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Critical tangents and local multifractal spectra of tempered multifractional multistable motion

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Abstract

We study a tempered moving average with Hurst parameter $H(t)$ at the observation time and stability index $\alpha(x)$ at the noise location. Suppose α is continuous at u and the increments of H have a profile at scale $r^{H(u)}$. The tangent is the sum of a linear fractional stable motion independent of the original noise and the profile multiplied by the Hurst derivative of the original field. Convergence holds stably in finite-dimensional distributions and in $L^p(K)$ on each compact interval K for $0 < p < \alpha(u)$. At $H(u)\alpha(u) = 1$ we give conditions for convergence in the Skorokhod M_1 topology. For the spectrum, assume H is continuous and locally Lipschitz off u, with $q = H - 1/\alpha$ positive off u and $q(u) \geq 0$. If $1/\alpha$ is γ-Hölder for $\gamma \geq 1/2$, a comparison with a fractional integral gives the exact local spectrum $\alpha(u)(s - q(u))$ for $q(u) < s < H(u)$. If $3\gamma > 1 + 1/\alpha(u)$ and the pointwise exponent at u equals $H(u)$ almost surely, the local behavior of H determines the upper endpoint. We exhibit critical contacts with pointwise exponent $H(u)$ and local uniform exponent zero, although all nearby pointwise exponents are positive.

Keywords. Multistable random measure, multifractional motion, stable convergence, Young integral, Skorokhod topology, multifractal spectrum.

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1 Introduction

Consider a moving average $Y(t,h)$ driven by an independently scattered multistable random measure, and let $X(t) = Y(t, H(t))$. Two terms contribute to the increment of X near u. Translating the kernel samples fresh noise near u, while changing its order evaluates the same random field at a new parameter. When the change in H is of order $r^{H(u)}$ at distance r, both terms can appear in the tangent process. They have different limits even though the same noise drives them at every scale.

1.1 The tangent process

We use the multistable random measure of Falconer and Liu [FL12]. Its stability index $\alpha(x)$ enters both the characteristic functional and the kernel at the source location x. Fan and Lévy Véhel studied the tempered fractional multistable and tempered multifractional stable models separately and proposed combining them [FLV19]. For constant Hurst order, their proof of localisability already exhibits the cancellation between the rescaled kernel and the control measure [FLV19, Propositions 8.1–8.2]. Ayache and Hamonier [AH19] combined a spatial index with a variable order in a Riemann–Liouville model and obtained a wavelet representation for $C^{1+\rho}$ indices. Biermé and Lacaux [BL13] studied a different coupling through a parameter-indexed LePage series. Mies and Sikkens [MS26, Proposition 3.4] place a random Hurst exponent at the source and keep the stability index constant. Their tangent is a fractional stable integral with the Hurst exponent frozen at the base point.

Write $h = H(u)$ and $A = \alpha(u)$. If $r^{-h}(H(u+rv) - h) \to g(v)$, we prove that

$$r^{-h}\{X(u+rv) - X(u)\} \xrightarrow[\text{fdd}]{\text{st}} L_{h,A}(v) + g(v)D_u, \qquad D_u = \partial_h Y(u,h).$$

The convergence is stable relative to the original noise. The linear fractional stable motion $L_{h,A}$ is independent of that noise, whereas D_u is a functional of it. In the constant-index theory, Stoev and Taqqu developed the parameter expansion [ST04] and studied the regimes in which either the frozen-order increment or the parameter derivative dominates [ST05, Theorems 1 and 2]. Here continuity of α at u suffices for the balanced finite-dimensional limit at every

$0 < h < 1$. On compact time intervals the limit holds in L^p for $0 < p < A$ and, when $hA > 1$ and H is continuous near u, in the uniform topology.

At $hA = 1$, $L_{h,A}$ is a Lévy motion. Under the hypotheses of Theorem 4.1, each large noise atom produces a short transition that converges to a jump in the scalar Skorokhod M_1 topology. The theorem allows either a positive contact $H(u+s) - 1/\alpha(u+s) \geq c|s|^h$ or a nonnegative difference together with the modulus condition $\omega_H(s)|\log s| \to 0$. A continuous Hurst function can violate M_1 tightness even when the finite-dimensional tangent and the pointwise exponent persist. Related M_1 limits for Lévy-driven convolution integrals appear in [PR15].

1.2 Path regularity and local spectra

Put $\beta = 1/\alpha$ and $q = H - \beta$. Let B denote the additive noise on a compact source interval J and write $b_J = \sup_J \alpha$. For $\beta \in C^\gamma(J)$ with $\gamma > 1 - 1/b_J$, the kernel is Young integrable against B wherever $q > 0$. The small jumps of B satisfy

$$\mathbb{E}V_p(B^{\leq\varepsilon}; J)^p \leq C|J|\varepsilon^{p-b_J}, \qquad \max(1, b_J) < p < 2.$$

This estimate controls the error from removing small marks in the critical M_1 limit. It also gives the finite p-variation needed for the path representation. For general variation criteria for Markov processes, see [Man04]. At the critical boundary we also allow continuous indices of bounded variation under a Dini condition on their variation measures.

On a rectangle with $h' > \beta(t)$, the leading term of $Y(t, h')$ is $\Gamma(h' - \beta(t) + 1) I_{\ell+}^{h'-\beta(t)} B(t)$. The remainder and its derivatives in h' are uniformly γ-Hölder in t. To compare pointwise exponents we freeze the order at the observation point before estimating the difference of the two kernels. The oscillation of B then gives a sharper error bound than the uniform Hölder estimate alone.

Set $q_0 = h - 1/A$. Suppose that H is continuous, locally Lipschitz away from u, and that $q > 0$ on a punctured neighborhood of u with $q_0 \geq 0$. Under the local Young condition, the exact local spectrum is

$$d_X(s, u) = A(s - q_0), \qquad q_0 < s < \min\left\{h, q_0 + \frac{\gamma}{A(1-\gamma)}\right\}.$$

We read the quotient as infinity for $\gamma = 1$. Thus $\gamma \geq 1/2$ gives the entire open interval (q_0, h). If $3\gamma > 1 + \beta(u)$, the pointwise exponents on the punctured neighborhood agree with those of the corresponding fixed-order fractional integrals. If also $h_X(u) = h$, the exact upper endpoint has dimension one whenever H exceeds h arbitrarily close to u. At a local maximum it has the local Hausdorff dimension of $\{H = h\}$, which can take any value in $[0, 1]$. The proof uses the localized approximation theorem of Barral and Seuret [BS11] and an isolated-jump argument for the primitive of B, related to the jump estimates of Balança [Bal14].

Section 2 proves the tangent limits and Section 3 develops the Young representation and the fixed-order comparison. Section 4 treats the critical path limit. Sections 5 and 6 give the spectra and the critical-contact examples. The Poisson estimates for continuous α are proved in Appendix A.

We use C^γ, $0 < \gamma < 1$, for the ordinary uniform Hölder class. Whenever the endpoint $\gamma = 1$ is included in a Hölder condition, it denotes the Lipschitz class. Explicit assumptions $H \in C^1$ retain their usual meaning of continuous differentiability. All pathwise assertions are made on the common events specified in the proofs. Exponent and dimension statements at a fixed deterministic base point are almost-sure statements for that point, unless simultaneity is stated.

2 The common-noise field and its critical tangent

2.1 Stochastic integration and parameter differentiation

Let $\alpha : \mathbb{R} \to [a, b]$ be measurable, where $0 < a \le b < 2$, and let M_α be a symmetric independently scattered random measure satisfying

$$\mathbb{E} \exp\left(i \int f(x) M_\alpha(\,\mathrm{d}x) \right) = \exp\left(- \int_{\mathbb{R}} |f(x)|^{\alpha(x)} \,\mathrm{d}x \right). \tag{1}$$

The admissible integrands are the measurable functions f with finite modular $\rho_\alpha(f) = \int_{\mathbb{R}} |f(x)|^{\alpha(x)} \,\mathrm{d}x$. Put $\beta = 1/\alpha$ and fix $\lambda > 0$. For $h \in (0, 1)$ set

$$\begin{aligned} \phi_{h,x}(y) &= e^{-\lambda y_+} y_+^{h-\beta(x)}, \\ K(t, h; x) &= \phi_{h,x}(t - x) - \phi_{h,x}(-x), \qquad (2) \\ Y(t, h) &= \int_{\mathbb{R}} K(t, h; x) M_\alpha(\,\mathrm{d}x), \qquad X(t) = Y(t, H(t)). \qquad (3) \end{aligned}$$

Here $H : \mathbb{R} \to (0, 1)$ is deterministic and measurable. For every real ζ, we define $y_+^\zeta = y^\zeta$ for $y > 0$ and zero for $y \le 0$. In particular $y_+^0 = \mathbf{1}_{\{y>0\}}$. Values of a deterministic kernel at its diagonal do not change its stochastic integral. Later, a right-continuous version fixes the values at random jump times.

Write $\mathcal{F}_M = \sigma(M_\alpha)$, completed in the usual way. Stable convergence relative to $\mathcal{F}_M$ means joint convergence with every bounded $\mathcal{F}_M$-measurable random variable, on an extension that carries the limiting independent noise.

Lemma 2.1 (Poisson realization and a local moment bound)**.** *There is a common Poisson realization of the random measure in* (1) *with intensity*

$$\nu(\,\mathrm{d}x, \,\mathrm{d}j) = c(\alpha(x)) |j|^{-1-\alpha(x)} \,\mathrm{d}x \,\mathrm{d}j, \qquad c(A) \int_{\mathbb{R}} (1 - \cos j) |j|^{-1-A} \,\mathrm{d}j = 1. \tag{4}$$

For every admissible deterministic f and $z > 0$,

$$\mathbb{P}\left(\left| \int f \,\mathrm{d}M_\alpha \right| > z \right) \le C \rho_\alpha(f/z). \tag{5}$$

If f is supported on a set U on which $\alpha \ge a_U > p > 0$, and $\rho_\alpha(f/\sigma) \le 1$, then

$$\mathbb{E} \left| \int f \,\mathrm{d}M_\alpha \right|^p \le \left(1 + \frac{Cp}{a_U - p} \right) \sigma^p. \tag{6}$$

In particular, modular convergence to zero of such kernels implies convergence to zero in L^p.

Proof. The normalization c is positive and continuous on every compact subset of $(0, 2)$. For an admissible f, use the finite Poisson sum over $|f(x)j| > 1$ and the compensated integral over $|f(x)j| \le 1$. The former region has finite intensity, and the latter integral has finite second moment, both bounded by constants times $\rho_\alpha(f)$. Symmetry removes the imaginary compensator terms. The change of variable $j \mapsto f(x)j$ in the characteristic exponent gives (1). The same construction for finitely many kernels gives their joint law and independent scattering on disjoint source sets.

