power. Hence, for large $x$ with $Q < \sqrt{x} < y < x$,

$$S_g(x) - S_v(x) = -\frac{1}{2} \sum_{\substack{\mathfrak{p} \\ \sqrt{x} < q \le x}} q^{-2} + \sum_{\substack{\mathfrak{p} \\ \sqrt{x} < q \le y}} q^{-2} - \lambda \sum_{\substack{\mathfrak{p} \\ \sqrt{x} < q \le y}} \log q + O_K(x^{-2/3}). \tag{4.9}$$

All three sums are over prime ideals, so equal norms are counted separately.

For fixed $K$, the prime ideal theorem gives $\vartheta_K(t) := \sum_{q \le t} \log q \sim t$. Indeed the difference between $\psi_K(t)$ and this sum is $O_K(\sqrt{t}\log^2 t)$ by summing the higher powers over $q \le \sqrt{t}$. Partial summation gives $P_K(t) := \#\{\mathfrak{p} : q \le t\} \sim t/\log t$. Using this last asymptotic in the tail identity for $\sum_{q>u} q^{-2}$ in Section 4 gives $\sum_{q>u} q^{-2} \sim 1/(u \log u)$. For example, the boundary term is asymptotic to $-1/(u \log u)$ and the integral is asymptotic to $2/(u \log u)$, by integration by parts. Also $y/\sqrt{x} \to \sqrt{2}$: for fixed $c > 0$ the ratio $(c\sqrt{x})^2 \log(c\sqrt{x})/(x \log x)$ tends to $c^2/2$, so the increasing function $t^2 \log t$ brackets $y$ between $c\sqrt{x}$ for any two $c$ straddling $\sqrt{2}$. Upon multiplying (4.9) by $\sqrt{x} \log x$, the three terms tend respectively to $-1$, $2-\sqrt{2}$, and $-(\sqrt{2}-1)$.

Thus

$$S_g(x) - S_v(x) = \frac{2 - 2\sqrt{2} + o_K(1)}{\sqrt{x}\log x}. \tag{4.10}$$

*Upper bound and a matching family of ideals.* The Mertens identity (4.2) gives

$$S_v(x) + \lambda x = \gamma + \log \kappa_K + \log\log x + H(x).$$

For any ideal with $x = h(\mathfrak{a}) > 1$, substitute this identity and (4.10) into (4.8). By (4.7), the resulting inequality is

$$\log \frac{\sigma_K(\mathfrak{a})}{N\mathfrak{a}} \le \gamma + \log \kappa_K + \log\log x + \frac{\mathcal{P}_K(\log x) + 2 - 2\sqrt{2} + o_K(1)}{\sqrt{x}\log x}.$$

Since $\mathcal{P}_K$ is bounded, exponentiating gives $U_K(\mathfrak{a}) \le \mathcal{P}_K(\log x) + 2 - 2\sqrt{2} + o_K(1)$, uniformly over the ideals as $h(\mathfrak{a}) \to \infty$.

For the equality ideal $\mathfrak{b}_x$, put $T_x = h(\mathfrak{b}_x)$. We have

$$T_x = x + O_K(\sqrt{x}\log^2 x). \tag{4.11}$$

To verify this, first compare with $\psi_K(x)$. The condition for an exact first increment can differ from $q \le x$ only in $x - 3 < q \le x$. This follows from $q^{-1} - 1/(2q^2) \le \log(1 + q^{-1}) \le q^{-1}$ and the margin estimate above. The contribution of the difference there is $O_K(\log x)$. For $j \ge 2$, every selected exact power satisfies $q^j < X = x\log x/\log 2$. Using $\vartheta_K(u) \ll_K u$, their total log norm is at most

$$\sum_{2 \le j \le \log X/\log 2} \vartheta_K(X^{1/j}) \ll_K \sqrt{X} + X^{1/3}\log X \ll_K \sqrt{X}.$$

The comparison powers with $j \ge 2$ have the same bound. Thus $T_x - \psi_K(x) = O_K(\sqrt{x\log x})$. Now $\psi_K(x) - x = O_K(\sqrt{x}\log^2 x)$ under Dedekind RH by [20, Corollary 1.2]. This proves (4.11).

The equality in (4.8) includes $\lambda(T_x - x)$ in addition to the preceding expansion. Because $\lambda$ is the derivative of $\log\log x$, Taylor's theorem and (4.11) give

$$\log\log T_x - \log\log x - \lambda(T_x - x) = O_K\left(\frac{(T_x - x)^2}{x^2\log x}\right) = o_K\left(\frac{1}{\sqrt{x}\log x}\right).$$

Also $\sqrt{T_x}\log T_x/(\sqrt{x}\log x) \to 1$. Exponentiating the equality at the actual height gives

$$U_K(\mathfrak{b}_x) = \mathcal{P}_K(\log x) + 2 - 2\sqrt{2} + o_K(1).$$

Taking $x$ along a sequence for which $\mathcal{P}_K(\log x) \to B_K$ establishes the matching lower bound.

Finally, by the functional equation $\xi_K(1) - \xi_K(0) = 2\xi_K(1)$, where $\xi_K = \Xi_K'/\Xi_K$. The difference of the Hadamard expansions at 1 and 0 is the absolutely convergent sum of $1/[\rho(1-\rho)]$. Under Dedekind RH it equals $B_K$. This identifies the coefficient in (4.6). If Dedekind RH fails, the converse proof of Theorem 4.1 applies to any fixed positive coefficient: only the order $O_K(1/(\sqrt{x}\log x))$ is used in the upper bound on $H(x)$. If the limsup were finite or $-\infty$, one such coefficient would work eventually, a contradiction. The equivalence with $C_K^\sharp + \varepsilon$ follows. □

Evaluation and dependence on the field. If $\zeta_K(s) = \kappa_K/(s-1) + a_K + O_K(s-1)$, write $\gamma_K = a_K/\kappa_K$ for the Euler–Kronecker constant. Logarithmically differentiating the completion at 1 and using $\psi_\Gamma(1) = -\gamma$ and $\psi_\Gamma(1/2) = -\gamma - 2\log 2$ gives

$$C_K^\sharp = \log|D_K| + 2\gamma_K - r_1(\gamma + \log(4\pi)) - 2r_2(\gamma + \log(2\pi)) + 4 - 2\sqrt{2}. \tag{4.12}$$

For $K = \mathbb{Q}$, $\gamma_K = \gamma$ and $\kappa_K = 1$. The normalized coefficient is $4 + \gamma - \log(4\pi) - 2\sqrt{2}$, approximately $-0.782236$. Multiplying by $e^\gamma$ recovers Ramanujan's upper constant $-1.393218\ldots$ for the unnormalized divisor ratio [22].

For $L = \log|D_K| \to \infty$, Ihara's upper bound $\gamma_K \le ((L/2+1)/(L/2-1))(2\log(L/2)+1)$ under Dedekind RH for sufficiently large $L$ gives $C_K^\sharp \le L + 4\log L + O(1)$ with an absolute implied constant. The signature terms in (4.12) are nonpositive. Along quadratic fields, Ihara's fixed-degree lower bound also gives $\gamma_K \gg -\log L$ under Dedekind RH, so $C_K^\sharp = L + O(\log L)$ there. Consequently the leading factor 1 of $L$ is sharp for a discriminant-only upper bound under ERH. [21, Section 1.6, Theorem 1 and Corollary 1 to Theorem 3]

The sharp coefficient applies to the Dirichlet specialization by using each cyclotomic field in Theorem 5.1. Theorem 4.2 and (4.12) are manuscript results. The checked ideal and cyclotomic equivalences use the sufficient coefficient in (4.1).

## 5. Dirichlet GRH and Dedekind ERH

**Theorem 5.1** (Dirichlet GRH)**.** *Every zero in $0 < \Re s < 1$ of every Dirichlet $L(s,\chi)$, over all positive moduli and all characters, has real part $1/2$ if and only if (4.1) holds in every cyclotomic field $K = \mathbb{Q}(\zeta_m)$, $m \ge 1$, with its own norm cutoff.*

*Proof.* For $G = (\mathbb{Z}/m\mathbb{Z})^\times$ and $\chi \in \widehat{G}$, let $\chi^*$ denote the primitive character inducing $\chi$. The abelian zeta factorization in this case is

$$\zeta_{\mathbb{Q}(\zeta_m)}(s) = \prod_{\chi \in \widehat{G}} L(s,\chi^*). \tag{5.1}$$

We check the Euler factors, including primes dividing $m$. For a rational prime $\ell$, write $m = \ell^a m'$ with $(m',\ell) = 1$. The inertia quotient is $G/I_\ell = (\mathbb{Z}/m'\mathbb{Z})^\times$. The class of $\ell$ in this quotient is the Frobenius and has order $f$, the residue degree. There are $g = |G/I_\ell|/f$ primes over $\ell$, each of norm $\ell^f$. These assertions follow from cyclotomic ramification and the Frobenius action on roots of unity [16, Ch. 2 and 4]. The primitive character contributes a factor at $\ell$ exactly when $\chi$ is trivial on $I_\ell$. Among these characters every $f$-th root of unity occurs $g$ times at Frobenius, by restriction of characters to its cyclic subgroup. The local product is thus $(1-\ell^{-fs})^{-g}$, the Dedekind Euler factor. A trivial inertia quotient has $f = g = 1$. Equality for $\Re s > 1$ gives (5.1) by continuation.

All the factors in (5.1) are holomorphic in the open strip. A zero of any one of them is a zero of the product, and every zero of the product comes from a factor. Every primitive character occurs in (5.1) at its conductor. If $\chi$ is imprimitive, then

$$L(s,\chi) = L(s,\chi^*) \prod_{\ell \mid m,\ \ell \nmid \mathrm{cond}(\chi^*)} (1 - \chi^*(\ell)\ell^{-s}).$$

The extra factors have no zeros for $\Re s > 0$. Thus the open-strip assertion for all the cyclotomic zeta functions is equivalent to Dirichlet GRH. Apply Theorem 4.1 in each cyclotomic field. □

In Lean, the bridge `between Dirichlet and cyclotomic RH` has the same two quantifiers used in this proof. The argument is composed with the fieldwise ideal equivalence in `the advertised Dirichlet theorem`. In particular the cyclotomic statement enters as a proved intermediate result, while the theorem still speaks directly about Dirichlet $L$-functions.

**Theorem 5.2** (Dedekind ERH)**.** *Every zero in $0 < \Re s < 1$ of $\zeta_K(s)$, for every number field $K$, has real part $1/2$ if and only if (4.1) holds in every number field.*

*Proof.* Apply Theorem 4.1 for each $K$. The threshold in each inequality may depend on $K$. □

In particular Dedekind ERH implies Dirichlet GRH. Over a general base field $K$, a finite-order Hecke character factors through the Galois group of a finite abelian extension $E/K$ by global

reciprocity [6]. The Artin factorization is

$$\zeta_E(s) = \prod_{\eta \in \widehat{\mathrm{Gal}(E/K)}} L(s, \eta),$$

where the factors are the Hecke $L$-functions with their primitive local factors, and the trivial factor is $\zeta_K(s)$. All the factors are holomorphic in $0 < \Re s < 1$. Thus a zero off the critical line of any one factor would be a zero of $\zeta_E$ there. The trivial Hecke character over each $K$ gives the converse. Dedekind ERH is therefore equivalent to the critical-line assertion over all fields for finite-order Hecke characters. Brauer induction and induction invariance of Artin $L$-functions write each Artin $L$-function as a quotient of products of such Hecke factors over intermediate fields [7]. This also excludes Artin zeros and poles off the critical line in the open critical strip under Dedekind ERH. The next criterion includes Hecke characters of infinite order and higher-rank standard functions.

## 6. The exact classical criteria

For a positive integer $n$, let

$$\sigma(n) = \sum_{d|n} d, \qquad \mathcal{A}(n) = \frac{\sigma(n)}{n},$$

and let $\gamma$ be the Euler–Mascheroni constant. Robin's theorem is the following exact criterion. Throughout, RH denotes the standard zero-locus formulation: every zero $s$ of the analytically continued Riemann zeta function which is neither a negative even integer nor the pole at 1 satisfies $\Re s = 1/2$. This is also Mathlib's definition of `RiemannHypothesis`; it is stronger in presentation, but equivalent in the usual way, to saying that zeta has no zero with $\Re s > 1/2$.

**Theorem 6.1** (Robin [46])**.** *The Riemann hypothesis holds if and only if*

$$\sigma(n) < e^{\gamma} n \log\log n \tag{6.1}$$

*for every integer $n > 5040$.*

The cutoff is part of the theorem. In particular, the result is not merely an eventual estimate. It combines an analytic argument above an explicit height with a finite proof below that height.

The second principal result makes Robin's extremal reduction explicit. For $\varepsilon > 0$, put

$$Q_\varepsilon(m) = \frac{\sigma(m)}{m^{1+\varepsilon}} = \frac{\mathcal{A}(m)}{m^{\varepsilon}}. \tag{6.2}$$

We say that $N > 1$ is colossally abundant with parameter $\varepsilon$ if $Q_\varepsilon(m) \le Q_\varepsilon(N)$ for every $m > 1$.

**Theorem 6.2.** *The Riemann hypothesis holds if and only if (6.1) holds for every $N > 5040$ which is colossally abundant with some positive parameter.*

Write $H_n = \sum_{k=1}^{n} 1/k$. Lagarias proved the following elementary reformulation of the Riemann hypothesis [38].

**Theorem 6.3** (Lagarias)**.** *The Riemann hypothesis holds if and only if every positive integer $n$ satisfies*

$$\sigma(n) \le H_n + e^{H_n} \log H_n. \tag{6.3}$$

Colossally abundant numbers were introduced by Alaoglu and Erdős [34]; their use in the criterion is due to Robin. The exact threshold 5040 requires a separate finite argument. Appendix C gives the proof roadmap and checked dependencies for the classical criteria. It includes the extremal reduction, the finite certificates, the explicit large-height estimate and the Nicolas–Landau converse. Appendix L gives the transfer to the harmonic-number bound.

## 7. Critical-line location with prescribed multiplicities

The triangular bound controls the locations of the zeros. A second test records the order at each point, including zeros shared by several factors. We use the sublevel-area test for simple zeros developed in *Prime Factor Oscillations* and extend its threshold to any nonnegative integer.[1] For a holomorphic function $F$ with a nonzero germ at $w$, let

$$A_F(w,R,\eta)=\operatorname{area}\{z\in\mathbb{C}:|z-w|<R,\ |F(z)|\le\eta\}.$$

For every integer $B\ge 0$,

$$\operatorname{ord}_w F\le B \iff \exists R>0:\quad A_F(w,R,\varepsilon^{B+1})=o(\varepsilon^2)\quad(\varepsilon\downarrow 0). \tag{7.1}$$

The disk on the right is taken inside the domain of $F$. To prove (7.1), write $F(z)=(z-w)^m g(z)$, with $m=\operatorname{ord}_w F$ and $g(w)\neq 0$. Choose $r>0$ so the closed disk of radius $r$ lies in the domain and $0<c\le|g(z)|\le C$ on it. For $m=0$ the sublevel area in that disk is zero for $\eta<c$. For $m>0$ and $0<\eta<cr^m$, the sublevel set contains the disk of radius $(\eta/C)^{1/m}$ and is contained in the closed disk of radius $(\eta/c)^{1/m}$. Hence

$$\pi(\eta/C)^{2/m}\le A_F(w,r,\eta)\le\pi(\eta/c)^{2/m}.$$

Substituting $\eta=\varepsilon^{B+1}$ makes the area divided by $\varepsilon^2$ comparable to $\varepsilon^{2(B+1)/m-2}$. Its exponent is positive exactly when $m\le B$. If $m>B$, start with any proposed disk of radius $R$ and use the lower bound in a smaller disk of radius at most $r$ and $R$. The area ratio in the proposed disk is then bounded below by a positive constant or diverges. This proves both directions. Taking $B=1$ and putting $\eta=\varepsilon^2$ gives $A_F(w,R,\eta)=o(\eta)$.

Suppose a chosen factorization on the strip has the form

$$F(s)=U(s)\prod_{j=1}^{r}F_j(s)^{e_j},$$

where $U$ is holomorphic and nowhere zero, the $F_j$ are holomorphic with nonzero germs, and $e_j$ are positive integers. Fix lower bounds $b_j(w)\le\operatorname{ord}_w F_j$. The basic choice is $b_j(w)=1$ at a zero of $F_j$ and 0 elsewhere. A known order or a larger proved lower bound can be used at an exceptional point. Set $B_F(w)=\sum_{j=1}^{r}e_j b_j(w)$. Factoring each germ gives a product of units times a power of $s-w$, whose exponent is the sum of the factor orders with their weights. In particular,

$$\operatorname{ord}_w F-B_F(w)=\sum_{j=1}^{r}e_j\bigl(\operatorname{ord}_w F_j-b_j(w)\bigr).$$

All terms are nonnegative and each $e_j>0$. The area test with cap $B_F(w)$ therefore holds exactly when every factor attains its lower bound. At a common zero the contributions add. If simplicity is required away from a factor's intrinsic center, its condition at that center can instead be omitted in the factorwise test. If an exact central order is prescribed, it can be tested by applying (7.1) at that order and, for positive order, requiring failure at the preceding integer.

Thus the combined location and multiplicity assertion is equivalent to (7.1) at every point of the open critical strip with cap 0 off the line and cap $B_F(w)$ on the line. Equivalently, one can use the corrected Robin inequality or the triangular inequality from the corresponding theorem for location, and test area only on the line.

For a completed function, apply the test to its holomorphic germ after clearing the poles of the completion. In particular the Dedekind norm twist uses $\Xi_K(s+it)$, which clears the translated boundary poles. At a pole of an individual Gamma factor, use the completed germ so that cancellation is accounted for. For meromorphic germs the identity $\operatorname{ord}_w(GF)=\operatorname{ord}_w G+\operatorname{ord}_w F$

[1] See `ZeroSmallValueArea.lean` and the coprimality application `CarmichaelSimplicityCriterion.lean`.

counts poles with negative order. For a standard unitary completion on $\Re w = 1/2$, the Gamma shift bound in Section 3 makes the Gamma factors holomorphic units at $w$. For Dedekind and primitive Dirichlet completions these factors are units throughout the open strip. Multiplication by a holomorphic unit preserves the order and either area test. We may therefore use the following arithmetic carriers, whose identities hold first in the half-plane of absolute convergence and then by continuation:

$$C_\chi(s) = \frac{L(s,\chi)}{L(s+1,\chi)} = \sum_{n\ge 1} \frac{\chi(n)\varphi(n)}{n^{s+1}},$$

$$C_K(s) = \frac{\zeta_K(s)}{\zeta_K(s+1)} = \sum_{\mathfrak{a}\neq(0)} \frac{\varphi_K(\mathfrak{a})}{(N\mathfrak{a})^{s+1}}, \qquad \varphi_K(\mathfrak{a}) = N\mathfrak{a}\prod_{\mathfrak{p}\mid\mathfrak{a}}(1-(N\mathfrak{p})^{-1}).$$

The ideal sum is over nonzero integral ideals. Both identities follow by taking local geometric series with the coprimality weight at each positive exponent. The denominators are nonzero on the strip. For an individual automorphic $L$-function use $C_\pi(s) = Z_\pi(s)/L(s+1,\pi)$, where $Z_\pi$ is its individual pole-cleared completion. The finite part $L(s,\pi)/L(s+1,\pi)$ has local factors

$$\prod_{j=1}^{n} \frac{1-\alpha_{\pi,j}(\mathfrak{p})q^{-s-1}}{1-\alpha_{\pi,j}(\mathfrak{p})q^{-s}}, \qquad q = N\mathfrak{p}.$$

For each Satake coordinate the factor at exponent $k \ge 1$ is $(1-q^{-1})\alpha_{\pi,j}(\mathfrak{p})^k$, with constant term 1. Use the ramified parameters from the local standard factor, padded by zeros as above. The individual completions measure the orders of the original functions. Testing a paired completion instead adds the orders of both members.

