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BERNOULLI DOMINATION OF DETERMINANTAL MEASURES AND THE FUGLEDE–KADISON BOUNDS

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Abstract. We characterize the Bernoulli product measures stochastically dominated by a determinantal measure on an arbitrary countable set. The necessary inclusion inequalities on finite subsets are sufficient, including for inhomogeneous parameters and singular kernels. The proof rounds logarithmic parameters in a submodular base polytope; one rounding distribution works for every increasing event. For equivariant kernels on countable groups, a quantitative compression inequality proves the two Fuglede–Kadison comparison bounds proposed by Lyons. When the determinant is positive, its parameter is optimal exactly when the subgroup generated by the kernel support is amenable. Thus the optimality assertion in Lyons's conjecture fails in general. We also obtain invariant monotone couplings at the Fuglede–Kadison parameters on sofic groups. For adjacency kernels on regular trees we compute the exact Bernoulli thresholds by combining greedy-tree moment inequalities with leaf elimination.

Because of time constraints, we consulted only part of the relevant literature, and we ask readers who find substantial overlap between this manuscript and their own work to contact the author as soon as possible.

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1. Introduction and main results

Let E be a nonempty finite or countably infinite set and let Q be a positive contraction on $\ell^2(E)$. The associated determinantal measure $\mathbb{P}^Q$ is the law of a random subset $X \subset E$ characterized by

$$\mathbb{P}^Q[S \subset X] = \det Q_S \qquad (S \subset E \text{ finite}), \tag{1.1}$$

where $Q_S = P_S Q|_{\ell^2(S)}$ is the finite principal compression. The empty determinant is one. The existence, conditioning properties, and stochastic comparison of these measures are developed in [7, 8]. For $p = (p_i)_{i\in E} \in [0,1]^E$, write

$$\pi_p = \bigotimes_{i\in E} \mathrm{Ber}(p_i).$$

For constant parameters, the same notation π_p denotes the homogeneous product.

We use $\mu \preccurlyeq \nu$ for stochastic domination: the expectation of every bounded increasing measurable function under μ is at most its expectation under ν. Equivalently, there is a coupling (Y, X) with these marginals such that $Y \subset X$ almost surely [11]. Our first theorem reduces comparison with a product measure to the individual inclusion probabilities in (1.1).

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Theorem 1.1 (Product comparison criterion)**.** *Let $0 \le Q \le I$ on $\ell^2(E)$ and let $p, q \in [0,1]^E$. Then*

$$\pi_p \preccurlyeq \mathbb{P}^Q \quad \Longleftrightarrow \quad \prod_{i \in S} p_i \le \det Q_S \quad \text{for every finite } S \subset E. \tag{1.2}$$

Likewise,

$$\mathbb{P}^Q \preccurlyeq \pi_q \quad \Longleftrightarrow \quad \prod_{i \in S} (1 - q_i) \le \det(I - Q)_S \quad \text{for every finite } S \subset E. \tag{1.3}$$

There is no invariance hypothesis in this theorem. In particular it gives the exact homogeneous comparison parameters.

Corollary 1.2 (Exact thresholds)**.** *The largest p and smallest q for which $\pi_p \preccurlyeq \mathbb{P}^Q \preccurlyeq \pi_q$ are*

$$p_*(Q) = \inf_{\varnothing \ne S \Subset E} (\det Q_S)^{1/|S|}, \qquad q_*(Q) = 1 - p_*(I - Q). \tag{1.4}$$

Both endpoint comparisons hold. Here $S \Subset E$ means that S is finite.

The logarithms of the inequalities in (1.2) define a submodular polyhedron. We use an exchange rounding in that polyhedron, related to pipage rounding [1, 3], to compare with product laws whose parameters are sequential conditional probabilities. The convexity used in the exchange step holds for every increasing function simultaneously. Section 2 gives the complete finite argument and then treats zero parameters, singular kernels, and countable sets.

1.1. **Equivariant kernels and amenability.** Let Γ be a countable group. All group actions on configurations in this paper are by left translation. Suppose Q commutes with the left regular representation on $\ell^2(\Gamma)$. Write

$$\tau(A) = \langle A\delta_e, \delta_e \rangle, \qquad \mathrm{FK}(Q) = \exp\left(\int_{[0,1]} \log t \, d\nu_Q(t) \right),$$

where ν_Q is the spectral probability measure of Q at δ_e. The value is zero if the integral is $-\infty$. This is the Fuglede–Kadison determinant [5]. Define

$$\begin{aligned} H_Q &= \langle g \in \Gamma : Q(e, g) \ne 0 \rangle, \\ \beta_Q &= \inf_{\varnothing \ne S \Subset \Gamma} \frac{\| P_{S^c} Q P_S \|_{\mathrm{HS}}^2}{|S|}. \end{aligned} \tag{1.5}$$

The Hilbert–Schmidt norm here is the ordinary, unnormalized norm from $\ell^2(S)$ to $\ell^2(S^c)$.