For the tail estimate, split at $|f(x)j| = z/2$. The probability of a large jump is at most $C\rho_\alpha(f/z)$. The variance of the remaining compensated sum is at most $Cz^2\rho_\alpha(f/z)$, so Chebyshev's inequality proves (5). For $s \geq 1$, the assumptions of the moment assertion imply

$$\mathbb{P}\left(\left|\int f \, \mathrm{d}M_\alpha\right| > s\sigma\right) \leq Cs^{-a_U}.$$

Integration of this bound proves (6). If $0 < \rho_\alpha(f_n) \leq 1$, then $\sigma_n = \rho_\alpha(f_n)^{1/b}$ satisfies $\rho_\alpha(f_n/\sigma_n) \leq 1$, and $\sigma_n \to 0$ when the modular tends to zero. □

Lemma 2.2 (Existence and parameter derivatives)**.** *The integrals in (3) exist for every $(t, h) \in \mathbb{R} \times (0, 1)$. For every integer $m \geq 0$, differentiation m times with respect to h is valid in probability and gives the stochastic integral of $\partial_h^m K$. These derivatives are continuous in probability in (t, h). In particular, for $h = H(u)$,*

$$D_u = \partial_h Y(u, h) = \int_{\mathbb{R}} Q_u(x) M_\alpha(\, \mathrm{d}x), \tag{7}$$

where

$$\begin{aligned} Q_u(x) = {} & \mathbf{1}_{\{x<u\}} e^{-\lambda(u-x)}(u-x)^{h-\beta(x)} \log(u-x) \\ & - \mathbf{1}_{\{x<0\}} e^{\lambda x}(-x)^{h-\beta(x)} \log(-x). \end{aligned}$$

One has $D_0 = 0$. If $u \neq 0$, D_u is nondegenerate and has a smooth density.

Proof. On a compact order interval with lower endpoint $h_* > 0$, the modular of a logarithmic derivative near a moving singularity is bounded by a constant times

$$\int_0^1 y^{ah_*-1}(1 + |\log y|^{mb}) \, \mathrm{d}y < \infty.$$

Exponential tempering controls the left tail, with all polynomial and logarithmic factors. Splitting off small neighborhoods of t and 0, followed by dominated convergence on their complement, proves modular continuity. For divided differences in h, the ordinary mean-value theorem applied to the deterministic kernel gives the same envelope. Modular convergence implies convergence in probability by (1). This argument uses pointwise bounds on deterministic kernels and thus also applies when $a < 1$.

If $u \neq 0$, Q_u is bounded away from zero on a positive-measure subset of the interval between 0 and u. Its characteristic function is consequently bounded by $\exp(-c|\theta|^a)$ for large $|\theta|$. Fourier inversion, with arbitrary polynomial weights, proves the last assertion. □

Remark 2.3 (Measurable versions). We use a jointly measurable version of Y and its parameter derivatives. On each compact parameter rectangle, the fields are uniformly continuous in probability. Choose deterministic finite grids with mesh tending to zero so rapidly that the probability of an approximation error larger than 2^{-n} is at most 2^{-n}, uniformly over parameters. The grid fields are jointly measurable. For each fixed parameter, Borel–Cantelli identifies their measurable pointwise limit with the given stochastic integral. A countable exhaustion supplies the version on the whole parameter domain. The pathwise versions constructed below agree with it almost surely at each fixed parameter. Continuous local versions coincide on overlaps by continuity from a countable dense set.

2.2 The critical stable tangent

For $A \in (0,2)$ and $h \in (0,1)$, let

$$L_{h,A}(v) = \int_{\mathbb{R}} F_{h,A}(v,z)\widetilde{M}_A(\,\mathrm{d}z), \qquad F_{h,A}(v,z) = (v-z)_+^{h-1/A} - (-z)_+^{h-1/A}, \tag{8}$$

where $\widetilde{M}_A$ is a symmetric A-stable random measure with Lebesgue control measure, independent of $\mathcal{F}_M$.

Theorem 2.4 (Critical stable tangent)**.** *Fix $u \in \mathbb{R}$, put $h = H(u)$ and $A = \alpha(u)$, and assume that α is continuous at u. Suppose that, for every $v \in \mathbb{R}$,*

$$g_r(v) := \frac{H(u+rv)-h}{r^h} \longrightarrow g(v), \qquad r \downarrow 0. \tag{9}$$

Then, relative to $\mathcal{F}_M$,

$$\left\{\frac{X(u+rv)-X(u)}{r^h}\right\}_v \xrightarrow[\text{fdd}]{\text{st}} \{L_{h,A}(v) + g(v)D_u\}_v. \tag{10}$$

In particular, the stable motion in the limit is independent of D_u. For $v_1,\dots,v_m$ and $\theta_1,\dots,\theta_m$, the limiting characteristic function is

$$\exp\left\{-\int_{\mathbb{R}}\left|\sum_j \theta_j F_{h,A}(v_j,z)\right|^A \mathrm{d}z - \int_{\mathbb{R}}\left|\Big(\sum_j \theta_j g(v_j)\Big)Q_u(x)\right|^{\alpha(x)} \mathrm{d}x\right\}. \tag{11}$$

Proof. Decompose the increment as $A_r(v) + C_r(v)$, where

$$\begin{aligned} A_r(v) &= r^{-h}\{Y(u+rv,h) - Y(u,h)\},\\ C_r(v) &= r^{-h}\{Y(u+rv,H(u+rv)) - Y(u+rv,h)\}. \end{aligned}$$

With $x = u + rz$ and $\alpha_r(z) = \alpha(u+rz)$, the kernel of $A_r(v)$ is $r^{-1/\alpha_r(z)}F_{r,v}(z)$, where

$$F_{r,v}(z) = e^{-\lambda r(v-z)_+}(v-z)_+^{h-1/\alpha_r(z)} - e^{-\lambda r(-z)_+}(-z)_+^{h-1/\alpha_r(z)}.$$

The positive-support convention is retained for every real power. Consequently its joint characteristic exponent is exactly

$$\int_{\mathbb{R}}\left|\sum_j \theta_j F_{r,v_j}(z)\right|^{\alpha_r(z)} \mathrm{d}z. \tag{12}$$

The factor r from the control measure cancels $|r^{-1/\alpha_r(z)}|^{\alpha_r(z)}$. Near each kernel singularity, an integrable envelope is a constant times $1 + |z-v_j|^{ah-1} + |z|^{ah-1}$, interpreted only on unit neighborhoods. On the left tail, the mean-value theorem gives the envelope $C|z|^{-1-a(1-h)}$. The term produced by tempering satisfies the same bound because $(r|z|)e^{-cr|z|}$ is bounded. Thus dominated convergence in (12) gives the frozen stable limit for all $0 < h < 1$.

For stable convergence, first fix $\delta > 0$. The part of A_r outside $(u-\delta, u+\delta)$ tends to zero in probability because the time derivative of the moving kernel has a common envelope with finite multistable modular there, and the normalization leaves a factor r^{1-h}. The remaining

part is independent of the noise outside this interval. As $\delta \downarrow 0$, these outside-noise sigma fields generate $\mathcal{F}_M$. Indeed, truncating any admissible kernel to $|x-u| \geq \delta$ recovers the full kernel in the modular as $\delta \downarrow 0$, hence its integral in probability. Conditional-expectation approximation for a bounded $\mathcal{F}_M$-measurable variable proves the asserted mixing, hence stable, convergence of A_r.

Lemma 2.2, applied to divided differences, gives

$$C_r(v) = g_r(v)\,[D_u + o_{\mathbb{P}}(1)]\,.$$

This holds jointly for any finite set of v's. Stable Slutsky convergence proves (10), and independence gives (11). □

Remark 2.5. If (9) holds along all $r \downarrow 0$, then $g(cv) = c^h g(v)$ for every $c > 0$. Thus $g(v) = c_+ v_+^h + c_-(-v)_+^h$, where $c_\pm = g(\pm 1)$. Local uniform convergence follows since for $0 < |v| \leq T$ the error is $|v|^h |g_{r|v|}(\operatorname{sgn} v) - g(\operatorname{sgn} v)|$. Subsequential limits may have other profiles. The law of D_u is determined by the spatial index on the support of Q_u. The tempering parameter disappears from $L_{h,A}$ but remains in D_u. At the anchored point $u = 0$, this derivative term vanishes.

Corollary 2.6 (Competing powers)**.** *Suppose $u \neq 0$ and replace (9) by $H(u+rv) - h = r^\eta g(v) + o(r^\eta)$, where $\eta > 0$. For $\eta > h$, normalization by r^h gives $L_{h,A}$. At $\eta = h$ the limit is given by (10). For $\eta < h$, normalization by r^η gives gD_u. These assertions concern finite-dimensional distributions and have the same stable interpretation.*

2.3 Functional convergence away from the path boundary

We record the localization fact used to retain the local moment range even when α is smaller far from u.

Lemma 2.7 (Smooth remote fields)**.** *Restrict the moving part of the kernel to sources at positive distance from a compact interval of observation times. The resulting field, together with the anchored term, has a modification smooth in (t,h) on compact order intervals in $(0,1)$. No continuity of α on the remote set is needed.*

Proof. Every derivative of a prescribed finite order is bounded by a deterministic envelope $w(x)$ with $\int w(x)^{\alpha(x)}\,\mathrm{d}x < \infty$. Use the Poisson realization of Lemma 2.1. The atoms with $|j|w(x) > 1$ are finite in number. The compensated remaining sum has finite second moment in every prescribed finite-order Sobolev space, since its Hilbert-valued variance is bounded by $C\int w(x)^{\alpha(x)}\,\mathrm{d}x$. Sobolev embedding, with the order chosen arbitrarily large, gives the assertion on a countable exhaustion. The anchor depends only on h and is treated by the same logarithmic envelopes. □

Theorem 2.8 (Functional critical limits)**.** *Under the hypotheses of Theorem 2.4, on every compact interval K, (10) holds stably in $L^p(K)$ for every $0 < p < A$. For $p < 1$, this space is equipped with the metric $\int_K |f-g|^p$. If $hA > 1$ and H is continuous in a neighborhood of u, the convergence also holds stably in $C(K)$ with its uniform norm.*

Proof. Choose a fixed neighborhood U of u with $a_U = \inf_U \alpha > p$, and split

$$Y^N(t,h') = \int_U \phi_{h',x}(t-x) M_\alpha(\mathrm{d}x), \qquad S = Y - Y^N.$$

The anchor belongs entirely to S, which is smooth near (u,h) by Lemma 2.7. Put $A_r^N(v) = r^{-h}(Y^N(u+rv,h) - Y^N(u,h))$. Lemma 2.1 bounds every near-field pth moment by a constant times the pth power of its modular scale. For $v \neq w$, substitute $x = u + rw + r|v-w|z$ in the modular of the increment divided by $|v-w|^h$. The Jacobian cancels the variable power of $r|v-w|$ as in (12). The two kernels now have singularities at 0 and $\operatorname{sgn}(v-w)$, so the same envelopes give a bound uniform in v, w and small r. The moment estimate yields

$$\mathbb{E}|A_r^N(v) - A_r^N(w)|^p \le C|v-w|^{ph}, \qquad v, w \in K. \tag{13}$$

Let $\pi_\delta v$ denote the left endpoint of the mesh cell containing v, for a mesh of width at most δ in K. Integrating (13) gives

$$\mathbb{E}\int_K |A_r^N(v) - A_r^N(\pi_\delta v)|^p \,\mathrm{d}v \le C_K \delta^{ph}.$$

The same bound holds for $L_{h,A}$. Finite-dimensional stable convergence and the converging-together argument therefore give the L^p assertion for the frozen part. Modular continuity of the divided differences in Lemma 2.2, restricted to U, gives $\sup_{v\in K}\mathbb{E}|Q_r^N(v) - \partial_h Y^N(u,h)|^p \to 0$, where Q_r^N is the near-field parameter divided difference (the derivative when the denominator vanishes). This proves the near parameter assertion. The jointly measurable versions from Remark 2.3 and Fubini's theorem give measurable $L^p(K)$-valued variables. The deterministic mesh approximations converge in the L^p metric in probability, which also identifies these versions with the variables used in the finite-dimensional approximation argument. Smoothness of S gives $r^{-h}\Delta_t S \to 0$ uniformly almost surely and its normalized parameter correction converges uniformly to $g\partial_h S(u,h)$. No remote pth moment is required.