For Dirichlet GRH the natural additional requirement is simplicity for each character. Finite Euler factors for imprimitive characters are units in the open strip. For Dedekind ERH use the exponents of the specified holomorphic factorization, for example the character factorization of an abelian field. In a full irreducible Artin factorization, the exponents are the multiplicities in the permutation representation. Using all these factors as holomorphic functions for an arbitrary field requires their holomorphy. Even in the known holomorphic cases, the exponents can exceed one. For the $S_3$ splitting field of $X^3-2$, the standard two-dimensional constituent occurs twice in the regular representation. A zero of its holomorphic $L$-function therefore contributes at least two to the order of the Dedekind zeta function. In the standard-family case, apply the criterion to each individual cuspidal standard completion and specify any intrinsic central exceptions. Higher central orders can occur already in degree two; the level 389 elliptic curve `389a1` supplies an example.[2] The location equivalences above and the area threshold together give criteria with the chosen multiplicity requirements. Proving the resulting area bounds is an additional assertion.

7.1. **Multiplicity from the same prime-power remainder.** For a triangular remainder $R_Z$ as in (3.3), its square-root bound is equivalent to the location condition, and the forward direction of that equivalence gives $|R_Z(x)| = O_Z(\sqrt{x})$. Under these equivalent conditions, $\mathcal{M}R_Z(s)$ is an absolutely convergent integral for $\Re s > 1/2$. Equation (3.3) holds there by analytic continuation. Let $w = 1/2 + i\gamma$, $s = w + \delta$ with $\delta > 0$, and set

$$D_Z(w,\delta) = \delta\, s(s+1)\int_1^\infty R_Z(x)x^{-s-1}\,dx.$$

The logarithmic derivative of $F_Z$ is $\operatorname{ord}_w F_Z/(s-w)$ plus a function holomorphic in a neighborhood of $w$. Thus

$$\lim_{\delta\downarrow 0} D_Z(w,\delta) = \operatorname{ord}_w F_Z.$$

[2] LMFDB entry for `389a1` records analytic rank two.

The limit exists also at a nonzero value of $F_Z$, where it is zero. Consequently the area bound at a prescribed maximum order $B \in \mathbb{Z}_{\geq 0}$ has a test involving the very same arithmetic remainder:

$$\Re D_Z(w,\delta) \;\leq\; B + \tfrac{1}{2} \qquad \text{for every sufficiently small } \delta > 0. \tag{7.2}$$

The half-integer threshold uses the fact that the limit is an integer. Changing variables $x = e^u$ identifies the integral as the exponentially damped Fourier transform of $e^{-u/2}R_Z(e^u)$ at frequency $\gamma$. Thus the square-root bound and the boundary tests (7.2) use one arithmetic input. The order of the full completion agrees with this limit when the additional completion factor is a unit at $w$. In general, if that factor is $G_Z$, the expression for the full completion includes $\delta G_Z'(s)/G_Z(s)$. For a holomorphic product, the boundary test with the sum of proved lower bounds as cap enforces equality factor by factor because each excess order is nonnegative.

7.2. **A global mean-square criterion.** One can test all ordinates together. Assume the location inequality for $R_Z$ and put $A_Z(t) = e^{-t/2}R_Z(e^t)$ for $t \geq 0$, extended by zero for $t < 0$. Write $\rho_\gamma = 1/2 + i\gamma$ and $m_\gamma = \operatorname{ord}_{\rho_\gamma} F_Z$. In the following sums, the ordinates with $m_\gamma > 0$ are distinct. The inversion (3.4) gives

$$A_Z(t) = \sum_\gamma \frac{m_\gamma}{\rho_\gamma(\rho_\gamma + 1)}\, e^{i\gamma t} + o(1) \qquad (t \to \infty). \tag{7.3}$$

The sum converges absolutely and uniformly. Zeros strictly to the left of the line contribute $o(1)$ by dominated convergence using the summable inverse Mellin weights. For the real paired data, the conjugate zeros make the displayed series real. For an individual complex factor, use its own complex prime-power coefficients and omit the real part in the inversion, which follows from the same absolute convergence and uniqueness. Pure norm translates use the corresponding translated ordinate and polar term.

Use the Fourier convention $\widehat{k}(\gamma) = \int_{\mathbb{R}} k(v)e^{-i\gamma v}\,dv$. For $a = 1, 2$, put

$$g_a(v) = \frac{e^{-v^2/(4a)}}{\sqrt{4\pi a}}, \qquad k_a = \Big(\frac{d}{dv} + \tfrac{1}{2}\Big)\Big(\frac{d}{dv} + \tfrac{3}{2}\Big)g_a, \qquad H_a(t) = \int_{\mathbb{R}} k_a(v)A_Z(t-v)\,dv.$$

In particular, $\widehat{k}_a(\gamma) = \rho_\gamma(\rho_\gamma + 1)e^{-a\gamma^2}$. Let $E \subset \mathbb{R}$ be a finite set of ordinates at which a different order is allowed. Define $\widetilde{H}_a(t)$ by subtracting, for $\gamma \in E$, the terms $h_a(\gamma)e^{i\gamma t}$, where

$$h_a(\gamma) = \lim_{T\to\infty} \frac{1}{T}\int_0^T H_a(t)e^{-i\gamma t}\,dt.$$

The basic case is $E = \varnothing$. All transforms use the arithmetic remainder and the chosen exceptional ordinates.

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\widetilde{H}_1(t)|^2\,dt \;\leq\; \limsup_{t\to\infty} \Re\widetilde{H}_2(t) \tag{7.4}$$

holds if and only if every zero on the line outside $E$ is simple. To see this, convolve (7.3) termwise with $k_a \in L^1(\mathbb{R})$. The remainder is bounded and tends to zero at positive infinity, so its convolution also tends to zero. The smoothed series has coefficient $m_\gamma e^{-a\gamma^2}$. Averaging its product with $e^{-i\gamma t}$ recovers that coefficient, giving the asserted exception removal.[3]

[3]For logarithmic frequency recovery in the zeta setting, see `XiZeroLogSpectrum.lean`.

Error bounds and nearby ordinates. For one individual function, let $r_Z(t)$ be the difference between $A_Z(t)$ and the line series in (7.3), defined also for $t<0$. By (3.4) there is a constant $C_Z$ for which, at $t\ge 0$,

$$|r_Z(t)|\le \mathcal{E}_Z(t):=C_Z(1+t)e^{-t/2}+\sum_{\substack{\rho\notin\{0,-1\}\\ \Re\rho<1/2}}\frac{e^{-(1/2-\Re\rho)t}}{|\rho(\rho+1)|}.$$

The sum repeats a zero of order $m$ exactly $m$ times. The weights have finite total mass. At any $d>0$, the part with $1/2-\Re\rho\ge d$ is bounded by that total mass times $e^{-dt}$. The other part is bounded by the weight of $0<1/2-\Re\rho<d$, which tends to zero as $d\downarrow 0$. This proves decay even when no uniform positive horizontal gap is available.

Write the smoothed signal as the line series with coefficients $m_\gamma e^{-a\gamma^2}$ plus $r_{Z,a}=k_a*r_Z$. The explicit kernel is

$$k_a(v)=\left(\frac{v^2}{4a^2}-\frac{v}{a}+\frac{3}{4}-\frac{1}{2a}\right)g_a(v).$$

Let $M_Z=\sup_{\mathbb{R}}|r_Z|$, which is finite by the series bound and the zero extension of $A_Z$, and choose $C_a'$ so $|k_a(v)|\le C_a'(1+v^2)e^{-v^2/(4a)}$. Splitting the integral at $|v|=t/2$ gives, for $t\ge 2$,

$$|r_{Z,a}(t)|\le \|k_a\|_1\,\mathcal{E}_Z(t/2)+M_Z C_a''e^{-t^2/(32a)},\quad C_a''=C_a'\int_{\mathbb{R}}(1+v^2)e^{-v^2/(8a)}\,dv. \tag{7.5}$$

For the inner part, $\mathcal{E}_Z$ is decreasing on $[1,\infty)$; for the outer part the Gaussian supplies the displayed tail. For norm twists these bounds apply in the translated ordinate coordinates used by the Dedekind test.

For an ordinate $u$, the finite-time average used to compute $h_a(u)$ differs from $m_u e^{-au^2}$ by a quantity of modulus at most

$$\frac{1}{T}\int_0^T|r_{Z,a}(t)|\,dt+\sum_{\gamma\ne u}m_\gamma e^{-a\gamma^2}\min\left(1,\frac{2}{T|\gamma-u|}\right).$$

Indeed, the average of $e^{i(\gamma-u)t}$ is bounded by the displayed minimum. The error converges to zero by (7.5) and dominated convergence over the absolutely summable Gaussian weights. This argument needs no global minimum spacing of distinct zeros. Nearby frequencies can slow finite-time recovery, while coincident frequencies contribute a single coefficient with their total order.

The energy computation has the same control. Put $v_\gamma=m_\gamma e^{-\gamma^2}$ for the retained ordinates and $V=\sum_\gamma v_\gamma$. The cross terms at distinct frequencies in the average squared modulus are bounded by

$$\sum_{\gamma\ne\gamma'}v_\gamma v_{\gamma'}\min\left(1,\frac{2}{T|\gamma-\gamma'|}\right),$$

with both indices restricted to the retained set. The contribution from replacing the line series by the smoothed signal is at most $T^{-1}\int_0^T(2V|r_{Z,1}(t)|+|r_{Z,1}(t)|^2)\,dt$. Both errors tend to zero. In particular high orders enter with full weight $m_\gamma$ in the recovered coefficient and with $m_\gamma^2$ in the energy.

Orthogonality of distinct frequencies in the mean and absolute convergence give the left side of (7.4) as $\sum_{\gamma\notin E}m_\gamma^2e^{-2\gamma^2}$. The real part of the $a=2$ series is at most the sum of its nonnegative coefficients. It approaches that bound arbitrarily far out. For a finite set of frequencies, take the difference of two arbitrarily distant indices in a convergent subsequence of $(e^{i\gamma n})_\gamma$ in the finite torus. At those differences all the phases approach 1. Absolute convergence makes the tail of the

infinite series arbitrarily small. The right side is therefore $\sum_{\gamma\notin E} m_\gamma e^{-2\gamma^2}$. Their difference is the sum of $m_\gamma(m_\gamma-1)e^{-2\gamma^2}$, which vanishes exactly when every retained zero is simple.

For a chosen positive factorization, apply (7.4) to the individual factors and sum the two sides with the factorization weights $e_j$. This gives one inequality for the product. A common zero contributes once to each factor's test, and its order in the product is the weighted sum. To test a prescribed exact order $r$ at an exceptional ordinate $\gamma$, add $|h_1(\gamma)-re^{-\gamma^2}|^2$ to the left side for that factor. Under the location criterion, the resulting inequality requires those exact orders and simplicity at every remaining zero. If $G_Z$ is the additional factor in the full completion, set $g_\gamma=\operatorname{ord}_{\rho_\gamma} G_Z$, allowing negative orders at poles. Before removing exceptional frequencies, add

$$\sum_{\gamma:\, g_\gamma\neq 0} g_\gamma e^{-a\gamma^2}e^{i\gamma t}$$

to $H_a$. The sum is finite for the gamma, exponential, and polynomial factors used here. The resulting coefficients use the orders in the full completion. When $G_Z$ is a unit on the line, the inversion in (7.3) gives these multiplicities directly.

7.3. **Four conditional multiplicity equivalences.** The area calculation above detects a local order bound on a holomorphic germ. The advertised Lean theorems instead test the same zero-order assertions by the arithmetic mean-square criterion. The `standard-family multiplicity proof` uses the individual completed functions and their logarithmic data. The `Dedekind prescribed-order proof` uses the prime ideals of the field directly. The `rank-one prescribed-order theorem` combines this Dedekind test with the individual pole-free Hecke tests and shifts the norm-character ordinates.

Write $\mathcal{E}(Z;E)$ for the mean-square limit on the left of (7.4), including the completion correction where needed. Write $\mathcal{S}(Z;E)$ for the limit superior on the right. For a finite set $J$ and prescribed orders $r(\gamma)\geq 0$ on $J$, put

$$\mathcal{P}(Z;J,r)=\sum_{\gamma\in J}|h_1(\gamma)-r(\gamma)e^{-\gamma^2}|^2.$$

Here $h_1$ is computed before removing the ordinates in $E$. Thus $E\setminus J$ permits arbitrary orders, and $J$ prescribes exact orders. Orders prescribed outside $E$ are also subject to the simplicity requirement.

**Theorem 7.1** (RH and simplicity)**.** *Assume the Riemann hypothesis. Every nontrivial zero of $\zeta$ is simple if and only if*

$$\mathcal{E}(\zeta;\varnothing)\leq\mathcal{S}(\zeta;\varnothing).$$

*The input is $R_Z$ for $Z=\zeta_{\mathbb{Q}}$.*

**Theorem 7.2** (Dirichlet GRH and simplicity)**.** *Assume Dirichlet GRH. Every nontrivial zero of every Dirichlet $L$-function is simple if and only if, for every Dirichlet character $\chi$,*

$$\mathcal{E}(L(\cdot,\chi);\varnothing)\leq\mathcal{S}(L(\cdot,\chi);\varnothing).$$

*It suffices on both sides to take primitive characters.*

**Theorem 7.3** (Dedekind ERH and simplicity)**.** *Let $K$ be any number field and assume Dedekind RH for $K$. Every nontrivial zero of $\zeta_K$ is simple if and only if*

$$\mathcal{E}(\zeta_K;\varnothing)\leq\mathcal{S}(\zeta_K;\varnothing).$$

*The arithmetic input is $R_{\zeta_K}$, with one term of weight $\log N\mathfrak{p}$ for each prime ideal power in the triangular sum.*

**Theorem 7.4** (Multiplicity in a positive Dedekind factorization)**.** *Assume Dedekind ERH. Fix a positive holomorphic factorization $\zeta_K = U_K \prod_{j\in I_K} L_{K,j}^{e_{K,j}}$ on the strip for each field in the test, with $U_K$ a unit and factors admitting the prime-power inversion above. Then for every such $K$ and $w$ on the line,*

$$\operatorname{ord}_w \zeta_K = \sum_{j\in I_K} e_{K,j}\, \mathbf{1}_{\{L_{K,j}(w)=0\}}$$

*if and only if, for every such $K$,*

$$\sum_{j\in I_K} e_{K,j}\mathcal{E}(L_{K,j};\varnothing) \le \sum_{j\in I_K} e_{K,j}\mathcal{S}(L_{K,j};\varnothing).$$

*In a full holomorphic Artin factorization the weights are the irreducible multiplicities of the permutation representation.*

**Theorem 7.5** (Grand RH and multiplicity of standard $L$-functions)**.** *Assume the critical-line condition for the standard family. For each individual unitary cuspidal standard $L$-function over a number field, fix finite sets $E_\pi, J_\pi$ and orders $r_\pi$ as above. All nontrivial zeros outside $E_\pi$ are simple and all orders on $J_\pi$ equal $r_\pi$, for every $\pi$, if and only if, for every $\pi$,*

$$\mathcal{E}(L(\cdot,\pi);E_\pi) + \mathcal{P}(L(\cdot,\pi);J_\pi,r_\pi) \le \mathcal{S}(L(\cdot,\pi);E_\pi).$$

*An intrinsic center can be included in $E_\pi$ with its order left free or placed in $J_\pi$ with an exact order specified. For a pure norm twist of the trivial character use the untranslated zeta function, shifting the prescribed ordinates accordingly.*

*Proof.* The RH and Dirichlet statements apply (7.4) to the individual functions. The additional Euler factors of an imprimitive character are units in the open strip.

For Theorem 7.3, use $F_K(s) = (s-1)\zeta_K(s)$. Its logarithmic derivative for $\Re s > 1$ is

$$\frac{F_K'}{F_K}(s) = \frac{1}{s-1} - \sum_{\mathfrak{p}}\sum_{j\ge 1} \log N\mathfrak{p}\,(N\mathfrak{p})^{-js}.$$

The series is absolutely convergent in this half-plane. The completed pole-cleared function $sZ_\infty(K,s)F_K(s)$ is entire of order at most one. The reciprocal of $Z_\infty(K,s)$ is a product of reciprocal Gamma factors and exponentials of linear functions, each entire of order at most one. Multiplying the two functions gives $sF_K(s)$ by analytic continuation. Removing the factor $s$ preserves the order-one bound, so the inversion applies to $F_K$. The simple pole at 1 and nonvanishing of $\zeta_K$ on $\Re s = 1$ make $F_K$ nonvanishing on $\Re s \ge 1$. Dedekind RH then gives $\Re s \le 1/2$ at every zero of $F_K$. The factor $s-1$ is a unit throughout the open strip, so the order at each critical-line zero is the order of $\zeta_K$.

In Theorem 7.4, positivity of the factorization makes every zero of a factor in the open strip a zero of $\zeta_K$, hence a critical-line zero under ERH. The difference of the two weighted sums is nonnegative and is zero precisely when each factor has only simple zeros. Additivity of orders gives the displayed equality, including coincidences between different factors. For Grand RH the difference of the two sides is again a sum of nonnegative multiplicity terms plus the squares in $\mathcal{P}$. Each vanishes precisely at the claimed orders. □

The four equivalences assume the corresponding location hypothesis. Combining any of them with the location equivalence gives a simultaneous arithmetic criterion for location and the specified orders. The direct Dedekind test applies to every number field using its own prime ideals. The factorwise refinement uses a supplied holomorphic factorization. An application of that refinement with all irreducible Artin factors requires their holomorphy.

## 8. Related zero-location and multiplicity criteria

*The Riemann zeta function.* Write $\psi(t) = \sum_{n \le t} \Lambda(n)$. For $x \ge 2$ the triangular remainder for $\zeta_{\mathbb{Q}}$ satisfies

$$R_{\zeta_{\mathbb{Q}}}(x) = \frac{2}{x} - \frac{1}{x}\int_2^x (\psi(t) - t)\, dt.$$

Johnston [19, Lemmas 2.6 and 2.8 in the arXiv version] proves an absolute $O(x^{3/2})$ bound on the integral under RH. If RH fails, his negative oscillation result with $c = 0$ gives values of the integral below $-c_1 x^{3/2+\eta}$ at arbitrarily large $x$, for some $c_1, \eta > 0$. Thus the exact eventual one-sided $O(\sqrt{x})$ criterion in the zeta case already follows from Johnston's results.