For stationary determinantal measures on $\mathbb{Z}^d$, Lyons and Steif identify the optimal homogeneous parameters with the geometric means of the symbol and its complement [9, Theorems 5.3 and 5.11]. Cyr studies stationary kernels with several labelled objects at each site, with an emphasis on full domination and uniform insertion tolerance [4]. Lyons's Conjecture 5.7 in [8] proposes, on every countable group, both the comparisons

$$\pi_{\mathrm{FK}(Q)} \preccurlyeq \mathbb{P}^Q \preccurlyeq \pi_{1 - \mathrm{FK}(I - Q)} \tag{1.6}$$

and the optimality of their parameters. The next theorem proves the comparisons and determines the role of amenability in the optimality assertion.

Theorem 1.3 (Fuglede–Kadison bounds and sharpness)**.** *Let Q be an equivariant positive contraction on $\ell^2(\Gamma)$. Then* (1.6) *holds, and more precisely*

$$p_*(Q) \ge \mathrm{FK}(Q) \exp(\beta_Q / 2). \tag{1.7}$$

Moreover, $\beta_Q = 0$ if and only if H_Q is amenable. If H_Q is amenable, then $p_*(Q) = \mathrm{FK}(Q)$, also when $\mathrm{FK}(Q) = 0$. If $\mathrm{FK}(Q) > 0$, then

$$p_*(Q) = \mathrm{FK}(Q) \quad \Longleftrightarrow \quad H_Q \text{ is amenable.} \tag{1.8}$$

The corresponding assertions for the upper parameter follow by replacing Q with $I - Q$.

When H_Q is nonamenable and $\mathrm{FK}(Q) = 0$, the theorem does not assert that $p_*(Q)$ is positive. The positivity assumption in (1.8) is part of its statement.

The two Fuglede–Kadison parameters are optimal for every equivariant positive contraction on Γ exactly when Γ is amenable; see Corollary 3.4. In particular the general optimality assertion of [8, Conjecture 5.7] is false. Corollary 3.5 gives a strictly positive, finite-range example on the free group on two generators.

1.2. **Invariant couplings and tree kernels.** Stochastic domination alone does not supply invariance of a coupling on a nonamenable group. Lyons and Thom prove invariant monotone coupling results on sofic groups when the kernels are ordered as operators [10]. Their finite-model and random-root method also underlies the following result.

Theorem 1.4 (Invariant Fuglede–Kadison couplings)**.** *If Γ is countable and sofic, both comparisons in* (1.6) *admit Γ-invariant monotone couplings.*

Here the product parameters are the Fuglede–Kadison parameters. The ordinary optimal parameters are given by Corollary 1.2. The comparison in Theorem 1.4 need not arise from an operator inequality $\mathrm{FK}(Q)I \le Q$. The additional finite input is a product comparison using the diagonal of $\log Q$, proved in Corollary 3.2. Section 4 includes the approximation and invariance arguments, in order to specify this extension of the method of [10].

Finally, for $Q = aI + bA$ on an infinite regular tree, we determine $p_*(Q)$ explicitly. The formula in Theorem 5.1 is a convergent weighted sum of the logarithms of the leaf-elimination pivots. The graph-theoretic input is the greedy-tree spectral-moment theorem of Andriantiana and Wagner [2]. We prove the reduction from arbitrary finite principal compressions to tree balls and include the spectral endpoints.

2. Logarithmic rounding and product comparison

2.1. **The conditional probability property.** We first work on a finite set E. A law μ on $\{0,1\}^E$ has *full support* if every configuration has positive probability. The property needed below is

$$\mu[X_i = 1 \mid X|_T = \eta] \;\ge\; \mu[X_i = 1 \mid X_j = 1 \ (j \in T)] \tag{2.1}$$

for every $T \subset E$, $i \notin T$, and $\eta \in \{0,1\}^T$. Thus the all-occupied pattern minimizes each remaining one-site occupation probability.

Lemma 2.1. *If $0 < Q < I$ on a finite set, then $\mathbb{P}^Q$ has full support and satisfies* (2.1)*.*

Proof. We recall the conditioning formulas, with the conventions needed here; see also [7, Sections 6 and 8]. Full support follows from the L-ensemble expression

$$\mathbb{P}^Q[X = A] = \frac{\det L_A}{\det(I + L)}, \qquad L = Q(I - Q)^{-1} > 0.$$

After conditioning on occupations on A, the kernel on the remaining coordinates is

$$K = Q_{A^c,A^c} - Q_{A^c,A} Q_A^{-1} Q_{A,A^c}.$$

It satisfies $0 < K < I$. For example, positivity is the Schur-complement criterion; the upper bound follows because $K \le Q_{A^c,A^c} < I$. If B is a further set and $C = E \setminus (A \cup B)$, conditioning K on vacancies or occupations on B gives, respectively,

$$K_C^{0_B} = K_C + K_{C,B}(I - K_B)^{-1}K_{B,C},$$
$$K_C^{1_B} = K_C - K_{C,B}K_B^{-1}K_{B,C}.$$

These identities follow from block determinants in the conditional inclusion probabilities. Their difference is positive semidefinite. Write $T = A \cup B$ for the occupied and vacant parts of a given pattern. The diagonal comparison just proved gives (2.1). □

Lemma 2.2. *Let μ have full support and satisfy* (2.1)*. Then*

$$s(S) = \log \mu[S \subset X], \qquad s(\varnothing) = 0,$$

is submodular.

