On the planes of Cartan's cubics and the missing seven

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Abstract

A complex field z with the interaction $W = \operatorname{Re} z^3$ has three preferred directions, yet its energy $|\nabla W|^2$ depends only on the length of the field. The question is when several such fields can be coupled only through terms in three different fields, so that each can still be excited alone, with the energy of the whole again depending only on length. This happens only for four or thirteen fields, with the cubic of $\mathfrak{su}(3)$ or of the exceptional Jordan algebra. The question names no number system, and beyond four fields its only answer is built on the octonions. Seven, the count the quaternions would give, is missing. Conditions on pairs of fields allow it, and the obstruction lies in triples: any two fields pass a fixed total of two units of coupling to the others, and over the quaternions each of the five others takes at least $1 - 1/\sqrt{3} > 2/5$. The fields span planes that are three-outcome tests in a Jordan algebra, and orthogonal planes are unbiased tests. Hence $\mathbb{H}^3$ does not have the seven mutually unbiased bases that a count of dimensions allows, and compact $\mathfrak{sp}(3)$ (type C_3) has no orthogonal decomposition into Cartan subalgebras. Any six such bases would extend to a seventh, so $\mathbb{H}^3$ has at most five. With two units per pair, thirteen planes carry total coupling 52 on at least 52 triples. Equality holds exactly when the closed sets of four planes form the projective plane of order three, and such a system is unique up to automorphism. With the three-field terms scaled by $t > 0$, $t \neq 1$, the energy shows the planes and, among such cubics, determines the cubic up to sign. In $4 - \varepsilon$ dimensions with $\varepsilon > 0$, the one-loop renormalization group flow with nonzero coupling constant carries every $t > 0$ to $t = 1$, where the planes are no longer visible.

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1 Introduction

One complex field z with the interaction $W(z) = \operatorname{Re} z^3$ has energy $|\nabla W|^2 = 9|z|^4$. Here and throughout, the energy is the squared length of the gradient, which is twice the scalar potential when W is the superpotential of real superfields (Section 12.1). The cubic has three maxima on the unit circle, but its energy depends only on the length of the field. Take two such fields and do not couple them. A unit field configuration split equally between them has energy $9/2$, half that of a unit field in one of them, since each half contributes a quarter. Couplings through terms that contain three different fields, such as $\operatorname{Re}(\bar{z}_1 z_2 z_3)$, can supply the missing half, and they leave each field a consistent truncation. A field configuration that starts in one coordinate plane keeps its gradient in that plane and sees only $\operatorname{Re} z^3$. This paper asks when such couplings make the energy of every field configuration depend only on its length.

Margin notes name the Lean declarations that prove the adjacent statement and say how the Lean statement relates to the printed one; the conventions are in Appendix D.

Without the fields, the question concerns a cubic form W on a Euclidean space that solves the eikonal equation $|\nabla W|^2 = 9|x|^4$ (eikonal constant $c = 1$ below). The cubic solutions of this equation are known: Cartan classified the harmonic ones, his isoparametric cubics, and Tkachev showed that the others have a single reducible form[1]. The question is when the space of such a cubic splits into pairwise orthogonal two-dimensional subspaces, each mapped to itself by ∇W.

Such couplings exist for four fields and for thirteen, and for no other number greater than one (Theorem A). For four the interaction is the cubic invariant of the traceless Hermitian 3×3 matrices, the observables of a three-level quantum system (a qutrit). For thirteen it is the same invariant of Hermitian matrices whose entries are octonions, the eight-dimensional number system with a nonassociative product; these matrices form the exceptional Jordan algebra, also called the Albert algebra. The question names no number system (normed division algebra), and its two answers are built on the complex numbers and on the octonions.

The quaternions lie between, and the count that gives four and thirteen gives seven for them. Seen from one field, the rest of the space splits into three parts of equal dimension joined by a product that multiplies lengths, and by Hurwitz's theorem on bilinear maps that dimension is 2, 4 or 8, which leaves four, seven or thirteen fields. Seven such fields do not exist. Conditions on pairs of fields allow seven, and the obstruction is a condition on triples. In any solution the shares that each pair of fields gives the others add up to two (the sum rule, Theorem 5.9), and over the quaternions each of the five others must take a share of more than two fifths (Theorem 6.4; shares are defined in Section 1.1). Five such shares add up to more than two. Over the complex numbers the two others take share one each, and over the octonions a share can be zero.

Each field spans a plane, the two real dimensions of its values, and these planes are the subject of the paper. In the matrix models a plane is a test with three outcomes, namely the traceless span of a Jordan frame (three primitive

[1] É. Cartan, Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques, *Math. Z.* **45** (1939) 335–367. doi:10.1007/BF01580289; V. G. Tkachev, A generalization of Cartan's theorem on isoparametric cubics, *Proc. Amer. Math. Soc.* **138** (2010) 2889–2895. doi:10.1090/S0002-9939-10-10385-2. arXiv:1001.2387v2.

idempotents with sum the unit; in the Jordan-algebraic interpretation, a rank-one projective measurement with three outcomes), and two planes are orthogonal when the tests are unbiased: if one has a definite outcome, the three outcomes of the other are equally likely. Over the complex numbers and the quaternions a test is an orthonormal basis of $\mathbb{C}^3$ or $\mathbb{H}^3$, up to unit scalars, and unbiased tests are mutually unbiased bases. Four or thirteen fields are a complete set of such tests. The missing seven says that the three-level system over the quaternions has no complete set (Corollary 6.6). Six pairwise unbiased tests there would extend to a seventh, so $\mathbb{H}^3$ has at most five mutually unbiased bases (Theorem B below), where a count of dimensions allows seven. In the language of Lie algebras, compact $\mathfrak{sp}(3)$ (type C_3, dimension 21) has no orthogonal decomposition into Cartan subalgebras (Corollary 10.10).

Over the octonions the sum rule leaves freedom, and thirteen planes can be arranged in more than one way. Their shares total 52 and no share exceeds one, so at least 52 triples of planes are coupled. The number is 52 exactly when the sets of four planes that close among themselves are the projective lines of the projective plane of order three, an arrangement unique up to automorphism (Theorems 9.1 and 9.4).

The energy of a solution does not show its planes, but it can be made to. Scale the couplings between different fields by a factor $t > 0$ other than 1. The energy of a unit field is then greatest on the planes for $t < 1$ and least on them for $t > 1$ (Proposition 11.7), and among interactions scaled in this way it determines the interaction up to sign (Theorem 11.10). The one-loop renormalization group flow in $4 - \varepsilon$ dimensions, $\varepsilon > 0$, with nonzero coupling constant returns every $t > 0$ to 1, where the energy depends on length alone (Theorem 12.4).

For two or more fields, couplings that are holomorphic in the fields never make the energy depend on length alone (Remark 3.9). With conjugates allowed, four fields succeed. The cubic

$$W_4(z) = \operatorname{Re}\Big[z_0^3 + z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 3z_0\big(\bar{z}_1z_2 + \bar{z}_2z_3 + \bar{z}_3z_1\big)\Big] \qquad (1.1)$$

on $\mathbb{C}^4$ has $|\nabla W_4|^2 = 9|z|^4$. It is the cubic invariant $\frac{1}{6}\operatorname{tr} H^3$ of SU(3), for a real linear bijection $z \mapsto H(z)$ onto the traceless Hermitian 3×3 matrices with $\operatorname{tr} H^2 = 6|z|^2$ (Example 3.6), and its coordinate planes are the planes of the four mutually unbiased bases of $\mathbb{C}^3$ (Theorem 5.10). On real z it is the four-line cubic (Example 3.4).

To state the answer, write $W(x) = d(x, x, x)$ for a symmetric trilinear form d on a Euclidean space V and define a commutative product, the *cubic product*, by $\langle x \circ y, z\rangle = d(x, y, z)$, so that $\nabla W = 3\, x \circ x$. The cubic is *eikonal* if $|x \circ x|^2 = c|x|^4$ for a constant $c > 0$, the *eikonal constant*, that is, if it solves the cubic eikonal equation $|\nabla W|^2 = 9c|x|^4$. A subspace U is *closed* if $x \circ x \in U$ for every $x \in U$, the consistent truncation above. A *plane* is a closed two-dimensional subspace, on which an eikonal cubic is a multiple of $\operatorname{Re} z^3$ (Lemma 3.3), and an *octet* is a closed subspace on which the cubic is, after an isometry, a positive multiple of W_4.

Theorem A. (i) *Let $m \geq 1$ and $W(z) = \sum_{i=1}^{m} \operatorname{Re} z_i^3 + \sum_{i<j<k} T_{ijk}(z_i, z_j, z_k)$ on $\mathbb{C}^m = \mathbb{R}^{2m}$, with each T_{ijk} real trilinear over $\mathbb{R}$ (conjugates allowed). If the length of the real gradient $\nabla W(z)$ depends only on $|z|$, then m is 1, 4 or 13. For $m = 4$ and $m = 13$ there is a linear isometry Φ of $\mathbb{R}^{2m}$ onto the traceless Hermitian 3×3 matrices over $\mathbb{C}$ or over the octonions $\mathbb{O}$, with the trace form $\langle X, Y \rangle = \operatorname{tr}(X \bullet Y)$, such that $W(z) = \sqrt{6}\, \operatorname{tr}\big((\Phi z)^3\big)$, and both values occur.*

Formalized. `field_polynomial_count_and_model`, `four_and_thirteen_radial_field_polynomials`.

(ii) *An eikonal cubic has an orthogonal decomposition into two or more nonzero closed subspaces only in dimension 4, as four one-dimensional summands, with a cubic that is not harmonic; in dimension 8, as four planes; and in dimension 26, as thirteen planes or as an octet and nine planes. All four occur.*

Formalized. `classifyRoundDecompositions`; existence: `roundDecompositionRealizations`.

The three-field form in (i) says that the coordinate planes are closed, so for $m \geq 2$ part (i) is the case of (ii) in which every summand is a plane. Part (i) is Theorem 8.11. It rests on the count of Theorem 4.9 and the identification of Theorem 4.7, quotes Hurwitz's theorem on bilinear maps[2]and the algebra classification in Theorem 4.4, the associativity of the trace form of $H_3(\mathbb{O})$, the spectral decomposition in $H_3(\mathbb{O})$ and the transitivity of its automorphisms on Jordan frames (properties (J2) and (J3) of Section 3), and the spectral theorem for quaternionic Hermitian matrices[3], uses the normal form of three unbiased bases of $\mathbb{H}^3$ derived from the proof of a theorem of Chterental and Đoković[4], and removes seven with the quaternionic lower bound of Section 6. The identification re-proves Cartan's classification of the harmonic eikonal cubics by way of the identification at an idempotent (Theorem 4.6, with the normal form in Appendix A), so the quoted classification theorem is not used in (i); Appendix C lists the results that the proofs use without proof; the existence of the thirteen-field solution rests on Tits's construction of the Albert algebra (Theorem 7.4). Since the couplings range over a finite-dimensional space, compactness turns (i) into a positive lower bound on how far a cubic of the form in (i) with $m \notin \{1, 4, 13\}$ is from having an energy that depends only on length (Proposition 8.12).

Part (ii) for harmonic cubics is Theorem 8.8, proved with the same tools and the decompositions of Cartan's cubics (Theorem 8.7), which are found in the coordinates of Tits's construction of the Albert algebra (Theorem 7.4). For cubics that are not harmonic it quotes Tkachev's theorem that such a cubic, scaled to eikonal constant 1, is $s^3 - 3s|u|^2$ in suitable orthonormal coordinates[5], which splits only into four one-dimensional summands (Lemma 8.6).

[2] A. Hurwitz, Über die Composition der quadratischen Formen von beliebig vielen Variabeln, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* (1898) 309–316.

[3] L. Rodman, *Topics in Quaternion Linear Algebra*, Princeton University Press, Princeton, 2014; F. Zhang, Quaternions and matrices of quaternions, *Linear Algebra Appl.* **251** (1997) 21–57. doi:10.1016/0024-3795(95)00543-9.

[4] O. Chterental and D. Ž. Đoković, On orthostochastic, unistochastic and qustochastic matrices, *Linear Algebra Appl.* **428** (2008) 1178–1201. doi:10.1016/j.laa.2007.09.022. arXiv:0708.4051; theorem and page numbers as in arXiv version 1. proof of Thm. 5.4, pp. 24–25.

[5] Tkachev, "A generalization of Cartan's theorem", note 1. Thm. 1.1.

Where exceptional structure is forced. Beyond four fields the hypotheses of Theorem A force the octonions and the exceptional Jordan algebra, but not the arrangement of the thirteen planes inside that algebra. Cohn, Kumar and Minton give a four-parameter family of thirteen pairwise unbiased frames[6], and rotating one quartet (four planes whose sum is closed) of the graded system of Proposition 7.3 gives systems that no automorphism carries to it (Proposition 9.5). Among the arrangements the sum rule singles out the sparsest one (Section 9.1), with the fewest triples of positive share.

1.1 The sum rule against the lower bound

Let U_i and U_j be orthogonal planes of an eikonal cubic. Their products $x \circ y$ are orthogonal to both planes and satisfy $2|x \circ y|^2 = c|x|^2|y|^2$ (Lemma 3.11), and they span a four-dimensional *product space* $U_i \circ U_j$ (Lemma 4.10). When planes $U_1, \dots, U_m$ fill V, the product space of a pair lies in the sum of the other $m-2$ planes, and each of them takes a part of it. With P_k and Q_{ij} the orthogonal projections onto U_k and $U_i \circ U_j$, the *share* of U_k is $s_{ijk} = \frac{1}{2}\operatorname{Tr}(P_k Q_{ij})$. It lies in $[0,1]$ and is symmetric in i, j, k. For $c = 1$ it is, in complex coordinates and after factoring out -3, the sum of the squared moduli of the four coefficients of the three-field term of the three planes (Lemma 5.8). Since $U_i \circ U_j$ has dimension four,

$$\sum_{k \neq i,j} s_{ijk} = 2 \qquad \text{for every pair } i \neq j$$

(Theorem 5.9).

For $c = 1$ the point $z = 1$ of a plane is an idempotent of unit length for the cubic product (Section 4). Its multiplication operator has eigenvalues among 1, -1 and $\frac{1}{2}$, whose eigenspaces are the Peirce spaces of the cubic; harmonicity fixes their multiplicities, and for $m > 1$ off the plane the eigenspaces split into three spaces of dimension $k = 2(m-1)/3$, joined by a bilinear map that multiplies lengths. Hurwitz's theorem on bilinear maps leaves $k = 2$, 4 or 8, that is $m = 4$, 7 or 13 besides $m = 1$ (Corollary 4.5), and the isomorphism part of Theorem 4.4 identifies the cubic with Cartan's cubic on the traceless Hermitian matrices over $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (Theorem 4.7). In these models a plane is the traceless part of the span of a Jordan frame, and two planes are orthogonal exactly when their frames are *unbiased*, meaning that every Jordan idempotent of one has trace product $\frac{1}{3}$ with every idempotent of the other (Proposition 5.1). For the planes of three frames $\{p_a\}$, $\{p'_b\}$, $\{p''_c\}$ the share is a sum of squared centred triple traces,

$$s_{ijk} = 6 \sum_{a,b,c} \Big(\operatorname{tr}\big((p_a \bullet p'_b) \bullet p''_c\big) - \tfrac{1}{9} \Big)^2$$

(Lemma 5.8(iii)), and for orthonormal bases (u_a), (v_b), (w_c) of $\mathbb{C}^3$ or $\mathbb{H}^3$ the triple trace is $\operatorname{Re}\big((u_a^* v_b)(v_b^* w_c)(w_c^* u_a)\big)$. Over $\mathbb{C}$ the product inside the real part is a

[6] H. Cohn, A. Kumar and G. Minton, Optimal simplices and codes in projective spaces, *Geom. Topol.* **20** (2016) 1289–1357. doi:10.2140/gt.2016.20.1289. arXiv:1308.3188. §8.2.

Bargmann invariant; over $\mathbb{H}$ a change of unit representatives conjugates it by a unit quaternion, which leaves its real part unchanged. So the sum rule constrains triple traces of three bases, which conditions on pairs do not see. It decides among 4, 7 and 13 by setting the average share it demands against the least share possible (Table 1).

Table 1: The sum rule against the lower bound. The average share needed is $2/(m-2)$; the least shares are those of Theorem 5.10, Theorem 6.4 and Proposition 7.3.

number system	planes m	average share needed	least share possible
$\mathbb{C}$	4	1	1
$\mathbb{H}$	7	$2/5$	$1 - 1/\sqrt{3} \approx 0.4226$
$\mathbb{O}$	13	$2/11 \approx 0.18$	0

Over $\mathbb{C}$ the two numbers agree. Both other shares of a pair are one, and the four planes are unique up to unitary conjugation and relabelling (Theorem 5.10).

Over $\mathbb{H}$ the least share exceeds the average. Every three pairwise unbiased bases of $\mathbb{H}^3$ have share at least $1 - 1/\sqrt{3}$, with equality attained (Theorem 6.4). The five other shares of a pair would add up to at least $5(1 - 1/\sqrt{3}) \approx 2.11$, more than the 2 of the sum rule, so $\mathbb{H}^3$ has no seven pairwise unbiased bases and its cubic no seven orthogonal planes (Corollary 6.6). Any six pairwise unbiased bases extend to a seventh (Proposition 6.7), and this gives the bound of Corollary 6.9.

Theorem B. *Every set of pairwise unbiased bases of $\mathbb{H}^3$ has at most five members.*

Formalized. `Quaternionic MUB.quaternionicFive Bound`.

Sets of four exist, so whether five exist is the open question (Section 13). A count of dimensions alone gives seven, the bound $2n + 1$ of Chterental and Đoković for $\mathbb{H}^n$[7].

Over $\mathbb{O}$ a share can vanish. Twelve pairwise unbiased frames of the Albert algebra have exactly one common unbiased frame (Proposition 6.8), so a maximal set of pairwise unbiased frames there has at most eleven or exactly thirteen members.

The least shares differ because of what squares back into the complex numbers. Write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. Since $jz = \bar{z}j$ for $z \in \mathbb{C}$ and $j^2 = -1$, the product of two elements of $\mathbb{C}j$ lies in $\mathbb{C}$, and this parity keeps every share of three unbiased bases of $\mathbb{H}^3$ at or above $1 - 1/\sqrt{3}$ (Section 6). In $\mathbb{O}$ there is no such parity, since for a unit l orthogonal to a quaternion subalgebra containing $\mathbb{C}$ and j, the product jl is again orthogonal to $\mathbb{C}$.

1.2 Planes in fields, measurements, geometry and Lie algebras

For $\mathbb{K} = \mathbb{C}$ a Jordan frame is an orthonormal basis of $\mathbb{C}^3$ up to phases, and $H_3(\mathbb{C})$ is the algebra of observables of a qutrit, an ordinary three-level quantum system. In Table 2 the first two rows are Sections 3–9. The Lie-algebra row reads from left to

[7] Chterental and Đoković, "On orthostochastic, unistochastic and qustochastic matrices", note 4. Thm. 5.1.

right and concerns the Cartan subalgebras of one orthogonal decomposition into Cartan subalgebras (Section 10); the zero weight plane of a Cartan subalgebra is the two-dimensional subspace of $H_3^0(\mathbb{K})$ that it kills (Definition 10.1). Two orthogonal Cartan subalgebras that are not part of a decomposition can have the same zero weight plane (Example 10.9), and the converse is open. The geometric row is the standard description of the maximal flats through the base point of SU(3), SU(6)/Sp(3) and E_6/F_4, whose tangent spaces there are the spaces $H_3^0(\mathbb{K})$[8]; no proof uses it.

Table 2: Four readings of a plane of Cartan's cubic on $H_3^0(\mathbb{K})$, $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$.

reading	a plane is	two orthogonal planes are	a complete system is
fields	a consistent truncation to one field with $\operatorname{Re} z^3$	coupled only by three-field terms	an eikonal interaction tiled by truncations
measurements	a three-outcome test (Jordan frame)	unbiased tests	a complete set of unbiased tests
geometry	the tangent plane of a maximal flat	orthogonal tangent planes	the tangent space split into such planes
Lie algebras	the zero weight plane of a Cartan subalgebra	from two Cartan subalgebras of an orthogonal decomposition into Cartan subalgebras	from an orthogonal decomposition into Cartan subalgebras (none for compact $\mathfrak{sp}(3)$)

1.3 Thirteen planes, Cartan subalgebras, energy and flow

Thirteen planes. A grading of the complexified Albert algebra by $\mathbb{F}_3^3$ with one-dimensional components, due to Griess[9], gives thirteen planes, one for each point of the projective plane PG(2, 3) (Proposition 7.3). In these coordinates the cubic of $H_3(\mathbb{O})$, whose automorphism group is compact F_4[10], is a sum of thirteen terms $\operatorname{Re} \zeta_p^3$ minus three times a sum of 52 monomials, one for each collinear triple of points, each carrying a cube root of unity (7.4). No choice of complex coordinates on these thirteen planes, normalized as in (1.1), makes all the coefficients real (Proposition 7.7), whereas the coefficients of W_4 are real.

For any thirteen-plane system the 286 shares total 52, so at least 52 are positive. Exactly 52 are positive if and only if every share is 0 or 1, if and only if every pair of planes lies in a quartet whose sum is an octet, if and only if the quartets are the projective lines of a projective plane of order three, which is then isomorphic to PG(2, 3) (Theorem 9.1). Such a system is the graded one up to an automorphism

[8] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[9] R. L. Griess, Jr., A Moufang loop, the exceptional Jordan algebra, and a cubic form in 27 variables, *J. Algebra* **131** (1990) 281–293. doi:10.1016/0021-8693(90)90176-O. (4.3)–(4.6).

[10] J. C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002) 145–205. arXiv:math/0105155.

of $H_3(\mathbb{O})$ (Theorem 9.4). Three orthogonal planes generate a copy of $H_3(\mathbb{C})$, $H_3(\mathbb{H})$ or $H_3(\mathbb{O})$, and a third plane inside a copy of $H_3(\mathbb{H})$ has share at least $1 - 1/\sqrt{3}$, so for every pair at least seven of the other eleven planes generate $H_3(\mathbb{O})$ with it (Theorem 9.9). An octet forces the other summands to be nine planes (Proposition 8.2), and no summand of dimension 5 or 14 occurs (Proposition 8.5).

Cartan subalgebras. The derivation algebras of $H_3(\mathbb{C})$, $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$ are compact $\mathfrak{su}(3)$, $\mathfrak{sp}(3)$ and $\mathfrak{f}_4$. In an orthogonal decomposition into Cartan subalgebras, the compact analogue of the notion for complex simple Lie algebras treated by Kostrikin and Tiep[11], the zero weight planes are pairwise orthogonal planes that fill the traceless part (Theorem 10.7; for $\mathfrak{f}_4$ through the conjugacy of maximal tori). The proof splits the Casimir operator along the decomposition. With the missing seven it shows that compact $\mathfrak{sp}(3)$ has no orthogonal decomposition into Cartan subalgebras (Corollary 10.10), the part stable under a compact conjugation of the open case C_3 of a conjecture of Kostrikin, Kostrikin and Ufnarovskii (Section 1.4). The grading gives the known orthogonal decomposition into Cartan subalgebras of $\mathfrak{f}_4$, with the graded planes as zero weight planes (Proposition 10.12).

The energy away from the eikonal cubic. Split an eikonal cubic with a complete system of planes (pairwise orthogonal planes that fill V, Section 5) as $d = D + M$, the sum of the cubics of the planes and the mixed part, and follow the normalized members $d_t = D + tM$ of the pencil spanned by D and M, with the ratio t, from the uncoupled planes at $t = 0$ to the eikonal cubic at $t = 1$. The normalized energy of a unit field configuration, that is its energy in units of the energy of a unit field in one plane, differs from 1 by $t - 1$ times a quartic that, for $t \geq 0$, vanishes exactly on the planes (Theorem 11.5). For $0 \leq t < 1$ its maxima on the unit sphere are exactly the unit circles of the planes, and for $t > 1$ these circles are its minima (Proposition 11.7). For $t > 0$, $t \neq 1$, the energy determines the planes and t, and among the cubics d_t of such pencils, for a fixed metric and eikonal constant c, it determines the cubic up to sign (Theorem 11.10). For such t the symmetry group of the energy is finite; at $t = 1$ it is the whole orthogonal group (Proposition B.3).

The one-loop flow. Read d_t, up to the factor $g/6$ with g the coupling constant, as the superpotential of $N = 2m$ real superfields in the continuation to $4-\varepsilon$ dimensions of the three-dimensional $\mathcal{N} = 1$ (two real supercharges) Wess–Zumino model. The scalar potential is then proportional to the energy; for the superpotential $d(\phi, \phi, \phi)$ the energy is twice the scalar potential (Section 12.1). The one-loop renormalization group equations of Liendo and Rong[12]keep the pencil of every complete system of planes (Proposition 12.3). On it, with $h = g\, d_t$,

$$16\pi^2 \frac{dt}{d\ell} = 6c\, g^2\, t^2(\lambda t + 1)(1 - t), \qquad \lambda = \frac{N-2}{12},$$

[11] A. I. Kostrikin and P. H. Tiep, *Orthogonal Decompositions and Integral Lattices*, de Gruyter Exp. Math. 15, de Gruyter, Berlin, 1994. doi:10.1515/9783110901757. pp. 1–3.

[12] P. Liendo and J. Rong, Seeking SUSY fixed points in the $4 - \epsilon$ expansion, *J. High Energy Phys.* **2021**(12) (2021) 033. doi:10.1007/JHEP12(2021)033. arXiv:2107.14515. Eqs. (2.1)–(2.2).

and for $\varepsilon > 0$, $g(0) \neq 0$ and $t(0) > 0$ the ratio runs monotonically to $t = 1$ (Theorem 12.4).

On each pencil the excess of the A-function over its minimum is a function of the squared norm of the rescaled coupling $\kappa = h/(4\pi\sqrt{\varepsilon})$ plus a positive multiple of the variance of its scalar potential over the unit sphere, and the relative variance decreases strictly while the ratio t moves (Proposition 12.5). On the space of all cubics the identity has further terms, and along the flow on all cubics it is the A-function, not the variance, that decreases. Whether the three-dimensional theory behaves in the same way is open (Remark 12.8).

1.4 Relation to earlier work

Eikonal cubics. Cartan classified the harmonic eikonal cubics, and Tkachev showed that every other cubic solution of $|\nabla W|^2 = 9|x|^4$ is $s^3 - 3s|u|^2$ in suitable orthonormal coordinates[13,14]; in classifying cubic minimal cones under a harmonicity assumption, Peng and Xiao had met the same alternative earlier, as Cui reports[15,16]. The argument from one plane in Section 4 is the beginning of Cartan's proof, and Tkachev's classification reduces to the same equations of Hurwitz type[17]; the dimensions 2, 4 and 8 are those of Hurwitz's theorem[18]. These classifications list the cubics. Theorem A says which of them split into closed subspaces, and its proof adds the sum rule and the quaternionic lower bound.

Unbiased bases. That unbiased bases of $\mathbb{C}^d$ give orthogonal subspaces of the traceless Hermitian matrices goes back to Ivanović and to Wootters and Fields[19], and the uniqueness of the four bases of $\mathbb{C}^3$ in Theorem 5.10 is quoted from Brierley, Weigert and Bengtsson[20]. For $\mathbb{H}^n$ Chterental and Đoković proved the bound $2n + 1$, derived in the proof of their Theorem 5.4 the normal form of three unbiased bases of $\mathbb{H}^3$ on which Theorem 6.4 rests, exhibited sets of four in $\mathbb{H}^3$ that admit no fifth, and asked for all maximal sets[21]. McNulty and Weigert report no construction

[13] Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1.

[14] Tkachev, "A generalization of Cartan's theorem", note 1. Thm. 1.1.

[15] H. Cui, Laplacian algebras and subcalibrations for minimal hypercones. arXiv:2609.17217, 2026. §9.1, p. 83.

[16] C.-K. Peng and L. Xiao, On the classification of cubic minimal cones (in Chinese), *J. Grad. School Academia Sinica* **10** (1993), no. 1, 1–19.

[17] Tkachev, "A generalization of Cartan's theorem", note 1. §3.

[18] Hurwitz, "Über die Composition der quadratischen Formen", note 2.

[19] I. D. Ivanović, Geometrical description of quantal state determination, *J. Phys. A* **14** (1981) 3241–3245; W. K. Wootters and B. D. Fields, Optimal state-determination by mutually unbiased measurements, *Ann. Phys.* **191** (1989) 363–381. doi:10.1016/0003-4916(89)90322-9; I. Bengtsson, Three ways to look at mutually unbiased bases, in *Foundations of Probability and Physics – 4*, AIP Conf. Proc. **889** (2007) 40–51. arXiv:quant-ph/0610216.

[20] S. Brierley, S. Weigert and I. Bengtsson, All mutually unbiased bases in dimensions two to five, *Quantum Inf. Comput.* **10** (2010) 803–820. arXiv:0907.4097. §2.2, Eqs. (6)–(11), pp. 4–5.

[21] Chterental and Đoković, "On orthostochastic, unistochastic and qustochastic matrices", note 4. Thms. 5.1, 5.4, Exs. 5.5, 5.6, Problem 6.3.

of more than four[22], and Cohn, Kumar and Minton mark the quaternionic row of their table of previously known universal optima, to which seven bases would belong, with a question mark[23]. Theorem B lowers the bound from seven to five.

Thirteen frames and the grading. Thirteen pairwise unbiased frames in $H_3(\mathbb{O})$, and a four-parameter family of them, are due to Cohn, Kumar and Minton[24]. Near their original configuration the systems form patches of dimension ten modulo F_4, by an earlier AI-generated result of the author[25], and the coordinates of that paper give the model of $H_3(\mathbb{O})$ used in Section 7[26]. The grading of the complexified Albert algebra by $\mathbb{F}_3^3$ is due to Griess and is described by Elduque and Kochetov, and its uniqueness up to equivalence (an automorphism carrying homogeneous components to homogeneous components) is known[27,28]. Theorems 9.1 and 9.4 start instead from thirteen frames, with no group given, and characterize the graded system by its 52 positive shares.

Orthogonal decompositions. The notion, for complex simple Lie algebras, and its theory are in the monograph of Kostrikin and Tiep[29], and the orthogonal decomposition into Cartan subalgebras of the complex Lie algebra of type F_4 attached to the grading is known, in the forms given by Kostrikin and Tiep and by Elduque[30,31]. Thiel and Williams split the Casimir operator along a decomposition and take its trace[32]; Theorem 10.7 keeps the operator and obtains the planes. Kostrikin, Kostrikin and Ufnarovskii conjectured that the complex Lie algebra of type C_n has an orthogonal decomposition into Cartan subalgebras exactly when n is a power of two, and Kantor records the case $n = 3$ as open[33]. Corollary 10.10 settles the part of that case stable under a compact conjugation.

[22] D. McNulty and S. Weigert, Mutually unbiased bases in composite dimensions – a review, *Quantum* **10** (2026) 2051. doi:10.22331/q-2026-04-01-2051. arXiv:2410.23997. §9.3.

[23] Cohn, Kumar and Minton, "Optimal simplices and codes", note 6. Table 2.3, p. 14.

[24] Ibid. Thm. 8.3, §8.2.

[25] Earlier AI-generated work of the author: On the local geometry of the tight 39-point code in the octonionic projective plane, Hexagon preprint hexagon:2611.00340v2, 2026. Cor. 1.4.

[26] Ibid. §4.2, Eqs. (10)–(11), proved in App. A.1.

[27] Griess, "A Moufang loop, the exceptional Jordan algebra, and a cubic form in 27 variables", note 9. (4.3)–(4.6).

[28] A. Elduque and M. Kochetov, Weyl groups of fine gradings on matrix algebras, octonions and the Albert algebra, *J. Algebra* **366** (2012) 165–186. doi:10.1016/j.jalgebra.2012.05.008. arXiv:1009.1462. Def. 1.2, §§4.1, 4.5.

[29] Kostrikin and Tiep, *Orthogonal decompositions*, note 11. pp. 1–3.

[30] Ibid. Thm. 3.1.2, pp. 85–86.

[31] A. Elduque, Jordan gradings on exceptional simple Lie algebras, *Proc. Amer. Math. Soc.* **137** (2009) 4007–4017. arXiv:0810.2737. Main Theorem, Thm. 3.5.

[32] M. Thiel and N. Williams, Strange expectations and the Winnie-the-Pooh problem, *J. Combin. Theory Ser. A* **176** (2020) 105298. arXiv:1811.02550. §6.1.

[33] W. M. Kantor, Note on Lie algebras, finite groups and finite geometries, Ohio State Univ. Math. Res. Inst. Publ. **4** (1996) 73–81; page numbers refer to the preprint at `https://pages.uoregon.edu/kantor/PAPERS/LieAlgebras.pdf`. p. 2.

The flow. The one-loop equations are those of Liendo and Rong[34]. The lower bound on the A-function, with Cartan's cubics as its equality case, is due to Osborn and Stergiou and to Jack, Osborn and Steudtner[35], and the limiting cubics are the fixed points with SU(3) and F_4 symmetry studied by Rong and Su[36]. The pencil from the uncoupled planes to these fixed points and its equation for t are the additions.

A good faith effort was made to search for prior work with AI tools on each of these subjects, and the sources found are cited above and where they are used. The search found no earlier statement of the restriction to four or thirteen planes in Theorem A, of the quaternionic lower bound or of the bound of five unbiased bases in $\mathbb{H}^3$. A search is not a certificate that none exists.

1.5 Guide to the sections

Section 3 sets up eikonal cubics, quotes the classification theorem of Cartan and Tkachev, and ends with a table of notation. Section 4 obtains 1, 4, 7, 13 and the identification with Cartan's cubics from one plane and Hurwitz's theorem; Appendix A holds the normal form of the cubic used there. Section 5 identifies planes with frames and proves the sum rule, and Section 6 the quaternionic lower bound and the two completions. Section 7 builds the graded system of thirteen planes, Section 8 lists the summands, proves Theorem A and bounds the defect, and Section 9 characterizes the sparsest systems and treats the rotated systems and the subalgebras generated by three planes. Sections 10, 11 and 12 read the planes in Lie algebras, in the energy and in the one-loop flow; nothing earlier depends on them, and the physics enters only through the definition of the flow. Section 13 discusses the results and collects open questions. Appendix B holds the second row of the magic square, the symmetry groups along the pencil and the stability of the eikonal cubic. Appendix C lists the imported results, and Appendix D describes the formalization. Every proof is by hand, and every example is given with data that can be checked by hand.

2 Sketch of the proofs

One plane. For a complete system of $m > 1$ planes, scale the cubic to eikonal constant 1. The point $z = 1$ of a plane is then a unit idempotent a of the cubic product, with $(2L_a - 1)(L_a + 1) = 0$ on $a^\perp$ by (4.1). The point $z = i$ splits the eigenspaces of L_a (the Peirce spaces of the cubic) off the plane into three spaces of dimension k,

[34] Liendo and Rong, "Seeking SUSY fixed points", note 12. Eqs. (2.1)–(2.2).

[35] H. Osborn and A. Stergiou, Seeking fixed points in multiple coupling scalar theories in the ϵ expansion, *J. High Energy Phys.* **2018**(05) (2018) 051. doi:10.1007/JHEP05(2018)051. arXiv:1707.06165v7; I. Jack, H. Osborn and T. Steudtner, Explorations in scalar fermion theories: β-functions, supersymmetry and fixed points, *J. High Energy Phys.* **2024**(02) (2024) 038. doi:10.1007/JHEP02(2024)038. arXiv:2301.10903v3.

[36] J. Rong and N. Su, Bootstrapping the $\mathcal{N} = 1$ Wess–Zumino models in three dimensions, *J. High Energy Phys.* **2021**(06) (2021) 153. doi:10.1007/JHEP06(2021)153. arXiv:1910.08578; equation numbers as in arXiv version 1. Eq. (2.10), Table III.

joined by a bilinear map that multiplies lengths (Lemma 4.3). Hurwitz's theorem on bilinear maps gives $k \in \{2, 4, 8\}$, so $m \in \{4, 7, 13\}$ (Corollary 4.5), and the isomorphism part of Theorem 4.4, applied in the normal form of Lemma A.1, identifies the cubic with Cartan's (Theorem 4.6).

Frames and the sum rule. A plane of $H_3^0(\mathbb{K})$ is the traceless span of a Jordan frame, and orthogonal planes have unbiased frames (Proposition 5.1). The product space of two orthogonal planes has dimension four (Lemma 4.10) and lies in the sum of the other planes (Lemma 3.11), so $\sum_{k \neq i,j} P_k Q_{ij} = Q_{ij}$, and half its trace is the sum rule (Theorem 5.9).

The missing seven and Theorem B. In the normal form derived from the proof of a theorem of Chterental and Đoković the third of three unbiased bases of $\mathbb{H}^3$ is $\mathrm{diag}(q_0, q_1, q_2)\mathsf{F}$ with $q_m = a_m + b_m j$. Since $jz = \bar{z}j$, unbiasedness splits into a condition on the a_m and one on the b_m, each making the products $|a_m||a_n|$, respectively $|b_m||b_n|$, the sides of a possibly degenerate triangle. Heron's inequality for the two triangles gives share at least $1 - 1/\sqrt{3}$ (Theorem 6.4), and five such shares exceed 2 (Corollary 6.6). For six bases, an orthonormal pair x, y spanning the complement of their planes has $x^2 + y^2 = \frac{2}{3}e$, and the quartic trace identity of $H_3^0(\mathbb{H})$ then gives $\mathrm{tr}(x^2y^2) = \mathrm{tr}(xyxy)$, so $[x, y] = 0$ and a common eigenbasis is a seventh basis (Proposition 6.7). Hence at most five (Corollary 6.9).

Thirteen planes. In the coordinates of Tits's construction of the Albert algebra, the grading of $H_3(\mathbb{O}) \otimes \mathbb{C}$ by $\mathbb{F}_3^3$ gives planes indexed by the points of the projective plane PG(2, 3), with share 1 on collinear triples and 0 otherwise (Proposition 7.3). In any system the 286 shares lie in $[0, 1]$ and total 52, so at least 52 are positive, with equality exactly when every share is 0 or 1. The pairs then close into quartets, the projective lines of a projective plane of order three, hence of PG(2, 3) (Theorem 9.1), and products of clock elements reach every homogeneous component of the grading, which recognizes the graded system up to automorphism (Theorem 9.4).

Cartan subalgebras. For a Cartan subalgebra $\mathfrak{t}$ the partial Casimir operator is $\kappa(1 - P_{\mathfrak{t}})$, with $P_{\mathfrak{t}}$ the projection onto its zero weight plane (Proposition 10.5; for $\mathfrak{f}_4$ by the conjugacy of maximal tori, Proposition 10.6). The full Casimir operator is scalar, so along an orthogonal decomposition into Cartan subalgebras $\sum_r P_r$ is scalar, and a trace gives $\sum_r P_r = 1$ (Theorem 10.7). The missing seven then excludes compact $\mathfrak{sp}(3)$ (Corollary 10.10).

Energy and flow. For the normalized members $d_t = D + tM$ of the pencil the eikonal equation of d gives, for unit x,

$$e_t(x) - 1 = (t - 1)\Big(1 - \sum_i |x_i|^4 + \tfrac{t}{c}\,|M(x, x)|^2\Big)$$

(Theorem 11.5), and for $t \geq 0$ the bracket bounds the squared distance to the union of the planes (Proposition 11.7). With the planes given and $t \notin \{0, 1\}$, the energy fixes the products up to a rotation of each plane, and a nontrivial rotation would force $\operatorname{Tr} L_x^3 = 0$, against $\operatorname{Tr} L_x^3 \propto W(x)$ (Theorem 11.10). The one-loop triangle of $aD + bM$ lies in the pencil (Proposition 12.3). For $\varepsilon > 0$, $a(0) \neq 0$ and $t(0) = b(0)/a(0) > 0$ the ratio moves with the sign of $1 - t$ while the norm of the coupling stays in a compact interval of $(0, \infty)$, so $t \to 1$ (Theorem 12.4).

3 Eikonal cubics and closed summands

This section fixes the objects of the paper and collects the facts about them that later proofs use. Readers who know Cartan's isoparametric cubics can read Definitions 3.1 and 3.10 and Lemma 3.11, and skip the rest on first reading.

3.1 Cubics, products and the eikonal equation

Let V be a real Euclidean space of dimension N, with inner product $\langle x, y\rangle$ and length $|x|$. A symmetric trilinear form d on V defines the cubic $W(x) = d(x, x, x)$, the *interaction*, and a commutative bilinear product by

$$\langle x \circ y, z\rangle = d(x, y, z) \qquad (x, y, z \in V). \tag{3.1}$$

This product is the *cubic product*; it carries the same information as the cubic. In the conventions of Tkachev and Fox, the metrized algebra of the same cubic and inner product is defined by $\langle xy, z\rangle = D^3W(x, y, z)$, so its product is $6\,x \circ y$[37,38]. Differentiating W gives $\nabla W(x) = 3\,x \circ x$, so the *energy* (squared norm of the gradient) of a point x of V (a field configuration) is $|\nabla W(x)|^2 = 9|x \circ x|^2$, a quartic form on V. In the supersymmetric model of Section 12, whose real superfields have the real superpotential W, the scalar potential is $\frac{1}{2}|\nabla W|^2$, so the energy is twice the scalar potential of this real-superfield model. Let L_x denote the self-adjoint operator $y \mapsto x \circ y$.

Definition 3.1. The cubic W (or the form d) is *eikonal* with *eikonal constant* $c > 0$ if

$$|x \circ x|^2 = c|x|^4 \qquad \text{for all } x \in V, \tag{3.2}$$

that is, if its energy $9c|x|^4$ depends only on the length of x.

The name comes from the equation: (3.2) is the cubic eikonal equation $|\nabla W|^2 = 9c|x|^4$, whose cubic solutions Tkachev studied under that name, spelled "eiconal" there[39]. The harmonic eikonal cubics in dimension $N \geq 3$ are Cartan's isoparametric

[37] V. G. Tkachev, Hsiang cubics as spherical Chebyshev polynomials: equioscillation and idempotent quadrature. arXiv:2610.13532v1, 2026. §2.2, Eq. (2.1).

[38] D. J. F. Fox, Commutative algebras with nondegenerate invariant trace form and trace-free multiplication endomorphisms. arXiv:2004.12343. Eq. (1.2).

[39] Tkachev, "A generalization of Cartan's theorem", note 1; V. G. Tkachev, A Jordan algebra approach to the cubic eiconal equation, *J. Algebra* **419** (2014) 34–51. doi:10.1016/j.jalgebra.2014.06.039. arXiv:1211.7223. Addendum, *J. Algebra* **449** (2016) 774, doi:10.1016/j.jalgebra.2015.12.010.

cubics: their regular level sets on the unit sphere are the isoparametric hypersurfaces of the sphere (hypersurfaces with constant principal curvatures) that have three distinct principal curvatures[40]. Scaling the inner product and d by the same factor $\lambda > 0$ leaves the product (3.1) unchanged and replaces c by c/λ. The paper uses two normalizations. In complex coordinates, where each plane carries the cubic $\operatorname{Re} z^3$, the constant is $c = 1$. In the Jordan models of Example 3.5, with the trace form as inner product, it is $c = 1/6$. Scaling by $\lambda = 1/6$ turns the second into the first: on traceless Hermitian matrices X one uses $|X|^2 = \operatorname{tr} X^2/6$ and $W(X) = \operatorname{tr} X^3/6$. Equivalently, keeping the trace form $\langle X, Y\rangle = \operatorname{tr}(X \bullet Y)$, the cubic rescaled to constant 1 is $\sqrt{6}\ \operatorname{tr} X^3$ (Theorem 4.6, Lemma A.2); the map $X \mapsto X/\sqrt{6}$ carries one presentation to the other.

Example 3.2 (One plane)**.** On $V = \mathbb{C}$ with $\langle z, w\rangle = \operatorname{Re}(z\bar{w})$ take $W(z) = \operatorname{Re} z^3$. Then $d(z, w, u) = \operatorname{Re}(zwu)$, the cubic product is $z \circ w = \bar{z}\bar{w}$ (the para-Hurwitz product of $\mathbb{C}$[41]), and $|z \circ z|^2 = |z|^4$. So $\operatorname{Re} z^3$ is eikonal with $c = 1$, although the cubic has three maxima on the unit circle.

Lemma 3.3. *An eikonal cubic on $\mathbb{R}^2$ with constant c is $\sqrt{c}\ \operatorname{Re} z^3$ in a suitable orthonormal complex coordinate z. In particular it is harmonic.*

Formalized. `binaryPlaneNormalForm`.

This is the case of two variables in Tkachev's classification[42](Theorem 3.7 below): in suitable orthonormal coordinates $W/\sqrt{c}$ is the cubic $x_2^3 - 3x_2x_1^2 = \operatorname{Re}(x_2 + ix_1)^3$ of that equation, which for two variables is harmonic. The direct proof below does not use the classification, so neither does the count in Section 4.

Proof. Every real cubic on $\mathbb{C} = \mathbb{R}^2$ is $W = \operatorname{Re}(\alpha z^3 + \beta z^2\bar{z})$ with $\alpha, \beta \in \mathbb{C}$. Its gradient, written as a complex number, is $\partial_x W + i\partial_y W = 2\,\partial W/\partial\bar{z} = \beta z^2 + 3\bar{\alpha}\bar{z}^2 + 2\bar{\beta}|z|^2$. On the unit circle $z = e^{i\theta}$ its squared length is

$$|\beta|^2 + 4|\beta|^2 + 9|\alpha|^2 + 2\operatorname{Re}\big(3\alpha\beta\, e^{4i\theta}\big) + 2\operatorname{Re}\big((2\beta^2 + 6\alpha\bar{\beta})\, e^{2i\theta}\big),$$

which is constant only if $\alpha\beta = 0$ and $2\beta^2 + 6\alpha\bar{\beta} = 0$. If $\beta \neq 0$ the first condition gives $\alpha = 0$ and the second fails, so $\beta = 0$. Then $W = \operatorname{Re}(\alpha z^3)$, and rotating z makes $\alpha = |\alpha| > 0$. Comparing with Example 3.2 gives $|\alpha|^2 = c$. □

Example 3.4 (The four-line cubic)**.** On $\mathbb{R}^4$ put

$$Q(x) = x_1^3 + x_2^3 + x_3^3 + x_4^3 - 3\sum_{i<j<k} x_ix_jx_k. \tag{3.3}$$

With $\sigma_1 = \sum_i x_i$ and $\sigma_2 = \sum_{i<j} x_ix_j$, Newton's identity gives $\sum_i x_i^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sum_{i<j<k} x_ix_jx_k$, so $Q = \sigma_1^3 - 3\sigma_1\sigma_2$. Write $x = s\,a + u$ with $a = \frac{1}{2}(1,1,1,1)$, $s = \langle x, a\rangle = \sigma_1/2$ and $u \perp a$. Since $2\sigma_2 = \sigma_1^2 - |x|^2$ and $|x|^2 = s^2 + |u|^2$,

$$Q = 3s|u|^2 - s^3. \tag{3.4}$$

[40]Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1.

[41]D. J. F. Fox, Harmonic cubic homogeneous polynomials such that the norm-squared of the Hessian is a multiple of the Euclidean quadratic form, *Anal. Math. Phys.* **11** (2021) 43; correction **11** (2021) 172. doi:10.1007/s13324-020-00462-4. arXiv:1905.00071. Ex. 1.13.

[42]Tkachev, "A generalization of Cartan's theorem", note 1. Thm. 1.1 and Eq. (3).

Hence $\partial Q/\partial s = 3|u|^2 - 3s^2$ and $\nabla_u Q = 6su$, so $|\nabla Q|^2 = 9(s^2-|u|^2)^2 + 36s^2|u|^2 = 9|x|^4$, and Q is eikonal with $c = 1$. It is not harmonic: $\Delta Q = 6\sigma_1$. Each coordinate axis is closed under the cubic product, since $\nabla Q(te_i) = 3t^2 e_i$, and Q restricts to t^3 on it.

Form (3.4) is, after the reflection $x \mapsto -x$, the reducible cubic $s^3 - 3s|u|^2$ that appears in Tkachev's classification theorem below; following Agrachev, Kozhasov and Uschmajew, Tkachev calls the m-variable form $s^3 - 3s|y|^2$ a cubic Chebyshev form[43]. Below it is called the reducible cubic, and the four-line cubic for $N = 4$; for two variables it is the harmonic cubic $\operatorname{Re} z^3$ of Lemma 3.3.

Example 3.5 (Cartan's cubics)**.** Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, of real dimension $k = 1, 2, 4, 8$, and let $H_3(\mathbb{K})$ be the Hermitian 3×3 matrices over $\mathbb{K}$ with the Jordan product $X \bullet Y = \frac{1}{2}(XY + YX)$ and the trace form $\langle X, Y\rangle = \operatorname{tr}(X \bullet Y)$, which is positive definite. Three facts are used throughout:

(J1) the trace form is associative (invariant): $\operatorname{tr}((X \bullet Y) \bullet Z) = \operatorname{tr}(X \bullet (Y \bullet Z))$;

(J2) every X is $\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3$ for a *Jordan frame* p_1, p_2, p_3 (three orthogonal idempotents of the Jordan product of trace one, that is, primitive idempotents, with sum e, the identity; in the Jordan-algebraic interpretation, a rank-one projective measurement with three outcomes), and then $\operatorname{tr} X^n = \sum_i \lambda_i^n$;

(J3) the automorphism group of $H_3(\mathbb{K})$ acts transitively on Jordan frames.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ they are linear algebra. For $\mathbb{K} = \mathbb{H}$, (J1) is the cyclicity $\operatorname{Re}\operatorname{tr}(XY) = \operatorname{Re}\operatorname{tr}(YX)$ of the real part of the matrix trace, and (J2) and (J3) follow from the quaternionic spectral theorem, in the form of Lemma 6.1, and conjugation by unitary matrices. For $\mathbb{K} = \mathbb{O}$ they are quoted from Yokota. (J1) and positivity of the trace form are a lemma of Yokota[44]. Every automorphism preserves the trace[45], and every $X \in H_3(\mathbb{O})$ is carried to a real diagonal matrix by an automorphism[46]; this gives (J2). For (J3), diagonalize $p_1 + 2p_2$ in this way: the Jordan idempotents p_a are polynomials in it, so they become diagonal idempotents of trace one, that is the E_{aa} in some order, and permutation matrices act by automorphisms. Baez sketches the same diagonalization[47]. The automorphisms of $H_3(\mathbb{O})$ that fix the diagonal frame (Lemma 5.3) are also quoted from Yokota. Write $L_X^\bullet$ for the operator $Y \mapsto X \bullet Y$. Below, frame means Jordan frame.

On the traceless part $H_3^0(\mathbb{K})$, of dimension $N = 3k + 2 = 5, 8, 14, 26$, put $d(X, Y, Z) = \operatorname{tr}((X \bullet Y) \bullet Z)$, which is symmetric by (J1), and $W(X) = \operatorname{tr} X^3$. The cubic product (3.1) of this form is the traceless part of the Jordan product (Fox's deunitalization[48]),

$$X \circ Y = X \bullet Y - \tfrac{1}{3}\langle X, Y\rangle\, e.$$

Throughout the paper $X \bullet Y$ denotes the Jordan product, and for matrix arguments $X, Y \in H_3^0(\mathbb{K})$ the symbol $X \circ Y$ denotes this traceless part of the Jordan product, the cubic product of $\operatorname{tr} X^3$ for the trace form. The cubic $\sqrt{6}\ \operatorname{tr} X^3$, rescaled to constant 1

[43] Tkachev, "Hsiang cubics as spherical Chebyshev polynomials", note 37. Rem. 5.4.

[44] I. Yokota, Exceptional Lie groups, 2009. arXiv:0902.0431. Lemma 2.1.1(2).

[45] Ibid. Lemma 2.2.1(2).

[46] Ibid. Prop. 2.8.2.

[47] Baez, "The octonions", note 10. §3.4.

[48] Fox, "Harmonic cubic homogeneous polynomials", note 41. §8.

and used in Section 4 and Lemma A.2, has cubic product $\sqrt{6}\,X \circ Y$. For traceless X with eigenvalues λ_i as in (J2), put $q = \operatorname{tr} X^2$. Since $\sum_i \lambda_i = 0$, the second elementary symmetric function is $\sigma_2 = -q/2$ and $\sum_{i<j} \lambda_i^2 \lambda_j^2 = \sigma_2^2$, so $\sum_i \lambda_i^4 = q^2 - 2\sigma_2^2 = q^2/2$. Hence

$$|X \circ X|^2 = \operatorname{tr} X^4 - \tfrac{1}{3}(\operatorname{tr} X^2)^2 = \tfrac{1}{6}|X|^4,$$

so Cartan's cubic $\operatorname{tr} X^3$ is eikonal with $c = 1/6$ for the trace form; for $\mathbb{K} = \mathbb{O}$ this rests on the quoted fact (J2). It is harmonic. For an orthonormal basis (E_a) the sum $\sum_a E_a \bullet E_a$ does not depend on the basis; in the basis made of the two normalized traceless diagonal matrices and, for each off-diagonal position and each unit ϵ of a real orthonormal basis of $\mathbb{K}$, the matrix $(\epsilon E_{ij} + \bar{\epsilon} E_{ji})/\sqrt{2}$, it equals $\frac{2}{3}e + ke = \frac{N}{3}e$, whose traceless part vanishes.

The cubic of $H_3^0(\mathbb{C})$ has a form in four complex coordinates in which the four planes of Section 5 are the coordinate planes.

Example 3.6 (Four complex fields)**.** Let $\omega = e^{2\pi i/3}$ and, for $z = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4$,

$$W_4(z) = \operatorname{Re}\Big[z_0^3 + z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 3z_0\big(\bar{z}_1z_2 + \bar{z}_2z_3 + \bar{z}_3z_1\big)\Big]. \tag{3.5}$$

Define a traceless Hermitian matrix $H(z)$, indices modulo 3, by the diagonal entries $h_j = 2\operatorname{Re}(\omega^{-j}z_0)$ and the off-diagonal entries $H_{j+1,j+2} = u_j = \sum_{k=0}^{2} \omega^{jk} z_{k+1}$ (the entry opposite the vertex j), with $H_{j+2,j+1} = \bar{u}_j$. Then

$$\operatorname{tr} H^2 = 6|z|^2, \qquad \det H = 2W_4(z), \qquad \text{so} \qquad W_4 = \tfrac{1}{6}\operatorname{tr} H^3. \tag{3.6}$$

The first identity is Parseval's identity for the discrete Fourier transform, $\sum_j |u_j|^2 = 3\sum_{k\geq 1}|z_k|^2$, together with $\sum_j h_j^2 = 6|z_0|^2$. For the second, expand $\det H = h_0h_1h_2 + 2\operatorname{Re}(u_0u_1u_2) - \sum_j h_j|u_j|^2$: the factorizations $\prod_j(a + \omega^j b) = a^3 + b^3$ and $\prod_j(x + \omega^j y + \omega^{2j}w) = x^3 + y^3 + w^3 - 3xyw$ give the first two terms, and orthogonality of characters gives $\sum_j h_j|u_j|^2 = 6\operatorname{Re}\big(z_0(\bar{z}_1z_2 + \bar{z}_2z_3 + \bar{z}_3z_1)\big)$. For a traceless matrix Newton's identity gives $\operatorname{tr} H^3 = 3\det H$. The map $z \mapsto H(z)$ is a real linear bijection $\mathbb{C}^4 \to H_3^0(\mathbb{C})$ that multiplies lengths by $\sqrt{6}$ for the trace form, so W_4 is Cartan's cubic of $H_3^0(\mathbb{C})$ in the normalization $c = 1$: $|\nabla W_4|^2 = 9|z|^4$.

W_4 restricts to $\operatorname{Re} z_p^3$ on each coordinate plane, and its other terms each contain three different fields. On real z it is the four-line cubic (3.3).

Theorem 3.7 (Cartan; Tkachev; quoted)**.** *Let W be an eikonal cubic on a Euclidean space V of dimension $N \geq 1$, with constant c.*

(i) If W is harmonic, then either $N = 2$ and W is as in Lemma 3.3, or $N = 5, 8, 14, 26$ and there is an isometry $\phi\colon V \to H_3^0(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, with $W(x) = \sqrt{6c}\ \operatorname{tr}\big(\phi(x)^3\big)$.

(ii) If W is not harmonic, then $N \neq 2$ and, for a unit vector a, $W(x) = \sqrt{c}\,(s^3 - 3s|u|^2)$ with $s = \langle x, a\rangle$ and $u = x - sa$.

Formalized, different route. (i) $N = 2$: `binaryPlaneNormalForm`; $N \geq 3$: `harmonicRoundCubic_identification`, proved at a unit idempotent of the cubic product as in Theorem 4.6; (ii) `tkachevNonharmonic`, by a trace identity at a unit idempotent of the cubic product instead of Tkachev's argument.