*Smoothed prime ideals.* For $Z = \zeta_K$, the sum subtracted in (3.2) is $x^{-1}\int_1^x \psi_K(t)\, dt$, with $\psi_K$ defined in Section 4. This is the first Riesz mean of Grenié and Molteni divided by $x$. Their explicit two-sided estimate under Dedekind RH implies the one-sided bound used here [18, Theorem 1.1]. The converse can be obtained from the same first-mean Mellin formula by Landau's argument, as in the proof of Theorem 3.1. Consequently the fieldwise one-sided condition and their explicit estimates are equivalent through Dedekind RH in the stated range. The explicit two-sided estimate also supplies quantitative information. The ideal-divisor inequality (4.1) gives a separate norm-budget form with an explicit correction coefficient and a field-dependent cutoff.

*Inverse Dirichlet series.* Dixit, Gupta, and Vatwani [29, Theorem 1.4] use coefficients $c_K(n)$ of $1/\zeta_K(s)$ and the Mellin–Barnes kernel $Z_{r_1,r_2}$ associated with $\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}$. Their statistic is

$$P_K(y) = \sum_{n \ge 1} \frac{c_K(n)}{n} Z_{r_1,r_2}(\sqrt{y}/n).$$

The estimate $P_K(y) = O_{K,\delta}(y^{-1/4+\delta})$ for every $\delta > 0$ implies Dedekind RH. Their GRH asymptotic has a signature-dependent logarithmic term when $r_1 + r_2 - 1 > 0$. For unit rank zero, which includes $\mathbb{Q}$ and the imaginary quadratic fields, the decay estimate and our triangular test are equivalent through Dedekind RH. In higher unit rank the triangular test implies the RH asymptotic with its extra term. The inverse-series theorem gives the implication from the raw decay estimate to RH. These directions give no strict separation between the arithmetic tests in general. Garg and Maji give criteria for a class of $L$-functions, including full-level holomorphic Hecke eigenforms, using exponential weights on inverse Dirichlet coefficients [30, Theorem 3.10 and Corollary 3.12]. Our triangular statistic is a finite sum at each $x$. Its Mellin transform contains the logarithmic derivative $Z'/Z$ with denominator $s(s+1)$, so a zero of any order contributes a simple pole with residue proportional to its order. Section 7 uses this fact to detect multiplicity.

*Li coefficients and the choice of test.* For the unitary cuspidal standard family over $\mathbb{Q}$ in Lagarias's formulation, nonnegativity of the real parts of all the Li coefficients is equivalent to the critical-line condition [31, Theorem 2.4]. For a common standard test, both criteria therefore have the same logical strength. The Li test uses one inequality at each positive integer index of the sequence attached to the completed function. The triangular test uses the finite sum of local logarithmic coefficients at each real cutoff $x$. Its forward proof is an explicit formula with absolutely summable zero weights $1/|\rho(\rho+1)|$, and the converse uses only an eventual upper bound on that sum's signed remainder. This is useful when an arithmetic estimate naturally gives a lower bound on the smoothed prime-power total. The same logarithmic derivative also records zero orders through simple residues, which Section 7 uses for a separate test.

*Other one-sided prime-power criteria.* Suzuki's criterion for self-dual $L$-functions uses eventual constancy of sign for sums associated to a quotient of the completions shifted by $-\omega$ and $+\omega$, for every $0 < \omega < 1/2$. The converse assumes real-axis nonvanishing to the right of the tested threshold [26, Theorem 2.3 and Eq. (4.3)]. In a separate result for Dirichlet characters, Suzuki

studies the weight $n^{-1/2}\log(x/n)$, deriving a zero-location implication from an eventual sign and equivalences from limiting conditions under their stated hypotheses [28, Theorems 6 and 8].

In the Alpöge–Claude argument, the one-sided estimates concern positive combinations of centered twisted prime-power sums on arbitrarily large power windows $[Y, Y^\lambda]$. The raw sum has weight $x/n-1$ and Mellin denominator $s(s-1)$ [5, Eq. (2.3) and Proposition 2.2] [27, Proposition 2.1 and Eq. (2.1)]. For a raw sum $S(x)=\sum_{n\le x} a(n)(x/n-1)$, the triangular sum with the same coefficients is

$$\sum_{n\le x} a(n)(1-n/x) = S(x) - \frac{2}{x}\int_1^x S(t)\,dt.$$

The negative coefficient on the integral means that a one-sided upper bound for $S$ alone does not directly bound this difference.

*Contribution for the standard automorphic family.* Theorem 3.1 gives an equivalence with a single eventual upper bound $R_Z(x)\le C_{K,Z}\sqrt{x}$ for each prescribed test $Z$, using the fixed prime-power weight $1-q^j/x$. Both the constant and the cutoff may depend on the field and the test. The family covers standard $L$-functions of unitary cuspidal representations of $\mathrm{GL}_n$ over every number field and in every rank. Pairing a factor with its contragredient gives real coefficients that may have either sign. At a real zero $\beta>1/2$ of order $r$, the triangular remainder has Mellin residue $r/[\beta(\beta+1)]>0$. It enters the Mellin transform of the nonnegative comparison with a minus sign, so Landau's argument also excludes real off-line zeros. The criterion combines this treatment of real zeros with the exact square-root exponent for one fixed weight across the whole standard family.

8.1. **Comparison with earlier multiplicity criteria.** The area and Mellin tests in Section 7 use classical local facts. A zero of order $m>0$ gives sublevel area of order $\eta^{2/m}$, and $F'/F$ has residue $m$ at that zero. The arithmetic feature of (7.2) is that it recovers the residue from the triangular remainder used in the location theorem.

*Mean square and the simplicity defect.* Akbary, Ng, and Shahabi compute mean squares of logarithmically normalized prime-counting errors by almost-periodic Parseval. Their automorphic formula under the critical-line hypothesis already has squared multiplicities [24, Theorem 1.14 and Corollary 1.15(i)]. Caceres goes further for $\zeta$. He computes the log-scale mean square of a slope obtained from a logarithmically weighted prime-power sum. Under RH the energy is $2\sum_{\gamma>0} m_\gamma^2/\gamma^2$, with the sum over distinct positive zero ordinates. He compares it with $(\log\xi)''(1/2) = 2\sum_{\gamma>0} m_\gamma/\gamma^2$. His Theorem 4.6 gives the joint equivalence with RH and simplicity. If RH fails, the log-scale mean square has infinite limit superior [23, Proposition 4.1 and Theorem 4.6]. This is a precedent for a global arithmetic simplicity criterion based on the defect $m_\gamma^2-m_\gamma$.

*The multiplicity spectrum from screw functions.* Suzuki's screw function in the extended Selberg class already contains the multiplicities. In the semi-extended Selberg class he computes it from logarithmic Dirichlet coefficients and completion data, without a list of zeros [25, Eq. (1.1) and Theorem 4.1]. Write $\Psi_F=-g_F$ in Suzuki's notation and use the ordinate convention $\rho_\gamma=1/2+i\gamma$ of this paper. Under the critical-line condition, $\Psi_F$ is the sum of a linear term, $m_0t^2/2$, and the series with terms $m_\gamma(1-e^{i\gamma t})/\gamma^2$ for distinct nonzero ordinates. Its convolution with the second derivative of the Gaussian $g_a$ from Section 7 is

$$(g_a''*\Psi_F)(t) = m_0 + \sum_{\substack{\gamma\neq 0\\ m_\gamma>0}} m_\gamma e^{-a\gamma^2}e^{i\gamma t}.$$

Indeed, the convolution kills the affine terms, sends $t^2/2$ to 1, and has Fourier multiplier $-\gamma^2e^{-a\gamma^2}$ on the nonconstant modes. The series before convolution is absolutely summable and $g_a''$ is

integrable, so the convolution can be taken termwise. This gives the same limiting multiplicity spectrum as our Gaussian-smoothed triangular signal. Suzuki also uses the screw function in an eventual sign criterion for location under a real-axis nonvanishing hypothesis [25, Theorem 1.1].

*Contribution of the triangular test.* Equation (7.4) gives both sides of the simplicity comparison from $e^{-t/2}R_Z(e^t)$, using the fixed weight $1-q^j/x$ and the completion correction where required. The log-scale mean square of the first smoothing and the limit superior of the real part of the second have matching weights. In the notation of Section 7, their difference is

$$\mathcal{E}(Z;E)-\mathcal{S}(Z;E)=\sum_{\substack{\gamma\notin E\\ m_\gamma>0}} m_\gamma(m_\gamma-1)e^{-2\gamma^2},$$

where the sum is over distinct retained ordinates. The basic simplicity test uses no zero ordinates as inputs. For prescribed orders, the only ordinates supplied are the finite exceptions and the points at which an order is prescribed.

The location theorem uses paired coefficients where necessary to give a real remainder. The multiplicity comparison uses the same triangular construction for each individual factor, so a zero shared by two factors contributes separately to the two tests. Theorem 7.5 applies this formulation to the standard unitary cuspidal family in every rank over every number field and allows finite prescribed orders, including at the center. Theorem 7.3 uses the field's own prime ideals, and Theorem 7.4 gives the corresponding weighted comparison when a positive holomorphic factorization is supplied.

8.2. **Comparison with formal proofs and other Lean criteria.** The Alpöge–Claude development proves the classical and Dirichlet zero-location assertions themselves. The five theorem statements in Sutherland's verification record range over the Riemann zeta function and Dirichlet $L$-functions, including the statement for all characters over all positive moduli [1, 2]. Alpöge describes a Stepanov-inspired argument that uses chains of theta lifts to control von Mangoldt sums [1]. The scope of the present formalization is arithmetic equivalence. Its proof relates the zero assertions to divisor and prime-power bounds through Mellin transforms, and its Dedekind theorem quantifies over every number field. For an actual higher-rank standard family, the generic Lean equivalence uses the analytic inputs listed in Section 3.2.

The two developments can be composed at the level of theorem types. Sutherland's verification record combines Alpöge's proofs with an earlier Robin1984 revision to obtain the classical Robin and Lagarias inequalities and the corrected ideal inequality for every cyclotomic field [2]. The all-field and standard automorphic zero assertions are outside the scope of that composition. The reusable part of the infrastructure here is the proof that a bound on a real prime-power remainder controls poles of a Mellin transform under explicit analytic hypotheses. The individual version of the arithmetic data also supports the prime-power multiplicity tests.

The Jensen–Pólya criterion concerns real-rootedness of polynomials formed from the central Taylor coefficients of a completed zeta function [3]. The related *RiemannJensen* Lean project states and proves root-count bounds for the unshifted Jensen polynomials [4]. Its Dedekind theorem gives at least $d-C_K\sqrt{d\log d}$ distinct negative real roots in degree $d$ for sufficiently large $d$, with constants depending on $K$. This is a quantitative approximation to real-rootedness. The equivalence here uses the prime ideals of $K$ and uses the logarithmic derivative instead of Taylor coefficients. It gives an exact criterion for the zero assertion over $K$.

## 9. Formal verification and source availability

The formalization used here is release `v1.3.0` at `b359826`. Appendix B reproduces the exact types of all ten headline declarations. The `Challenge.lean` file supplies Mathlib-only definitions and the statements to be checked. It uses proof placeholders in those statements. The proofs

are in `Solution.lean`. All ten entries in `comparator.json` are proved in Lean and replayed at their displayed types. The table gives their mathematical content and the names checked by Comparator. Rows 1–6 concern location. Rows 7–10 test multiplicity assuming the indicated location condition. Rows 1–3 are in namespace `Robin1984` and rows 4–10 are in `Robin1984.GRH`. Each declaration links to its source and uses the same number as the exact type in Appendix B.

| | Headline equivalence and Lean declaration |
|---|---|
| 1 | Classical RH $\iff$ Robin's strict inequality for every integer $n > 5040$ (Theorem 6.1). `robin_inequality_iff_riemannHypothesis` |
| 2 | Classical RH $\iff$ Robin's inequality for every colossally abundant integer $n > 5040$ (Theorem 6.2). `riemannHypothesis_iff_colossallyAbundant_robin` |
| 3 | Classical RH $\iff$ Lagarias's harmonic-number inequality for every $n \geq 1$ (Theorem 6.3). `riemann_hypothesis_iff_lagarias_elementary_criterion` |
| 4 | Dirichlet GRH for all moduli and characters $\iff$ the corrected ideal inequality (4.1) in every cyclotomic field. `dirichletGRH_iff_cyclotomicCorrectedIdealRobin` |
| 5 | Dedekind ERH $\iff$ the corrected ideal inequality (4.1) in every number field, with its own cutoff. `dedekindERH_iff_correctedIdealRobin` |
| 6 | All supplied completed zeros lie on the line $\iff$ each paired test satisfies the eventual triangular upper bound (3.2). Inputs are A1–A6. `grandRH_iff_triangularPrimePowerBound` |
| 7 | Under RH, simplicity of the nontrivial zeta zeros $\iff$ the zeta mean-square inequality (7.4) with no exceptions. `riemannRH_iff_globalPrimePowerMultiplicityInequality` |
| 8 | Under Dirichlet GRH, simplicity of the nontrivial zeros of every Dirichlet $L$-function $\iff$ the individual inequality (7.4) for every character. `dirichletGRH_iff_globalPrimePowerMultiplicityInequality` |
| 9 | For a field $K$ under Dedekind RH, simplicity of the nontrivial zeros of $\zeta_K \iff$ the prime-ideal mean-square inequality (7.4). `dedekindERH_iff_globalPrimePowerMultiplicityInequality` |
| 10 | With supplied individual data and the line condition, simplicity outside $E$ and the prescribed orders on $J$ for every member $\iff$ the mean-square inequality with the squared order errors of Theorem 7.5. `grandRH_iff_globalPrimePowerMultiplicityInequality` |

Rows 1–5 prove unconditional equivalences between the stated propositions. In particular the location assertion in an equivalence is itself one side of the theorem. Rows 7–10 take the location assertion as an input to a separate multiplicity equivalence. Only rows 6 and 10 take a standard-family analytic record as an input. The properties of this record for actual higher-rank representations come from the cited analytic theorems in Sections 3 and 3.2.

The fieldwise helper `dedekindRH_iff_correctedIdealRobin` underlies statements 4 and 5. Statement 9, for Dedekind simplicity, uses prime-ideal coefficients for each field. The separate result for a positive holomorphic factorization takes its factors and their analytic properties as inputs. In the standard family, statement 6 uses paired Euler logarithms, and statement 10 uses the individual ones. The `Robin1984/GRH/HeckeStandardRankOne.lean` module constructs

both sets of analytic inputs for the actual standard $\mathrm{GL}_1$ family, as detailed in Appendix N. For higher ranks the paper uses the cited analytic theorems to instantiate the reasoning.

9.1. **What was checked.**
Scope for the correction constants. Rows 4 and 5 verify the corrected ideal inequality with $12 + 3\log|D_K|$, and row 6 verifies the existence of a triangular coefficient and cutoff for each test with the supplied analytic data. The sharp limsup formulas in Theorems 3.3 and 4.2, including the refined cyclotomic coefficient, are proved in the manuscript. Neither is among the ten exported theorem types or part of the reported kernel replays.

The Lean build job and the Comparator job both succeeded on the same source commit. The latter job exports Challenge and Solution separately. Its transcript lists the same ten theorem names in both exports. Comparator checks the exported theorem types and all fixed definitions they reach. The `comparator.json` file allows no definition holes. Comparator also traverses the Solution proof dependencies and permits only `propext`, `Classical.choice`, and `Quot.sound`. This rejects, in particular, any transitive use of `sorryAx` in a solution.
Analytic hypotheses and the verification boundary. A1–A6 are fields of the proposition `HasStandardAutomorphicAnalyticsfamily`. The location theorem takes a proof of that proposition as an explicit argument. These fields assert properties of the data supplied to the theorem. In a concrete instantiation, Lean must receive proofs of these properties before the equivalence can be applied to that family. The axiom audit checks the dependencies of the proof of the implication with its stated arguments.
Trusted components of the replay. The reference point is the pinned Mathlib-only statement environment and its fixed definitions. The proof export, Comparator's statement and axiom checks, and the isolated build environment must preserve that reference point. The replay submits the resulting Solution export to Lean's default kernel and to NanoDa, an independently implemented checker of the same type theory. This gives two checks of the proof terms against their types. The pinned Comparator documentation spells out the assumptions on the statement environment and isolation. Sutherland's separate replay in Section 9.3 checks concrete compositions with zero-location proofs, using a different challenge consisting of the three arithmetic consequences. The last three acceptance lines in the hosted Robin1984 job transcript of 26 November 2026 are:

```
Nanoda kernel accepts the solution
Lean default kernel accepts the solution
Your solution is okay!
```

The release supplies the source commit, the two job records and the matching compiled archive. The archive is a Linux build artifact, and the checkers used source exports. The record validates the types printed in Appendix B. For supplied-data theorems it verifies the implication from those data. For actual higher-rank representations, the mathematical identification of the data is the argument in Section 3.2. The concrete rank-one instantiation is a separate library result documented in Appendix N. The hosted statement replay is for the ten declarations listed here. It does not export the specialized rank-one declaration as an additional headline.

9.2. **Source and replay recipe.** Use Lean 4.33.1 and the package revisions in `lake-manifest.json`, listed in Appendix A. In a Linux checkout with Git, Elan, Go, Cargo and Python 3, the build and statement replay are:

```
git clone https://github.com/kimihiro64/Robin1984.git
cd Robin1984
git checkout b359826e62c50b6cef55db4b2fcab793232498cc
lake exe cache get
lake build
```

`bash scripts/verify-comparator.sh`

The cache step retrieves the artifacts matching the pinned Mathlib dependency. The `scripts/verify-comparator.sh` recipe fixes the export and checker versions, together with the Landrun sandbox. The `comparator.json` file specifies both modules and the entire theorem list. The linked hosted jobs document the actual execution of the build and replay. The binary archive `robin1984-linux-lean-v1.3.0.tar.zst` has SHA-256

`fd95a4de7c8999053672b3e2bdfc11b4bc085e256ec2fd402acc439c7c6f0896`

9.3. **Use in Sutherland's independent GRH verification.** Andrew Sutherland's verification report [2] records a separate check of Robin1984 at commit `5ce18bf3`. That revision exposed five theorems to Comparator. Sutherland reports acceptance of all five by unmodified Comparator with NanoDa and Lean's default kernel. A separate replay from exports also used Lean's paranoid kernel. The Comparator transcript records the former check.

The report publishes the three consequences as a separate Mathlib-only challenge. The proofs in `ArithSolution.lean` import `Main` from the Alpöge–Claude submission and `Solution` from Robin1984. They apply three of the biconditionals in the headline table as follows.

(i) `robin_inequality` proves $\sigma(n) < e^{\gamma} n \log\log n$ for every integer $n > 5040$. It applies the right-to-left implication of row 1 to `riemannHypothesis` from `Main`.