Proof. For $S \subset T$ and $i \notin T$, the conditional probability at i given occupations on S is the average of the conditional probabilities given the patterns on T which extend those occupations. By (2.1), this average is at least the conditional probability given occupations on all of T. Taking logarithms gives

$$s(S \cup \{i\}) - s(S) \ge s(T \cup \{i\}) - s(T).$$

This is the diminishing-returns form of submodularity. For $\mathbb{P}^Q$ it is also the Hadamard–Fischer determinant inequality. □

2.2. **The exchange argument.** For $y \in \mathbb{R}^E$, write $y(S) = \sum_{i \in S} y_i$ and $\pi_{e^y} = \bigotimes_i \mathrm{Ber}(e^{y_i})$ whenever $y_i \le 0$.

Lemma 2.3 (Logarithmic rounding)**.** *Let μ have full support and satisfy* (2.1)*. If $x(S) \le s(S)$ for every $S \subset E$, then*

$$\pi_{e^x} \preccurlyeq \mu.$$

The singleton constraints imply $x_i < 0$.

Proof. Passage to the base. Enlarge x coordinatewise to a point y in

$$B(s) = \{y : y(S) \le s(S) \ (S \subset E),\ y(E) = s(E)\}.$$

Here is an elementary existence argument. Maximize $y(E)$ over the constraints $y(S) \le s(S)$ and $y_i \ge x_i$. The feasible region is nonempty and compact: the singleton constraints give upper bounds and x gives lower bounds. At a maximizer, every coordinate belongs to a tight set, since otherwise that coordinate could be increased. Submodularity makes unions and intersections of tight sets tight. Their union is E, which proves $y(E) = s(E)$. The set $B(s)$ is compact as well, since

$$s(E) - \sum_{j \ne i} s(\{j\}) \le y_i \le s(\{i\}) < 0.$$

Convexity for all increasing functions. Let F be any increasing real function on $\{0,1\}^E$ and put $\Phi_F(y) = \mathbb{E}_{\pi_{e^y}} F$. Holding the other coordinates fixed, write $p_i = e^{y_i}$ and $p_j = e^{y_j}$. Expansion in these two Bernoulli variables gives

$$\Phi_F(y) = c_0 + c_i p_i + c_j p_j + c_{ij} p_i p_j, \qquad c_i, c_j \ge 0.$$

The coefficients c_i, c_j are averages of $F(1,0,\cdot) - F(0,0,\cdot)$ and $F(0,1,\cdot) - F(0,0,\cdot)$. On the line $y + t(e_i - e_j)$, the product $p_i p_j$ is constant. Consequently Φ_F is convex along that line. There is no sign condition on c_{ij}, and none is needed.

A common rounding distribution. At the current point of $B(s)$ choose a maximal chain of tight sets from $\varnothing$ to E. If an increment of the chain has at least two elements, choose distinct i, j in that increment. Every tight set contains both or neither of i, j. Indeed, if a tight set separates them, intersect it with the upper member of the increment and unite with the lower member; the resulting tight set strictly refines the chain. Thus both signs of a sufficiently small movement in direction $e_i - e_j$ are feasible.

Move randomly to one of the two endpoints of the feasible interval, with probabilities making the conditional mean equal to the current point. Explicitly, if the interval is $[t_-, t_+]$ with $t_- < 0 < t_+$, choose t_- with probability $t_+/(t_+ - t_-)$ and t_+ with probability $-t_-/(t_+ - t_-)$. All old tight constraints remain tight and at least one new independent constraint becomes tight at each endpoint. Hence the dimension of the minimal face decreases. After finitely many steps the chain is complete. The endpoint choices depend only on s and the current point, not on F. If Y denotes the resulting terminal point, the convexity above therefore gives, for every increasing F,

$$\mathbb{E}_{\pi_{e^y}} F \le \mathbb{E}\big[\mathbb{E}_{\pi_{e^Y}} F\big]. \tag{2.2}$$

This is the exchange-rounding principle of [1, 3], applied to logarithmic parameters; the argument above establishes the form used here directly.

Comparison at a terminal point. A complete tight chain is given by prefixes $S_k = \{i_1, \dots, i_k\}$. The terminal parameters satisfy

$$e^{Y_{i_k}} = \exp(s(S_k) - s(S_{k-1})) = \mu[X_{i_k} = 1 \mid S_{k-1} \subset X].$$

Use independent uniform variables to sample μ sequentially in this order. With the same variable at step k, sample the Bernoulli bit with the displayed, fixed parameter. Property (2.1) makes that Bernoulli bit at most the sampled bit of μ. The Bernoulli bits are independent because their thresholds are fixed and the uniforms are independent. Thus $\pi_{e^Y} \preccurlyeq \mu$ at every terminal point. Integrating this comparison in (2.2) gives $\pi_{e^y} \preccurlyeq \mu$. Finally $x \le y$ gives $\pi_{e^x} \preccurlyeq \pi_{e^y}$. □

2.3. **Boundary cases and countable sets.**

Proof of Theorem 1.1. Necessity in (1.2) follows by testing $\{S \subset X\}$. For sufficiency on a finite set, first suppose that every $p_i > 0$ and $0 < Q < I$. Apply Lemmas 2.1–2.3 to $x_i = \log p_i$ and $s(S) = \log\det Q_S$.