Cartan classified the harmonic solutions, which are his isoparametric cubics, and wrote them with triples of real numbers, complex numbers, quaternions and octonions[49], with the completeness of the list in Part II, §§VII–IX, pp. 359–367. Fox writes them, up to scale, as $\operatorname{tr} X^3$ on traceless Hermitian matrices, the octonionic case included[50]. Tkachev proved that every cubic solution of $|\nabla W|^2 = 9|x|^4$ is either one of Cartan's cubics or the reducible cubic $x_N^3 - 3x_N(x_1^2 + \cdots + x_{N-1}^2)$, whose Laplacian is $6(2-N)x_N$[51]. Peng and Xiao had earlier classified cubic minimal cones under a harmonicity assumption, and Cui reports that their Theorem 2 gives the same alternative, reducible or Cartan, for the cubic with this gradient equation that occurs in the normal form[52,53]. Their paper could not be consulted here. The factor $\sqrt{6c}$ in (i) matches the constants, since $\operatorname{tr} X^3$ has $c = 1/6$ for the trace form, and multiplying a cubic by κ multiplies c by κ^2. Tkachev also showed that cubic solutions correspond to cubic Jordan algebras[54].

This theorem is called the classification theorem below. Its harmonic half is proved again: for $N = 2$ by Lemma 3.3, and for $N \geq 3$ at a unit idempotent of the cubic product and along Cartan's lines, in Section 4 (Theorem 4.6).

Lemma 3.8. *Let W be eikonal with constant c. For all $x, y, z, w \in V$,*

Formalized. `polarization_and_contractions`.

$$\langle x\circ y, z\circ w\rangle + \langle x\circ z, y\circ w\rangle + \langle x\circ w, y\circ z\rangle = c\big(\langle x,y\rangle\langle z,w\rangle + \langle x,z\rangle\langle y,w\rangle + \langle x,w\rangle\langle y,z\rangle\big). \tag{3.7}$$

In particular $\langle x\circ x, z\circ z\rangle + 2|x\circ z|^2 = c(|x|^2|z|^2 + 2\langle x,z\rangle^2)$. If moreover W is harmonic and (e_a) is an orthonormal basis, then

$$\sum_a L_{e_a}^2 = \tfrac{c(N+2)}{2}\, I, \qquad \operatorname{Tr} L_x^3 = -\tfrac{c(N-2)}{4}\, W(x). \tag{3.8}$$

Proof. Both sides of (3.7) are symmetric four-linear forms, with values $3|x\circ x|^2$ and $3c|x|^4$ on (x,x,x,x); a symmetric multilinear form is determined by its diagonal. Setting $y = x$, $w = z$ gives the second identity. Write $\tau(V) = \sum_a e_a \circ e_a$ for the *trace vector*, so that $\operatorname{Tr} L_x = \langle x, \tau(V)\rangle$; since $\Delta W(x) = 6\, d(e_a, e_a, x)$ summed over a, W is harmonic exactly when $\tau(V) = 0$. Put $x = y = e_a$ in (3.7) and sum over a: the first

[49] Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1. Part I, §I, nos. 2–6, pp. 337–341, Eqs. (13)–(17).

[50] Fox, "Harmonic cubic homogeneous polynomials", note 41. §8, p. 36, Eqs. (8.4)–(8.6).

[51] Tkachev, "A generalization of Cartan's theorem", note 1. Thm. 1.1.

[52] Cui, "Laplacian algebras and subcalibrations for minimal hypercones", note 15. §9.1, p. 83.

[53] Peng and Xiao, "On the classification of cubic minimal cones", note 16.

[54] Tkachev, "A Jordan algebra approach to the cubic eiconal equation", note 39. Thm. 1.

term gives $\langle \tau(V), z \circ w\rangle = 0$, the other two give $2\sum_a \langle e_a \circ z, e_a \circ w\rangle = 2\langle \sum_a L_{e_a}^2 z, w\rangle$, and the right side is $c(N+2)\langle z, w\rangle$. For the second formula write $d_{ijk} = d(e_i, e_j, e_k)$, so that (3.7) reads $d_{ijm}d_{klm} + d_{ikm}d_{jlm} + d_{ilm}d_{jkm} = c(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with summation over m. Multiply by d_{jkp} and sum over j, k. The first two terms each give the triangle $\mathrm{T}(d)_{ilp} = \sum d_{iam}d_{pab}d_{lbm}$; by the first formula the third gives $\frac{c(N+2)}{2} d_{ilp}$; harmonicity reduces the right side to $2c\, d_{ilp}$. Hence $\mathrm{T}(d) = \frac{c(2-N)}{4} d$, and $\operatorname{Tr} L_x^3 = \mathrm{T}(d)(x, x, x)$. □

The second formula in (3.8) is the contraction identity of the one-loop computations in Section 12.

Remark 3.9. For $n \geq 1$, a cubic $W = \operatorname{Re} f$ with f a holomorphic cubic polynomial on $\mathbb{C}^n$ is eikonal, for the Euclidean metric of $\mathbb{R}^{2n}$, only if $n = 1$. Here $|\nabla W|^2 = \sum_k |\partial f/\partial z_k|^2 = 9|B(z, z)|^2$ for a symmetric complex bilinear map $B\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$. Suppose $|B(z, z)|^2 = \kappa |z|^4$ with $\kappa > 0$. Put $z = u + \lambda v$ with $\lambda \in \mathbb{C}$ and compare the coefficients of $\lambda\bar{\lambda}$; holomorphy keeps $B(u, u)\overline{B(v, v)}$ out of this coefficient, and with $\eta(u, v) = \sum_k u_k \bar{v}_k$ one gets

$$|B(u, v)|^2 = \tfrac{\kappa}{2}\big(|u|^2|v|^2 + |\eta(u, v)|^2\big) = \kappa\, |u \odot v|^2,$$

where $u \odot v = \frac{1}{2}(u \otimes v + v \otimes u)$ in $\operatorname{Sym}^2 \mathbb{C}^n$ with its induced Hermitian product. So the Hermitian form $F(\xi) = |\beta\xi|^2 - \kappa|\xi|^2$ on $\operatorname{Sym}^2 \mathbb{C}^n$, where β is the linear map induced by B, vanishes on every $u \odot v$. Replacing u by $a + tb$ and then v by $v + sw$, with $s, t \in \mathbb{C}$, shows that its polarization vanishes on all pairs $a \odot v$, $b \odot w$. These span $\operatorname{Sym}^2 \mathbb{C}^n$, so $\beta^*\beta = \kappa I$ and β is injective: $n(n+1)/2 \leq n$, hence $n = 1$.

3.2 Closed summands and the mixed product law

Definition 3.10. A subspace $U \subset V$ is *closed* if $x \circ x \in U$ for every $x \in U$; by polarization this is $U \circ U \subset U$, that is, U is a subalgebra of the commutative algebra $(V, \circ)$. An *orthogonal decomposition into closed subspaces* is an orthogonal direct sum $V = U_1 \oplus \cdots \oplus U_m$ of nonzero closed subspaces; a *summand* is a member of one, and the decomposition is *proper* if $m \geq 2$. A *plane* is a closed two-dimensional subspace.

For an eikonal cubic and distinct summands U_i, U_j, Lemma 3.11 gives $x \circ y \neq 0$ whenever $0 \neq x \in U_i$ and $0 \neq y \in U_j$. So in a proper decomposition no summand is an ideal.

In field theory a closed subspace is a consistent truncation: since $\nabla W = 3\, x \circ x$, a field configuration in U has its gradient in U, so the gradient flow of W keeps it in U. The gradient $36\, x \circ (x \circ x)$ of the energy also lies in U, so the classical equations of motion keep a configuration in U when it starts in U with its time derivative in U. By Proposition 3.12 and Lemma 3.3, the cubic on a plane is $\sqrt{c}\ \operatorname{Re} z^3$ in a suitable complex coordinate, that is $\operatorname{Re} z^3$ for $c = 1$.

For an orthogonal splitting of V, closedness says which terms of the cubic are allowed. Let $V = U_1 \oplus \cdots \oplus U_m$ be orthogonal and write x_i for the component of x in U_i. Expanding $d(x, x, x)$ by components, the terms with two entries in one U_i

and the third in another U_j are $3\,d(x_i, x_i, x_j) = 3\langle x_i \circ x_i, x_j\rangle$. So every U_i is closed exactly when W has no such terms, that is when

$$W(x) = \sum_i W(x_i) + 6 \sum_{i<j<k} d(x_i, x_j, x_k). \tag{3.9}$$

The summands are then coupled only through three-field terms, the terms containing three different fields. When the U_i are coordinate planes with complex coordinates z_i, the three-field term of U_i, U_j, U_k in Theorem A(i) and Lemma 8.10 is $T_{ijk}(z_i, z_j, z_k) = 6\,d(x_i, x_j, x_k)$. The four coordinate axes of Example 3.4 (one-dimensional summands) and the four coordinate planes of Example 3.6 are summands in this sense.

Lemma 3.11 (Mixed product law)**.** *Let W be eikonal with constant c and let U, U' be orthogonal closed subspaces, for instance two different summands. For $x \in U$ and $y \in U'$,*

Formalized. `mixedProduct Law`.

$$x \circ y \perp U \oplus U', \qquad 2|x \circ y|^2 = c|x|^2|y|^2.$$

Consequently, if U and U' are both nonzero, $N \geq \dim U + \dim U' + \max(\dim U, \dim U')$.

Proof. For $u \in U$, $\langle x \circ y, u\rangle = \langle x \circ u, y\rangle = 0$ because $x \circ u \in U$; likewise $x \circ y \perp U'$. In the second identity of Lemma 3.8 with $z = y$, the term $\langle x \circ x, y \circ y\rangle$ vanishes because $x \circ x \in U$ and $y \circ y \in U'$, and $\langle x, y\rangle = 0$. For a fixed nonzero x the map $y \mapsto x \circ y$ is therefore injective from U' into $(U \oplus U')^\perp$, so $N - \dim U - \dim U' \geq \dim U'$; exchanging the roles gives the inequality. □

So the product of two summands is never zero, and it lands outside both. This separates eikonal cubics from the orthogonally decomposable cubics studied by Fox, which are sums of cubics on orthogonal subspaces with no cross terms at all[55], so that the pieces are ideals with zero products between them. Fox's "orthogonally decomposable" therefore describes the opposite situation to a proper orthogonal decomposition into closed subspaces of an eikonal cubic here. Fox describes an eight-dimensional algebra built from the affine plane of order three that is the orthogonal sum of four two-dimensional algebras of the type of Example 3.2, with zero products between them[56]; by Lemma 3.11 its cubic is not eikonal, and it differs from the four planes of W_4, whose mutual products are nonzero.

At a nonzero idempotent a of the metrized algebra of a cubic minimal cone (a Hsiang algebra, whose product is six times the cubic product of the same cubic), Tkachev obtains a multiple of $\cos 3\phi$ on the unit circle of each plane spanned by a and a nonzero vector of its Peirce space for -1, and a scaled cubic Chebyshev form on the sum of that Peirce space and $\mathbb{R}a$[57]. For orthogonal sums of ideals in Killing-metrized exact commutative algebras, see Fox[58]. The question here is which proper orthogonal decompositions into closed subspaces an eikonal cubic has.

[55] Fox, "Harmonic cubic homogeneous polynomials", note 41. §1, p. 3, Eq. (1.4).

[56] D. J. F. Fox, Killing metrized commutative nonassociative algebras associated with Steiner triple systems, *J. Algebra* **608** (2022) 186–213. arXiv:2205.08838. Ex. 5.6.

[57] Tkachev, "Hsiang cubics as spherical Chebyshev polynomials", note 37. Thm. 5.1(iii), Rem. 5.4.

[58] Fox, "Commutative algebras with nondegenerate invariant trace form", note 38. Lemma 2.5.

Proposition 3.12. *Let W be eikonal with constant c.*

Formalized. `closed Pieces_round`.

(i) On a closed subspace U the restriction of d has the restricted product, and is eikonal with constant c.

(ii) Let $V = U_1 \oplus \cdots \oplus U_m$ be an orthogonal decomposition into closed subspaces. Then W is harmonic if and only if every restriction $W|_{U_i}$ is harmonic.

(iii) If W is harmonic, no summand of such a decomposition is one-dimensional.

Proof. (i) For $x, y \in U$ the vector $x \circ y$ lies in U, so it is also the product of the restricted form, and (3.2) holds on U. (ii) With the trace vector $\tau(\cdot)$ of the proof of Lemma 3.8, computed in an orthonormal basis adapted to the decomposition, $\tau(V) = \sum_i \tau(U_i)$ with $\tau(U_i) \in U_i$ by closedness. These vectors are orthogonal, so $\tau(V) = 0$ if and only if every $\tau(U_i) = 0$. (iii) By (i) and (ii) a one-dimensional summand carries a harmonic eikonal cubic; but an eikonal cubic on a one-dimensional space is $\pm\sqrt{c}\, t^3$, with nonzero Laplacian. □

Parts (i) and (ii) say more once the harmonic eikonal cubics are known. If W is harmonic, every summand carries a harmonic eikonal cubic, so by the classification theorem, or by Theorem 4.6 applied to the summand, it has dimension 2, 5, 8, 14 or 26 and is a plane or is isometric to Cartan's cubic on some $H_3^0(\mathbb{K}')$, scaled to the constant c. All these dimensions, and N itself, are 2 modulo 3, so the number of summands is 1 modulo 3.

Inside $H_3^0(\mathbb{K})$ the identification is with a subalgebra. Call a subalgebra of $H_3(\mathbb{K})$ that contains e and is isomorphic to $H_3(\mathbb{K}')$ a *copy of* $H_3(\mathbb{K}')$, and call a subspace $U \subset H_3^0(\mathbb{K})$ a *copy of* $H_3^0(\mathbb{K}')$ if $\mathbb{R}e \oplus U$ is a copy of $H_3(\mathbb{K}')$. An octet, as defined in the introduction, is a closed subspace on which the cubic is, after a linear isometry, a positive multiple of Cartan's cubic on $H_3^0(\mathbb{C})$; the name is the one particle physics gives to the eight-dimensional adjoint representation of $\mathfrak{su}(3)$, which acts on $H_3^0(\mathbb{C}) = i\,\mathfrak{su}(3)$. In $H_3^0(\mathbb{K})$ the octets are the copies of $H_3^0(\mathbb{C})$. A closed $U \subset H_3^0(\mathbb{K})$ whose cubic is carried by an isometry $\psi\colon U \to H_3^0(\mathbb{K}')$ to $\operatorname{tr}\bigl(\psi(x)^3\bigr)$ is such a copy: ψ carries the cubic product $\circ$ to $\circ$, and since $x \bullet y = \frac{1}{3}\langle x, y\rangle e + x \circ y$ in both algebras, ψ extended by $e \mapsto e$ is an isomorphism $\mathbb{R}e \oplus U \cong H_3(\mathbb{K}')$.

Both completeness of the decomposition and harmonicity of W are needed here. In $H_3^0(\mathbb{K})$ let p be a primitive idempotent of the Jordan product and $u_0 = (3p - e)/\sqrt{6}$. Any subspace L of traceless matrices in the corner of $e - p$ (the Jordan Peirce space of p for the eigenvalue 0) gives a closed subspace $\mathbb{R}u_0 \oplus L$: for $f \in L$ one has $f \circ f = -|f|^2 u_0/\sqrt{6}$, $u_0 \circ u_0 = u_0/\sqrt{6}$ and $u_0 \circ f = -f/\sqrt{6}$, and the restricted cubic is $(s^3 - 3s|f|^2)/\sqrt{6}$, the reducible cubic of the classification theorem. Its dimension $1 + \dim L$ takes every value up to $k + 2$. The traceless parts of $H_3(\mathbb{R})$ and $H_3(\mathbb{H})$ inside $H_3^0(\mathbb{O})$ are closed, harmonic and of dimensions 5 and 14; Section 8 shows that neither can be a summand.

3.3 Notation

Table 3 lists the symbols that occur in more than one section, with the place where each is defined. Two products occur: the cubic product $x \circ y$ of a cubic on V, and the Jordan product $X \bullet Y$ of $H_3(\mathbb{K})$. On $H_3^0(\mathbb{K})$ with the cubic $\operatorname{tr} X^3$ the first is the

traceless part of the second, and the cubic $\sqrt{6}\ \mathrm{tr}\, X^3$ has cubic product $\sqrt{6}\, X \circ Y$ (Example 3.5). Baez and Yokota write the Jordan product as $X \circ Y$[59,60]; here $\circ$ is never the Jordan product. In the conventions of Tkachev and Fox the metrized algebra of a cubic has product $6\, x \circ y$. An idempotent is always qualified by its product: a unit idempotent is an idempotent of norm one for the cubic product, and a Jordan frame consists of primitive idempotents of the Jordan product.

[59] Baez, "The octonions", note 10. §3.

[60] Yokota, "Exceptional Lie groups", note 44. §2.1.

Table 3: Notation used in more than one section.

symbol	meaning	defined in
V, N, $\langle x, y\rangle$, $\lvert x\rvert$	Euclidean space, its dimension, inner product and length	§3
d, $W(x) = d(x, x, x)$	symmetric trilinear form and its cubic	§3
$x \circ y$, L_x	cubic product, $\langle x \circ y, z\rangle = d(x, y, z)$ (six times it is the metrized-algebra product of Tkachev and Fox); $L_x y = x \circ y$	(3.1)
$\lvert\nabla W(x)\rvert^2 = 9\lvert x \circ x\rvert^2$	energy, twice the scalar potential of the real-superfield model	§3
c	eikonal constant; $c = 1$ for the normalized cubic with plane restrictions $\operatorname{Re} z_i^3$; on $H_3^0(\mathbb{K})$ with the trace form, $c = 1$ for $\sqrt{6}\ \operatorname{tr} X^3$ and $c = 1/6$ for $\operatorname{tr} X^3$	Definition 3.1
$\tau(V)$	trace vector $\sum_a e_a \circ e_a$, zero exactly for harmonic cubics	proof of Lemma 3.8
closed, summand, plane	subalgebra of $(V, \circ)$; member of an orthogonal decomposition into closed subspaces; closed subspace of dimension two	Definition 3.10
T_{ijk}	three-field term of three coordinate planes, $T_{ijk}(z_i, z_j, z_k) = 6\, d(x_i, x_j, x_k)$, written $-3 \operatorname{Re}(a_0 z_i z_j z_k + \dots)$ in Lemma 5.8(ii)	(3.9), Lemma 8.10
$\mathbb{K}$, k	$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$; $k = \dim_{\mathbb{R}} \mathbb{K}$	Example 3.5
$H_3(\mathbb{K})$, $H_3(\mathbb{O})$, J	Hermitian 3×3 matrices over $\mathbb{K}$; the Albert algebra $H_3(\mathbb{O})$; the algebra at hand in §§5–10	Example 3.5, §5
$X \bullet Y$, $L_X^\bullet$	Jordan product $\frac{1}{2}(XY + YX)$; $L_X^\bullet Y = X \bullet Y$	Example 3.5
E_{ij}; $F_i(a)$	matrix units; Yokota's off-diagonal elements of $H_3(\mathbb{O})$	Example 3.5; Lemma 5.3
e, tr, $\langle X, Y\rangle = \operatorname{tr}(X \bullet Y)$	unit, trace and trace form of $H_3(\mathbb{K})$	Example 3.5
$H_3^0(\mathbb{K})$, $X \circ Y$	traceless part; $X \circ Y = X \bullet Y - \frac{1}{3}\langle X, Y\rangle e$	Example 3.5
$\{p_1, p_2, p_3\}$, $q_a = p_a - \frac{1}{3}e$	frame (Jordan frame: three primitive idempotents of the Jordan product with sum e); its centered frame elements	Example 3.5, §5
U_F	plane of a frame F, the traceless part of its span	§5
$\mathbb{C}_u$, ω_u; F, p_k^{F}	$\mathbb{R} + \mathbb{R}u$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}u$ for a unit imaginary u; Fourier matrix and Fourier frame	Lemma 5.4, §6.2
$q_m = a_m + b_m j$	unit quaternions of the normal form of three unbiased bases	Lemma 6.2
copy of $H_3(\mathbb{K}')$, copy of $H_3^0(\mathbb{K}')$, octet	subalgebra containing e isomorphic to $H_3(\mathbb{K}')$; its traceless part; closed subspace carrying a positive multiple of Cartan's cubic on $H_3^0(\mathbb{C})$, in $H_3^0(\mathbb{K})$ a copy of $H_3^0(\mathbb{C})$	§1, §3
a; V_{-1}, $V_{1/2}$; B_v	unit idempotent of the cubic product ($c = 1$); its Peirce spaces of the cubic (eigenspaces of L_a) for -1 and $\frac{1}{2}$; $\frac{2}{\sqrt{3}}L_v$ on $V_{1/2}$	Lemma 4.1, §4
S_1, S_2, S_3; ϕ	joint eigenspaces; orthogonal multiplication	Lemma 4.3
a', b'; ι_{ij}	standard unit idempotent of the cubic product of $\sqrt{6}\ \operatorname{tr} X^3$, and unit vector, of $H_3^0(\mathbb{K})$; isometry of $\mathbb{K}$ onto position (i, j)	Theorem 4.6
Φ	linear isometry onto $H_3^0(\mathbb{K})$ with $\Phi(x \circ y) = \sqrt{6}\,\Phi x \circ \Phi y$ ($c = 1$)	Lemma 4.8

Table 3, continued.

symbol	meaning	defined in
$U_1, \dots, U_m$; P_i	complete system of planes; orthogonal projection onto U_i	§5
$U_i \circ U_j$, $\mathcal{M}$, Q_{ij}	product space, the span of all $x \circ y$ with $x \in U_i$, $y \in U_j$; product map $x \otimes y \mapsto x \circ y$; orthogonal projection onto $U_i \circ U_j$	Lemma 4.10, §5
s_{ijk}, Tr	share of U_k on the pair U_i, U_j; trace of operators on V	Definition 5.7
ω, μ_3	$e^{2\pi i/3}$; $\{1, \omega, \omega^2\}$	Example 3.6, §7
thirteen-plane system, quartet	thirteen pairwise unbiased frames of J with their planes; four of its planes whose sum is closed	§7
u_f, $\overline{u}_f$	clock element $\sum_a \omega^a p_{fa}$ of an ordered frame f; conjugate clock element	§7.1, §9.1
X, Z, D_v	shift and clock matrices; displacement operators (generalized Pauli matrices)	§7.1
E; graded system	ternary grading group; its thirteen frames of Proposition 7.3(iii) with their planes	§7.2
U_p, J_ℓ	planes of the graded system, p a point of the projective plane PG(2, 3); the copy of $H_3(\mathbb{C})$ of a projective line ℓ	Proposition 7.3
D, M; d_t, $x \circ_t y$	diagonal and mixed parts; the pencil, the span of D and M, with its normalized member $d_t = D + tM$ (ratio t) and the cubic product of d_t	§§11.1–11.2
f_t, e_t	normalized energy of d_t, on V and on the unit sphere	§11.2
$C_c(V)$	normalized pencil class	§11.4
$h(x, y)$, $\mathrm{K}(h)$, $\mathrm{T}(h)$	vector $\langle h(x, y), z\rangle = h(x, y, z)$; contractions $h_{iab}h_{jab}$ and $h_{iab}h_{jac}h_{kbc}$	proof of Lemma 3.8, §11.1
$\mathfrak{sp}(3)$; $\mathfrak{sp}_6(\mathbb{C})$	compact symplectic Lie algebra of type C_3, dimension 21 (index: the rank); the complex one (index: the matrix size)	§10
$\mathcal{W}$, $\mathcal{V}$, g	superpotential, scalar potential, coupling constant	§12.1

4 One plane gives the count 1, 4, 7, 13

An eikonal cubic split into m orthogonal planes has $m = 1$, 4 or 13, and for $m = 4$ and $m = 13$ it is Cartan's cubic of $H_3^0(\mathbb{C})$ or of $H_3^0(\mathbb{O})$. The classification theorem (Theorem 3.7) gives $m \in \{1, 4, 7, 13\}$ at once. This section obtains the same list, and the identification with Cartan's cubics, from one plane and Hurwitz's theorem, which shows where the numbers come from, and turns planes into frames. Only the exclusion of seven needs the sum rule. Readers who accept the classification theorem need only Theorem 4.9, Lemma 4.8 and Lemma 4.10 on first reading.

The argument from one plane is the beginning of Cartan's proof[61]. Tkachev starts with the same steps for cubic solutions of the eikonal equation, harmonic or not[62], and the language of Peirce spaces is that of Tkachev and of Nadirashvili,

[61] Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1. Part I, §I, nos. 2–6, pp. 337–341.

[62] Tkachev, "A generalization of Cartan's theorem", note 1. §3, Eqs. (11)–(16).

Tkachev and Vlăduţ[63,64]. The normal form of the cubic used for the identification is in Appendix A.

Throughout, W is eikonal with constant $c = 1$; dividing W by $\sqrt{c}$ achieves this without changing closed subspaces. By Lemma 3.3 and Example 3.2, each plane has an orthonormal complex coordinate z in which $W = \operatorname{Re} z^3$ and $z \circ w = \bar{z}\bar{w}$. Setting $y = x$ in (3.7) gives, for every $x \in V$,

$$L_{x\circ x} + 2L_x^2 = |x|^2 I + 2\, x\langle x, \cdot\rangle, \tag{4.1}$$

where $x\langle x, \cdot\rangle$ is the operator $z \mapsto \langle x, z\rangle\, x$.

4.1 Peirce spaces and joint eigenspaces

A *unit idempotent* is a vector a with $|a| = 1$ and $a \circ a = a$ for the cubic product. In a plane the vectors with coordinates $z = 1, \omega, \omega^2$, $\omega = e^{2\pi i/3}$, are unit idempotents, since $\bar{z}^2 = z$ there; their sum is zero.

Lemma 4.1 (Peirce decomposition)**.** *Let a be a unit idempotent.*

(i) $L_a a = a$, *and on $a^\perp$ the operator L_a has no eigenvalues other than -1 and $\frac{1}{2}$. So $V = \mathbb{R}a \oplus V_{-1} \oplus V_{1/2}$, orthogonally, where V_{-1} and $V_{1/2}$ are the eigenspaces of L_a in $a^\perp$.*

(ii) For $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{V_{-1}, V_{1/2}\}$, the restriction of d to $\varepsilon_1 \times \varepsilon_2 \times \varepsilon_3$ vanishes unless exactly one ε_i is V_{-1}. Hence $u \circ v = -\langle u, v\rangle\, a$ for $u, v \in V_{-1}$, $V_{-1} \circ V_{1/2} \subset V_{1/2}$, and $x \circ y - \frac{1}{2}\langle x, y\rangle\, a \in V_{-1}$ for $x, y \in V_{1/2}$.

(iii) If $\operatorname{Tr} L_a = 0$, *then* $\dim V_{-1} = (N+1)/3$ *and* $\dim V_{1/2} = 2(N-2)/3$.

Formalized, details in Lean. `peirceDecomposition`; the identities $3\dim V_{-1} = N + 1$ and $3\dim V_{1/2} = 2(N-2)$ in (iii): `finrank_exact_of_trace_zero`.

Proof. (i) At $x = a$, (4.1) reads $L_a + 2L_a^2 = I + 2a\langle a, \cdot\rangle$, so on the invariant subspace $a^\perp$, $(2L_a - 1)(L_a + 1) = 0$. (ii) For eigenvectors $y, z, w \in a^\perp$ of L_a with eigenvalues β, γ, δ, (3.7) at (a, y, z, w) gives $(\beta + \gamma + \delta)\, d(y, z, w) = 0$, and a sum of three numbers from $\{-1, \frac{1}{2}\}$ vanishes only for $-1, \frac{1}{2}, \frac{1}{2}$. The product rules (Peirce multiplication rules) follow by taking components, using $\langle x\circ y, a\rangle = \langle L_a x, y\rangle$. (iii) $1 + \dim V_{-1} + \dim V_{1/2} = N$ and $\operatorname{Tr} L_a = 1 - \dim V_{-1} + \frac{1}{2}\dim V_{1/2} = 0$. □

In Tkachev's proof the same eigenvalues appear in the quadratic part of the cubic at a maximum on the unit sphere[65]. The spaces V_{-1} and $V_{1/2}$ are the Peirce spaces of the cubic. Tkachev's Peirce decomposition at an idempotent[66]has a fourth space, for the eigenvalue $-\frac{1}{2}$, which vanishes for eikonal cubics. The Jordan Peirce spaces of a Jordan idempotent carry the eigenvalues $1, \frac{1}{2}, 0$ instead. In $H_3^0(\mathbb{K})$ with the cubic $\sqrt{6}\ \operatorname{tr} X^3$ and $a = a'$ (defined before Theorem 4.6), $V_{1/2}$ is the Jordan Peirce $\frac{1}{2}$-space of $E_{11} = \frac{1}{3}(e + \sqrt{6}\, a')$ and V_{-1} is the traceless part of its Jordan Peirce 0-space.

[63]Tkachev, "Hsiang cubics as spherical Chebyshev polynomials", note 37. Thm. 2.5.

[64]N. Nadirashvili, V. Tkachev and S. Vlăduţ, *Nonlinear Elliptic Equations and Nonassociative Algebras*, Math. Surveys Monogr. **200**, Amer. Math. Soc., Providence, 2014. doi:10.1090/surv/200. Ch. 6, as cited in Tkachev, "Hsiang cubics as spherical Chebyshev polynomials", note 37, Thm. 2.5.

[65]Tkachev, "A generalization of Cartan's theorem", note 1. Eq. (15), p. 4.

[66]Tkachev, "Hsiang cubics as spherical Chebyshev polynomials", note 37. Thm. 2.5.

Lemma 4.2. *Let $V = U_1 \oplus \cdots \oplus U_m$ be an orthogonal decomposition into planes, and let a be a unit idempotent of U_1. Then $\operatorname{Tr} L_x = 0$ for every $x \in V$. Consequently $m \equiv 1 \pmod 3$, and with $k = 2(m-1)/3$, which is even, $\dim V_{-1} = k+1$, $\dim V_{1/2} = 2k$ and $N = 3k+2$.*

Formalized. `closedPlaneDimensions`.

Proof. The cubic of each plane is harmonic (Lemma 3.3), so W is harmonic by Proposition 3.12(ii), and harmonicity is $\operatorname{Tr} L_x = \langle x, \tau(V)\rangle = 0$. With $N = 2m$, Lemma 4.1(iii) shows that 3 divides $2m+1$ and gives the dimensions. □

A second vector splits $V_{1/2}$. For $v \in V_{-1}$ let B_v be $\frac{2}{\sqrt{3}}$ times the restriction of L_v to $V_{1/2}$; it maps $V_{1/2}$ to itself by Lemma 4.1(ii). For a unit idempotent a and a unit vector $b \in V_{-1}$, put

$$S_1 = V_{-1} \cap b^{\perp}, \qquad S_2 = \{y \in V_{1/2} : B_b y = -y\}, \qquad S_3 = \{y \in V_{1/2} : B_b y = y\}.$$

By Lemma 4.3(ii) these are the joint eigenspaces of L_a and L_b in $(\mathbb{R}a \oplus \mathbb{R}b)^{\perp}$. In a plane, the vectors a, b with coordinates $1, i$ are such a pair, since $a \circ b = \bar{1}\,\bar{i} = -b$.

Lemma 4.3 (Joint eigenspaces)**.** *Let a be a unit idempotent and $b \in V_{-1}$ a unit vector.*

(i) $a \circ b = -b$ and $b \circ b = -a$. So $\mathbb{R}a \oplus \mathbb{R}b$ is a plane, and $W(\alpha a + \beta b) = \alpha^3 - 3\alpha\beta^2$.

(ii) B_b is a self-adjoint involution of $V_{1/2}$, and $(\mathbb{R}a \oplus \mathbb{R}b)^{\perp} = S_1 \oplus S_2 \oplus S_3$ orthogonally. On S_1, S_2, S_3 the operator L_a is -1, $\frac{1}{2}$, $\frac{1}{2}$ and the operator L_b is 0, $-\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$. In particular L_a and L_b commute on $(\mathbb{R}a \oplus \mathbb{R}b)^{\perp}$.

(iii) For $u \in S_1$, L_u maps S_2 into S_3 and S_3 into S_2, and the orthogonal multiplication *$\phi\colon S_1 \times S_2 \to S_3$, $\phi(u,y) = \frac{2}{\sqrt{3}}\, u \circ y$, satisfies $|\phi(u,y)| = |u||y|$.*

(iv) If $\operatorname{Tr} L_a = 0$, then $\dim S_1 = (N-2)/3$, and if this number k is positive, the three joint eigenspaces *S_1, S_2, S_3 all have dimension k. Under the hypotheses of Lemma 4.2 with $m > 1$, $k = 2(m-1)/3 \geq 2$.*

Formalized, details in Lean. (i) `markedPlane_normalForm`; (ii) `marked_action_involution`, `slots_span_perp`, `slots_commute`; (iii) `slotOne_mul_slotTwo`, `slotProduct_norm`; (iv) `slot_finranks`, `slot_dimension_of_closed_planes`.

Proof. The decisive step is a Clifford relation: the operators B_u are self-adjoint, and on an orthonormal basis of V_{-1} the identity below gives the matrix relations of a symmetric Clifford system[67]. For $u \in V_{-1}$ and $y \in V_{1/2}$, (4.1) at $x = u$, with $u \circ u = -|u|^2 a$ and $L_a y = \frac{1}{2} y$, gives $2L_u^2 y = \frac{3}{2}|u|^2 y$; polarizing in u,

$$B_u B_v + B_v B_u = 2\langle u, v\rangle\, I \qquad (u, v \in V_{-1}).$$

(i) is Lemma 4.1(ii). (ii) $B_b^2 = I$; on S_1, $b \circ u = -\langle b, u\rangle\, a = 0$; the rest is the Peirce decomposition. (iii) For $u \in S_1$, B_u anticommutes with B_b, so it exchanges S_2 and S_3, and $|B_u y|^2 = \langle B_u^2 y, y\rangle = |u|^2|y|^2$. (iv) Lemma 4.1(iii) gives $\dim S_1 = (N-2)/3$ and $\dim S_2 + \dim S_3 = 2k$; for a unit $u \in S_1$ the involution B_u exchanges S_2 and S_3, so they have equal dimension. Under Lemma 4.2, $k = 2(m-1)/3$, and $m \equiv 1 \pmod 3$ with $m > 1$ gives $k \geq 2$. □

In $H_3^0(\mathbb{K})$, with a and b diagonal, the joint eigenspaces are the three off-diagonal positions (the off-diagonal Jordan Peirce spaces of the diagonal frame). Cartan

[67] Ibid. Def. 2.7, Eq. (2.7).

splits the same space by the same sign and finds the same multiplication[68]. It is a composition formula of size $[k, k, k]$ in Tkachev's terminology[69]. In his argument the matrices of Eqs. (17)–(19) form a symmetric system on a $2k$-dimensional space, and after one generator is diagonalized the others are off-diagonal in k-dimensional blocks[70].

4.2 Hurwitz's theorem and the count

Ordinary multiplication gives the case $n = 1$. For $n \geq 2$, Hurwitz proved that a bilinear map $\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $|\mu(x, y)| = |x||y|$ exists only for $n = 2, 4, 8$[71]. In this form, Hurwitz's theorem on bilinear maps, it applies to the orthogonal multiplication directly. For the identification of the cubic the algebra form, Theorem 4.4, is needed.

Theorem 4.4 (Hurwitz; quoted)**.** *Let A be a nonzero finite-dimensional real algebra with unit 1 and an inner product such that $|xy| = |x||y|$ for all $x, y \in A$ (a Euclidean composition algebra, or normed division algebra). Then there is an isomorphism of algebras from A onto one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ that is an isometry for their standard inner products. In particular* $\dim A \in \{1, 2, 4, 8\}$.

Formalized. `CompositionAlgebra.hurwitz_classification`, `CompositionAlgebra.finrank_eq_one_or_two_or_four_or_eight`, in the composition-algebra library on which the development builds.

The dimension statement also follows from Hurwitz's theorem on bilinear maps; the identification uses the isomorphism part. Baez states the whole theorem as the classification of normed division algebras[72]; Springer and Veldkamp prove that every such algebra is obtained from $\mathbb{R}$ by doubling and so is one of the four[73].

Corollary 4.5. *Let W be eikonal on $V = \mathbb{R}^{2m}$, $m \geq 1$, with an orthogonal decomposition of V into m planes. Then $m \in \{1, 4, 7, 13\}$.*

Formalized, different route. `planeCount_one_four_seven_thirteen`: the formal proof builds a unital algebra with multiplicative norm on S_3 and applies Theorem 4.4 to it, instead of Hurwitz's theorem on bilinear maps.

Proof. Scale W to constant 1 and let $m > 1$. By Lemmas 4.2 and 4.3, the joint eigenspaces of the vectors with coordinates $1, i$ in U_1 have a common dimension $k = 2(m-1)/3 \geq 2$. Identifying each with $\mathbb{R}^k$ by an isometry turns ϕ into a bilinear map $\mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ with $|\phi(u, y)| = |u||y|$, so $k \in \{2, 4, 8\}$ by Hurwitz's theorem on bilinear maps[74], that is $m \in \{4, 7, 13\}$. □

The isomorphism part of Theorem 4.4 goes further and identifies the cubic with the division-algebra models in Cartan's formulas[75]. The identification needs only one unit idempotent and harmonicity, so it is stated in that generality; Section 8 uses it for summands that are not planes. Write $a' = \frac{1}{\sqrt{6}}\operatorname{diag}(2, -1, -1)$ and

[68] Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1. pp. 337–340, Eq. (14).

[69] Tkachev, "A generalization of Cartan's theorem", note 1. §2.

[70] Ibid. Prop. 2.1, Eq. (19), Table 1.

[71] Hurwitz, "Über die Composition der quadratischen Formen", note 2. p. 309, Eq. (2).

[72] Baez, "The octonions", note 10. Thm. 1 in §1.1.

[73] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monogr. Math., Springer, Berlin, 2000. doi:10.1007/978-3-662-12622-6.

[74] Hurwitz, "Über die Composition der quadratischen Formen", note 2.

[75] Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1. p. 341, Eqs. (16)–(17).

$b' = \frac{1}{\sqrt{2}}\operatorname{diag}(0,1,-1)$ in $H_3^0(\mathbb{K})$, and $\iota_{ij}(p) = (pE_{ij} + \bar{p}E_{ji})/\sqrt{2}$ for the isometry of $\mathbb{K}$ onto the matrices supported at (i,j) and (j,i). Here $H_3^0(\mathbb{K})$ carries the trace form $\langle X, Y\rangle = \operatorname{tr}(X \bullet Y)$. In Yokota's notation of Section 5, $\iota_{23}(p) = F_1(p)/\sqrt{2}$, $\iota_{13}(p) = F_2(\bar{p})/\sqrt{2}$ and $\iota_{12}(p) = F_3(p)/\sqrt{2}$.

Formalized, details in Lean. (i) `unitIdempotent_of_isMaxOn`; (ii) `harmonicRoundCubic_identification`.

Theorem 4.6 (Identification at an idempotent)**.** *Let W be a harmonic eikonal cubic with constant 1 on V, with $N \geq 3$.*

(i) Every maximum point of W on the unit sphere is a unit idempotent.

(ii) Let a be a unit idempotent. Then $V_{-1} \neq 0$ and $N = 3k+2$ with $k \in \{1,2,4,8\}$. For every unit vector $b \in V_{-1}$ there is a linear isometry $\Phi\colon V \to H_3^0(\mathbb{K})$, where $\mathbb{K}$ is the number system of dimension k, with

$$d(x,y,z) = \sqrt{6}\,\operatorname{tr}\big((\Phi x \bullet \Phi y) \bullet \Phi z\big) \qquad (x,y,z \in V),$$

$\Phi a = a'$ and $\Phi b = b'$.

Proof. (i) The cubic is odd and not zero, since $|x \circ x| = 1$ on the unit sphere, so its maximum $W(x_0)$ there is positive. By Lagrange, $3\,x_0 \circ x_0 = \nabla W(x_0) = \lambda x_0$ with $\lambda = \langle x_0, \nabla W(x_0)\rangle = 3W(x_0) > 0$, and $|x_0 \circ x_0| = 1$ forces $\lambda = 3$. This is the first step of Tkachev's proof[76].

(ii) Harmonicity gives $\operatorname{Tr} L_a = 0$, so $\dim V_{-1} = (N+1)/3 > 0$ by Lemma 4.1(iii), and the joint eigenspaces of (a,b) have dimension $k = (N-2)/3 \geq 1$ by Lemma 4.3(iv). Fix unit vectors $u_0 \in S_1$, $y_0 \in S_2$. The maps $u \mapsto \phi(u, y_0)$ and $y \mapsto \phi(u_0, y)$ are isometries onto S_3, so $\phi(u, y_0)\,\phi(u_0, y) = \phi(u, y)$ defines on S_3 a product with unit $\phi(u_0, y_0)$ and multiplicative norm. By Hurwitz's theorem (Theorem 4.4) $k \in \{1,2,4,8\}$ and there is an isometric isomorphism ρ_3 of this algebra onto $\mathbb{K}$; with $\rho_1(u) = \rho_3\phi(u, y_0)$ and $\rho_2(y) = \rho_3\phi(u_0, y)$ the isometries $\rho_r\colon S_r \to \mathbb{K}$ satisfy $\rho_3\phi(u,y) = \rho_1(u)\rho_2(y)$.

Define Φ by $a \mapsto a'$, $b \mapsto b'$ and

$$\Phi u = \iota_{23}(\rho_1 u), \qquad \Phi y_2 = \iota_{13}\big(\overline{\rho_2 y_2}\big), \qquad \Phi y_3 = \iota_{12}\big(\overline{\rho_3 y_3}\big)$$

on S_1, S_2, S_3. It is an isometry onto $H_3^0(\mathbb{K})$, since it maps the summands of $\mathbb{R}a \oplus \mathbb{R}b \oplus S_1 \oplus S_2 \oplus S_3$ isometrically onto the diagonal and the three off-diagonal positions. Since conjugation is an isometry that reverses products[77],

$$\phi_{\mathbb{K}}(\Phi u, \Phi y_2) = \iota_{12}\big(\overline{\rho_2 y_2}\;\overline{\rho_1 u}\big) = \iota_{12}\big(\overline{\rho_3\phi(u, y_2)}\big) = \Phi\,\phi(u, y_2),$$

with $\phi_{\mathbb{K}}$ the orthogonal multiplication of Cartan's cubic $W_{\mathbb{K}}(X) = \sqrt{6}\,\operatorname{tr} X^3$ at (a', b'). The normal form (Lemma A.1) holds for W at (a,b) and, by Lemma A.2, for $W_{\mathbb{K}}$ at (a', b'); Φ preserves every term, so $W_{\mathbb{K}}(\Phi x) = W(x)$. Both d and $\sqrt{6}\,\operatorname{tr}((\Phi x \bullet \Phi y) \bullet \Phi z)$ are symmetric, the second by (J1), and have the same cubic, so they agree. □

[76] Tkachev, "A generalization of Cartan's theorem", note 1. §3, p. 4.

[77] Baez, "The octonions", note 10. §2.2.

Theorem 4.7 (Identification)**.** *Let W be eikonal with constant 1 and let $V = U_1 \oplus \cdots \oplus U_m$ be an orthogonal decomposition into planes with $m > 1$. Let $\mathbb{K}$ be the number system of dimension $k = 2(m-1)/3$. There is a linear isometry $\Phi\colon V \to H_3^0(\mathbb{K})$ with*

$$W(x) = \sqrt{6}\ \mathrm{tr}\big((\Phi x)^3\big) \quad \textit{and} \quad d(x,y,z) = \sqrt{6}\ \mathrm{tr}\big((\Phi x \bullet \Phi y) \bullet \Phi z\big)$$

for all $x, y, z \in V$, which maps U_1 onto the traceless diagonal matrices. This is the plane case of Theorem 4.6.

Formalized. `closedPlaneSystem_identification`.

Proof. By Lemma 4.2, W is harmonic and $N = 2m \geq 8$. The vectors a, b with coordinates $1, i$ in U_1 are a unit idempotent and a unit vector of V_{-1}, and Theorem 4.6 gives Φ with $k = (N-2)/3 = 2(m-1)/3$. It maps U_1 onto the span of a' and b'. □

An isometry carrying d to $\sqrt{6}\ \mathrm{tr}((X \bullet Y) \bullet Z)$ carries products to products, $\Phi(x \circ y) = \sqrt{6}\,\Phi x \circ \Phi y$, so it maps the planes U_i to m pairwise orthogonal planes of $H_3^0(\mathbb{K})$.

Cartan's cubic on $H_3^0(\mathbb{H})$ is eikonal, harmonic and contains planes, but seven of them cannot be orthogonal, because of the lower bound on shares of Section 6. To use that bound the planes are turned into frames, directly from their unit idempotents. The unit idempotents a_r of the cubic product and the primitive idempotents p_r (trace one) of the Jordan product are different objects; Lemma 4.8 passes from the first to the second.

Lemma 4.8. *Let $\Phi\colon V \to H_3^0(\mathbb{K})$ be a linear isometry with $\Phi(x \circ y) = \sqrt{6}\,\Phi x \circ \Phi y$ for all $x, y \in V$. Let $U \subset V$ be a plane, with its three unit idempotents a_1, a_2, a_3 (so $a_1 + a_2 + a_3 = 0$), and put $p_r = \frac{1}{3}(e + \sqrt{6}\,\Phi a_r)$. Then $\{p_1, p_2, p_3\}$ is a Jordan frame of $H_3(\mathbb{K})$. If $U' \perp U$ is another plane, with frame $\{p_1', p_2', p_3'\}$ obtained in the same way, then the two frames are unbiased (Section 5).*

Formalized. Frames: `realFrame_of_three_unit_idempotents`, `complexFrame_of_three_unit_idempotents`, `quaternionicFrame_of_three_unit_idempotents`, `nativeFrame_of_three_unit_idempotents_of_intertwines`; unbiased: `complexFrame_formula_unbiased` and its real, quaternionic and native analogues. For any three unit idempotents with sum 0.

Proof. Put $X_r = \Phi a_r$. Then $\sqrt{6}\,X_r \circ X_r = X_r$ and $\mathrm{tr}\,X_r^2 = 1$, so $X_r \bullet X_r = \frac{1}{\sqrt{6}} X_r + \frac{1}{3} e$, and expanding gives $p_r \bullet p_r = p_r$; also $\mathrm{tr}\,p_r = 1$ and $p_1 + p_2 + p_3 = e$. Multiplying $p_1 + p_2 + p_3 = e$ by each p_r shows that the three products $p_r \bullet p_s$, $r \neq s$, have pairwise sums zero, so they vanish. For the second frame, $\langle X_r, X_s' \rangle = 0$ and $\mathrm{tr}\,X_r = \mathrm{tr}\,X_s' = 0$ give $\mathrm{tr}(p_r \bullet p_s') = \frac{1}{9}\,\mathrm{tr}\,e = \frac{1}{3}$. □

Theorem 4.9 (Count)**.** *Let W be an eikonal cubic on $V = \mathbb{R}^{2m}$, $m \geq 1$, with an orthogonal decomposition of V into m planes. Then $m \in \{1, 4, 13\}$. The proof uses Hurwitz's theorem, through Corollary 4.5 and Theorem 4.7, and Corollary 6.6.*

Formalized. `planeCount_one_four_thirteen`.

The exclusion of seven, Corollary 6.6, is proved in Section 6 from Lemmas 3.11 and 4.10, Proposition 5.1, the sum rule, and the normal form and lower bound of that section. None of these depends on Theorem 4.9.

Proof. Scale W to constant 1. By Corollary 4.5, $m \in \{1, 4, 7, 13\}$. Suppose $m = 7$. Then $k = 4$, and Theorem 4.7 gives an isometry $\Phi\colon V \to H_3^0(\mathbb{H})$ with $\Phi(x \circ y) = \sqrt{6}\,\Phi x \circ \Phi y$. By Lemma 4.8 the seven planes give seven pairwise unbiased frames of $H_3(\mathbb{H})$; they are distinct, since $\mathrm{tr}(p \bullet p) = 1 \neq \frac{1}{3}$. This contradicts Corollary 6.6. □

4.3 The product space of two planes

Lemma 4.10. *Let U, U' be orthogonal planes of an eikonal cubic with constant* 1. *For $x, x' \in U$ and $y, y' \in U'$,*

$$\langle x \circ y, x' \circ y' \rangle = \tfrac{1}{2}\langle x, x' \rangle \langle y, y' \rangle.$$

So the map $\mathcal{M}\colon U \otimes U' \to V$, $x \otimes y \mapsto x \circ y$, satisfies $\mathcal{M}^*\mathcal{M} = \frac{1}{2}I$, and its image $U \circ U'$ is four-dimensional and orthogonal to U and U'.

Formalized. `mixedGram`, at any constant c, with $\frac{c}{2}$ in place of $\frac{1}{2}$.

Proof. Apply (4.1) with $x \in U$ and pair it with $y, y' \in U'$. The first term gives $\langle x \circ x, y \circ y' \rangle = 0$, because $x \circ x \in U$ and $y \circ y' \in U'$, and $\langle x, y \rangle = 0$. So $2\langle x \circ y, x \circ y' \rangle = |x|^2 \langle y, y' \rangle$, and polarizing in x,

$$\langle L_{x'} L_x y, y' \rangle + \langle L_x L_{x'} y, y' \rangle = \langle x, x' \rangle \langle y, y' \rangle.$$

Let a, b be the vectors with coordinates $1, i$ in U. By Lemma 4.3(ii), L_a and L_b preserve $U^\perp \supset U'$ and commute there, so the two terms on the left are equal. Orthogonality to U and U' is Lemma 3.11. □

The mixed product law gives the case $x = x'$, $y = y'$. That identity alone allows a two-dimensional image, as complex multiplication shows; the commuting of L_a and L_b off the plane makes the image four-dimensional.

5 Planes, frames and the sum rule

This section identifies the planes of Cartan's cubics with Jordan frames, shows that two orthogonal planes generate a copy of $H_3(\mathbb{C})$, and proves the sum rule for every eikonal cubic that splits into planes. Lemmas 5.3 and 5.4, which move two unbiased frames to a standard position, can be skipped on first reading.

5.1 Planes and frames

In this subsection and the next, except in Corollary 5.6, $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, $J = H_3(\mathbb{K})$ and $V = H_3^0(\mathbb{K})$ carries the trace form, the cubic product $X \circ Y$ (the traceless part of the Jordan product $X \bullet Y$) and the eikonal constant $c = 1/6$ of Example 3.5. For a Jordan frame $F = \{p_1, p_2, p_3\}$ write $q_a = p_a - \frac{1}{3}e$ for its centered frame elements and

$$U_F = \mathbb{R}q_1 + \mathbb{R}q_2 + \mathbb{R}q_3 = \left\{ \textstyle\sum_a \lambda_a p_a : \sum_a \lambda_a = 0 \right\}$$

for the traceless part of its span. In quantum language a frame is a test with three outcomes, and the trace-one section of the self-dual cone of squares of J is the state space; for $\mathbb{K} = \mathbb{C}$ the frames are the orthonormal bases of $\mathbb{C}^3$ up to phases. Two frames F and $G = \{p'_1, p'_2, p'_3\}$ are *unbiased* if $\operatorname{tr}(p_a \bullet p'_b) = \frac{1}{3}$ for all a, b: each outcome of one test has probability $1/3$ in every state p'_b in which an outcome of the other is certain. Cohn, Kumar and Minton use the same term for orthogonal

triples of points in the octonionic projective plane, calling their 13 triples mutually unbiased bases[78]. Since $\operatorname{tr}(p_a \bullet e) = 1$ and $\operatorname{tr} e = 3$,

$$\langle q_a, q'_b\rangle = \operatorname{tr}(p_a \bullet p'_b) - \tfrac{1}{3}, \qquad \langle q_a, q_b\rangle = \delta_{ab} - \tfrac{1}{3}, \qquad \sum_a q_a = 0. \tag{5.1}$$

The last two relations give $\sum_a \langle q_a, x\rangle\, q_a = x$ for every $x \in U_F$.

Proposition 5.1. *A two-dimensional subspace $U \subset V$ is closed if and only if $U = U_F$ for a Jordan frame F, and then F is determined by U. Two such planes U_F, U_G are orthogonal if and only if F and G are unbiased.*

Formalized. Planes: `real_closedPlane_iff_frame`, `complex_closedPlane_iff_frame`, `quaternionic_closedPlane_iff_frame`, `native_closedPlane_iff_frame`; orthogonality: `real_framePlane_orthogonal_iff`, `complex_framePlane_orthogonal_iff`, `centeredPlane_orthogonal_iff`, `native_framePlane_orthogonal_iff`.

Proof. The span of a frame is closed under the Jordan product, since $p_a \bullet p_b = \delta_{ab} p_a$, so its traceless part U_F is closed under $\circ$. Conversely let U be a plane. The function $x \mapsto \operatorname{tr} x^3$ is odd and continuous on the unit circle of U, so it vanishes at some unit x. By (J2) the eigenvalues of x have sum and product zero, so they are $\lambda, 0, -\lambda$ with $\lambda > 0$, and $x = \lambda(p_1 - p_3)$ for a frame $\{p_1, p_2, p_3\}$. Then $x \circ x = \lambda^2(p_1 + p_3) - \frac{2}{3}\lambda^2 e = \frac{\lambda^2}{3}(p_1 - 2p_2 + p_3)$ lies in U, is nonzero, and is orthogonal to x. Both vectors lie in the plane U_F, so $U = U_F$. The algebra $\mathbb{R}e \oplus U_F$ is the span of F, isomorphic to $\mathbb{R}^3$, and its primitive (minimal nonzero) idempotents for the Jordan product are exactly p_1, p_2, p_3; so U determines F. The last statement is the first relation in (5.1). □

Since $H_3^0(\mathbb{R})$ has dimension five, Lemma 3.11 rules out two orthogonal planes there; from now on $\mathbb{K} \neq \mathbb{R}$.

For an eikonal cubic on a space V, an orthogonal decomposition $V = U_1 \oplus \cdots \oplus U_m$, $m \geq 2$, in which every U_i is a plane is a *complete system of planes*. A set of pairwise unbiased frames is a complete system when its planes form one. This is the analogue of a complete set of mutually unbiased bases. It is not a complete system of orthogonal idempotents in the sense of Jordan theory (orthogonal idempotents summing to e; a Jordan frame is one whose idempotents are primitive).

Corollary 5.2. *A system of pairwise orthogonal planes of V that fills V has $m = (3 \dim_{\mathbb{R}} \mathbb{K} + 2)/2$ planes, that is 4, 7 or 13 for $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$. Equivalently, a set of pairwise unbiased Jordan frames of J has at most m members, and a set of m of them is a complete system.*

Formalized. `complex_completePlane_count`, `quaternionic_completePlane_count`, `native_completePlane_count`, for any orthogonal two-dimensional subspaces, without a closedness hypothesis; frames: `complex_frameSet_capacity`, `quaternionic_frameSet_capacity`, `native_frameSet_capacity`.

Proof. The planes are two-dimensional, orthogonal and fill V, of dimension $3 \dim_{\mathbb{R}} \mathbb{K} + 2$. Proposition 5.1 translates planes into frames. □

5.2 Two planes generate a copy of $H_3(\mathbb{C})$

Two unbiased frames can be moved to a standard position (for bases of $\mathbb{C}^3$, the dephased or standard form of Brierley, Weigert and Bengtsson[79]). Over $\mathbb{C}$ and $\mathbb{H}$ the

[78] Cohn, Kumar and Minton, "Optimal simplices and codes", note 6. §8.2, remark after Thm. 8.3, p. 1352.

[79] Brierley, Weigert and Bengtsson, "All mutually unbiased bases in dimensions two to five", note 20. §1.

automorphisms needed are conjugations $X \mapsto \Delta X \Delta^*$ by diagonal unitary matrices. Over $\mathbb{O}$ such a conjugation is not an automorphism in general; the automorphisms fixing the diagonal frame are given by triality. Following Yokota, write $F_1(a)$, $F_2(a)$, $F_3(a)$, for $a \in \mathbb{O}$, for the elements of $H_3(\mathbb{O})$ with entry a in position $(2,3)$, $(3,1)$, $(1,2)$ respectively, $\bar{a}$ in the transposed position and zeros elsewhere[80].

Lemma 5.3. *The automorphisms of $H_3(\mathbb{O})$ that fix E_{11}, E_{22}, E_{33} are exactly the maps ϑ_α given by*

$$\vartheta_\alpha\Big(\textstyle\sum_i \xi_i E_{ii} + \sum_i F_i(a_i)\Big) = \sum_i \xi_i E_{ii} + \sum_i F_i(\alpha_i a_i),$$

for the triples $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in SO(8)^3$ with $(\alpha_1 a)(\alpha_2 b) = \overline{\alpha_3(\overline{ab})}$ for all $a, b \in \mathbb{O}$.

Formalized, different route. `diagonalFrame Stabilizer_iff_yokota Triality`: the maps $\alpha_1, \alpha_2, \alpha_3$ on the three off-diagonal positions, their triality relation and their determinants are derived from the automorphism, instead of quoting Yokota's description of Spin(8).

Proof. This is Yokota's description of the pointwise stabilizer of the diagonal frame as Spin(8)[81], where Spin(8) is the group of such triples[82]. □

Lemma 5.4. *Let $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and let F, G be unbiased Jordan frames of J.*

(i) There are an automorphism Ψ of J and a unit imaginary $u \in \mathbb{K}$ such that $\Psi(F)$ is the diagonal frame and $\Psi(G)$ is the Fourier frame $\{p_0^{\mathsf{F}}, p_1^{\mathsf{F}}, p_2^{\mathsf{F}}\}$, $(p_k^{\mathsf{F}})_{ij} = \frac{1}{3}\omega_u^{k(i-j)}$, where $\omega_u = -\frac{1}{2} + \frac{\sqrt{3}}{2}u$. All these matrices lie in the subalgebra $H_3(\mathbb{C}_u)$, $\mathbb{C}_u = \mathbb{R} + \mathbb{R}u$.