(ii) `lagarias_inequality` proves $\sigma(n) \le H_n + e^{H_n} \log H_n$ for every $n \ge 1$. It applies the left-to-right implication of row 3 to the same classical RH proof.

(iii) `cyclotomic_corrected_ideal_robin` proves that for every $m \ge 1$ there is a cutoff $A_m \in \mathbb{N}$ such that every nonzero integral ideal of $\mathbb{Q}(\zeta_m)$ with norm at least $A_m$ satisfies (4.1). It applies the left-to-right implication of row 4 to `LFunction_eq_zero_re`, the theorem of `Main` for zeros in the open strip of every Dirichlet $L$-function, with all characters allowed.

In this composition the zero-location theorems are proof inputs, so the three arithmetic conclusions have no remaining RH or GRH assumption. The Mathlib-only challenge spells out the integer inequalities and gives the fixed ideal-divisor and cutoff definitions for the third result. The report records statement comparison by con-leche and con-ron against that challenge, a per-theorem axiom audit, and an independent NanoDa replay of the combined export. The scope of this composition is the three displayed consequences. The ten-theorem evidence for v1.3.0 comes from the separate hosted replay in Section 9.1.

## Appendix A. Pinned dependency and checker versions

The Lean toolchain is `leanprover/lean4:v4.33.1`. The package pins below are from `lake-manifest.json` at commit `b359826`. The last four entries are the checker's tooling pins from `scripts/verify-comparator.sh`.

| Dependency | Git commit |
|---|---|
| `mathlib` | `0df444a360eaa60ab8c11dca51a86af692955474` |
| `leancert` | `621a43d7cf21f87872392a01e874f2f1dbddc926` |
| `PrimeNumberTheoremAnd` | `f8f58c749d6cde8a641348fcd5e4702993651cd6` |
| `plausible` | `b7eb3304aeae834b12dda98993a37f6a41f6f0bb` |
| `LeanSearchClient` | `5f4d51b81cbd3f6b32b156bfad9056621a040404` |
| `importGraph` | `16f02aa7642864af59f1ff0e384a015994db9118` |
| `proofwidgets` | `4be2e3d5087eeb272cf5a8853b8f9dd025ef5957` |
| `aesop` | `3448c0bcc5ce01b2d1546e483ec3620e32df3d0e` |
| `Qq` | `92c15be17b7caf78c2ad767ec40f89052d908d81` |

| Dependency | Git commit |
|---|---|
| `batteries` | `4488d40d070b9700d4d5a6aa342f0d40c31b2a2d` |
| `PrimeCert` | `916ee9c35a57af128d5089a69172415051e700ca` |
| `LeanArchitect` | `d9013cc08bd2b5483e837368dfa4cc7ead92a5c2` |
| `Cli` | `6130a47896ce867c6a4a55373441e59e565bad0f` |
| Comparator | `68a064109f01c08f47c8edc9f51d6a2bbffaa188` |
| lean4export | `15f6055e299ad5b89345e533cc2192f4cc00f659` |
| Landrun | `811cfff51ceaf3d9843708aa6d22e9b84ccac8b4` |
| NanoDa | `68d5ca9db226849b41a6fff59d796ff19d0a8840` |

## Appendix B. Exact headline Lean statements

These are verbatim statement excerpts from `Challenge.lean` at the verified revision in Section 9, with the proof markers omitted. Statements 1–3 are in namespace `Robin1984`; statements 4–10 are in `Robin1984.GRH`. The links give the original line ranges and their definition context. The prefixes of statements 6 and 10 display the supplied standard family and its analytic hypotheses. Comparator compares the elaborated versions of these types and the fixed definitions on which they depend.

Statement 1.

```
theorem robin_inequality_iff_riemannHypothesis :
    (forall n : Nat, 5040 < n -> robinInequality n) <->
      RiemannHypothesis
```

Statement 2.

```
theorem riemannHypothesis_iff_colossallyAbundant_robin :
    RiemannHypothesis <->
      (forall n : Nat, forall eps : Real, 5040 < n ->
        IsColossallyAbundantWith n eps -> robinInequality n)
```

Statement 3.

```
theorem riemann_hypothesis_iff_lagarias_elementary_criterion :
    RiemannHypothesis <-> LagariasElementaryCriterion
```

Statement 4.

```
theorem dirichletGRH_iff_cyclotomicCorrectedIdealRobin :
    (forall (modulus : Nat) [NeZero modulus]
      (chi : DirichletCharacter Complex modulus) (s : Complex),
      0 < s.re -> s.re < 1 -> DirichletCharacter.LFunction chi s = 0 ->
        s.re = 1 / 2) <->
    (forall (modulus : Nat) [NeZero modulus],
      CorrectedIdealRobin (CyclotomicField modulus Rat))
```

Statement 5.

```
theorem dedekindERH_iff_correctedIdealRobin :
    (forall (K : Type u) [Field K] [NumberField K], DedekindRH K) <->
      (forall (K : Type u) [Field K] [NumberField K], CorrectedIdealRobin K)
```

Statement 6.

```
theorem grandRH_iff_triangularPrimePowerBound {index : Type u} {powers : index -> Type u}
    [forall pi, Countable (powers pi)]
    (family : StandardAutomorphicFamilyData index powers)
    (analytics : HasStandardAutomorphicAnalytics family) :
    (forall (pi : index) (s : Complex), family.completedL pi s = 0 -> s.re = 1 / 2) <->
      (forall pi, StandardFamilyTriangularBound family pi)
```

Statement 7.

```
theorem riemannRH_iff_globalPrimePowerMultiplicityInequality
    (hRH : RiemannHypothesis) :
    (forall z : Complex, 0 < z.re -> z.re < 1 -> riemannZeta z = 0 ->
      analyticOrderNatAt riemannZeta z = 1) <-> RiemannPrimePowerMultiplicityInequality
```

Statement 8.

```
theorem dirichletGRH_iff_globalPrimePowerMultiplicityInequality (hGRH : DirichletGRH) :
    (forall (N : Nat) [NeZero N] (chi : DirichletCharacter Complex N) (z : Complex),
      0 < z.re -> z.re < 1 -> DirichletCharacter.LFunction chi z = 0 ->
        analyticOrderNatAt (DirichletCharacter.LFunction chi) z = 1) <->
    (forall (N : Nat) [NeZero N] (chi : DirichletCharacter Complex N),
      DirichletPrimePowerMultiplicityInequality chi)
```

Statement 9.

```
theorem dedekindERH_iff_globalPrimePowerMultiplicityInequality
    (K : Type u) [Field K] [NumberField K] (hERH : DedekindRH K) :
    DedekindPrimeIdealSimpleZeros K <-> DedekindPrimePowerMultiplicityInequality K
```

Statement 10.

```
theorem grandRH_iff_globalPrimePowerMultiplicityInequality
    {index : Type u} {powers : index -> Type v} [forall pi, Countable (powers pi)]
    (family : StandardMultiplicityFamilyData index powers)
    (analytics : HasStandardMultiplicityAnalytics family)
    (E J : index -> Finset Real) (order : index -> Real -> Nat)
    (hRH : forall pi s, family.completedL pi s = 0 -> s.re = 1 / 2) :
    (forall pi, HasPrescribedZeroOrders (family.completedL pi) (E pi) (J pi) (order pi)) <->
    (forall pi, GrandPrimePowerMultiplicityInequality (standardMultiplicityLogSignal family pi)
      (E pi) (J pi) (order pi))
```

## Appendix C. The exact classical criteria: proof roadmap

The rational corrected criterion is eventual. The exact classical theorem additionally proves the strict inequality for every integer above the cutoff 5040. The following graph identifies the proof of that stronger arithmetic conclusion under RH and its converse.

The proof and its checked endpoints can be read as the following directed graph. The declaration names are printed here so that a reader can move from the mathematical route to the exact Lean statement without searching the repository.

| mathematical stage | paper endpoint and checked declaration |
|---|---|
| CA extremal reduction | Proposition F.5; `nativeRobinInequalityAll_iff_colossallyAbundant` in `CAReduction` |
| finite range | Theorem G.5; `nativeRobinInequality_finite_log` in `FiniteComplete` |
| large-height RH branch | Theorem H.4; `nativeRobinInequality_of_RH_large_log` in `LargeHeightRobin` |
| false-RH converse | Theorem J.3; `riemannHypothesis_of_nativeRobinInequalityAll` in `LCM bridge` |
| public biconditionals | Theorems 6.1 and 6.2; `Robin1984.robin_inequality_iff_riemannHypothesis` and `Robin1984.riemannHypothesis_iff_colossallyAbundant_robin` in `Solution` |

C.1. **What Robin's inequality measures.** The ratio $\mathcal{A}(n) = \sigma(n)/n$ measures how rich $n$ is in divisors relative to its size. If

$$n = \prod_p p^{a_p},$$

then $\mathcal{A}(n)$ is a product of local factors, one for each prime dividing $n$. Small primes increase this ratio most efficiently, and raising a prime to a higher power gives progressively smaller additional gains. Integers built from many small prime powers are therefore the difficult cases for an upper bound on $\mathcal{A}(n)$.

Robin's inequality says that even these difficult integers satisfy

$$\mathcal{A}(n) < e^{\gamma} \log\log n$$

once $n > 5040$, exactly when RH is true. Taking logarithms turns the comparison into the sign of

$$\mathfrak{r}(n) = \log \mathcal{A}(n) - \gamma - \log\log\log n.$$

The proof is consequently a search for structural reasons why this margin is negative, or, if RH fails, why it must sometimes be positive.

C.2. **Why the extremal integers suffice.** For a positive penalty $\varepsilon$, the quantity

$$Q_{\varepsilon}(m) = \mathcal{A}(m) m^{-\varepsilon}$$

rewards divisors but charges for size. Its maximizers are the colossally abundant integers. They are useful because the penalty separates into independent prime-power decisions: include a prime-power event when its marginal gain exceeds its size cost, exclude it when the cost is larger.

Given a general integer $n$, choose the penalty

$$\varepsilon = \frac{1}{\log n \, \log\log n}.$$

Let $q$ maximize $Q_{\varepsilon}$. Maximality says that the logarithmic abundancy of $n$ lies below the affine line determined by $q$ and this penalty. The function appearing in Robin's bound is concave, and the chosen penalty is its tangent slope at $n$. The tangent inequality therefore transfers Robin's inequality from $q$ back to $n$. A short exact calculation at the transition $5040 \leftrightarrow 55440$ ensures that the chosen maximizer lies above the exceptional cutoff whenever this reduction is used. The bounded interval not covered by that argument is settled separately. This explains why checking colossally abundant integers is both sufficient and necessary.

C.3. **Why the Riemann hypothesis implies the inequality.** Write $h = \log n$. The prime-power factorization of $n$ expresses $\log \mathcal{A}(n)$ as a sum of marginal gains. Completing this sum to a convenient set of small primes produces two competing quantities. One is an analytic error term, controlled through weighted estimates for the Chebyshev functions $\vartheta$ and $\psi$. The other is a definite arithmetic deduction: an integer of height $h$ cannot take every small-prime gain for free, so omitted primes or finite exponents impose a cost.

Under RH, the weighted explicit formula bounds the analytic error on the natural square-root scale

$$S(h) = h^{-1/2} (\log h)^{-1}.$$

For $h \geq 74500$, the proof shows that the total error is strictly less than $(113/50)S(h)$, while a complete block of prime squares guarantees a cost of at least $(113/50)S(h)$. Hence error minus cost is negative, which is exactly $\mathfrak{r}(n) < 0$.

The square-root scale has a simple spectral interpretation. The explicit formula rewrites the weighted prime-counting discrepancy as a superposition of contributions resembling $h^{\rho}$, one for each nontrivial zero $\rho$ of zeta. Under RH, every contribution has magnitude on the $h^{1/2}$ scale; the weight damps the oscillation and the inverse-square zero sum controls the aggregate amplitude. The displayed error term is therefore a rigorous envelope for all of those waves, not a numerical fit.

The prime cost is the complementary combinatorial picture. An integer with logarithmic size $h$ has a finite budget of prime exponents. It cannot collect every favorable small-prime contribution

without spending that budget. The minimum in (H.9) records whichever of two unavoidable losses is cheaper for a prime. The strict coefficient gap says that even this conservative total loss is larger than the worst analytic envelope.

Below that analytic height, the argument is finite but not a decimal computation. Exact factorizations handle the first interval. The remaining range is covered by rational log-height rows. Each row lists a finite box of possible profitable prime powers and proves, by exact integer products and rational logarithm bounds, that the Robin margin is negative throughout the whole interval. Thus the finite and analytic halves meet at $h = 74500$ without an unverified numerical gap.

C.4. **Why the inequality implies the Riemann hypothesis.** The converse is best read contrapositively. If RH is false, symmetry of the zeta function supplies a nontrivial zero to the right of the critical line. That zero becomes a pole of a Mellin transform built from a weighted prime-counting error. Landau's positivity principle says that this transform cannot remain analytic as far as that pole if the underlying error is eventually of one sign. Consequently the error has negative excursions of a fixed power size arbitrarily far out.

Geometrically, the real axis of convergence acts like a wall. Eventual one-sidedness turns the Mellin transform, after $x = e^y$, into the Laplace transform of a positive measure. Positivity forces convergence to advance whenever the transform analytically crosses that wall. An off-critical zero, however, creates a genuine pole just beyond it. The two statements are incompatible, so the assumed one-sided tail must fail repeatedly. Section I replaces this picture by the exact continuation, integrability, and pole calculations.

Nicolas's finite product $F(P)$ converts those analytic excursions into negative excursions of $\log F(P)$. To obtain actual integers, take

$$L_P = \operatorname{lcm}(1, 2, \dots, P).$$

This least common multiple contains precisely the prime powers up to $P$ and satisfies $\log L_P = \psi(P)$. An exact identity expresses its Robin margin as

$$\mathfrak{r}(L_P) = -\log F(P) - D(P) - T(P),$$

where the height loss $D(P)$ and the finite-prime-power reserve $T(P)$ are nonnegative but smaller than the oscillation. The negative excursions of $\log F(P)$ therefore give positive values of $\mathfrak{r}(L_P)$ for arbitrarily large $P$. These are violations of Robin's inequality above 5040. Hence universal validity of the inequality forces RH.

The technical proof that follows adds the exact thresholds, constants, convergence arguments, and finite data needed to make this outline rigorous; it does not change the four-step logical route.

## Appendix D. Historical comparison and scope of novelty

The theorem, the use of colossally abundant extrema, the weighted prime-function estimates, and the Nicolas–Landau oscillation mechanism are classical. The contributions to the classical proof are its reconstruction, exact certificate bounds, and machine-checked implementation. Its contributions occur at three different levels: an English reconstruction of Robin's proof; several exact decompositions and interfaces chosen to make that proof formalizable; and a checked Lean artifact with independent statement and kernel replay. The distinction is summarized below.

| stage | classical source | present reconstruction |
|---|---|---|
| criterion and cutoff | Robin's Theorem 1 [46] | The same theorem and cutoff, stated directly over Mathlib objects; no new mathematical claim. |
| CA reduction | Robin's Section 3, Proposition 1, compares successive CA integers using the convexity of $x \mapsto \varepsilon x - \log\log x$. | Proposition F.2 expresses the same supporting-line geometry at an arbitrary target and separates the exact $5040 \leftrightarrow 55440$ startup transition. The reformulation and its quantifier-exact assembly are project work; the reduction is Robin's. |
| RH implication | Robin's Section 3, especially Lemmas 2–8 and Proposition 3, combines a weighted explicit formula with prime-product estimates, following Nicolas but requiring finer bounds. | The complete exponent budget packages every actual exponent into an error term and a nonnegative prime cost, then proves a single exact coefficient gap at $h \geq 74500$. This is a project-authored formal decomposition and explicit cutoff, not a claim of a new asymptotic estimate or an optimal constant. |
| finite range | Robin reports computing 2347 CA integers below the analytic range and then checking the remaining integers below 55440. | Exact startup factorizations, a tangent interval, and rational certificate rows replace the historical computer calculation by kernel-checkable data plus a general soundness theorem. |
| false-RH converse | Robin invokes Nicolas's oscillation theorem, itself based on Landau's principle, and transfers the conclusion directly to CA integers. | Sections I and J reconstruct the transform argument, isolate the positive-transform theorem, and use least common multiples to obtain literal integer counterexamples before applying the CA equivalence. This is an alternative formal proof architecture, not a new oscillation theorem attributed to this project. |

Two formal interfaces merit a more precise explanation. First, a classical sum over zeta zeros is normally understood to repeat a zero according to its analytic multiplicity. That convention is not enough for the interchanges needed in Lean. The multiplicity-aware infrastructure supplies an explicit index type for the divisor of $\xi$, a value map from indices to nonzero zeros, inverse-norm summability, and the hypotheses licensing each exchange of an integral with an infinite sum. It proves no new fact about where the zeros lie; it turns the classical convention into a reusable formal interface.

Second, the complete prime cost in (H.9) is a bookkeeping device. Robin estimates several prime products separately. The present proof first sums the actual exponent choices of an arbitrary integer, then completes to all relevant primes. Each missing prime or prematurely terminated exponent pays the smaller of two explicit penalties, so the analytic error and arithmetic loss can be bounded in independent modules. This modularity is useful for verification, but no theorem here says that it is computationally minimal or analytically more efficient than every earlier argument.

Third, Lemma I.1 is called a "finite Nicolas comparison" because it connects Nicolas's finite product to the analytic error used in his oscillation theorem. The product, Mertens series, and partial-summation idea are classical. The repository-owned contribution is the exact identity $\log F = R_M + R_\vartheta + K$, the explicit $4/x$ tail, and the separately proved signs $R_\vartheta \leq 0$ and

$K \le J$ that make the comparison usable without asymptotic shorthand. This is again a formal reconstruction and reusable interface, not a new oscillation phenomenon.

## Appendix E. Arithmetic preliminaries

For $n > e^e$, set

$$\mathfrak{r}(n) = \log \mathcal{A}(n) - \gamma - \log\log\log n. \tag{E.1}$$

All logarithms in (E.1) are then defined, and Robin's inequality is equivalent to $\mathfrak{r}(n) < 0$.

Write $n = \prod_p p^{a_p}$. For a prime $p$ and an integer $j \ge 1$, define the marginal gain and threshold

$$g_{p,j} = \log\left(\frac{1-p^{-(j+1)}}{1-p^{-j}}\right), \qquad \tau_{p,j} = \frac{g_{p,j}}{\log p}. \tag{E.2}$$

The event $(p, j)$ records the change from exponent $j-1$ to exponent $j$.