If every $p_i > 0$ but Q is an arbitrary positive contraction, the constraint for $S = E$ forces Q to be positive definite. For $0 < \varepsilon < 1$, put

$$Q_\varepsilon = (1-\varepsilon)Q, \qquad p_i^\varepsilon = (1-\varepsilon)p_i.$$

The principal determinants and the parameter products on S both acquire the factor $(1-\varepsilon)^{|S|}$. The strict-kernel case applies, and the finite measures converge as $\varepsilon \downarrow 0$. Thus the domination persists in the limit. This argument also allows $p_i = 1$.

If some $p_i = 0$, put $A = \{i : p_i > 0\}$. If $A = \varnothing$, the lower product is the constant empty configuration and the comparison is immediate. Otherwise apply the preceding argument to the compression on A. Couple its determinantal marginal with the product on A, extend the latter by zeros, and sample the remaining determinantal coordinates from their conditional law. This yields the finite comparison without an invertibility assumption on Q.

Now let E be countably infinite. Every finite restriction has the comparison just proved. Consider the compact space of joint laws on $\{0,1\}^E \times \{0,1\}^E$ with the prescribed marginals. For each finite $F \subset E$, the requirement of coordinatewise inclusion on F is closed and can be realized by extending a finite coupling with conditional laws on the remaining coordinates.

These closed sets have the finite-intersection property. A joint law in their intersection is a monotone coupling on all of E. This is also the compactness form of Strassen's comparison theorem [11].

The complement of a determinantal configuration with kernel Q has kernel $I - Q$, as follows by inclusion–exclusion. Taking complements reverses the order and proves (1.3). □

Proof of Corollary 1.2. For constant p, the inequalities in (1.2) are exactly $p \le (\det Q_S)^{1/|S|}$ for every nonempty finite S. Their simultaneous endpoint is their infimum. The upper threshold follows by complementation. □

Remark 2.4. Product comparison can hold beyond the range obtained from operator order. For example, for

$$Q = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$$

Corollary 1.2 gives $p_*(Q) = \sqrt{3}/4$. The largest p with $pI \le Q$ is only $1/4$. This distinction is relevant to the invariant couplings in Section 4.

3. Compression gaps and amenability

3.1. **A quantitative logarithmic compression inequality.** We use the following scalar Jensen estimate so that no strict-positivity case is implicit in the argument.

Proposition 3.1. *Suppose $\delta I \le Q \le I$ on $\ell^2(E)$ with $\delta > 0$. For each finite $S \subset E$,*

$$\frac{1}{2}\|P_{S^c} Q P_S\|_{\mathrm{HS}}^2 \le \log\det Q_S - \mathrm{Tr}(P_S(\log Q)P_S) \le \frac{1}{2\delta^2}\|P_{S^c} Q P_S\|_{\mathrm{HS}}^2. \tag{3.1}$$

Proof. Let u be a unit eigenvector of Q_S with eigenvalue λ, extended by zero outside S. Its spectral probability measure ν_u for Q has mean λ. Since $1 \le -(\log t)'' \le \delta^{-2}$ on $[\delta, 1]$, Taylor's inequality about λ, followed by integration, gives

$$\frac{1}{2}\int (t-\lambda)^2 \, d\nu_u(t) \le \log\lambda - \langle (\log Q)u, u\rangle \le \frac{1}{2\delta^2}\int (t-\lambda)^2 \, d\nu_u(t).$$

The linear term integrates to zero. Also,

$$\int (t-\lambda)^2 \, d\nu_u(t) = \|Qu\|^2 - \lambda^2 = \|P_{S^c} Q u\|^2.$$

Sum over an orthonormal eigenbasis of Q_S. The sum of these variances is the Hilbert–Schmidt norm squared in (3.1). □

Corollary 3.2 (Diagonal-log product comparison)**.** *If $0 < K \le I$ on a finite set, then*

$$\pi_p \preccurlyeq \mathbb{P}^K, \qquad p_i = \exp((\log K)_{ii}).$$

Proof. Use the lower bound in Proposition 3.1 with δ equal to the smallest eigenvalue of K. For every S it gives $\log\det K_S \ge \sum_{i\in S}(\log K)_{ii}$. Theorem 1.1 applies. □

3.2. **The support subgroup.** We use the Følner criterion for amenability: a countable group H is amenable if for every finite $F \subset H$ and $\eta > 0$ there is a nonempty finite $T \subset H$ such that $\sum_{g\in F} |Tg \triangle T| < \eta |T|$.

Lemma 3.3. *For the equivariant kernel in* (1.5)*, $\beta_Q = 0$ if and only if H_Q is amenable.*

Proof. Put $q(g) = Q(e, g)$. Equivariance gives $Q(x, y) = q(x^{-1}y)$, and self-adjointness gives $|q(g^{-1})| = |q(g)|$. Reindexing the squared matrix entries therefore gives

$$\|P_{S^c} Q P_S\|_{\mathrm{HS}}^2 = \sum_{g \in \Gamma} |q(g)|^2 \, |Sg \setminus S|. \tag{3.2}$$

The weights are summable because $\sum_g |q(g)|^2 = \|Q\delta_e\|^2 \le 1$. All nonzero weights are in H_Q.