(ii) Let $x, y \in \mathbb{O}$ have norm one. There is an automorphism of $H_3(\mathbb{O})$ fixing the diagonal idempotents and sending $xE_{12} + \bar{x}E_{21}$ to $E_{12} + E_{21}$ and $yE_{23} + \bar{y}E_{32}$ to $E_{23} + E_{32}$.

Formalized, different route. (i) `complex_normalize_unbiased_pair`, `quaternionic_normalize_unbiased_pair`, `native_normalize_unbiased_pair`, entries `native Fourier_matrix_entry`, `nativeFourier_entry_in_complex_slice`; (ii) `native_normalize_two_slots`: the automorphism is composed of explicit automorphisms of $H_3(\mathbb{O})$, instead of Yokota's transitivity results.

Proof. (ii) In the notation of Lemma 5.3 the two given elements are $F_3(x)$ and $F_1(y)$, and $F_3(1) = E_{12} + E_{21}$, $F_1(1) = E_{23} + E_{32}$. The first components β_1 of the triples act transitively on unit octonions[83]; choose a triple β with $\beta_1 y = 1$. If a triple γ has $\gamma_1(1) = 1$, putting $a = 1$ in the triality relation gives $\gamma_2 b = \overline{\gamma_3(\bar{b})}$, and the components γ_3 that occur form the group $\widetilde{B}_3 \cong \mathrm{Spin}(7)$ or its conjugate by $b \mapsto \bar{b}$, either of which acts transitively on unit octonions[84]. Choose such a γ with $\gamma_3\beta_3 x = 1$. Then $\vartheta_\gamma\vartheta_\beta$ fixes the E_{ii} and sends $F_1(y)$ to $F_1(1)$ and $F_3(x)$ to $F_3(1)$.

(i) By (J3), F may be taken diagonal. Let g be an idempotent of trace one unbiased to the diagonal frame. Unbiasedness gives $g_{ii} = \mathrm{tr}(E_{ii} \bullet g) = \frac{1}{3}$. The diagonal entries of $g^2 = g$ give $\frac{1}{9} + |g_{ij}|^2 + |g_{ik}|^2 = \frac{1}{3}$ for each i, so every off-diagonal entry of $3g$ has norm one. Write x, y, z for the entries of $3g$ at $(1,2)$, $(2,3)$, $(1,3)$; the $(1,3)$ entry of $g^2 = g$ reads $(2z + xy)/9 = z/3$, so $z = xy$. Now take $g \in G$. For $\mathbb{K} = \mathbb{O}$ let χ be the automorphism of (ii); for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ let χ be conjugation by $\Delta = \mathrm{diag}(d_1, d_2, d_3) = \mathrm{diag}(1, x, xy)$, which multiplies the (i,j) entry by d_i on the left and $\bar{d}_j$ on the right and so sends x and y to $x\bar{x} = 1$ and $xy\,\overline{xy} = 1$. In both cases χ fixes F and preserves the trace form, so $\chi(g)$ is again an idempotent of trace one unbiased to F, now with $x = y = 1$; hence $z = 1$ and $\chi(g) = \frac{1}{3}\mathbf{1}$, where $\mathbf{1}$ is the matrix of ones.

[80] Yokota, "Exceptional Lie groups", note 44. §2.1.
[81] Ibid. Thm. 2.7.1.
[82] Ibid. §1.16, Thm. 1.16.2.
[83] Ibid. Prop. 1.16.1.
[84] Ibid. §1.15, Thm. 1.15.1.

Now let $g' \in G$ be a second idempotent, and write a_1, a_2, a_3 for the entries of $3g'$ at $(1,2)$, $(2,3)$, $(3,1)$; they have norm one, as for g. The relation $g \bullet g' = 0$ reads $\mathbf{1}g' + g'\mathbf{1} = 0$, with real $\mathbf{1}$. The (i,j) entry of $g'\mathbf{1}$ is the row sum r_i of g', and that of $\mathbf{1}g'$ is the column sum $\bar{r}_j$, so $r_i + \bar{r}_j = 0$ for all i, j. Hence all r_i are equal to one imaginary number, while their sum, the sum of all entries of g', is real; so every $r_i = 0$. The first two row sums of $3g'$ give $a_1 + \bar{a}_3 = -1$ and $\bar{a}_1 + a_2 = -1$. Two unit octonions with real sum -1 have real parts $-\frac{1}{2}$ and opposite imaginary parts, so $a_3 = a_1 = a_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}u$ for a unit imaginary u. Then $g' = p_k^{\mathsf{F}}$ for the $k \in \{1,2\}$ with $\omega_u^{-k} = a_1$, and the third idempotent of G is $e - g - g'$, the remaining p_k^{F}. All entries lie in $\mathbb{C}_u$, which is closed under multiplication and conjugation. □

For $\mathbb{K} = \mathbb{C}$, part (i) is the classical fact that every complex Hadamard matrix of order three is equivalent to the Fourier matrix, a case of Haagerup's classification in orders up to five[85]; see also Brierley, Weigert and Bengtsson[86].

As in Lemma 4.10, the *product space* $U \circ U'$ of two orthogonal planes U, U' is the image of the product map $\mathcal{M}\colon U \otimes U' \to V$, $x \otimes y \mapsto x \circ y$, that is, the span of all $x \circ y$ with $x \in U$ and $y \in U'$.

Proposition 5.5. *Let $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and let U_F, U_G be orthogonal planes of V.*

(i) The unital subalgebra A of J generated by $F \cup G$ is isomorphic to $H_3(\mathbb{C})$, by an isomorphism taking F to the diagonal frame and G to the Fourier frame.

(ii) Its traceless part is $A^0 = U_F \oplus U_G \oplus (U_F \circ U_G)$, orthogonally, where the product space $U_F \circ U_G$ is four-dimensional. More precisely, for $x, x' \in U_F$ and $y, y' \in U_G$,

$$\langle x \circ y, x' \circ y' \rangle = \tfrac{c}{2}\langle x, x' \rangle \langle y, y' \rangle, \qquad c = \tfrac{1}{6}. \tag{5.2}$$

Formalized. `complex_frame Pair_generated_qutrit`, `quaternionic_frame Pair_generated_qutrit`, `native_framePair_generated_qutrit`.

Proof. Apply the automorphism Ψ of Lemma 5.4(i); automorphisms preserve the trace (for $H_3(\mathbb{O})$ by Yokota's Lemma 2.2.1(2)[87]), hence the trace form and the products. So F is diagonal, G is the Fourier frame, and the generators of A lie in the subalgebra $H_3(\mathbb{C}_u) \cong H_3(\mathbb{C})$, of dimension nine.

Identity (5.2) is Lemma 4.10, scaled from constant 1 to constant c: replacing W by $W/\sqrt{c}$ divides all products by $\sqrt{c}$. So $U_F \circ U_G$ has dimension four; it lies in V and is orthogonal to U_F and U_G. It lies in A, since $x \circ y = x \bullet y - \frac{1}{3}\langle x, y \rangle e$ and A is a unital subalgebra containing U_F and U_G. Hence $\mathbb{R}e \oplus U_F \oplus U_G \oplus (U_F \circ U_G)$ is a nine-dimensional subspace of A, while A lies in the nine-dimensional $H_3(\mathbb{C}_u)$. So all three coincide, which proves (ii), and (i) follows with $\mathbb{C}_u \cong \mathbb{C}$, $u \mapsto i$. □

For an eikonal cubic that splits into planes the same holds without naming a Jordan algebra.

[85] U. Haagerup, Orthogonal maximal abelian $*$-subalgebras of the $n \times n$ matrices and cyclic n-roots, in *Operator Algebras and Quantum Field Theory (Rome, 1996)*, International Press, Cambridge MA, 1997, 296–322.

[86] Brierley, Weigert and Bengtsson, "All mutually unbiased bases in dimensions two to five", note 20. §2.2, Eq. (6), p. 4.

[87] Yokota, "Exceptional Lie groups", note 44.

Corollary 5.6. *Let W be eikonal and $V = U_1 \oplus \cdots \oplus U_m$ an orthogonal decomposition into planes. For $i \neq j$ the subspace $U_i \oplus U_j \oplus (U_i \circ U_j)$ is closed, and W restricted to it is, after a linear isometry, a positive multiple of Cartan's cubic on $H_3^0(\mathbb{C})$.*

Formalized. `pairOctet_of RoundPlaneSystem`.

Proof. For $m = 1$ there is nothing to prove, so let $m > 1$ and scale W to constant 1. Theorem 4.7 gives an isometry Φ onto some $H_3^0(\mathbb{K})$, with the trace form $\langle X, Y\rangle = \operatorname{tr}(X \bullet Y)$, such that $\Phi(x \circ y) = \sqrt{6}\,\Phi x \circ \Phi y$; it carries U_i, U_j to orthogonal planes U_F, U_G and $U_i \circ U_j$ to $U_F \circ U_G$. By Proposition 5.5, $U_F \oplus U_G \oplus (U_F \circ U_G)$ is the traceless part of a subalgebra isomorphic to $H_3(\mathbb{C})$; it is closed, and Cartan's cubic on $H_3^0(\mathbb{K})$ restricts to Cartan's cubic of $H_3(\mathbb{C})$. □

5.3 Shares and the sum rule

From here on W is any eikonal cubic, with eikonal constant c. Fix three pairwise orthogonal planes U_i, U_j, U_k, with orthogonal projections P_i, P_j, P_k, product space $U_i \circ U_j$ of the first two, and Q_{ij} the projection onto $U_i \circ U_j$. By Lemma 4.10, applied to $W/\sqrt{c}$, which has constant 1, the product map of each pair of the three planes satisfies $\mathcal{M}^*\mathcal{M} = \frac{c}{2} I$, so $\mathcal{M}\mathcal{M}^* = \frac{c}{2} Q$.

Definition 5.7. The *share* of U_k on the pair U_i, U_j is

$$s_{ijk} = \tfrac{1}{2} \operatorname{Tr}(P_k Q_{ij}),$$

where Tr is the trace of operators on V.

Since U_k is a plane, $\operatorname{Tr}(P_k Q_{ij}) = \cos^2\theta_1 + \cos^2\theta_2$ for the two principal angles θ_1, θ_2 between U_k and $U_i \circ U_j$, so the share is the mean of their squared cosines. By Lemma 5.8 it is also, for eikonal constant 1, the sum of the squared moduli of the four coefficients of the three-field term of U_i, U_j, U_k in complex coordinates after factoring out -3, as in the normal form of Lemma 5.8(ii), and, in the Jordan models $H_3^0(\mathbb{K})$, a sum of squares of centered triple traces of the three frames.

Lemma 5.8. *Let (e_α), (f_β), (g_γ) be orthonormal bases of U_i, U_j, U_k.*

(i) $s_{ijk} = \frac{1}{c} \sum_{\alpha,\beta,\gamma} d(e_\alpha, f_\beta, g_\gamma)^2$. *In particular the share is symmetric in i, j, k.*

(ii) Let $c = 1$ and let z_i, z_j, z_k be complex coordinates on the three planes, orthonormal for the Euclidean metric, in which the three-field term of W is

$$-3\,\mathrm{Re}\big(a_0\, z_i z_j z_k + a_1\, \bar{z}_i z_j z_k + a_2\, z_i \bar{z}_j z_k + a_3\, z_i z_j \bar{z}_k\big).$$

Then $s_{ijk} = |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2$.

(iii) In $V = H_3^0(\mathbb{K})$, if the planes are U_F, U_G, U_H with frames $\{p_a\}, \{p'_b\}, \{p''_h\}$, then

$$s_{ijk} = 6 \sum_{a,b,h} \Big(\operatorname{tr}\big((p_a \bullet p'_b) \bullet p''_h\big) - \tfrac{1}{9} \Big)^2.$$

Formalized. (i) `shareOf ThreePlanes_eq_sum_ cubic_sq`, symmetry `shareOfThreePlanes_ swap_first`, `shareOf ThreePlanes_swap_last`; (ii) `shareOfThree Planes_eq_sum_ complex_coefficients`, the three-field term being *6d* by `three InputCubic_eq`; (iii) `complex_shareOfThree Planes_eq_frame_trace`, `frameShare_eq_frameS`, `native_shareOfThree Planes_eq_frame_trace`.

Proof. (i) From $\mathcal{M}_{ij}\mathcal{M}_{ij}^* = \frac{c}{2} Q_{ij}$, $\operatorname{Tr}(P_k Q_{ij}) = \frac{2}{c} \operatorname{Tr}(P_k \mathcal{M}_{ij} \mathcal{M}_{ij}^*) = \frac{2}{c} \sum_{\alpha,\beta} |P_k(e_\alpha \circ f_\beta)|^2$, and $\langle e_\alpha \circ f_\beta, g_\gamma\rangle = d(e_\alpha, f_\beta, g_\gamma)$.

(ii) The term of $W(x) = d(x,x,x)$ with one entry in each plane is $6\,d(x_i, x_j, x_k)$, so $d(x_i, x_j, x_k) = -\frac{1}{2}\operatorname{Re}\sum_\epsilon a_\epsilon m_\epsilon$, where m_ϵ runs over the four monomials and a_ϵ over $a_0, \dots, a_3$. Take the real bases $\{1, i\}$ in each plane, so that the eight evaluation points are $z_p = i^{t_p}$, $t \in \{0,1\}^3$. For two patterns ϵ, η of conjugations, $\sum_t m_\epsilon \overline{m_\eta} = \prod_p \bigl(1 + i^{\epsilon_p - \eta_p}\bigr)$ and $\sum_t m_\epsilon m_\eta = \prod_p \bigl(1 + i^{\epsilon_p + \eta_p}\bigr)$, with $\epsilon_p, \eta_p = \pm 1$. The first is $8\delta_{\epsilon\eta}$; the second vanishes, because no pattern in the list is the reverse of another. Hence the eight functions $\operatorname{Re} m_\epsilon$, $\operatorname{Im} m_\epsilon$ on the evaluation points are orthogonal with squared norm 4, and $\sum_t \bigl(\operatorname{Re}\sum_\epsilon a_\epsilon m_\epsilon\bigr)^2 = 4\sum_\epsilon |a_\epsilon|^2$. By (i) with $c = 1$, $s_{ijk} = \frac{1}{4}\cdot 4\sum_\epsilon |a_\epsilon|^2$.

(iii) By the identity $\sum_a \langle q_a, x\rangle\, q_a = x$ on U_F and its analogues for G and H, the sum in (i) may be taken over (q_a, q'_b, q''_h) in place of orthonormal bases. Expanding $p = q + \frac{1}{3}e$ and using the orthogonality of the planes and $\operatorname{tr} q = 0$ gives $d(q_a, q'_b, q''_h) = \operatorname{tr}((q_a \bullet q'_b) \bullet q''_h) = \operatorname{tr}((p_a \bullet p'_b) \bullet p''_h) - \frac{1}{9}$; and the eikonal constant of $H^0_3(\mathbb{K})$ is $1/6$. □

For $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ the idempotents of trace one are the rank-one projections. For frames $p_a = u_a u_a^*$, $p'_b = v_b v_b^*$, $p''_h = w_h w_h^*$, the triple trace in (iii) is $\operatorname{tr}((p_a \bullet p'_b) \bullet p''_h) = \operatorname{Re}\bigl((u_a^* v_b)(v_b^* w_h)(w_h^* u_a)\bigr)$, for $\mathbb{K} = \mathbb{C}$ the real part of a Bargmann invariant[88]. For $\mathbb{K} = \mathbb{H}$, changing the unit representatives conjugates the product by a unit quaternion and leaves its real part unchanged. So the share is 6 times the sum of the squared deviations of these real parts from $1/9$. In W_4, which has $c = 1$, each triple of coordinate planes has, after factoring out -3, one coefficient of modulus one and the other three zero, so every share is 1.

Theorem 5.9 (Sum rule)**.** *Let W be an eikonal cubic on V and $V = U_1 \oplus \cdots \oplus U_m$ an orthogonal decomposition into planes, $m \geq 3$. Every share lies in $[0,1]$, with $s_{ijk} = 1$ exactly when $U_k \subset U_i \circ U_j$ and $s_{ijk} = 0$ exactly when $U_k \perp U_i \circ U_j$, and*

$$\sum_{k \neq i,j} s_{ijk} = 2 \qquad \text{for every pair } i \neq j. \tag{5.3}$$

The total share $\sum_{i<j<k} s_{ijk}$ is $m(m-1)/3$.

Formalized. `plane Share_sum_rule`. Self-contained statement: `Challenge.sum_rule`.

Proof. By Lemma 4.10, $U_i \circ U_j$ has dimension four, and by Lemma 3.11 $U_i \circ U_j \subset (U_i \oplus U_j)^\perp = \bigoplus_{k \neq i,j} U_k$, so $\sum_{k \neq i,j} P_k Q_{ij} = Q_{ij}$, and taking half the trace gives (5.3). For orthogonal projections, $0 \leq \operatorname{Tr}(P_k Q_{ij}) \leq \operatorname{Tr} P_k = 2$, with equality on the right exactly when $\operatorname{Tr}(P_k(I - Q_{ij})P_k) = 0$, that is $U_k \subset U_i \circ U_j$, and on the left exactly when $Q_{ij}P_k = 0$. Summing (5.3) over the $\binom{m}{2}$ pairs counts each triple three times, by the symmetry in Lemma 5.8(i). □

[88] V. Bargmann, Note on Wigner's theorem on symmetry operations, *J. Math. Phys.* **5** (1964) 862–868. doi:10.1063/1.1704188. §1.5, p. 864.

Theorem 5.10. *(i) An eikonal cubic with a proper orthogonal decomposition into planes has at least four of them.*

(ii) Every orthogonal decomposition of $H_3^0(\mathbb{C})$ into planes has four planes, every share equals 1, and $U_i \circ U_j = U_k \oplus U_l$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

(iii) Such a decomposition of $H_3^0(\mathbb{C})$ exists and is unique up to conjugation by a unitary matrix and relabelling. The four coordinate planes of W_4 in (3.5) form one.

Formalized, different route. (i) `round_plane System_at_least_four`; (ii) `complex_closed PlaneSystem_four`; (iii) `complex_closed PlaneSystem_unitary`, `toMatrix_unitary TraceIsometry`, for W_4 `tensor_diagonal`, `matrixCoordinate Planes_isClosed`: the formal proof normalizes two frames and shows that the other two planes are determined, instead of quoting Brierley, Weigert and Bengtsson.

Proof. (i) Two planes cannot fill V: their products would have nowhere to go (Lemma 3.11). Three planes would give $s_{123} = 2$ in (5.3), while a share is at most 1.

(ii) By Corollary 5.2 there are four planes. For each pair the two remaining shares are at most 1 and add up to 2, so both equal 1 and $U_k, U_l \subset U_i \circ U_j$, which has dimension four.

(iii) By (ii) and Proposition 5.1, such a decomposition is the same as four pairwise unbiased frames, that is four mutually unbiased orthonormal bases of $\mathbb{C}^3$ taken up to phases of their vectors. Brierley, Weigert and Bengtsson classify these[89]: up to a unitary, every pair is the standard basis and the Fourier basis (their matrix F_3); exactly six unit vectors up to phase are unbiased to both (their Eq. (9)); and these form two orthonormal bases in only one way, DF_3 and D^2F_3 with their $D = \mathrm{diag}(1, \omega, \omega)$ (their Eq. (10)). Their equivalence allows a unitary, rephasing and reordering of vectors, reordering of bases, and an overall complex conjugation[90]. Their derivation in dimension three uses only the first four operations (their Eq. (11)); on frames these are conjugation by a unitary and relabelling, since rephasing a vector does not change its projector. So the decomposition is unique in the stated sense. The coordinate planes of W_4 are closed and orthogonal (Example 3.6), so they form one. □

The four bases are the complete set of mutually unbiased bases of $\mathbb{C}^3$ found by Ivanović[91]and by Wootters and Fields[92]. By the six-vector count of Brierley, Weigert and Bengtsson, once two planes of the decomposition are chosen, the other two are determined.

6 The missing seven

A complete system of planes in $H_3^0(\mathbb{H})$ would have seven planes. For each pair, the shares of the five other planes would add up to two, an average share of $\frac{2}{5}$. This section proves that every share in $H_3^0(\mathbb{H})$ is at least $1 - 1/\sqrt{3} \approx 0.4226$, so seven planes do not exist.

The reason the quaternions fail where the complex numbers and the octonions succeed lies in how much of the algebra squares back into $\mathbb{C}$. Write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. Since $jz = \bar{z}j$ for $z \in \mathbb{C}$ and $j^2 = -1$, the product of two elements of $\mathbb{C}j$ lies in $\mathbb{C}$. In

[89] Brierley, Weigert and Bengtsson, "All mutually unbiased bases in dimensions two to five", note 20. §2.2, Eqs. (6)–(11), pp. 4–5.

[90] Ibid. App. A, Eqs. (53)–(57).

[91] Ivanović, "Geometrical description of quantal state determination", note 19.

[92] Wootters and Fields, "Optimal state-determination by mutually unbiased measurements", note 19.

the normal form of Lemma 6.2, a third basis is described by three unit quaternions, and their $\mathbb{C}$-parts and $\mathbb{C}j$-parts each obey a triangle condition. The share is a quadratic function of the balance between the two parts (Lemma 6.3). The balance alone forces $s \geq \frac{1}{3}$, which is below the average of $\frac{2}{5}$ that seven planes need. The two triangle conditions raise the bound to $1 - 1/\sqrt{3}$. In $\mathbb{O}$ the complement of $\mathbb{C}$ has no such parity, since for a unit l orthogonal to the quaternions $\mathbb{H} \subset \mathbb{O}$ the product jl is again orthogonal to $\mathbb{C}$. Shares in $H_3^0(\mathbb{O})$ can vanish, and Section 7 has many that do.

6.1 Bases, frames and the spectral theorem

Let $\mathbb{H}^3$ be the right quaternionic vector space of columns (not the matrix algebra $H_3(\mathbb{H})$), with the quaternion-valued inner product $\langle u, v\rangle = u^* v$. Two orthonormal bases (u_a) and (v_b) are *mutually unbiased* (unbiased for short) if $|\langle u_a, v_b\rangle|^2 = \frac{1}{3}$ for all a, b. On $H_3(\mathbb{H})$ use the real trace $\operatorname{tr} x = \operatorname{Re}\operatorname{Tr} x$, the Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$ and $\langle x, y\rangle = \operatorname{tr}(x \bullet y) = \operatorname{Re}\operatorname{Tr}(xy)$, as in Example 3.5. The real trace is cyclic, $\operatorname{Re}\operatorname{Tr}(xy) = \operatorname{Re}\operatorname{Tr}(yx)$, because $\operatorname{Re}(\alpha\beta) = \operatorname{Re}(\beta\alpha)$ for quaternions.

An orthonormal basis (u_a) gives the Jordan frame $p_a = u_a u_a^*$. For unit vectors u, v one has $\operatorname{tr}(uu^* vv^*) = \operatorname{Re}(u^* v\, v^* u) = |\langle u, v\rangle|^2$. Hence two bases are unbiased exactly when their frames are, which by Proposition 5.1 means that their planes are orthogonal. A unitary g (a quaternionic matrix with $g^* g = 1$, an element of the compact symplectic group $\mathrm{Sp}(3)$) acts on $H_3(\mathbb{H})$ by $x \mapsto gxg^*$, which preserves the trace and the Jordan product and so preserves planes, product spaces and shares. That every frame comes from a basis follows from the quaternionic spectral theorem (Lemma 6.1). A right subspace of $\mathbb{H}^3$ is a subspace closed under right multiplication by quaternions; by Gram–Schmidt it has an orthonormal basis.

Lemma 6.1 (Spectral theorem)**.** (i) *Let $S \subset \mathbb{H}^3$ be a right subspace and $x \in H_3(\mathbb{H})$ with $xS \subset S$. Then S has an orthonormal basis of vectors u with $xu = u\lambda$, $\lambda \in \mathbb{R}$. For $S = \mathbb{H}^3$ this gives $x = \sum_a \lambda_a u_a u_a^*$ for an orthonormal basis (u_a).*

(ii) *Two commuting matrices $x, y \in H_3(\mathbb{H})$ have a common orthonormal basis of eigenvectors with real eigenvalues.*

(iii) *Every frame of $H_3(\mathbb{H})$ is $(u_a u_a^*)_a$ for an orthonormal basis (u_a) of $\mathbb{H}^3$.*

Formalized, different route. `QuaternionicMU B.spectralTheorem`: the formal proof takes a real eigenvector of x acting real-linearly on S and splits off its quaternionic line by induction, instead of citing the quaternionic spectral theorem.

Proof. (i) In an orthonormal basis of S the restriction of x is a quaternionic Hermitian matrix, and the spectral theorem for such matrices[93,94] gives the eigenbasis. For $S = \mathbb{H}^3$ the matrix g of columns u_a is unitary and $xg = g\,\mathrm{diag}(\lambda_a)$.

(ii) Eigenspaces of x for distinct real eigenvalues are orthogonal right subspaces, they span $\mathbb{H}^3$ by (i), and y preserves each of them. Apply (i) to y on each.

(iii) By (i) an idempotent of trace one is uu^* for a unit vector u, and two orthogonal such idempotents uu^*, vv^* have $|\langle u, v\rangle|^2 = \operatorname{tr}(uu^* \bullet vv^*) = 0$. □

[93] Rodman, *Topics in quaternion linear algebra*, note 3. Thm. 5.3.6, as cited in Macías-Virgós, Pereira-Sáez and Tarrío-Tobar, arXiv:2012.03621, §3.1.

[94] Zhang, "Quaternions and matrices of quaternions", note 3.

By Lemma 5.8(iii), the share of three pairwise unbiased frames (p_a), (p'_b), (p''_h) is

$$s = 6 \sum_{a,b,h} \left(\langle p_a \bullet p'_b, p''_h \rangle - \tfrac{1}{9} \right)^2. \tag{6.1}$$

6.2 The normal form of three unbiased bases

Fix unit imaginary quaternions i, j with $ij = -ji$, write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ with $\mathbb{C} = \mathbb{R} + \mathbb{R}i$, and put $\omega = e^{2\pi i/3}$ and $\mathsf{F} = \frac{1}{\sqrt{3}}(\omega^{mk})_{m,k=0}^{2}$, the Fourier matrix. Quaternions are written $a + bj$ with $a, b \in \mathbb{C}$. Section 10.3 writes them as $z_1 + jz_2$ instead; since $bj = j\bar{b}$, the two forms agree when $z_1 = a$ and $z_2 = \bar{b}$. A matrix whose columns are an orthonormal basis is a unitary matrix, and $g^*g = 1$ implies $gg^* = 1$ for square quaternionic matrices, as one sees in their complex 6×6 form. Multiplying a column on the right by a unit quaternion does not change its projector, and reordering the columns of a matrix does not change its frame. Call two ordered triples of bases *equivalent as ordered triples*, or simply equivalent, if a unitary matrix, together with such column phases (rephasing) and reorderings, carries each basis of one to the corresponding basis of the other. Equivalent triples have the same frames up to a unitary, hence the same share.

Lemma 6.2 (Normal form)**.** *Every ordered triple of pairwise unbiased bases of $\mathbb{H}^3$ is equivalent to the columns of*

$$1, \qquad \mathsf{F}, \qquad \operatorname{diag}(q_0, q_1, q_2)\,\mathsf{F} \tag{6.2}$$

for unit quaternions $q_m = a_m + b_m j$, $a_m, b_m \in \mathbb{C}$. Conversely, (6.2) *is an unbiased triple exactly when*

$$a_1\bar{a}_0 + a_0\bar{a}_2 + a_2\bar{a}_1 = 0, \qquad b_1\bar{b}_0 + b_0\bar{b}_2 + b_2\bar{b}_1 = 0. \tag{6.3}$$

Formalized, different route. `quaternionicNormalForm_withPermutations`, with `fourierBasis_entry` for F: the formal proof reaches the normal form by dephasing, with no reordering of columns, instead of through the theorem of Chterental and Đoković.

The normal form is derived from the proof of Theorem 5.4 of Chterental and Đoković, whose statement puts the first two bases in the form 1, F and describes the third, through the solutions of the unbiasedness condition, in six families[95]; condition (6.3) is the form of that condition used here.

Proof. Every quaternionic Hadamard matrix of order three, a matrix H with unit entries and $H^*H = 3 \cdot 1$, is equivalent to $\sqrt{3}\mathsf{F}$ under left and right multiplication by monomial unitary matrices (a permutation matrix times a diagonal unitary)[96]. Using this, the proof of Theorem 5.4 of Chterental and Đoković[97] shows that one unitary applied to all three matrices, together with a monomial unitary right factor on each, carries the triple to 1, F and $\operatorname{diag}(1, \alpha, \beta)\, X\mathsf{F}X^{-1}$ with $X = x1$ for unit quaternions α, β, x. Their notion of equivalence also allows the three bases to be permuted, but the proof does not permute them. A monomial right factor changes

[95] Chterental and Đoković, "On orthostochastic, unistochastic and qustochastic matrices", note 4. Thm. 5.4, pp. 24–25.

[96] Ibid. §4, p. 15.

[97] Ibid. pp. 24–25.

a basis by column phases and a reordering, so the result is equivalent to the given triple in the sense above. Multiplying the columns of the third matrix on the right by x gives $\operatorname{diag}(x, \alpha x, \beta x)\mathsf{F}$, the form (6.2).

For the condition, let g_l and h_k be the l-th column of F and the k-th column of $\operatorname{diag}(q)\mathsf{F}$. Since $jz = \bar{z}j$ for $z \in \mathbb{C}$,

$$3\langle g_l, h_k\rangle = \sum_m \omega^{-ml} q_m \omega^{mk} = \hat{a}(k-l) + \hat{b}(-k-l)\, j,$$

where

$$\hat{a}(t) = \sum_m a_m \omega^{mt}, \qquad \hat{b}(t) = \sum_m b_m \omega^{mt}.$$

So $9|\langle g_l, h_k\rangle|^2 = |\hat{a}(k-l)|^2 + |\hat{b}(-k-l)|^2$. The map $(l,k) \mapsto (k-l, -k-l)$ is a bijection of $\mathbb{F}_3^2$, its determinant being 2. The triple is unbiased exactly when $|\hat{a}(t)|^2 + |\hat{b}(t')|^2 = 3$ for all $t, t' \in \mathbb{F}_3$, that is, when $|\hat{a}|^2$ and $|\hat{b}|^2$ are constant with constants summing to 3. The sum is automatic, since $\sum_t |\hat{a}(t)|^2 + |\hat{b}(t)|^2 = 3\sum_m |q_m|^2 = 9$. Finally $|\hat{a}(t)|^2 = \sum_m |a_m|^2 + 2\operatorname{Re}\big(\gamma\, \omega^t\big)$ with $\gamma = a_1\bar{a}_0 + a_2\bar{a}_1 + a_0\bar{a}_2$, which is constant in t exactly when $\gamma = 0$. The same holds for $\hat{b}$. □

Put $x_m = |a_m|^2$, $y_m = |b_m|^2$ and $z_m = x_m - y_m \in [-1, 1]$, so that $x_m = \frac{1}{2}(1 + z_m)$ and $y_m = \frac{1}{2}(1 - z_m)$. The numbers z_m measure the balance between the $\mathbb{C}$-part and the $\mathbb{C}j$-part of q_m. Write

$$\sigma_1 = z_0 + z_1 + z_2, \qquad \sigma_2 = z_0 z_1 + z_1 z_2 + z_2 z_0, \qquad \sigma_3 = z_0 z_1 z_2$$

for the elementary symmetric functions of z.

Lemma 6.3 (The share in the normal form)**.** *For unit quaternions q_m satisfying* (6.3)*, the triple* (6.2) *has share*

$$s = \tfrac{1}{3} \sum_{m<n} \big(x_m x_n + y_m y_n\big) = \frac{3 + \sigma_2}{6}. \tag{6.4}$$

Formalized. `quaternionic NormalFormShare`.

Proof. In this proof the indices of the standard basis and of the columns of F are $r, l \in \mathbb{F}_3$, to keep them apart from the complex numbers a_m and b_m. Let e_r be the standard basis. For unit vectors u, v, w the definition of the trace gives $\operatorname{tr}\big((uu^* \bullet vv^*)ww^*\big) = \operatorname{Re}\big(\langle u, v\rangle\langle v, w\rangle\langle w, u\rangle\big)$: the two terms of the Jordan product give conjugate quaternions up to a cyclic reordering, which does not change the real part. With $\langle e_r, g_l\rangle = \omega^{rl}/\sqrt{3}$ and $\langle h_k, e_r\rangle = \omega^{-rk}\bar{q}_r/\sqrt{3}$, the entries of (6.1), with p_r the projector of e_r, are

$$\langle p_r \bullet p'_l, p''_k\rangle = \tfrac{1}{9} \sum_m \operatorname{Re}\big(\omega^{(r-m)l} q_m \omega^{(m-r)k} \bar{q}_r\big).$$

The term $m = r$ is $\frac{1}{9}$. For a complex number ζ and a quaternion ξ, $\operatorname{Re}(\zeta\xi) = \operatorname{Re}(\zeta\pi(\xi))$, where $\pi(\xi)$ is the $\mathbb{C}$-part of ξ. Put

$$\alpha_k(m, n) = \pi\big(q_m \omega^k \bar{q}_n\big) = a_m \bar{a}_n \omega^k + b_m \bar{b}_n \omega^{-k}.$$

The term $m = r + 1$ is $\frac{1}{9}\operatorname{Re}\big(\omega^{-l}\alpha_k(r+1, r)\big)$, and the term $m = r - 1$ is its analogue $\frac{1}{9}\operatorname{Re}\big(\omega^{-l}\alpha_k(r, r-1)\big)$, after conjugating inside the real part. By (6.3), the three numbers $\alpha_k(1,0)$, $\alpha_k(0,2)$, $\alpha_k(2,1)$ sum to zero. Hence

$$\langle p_r \bullet p'_l, p''_k\rangle - \tfrac{1}{9} = -\tfrac{1}{9}\operatorname{Re}\big(\omega^{-l}\alpha_k(r+2, r+1)\big).$$

As r runs over $\mathbb{F}_3$, the pair $(r+2, r+1)$ runs over the three cyclic pairs. Summing over l with $\sum_l \operatorname{Re}(\omega^{-l}\alpha)^2 = \frac{3}{2}|\alpha|^2$, then over k with $\sum_k |\alpha_k(m,n)|^2 = 3\big(x_m x_n + y_m y_n\big)$, (6.1) gives

$$s = 6 \cdot \tfrac{1}{81} \cdot \tfrac{3}{2} \sum_k \sum_{(m,n)} |\alpha_k(m,n)|^2 = \tfrac{1}{3} \sum_{m<n} \big(x_m x_n + y_m y_n\big).$$

Finally $x_m x_n + y_m y_n = \frac{1}{2}(1 + z_m z_n)$, and summing over the three pairs gives $(3 + \sigma_2)/6$. □

The function σ_2 is affine in each z_m separately, so its minimum over the cube $[-1,1]^3$ is attained at a vertex, where σ_2 is -1 or 3. Hence $s \geq \frac{1}{3}$ for every triple, the balance bound of the opening paragraph, which uses only (6.4) and not the two triangle conditions below.

6.3 The quaternionic lower bound

Theorem 6.4 (The quaternionic lower bound)**.** *Every three pairwise unbiased bases of $\mathbb{H}^3$ have share*

$$s \;\geq\; 1 - \frac{1}{\sqrt{3}}. \tag{6.5}$$

Equality holds for the triple (6.2) *with the quaternions q_m of* (6.8) *below. In every case of equality, for every normal form* (6.2) *of the triple, the numbers (z_0, z_1, z_2) are, up to order, $(\mu, 0, -\mu)$ with $\mu^2 = 2\sqrt{3} - 3$.*

Formalized, details in Lean. `quaternionicFloor`, with `attainmentPhases_formula` for (6.8); normal forms with reordered columns: `quaternionicFloor_equality_withPermutations`, which also gives the converse. Self-contained statement: `Challenge.Quaternionic.quaternionicLowerBound`.

If every q_m is complex, then $z_m = 1$, $\sigma_2 = 3$ and $s = 1$: the triple lies in $H_3(\mathbb{C})$, where every share is one (Theorem 5.10). Moving the q_m out of $\mathbb{C}$ lowers σ_2, and the triangle conditions keep it at least $3 - 2\sqrt{3}$.

Proof. For $\xi_0, \xi_1, \xi_2 \geq 0$ put

$$\Delta(\xi) = 4\sigma_1(\xi)\sigma_3(\xi) - \sigma_2(\xi)^2 = 2\xi_0\xi_1\xi_2(\xi_0 + \xi_1 + \xi_2) - (\xi_0^2\xi_1^2 + \xi_1^2\xi_2^2 + \xi_2^2\xi_0^2),$$

with $\sigma_k(\xi)$ the elementary symmetric functions of ξ. If $\ell_0, \ell_1, \ell_2 \geq 0$ are the side lengths of a triangle, possibly degenerate, Heron's formula gives $(\ell_0 + \ell_1 + \ell_2)(-\ell_0 + \ell_1 + \ell_2)(\ell_0 - \ell_1 + \ell_2)(\ell_0 + \ell_1 - \ell_2) = 2\sum \ell_m^2\ell_n^2 - \sum \ell_m^4 \geq 0$. The three terms of the first sum in (6.3) add up to zero, so their moduli $\sqrt{x_1x_0}$, $\sqrt{x_0x_2}$, $\sqrt{x_2x_1}$ are the sides of such a triangle, and the left side of Heron's identity, written in their squares, is $\Delta(x)$. With the second sum the same holds for y. Hence

$$\Delta_+ = \Delta\big(\tfrac{1+z_0}{2}, \tfrac{1+z_1}{2}, \tfrac{1+z_2}{2}\big) \geq 0, \qquad \Delta_- = \Delta\big(\tfrac{1-z_0}{2}, \tfrac{1-z_1}{2}, \tfrac{1-z_2}{2}\big) \geq 0.$$

Writing σ_k for $\sigma_k(z)$, the substitution $\xi_m = \frac{1}{2}(1 \pm z_m)$ and a direct expansion give

$$16\Delta_\pm = 4\sigma_1\sigma_3 - \sigma_2^2 + 6\sigma_2 + 3 \pm 4(\sigma_1 + 3\sigma_3). \tag{6.6}$$

Both are nonnegative, hence

$$\sigma_2^2 - 6\sigma_2 - 3 \;\le\; 4\big(\sigma_1\sigma_3 - |\sigma_1 + 3\sigma_3|\big) \;\le\; 0. \tag{6.7}$$

For the second inequality, there is nothing to prove if $\sigma_1\sigma_3 \le 0$. If $\sigma_1\sigma_3 > 0$, then σ_1 and σ_3 have the same sign, $|\sigma_1 + 3\sigma_3| = |\sigma_1| + 3|\sigma_3|$, and $\sigma_1\sigma_3 - |\sigma_1 + 3\sigma_3| = |\sigma_1|(|\sigma_3| - 1) - 3|\sigma_3| < 0$ because $0 < |\sigma_3| \le 1$. The roots of $\sigma_2^2 - 6\sigma_2 - 3$ are $3 \pm 2\sqrt{3}$, so $\sigma_2 \ge 3 - 2\sqrt{3}$, and Lemma 6.3 gives $s \ge (6 - 2\sqrt{3})/6 = 1 - 1/\sqrt{3}$.

If equality holds, then $\sigma_2^2 - 6\sigma_2 - 3 = 0$, so both inequalities in (6.7) are equalities. The argument above shows $\sigma_1\sigma_3 \le 0$, and then $\sigma_1\sigma_3 = 0$ and $\sigma_1 + 3\sigma_3 = 0$, so $\sigma_1 = \sigma_3 = 0$. One z_m vanishes, the other two are opposite, and $\sigma_2 = -\mu^2 = 3 - 2\sqrt{3}$.

For attainment let $\rho = 3^{1/4}$ and

$$A = \tfrac{1}{2}\Big(\rho + \tfrac{\rho^2 - 1}{\sqrt{2}}\Big), \qquad C = \tfrac{1}{2}\Big(\rho - \tfrac{\rho^2 - 1}{\sqrt{2}}\Big), \qquad B_0 = \tfrac{1}{\sqrt{2}}.$$

Then $A^2 + C^2 = \frac{1}{4}\big(\rho^4 + 1\big) = 1$ and $AC = \frac{1}{4}\big(\rho^2 - \frac{1}{2}(\rho^2 - 1)^2\big) = \frac{1}{2}(\sqrt{3} - 1) = B_0(A - C)$. Take

$$q_0 = A + Cj, \qquad q_1 = \omega^2 B_0(1 - j), \qquad q_2 = -\omega(C + Aj). \tag{6.8}$$

These are unit quaternions, with $a = (A, \omega^2 B_0, -\omega C)$ and $b = (C, -\omega^2 B_0, -\omega A)$. The two sums in (6.3) are $\omega^2(AB_0 - AC - CB_0)$ and $\omega^2(AB_0 - AC - B_0C)$, both zero, so Lemma 6.2 gives an unbiased triple. Its balances are $z = (A^2 - C^2, 0, C^2 - A^2)$, and $(A^2 - C^2)^2 = \rho^2(\rho^2 - 1)^2/2 = 2\sqrt{3} - 3$. So $\sigma_2 = 3 - 2\sqrt{3}$ and equality holds. □

The proof uses only that the triple is written in the form (6.2) with (6.3). So the description of the equality case holds for every such presentation of the ordered triple, and for every ordering of the three bases, since the share (6.1) does not depend on the order. Normal forms are not unique. Conjugating all three matrices by the scalar unitary j, an equivalence, fixes 1, maps F to $\bar{\mathsf{F}}$, whose columns are those of F in another order, and replaces $q_m = a_m + b_m j$ by $jq_m j^{-1} = \bar{a}_m + \bar{b}_m j$. This gives another normal form of the triple with the same numbers z_m. A normal form of the triple with its bases reordered is obtained afresh from Lemma 6.2, and the theorem applies to it as well.

Remark 6.5. The shares of unbiased triples in $\mathbb{H}^3$ fill the interval $[1 - 1/\sqrt{3}, 1]$. For $z = (\lambda, \lambda, \lambda)$ with $\lambda \in [0, 1]$ both triangles are equilateral and $s = (1 + \lambda^2)/2$. For $z = (\mu, 0, -\mu)$ both quantities $16\Delta_\pm$ equal $3 - 6\mu^2 - \mu^4$, which is nonnegative for $\mu^2 \le 2\sqrt{3} - 3$, and $s = (3 - \mu^2)/6$. In each case $\Delta_\pm \ge 0$, so the moduli of the three terms of each sum in (6.3) satisfy the triangle inequalities. When the three moduli are positive, phases of the a_m that make the first sum vanish exist: rotate a closed triangle with these side lengths, read as three complex numbers with sum zero, until the product of the three is positive, and take phases θ_m of the a_m whose differences $\theta_1 - \theta_0$, $\theta_0 - \theta_2$, $\theta_2 - \theta_1$ are the arguments of the three sides. The same holds for the b_m. All moduli are positive in both families except at the endpoint $\lambda = 1$. There $b_m = 0$, the second sum vanishes for every choice, the $q_m = a_m$ are complex, and the triple is a complex unbiased triple with share one.

6.4 At most five unbiased bases in $\mathbb{H}^3$

Corollary 6.6. *The algebra $H_3(\mathbb{H})$ contains no seven pairwise unbiased Jordan frames. Equivalently, $\mathbb{H}^3$ contains no seven pairwise unbiased orthonormal bases, and $H_3^0(\mathbb{H})$ contains no seven pairwise orthogonal planes: the cubic of $H_3^0(\mathbb{H})$ has no orthogonal decomposition into closed subspaces of dimension two.*

Formalized. `quaternionicNoSeven`.

Proof. Frames are the frames of orthonormal bases (Lemma 6.1(iii)). The planes of seven pairwise unbiased frames are pairwise orthogonal and fill the 14-dimensional space $H_3^0(\mathbb{H})$, so the sum rule (Theorem 5.9) applies. Planes are frame planes, orthogonal exactly when the frames are unbiased (Proposition 5.1), which gives the last two forms. For a fixed pair, the five other shares add up to 2, while Theorem 6.4 makes their sum at least $5 - 5/\sqrt{3}$, which exceeds 2 because $3\sqrt{3} > 5$. □

Without Theorem 6.4, the count of dimensions gives seven as the bound. Chterental and Đoković proved that $\mathbb{H}^n$ contains at most $2n + 1$ pairwise unbiased bases for $n \geq 2$, by the same count of orthogonal subspaces of the traceless Hermitian matrices[98]. The linear programming bound of Levenshtein, specialized to quaternionic projective space, gives the same number[99,100]. Whether the bound is attained was left open. Cohn, Kumar and Minton mark the quaternionic row of their table of tight codes, to which the corresponding code would belong, with a question mark[101], and a recent review describes a complete set of seven bases in $\mathbb{H}^3$ as expected but elusive[102]. Corollary 6.6 shows that it does not exist. Buckley and Waldron report that Hoggar gave five pairwise unbiased bases of $\mathbb{H}^2$ and nine of $\mathbb{H}^4$, so that the bound $2n + 1$ is attained for $n = 2$ and $n = 4$, and they determine reflection groups preserving the five bases of $\mathbb{H}^2$[103,104].

Proposition 6.7 (Completion)**.** *For any six pairwise unbiased Jordan frames of $H_3(\mathbb{H})$ there is a seventh frame unbiased to all six.*

Formalized, details in Lean. `quaternionicSixFrameCompletion`.

Proof. The six planes span a 12-dimensional subspace of $V = H_3^0(\mathbb{H})$. Let x, y be an orthonormal basis of its two-dimensional complement.

The proof shows that x and y commute, so that their span is a frame plane. Write $z^2 = z \bullet z$. For an orthonormal basis (f_α) of V the sum $\sum_\alpha f_\alpha^2 = \frac{14}{3}e$ does not depend on the basis (Example 3.5). The same contraction over the plane of a frame (p_a) can be taken with the centered frame elements $q_a = p_a - \frac{1}{3}e$ in place of

[98] Ibid. Thm. 5.1.

[99] V. I. Levenshtein, Bounds on the maximal cardinality of a code with bounded modulus of the inner product (in Russian), *Dokl. Akad. Nauk SSSR* **263**(6) (1982) 1303–1308. Math-Net.Ru dan45246. Eqs. (12)–(13), (16).

[100] P. Boyvalenkov and D. Danev, On linear programming bounds for codes in polynomial metric spaces (in Russian), *Problemy Peredachi Informatsii* **34**(2) (1998) 16–31. Math-Net.Ru ppi400. Thm. 2.1.

[101] Cohn, Kumar and Minton, "Optimal simplices and codes", note 6. Table 2.3, p. 14.

[102] McNulty and Weigert, "Mutually unbiased bases in composite dimensions", note 22. §9.3, pp. 65–66.

[103] Z. Buckley and S. Waldron, Quaternionic MUBs in $\mathbb{H}^2$ and their reflection symmetries. arXiv:2509.01859, 2025. §1, p. 2, and §§2–4.

[104] S. G. Hoggar, t-designs in projective spaces, *European J. Combin.* **3** (1982) 233–254.

an orthonormal basis, since $w = \sum_a \langle q_a, w\rangle\, q_a$ for w in the plane (Section 5), and it equals $\sum_a q_a^2 = \sum_a \left(\frac{1}{3}p_a + \frac{1}{9}e\right) = \frac{2}{3}e$. Subtracting the six frame planes gives

$$x^2 + y^2 = \tfrac{2}{3}e. \tag{6.9}$$

By Example 3.5, $\operatorname{tr} z^4 = \frac{1}{2}(\operatorname{tr} z^2)^2$ on V. The coefficient of s^2 at $z = x + sy$, together with the trace of the square of (6.9) and $\operatorname{tr} x^4 = \operatorname{tr} y^4 = \frac{1}{2}$, gives

$$\operatorname{tr}(x^2y^2) = \operatorname{tr}(xyxy) = \tfrac{1}{6}. \tag{6.10}$$

The commutator $[x, y]$ is skew-Hermitian, so the sum of the squared norms of its entries is $-\operatorname{tr}([x, y]^2) = 2\operatorname{tr}(x^2y^2) - 2\operatorname{tr}(xyxy) = 0$, and $xy = yx$. By Lemma 6.1(ii) there is an orthonormal basis in which x and y are both real diagonal. Their span is the plane of the frame of that basis, and it is orthogonal to the six given planes, so by Proposition 5.1 the new frame is unbiased to all six. □

Proposition 6.8 (Completion of twelve)**.** *For any twelve pairwise unbiased Jordan frames of $H_3(\mathbb{O})$ there is exactly one frame unbiased to all twelve. Equivalently, for any twelve pairwise orthogonal planes of $H_3^0(\mathbb{O})$ there is exactly one plane orthogonal to all twelve.*

Formalized, details in Lean. `completion O_frames`, `completion O_uniqueClosedPlane`. Self-contained statement: `Challenge.albert Completion`.

Proof. Let $V = H_3^0(\mathbb{O})$. The twelve planes span a 24-dimensional subspace of the 26-dimensional space V. Let x, y be an orthonormal basis of its two-dimensional complement.

Write $z^2 = z \bullet z$. The first two steps of the proof of Proposition 6.7 hold in $H_3(\mathbb{O})$ with the trace form in place of tr, by (J2) and Example 3.5: here $\sum_\alpha f_\alpha^2 = \frac{26}{3}e$, so

$$x \bullet x + y \bullet y = \tfrac{26}{3}e - 12 \cdot \tfrac{2}{3}e = \tfrac{2}{3}e, \tag{6.11}$$

and $\langle z^2, z^2\rangle = \frac{1}{2}\langle z, z\rangle^2$ on V, so the same expansion at $z = x + sy$ gives

$$\langle x^2, y^2\rangle = \langle x \bullet y, x \bullet y\rangle = \tfrac{1}{6}. \tag{6.12}$$

Next x and y are moved into a copy of $H_3(\mathbb{C})$. By (J2), $x = \sum_a \lambda_a p_a$ for a frame (p_a), and by (J3) an automorphism ψ carries p_a to E_{aa}. Let $\eta_{12}, \eta_{23}, \eta_{13}$ be the off-diagonal entries of $\psi(y)$, and let ξ and ξ' be norm-one octonions with $\eta_{12} = |\eta_{12}|\xi$ and $\eta_{23} = |\eta_{23}|\xi'$ (any norm-one octonion if the entry is zero). Lemma 5.4(ii) gives an automorphism χ that fixes each E_{aa} and sends $\xi E_{12} + \bar{\xi}E_{21}$ to $E_{12} + E_{21}$ and $\xi' E_{23} + \bar{\xi}' E_{32}$ to $E_{23} + E_{32}$; it preserves each Jordan Peirce space $\{w : E_{ii} \bullet w = E_{jj} \bullet w = \frac{1}{2}w\}$ of off-diagonal positions $(i, j), (j, i)$. So $\kappa = \chi\psi$ sends x to the real diagonal matrix $\sum_a \lambda_a E_{aa}$, and y to a matrix with real entries at $(1, 2)$ and $(2, 3)$ and some octonion ζ at $(1, 3)$. Choose a unit imaginary u with $\zeta \in \mathbb{C}_u = \mathbb{R} + \mathbb{R}u$. Since $\mathbb{C}_u$ is an associative subalgebra closed under conjugation, $H_3(\mathbb{C}_u)$ is a subalgebra of $H_3(\mathbb{O})$ on which $\bullet$ is the symmetrized matrix product and tr = Re Tr; with $u \mapsto i$ it is $H_3(\mathbb{C})$. It contains $X = \kappa(x)$ and $Y = \kappa(y)$.

The automorphism κ preserves the trace[105], hence the trace form, and (6.12) holds for X, Y. With ordinary matrix multiplication in $M_3(\mathbb{C}_u)$ this is (6.10), since

[105] Yokota, "Exceptional Lie groups", note 44. Lemma 2.2.1(2).

$\langle X^2, Y^2\rangle = \operatorname{tr}(X^2Y^2)$ and $\langle X \bullet Y, X \bullet Y\rangle = \frac{1}{2}\operatorname{tr}(XYXY) + \frac{1}{2}\operatorname{tr}(X^2Y^2)$, so $XY = YX$ as in the proof of Proposition 6.7. A common orthonormal eigenbasis (v_a) of $\mathbb{C}^3_u$ gives a frame $P_a = v_a v_a^*$ of $H_3(\mathbb{C}_u)$, hence of $H_3(\mathbb{O})$, whose plane contains, and so is spanned by, X and Y. The frame $\kappa^{-1}(P_a)$ has plane spanned by x, y, which is orthogonal to the twelve given planes, so by Proposition 5.1 it is unbiased to all twelve frames.

The plane of any frame unbiased to the twelve is orthogonal to their planes (Proposition 5.1), so it is the span of x, y, and the plane determines the frame. This gives uniqueness. □

Thirteen orthogonal planes fill the 26-dimensional space $H_3^0(\mathbb{O})$, and twelve always extend by Proposition 6.8, so a maximal (unextendible) set of pairwise unbiased frames of $H_3(\mathbb{O})$, one to which no further unbiased frame can be added, has at most eleven or exactly thirteen members.

The two completions are analogues of Weiner's theorem that n pairwise unbiased bases of $\mathbb{C}^n$ extend to $n + 1$[106]. The arguments are different and use the quartic trace identity of traceless 3×3 matrices.

Corollary 6.9. *Every set of pairwise unbiased bases of $\mathbb{H}^3$ has at most five members.*

Formalized. `QuaternionicMUB.quaternionicFiveBound`. Self-contained statement: `Challenge.Quaternionic.quaternionicFiveBound`.

Proof. Seven are excluded by Corollary 6.6. Six would extend to seven by Proposition 6.7. □

In particular $H_3(\mathbb{H})$ contains no six pairwise unbiased frames, so in retrospect Proposition 6.7 holds vacuously. Its role is the step from Corollary 6.6 to Corollary 6.9.

Sets of four exist. The four unbiased bases of $\mathbb{C}^3$ are unbiased in $\mathbb{H}^3$, and Chterental and Đoković give families of four that admit no fifth basis[107]. So the largest size is four or five.

Whether five unbiased bases exist in $\mathbb{H}^3$ is open. Neither argument excludes five. With five bases the planes span only 10 of the 14 dimensions, the three other shares of a pair total at most 2, the rest of the product space lying in the four-dimensional complement, and Theorem 6.4 gives them at least $3 - \sqrt{3} \approx 1.27$. Chterental and Đoković asked for all maximal sets of unbiased bases in $\mathbb{H}^3$[108], and McNulty and Weigert report no construction of more than four[109].

7 Thirteen planes in the Albert algebra

Over the octonions the eleven shares of a pair need only average $\frac{2}{11}$, and a share may vanish. This section builds thirteen planes from a grading of the complexified Albert algebra by $\mathbb{F}_3^3$ and writes the cubic in their coordinates; Section 9 returns to these

[106] M. Weiner, A gap for the maximum number of mutually unbiased bases, *Proc. Amer. Math. Soc.* **141** (2013) 1963–1969. doi:10.1090/S0002-9939-2013-11487-5. arXiv:0902.0635. Cor. 3.4.

[107] Chterental and Đoković, "On orthostochastic, unistochastic and qustochastic matrices", note 4. Exs. 5.5, 5.6.

[108] Ibid. Problem 6.3.

[109] McNulty and Weigert, "Mutually unbiased bases in composite dimensions", note 22. §9.3.

systems after the proof of Theorem A. The existence of thirteen pairwise unbiased frames in $H_3(\mathbb{O})$ is due to Cohn, Kumar and Minton, who call their configuration "a system of 13 mutually unbiased bases"[110]. Here existence comes from Tits's second construction of the Albert algebra in the coordinates of Lemma 7.5, whose explicit formulas can be skipped on first reading.

Throughout, $J = H_3(\mathbb{O})$ with Jordan product $x \bullet y$, unit e, trace tr and $\langle x, y\rangle = \operatorname{tr}(x \bullet y)$, and $V = H_3^0(\mathbb{O})$ with the cubic product $x \circ y = x \bullet y - \frac{1}{3}\langle x, y\rangle e$, as in Example 3.5. Let $J_{\mathbb{C}} = J \otimes \mathbb{C}$, extend $\bullet$ and $\langle\ ,\ \rangle$ complex-bilinearly, and write $x \mapsto \overline{x}$ for the conjugation fixing J. The form $\langle x, \overline{y}\rangle$ is Hermitian and positive definite. A *thirteen-plane system* is a set of thirteen pairwise unbiased frames of J together with their planes. By Proposition 5.1 the planes are closed and pairwise orthogonal, and every set of thirteen pairwise orthogonal planes arises in this way. A *quartet* of a thirteen-plane system is a set of four of its planes whose sum is closed. Put $\omega = e^{2\pi i/3}$ and $\mu_3 = \{1, \omega, \omega^2\}$.