When $hA > 1$, choose p with $1/h < p < \inf\alpha$ locally. Estimate (13) gives tightness of the frozen part in $C(K)$. To control the parameter increment, take a compact order interval Q about h with $h_* = \inf Q > 1/p$, and fix $1/p < \xi < h_*$. On U, the moment estimate makes stochastic integration a bounded linear map from the variable-exponent space with its Luxemburg norm into $L^p(\Omega)$. The kernel is differentiable in that norm at every order by the bounds in Lemma 2.2. Its stochastic derivatives therefore satisfy the fundamental theorem of calculus in $L^p(\Omega)$.

For $m = 0, 1$ the increment estimate for the logarithmic kernels of orders m and $m+1$, integrated over Q, yields

$$\mathbb{E}\big\|\partial_h^m Y^N(t,\cdot) - \partial_h^m Y^N(s,\cdot)\big\|_{W^{1,p}(Q)}^p \le C|t-s|^{p\xi}.$$

Indeed, rescaling the time increment produces powers of $1 + |\log|t-s||$, which are absorbed by the strict inequality $\xi < h_*$. The one-dimensional Sobolev embedding and Kolmogorov's criterion give jointly continuous versions of Y^N and its first derivative. The integral identity in the order variable holds simultaneously by continuity from rational triples. The parameter increment thus converges uniformly in probability to gD_u after the remote term is restored. Joint tightness with each bounded $\mathcal{F}_M$-measurable variable and the finite-dimensional stable limit identify the limit in $C(K)$. □

Remark 2.9. Uniform convergence of g_r does not imply that the prelimit paths are continuous if H is not continuous near u. The continuity assumption in the $C(K)$ assertion is therefore separate. No assertion of C- or D-valued convergence is made for the full subcritical region $hA < 1$.

3 Path representations with a rough spatial index

Where $h > \beta(t)$ the power kernel vanishes at the diagonal. This permits integration by parts for a BV index and Young integration for a Hölder index. Choose a finite source interval $J = [\ell, d]$ and a compact observation interval $I \Subset (\ell, d)$, and set $B(x) = M_\alpha((\ell, x])$. We take its càdlàg version. Sources to the left of ℓ, and the anchor, are handled by Lemma 2.7. Write

$$k(t,h;x) = \mathbf{1}_{\{x<t\}} e^{-\lambda(t-x)}(t-x)^{h-\beta(x)}, \qquad \mathcal{D} = \{(t,h) : \ell < t < d,\ 0 < h < 1,\ h > \beta(t)\}.$$

No spatial derivative is required for the first representation.

Proposition 3.1 (Continuous bounded-variation indices)**.** *Suppose β is continuous and of bounded variation on J. On compact subsets of $\mathcal{D}$, the local field has the pathwise formula*

$$\int_\ell^t k(t,h;x)\,\mathrm{d}B(x) = \int_\ell^t B(x)k(t,h;x)\left(\frac{h-\beta(x)}{t-x} - \lambda\right)\mathrm{d}x + \int_{(\ell,t)} B(x)k(t,h;x)\log(t-x)\,\mathrm{d}\beta(x). \tag{14}$$

The second integral is a Lebesgue–Stieltjes integral. The field and all its h-derivatives are jointly continuous there. The formula gives a modification of the multistable integral.

Proof. Put $q = h - \beta(t)$. On a compact subset of $\mathcal{D}$, q has a positive lower bound. Uniform continuity of β gives $h - \beta(x) \geq q_*/2 > 0$ sufficiently close to the diagonal. Away from it, the continuous-BV chain rule gives

$$\mathrm{d}_x k = k\left[\left(\lambda - \frac{h-\beta(x)}{t-x}\right)\mathrm{d}x - \log(t-x)\,\mathrm{d}\beta(x)\right]. \tag{15}$$

The Lebesgue part of the variation up to the diagonal is bounded by an integrable multiple of $y^{q_*/2-1}(1+|\log y|^m)$, and the Stieltjes part by a bounded multiple of $y^{q_*/2}(1+|\log y|^{m+1})\,\mathrm{d}|D\beta|$. The same bounds apply to every fixed logarithmic derivative. Since $k(t,h;t) = 0$ and $B(\ell) = 0$, integration by parts proves (14). Continuity follows by splitting off a small diagonal strip, on which these bounds vanish uniformly, and using dominated convergence elsewhere. Deterministic Riemann sums converge both pathwise by integration by parts and in probability by modular convergence, identifying the version. □

3.1 Young integration and finite variation of order p

For $p \geq 1$ let $V_p(f;J) = (\sup_\pi \sum_i |f(t_i) - f(t_{i-1})|^p)^{1/p}$.

Lemma 3.2 (A quantitative variation estimate for small jumps)**.** *Let N be a Poisson random measure on $[0,T]\times\mathbb{R}$ with deterministic compensator $\mathrm{d}t\,\nu_t(\mathrm{d}j)$, where each ν_t is symmetric. Suppose that, for some $b < 2$ and $C_0 < \infty$, uniformly in t and $0 < a \leq 1$,*

$$\nu_t(\{|j| > a\}) \leq C_0 a^{-b}, \qquad \int_{|j|\leq a} j^2 \nu_t(\mathrm{d}j) \leq C_0 a^{2-b}.$$

For $0 < \varepsilon \leq 1$, let

$$Z^\varepsilon(t) = \int_{(0,t]\times\{|j|\leq\varepsilon\}} j\,(N - \mathrm{d}s\,\nu_s)(\mathrm{d}s, \mathrm{d}j).$$

For every $\max(1,b) < p < 2$ *there is a constant* $C = C(p,b,C_0)$ *such that*

$$\mathbb{E}V_p(Z^\varepsilon;[0,T])^p \le CT\varepsilon^{p-b}. \tag{16}$$

In particular, the small-jump part tends to zero in p-variation in probability. The estimate is uniform over families of compensators satisfying the displayed bounds with the same constants.

Proof. For a càdlàg function f, start at $\tau_0 = 0$ and define successive exit times from a ball of radius a about $f(\tau_k)$. Write $N_a(f)$ for the number of completed exits up to and including T. These times are finite in number for fixed $a > 0$, since a càdlàg function is regulated. At an exit, $|f(\tau_{k+1}) - f(\tau_k)| \ge a$, and within a greedy interval, values remain within a of its initial value. Hence every interval of an arbitrary partition on which $|\Delta f| > 2a$ contains a greedy exit time. Assigning increments to dyadic size classes gives the deterministic bound

$$V_p(f;[0,T])^p \le C_p \sum_{n\in\mathbb{Z}} 2^{-np} N_{2^{-n}}(f). \tag{17}$$

The half-open intervals $(\tau_k, \tau_{k+1}]$ ensure that a jump at a partition endpoint is counted only once.

Apply this construction to Z^ε. For $a \le \varepsilon$ split $Z^\varepsilon = S_a + J_a$, where S_a retains marks $|j| \le a/4$, and J_a retains $a/4 < |j| \le \varepsilon$. Symmetry makes the compensating drift of J_a zero. The number of completed greedy intervals containing a J_a jump is at most its Poisson jump count, whose expectation is at most CTa^{-b}. On every remaining completed interval the increment of S_a has magnitude at least a. The stopped increments of the square-integrable martingale S_a, over all the greedy intervals, are orthogonal in L^2. Consequently

$$a^2\mathbb{E}N_a^{\text{no large jump}} \le \mathbb{E}\sum_k |S_a(\tau_{k+1}\wedge T) - S_a(\tau_k \wedge T)|^2 \le \mathbb{E}[S_a]_T \le CTa^{2-b}.$$

Here one first sums finitely many stopped intervals and then uses monotone convergence. Thus

$$\mathbb{E}N_a(Z^\varepsilon) \le CTa^{-b}, \qquad a \le \varepsilon. \tag{18}$$

For $a \ge \varepsilon$ the same stopped-increment identity applied directly to Z^ε gives

$$\mathbb{E}N_a(Z^\varepsilon) \le a^{-2}\mathbb{E}[Z^\varepsilon]_T \le CT\varepsilon^{2-b}a^{-2}. \tag{19}$$

Taking expectations in (17), and splitting its dyadic sum at $a = \varepsilon$, gives

$$\mathbb{E}V_p(Z^\varepsilon)^p \le CT\left(\sum_{2^{-n}\le\varepsilon} 2^{-n(p-b)} + \varepsilon^{2-b}\sum_{2^{-n}>\varepsilon} 2^{-n(p-2)}\right) \le CT\varepsilon^{p-b}.$$

The first sum converges because $p > b$, and the second because $p < 2$. □

Corollary 3.3 (Variation of the additive noise)**.** *For measurable* $\alpha : J \to [a_J, b_J] \subset (0,2)$*, the càdlàg additive noise* $B(t) = M_\alpha((\ell,t])$ *has finite p-variation almost surely for every* $p > \max(1,b_J)$*, simultaneously. Its small-mark part satisfies* (16) *for* $\max(1,b_J) < p < 2$*, with* $T = |J|$*.*

Proof. The intensity $c(\alpha(t))|j|^{-1-\alpha(t)}\,\mathrm{d}j$ satisfies the bounds of Lemma 3.2, uniformly on J. The compensated marks of size at most one have finite p-variation by that lemma. The finitely many marks larger than one have a cumulative sum of bounded variation almost surely. Intersect the resulting events over rational $p \in (\max(1, b_J), 2)$ and use monotonicity of variation classes for the remaining orders. A further countable intersection over rational source intervals gives an event usable for all local representations. □

Proposition 3.4 (Young representation)**.** *Suppose* $1 < a_J \le \alpha \le b_J < 2$ *on* J*, and*

$$\beta \in C^\gamma(J), \qquad 1 - 1/b_J < \gamma \le 1. \tag{20}$$

The local field on $\mathcal{D}$ *has the pathwise representation*

$$\mathcal{Y}(t,h) = \int_J k(t,h;x)\,\mathrm{d}B(x) \tag{21}$$

as a Young integral. On one event of probability one it is jointly continuous, is infinitely differentiable in h*, and all these derivatives are jointly continuous and obtained by inserting* $\log^m(t-x)$ *into the kernel. It is a modification of the corresponding stochastic field.*

Proof. On a compact subset of $\mathcal{D}$, factor the mth derivative as

$$\mathbf{1}_{\{x<t\}} e^{-\lambda y} y^{h-\beta(t)} \log^m y \, \exp\{(\beta(t)-\beta(x))\log y\}, \qquad y = t - x.$$

The first factor has uniformly bounded variation because its power is bounded below by a positive constant. Extend the exponential factor by one for $x \ge t$. It is uniformly η-Hölder for every $\eta < \gamma$ because the exponent has modulus at most $Cs^\gamma(1 + |\log s|)$. To check this bound, separate pairs at distance at most half their distance to t from pairs that meet the diagonal strip. In the latter case use $|\beta(t) - \beta(x)| \le C|t-x|^\gamma$ directly.