If H_Q is amenable, choose a Følner sequence there. For each fixed $g \in H_Q$ its normalized contribution in (3.2) tends to zero and is at most $|q(g)|^2$. Dominated convergence proves $\beta_Q = 0$.

Conversely, let nonempty finite $S_j \subset \Gamma$ have normalized boundary in (3.2) tending to zero. For every g with $q(g) \neq 0$,

$$|S_j g \triangle S_j| = 2|S_j g \setminus S_j| = o(|S_j|).$$

The symmetric-difference triangle inequality, preserved by right translation, extends this to every $g \in H_Q$ and hence to any fixed finite collection $F \subset H_Q$. Decompose S_j over left cosets of H_Q. Right multiplication by F preserves each coset, so

$$\frac{\sum_{g \in F} |S_j g \triangle S_j|}{|S_j|}$$

is the size-weighted average of the corresponding ratios on its nonempty coset pieces. One piece has ratio at most that average. Translate this piece on the left into H_Q. For large j it is a Følner set for F with any prescribed tolerance. Thus H_Q is amenable. □

3.3. **Proof of the group comparison.**

Proof of Theorem 1.3. If $\delta I \le Q \le I$, the diagonal of $\log Q$ is constant and equal to $\tau(\log Q)$. Proposition 3.1 becomes

$$\frac{1}{2}\|P_{S^c} Q P_S\|_{\mathrm{HS}}^2 \le \log\det Q_S - |S|\tau(\log Q) \le \frac{1}{2\delta^2}\|P_{S^c} Q P_S\|_{\mathrm{HS}}^2. \tag{3.3}$$

After division by $|S|$, the lower bound and Corollary 1.2 imply (1.7).

For a general positive contraction set $Q_\varepsilon = (1-\varepsilon)Q + \varepsilon I$. As $\varepsilon \downarrow 0$, its spectral logarithms decrease to that of Q. If $\mathrm{FK}(Q) > 0$, their integrals have a finite limit. The lower bound in (3.3) forces $\det(Q_\varepsilon)_S$ to stay bounded away from zero for every fixed S. It therefore proves $\det Q_S > 0$ and passes to the limit. The boundary norm is multiplied by $(1-\varepsilon)^2$. This gives (1.7). If $\mathrm{FK}(Q) = 0$, the same bound is immediate because its right side is zero. Thus the lower comparison in (1.6) always holds. Apply it to $I - Q$ and take complements for the upper comparison.

Suppose now that H_Q is amenable. If $Q \ge \delta I$, insert Følner sets in H_Q into the upper bound in (3.3). Equation (3.2) makes their normalized boundary tend to zero, so (1.4) gives $p_*(Q) = \mathrm{FK}(Q)$. For general Q, the off-diagonal support of Q_ε generates the same subgroup as that of Q. Moreover $Q_S \le (Q_\varepsilon)_S$, and finite positive determinants are monotone in the positive-semidefinite order. Consequently,

$$\mathrm{FK}(Q) \le p_*(Q) \le p_*(Q_\varepsilon) = \mathrm{FK}(Q_\varepsilon).$$

Let $\varepsilon \downarrow 0$. This proves equality, also when $\mathrm{FK}(Q) = 0$. If H_Q is nonamenable and $\mathrm{FK}(Q) > 0$, Lemma 3.3 and (1.7) give strict inequality. Finally, $I - Q$ has the same off-diagonal support and the same boundary quantity as Q; its support subgroup is therefore the same. □

Corollary 3.4. *A countable group is amenable if and only if the two Fuglede–Kadison parameters are optimal for every equivariant positive contraction on that group.*

Proof. For an amenable group, every subgroup is amenable, and Theorem 1.3 applies. If Γ is nonamenable, some finite symmetric set F generates a nonamenable subgroup. Indeed, if every finitely generated subgroup were amenable, the Følner criterion would hold for every finite subset of Γ. Choose $\varepsilon > 0$ with $\varepsilon|F \setminus \{e\}| < 1/4$ and set

$$Q = \tfrac{1}{2}I + \varepsilon \sum_{g \in F\setminus\{e\}} R_g, \qquad (R_g f)(x) = f(xg).$$

It satisfies $\frac{1}{4}I < Q < \frac{3}{4}I$. Both determinants are positive, and the support subgroup is nonamenable. Both comparisons are strictly improvable. □

Corollary 3.5 (A finite-range counterexample to optimality)**.** *Let A be adjacency on the standard Cayley tree of the free group F_2 and put*

$$Q = \tfrac{1}{2}I + \tfrac{1}{16}A.$$

Both Fuglede–Kadison parameters are strictly nonoptimal. More precisely,

$$p_*(Q) \geq e^{1/256}\,\mathrm{FK}(Q) > \mathrm{FK}(Q), \qquad q_*(Q) \leq 1 - e^{1/256}\,\mathrm{FK}(I-Q) < 1 - \mathrm{FK}(I-Q).$$

Proof. The elementary bound $\|A\| \leq 4$ gives $\frac{1}{4}I \leq Q \leq \frac{3}{4}I$. For a finite set S its induced graph is a forest. If it has $c(S)$ components, the number of directed edges from S to S^c is $2|S| + 2c(S)$. Hence

$$\|P_{S^c} Q P_S\|_{\mathrm{HS}}^2 = \frac{2|S| + 2c(S)}{256}.$$

Connected sets of arbitrarily large cardinality show that $\beta_Q = 1/128$. Theorem 1.3 and its complementary version prove the assertions. □

4. Invariant couplings on sofic groups

We give the approximation argument in full. The use of finite models and a uniform root follows [10]; the finite comparison used here is Corollary 3.2.