**Lemma E.1** (Prime-power event decomposition)**.** *For every positive integer $n$ and every $\varepsilon > 0$,*

$$\log \mathcal{A}(n) = \sum_p \sum_{1 \le j \le a_p} g_{p,j}, \tag{E.3}$$

$$\log Q_\varepsilon(n) = \sum_p \sum_{1 \le j \le a_p} (g_{p,j} - \varepsilon \log p). \tag{E.4}$$

*For fixed $p$, the sequences $g_{p,j}$ and $\tau_{p,j}$ are nonincreasing in $j$.*

*Proof.* The geometric-series identity gives

$$\frac{\sigma(p^a)}{p^a} = \frac{1-p^{-(a+1)}}{1-p^{-1}} = \prod_{j=1}^{a} \frac{1-p^{-(j+1)}}{1-p^{-j}}.$$

Multiplicativity and logarithms give (E.3); subtracting $\varepsilon \log n$ gives (E.4). If $q = p^{-1} \in (0,1)$, direct cross-multiplication shows that $(1-q^{j+2})/(1-q^{j+1}) \le (1-q^{j+1})/(1-q^j)$. Taking logarithms and then dividing by $\log p > 0$ proves the monotonicity. □

The reduced weight of an event at parameter $\varepsilon$ is

$$w_\varepsilon(p, j) = g_{p,j} - \varepsilon \log p = (\tau_{p,j} - \varepsilon) \log p. \tag{E.5}$$

Thus events above the threshold have nonnegative reduced weight and events below it have nonpositive reduced weight. This is the elementary complementary-slackness description of a colossally abundant factorization.

**Lemma E.2** (Existence of a maximizer)**.** *For every $\varepsilon > 0$ there exists an integer $N > 1$ maximizing $Q_\varepsilon$.*

*Proof.* The divisor bound

$$\mathcal{A}(n) = \sum_{d \mid n} \frac{1}{d} \le \sum_{1 \le d \le n} \frac{1}{d} \le 1 + \log n$$

implies $Q_\varepsilon(n) \le (1+\log n) n^{-\varepsilon} \to 0$. Beyond a finite point the objective is below its value at 2. A maximum over the remaining finite set is therefore a global maximum. □

## Appendix F. The colossally abundant tangent reduction

Put $H_n = \log n$ and

$$\lambda(n) = \frac{1}{H_n \log H_n}. \tag{F.1}$$

This is the derivative at $H_n$ of the concave function $\phi(h) = \log\log h$.

**Lemma F.1** (Global tangent inequality)**.** *If $L, U > 1$, then*

$$\log\log U - \log\log L \le \frac{U - L}{L \log L}. \tag{F.2}$$

*Proof.* Apply $\log y \le y - 1$ first to $y = (\log U)/(\log L)$ and then to $y = U/L$. Division by the positive quantities $L$ and $\log L$ gives (F.2). This proof applies whether $U$ lies to the left or to the right of $L$. □

**Proposition F.2** (Tangent transfer)**.** *Let $n, q > 5040$. Suppose that*

$$\log Q_{\lambda(n)}(n) \le \log Q_{\lambda(n)}(q). \tag{F.3}$$

*If Robin's inequality holds at $q$, then it holds at $n$.*

*Proof.* Writing $L = H_n$ and $U = H_q$, inequality (F.3) becomes

$$\log \mathcal{A}(n) - \lambda(n) L \le \log \mathcal{A}(q) - \lambda(n) U.$$

By Lemma F.1, $\phi(U) - \phi(L) \le \lambda(n)(U - L)$. Adding the two inequalities yields

$$\log \mathcal{A}(n) - \phi(L) \le \log \mathcal{A}(q) - \phi(U).$$

After subtracting $\gamma$, the left- and right-hand sides are precisely $\mathfrak{r}(n)$ and $\mathfrak{r}(q)$. Hence $\mathfrak{r}(q) < 0$ implies $\mathfrak{r}(n) < 0$. □

The remaining point is to ensure that a maximizer at the tangent parameter is itself above the exceptional cutoff. The exact startup transition is particularly useful. Let

$$\varepsilon_0 = \frac{\log(12/11)}{\log 11}. \tag{F.4}$$

The event $(11, 1)$ changes 5040 into $55440 = 11 \cdot 5040$ and has threshold $\varepsilon_0$.

**Lemma F.3** (The startup tie)**.** *Both* 5040 *and* 55440 *maximize $Q_{\varepsilon_0}$. Moreover*

$$\frac{1}{34} < \varepsilon_0 < \frac{1}{22}. \tag{F.5}$$

*Proof.* The factorization $5040 = 2^4 3^2 5 \cdot 7$ determines its included and first excluded events. Put

$$r_{p,j} = \frac{1 - p^{-(j+1)}}{1 - p^{-j}}, \qquad \tau_{p,j} = \frac{\log r_{p,j}}{\log p}.$$

The complete boundary calculation is small enough to display:

| event | $r_{p,j}$ | comparison with $\varepsilon_0$ |
|---|---|---|
| $(2,4)$ | $31/30$ | $\tau_{2,4} > \varepsilon_0$ |
| $(3,2)$ | $13/12$ | $\tau_{3,2} > \varepsilon_0$ |
| $(5,1)$ | $6/5$ | $\tau_{5,1} > \varepsilon_0$ |
| $(7,1)$ | $8/7$ | $\tau_{7,1} > \varepsilon_0$ |
| $(2,5)$ | $63/62$ | $\tau_{2,5} < \varepsilon_0$ |
| $(3,3)$ | $40/39$ | $\tau_{3,3} < \varepsilon_0$ |
| $(5,2)$ | $31/30$ | $\tau_{5,2} < \varepsilon_0$ |
| $(7,2)$ | $57/56$ | $\tau_{7,2} < \varepsilon_0$ |
| $(11,1)$ | $12/11$ | $\tau_{11,1} = \varepsilon_0$ |

Here is the complete arithmetic behind the strict signs. For an event with base $p$ and gain ratio $r$, positivity of logarithms gives

$$p < r^N \implies \frac{1}{N} < \frac{\log r}{\log p}, \qquad r^N < p \implies \frac{\log r}{\log p} < \frac{1}{N}.$$

The cutoff enclosure (F.5) follows from the two exact rational inequalities

$$11 < (12/11)^{34}, \qquad (12/11)^{22} < 11.$$

For the four included top events the kernel checks

$$2 < (31/30)^{22}, \quad 3 < (13/12)^{22}, \quad 5 < (6/5)^{22}, \quad 7 < (8/7)^{22};$$

hence their thresholds exceed $1/22 > \varepsilon_0$. For the four first-missing events it checks

$$(63/62)^{34} < 2, \quad (40/39)^{34} < 3, \quad (31/30)^{34} < 5, \quad (57/56)^{34} < 7;$$

hence their thresholds lie below $1/34 < \varepsilon_0$. These are integer comparisons after denominators are cleared; the tie $(11,1)$ is definitional.

For a fixed prime, monotonicity in $j$ propagates the displayed top-event and first-missing-event signs through every layer. Every prime not dividing 5040 is at least 11, and the first-layer threshold is nonincreasing in the prime; hence its reduced weight is nonpositive, with equality only at $(11,1)$. Equation (E.5) now compares the event packet of 5040 term by term with every competitor. Thus 5040 maximizes $Q_{\varepsilon_0}$. The tied event has weight zero, and adjoining it multiplies 5040 by 11, proving the same maximality for 55440. □

**Lemma F.4.** *If $n \geq 720720$ and $q$ maximizes $Q_{\lambda(n)}$, then $q > 5040$.*

*Proof.* The function $\lambda(n)$ is nonincreasing above 5040. Exact logarithmic bounds give

$$\lambda(720720) < \frac{1}{34} < \varepsilon_0,$$

and hence $\lambda(n) < \varepsilon_0$. Write $\lambda = \lambda(n)$. If $q \leq 5040$, then maximality at $\varepsilon_0$ gives

$$\log Q_\lambda(q) - \log Q_\lambda(5040) = \log Q_{\varepsilon_0}(q) - \log Q_{\varepsilon_0}(5040) + (\varepsilon_0 - \lambda)\log\frac{q}{5040} \leq 0.$$

On the other hand, the exact tie of Lemma F.3 gives

$$\log Q_\lambda(55440) - \log Q_\lambda(5040) = (\varepsilon_0 - \lambda)\log 11 > 0.$$

Thus no $q \leq 5040$ can maximize $Q_\lambda$, because 55440 has strictly larger objective value. □

**Proposition F.5** (Reduction to colossally abundant integers)**.** *Assume Robin's inequality for $5040 < n < 720720$. Then the following are equivalent:*

*(i) Robin's inequality holds for every $n > 5040$;*

*(ii) it holds for every colossally abundant $N > 5040$.*

*Proof.* Only (ii)⇒(i) needs proof. Let $n \geq 720720$. By Lemma E.2, choose a maximizer $q$ for $Q_{\lambda(n)}$. Lemma F.4 gives $q > 5040$, so (ii) gives Robin's inequality at $q$. Maximality gives (F.3), and Proposition F.2 transfers the inequality to $n$. The assumed finite interval handles the remaining integers. □

## Appendix G. The finite range

The analytic argument in Section H starts at $\log n = 74500$. The exact cutoff therefore requires a proof for all $5040 < n$ below that logarithmic height. We divide it into three pieces.

**Proposition G.1.** *For $5041 \leq n < 7560$,*

$$35\sigma(n) \leq 127n. \tag{G.1}$$

*Consequently Robin's inequality holds throughout this interval.*

*Proof.* For each integer in the interval, an exact prime-power factorization is given. Pairwise coprimality and the identity $\sigma(p^a) = 1 + p + \cdots + p^a$ reduce (G.1) to natural-number arithmetic. To pass from (G.1) to Robin's inequality, use the exact bounds

$$e^\gamma > \frac{177}{100}, \qquad \log\log n > \frac{103}{50}, \qquad \frac{127}{35} < \frac{177}{100}\frac{103}{50}.$$

□

**Proposition G.2.** *Robin's inequality holds for $7560 \leq n < 720720$.*

*Proof.* The maximality of 5040 at $\varepsilon_0$ gives

$$\log \mathcal{A}(n) - \varepsilon_0 \log n \leq \log \mathcal{A}(5040) - \varepsilon_0 \log 5040.$$

Set $y = \log n$. It is enough to prove positivity of the explicit gap

$$G(y) = \gamma + \log\log y - \log \mathcal{A}(5040) - \varepsilon_0(y - \log 5040),$$

because the preceding inequality then places $\log \mathcal{A}(n)$ strictly below $\gamma + \log\log y$. The curvature used at this step is explicit:

$$G''(y) = -\frac{\log y + 1}{y^2(\log y)^2} < 0 \qquad (y > 1).$$

Thus $G$ is concave, and its value at every point of a closed interval is at least the smaller of its two endpoint values. The endpoint checks use the following strict rational bounds:

| endpoint | certified inequalities |
|---|---|
| 7560 | $\gamma > 577/1000$, $\log\log\log 7560 > 3917/5000$, $\log \mathcal{A}(5040) < 269/200$, and $\varepsilon_0 \log(3/2) < 203/13500$ |
| 720720 | $\gamma > 72/125$, $\log\log\log 720720 > 191/200$, $\log \mathcal{A}(5040) < 269/200$, and $\varepsilon_0 \log 143 < 5/27$ |

Here $\mathcal{A}(5040) = 403/105$, and the arguments of the last two logarithms arise from $7560/5040 = 3/2$ and $720720/5040 = 143$. Substitution makes each endpoint gap a strict rational inequality, so concavity proves positivity throughout the interval. □

For the much larger interval above 720720, a compact rational certificate replaces integer-by-integer factorization. A certificate row consists of a log-height interval $[L, U]$, a rational slope $\lambda$, prime-layer cutoffs, an upper bound for the corresponding product, a lower height bound, and lower bounds for $\log L$ and $\log U$. The layer inequalities certify which prime-power events can have positive reduced weight. The product inequality bounds the maximum possible value of $\log \mathcal{A}(n) - \lambda \log n$. Finally, concavity reduces the Robin comparison on $[L, U]$ to two endpoint inequalities.

More precisely, for a cutoff vector $c = (c_1, \dots, c_d)$ define the layer box

$$\mathcal{B}(c) = \{(p, j) : p \text{ prime},\ 1 \le j \le d,\ p \le c_j\}.$$

The reduced weight of an event is

$$w_{p,j}(\lambda) = g_{p,j} - \lambda \log p = \log p\,(\tau_{p,j} - \lambda). \tag{G.2}$$

To state the integer boundary checks without pretending that a cutoff is prime, extend this expression to every real base $x > 1$ by

$$\widetilde{w}_j(x; \lambda) = \log\left(1 + \frac{x - 1}{x(x^j - 1)}\right) - \lambda \log x.$$

For a prime $p$, this is exactly $w_{p,j}(\lambda)$. For every layer $j$, the row proves

$$\widetilde{w}_j(c_j; \lambda) \ge 0, \qquad \widetilde{w}_j(c_j + 1; \lambda) \le 0,$$

using the rational logarithm bounds below. The gain decreases with the base, so these two boundary signs classify every prime in that layer. The extra check $\widetilde{w}_{d+1}(2; \lambda) \le 0$, together with monotonicity in the layer and base, excludes every event above the stored vector. Thus events inside $\mathcal{B}(c)$ have nonnegative weight and all events outside have nonpositive weight. Any integer's actual exponent packet is consequently bounded above by the sum over this finite box; this is the precise sense in which the box is a valid worst case.

Appendix P prints the complete first row, the scaled logarithm checker, and the two endpoint predicates. Appendix Q lists all 36 intervals and their main structural parameters. Keeping those arithmetic audits separate here leaves the main argument focused on the row schema and its soundness theorem.

The logarithm certificates used here are elementary and explicit. For a rational $q \ge 1$, put $z = (q - 1)/(q + 1)$. Then $0 \le z < 1$ and, for $N \ge 1$,

$$\begin{aligned} L_N(q) &= 2 \sum_{i=0}^{N-1} \frac{z^{2i+1}}{2i + 1}, \\ U_N(q) &= L_N(q) + \frac{2z^{2N+1}}{1 - z^2}, \\ L_N(q) &\le \log q \le U_N(q). \end{aligned} \tag{G.3}$$

The upper remainder is deliberately coarse: it follows by deleting the denominators $2i + 1$ from the positive tail of the atanh series. Consequently both bounds in (G.3) are rational and their soundness is proved once for arbitrary $q$ and $N$. Every row in this proof uses $N = 20$.

Certificate discovery versus certificate checking. A practical way to find a row is to choose rational $L, U, \lambda$ and then, for layer $j$, choose an integer cutoff $c_j$. The checker compares the threshold formula at $c_j$ and $c_j + 1$: the former must have nonnegative reduced weight and the latter nonpositive reduced weight. Monotonicity in the base then proves the signs for every prime in that layer. A final check at base 2 proves that no still higher layer can contribute positively. From the resulting finite box one computes the exact integer products

$$A = \prod (p^{j+1} - 1), \qquad B = \prod p(p^j - 1), \qquad Q = \prod p.$$

All three products are over $\mathcal{B}(c)$. Their logarithms give exactly

$$\sum_{(p,j)\in\mathcal{B}(c)} w_{p,j}(\lambda) = \log A - \log B - \lambda \log Q.$$

For example, Row 00 has $c = [17, 5, 3, 2, 2]$. Its layers contain respectively the prime bases

$$\{2, 3, 5, 7, 11, 13, 17\}, \quad \{2, 3, 5\}, \quad \{2, 3\}, \quad \{2\}, \quad \{2\}.$$

Direct multiplication gives

$$A = 8\,490\,305\,136\,604\,741\,632\,000, \qquad B = 1\,651\,300\,226\,181\,365\,760\,000,$$
$$Q = 367\,567\,200.$$

The kernel then verifies, among the remaining row predicates,

$$A \le \frac{5141587823941}{10^{12}} B, \qquad 2^{28}\frac{11486475}{8388608} \le Q.$$

Rational bounds of the forms $A \le (\text{gain upper})B$ and $2^e m \le Q$, together with (G.3), produce the objective and height bounds. If either endpoint inequality fails, one changes the slope or shortens the interval; accepted endpoints are then chained exactly. This search is only certificate discovery. It need not be trusted or even reproduced: the stored row contains all of its outputs, the general soundness lemma proves what successful checks imply, and Lean's kernel evaluates every row predicate from those visible rational and integer values.

The discovery process was iterative, not an optimization computation. An earlier checked cover contained 39 rows. The cancellation-aware coefficient revision in `208b447` lowered the analytic handoff from 100000 to 74500 and removed the last three rows, leaving the present 36-row witness. Rejected candidate rows were not retained as a stable trace, so no numerical count of trial proposals is claimed. Nor is 36 claimed minimal: it is an upper bound supplied by one accepted cover, and there is no nontrivial lower bound or optimality theorem for this certificate format. Different slopes, interval widths, or logarithm approximants could change the row count without changing the soundness argument.

The historical discovery runs were not instrumented per candidate, so the paper does not report invented search times or a trial count. Appendix Q instead reports a reproducible quantity: the elapsed time for each accepted row module in one identified clean CI build. Those times measure elaboration and kernel checking on that runner, not intrinsic certificate complexity; they vary with hardware, toolchain, and cache state.

The height data certify that the relevant prime-base product is at least $2^e m$; the checker turns this into a lower logarithmic height with a 20-term rational log series. The gain bound is likewise converted into an upper logarithmic objective. Thus this row checks the two literal rational inequalities displayed in Lemma G.3, not floating-point approximations. Its complete definition and kernel proof are `Robin1984/Finite/Certificates/Row00.lean`. The other rows have the same schema; the full list and gap-free cover proof are `Robin1984/Finite/Certificates/AllRows.lean`, with all individual records in `Robin1984/Finite/Certificates`.

**Lemma G.3** (Soundness of a finite row)**.** *Suppose a row $[L, U]$ satisfies its layer-sign, product, height, and endpoint inequalities. If $n > 5040$ and $L \le \log n \le U$, then $\mathfrak{r}(n) < 0$.*

*Proof.* The sign checks and (E.5) bound the actual factorization packet by the finite threshold box. The exact product bounds convert that box into an upper bound $B$ for $\log \mathcal{A}(n) - \lambda \log n$. The two endpoint checks state

$$B < \gamma + \log\log L - \lambda L, \qquad B < \gamma + \log\log U - \lambda U.$$

Concavity of $y \mapsto \log\log y - \lambda y$ gives the same strict inequality at $y = \log n$. Rearrangement is $\mathfrak{r}(n) < 0$. □

**Proposition G.4** (Certified finite cover)**.** *Robin's inequality holds whenever*

$$720720 \le n, \qquad \log n < 74500. \tag{G.4}$$

*Proof.* Thirty-six rational rows form a gap-free chain from $336/25$ to $37283397387/500000$, an endpoint strictly larger than 74500. Since $336/25 < \log 720720$, they cover (G.4). Every row satisfies the hypotheses of Lemma G.3. In the final retained row the first-layer product includes every prime at most 70849; it is split into 105 contiguous blocks, whose exact products concatenate to the required prefix. The blocks cover the half-open base range $[0, 70850)$ without a gap, and each block proves its numerator, denominator, and prime-base product. Concatenation therefore reconstructs the same first-layer factors in $A, B, Q$ that a monolithic product would contain; the split changes evaluation cost, not the mathematical packet. All comparisons are integer or rational inequalities. Thus the list of rows and their checks constitute a finite proof of the proposition. □

Combining Propositions G.1, G.2, and G.4 gives the following.