Choose $p > b_J$ and $1 - 1/p < \eta_0 < \eta_1 < \gamma$. The kernels have uniformly bounded $1/\eta_1$-variation. They are continuous in (t,h) in supremum norm, including across the moving diagonal, where $y^{q_*/2}|\log y|^m \to 0$. The interpolation inequality

$$V_{r_0}(f) \le (2\|f\|_\infty)^{1-r_1/r_0} V_{r_1}(f)^{r_1/r_0}, \qquad r_1 < r_0,$$

then gives continuity in $1/\eta_0$-variation. Young's theorem [You36], Lemma 3.3, and $\eta_0 + 1/p > 1$ construct the integrals and prove joint continuity. Taylor expansion in this variation norm proves the differentiation assertions.

For fixed parameters, deterministic left step approximations to the continuous kernel converge uniformly, hence in the multistable modular. Their stochastic integrals are the Young Riemann sums. The sums converge in probability to the stochastic integral and pathwise to the Young integral. Uniqueness of limits identifies the modification. A countable compact exhaustion of $\mathcal{D}$ provides the common pathwise event. □

3.2 A fractional-integral normal form

For a constant fractional order $q > 0$, put

$$P(t,q) = \int_\ell^t (t-x)^q\,\mathrm{d}B(x) = q\int_\ell^t (t-x)^{q-1} B(x)\,\mathrm{d}x = \Gamma(q+1) I^q_{\ell+} B(t), \tag{22}$$

where $I^q_{\ell+}$ is the left Riemann–Liouville integral. For $\gamma = 1$, C^γ below means the Lipschitz class.

Theorem 3.5 (Two-parameter Hölder normal form)**.** *Suppose $\beta \in C^\gamma(J)$, $0 < \gamma \le 1$, and choose $p > \max(1, b_J)$ with $\gamma + 1/p > 1$. Let $I \Subset J$ and let $Q = [h_-, h_+] \Subset (0,1)$ be sufficiently small that*

$$q_* := \inf_{x \in J,\, h \in Q} (h - \beta(x)) > 0.$$

With $\ell = \inf J$, the common-noise field has the representation

$$Y(t,h) = P(t, h - \beta(t)) + \mathcal{R}(t,h), \qquad (t,h) \in I \times Q. \tag{23}$$

On one event of probability one, the field $\mathcal{R}$ is infinitely differentiable in h, and for every $m \ge 0$,

$$\sup_{h \in Q} \|\partial_h^m \mathcal{R}(\cdot, h)\|_{C^\gamma(I)} < \infty. \tag{24}$$

The derivatives are jointly continuous. Equivalently, on smaller compact order intervals, $h \mapsto \mathcal{R}(\cdot, h)$ is a smooth map into $C^\gamma(I)$. When $\gamma = 1$, this notation denotes the Lipschitz class. These assertions hold on a common event for a countable local exhaustion of the supercritical parameter domain.

Proof. Work on the event $V_p(B; J) < \infty$. The local difference in (23) is the Young integral of

$$r(t,h,c;x) = \mathbf{1}_{\{x<t\}}\big(e^{-\lambda(t-x)}(t-x)^{h-\beta(x)} - (t-x)^{h-c}\big), \qquad c = \beta(t).$$

It suffices to estimate distances below one. On source intervals separated from I the kernel is smooth in (t,h,c) as a function with values in the source variation space. Substituting $c = \beta(t)$ gives a C^γ contribution in t. For an interval E of length at most d, Young's inequality gives

$$\left|\int_E f \,\mathrm{d}B\right| \le C\big(\|f\|_{\infty;E} + d^\gamma [f]_{\gamma;E}\big) V_p(B; E).$$

For a BV kernel we also use the integration-by-parts bound $C\|B\|_\infty(\|f\|_\infty + \operatorname{Var} f)$.

We record the kernel estimates for every $m \ge 0$. On a diagonal annulus $d \le t - x \le 2d$, the local error obeys

$$\|\partial_h^m r(t,h,\beta(t);\cdot)\|_\infty + d^\gamma [\partial_h^m r(t,h,\beta(t);\cdot)]_\gamma \le C_m d^{q_*+\gamma}(1 + |\log d|^{m+1}). \tag{25}$$

To see this, write $y = t - x$ and $z = \beta(t) - \beta(x)$, so that the undifferentiated expression is $y^{h-\beta(t)}(e^{-\lambda y}e^{z\log y} - 1)$ and $|z| \le Cd^\gamma$. The partial derivatives with respect to the smooth variable y and the index variable z, combined with $\beta \in C^\gamma$, give the displayed bound, with one additional logarithm for each h derivative. The estimate also shows that the diagonal error, extended by zero, is C^γ in its source variable.

For $s < t$, put $\delta = t - s$ and first freeze $c = \beta(s)$. Replacing this by $\beta(t)$ costs at most $C_m\|B\|_\infty \delta^\gamma$, since the h- and c-derivatives of the power integral P are uniformly bounded in the strictly positive order range. On the source interval $[s - 2\delta, t]$, use (25) on successive annuli. The remaining new-diagonal mismatch is a difference of two power kernels. Its mth h-derivative has BV norm at most

$$C_m|\beta(t) - \beta(s)|\delta^{q_*}(1 + |\log\delta|^{m+1}).$$

Thus the near contribution is bounded by $C_m(V_p(B) + \|B\|_\infty)\delta^{q_*+\gamma}(1 + |\log\delta|^{m+1})$.

On a far annulus $d \le s - x \le 2d$, $d \ge 2\delta$, differentiate in $v \in [s,t]$ while keeping h and c fixed. Since $|c - \beta(x)| \le Cd^\gamma$, the same two-variable calculation gives

$$\|\partial_v \partial_h^m r(v,h,c;\cdot)\|_\infty + d^\gamma [\partial_v \partial_h^m r(v,h,c;\cdot)]_\gamma \le C_m d^{q_*+\gamma-1}(1 + |\log d|^{m+2}). \tag{26}$$

Consequently the far contribution is bounded by

$$C_m V_p(B)\delta \sum_{2^k\delta \le C_I} (2^k\delta)^{q_*+\gamma-1}(1+|\log(2^k\delta)|^{m+2}).$$

This is $O_\omega(\delta^\gamma)$. Indeed, for $q_*+\gamma<1$ it is $O(\delta^{q_*+\gamma}(1+|\log\delta|^{m+2}))$. At equality it is $O(\delta(1+|\log\delta|^{m+3}))$, and for $q_*+\gamma>1$ it is $O(\delta)$. The first two bounds are $O(\delta^\gamma)$ because $q_*>0$, and $\gamma=1$ belongs only to the third case.

These estimates prove (24) for the local field. The remote part and the anchor are smooth in (t,h), so they preserve it. The Young representation already supplies all h-derivatives on a common event. Integrating the $(m+1)$st derivative in h shows continuity of the mth derivative in the C^γ norm. Iteration proves the stated smoothness into that space.

To obtain a single event for all (t,h), take rational source, observation and order intervals. Intersect the events of finite p-variation and smooth remote fields over this countable collection and rational admissible p. On each such event, deterministic Young estimates construct the whole local field. Young sums along fixed meshes are jointly measurable in (ω,t,h), and converge pointwise to the constructed field. For each deterministic parameter they also converge in probability to the multistable integral by modular convergence of the kernel approximants. Continuous versions from two local charts agree on a countable dense parameter set and hence throughout their overlap. This supplies a jointly measurable common modification on the supercritical domain. □

Corollary 3.6 (Composition with a rough Hurst function)**.** *Under the preceding assumptions, suppose $H: I\to Q$ is η-Hölder, $0<\eta\le 1$, and put $\kappa=\min(\gamma,\eta)$. Then*

$$X(t)=P(t,H(t)-\beta(t))+R(t),\qquad R\in C^\kappa(I)\quad \textit{almost surely.}$$

For every $t_0\in I$, simultaneously,

$$|X(t)-X(t_0)-P(t,q(t_0))+P(t_0,q(t_0))|\le C(\omega)|t-t_0|^\kappa.$$

For a continuous H with a general modulus ω_H, the same argument gives a remainder modulus bounded by $C(\omega)(|t-s|^\gamma+\omega_H(|t-s|))$.

Proof. Set $R(t)=\mathcal{R}(t,H(t))$. Split its increment into the increment in t at fixed h and the increment in h at fixed t. The first is bounded by $C|t-s|^\gamma$ and the second by $C|H(t)-H(s)|$, by (24) for $m=0,1$. The same bound holds for $q(t)-q(s)$, and the order derivative of P is uniformly bounded. These facts prove the displayed estimates. □

Remark 3.7 (The remainder need not be differentiable). Take $H\equiv h$ and $\beta(t)=\beta_0+c|t-u|^\gamma$ near u, where $0<\gamma<1$ and $h>\beta$. Assume $c\neq 0$ and the local Young condition. An atom $j\delta_x$ with $\ell<x<u$, away from u, contributes to the remainder the function

$$j\left(e^{-\lambda(t-x)}(t-x)^{h-\beta(x)}-(t-x)^{h-\beta(t)}\right).$$

Its nonsmooth leading increment is $jc(u-x)^{h-\beta_0}\log(u-x)|t-u|^\gamma$, which is nonzero for $u-x\neq 1$. Thus it is not differentiable at u. This also rules out almost-sure differentiability of the stochastic remainder at u. To see this, isolate a finite-intensity Poisson region with locations away from u and marks in $[1,2]$. Conditioning on zero and on one atom, independently of the remaining noise, would otherwise force almost every such one-atom contribution to be C^1.

3.3 Comparison with an order frozen at one point

Proposition 3.8 (Comparison with a fixed fractional order)**.** *Assume the local Young condition (20), let H be locally Lipschitz, and suppose $q = H - \beta > 0$. With P as in (22), the path version may be chosen so that the following assertion holds simultaneously at every interior point t_0. Put $q_0 = q(t_0)$ and*

$$E_{t_0}(t) = X(t) - X(t_0) - P(t, q_0) + P(t_0, q_0).$$

For every $\eta < \min\{1, q_0 + \gamma\}$,

$$E_{t_0}(t) = O_\omega(|t - t_0|^\eta), \qquad t \longrightarrow t_0. \tag{27}$$

In particular, with $\sigma(t_0) = \min\{1, q(t_0) + \gamma\}$,

$$\min\{h_X(t_0), \sigma(t_0)\} = \min\{h_{P(\cdot, q(t_0))}(t_0), \sigma(t_0)\}. \tag{28}$$

The constants can be chosen locally uniformly in t_0 whenever η is strictly below the infimum of σ on the compact interval in question.