A sofic approximation of a countable group is a sequence $\sigma_n : \Gamma \to \mathrm{Sym}(V_n)$ such that, for each fixed $g, h \in \Gamma$, the permutations $\sigma_n(gh)$ and $\sigma_n(g)\sigma_n(h)$ agree on a proportion tending to one, and, for $g \neq e$, the proportion of fixed points of $\sigma_n(g)$ tends to zero. We may take $\sigma_n(e)$ to be the identity. These conditions imply the corresponding statements for any fixed finite list of words, inverses, and distinct group elements.

4.1. Clipped finite-range operators.

Fix $0 < \delta < 1$ and let

$$c_\delta(t) = \min(1, \max(\delta, t)).$$

Let $R = \sum_g r_g R_g$ be a self-adjoint finite group-algebra operator, using the right shifts from Corollary 3.4. Set $Q = c_\delta(R)$.

On $\ell^2(V_n)$ let $U_{n,g}\delta_v = \delta_{\sigma_n(g)v}$. Replace the shifts in R by these permutation matrices and take the self-adjoint part:

$$R_n = \frac{1}{2}\left(\sum_g r_g U_{n,g} + \left(\sum_g r_g U_{n,g}\right)^*\right), \qquad K_n = c_\delta(R_n).$$

Then $\delta I \leq K_n \leq I$ and $\|R_n\| \leq M := \sum_g |r_g|$.

For a uniform root $v \in V_n$, use the coordinates

$$\phi_{n,v}(x) = \sigma_n(x^{-1})v, \qquad x \in \Gamma. \tag{4.1}$$

For each fixed finite $F \subset \Gamma$, these coordinates are distinct with probability tending to one. The sofic identities also imply

$$\begin{aligned} K_n(\phi_{n,v}(x), \phi_{n,v}(y)) &\longrightarrow Q(x,y) \qquad (x,y \in F), \\ (\log K_n)(\phi_{n,v}(x), \phi_{n,v}(x)) &\longrightarrow \tau(\log Q) \qquad (x \in F) \end{aligned} \tag{4.2}$$

in probability with respect to v. For a polynomial in R_n, this follows on the roots where the finitely many words in that polynomial act as in the group. The use of x^{-1} in (4.1) matches $R_g \delta_y = \delta_{yg^{-1}}$. Approximate c_δ and $\log c_\delta$ uniformly by polynomials on $[-M, M]$. The resulting operator-norm errors bound every matrix entry, which proves (4.2).

By Corollary 3.2, there is a finite monotone coupling (Y_n, X_n) with

$$Y_n \sim \bigotimes_{w \in V_n} \mathrm{Ber}(p_{n,w}), \qquad X_n \sim \mathbb{P}^{K_n}, \qquad p_{n,w} = \exp((\log K_n)_{ww}).$$

Choose v uniformly and independently of this coupling, and pull both configurations back by (4.1). The first marginal converges to $\pi_{\mathrm{FK}(Q)}$. Indeed, on every finite F its parameters converge in probability to the same constant, and its coordinates are distinct with probability tending to one. The second marginal converges to $\mathbb{P}^Q$, because the inclusion determinants converge on each finite set; inclusion–exclusion then gives all finite-pattern probabilities. The configuration-pair space is compact, so a joint subsequential limit exists and preserves inclusion.

We check invariance without any equivariance requirement on the selected finite couplings. For a fixed $\gamma \in \Gamma$ and finite F,

$$\phi_{n,v}(\gamma^{-1}x) = \sigma_n(x^{-1}\gamma)v = \phi_{n,\sigma_n(\gamma)v}(x) \quad (x \in F)$$

outside a set of roots of relative size tending to zero. Replacing the independent uniform root by $\sigma_n(\gamma)v$ leaves its joint law with (Y_n, X_n) unchanged. The limiting coupling is therefore invariant under left translation by γ. This proves Theorem 1.4 for $Q = c_\delta(R)$.

4.2. **General equivariant kernels.**

Completion of the proof of Theorem 1.4. First suppose $\delta I \le Q \le I$. Finite self-adjoint Fourier sums R_m approximate Q in the trace-L^2 norm $\|T\|_2 = \tau(T^*T)^{1/2}$. For example, truncate its square-summable coefficients to increasing finite symmetric sets. Put $Y_m = c_\delta(R_m)$. Then

$$\|Y_m - Q\|_2 \le \|R_m - Q\|_2 \longrightarrow 0. \tag{4.3}$$

To verify the contraction needed here, work in the real Hilbert space of self-adjoint trace-L^2 operators. For any $\delta I \le X \le I$,

$$\tau((R_m - Y_m)(X - Y_m)) \le 0.$$

On the spectral subspace $R_m > 1$, this is the trace of the positive part $R_m - 1$ against $X - I \le 0$. On $R_m < \delta$, it is the trace of the negative part $R_m - \delta$ against $X - \delta I \ge 0$. The other spectral subspace contributes zero. Thus Y_m is the metric projection of R_m onto the closed convex order interval, giving (4.3).