**Theorem G.5** (Finite completion)**.** *If* $5040 < n$ *and* $\log n < 74500$*, then Robin's inequality holds.*

In particular, Proposition F.5 is now unconditional.

## Appendix H. The implication under the Riemann hypothesis

Let

$$\vartheta(x) = \sum_{p \le x} \log p, \qquad \psi(x) = \sum_{p^k \le x} \log p.$$

The analytic estimate is most transparent after setting $h = \log n$ and

$$S(h) = h^{-1/2}(\log h)^{-1}. \tag{H.1}$$

The section is easiest read backward from its last line. We seek $\mathfrak{r}(n) < 0$. Proposition H.2 bounds this margin by an analytic debit $\mathcal{E}(h)$ minus an arithmetic credit $\mathcal{C}(h)$. The weighted explicit formula controls the debit; a complete block of prime squares supplies the credit; and Proposition H.3 proves that, once $h \ge 74500$, the credit is at least $(113/50)S(h)$ while the debit is strictly smaller. The formulas below establish these three statements but do not add another logical layer.

For a first reading, Lemma H.1 may be used as the analytic input and its kernel-estimate proof skipped; the narrative resumes at the definition of $\mathcal{C}(h)$ and $\mathcal{E}(h)$. The intervening integration-by-parts calculation is retained because it explains the exact constant $c_0$, the asymmetric error bound, and the otherwise delicate endpoint $m = 1$.

We first record the weighted explicit-formula input. For $m \ge 1$, define

$$w_m(t) = t^{-m-1}\frac{m\log t + 1}{(\log t)^2}, \qquad I_m(x) = \int_x^\infty (\psi(t) - t) w_m(t)\, dt. \tag{H.2}$$

For a nontrivial zero $\rho$ of $\zeta$, counted with multiplicity, put

$$K_m(\rho, x) = \int_x^\infty t^{\rho - m - 1}\frac{m\log t + 1}{(\log t)^2}\, dt.$$

**Lemma H.1** (Robin's weighted explicit formula)**.** *Assume* RH*. For* $m \ge 1$ *and* $x \ge 2$*,*

$$I_m(x) = -\sum_\rho \frac{K_m(\rho, x)}{\rho} - T_m(x), \tag{H.3}$$

*where the pole, gamma-factor, and trivial-zero correction is the explicit nonnegative integral*

$$T_m(x) = \int_x^\infty w_m(t)\left(\log(2\pi) + \frac{1}{2}\log(1-t^{-2})\right)\,dt. \tag{H.4}$$

*For $t \geq 2$ the parenthesis lies between 0 and $\log(2\pi)$, and hence*

$$0 \leq T_m(x) \leq \frac{\log(2\pi)x^{-m}}{\log x}. \tag{H.5}$$

*If*

$$c_0 = \gamma + 2 - \log(4\pi),$$

*then*

$$-B_m(x) - \frac{\log(2\pi)x^{-m}}{\log x} \leq I_m(x) \leq B_m(x), \tag{H.6}$$

*where*

$$B_m(x) = c_0 x^{1/2-m}(\log x)^{-1}\left(m + (\log x)^{-1} + \frac{4}{(2m-1)(\log x)^2}\right). \tag{H.7}$$

*Proof.* Apply the explicit formula to the continuous cutoff whose Mellin transform is $K_m(s,x)/s$. The arithmetic side is the von Mangoldt sum associated with $I_m(x)$; moving the vertical line separates the pole at 1, the nontrivial zeros, and the negative even zeros. Every interchange is justified by an explicit summable norm majorant, so the zero sum retains analytic multiplicity rather than choosing an enumeration without proof.

The kernel calculation is not a black box. Since

$$-\frac{d}{dt}\frac{t^{-m}}{\log t} = t^{-m-1}\frac{m\log t + 1}{(\log t)^2},$$

one integration by parts gives

$$\frac{K_m(\rho,x)}{\rho} = \frac{x^{\rho-m}}{\rho\log x}\int_x^\infty \frac{t^{\rho-m-1}}{\log t}\,dt. \tag{H.8}$$

A second integration by parts, with $d(t^{\rho-m}) = (\rho-m)t^{\rho-m-1}\,dt$, separates the leading term

$$\frac{m}{\rho(m-\rho)}\frac{x^{\rho-m}}{\log x}$$

from a remainder containing $(\log t)^{-2}$. Bounding that remainder once more by integration against $(\log t)^{-3}$ gives, under RH and hence $\Re\rho = 1/2$, the pointwise majorant

$$\left|\frac{K_m(\rho,x)}{\rho}\right| \leq \frac{1}{|\rho|^2}\,x^{1/2-m}(\log x)^{-1}\left(m + (\log x)^{-1} + \frac{4}{(2m-1)(\log x)^2}\right).$$

Hadamard factorization of the completed zeta function supplies the exact multiplicity-counted identity

$$\sum_\rho |\rho|^{-2} = \gamma + 2 - \log(4\pi) = c_0.$$

Summing the majorant gives $B_m(x)$. The combined pole/trivial-zero factor is exactly (H.4); its integrand bound and $\int_x^\infty w_m(t)\,dt = x^{-m}/\log x$ prove (H.5), explaining the asymmetry in (H.6).

The contour identity initially covers $m \geq 2$. For the endpoint let

$$q(t) = \frac{t(\log t + 1)}{2\log t + 1}, \qquad q'(t) = \frac{\log t(2\log t + 3)}{(2\log t + 1)^2}.$$

Exact integration by parts gives

$$I_1(x) = q(x)I_2(x) + \int_x^\infty q'(t)I_2(t)\,dt.$$

The same identity is proved separately for the zero series and for $T_m$, with absolute-integrability bounds licensing each interchange. Substitution of the $m = 2$ formula therefore yields (H.3) at $m = 1$, rather than obtaining it by a limiting convention. This is Robin's Lemma 2 with no omitted endpoint. □

Costa Pereira's estimates for $\psi - \vartheta$, with the corrected constants from [36], control the prime-power part separated from (H.3). Abel summation supplies the complementary lower bound for a complete block of prime squares.

Define the complete prime cost

$$\mathcal{C}(h) = \sum_{p \le h/2} \min\left\{ \frac{\log p}{h \log h}, \frac{1}{p^2} \right\}. \tag{H.9}$$

Also define

$$\begin{aligned} \mathcal{E}(h) = (2 + c_0)\frac{h^{-1/2}}{\log h} + (c_0 - 2)\frac{h^{-1/2}}{(\log h)^2} \\ + (8 + 4c_0)\frac{h^{-1/2}}{(\log h)^3} + 2\frac{h^{-2/3}}{\log h} + \log(2\pi)\frac{h^{-1}}{\log h}. \end{aligned} \tag{H.10}$$

Thus $\mathcal{E}(h)$ is a uniform upper bound assembled from the weighted prime-function errors, whereas $\mathcal{C}(h)$ is a nonnegative deduction forced by completing the exponent sum to all small primes. Their names record their roles in (H.11); neither is a new number-theoretic function outside this proof.

**Proposition H.2** (Complete exponent budget)**.** *Assume* RH. *If* $n \ge 1$ *and* $h = \log n \ge 20000$, *then*

$$\mathfrak{r}(n) \le \mathcal{E}(h) - \mathcal{C}(h). \tag{H.11}$$

*Proof.* For a prime $p$ occurring in $n$ with exponent $a$, the telescoping identity from Lemma E.1 and $\log y \le y - 1$ give

$$\log \frac{1 - p^{-(a+1)}}{1 - p^{-1}} \le \frac{1}{p} - p^{-(a+1)} - m_p, \qquad m_p = \log(1 - p^{-1}) + p^{-1} \le 0. \tag{H.12}$$

Subtract $a \log p/(h \log h)$ from each prime contribution. Completing the sum to all primes $p \le h$ does not lose the omitted primes: if $a = 0$ and $p \le h/2$, the benchmark contribution pays at least the minimum in (H.9). Summation therefore yields a prime-reciprocal expression minus $\mathcal{C}(h)$.

Partial summation writes the prime-reciprocal expression as $\log\log h$ plus the Meissel–Mertens constant and a weighted $\vartheta$-error. Replacing $\vartheta$ by $\psi$ separates the nonnegative prime-power tail. Lemma H.1 and the Costa Pereira bounds show that the remaining weighted error is at most (H.10). The Mertens constants cancel with the sum of the $m_p$, giving exactly (H.11). □

The numerical heart of the large-height proof is the following strict gap.

**Proposition H.3** (Explicit coefficient gap)**.** *Assume* RH. *For* $h \ge 74500$,

$$\mathcal{E}(h) < \frac{113}{50} S(h), \qquad \mathcal{C}(h) \ge \frac{113}{50} S(h). \tag{H.13}$$

*Proof.* For the first inequality, divide (H.10) by $S(h)$. The exact identity becomes

$$2 + c_0 + (c_0 - 2)(\log h)^{-1} + (8 + 4c_0)(\log h)^{-2} + 2H^{-1/6} + \log(2\pi)h^{-1/2}.$$

Uniform rational bounds valid for every $h \geq 74500$ control $(\log h)^{-1}$, $h^{-1/6}$, $h^{-1/2}$, $c_0$, and $\log(2\pi)$. The values actually used are

$$\begin{aligned} \log h \geq \frac{56}{5}, \quad (\log h)^{-1} \leq \frac{5}{56}, \quad (\log h)h^{-1/6} \leq \frac{87}{50}, \\ h^{-1/2} \leq \frac{1}{272}, \qquad c_0 \leq \frac{1}{20}, \qquad \log(2\pi) \leq \frac{46}{25}. \end{aligned} \tag{H.14}$$

There is no numerical conflict in the bound for $c_0$:

$$c_0 = \gamma + 2 - \log(4\pi) = 0.0461914179\ldots < 0.05 = \frac{1}{20}.$$

The proof does not depend on that decimal evaluation. The harmonic-number and logarithm bounds in Appendix O prove $c_0 < 154/3125 < 1/20$ rationally. Substitution into (H.14), while retaining the favorable negative coefficient $c_0 - 2$, gives the exact margin

$$\frac{\mathcal{E}(h)}{S(h)} \leq \frac{3010451}{1332800} = \frac{113}{50} - \frac{1677}{1332800} < \frac{113}{50}.$$

For the second inequality, take the complete prime-square block between $\sqrt{2H}$ and $h/2$. Abel summation gives a dominant normalized main term tending to $2\sqrt{2}$, while the prime-function error, endpoint, and prime-power terms are lower order. The calculation retains the RH error at the far endpoint instead of replacing the whole $\vartheta$ tail by a coarse global bound; only after that cancellation is the remaining nonnegative prime-power contribution discarded. Appendix O gives the complete normalized expression and every rational substitution. The resulting exact lower margin is

$$\frac{\mathcal{C}(h)}{S(h)} \geq \frac{15537277889}{6868000000} = \frac{113}{50} + \frac{15597889}{6868000000} > \frac{113}{50}.$$

Thus the two sides of the gap are proved independently and meet with positive rational margins. □

**Theorem H.4** (The RH implication)**.** *If* RH *holds, then Robin's inequality holds for every* $n > 5040$.

*Proof.* If $\log n < 74500$, use Theorem G.5. Otherwise combine Propositions H.2 and H.3:

$$\mathfrak{r}(n) < \left(\frac{113}{50} - \frac{113}{50}\right) S(\log n) = 0.$$

This is equivalent to Robin's inequality. □

## Appendix I. The Nicolas–Landau oscillation

Define Nicolas's finite product and normalized function by

$$\mathfrak{P}(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right), \qquad F(x) = e^{\gamma} \log \vartheta(x)\, \mathfrak{P}(x). \tag{I.1}$$

The signed quantity used below is $\log F(x)$.

Let

$$k(t) = \frac{(\log t)^{-1} + (\log t)^{-2}}{t^2},$$

and put

$$K(x) = \int_x^\infty (\vartheta(t) - t)k(t)\,dt, \qquad J(x) = \int_x^\infty (\psi(t) - t)k(t)\,dt. \tag{I.2}$$

For functions $g, h : \mathbb{R} \to \mathbb{R}$, write $g = \Omega_-(h)$ at infinity if there is $c > 0$ such that, for every $X$, some $x \geq X$ satisfies $g(x) \leq -ch(x)$.

Only two outputs of this section are needed later. First, $\log F(x)$ is bounded above by the weighted prime error $J(x)$ plus a term of order $x^{-1}$. Second, if RH fails, Landau's principle forces $J(x)$ below a negative multiple of $x^{-b}$ arbitrarily far out for some $b < 1/2$. Since $x^{-1} = o(x^{-b})$, the first output transfers the second one to $\log F$. Section J then absorbs the remaining LCM losses, which are only of order $P^{-1/2}\log P$.

**Lemma I.1** (Finite Nicolas comparison)**.** *For all sufficiently large $x$,*

$$\log F(x) \leq J(x) + \frac{4}{x}. \tag{I.3}$$

*Proof.* The exact decomposition, with no asymptotic notation, is

$$\begin{aligned} \log F(x) &= R_M(x) + R_\vartheta(x) + K(x), \\ R_M(x) &= \sum_{0<n\leq\lfloor x\rfloor} \Big(\log(1 - n^{-1}) + n^{-1}\Big) - (M - \gamma), \\ R_\vartheta(x) &= \log\log\vartheta(x) - \log\log x - \frac{\vartheta(x) - x}{x\log x}, \end{aligned} \tag{I.4}$$

where the summand is understood to be zero for nonprimes and $M$ is the Meissel–Mertens constant in the convergent prime summand series. The explicit tail estimate gives $|R_M(x)| \leq 4/x$. The inequality $\log y \leq y - 1$, applied to the ratio of the two logarithmic heights, is exactly $R_\vartheta(x) \leq 0$ after simplification.

Finally

$$J(x) - K(x) = \int_x^\infty (\psi(t) - \vartheta(t))k(t)\,dt \geq 0,$$

because both factors are nonnegative. Applying these three bounds to (I.4) gives (I.3). □

**Proposition I.2** (Nicolas–Landau negative oscillation)**.** *If* RH *is false, then there is $b$ with*

$$0 < b < \frac{1}{2} \tag{I.5}$$

*such that*

$$J(x) = \Omega_-(x^{-b}). \tag{I.6}$$

*Proof.* Failure of RH, together with the functional equation and conjugation symmetry, gives a nontrivial zero $\rho$ of $\zeta$ with $1/2 < \Re\rho < 1$. On its horizontal line choose a zero $\rho_*$ with maximal real part. Choose

$$1 - \Re\rho_* < b < \frac{1}{2}.$$

The contradiction is an instance of a simple positivity obstruction. If the corrected tail $J(x) + x^{-b}$ were eventually positive, its Mellin transform would be the Laplace transform, under $x = e^y$, of a positive measure. Positivity makes convergence propagate to every real point that the analytic continuation can cross. The zeta logarithmic derivative, however, gives that continuation a pole at $s = 1 - \rho_*$, strictly inside the propagated half-plane. The remainder of the proof makes each part of this obstruction precise.

Suppose (I.6) fails. Then, beyond some $X$,

$$J(x) + x^{-b} > 0. \tag{I.7}$$

Define the positive measure

$$d\mu_b(x) = \mathbf{1}_{(X,\infty)}(x)x^{-1}\big(J(x) + x^{-b}\big)\,dx$$

and use $Y(x) = \log x$ as the random variable. Its complex moment-generating function is the Mellin transform

$$M_b(s) = \int_X^\infty x^{s-1}\big(J(x) + x^{-b}\big)\,dx. \tag{I.8}$$

For $\Re s < 0$ the integral is computed directly. Integration by parts in the $J$ term expresses it through the Mellin transform of $\psi(x) - x$ and hence through $-\zeta'(1-s)/\zeta(1-s)$ after its pole at 1 is removed. The added power has the exact transform

$$\int_X^\infty x^{s-b-1}\,dx = \frac{X^{s-b}}{b-s} \qquad (\Re s < b).$$

These formulas define an analytic continuation $H_b(s)$ wherever the zeta factor is regular.

Here is the transform calculation used in that sentence. The Frullani identity

$$k(t) = \int_0^\infty (u+1)t^{-(u+2)}\,du$$

and Fubini's theorem give, for $\Re s < 0$,

$$\int_3^\infty x^{s-1}J(x)\,dx = \int_0^\infty (u+1)\frac{\Phi_3(u+1-s) - 3^s\Phi_3(u+1)}{s}\,du, \tag{I.9}$$

where

$$\begin{aligned}\Phi_3(z) = {} & \left(\frac{-\zeta'(z)}{z\zeta(z)} - \frac{1}{z-1}\right) \\ & - \int_1^3 (\psi(t) - t)t^{-z-1}\,dt.\end{aligned} \tag{I.10}$$

The apparent value at $s = 0$ is filled by analytic continuation. Absolute integrability of the three-variable carrier $3 < x < t$, $u > 0$ licenses the two exchanges of integration in (I.9); compact and large-$u$ majorants then prove analyticity of the filled expression. Formula (I.10) makes the singularity source explicit: the finite startup integral is entire, while the logarithmic derivative records the nontrivial zeros of $\zeta$.

We now separate the two roles of Landau's argument. First, positivity of $\mu_b$ makes the real convergence set of (I.8) convex. At an interior point $a$, the derivatives of $M_b$ are its nonnegative moments

$$\int (\log x)^k e^{a\log x}\,d\mu_b(x).$$

If an analytic continuation has a power-series radius reaching $a + d$, its Taylor series bounds the sum of these moments at $d \geq 0$; monotone convergence then proves actual integrability at $a + d$. Iterating this strict crossing shows that every real $\sigma < b$ lies in the interior of the true integrability set. Consequently $M_b$ itself is analytic throughout the half-plane $\Re s < b$ and agrees there with $H_b$ by the identity theorem.

Second, put $s_0 = 1 - \rho_*$. Since $\Re s_0 = 1 - \Re\rho_* < b$, the preceding conclusion makes $M_b$ analytic, and therefore locally bounded, at $s_0$. Approach that point along

$$s_\delta = 1 - \rho_* - \delta, \qquad \delta > 0.$$

Maximality of $\Re\rho_*$ on its horizontal line says that $\zeta(\rho_* + \delta) \neq 0$, so $H_b(s_\delta)$ is regular for every $\delta > 0$. At the endpoint, however, $1 - s_0 = \rho_*$ and the logarithmic derivative has the zero pole; the norm of $H_b(s_\delta)$ tends to infinity as $\delta \downarrow 0$. Since $M_b(s_\delta) = H_b(s_\delta)$, analytic boundedness and divergence contradict one another. Thus the assumed eventual positivity is impossible, proving (I.6). □

The proof just given is the strict form of the oscillation argument used by Nicolas and Robin. Its quantifiers are important: the excursions occur arbitrarily far out, rather than at one selected numerical frontier. The positive-transform step is not imported as an unnamed principle. Its reusable formal statement proves that if a nonnegative exponential transform converges at every real exponent below a frontier and has an analytic continuation across the frontier, then it actually converges at some strictly larger exponent. The proof expands the continuation into its power series, identifies its coefficients with nonnegative moments, and uses monotone convergence to recover exponential integrability. This generic theorem is the formal counterpart of Lemma 2.1. The subsequent Nicolas modules supply (I.9), the rightmost-zero ray, and the blow-up contradiction.