Proof. Work on a compact source and observation pair for which the joint Young field and its order derivative are continuous, and on the common event $V_p(B) < \infty$, where $p > b_J$ and $\gamma + 1/p > 1$. The remote and anchor fields are smooth in the observation variable and the order. Since H is locally Lipschitz and the order derivative of the local field is bounded on compact subsets of the strictly supercritical parameter domain,

$$Y(t, H(t)) - Y(t, H(t_0)) = O_\omega(|t - t_0|).$$

It is therefore enough to freeze H at $h_0 = H(t_0)$ and consider

$$r_v(x) = \mathbf{1}_{\{x<v\}} \left(e^{-\lambda(v-x)} (v - x)^{h_0 - \beta(x)} - (v - x)^{q_0} \right).$$

Writing $\delta = |t - t_0|$, we estimate $\int (r_t - r_{t_0})\, \mathrm{d}B$. Split the source interval at $t_0 - 2\delta$. This works for both signs of $t - t_0$.

On the near interval, write $q_v = h_0 - \beta(v)$ and decompose

$$r_v(x) = \mathbf{1}_{\{x<v\}} \left(e^{-\lambda(v-x)} (v - x)^{h_0 - \beta(x)} - (v - x)^{q_v} \right) + \mathbf{1}_{\{x<v\}} \left((v - x)^{q_v} - (v - x)^{q_0} \right).$$

For $v = t$ or $v = t_0$, the first bracket is the proper diagonal error. On an annulus $v - x \asymp d$, its supremum norm plus d^γ times its γ-Hölder seminorm is at most

$$C d^{q_v + \gamma} (1 + |\log d|).$$

The Young estimate and summation over $d \le C\delta$ bound its integral by $C_\omega \delta^{q_0 + \gamma} (1 + |\log \delta|)$, because $q_v - q_0 = O(\delta^\gamma)$ and $\delta^{q_v - q_0}$ is bounded uniformly as $\delta \downarrow 0$. The second bracket is a difference of two positive power kernels. Its supremum norm and total variation on $0 < v - x \le C\delta$ are bounded by

$$C |q_v - q_0| \delta^{\min(q_v, q_0)} (1 + |\log \delta|) \le C \delta^{q_0 + \gamma} (1 + |\log \delta|).$$

Integration by parts against the bounded path B gives the same bound for this contribution.

On a far annulus $t_0 - x \asymp d$, $d \geq 2\delta$, let v range between t_0 and t. Here $v - x \asymp d$ and $|\beta(x) - \beta(t_0)| \leq Cd^\gamma$. Differentiation is only in v, with h_0 and $\beta(t_0)$ fixed. The mean-value theorem for the index value and the Hölder bound in the source variable give

$$\|\partial_v r_v\|_\infty + d^\gamma [\partial_v r_v]_\gamma \leq Cd^{q_0+\gamma-1}(1 + |\log d|^2).$$

The exponent q_0 in this estimate follows from $d^{\beta(t_0)-\beta(x)} \leq \exp(Cd^\gamma |\log d|)$. This factor is uniformly bounded on the small annuli. The remaining annuli are bounded away from the diagonal and contribute $O_\omega(\delta)$. The far Young integrals are consequently at most

$$C_\omega \delta \sum_{2^k\delta \leq C_I} (2^k\delta)^{q_0+\gamma-1}(1 + |\log(2^k\delta)|^2).$$

For $q_0+\gamma < 1$ this is $O_\omega(\delta^{q_0+\gamma}(1+|\log\delta|^2))$. At equality it is $O_\omega(\delta(1+|\log\delta|^3))$, and for $q_0+\gamma > 1$ it is $O_\omega(\delta)$. These estimates prove (27). All estimates use the same finite-variation event and deterministic compact Hölder bounds. Countable compact exhaustion gives the simultaneous assertion. Equation (28) follows by the elementary stability of pointwise exponents below the regularity of an additive error. □

4 The critical path boundary for rough indices

Fix $u \in \mathbb{R}$, $A = \alpha(u) \in (1, 2)$, and $h = H(u) = 1/A$. Write $\beta = 1/\alpha$ and $q = H - \beta$. Choose compact intervals $J \Subset I$ around u, with $1 < a_I := \inf_I \alpha \leq b_I := \sup_I \alpha < 2$. If $u \neq 0$, take I disjoint from the anchor 0. We impose one of the following conditions on the spatial index.

(A) $\beta \in C^\gamma(I)$ for some $1 - 1/b_I < \gamma \leq 1$, where C^1 at this endpoint denotes the Lipschitz class.

(B) β is continuous and of bounded variation on I, and, with $\mu = |D\beta|$,

$$\omega_\mu(s) = \sup_{t\in J} \mu([t - s, t + s] \cap I), \qquad \int_0^{s_0} \omega_\mu(s)\frac{\mathrm{d}s}{s} < \infty. \tag{29}$$

The compact interval in this supremum may be enlarged slightly.

Condition (A) imposes no bounded-variation assumption. In both cases, if ω_β is the uniform modulus of continuity on a slightly smaller interval, then

$$\omega_\beta(s)|\log s| \longrightarrow 0. \tag{30}$$

For (B), this follows from the Dini condition and the inequality $\omega_\beta(s) \leq \omega_\mu(s)$.

We use the scalar M_1 topology on $D[-T, T]$. A completed graph contains the vertical segment between the left and right values at every jump. Its parametrizations traverse time in increasing order and each vertical segment in the order of that jump. The M_1 distance is the infimum of the maximum uniform discrepancy of the time and value coordinates over continuous ordered parametrizations of the two completed graphs. The J_1 topology instead uses increasing time changes of the paths themselves. The graph argument below uses only scalar M_1.

Theorem 4.1 (Critical M_1 limit for a rough spatial index)**.** *Assume either (A) or (B), continuity of H near u, and the profile limit (9). Suppose in addition that either*

(i) $q(u+s) \geq c|s|^h$ *for all sufficiently small nonzero* s*, for some* $c > 0$*, or*

(ii) $q \geq 0$ *near* u*, and the local uniform modulus of* H *satisfies*

$$\omega_H(s)|\log s| \longrightarrow 0 \qquad (s \downarrow 0). \tag{31}$$

Then X *has a local càdlàg modification and, for every* $T < \infty$*,*

$$r^{-h}\{X(u+r\cdot) - X(u)\} \xrightarrow{\text{st}} L_A + gD_u \quad \text{in } (D[-T,T], M_1). \tag{32}$$

Here $L_A = L_{1/A,A}$ *is the two-sided symmetric* A*-stable Lévy motion independent of* $\mathcal{F}_M$*. At* $u = 0$*, the derivative term vanishes because* $D_0 = 0$*. In case (i), the modification is continuous near* u*, and* (32) *cannot hold in* J_1*. In both cases,*

$$h_X(u) = 1/A \qquad \text{almost surely.} \tag{33}$$

Remark 4.2. Every positive Hölder modulus for H implies (31). Condition (B) includes $\beta \in W^{1,m}_{\text{loc}}$ for every fixed $m > 1$, because $\omega_\mu(s) \leq Cs^{1-1/m}\|\beta'\|_{L^m}$. It also includes singular continuous variation measures satisfying $\mu([t-s, t+s]) \leq Cs^\eta$ for some $\eta > 0$. A positive profile $g(v) > 0$ for $v \neq 0$ implies (i) if $\beta(u+s) - \beta(u) = o(|s|^h)$. In (ii), the sign condition is imposed on $q = H - \beta$.

Lemma 4.3 (The anchor at the base point)**.** *Suppose* β *is continuous at zero and* $h = \beta(0) \in (0,1)$*. For sufficiently small* $\delta, \eta > 0$*, put*

$$A_\delta(z) = \int_{-\delta}^{0} e^{\lambda x}(-x)^{z-\beta(x)} M_\alpha(\,\mathrm{d}x), \qquad |z-h| \leq \eta.$$

This field has a continuously differentiable version in z*. Its small-mark part converges to zero uniformly in* z *almost surely along a sequence of cutoffs. Furthermore, for each fixed* $R < \infty$*,*

$$\sup_{|z-h| \leq Cr^h} \left| \int_{-rR}^{0} e^{\lambda x}(-x)^{z-\beta(x)} \log(-x) M_\alpha(\,\mathrm{d}x) \right| \longrightarrow 0$$

in probability as $r \downarrow 0$*.*

Proof. Shrink δ, η until $z - \beta(x) > -1/4$ throughout the indicated rectangle, and put $a_0 = \inf_{[-\delta,0]} \alpha > 1$ and $b_0 = \sup_{[-\delta,0]} \alpha < 2$. For each $m \leq 2$, the squared deterministic kernel of $\partial_z^m A_\delta$ is bounded by $C|x|^{-1/2}(1 + |\log|x||^4)$, integrable on $[-\delta, 0]$. The compensated small-mark noise has pointwise variance density bounded by $C\varepsilon^{2-b_0}$. The one-dimensional Sobolev inequality in z, applied to the field and to its first derivative, therefore gives

$$\mathbb{E}\|A_\delta^{\leq\varepsilon}\|^2_{C^1([h-\eta,h+\eta])} \leq C\varepsilon^{2-b_0}.$$

Finite large-mark fields are smooth. The estimate yields the stated version and almost-sure convergence along a sequence. For the shrinking-interval assertion, choose $1 < \kappa < a_0$. The modulars of the first and second parameter derivative kernels on $[-rR, 0]$ tend to zero uniformly for $|z-h| \leq \eta$, after possibly reducing η. Their common bound is an integral of $y^{a_0(h-\eta)-1}(1 + |\log y|^{2b_0})$ over $0 < y < rR$. The modular moment estimate and the fundamental theorem of calculus in z now give convergence in L^κ of the displayed supremum. □

Proof of Theorem 4.1. Source kernels and the path modification. Choose $p \in (b_I, 2)$. In case (A), choose it close enough to b_I that there is an η satisfying

$$1 - 1/p < \eta < \gamma,$$

and put $\rho = 1/\eta$. In case (B), put $\rho = 1$. Thus $1/p + 1/\rho > 1$ in both cases. For $q(t) \geq 0$, the positive source kernel can be written

$$k_t(x) = \mathbf{1}_{\{x<t\}} e^{-\lambda(t-x)} (t-x)^{q(t)} a_t(x), \qquad a_t(x) = \exp\{(\beta(t) - \beta(x)) \log(t-x)\}. \tag{34}$$

Extend a_t by one for $x \geq t$. At the diagonal assign $k_t(t) = 1$ when $q(t) = 0$, and $k_t(t) = 0$ when $q(t) > 0$. These are the source left limits. All path integrals in this proof use the forward convention determined by these values, so a noise atom at $x = t$ is assigned weight one exactly when $q(t) = 0$.