Equivariance and (4.3) imply convergence of every matrix entry. Uniform polynomial approximation of log on $[\delta, 1]$ also gives $\tau(\log Y_m) \to \tau(\log Q)$. For clarity, for each fixed k the telescoping identity and the trace Cauchy–Schwarz inequality give

$$|\tau(Y_m^k - Q^k)| \le k\|Y_m - Q\|_2;$$

polynomial approximation then applies. For each Y_m the preceding subsection supplies an invariant monotone coupling with parameter $\mathrm{FK}(Y_m)$. A weak limit has the desired marginals for Q. Both inclusion and invariance are closed under this limit.

Finally replace a general positive contraction by $Q_\varepsilon = (1-\varepsilon)Q + \varepsilon I$. Its finite inclusion determinants converge to those of Q, and monotone convergence of the spectral logarithmic integrals gives $\mathrm{FK}(Q_\varepsilon) \to \mathrm{FK}(Q)$, possibly to zero. Take a further compactness limit. Applying the lower-coupling argument to $I - Q$ and taking complements proves the invariant upper comparison. □

5. Exact thresholds for regular-tree kernels

Let $\mathcal{T}_d$ be the infinite d-regular tree and let A be its adjacency operator. We use $d \geq 3$ throughout this section.

Theorem 5.1 (Regular-tree formula)**.** *Suppose $b \neq 0$ and*

$$2\sqrt{d-1}|b| \leq \min(a, 1-a). \tag{5.1}$$

For $Q = aI + bA$, define

$$r_0 = a, \qquad r_{j+1} = a - \frac{(d-1)b^2}{r_j}. \tag{5.2}$$

Then

$$\log p_*(Q) = \sum_{j=0}^{\infty} \frac{d-2}{(d-1)^{j+1}} \log r_j. \tag{5.3}$$

The upper threshold is $q_*(Q) = 1 - p_*(I-Q)$, computed by replacing a with $1-a$ in (5.2)–(5.3)*. The series converges absolutely, including at the spectral endpoints in* (5.1)*. If $b = 0$, both thresholds equal a for $0 \leq a \leq 1$.*

5.1. Spectral range and extremal finite trees.

The adjacency norm is

$$\|A\| = 2\sqrt{d-1}. \tag{5.4}$$

This is the classical regular-tree spectral bound; see Kesten's free-group calculation [6]. We recall a proof valid for every $d \geq 3$. For the upper bound orient all edges toward one end. The forward sum over the $d-1$ children has norm $\sqrt{d-1}$ and its sum with its adjoint is A. For the lower bound take the function $(d-1)^{-|x|/2}$ on a ball of radius R and zero outside. The squared norm grows linearly in R, and the Rayleigh quotient tends to $2\sqrt{d-1}$ by counting the edges between successive spheres. Bipartiteness makes the spectrum symmetric, so (5.1) is exactly the positive-contraction range.

Every finite forest of maximum degree at most d can be embedded as an induced forest in $\mathcal{T}_d$. One embeds a component successively along the tree and places different finite components at sufficiently large distances. Its adjacency norm is strictly below (5.4). Indeed, if equality held, a Perron eigenvector on a component, extended by zero, would attain the extremal Rayleigh quotient of A and hence be an eigenvector of A. At an exterior neighbor of a leaf its image under A is nonzero, contradicting that eigenvalue equation.

Let B_R denote a tree ball and let

$$N_R = |B_R| = 1 + d\,\frac{(d-1)^R - 1}{d-2} \qquad (R \geq 1). \tag{5.5}$$

Its degree sequence has $m = (N_R - 2)/(d-1)$ entries equal to d and all remaining entries equal to one. This sequence majorizes the degree sequence of every tree of order N_R and maximum degree at most d. In fact its first k entries sum to $\min(kd, N_R - 2 + k)$, while every competing sequence has that same upper bound.

Theorem 12 of Andriantiana and Wagner [2] maximizes all spectral moments at the greedy tree with a prescribed degree sequence. Their Theorem 18 makes these greedy-tree moments

monotone under degree-sequence majorization. For the degree sequence above the greedy tree is exactly B_R. It follows that

$$\operatorname{Tr}(A_T^{2k}) \le \operatorname{Tr}(A_{B_R}^{2k}) \qquad (k \ge 1) \tag{5.6}$$

for every tree T of cardinality N_R and maximum degree at most d. The same holds for forests of that cardinality: join their components using leaf endpoints, without exceeding the degree bound. All old closed walks are retained when edges are added.