7.1 Clock elements of frames

The complex coordinate on the plane of a frame is the discrete Fourier transform of its three idempotents. For an ordered frame $f = (p_{f0}, p_{f1}, p_{f2})$ define its *clock element*

$$u_f = \sum_{a \in \mathbb{F}_3} \omega^a p_{fa} \in J_{\mathbb{C}}.$$

For the diagonal frame of $H_3(\mathbb{C})$ it is the clock matrix Z of the Weyl–Heisenberg pair introduced below.

Lemma 7.1. (i) $u_f \bullet u_f = \overline{u}_f$, $u_f \bullet \overline{u}_f = e$, $\langle u_f, u_f\rangle = 0$, $\langle u_f, \overline{u}_f\rangle = 3$, *and* $p_{fa} = \frac{1}{3}\big(e + \omega^{-a} u_f + \omega^a \overline{u}_f\big)$. *The complexified plane of f is spanned by u_f and $\overline{u}_f$. A permutation of the outcomes replaces u_f by ηu_f or $\eta \overline{u}_f$ with $\eta \in \mu_3$.*

Formalized. `frame Contrast_native.`

(ii) *Two frames f, g are unbiased exactly when $\langle u_f, u_g\rangle = \langle u_f, \overline{u}_g\rangle = 0$.*

Proof. (i) The relations $p_{fa} \bullet p_{fb} = \delta_{ab} p_{fa}$ give $u_f \bullet u_f = \sum \omega^{2a} p_{fa} = \overline{u}_f$ and $u_f \bullet \overline{u}_f = \sum p_{fa} = e$, and $\operatorname{tr} p_{fa} = 1$ gives the two inner products. Fourier inversion gives p_{fa}. Every permutation of $\mathbb{F}_3$ is $a \mapsto \pm a + b$, which multiplies u_f by an element of μ_3 or turns it into $\eta \overline{u}_f$ with $\eta \in \mu_3$.

(ii) By (i), $9\langle p_{fa}, p_{gb}\rangle = 3 + \sum \omega^{-\epsilon a - \epsilon' b} \langle u_f^{\epsilon}, u_g^{\epsilon'}\rangle$, the sum over $\epsilon, \epsilon' = \pm 1$, with $u^{+1} = u$ and $u^{-1} = \overline{u}$. This equals 3 for all a, b exactly when the four coefficients vanish, since the four characters $(a, b) \mapsto \omega^{-\epsilon a - \epsilon' b}$ of $\mathbb{F}_3^2$ are linearly independent. The coefficients for $(-\epsilon, -\epsilon')$ are the conjugates of those for (ϵ, ϵ'). □

On $\mathbb{C}^3$ let $\mathsf{X} e_m = e_{m+1}$ (shift) and $\mathsf{Z} e_m = \omega^m e_m$ (clock), indices modulo 3, and for $v = (a, b) \in \mathbb{F}_3^2$ put $D_v = \omega^{-ab} \mathsf{X}^a \mathsf{Z}^b$ (the displacement operators, or generalized Pauli matrices). From $\mathsf{Z}\mathsf{X} = \omega \mathsf{X}\mathsf{Z}$,

$$D_v D_w = \omega^{[v,w]} D_{v+w}, \qquad [v, w] = v_1 w_2 - v_2 w_1. \tag{7.1}$$

[110] Cohn, Kumar and Minton, "Optimal simplices and codes", note 6. §8.2, Thm. 8.3 and p. 1352.

Hence $D_v^3 = 1$, $D_v^* = D_{-v}$ and $\operatorname{tr} D_v = 0$ for $v \neq 0$, so the eigenvalues of D_v are $1, \omega, \omega^2$ and D_v is the clock element of its eigenframe, labelled by the exponent of the eigenvalue. The matrices D_v and $D_{-v} = D_v^2$ have the same eigenframe, so the eight matrices D_v, $v \neq 0$, give four frames, one for each point of the projective line over $\mathbb{F}_3$. They are pairwise unbiased by Lemma 7.1(ii), since $\operatorname{tr}(D_v D_w) = 3\delta_{v+w,0}$; the eigenframes of Z and X are the diagonal and the Fourier frame. In the complexification $M_3(\mathbb{C})$ of $H_3(\mathbb{C})$, with the product $\frac{1}{2}(xy + yx)$ and the conjugation $x \mapsto x^*$, (7.1) gives

$$D_v \bullet D_w = D_{v+w} \ \ (v, w \text{ dependent}), \qquad D_v \bullet D_w = -\tfrac{1}{2} D_{v+w} \ \ (v, w \text{ independent}), \tag{7.2}$$

because $\frac{1}{2}(\omega^t + \omega^{-t}) = -\frac{1}{2}$ for $t \neq 0$. These four bases are the standard unbiased bases of $\mathbb{C}^3$, also used by Cohn, Kumar and Minton[111].

Lemma 7.2 (Four unbiased frames in a copy of $H_3(\mathbb{C})$)**.** *Let $J' \subset J$ be a unital subalgebra isomorphic to $H_3(\mathbb{C})$ that contains four pairwise unbiased frames. Their clock elements u_f and conjugate clock elements $\overline{u}_f$ can be labelled u_v, $v \in \mathbb{F}_3^2 \setminus \{0\}$, with $u_{-v} = \overline{u}_v$, and with $u_0 = e$, so that $u_v \bullet u_w$ is a nonzero multiple of u_{v+w} for all v, w: the multiple is 1 if v, w are dependent and lies in $-\frac{1}{2}\mu_3$ if they are independent.*

Formalized. `qutrit ContrastTable_from NativeFrames`.

Proof. Three nonzero orthogonal idempotents of J with sum e each have trace 1, since by (J2) the trace of a nonzero idempotent is a positive integer and the three traces sum to 3. So the trace of J restricted to J' is the trace of J', and an isomorphism $J' \to H_3(\mathbb{C})$ preserves traces, frames and unbiasedness. By Theorem 5.10(iii) it may be chosen to carry the four frames to the eigenframes of the D_v. Its complex extension carries clock elements to clock elements. By Lemma 7.1(i) each clock element or conjugate clock element of the four frames is ηD_v for some $v \neq 0$ and $\eta \in \mu_3$. Label it u_v. Then (7.2) gives the products. □

7.2 The graded system

Call a subgroup E of $\operatorname{Aut}(J)$ a *ternary grading group* if $E \cong (\mathbb{Z}/3)^3$ and each of its 27 characters has a one-dimensional eigenspace on $J_{\mathbb{C}}$. Over $\mathbb{C}$ this is the grading of the Albert algebra by $\mathbb{F}_3^3$ with one-dimensional components, called fine and nontoral in the grading literature, constructed by Griess from a Moufang loop[112]and described by Draper and Martín González and by Elduque and Kochetov[113,114]; in the literature on orthogonal decompositions the image of E in the complex group

[111] Ibid. §8.2.

[112] Griess, "A Moufang loop, the exceptional Jordan algebra, and a cubic form in 27 variables", note 9. (4.3)–(4.6).

[113] C. Draper Fontanals and C. Martín González, Gradings on the Albert algebra and on $\mathfrak{f}_4$, *Rev. Mat. Iberoam.* **25** (2009) 841–908. doi:10.4171/RMI/585. arXiv:math/0703840. §3.4.

[114] Elduque and Kochetov, "Weyl groups of fine gradings", note 28. §4.1; §4.5, Thm. 4.8.

$F_4(\mathbb{C})$ is the Jordan subgroup[115,116,117].

Fix a ternary grading group E. Identify its character group with $\mathbb{F}_3^3$, and let $A_g \subset J_{\mathbb{C}}$ be the character space of g (the homogeneous component of the grading), the one-dimensional subspace on which E acts by the character g. A point $p = \{g, -g\}$ of the projective plane PG(2, 3) is a pair of opposite nonzero vectors. The two-dimensional subspaces of $\mathbb{F}_3^3$ give the thirteen projective lines, each with four points, and three distinct points are collinear when they lie on a projective line.

The proofs below use the polarized identity (3.7) of Lemma 3.8 with the eikonal constant $c = \frac{1}{6}$ of Example 3.5, which for $\mathbb{O}$ rests on the quoted fact (J2). It holds on the complexification $V_{\mathbb{C}}$ because both sides are polynomial.

Proposition 7.3 (The graded system)**.** *Let E be a ternary grading group with character spaces A_g, $g \in \mathbb{F}_3^3$.*

Formalized. `nativeGradedSystem`.

(i) *$A_0 = \mathbb{C}e$, $\overline{A_g} = A_{-g}$, $A_g \bullet A_h \subset A_{g+h}$, and $\langle A_g, A_h\rangle = 0$ unless $g + h = 0$.*

(ii) *Each A_g with $g \neq 0$ contains an element u_g with $u_g \bullet u_g = \overline{u}_g$ and $u_g \bullet \overline{u}_g = e$, unique up to a factor in μ_3. Put $u_{-g} = \overline{u}_g$. Then $p_{g,a} = \frac{1}{3}(e + \omega^{-a}u_g + \omega^a u_{-g})$, $a \in \mathbb{F}_3$, is a frame of J with clock element u_g, and its plane is $U_p = J \cap (A_g \oplus A_{-g})$ for $p = \{g, -g\}$.*

(iii) *The thirteen frames $(p_{g,a})_a$, one for each point, are pairwise unbiased, and their planes U_p form an orthogonal decomposition of V into closed subspaces.*

(iv) *For distinct points p, q the product space is $U_p \circ U_q = U_r \oplus U_{r'}$, where r, r' are the other two points of the projective line through p and q. So $s_{pqr} = 1$ when p, q, r are collinear and $s_{pqr} = 0$ otherwise.*

(v) *For each projective line ℓ, $J_\ell = \mathbb{R}e \oplus \bigoplus_{p\in\ell} U_p$ is a unital subalgebra isomorphic to $H_3(\mathbb{C})$, containing the four frames of ℓ. So every pair of planes lies in a quartet whose sum J_ℓ^0 is closed, an octet.*

Proof. (i) The unit is fixed by E, and A_0 is one-dimensional, so $A_0 = \mathbb{C}e$. Elements of E commute with the conjugation, so $x \in A_g$ gives $\overline{x} \in A_{-g}$. Since E acts by automorphisms preserving $\langle\ ,\ \rangle$, for $x \in A_g$, $y \in A_h$ and every $s \in E$, writing characters additively, $s(x \bullet y) = (g + h)(s)\, x \bullet y$ and $\langle x, y\rangle = (g + h)(s)\langle x, y\rangle$, which gives the last two assertions.

(ii) Let $0 \neq x \in A_g$. Then $x \bullet \overline{x} = \lambda e$ with $3\lambda = \langle x, \overline{x}\rangle > 0$, and $x \bullet x \in A_{2g} = A_{-g}$, so $x \bullet x = \kappa\overline{x}$. Since $\operatorname{tr} x = \langle x, e\rangle = 0$, the elements $x, \overline{x}$ lie in $V_{\mathbb{C}}$, and $\langle x, x\rangle = 0$ gives $x \circ x = \kappa\overline{x}$ and $x \circ \overline{x} = \lambda e - \frac{1}{3}\langle x, \overline{x}\rangle e = 0$. So (3.7) for $(x, x, \overline{x}, \overline{x})$ reads

[115] A. V. Alekseevskii, Finite commutative Jordan subgroups of complex simple Lie groups, *Funct. Anal. Appl.* **8** (1974) 277–279 (*Funkts. Anal. Prilozh.* **8** (1974) no. 4, 1–4). doi:10.1007/BF01075483.

[116] R. L. Griess, Jr., Elementary abelian p-subgroups of algebraic groups, *Geom. Dedicata* **39** (1991) 253–305. doi:10.1007/BF00150757. Thm. (7.4), p. 278.

[117] J. An, H. Dietrich and A. J. Litterick, Elementary abelian subgroups: from algebraic groups to finite groups, *Trans. Amer. Math. Soc.* **377** (2024) 8335–8380. doi:10.1090/tran/9140. arXiv:2303.02364; section numbers as in arXiv version 2. Prop. 6.1.

$3\lambda|\kappa|^2 = \frac{1}{6}\cdot 2(3\lambda)^2 = 3\lambda^2$, so $|\kappa|^2 = \lambda$ and $\kappa \neq 0$. For $u = tx$ the two conditions read $t^2\kappa = \bar{t}$ and $|t|^2\lambda = 1$. Both hold exactly when $t^3 = 1/(\lambda\kappa)$, since this equation forces $|t|^6 = \lambda^{-3}$, and then $t^3\kappa = \lambda^{-1} = |t|^2$. Its three solutions differ by factors in μ_3. Then $u_{-g} \bullet u_{-g} = \overline{u_g \bullet u_g} = u_g$, so e, u_g, u_{-g} multiply as the group algebra of the cyclic group of order three, which is associative. In it the $p_{g,a}$ are the primitive idempotents, so they are pairwise orthogonal idempotents with sum e. They are fixed by the conjugation, so they lie in J, and $\operatorname{tr} p_{g,a} = \frac{1}{3}\operatorname{tr} e = 1$. Their centered frame elements span $J \cap (A_g \oplus A_{-g})$.

(iii) For distinct points with representatives g, h, the sums $\pm g \pm h$ are nonzero, so the planes are orthogonal by (i), and the frames are unbiased by Proposition 5.1. Thirteen orthogonal planes fill the 26-dimensional space V.

(iv) Let $p = \{\pm g\}$ and $q = \{\pm h\}$. By (i) the complexified product space lies in $A_{g+h} \oplus A_{-g-h} \oplus A_{g-h} \oplus A_{h-g}$, whose real part is $U_r \oplus U_{r'}$ with $r = \{\pm(g+h)\}$ and $r' = \{\pm(g-h)\}$, the other two points of the projective line spanned by g, h. Both spaces have dimension four, the product space by Lemma 4.10 applied to the cubic $\sqrt{6}\,d$, which has eikonal constant 1. So they are equal. Theorem 5.9 gives the shares.

(v) If $\hat{\ell} \subset \mathbb{F}_3^3$ is the two-dimensional subspace of the projective line ℓ, then $\bigoplus_{g\in\hat{\ell}} A_g$ is closed under $\bullet$ and conjugation by (i), so its real part J_ℓ is a nine-dimensional unital subalgebra. It contains the subalgebra generated by the frames of two points of ℓ, a copy of $H_3(\mathbb{C})$ of dimension nine (Proposition 5.5), so the two are equal. Its traceless part J_ℓ^0 is closed, since $x \circ y$ is the traceless part of $x \bullet y \in J_\ell$. □

The thirteen frames of (iii) with their planes form the *graded system* of E. It has share one on each of the 52 collinear triples and share zero on the other 234. The elements e and u_g are a joint eigenbasis of E on $J_{\mathbb{C}}$ (compare the basis indexed by $\mathbb{F}_3^3$ in[118], and the conjugate-pair real structure on the nonzero grades in[119]). By Proposition 7.3(ii) the group E permutes the three idempotents of each frame, and each of the thirteen planes is the real fixed subspace of $A_g \oplus A_{-g}$ for a pair of opposite nonzero characters.

A ternary grading group can be written down in a real form of Tits's first construction of the Albert algebra from $M_3(\mathbb{C})$. In this model the symmetries needed below are operations on associative complex 3×3 matrices: conjugation of all three entries of a triple by a unitary, multiplication of the second entry on the right by a special unitary and of the third on the left by its inverse, and the $\mathbb{Z}/3$-grading by position. Three of them generate the ternary grading group (Lemma 7.5). On triples $x = (a_0, a_1, a_2)$ of complex 3×3 matrices put $e = (1,0,0)$, $T(x) = \operatorname{tr} a_0$, $T(x,y) = \operatorname{tr}(a_0b_0) + \operatorname{tr}(a_1b_2) + \operatorname{tr}(a_2b_1)$, and $a\#b = ab + ba - (\operatorname{tr} a)b - (\operatorname{tr} b)a + \big(\operatorname{tr} a \operatorname{tr} b - \operatorname{tr}(ab)\big)1$,

[118] Kostrikin and Tiep, *Orthogonal decompositions*, note 11. §14.1B–C, pp. 473–475.
[119] Ibid. §14.1F, p. 478.

so that $\frac{1}{2}a\#a$ is the adjugate of a. The product $x \bullet y = (c_0, c_1, c_2)$ is

$$\begin{aligned} c_0 &= \tfrac{1}{2}(a_0b_0 + b_0a_0) - \tfrac{1}{2}(a_1b_2 + b_1a_2) + \tfrac{1}{2}\operatorname{tr}(a_1b_2 + b_1a_2)\,1, \\ c_1 &= \tfrac{1}{2}a_2\#b_2 - \tfrac{1}{2}(a_0b_1 + b_0a_1) + \tfrac{1}{2}(\operatorname{tr} a_0)b_1 + \tfrac{1}{2}(\operatorname{tr} b_0)a_1, \\ c_2 &= \tfrac{1}{2}a_1\#b_1 - \tfrac{1}{2}(a_2b_0 + b_2a_0) + \tfrac{1}{2}(\operatorname{tr} a_0)b_2 + \tfrac{1}{2}(\operatorname{tr} b_0)a_2. \end{aligned} \tag{7.3}$$

Let $\sigma(a_0, a_1, a_2) = (a_0^*, a_2^*, a_1^*)$, and let J_T be the real algebra of fixed points (A, B, B^*), A Hermitian, with the form $T(x, y) = \operatorname{tr} AA' + 2\operatorname{Re}\operatorname{tr}(BB'^*)$, which is positive definite. On the complexification $M_3(\mathbb{C})^3$ of J_T, the conjugation $x \mapsto \overline{x}$ fixing J_T is σ.

Theorem 7.4 (Tits; quoted)**.** *J_T is a Jordan algebra isomorphic to $H_3(\mathbb{O})$, by an isomorphism carrying the trace $T(x)$ to the trace, and hence the trace form $T(x, y)$ to $\langle\ ,\ \rangle$.*

Formalized. `nativeCompact TitsEquiv_product`, `nativeCompactTits Equiv_one`, `nativeCompact TitsEquiv_trace`, for an isometry of $H_3(\mathbb{O})$ onto J_T.

Over $\mathbb{C}$, (7.3) is Tits's first construction of the Albert algebra from $M_3(\mathbb{C})$[120,121,122], and J_T is Tits's second construction for $M_3(\mathbb{C})$ with conjugate transposition. For $x = (A, B, B^*) \in J_T$, (7.3) gives

$$x \bullet x = \big(A^2 - BB^* + \operatorname{tr}(BB^*)1,\ ((\operatorname{tr} A)1 - A)B + \operatorname{adj}(B)^*,\ *\big),$$

the third entry being the adjoint of the second. An earlier AI-generated paper of the author writes $H_3(\mathbb{O})$ in coordinates $x \mapsto (a, B) \in H_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ in which the unit is $(1, 0)$, the trace is $\operatorname{tr} a$, the trace form is T, and the square has the same two entries[123]; Yokota writes out the same decomposition[124]. A commutative product is determined by its squares, so $(A, B, B^*) \mapsto (A, B)$ identifies J_T with $H_3(\mathbb{O})$.

A ternary grading group of J_T is defined as for $H_3(\mathbb{O})$, with automorphisms the real-linear bijections preserving (7.3). Through Theorem 7.4 such a map preserves T, since every automorphism of $H_3(\mathbb{O})$ preserves the trace[125], and hence the form $T(x, y) = T(x \bullet y)$. Write $\iota_n(a)$ for the triple with a in position n and zeros elsewhere.

Lemma 7.5. (i) *For a unitary Y, the map $\mathrm{Ad}_Y(a_0, a_1, a_2) = (Ya_0Y^*, Ya_1Y^*, Ya_2Y^*)$; the map $\tau_\omega(a_0, a_1, a_2) = (a_0, \omega a_1, \omega^2 a_2)$; and, for $R \in \mathrm{SU}(3)$, the map $\psi_R(a_0, a_1, a_2) = (a_0, a_1R, R^*a_2)$ preserve e, $T(\ ,\ )$ and the product* (7.3)*, and commute with σ. They restrict to automorphisms of J_T.*

Formalized, details in Lean. `titsLemma`.

(ii) *For $g = (v, n) \in \mathbb{F}_2^2 \times \mathbb{F}_3 = \mathbb{F}_3^3$ put $u_g = \iota_n(D_v)$, so $u_0 = e$. Then $\sigma u_g = u_{-g}$, $T(u_g, u_h) = 3\delta_{g+h,0}$, $u_g \bullet u_h = u_{g+h}$ when g, h are dependent, and*

$$u_g \bullet u_h = -\tfrac{1}{2}\,\omega^{[v,w](n'-n)}\,u_{g+h} \qquad \text{for independent } g = (v, n),\ h = (w, n').$$

[120] T. Moran and S. Pumplün, A generalization of the first Tits construction, arXiv:2403.16342, 2024. §1.4; §2.1, Thms. 3(i), 4(i).

[121] N. Jacobson, *Structure and Representations of Jordan Algebras*, Amer. Math. Soc. Colloq. Publ. 39, Providence, 1968. doi:10.1090/coll/039. p. 412, as cited in A. Elduque and M. Kochetov, Gradings on the exceptional Lie algebras F_4 and G_2 revisited, *Rev. Mat. Iberoam.* **28** (2012). arXiv:1009.1218; numbering as in arXiv version 3, Rem. 6.5.

[122] J. C. Baez and S. Pumplün, Three-dimensional geometry in exceptional mathematics. arXiv:2609.07538, 2026. §5.

[123] Earlier AI-generated work, "On the local geometry of the tight 39-point code", note 25. §4.2, Eqs. (10)–(11), proved in App. A.1.

[124] Yokota, "Exceptional Lie groups", note 44. §2.12.

[125] Ibid. Lemma 2.2.1(2).

(iii) *The maps* Ad_{Z}, $\mathrm{Ad}_{\mathsf{X}^{-1}}$ *and* τ_ω *generate a ternary grading group of* J_T *whose character spaces are the one-dimensional subspaces* $\mathbb{C}u_g$.

Proof. (i) These are the standard automorphisms of Tits's first construction, and each identity is read off from (7.3). For Ad_Y and τ_ω every term of c_n is conjugated by Y, respectively multiplied by ω^n. For ψ_R the only term that is not visibly equivariant is $a\#b$, and $(R^*a)\#(R^*b) = (a\#b)R$ because $\mathrm{adj}(R^*a) = \mathrm{adj}(a)\,\mathrm{adj}(R^*) = \mathrm{adj}(a)R$ for $R \in \mathrm{SU}(3)$. The commutation with σ and the invariance of $T(\ ,\ )$ are direct.

(ii) The first two assertions follow from $D_v^* = D_{-v}$ and $\mathrm{tr}(D_vD_w) = 3\delta_{v+w,0}$. For the products, put $a = D_v$ in position n and $b = D_w$ in position n'; in each case one term of (7.3) survives. For $n = n' = 0$ it is $D_v \bullet D_w$, given by (7.2). For $n = 0$, $n' = 1$ it is $c_1 = \frac{1}{2}(\mathrm{tr}\,D_v)D_w - \frac{1}{2}D_vD_w$, which is D_w if $v = 0$ and $-\frac{1}{2}\omega^{[v,w]}D_{v+w}$ if $v \neq 0$; for $n' = 2$ the product is D_wD_v and the exponent $-[v,w] = 2[v,w]$. For $n = n' = 1$ it is $c_2 = \frac{1}{2}D_v\#D_w$, which is D_{2v} if $v = w$ and $-\frac{1}{2}D_{v+w}$ otherwise (for $w = -v \neq 0$ it is $\frac{1}{2}(2-3)1$). For $n = 1$, $n' = 2$ it is $c_0 = -\frac{1}{2}D_vD_w + \frac{1}{2}\,\mathrm{tr}(D_vD_w)1$, which is e if $w = -v$ and $-\frac{1}{2}\omega^{[v,w]}D_{v+w}$ otherwise. In each case g, h are dependent exactly when the coefficient is 1, and the exponent is $[v,w](n'-n)$. The case $n = n' = 2$ follows from $n = n' = 1$ by the conjugate-linear automorphism σ, and the remaining cases by commutativity.

(iii) From $\mathsf{ZXZ}^{-1} = \omega\mathsf{X}$ and $\mathsf{X}^{-1}\mathsf{ZX} = \omega\mathsf{Z}$, the three maps multiply $u_{(a,b,n)}$ by ω^a, ω^b and ω^n. They commute, since ZX^{-1} is a scalar multiple of $\mathsf{X}^{-1}\mathsf{Z}$, and have order three. By (i) they are automorphisms of J_T, and the 27 characters on the basis u_g are distinct. □

Lemma 7.5 gives directly, without the isomorphism with $H_3(\mathbb{O})$, the facts about J_T used in Proposition 7.3(i)–(ii) and in Proposition 7.6 below. In particular $T(u_g \bullet u_h, u_k)$ is symmetric in g, h, k, since the exponent $[v,w](n'-n)$ does not change under permutations of a triple with zero sum. The frames of the graded system of J_T are $p_{g,a} = \frac{1}{3}\big(e + \omega^{-a}\iota_n(D_v) + \omega^a\iota_{-n}(D_{-v})\big)$ for $g = (v,n)$.

By Theorem 7.4, an isomorphism $H_3(\mathbb{O}) \to J_T$ carrying the trace to T pulls the group of Lemma 7.5(iii) back to a ternary grading group of $H_3(\mathbb{O})$. So $H_3(\mathbb{O})$ has a ternary grading group, and the existence of thirteen planes in Proposition 8.2, Theorem 8.7(iii), Theorem 8.11 and Theorem 9.4 rests on this group and on Theorem 7.4.

7.3 The cubic in thirteen coordinates

Let E be a ternary grading group with elements u_g as in Proposition 7.3. Choose a representative g_p of each point p, write the field configuration as $X = \frac{1}{\sqrt{6}}\sum_p\big(\zeta_p u_{g_p} + \bar\zeta_p u_{-g_p}\big)$, and put $\zeta_g = \zeta_p$ for $g = g_p$ and $\zeta_g = \bar\zeta_p$ for $g = -g_p$. Lemma 7.1 gives $|X|^2 = \sum_p|\zeta_p|^2$. For independent g, h the product $u_g \bullet u_h$ lies in A_{g+h}, and by Lemma 7.2 applied to the copy J_ℓ of $H_3(\mathbb{C})$ of their projective line it is $-\frac{1}{2}\omega^\varphi u_{g+h}$ for some $\varphi \in \mathbb{F}_3$. Here the labels of Lemma 7.2 differ from the u_g by factors in μ_3 and conjugation, and distinct points of ℓ correspond to distinct frames, hence to independent labels. Since $\langle u_g \bullet u_h, u_k\rangle = d(u_g, u_h, u_k)$ is symmetric in its three

arguments, the exponent depends only on the unordered triple $\{g, h, k\}$ with sum zero:

$$\langle u_g \bullet u_h, u_k \rangle = -\tfrac{3}{2}\,\omega^{\varphi_{ghk}}, \qquad g + h + k = 0,$$

and $\varphi_{-g,-h,-k} = -\varphi_{ghk}$ by conjugation. Three distinct collinear points have exactly two choices of representatives with zero sum, (g, h, k) and $(-g, -h, -k)$.

Proposition 7.6. *In the coordinates ζ_p, the cubic rescaled to eikonal constant 1, $W = \sqrt{6}\,d(X, X, X)$, is*

Formalized. `thirteenPlaneCubicFull`.

$$W = \sum_p \operatorname{Re}\zeta_p^3 \;-\; 3 \sum_{\{p,q,r\}} \operatorname{Re}\big(\omega^{\varphi_{ghk}}\zeta_g\zeta_h\zeta_k\big), \tag{7.4}$$

the second sum over the 52 collinear triples, with $g \in p$, $h \in q$, $k \in r$ and $g + h + k = 0$. For the group of Lemma 7.5 and its clock elements $u_g = \iota_n(D_v)$, $\varphi_{ghk} = [v, w](n' - n)$ for $g = (v, n)$, $h = (w, n')$.

Proof. Write $z_g = \zeta_g/\sqrt{6}$. By Proposition 7.3(i), $d(u_g, u_h, u_k)$ vanishes unless $g + h + k = 0$, and $g + h + k = 0$ with nonzero entries means either $g = h = k$ or three distinct collinear points. On one plane, $d(u_g, u_g, u_g) = \langle \overline{u}_g, u_g \rangle = 3$, so the terms of $d(X, X, X)$ within the plane of p are $3z_g^3 + 3\bar{z}_g^3 = 6\operatorname{Re} z_g^3$. For a collinear triple the six orders of (g, h, k) and the six of $(-g, -h, -k)$ contribute $6\big(-\frac{3}{2}\omega^{\varphi} z_g z_h z_k\big)$ and its conjugate, together $-18\operatorname{Re}(\omega^{\varphi} z_g z_h z_k)$. Multiplying by $\sqrt{6}$ and substituting $z = \zeta/\sqrt{6}$ gives (7.4). For the group of Lemma 7.5 the same computation in J_T, with the cubic $T(X \bullet X, X)$, uses only Lemma 7.5(ii) and gives the stated exponents. □

The cubic W_4 of Example 3.6 has real coefficients, and its restriction to real coordinates is the four-line cubic of Example 3.4. The real coordinates form a closed subspace consisting of a one-dimensional subspace in each of the four planes. For the thirteen planes of the graded system, the coordinates ζ_p are determined by the planes up to the symmetries of $\operatorname{Re}\zeta_p^3$, and no such choice makes all the coefficients of the three-field terms of (7.4) real (Proposition 7.7).

Equivalently, no sum $L = \bigoplus_p L_p$ of a one-dimensional subspace L_p in each of the thirteen planes is closed. If L is closed, then $x \circ x \in L \cap U_p = L_p$ for $x \in L_p$. Since $\nabla W(x) = 3\sqrt{6}\,x \circ x$ for $W = \sqrt{6}\,d(X, X, X)$ and the three-field terms have zero gradient on a single plane, this means that ζ_p^3 is real on L_p, so the coordinates can be chosen with L_p the real axis of ζ_p. Closedness of L then says that the derivative of W in each direction $i\zeta_p$ vanishes on the real locus. The plane terms contribute nothing to it there, and a three-field term $\operatorname{Re}(\alpha m)$ with $\alpha \notin \mathbb{R}$ contributes $\pm \operatorname{Im}\alpha$ times a product of two other real coordinates, a monomial that no other term produces. Conversely, if all coefficients are real, the conjugation $\zeta \mapsto \bar{\zeta}$ is an isometry preserving W, hence the product $\circ$, and its fixed space, the real locus, is closed.

Proposition 7.7. *Let E be a ternary grading group. No choice of complex coordinates ζ_p on the thirteen planes of the graded system, with $|X|^2 = \sum_p |\zeta_p|^2$ and $W = \operatorname{Re}\zeta_p^3$ on each*

Formalized. `noRealPlaneCoordinates`.

plane, makes all the coefficients of the three-field terms of the cubic $W = \sqrt{6}\, d(X, X, X)$ *real.*

Proof. An orthogonal map of a plane that preserves $\operatorname{Re}\zeta^3$ is $\zeta \mapsto \eta\zeta$ or $\zeta \mapsto \eta\bar{\zeta}$ with $\eta \in \mu_3$, the symmetries of $r^3 \cos 3\theta$. So admissible coordinates arise from those of Proposition 7.6 by replacing, for each point p, the pair u_{g_p}, u_{-g_p} by a pair $c_p, \overline{c}_p$ with $c_p \in \mu_3 u_{g_p} \cup \mu_3 u_{-g_p}$. The thirteen pairs form a basis $\mathcal{B}$ of $V_{\mathbb{C}}$ with one element in each character space A_g, $g \neq 0$, and for basis elements $b \in A_g$, $b' \in A_h$ with g, h independent, $b \bullet b' = \kappa\, b''$ with $b'' \in \mathcal{B} \cap A_{g+h}$ and $\kappa \in -\frac{1}{2}\mu_3$, and $b \circ b' = b \bullet b'$ since $\langle b, b' \rangle = 0$. In admissible coordinates the three-field term of three distinct planes is $\operatorname{Re}(\alpha m)$ for a single monomial m in the coordinates and their conjugates; whether α is real does not depend on the choice between m and $\bar{m}$, and α is a positive multiple of κ or of $\bar{\kappa}$. Suppose all these coefficients were real. Then every such κ equals $-\frac{1}{2}$.

Let g_1, g_2, g_3 be a basis of $\mathbb{F}_3^3$, put $g_4 = -(g_1 + g_2 + g_3)$, and let b_m be the element of $\mathcal{B}$ in A_{g_m}. No three of $g_1, \ldots, g_4$ are linearly dependent, so the four points are distinct and the right side of (3.7) vanishes for (b_1, b_2, b_3, b_4). On the left, $b_1 \circ b_2 = -\frac{1}{2}b$ and $b_3 \circ b_4 = -\frac{1}{2}b'$ with $b \in \mathcal{B} \cap A_{g_1+g_2}$ and $b' \in \mathcal{B} \cap A_{-g_1-g_2}$, so $b' = \overline{b}$ and the pairing is $\frac{1}{4}\langle b, \overline{b} \rangle = \frac{3}{4}$. The same holds for the other two pairings, so the left side equals $\frac{9}{4} \neq 0$, a contradiction. □

On the four planes of one projective line the phases can be removed, since in the coordinates of $J_\ell \cong H_3(\mathbb{C})$ given by the D_v, all coefficients are $-\frac{1}{2}$ by (7.2). The exponents $[v, w](n' - n)$ agree with Griess's product rule after negating the labels and rescaling by cube roots of unity[126].

Remark 7.8. In J_T the fields of the graded system with $n = 0$ form a traceless Hermitian matrix A, and those with $n = 1$ an arbitrary complex matrix B, so the 26 real coordinates split as $8 + 18$, an octet plus a 3×3 complex matrix. The cubic norm of Tits's first construction is $N(a_0, a_1, a_2) = \det a_0 + \det a_1 + \det a_2 - \operatorname{tr}(a_0 a_1 a_2)$[127,128], which on (A, B, B^*) is $\det A + 2 \operatorname{Re} \det B - \operatorname{tr}(ABB^*)$, and on traceless elements the cubic $d(X, X, X)$ is three times the norm.

Remark 7.9 (The Hesse cubic). Take the projective line $n = 0$ as the line at infinity. The other nine points are $(x, 1)$, $x \in \mathbb{F}_3^2$, the affine plane of order three, and three of them are collinear exactly when they lie on one of its twelve lines. With the representatives $(x, 1)$, three distinct affine points have zero sum exactly when they are collinear, and by Proposition 7.6 their exponent is $[x, y](1 - 1) = 0$. So setting the four fields of the projective line $n = 0$ to zero leaves

$$W = \operatorname{Re} F(\zeta), \qquad F(\zeta) = \sum_{x \in \mathbb{F}_3^2} \zeta_x^3 - 3 \sum_{\text{affine lines } \{x,y,w\}} \zeta_x \zeta_y \zeta_w .$$

[126] Griess, "A Moufang loop, the exceptional Jordan algebra, and a cubic form in 27 variables", note 9. (2.9), (4.3), (4.6).

[127] Draper and Martín González, "Gradings on the Albert algebra and on $\mathfrak{f}_4$", note 113. §2.4, Eq. (4).

[128] Moran and Pumplün, "A generalization of the first Tits construction", note 120. §1.4.

The nine points and twelve lines are the Hesse configuration of the inflection points of a plane cubic[129], which in quantum information describes the nine states of the Hesse SIC and the twelve states of the four unbiased bases of $\mathbb{C}^3$[130,131,132]. The same cubic polynomial appears for the SIC probabilities of a qutrit state. Tabia and Appleby write the condition for a pure state as a sum of cubes against a sum over the twelve lines, with ordered triples and a factor $\frac{1}{2}$[133], and Stacey gives the version over unordered lines with its measurement interpretation[134]. Here the variables are the nine complex amplitudes ζ_x, not probabilities.

8 The decompositions of an eikonal cubic into closed summands

Planes are not the only closed subspaces of an eikonal cubic. This section determines which closed summands can occur in a proper orthogonal decomposition of Cartan's cubics and of the reducible cubic, and then proves Theorems A(i) and A(ii) of the introduction. A reader interested only in Theorem A(i) can go directly to Lemma 8.10 and Theorem 8.11 at the end of the section.

In the first two subsections, $J = H_3(\mathbb{O})$ and $V = H_3^0(\mathbb{O})$ with the trace form and $c = 1/6$. If $A = \mathbb{R}e \oplus U$ is a subalgebra of J isomorphic to some $H_3(\mathbb{K}')$, then the frames of A are frames of J and the trace of J restricts to the trace of A (proof of Lemma 7.2); so the inner product on U is the trace form of A, and unbiased frames of A are unbiased in J.

8.1 An octet forces nine planes

Lemma 8.1. *Let S be a real subspace of $M_3(\mathbb{C})$ such that Z^*Z is a real multiple of I for every $Z \in S$. Then* $\dim_{\mathbb{R}} S \le 2$.

Formalized. `scalarUnitary Subspace_finrank_le_two`.

Proof. Suppose $\dim_{\mathbb{R}} S \ge 3$. Pick $Z_0 \in S$ with $Z_0^* Z_0 = I$. The space SZ_0^* has the same property and the same dimension, and contains I; replace S by it. Use the real inner product $\operatorname{Re}\operatorname{tr}(X^*Y)$ and choose orthonormal $Y_1, Y_2 \in S$ orthogonal to I. The polynomial identity $(I + tY_1)^*(I + tY_1) \in \mathbb{R}I$ in t gives $Y_1 + Y_1^* \in \mathbb{R}I$; its trace is $2\operatorname{Re}\operatorname{tr} Y_1 = 0$, so $Y_1^* = -Y_1$, and likewise $Y_2^* = -Y_2$. Then $Y_1^* Y_1 = -Y_1^2$ is a positive multiple of I, so Y_1 is invertible, and so is Y_2. From $(Y_1 + tY_2)^*(Y_1 + tY_2) \in \mathbb{R}I$ follows $Y_1^* Y_2 + Y_2^* Y_1 = -(Y_1 Y_2 + Y_2 Y_1) \in \mathbb{R}I$, whose trace is $2\operatorname{Re}\operatorname{tr}(Y_1^* Y_2) = 0$; so

[129] O. Hesse, Über die Wendepunkte der Curven dritter Ordnung, *J. Reine Angew. Math.* **28** (1844) 97–107.

[130] L. P. Hughston, $d = 3$ SIC-POVMs and elliptic curves, talk at the Perimeter Institute, October 2007, PIRSA 07100040. `http://pirsa.org/07100040/`.

[131] I. Bengtsson, From SICs and MUBs to Eddington, *J. Phys. Conf. Ser.* **254** (2010) 012007. arXiv:1103.2030. §3.

[132] H. B. Dang, K. Blanchfield, I. Bengtsson and D. M. Appleby, Linear dependencies in Weyl–Heisenberg orbits, *Quantum Inf. Process.* **12** (2013) 3449–3475. arXiv:1211.0215. §3.

[133] G. N. M. Tabia and D. M. Appleby, Exploring the geometry of qutrit state space using symmetric informationally complete probabilities, *Phys. Rev. A* **88** (2013) 012131. arXiv:1304.8075. Eqs. (27)–(28).

[134] B. C. Stacey, SIC-POVMs and compatibility among quantum states, *Mathematics* **4** (2016) 36. arXiv:1404.3774. §§II–III.

$Y_1 Y_2 = -Y_2 Y_1$. Taking determinants of 3×3 matrices gives $\det(Y_1 Y_2) = -\det(Y_1 Y_2)$, which contradicts invertibility. □

The lemma is a special case, for $n = 3$, of the theorem of Adams, Lax and Phillips on real spaces of complex $n \times n$ matrices whose nonzero members are all invertible, for odd n[135].

Proposition 8.2. *Let $U \subset V = H_3^0(\mathbb{O})$ be an octet. Every closed subspace of V orthogonal to U has dimension at most two. In every orthogonal decomposition of V into closed subspaces that has U as a summand, the other summands are nine planes. Such decompositions exist.*

Formalized, different route. `complementsOfOctet`: the copy of $H_3(\mathbb{C})$ is moved onto the standard one and its orthogonal complement is given complex coordinates directly, instead of through the coordinates of Theorem 7.4.

Proof. Let $A = \mathbb{R}e \oplus U$. In $H_3^0(\mathbb{O})$ the octets are the copies of $H_3^0(\mathbb{C})$ (Section 3), so $A \cong H_3(\mathbb{C})$. The diagonal and Fourier frames of A generate A (Proposition 5.5 with $\mathbb{K} = \mathbb{C}$), and by the remark above they are unbiased frames of J. Lemma 5.4(i) gives an automorphism of J sending them into $H_3(\mathbb{C}_u)$, hence sending A onto $H_3(\mathbb{C}_u)$, which has the same dimension. The automorphism group of $\mathbb{O}$ is transitive on unit imaginary octonions[136], and an automorphism carrying u to i, applied to every entry, carries $H_3(\mathbb{C}_u)$ onto $H_3(\mathbb{C})$, $\mathbb{C} = \mathbb{R} + \mathbb{R}i$. So it may be assumed that $A = H_3(\mathbb{C})$. In the coordinates $x \mapsto (a, B) \in H_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ of Theorem 7.4, A is the set of pairs $(a, 0)$, its orthogonal complement is the set of pairs $(0, B)$, and $(0, B)^2 = \big(\mathrm{tr}(BB^*)I - BB^*, \mathrm{adj}(B)^*\big)$[137].

Let $S \subset V$ be closed and orthogonal to U; then S lies in that complement. For $x = (0, B) \in S$, $x \bullet x = \frac{1}{3}|x|^2 e + x \circ x$ with $x \circ x \in S \perp A$, so the component $\mathrm{tr}(BB^*)I - BB^*$ of $x \bullet x$ in A is a multiple of I, and so is BB^*. The map $x \mapsto B^*$ is real linear and injective, so Lemma 8.1 gives $\dim S \leq 2$.

The other summands are closed and orthogonal to U, so each has dimension at most two. Cartan's cubic is harmonic (Example 3.5), so none is one-dimensional (Proposition 3.12(iii)); so they are planes, and $26 - 8 = 18$ gives nine of them. For existence, the four planes of a projective line ℓ of the graded system span the octet J_ℓ^0 (Proposition 7.3(v)); merging them gives an octet and nine planes. □

8.2 No summands of dimension five or fourteen

The proof that two copies of $H_3^0(\mathbb{R})$ cannot be orthogonal compares two rotations: one of $\mathbb{R}^3$ acting on octonionic entries, compressed to the second copy, and the infinitesimal rotations that are the only derivations of $H_3(\mathbb{R})$.

[135] J. F. Adams, P. D. Lax and R. S. Phillips, On matrices whose real linear combinations are nonsingular, *Proc. Amer. Math. Soc.* **16** (1965) 318–322; correction **17** (1966) 945–947.

[136] Baez, "The octonions", note 10. §4.1.

[137] Earlier AI-generated work, "On the local geometry of the tight 39-point code", note 25. §4.2, Eqs. (10)–(11); App. A.1.

Lemma 8.3. *Every derivation of $H_3(\mathbb{R})$ is $\delta_\Omega(x) = \Omega x - x\Omega$ for a real skew-symmetric Ω. If $\Omega \neq 0$, the singular values of δ_Ω on $H_3^0(\mathbb{R})$, with the trace form, are $0, \lambda, \lambda, 2\lambda, 2\lambda$ for some $\lambda > 0$.*

Formalized, different route. `real_symmetric_derivation_inner`, `tracelessCommutator_singular_spectrum`: the derivations are computed on matrix units and the singular values from the characteristic polynomial of $\delta_\Omega^*\delta_\Omega$, instead of the cited derivation algebra and a normal form for Ω.

Proof. As a real vector space, the derivation algebra of $H_3(\mathbb{K})$ is $\mathrm{der}(\mathbb{K}) \oplus \mathfrak{sa}_3(\mathbb{K})$, the second summand acting by commutators[138], where $\mathfrak{sa}_3(\mathbb{K})$ is the space of traceless skew-Hermitian 3×3 matrices over $\mathbb{K}$; for $\mathbb{K} = \mathbb{R}$ the first summand is zero and the second is the real skew-symmetric matrices. Conjugating by a rotation, which is an isometry of $H_3^0(\mathbb{R})$, one may take $\Omega = \lambda(E_{21} - E_{12})$. Then δ_Ω kills $\mathrm{diag}(1, 1, -2)$, rotates the pair $E_{13} + E_{31}$, $E_{23} + E_{32}$ at speed λ, and the pair $\mathrm{diag}(1, -1, 0)$, $E_{12} + E_{21}$ at speed 2λ; for instance $\delta_\Omega(\mathrm{diag}(1, -1, 0)) = 2\lambda(E_{12} + E_{21})$. □

Lemma 8.4. *There are no two orthogonal copies U_1, U_2 of $H_3^0(\mathbb{R})$ in $V = H_3^0(\mathbb{O})$.*

Formalized. `noTwoOrthogonalRealQutrits`.

Proof. Suppose there are, and let $A_r = \mathbb{R}e \oplus U_r \cong H_3(\mathbb{R})$ for $r = 1, 2$.

Normal form of A_1. By the remark at the start of this section and (J3) the frame of A_1 may be taken to be the diagonal frame. The Jordan Peirce spaces of A_1 relative to this frame at the off-diagonal positions $(1, 2)$ and $(2, 3)$ are one-dimensional, spanned by elements $xE_{ij} + \bar{x}E_{ji}$ with $|x| = 1$, and Lemma 5.4(ii) moves them to $\mathbb{R}(E_{12} + E_{21})$ and $\mathbb{R}(E_{23} + E_{32})$ while fixing the diagonal frame. These elements and the E_{ii} generate $H_3(\mathbb{R})$, so $A_1 = H_3(\mathbb{R})$ by dimension. Its orthogonal complement in J is the space of Hermitian matrices with zero diagonal and imaginary off-diagonal entries,

$$y = \sum_{k=1}^{3} v_k \otimes \Lambda_k, \qquad (\Lambda_k)_{ij} = \epsilon_{ijk}, \qquad v_k \in \mathrm{Im}\,\mathbb{O},$$

with $\langle y, y' \rangle = 2\sum_k \langle v_k, v_k' \rangle$.

Three isometries. Let $y \in U_2$. As in the proof of Proposition 8.2, $U_2 \perp A_1$ forces the component of $y \bullet y$ in A_1, which is the real part of y^2 entry by entry, to be a multiple of e. Squaring y shows that this says exactly that the $v_k(y)$ have equal lengths and are pairwise orthogonal (the diagonal entries of y^2 are $|v_k|^2 + |v_l|^2$, and the off-diagonal ones are products $v_k v_l$, $k \neq l$, with real part $-\langle v_k, v_l \rangle$). The maps $F_k = \sqrt{6}\, v_k \colon U_2 \to \mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ are therefore isometries with $\langle F_k y, F_l y \rangle = 0$ for $k \neq l$. Polarizing the last relation shows that $G = F_1^* F_2$ is skew-adjoint on U_2. The images of F_1 and F_2 are five-dimensional in $\mathbb{R}^7$, so they meet in a subspace of dimension at least three. On its preimage under F_2, $\langle G^*Gy, y \rangle = |\Pi_1 F_2 y|^2 = |y|^2$, where $\Pi_1 = F_1 F_1^*$ is the projection onto the image of F_1; since $0 \leq G^*G \leq I$, this gives $G^*Gy = y$ there. The nonzero eigenvalues of $G^*G = -G^2$ for a skew G have even multiplicity, and $\dim U_2 = 5$ is odd, so the singular values of G are $1, 1, 1, 1, 0$.

A rotation compressed to A_2. For $S \in \mathrm{SO}(3)$ the map $Y \mapsto SYS^{\mathsf{T}}$ is an automorphism of J, because real matrices associate with octonionic ones, and on the complement of A_1 it acts by $v \mapsto Sv$, since $S\Lambda_k S^{\mathsf{T}} = \sum_l S_{lk}\Lambda_l$. Differentiating the rotations of the first two coordinates gives a derivation δ of J, skew for the trace form, with $\delta e = 0$ and $v \mapsto (-v_2, v_1, 0)$ on the complement of A_1. Let π be the orthogonal projection onto A_2. For $b \in A_2$ the self-adjoint operator $L_b^\bullet$ preserves

[138] Baez, "The octonions", note 10. §4.2, Eq. (19).

A_2, hence also its orthogonal complement, so $\pi L_b^\bullet = L_b^\bullet \pi$. Then $\delta_2 = \pi\delta|_{A_2}$ is a derivation of A_2: $\delta_2(b \bullet b') = \pi L_{b'}^\bullet \delta b + \pi L_b^\bullet \delta b' = \delta_2 b \bullet b' + b \bullet \delta_2 b'$. It kills e and maps U_2 to itself, since $\langle \delta y, e\rangle = -\langle y, \delta e\rangle = 0$. For $y, y' \in U_2$,

$$\begin{aligned}\langle \delta_2 y, y'\rangle = \langle \delta y, y'\rangle &= 2\big(\langle v_1, v_2'\rangle - \langle v_2, v_1'\rangle\big)\\ &= \tfrac{1}{3}\big(\langle F_1 y, F_2 y'\rangle - \langle F_2 y, F_1 y'\rangle\big) = -\tfrac{2}{3}\langle Gy, y'\rangle.\end{aligned}$$

So δ_2 is a derivation of $A_2 \cong H_3(\mathbb{R})$ whose singular values on U_2 are $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0$. The inner product on U_2 is the trace form of A_2, so this contradicts Lemma 8.3. □

Proposition 8.5. *In a proper orthogonal decomposition of $V = H_3^0(\mathbb{O})$ into closed subspaces, each a plane or a copy of $H_3^0(\mathbb{K}')$ for some $\mathbb{K}'$, no summand is a copy of $H_3^0(\mathbb{R})$ or of $H_3^0(\mathbb{H})$.*

Formalized, stronger. `no RealOrQuaternionicBlock`. The summands need not be planes or copies, and none has dimension 5 or 14.

Proof. A copy of $H_3^0(\mathbb{H})$, of dimension 14, next to any other summand U would by Lemma 3.11 need $26 \geq 14 + \dim U + 14$, which is false. Suppose a summand is a copy of $H_3^0(\mathbb{R})$. The other summands have dimension 2, 5 or 8. If one is an octet, the others are planes by Proposition 8.2, a contradiction. Otherwise $5a + 2b = 26$, where $a \geq 1$ counts the copies of $H_3^0(\mathbb{R})$, forces a to be even, and two of these summands contradict Lemma 8.4. □

8.3 The decompositions of the models and the proof of Theorem A

The reducible cubic of the classification theorem splits only into four one-dimensional summands.

Lemma 8.6. *Let $W = s^3 - 3s|u|^2$ on $\mathbb{R}^N$, $N \neq 2$, with $s = \langle x, a\rangle$ for a unit vector a and $u = x - sa$. A proper orthogonal decomposition of $\mathbb{R}^N$ into closed subspaces exists only for $N = 4$, and then it consists of four one-dimensional summands spanned by $\frac{1}{2}a + \frac{\sqrt{3}}{2}w_i$, where $w_1, \dots, w_4$ are unit vectors in $a^\perp$ with pairwise inner products $-\frac{1}{3}$ (the vertices of a regular tetrahedron).*

Formalized. `cubic_diagonal`, `four_lines`; existence for $N = 4$: `exists_four_lines`.

Proof. Here $x \circ x = \frac{1}{3}\nabla W = (s^2 - |u|^2)a - 2su$. For $u \perp a$, $u \circ u = -|u|^2 a$; so a closed subspace of dimension at least two, which contains a nonzero $u \perp a$, contains a, and a one-dimensional subspace orthogonal to a is not closed. If a summand of a proper decomposition contained a, the other summands would be orthogonal to a, hence neither of dimension at least two nor one-dimensional. So no summand contains a, and every summand is one-dimensional, spanned by a vector $sa + u$ with $u \neq 0$. A unit vector $v = sa + u$ with $u \neq 0$ spans a closed one-dimensional subspace exactly when $v \circ v = -2sv$ and $s^2 = \frac{1}{4}$, by comparing components. With signs chosen so that $s = \frac{1}{2}$, an orthonormal basis of such vectors gives $1 = |a|^2 = \sum_i \langle v_i, a\rangle^2 = N/4$, and $v_i = \frac{1}{2}a + \frac{\sqrt{3}}{2}w_i$ with unit $w_i \perp a$ satisfying $\frac{1}{4} + \frac{3}{4}\langle w_i, w_j\rangle = 0$. Conversely such vectors are orthonormal and span closed one-dimensional subspaces. □

With $a = \frac{1}{2}(1,1,1,1)$ and $x \mapsto -x$ these are the coordinate axes of Example 3.4.

Theorem 8.7. *Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. The proper orthogonal decompositions of $H_3^0(\mathbb{K})$ into closed subspaces, each of which is a plane or a copy of some $H_3^0(\mathbb{K}')$, are as follows.*

(i) For $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{H}$ there are none.

(ii) For $\mathbb{K} = \mathbb{C}$ they are the decompositions into four planes of Theorem 5.10.

(iii) For $\mathbb{K} = \mathbb{O}$ they are the decompositions into thirteen planes and the decompositions into an octet and nine planes. Both kinds exist.

In particular $H_3^0(\mathbb{K})$ has an orthogonal decomposition into planes exactly for $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{O}$.

Formalized, stronger. `classifyHermitianBlockDecompositions`. The summands need only have dimension 2, 5, 8, 14 or 26, and for $\mathbb{O}$ nothing is assumed of them.

Proof. A copy of $H_3^0(\mathbb{K}')$ has dimension 5, 8, 14 or 26, and a proper decomposition has no summand of dimension N. For $\mathbb{K} = \mathbb{R}$, $N = 5$ is odd and the only summand dimension below 5 is 2. For $\mathbb{K} = \mathbb{C}$, $N = 8$ allows only planes, and Theorem 5.10 applies. For $\mathbb{K} = \mathbb{H}$ the patterns summing to 14 are 2^7, $8+2^3$ and $5+5+2+2$. Seven orthogonal planes would be seven pairwise unbiased frames (Proposition 5.1), which Corollary 6.6 excludes; the other two patterns violate Lemma 3.11, since $14 < 8+2+8$ and $14 < 5+5+5$. For $\mathbb{K} = \mathbb{O}$, Proposition 8.5 leaves planes and octets, and with an octet the other summands are nine planes by Proposition 8.2. Thirteen planes exist by Proposition 7.3, and an octet with nine planes by Proposition 8.2. □

The two kinds of decomposition in (iii) are related. An octet U of $V = H_3^0(\mathbb{O})$ is closed, so the cubic product of the cubic restricted to U is the restriction of $\circ$, and Theorem 5.10(iii) splits U into four orthogonal planes that are closed in V. So every decomposition into an octet and nine planes refines to a thirteen-plane system in which the four planes of the octet form a quartet. Conversely, if four planes U_i, U_j, U_k, U_l of a thirteen-plane system have closed sum S, then $U_i \circ U_j \subset S$, and the eight-dimensional octet $U_i \oplus U_j \oplus (U_i \circ U_j)$ of Corollary 5.6 lies in S, hence equals it. So a thirteen-plane system coarsens to an octet and nine planes, by merging four of its planes, exactly when it contains a quartet.

Theorem 8.8 (Theorem A(ii), harmonic case)**.** *Let W be a harmonic eikonal cubic on a Euclidean space V of dimension N with a proper orthogonal decomposition into closed subspaces. Then either $N = 8$ and the summands are four planes, or $N = 26$ and the summands are thirteen planes, or an octet and nine planes; here an octet is a summand on which W is, after a linear isometry, a positive multiple of Cartan's cubic on $H_3^0(\mathbb{C})$. In dimensions 2, 5 and 14 there is no such decomposition.*

Formalized. `classifyHarmonicRoundDecompositions`.

Proof. Scale W to constant 1. By Proposition 3.12 each summand is closed, eikonal and harmonic, and not one-dimensional, so $N \geq 4$. Theorem 4.6, applied at a unit idempotent of the cubic product from its part (i), gives $N = 3k + 2$ with $k \in \{1, 2, 4, 8\}$ and a linear isometry $\Phi\colon V \to H_3^0(\mathbb{K})$ with $W = \sqrt{6}\,\mathrm{tr}\big((\Phi x)^3\big)$, Cartan's cubic rescaled to constant 1 for the trace form $\langle X, Y\rangle = \mathrm{tr}(X \bullet Y)$. The isometry Φ carries cubic products to $\sqrt{6}$ times cubic products, hence summands to closed subspaces of $H_3^0(\mathbb{K})$ with harmonic cubic. A summand of dimension two is a plane. A summand of dimension at least three carries a harmonic eikonal cubic, so Theorem 4.6 gives an isometry of it onto some $H_3^0(\mathbb{K}')$ carrying its cubic to Cartan's cubic, and by Section 3 the image of the summand under Φ is a copy of $H_3^0(\mathbb{K}')$. Theorem 8.7 now gives the list. □

Theorem 8.8 is the harmonic case of Theorem A(ii). The general case uses Theorem 3.7(ii).

Corollary 8.9 (Theorem A(ii))**.** *An eikonal cubic has a proper orthogonal decomposition into closed subspaces only in dimension* 4*, as four one-dimensional summands, with a cubic that is not harmonic; in dimension* 8*, as four planes; and in dimension* 26*, as thirteen planes or as an octet and nine planes. All four occur.*

Formalized. `classifyRoundDecompositions`; existence: `roundDecompositionRealizations`. In the self-contained file: `Challenge.closed_decomposition_classification`, `Challenge.closed_decomposition_realizations`.