**Corollary I.3.** *If* RH *is false, there exists* $0 < b < 1/2$ *such that*

$$\log F(x) = \Omega_-(x^{-b}). \tag{I.11}$$

*Proof.* Use Lemma I.1 and $x^{-1} = o(x^{-b})$, which holds because $b < 1$. □

## Appendix J. Least common multiples and the converse

Let

$$L_P = \operatorname{lcm}(1, 2, \dots, P).$$

The classical identity $\log L_P = \psi(P)$ connects this finite integer with the analytic frontier. Define the logarithmic height loss

$$D(P) = \log\log\psi(P) - \log\log\vartheta(P). \tag{J.1}$$

The saturated Euler product differs from the actual abundancy of $L_P$ by the nonnegative prime-power reserve

$$T(P) = \sum_{p \le P} -\log\Big(1 - p^{-a_p - 1}\Big), \qquad a_p = \max\{a : p^a \le P\}. \tag{J.2}$$

**Lemma J.1** (Exact LCM identity)**.** *For all sufficiently large* $P$*,*

$$\mathfrak{r}(L_P) = -\log F(P) - D(P) - T(P). \tag{J.3}$$

*Proof.* The finite Euler product gives

$$\sum_{p \le P} -\log(1 - p^{-1}) = -\log\mathfrak{P}(P),$$

where $\mathfrak{P}(P)$ denotes the product in (I.1). The prime-power factorization of $L_P$ gives

$$-\log\mathfrak{P}(P) = \log\mathcal{A}(L_P) + T(P).$$

Now expand $\log F(P)$ and use $\log L_P = \psi(P)$. Rearrangement is exactly (J.3). □

**Lemma J.2** (The transfer losses)**.** *For every* $b < 1/2$*,*

$$D(P) + T(P) = o(P^{-b}). \tag{J.4}$$

*Proof.* The elementary bound

$$0 \le \psi(P) - \vartheta(P) \le 2\sqrt{P}\log P$$

and a linear lower bound for $\vartheta(P)$ give

$$0 \le D(P) \ll \frac{\log P}{\sqrt{P}}.$$

Splitting the reserve at $\sqrt{P}$ and summing the geometric tails gives $T(P) \ll P^{-1/2}$. Since $P^{-1/2}\log P = o(P^{-b})$ whenever $b < 1/2$, the claim follows. □

**Theorem J.3** (The converse implication)**.** *If Robin's inequality holds for every $n > 5040$, then* RH *holds.*

*Proof.* Suppose RH is false. Choose $b$ as in Corollary I.3. Negative excursions of $\log F(P)$ give positive excursions of $-\log F(P)$ of order $P^{-b}$. By Lemma J.2, subtracting $D(P) + T(P)$ preserves such excursions. Lemma J.1 therefore gives arbitrarily large $P$ with $\mathfrak{r}(L_P) > 0$. Since $L_P \ge P$, these least common multiples eventually exceed 5040, and $\mathfrak{r}(L_P) > 0$ is a literal failure of Robin's inequality. This contradicts the hypothesis. □

## Appendix K. Assembly of the classical equivalences

*Proof of Theorem 6.1.* The implication RH ⇒(6.1) is Theorem H.4. The reverse implication is Theorem J.3. □

*Proof of Theorem 6.2.* If RH holds, Theorem 6.1 gives Robin's inequality for every integer above 5040, hence in particular for every colossally abundant one. Conversely, assume it for every colossally abundant integer above 5040. The finite startup theorem and Proposition F.5 extend it to every integer above 5040. Theorem 6.1 then gives RH. □

## Appendix L. Lagarias's elementary criterion

The proof of Theorem 6.3 uses the classical Robin implication above the cutoff and the same oscillation in the converse. For the forward implication, Lagarias's Lemma 3.1 gives, for $n \ge 3$,

$$e^{\gamma} n \log\log n \le e^{H_n} \log H_n. \tag{L.1}$$

Indeed $\gamma + \log n \le H_n$, exponentiation gives $e^{\gamma} n \le e^{H_n}$, and $\log n \le H_n$ gives $\log\log n \le \log H_n$; all factors have the required signs. Under RH, Theorem 6.1 and (L.1) prove (6.3) for $n > 5040$. The case $n = 1$ is equality. The interval $2 \le n \le 5040$ is split into 46 consecutive blocks. Each block records an exact upper bound for $\sigma(n)$ throughout the block and rational lower bounds for $H_n$, $e^{H_n}$, and $\log H_n$. The reusable block soundness theorem converts those checks into (6.3); all arithmetic checks use `decide+kernel`, not `native_decide`.

For the converse, a relaxed form of Lagarias's Lemma 3.2 suffices: for $n \ge 20$,

$$H_n + e^{H_n}\log H_n \le e^{\gamma} n \log\log n + \frac{24n}{\log n}. \tag{L.2}$$

The published lemma has the sharper constant 7; no sharp constant is needed here. Apply (L.2) to $L_P = \operatorname{lcm}(1,\dots,P)$. If (6.3) holds, the positive logarithmic Robin margin at $L_P$ is at most $24/\log L_P$. The elementary Chebyshev lower bound $\log L_P \gg P$ makes this $O(P^{-1})$. The already formalized Nicolas–Landau argument under ¬RH supplies positive margin excursions of order $P^{-b}$ for some $0 < b < 1/2$, while $P^{-1} = o(P^{-b})$. The two estimates contradict one another for arbitrarily large $P$.

The corresponding Lean declarations are `robinBound_le_exp_harmonic_mul_log_harmonic`, `lagarias_upper_bound`, `lagarias_finite_two_to_5040`, and `riemannHypothesis_iff_`

`lagariasElementaryCriterion` in namespace `Robin1984`. They are in `Robin1984/Lagarias`. The forward argument combines the harmonic estimate with the bounded verification. The converse calls the LCM oscillation theorem directly.

## Appendix M. Degree-one primes and a nonabelian example

Write $a_K(p)$ for the number of prime ideals of $K$ above a rational prime $p$ with residue degree one. Put

$$\Theta_K^{(1)}(x) = \sum_{\substack{p \le x \\ p \text{ prime}}} a_K(p) \log p \, (1 - p/x).$$

**Proposition M.1.** *For a fixed number field $K$, the corrected ideal Robin criterion (4.1) is equivalent to the existence of a constant $C_K$ such that*

$$\Theta_K^{(1)}(x) \;\ge\; x/2 - C_K\sqrt{x}$$

*for all sufficiently large $x$. Consequently Dedekind ERH is equivalent to this lower bound in every number field.*

*Proof.* Write $d = [K : \mathbb{Q}]$, and let $f_{\mathfrak{p}}$ and $e_{\mathfrak{p}}$ be the residue degree and ramification index of $\mathfrak{p}$ over $p$. At any given power $p^k$ the coefficient of the prime-ideal sum for $Z = \zeta_K$ is

$$\log p \sum_{\substack{\mathfrak{p} \mid p \\ f_{\mathfrak{p}} \mid k}} f_{\mathfrak{p}} \;\le\; d \log p$$

by $\sum_{\mathfrak{p} \mid p} e_{\mathfrak{p}} f_{\mathfrak{p}} = d$. The coefficient at $k = 1$ is $a_K(p) \log p$. Take an absolute Chebyshev constant $C_\vartheta$ such that $\sum_{p \le y,\, p \text{ prime}} \log p \le C_\vartheta y$ for $y \ge 2$. The total contribution $\mathcal{P}_{\ge 2,K}(x)$ of $k \ge 2$ in the triangular sum satisfies, for $x \ge 2$,

$$0 \le \mathcal{P}_{\ge 2,K}(x) \le dC_\vartheta \sum_{k=2}^{\lfloor \log_2 x \rfloor} x^{1/k} \le dC_\vartheta \left(1 + \frac{6}{e \log 2}\right) \sqrt{x}.$$

For the last inequality the terms with $k \ge 3$ are bounded by $(\log_2 x) x^{1/3}$, and $ue^{-u/6} \le 6/e$ for $u \ge 0$. Since $R_{\zeta_K}(x) = x/2 - \Theta_K^{(1)}(x) - \mathcal{P}_{\ge 2,K}(x)$, the two one-sided bounds are equivalent after a change in their constants. The specialization of Theorem 3.1 and Theorem 4.1 complete the proof. □

It is enough to verify the degree-one criterion in finite Galois extensions $L/\mathbb{Q}$. For the normal closure $L$ of any $K$, the Aramata–Brauer theorem makes $\zeta_L/\zeta_K$ entire, so every open-strip zero of $\zeta_K$ is a zero of $\zeta_L$ [14]. For a Galois $L/\mathbb{Q}$ an unramified prime has degree-one factors precisely when it splits completely, in which case it has $[L : \mathbb{Q}]$ such factors. The finitely many ramified primes contribute $O_L(1)$. Consequently Dedekind ERH is also equivalent to

$$\sum_{\substack{p \le x \\ p \text{ prime splitting completely in } L}} \log p \, (1 - p/x) \;\ge\; \frac{x}{2[L : \mathbb{Q}]} - C_L\sqrt{x}$$

eventually for every finite Galois $L/\mathbb{Q}$, with a constant and a threshold depending on $L$.

For $K = \mathbb{Q}(\sqrt[3]{2})$, the Dedekind–Kummer theorem identifies $a_K(p)$ for $p \nmid 6$ with the number of roots of $T^3 - 2$ in $\mathbb{F}_p$ [15, Theorem 2.1]. The cubic map on $\mathbb{F}_p^\times$ is bijective for $p \equiv 2 \bmod 3$. For $p \equiv 1 \bmod 3$, the root count is three if $2^{(p-1)/3} \equiv 1 \bmod p$, and zero otherwise. Under rational RH, $\Theta_{\mathbb{Q}}^{(1)}(x) = x/2 + O(\sqrt{x})$ by the two-sided estimate from (3.4) and the same bound for $E_{\mathbb{Q}}$.

The finitely many excluded primes contribute $O(1)$. Thus RH for this cubic field is equivalent, given rational RH, to the bound

$$\sum_{\substack{p\le x,\ p\ \text{prime}\\ p\equiv 1 \bmod 3}} \Big(3\mathbf{1}_{\{2^{(p-1)/3}\equiv 1 \bmod p\}} - 1\Big)\log p\,(1-p/x) \ \ge\ -C\sqrt{x}$$

with some $C$ for all sufficiently large $x$. This gives an explicit nonabelian prime-splitting sum on which to test an extension of the Dirichlet method.

For a paired factor $Z = L(s,\pi)L(s,\widetilde{\pi})$, there is a related exact comparison of nonnegative sums. Let $I_\pi(x) = \{(\mathfrak{p},j) : \mathfrak{p}\ \text{unramified for}\ \pi,\ j\ge 1,\ q^j\le x\}$, where $q = N\mathfrak{p}$, and put $a_{\mathfrak{p},j} = \sum_k \alpha_{\mathfrak{p},k}^j$. The identity

$$2\Re a_{\mathfrak{p},j} = |1+a_{\mathfrak{p},j}|^2 - |a_{\mathfrak{p},j}|^2 - 1$$

gives the equivalent bound

$$\sum_{I_\pi(x)} |1+a_{\mathfrak{p},j}|^2 \log q\,(1-q^j/x) \ \ge\ \sum_{I_\pi(x)} (|a_{\mathfrak{p},j}|^2+1)\log q\,(1-q^j/x) - C_\pi\sqrt{x}. \tag{M.1}$$

Indeed, choose $\theta_n < \eta < 1/2$ with $\theta_n$ from [33, (2.2)]. Each omitted finite prime contributes at most $2n(\log q)x^\eta/(1-q^{-\eta})$ in absolute value, by a geometric sum over its powers. The omitted contribution is therefore $O_\pi(\sqrt{x})$. The two squares in (M.1) are the unramified self Rankin–Selberg power sums for $1\boxplus\pi$ and $\pi$. The estimate concerns their difference after including the trivial factor. Positivity of the individual sums alone gives no bound of this size.

If a paired standard factor has $|\alpha_{\mathfrak{p},k}|\le 1$ at all finite places, its coefficients satisfy $|b_{\mathfrak{p},j}|\le 2n\log N\mathfrak{p}$. The same calculation then bounds the absolute contribution of the terms of rational norm $p^k$, $k\ge 2$, by $2ndC_\vartheta(1+6/(e\log 2))\sqrt{x}$. For that factor the degree-one prime terms also suffice. The general triangular criterion retains all prime powers and does not require these bounds on the local parameters.

In either setting one can first seek a lower bound with error $C_\varepsilon x^{1/2+\varepsilon}$ for every $\varepsilon>0$, holding eventually with the constant and threshold allowed to depend on the fixed field or factor. Lemma 2.1 applied with $\alpha = 1/2+\varepsilon$ excludes any pole to the right of that line. Taking all $\varepsilon>0$ gives the square-root criterion by the already proved reverse implication.

## Appendix N. Checked specializations and the Hecke construction

Degree one includes unitary Hecke $L$-functions. For the trivial Hecke character of a field $K$, Lean verifies that the triangular bound is equivalent to Dedekind RH and to the corrected ideal criterion for that field. Both proofs are in `Robin1984/GRH/DedekindTriangular.lean`. For a unitary Hecke character $\chi$, Lean constructs its local parameters $\alpha_\chi(\mathfrak{p})$ from the finite ideles, with $\alpha_\chi(\mathfrak{p}) = 0$ at ramified primes. It proves the Euler product on $\Re s>1$ and the paired coefficient $b_{\mathfrak{p},j} = 2\Re(\alpha_\chi(\mathfrak{p})^j)\log N\mathfrak{p}$ for every $j\ge 1$ in `Robin1984/GRH/HeckeEulerProduct.lean`. The same file evaluates the finite local Tate integrals on $\Re s>0$, using multiplicative Haar measure normalized by the unit subgroup. For each $t\in\mathbb{R}$ the unitary idelic norm character $\chi_t = |\cdot|^{it}$ gives $L(s,\chi_t) = \zeta_K(s+it)$ on $\Re s>1$. Lean constructs its shifted continuation and proves that its critical-line property is equivalent to both the corrected ideal Robin criterion and the triangular prime-power bound for $K$. The local exponents give the conductor ideal in `Robin1984/GRH/HeckeConductor.lean`. The real and complex shifts and their radial Gaussian Mellin evaluations on $\Re s>0$ are verified in `Robin1984/GRH/HeckeArchimedean.lean`. The resulting Hecke triangular criterion uses those local data.

The prescribed-order criterion uses the growth of each individual completed function. The Euler product and archimedean factors bound it in the right half-plane. The completed functional equation transfers the dual bound to the left, and the Tate integral bounds the middle strip. These

estimates give order at most one for each pole-free completion. After removing the conductor and Gamma factors, the logarithmic derivative has coefficients $\alpha_\chi(\mathfrak{p})^j \log N\mathfrak{p}$. Thus the Gaussian inequality measures the orders of the individual Hecke function. The norm characters use the Dedekind signal, with the order at ordinate $u$ transported to $u-t$ for $\chi_t$. The combined Lean results are in `Robin1984/GRH/HeckeStandardRankOne.lean`.

The adelic Poisson formula for a finite coset. Lean verifies that the diagonal is discrete and that $\mathbb{A}_K/K$ is compact. It constructs the continuous additive character $\psi$ that is trivial on the diagonal and proves that the finite character annihilator for $d\widehat{\mathcal{O}}_K$ is the trace-dual ideal $(d)^\vee$. With mass one on $\widehat{\mathcal{O}}_K$, the finite additive Haar measure satisfies $N((d))\mu_f(d\widehat{\mathcal{O}}_K) = 1$.

For every $a \in \mathbb{A}_{K,f}$, $0 \neq d \in \mathcal{O}_K$, and Schwartz function $f$ on the weighted Minkowski space, `Robin1984/GRH/HeckeAdelePoisson.lean` proves

$$\sum_{\substack{x\in K\\ x_f\in a+d\widehat{\mathcal{O}}_K}} f(e_K(x)) = \frac{1}{N((d))\sqrt{|D_K|}} \sum_{\xi\in(d)^\vee} \psi_f(-a\xi)\widehat{f}(c_K(e_K(\xi))).$$

Here $x_f$ is the finite component of the principal adele, $e_K$ weights the complex coordinates by $\sqrt{2}$, and $c_K$ conjugates those coordinates.

The finite Tate test $T_\chi$ is the product of the integral-coordinate indicators at unramified places and the inverse local character on the unit group at ramified places, extended by zero there off the units. For every $b \in \mathbb{A}_{K,f}^\times$, clearing a principal denominator reduces $z \mapsto T_\chi(zb)$ to a function supported on $\widehat{\mathcal{O}}_K$ and constant modulo a nonzero principal ideal. Pulling the finite residue-class expansion back through the rescaling gives a finite sum of principal coset indicators. For every such $b$ and every archimedean Schwartz function $f$, `Robin1984/GRH/HeckeFiniteTateTest.lean` proves

$$\sum_{x\in K} f(e_K(x))T_\chi(x_f b) = \frac{1}{\sqrt{|D_K|}} \sum_{\xi\in K} \widehat{f}(c_K(e_K(\xi))) \int_{\mathbb{A}_{K,f}} T_\chi(zb)\psi_f(-\xi_f z)\, d\mu_f(z).$$

`Robin1984/GRH/HeckeFiniteHaarDilation.lean` identifies the additive Haar modulus of a finite idele with its finite idelic norm. `Robin1984/GRH/HeckeAdelicFourier.lean` applies the resulting change of variables together with the archimedean Fourier scaling to prove the Poisson formula with both idele components dilated.

For the multiplicative quotient, `Robin1984/GRH/HeckeFiniteIdealClass.lean` proves the surjective finite-idele map onto nonzero fractional ideals and identifies its compact kernel as the integral finite units. Using Dirichlet's unit theorem and the finiteness of the ideal class group, `Robin1984/GRH/HeckeIdeleClassCompactness.lean` constructs a compact set of norm-one ideles meeting every norm-one idele class. It also proves that the principal subgroup is closed and discrete. On those compact representatives, `Robin1984/GRH/HeckeTateThetaDecay.lean` proves an absolute theta-sum estimate by clearing one common finite denominator and then summing the archimedean Schwartz bound over a lattice. `Robin1984/GRH/HeckeTateDualSupport.lean` verifies compact support and boundedness for the finite Fourier dual. `Robin1984/GRH/HeckeTateThetaQuotient.lean` descends the sum with its zero term removed to the norm-one class quotient. It proves uniform exponential decay in the logarithmic norm parameter for the original and dual tests. These are the large-norm estimates used in Tate's Mellin argument [13]. `Robin1984/GRH/HeckeTateMellin.lean` proves joint continuity on the norm-one class quotient and absolute Mellin integrability at every real exponent for both tests. A unitary Hecke character descends to this quotient. Its integral against invariant measure vanishes if the restriction is nontrivial. The module also proves that the character-weighted positive tail defines an entire function of the Mellin parameter. The module `HeckeTateQuotientIntegration.lean` proves the product formula for the full quotient measure using a Borel fundamental domain. The module

`HeckeTateZetaUnfolding.lean` identifies the principal-idele orbit sum with the actual theta series and unfolds the weighted zeta integral. For the canonical polynomial Gaussian and finite Tate test, `HeckeTateAdelicEuler.lean` proves its equality on $\Re s > 1$ with the Gamma factors times the Hecke Euler product. The quotient Poisson identity in `HeckeTateClassPoisson.lean` keeps the two terms at zero. Averaging against a character nontrivial on norm-one idele classes makes both terms vanish.