5.2. **From arbitrary compressions to balls.**

Proof of Theorem 5.1. A finite principal compression corresponds to an induced forest T. By its strict spectral bound, the trace-log expansion is convergent even when equality holds in (5.1):

$$\log\det(aI + bA_T) = |T|\log a - \sum_{k\ge1} \frac{(b/a)^{2k}}{2k} \operatorname{Tr}(A_T^{2k}). \tag{5.7}$$

Odd traces vanish by bipartiteness. Equations (5.6) and (5.7) show that B_R minimizes the determinant among forests of cardinality N_R.

Fix any nonempty finite forest T. At size N_R, form a forest of $\lfloor N_R/|T| \rfloor$ disjoint copies of T and at most $|T|-1$ isolated vertices. The determinant comparison and block multiplicativity give

$$\limsup_{R\to\infty} \frac{1}{N_R} \log\det Q_{B_R} \le \frac{1}{|T|} \log\det Q_T. \tag{5.8}$$

Conversely, the infimum in (1.4) is at most the normalized logarithm for each ball. Thus, once the ball limit is computed, it equals the infimum over all finite compressions.

Eliminate the leaves of B_R toward the root. Every nonroot vertex at height j above the leaf layer has pivot r_j from (5.2). The root pivot is

$$s_R = a - \frac{db^2}{r_{R-1}}.$$

Let

$$r_+ = \frac{a + \sqrt{a^2 - 4(d-1)b^2}}{2}. \tag{5.9}$$

The recurrence gives $a \ge r_j \ge r_+ > 0$. Since $a = r_+ + (d-1)b^2/r_+$,

$$s_R \ge r_+ - \frac{b^2}{r_+} > 0.$$

The last strict inequality follows from $b^2/r_+^2 \le 1/(d-1) < 1$. All pivots are therefore bounded above and bounded away from zero, uniformly in R. This includes the spectral endpoint.

There are $d(d-1)^{R-j-1}$ nonroot vertices at height j, for $0 \le j < R$. Their proportion tends to $(d-2)(d-1)^{-j-1}$. The root proportion tends to zero, and the sum of the proportions at heights at least J is bounded by a constant times $(d-1)^{-J}$. The logarithms of all pivots have a common absolute bound. The product of the pivots is the determinant, so

$$\lim_{R\to\infty} \frac{1}{N_R} \log\det Q_{B_R} = \sum_{j\ge0} \frac{d-2}{(d-1)^{j+1}} \log r_j.$$

Together with (5.8) this proves (5.3). The upper comparison follows by complementation, and $b = 0$ is the independent case. □

5.3. **Comparison with the Fuglede–Kadison value.** Realize $\mathcal{T}_d$ as the Cayley graph of the free product of d copies of $\mathbb{Z}/2\mathbb{Z}$. The canonical group trace is the diagonal entry at a vertex, so the Fuglede–Kadison determinant below is unambiguous.

Proposition 5.2. *Under* (5.1) *with* $b \neq 0$,

$$\log \mathrm{FK}(aI + bA) = \log r_+ + \left(1 - \frac{d}{2}\right) \log \left(1 - \frac{b^2}{r_+^2}\right). \tag{5.10}$$

In particular this determinant is positive, including at the spectral endpoint, and

$$p_*(Q) \geq \mathrm{FK}(Q) \exp\left(\frac{d-2}{2} b^2\right) > \mathrm{FK}(Q). \tag{5.11}$$

Proof. First let $a > 2\sqrt{d-1}|b|$. The same spectral definition of the determinant is available for positive invertible operators without the upper bound by I. Its logarithmic derivative with respect to a is the diagonal of $(aI + bA)^{-1}$. The finite-ball Schur complements give

$$((aI + bA)^{-1})(e, e) = \left(a - \frac{db^2}{r_+}\right)^{-1}.$$

Indeed, the finite compressions, extended by aI off the ball, converge strongly with a uniform positive lower bound. Their inverses converge strongly, and $r_j \downarrow r_+$ in their root Schur complements.

Differentiate the right side of (5.10), using $a = r_+ + (d-1)b^2/r_+$. The result is the same reciprocal. Both expressions are $\log a + o(1)$ as $a \to \infty$, which fixes the integration constant. As a decreases to $2\sqrt{d-1}|b|$, monotone convergence of the spectral logarithmic integrals gives the endpoint formula. Its right side is finite because $b^2/r_+^2 = 1/(d-1) < 1$ there.

For a nonempty finite set S, the number of edges from S to its complement is $(d-2)|S| + 2c(S)$, where $c(S)$ is the number of components in its induced forest. Consequently

$$\beta_Q = (d-2)b^2;$$

connected sets of arbitrarily large cardinality attain this infimum in the limit. Apply Theorem 1.3. □

At $a = 2\sqrt{d-1}|b|$, the pivots have the simple form

$$r_j = r_+ \frac{j+2}{j+1}.$$

This follows directly from (5.2). The threshold in (5.3) therefore remains explicit at the edge of the positive spectrum.

AI use declaration

OpenAI Codex assisted in developing the arguments, checking calculations, searching the literature, and preparing and revising this manuscript. Its mathematical contributions include the logarithmic exchange-rounding argument in Section 2, the compression and amenability arguments in Section 3, and assistance with the sofic approximation and regular-tree calculations in Sections 4–5. Sihan Chen is the sole human author and takes responsibility for the mathematical content and the accuracy of the references.

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