In case (A), a_t is uniformly η-Hölder. Indeed, write $\psi_t(x) = (\beta(t) - \beta(x)) \log(t-x)$ on $x < t$ and zero otherwise. For $d = |x - y|$, if one of x, y is within $2d$ of t, then

$$|\psi_t(x) - \psi_t(y)| \leq C d^\gamma (1 + |\log d|).$$

Otherwise both points are on the same side of the diagonal. On its left, split the difference into a difference of β and a difference of logarithms. With $s = \min(t - x, t - y) > 2d$, their respective bounds are $Cd^\gamma |\log s|$ and $Cds^{\gamma-1}$. Combining the two estimates gives the desired bound. Exponentiation preserves the estimate because ψ_t is uniformly bounded. The first factor in (34) has uniformly bounded variation, including its possible diagonal jump. The variation product inequality therefore gives

$$\sup_{t \in J} \big(\|k_t\|_\infty + V_\rho(k_t; I) \big) < \infty. \tag{35}$$

For case (B), the Dini condition gives

$$\omega_\mu(s) |\log s| \longrightarrow 0, \qquad \sup_{t \in J} \int_{0 < |x-t| \leq s} |\log |x - t|| \mu(\mathrm{d}x) \longrightarrow 0. \tag{36}$$

The second assertion follows by integration by layers from $\omega_\mu(s) \log(1/s) + \int_0^s \omega_\mu(y)\, \mathrm{d}y/y$. The continuous-BV chain rule gives the uniform variation bound

$$C \int_0^{s_0} y^{q(t)} \left(\lambda + \frac{q(t)}{y} \right) \mathrm{d}y + C \int_0^{s_0} \omega_\mu(y) \frac{\mathrm{d}y}{y} + C \int_{(t-s_0, t)} |\log(t - x)| \mu(\mathrm{d}x).$$

The contribution of the first integral is uniformly bounded for $q(t) \geq 0$. This proves (35) with $\rho = 1$.

For a finite atomic integrator Z, use the specified source values and set

$$\mathcal{I}_f(Z) = \sum_x f(x) \Delta Z(x).$$

The discrete Young inequality applied to the ordered atom locations gives

$$|\mathcal{I}_f(Z)| \leq C_{p,\rho} \big(\|f\|_\infty + V_\rho(f) \big) V_p(Z). \tag{37}$$

The sequence of specified kernel values has variation no larger than that of the kernel itself. We define the full path integral as the limit of these finite symmetric Poisson sums. At a common jump, the source left limit used in the sum gives the forward integral convention. Apply (37) to differences of finite symmetric Poisson truncations. Lemma 3.2 bounds a band of marks $\varepsilon' < |j| \le \varepsilon$ by the same estimate as all marks $|j| \le \varepsilon$. Consequently these integrals are uniformly Cauchy in probability over $t \in J$, and their small-mark limit satisfies

$$\mathbb{E} \sup_{t \in J} |X^{\le \varepsilon}_{\text{moving,local}}(t)|^p \le C \varepsilon^{p - b_I}. \tag{38}$$

For $u \neq 0$, local anchor kernels have bounded source variation norms of the required order, uniformly over a compact interval of Hurst parameters, and satisfy the same bound. At $u = 0$ their source kernels may be unbounded. Lemma 4.3 instead gives uniform C^1 control in the Hurst parameter. The continuity of H then supplies uniform convergence of their compositions with H.

Finite Poisson truncations have càdlàg paths under (ii). At a mark location x with $q(x) = 0$, its kernel weight tends to one from the right, by $(H(x+s) - H(x)) \log s \to 0$. Under (i), every mark location has $q(x) > 0$ almost surely, so the finite truncations are continuous. The estimates just established give uniform convergence almost surely along a sufficiently fast sequence of cutoffs. The limit is a modification of the stochastic integral. At every fixed deterministic t, the noise has no atom at t almost surely, so the diagonal convention does not affect the stochastic integral. Symmetric Poisson truncation converges to it in probability. Adding the remote and anchor fields establishes the asserted path modification. Their composition with the continuous function H is continuous.

The rescaled near noise. Fix $R > T + 1$, put $W_r = [u - rR, u + rR]$, and define $M_r(\mathrm{d}z) = r^{-h} M_\alpha(u + r\,\mathrm{d}z)$ on $[-R, R]$. Its Poisson intensity is

$$c(\alpha(u + rz)) r^{1 - \alpha(u+rz)/A} |j|^{-1-\alpha(u+rz)}\,\mathrm{d}z\,\mathrm{d}j. \tag{39}$$

Equation (30) gives $\sup_{|z| \le R} |\alpha(u + rz) - A| |\log r| \to 0$. Consequently the intensity factors in (39) are uniformly bounded, and, on $|j| > \varepsilon$, the intensities converge in total variation to $c(A)|j|^{-1-A}\,\mathrm{d}z\,\mathrm{d}j$.

The moving near increment is

$$\begin{aligned} N_r(v) &= \int_{-R}^{R} (w_r(v, z) - w_r(0, z)) M_r(\mathrm{d}z), \\ w_r(v, z) &= \mathbf{1}_{\{z < v\}} e^{-\lambda r(v-z)} [r(v - z)]^{H(u+rv) - \beta(u+rz)}. \end{aligned} \tag{40}$$

The diagonal convention in (34) gives the right-continuous time version. The source variation estimate has the rescaled form

$$\sup_{r \text{ small},\ |v| \le T} \big(\|w_r(v, \cdot)\|_\infty + V_\rho(w_r(v, \cdot); [-R, R]) \big) < \infty. \tag{41}$$

Indeed $w_r(v, z) = k_{u+rv}(u + rz)$, including the value prescribed at the diagonal. Restriction to W_r can only decrease variation, and an increasing affine change of variable preserves it. Thus (35) gives (41) in both cases.

By (37) and Lemma 3.2,

$$\sup_{r \text{ small}} \mathbb{E} \|N_r^{\le \varepsilon}\|_\infty^p \le C_R \varepsilon^{p - b_I}. \tag{42}$$

Thus the small marks can be removed uniformly in probability. Couple the finite large-mark configurations by their total variation convergence. The limiting locations are distinct and avoid $-T, 0, T$ almost surely. For one such atom at $z \neq 0$, put $q_{z,r} = q(u + rz)$.

In case (ii), write $s = r(v - z) > 0$. The mean-value theorem, the inequality $H(u + rv) - \beta(u + rz) \geq -\omega_\beta(s)$, and $q_{z,r} \geq 0$ give

$$\left| s^{H(u+rv)-\beta(u+rz)} - s^{q_{z,r}} \right| \leq \omega_H(s) |\log s| \exp\{\omega_\beta(s) |\log s|\}. \tag{43}$$

Its right side tends to zero uniformly for $0 < s \leq r(R + T)$ by (30) and (31). Moreover, $q_{z,r} |\log r| \to 0$. The frozen power is a monotone ramp when $q_{z,r} > 0$ and a step when $q_{z,r} = 0$. In either case it converges in scalar M_1 to $\mathbf{1}_{[z,\infty)}$.

In case (i), let $e_r(v, z) = H(u + rv) - \beta(u + rz)$ denote the exponent at the atom z. Its variation in v is given by

$$e_r(v, z) - q_{z,r} = r^h \{g_r(v) - g_r(z)\}, \qquad q_{z,r} \geq c r^h |z|^h.$$

For each fixed $z \neq 0$, local uniform convergence and continuity of g therefore give

$$\lim_{\delta \downarrow 0} \limsup_{r \downarrow 0} \sup_{|v-z| \leq \delta} \left| \frac{e_r(v, z)}{q_{z,r}} - 1 \right| = 0.$$

The elementary bound

$$\sup_{0 \leq a \leq 1} |a^\theta - a| \leq C |\theta - 1| \quad (1/2 \leq \theta \leq 3/2)$$

therefore gives the same approximation by a monotone frozen ramp. Away from z, convergence to the step is uniform by (30). Tempering changes these conclusions by $o(1)$.

For a finite sum, choose disjoint shrinking intervals around the distinct limiting jump times. Outside these intervals convergence is uniform. On each interval there is one limiting jump. Up to an error tending uniformly to zero, its approximating contribution is monotone between the two endpoint values. Parametrize that monotone graph and the corresponding limiting vertical segment in the same order. The time coordinates differ by at most the interval length, and the value coordinates by a quantity tending to zero. Concatenating these parametrizations proves the assertion for the finite sum. Negative marks give downward parametrizations and cause no change in this construction. Removal of the cutoff by (42) and the corresponding estimate for stable noise yields

$$N_r \Rightarrow L_A \quad \text{in } M_1. \tag{44}$$

The near anchor correction is uniformly negligible. Its normalized parameter change is bounded by $\|g_r\|_\infty$. If $u \neq 0$, its derivative kernel is bounded on W_r, and its stochastic integral converges to zero uniformly over the shrinking parameter interval by the modular estimate and one further parameter derivative. If $u = 0$, the same assertion is precisely the shrinking-interval conclusion of Lemma 4.3.

The far increment and the derivative term. Localize first to a fixed interval with $\inf \alpha > \kappa > 1$. The fixed remote field is smooth in (t, h'). Put $E_r = I \setminus W_r$, $q_0(x) = h - \beta(x)$, and let C_r be the normalized frozen increment over E_r. Only $x < u - rR$ contributes. For $x = u + rz$ and $y = v - z > 1$, the kernel of $\partial_v C_r(v)$ is

$$r^{-\beta(x)} e^{-\lambda r y} y^{q_0(x)-1} (q_0(x) - \lambda r y).$$

Its modular is exactly

$$\int e^{-\lambda\alpha(u+rz)ry} y^{-1-\alpha(u+rz)(1-h)} |q_0(u+rz) - \lambda ry|^{\alpha(u+rz)} \, \mathrm{d}z. \tag{45}$$

The integrand is bounded by $C(1+|z|)^{-1-a_I(1-h)}$, uniformly in $v \in [-T,T]$, and tends to zero pointwise. The modular moment estimate and Minkowski's inequality therefore give

$$\big\| \|C_r\|_\infty \big\|_{L^\kappa(\Omega)} \longrightarrow 0. \tag{46}$$

For the far derivative field $D_r(t,h') = \int_{E_r} \partial_h K(t,h';x) M_\alpha(\,\mathrm{d}x)$, let $|h'-h| \le Q_r = Cr^h$ and $|t-u| \le Tr$. Scaling as in (45), with the logarithmic factors retained, gives

$$\begin{aligned} \sup \|r\partial_t D_r(t,h')\|_\kappa &\le Cr^h(1+|\log r|), \\ \sup \|r\partial_t\partial_h D_r(t,h')\|_\kappa &\le Cr^h(1+|\log r|)^2, \\ \sup_{|h'-h|\le Q_r} \|\partial_h D_r(u,h')\|_\kappa &\le C. \end{aligned} \tag{47}$$

After division by the displayed factors, an integrable modular envelope is

$$C(1+|z|)^{-1-a_I(1-h)/2}(1+\log(2+|z|))^{2b_I}.$$

The factor $r^{h'-h}$ is uniformly bounded because $Q_r|\log r| \to 0$. The last estimate follows from the integrability of the second parameter derivative at u. At $u=0$, one has $D_r(0,h') = 0$ identically because $K(0,h';x) = 0$. The elementary two-parameter identity

$$\begin{aligned} D_r(t,h') - D_r(u,h) = &\int_u^t \partial_t D_r(s,h)\,\mathrm{d}s + \int_h^{h'} \partial_h D_r(u,z)\,\mathrm{d}z \\ &+ \int_u^t \int_h^{h'} \partial_t\partial_h D_r(s,z)\,\mathrm{d}z\,\mathrm{d}s \end{aligned}$$

and Minkowski's inequality show that

$$\Big\| \sup_{v,h'} |D_r(u+rv,h') - D_r(u,h)| \Big\|_\kappa \longrightarrow 0.$$

The omitted derivative integral on W_r tends to zero by modular convergence. Restoring the remote derivative proves uniform convergence in probability of the far parameter contribution to gD_u.