Proof. If the cubic is not harmonic, Theorem 3.7(ii) and Lemma 8.6 give four one-dimensional summands. If it is harmonic, Theorem 8.8 applies. Existence is Example 3.4 and Theorem 8.7. □

Theorem A(i) does not use the quoted Theorem 3.7. Its identification step is Theorem 4.7, the plane case of Theorem 4.6, which re-proves Cartan's classification of harmonic eikonal cubics in dimension $N \geq 3$ at a unit idempotent of the cubic product from Hurwitz's theorem (Theorem 4.4) and the normal form of Appendix A. Its hypothesis, complex fields coupled only through three-field terms, is restated by the next lemma in terms of planes.

Lemma 8.10. *Let $m \geq 1$ and $W(z) = \sum_{i=1}^{m} \operatorname{Re} z_i^3 + \sum_{i<j<k} T_{ijk}(z_i, z_j, z_k)$ on $\mathbb{C}^m = \mathbb{R}^{2m}$, with each T_{ijk} real trilinear over $\mathbb{R}$ (conjugates allowed), and suppose that the length of the real gradient $\nabla W(z)$ depends only on $|z|$. Then W is eikonal with $c = 1$ and harmonic, and the coordinate planes form an orthogonal decomposition of $\mathbb{R}^{2m}$ into planes.*

Formalized. `radial_polynomial_isRound`.

Proof. By (3.9), where T_{ijk} is $6d$ restricted to the three coordinate planes, the coordinate planes are closed; they are orthogonal and fill $\mathbb{R}^{2m}$. The energy is a quartic form that depends only on $|z|$, so it is $\kappa|z|^4$. At a point of the first coordinate plane every three-field term has zero gradient, because at least two of its three fields vanish there, so the energy is that of $\operatorname{Re} z_1^3$ and $\kappa = 9$. Each $\operatorname{Re} z_i^3$ is harmonic, and each T_{ijk} is linear in the coordinates of each plane, hence harmonic. □

Theorem 8.11 (Theorem A(i))**.** *For a cubic W as in Lemma 8.10 whose energy depends only on $|z|$, m is* 1*,* 4 *or* 13*. For $m = 4$ and $m = 13$ there is a linear isometry Φ of $\mathbb{R}^{2m}$ onto $H_3^0(\mathbb{C})$ or $H_3^0(\mathbb{O})$ with $W(z) = \sqrt{6}\ \operatorname{tr}\big((\Phi z)^3\big)$ (Cartan's cubic rescaled to constant* 1 *for the trace form $\langle X, Y\rangle = \operatorname{tr}(X \bullet Y)$), and both values occur.*

Formalized. `field_polynomial_count_and_model`; both values occur: `four_and_thirteen_radial_field_polynomials`. In the self-contained file: `Challenge.fields_classification`, `Challenge.fields_realizations`.

Proof. By Lemma 8.10, W is eikonal with constant 1 and the coordinate planes form an orthogonal decomposition into planes. Theorem 4.9 gives $m \in \{1, 4, 13\}$, and Theorem 4.7 gives Φ. For existence, the cubics W_4 of (3.5) and (7.4) have the form of Lemma 8.10, and they are Cartan's cubics on $H_3^0(\mathbb{C})$ and $H_3^0(\mathbb{O})$ in coordinates normalized as in the statement (by (3.6) and the last assertion of Proposition 7.6, whose proof uses only Lemma 7.5(ii)). So their energy is $9|z|^4$ by the eikonal identity of Example 3.5, which rests on the facts (J1)–(J2) about $H_3(\mathbb{K})$ quoted there. For $m = 13$ the coordinates require a ternary grading group of $H_3(\mathbb{O})$, which comes from Tits's second construction of the Albert algebra through the quoted Theorem 7.4 and Lemma 7.5. □

For $m \geq 1$ let $\mathcal{W}_m$ be the set of cubics W on $\mathbb{C}^m$ of the form in Lemma 8.10, without the hypothesis on the gradient, and define the *defect* of W and its infimum by

$$\operatorname{def}(W) = \max_{|z|=1} \left| \tfrac{1}{9}|\nabla W(z)|^2 - 1 \right|, \qquad \delta_m = \inf_{W \in \mathcal{W}_m} \operatorname{def}(W).$$

By Lemma 8.10 and homogeneity, $\operatorname{def}(W) = 0$ exactly when the length of $\nabla W(z)$ depends only on $|z|$. For frames, let $\mathcal{F}$ be the set of frames of $H_3(\mathbb{H})$ (ordered orthonormal bases of $\mathbb{H}^3$ modulo independent right multiplication of the basis vectors by unit quaternions) and, for frames $(p_a^1), \dots, (p_a^k)$,

$$\mu\big(p^1, \dots, p^k\big) = \max_{i<j} \max_{a,b} \left| \operatorname{tr}\big(p_a^i \bullet p_b^j\big) - \tfrac{1}{3} \right|.$$

Proposition 8.12. (i) *If $m \notin \{1, 4, 13\}$, then* def *attains the value δ_m on $\mathcal{W}_m$, and $\delta_m > 0$.*

Formalized. `nativeField DefectAndFrameGap`; that the defect is a maximum: `defect_isMax`.

(ii) *$\delta_7 \leq \frac{3}{4}$, with $\operatorname{def}(W) = \frac{3}{4}$ for $W(z) = W_4(z_1, z_2, z_3, z_4) + \operatorname{Re} z_5^3 + \operatorname{Re} z_6^3 + \operatorname{Re} z_7^3$.*

(iii) *The function μ attains a positive minimum ϵ_6 on $\mathcal{F}^6$, and $\mu \geq \epsilon_6$ on $\mathcal{F}^k$ for every $k \geq 6$.*

Proof. (i) The cubics in $\mathcal{W}_m$ are $W_T = D + T$, where $D(z) = \sum_i \operatorname{Re} z_i^3$ and the three-field part $T = \sum_{i<j<k} T_{ijk}$ ranges over a finite-dimensional real vector space $\mathcal{T}$. The defect $f(T) = \operatorname{def}(W_T)$ is a maximum over the unit sphere S of a function continuous in (T, z), so it is continuous in T.

A bounded defect bounds T, because the gradient controls T: $N(T) = \max_S |\nabla T|$ is a norm on $\mathcal{T}$. Indeed, if $N(T) = 0$, then ∇T vanishes everywhere, being homogeneous, so T is constant and equal to $T(0) = 0$; and each T_{ijk} is recovered from T by setting all coordinates other than z_i, z_j, z_k to zero. On S one has $|\nabla D| \leq 3$, and $|\nabla W_T| \leq 3\sqrt{1 + f(T)}$ by the definition of the defect, so $N(T) \leq 3\sqrt{1 + f(T)} + 3$. Hence the sublevel set $\{f \leq f(0)\}$ is topologically closed, bounded and nonempty, hence compact, and f attains its infimum δ_m there. If $\delta_m = 0$, a minimizer has $|\nabla W(z)|^2 = 9|z|^4$ by homogeneity, and Theorem 8.11 gives $m \in \{1, 4, 13\}$, contrary to the hypothesis.

(ii) The cubic W_4 of Example 3.6 restricts to $\operatorname{Re} z_p^3$ on each coordinate plane and its other terms contain three different fields, so $W \in \mathcal{W}_7$. The gradient of W is the orthogonal sum of the gradients of its four summands, which depend on disjoint sets of coordinates. With $z' = (z_1, \dots, z_4)$ and $|\nabla W_4(z')|^2 = 9|z'|^4$,

$$\tfrac{1}{9}|\nabla W(z)|^2 = m_0^2 + m_5^2 + m_6^2 + m_7^2, \qquad m_0 = |z'|^2,\ m_p = |z_p|^2.$$

On S the four masses are nonnegative with sum one, so this lies in $[\frac{1}{4}, 1]$; the value $\frac{1}{4}$ is taken at equal masses and the value 1 on the plane of z_5. So $\operatorname{def}(W) = \frac{3}{4}$.

(iii) A frame consists of three idempotents of trace one with pairwise products zero and sum e, so $\mathcal{F}$ is topologically closed in $H_3(\mathbb{H}) \times H_3(\mathbb{H}) \times H_3(\mathbb{H})$, and it is bounded, since an idempotent p of trace one has $|p|^2 = \operatorname{tr}(p \bullet p) = 1$. So $\mathcal{F}^6$ is compact and the continuous function μ attains its minimum ϵ_6 there. If $\epsilon_6 = 0$, the six frames of a minimizer are pairwise unbiased, which Corollary 6.9 excludes, by Lemma 6.1(iii). For $k \geq 6$, μ of k frames is at least μ of the first six. □

By Corollary 6.9 no six frames of $H_3(\mathbb{H})$ are pairwise unbiased, so μ is positive at every point of $\mathcal{F}^k$ for $k \geq 6$, and ϵ_6 is the minimum of a function that vanishes nowhere. Part (iii) says that this positivity is uniform: six frames of $H_3(\mathbb{H})$ always miss unbiasedness by at least ϵ_6.

9 Sparsest systems, rotated quartets and generating triples

This section returns to the thirteen-plane systems of Section 7, with its notation. It shows that the sum rule forces the incidence of the projective plane of order three on the sparsest systems, that the sparsest system is unique up to automorphism, that rotating one quartet of the graded system gives systems that no automorphism carries to it, and that for every pair of planes of a system, for at least seven of the other eleven planes the frames of that plane and of the pair generate the Albert algebra. Lemma 9.2 and Proposition 9.3, which work in the Tits model, are used only for the uniqueness in Theorem 9.4 and can be skipped on first reading.

9.1 Sparsest systems and PG(2, 3)

A thirteen-plane system is *sparsest* if exactly 52 of its shares are positive, the fewest that part (i) of the next theorem allows.

Theorem 9.1. *Let $U_1, \ldots, U_{13}$ be the planes of a thirteen-plane system.*

(i) *The 286 shares have total 52, and at least 52 of them are positive.*

(ii) *The following are equivalent.*

- (a) *Exactly 52 shares are positive.*
- (b) *Every share is 0 or 1.*
- (c) *Every pair of planes lies in a quartet.*
- (d) *The quartets are the projective lines of a projective plane of order three whose points are the thirteen planes.*

In this case each pair lies in exactly one quartet, this projective plane is isomorphic to PG(2, 3), and s_{ijk} is 1 for collinear triples and 0 otherwise.

Formalized, different route. `thirteenPlane_share Support_and_incidence`, for thirteen orthogonal planes of any eikonal cubic: a quartet is closed by Corollary 5.6 instead of Proposition 5.5, and the identification with PG(2, 3) is proved instead of cited. In the self-contained file, the bound of 52 and shares 0 or 1 at equality: `Challenge.thirteen_plane_share_support`.

Proof. (i) By Theorem 5.9 the total is $13 \cdot 12/3 = 52$. Every share lies in $[0, 1]$, so at least 52 are positive.

(ii) Shares lie in $[0, 1]$ and total 52, so exactly 52 are positive if and only if every positive share equals one, which is (a)⇔(b).

(b)⇒(c). For a pair i, j, the shares s_{ijk} sum to 2, so exactly two of them, say for k, l, equal one, which means $U_k, U_l \subset U_i \circ U_j$ (Theorem 5.9). Hence $U_i \circ U_j = U_k \oplus U_l$, and $U_i \oplus U_j \oplus U_k \oplus U_l$ is the traceless part of the copy J_{ij} of $H_3(\mathbb{C})$ generated by the frames of U_i, U_j (Proposition 5.5). It is closed, because $x \circ y$ is the traceless part of $x \bullet y \in J_{ij}$.

(c)⇒(b). Let $O = U_i \oplus U_j \oplus U_k \oplus U_l$ be closed. Then $\mathbb{R}e \oplus O$ is a unital subalgebra, because $x \bullet y = \frac{1}{3}\langle x, y\rangle e + x \circ y$. It contains the nine-dimensional subalgebra

generated by the frames of U_i, U_j, so the two coincide, and $U_i \circ U_j = O \ominus (U_i \oplus U_j) = U_k \oplus U_l$. Then $s_{ijk} = s_{ijl} = 1$, the other shares of the pair vanish, and the quartet containing i, j is unique. This holds for every pair.

(c)⇒(d). By the last step each pair of planes lies in exactly one quartet. So the quartets are the blocks of a 2-$(13, 4, 1)$ design, which is a projective plane of order three, and every projective plane of order three is isomorphic to $\mathrm{PG}(2, 3)$[139,140].

(d)⇒(c) is immediate, and the last sentence follows from the proof of (c)⇒(b). □

Uniqueness is proved in the model J_T, in two steps that use only (7.3). A *frame* of J_T is a triple of idempotents p_a with $p_a \bullet p_b = 0$ for $a \neq b$, $T(p_a) = 1$ and $\sum_a p_a = e$; its plane is the span of the $p_a - \frac{1}{3}e$. In J_T the bar is σ, the conjugation of $M_3(\mathbb{C})^3$ fixing J_T, and the *conjugate clock element* of a frame f is $\overline{u}_f = \sigma(u_f)$. Let ℓ_0 be the projective line $n = 0$ of the graded system of Lemma 7.5, let $J_{\ell_0} = \{(A, 0, 0)\}$, a unital subalgebra equal to $H_3(\mathbb{C})$ with its Jordan product by (7.3), and let $J^0_{\ell_0}$ be its traceless part.

Lemma 9.2 (An exterior frame)**.** *Let (p_a) be a frame of J_T whose plane is orthogonal to $J^0_{\ell_0}$. After relabelling its outcomes, $p_a = \frac{1}{3}(1, \omega^a R, \omega^{-a} R^*)$ with $R \in \mathrm{SU}(3)$, and ψ_{R^*} carries it to the frame $\frac{1}{3}(1, \omega^a, \omega^{-a})$, whose clock element is $\iota_2(1)$.*

Formalized. `exterior Frame`.

Proof. An idempotent $p = (A, B, B^*)$ of the frame has $T(p) = \operatorname{tr} A = 1$ and $\operatorname{tr}\big((A - \frac{1}{3})A'\big) = 0$ for all traceless Hermitian A', so $A = \frac{1}{3}$, and $p_a = \frac{1}{3}(1, R_a, R_a^*)$. By (7.3), $p_a \bullet p_a = p_a$ reads $1 - R_a R_a^* + t1 = 3 \cdot 1$ in position 0, with $t = \operatorname{tr}(R_a R_a^*)$, so $R_a R_a^* = (t - 2)1$; taking traces gives $t = 3$, so $R_a R_a^* = 1$. In position 1 it reads $\frac{1}{2} R_a^* \# R_a^* + 2R_a = 3R_a$, so $\operatorname{adj}(R_a^*) = R_a$, which for unitary R_a gives $\det R_a = 1$. The relation $\sum_a p_a = e$ gives $R_0 + R_1 + R_2 = 0$. Put $K = R_0^* R_1$. Both K and $1 + K = -R_0^* R_2$ are unitary, so $K + K^* = -1$, every eigenvalue of K is ω or ω^2, and since $\det K = 1$ they are all equal, so K is scalar. After relabelling the outcomes, $R_a = \omega^a R_0$, and $R = R_0$. The automorphism ψ_{R^*} (Lemma 7.5(i)) carries p_a to $\frac{1}{3}(1, \omega^a, \omega^{-a})$, and $\sum_a \omega^a \frac{1}{3}(1, \omega^a, \omega^{-a}) = (0, 0, 1)$. So the clock element of this frame is $\iota_2(1)$, and its conjugate clock element is $\sigma\iota_2(1) = \iota_1(1)$. □

Proposition 9.3 (Recognition in J_T)**.** *Let $\mathcal{F}$ be a set of thirteen frames of J_T that contains the four unordered frames of spectral idempotents of the $\iota_0(D_v)$, one for each pair $\{v, -v\}$ with $v \neq 0$, and the frame $\frac{1}{3}(1, \omega^a, \omega^{-a})$. Suppose that for any two frames $f \neq g$ of $\mathcal{F}$ the product of u_f or $\overline{u}_f$ with u_g or $\overline{u}_g$ is a nonzero multiple of e or of a clock element or conjugate clock element of a frame of $\mathcal{F}$. Then $\mathcal{F}$ is the graded system of Lemma 7.5: each of its frames consists of the idempotents $p_{g,a}$ of one point $\{\pm g\}$, up to relabelling the outcomes.*

Formalized. `recognition_entire_graded_system`.

Proof. Let $\mathcal{N}$ be the set of at most 27 one-dimensional subspaces $\mathbb{C}e$, $\mathbb{C}u_f$, $\mathbb{C}\overline{u}_f$, $f \in \mathcal{F}$, of $J_T \otimes \mathbb{C}$. The product of nonzero elements of two members of $\mathcal{N}$ is a nonzero element of a member of $\mathcal{N}$: for one frame by Lemma 7.1(i), whose proof

[139] M. Hall, Jr., *Combinatorial Theory*, 2nd ed., Wiley, New York, 1986.

[140] D. R. Hughes and F. C. Piper, *Projective Planes*, Graduate Texts in Mathematics 6, Springer, New York, 1973.

uses only the frame relations, and for two frames by hypothesis. The subspaces $\mathbb{C}\iota_0(D_v)$, $v \neq 0$, lie in $\mathcal{N}$, and so does $\mathbb{C}\iota_1(1)$, because $\iota_1(1)$ is the conjugate clock element of the frame $\frac{1}{3}(1, \omega^a, \omega^{-a})$ (proof of Lemma 9.2). By Lemma 7.5(ii), $\iota_0(D_v) \bullet \iota_1(1) = -\frac{1}{2}\iota_1(D_v)$ for $v \neq 0$, so the nine subspaces $\mathbb{C}\iota_1(D_v)$, $v \in \mathbb{F}_3^2$, lie in $\mathcal{N}$, and applying σ, so do the nine subspaces $\mathbb{C}\iota_2(D_v)$. These are 27 subspaces, so $\mathcal{N}$ is the set of subspaces $\mathbb{C}u_g$. A frame f gives the conjugate pair $\mathbb{C}u_f, \mathbb{C}\overline{u}_f$, so $u_f = \lambda u_g$ for some g, after replacing g by $-g$ if needed. The relations $u_f \bullet u_f = \overline{u}_f$ and $u_f \bullet \overline{u}_f = e$ give $\lambda^2 = \bar{\lambda}$ and $|\lambda| = 1$, so $\lambda = \omega^k$, and Fourier inversion (Lemma 7.1(i)) gives $p_{fa} = p_{g,a-k}$. □

Theorem 9.4 (The sparsest system)**.** *The algebra $H_3(\mathbb{O})$ has a thirteen-plane system with exactly 52 positive shares, and any two such systems are carried to each other by an automorphism of $H_3(\mathbb{O})$, which carries the 39 idempotents of the frames of one to those of the other up to relabelling the frames and the outcomes within each frame.*

Formalized. `native Sparsest`; for systems given by their planes, `sparsest_closed PlaneSystems`. In the self-contained file, existence and uniqueness: `Challenge.albert SparsestSystem`.

Proof. By Theorem 7.4, choose an isomorphism $\Psi\colon H_3(\mathbb{O}) \to J_T$ carrying the trace to T; it carries frames, planes, shares and automorphisms to the corresponding objects. The graded system of Lemma 7.5, pulled back by Ψ, has 52 positive shares by Proposition 7.3(iv), which proves existence. For uniqueness it suffices to carry any system $\mathcal{F}$ with 52 positive shares, transported to J_T, to the graded system by an automorphism of J_T.

Normalizing a quartet. The automorphism group of $\mathbb{O}$ is transitive on unit imaginary octonions[141]. Apply Lemma 5.4(i), and then, to every entry, an automorphism of $\mathbb{O}$ carrying the unit imaginary u of that lemma to i. This carries any ordered pair of unbiased frames of $H_3(\mathbb{O})$ to the pair (diagonal frame, Fourier frame) of the standard $H_3(\mathbb{C})$, up to relabelling outcomes. Hence the automorphism group of J_T is transitive on unbiased pairs, up to labels. Let ℓ be a quartet of $\mathcal{F}$ (Theorem 9.1), read also as the set of the four frames of its planes, and F, G two of its frames. An automorphism carries F, G to the eigenframes of $\iota_0(\mathsf{Z})$ and $\iota_0(\mathsf{X})$. The algebra they generate has dimension nine (Proposition 5.5) and lies in J_{ℓ_0}, so it is J_{ℓ_0}. The sum of the planes of ℓ is the traceless part of this algebra (proof of Theorem 9.1, (c)⇒(b)), so the other two planes of ℓ fill $J^0_{\ell_0} \ominus (U_F \oplus U_G) = U_{\mathsf{XZ}} \oplus U_{\mathsf{XZ}^2}$, where $U_{\mathsf{XZ}}, U_{\mathsf{XZ}^2}$ are the planes of the eigenframes of $\iota_0(\mathsf{XZ})$ and $\iota_0(\mathsf{XZ}^2)$ (Theorem 5.10(ii)). A plane $U \subset U_{\mathsf{XZ}} \oplus U_{\mathsf{XZ}^2}$ equals one of these two: for $x = x_3 + x_4 \in U$ with $x_3 \in U_{\mathsf{XZ}}$ and $x_4 \in U_{\mathsf{XZ}^2}$, the component $2x_3 \circ x_4$ of $x \circ x$ lies in $U_{\mathsf{XZ}} \circ U_{\mathsf{XZ}^2} = U_F \oplus U_G$, so it vanishes, and by Lemma 3.11 $x_3 = 0$ or $x_4 = 0$. So the four frames of ℓ are the eigenframes of the $\iota_0(D_v)$.

An exterior frame. Let H be a frame outside ℓ. By Theorem 9.1, $s_{FGH} = 0$, so $U_H \perp U_F \circ U_G$ (Theorem 5.9), and $U_H \perp J^0_{\ell_0}$. Lemma 9.2 gives an automorphism ψ_{R^*}, fixing J_{ℓ_0} pointwise, after which H is the frame $\frac{1}{3}(1, \omega^a, \omega^{-a})$.

Recognition. Every pair of frames of $\mathcal{F}$ lies in a quartet spanning a copy of $H_3(\mathbb{C})$ (Theorem 9.1 and its proof), so by Lemma 7.2 the product of their clock elements or conjugate clock elements is a nonzero multiple of e or of a clock element or conjugate clock element of a frame of that quartet. Proposition 9.3 applies. □

[141] Baez, "The octonions", note 10. §4.1.

The proof of Theorem 9.4 uses only that each pair lies in a quartet; the incidence of PG(2, 3) in Theorem 9.1 describes the configuration. For a sparsest system the 27 one-dimensional subspaces spanned by e, the clock elements and the conjugate clock elements are the homogeneous components of a grading of $J_{\mathbb{C}}$ by $\mathbb{F}_3^3$. That grading is known to be unique up to equivalence, and the corresponding Jordan subgroup of the complex group $F_4(\mathbb{C})$ up to conjugacy[142,143,144,145]. Theorem 9.4 starts instead from thirteen frames and the number of positive shares, with no group given.

9.2 Rotating one quartet of the graded system

The graded system is not the only thirteen-plane system. Cohn, Kumar and Minton exhibit a four-parameter family of thirteen pairwise unbiased frames in $H_3(\mathbb{O})$ (in their terms, a four-dimensional family of tight codes of 39 points in the octonionic projective plane)[146], and near their original configuration the systems form patches of dimension ten modulo F_4, by an earlier AI-generated result of the author[147]. A simpler construction rotates one quartet of the graded system.

Let E be a ternary grading group, with its graded system (Proposition 7.3). Fix a projective line ℓ of PG(2, 3) and an automorphism ϕ of the copy $J_\ell \cong H_3(\mathbb{C})$ of Proposition 7.3(v), for instance conjugation by a unitary matrix. The *rotated system* replaces the four frames of ℓ by their images under ϕ and keeps the other nine.

Proposition 9.5. (i) *Every rotated system is a thirteen-plane system.*

Formalized. `lineRotatedPlaneSystems`.

(ii) *A rotated system has exactly* 52 *positive shares if and only if ϕ permutes the four planes U_p, $p \in \ell$. No other rotated system is the image of the graded system under an automorphism of J.*

Proof. (i) An automorphism of J_ℓ carries frames of J_ℓ to frames, and a frame of the subalgebra J_ℓ is a frame of J, since its idempotents have trace one in J (proof of Lemma 7.2). The images are pairwise unbiased, and their planes lie in J_ℓ^0, which is orthogonal to the nine other planes.

(ii) Let s, s' be points off ℓ, let r be the point where the projective line ss' meets ℓ, and let s'' be its fourth point. By Proposition 7.3(iv), $U_s \circ U_{s'} = U_r \oplus U_{s''}$, and this space does not change under the rotation. The plane $\phi(U_p) \subset J_\ell^0$ is orthogonal to $U_{s''}$, so its share with $U_s, U_{s'}$ is

$$A_{rp} = \tfrac{1}{2} \operatorname{Tr}\bigl(\Pi_r \Pi'_p\bigr),$$

where Π_r and Π'_p are the projections onto U_r and $\phi(U_p)$. Since the four planes U_r, $r \in \ell$, fill J_ℓ^0, $\sum_r A_{rp} = \frac{1}{2} \operatorname{Tr} \Pi'_p = 1$, and $A_{rp} = 1$ only if $\phi(U_p) = U_r$, because

[142] Alekseevskii, "Finite commutative Jordan subgroups", note 115.
[143] Griess, "Elementary abelian p-subgroups", note 116. Thm. (7.4), p. 278.
[144] An, Dietrich and Litterick, "Elementary abelian subgroups", note 117. Prop. 6.1.
[145] Elduque and Kochetov, "Weyl groups of fine gradings", note 28. §4.1, after Eq. (16).
[146] Cohn, Kumar and Minton, "Optimal simplices and codes", note 6. §8.2, after Thm. 8.3, p. 1352.
[147] Earlier AI-generated work, "On the local geometry of the tight 39-point code", note 25. Cor. 1.4.

$\mathrm{Tr}(\Pi_r \Pi'_p) = 2$ forces the range of Π'_p into that of Π_r. If $\phi(U_p)$ is none of the original planes, then some A_{rp} lies strictly between 0 and 1, and every $r \in \ell$ arises from a pair s, s', so the rotated system has a share strictly between 0 and 1 and, by Theorem 9.1, more than 52 positive shares. If ϕ permutes the four planes, the system is unchanged. An automorphism of J preserves inner products and products, hence shares, so a system with more than 52 positive shares is not the image of the graded system. □

Remark 9.6. For ϕ conjugation by $u \in \mathrm{SU}(3)$, outside a closed subset of SU(3) with empty interior, the rotated system has exactly 160 positive shares. Classify the triples. Three rotated planes have share one, since two of them generate a copy of $H_3(\mathbb{C})$ inside J_ℓ, hence equal to it, and its product space is the sum of the other two. Two rotated planes and one other plane have share zero, since their product space lies in J^0_ℓ. One rotated plane $\phi(U_p)$ and two planes $U_s, U_{s'}$ off ℓ have share A_{rp}, where r is the point of ℓ on the projective line ss'. Three planes off ℓ keep their graded shares, one for the three points off ℓ of each of the twelve projective lines other than ℓ, zero otherwise. Each A_{rp}, as a function of u, is a polynomial in the entries of u and $\bar{u}$, so it is real analytic on the connected manifold SU(3). It is not identically zero, because some u carries the frame of U_p to that of U_r, where $A_{rp} = 1$. So its zero set is closed with empty interior, and so is the union of the sixteen zero sets. Off that union all A_{rp} are positive, and the number of positive shares is $4 + 12 + 4\binom{9}{2} = 160$.

Remark 9.7. Identify J_ℓ with $H_3(\mathbb{C})$ as in Lemma 7.2, so that the four frames of ℓ are the eigenframes of Z, X, XZ and XZ^2, and order the four planes of ℓ in this way. Let ϕ be conjugation by $u = \frac{1}{7} v v^{\mathsf{T}} - 1$, $v = (1, 2, 3)^{\mathsf{T}}$, a real symmetric orthogonal matrix of determinant one. The centered frame elements $q_a = p_a - \frac{1}{3} e$ of a frame satisfy $\sum_a \langle q_a, x \rangle q_a = x$ for every x in its plane, so the projection onto the plane is $\Pi = \sum_a q_a \langle q_a, \cdot \rangle$ and $\mathrm{Tr}(\Pi_r \Pi'_p) = \sum_{a,b} \langle q_a, q'_b \rangle^2$. With $\langle q_a, q'_b \rangle = |x_a^* u y_b|^2 - \frac{1}{3}$ this gives

$$A_{rp} = \tfrac{1}{2} \sum_{a,b} \Big(|x_a^* u y_b|^2 - \tfrac{1}{3} \Big)^2$$

for orthonormal eigenbases (x_a) of the r-th frame and (y_b) of the p-th. For $r = p$ the diagonal frame, $x_a = y_a = e_a$ and $A_{\mathsf{ZZ}} = \frac{1}{2} \big(\sum_{a,b} u_{ab}^4 - 1 \big) = \frac{889}{2401}$ from the entries $\frac{1}{7}(-6, -3, 2)$ on the diagonal and $\frac{1}{7}(2, 3, 6)$ off it. The other entries are computed in the same way, by exact arithmetic in $\mathbb{Q}(\omega)$, and give

$$2401\,(A_{rp}) = \begin{pmatrix} 889 & 672 & 420 & 420 \\ 672 & 697 & 516 & 516 \\ 420 & 516 & 337 & 1128 \\ 420 & 516 & 1128 & 337 \end{pmatrix}.$$

Each column sums to 2401, as $\sum_r A_{rp} = 1$ requires. Every entry lies strictly between 0 and 1, so this rotated system has 160 positive shares (Remark 9.6).

9.3 Subalgebras generated by three planes

A third plane can enlarge the copy of $H_3(\mathbb{C})$ generated by two planes in only two ways, and the quaternionic lower bound of Section 6 limits how often it fails to reach all of J.

Lemma 9.8. *Let $u \in \mathbb{O}$ be a unit imaginary, $\mathbb{C}_u = \mathbb{R} + \mathbb{R}u$, and let $S \subset H_3(\mathbb{O})$ be a unital subalgebra containing $H_3(\mathbb{C}_u)$. Then $S = H_3(\mathbb{K}')$ for a subalgebra $\mathbb{C}_u \subset \mathbb{K}' \subset \mathbb{O}$ of dimension 2, 4 or 8. So $\dim S \in \{9, 15, 27\}$.*

Formalized. `subalgebrasContainingComplexWindow`.

Proof. Let E_{ac} be the matrix units. Since S contains the diagonal idempotents E_{aa}, it contains the Jordan Peirce components of each of its elements, because the component at the positions $(a, c), (c, a)$ of x is $4E_{cc} \bullet (E_{aa} \bullet x)$, and the diagonal entries are obtained similarly. The diagonal part of S is $\mathbb{R}^3$. For $a \neq c$ let $K_{ac} \subset \mathbb{O}$ be the set of x with $xE_{ac} + \bar{x}E_{ca} \in S$, so $K_{ca} = \overline{K_{ac}}$ and $\mathbb{C}_u \subset K_{ac}$. For distinct a, c, d,

$$(xE_{ac} + \bar{x}E_{ca}) \bullet (yE_{cd} + \bar{y}E_{dc}) = \tfrac{1}{2}\big(xy\,E_{ad} + \overline{xy}\,E_{da}\big).$$

With $y = 1$ this gives $K_{ac} \subset K_{ad}$ for all distinct a, c, d, so $K_{12} = K_{13}$, $K_{21} = K_{23}$ and $K_{31} = K_{32}$. Then $K_{31} = \overline{K_{13}} = \overline{K_{12}} = K_{21} = K_{23} = \overline{K_{32}} = \overline{K_{31}}$, so all six sets equal one real subspace $\mathbb{K}'$ closed under conjugation. The display shows that $\mathbb{K}'$ is closed under multiplication. A subalgebra of $\mathbb{O}$ containing 1 is a composition algebra, so by Hurwitz's theorem[148](Theorem 4.4) it has dimension 1, 2, 4 or 8, and it contains $\mathbb{C}_u$. Then S is the set of Hermitian matrices with entries in $\mathbb{K}'$. □

Theorem 9.9 (Generation)**.** (i) *The frames of two orthogonal planes of V generate a copy of $H_3(\mathbb{C})$, whose traceless part has dimension 8.*

Formalized. (i) `nativeGenerationTwoPlanes`; (ii) `nativeGenerationThreePlanes`; (iii) `generationCount_closedPlanes`.

(ii) *The frames of three pairwise orthogonal planes generate a copy of $H_3(\mathbb{C})$, $H_3(\mathbb{H})$ or $H_3(\mathbb{O})$, with traceless part of dimension 8, 14 or 26.*

(iii) *In a thirteen-plane system, for every pair of planes and for at least seven of the other eleven planes, the frames of that plane and of the pair generate $H_3(\mathbb{O})$.*

Proof. (i) is Proposition 5.5. For (ii), apply the automorphism of Lemma 5.4(i) to the first two frames; the algebra they generate becomes $H_3(\mathbb{C}_u)$ (proof of Proposition 5.5), and Lemma 9.8 applies to the algebra generated by all three, since by Hurwitz's theorem (Theorem 4.4) a composition algebra of dimension 2, 4 or 8 is isomorphic to $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ by an isometry, which induces an isomorphism of the Hermitian matrix algebras preserving traces.

(iii) Fix a pair i, j, and let J_{ij} be the copy of $H_3(\mathbb{C})$ generated by their frames. Let $n_{\mathbb{C}}$ count the planes U_k with $U_k \subset J_{ij}$, and $n_{\mathbb{H}}$ those for which the three frames generate a copy S of $H_3(\mathbb{H})$. If $U_k \subset J_{ij}$, then $U_k \subset J_{ij}^0 \ominus (U_i \oplus U_j) = U_i \circ U_j$ and $s_{ijk} = 1$. In the second case the three frames are pairwise unbiased frames of S, the product space and the share are the same whether computed in S or in J, and Theorem 6.4 gives $s_{ijk} \geq 1 - 1/\sqrt{3}$. The sum rule gives $n_{\mathbb{C}} + n_{\mathbb{H}}\,(1 - 1/\sqrt{3}) \leq 2$. Also $n_{\mathbb{C}} \leq 2$, since J_{ij}^0 has room for only two planes besides U_i, U_j. For $n_{\mathbb{C}} = 0, 1, 2$ this

[148] Baez, "The octonions", note 10. Thm. 1 in §1.1.

forces $n_{\mathbb{H}} \le 4, 2, 0$, because $5(1 - 1/\sqrt{3}) > 2$ and $3(1 - 1/\sqrt{3}) > 1$. So $n_{\mathbb{C}} + n_{\mathbb{H}} \le 4$, and by (ii) at least seven of the eleven remaining planes generate J. □

In the graded system every pair has $n_{\mathbb{C}} = 2$ and $n_{\mathbb{H}} = 0$, and all nine noncollinear third planes generate J. Summing over the 78 pairs, at least $78 \cdot 7/3 = 182$ triples of any thirteen-plane system generate J. Whether seven is attained is open.

10 Planes from Cartan subalgebras

The planes of Sections 5–9 also arise from Lie algebras. The compact Lie algebras $\mathfrak{su}(3)$, $\mathfrak{sp}(3)$ and $\mathfrak{f}_4$ are the derivation algebras of $H_3(\mathbb{C})$, $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$, and every Cartan subalgebra fixes exactly one frame plane in V. This section proves that the planes fixed by the summands of an orthogonal decomposition of the Lie algebra into Cartan subalgebras form an orthogonal decomposition of V into closed subspaces, all of them planes. Of the later text, only Appendix B.1 depends on this section.

10.1 Orthogonal decompositions

Let $J = H_3(\mathbb{K})$ with $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, with Jordan product $\bullet$, and $V = H_3^0(\mathbb{K})$ with $\langle x, y\rangle = \operatorname{tr}(x \bullet y)$. Let $G_{\mathbb{K}}$ be SU(3) or Sp(3) acting on J by $X \mapsto gXg^*$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$, and $F_4 = \operatorname{Aut}(J)$ for $\mathbb{K} = \mathbb{O}$. Each $g \in G_{\mathbb{K}}$ is an automorphism of J preserving the trace, visibly for SU(3) and Sp(3), and for F_4 because every automorphism of $H_3(\mathbb{O})$ does[149]. So g preserves V and $\langle \cdot, \cdot \rangle$.

Let $\mathfrak{k}_{\mathbb{K}}$ be the space of operators $X \mapsto [A, X]$ on J with A in $\mathfrak{su}(3)$ or $\mathfrak{sp}(3)$ (the anti-Hermitian matrices, traceless for $\mathbb{C}$) for $\mathbb{K} = \mathbb{C}, \mathbb{H}$, and the space of derivations of J skew for the trace form for $\mathbb{K} = \mathbb{O}$. Its elements are derivations of J, skew for the trace form, and kill e. They are regarded as skew operators on V, with the inner product

$$\langle X, Y\rangle_{\mathfrak{k}} = -\operatorname{Tr}_V(XY).$$

In Lie theory $\mathfrak{k}_{\mathbb{K}}$ is the Lie algebra $\mathfrak{su}(3)$, $\mathfrak{sp}(3)$, compact $\mathfrak{f}_4$ of $G_{\mathbb{K}}$; it is all of $\operatorname{Der}(J)$, of dimension 8, 21, 52[150,151], and $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is a positive multiple of minus the Killing form, since $\mathfrak{k}_{\mathbb{K}}$ is simple. A *Cartan subalgebra* of $\mathfrak{k}_{\mathbb{K}}$ is a maximal commutative subalgebra; all have the same dimension $\ell = 2, 3, 4$, the rank (for $\mathfrak{f}_4$ by Yokota[152]), and any two are conjugate under $G_{\mathbb{K}}$[153].

Definition 10.1. (a) An *orthogonal decomposition into Cartan subalgebras* of $\mathfrak{k}_{\mathbb{K}}$ is a splitting $\mathfrak{k}_{\mathbb{K}} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_m$ into Cartan subalgebras that are pairwise orthogonal for $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$, equivalently for the Killing form.

[149] Yokota, "Exceptional Lie groups", note 44. Lemma 2.2.1(2).

[150] Baez, "The octonions", note 10. §4.2, Eq. (19).

[151] Yokota, "Exceptional Lie groups", note 44. Thms. 2.3.2, 2.3.8, 2.10.2.

[152] Ibid. Thm. 2.6.1.

[153] A. W. Knapp, Structure theory of semisimple Lie groups, *Proc. Sympos. Pure Math.* **61**, Amer. Math. Soc., Providence, 1997, 1–27. `https://www.math.stonybrook.edu/~aknapp/pdf-files/1-27.pdf`. Thm. 3.6, p. 16.

(b) A commutative subspace $\mathfrak{t} \subset \mathfrak{k}_{\mathbb{K}}$ of dimension ℓ is of *frame type* if $\dim V^{\mathfrak{t}} = 2$ and $\Omega_{\mathfrak{t}} = \kappa(1 - P_{\mathfrak{t}})$ with $\kappa = \ell/(N-2)$, that is $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ (notation of Section 10.2). In terms of weights: the zero weight space is two-dimensional and all nonzero weights have squared length κ.

For a commutative subspace $\mathfrak{t}$ with $\dim V^{\mathfrak{t}} = 2$, the subspace $V^{\mathfrak{t}}$, the real form of the zero weight space, is the *zero weight plane* of $\mathfrak{t}$ (a plane by Proposition 10.5(a)); every Cartan subalgebra of $\mathfrak{k}_{\mathbb{K}}$ has one (Propositions 10.5 and 10.6).

Counting dimensions gives $m = \dim \mathfrak{k}/\ell$, which is $8/2 = 4$, $21/3 = 7$ and $52/4 = 13$, the numbers of planes in Corollary 5.2. The notion is due to Kostrikin, Kostrikin and Ufnarovskii, who posed it for complex simple Lie algebras; the monograph of Kostrikin and Tiep collects the theory[154]. Definition 10.1(a) is its analogue for the compact form. The smallest case is classical in another language: Boykin, Sitharam, Tiep and Wocjan showed that orthogonal decompositions of $\mathfrak{sl}_n(\mathbb{C})$ into Cartan subalgebras closed under the conjugate transpose correspond to complete sets of $n+1$ mutually unbiased bases of $\mathbb{C}^n$[155]. By Lemma 10.2 below these are the complexifications of the orthogonal decompositions of compact $\mathfrak{su}(n)$ into Cartan subalgebras, the form in which Kramer states the correspondence[156]. For $n = 3$ these are the four planes of Theorem 5.10.

The compact and the complex problems are related as follows. Let $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ and let σ be the complex conjugation of $\mathfrak{g}$ fixing $\mathfrak{k}$ (a *compact conjugation*). Then $\eta(x, y) = -B(x, \sigma y)$, with B the Killing form of $\mathfrak{g}$, is a positive definite Hermitian form. For any compact semisimple $\mathfrak{k}$, Cartan subalgebras and orthogonal decompositions into Cartan subalgebras are defined as for $\mathfrak{k}_{\mathbb{K}}$ (maximal commutative subalgebras; Definition 10.1(a)), with minus the Killing form as inner product. In the complex semisimple algebra $\mathfrak{g}$ a *Cartan subalgebra* is a nilpotent subalgebra equal to its own normalizer, equivalently a maximal commutative subalgebra on which every $\operatorname{ad} H$ is diagonalizable[157]; a maximal commutative subalgebra alone need not be one. An *orthogonal decomposition of* $\mathfrak{g}$ *into Cartan subalgebras* is a direct sum of Cartan subalgebras pairwise orthogonal for B, the notion of Kostrikin, Kostrikin and Ufnarovskii.

Lemma 10.2. *Let $\mathfrak{k}$ be compact semisimple. The complexifications of the summands of an orthogonal decomposition of $\mathfrak{k}$ into Cartan subalgebras form an orthogonal decomposition of $\mathfrak{g}$ into Cartan subalgebras. Conversely, if σ permutes the summands of an orthogonal decomposition $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_m$ into Cartan subalgebras, then it fixes each of them, and $\mathfrak{k} = \bigoplus_r (\mathfrak{h}_r \cap \mathfrak{k})$ is an orthogonal decomposition of $\mathfrak{k}$ into Cartan subalgebras.*

Formalized, different route. `compactCartanOD_complexification`, `compactCartanOD_descent`: that the complexified summands are Cartan subalgebras is proved in the development, instead of quoting Knapp.

Proof. The complexification of a maximal commutative subalgebra of a compact Lie algebra is a Cartan subalgebra of its complexification[158]. The Killing form of $\mathfrak{g}$

[154] Kostrikin and Tiep, *Orthogonal decompositions*, note 11. pp. 1–3.

[155] P. O. Boykin, M. Sitharam, P. H. Tiep and P. Wocjan, Mutually unbiased bases and orthogonal decompositions of Lie algebras, *Quantum Inf. Comput.* **7** (2007) 371–382. doi:10.26421/QIC7.4-6. arXiv:quant-ph/0506089. §4, Thm. 1 and Cor. 1, pp. 374–375.

[156] L. Kramer, A note on commutators in compact semisimple Lie algebras, *Innov. Incidence Geom.* **20** (2023), no. 2–3, 317–324. doi:10.2140/iig.2023.20.317. arXiv:2204.09913. p. 322.

[157] Knapp, "Structure theory of semisimple Lie groups", note 153. p. 2.

[158] Ibid. §3, pp. 16–17.

is the complex bilinear extension of that of $\mathfrak{k}$, so orthogonality and the dimension count carry over.

Conversely, suppose $\sigma\mathfrak{h}_r = \mathfrak{h}_s$ with $s \neq r$. For $0 \neq x \in \mathfrak{h}_r$ this gives $B(x, \sigma x) = 0$, against $\eta(x, x) > 0$. So $\sigma\mathfrak{h}_r = \mathfrak{h}_r$, and $\mathfrak{h}_r = (\mathfrak{h}_r \cap \mathfrak{k}) \oplus i(\mathfrak{h}_r \cap \mathfrak{k})$. The real part is commutative of dimension $\dim_{\mathbb{C}} \mathfrak{h}_r = \operatorname{rank} \mathfrak{k}$, hence a Cartan subalgebra of $\mathfrak{k}$. These parts are pairwise orthogonal, and their dimensions add up to $\dim \mathfrak{k}$. □

10.2 The Casimir lemma

For a subspace $\mathfrak{s}$ of skew operators on a Euclidean space V, with an orthonormal basis $X_1, \dots, X_k$ for the inner product $-\operatorname{Tr}(XY)$, put $\Omega_{\mathfrak{s}} = -\sum_a X_a^2$, a positive semidefinite symmetric operator independent of the basis. For a subspace $\mathfrak{t}$ let $V^{\mathfrak{t}}$ be the subspace killed by $\mathfrak{t}$ and $P_{\mathfrak{t}}$ the orthogonal projection onto it. If $\mathfrak{t}$ is commutative, $V \otimes \mathbb{C}$ is the orthogonal sum of joint eigenspaces V_μ on which $H \in \mathfrak{t}$ acts by $i\mu(H)$, with real *weights* $\mu \in \mathfrak{t}^*$, and $\Omega_{\mathfrak{t}}$ acts on V_μ by $|\mu|^2$. So if every nonzero weight has squared length κ, then $\Omega_{\mathfrak{t}} = \kappa(1 - P_{\mathfrak{t}})$.

Lemma 10.3 (Casimir lemma)**.** *Let V be a Euclidean space of dimension N and $\mathfrak{k}$ a space of skew operators on V with $\Omega_{\mathfrak{k}} = \lambda \cdot 1$. Let $\mathfrak{k} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_m$ be a splitting into pairwise orthogonal subspaces, and let $\kappa > 0$ with $\Omega_{\mathfrak{t}_r} = \kappa(1 - P_r)$ for every r, where $P_r = P_{\mathfrak{t}_r}$. If $\sum_r \dim V^{\mathfrak{t}_r} = N$, the spaces $V^{\mathfrak{t}_r}$ are pairwise orthogonal and $V = V^{\mathfrak{t}_1} \oplus \cdots \oplus V^{\mathfrak{t}_m}$.*

Formalized. `fixedSpaces_orthSplit_of_casimir`.

Proof. The orthonormal bases of the $\mathfrak{t}_r$ together form an orthonormal basis of $\mathfrak{k}$, so $\lambda \cdot 1 = \sum_r \Omega_{\mathfrak{t}_r} = \kappa\big(m \cdot 1 - \sum_r P_r\big)$. Hence $\sum_r P_r = (m - \lambda/\kappa) \cdot 1$. Taking traces, $\sum_r \dim V^{\mathfrak{t}_r} = (m - \lambda/\kappa)N$, so $\sum_r P_r = 1$ (for $N > 0$; for $N = 0$ there is nothing to prove). Orthogonal projections with sum 1 are mutually orthogonal: $P_r = P_r\big(\sum_s P_s\big)P_r = P_r + \sum_{s \neq r}(P_s P_r)^*(P_s P_r)$, and a sum of positive semidefinite operators vanishes only if each term does, so $P_s P_r = 0$ for $s \neq r$. □

Thiel and Williams split the Casimir operator along an orthogonal decomposition into Cartan subalgebras in the same way and take its trace on every irreducible representation, which gives the mean squared length of its weights[159]. Lemma 10.3 keeps the operators $\Omega_{\mathfrak{t}_r}$ and reads off their kernels. For $\mathfrak{k} = \mathfrak{k}_{\mathbb{K}}$ the operator $\Omega_{\mathfrak{k}}$ is the Casimir operator, scalar because V is irreducible; the next lemma gives a direct reason.

[159] Thiel and Williams, "Strange expectations and the Winnie-the-Pooh problem", note 32. §6.1, pp. 12–13.

Lemma 10.4. *Every symmetric operator S on V commuting with $G_{\mathbb{K}}$ is a multiple of the identity. In particular $\Omega_{\mathfrak{k}_{\mathbb{K}}} = (\dim \mathfrak{k}_{\mathbb{K}}/N)\cdot 1$.*

Formalized, different route. `complex_scalar_commutant`, `quaternionJordan_commuting_eq_smul_id`, `albertJordan_commuting_eq_smul_id`, for every operator, symmetric or not; the Casimir: `complex_full_casimir`, `quaternionFullCasimir`, `albertFullCasimir`: the commutant is computed on the diagonal with sign changes and permutations, instead of irreducibility and Höhn's invariant ring.

Proof. For SU(3) and Sp(3), V is irreducible: it is the adjoint representation of the simple algebra $\mathfrak{su}(3)$, respectively the real form of the 14-dimensional representation $\Lambda^2_0\mathbb{C}^6$. The eigenspaces of S are invariant, so S has a single eigenvalue. For F_4, $x \mapsto \langle Sx, x\rangle$ is an invariant quadratic form, and by the spectral theorem, as Höhn records in Lemma 5.1[160], the F_4-invariant polynomials on V are generated by $|x|^2$ and the cubic, so this form is a multiple of $|x|^2$ and S is scalar.

The group $G_{\mathbb{K}}$ normalizes $\mathfrak{k}_{\mathbb{K}}$ and preserves $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$, so it maps orthonormal bases to orthonormal bases and commutes with $\Omega_{\mathfrak{k}_{\mathbb{K}}}$. The scalar is fixed by $\operatorname{Tr}\Omega_{\mathfrak{k}} = \sum_a \langle X_a, X_a\rangle_{\mathfrak{k}} = \dim \mathfrak{k}$. □

10.3 Zero weight planes and compact $\mathfrak{sp}(3)$

For $\mathbb{K} = \mathbb{H}$, Corollary 10.11 below uses complex coordinates. Write quaternions as $q = z_1 + jz_2$ with $z_1, z_2 \in \mathbb{C}$, matched with the form $a + bj$ of Section 6 at the start of Section 6.2. This makes $\mathbb{H}^3$ a right complex vector space $\mathbb{C}^6$, and for $u, w \in \mathbb{H}^3$ the product $\langle u, w\rangle_{\mathbb{H}} = \sum_s \bar{u}_s w_s = \alpha + j\beta$ ($\alpha, \beta \in \mathbb{C}$) has α equal to the standard Hermitian product $\langle u, w\rangle_{\mathbb{C}}$ of $\mathbb{C}^6$. Let $\mathcal{J}w = wj$, an antilinear map with $\mathcal{J}^2 = -1$. Since $zj = j\bar{z}$ for $z \in \mathbb{C}$, $\langle u, \mathcal{J}w\rangle_{\mathbb{H}} = (\alpha + j\beta)j = -\bar{\beta} + j\bar{\alpha}$, so $\langle u, \mathcal{J}w\rangle_{\mathbb{C}} = -\bar{\beta}$ and

$$|\langle u, w\rangle_{\mathbb{H}}|^2 = |\langle u, w\rangle_{\mathbb{C}}|^2 + |\langle u, \mathcal{J}w\rangle_{\mathbb{C}}|^2. \tag{10.1}$$

Proposition 10.5. (a) *For every commutative subspace $\mathfrak{t} \subset \mathfrak{k}_{\mathbb{K}}$, $\mathbb{R}e \oplus V^{\mathfrak{t}}$ is a subalgebra of J; if $\dim V^{\mathfrak{t}} = 2$, then $V^{\mathfrak{t}}$ is a plane (Definition 3.10), the plane of a Jordan frame.*

(b) *If $\mathfrak{t}$ is of frame type and $g \in G_{\mathbb{K}}$, then $g\mathfrak{t}g^{-1}$ is of frame type, with $V^{g\mathfrak{t}g^{-1}} = gV^{\mathfrak{t}}$.*

(c) *For $\mathbb{K} = \mathbb{C}, \mathbb{H}$, every Cartan subalgebra of $\mathfrak{k}_{\mathbb{K}}$ is $g\mathfrak{t}_0 g^{-1}$ with $g \in G_{\mathbb{K}}$, where $\mathfrak{t}_0$ consists of the maps $[A, \cdot\,]$ with $A = i\operatorname{diag}(a)$, $\sum_s a_s = 0$ (for $\mathbb{C}$), or $A = \operatorname{diag}(a_1 i, a_2 i, a_3 i)$ (for $\mathbb{H}$); and every Cartan subalgebra is of frame type.*

Formalized, different route. (a) `complex_fixedSubalgebra_and_framePlane`, `quaternion_fixedSubalgebra_and_framePlane`, `albert_fixedSubalgebra_and_framePlane`, for every subspace; (b) `conjugate_frameType_fixed`, for every orthogonal g; (c) `complexOperatorCartan_conjugateStandard`, `complexOperatorCartan_frameType`, `quaternionicOperatorCartan_conjugateStandard`, `quaternionicOperatorCartan_frameType`: the conjugacy to $\mathfrak{t}_0$ is proved by simultaneous diagonalization, instead of quoting Knapp.

Proof. (a) A derivation killing x and y kills $x \bullet y$, and derivations kill e; so the kernel of $\mathfrak{t}$ in J, which is $\mathbb{R}e \oplus V^{\mathfrak{t}}$, is a subalgebra, and $V^{\mathfrak{t}}$ is closed under the traceless product. A plane is a frame plane by Proposition 5.1.

(b) Conjugation by g preserves $\langle\cdot,\cdot\rangle_{\mathfrak{t}}$, so $\Omega_{g\mathfrak{t}g^{-1}} = g\Omega_{\mathfrak{t}}g^{-1}$, and $V^{g\mathfrak{t}g^{-1}} = gV^{\mathfrak{t}}$.

(c) The space $\mathfrak{t}_0$ is commutative of dimension $\ell = 2, 3$, the common dimension of all Cartan subalgebras, so it is contained in a Cartan subalgebra and equals it. Any two Cartan subalgebras are conjugate under $G_{\mathbb{K}}$ (Section 10.1), and conjugation by g carries $[A, \cdot\,]$ to $[gAg^*, \cdot\,]$.

By (b) it remains to show that $\mathfrak{t}_0$ is of frame type. For $\mathbb{C}$, $[i\operatorname{diag}(a), \cdot\,]$ multiplies the entry in place (s, t) by $i(a_s - a_t)$. For $\mathbb{H}$, write an off-diagonal entry as $x = z + wj$ with $z, w \in \mathbb{R} + \mathbb{R}i$; from $wji = -iwj$,

$$a_s i\, x - x\, a_t i = (a_s - a_t)\, iz + (a_s + a_t)\, iw\, j.$$

[160] G. Höhn, The unique extremal threemodular lattice of rank 26, the generalized hexagon $(2, 8)$, and the tight Cayley-plane 5-design. arXiv:2609.25086, 2026. pp. 20–21.

So $V^{\mathfrak{t}_0} = V_0$, the traceless real diagonal, and the nonzero weights of $\delta_a = [A, \cdot\,]$ are $\pm(a_s - a_t)$, and for $\mathbb{H}$ also $\pm(a_s + a_t)$. Here $\langle \delta_a, \delta_b \rangle_{\mathfrak{t}}$ is $2\sum_{s<t}(a_s - a_t)(b_s - b_t) = 6\sum_s a_s b_s$ for $\mathbb{C}$ and $2\sum_{s<t}\big((a_s - a_t)(b_s - b_t) + (a_s + a_t)(b_s + b_t)\big) = 8\sum_s a_s b_s$ for $\mathbb{H}$, so each weight has squared length $2/6 = \frac{1}{3}$, respectively $2/8 = \frac{1}{4}$. □

For $\mathbb{K} = \mathbb{O}$ the same holds, with a Cartan subalgebra of the Lie algebra of the frame stabilizer Spin(8) in place of the diagonal matrices.

Proposition 10.6. *Every Cartan subalgebra of compact $\mathfrak{f}_4$ is of frame type.*

Formalized, different route. `compactF4Tori_frameType`, `nativeAlbertCartan_frameType`: frame type is proved for a Cartan subalgebra of a ternary grading, `nativeGradedCartan_frameType`, and carried to every other one by a conjugacy proved in the development, instead of from the weights of Spin(8).

Proof. By Proposition 10.5(b) and the conjugacy of Cartan subalgebras (Section 10.1) it suffices to treat one Cartan subalgebra. The automorphisms fixing E_{11}, E_{22}, E_{33} form Spin(8), which acts on the three off-diagonal octonion entries by a related triple $(\alpha_1, \alpha_2, \alpha_3) \in \mathrm{SO}(8)^3$[161], that is, by its vector representation and its two half-spin representations[162]. Its Lie algebra is a copy of $\mathfrak{so}(8)$ of rank four[163], so a Cartan subalgebra $\mathfrak{t}$ of this $\mathfrak{so}(8)$ is commutative of dimension $4 = \operatorname{rank}\mathfrak{f}_4$ and is a Cartan subalgebra of $\mathfrak{f}_4$. It kills the traceless real diagonal V_0. In standard coordinates on $\mathfrak{t}$ the weights on the three entries are the eight weights $\pm\epsilon_s$ of the vector representation and the sixteen weights $\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ of the half-spin representations, none zero, so $V^{\mathfrak{t}} = V_0$ is two-dimensional. The restriction of $\langle \cdot, \cdot \rangle_{\mathfrak{t}}$ to the simple algebra $\mathfrak{so}(8)$ is invariant, hence a multiple of the standard form, for which all 24 weights have the same length. So $\Omega_{\mathfrak{t}} = \kappa(1 - P_{\mathfrak{t}})$ (Section 10.2), and taking traces, $\operatorname{Tr}\Omega_{\mathfrak{t}} = \dim\mathfrak{t} = 4$ equals 24κ, so $\kappa = \frac{1}{6}$, the value in Definition 10.1. □

Theorem 10.7. *Let $\mathfrak{t}_{\mathbb{K}} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_m$ ($m = 4, 7, 13$) be a splitting into pairwise orthogonal commutative subspaces of frame type. Then the subspaces $U_r = V^{\mathfrak{t}_r}$ are pairwise orthogonal planes, $V = U_1 \oplus \cdots \oplus U_m$, and their Jordan frames are pairwise unbiased. For $\mathbb{K} = \mathbb{C}, \mathbb{H}$, that is for $\mathfrak{su}(3)$ and $\mathfrak{sp}(3)$, every orthogonal decomposition into Cartan subalgebras is such a splitting.*

Formalized. `complex_operatorOD_fixedPlanes`, `quaternion_frameType Splitting_fixedPlanes`, `albert_frameType Splitting_fixedPlanes`; last sentence: `complex_cartanOD_fixedPlanes`, `quaternionicOperator Cartan_frameType`.