The conductor in the functional equation. Write $Q_\chi = |D_K| N\mathfrak{f}_\chi$. Choose finite ideles $d, c$ representing the different and conductor ideals and put $b = dc$. `HeckeTateFiniteFourier.lean` proves

$$T_\chi^{\mathrm{D}}(y) = T_\chi^{\mathrm{D}}(b^{-1}) T_{\chi^{-1}}(by), \qquad |b|_f = Q_\chi^{-1},$$

where $T_\chi^{\mathrm{D}}$ is the finite Fourier test with the discriminant normalization and trace sign of the Poisson formula. To prove the first identity, trace duality identifies the annihilator of the integral adeles with the inverse different. Translation by the conductor forces $by$ to be integral on the support of the transform. At every ramified place, covariance under local units forces $by$ to be a unit. On that support one can match the ramified coordinates by an integral finite idele, and covariance determines the transform from its value at $b^{-1}$. `HeckeTateInfiniteFourier.lean` computes the archimedean Fourier test as $P_\chi f_{\chi^{-1}}$, where

$$P_\chi = (-i)^{\sum_{v \text{ real}} \epsilon_v + \sum_{v \text{ complex}} |m_v|}$$

for the real parities $\epsilon_v$ and complex angular weights $m_v$.

Let $a(t)$ be the positive archimedean section of norm $e^t$ with finite component one, and let $G_\chi(t)$ be the canonical theta sum averaged against $\chi$ on the norm-one class group. With $\beta = \log |b|_f = -\log Q_\chi$, the idele $a(-\beta)b$ has norm one. Haar invariance under multiplication by its class $w$ gives the shifted identity

$$G_\chi(t) = k_\chi e^{-t} G_{\chi^{-1}}(\beta - t), \qquad k_\chi = P_\chi T_\chi^{\mathrm{D}}(b^{-1}) \chi^{-1}(w^{-1}).$$

The finite translation changes the logarithmic norm by $\beta$. The factor $\chi^{-1}(w^{-1})$ comes from translating the character in the averaging integral.

The completed family. Write $\chi(a(t)) = e^{i\tau_\chi t}$ with $\tau_\chi \in \mathbb{R}$. The entire completion is

$$\Lambda_\chi(s) = e^{-s\beta/2} \int_{\mathbb{R}} e^{st} \chi(a(t)) G_\chi(t)\, dt.$$

Fubini's theorem and reflection express the full integral as two positive Mellin tails at $s + i\tau_\chi$ and $1 - s - i\tau_\chi$. The absolute moments for all real exponents justify both tails and their derivatives. On $\Re s > 1$, the unfolded Euler integral identifies $\Lambda_\chi$ with $Q_\chi^{s/2}$ times the Gamma factors and $L(s, \chi)$. Changing variables $u = \beta - t$ in the full integral gives

$$\Lambda_\chi(s) = \omega_\chi \Lambda_{\chi^{-1}}(1 - s), \qquad \omega_\chi = k_\chi \exp(-\beta/2 + i\tau_\chi \beta).$$

The canonical tests and norm-section character for $\chi^{-1}$ are the complex conjugates of those for $\chi$. Consequently, $\Lambda_{\chi^{-1}}(\bar{s}) = \overline{\Lambda_\chi(s)}$. Applying the reflected conjugate identity twice gives $\Lambda_\chi(s) = |\omega_\chi|^2 \Lambda_\chi(s)$. The completed Euler product is nonzero for $\Re s > 1$, so $|\omega_\chi| = 1$.

On $-2 \le \Re s \le 3$ the moments at exponent 3 bound both tails, and the conductor exponential is bounded by $\exp(3|\beta|/2)$. The paired product has order at most one by the Euler and Gamma bounds and the completed functional equation. The paired triangular criterion therefore applies without an additional analytic premise for these characters. If the restriction to norm-one classes is trivial, the character is determined by its values on $a(t)$ and is a norm twist. The pole-cleared completion is then the translated Dedekind completion, for which the Dedekind triangular criterion applies. The full degree-one result is in `HeckeTateFunctionalEquation.lean`.

A weighted all-ideal bound can also be obtained when the logarithmic Euler coefficients are nonnegative. For unitary Hecke characters other than pure norm twists, the coefficients of $\zeta_K(s)^2 L(s,\chi)L(s,\bar{\chi})$ divided by $\log N\mathfrak{p}$ are $2 + 2\Re(\alpha_\chi(\mathfrak{p})^j) \geq 0$ for every $j \geq 1$. Subtracting a price $1/(x\log x)$ for each unit of weighted log norm then cancels the Mertens boundary term exactly. The details of the weighted comparison are recorded in `docs/GRH.md`.

## Appendix O. Exact rational coefficient audit

This appendix collects the arithmetic intentionally omitted from the main narrative of Proposition H.3. First, the formal bound for the zero constant uses

$$\gamma < H_{256} - \log 256, \qquad H_{256} \leq \frac{15311}{2500}, \qquad \log 2 \geq \frac{69314}{100000}, \qquad \log(4\pi) \geq \frac{253}{100}.$$

Consequently

$$c_0 < \frac{15311}{2500} - 8\frac{69314}{100000} + 2 - \frac{253}{100} = \frac{154}{3125} < \frac{1}{20}.$$

This is the rational proof behind the decimal orientation $c_0 = 0.0461914179\ldots$.

For the error bound, put $q = (\log h)^{-1}$. The $h^{-1/6}$ estimate in (H.14) turns $2H^{-1/6}$ into at most $(87/25)q$. Combining this with the favorable term $-2q$ and applying the remaining bounds gives

$$\begin{aligned}\frac{\mathcal{E}(h)}{S(h)} &\leq 2 + \frac{37}{25}\frac{5}{56} + 8\left(\frac{5}{56}\right)^2 \\ &\quad + \frac{1}{20}\left(1 + \frac{5}{56} + 4\left(\frac{5}{56}\right)^2\right) + \frac{46}{25}\frac{1}{272} \\ &= \frac{3010451}{1332800} = \frac{113}{50} - \frac{1677}{1332800} < \frac{113}{50}.\end{aligned}$$

For the complete prime cost, Abel summation over the prime-square block $\sqrt{2H} < p \leq h/2$ gives the following normalized lower expression:

$$\begin{aligned}&2\sqrt{2}\frac{\log h}{\log h + \log 2}\left(1 - \frac{1}{\log h + \log 2}\right) \\ &- \frac{9}{4}\,2^{1/4}h^{-1/4}\frac{\log h}{\log h + \log 2} - \log(2\pi)h^{-1/2}\frac{\log h}{\log h + \log 2} \\ &- 4H^{-1/2}\frac{\log h}{\log h - \log 2} \\ &- 2\sqrt{2}\,c_0 H^{-1}\frac{\log h}{\log h - \log 2}\left(2 + \frac{1}{\log h - \log 2} + \frac{4}{3(\log h - \log 2)^2}\right).\end{aligned}$$

The last line is the far-end RH term retained through the cancellation. The rational bounds used for $h \geq 74500$ are

$$\begin{aligned}\sqrt{2} &\geq \frac{7071}{5000}, & 2^{1/4} &\leq \frac{119}{100}, \\ \frac{\log h}{\log h + \log 2} &\geq \frac{16}{17}, & \frac{1}{\log h + \log 2} &\leq \frac{17}{202}, \\ h^{-1/4}\frac{\log h}{\log h + \log 2} &\leq \frac{573}{10000}, & h^{-1/2}\frac{\log h}{\log h + \log 2} &\leq \frac{7}{2000}, \\ h^{-1/2}\frac{\log h}{\log h - \log 2} &\leq \frac{197}{50000}, & \frac{\log h}{\log h - \log 2} &\leq \frac{107}{100}.\end{aligned} \tag{O.1}$$

The normalized far-end term is at most $1/10000$, so the final exact substitution is

$$\begin{aligned} 2\frac{7071}{5000}\frac{16}{17}\left(1-\frac{17}{202}\right)-\frac{9}{4}\frac{119}{100}\frac{573}{10000}-\frac{46}{25}\frac{7}{2000} \\ -4\frac{197}{50000}-\frac{1}{10000} \\ =\frac{15537277889}{6868000000}=\frac{113}{50}+\frac{15597889}{6868000000}>\frac{113}{50}. \end{aligned}$$

## Appendix P. A finite certificate in full

The first of the 36 accepted rows stores the following exact record:

| field | stored rational value |
|---|---|
| $[L,U]$ | $[336/25,\ 22117363/1000000]$ |
| $\lambda$ | 9771914406623/500000000000000 |
| layer cutoffs | $[17,5,3,2,2]$ |
| gain upper bound | 5141587823941/1000000000000 |
| height data $(e,m)$ | $(28,\ 11486475/8388608)$ |
| lower bounds for $\log L, \log U$ | $1299117667/500000000,\ 619272591/200000000$ |

The essential checked interface, with one proof body elided, is

```
theorem RobinFiniteRowChecks.sound {r : RobinFiniteRow}
    (h : RobinFiniteRowChecks r) {n : Nat} (hn : 5040 < n)
    (hLo : (r.lo : Real) <= Real.log (n : Real))
    (hHi : Real.log (n : Real) <= (r.hi : Real)) :
    Robin1984.Core.NativeRobinInequality n := by ...

theorem robinFiniteRow00_checks :
    RobinFiniteRowChecks robinFiniteRow00 := by
  decide +kernel
```

Let $g = 5141587823941/10^{12}$ be the stored gain bound and $k(q) = \lfloor \log_2 q \rfloor$. In terms of the functions in (G.3), the scaled upper checker is

$$U_{20}^{\mathrm{sc}}(q) = k(q)\frac{693147180559945310}{10^{18}} + U_{20}\left(\frac{q}{2^{k(q)}}\right),$$

and the lower checker uses $693147180559945309/10^{18}$ and $L_{20}$. With the stored $\lambda$, exponent $e = 28$, and mantissa $m = 11486475/8388608$, the objective supplied to both endpoints is

$$B_0 = U_{20}^{\mathrm{sc}}(g) - \lambda\left(28\frac{693147180559945309}{10^{18}} + L_{20}(m)\right).$$

The kernel decision verifies the integer product inequalities $A \le gB$ and $2^{28}m \le Q$, every adjacent layer-sign comparison, and

$$B_0 < \frac{57721}{100000} + L_{20}^{\mathrm{sc}}(\ell_L) - \lambda L, \qquad B_0 < \frac{57721}{100000} + L_{20}^{\mathrm{sc}}(\ell_U) - \lambda U.$$

The general logarithm and Euler-constant theorems then convert these rational facts into the real endpoint inequalities used by Lemma G.3. The record and its kernel proof are `Row00`; the reusable soundness theorem is `FiniteRowCertificate`.

## Appendix Q. Complete finite-cover inventory

Table 5 lists every accepted row. The columns $L, U$ and $\lambda$ are the exact stored rationals. The entry $c_1/d$ gives the first-layer cutoff and the number of stored layers; "blocks" is the number of separately checked first-layer prime-product blocks, with zero denoting the direct monolithic calculation used by Row 00. The final column is the elapsed module time reported by the clean CI build of commit `4d0b59e`. These are observed elaboration and kernel-checking times on that runner, not discovery times or hardware-independent complexity bounds.

Table 5: The complete 36-row certificate cover.

| row | $L$ | $U$ | $\lambda$ | $c_1/d$ | blocks | seconds |
|---|---|---|---|---|---|---|
| 00 | $\frac{336}{25}$ | $\frac{22117363}{1000000}$ | $\frac{9771914406623}{500000000000000}$ | 17/5 | 0 | 2.3 |
| 01 | $\frac{22117363}{1000000}$ | $\frac{35788699}{1000000}$ | $\frac{2565510579241}{250000000000000}$ | 28/6 | 2 | 2.3 |
| 02 | $\frac{35788699}{1000000}$ | $\frac{29149999}{500000}$ | $\frac{5519615043151}{1000000000000000}$ | 46/7 | 3 | 2.4 |
| 03 | $\frac{29149999}{500000}$ | $\frac{47544747}{500000}$ | $\frac{3004424868557}{1000000000000000}$ | 76/7 | 4 | 2.4 |
| 04 | $\frac{47544747}{500000}$ | $\frac{146801491}{1000000}$ | $\frac{1724212981141}{1000000000000000}$ | 120/8 | 5 | 2.5 |
| 05 | $\frac{146801491}{1000000}$ | $\frac{111686563}{500000}$ | $\frac{8278925801}{8000000000000}$ | 184/9 | 6 | 2.5 |
| 06 | $\frac{111686563}{500000}$ | $\frac{163594351}{500000}$ | $\frac{80829194299}{125000000000000}$ | 274/10 | 7 | 2.6 |
| 07 | $\frac{163594351}{500000}$ | $\frac{464815043}{1000000}$ | $\frac{422180858753}{1000000000000000}$ | 395/10 | 8 | 2.6 |
| 08 | $\frac{464815043}{1000000}$ | $\frac{130165559}{200000}$ | $\frac{70868018799}{250000000000000}$ | 557/11 | 9 | 2.7 |
| 09 | $\frac{130165559}{200000}$ | $\frac{893036263}{1000000}$ | $\frac{38967393937}{200000000000000}$ | 771/11 | 10 | 2.7 |
| 10 | $\frac{893036263}{1000000}$ | $\frac{240770673}{200000}$ | $\frac{137136588803}{1000000000000000}$ | 1048/12 | 12 | 2.8 |
| 11 | $\frac{240770673}{200000}$ | $\frac{63176719}{40000}$ | $\frac{99275427189}{1000000000000000}$ | 1391/12 | 13 | 2.8 |
| 12 | $\frac{63176719}{40000}$ | $\frac{2035337871}{1000000}$ | $\frac{73775311097}{1000000000000000}$ | 1806/13 | 14 | 2.9 |
| 13 | $\frac{2035337871}{1000000}$ | $\frac{518900301}{200000}$ | $\frac{11152011487}{200000000000000}$ | 2314/13 | 16 | 2.9 |
| 14 | $\frac{518900301}{200000}$ | $\frac{815385537}{250000}$ | $\frac{42786768497}{1000000000000000}$ | 2927/14 | 17 | 3.1 |
| 15 | $\frac{815385537}{250000}$ | $\frac{406392661}{100000}$ | $\frac{6654185407}{200000000000000}$ | 3662/14 | 19 | 3.1 |
| 16 | $\frac{406392661}{100000}$ | $\frac{1248690297}{250000}$ | $\frac{2622639723}{100000000000000}$ | 4528/14 | 21 | 3.1 |
| 17 | $\frac{1248690297}{250000}$ | $\frac{3045165617}{500000}$ | $\frac{20930175389}{1000000000000000}$ | 5542/15 | 23 | 3.3 |
| 18 | $\frac{3045165617}{500000}$ | $\frac{458996953}{62500}$ | $\frac{16893531041}{1000000000000000}$ | 6716/15 | 25 | 3.4 |
| 19 | $\frac{458996953}{62500}$ | $\frac{4387795691}{500000}$ | $\frac{13794103247}{1000000000000000}$ | 8059/15 | 27 | 3.5 |
| 20 | $\frac{4387795691}{500000}$ | $\frac{10433152137}{1000000}$ | $\frac{11354367573}{1000000000000000}$ | 9603/15 | 30 | 3.6 |
| 21 | $\frac{10433152137}{1000000}$ | $\frac{12279993269}{1000000}$ | $\frac{2357543377}{250000000000000}$ | 11356/16 | 33 | 3.8 |
| 22 | $\frac{12279993269}{1000000}$ | $\frac{179906601}{12500}$ | $\frac{1973616211}{250000000000000}$ | 13335/16 | 36 | 4.0 |
| 23 | $\frac{179906601}{12500}$ | $\frac{16784073643}{1000000}$ | $\frac{664793849}{1000000000000000}$ | 15587/16 | 39 | 4.3 |
| 24 | $\frac{16784073643}{1000000}$ | $\frac{9718846411}{500000}$ | $\frac{1126354703}{200000000000000}$ | 18110/16 | 42 | 4.6 |
| 25 | $\frac{9718846411}{500000}$ | $\frac{22433188173}{1000000}$ | $\frac{1200244709}{250000000000000}$ | 20934/17 | 46 | 4.9 |
| 26 | $\frac{22433188173}{1000000}$ | $\frac{1287242339}{50000}$ | $\frac{2057220229}{500000000000000}$ | 24088/17 | 50 | 5.4 |
| 27 | $\frac{1287242339}{50000}$ | $\frac{7349340853}{250000}$ | $\frac{3547339421}{1000000000000000}$ | 27570/17 | 54 | 5.8 |
| 28 | $\frac{7349340853}{250000}$ | $\frac{1338267981}{40000}$ | $\frac{3072760897}{1000000000000000}$ | 31426/17 | 59 | 6.4 |
| 29 | $\frac{1338267981}{40000}$ | $\frac{9481137173}{250000}$ | $\frac{2672852847}{1000000000000000}$ | 35690/18 | 64 | 7.1 |
| 30 | $\frac{9481137173}{250000}$ | $\frac{1070418171}{25000}$ | $\frac{2335546837}{1000000000000000}$ | 40370/18 | 70 | 8.2 |
| 31 | $\frac{1070418171}{25000}$ | $\frac{48110318623}{1000000}$ | $\frac{2050940953}{1000000000000000}$ | 45463/18 | 76 | 9.6 |
| 32 | $\frac{48110318623}{1000000}$ | $\frac{10789326747}{200000}$ | $\frac{1807809003}{1000000000000000}$ | 51028/18 | 82 | 10 |
| 33 | $\frac{10789326747}{200000}$ | $\frac{30165067587}{500000}$ | $\frac{1597827091}{1000000000000000}$ | 57137/18 | 89 | 12 |
| 34 | $\frac{30165067587}{500000}$ | $\frac{33566442549}{500000}$ | $\frac{1418388113}{1000000000000000}$ | 63731/18 | 97 | 15 |
| 35 | $\frac{33566442549}{500000}$ | $\frac{37283397387}{500000}$ | $\frac{157973183}{125000000000000}$ | 70849/19 | 105 | 19 |

The endpoint list satisfies $U_i = L_{i+1}$ for every adjacent pair. Two declarations in the linked `AllRows module` complete the aggregate check:

- `robinFiniteRows_cover` checks the gap-free equalities and outer endpoints by kernel reduction;
- `robinFiniteRows_checks` dispatches every row to its separately checked predicate.

The exact remaining cutoffs, gain bound, height mantissa and exponent, and endpoint logarithm bounds are stored in the linked `certificate directory`.

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