For stable convergence, fix $\delta > 0$. For small r, the near noise is independent of the noise outside $(u-\delta, u+\delta)$. Approximate a bounded $\mathcal{F}_M$-measurable random variable by its conditional expectation on this outside-noise sigma field. These approximations converge in L^1 as $\delta \downarrow 0$, because the fixed point u carries no noise atom. First let $r \downarrow 0$ and then $\delta \downarrow 0$. This upgrades (44) to mixing convergence and yields the joint limit with D_u. Addition of a uniformly convergent continuous function is continuous in scalar M_1, proving (32). Since continuous paths form a J_1-closed subset of D, the jumps of the limit preclude J_1 convergence in case (i).

To prove (33), choose $1 < \kappa < a_I$ and retain the variation order $p > b_I$ above. The cumulative near noise has uniformly bounded κth moment of its p-variation. Indeed, Lemma 3.2 controls marks of size at most one in L^p, and the total variation of the finitely many larger

marks has a uniformly bounded κth moment since $\kappa < a_I$. Thus (37), (41), (45), and (47) give a uniform L^κ bound for the normalized local increment supremum. The anchor estimates, including Lemma 4.3 at zero, have the same required moment bound. The fixed remote part has a pathwise finite bound after division by r^h, using the bounded critical profile and smoothness in (t, h'). Markov's inequality and Borel–Cantelli on $r_n = 2^{-n}$ imply, for every $\varepsilon > 0$,

$$\sup_{|t-u|\le r_n} |X(t) - X(u)| = O_\omega(r_n^{h-\varepsilon}).$$

Hence $h_X(u) \ge h$. For fixed $v \neq 0$, the limiting variable $L_A(v) + g(v)D_u$ has no atom at zero, by independence and the stable density. On the event $\{h_X(u) > h\}$, the normalized increment tends to zero, since polynomial terms of positive degree also vanish after this normalization when $h < 1$. For each $\eta > 0$, that event has probability at most $\liminf_r \mathbb{P}(|r^{-h}(X(u+rv)-X(u))| \le \eta)$, and Portmanteau bounds this by $\mathbb{P}(|L_A(v) + g(v)D_u| \le \eta)$. Letting $\eta \downarrow 0$ proves the reverse inequality.

□

4.1 Continuity of the Hurst function alone is insufficient

Proposition 4.4. *For every $A \in (1, 2)$ there is a continuous H, with constant $\alpha \equiv A$, $H(1) = h = 1/A$, and $H(t) > h$ for $t \neq 1$ near 1, such that $g_r \to c(-v)_+^h$ uniformly on compact intervals, while the normalized processes are not M_1-tight on $[-3, 3]$. The process has continuous paths near 1 and $h_X(1) = h$ almost surely.*

Proof. Let $r_n = 4^{-n}$, $a_n = r_n^{2h}$, $b_n = a_n^2$, and $\ell_n = \exp(-1/a_n)$. On $[1.2r_n, 1.8r_n]$ inside each disjoint shell $[r_n, 2r_n]$, let $q(1+s) = H(1+s) - h$ alternate continuously between b_n and a_n with period ℓ_n and with flat low and high plateaus of fixed positive relative length. Interpolate at the edges to the baseline s^{2h}, maintaining $0 < q(1+s) \le Cs^{2h}$. For $s < 0$ take $q(1+s) = c|s|^h$, and set $q(1) = 0$. Shrink the neighborhood to keep $H < 1$. This gives the stated uniform profile. The fixed-window variation argument above constructs continuous paths, as $q > 0$ at every random mark. The boundary exponent proof uses only bounded critical profiles and $q \ge 0$ for its lower bound, so it still applies. The upper bound follows from Theorem 2.4.

At scale r_n, the rescaled noise is exactly A-stable. Fix a window $[-R, R]$ with $R > 4$. For a fixed small cutoff ε, the event that there is precisely one mark larger than ε in absolute value, with size in $[1, 2]$ and location in $[1.3, 1.7]$, has positive probability independent of n. After its physical time, choose three low/high/low plateau points at distances between ℓ_n and $C\ell_n$. Its responses converge to

$$j, \qquad e^{-1}j, \qquad j,$$

because $\ell_n^{b_n} \to 1$ and $\ell_n^{a_n} = e^{-1}$. The three rescaled times lie in an interval of length $C\ell_n/r_n \to 0$. By (42), choose ε so that the supremum of the small-mark term is less than $(1 - e^{-1})/8$ with probability at least $1/2$. This event is independent of the specified large-mark configuration, whose probability is a fixed $p_\varepsilon > 0$. The far and anchor errors tend uniformly to zero in probability on the positive inner shell, where $g = 0$. The three events therefore occur together with probability at least $p_\varepsilon/2 - o(1)$. Hence, with probability bounded away from zero, the distance of the middle value from the segment joining the other two is bounded below by a positive constant. This violates the M_1 oscillation tightness criterion. The constructed H violates (31) because its oscillations have size a_n at distances comparable to ℓ_n, and $a_n|\log \ell_n| = 1$. □

5 Pointwise exponents and the local spectrum

For a locally bounded function f, $h_f(t)$ denotes the supremum of $s \geq 0$ for which $f(t+v)-P(v) = O(|v|^s)$ with a polynomial P of degree less than s. At a jump we set $h_f(t) = 0$. Define

$$\begin{aligned} E_f(s) &= \{t : h_f(t) = s\}, \\ d_f(s,u) &= \lim_{\rho \downarrow 0} \dim_{\mathrm{H}}(E_f(s) \cap (u-\rho, u+\rho)), \qquad (48) \\ \overline{d}_f(s,u) &= \lim_{\varepsilon \downarrow 0} \lim_{\rho \downarrow 0} \dim_{\mathrm{H}}\{t \in (u-\rho, u+\rho) : |h_f(t) - s| < \varepsilon\}. \qquad (49) \end{aligned}$$

We use $\dim_{\mathrm{H}} \varnothing = -\infty$. The two definitions distinguish an exact level from exponents approaching that level.

The comparison with a fractional integral uses the exponents of B and of its ordinary primitive. For the independent-increments multistable motion with differentiable index, see [LL20]. The Poisson estimates in Appendix A give the following bounds and level-set dimensions for continuous α.

Proposition 5.1 (Additive-noise geometry)**.** *Let α be continuous on an open interval with values in $(1,2)$, let B be its additive noise, and let $J_B(t) = \int_\ell^t B(x)\,\mathrm{d}x$. On one event of probability one, simultaneously at all interior points,*

$$h_B(t) \leq \beta(t), \qquad h_{J_B}(t) \leq 1 + \beta(t). \qquad (50)$$

On a nondegenerate compact interval I, put $b_I = \sup_I \alpha$. For $0 \leq \kappa < 1/b_I$, the set

$$\mathcal{O}_\kappa(I) = \{t \in I : h_B(t) \leq \kappa,\ h_{J_B}(t) > h_B(t) + 1\}$$

has dimension at most $\max(0, 2b_I\kappa - 1)$, and is empty if $2b_I\kappa < 1$. Moreover,

$$\dim_{\mathrm{H}}\{t \in I : h_B(t) \leq \kappa\} \leq b_I\kappa. \qquad (51)$$

For every continuous $\eta : I \to (0,\infty)$ with $\alpha(t)\eta(t) < 1$ throughout I,

$$\dim_{\mathrm{H}}\{t \in I : h_B(t) = \eta(t)\} = \sup_{t \in I} \alpha(t)\eta(t). \qquad (52)$$

One may use compact intervals contained in a slightly larger open interval.

Lemma 5.2 (Fractional integration at an interior point)**.** *Let f be bounded and measurable on $[\ell, d]$ and $t \in (\ell, d)$. For $\eta > 0$, its left fractional integral satisfies $h_{I^\eta_{\ell+} f}(t) \geq h_f(t) + \eta$. Consequently, for $0 < q < 1$,*

$$h_B(t) + q \leq h_{I^q_{\ell+} B}(t) \leq h_{J_B}(t) - 1 + q. \qquad (53)$$

If $h_{J_B}(t) = h_B(t) + 1$, equality holds on both sides.

Proof. Fix t and an exponent $0 < \sigma < h_f(t)$, and choose a local polynomial P such that $|f(x) - P(x-t)| \leq C|x-t|^\sigma$ near t. Let a smooth cutoff χ equal one near t, with support in (ℓ, d). The fractional integrals of χP and of $(1-\chi)f$ are smooth near t. For the first term, derivatives act on the smooth compactly supported factor. For the second, the source is separated from the observation point. It is enough to treat $r(x) = \chi(x)(f(x) - P(x-t))$.

Write $a = \sigma + \eta$, choose σ so that a is not an integer, and put $m = \lfloor a \rfloor$. The coefficients

$$c_k = \frac{1}{\Gamma(\eta)k!} \int_\ell^t \partial_v^k (v-x)^{\eta-1}\big|_{v=t}\, r(x)\,\mathrm{d}x, \qquad 0 \le k \le m,$$

are finite because $|r(x)| \le C|x-t|^\sigma$ and $k < a$. For $v = t+z$ put $\delta = |z|$. On the source region $x < t - 2\delta$, Taylor's theorem bounds the error after subtracting $\sum_{k=0}^m c_k z^k$ by a constant times

$$\delta^{m+1} \int_{2\delta}^C y^{a-m-2}\,\mathrm{d}y \le C\delta^a.$$

The actual near-source integral is at most $C\delta^\sigma \int_0^{3\delta} y^{\eta-1}\,\mathrm{d}y \le C\delta^a$. The omitted near-source portion of each Taylor coefficient is bounded by $C\delta^k \int_0^{2\delta} y^{a-k-1}\,\mathrm{d}y \le C\delta^a$. These estimates hold for either sign of z. Thus $h_{I^\eta_{\ell+} f}(t) \ge \sigma + \eta$. Letting $\sigma \uparrow h_f(t)$ proves the assertion when $h_f(t) > 0$. If $h_f(t) = 0$, use $r = \chi f$ and the boundedness of f in the same calculation with $\sigma = 0$ for noninteger η. For an integer $\eta = n \ge 1$, take $m = n-1$. The kernel on the far interval is polynomial of degree $n-1$, so its Taylor remainder vanishes. The near integral and the omitted coefficient terms are $O(\delta^n)$. The same argument treats arbitrarily large σ when $h_f(t) = \infty$.