Proof. Lemma 10.4 makes $\Omega_{\mathfrak{t}}$ scalar, frame type gives $\Omega_{\mathfrak{t}_r} = \kappa(1 - P_r)$ with one κ, and $\sum_r \dim U_r = 2m = N$. Lemma 10.3 gives the orthogonal splitting, Proposition 10.5(a) makes each U_r a plane, and orthogonal frame planes belong to unbiased frames by Proposition 5.1. The last sentence is Proposition 10.5(c). □

Corollary 10.8 (Orthogonal decompositions of compact $\mathfrak{f}_4$ into Cartan subalgebras)**.** *For every orthogonal decomposition of compact $\mathfrak{f}_4$ into Cartan subalgebras the thirteen zero weight planes of its summands are pairwise orthogonal planes filling $H_3^0(\mathbb{O})$; their frames are thirteen pairwise unbiased frames.*

Formalized. `compactF4_cartanDecomposition_byFramePlanes`, which also derives the number thirteen.

Proof. By Proposition 10.6 every summand is of frame type; apply Theorem 10.7. □

For $\mathfrak{su}(3)$ the plane U_r is i times the matrices of $\mathfrak{t}_r$, so the theorem restates the decomposition. It has content for $\mathfrak{sp}(3)$ and $\mathfrak{f}_4$, where V is not the adjoint representation. For $\mathfrak{f}_4$ the 39 idempotents of the thirteen frames have pairwise

[161] Yokota, "Exceptional Lie groups", note 44. Thms. 1.14.2, 2.7.1.
[162] Baez, "The octonions", note 10.
[163] Yokota, "Exceptional Lie groups", note 44. Prop. 2.3.7.

overlaps $\operatorname{tr}(p \bullet q) \in \{0, \frac{1}{3}\}$, so every orthogonal decomposition of compact $\mathfrak{f}_4$ into Cartan subalgebras yields a tight 39-point code in the Cayley plane in the sense of Cohn, Kumar and Minton[164]. The theorem needs the summands to fill $\mathfrak{k}_{\mathbb{K}}$, which enters through $\sum_r \dim U_r = N$: two orthogonal Cartan subalgebras can fix the same plane.

Example 10.9. In $\mathfrak{sp}(3)$ let $\mathfrak{t}_u$ consist of the maps $[A, \cdot\,]$ with $A = \operatorname{diag}(a_1 u, a_2 u, a_3 u)$, $a \in \mathbb{R}^3$, for $u = i$ and $u = j$. Each is a Cartan subalgebra, conjugate to $\mathfrak{t}_i$ by $\operatorname{diag}(q, q, q)$ with $qiq^{-1} = j$. They are orthogonal: on each off-diagonal entry the two act by $x \mapsto a_s i x - x a_t i$ and $x \mapsto b_s j x - x b_t j$, and the composites $x \mapsto ijx$, ixj, jxi, xji have trace 0 on $\mathbb{H}$, since $x \mapsto pxq$ has trace $4 \operatorname{Re} p \operatorname{Re} q$ for imaginary units and $\operatorname{Re}(ij) = 0$. An off-diagonal entry x fixed by $\mathfrak{t}_u$ satisfies $a_s u x - x a_t u = 0$ for all a; taking $a_t = 0$ gives $ux = 0$, so $x = 0$. Hence both fix the same plane V_0.

Corollary 10.10. *Compact* $\mathfrak{sp}(3)$ (*type* C_3, *dimension* 21) *has no orthogonal decomposition into Cartan subalgebras. Equivalently, no orthogonal decomposition of* $\mathfrak{sp}_6(\mathbb{C})$ (*type* C_3) *into Cartan subalgebras has its seven summands permuted by a compact conjugation.*

Formalized. `no_compactSp3_cartanOD`; second sentence: `no_sp6C_OD_permuted_by_compactConjugation`.

Proof. By Theorem 10.7 an orthogonal decomposition of $\mathfrak{sp}(3)$ into Cartan subalgebras would split $H^0_3(\mathbb{H})$ into seven pairwise orthogonal planes, which Corollary 6.6 excludes. The second sentence is equivalent to the first by Lemma 10.2, since $\mathfrak{sp}(3) \otimes \mathbb{C} = \mathfrak{sp}_6(\mathbb{C})$. □

Kostrikin, Kostrikin and Ufnarovskii conjectured that the complex simple Lie algebra of type C_n has an orthogonal decomposition into Cartan subalgebras exactly when n is a power of two, and the case $n = 3$ is open, as recorded by Kantor[165]and by Kostrikin and Tiep[166]. Torstensson showed that C_3 has no monomial orthogonal decomposition into Cartan subalgebras containing the standard Cartan subalgebra[167], and Tiep excluded irreducible orthogonal decompositions into Cartan subalgebras, those whose automorphism group acts irreducibly[168](see also Theorem 4.5.1 of Kostrikin and Tiep[169]). Corollary 10.10 settles the part of the case $n = 3$ that is stable under a compact conjugation; the general complex case remains open (Remark 10.16). Kramer asks which compact semisimple Lie algebras have an orthogonal decomposition into Cartan subalgebras[170]; Corollary 10.10 answers this for $\mathfrak{sp}(3)$.

Call an orthonormal basis of $\mathbb{C}^6$ *$\mathcal{J}$-invariant up to phase* if $\mathcal{J}$ maps each of its lines onto one of its lines.

[164]Cohn, Kumar and Minton, "Optimal simplices and codes", note 6. Thm. 8.3.

[165]Kantor, "Note on Lie algebras, finite groups and finite geometries", note 33. p. 2.

[166]Kostrikin and Tiep, *Orthogonal decompositions*, note 11. Conj. 2.1.1, p. 56.

[167]A. Torstensson, On the existence of orthogonal decompositions of the simple Lie algebra of type C_3, *Comput. Sci. J. Moldova* **8** (2000) no. 1, 16–41. `https://www.math.md/files/csjm/v8-n1/v8-n1-(pp16-41).pdf`.

[168]P. H. Tiep, Irreducible orthogonal decompositions in Lie algebras, *Mat. Sb.* **180** (1989) 1396–1414; English transl. *Math. USSR-Sb.* **68** (1991) 257–275. doi:10.1070/SM1991v068n01ABEH002103. Math-Net.Ru sm1666.

[169]Kostrikin and Tiep, *Orthogonal decompositions*, note 11. p. 125.

[170]Kramer, "A note on commutators in compact semisimple Lie algebras", note 156. Problem 10, p. 322.

Corollary 10.11. *A family of pairwise unbiased orthonormal bases of $\mathbb{C}^6$, each $\mathcal{J}$-invariant up to phase, has at most five members. In particular no complete set of seven unbiased bases of $\mathbb{C}^6$ consists of such bases.*

Formalized. `atMostFive QuaternionicInvariant Bases`.

Proof. No line is mapped to itself: $\mathcal{J}v = \lambda v$ would give $-v = \mathcal{J}^2 v = |\lambda|^2 v$. So the six lines of a $\mathcal{J}$-invariant basis form three pairs $\{\mathbb{C}u, \mathbb{C}\mathcal{J}u\}$, and the basis is $\{u_s, \mathcal{J}u_s\}_{s=1}^{3}$ up to phases. By (10.1) the u_s are an orthonormal basis of $\mathbb{H}^3$. For two such bases $\{u_s, \mathcal{J}u_s\}$ and $\{w_t, \mathcal{J}w_t\}$, unbiasedness in $\mathbb{C}^6$ gives $|\langle u_s, w_t\rangle_{\mathbb{C}}|^2 = |\langle u_s, \mathcal{J}w_t\rangle_{\mathbb{C}}|^2 = \tfrac{1}{6}$, hence $|\langle u_s, w_t\rangle_{\mathbb{H}}|^2 = \tfrac{1}{3}$ by (10.1). So the quaternionic bases are pairwise unbiased, and Corollary 6.9 allows at most five. □

For arbitrary orthonormal bases of $\mathbb{C}^6$, OpenAI gives the bound of three mutually unbiased bases[171], which contains this corollary; the proof above does not use it.

The second row of the magic square (the row of $\mathbb{C}$), the compact Lie algebras $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$, $\mathfrak{su}(6)$ and compact $\mathfrak{e}_6$ built from $\mathfrak{f}_{\mathbb{K}}$ and the Jordan multiplications by V, carries the same decompositions, with one Cartan subalgebra for each plane, stable under the involution θ defined in Appendix B.1.

10.4 Thirteen Cartan subalgebras of $\mathfrak{f}_4$

Orthogonal decompositions of $\mathfrak{f}_4 \otimes \mathbb{C}$ into Cartan subalgebras exist. Kostrikin, Kostrikin and Ufnarovskii constructed one, and one of the complex Lie algebra of type E_6, from root vectors and cube roots of unity[172]. Burichenko later arrived at a decomposition of the same class[173], constructed from a Jordan subgroup $\mathbb{Z}_3^3$ (in the sense of[174]) of the complex Lie group of type F_4, with automorphism group $3^3{:}\mathrm{SL}_3(3)$[175]; Borovik showed that every Jordan subgroup gives such a decomposition[176]; Kostrikin and Tiep give the general construction[177]; and Elduque states it as the orthogonal decomposition into Cartan subalgebras attached to the $\mathbb{Z}_3^3$-grading of $\mathfrak{f}_4 \otimes \mathbb{C}$[178]. For Jordan subgroups of exponent three, Cohen and Tiep show that the Lie algebra has a rational form invariant under the normalizer of the subgroup[179], and gradings on the real forms of the Albert algebra are treated

[171] OpenAI, The maximum number of mutually unbiased bases in dimension six, preprint, 24 September 2026. github.com/openai/math.

[172] A. I. Kostrikin, I. A. Kostrikin and V. A. Ufnarovskii, On decompositions of classical Lie algebras (in Russian), *Trudy Mat. Inst. Steklov.* **166** (1984) 107–122; English transl.: Decompositions of classical Lie algebras, *Proc. Steklov Inst. Math.* **166** (1986) 117–133. Math-Net.Ru tm2255. §4, Thms. 4–5, pp. 119–121.

[173] Kostrikin and Tiep, *Orthogonal decompositions*, note 11. Ch. 3, Commentary, p. 103.

[174] Ibid. Def. 3.1.1, p. 84.

[175] V. P. Burichenko, Transitive orthogonal decompositions of simple complex Lie algebras of types F_4 and E_6, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 1988, no. 4, 78–80.

[176] A. V. Borovik, Jordan subgroups and orthogonal decompositions, *Algebra i Logika* **28** (1989) 382–392; English transl. *Algebra and Logic* **28** (1989). doi:10.1007/BF01982015.

[177] Kostrikin and Tiep, *Orthogonal decompositions*, note 11. Thm. 3.1.2, pp. 85–86.

[178] Elduque, "Jordan gradings on exceptional simple Lie algebras", note 31. §1, Eq. (1.2); Main Theorem (ii), Thm. 3.5.

[179] A. M. Cohen and P. H. Tiep, Splitting fields for Jordan subgroups, in *Finite Reductive Groups* (Luminy, 1994), Progr. Math. **141**, Birkhäuser, Boston, 1997, 165–183.

by Calderón, Draper and Martín[180]. The proof below identifies the grading of $\mathfrak{f}_4$ induced by a ternary grading group with Elduque's and identifies the planes.

Fix a ternary grading group E of $J = H_3(\mathbb{O})$ (Section 7.2). By Proposition 7.3, E acts on $J_{\mathbb{C}} = \bigoplus_{g\in\mathbb{F}_3^3} A_g$, each A_g one-dimensional, with E acting on it by the character g, with $A_0 = \mathbb{C}e$, and A_g is spanned by u_g with $u_g \bullet u_g = u_{-g}$ and $u_g \bullet u_{-g} = e$. For $p = \{\pm g\}$ the plane is $U_p = J \cap (A_g \oplus A_{-g})$. One more fact is needed.

(G) $u_g \bullet u_h$ is a nonzero multiple of u_{g+h} for linearly independent g, h. Indeed it lies in A_{g+h}, which is traceless. Let $p \ni g$, $q \ni h$. By Lemma 4.10, applied to the cubic $\sqrt{6}\,d$ of constant 1, the product map $U_p \otimes U_q \to V$ is injective, and so is its complexification. Since $u_g \bullet u_h$ is traceless, it equals $u_g \circ u_h$, the image of $u_g \otimes u_h \neq 0$.

Hence, if g, h, k is a basis of $\mathbb{F}_3^3$, the elements u_g, u_h, u_k generate $J_{\mathbb{C}}$: the grades reached contain 0, are closed under negation (squares) and under sums of independent pairs, hence contain $\operatorname{span}(g, h)$ and then every $w + ck$. Likewise u_g, u_h generate $\bigoplus_{w\in\operatorname{span}(g,h)} A_w$.

Proposition 10.12. *Let E be a ternary grading group of $H_3(\mathbb{O})$, let $\mathfrak{f}_{4,\gamma}^{\mathbb{C}}$ be the subspace of $\mathfrak{f}_4 \otimes \mathbb{C}$ on which conjugation by E acts by the character γ, and for $p = \{\pm\gamma\}$ put $\mathfrak{t}_p = (\mathfrak{f}_{4,\gamma}^{\mathbb{C}} \oplus \mathfrak{f}_{4,-\gamma}^{\mathbb{C}}) \cap \mathfrak{f}_4$. Then each $\mathfrak{t}_p$ is a Cartan subalgebra, $\mathfrak{f}_4 = \bigoplus_{p\in\mathrm{PG}(2,3)} \mathfrak{t}_p$ is an orthogonal decomposition into Cartan subalgebras, and $V^{\mathfrak{t}_p} = U_p$.*

Formalized, different route. `ternaryGrading_derivationCartanOD`, with $\mathfrak{f}_{4,\gamma}^{\mathbb{C}}$ the character space by `albertComplexGrade_mem_iff_character`: the orthogonality comes from the pairing invariant under E and the commutativity from the grades of dimension two, instead of Elduque's theorem.

Proof. Grading. Since E is abelian and normalizes $\mathfrak{f}_4$, $\mathfrak{f}_4 \otimes \mathbb{C} = \bigoplus_\gamma \mathfrak{f}_{4,\gamma}^{\mathbb{C}}$ and $[\mathfrak{f}_{4,\alpha}^{\mathbb{C}}, \mathfrak{f}_{4,\beta}^{\mathbb{C}}] \subset \mathfrak{f}_{4,\alpha+\beta}^{\mathbb{C}}$. An element of $\mathfrak{f}_{4,\gamma}^{\mathbb{C}}$ maps A_g into $A_{g+\gamma}$. The group E acts by real automorphisms, so complex conjugation τ maps $\mathfrak{f}_{4,\gamma}^{\mathbb{C}}$ onto $\mathfrak{f}_{4,-\gamma}^{\mathbb{C}}$.

No zero grade. Let $\delta \in \mathfrak{f}_{4,0}^{\mathbb{C}}$, so $\delta u_g = \lambda_g u_g$ and $\delta e = 0$. Applying δ to $u_g \bullet u_{-g} = e$ gives $\lambda_g + \lambda_{-g} = 0$; applying it to $u_g \bullet u_g = u_{-g}$ gives $2\lambda_g = \lambda_{-g}$. So $\lambda_g = 0$ and $\mathfrak{f}_{4,0}^{\mathbb{C}} = 0$.

Two dimensions per grade. Let $\gamma \neq 0$ and $\delta \in \mathfrak{f}_{4,\gamma}^{\mathbb{C}}$. Then $\delta u_\gamma = a u_{-\gamma}$ and $\delta u_{-\gamma} = be$. Applying δ to $u_\gamma \bullet u_{-\gamma} = e$ gives $a u_\gamma + b u_\gamma = 0$; applying it to $u_{-\gamma} \bullet u_{-\gamma} = u_\gamma$ gives $2b u_{-\gamma} = a u_{-\gamma}$. So $a = b = 0$: every element of $\mathfrak{f}_{4,\gamma}^{\mathbb{C}}$ kills $A_\gamma \oplus A_{-\gamma}$. Complete γ to a basis γ, h, k. A derivation is determined by its values on generators, so δ is determined by the coefficients c_h, c_k in $\delta u_h = c_h u_{h+\gamma}$, $\delta u_k = c_k u_{k+\gamma}$, and $\dim \mathfrak{f}_{4,\gamma}^{\mathbb{C}} \leq 2$. These dimensions add up over the 26 nonzero grades to $\dim \mathfrak{f}_4 = 52$[181], so each is 2, and $\delta \mapsto (c_h, c_k)$ is bijective.

Cartan subalgebras. So the grading of $\mathfrak{f}_4 \otimes \mathbb{C}$ by $\mathbb{F}_3^3$ has zero neutral component and all other components of dimension two. By Elduque's theorem such a grading is unique up to equivalence, and for it each $\mathfrak{f}_{4,\gamma}^{\mathbb{C}} \oplus \mathfrak{f}_{4,-\gamma}^{\mathbb{C}}$ is a Cartan subalgebra and

[180] A. J. Calderón Martín, C. Draper and C. Martín González, Gradings on the real forms of the Albert algebra, of $\mathfrak{g}_2$, and of $\mathfrak{f}_4$, *J. Math. Phys.* **51** (2010) 053516. doi:10.1063/1.3326238.

[181] Yokota, "Exceptional Lie groups", note 44. Thm. 2.3.8.

the thirteen of them are pairwise orthogonal for the Killing form[182], the uniqueness being Theorem 3.5 and the two properties being recalled in the introduction there; an equivalence of gradings is an automorphism carrying components to components, so it preserves both properties. Kostrikin and Tiep obtain the same decomposition from the Jordan subgroup[183]. Each $\mathfrak{f}_{4,\gamma}^{\mathbb{C}} \oplus \mathfrak{f}_{4,-\gamma}^{\mathbb{C}}$ is τ-stable, so by Lemma 10.2 each $\mathfrak{t}_p$ is a Cartan subalgebra of $\mathfrak{f}_4$ and $\mathfrak{f}_4 = \bigoplus_p \mathfrak{t}_p$ is an orthogonal decomposition into Cartan subalgebras.

The planes. By the third step $\mathfrak{t}_p$ kills U_p. The fixed space of $\mathfrak{t}_p$ in $V \otimes \mathbb{C}$ is E-stable, since E normalizes $\mathfrak{t}_p \otimes \mathbb{C}$, so it is a sum of spaces A_g. For $g \notin \{0, \pm\gamma\}$ use the basis γ, g, k: bijectivity gives $\delta \in \mathfrak{f}_{4,\gamma}^{\mathbb{C}}$ with $\delta u_g \neq 0$. So the fixed space is $A_\gamma \oplus A_{-\gamma}$, and $V^{\mathfrak{t}_p} = U_p$. □

Uniqueness is known only in restricted classes: the irreducible orthogonal decomposition of $\mathfrak{f}_4 \otimes \mathbb{C}$ into Cartan subalgebras is unique up to conjugacy[184], and the $\mathbb{Z}_3^3$-grading of $\mathfrak{f}_4 \otimes \mathbb{C}$ with two-dimensional components is unique up to equivalence[185]. Without these restrictions the decomposition is not unique.

Proposition 10.13. *Let $\mathcal{L}$ be a projective line of the projective plane* PG(2, 3), *$\hat{\mathcal{L}} \subset \mathbb{F}_3^3$ the corresponding two-dimensional subspace, $J_{\mathcal{L}} = \mathbb{R}e \oplus \bigoplus_{p \in \mathcal{L}} U_p$ with a fixed isomorphism $\psi \colon J_{\mathcal{L}} \to H_3(\mathbb{C})$ (Proposition 7.3(v)), and $\mathfrak{k}_{\mathcal{L}} = \bigoplus_{p \in \mathcal{L}} \mathfrak{t}_p$.*

Formalized. `ternary Grading_lineRecut`, for every such ψ; one exists by `nativeGraded Line_has_hermitian Model`.

(i) *For every $R \in \mathrm{SU}(3)$ there is $X \in \mathfrak{k}_{\mathcal{L}}$ such that $\phi = \exp X$ is an automorphism of J preserving $J_{\mathcal{L}}$ and acting on $J_{\mathcal{L}}$ by $Y \mapsto \psi^{-1}(R\psi(Y)R^*)$, and such that $\phi \mathfrak{k}_{\mathcal{L}} \phi^{-1} = \mathfrak{k}_{\mathcal{L}}$.*

(ii) *Replacing $\mathfrak{t}_p$ by $\phi \mathfrak{t}_p \phi^{-1}$ for $p \in \mathcal{L}$ gives an orthogonal decomposition of $\mathfrak{f}_4$ into Cartan subalgebras with planes ϕU_p ($p \in \mathcal{L}$) and U_q ($q \notin \mathcal{L}$). If ϕ does not permute the four planes U_p, $p \in \mathcal{L}$, this decomposition is not the image of the graded one under conjugation by an automorphism of J.*

Proof. (i) $\mathfrak{k}_{\mathcal{L}}$ is closed under commutators, since $\hat{\mathcal{L}}$ is closed under addition and $\mathfrak{f}_{4,0}^{\mathbb{C}} = 0$, and its elements preserve $J_{\mathcal{L}} \otimes \mathbb{C} = \bigoplus_{w \in \hat{\mathcal{L}}} A_w$. By Proposition 7.3(v), $J_{\mathcal{L}} \cong H_3(\mathbb{C})$. For $0 \neq \alpha \in \hat{\mathcal{L}}$ choose $h \in \hat{\mathcal{L}}$ independent of α. An element of $\mathfrak{f}_{4,\alpha}^{\mathbb{C}}$ kills u_α, and u_α, u_h generate $J_{\mathcal{L}} \otimes \mathbb{C}$; so it acts trivially there exactly when $c_h = 0$, a one-dimensional subspace of $\mathfrak{f}_{4,\alpha}^{\mathbb{C}}$ by the bijectivity in Proposition 10.12. So the restrictions to $J_{\mathcal{L}} \otimes \mathbb{C}$ have complex dimension 8, one in each grade α.

On the other hand the derivations of $J_{\mathcal{L}} \otimes \mathbb{C}$ have dimension at most 8: the steps "no zero grade" and "two dimensions per grade" of Proposition 10.12, applied to $J_{\mathcal{L}} \otimes \mathbb{C}$ with its grading by $\hat{\mathcal{L}}$, give grade 0 zero and at most one dimension in each nonzero grade. So the restriction map from $\mathfrak{k}_{\mathcal{L}}$ onto the derivations of $J_{\mathcal{L}}$ is surjective, and these include $Y \mapsto [A, Y]$ for every $A \in \mathfrak{su}(3)$.

Write $R = \exp A$ with $A \in \mathfrak{su}(3)$ (diagonalize R and choose the logarithms of its eigenvalues with sum zero), and choose $X \in \mathfrak{k}_{\mathcal{L}}$ restricting to $[A, \cdot\,]$. The exponential

[182] Elduque, "Jordan gradings on exceptional simple Lie algebras", note 31. §1, Eqs. (1.1)–(1.2), and Thm. 3.5.

[183] Kostrikin and Tiep, *Orthogonal decompositions*, note 11. Thm. 3.1.2, pp. 85–86.

[184] Ibid. Thm. 4.5.1, p. 125.

[185] Elduque, "Jordan gradings on exceptional simple Lie algebras", note 31. Thm. 3.5.

of a derivation is an automorphism, $\phi = \exp X$ restricts to $\exp([A, \cdot\,]) = R(\cdot)R^*$, and $\phi\mathfrak{k}_{\mathcal{L}}\phi^{-1} = \exp(\operatorname{ad} X)\mathfrak{k}_{\mathcal{L}} = \mathfrak{k}_{\mathcal{L}}$.

(ii) The four spaces $\phi\mathfrak{t}_p\phi^{-1}$ ($p \in \mathcal{L}$) are pairwise orthogonal Cartan subalgebras of $\mathfrak{f}_4$ contained in $\mathfrak{k}_{\mathcal{L}}$ that span it, and they are orthogonal to the nine $\mathfrak{t}_q$ ($q \notin \mathcal{L}$), which span $\mathfrak{k}_{\mathcal{L}}^{\perp}$. Their fixed planes are ϕU_p, since $V^{\phi\mathfrak{t}\phi^{-1}} = \phi V^{\mathfrak{t}}$. If the new decomposition were the conjugate of the graded one by an automorphism of J, its planes would be the image of the graded system under that automorphism, which Proposition 9.5(ii) excludes. □

10.5 Other Lie algebras and the complex case

Remark 10.14 (Why the pattern stops at rank three). For a traceless x in a Jordan algebra of rank r, with eigenvalues λ_a, power sums $p_k = \sum \lambda_a^k$ and elementary symmetric functions σ_k, Newton's identity with $\sigma_1 = 0$ gives $p_4 = \frac{1}{2}p_2^2 - 4\sigma_4$. At rank three $\sigma_4 = 0$, so the traceless part of $x \bullet x$ has squared length $p_4 - \frac{1}{3}p_2^2 = \frac{1}{6}p_2^2$, which is the eikonal equation with $c = \frac{1}{6}$. At rank $r \geq 4$ the traceless diagonals $(1, -1, 0, 0, \dots)/\sqrt{2}$ and $(1, 1, -1, -1, 0, \dots)/2$ have $p_2 = 1$ and $p_4 = \frac{1}{2}$, $\frac{1}{4}$, so $\operatorname{tr} x^3$ is not eikonal on $H_r(\mathbb{K})_0$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Above $\mathfrak{e}_6$ a quartic law on the adjoint representation takes its place: $\operatorname{Tr}(\operatorname{ad} X)^4$ is a multiple of $\left(\operatorname{Tr}(\operatorname{ad} X)^2\right)^2$ exactly for Deligne's exceptional series $A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8$[186].

Remark 10.15 (E_7 and G_2). The construction does not extend to E_7. The Lie group of type E_7 has no Jordan subgroup[187], the only orthogonal decomposition of E_7 into Cartan subalgebras known to Kostrikin and Tiep is a root orthogonal decomposition into Cartan subalgebras, built from root vectors and containing the standard Cartan subalgebra[188], and E_7 has no irreducible orthogonal decomposition into Cartan subalgebras, by the same theorem that excludes them for C_3 and A_5[189].

For $\mathfrak{g}_2$ acting on the imaginary octonions, the nonzero weights are the six short roots and zero has multiplicity one, and $7 = 7 \cdot 1$ with $7 = \dim \mathfrak{g}_2/\operatorname{rank} \mathfrak{g}_2$. Lemma 10.3, with these weights as input, shows that an orthogonal decomposition of compact $\mathfrak{g}_2$ into Cartan subalgebras yields seven pairwise orthogonal one-dimensional zero weight spaces, spanned by an orthonormal basis of $\operatorname{Im}\mathbb{O}$. Such decompositions exist: the $\mathbb{Z}_2^3$-grading of $\mathfrak{g}_2$ gives seven Cartan subalgebras of the complex algebra[190], and since the grading group acts on the octonions by real automorphisms with real characters, each summand is defined over the compact form. These seven summands of $\mathbb{R}^7$ are one-dimensional, not planes, and $\mathfrak{g}_2$ has no cubic invariant on $\operatorname{Im}\mathbb{O}$.

[186] P. Deligne, La série exceptionnelle de groupes de Lie, *C. R. Acad. Sci. Paris Sér. I* **322** (1996) 321–326; P. Cvitanović, *Group Theory: Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press, 2008.

[187] Kostrikin and Tiep, *Orthogonal decompositions*, note 11. p. 92.

[188] Ibid. Introduction, p. 7; §3.2.

[189] Ibid. Thm. 4.5.1, p. 125.

[190] Elduque, "Jordan gradings on exceptional simple Lie algebras", note 31. §1, item (1) and Eq. (1.1); Main Theorem (i).

Remark 10.16 (The complex question). The splitting in Lemma 10.3 survives over $\mathbb{C}$. Let $\mathfrak{sp}_6(\mathbb{C}) = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_7$ be an orthogonal decomposition into Cartan subalgebras, regarded as operators on $V \otimes \mathbb{C}$, and extend $\langle \cdot, \cdot \rangle_{\mathfrak{t}}$ and the trace form of V complex-bilinearly; every element of $\mathfrak{sp}_6(\mathbb{C})$ is skew for the latter. Cartan subalgebras of a complex semisimple Lie algebra are conjugate under inner automorphisms[191], and an inner automorphism $\exp(\operatorname{ad} X)$ is induced on $V \otimes \mathbb{C}$ by conjugation with $\exp X$, an isometry of the bilinear trace form. So each $\mathfrak{h}_r$ is $g_r(\mathfrak{t}_0 \otimes \mathbb{C})g_r^{-1}$ for such an isometry g_r, with $\mathfrak{t}_0$ as in Proposition 10.5(c). Hence $\langle \cdot, \cdot \rangle_{\mathfrak{t}}$ is nondegenerate on $\mathfrak{h}_r$, and the toral Casimir operator $\Omega_r = -\sum_a X_a X^a$, built from dual bases of $\mathfrak{h}_r$, is $g_r \Omega_{\mathfrak{t}_0} g_r^{-1} = \kappa(1 - \Pi_r)$, where $\Pi_r = g_r P_{\mathfrak{t}_0} g_r^{-1}$ is the projection of $V \otimes \mathbb{C}$ onto the two-dimensional zero weight space of $\mathfrak{h}_r$ along its nonzero weight spaces.

The dual bases of the $\mathfrak{h}_r$ together form dual bases of $\mathfrak{sp}_6(\mathbb{C})$, so $\sum_r \Omega_r$ is the complexified Casimir operator, a scalar by Lemma 10.4. As in the proof of Lemma 10.3, taking traces gives $\sum_r \Pi_r = 1$. Idempotents with sum 1 whose ranks add up to $\dim V \otimes \mathbb{C} = 14$ have images forming a direct sum, so $\Pi_r \Pi_s = 0$ for $r \neq s$. Each Π_r is symmetric for the bilinear trace form, since Ω_r is (the X_a and X^a are skew and commute), so the seven two-dimensional zero weight spaces are pairwise orthogonal for it, and each is closed under the product because derivations kill products. What fails is positivity: the share of Lemma 5.8(iii), computed by the same formula, becomes a sum of squares of complex numbers, and Example 10.17 gives three pairwise unbiased frames of $H_3(\mathbb{H}) \otimes \mathbb{C}$ with share 0. So the proof of Corollary 10.10, through the quaternionic lower bound, uses the compact form.

Example 10.17 (Three complex frames with share zero)**.** Let $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ and write ι for the imaginary unit of the second factor, which commutes with $\mathbb{H}$. Extend the conjugation of $\mathbb{H}$, the real part Re and the trace tr complex-linearly. For X, Y in $A_{\mathbb{C}} = H_3(\mathbb{H}) \otimes \mathbb{C}$, the 3×3 matrices over $\mathbb{H}_{\mathbb{C}}$ equal to their conjugate transpose X^*, put $X \bullet Y = \frac{1}{2}(XY + YX)$. A *frame* of $A_{\mathbb{C}}$ is a triple of idempotents of trace one with pairwise products zero (their sum is then the unit, since $1 - \sum_a p_a$ is an idempotent of trace zero, hence zero), two frames are *unbiased* if $\operatorname{tr}(p_a \bullet p'_b) = \frac{1}{3}$ for all a, b (a complex-bilinear condition), and the share of three frames is given by the formula of Lemma 5.8(iii).

If M is a 3×3 matrix over $\mathbb{H}_{\mathbb{C}}$ with $M^*M = 1$, its columns h_a give a frame $(h_a h_a^*)_a$, since $h_a^* h_b$ is 1 for $a = b$ and 0 otherwise and $\operatorname{tr}(h_a h_a^*) = \operatorname{Re}(h_a^* h_a) = 1$.

Let $\alpha, \beta \in \mathbb{R} + \mathbb{R}\iota$, scalars of the second factor, central in $\mathbb{H}_{\mathbb{C}}$ and fixed by its conjugation, with $\alpha^2 + \beta^2 = 1$, and put $u = \alpha^2$, $v = \beta^2$, $q = \alpha + \beta j$ and $q_m = \omega^{m^2} q$, where $\omega = e^{2\pi i/3}$ is formed with the quaternion unit i, not with ι. Take the three frames of the columns of

$$1, \qquad \mathsf{F}, \qquad \operatorname{diag}(q_0, q_1, q_2)\,\mathsf{F},$$

the normal form (6.2) with complex coefficients. For real α, β the q_m are unit quaternions with $a_m = \omega^{m^2}\alpha$ and $b_m = \omega^{m^2}\beta$, both sums in (6.3) equal $(1 + \omega + \omega^2)$ times u or v, which is zero, and every z_m equals $u - v$. So Lemmas 6.2 and 6.3 give three pairwise unbiased frames with share $(3 + 3(u - v)^2)/6 = u^2 + v^2$, and the

[191] Knapp, "Structure theory of semisimple Lie groups", note 153. p. 7.

formula in the proof of Lemma 6.3 gives the entry $\langle p_0 \bullet p_0', p_0'' \rangle = \frac{1}{9} - \frac{1}{9}(u+v) = 0$ of (6.1), where p_0, p_0', p_0'' are the idempotents of the first columns.

Every quantity in these statements (the entries of M^*M for the three matrices, the traces of the products of the idempotents, the share) is given by one polynomial in α and β for real and for complex values, because the conjugation of $\mathbb{H}_{\mathbb{C}}$ fixes α and β and all operations are complex-linear. The identities hold on the real circle $\alpha^2 + \beta^2 = 1$, which is Zariski dense in the irreducible complex conic with the same equation. So for all complex α, β with $\alpha^2 + \beta^2 = 1$ the three frames are pairwise unbiased, with share $u^2 + v^2$ and with $\langle p_0 \bullet p_0', p_0'' \rangle = 0$. For $u = \frac{1}{2}(1+\iota)$ and $v = \frac{1}{2}(1-\iota)$ the share is 0, although that entry differs from $\frac{1}{9}$ by $-\frac{1}{9}$.

11 Away from the eikonal cubic the energy determines the planes

This section follows the energy along the segment from the uncoupled cubic D to the eikonal cubic d (Definition 11.4). In single-plane units the equal split of the introduction has normalized energy $(1+t^2)/2$ at ratio t (Example 11.6). The section uses only Sections 3–5.

Throughout this section and the next, d is an eikonal cubic on V with eikonal constant c (Definition 3.1), and $U_1, \dots, U_m$, $m \geq 2$, is a complete system of planes (Section 5), so that $V = U_1 \oplus \cdots \oplus U_m$ and $N = 2m$. Let P_i be the orthogonal projection onto U_i and $x_i = P_i x$. By Theorem 4.9, m is 4 or 13, so N is 8 or 26, but the arguments below do not use this. Where they need a third plane, they use that for $j \neq k$ the product of nonzero vectors of U_j and U_k is nonzero and orthogonal to both planes (Lemma 3.11), so it has a nonzero component in some third plane. In particular $m \geq 3$.

11.1 The diagonal and mixed parts

The *diagonal part* of d is $D(x,y,z) = \sum_i d(x_i, y_i, z_i)$, the sum of the cubics of the planes, and the *mixed part* is $M = d - D$. For a symmetric trilinear form h on V, $h(x,y)$ denotes the vector with $\langle h(x,y), z \rangle = h(x,y,z)$, so that $d(x,y) = x \circ y$.

Lemma 11.1. (a) *On each plane U_i there is an orthonormal complex coordinate ζ with $x \circ x = \sqrt{c}\,\bar{\zeta}^2$ for $x \in U_i$. In particular the squares $x \circ x$, $x \in U_i$, fill U_i, and $u_1 \circ u_1 + u_2 \circ u_2 = 0$ for every orthonormal basis u_1, u_2 of U_i.*

Formalized. `planePencilSplit`.

(b) *$D(x,x) = \sum_i x_i \circ x_i$, with $x_i \circ x_i \in U_i$ and $|x_i \circ x_i|^2 = c|x_i|^4$. The form M vanishes on every triple of vectors of which two lie in one plane and the third lies in any plane, so $M(x,x) = \sum_{i \neq j} x_i \circ x_j$.*

(c) *The forms d, D and M are harmonic (traceless): $\sum_a h(e_a, e_a, y) = 0$ for every orthonormal basis (e_a) and every y.*

Proof. (a) By closedness the product of the restricted cubic $W|_{U_i}$ is the restriction of $\circ$, so $W|_{U_i}$ is eikonal with constant c (Proposition 3.12(i)). By Lemma 3.3,

$W|_{U_i} = \sqrt{c}\ \mathrm{Re}(\zeta^3)$ in a suitable orthonormal complex coordinate ζ. Its gradient, written as a complex number, is $3\sqrt{c}\,\bar{\zeta}^2$, and $\nabla W = 3\,x \circ x$. The map $\zeta \mapsto \bar{\zeta}^2$ is onto, and $\bar{1}^2 + \bar{i}^2 = 0$; a rotation of the basis changes neither statement.

(b) For $u, u' \in U_i$ and $y \perp U_i$, closedness gives $d(u, u', y) = \langle u \circ u', y\rangle = 0$. Hence $\langle D(x,x), z\rangle = \sum_i \langle x_i \circ x_i, z_i\rangle = \langle \sum_i x_i \circ x_i, z\rangle$, and the eikonal equation on U_i gives the norm. On a triple with two vectors in U_i and the third in U_j, $j \neq i$, both d and D vanish, and on a triple inside U_i they agree. Expanding $x \circ x = \sum_{i,j} x_i \circ x_j$ gives $M(x,x)$.

(c) Each plane cubic is harmonic (Lemma 3.3), so d is harmonic by Proposition 3.12(ii), D is harmonic as the orthogonal sum of the plane cubics, and so is $M = d - D$. □

By part (b), $M(x,x,x) = 6\sum_{i<j<k} d(x_i, x_j, x_k)$, so the mixed part is the sum of the three-field terms of (3.9).

Lemma 11.2. *Let $j \neq k$. For $x, x' \in U_j$ and $y, y' \in U_k$,*

$$\langle x \circ y, x' \circ y'\rangle = \tfrac{c}{2}\,\langle x, x'\rangle\langle y, y'\rangle.$$

Hence $x \otimes y \mapsto x \circ y$ is $\sqrt{c/2}$ times an isometry of $U_j \otimes U_k$ onto the product space $U_j \circ U_k$, which is four-dimensional and orthogonal to U_j and U_k, and

$$s_{ijk} = \frac{1}{c}\sum_{\beta,\gamma} |P_i(y_\beta \circ z_\gamma)|^2 \tag{11.1}$$

for every $i \neq j, k$ and all orthonormal bases (y_β) of U_j and (z_γ) of U_k.

Formalized. `plane Share_tensor_isometry`.

Proof. The cubic $d/\sqrt{c}$ has constant 1, product $(x,y) \mapsto x \circ y/\sqrt{c}$ and the same planes, so Lemma 4.10 applied to it gives the identity, the isometry and the dimension; Lemma 3.11 gives the orthogonality to U_j and U_k. Since $\langle y_\beta \circ z_\gamma, w\rangle = d(y_\beta, z_\gamma, w)$, expanding $P_i(y_\beta \circ z_\gamma)$ in an orthonormal basis of U_i turns Lemma 5.8(i), with the symmetry of the share stated there, into (11.1). □

In an orthonormal basis (e_p) adapted to the planes write $h_{pqr} = h(e_p, e_q, e_r)$, sum repeated indices, and put

$$\mathrm{K}(h)_{pq} = h_{pab}h_{qab}, \qquad \mathrm{T}(h)_{pqr} = h_{pab}h_{qac}h_{rbc}.$$

The matrix $\mathrm{K}(h)$ is symmetric. The *triangle* $\mathrm{T}(h)$ is a symmetric cubic: in a picture with one vertex for each factor and one edge for each summed index, its three vertices are pairwise joined.

Lemma 11.3. *Let h be a harmonic eikonal cubic with eikonal constant c on a Euclidean space of dimension n. Then*

$$h_{pqa}h_{rsa} + h_{pra}h_{qsa} + h_{psa}h_{qra} = c\,(\delta_{pq}\delta_{rs} + \delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}), \tag{11.2}$$

$\mathrm{K}(h) = \frac{c}{2}(n+2)\,I$ and $\mathrm{T}(h) = -\frac{c}{4}(n-2)\,h$. In particular $\mathrm{K}(d) = \frac{c}{2}(N+2)I$, $\mathrm{T}(d) = -\frac{c}{4}(N-2)d$, $\mathrm{K}(D) = 2c\,I$ and $\mathrm{T}(D) = 0$.

Formalized. `round Contractions`; d is harmonic by `trace Free_of_isClosedPlane System`.

Proof. Equation (11.2) is (3.7) in coordinates, and the formulas for $\mathrm{K}(h)$ and $\mathrm{T}(h)$ are (3.8), since $\mathrm{K}(h) = \sum_a L_{e_a}^2$ and $\operatorname{Tr} L_x^3 = \mathrm{T}(h)(x,x,x)$ for the symmetric cubic $\mathrm{T}(h)$. For d use Lemma 11.1(c). The cubic D is the orthogonal sum of the plane cubics, which are harmonic and eikonal in dimension $n = 2$ by Lemma 11.1, and the contractions of an orthogonal sum are the sums of the contractions. □

11.2 The energy identity

Definition 11.4. The *pencil* of d along the complete system of planes is the span of D and M, the space of cubics $aD + bM$ with $a, b \in \mathbb{R}$. For $t \in \mathbb{R}$ its *normalized member* with ratio t is $d_t = D + tM$ (the member with $a = 1$, so that $t = b/a$), with product $x \circ_t y = d_t(x, y)$. The *normalized energy* of d_t is the quartic $f_t(x) = |x \circ_t x|^2/c$, and $e_t(x) = f_t(x)$ for unit x.

At $t = 0$ the planes do not interact and $d_0 = D$; at $t = 1$ the cubic is d. The energy $|\nabla W_t|^2$ of $W_t(x) = d_t(x,x,x)$ is $9c\, f_t(x)$, so e_t measures it in units of the energy of a unit vector in one plane. Such a vector has energy $9c$, so its normalized energy is 1 for every t, by Lemma 11.1(b). Since c is fixed, two such cubics have the same energy exactly when they have the same normalized energy.

Theorem 11.5. *For every real t and every unit vector x,*

Formalized. `planePencilEnergy`.

$$e_t(x) - 1 = (t-1)\Big(1 - \sum_i |x_i|^4 + \frac{t}{c}\,|M(x,x)|^2\Big). \tag{11.3}$$

The normalized energy e_t is constant on the unit sphere if and only if $t = 1$.

Proof. Write $D_x = D(x,x)$ and $M_x = M(x,x)$. The eikonal equation for d gives $|D_x|^2 + 2\langle D_x, M_x\rangle + |M_x|^2 = c|x|^4$, so for all x

$$f_t(x) = t^2|x|^4 + \frac{1-t^2}{c}|D_x|^2 + \frac{2t(1-t)}{c}\langle D_x, M_x\rangle. \tag{11.4}$$

By Lemma 11.1(b), $|D_x|^2 = c\sum_i |x_i|^4$. Eliminating $\langle D_x, M_x\rangle$ instead of $|M_x|^2$ gives $e_t = t(1 - \sum_i|x_i|^4) + \sum_i|x_i|^4 + (t^2 - t)|M_x|^2/c$ on the unit sphere, which is (11.3).

At $t = 1$, e_t is 1 everywhere. At $t = 0$ it is 1 on a plane and $\frac{1}{2}$ on the equal split of Example 11.6. Let $t \notin \{0, 1\}$. Take unit vectors $y \in U_j$ and $z \in U_k$, $j \neq k$. By Lemma 3.11, $y \circ z$ is nonzero and orthogonal to U_j and U_k, so $P_i(y \circ z) \neq 0$ for some $i \neq j, k$, and by Lemma 11.1(a) there is $u \in U_i$ with $\langle u \circ u, y \circ z\rangle \neq 0$. In $f_t(su + py + rz)$ the coefficient of s^2pr comes only from the last term of (11.4), since $|x|^4$ and $\sum_i|x_i|^4$ have no such monomial, and by Lemma 11.1(b) it equals

$$\frac{4t(1-t)}{c}\,\langle u \circ u, y \circ z\rangle \neq 0. \tag{11.5}$$

A multiple of $|x|^4$ has no monomial s^2pr, so f_t is not a multiple of $|x|^4$. □

Example 11.6. Let $y \in U_j$ and $z \in U_k$ be unit vectors, $j \neq k$, and $x = (y + z)/\sqrt{2}$. Then $\sum_i |x_i|^4 = \frac{1}{2}$ and $M(x,x) = y \circ z$, of squared length $c/2$ by Lemma 3.11. By (11.3),

$$e_t(x) = 1 + (t-1)\big(\tfrac{1}{2} + \tfrac{t}{2}\big) = \frac{1+t^2}{2}.$$

The equal split does not distinguish $t = 1$ from $t = -1$; the theorem separates them through three planes. In the form (11.2) the eikonal equation is the relation of Cvitanović[192]and of Rong and Su[193], and Jack, Osborn and Steudtner derive it from the degree-three relation of the traceless Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$[194]. Cvitanović's chapter calls the invariance groups of this relation the F_4 family. Among the normalized members d_t of the pencil of the ternary system with thirteen planes, Griess identified $t = 1$ as the Jordan member[195]: on the traceless part his parameter α corresponds to $t = -2\alpha$, and his algebra is Jordan exactly for $\alpha = -\frac{1}{2}$.

11.3 The planes as extrema of the energy

Proposition 11.7. *Let $t \geq 0$, $t \neq 1$, and let x be a unit vector. Then*

Formalized. `planePencilExtrema`.

$$\operatorname{dist}\Big(x, \bigcup_i U_i\Big)^2 \leq \frac{|e_t(x) - 1|}{|t - 1|},$$

and $e_t(x) = 1$ if and only if x lies in one of the planes. Hence for $0 \leq t < 1$ the maximum of e_t on the unit sphere is 1, attained exactly on the unit circles of the planes, and for $t > 1$ these circles are exactly where e_t attains its minimum 1.

Proof. Put $Q(x) = 1 - \sum_i |x_i|^4 = \sum_i |x_i|^2(1 - |x_i|^2)$. The distance from x to U_i is $(1 - |x_i|^2)^{1/2}$, so $Q(x) \geq (1 - \max_i |x_i|^2) \sum_i |x_i|^2 = \operatorname{dist}(x, \bigcup_i U_i)^2$, with $Q(x) = 0$ exactly when x lies in one plane. The second term inside the bracket of (11.3) is nonnegative for $t \geq 0$ and vanishes when x lies in one plane, because then $M(x, x) = 0$ by Lemma 11.1(b). So the bracket is at least $Q(x)$, and it vanishes exactly on the planes. The sign of $t - 1$ gives the rest. □

Remark 11.8. Theorem 11.5, apart from its last sentence, and Proposition 11.7 hold verbatim for any orthogonal decomposition of V into closed subspaces in place of the planes, with "plane" replaced by "summand": the proofs use only closedness of the summands and the eikonal equation for d.

The normalized energy also measures the shares. Fix distinct planes U_i, U_j, U_k and let $\rho^t_{i;jk}(u, y, z)$, $u \in U_i$, $y \in U_j$, $z \in U_k$, be the part of $f_t(u + y + z)$ of degree two in u and one in each of y and z. By the proof of Theorem 11.5, $\rho^t_{i;jk}(u, y, z) = \rho^t(u, u; y, z)$ with $\rho^t(u, u'; y, z) = \frac{4t(1-t)}{c}\langle u \circ u', y \circ z\rangle$. Write $|\rho^t_{i;jk}|^2$ for the sum of the squares of ρ^t over orthonormal bases of U_i (in both arguments), U_j and U_k.

[192]Cvitanović, *Group Theory*, note 186. Ch. 19, Eq. (19.16).

[193]Rong and Su, "Bootstrapping the $\mathcal{N} = 1$ Wess–Zumino models", note 36. Eq. (2.10), p. 9.

[194]Jack, Osborn and Steudtner, "Explorations in scalar fermion theories", note 35. §8, Eqs. (8.42)–(8.46), p. 31.

[195]Griess, "A Moufang loop, the exceptional Jordan algebra, and a cubic form in 27 variables", note 9. (4.3), (4.5), p. 288.

Proposition 11.9. *For distinct i, j, k, $|\rho^t_{i;jk}|^2 = 32\,t^2(1-t)^2 s_{ijk}$. Hence, for $t \notin \{0,1\}$,*

$$s_{ijk} = \frac{2\,|\rho^t_{i;jk}|^2}{\sum_{l\neq j,k}|\rho^t_{l;jk}|^2}.$$

Formalized. `planePencilResponses`; $\rho^t_{i;jk}$ is the part of f_t of degree $(2,1,1)$, polarized by ρ^t: `quarticResponse_normalizedPencilEnergy`, `planeResponseForm_eq_quartic_polarization`.

Proof. For $w \in U_i$ and an orthonormal basis u_1, u_2 of U_i, $\sum_{\alpha,\alpha'}\langle u_\alpha \circ u_{\alpha'}, w\rangle^2 = 2c|w|^2$, since in the coordinate ζ of Lemma 11.1(a) the product is $u \circ u' = \sqrt{c}\,\bar\zeta\bar\zeta'$, where ζ, ζ' are the coordinates of u, u'. Apply this with $w = P_i(y_\beta \circ z_\gamma)$ and use (11.1). For the denominator, $\sum_{l\neq j,k} s_{ljk} = 2$ by the sum rule (Theorem 5.9), so the denominator is $64t^2(1-t)^2$. □

11.4 The energy determines the cubic

The energy of a cubic does not in general determine the cubic. At $t = 1$ every eikonal cubic $d(g\cdot, g\cdot, g\cdot)$, with g orthogonal and preserving each plane, has normalized energy $|x|^4$. The Euclidean metric of V and the constant c are fixed throughout. The *normalized pencil class* $C_c(V)$ consists of the cubics $D + tM$ with $t > 0$ and $t \neq 1$, where $d = D + M$ is an eikonal cubic on V with the fixed constant c, split along one of its complete systems of $m \geq 2$ planes.

Theorem 11.10. (i) (Labelled planes.) *Let d and d' be eikonal cubics on V with eikonal constant c for which $U_1, \dots, U_m$ is a complete system of planes, and let t, t' be real. If d_t and $d'_{t'}$ have the same energy on every sum of three of the planes, then $t = t'$. If moreover $t \notin \{0,1\}$, then $d'_{t'} = \pm d_t$ and $d' = \pm d$, with the same sign.*

(ii) *For $t \in \{0,1\}$ the conclusion fails: there is such a d' with $d'_t \neq \pm d_t$ and the same energy on V.*

(iii) (Unlabelled planes.) *If two cubics in $C_c(V)$ have the same energy on V, they come from the same set of planes and the same t, and they are equal up to sign.*

Formalized, different route. (i) `planePencil_labeled_recovery`; (ii) `planePencil_endpoint_nonunique`: the witness reverses the sign on one plane instead of rotating it by $\pi/6$; (iii) `planePencil_unlabelled_recovery`. Self-contained statements: `Challenge.energy_recovery_labeled`, `Challenge.energy_recovery_endpoints`, `Challenge.energy_recovery_unlabelled`.

Proof. Three planes suffice. In (11.4) each monomial of $|x|^4$ and of $|D_x|^2 = c\sum_i |x_i|^4$ involves at most two planes, and each term $\langle x_i \circ x_i, x_j \circ x_k\rangle$ of $\langle D_x, M_x\rangle$ involves at most three. So two energies that agree on all sums of three planes agree on V.

The ratio. A unit vector in one plane has normalized energy 1 for every t, and the equal split of Example 11.6 has normalized energy $(1+t^2)/2$, so the data give t^2. Fix unit $y \in U_j$, $z \in U_k$, and in each other plane U_i (there is at least one, since $m \geq 3$) two unit vectors u_i, u'_i at angle $\pi/4$, so that $u_i \circ u_i$ and $u'_i \circ u'_i$ are orthogonal of length $\sqrt{c}$, for d and for d' (Lemma 11.1(a), applied to each). By (11.5) and $\sum_i |P_i(y \circ z)|^2 = |y \circ z|^2 = c/2$, the corresponding coefficients λ_i, λ'_i satisfy

$$\sum_{i\neq j,k}\left(\lambda_i^2 + \lambda_i'^2\right) = \frac{16t^2(1-t)^2}{c^2}\cdot c \cdot \frac{c}{2} = 8t^2(1-t)^2.$$

So the data give t^2 and $t^2(1-t)^2$, hence $t(1-t)$ up to sign. If $t' = -t \neq 0$, then $(1-t)^2 = (1+t)^2$, which forces $t = 0$. Hence $t' = t$.

The cubic. Let $t \notin \{0, 1\}$. In the coordinate ζ of Lemma 11.1(a) for d, the restriction of d' to U_i is again eikonal with constant c, hence of the form $\sqrt{c}\ \mathrm{Re}(\xi_i \zeta^3)$ with $|\xi_i| = 1$ (an orthonormal coordinate is $\eta\zeta$ or $\eta\bar{\zeta}$ with $|\eta| = 1$, and $\mathrm{Re}(\bar{\zeta}^3) = \mathrm{Re}(\zeta^3)$). Write $x \circ' y = d'(x, y)$ for the product of d' and L'_x for its multiplication by x. So $u \circ' u = R_i(u \circ u)$ on U_i, where R_i is the rotation of U_i by $\arg \bar{\xi}_i$. Equality of the coefficients (11.5) for d and d' at every $u \in U_i$, together with $t(1-t) \neq 0$ and the fact that the squares fill U_i, gives $P_i(y \circ' z) = R_i P_i(y \circ z)$ for $y \in U_j$, $z \in U_k$ and distinct i, j, k. With $R = \bigoplus_i R_i$ and Lemma 11.1(b), it follows that $x \circ' y = R(x \circ y)$ for all x, y.

Both $L'_x = RL_x$ and L_x are self-adjoint, so $RL_x = L_x R^{\mathsf{T}}$, and multiplying by R^{T} on the left and by R on the right gives $R^{\mathsf{T}} L_x = L_x R$. Hence $S = \frac{1}{2}(R + R^{\mathsf{T}})$ commutes with every L_x. On U_i, S is $\cos\theta_i$ times the identity, where θ_i is the angle of R_i. For unit $x \in U_i$ and $y \in U_j$, $i \neq j$, the vector $x \circ y \neq 0$ satisfies $S(x \circ y) = L_x S y = \cos\theta_j\, x \circ y$ and $S(x \circ y) = L_y S x = \cos\theta_i\, x \circ y$. So all $\cos\theta_i$ equal one number γ.

Suppose $|\gamma| < 1$. Then $\mathcal{J} = (R - \gamma I)/\sqrt{1-\gamma^2}$ is orthogonal with $\mathcal{J}^2 = -I$, since each $R_i - \gamma I$ is $\pm\sqrt{1-\gamma^2}$ times a rotation by a right angle, and $R^{\mathsf{T}} = \gamma I - \sqrt{1-\gamma^2}\,\mathcal{J}$. The relation $RL_x = L_x R^{\mathsf{T}}$ becomes $\mathcal{J} L_x = -L_x \mathcal{J}$. Then L_x and $-L_x$ are similar, and $\mathrm{Tr}(L_x^3) = 0$ for every x. But $\mathrm{Tr}(L_x^3) = \mathrm{T}(d)(x, x, x) = -\frac{c}{4}(N-2)\, d(x, x, x)$ by Lemma 11.3, which is not identically zero. So $\gamma = \pm 1$, every R_i is γI, and $d' = \gamma d$; then $D' = \gamma D$, $M' = \gamma M$ and $d'_t = \gamma d_t$.

Part (ii). Let g be the rotation of U_1 by $\pi/6$, extended by the identity on the other planes, and $d'(x, y, z) = d(gx, gy, gz)$. Then d' is eikonal with constant c with the same planes, $d'_t(x, y, z) = d_t(gx, gy, gz)$, and the normalized energy of d'_t is $x \mapsto f_t(gx)$. At $t = 0$ this is $\sum_i |x_i|^4$, and at $t = 1$ it is $|x|^4$; both are invariant under g. On U_1 the cubic of d'_t is $\sqrt{c}\ \mathrm{Re}(i\zeta^3)$, which is not $\pm\sqrt{c}\ \mathrm{Re}(\zeta^3)$.

Part (iii). Let d_t and d'_t, in $C_c(V)$ have the same energy. By Proposition 11.7, the set of unit vectors where the normalized energy equals 1 is the union of the unit circles of the planes of each. So every plane U' of the second system lies in the finite union $\bigcup_i U_i$; a real vector space is not a finite union of proper subspaces, so U' lies in one U_i and equals it by dimension. The two systems therefore coincide, and part (i) applies after labelling. □

The coefficient (11.5) carries the factor $t(1-t)$, so the data determining the cubic become small near both $t = 0$ and $t = 1$; the theorem is about exact polynomials and gives no estimate under noise. The uniqueness step shows that a harmonic eikonal cubic in dimension greater than two has no orthogonal complex structure anticommuting with its product. Remark 3.9 treats the same situation, the real part of a holomorphic cubic, where one complex variable is the exception.

The stabilizers of d_t and of its normalized energy are determined in Appendix B.2: for $t > 0$, $t \neq 1$, they are finite, and at $t = 1$ the stabilizer of the energy is the whole orthogonal group (Proposition B.3).

Does a natural dynamics on cubics keep t away from 1, so that the planes stay visible?

12 The one-loop flow returns to the eikonal cubic

A cubic interaction of real fields defines a quantum field theory, and the renormalization group flow moves its coupling as the length scale grows. Near four dimensions, at one loop, this flow is a system of ordinary differential equations on symmetric cubic tensors. On every pencil of Section 11 it reduces to one equation for the ratio t, and every positive ratio runs to $t = 1$, where the energy is $\mathrm{O}(N)$-invariant and no longer shows the planes. The physics enters only through Definition 12.1. Perturbations off the pencil are treated in Appendix B.3.

12.1 The one-loop equations

Take N real superfields Φ_i in the $\mathcal{N} = 1$ (two real supercharges) Wess–Zumino model with superpotential $\mathcal{W} = \frac{1}{6} h_{ijk} \Phi_i \Phi_j \Phi_k$, a real symmetric cubic h, continued to $4 - \varepsilon$ dimensions, $\varepsilon > 0$, with the relevant deformations tuned away (quadratic terms break time reversal and are not added to the superpotential, and the boson mass term is tuned to zero)[196,197]. The scalar components ϕ have the potential $\mathcal{V}(\phi) = \frac{1}{2}|\nabla \mathcal{W}|^2 = \frac{1}{8}|h(\phi, \phi)|^2$, in the notation $h(x, y)$ of Section 11.1. For $h = 6d$ the superpotential $\mathcal{W}$ is the cubic $W(\phi) = d(\phi, \phi, \phi)$, and $\mathcal{V} = \frac{1}{2}|\nabla W|^2$: in this convention the energy $|\nabla W|^2$ of Section 3 is twice the scalar potential. For $h = g\, d_t$ the superpotential is $\frac{g}{6} W_t$ with $W_t(\phi) = d_t(\phi, \phi, \phi)$, and $\mathcal{V} = \frac{cg^2}{8} f_t(\phi)$, so when $g \neq 0$ the normalized energy e_t measures the potential, as it measures the energy, in single-plane units. Let $\ell = \ln(\mu_0/\mu)$, for the renormalization scale μ and a reference scale μ_0; then ℓ grows toward long distances. The one-loop equations of Liendo and Rong[198], written for $dh/d\ell = -\beta(h)$, are the physical source of the following definition; their anomalous dimension matrix, the correction to the scaling dimensions of the fields, is $\mathrm{K}(h)/32\pi^2$.

Definition 12.1. Let $\varepsilon > 0$. The *one-loop flow* is the ordinary differential equation on the space $\mathrm{Sym}^3(V^*)$ of symmetric cubics, with real time ℓ,

$$16\pi^2 \frac{dh_{ijk}}{d\ell} = 8\pi^2 \varepsilon\, h_{ijk} - \tfrac{1}{2}\big(h_{ijm}\mathrm{K}_{mk} + h_{ikm}\mathrm{K}_{mj} + h_{jkm}\mathrm{K}_{mi}\big) - 2\,\mathrm{T}(h)_{ijk}, \qquad (12.1)$$

with $\mathrm{K} = \mathrm{K}(h)$ and T as in Section 11.1. The space $\mathrm{Sym}^3(V^*)$ carries the inner product $\langle h, h' \rangle = h_{ijk} h'_{ijk}$.

Every statement below is about this equation. Three dimensions is $\varepsilon = 1$, outside the range where one loop is the leading term (Remark 12.8).

Rescale by $h = 4\pi\sqrt{\varepsilon}\,\kappa$ and $s = \varepsilon\ell$, and write a dot for d/ds. Then (12.1) becomes $\dot{\kappa} = \frac{1}{2}\kappa - \frac{1}{2}(\kappa_{ijm}\mathrm{K}_{mk} + \kappa_{ikm}\mathrm{K}_{mj} + \kappa_{jkm}\mathrm{K}_{mi}) - 2\mathrm{T}(\kappa)$.

[196] Liendo and Rong, "Seeking SUSY fixed points", note 12. Eqs. (1.1)–(1.2), p. 3, n. 1, and p. 4.

[197] L. Fei, S. Giombi, I. R. Klebanov and G. Tarnopolsky, Yukawa CFTs and emergent supersymmetry, *Prog. Theor. Exp. Phys.* **2016** (2016) 12C105. doi:10.1093/ptep/ptw120. arXiv:1607.05316. §4.2.

[198] Liendo and Rong, "Seeking SUSY fixed points", note 12. Eqs. (2.1)–(2.2), p. 5.

Lemma 12.2. *The rescaled flow is $\dot\kappa = -\nabla A(\kappa)$ for*

$$A(\kappa) = -\tfrac{1}{4}|\kappa|^2 + \tfrac{3}{8}\,\mathrm{tr}\,\mathrm{K}(\kappa)^2 + \tfrac{1}{2}\mathrm{P}(\kappa), \qquad \mathrm{P}(\kappa) = \langle \kappa, \mathrm{T}(\kappa)\rangle = \kappa_{ijk}\kappa_{iab}\kappa_{jac}\kappa_{kbc}.$$

Along it $\dot A = -|\nabla A|^2$.

Formalized. Rescaling: `gradientScaling`; A and both forms of P: `gradientPotential`; the gradient: `gradientOf OneLoopPotential`; $\dot A$: `gradientDissipation`.

Proof. The gradient of $|\kappa|^2$ is 2κ. From $d\,\mathrm{tr}\,\mathrm{K}^2 = 4\mathrm{K}_{ij}\kappa_{jab}\,d\kappa_{iab}$ and symmetrization over the three indices, $\nabla\,\mathrm{tr}\,\mathrm{K}^2 = \frac{4}{3}(\kappa_{ijm}\mathrm{K}_{mk} + \kappa_{ikm}\mathrm{K}_{mj} + \kappa_{jkm}\mathrm{K}_{mi})$. Every two of the four factors of P share one summed index, so P is a tetrahedral contraction, and replacing any one factor by $d\kappa$ gives the same term $\langle d\kappa, \mathrm{T}(\kappa)\rangle$, and $\nabla\mathrm{P} = 4\mathrm{T}$. □

That the one-loop flow of such theories is a gradient is due to Jack and Osborn[199], and Jack and Jones treat chiral superfield theories[200]. Liendo and Rong state the gradient property for this model[201]; the quadratic term of their Eq. (2.4) requires a factor of $1/4$ to give Eq. (2.1). In scalar–fermion theories, Pannell, Ronayne and Stergiou check the gradient property through four loops with respect to a metric on the space of couplings, using a shift of the beta function that first appears at three loops[202]. The function A is the *A-function*, a rescaled form of the a-function of the literature[203]: with $h = 4\pi\sqrt{\varepsilon}\,\kappa$, the gradient of $16\pi^2\varepsilon^2 A(\kappa)$ with respect to h is the beta function of Liendo and Rong[204]. It is unchanged by orthogonal changes of the fields.

Form the symmetric quartic tensor $S_{ijkl} = \kappa_{ijm}\kappa_{klm} + \kappa_{ikm}\kappa_{jlm} + \kappa_{ilm}\kappa_{jkm}$. Each of its three terms has squared norm $\mathrm{tr}\,\mathrm{K}(\kappa)^2$ and any two have inner product $\mathrm{P}(\kappa)$, so $|S|^2 = 3\,\mathrm{tr}\,\mathrm{K}(\kappa)^2 + 6\mathrm{P}(\kappa)$ and

$$A(\kappa) = -\tfrac{1}{4}|\kappa|^2 + \tfrac{1}{8}\,\mathrm{tr}\,\mathrm{K}(\kappa)^2 + \tfrac{1}{12}|S|^2. \tag{12.2}$$

12.2 The pencil is invariant

Proposition 12.3. *Let d, $U_1, \dots, U_m$, D, M be as in Section 11, $N = 2m$ and $\lambda = (N-2)/12$. For all real a, b,*

$$\mathrm{K}(aD + bM) = 2c\,(a^2 + 3\lambda b^2)\,I, \qquad \mathrm{T}(aD + bM) = -3c\lambda b^3\,D - 3cb^2(a - b + \lambda b)\,M.$$

Hence (12.1) *is tangent to the pencil, the span of D and M, and the pencil is invariant.*

Formalized. `pencilContractions`, `pencilInvariant`.

Proof. Use an orthonormal basis adapted to the planes. An entry D_{irs} is nonzero only if i, r, s belong to one plane, and then $d_{jrs} = D_{jrs}$ for every j, since d vanishes on two arguments in a plane and one outside (Lemma 11.1(b)). Hence

[199] I. Jack and H. Osborn, Analogs of the c-theorem for four-dimensional renormalisable field theories, *Nucl. Phys. B* **343** (1990) 647–688. doi:10.1016/0550-3213(90)90584-Z.

[200] I. Jack and D. R. T. Jones, The a-function in fractional dimensions, arXiv:2611.27464, 2026.

[201] Liendo and Rong, "Seeking SUSY fixed points", note 12. Eqs. (2.1) and (2.3), p. 5.

[202] W. H. Pannell, W. P. Ronayne and A. Stergiou, Gradient RG flow in scalar-fermion QFTs, *J. High Energy Phys.* **2026**(05) (2026) 064. doi:10.1007/JHEP05(2026)064. arXiv:2511.01971. §1, Eqs. (1.3)–(1.4), pp. 2–3.

[203] Osborn and Stergiou, "Seeking fixed points in multiple coupling scalar theories", note 35. §1, p. 3.

[204] Liendo and Rong, "Seeking SUSY fixed points", note 12. Eqs. (2.1), (2.3), p. 5.

$D_{irs}d_{jrs} = \mathrm{K}(D)_{ij} = 2c\,\delta_{ij}$ by Lemma 11.3. Write $h = aD + bM = (a-b)D + b\,d$. Then $\mathrm{K}(h) = \big((a-b)^2 + 2(a-b)b\big)\mathrm{K}(D) + b^2\mathrm{K}(d) = 2c(a^2-b^2)I + \frac{c}{2}(N+2)b^2 I$, which is the first formula.

Expand $\mathrm{T}(h)$ as a sum of triangles whose three vertices carry D or d. If two vertices carry D, they share a summed index, so all their indices lie in one plane U_p. The third vertex then has two indices in U_p, and its entry is an entry of D on U_p; the triangle is the triangle of the plane cubic, which is zero by Lemma 11.3 with $n = 2$. If one vertex carries D, contract (11.2) for d, with indices (j, r, k, s), against D_{irs}. Two of the three terms on the left become the triangle $X_{ijk} = D_{irs}d_{jru}d_{ksu}$, by the symmetry of D in r, s, and the third becomes $D_{irs}d_{rsu}d_{jku} = 2c\,d_{ijk}$. On the right, harmonicity of D leaves $2c\,D_{ijk}$. So $X = -cM$, and the same holds with D at either other vertex. The triangle with no D is $b^3\mathrm{T}(d)$. Therefore

$$\mathrm{T}(h) = 3(a-b)b^2(-cM) - \tfrac{c}{4}(N-2)b^3(D+M),$$

which is the second formula. With K scalar, the middle term of (12.1) is a multiple of h, and every term lies in the pencil. □

The cubics D and M are orthogonal and nonzero, with $|D|^2 = \operatorname{tr}\mathrm{K}(D) = 2cN$ and $|M|^2 = \operatorname{tr}\mathrm{K}(M) = 6c\lambda N$.

12.3 The ratio runs to one

Theorem 12.4. *Let $\varepsilon > 0$, let $h(\ell) = a(\ell)D + b(\ell)M$ solve* (12.1)*, and put $\lambda = (N-2)/12$.*

Formalized. (i) `coefficientEquations`, `fixedRatios`; (ii) `Proofs.positiveMixing_tendsToRound`, the equal split: `equalSplitEnergy`; (iii) `infraredExponent`.

(i) $16\pi^2(a\,\dot b - b\,\dot a) = 6c\,b^2(a-b)(a+\lambda b)$*, where the dot is $d/d\ell$. In the chart $a = g \neq 0$, $t = b/a$,*

$$16\pi^2\frac{dt}{d\ell} = 6c\,g^2\,t^2(\lambda t+1)(1-t), \qquad 16\pi^2\frac{dg}{d\ell} = 8\pi^2\varepsilon\,g - 3c\,g^3(1+3\lambda t^2 - 2\lambda t^3). \tag{12.3}$$

At $a = 0$, $b \neq 0$, $16\pi^2\,da/d\ell = 6c\lambda b^3 \neq 0$. The fixed ratios are $t = 0$ (a double root), $t = 1$ and $t = -1/\lambda$.

(ii) *If $g(0) \neq 0$ and $t(0) > 0$, the solution exists for all $\ell \geq 0$, $t(\ell)$ tends monotonically to 1 as $\ell \to \infty$, and*

$$h(\ell) \to \operatorname{sgn} g(0) \cdot 4\pi\sqrt{\varepsilon}\,\sqrt{\frac{2}{c(N+10)}}\;d.$$

The normalized energy of the equal split of Example 11.6, $(1+t^2)/2$, tends to 1.

(iii) *Write $F(g,t)$ for the right side of the t-equation in* (12.3) *divided by $16\pi^2$. At $t = 1$ and $g^2/16\pi^2 = 2\varepsilon/(c(N+10))$, the value of g at the limit, $\partial_t F = -\varepsilon$ and $\partial_g F = 0$: the ratio has the infrared exponent $+\varepsilon$ (the corresponding eigenvalue of $D\beta$).*

Proof. (i) By Proposition 12.3, the D- and M-components of (12.1) are $16\pi^2\dot a = (8\pi^2\varepsilon - \frac{3}{2}\theta)a - 2\alpha$ and $16\pi^2\dot b = (8\pi^2\varepsilon - \frac{3}{2}\theta)b - 2\rho$, with $\theta = 2c(a^2+3\lambda b^2)$, $\alpha = -3c\lambda b^3$ and $\rho = -3cb^2(a - b + \lambda b)$. So $16\pi^2(a\dot b - b\dot a) = -2(a\rho - b\alpha) = 6cb^2\big(a(a-b+\lambda b) -$

$\lambda b^2\big) = 6cb^2(a-b)(a+\lambda b)$. Divide by a^2 for $\dot t$; the equation for g is the D-component, and the value at $a = 0$ is -2α.

(ii) First a bound on the squared norm of the coupling, valid on all of $\mathrm{Sym}^3(V^*)$. Write $\kappa = r\,u$ with $|u| = 1$ and $\nu = r^2$. By Lemma 12.2 and Euler's identity for the quartic part $r^4 q(u)$ of A, $\dot\nu = -2\langle\kappa, \nabla A\rangle = \nu - 8\nu^2 q(u)$, so $z = 1/\nu$ satisfies $\dot z = -z + 8q(u)$. By (12.2), with S formed from u, $q(u) = \frac{1}{8}\operatorname{tr}\mathrm{K}(u)^2 + \frac{1}{12}|S|^2 \geq \frac{1}{8}(\operatorname{tr}\mathrm{K}(u))^2/N = \frac{1}{8N}$, and q is bounded on the unit sphere. So z stays between $\min(z(0), 1/N)$ and $\max(z(0), 8\max q)$: the norm of κ stays in a compact interval of $(0,\infty)$, and the solution exists for all $s \geq 0$.

Let $[0, \ell_1)$ be the maximal interval on which $a \neq 0$. On it the chart is valid, $t = 1$ is a fixed point of the first equation in (12.3), and for $t > 0$ the right side has the sign of $1 - t$. So t moves monotonically toward 1 inside the interval I_t between $t(0)$ and 1, and $t \geq \min(t(0), 1) > 0$. Since $g^2|D + tM|^2 = |h|^2 = 16\pi^2\varepsilon\nu$, ν is bounded below by a positive constant and $|D + tM|$ is bounded on I_t, g^2 is bounded below on $[0, \ell_1)$ by a positive constant. If ℓ_1 were finite, continuity would give $a(\ell_1) = 0$, which this bound excludes. Hence $\ell_1 = \infty$, g keeps its sign, and $|dt/d\ell|$ is bounded below by a positive multiple of $|1 - t|$, so $t \to 1$.

Now $u = \operatorname{sgn}(g)(D + tM)/|D + tM| \to u_\infty = \operatorname{sgn} g(0)\, d/|d|$. By Lemma 11.3, $|d|^2 = \operatorname{tr}\mathrm{K}(d) = \frac{c}{2}N(N+2)$, $\mathrm{K}(u_\infty) = I/N$ and $\mathrm{P}(u_\infty) = -\frac{c}{4}(N-2)/|d|^2 = -\frac{N-2}{2N(N+2)}$, so $q(u_\infty) = \frac{3}{8N} - \frac{N-2}{4N(N+2)} = \frac{N+10}{8N(N+2)}$. The linear equation $\dot z = -z + 8q(u(s))$ has a forcing term that tends to $8q(u_\infty)$, so its solutions tend to $8q(u_\infty)$ (by variation of constants). So $\nu \to N(N+2)/(N+10)$ and $\kappa \to \sqrt{N(N+2)/(N+10)}\,u_\infty$, which is the stated limit.

(iii) At $t = 1$, $\partial_g F = 0$ because of the factor $1 - t$, and $\partial_t F = -6c(1+\lambda)g^2/16\pi^2$. With $g^2/16\pi^2 = 2\varepsilon/(c(N+10))$ and $12(1+\lambda) = N + 10$ this is $-\varepsilon$. □

In part (iii) the linearization of the (t, g)-equations at the limit has the eigenvalue $-\varepsilon$, so $t = 1$ attracts nearby ratios as $\ell \to \infty$. In physics terms the ratio is an irrelevant direction at the fixed point, and $+\varepsilon$ is its infrared exponent, an eigenvalue of the stability matrix[205], which is positive at an infrared-stable fixed point.

The ratio equation begins with t^2: a small ratio does not move t at linear order. The term of $\mathrm{T}(D + tM)$ linear in t has two vertices carrying D, and in a triangle any two vertices are joined, so both lie in one plane and the vertex carrying M has two indices in that plane, where M vanishes. Near $t = 0$, with g at its value $g^2/16\pi^2 = \varepsilon/(6c)$ at the decoupled fixed point, $dt/d\ell = \varepsilon t^2 + O(\varepsilon t^3)$. With g held at that value, reaching a fixed ratio in $(0, 1)$ from $t_0 \to 0^+$ takes an interval of ℓ (a logarithmic scale interval) with leading term $1/(\varepsilon t_0)$.

The limiting coupling (the eikonal fixed point), with $g^2/16\pi^2 = 12\varepsilon/(N+10)$ in the Jordan normalization $c = 1/6$, is the member of the family of Wess–Zumino fixed points with SU(3) or F_4 symmetry studied by Rong and Su[206]. At $t = 0$ each plane is the $\mathcal{N} = 2$ (four real supercharges) cubic Wess–Zumino model, one

[205] Osborn and Stergiou, "Seeking fixed points in multiple coupling scalar theories", note 35. Eq. (1.5), p. 3.

[206] Rong and Su, "Bootstrapping the $\mathcal{N} = 1$ Wess–Zumino models", note 36. Eq. (2.10), Table III, p. 9.

complex chiral superfield with a cubic superpotential, in which the scalar has dimension $(3-\varepsilon)/3$ in $4-\varepsilon$ dimensions[207]. Liendo and Rong call it the S_3 super Potts model[208], and their Eqs. (4.13)–(4.16), p. 9, treat two decoupled copies.

12.4 The A-function and the variance of the potential

For a cubic κ on $\mathbb{R}^N$ let $\mathcal{V}_\kappa(\phi) = \frac{1}{8}|\kappa(\phi,\phi)|^2$ be its potential, and let Var and $\mathbb{E}$ refer to the uniform probability measure on the unit sphere. With S as in (12.2), $\mathcal{V}_\kappa(\phi) = \frac{1}{24}S(\phi,\phi,\phi,\phi)$. Write $\nu = |\kappa|^2$, $v_i = \kappa_{iaa}$ for the trace vector, $\mathrm{K}_0 = \mathrm{K}(\kappa) - \frac{\nu}{N}I$, $R_{ij} = S_{ijkk}$ and $R_0 = R - \frac{1}{N}(\operatorname{tr} R)I$. Let $\Pi_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$; for a symmetric matrix B let $\delta \vee B$ be the sum of the six products $\delta_{ij}B_{kl}$ over the ways of splitting $ijkl$ into two pairs; and put

$$H = S - \tfrac{1}{N+4}\,\delta \vee R_0 - \frac{\operatorname{tr} R}{N(N+2)}\,\Pi.$$

For harmonic (traceless) cubics with scalar K, the bound $A \geq A_*$ of the next proposition and its equality case are known. In the equivalent form $\mathrm{P} \geq -\frac{N-2}{2(N+2)}\operatorname{tr}\mathrm{K}^2$, with equality exactly when S is a multiple of Π, that is, when the traceless part of S (the tensor S of Osborn and Stergiou) vanishes, it is due to Osborn and Stergiou, for cubics that in addition have triangle proportional to the cubic, in the cubic scalar theory near six dimensions[209]. Jack, Osborn and Steudtner state it for the real-superfield model and identify the equality case with the traceless Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$[210], and Harribey, Pannell and Stergiou observe, for interface-localized cubic interactions, that the members with $N = 5, 8, 14, 26$ saturate it[211]. The splitting of a quartic coupling into its $\mathrm{O}(N)$-irreducible parts is that of Hogervorst and Toldo[212], who call the squared norm a_2 of the rank-two part, here $\frac{1}{N+4}\delta \vee R_0$, the anisotropy[213].

[207] Fei, Giombi, Klebanov and Tarnopolsky, "Yukawa CFTs and emergent supersymmetry", note 197. §4.1, Eq. (4.3).

[208] Liendo and Rong, "Seeking SUSY fixed points", note 12. p. 6, after Eq. (3.4).

[209] Osborn and Stergiou, "Seeking fixed points in multiple coupling scalar theories", note 35. App. C, Eqs. (C.5)–(C.7), p. 57.

[210] Jack, Osborn and Steudtner, "Explorations in scalar fermion theories", note 35. §8, Eqs. (8.18)–(8.19), p. 27; Eqs. (8.42)–(8.48), pp. 31–32.

[211] S. Harribey, W. H. Pannell and A. Stergiou, Multiscalar critical models with localised cubic interactions, *J. High Energy Phys.* **2024**(12) (2024) 092. doi:10.1007/JHEP12(2024)092. arXiv:2407.20326v3. Eq. (3.5), p. 8; Eqs. (3.9)–(3.10), p. 9.

[212] M. Hogervorst and C. Toldo, Bounds on multiscalar CFTs in the ϵ expansion, *J. High Energy Phys.* **2021**(04) (2021) 068. doi:10.1007/JHEP04(2021)068. arXiv:2010.16222. Eqs. (3.11)–(3.16).

[213] Ibid. text before Eq. (3.30).

Proposition 12.5. *Put* $\alpha_N = \frac{N+10}{8N(N+2)}$, $\nu_* = \frac{N(N+2)}{N+10}$ *and* $A_* = -\nu_*/8$.

Formalized, different route. (a) `potentialFloor`; (b) `harmonicIsotropicFloor`, on the pencils `pencil FloorConditions`; (c) `relativeVarianceDescent`: in (a) the orthogonality and norms of the three parts are checked directly, instead of quoting Hogervorst and Toldo.

(a) *For every symmetric cubic* κ *on* $\mathbb{R}^N$,

$$A(\kappa) - A_* = \alpha_N(\nu - \nu_*)^2 + \frac{|\mathrm{K}_0|^2}{8} + \frac{|v|^2(4\nu + |v|^2)}{4N(N+2)} + \frac{|R_0|^2}{2(N+4)} + \frac{|H|^2}{12},$$

and the last two terms are $4N(N+2)(N+4)\,\mathrm{Var}_2$ *and* $2N(N+2)(N+4)(N+6)\,\mathrm{Var}_4$, *where* $\mathrm{Var}_2 + \mathrm{Var}_4 = \mathrm{Var}(\mathcal{V}_\kappa)$ *are the variances of the parts of* $\mathcal{V}_\kappa$ *of degree two and four on the unit sphere. So* $A \geq A_*$, *with equality exactly when* $\nu = \nu_*$, $v = 0$, $\mathrm{K}(\kappa)$ *is scalar and* $\mathcal{V}_\kappa$ *is constant on the sphere.*

(b) *If* $v = 0$ *and* $\mathrm{K}(\kappa)$ *is scalar, in particular on every pencil of Proposition 12.3, then*

$$A(\kappa) - A_* = \alpha_N(\nu - \nu_*)^2 + 2N(N+2)(N+4)(N+6)\ \mathrm{Var}(\mathcal{V}_\kappa).$$

(c) *Let* $\varepsilon > 0$ *and let* $h = aD + bM$ *solve* (12.1) *in such a pencil, with* $\kappa = h/(4\pi\sqrt{\varepsilon})$. *At every time with* $a \neq 0$, *writing* $t = b/a$, *the relative variance* $\mathrm{Var}(\mathcal{V}_\kappa)/(\mathbb{E}\,\mathcal{V}_\kappa)^2$ *equals that of* e_t. *Its derivative with respect to* ℓ *is negative whenever* $dt/d\ell \neq 0$.

Proof. (a) Contracting S over $k = l$ gives $R_{ij} = 2\mathrm{K}_{ij} + \kappa_{ijm}v_m$, so $\mathrm{tr}\,R = 2\nu + |v|^2$. The decomposition $S = H + \frac{1}{N+4}\delta \vee R_0 + \frac{\mathrm{tr}\,R}{N(N+2)}\Pi$ is the splitting of a symmetric quartic tensor into its $\mathrm{O}(N)$-irreducible parts[214]: H has every trace zero, the three parts are orthogonal, and their squared norms are $|H|^2$, $\frac{6}{N+4}|R_0|^2$ and $\frac{3(2\nu+|v|^2)^2}{N(N+2)}$, by $|\delta \vee B|^2 = 6(N+4)|B|^2$ for traceless B and $|\Pi|^2 = 3N(N+2)$. With $\mathrm{tr}\,\mathrm{K}^2 = |\mathrm{K}_0|^2 + \nu^2/N$, (12.2) gives

$$A = -\tfrac{\nu}{4} + \alpha_N\nu^2 + \frac{|\mathrm{K}_0|^2}{8} + \frac{|v|^2(4\nu + |v|^2)}{4N(N+2)} + \frac{|R_0|^2}{2(N+4)} + \frac{|H|^2}{12},$$

and $-\frac{\nu}{4} + \alpha_N\nu^2 = \alpha_N(\nu - \nu_*)^2 + A_*$.

On the sphere $\mathcal{V}_\kappa = \frac{1}{24}H(\phi,\phi,\phi,\phi) + \frac{1}{4(N+4)}R_0(\phi,\phi) + \frac{\mathrm{tr}\,R}{8N(N+2)}$, a sum of spherical harmonics of degrees four, two and zero, which are orthogonal in L^2 of the sphere. Their second moments follow from $\mathbb{E}[\phi_{i_1}\cdots\phi_{i_{2n}}] = \Pi^{(2n)}_{i_1\ldots i_{2n}}/\big(N(N+2)\cdots(N+2n-2)\big)$, with $\Pi^{(2n)}$ the sum over all pairings of the indices: a pairing that joins two indices of one traceless factor gives zero, so only the 4!, resp. 2, pairings across the two copies survive, and

$$\mathrm{Var}_4 = \frac{|H|^2}{24N(N+2)(N+4)(N+6)}, \qquad \mathrm{Var}_2 = \frac{|R_0|^2}{8N(N+2)(N+4)^2}.$$

Every term of the identity is nonnegative. Equality forces $\nu = \nu_*$, $v = 0$, $\mathrm{K}_0 = 0$ and $\mathrm{Var}_2 = \mathrm{Var}_4 = 0$, and conversely.

(b) If $v = 0$ and $\mathrm{K}_0 = 0$, then $R = \frac{2\nu}{N}I$, $R_0 = 0$ and $\mathrm{Var}_2 = 0$. On a pencil, $v = 0$ by Lemma 11.1(c) and K is scalar by Proposition 12.3.

[214] Ibid. Eqs. (3.11)–(3.16).

(c) On the pencil write $\kappa = r\,u$ as in the proof of Theorem 12.4, so that $A = -\frac{1}{4}r^2 + r^4 q(u)$. At $\nu = 1$, part (b) and $\mathbb{E}\,\mathcal{V}_u = \frac{1}{4N(N+2)}$ give $q(u) = \alpha_N + \frac{(N+4)(N+6)}{8N(N+2)}\,\mathrm{Var}(\mathcal{V}_u)/(\mathbb{E}\,\mathcal{V}_u)^2$, and the relative variance is unchanged by scaling. By Proposition 12.3, ∇A lies in the pencil, and its component along the unit circle of the pencil is $r^3\nabla_{\mathrm{sph}} q(u)$, with ∇_{sph} the gradient on that circle. With the time s of Lemma 12.2, $\dot{u} = -r^2\nabla_{\mathrm{sph}} q$ and $\dot{q} = -r^2|\nabla_{\mathrm{sph}} q|^2$, which is negative whenever u moves. At a time with $a \neq 0$, κ is a nonzero multiple of d_t with $t = b/a$, so the potential is a constant multiple of f_t, and u moves exactly when t changes. □

On a pencil the variance for a nonzero cubic vanishes only at $t = 1$, by Theorem 11.5, and at no point with $a = 0$, $b \neq 0$, where the potential is a multiple of $|M(x,x)|^2$, which is 0 on the planes and $c/2$ on the equal split. So on the pencil equality $A = A_*$ holds only at the two eikonal cubics $\pm\sqrt{2/(c(N+10))}\,d$ of Theorem 12.4. The two-term form of part (b) fails on the whole space. A cubic with nonzero trace vector can have $\nu = \nu_*$ and constant potential, as do the eikonal cubics of Tkachev's non-harmonic family[215], and then $A > A_*$ by part (a). Along the full flow (12.1) the quantity that decreases is A, not the variance.

Off the pencil the eikonal cubic is a stable fixed point of the rescaled flow up to rotations: its Hessian is computed in Appendix B.3 (Proposition B.5).

Remark 12.6. Every decomposition of an eikonal cubic into closed summands, as listed in Theorem A(ii), gives a segment $D + tM$ from the uncoupled cubic D, $t = 0$, to the eikonal cubic, and Theorem 11.5 and Proposition 11.7 apply to it (Remark 11.8). Only complete systems of planes are treated above. For a decomposition into an octet and nine planes $\mathrm{K}(D)$ is not scalar, and the pencil need not be invariant.

Remark 12.7. Jack, Osborn and Steudtner give the two-loop terms of the beta function of the real-superfield model in minimal subtraction[216]. Any term built from insertions of K, triangles and the derivative of the triangle maps the pencil into itself: on a pencil K is scalar, T takes values in the pencil, and the derivative $\mathrm{T}'(h)$ of T at $h = D + tM$ satisfies $\mathrm{T}'(h)[h] = 3\,\mathrm{T}(h)$ and $\mathrm{T}'(h)[M] = \partial_t \mathrm{T}(h)$. Whether every two-loop term has this form, and so whether the pencil stays invariant at two loops, is not established here.

Remark 12.8. Whether the three-dimensional ($\varepsilon = 1$) $\mathcal{N} = 1$ theory with interaction $D + tM$ and small $t > 0$ flows to the eikonal fixed point is open; the one-loop flow above does not decide it.

13 Discussion and open questions

The hypothesis of Theorem A(i) speaks of fields, of couplings among three of them and of the length of a gradient. Its conclusion is the traceless Hermitian 3×3 matrices over $\mathbb{C}$ and over $\mathbb{O}$. The quaternions enter as the case to be excluded.

[215] Tkachev, "A generalization of Cartan's theorem", note 1. Thm. 1.1.

[216] Jack, Osborn and Steudtner, "Explorations in scalar fermion theories", note 35. §7.2, Eqs. (7.13)–(7.14), (7.23)–(7.24), pp. 20–22.

Hurwitz's theorem admits them, Cartan's cubic on $H_3^0(\mathbb{H})$ is eikonal and has planes, and conditions on two planes do not separate it from the other two: orthogonal planes have unbiased frames in all three models, and the count of dimensions allows four, seven and thirteen alike. The three cases differ in the shares, which involve three planes.

The sum rule is the one constraint behind four of the conclusions. With the lower bound of Theorem 6.4 it excludes seven orthogonal planes in $H_3^0(\mathbb{H})$ (Corollary 6.6). With the completion of six bases it gives the bound of five unbiased bases in $\mathbb{H}^3$ (Corollary 6.9). Through Theorem 10.7 it excludes an orthogonal decomposition of compact $\mathfrak{sp}(3)$ into Cartan subalgebras (Corollary 10.10). Added over the 78 pairs of a thirteen-plane system it gives the total share 52, and so the 52 triples with positive share of the sparsest system (Theorem 9.1).

At $t = 1$ the energy carries no record of the planes. The cubic on $H_3^0(\mathbb{O})$ has thirteen-plane systems that no automorphism relates (Proposition 9.5), and its energy is the same function for all of them. For $t > 0$ other than 1 the energy determines the planes (Theorem 11.10) and has a finite symmetry group (Proposition B.3). The one-loop flow takes every such t to 1 (Theorem 12.4), and its equation for t depends on the system only through the number of fields, so at this order neither the flow nor its endpoint distinguishes one system from another.

The results have the following limits. Several inputs are quoted and not proved: Hurwitz's theorem, properties (J1)–(J3) of the Albert algebra, Tits's construction of the Albert algebra, the spectral theorem for quaternionic Hermitian matrices, the normal form of three unbiased bases of $\mathbb{H}^3$, derived from the proof of Theorem 5.4 of Chterental and Đoković, the classification of the unbiased bases of $\mathbb{C}^3$ by Brierley, Weigert and Bengtsson, the conjugacy of Cartan subalgebras, Elduque's theorem on the grading of $\mathfrak{f}_4$, and, for the cubics of Theorem A(ii) that are not harmonic, Tkachev's theorem. The flow is the one-loop flow, taken from Liendo and Rong as a definition (Section 12.1), and only complete systems of planes are treated (Remark 12.6). Whether the pencil stays invariant at two loops is not established (Remark 12.7), and the one-loop flow does not decide the three-dimensional theory (Remark 12.8). The bound of five in $\mathbb{H}^3$ is not known to be attained, Corollary 10.10 covers only the decompositions stable under a compact conjugation, the positivity of the defect of Proposition 8.12 comes without an explicit bound, and the thirteen-plane systems are not classified.

13.1 Open questions

Each paragraph below starts from a result of the paper and states what it leaves open.

Four or five bases in $\mathbb{H}^3$. By Corollary 6.9 a set of pairwise unbiased bases of $\mathbb{H}^3$ has at most five members. Sets of four exist, and for five bases neither the sum rule nor the quaternionic lower bound decides (end of Section 6). Do five exist? Every value in $[1 - 1/\sqrt{3}, 1]$ is the share of a triple of pairwise unbiased bases (Remark 6.5), so single triples do not exclude five. Which compatibility conditions

relate the triples that share a pair of bases?

Thirteen-plane systems and $\mathfrak{f}_4$. Every orthogonal decomposition of compact $\mathfrak{f}_4$ into Cartan subalgebras gives a thirteen-plane system (Section 10), and the graded system and the rotated systems, obtained by moving the four frames of one of its quartets by an automorphism of the copy of $H_3(\mathbb{C})$ that contains them, arise in this way (Propositions 10.12 and 10.13; the latter treats unitary conjugations, and an automorphism of transpose type gives the same rotated system as a unitary conjugation, since a complex conjugation of that copy permutes the four frames). Does every thirteen-plane system? The derivations of $H_3(\mathbb{O})$ that fix a frame form a copy of $\mathfrak{so}(8)$, so a plane does not determine its Cartan subalgebra, and the question is whether thirteen pairwise orthogonal choices exist. A related problem is the classification of thirteen-plane systems up to F_4. The sparsest system is unique (Theorem 9.4), and Section 9.2 describes the four-parameter family of Cohn, Kumar and Minton, the patches of dimension ten modulo F_4 near it, and the rotated systems. Whether these exhaust the space of systems, and what its components are, is not known.

The complex case of type C_3. Corollary 10.10 excludes only the orthogonal decompositions into Cartan subalgebras of $\mathfrak{sp}_6(\mathbb{C}) = \mathfrak{sp}(3) \otimes \mathbb{C}$ whose summands a compact conjugation permutes, the part of the open case C_3 of the conjecture of Kostrikin, Kostrikin and Ufnarovskii (Section 10.3) that is stable under such a conjugation. By Remark 10.16, for an arbitrary decomposition the complexified space still splits into seven pairwise orthogonal planes, but the shares, given by the same formula, become sums of squares of complex numbers and can vanish (Example 10.17). Is there a substitute for the quaternionic lower bound that survives complexification, or a decomposition of $\mathfrak{sp}_6(\mathbb{C})$ that uses this freedom?

Unbiased frames of the Albert algebra. By Proposition 6.8 every twelve pairwise unbiased frames of $H_3(\mathbb{O})$ extend to thirteen, so no set of twelve is maximal. Which sizes below twelve occur for maximal sets of pairwise unbiased frames? The graded system and the rotated systems all contain a quartet (Proposition 7.3(v), Section 9.2). Is there a thirteen-plane system with no quartet, that is, one that does not coarsen to an octet and nine planes?

Near the sparsest system. Shares lie in $[0, 1]$, so $\sum s_{ijk}(1 - s_{ijk})$, summed over the 286 triples, vanishes exactly when every share is 0 or 1, that is, for the sparsest system (Theorems 9.1 and 9.4). Does a small value force the system to be close to the graded system modulo F_4, and at what rate?

Quantifying the missing seven. The seven-field defect δ_7 of Proposition 8.12 is positive and attained, and $\delta_7 \leq \frac{3}{4}$. Its value, the cubics that attain it and an explicit positive lower bound are not known; the proof of positivity is by compactness and gives no bound. The same is true of ϵ_6, the least amount by which six frames of $H_3(\mathbb{H})$ fail to be pairwise unbiased.

Generating triples. By Theorem 9.9, for every pair of planes of a thirteen-plane system at least seven of the other eleven generate $H_3(\mathbb{O})$ with the pair, so at least 182 of the 286 triples generate $H_3(\mathbb{O})$. In the graded system every pair has nine such planes, and 234 triples generate $H_3(\mathbb{O})$. Is there a system with a pair for which only seven do? What is the least number of generating triples?

Stability of the recovery. By Theorem 11.10, for $t > 0$, $t \neq 1$, the energy of d_t determines the planes, t and the cubic up to sign, and the recovery fails at $t = 0$ and $t = 1$. If the energy is known only approximately, how well are the planes and the cubic determined, and how does the constant degenerate as t approaches 0 or 1?

The three-dimensional theory. Whether the pencil stays invariant at two loops is not established (Remark 12.7). Does the three-dimensional theory of Remark 12.8 flow to the eikonal fixed point, the limit of Theorem 12.4(ii) at $t = 1$? Its endpoint would be a fixed point with SU(3) or F_4 symmetry reached from decoupled copies of the $\mathcal{N} = 2$ cubic Wess–Zumino model (Section 12.3). The one-loop equation for t depends on the complete system of planes only through N; does any quantity of the theory distinguish two thirteen-plane systems that F_4 does not relate?

Excluding seven from one plane. Section 4 reaches 1, 4, 7, 13 from the joint eigenspaces S_1, S_2, S_3 of one plane, and seven is removed only after passing to quaternionic bases and the quaternionic lower bound. Cartan's cubic on $H_3^0(\mathbb{H})$ satisfies every condition that can be checked at a single idempotent (Theorem 4.6), and following the six other planes through $S_1 \oplus S_2 \oplus S_3$ leads back to their frames (Proposition 5.1) and so to the same proof. Is there an obstruction of a different kind, representation-theoretic, from Clifford modules or topological, to splitting $S_1 \oplus S_2 \oplus S_3$ into six planes for the quaternionic orthogonal multiplication, without the normal form of three bases?

Quartic and sextic analogues. Cartan's isoparametric hypersurfaces with three principal curvatures are level sets on the unit sphere of the harmonic eikonal cubics. Isoparametric polynomials of degree four and six also exist. Is there an analogue of Theorem A(i) for them, with complex fields on which the polynomial restricts to a multiple of $\operatorname{Re} z^4$ or $\operatorname{Re} z^6$ and couplings that each involve several fields?

A Cartan's normal form at an idempotent

This appendix shows that the orthogonal multiplication of Section 4 determines the cubic, and computes it for Cartan's cubics; the proof of Theorem 4.6 uses both lemmas. This is how Cartan's proof of the classification concludes[217,218].

[217] Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1. p. 341, Eqs. (16)–(17).

[218] Tkachev, "A generalization of Cartan's theorem", note 1. §3, Eqs. (17)–(19).

Throughout, W is eikonal with constant 1, and unit idempotents are those of the cubic product of W.

Lemma A.1 (Normal form)**.** *Let a be a unit idempotent, $b \in V_{-1}$ a unit vector, and S_1, S_2, S_3, ϕ as in Lemma 4.3. For $x = \alpha a + \beta b + u + y_2 + y_3$ with $u \in S_1$, $y_2 \in S_2$, $y_3 \in S_3$,*

Formalized, details in Lean. `roundSlot_normalForm`.

$$W(x) = \alpha^3 - 3\alpha\big(\beta^2 + |u|^2\big) + \tfrac{3}{2}\alpha\big(|y_2|^2 + |y_3|^2\big) + \tfrac{3\sqrt{3}}{2}\beta\big(|y_3|^2 - |y_2|^2\big) + 3\sqrt{3}\,\langle\phi(u, y_2), y_3\rangle. \tag{A.1}$$

Proof. Write $x = \alpha a + v + y$ with $v = \beta b + u \in V_{-1}$ and $y = y_2 + y_3 \in V_{1/2}$. The terms with a are $\alpha^3 + 3\alpha\langle L_a(v+y), v+y\rangle$, and by Lemma 4.1(ii) the only remaining term is $3\,d(v, y, y)$. By Lemma 4.3, $d(b, y, y) = \langle L_b y, y\rangle = \frac{\sqrt{3}}{2}(|y_3|^2 - |y_2|^2)$, and since L_u exchanges S_2 and S_3, $d(u, y, y) = 2\,d(u, y_2, y_3) = \sqrt{3}\,\langle\phi(u, y_2), y_3\rangle$. □

The models are Cartan's cubics in the normalization $c = 1$. On $V_{\mathbb{K}} = H_3^0(\mathbb{K})$ with the trace form $\langle X, Y\rangle = \operatorname{tr}(X \bullet Y)$ put $d_{\mathbb{K}}(X, Y, Z) = \sqrt{6}\,\operatorname{tr}\big((X \bullet Y) \bullet Z\big)$ and $W_{\mathbb{K}}(X) = \sqrt{6}\,\operatorname{tr} X^3$, eikonal with constant 1 by Example 3.5, and let ι_{ij} be as before Theorem 4.6. The cubic product of $d_{\mathbb{K}}$ is $\sqrt{6}\,X \circ Y$, with $X \circ Y$ the traceless part of the Jordan product; in this appendix L_X, unit idempotents, joint eigenspaces and $\phi_{\mathbb{K}}$ in $V_{\mathbb{K}}$ refer to this product. Thus $a' = (3E_{11} - e)/\sqrt{6}$ below is a rescaled, centred Jordan idempotent, not a Jordan idempotent.

Lemma A.2 (The models)**.** *In $V_{\mathbb{K}}$ the matrix $a' = \frac{1}{\sqrt{6}}\operatorname{diag}(2, -1, -1)$ is a unit idempotent of the cubic product $\sqrt{6}\,X \circ Y$ of $d_{\mathbb{K}}$ and $b' = \frac{1}{\sqrt{2}}\operatorname{diag}(0, 1, -1)$ a unit vector of V_{-1}. Their joint eigenspaces are $S_1' = \iota_{23}(\mathbb{K})$, $S_2' = \iota_{13}(\mathbb{K})$, $S_3' = \iota_{12}(\mathbb{K})$, and their orthogonal multiplication is*

$$\phi_{\mathbb{K}}\big(\iota_{23}(p), \iota_{13}(q)\big) = \iota_{12}(q\bar{p}).$$

So $W_{\mathbb{K}}$ is given by (A.1) *with $\phi_{\mathbb{K}}$.*

Formalized, details in Lean. `realCartan_model AtDiagonal`, `complex Cartan_modelAtDiagonal`, `quaternionCartan_model AtDiagonal`, `native Cartan_modelAtDiagonal`.

Proof. For a real diagonal D and Z supported at (i, j) and (j, i), $D \bullet Z = \frac{1}{2}(D_i + D_j)Z$; with the factor $\sqrt{6}$ of $d_{\mathbb{K}}$ this gives the eigenvalues of $L_{a'}$ and $L_{b'}$ in Lemma 4.3(ii) on the three positions, and $\sqrt{6}\,a' \circ a' = a'$, $\sqrt{6}\,a' \circ b' = -b'$ are diagonal computations. For $u = \iota_{23}(p)$ and $y = \iota_{13}(q)$ a direct multiplication gives $u \bullet y = \frac{\sqrt{2}}{4}\,\iota_{12}(q\bar{p})$, which is traceless, so $\frac{2}{\sqrt{3}}\sqrt{6}\,u \circ y = \iota_{12}(q\bar{p})$. Only products of two entries occur, so no associativity of $\mathbb{K}$ is used. Lemma A.1 applies. □

Cartan gives the corresponding form of his cubics in his Eq. (17)[219].

B The second row, symmetry groups and stability

This appendix holds three results that complete Sections 10, 11 and 12; nothing in the main text depends on them.

[219] Cartan, "Sur des familles remarquables d'hypersurfaces isoparamétriques", note 1. p. 341.

B.1 The second row of the magic square

The notation is that of Section 10.

Write $L^\bullet_x(y) = x \bullet y$ and consider, in the operators on $J \otimes \mathbb{C}$, the space

$$\mathfrak{g}_{\mathbb{K}} = \mathfrak{k}_{\mathbb{K}} \oplus iL^\bullet_V, \qquad L^\bullet_V = \{L^\bullet_x : x \in V\},$$

with $\theta(\delta + iL^\bullet_x) = \delta - iL^\bullet_x$, which is complex conjugation of operators. A derivation δ satisfies $[\delta, L^\bullet_x] = L^\bullet_{\delta x}$, so $x \mapsto iL^\bullet_x$ is an isomorphism of $\mathfrak{k}_{\mathbb{K}}$-modules from V onto $iL^\bullet_V$. Since $[L^\bullet_x, L^\bullet_y]$ is a derivation (Schafer[220]; for $H_3(\mathbb{O})$, Yokota[221]), $\mathfrak{g}_{\mathbb{K}}$ is closed under commutators. It is the Lie algebra that Tits's magic-square construction attaches to the composition algebra $\mathbb{C}$ and the Jordan algebra $H_3(\mathbb{K})$ (the row of $\mathbb{C}$ in the compact magic square, without its entry for $H_3(\mathbb{R})$), and the construction identifies it with $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$, $\mathfrak{su}(6)$ and compact $\mathfrak{e}_6$[222]. For $\mathfrak{e}_6$ the description goes back to Chevalley and Schafer[223], and Yokota writes compact $\mathfrak{e}_6$ in this form[224]. The next proposition is stated for the operator space and uses none of this.

Proposition B.1. *Let $\beta(X, Y) = -\operatorname{Tr}_{J\otimes\mathbb{C}}(XY)$ on the operator space $\mathfrak{g}_{\mathbb{K}} = \mathfrak{k}_{\mathbb{K}} \oplus iL^\bullet_V$.*

Formalized. (a) `complex_secondRow_centralizer`, `quaternion_secondRow_centralizer`, `albert_secondRow_centralizer`; (b) `secondRow_traceForm_pos`, `complex_secondRow_splitting`, `quaternion_secondRow_splitting`, `albert_secondRow_splitting`.

(a) *Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}_{\mathbb{K}}$ with $U = V^{\mathfrak{t}}$ two-dimensional. The operators in $\mathfrak{g}_{\mathbb{K}}$ commuting with $\mathfrak{t}$ form $\mathfrak{t} \oplus iL^\bullet_U$. This space is commutative and θ-stable, and contains every commutative subspace of $\mathfrak{g}_{\mathbb{K}}$ containing $\mathfrak{t}$.*

(b) *The form β is positive definite on $\mathfrak{g}_{\mathbb{K}}$. If $\mathfrak{k}_{\mathbb{K}} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_m$ is a splitting into pairwise orthogonal Cartan subalgebras of frame type with planes U_r, then $\mathfrak{g}_{\mathbb{K}}$ is the direct sum of the spaces $\mathfrak{t}_r \oplus iL^\bullet_{U_r}$, which are pairwise β-orthogonal and of dimension $\ell + 2$.*

Proof. (a) An element of $\mathfrak{g}_{\mathbb{K}}$ is uniquely $\delta + iL^\bullet_x$, because δ and $L^\bullet_x$ are real operators on J and $L^\bullet_x e = x$. For $H \in \mathfrak{t}$, $[H, \delta + iL^\bullet_x] = [H, \delta] + iL^\bullet_{Hx}$. This vanishes for all H exactly when δ commutes with $\mathfrak{t}$, so $\delta \in \mathfrak{t}$ by maximality, and $x \in U$. Within $\mathfrak{t} \oplus iL^\bullet_U$, $[\mathfrak{t}, iL^\bullet_U] = iL^\bullet_{\mathfrak{t}U} = 0$. By Proposition 10.5(a) and (J3), $U = gV_0$ for an automorphism g of J, and $L^\bullet_{gx} = gL^\bullet_x g^{-1}$. For a real diagonal $x = \operatorname{diag}(\lambda)$, $L^\bullet_x$ multiplies the entry in place (s, t) of every element of J by $\frac{1}{2}(\lambda_s + \lambda_t)$, so these operators commute. Hence $[iL^\bullet_x, iL^\bullet_y] = -[L^\bullet_x, L^\bullet_y] = 0$ for $x, y \in U$. A commutative subspace containing $\mathfrak{t}$ lies in the centralizer of $\mathfrak{t}$. The space is θ-stable by its form.

(b) The form β restricted to $\mathfrak{k}_{\mathbb{K}}$ is $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$, since derivations kill e; $\beta(\delta, iL^\bullet_x) = -i \operatorname{Tr}(\delta L^\bullet_x) = 0$, since δ is skew and $L^\bullet_x$ symmetric; and on $iL^\bullet_V$, $\beta(iL^\bullet_x, iL^\bullet_y) = \operatorname{Tr}_J(L^\bullet_x L^\bullet_y)$ is a symmetric form invariant under $G_{\mathbb{K}}$, hence by Lemma 10.4 a multiple of $\langle x, y \rangle$, with positive factor since $\operatorname{Tr}_J\big((L^\bullet_x)^2\big) > 0$ for $x \neq 0$. So β is positive definite. The commutativity argument of (a) uses only that $\mathfrak{t}_r$ is commutative with plane U_r. By Theorem 10.7 the $\mathfrak{t}_r$ and the U_r are pairwise orthogonal, so the spaces $\mathfrak{t}_r \oplus iL^\bullet_{U_r}$ are pairwise β-orthogonal, hence independent, and their dimensions add up to $m\ell + N = \dim \mathfrak{g}_{\mathbb{K}}$. □

[220] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.

[221] Yokota, "Exceptional Lie groups", note 44. Prop. 2.4.1(2).

[222] Baez, "The octonions", note 10. §4.3, Table 5.

[223] C. Chevalley and R. D. Schafer, The exceptional simple Lie algebras F_4 and E_6, *Proc. Nat. Acad. Sci. USA* **36** (1950) 137–141. doi:10.1073/pnas.36.2.137.

[224] Yokota, "Exceptional Lie groups", note 44. Thm. 3.2.4(2).

Proposition B.1 gives the following statement in the language of Lie theory.

Corollary B.2 (The second row)**.** *Grant the identification given by Tits's magic-square construction*[225]*: the space $\mathfrak{g}_{\mathbb{K}}$ is a compact Lie algebra, isomorphic to $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ for $\mathbb{K} = \mathbb{C}$ and to $\mathfrak{su}(6)$ for $\mathbb{K} = \mathbb{H}$, and for $\mathbb{K} = \mathbb{O}$ it is simple with complexification isomorphic to the complex simple Lie algebra $\mathfrak{e}_6(\mathbb{C})$*[226]*. Then $\mathfrak{g}_{\mathbb{O}}$ is isomorphic to compact $\mathfrak{e}_6$, and θ is an involution with fixed algebra $\mathfrak{f}_{\mathbb{K}}$. Cartan subalgebras of $\mathfrak{g}_{\mathbb{K}}$ (maximal commutative subalgebras) and orthogonal decompositions of $\mathfrak{g}_{\mathbb{K}}$ into Cartan subalgebras are defined as for $\mathfrak{f}_{\mathbb{K}}$ (Definition 10.1(a)), with the Killing form of $\mathfrak{g}_{\mathbb{K}}$, equivalently β. For a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{f}_{\mathbb{K}}$ with plane U, $\mathfrak{t} \oplus iL^\bullet_U$ is the unique Cartan subalgebra of $\mathfrak{g}_{\mathbb{K}}$ containing $\mathfrak{t}$; it is θ-stable, and $\operatorname{rank} \mathfrak{g}_{\mathbb{K}} = \ell + 2$. The map $(\mathfrak{t}_r) \mapsto (\mathfrak{t}_r \oplus iL^\bullet_{U_r})$ is a bijection from orthogonal decompositions of $\mathfrak{f}_{\mathbb{K}}$ into Cartan subalgebras onto orthogonal decompositions of $\mathfrak{g}_{\mathbb{K}}$ into Cartan subalgebras that are each θ-stable.*

Formalized. `compactSecondRowCorollary_realE6_ofMagicSquare`, with Tits's magic-square construction as the granted hypothesis and compact $\mathfrak{e}_6$ as the fixed algebra of a compact conjugation of $\mathfrak{e}_6(\mathbb{C})$; θ: `secondRowTheta_fixed_iff_real`; the Cartan subalgebra: `albert_secondRowCartanExtension`; rank: `compactCartanSubalgebras_finrank_eq`; bijection: `secondRowLieODCorrespondence_of_operators`.

Proof. The identifications for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ are granted. For $\mathbb{K} = \mathbb{O}$, $\mathfrak{g}_{\mathbb{O}}$ is a compact real form of $\mathfrak{e}_6(\mathbb{C})$, and any two compact real forms of a complex semisimple Lie algebra are conjugate[227]. The uniqueness and the rank follow from Proposition B.1(a), since every Cartan subalgebra of $\mathfrak{f}_{\mathbb{K}}$ has a two-dimensional plane (Propositions 10.5(c) and 10.6). For the forward map, Proposition B.1(b) applies, and β is a positive multiple of minus the Killing form on each simple factor (for $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$, θ exchanges the two factors and preserves β). Conversely, let $\mathfrak{g}_{\mathbb{K}} = \mathfrak{c}_1 \oplus \cdots \oplus \mathfrak{c}_{m'}$ be an orthogonal decomposition into Cartan subalgebras with each $\mathfrak{c}_r$ θ-stable. Since the rank is $\ell + 2$, $m' = m$. Each $\mathfrak{c}_r$ splits as $(\mathfrak{c}_r \cap \mathfrak{f}_{\mathbb{K}}) \oplus (\mathfrak{c}_r \cap iL^\bullet_V)$, so $\mathfrak{f}_{\mathbb{K}} = \bigoplus_r (\mathfrak{c}_r \cap \mathfrak{f}_{\mathbb{K}})$ with each summand commutative of dimension at most ℓ. The dimensions add up to $m\ell$, so each $\mathfrak{c}_r \cap \mathfrak{f}_{\mathbb{K}}$ is a Cartan subalgebra $\mathfrak{t}_r$, these are pairwise orthogonal, and $\mathfrak{c}_r = \mathfrak{t}_r \oplus iL^\bullet_{U_r}$ by uniqueness. □

Under $x \mapsto iL^\bullet_x$ the plane U is the part of $\mathfrak{t} \oplus iL^\bullet_U$ on which θ acts by -1. In the language of Cartan involutions[228], $\mathfrak{f}_{\mathbb{K}} \oplus L^\bullet_V$ is the noncompact dual $\mathfrak{sl}_3(\mathbb{C})$, $\mathfrak{su}^*(6)$ or $\mathfrak{e}_{6(-26)}$, θ comes from its Cartan involution, and $L^\bullet_U$ is a maximal abelian subspace of $L^\bullet_V$; the symmetric pairs $(\mathfrak{g}_{\mathbb{K}}, \mathfrak{f}_{\mathbb{K}})$ are of types II, AII and EIV in Cartan's classification of symmetric spaces, of rank two. By Corollary B.2 and Corollary 10.10, $\mathfrak{su}(6)$ has no orthogonal decomposition into θ-stable Cartan subalgebras. The correspondence of Corollary B.2 sends the decomposition of Proposition 10.12 to an orthogonal decomposition of compact $\mathfrak{e}_6$ into thirteen θ-stable Cartan subalgebras. Burichenko constructed an orthogonal decomposition of complex $\mathfrak{e}_6$ into Cartan subalgebras from the same Jordan subgroup[229].

B.2 Symmetry groups along the pencil

The eikonal cubic d, its complete system of planes $U_1, \ldots, U_m$, the normalized members $d_t = D + tM$ of its pencil (the span of D and M) and the normalized energy e_t are as in Section 11.

[225] Baez, "The octonions", note 10. §4.3, Table 5.
[226] Chevalley and Schafer, "The exceptional simple Lie algebras F_4 and E_6", note 223.
[227] Knapp, "Structure theory of semisimple Lie groups", note 153. §4, Cor. of Thm. 4.2(c), p. 21.
[228] Ibid. §4, pp. 21–22.
[229] Burichenko, "Transitive orthogonal decompositions", note 175.

Let Γ be the group of orthogonal maps g of V that preserve d and permute the planes U_i, and for a function or cubic ϕ on V let $\mathrm{Stab}(\phi)$ be its stabilizer in $\mathrm{O}(V)$.

Proposition B.3. (a) *The group Γ is finite. For $t > 0$, $t \neq 1$, $\mathrm{Stab}(d_t) = \Gamma$ and $\mathrm{Stab}(e_t) = \Gamma \times \{\pm I\}$.*

(b) *$\mathrm{Stab}(e_t) = \mathrm{O}(V)$ if and only if $t = 1$.*

(c) *At $t = 0$, $\mathrm{Stab}(e_t)$ contains every orthogonal map that preserves each plane.*

(d) *Let d be Cartan's cubic $\mathrm{tr}((x \bullet y) \bullet z)$ on $H_3^0(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, with the trace form, where $\bullet$ is the Jordan product. Then $\mathrm{Stab}(d)$ is the group of restrictions to $H_3^0(\mathbb{K})$ of the automorphisms of the Jordan algebra $H_3(\mathbb{K})$.*

Formalized. (a) `planeCubicStabilizer_finite`, `pencilCubic_stabilizer_eq`, `pencilEnergy_stabilizer_mulEquiv`; (b) `pencilEnergy_stabilizer_top_iff`; (c) `pencilEnergy_zero_stabilizer`; (d) `hermitianCartan_stabilizer_iff`, `hermitianAutomorphism_restricts`, on the copies of $H_3(\mathbb{K})$ in $H_3(\mathbb{O})$, identified with the matrix algebras by `matrixHermitianCoordinates_preserves`.

Proof. (a) An element of Γ carries $W|_{U_i}$ to $W|_{U_{\pi(i)}}$ for a permutation π, so it permutes the $3m$ points where W attains its maximum on the unit circles of the planes. These points span V, three on each plane at angles $2\pi/3$. So g is determined by a permutation of a finite set.

Let $t > 0$, $t \neq 1$, and $g \in \mathrm{Stab}(e_t)$. The cubic $d''(x, y, z) = d_t(gx, gy, gz)$ belongs to $C_c(V)$: it is the normalized member with the same ratio t of the pencil of the eikonal cubic $d(g\cdot, g\cdot, g\cdot)$ split along the planes $g^{-1}U_i$. Its normalized energy is $x \mapsto e_t(gx)$, which is e_t on the unit sphere. By Theorem 11.10(iii), $\{g^{-1}U_i\} = \{U_i\}$ and $d'' = \pm d_t$. Replacing g by $-g$ if needed, g preserves d_t and permutes the planes. Then g preserves D, which is the sum of the restrictions of d_t to the planes, hence M and d, so $g \in \Gamma$. Conversely Γ preserves D and M, and $-I$ preserves the even function e_t while it changes the sign of d_t. This gives both stabilizers.

(b) The group $\mathrm{O}(V)$ acts transitively on the unit sphere, so e_t is invariant under it exactly when it is constant, which by Theorem 11.5 happens exactly at $t = 1$.

(c) At $t = 0$, $e_t = \sum_i |x_i|^4$.

(d) For traceless x, y, $x \bullet y = x \circ y + \frac{1}{3}\langle x, y\rangle e$, since $\langle x \bullet y, z\rangle = d(x, y, z)$ for traceless z and $\mathrm{tr}(x \bullet y) = \langle x, y\rangle$. So an orthogonal map of $H_3^0(\mathbb{K})$ that preserves d, extended by $e \mapsto e$, preserves the Jordan product. Conversely an automorphism fixes e and maps Jordan frames to Jordan frames (proof of Lemma 7.2), so by the spectral theorem (J2) it preserves the trace, the sum of the eigenvalues, and the trace form; for $H_3(\mathbb{O})$ this is Lemma 2.2.1(2) of Yokota[230]. It therefore preserves $H_3^0(\mathbb{K})$ and d. □

The automorphism groups in part (d) are known. Their Lie algebras are the derivation algebras $\mathrm{der}(\mathbb{K}) \oplus \mathfrak{sa}_3(\mathbb{K})$, with $\mathfrak{sa}_3(\mathbb{K})$ the traceless skew-Hermitian 3×3 matrices over $\mathbb{K}$[231], of dimensions 3, 8, 21, 52 for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The automorphism group of $H_3(\mathbb{O})$ is the compact simply connected Lie group F_4[232]; Chevalley and Schafer identified its Lie algebra as $\mathfrak{f}_4$[233]. An automorphism of $H_3(\mathbb{C})$ maps rank-one projections to rank-one projections and preserves $\mathrm{tr}(pq)$, so by Wigner's theorem[234]it is $x \mapsto uxu^*$ or $x \mapsto u\bar{x}u^*$ with u unitary and determined

[230] Yokota, "Exceptional Lie groups", note 44.
[231] Baez, "The octonions", note 10. §4.2, Eq. (19).
[232] Yokota, "Exceptional Lie groups", note 44. Thm. 2.10.2.
[233] Chevalley and Schafer, "The exceptional simple Lie algebras F_4 and E_6", note 223.
[234] Bargmann, "Note on Wigner's theorem", note 88. §1.3, p. 863.

up to a scalar, and the group is $\mathrm{PSU}(3) \rtimes C_2$.

Remark B.4. For an eikonal cubic with a complete system of four or thirteen planes, Theorem 4.7 identifies V and d, after an isometry and a scaling, with Cartan's cubic on $H_3^0(\mathbb{C})$ or $H_3^0(\mathbb{O})$, so the stabilizer of the cubic jumps at $t = 1$ from the finite group Γ to $\mathrm{PSU}(3) \rtimes C_2$ or F_4, of dimension 8 or 52.

For $t < 0$ the stabilizer statements of part (a) are not proved, since the planes need not be the set where e_t attains its maximum or its minimum (compare Proposition 11.7).

B.3 Stability of the eikonal cubic under the one-loop flow

The rescaled flow, the A-function and the tensor S are as in Section 12.

In the field theory the rescaled equation of Lemma 12.2 describes the flow of the cubic couplings, with the relevant deformations tuned away as in Section 12.1. The proposition uses only Lemma 11.3, so it holds for $C = \sqrt{2/(c(N+10))}\, d$ with d any harmonic eikonal cubic with constant c on $\mathbb{R}^N$; for a complete system of planes with cubic d, the limit of κ in Theorem 12.4 is $\pm C$, with the sign of $g(0)$, and $-C$ lies in the orbit $\mathrm{O}(N) \cdot C$. Since A is invariant under $\mathrm{O}(N)$, every rotation of C is again a zero of the flow. For a skew matrix Ω let $L\Omega = \Omega_{ia}C_{ajk} + \Omega_{ja}C_{iak} + \Omega_{ka}C_{ija}$ be the tangent vector of the orbit $\mathrm{O}(N) \cdot C$; let τ map a cubic to its trace vector, and let $\tau^* w = \frac{1}{3}(w_i\delta_{jk} + w_j\delta_{ik} + w_k\delta_{ij})$, so that $\tau\tau^* = \frac{N+2}{3} I$.

Proposition B.5. *Let* $\mu = \frac{2}{N+10}$.

Formalized, details in Lean. (a) `roundPointHessianSpectrum`; (b) `roundPointHessianMultiplicities`; (c) `roundOrbitLocalConvergence`.

(a) *The cubic C is a zero (fixed point) of the rescaled flow, and the Hessian operator of A at C, the stability matrix for the beta function in rescaled variables (the linearization of the rescaled flow at C is $-\mathcal{H}$), is $\mathcal{H} = I + 3\mu\tau^*\tau - \frac{1}{3}LL^*$. It vanishes exactly on the space $\{L\Omega\}$, equals $\frac{3N+14}{N+10}$ on the N-dimensional space of cubics $\tau^* w$, and equals 1 on the orthogonal complement of these two spaces. The operator $\frac{1}{3}LL^*$ is the orthogonal projection onto $\{L\Omega\}$.*

(b) *The space $\{L\Omega\}$ has dimension $\frac{N(N-1)}{2} - \dim\ker L$, where $\ker L$, the Lie algebra of $\mathrm{Stab}(C)$, consists of the skew matrices Ω with $L\Omega = 0$; the eigenvalue 1 has multiplicity $\binom{N+2}{3} - N - \dim\{L\Omega\}$.*

(c) *Every solution of the rescaled flow that starts close enough to the orbit $\mathrm{O}(N) \cdot C$ converges, as $s \to \infty$, to a point of the orbit.*

Proof. (a) By Lemma 11.3, $\mathrm{K}(C) = k_* I$ and $S(C) = \mu\Pi$, with $k_* = \frac{N+2}{N+10}$, where S is formed from C as in Section 12.4; then the right side of the rescaled flow at C is $C\left(\frac{1}{2} - \frac{3}{2}k_* + \frac{N-2}{2}\mu\right)$, which is zero. For a symmetric cubic Y put $\Xi_{ij} = C_{iab}Y_{jab}$, $w = \tau Y$ and $Z = C_{ijm}C_{klm}Y_{ikn}Y_{jln}$. Expanding $A(C + sY)$ term by term, with $\mathrm{K}(C + sY) = k_* I + s(\Xi + \Xi^{\mathsf{T}}) + s^2\mathrm{K}(Y)$ and the six placements of two copies of Y among the four factors of P, each giving Z, the second derivative at $s = 0$ is $-\frac{1}{2}|Y|^2 + \frac{3}{2}k_*|Y|^2 + \frac{3}{4}|\Xi + \Xi^{\mathsf{T}}|^2 + 6Z$. Contracting $S(C) = \mu\Pi$ with $Y_{ijn}Y_{kln}$ gives

$|\Xi|^2 + 2Z = \mu(|w|^2 + 2|Y|^2)$. Substituting Z and using $|\Xi + \Xi^{\mathsf{T}}|^2 = 4|\Xi_{\mathrm{s}}|^2$, $|\Xi|^2 = |\Xi_{\mathrm{s}}|^2 + |\Xi_{\mathrm{a}}|^2$ for the symmetric and skew parts,

$$d^2A_C[Y,Y] = \big(\tfrac{3}{2}k_* - \tfrac{1}{2} + 6\mu\big)|Y|^2 + 3\mu|w|^2 - 3|\Xi_{\mathrm{a}}|^2 = |Y|^2 + 3\mu|\tau Y|^2 - 3|\Xi_{\mathrm{a}}|^2.$$

Since $\langle L\Omega, Y\rangle = 3\Omega_{ia}\Xi_{ai} = -3\langle\Omega, \Xi_{\mathrm{a}}\rangle$, the adjoint is $L^*Y = -3\Xi_{\mathrm{a}}$, and $\mathcal{H} = I + 3\mu\tau^*\tau - \frac{1}{3}LL^*$.

Differentiating $\nabla A(e^{s\Omega}\cdot C) = 0$ at $s = 0$ gives $\mathcal{H}L\Omega = 0$. The operator $\mathcal{H}$ is self-adjoint, so it preserves the orthogonal complement $\ker L^*$ of the orbit tangents, and there it is $I + 3\mu\tau^*\tau$. The trace of $L\Omega$ is zero, because Ω is skew and C is symmetric and harmonic, so $\tau L = 0$ and the cubics τ^*w lie in $\ker L^*$. As $\tau\tau^* = \frac{N+2}{3}I$, $\mathcal{H}$ is $1 + \mu(N+2) = \frac{3N+14}{N+10}$ on them and 1 on the rest of $\ker L^*$. So $\mathcal{H}$ is invertible on $\ker L^*$ and its kernel is $\{L\Omega\}$. Finally $\mathcal{H}L\Omega = 0$ and $\tau L = 0$ give $LL^*L = 3L$, so $\frac{1}{3}LL^*$ is a self-adjoint idempotent with image $\{L\Omega\}$.

(b) Apply rank and nullity to L on the $\frac{N(N-1)}{2}$-dimensional space of skew matrices, and use part (a) with $\dim \mathrm{Sym}^3(\mathbb{R}^N) = \binom{N+2}{3}$.

(c) The orbit is compact. Near it a cubic is a rotation of $C + Y$ with $Y \in \ker L^*$ small (a tubular neighbourhood), rotations change neither A nor $|\nabla A|$, and $\operatorname{dist}(\cdot, \mathrm{O}(N)\cdot C) \le |Y| \le c_1 \operatorname{dist}(\cdot, \mathrm{O}(N)\cdot C)$ there. Since A is a polynomial, Taylor's theorem and part (a) give $\delta > 0$ and K with $\frac{1}{4}|Y|^2 \le A - A_* \le K|Y|^2$ and $|\nabla A| \ge \frac{1}{2}|Y|$ for $|Y| < \delta$. There $\frac{d}{ds}\sqrt{A - A_*} = -|\nabla A|^2/(2\sqrt{A - A_*}) \le -|\nabla A|/(4\sqrt{K})$, so a piece of trajectory inside the neighbourhood has length at most $4\sqrt{K}\sqrt{A(0) - A_*} \le 4K|Y(0)|$.

Start with $|Y(0)| < \delta/\big(2c_1(1+4K)\big)$. At a first time σ with $|Y| = \delta/2$, if there were one, the distance to the orbit would be at most $(1+4K)|Y(0)|$, so $|Y(\sigma)| < \delta/2$, a contradiction. So the trajectory stays in the neighbourhood, has finite length and converges, and since $\int_0^\infty |\nabla A|^2 ds \le A(0) - A_*$ its limit has $\nabla A = 0$, hence $Y = 0$: it lies on the orbit. □

For Cartan's cubic on $H^0_3(\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ ($N = 5, 8, 14, 26$), $\mathrm{Stab}(C)$ is the automorphism group of $H_3(\mathbb{K})$ (Proposition B.3(d)), whose Lie algebra has dimension 3, 8, 21, 52 (Appendix B.2). So the orbit tangents have dimension 7, 20, 70, 273 and the eigenvalue 1 has multiplicity 23, 92, 476, 2977.

In the scale variable ℓ of Section 12.1, where $s = \varepsilon\ell$, the stability matrix $\partial_I\beta^J$ of Osborn and Stergiou[235] is $\varepsilon\mathcal{H}$. The direction of M at C, along the pencil, lies in the eigenspace 1, since M is harmonic and $C_{iab}M_{jab}$ is a multiple of $\mathrm{K}(M)$, which is symmetric. Its stability-matrix eigenvalue ε is the infrared exponent $+\varepsilon$ of Theorem 12.4(iii). Normal to the orbit the stability-matrix eigenvalues are ε and $\varepsilon\frac{3N+14}{N+10}$ at one loop; the orbit directions are exact zero modes, coming from the symmetry and not from marginal operators (Osborn and Stergiou describe such submanifolds of equivalent fixed points as containing apparently exactly marginal operators[236]).

[235] Osborn and Stergiou, "Seeking fixed points in multiple coupling scalar theories", note 35. §1, Eq. (1.5).

[236] Ibid. §1, after Eq. (1.3).

C Imported results

Tables 4 and 5 list the results from the literature that the proofs use without proof, each in the form used, with the source and locator printed at the point of use. Results cited only for context or attribution, and results the paper proves (for instance the identification at an idempotent, Theorem 4.6, and the existence of a ternary grading group, Lemma 7.5), are not listed. A locator marked "as cited in" was read in the secondary source named there. The rows write the Jordan product as $X \bullet Y$ (Section 3.3); Yokota and Baez write it $X \circ Y$, a symbol used here for the cubic product.

The one-loop flow of Section 12 is defined by Definition 12.1, whose physical source is the pair of one-loop equations of Liendo and Rong[237]; no proof uses anything about that source beyond the definition.

D Formalization

The accompanying Lean 4 development (`github.com/macgregor-sudo/cartan-cubic-planes`) is built with Lean[238] version 4.36.0 on Mathlib[239] at commit `fc4f59e`, unmodified. The file `PAPER-COVERAGE.tsv` of the repository lists every numbered statement of this paper with the declarations that prove it. Every such declaration depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`.

Every numbered theorem, proposition, lemma and corollary of this paper is proved in Lean, that is, its proof is checked by the Lean kernel. The printed proofs quote three theorems from the literature, Theorems 3.7, 4.4 and 7.4, which Appendix C lists with the other imported results, and the development does not assume them. Both parts of Theorem 3.7 are proved in it, part (i) at a unit idempotent as in Theorem 4.6 and part (ii), which the printed proof of Corollary 8.9 uses, by a trace identity at a unit idempotent. Theorem 4.4 is proved in the composition-algebra library on which the development builds, and the identification of Theorem 7.4 is proved in the development. Corollary B.2 grants the identification of $\mathfrak{g}_{\mathbb{K}}$ given by Tits's magic-square construction, and its Lean statement takes as hypothesis what that identification provides: each $\mathfrak{g}_{\mathbb{K}}$ is compact, $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{su}(3) \oplus \mathfrak{su}(3)$, $\mathfrak{g}_{\mathbb{H}} \cong \mathfrak{su}(6)$, and $\mathfrak{g}_{\mathbb{O}}$ is simple with complexification isomorphic to $\mathfrak{e}_6(\mathbb{C})$. The Lean input also supplies simplicity of each compact $\mathfrak{su}(3)$ factor and of the quaternionic operator algebra. The identification of $\mathfrak{g}_{\mathbb{O}}$ with compact $\mathfrak{e}_6$ is then proved. Theorem A(i) is Theorem 8.11 and Theorem A(ii) is Corollary 8.9. Examples and remarks are outside this claim.

[237] Liendo and Rong, "Seeking SUSY fixed points", note 12. Eqs. (2.1)–(2.2), p. 5.

[238] L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, in *Automated Deduction – CADE 28*, Lecture Notes in Comput. Sci. **12699**, Springer, Cham, 2021, 625–635. doi:10.1007/978-3-030-79876-5_37.

[239] The mathlib Community, The Lean mathematical library, in *Proc. 9th ACM SIGPLAN Int. Conf. on Certified Programs and Proofs (CPP 2020)*, ACM, 2020, 367–381. doi:10.1145/3372885.3373824. arXiv:1910.09336.

Table 4: Results used without proof: composition algebras, Jordan algebras and mutually unbiased bases.

Result, as used	Source	Used in
Hurwitz's theorem on bilinear maps: for $n \geq 1$, a bilinear map $\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $\lvert\mu(x,y)\rvert = \lvert x\rvert\lvert y\rvert$ (an orthogonal multiplication) exists only for $n = 1, 2, 4, 8$ ($n = 1$ is ordinary multiplication; the source treats $n > 1$)	[Hurwitz, "Über die Composition der quadratischen Formen", note 2]	Cor. 4.5, hence Thm. 4.9
Theorem 4.4: a nonzero finite-dimensional real unital algebra with an inner product and $\lvert xy\rvert = \lvert x\rvert\lvert y\rvert$ (a normed division algebra) is isometrically isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$	[Baez, "The octonions", note 10, Thm. 1 in §1.1]; [Springer and Veldkamp, *Octonions, Jordan algebras and exceptional groups*, note 73]	Thm. 4.6(ii), hence Thms. 4.7, 8.11; Lem. 9.8; Thm. 9.9
Theorem 3.7(ii): an eikonal cubic with constant c that is not harmonic is $\sqrt{c}\,(s^3 - 3s\lvert u\rvert^2)$, with $s = \langle x, a\rangle$, $u = x - sa$ for a unit vector a, and $N \neq 2$	[Tkachev, "A generalization of Cartan's theorem", note 1, Thm. 1.1]	the printed proof of Cor. 8.9 (Theorem A(ii)); the formal proof proves it (Appendix D)
(J1) and positivity of the trace form of $H_3(\mathbb{O})$; every automorphism of $H_3(\mathbb{O})$ preserves the trace; every element of $H_3(\mathbb{O})$ is carried to a real diagonal matrix by an automorphism, which gives (J2) and (J3)	[Yokota, "Exceptional Lie groups", note 44, Lemma 2.1.1(2), Lemma 2.2.1(2), Prop. 2.8.2]	Ex. 3.5, and through it every statement on $H_3(\mathbb{O})$, among them Prop. 5.1, Lem. 5.4(i), Prop. 5.5, Prop. 6.8, Lem. 7.2, Lem. 8.4, Thm. 8.11, Prop. B.3(d); the trace in Sec. 10.1
The automorphisms of $H_3(\mathbb{O})$ fixing E_{11}, E_{22}, E_{33} form Spin(8), acting on the three off-diagonal entries by the triples of Lemma 5.3	[Yokota, "Exceptional Lie groups", note 44, Thm. 2.7.1; §1.16, Thm. 1.16.2]	Lem. 5.3, hence Lem. 5.4(ii)
The first components of these triples act transitively on unit octonions; the third components of the triples with $\gamma_1(1) = 1$ form $\widetilde{B}_3 \cong$ Spin(7), up to conjugation by $b \mapsto \bar{b}$, which acts transitively on unit octonions	[Yokota, "Exceptional Lie groups", note 44, Prop. 1.16.1; §1.15, Thm. 1.15.1]	Lem. 5.4(ii)
Conjugation of $\mathbb{K}$ is an isometry that reverses products; the automorphism group of $\mathbb{O}$ is transitive on unit imaginary octonions	[Baez, "The octonions", note 10, §2.2; §4.1]	Thm. 4.6(ii); Prop. 8.2, Thm. 9.4
As a real vector space, the derivation algebra of $H_3(\mathbb{K})$ is $\mathrm{der}(\mathbb{K}) \oplus \mathfrak{sa}_3(\mathbb{K})$, where $\mathfrak{sa}_3(\mathbb{K})$ is the space of traceless skew-Hermitian 3×3 matrices over $\mathbb{K}$, acting by commutators	[Baez, "The octonions", note 10, §4.2, Eq. (19)]	Lem. 8.3
Theorem 7.4 (Tits's construction of the Albert algebra): J_T is a Jordan algebra isomorphic to $H_3(\mathbb{O})$, by an isomorphism carrying the trace form T to the trace form of $H_3(\mathbb{O})$; so $H_3(\mathbb{O})$ has coordinates $(a, B) \in H_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ with unit $(1, 0)$, trace $\mathrm{tr}\, a$ and the square of Section 7.2	[Earlier AI-generated work, "On the local geometry of the tight 39-point code", note 25, §4.2, Eqs. (10)–(11), proved in App. A.1]; [Yokota, "Exceptional Lie groups", note 44, §2.12]	Lem. 7.5 (transport to $H_3(\mathbb{O})$), Prop. 8.2, Thm. 8.11 (existence for $m = 13$), Thm. 9.4
A quaternionic Hermitian matrix has an orthonormal basis of eigenvectors with real eigenvalues	[Rodman, *Topics in quaternion linear algebra*, note 3, Thm. 5.3.6, as cited in Macías-Virgós, Pereira-Sáez and Tarrío-Tobar, arXiv:2012.03621, §3.1]; [Zhang, "Quaternions and matrices of quaternions", note 3]	Lem. 6.1(i), hence (J2), (J3) for $\mathbb{H}$, Cor. 6.6, Prop. 6.7, Prop. 8.12
Every quaternionic Hadamard matrix of order three is $\sqrt{3}\mathsf{F}$ up to left and right monomial unitary factors (a permutation matrix times a diagonal unitary; symplectic monomial matrices, in the source's terms); one unitary with a monomial right factor on each basis carries three pairwise mutually unbiased bases of $\mathbb{H}^3$ to $1, \mathsf{F}$, $\mathrm{diag}(1, \alpha, \beta)X\mathsf{F}X^{-1}$ with $X = x1$ for unit quaternions α, β, x	[Chterental and Đoković, "On orthostochastic, unistochastic and qustochastic matrices", note 4, §4, p. 15; proof of Thm. 5.4, pp. 24–25]	Lem. 6.2, hence Lem. 6.3, Thm. 6.4
Up to a unitary, two mutually unbiased bases of $\mathbb{C}^3$ are the standard and the Fourier basis; exactly six unit vectors up to phase are unbiased to both, and they form two orthonormal bases in only one way	[Brierley, Weigert and Bengtsson, "All mutually unbiased bases in dimensions two to five", note 20, §2.2, Eqs. (6)–(11), pp. 4–5]	Thm. 5.10(iii)
A 2-$(13, 4, 1)$ design is a projective plane of order three, and every projective plane of order three is isomorphic to the projective	[Hall, *Combinatorial theory*, note 139]; [Hughes and Piper, *Projective planes*, note 140]	Thm. 9.1, (c)⇒(d)

Table 5: Results used without proof: Lie algebras, invariants, tensors and the magic square.

Result, as used	Source	Used in
Cartan subalgebras (maximal commutative subalgebras) of $\mathfrak{su}(3)$, $\mathfrak{sp}(3)$ and compact $\mathfrak{f}_4$ are conjugate under $G_{\mathbb{K}}$; the rank of $\mathfrak{f}_4$ is four	[Knapp, "Structure theory of semisimple Lie groups", note 153, Thm. 3.6, p. 16]; [Yokota, "Exceptional Lie groups", note 44, Thm. 2.6.1]	Prop. 10.5(c), Prop. 10.6
The complexification of a maximal commutative subalgebra of a compact Lie algebra is a Cartan subalgebra of its complexification	[Knapp, "Structure theory of semisimple Lie groups", note 153, §3, pp. 16–17]	Lem. 10.2, hence Prop. 10.12, Cor. 10.10
$\mathrm{Der}(H_3(\mathbb{K}))$ has dimension 8, 21, 52 for $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$	[Baez, "The octonions", note 10, §4.2, Eq. (19)]; [Yokota, "Exceptional Lie groups", note 44, Thms. 2.3.2, 2.3.8, 2.10.2]	Sec. 10.1; Prop. 10.12
The stabilizer Spin(8) of the diagonal frame acts on the three off-diagonal entries by its vector and two half-spin representations, and its Lie algebra is a copy of $\mathfrak{so}(8)$ of rank four	[Yokota, "Exceptional Lie groups", note 44, Thms. 1.14.2, 2.7.1; Prop. 2.3.7]; [Baez, "The octonions", note 10]	Prop. 10.6, hence Cor. 10.8
The F_4-invariant polynomials on $H_3^0(\mathbb{O})$ are generated by $\lvert x\rvert^2$ and the cubic $\mathrm{tr}\, x^3$	[Höhn, "The unique extremal threemodular lattice of rank 26", note 160, pp. 20–21] (Lemma 5.1 there)	Lem. 10.4
A grading of $\mathfrak{f}_4 \otimes \mathbb{C}$ by $\mathbb{F}_3^3$ (a Jordan grading, in Elduque's terms) with zero neutral component and all other components of dimension two is unique up to equivalence; the sums of opposite components are thirteen Cartan subalgebras, pairwise orthogonal for the Killing form	[Elduque, "Jordan gradings on exceptional simple Lie algebras", note 31, §1 and Thm. 3.5]; [Kostrikin and Tiep, *Orthogonal decompositions*, note 11, Thm. 3.1.2, pp. 85–86]	Prop. 10.12, hence Prop. 10.13
A symmetric quartic tensor splits orthogonally into its $\mathrm{O}(N)$-irreducible parts, a totally traceless part, a traceless symmetric 2-tensor and a trace part	[Hogervorst and Toldo, Bounds on multiscalar CFTs, note 212, Eqs. (3.11)–(3.16)]	Prop. 12.5(a)
$[L_x^\bullet, L_y^\bullet]$ is a derivation; Tits's magic-square construction makes $\mathfrak{g}_{\mathbb{K}}$ a compact Lie algebra, isomorphic to $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$ for $\mathbb{K} = \mathbb{C}$ and to $\mathfrak{su}(6)$ for $\mathbb{K} = \mathbb{H}$, and for $\mathbb{K} = \mathbb{O}$ simple with complexification isomorphic to $\mathfrak{e}_6(\mathbb{C})$	[Schafer, *An introduction to nonassociative algebras*, note 220]; [Yokota, "Exceptional Lie groups", note 44, Prop. 2.4.1(2)]; [Baez, "The octonions", note 10, §4.3, Table 5]; [Chevalley and Schafer, "The exceptional simple Lie algebras F_4 and E_6", note 223]	Cor. B.2, as its stated hypothesis
Any two compact real forms of a complex semisimple Lie algebra are conjugate	[Knapp, "Structure theory of semisimple Lie groups", note 153, §4, Cor. of Thm. 4.2(c), p. 21]	Cor. B.2 ($\mathfrak{g}_{\mathbb{O}}$ is compact $\mathfrak{e}_6$)

Declarations are named by their last components, as many as identify them uniquely in the development; the full names are in `PAPER-COVERAGE.tsv`. In these names `round` and `radial` mean eikonal (Definition 3.1). The margin note beside a numbered statement names the declarations that prove it and begins with *Formalized* when the Lean statement is the printed one, *Formalized, stronger* when it has weaker hypotheses or a stronger conclusion, which the note names, *Formalized, different route* when the formal proof departs from the printed one, and *Formalized, details in Lean* when the printed proof summarizes computations that the formal proof carries out. A note also names a restriction of the setting where there is one. The notes beside the three quoted theorems say what the development proves of them.

Code the development builds on

The development imports Mathlib and the Lean package that accompanies an earlier AI-generated paper of the author on sequential ideal measurements in the exceptional Jordan algebra[240], at tag `v2.0.0` (commit `10e5ca8`), entry PALOMAR-2026-11-27-000005; its compact Albert algebra, octonions and Jordan operators are used from there. That package adapts its octonions and Albert algebra from an earlier AI-generated formalization of the author registered in Palomar as entry PALOMAR-2026-10-16-000013[241], and it supplies Theorem 4.4 through files adapted from the composition-algebra library of Ehrlich[242], entry PALOMAR-2026-08-29-000018.

Five groups of files are taken from elsewhere, all under the Apache License 2.0. The three modules for the coordinates, linear structure and squares of Tits's construction of the Albert algebra, and the trace-pairing and trace-cube lemmas, are adapted from the earlier AI-generated formalization of rational invariants of frame triples in the split Albert algebra[243], entry PALOMAR-2026-09-13-000001, at commit `dd06ef5`. The modules of the quaternionic lower bound (Section 6) are adapted from an earlier unpublished AI-generated Lean development of that bound, and the modules on the counting of coupled triples of frames and on the witness of the graded system from an earlier unpublished AI-generated Lean development of the bound of 52. The compact coordinates of Tits's construction of

[240] Earlier AI-generated work of the author: On sequential ideal measurements in the exceptional Jordan algebra, Hexagon preprint hexagon:2611.00620v2, 2026; Earlier AI-generated work of the author: `albert-sequential-measurements`, Lean 4 repository `macgregor-sudo/albert-sequential-measurements`, tag `v2.0.0`, commit `10e5ca8`, Palomar registry entry PALOMAR-2026-11-27-000005, 2026, Apache License 2.0.

[241] Earlier AI-generated work of the author: Finite observations and recognition of the compact Albert algebra, Palomar registry entry PALOMAR-2026-10-16-000013, 16 October 2026, Lean repository `macgregor-sudo/compact-albert-recognition`, commit `40d3b34`.

[242] B. Ehrlich, `composition-algebras`: Euclidean composition algebras over $\mathbb{R}$ in Lean 4, repository `ehrlich-b/composition-algebras`, 2026; Palomar registry entries PALOMAR-2026-08-29-000015 (commit `e37e22b0`) and PALOMAR-2026-08-29-000018 (commit `98d4437c`), Apache License 2.0.

[243] Earlier AI-generated work of the author: Rational invariants of three frames in the split Albert algebra, Palomar registry entry PALOMAR-2026-09-13-000001, 2026, Lean repository `macgregor-sudo/albert-frame-triple-invariants`, commit `dd06ef5`, Apache License 2.0.

the Albert algebra specialize a file of the earlier AI-generated formalization of the local geometry of the tight 39-point code[244], entry PALOMAR-2026-10-29-000034, at commit `46f01c1`. Four files of the TauCeti library[245], at commit `d91980b`, on finite indices, octonions, the Albert algebra and its determinant, are included with the upstream code intact and an attribution comment appended to each. The file `THIRD_PARTY_NOTICES.md` of the repository names each of these files, its source and its changes.

The main theorems stated over Mathlib alone

Thirteen theorems are stated in a separate file that imports Mathlib only (the self-contained file of the margin notes), and the development proves them with exactly these statements. The file's octonion product, conjugation and coordinates for Hermitian matrices follow the self-contained file of the package on sequential ideal measurements named above. They are the two halves of Theorem A(i), the count and the existence for $m = 4$ and $m = 13$ (Theorem 8.11); the two halves of Theorem A(ii), the list of decompositions and the existence of all four (Corollary 8.9); the sum rule (Theorem 5.9); the bound of 52 positive shares among thirteen planes, with every share 0 or 1 at equality (Theorem 9.1); the three parts of the recovery theorem (Theorem 11.10); the quaternionic lower bound with its attaining triple and the form of its equality case (Theorem 6.4); the bound of five unbiased bases (Corollary 6.9); the unique completion of twelve frames of $H_3(\mathbb{O})$, or of twelve planes, (Proposition 6.8); and the existence of a thirteen-plane system with exactly 52 positive shares together with its uniqueness up to automorphism (Theorem 9.4).

Formal proofs that differ from the printed ones

The printed proofs cite theorems from the literature where a working mathematician would; the formal proofs prove what is needed at those steps, or take it from the libraries named above. The difference is largest in the following statements, whose margin notes say *Formalized, different route*. For Corollary 4.5 the formal proof builds on the joint eigenspace S_3 a unital algebra with multiplicative norm and applies Theorem 4.4, where the printed proof applies Hurwitz's theorem on bilinear maps directly. For Theorem 5.10(iii) the formal proof normalizes the first pair of frames, identifies their four-dimensional product space from the shares and shows that it determines the other two planes up to interchange, where the printed proof cites the classification of Brierley, Weigert and Bengtsson. For Proposition 8.2 the formal proof moves the octet onto the standard copy of $H_3^0(\mathbb{C})$ and gives its orthogonal complement complex coordinates directly, where the printed proof uses the coordinates of Theorem 7.4. For Proposition 10.12 the formal proof

[244] Earlier AI-generated work, "On the local geometry of the tight 39-point code", note 25; Earlier AI-generated work of the author: On the local geometry of the tight 39-point code in $\mathbb{O}P^2$, Palomar registry entry PALOMAR-2026-10-29-000034, 2026, Lean repository `macgregor-sudo/tight-39-code`, commit `46f01c1`.

[245] The Tau Ceti contributors, TauCeti, Lean 4 library, repository `TauCetiProject/TauCeti`, commit `d91980b`, Apache License 2.0.

derives the orthogonality from the pairing invariant under the ternary grading group and the commutativity of each summand from the alternating form on a grade of dimension two, where the printed proof cites Elduque's theorem. For Proposition 12.5(a) the formal proof checks the orthogonality and the norms of the three parts directly, where the printed proof cites the splitting of Hogervorst and Toldo. For Proposition 10.5(c) the formal proof obtains the conjugacy by simultaneous diagonalization, where the printed proof cites the conjugacy of Cartan subalgebras. For Theorem 11.10(ii) the formal counterexample reverses the sign of the cubic on one plane, where the printed one rotates that plane. The printed proofs of Lemmas 5.3, 5.4(ii), 6.1, 6.2, 8.3, 10.2, 10.4, Theorem 9.1 and Proposition 10.6 cite the results listed against them in Appendix C, and the formal proofs reach the conclusions needed from those results by other arguments.

The printed proofs of Lemmas 4.1, 4.3, A.1, A.2 and 7.5, Theorems 4.6 and 6.4, and Propositions 6.7, 6.8 and B.5 summarize computations that the formal proofs carry out; their margin notes say *Formalized, details in Lean*.

Changelog

Version 2 (27 November 2026). No main theorem changed its conclusion. Mathematical changes:

- False as printed in version 1: the phase $\varphi_{ghk} = [v, w](n' - n)$ of Proposition 7.6 holds for the clock elements $u_g = \iota_n(D_v)$ (version 1: any clock elements); after it, $\nabla W = 3\sqrt{6}\, x \circ x$ (version 1: $3\, x \circ x$); the limit of κ in Appendix B.3 is C times the sign of $g(0)$ (version 1: C); the share is the squared-coefficient sum only for $c = 1$ after factoring out -3 (Sections 1, 5; version 1: nine times the share). In unnumbered sentences: the equations of motion need the initial velocity in the closed subspace (Section 3); Cartan's isoparametric description and the re-proof at an idempotent need $N \geq 3$ (Sections 3, 8); the quaternionic triple trace is not a Bargmann invariant (Section 5); Corollary 6.9 does not reduce the problem of Chterental and Đoković to five bases.

- Hypotheses now printed: $m \geq 1$ and the trace form (Theorem A(i)); nonzero summands (Theorem A(ii)); a nonzero algebra and $n \geq 1$ (Hurwitz's theorems, Theorem 4.4); norm one for "units" (Lemma 5.4(ii)); any Euclidean space V of dimension N (Theorem 8.8); unordered frames of spectral idempotents (Proposition 9.3); $\varepsilon > 0$, $a \neq 0$ and $t = b/a$ (Proposition 12.5(c)); $n \geq 1$ (Remark 3.9); in summaries, two or more fields for holomorphic couplings, $c = 1$, $m > 1$, and $\varepsilon > 0$, $g(0) \neq 0$, $t(0) > 0$ for the flow.

- Definition 11.4: the pencil is the span of D and M, with normalized members d_t (version 1: the family d_t). Corollary B.2 grants only that $\mathfrak{g}_{\mathbb{O}}$ is compact and simple with complexification $\mathfrak{e}_6(\mathbb{C})$, and derives compact $\mathfrak{e}_6$. The time to leave the decoupled point, with g held fixed, is stated as a logarithmic interval with leading term $1/(\varepsilon t_0)$.

- Cited results restated where proofs use them: the trace form in Theorem 7.4; the normal form of Chterental and Đoković, derived from the proof of their Theorem 5.4; complex conjugation in the equivalence of Brierley, Weigert and Bengtsson, unused in their derivation in dimension three.
- Added: definitions of zero weight planes, complex Cartan subalgebras and sparsest systems; Griess's parameter as $t = -2\alpha$ (Section 11).

Corrected notation and minor steps in several proofs. Added references and corrected locators and attributions, including statements of cited results not used in proofs. Adjusted terminology, notation and non-mathematical prose. Abstract: two further sentences reworded. Revised the front matter, the margin notes, the list of quoted inputs and Appendix D.

Version 1 (27 November 2026). First public version.

References

[1] J. F. Adams, P. D. Lax and R. S. Phillips, On matrices whose real linear combinations are nonsingular, *Proc. Amer. Math. Soc.* **16** (1965) 318–322; correction **17** (1966) 945–947.

[2] A. V. Alekseevskii, Finite commutative Jordan subgroups of complex simple Lie groups, *Funct. Anal. Appl.* **8** (1974) 277–279 (*Funkts. Anal. Prilozh.* **8** (1974) no. 4, 1–4). doi:10.1007/BF01075483.

[3] J. An, H. Dietrich and A. J. Litterick, Elementary abelian subgroups: from algebraic groups to finite groups, *Trans. Amer. Math. Soc.* **377** (2024) 8335–8380. doi:10.1090/tran/9140. arXiv:2303.02364; section numbers as in arXiv version 2.

[4] J. C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002) 145–205. arXiv:math/0105155.

[5] J. C. Baez and S. Pumplün, Three-dimensional geometry in exceptional mathematics. arXiv:2609.07538, 2026.

[6] V. Bargmann, Note on Wigner's theorem on symmetry operations, *J. Math. Phys.* **5** (1964) 862–868. doi:10.1063/1.1704188.

[7] I. Bengtsson, Three ways to look at mutually unbiased bases, in *Foundations of Probability and Physics – 4*, AIP Conf. Proc. **889** (2007) 40–51. arXiv:quant-ph/0610216.

[8] I. Bengtsson, From SICs and MUBs to Eddington, *J. Phys. Conf. Ser.* **254** (2010) 012007. arXiv:1103.2030.

[9] A. V. Borovik, Jordan subgroups and orthogonal decompositions, *Algebra i Logika* **28** (1989) 382–392; English transl. *Algebra and Logic* **28** (1989). doi:10.1007/BF01982015.

[10] P. O. Boykin, M. Sitharam, P. H. Tiep and P. Wocjan, Mutually unbiased bases and orthogonal decompositions of Lie algebras, *Quantum Inf. Comput.* **7** (2007) 371–382. doi:10.26421/QIC7.4-6. arXiv:quant-ph/0506089.

[11] P. Boyvalenkov and D. Danev, On linear programming bounds for codes in polynomial metric spaces (in Russian), *Problemy Peredachi Informatsii* **34**(2) (1998) 16–31. Math-Net.Ru ppi400.

[12] S. Brierley, S. Weigert and I. Bengtsson, All mutually unbiased bases in dimensions two to five, *Quantum Inf. Comput.* **10** (2010) 803–820. arXiv:0907.4097.

[13] Z. Buckley and S. Waldron, Quaternionic MUBs in $\mathbb{H}^2$ and their reflection symmetries. arXiv:2509.01859, 2025.

[14] V. P. Burichenko, Transitive orthogonal decompositions of simple complex Lie algebras of types F_4 and E_6, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 1988, no. 4, 78–80.

[15] A. J. Calderón Martín, C. Draper and C. Martín González, Gradings on the real forms of the Albert algebra, of $\mathfrak{g}_2$, and of $\mathfrak{f}_4$, *J. Math. Phys.* **51** (2010) 053516. doi:10.1063/1.3326238.

[16] É. Cartan, Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques, *Math. Z.* **45** (1939) 335–367. doi:10.1007/BF01580289.

[17] C. Chevalley and R. D. Schafer, The exceptional simple Lie algebras F_4 and E_6, *Proc. Nat. Acad. Sci. USA* **36** (1950) 137–141. doi:10.1073/pnas.36.2.137.

[18] O. Chterental and D. Ž. Đoković, On orthostochastic, unistochastic and qustochastic matrices, *Linear Algebra Appl.* **428** (2008) 1178–1201. doi:10.1016/j.laa.2007.09.022. arXiv:0708.4051; theorem and page numbers as in arXiv version 1.

[19] A. M. Cohen and P. H. Tiep, Splitting fields for Jordan subgroups, in *Finite Reductive Groups* (Luminy, 1994), Progr. Math. **141**, Birkhäuser, Boston, 1997, 165–183.

[20] H. Cohn, A. Kumar and G. Minton, Optimal simplices and codes in projective spaces, *Geom. Topol.* **20** (2016) 1289–1357. doi:10.2140/gt.2016.20.1289. arXiv:1308.3188.

[21] H. Cui, Laplacian algebras and subcalibrations for minimal hypercones. arXiv:2609.17217, 2026.

[22] P. Cvitanović, *Group Theory: Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press, 2008.

[23] H. B. Dang, K. Blanchfield, I. Bengtsson and D. M. Appleby, Linear dependencies in Weyl–Heisenberg orbits, *Quantum Inf. Process.* **12** (2013) 3449–3475. arXiv:1211.0215.

[24] P. Deligne, La série exceptionnelle de groupes de Lie, *C. R. Acad. Sci. Paris Sér. I* **322** (1996) 321–326.

[25] L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, in *Automated Deduction – CADE 28*, Lecture Notes in Comput. Sci. **12699**, Springer, Cham, 2021, 625–635. doi:10.1007/978-3-030-79876-5_37.

[26] C. Draper Fontanals and C. Martín González, Gradings on the Albert algebra and on $\mathfrak{f}_4$, *Rev. Mat. Iberoam.* **25** (2009) 841–908. doi:10.4171/RMI/585. arXiv:math/0703840.

[27] B. Ehrlich, `composition-algebras`: Euclidean composition algebras over $\mathbb{R}$ in Lean 4, repository `ehrlich-b/composition-algebras`, 2026; Palomar registry entries PALOMAR-2026-08-29-000015 (commit `e37e22b0`) and PALOMAR-2026-08-29-000018 (commit `98d4437c`), Apache License 2.0.

[28] A. Elduque, Jordan gradings on exceptional simple Lie algebras, *Proc. Amer. Math. Soc.* **137** (2009) 4007–4017. arXiv:0810.2737.

[29] A. Elduque and M. Kochetov, Gradings on the exceptional Lie algebras F_4 and G_2 revisited, *Rev. Mat. Iberoam.* **28** (2012). arXiv:1009.1218; numbering as in arXiv version 3.

[30] A. Elduque and M. Kochetov, Weyl groups of fine gradings on matrix algebras, octonions and the Albert algebra, *J. Algebra* **366** (2012) 165–186. doi:10.1016/j.jalgebra.2012.05.008. arXiv:1009.1462.

[31] L. Fei, S. Giombi, I. R. Klebanov and G. Tarnopolsky, Yukawa CFTs and emergent supersymmetry, *Prog. Theor. Exp. Phys.* **2016** (2016) 12C105. doi:10.1093/ptep/ptw120. arXiv:1607.05316.

[32] D. J. F. Fox, Harmonic cubic homogeneous polynomials such that the norm-squared of the Hessian is a multiple of the Euclidean quadratic form, *Anal. Math. Phys.* **11** (2021) 43; correction **11** (2021) 172. doi:10.1007/s13324-020-00462-4. arXiv:1905.00071.

[33] D. J. F. Fox, Commutative algebras with nondegenerate invariant trace form and trace-free multiplication endomorphisms. arXiv:2004.12343.

[34] D. J. F. Fox, Killing metrized commutative nonassociative algebras associated with Steiner triple systems, *J. Algebra* **608** (2022) 186–213. arXiv:2205.08838.

[35] R. L. Griess, Jr., A Moufang loop, the exceptional Jordan algebra, and a cubic form in 27 variables, *J. Algebra* **131** (1990) 281–293. doi:10.1016/0021-8693(90)90176-O.

[36] R. L. Griess, Jr., Elementary abelian p-subgroups of algebraic groups, *Geom. Dedicata* **39** (1991) 253–305. doi:10.1007/BF00150757.

[37] U. Haagerup, Orthogonal maximal abelian $*$-subalgebras of the $n \times n$ matrices and cyclic n-roots, in *Operator Algebras and Quantum Field Theory (Rome, 1996)*, International Press, Cambridge MA, 1997, 296–322.

[38] M. Hall, Jr., *Combinatorial Theory*, 2nd ed., Wiley, New York, 1986.

[39] S. Harribey, W. H. Pannell and A. Stergiou, Multiscalar critical models with localised cubic interactions, *J. High Energy Phys.* **2024**(12) (2024) 092. doi:10.1007/JHEP12(2024)092. arXiv:2407.20326v3.

[40] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[41] O. Hesse, Über die Wendepunkte der Curven dritter Ordnung, *J. Reine Angew. Math.* **28** (1844) 97–107.

[42] M. Hogervorst and C. Toldo, Bounds on multiscalar CFTs in the ϵ expansion, *J. High Energy Phys.* **2021**(04) (2021) 068. doi:10.1007/JHEP04(2021)068. arXiv:2010.16222.

[43] S. G. Hoggar, t-designs in projective spaces, *European J. Combin.* **3** (1982) 233–254.

[44] G. Höhn, The unique extremal threemodular lattice of rank 26, the generalized hexagon $(2, 8)$, and the tight Cayley-plane 5-design. arXiv:2609.25086, 2026.

[45] D. R. Hughes and F. C. Piper, *Projective Planes*, Graduate Texts in Mathematics 6, Springer, New York, 1973.

[46] L. P. Hughston, $d = 3$ SIC-POVMs and elliptic curves, talk at the Perimeter Institute, October 2007, PIRSA 07100040. `http://pirsa.org/07100040/`.

[47] A. Hurwitz, Über die Composition der quadratischen Formen von beliebig vielen Variabeln, *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* (1898) 309–316.

[48] I. D. Ivanović, Geometrical description of quantal state determination, *J. Phys. A* **14** (1981) 3241–3245.

[49] I. Jack and D. R. T. Jones, The a-function in fractional dimensions, arXiv:2611.27464, 2026.

[50] I. Jack and H. Osborn, Analogs of the c-theorem for four-dimensional renormalisable field theories, *Nucl. Phys. B* **343** (1990) 647–688. doi:10.1016/0550-3213(90)90584-Z.

[51] I. Jack, H. Osborn and T. Steudtner, Explorations in scalar fermion theories: β-functions, supersymmetry and fixed points, *J. High Energy Phys.* **2024**(02) (2024) 038. doi:10.1007/JHEP02(2024)038. arXiv:2301.10903v3.

[52] N. Jacobson, *Structure and Representations of Jordan Algebras*, Amer. Math. Soc. Colloq. Publ. 39, Providence, 1968. doi:10.1090/coll/039.

[53] W. M. Kantor, Note on Lie algebras, finite groups and finite geometries, Ohio State Univ. Math. Res. Inst. Publ. **4** (1996) 73–81; page numbers refer to the preprint at `https://pages.uoregon.edu/kantor/PAPERS/LieAlgebras.pdf`.

[54] A. W. Knapp, Structure theory of semisimple Lie groups, *Proc. Sympos. Pure Math.* **61**, Amer. Math. Soc., Providence, 1997, 1–27. `https://www.math.stonybrook.edu/~aknapp/pdf-files/1-27.pdf`.

[55] A. I. Kostrikin, I. A. Kostrikin and V. A. Ufnarovskii, On decompositions of classical Lie algebras (in Russian), *Trudy Mat. Inst. Steklov.* **166** (1984) 107–122; English transl.: Decompositions of classical Lie algebras, *Proc. Steklov Inst. Math.* **166** (1986) 117–133. Math-Net.Ru tm2255.

[56] A. I. Kostrikin and P. H. Tiep, *Orthogonal Decompositions and Integral Lattices*, de Gruyter Exp. Math. 15, de Gruyter, Berlin, 1994. doi:10.1515/9783110901757.

[57] L. Kramer, A note on commutators in compact semisimple Lie algebras, *Innov. Incidence Geom.* **20** (2023), no. 2–3, 317–324. doi:10.2140/iig.2023.20.317. arXiv:2204.09913.

[58] V. I. Levenshtein, Bounds on the maximal cardinality of a code with bounded modulus of the inner product (in Russian), *Dokl. Akad. Nauk SSSR* **263**(6) (1982) 1303–1308. Math-Net.Ru dan45246.

[59] P. Liendo and J. Rong, Seeking SUSY fixed points in the $4 - \epsilon$ expansion, *J. High Energy Phys.* **2021**(12) (2021) 033. doi:10.1007/JHEP12(2021)033. arXiv:2107.14515.

[60] Earlier AI-generated work of the author: Finite observations and recognition of the compact Albert algebra, Palomar registry entry PALOMAR-2026-10-16-000013, 16 October 2026, Lean repository `macgregor-sudo/compact-albert-recognition`, commit `40d3b34`.

[61] Earlier AI-generated work of the author: On sequential ideal measurements in the exceptional Jordan algebra, Hexagon preprint hexagon:2611.00620v2, 2026.

[62] Earlier AI-generated work of the author: `albert-sequential-measurements`, Lean 4 repository `macgregor-sudo/albert-sequential-measurements`, tag `v2.0.0`, commit `10e5ca8`, Palomar registry entry PALOMAR-2026-11-27-000005, 2026, Apache License 2.0.

[63] Earlier AI-generated work of the author: Rational invariants of three frames in the split Albert algebra, Palomar registry entry PALOMAR-2026-09-13-000001, 2026, Lean repository

`macgregor-sudo/albert-frame-triple-invariants`, commit `dd06ef5`, Apache License 2.0.

[64] Earlier AI-generated work of the author: On the local geometry of the tight 39-point code in the octonionic projective plane, Hexagon preprint hexagon:2611.00340v2, 2026.

[65] Earlier AI-generated work of the author: On the local geometry of the tight 39-point code in $\mathbb{O}P^2$, Palomar registry entry PALOMAR-2026-10-29-000034, 2026, Lean repository `macgregor-sudo/tight-39-code`, commit `46f01c1`.

[66] The mathlib Community, The Lean mathematical library, in *Proc. 9th ACM SIGPLAN Int. Conf. on Certified Programs and Proofs (CPP 2020)*, ACM, 2020, 367–381. doi:10.1145/3372885.3373824. arXiv:1910.09336.

[67] D. McNulty and S. Weigert, Mutually unbiased bases in composite dimensions – a review, *Quantum* **10** (2026) 2051. doi:10.22331/q-2026-04-01-2051. arXiv:2410.23997.

[68] T. Moran and S. Pumplün, A generalization of the first Tits construction, arXiv:2403.16342, 2024.

[69] N. Nadirashvili, V. Tkachev and S. Vlăduţ, *Nonlinear Elliptic Equations and Nonassociative Algebras*, Math. Surveys Monogr. **200**, Amer. Math. Soc., Providence, 2014. doi:10.1090/surv/200.

[70] OpenAI, The maximum number of mutually unbiased bases in dimension six, preprint, 24 September 2026. github.com/openai/math.

[71] H. Osborn and A. Stergiou, Seeking fixed points in multiple coupling scalar theories in the ϵ expansion, *J. High Energy Phys.* **2018**(05) (2018) 051. doi:10.1007/JHEP05(2018)051. arXiv:1707.06165v7.

[72] W. H. Pannell, W. P. Ronayne and A. Stergiou, Gradient RG flow in scalar-fermion QFTs, *J. High Energy Phys.* **2026**(05) (2026) 064. doi:10.1007/JHEP05(2026)064. arXiv:2511.01971.

[73] C.-K. Peng and L. Xiao, On the classification of cubic minimal cones (in Chinese), *J. Grad. School Academia Sinica* **10** (1993), no. 1, 1–19.

[74] L. Rodman, *Topics in Quaternion Linear Algebra*, Princeton University Press, Princeton, 2014.

[75] J. Rong and N. Su, Bootstrapping the $\mathcal{N} = 1$ Wess–Zumino models in three dimensions, *J. High Energy Phys.* **2021**(06) (2021) 153. doi:10.1007/JHEP06(2021)153. arXiv:1910.08578; equation numbers as in arXiv version 1.

[76] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.

[77] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monogr. Math., Springer, Berlin, 2000. doi:10.1007/978-3-662-12622-6.

[78] B. C. Stacey, SIC-POVMs and compatibility among quantum states, *Mathematics* **4** (2016) 36. arXiv:1404.3774.

[79] G. N. M. Tabia and D. M. Appleby, Exploring the geometry of qutrit state space using symmetric informationally complete probabilities, *Phys. Rev. A* **88** (2013) 012131. arXiv:1304.8075.

[80] The Tau Ceti contributors, TauCeti, Lean 4 library, repository `TauCetiProject/TauCeti`, commit `d91980b`, Apache License 2.0.

[81] M. Thiel and N. Williams, Strange expectations and the Winnie-the-Pooh problem, *J. Combin. Theory Ser. A* **176** (2020) 105298. arXiv:1811.02550.

[82] P. H. Tiep, Irreducible orthogonal decompositions in Lie algebras, *Mat. Sb.* **180** (1989) 1396–1414; English transl. *Math. USSR-Sb.* **68** (1991) 257–275. doi:10.1070/SM1991v068n01ABEH002103. Math-Net.Ru sm1666.

[83] V. G. Tkachev, A generalization of Cartan's theorem on isoparametric cubics, *Proc. Amer. Math. Soc.* **138** (2010) 2889–2895. doi:10.1090/S0002-9939-10-10385-2. arXiv:1001.2387v2.

[84] V. G. Tkachev, A Jordan algebra approach to the cubic eiconal equation, *J. Algebra* **419** (2014) 34–51. doi:10.1016/j.jalgebra.2014.06.039. arXiv:1211.7223. Addendum, *J. Algebra* **449** (2016) 774, doi:10.1016/j.jalgebra.2015.12.010.

[85] V. G. Tkachev, Hsiang cubics as spherical Chebyshev polynomials: equioscillation and idempotent quadrature. arXiv:2610.13532v1, 2026.

[86] A. Torstensson, On the existence of orthogonal decompositions of the simple Lie algebra of type C_3, *Comput. Sci. J. Moldova* **8** (2000) no. 1, 16–41. `https://www.math.md/files/csjm/v8-n1/v8-n1-(pp16-41).pdf`.

[87] M. Weiner, A gap for the maximum number of mutually unbiased bases, *Proc. Amer. Math. Soc.* **141** (2013) 1963–1969. doi:10.1090/S0002-9939-2013-11487-5. arXiv:0902.0635.

[88] W. K. Wootters and B. D. Fields, Optimal state-determination by mutually unbiased measurements, *Ann. Phys.* **191** (1989) 363–381. doi:10.1016/0003-4916(89)90322-9.

[89] I. Yokota, Exceptional Lie groups, 2009. arXiv:0902.0431.

[90] F. Zhang, Quaternions and matrices of quaternions, *Linear Algebra Appl.* **251** (1997) 21–57. doi:10.1016/0024-3795(95)00543-9.