For $0 < q < 1$, the lower inequality in (53) is the result just proved. Fubini's theorem and the beta integral give $I^{1-q}_{\ell+} I^q_{\ell+} B = I^1_{\ell+} B = J_B$. Applying the lower bound to $I^q_{\ell+} B$ with order $1-q$ gives $h_{J_B}(t) \ge h_{I^q_{\ell+}B}(t) + 1 - q$, which is the upper inequality. Equality follows when $h_{J_B}(t) = h_B(t) + 1$. The localization used above requires an interior point t. □

We call t a cusp point of B when $h_{J_B}(t) = h_B(t) + 1$, and a noncusp point otherwise.

Proposition 5.3 (A comparison adapted to the noise exponent)**.** *Under the hypotheses of Proposition 3.8, put*

$$\kappa_\gamma(t) = 1 - \alpha(t)(1-\gamma) > 0.$$

On the same event as in Proposition 3.8, simultaneously at every interior point t_0, the error in that proposition satisfies

$$h_{E_{t_0}}(t_0) \ge \min\{1, q(t_0) + \gamma + \kappa_\gamma(t_0) h_B(t_0)\}. \tag{54}$$

Proof. For $\gamma = 1$, Proposition 3.8 already gives the assertion. Suppose $\gamma < 1$. Write $q_0 = q(t_0)$, $\delta = |t - t_0|$, and $h_* = h_B(t_0)$. At a nonjump point with $h_* > 0$, fix $0 < \eta < h_*$. By Proposition 5.1, $h_* \le \beta(t_0) < 1$, so pointwise Hölder regularity therefore means

$$|B(x) - B(t_0)| \le C_\omega |x - t_0|^\eta$$

near t_0. When $h_* = 0$, including at jump points, use $\eta = 0$ and the local boundedness of B instead.

Choose a small source interval J about t_0 and numbers

$$\sup_J \alpha < p < p' < \frac{1}{1-\gamma}, \qquad \vartheta = 1 - \frac{p}{p'} > 0.$$

For every interval $E \subset J$, the elementary variation interpolation inequality is

$$V_{p'}(B; E) \le (\operatorname*{osc}_E B)^\vartheta V_p(B; E)^{1-\vartheta}. \tag{55}$$

Indeed, in any partition, bound each increment to the power $p' - p$ by the oscillation to that power and sum the remaining pth powers. On a far annulus at distance d from t_0, this yields $V_{p'}(B; E) \le C_\omega d^{\eta\vartheta}$. On every near interval contained in $[t_0 - C\delta, t_0 + C\delta]$, it gives $V_{p'}(B; E) \le C_\omega \delta^{\eta\vartheta}$.

Repeat the proof of Proposition 3.8 using Young integration with p' in place of p. The proper near-diagonal error is now bounded by

$$C_\omega \delta^{q_0+\gamma+\eta\vartheta}(1 + |\log \delta|).$$

For the BV difference of powers, integration by parts may use $B - B(t_0)$, whose supremum norm on the near interval is $O_\omega(\delta^\eta)$. This contribution is bounded by $C_\omega \delta^{q_0+\gamma+\eta}(1 + |\log \delta|)$, which is no larger for small δ. The far contribution is bounded by

$$C_\omega \delta \sum_{2^k\delta \le C_I} (2^k\delta)^{q_0+\gamma+\eta\vartheta-1}(1 + |\log(2^k\delta)|^2).$$

As before, this gives every exponent strictly below $\min\{1, q_0 + \gamma + \eta\vartheta\}$.

For fixed q_0, contributions outside a source neighborhood of t_0 are locally Lipschitz after composition with H. Changing the lower endpoint in $P(\cdot, q_0)$ also changes it by a smooth function near t_0. We may therefore shrink J to t_0, let $p \downarrow \alpha(t_0)$ and $p' \uparrow 1/(1-\gamma)$, and then let $\eta \uparrow h_*$ when $h_* > 0$. The limiting value of ϑ is $\kappa_\gamma(t_0)$. This proves (54).

For the common event, take all rational source intervals and rational orders $p > \sup_J \alpha$ in the finite-variation assertion, and then all rational p' with $p < p' < 1/(1-\gamma)$. The pointwise Hölder estimates apply to each fixed path on this event. Rational compact exhaustion and localization thus make the preceding reasoning simultaneous at every t_0. □

Theorem 5.4 (Spectral transfer for a Hölder index)**.** *Let u be fixed. On a neighborhood of u, suppose $\beta \in C^\gamma$, $0 < \gamma \le 1$, and $\gamma > 1 - 1/\sup \alpha$. Suppose that H is continuous, H is locally Lipschitz off u, and*

$$q(t) = H(t) - \beta(t) > 0 \quad (t \ne u), \qquad q(u) \ge 0.$$

Use the continuous path version that agrees with the stochastic integral at u. Put $h = H(u)$, $A = \alpha(u)$, and $q_0 = h - 1/A$. Then, almost surely,

$$d_X(s, u) = A(s - q_0), \qquad q_0 < s < \min\left\{h, q_0 + \frac{\gamma}{A(1-\gamma)}\right\}, \tag{56}$$

where the quotient is interpreted as $+\infty$ for $\gamma = 1$. In particular, $\gamma \ge 1/2$ gives the whole open interior $q_0 < s < h$.

If

$$\gamma > \frac{1 + \beta(u)}{3}, \tag{57}$$

then, on a sufficiently small punctured neighborhood, the actual exponent equals the frozen fractional-integral exponent at every point. Consequently,

$$q(t) + h_B(t) \le h_X(t) \le H(t), \tag{58}$$

and $h_X(t) = q(t) + h_B(t)$ whenever $h_{J_B}(t) = h_B(t) + 1$. If in addition $h_X(u) \in [q_0, h]$ almost surely, then

$$\overline{d}_X(s, u) = \begin{cases} A(s - q_0), & q_0 \le s \le h, \\ -\infty, & s \notin [q_0, h]. \end{cases}$$

Proof. For $q(u) > 0$, Proposition 3.4 gives the continuous version near u. When $q(u) = 0$, apply the fixed-interval construction leading to (38). That estimate requires only $q \geq 0$ and the local Young condition. Almost surely all local atoms lie off u, where $q > 0$, so each finite truncation is continuous. A uniformly convergent subsequence gives a continuous modification through u. Lemma 2.7 supplies the continuous anchor as a function of $H(t)$. In either case, cover the punctured neighborhood by countably many compact intervals on which Proposition 5.3 applies. Fractional integration and Proposition 5.1 give

$$q(t) + h_B(t) \leq h_{P(\cdot,q(t))}(t) \leq H(t) < 1.$$

The exceptional-set emptiness assertion in Proposition 5.1, applied on shrinking rational intervals, also implies the pointwise statement

$$h_{J_B}(t) > h_B(t) + 1 \quad \Longrightarrow \quad h_B(t) \geq \frac{\beta(t)}{2}. \tag{59}$$

Indeed, if the right inequality failed, a rational $\kappa > h_B(t)$ and a small interval with $2b_I\kappa < 1$ would contradict that emptiness assertion.

Under (57), shrink the neighborhood so that $3\gamma > 1 + \sup \beta$. Since $\beta > 1/2$, we also have $\gamma > 1/2$. At a cusp put $\eta = h_B(t) \leq \beta(t)$. The untruncated error lower bound exceeds $q(t) + \eta$, since

$$\gamma + \kappa_\gamma(t)\eta - \eta = \gamma - \alpha(t)(1-\gamma)\eta \geq 2\gamma - 1 > 0.$$

At a noncusp, (59) instead gives

$$q(t) + \gamma + \kappa_\gamma(t)h_B(t) \geq q(t) + \frac{3\gamma + \beta(t) - 1}{2} > H(t).$$

In both cases the cap at one in (54) remains strictly above the frozen exponent, which is at most $H(t) < 1$. The error therefore cannot change that exponent. This proves the all-point identification, (58), and the cusp equality.

For the interior spectrum, fix s in the open range of (56), and choose a small interval I about u on which

$$0 < \eta_s(t) := s - q(t) < \beta(t), \qquad \alpha(t)(1-\gamma)\eta_s(t) < \gamma, \qquad \kappa_I := \sup_I \eta_s < 1/b_I,$$

where $b_I = \sup_I \alpha$. The variable-level set $\{h_B = \eta_s\}$ has dimension $D_I = \sup_I \alpha(t)\eta_s(t)$. Its noncusp subset has dimension at most $\max\{0, 2b_I\kappa_I - 1\}$. As I shrinks to u, both D_I and $b_I\kappa_I$ tend to $d_0 = A(s - q_0) \in (0, 1)$. The exceptional dimension is strictly smaller than D_I for small I. At every remaining point, the frozen exponent equals s. Moreover,

$$q(t) + \gamma + \kappa_\gamma(t)\eta_s(t) > q(t) + \eta_s(t) = s,$$

so (54) transfers the exponent to X. Removing u does not change the positive dimension. This proves the lower bound.

Conversely, at a point of $I \setminus \{u\}$ with $h_X(t) = s$, one must have $h_B(t) \leq \eta_s(t)$. Otherwise the frozen exponent is at least $q(t) + h_B(t) > s$, and the error exponent is also larger than s, because $\kappa_\gamma(t) > 0$ and the last displayed inequality holds. Their sum would then have exponent larger than s, a contradiction. Thus the exact level set is contained in $\{h_B \leq s - \inf_I q\}$. Its dimension

is at most $b_I(s - \inf_I q)$. Shrinking I proves (56). If $\gamma \geq 1/2$, then $\gamma/[A(1-\gamma)] \geq 1/A$, which proves the full-interior assertion, including $\gamma = 1/2$.

For the window spectrum, assume (57) and the additional pointwise condition at u. The pointwise bounds exclude windows outside $[q_0, h]$ on sufficiently small neighborhoods, by continuity of q and H. Fix $q_0 < s < h$ and small $\varepsilon > 0$. The exact level s provides the lower bound $A(s - q_0)$. On a small interval I about u, the upper window $\{h_X < s + \varepsilon\}$ is contained, apart from u, in

$$\{h_B \leq s + \varepsilon - \inf_I q\}.$$

For small I and ε the threshold is below $1/b_I$, and its dimension is at most $b_I(s+\varepsilon-\inf_I q)$. First shrink I and then let $\varepsilon \downarrow 0$ to obtain the desired interior value. At $s = q_0$, the same estimate has limiting upper bound $A\varepsilon$, while exact levels in $(q_0, q_0 + \varepsilon)$ ensure nonempty windows in every neighborhood. The limiting dimension is therefore zero. At $s = h$, choose exact interior levels in $(h - \varepsilon, h)$. Their dimensions approach one, which is also the ambient upper bound. This proves the asserted window spectrum. □

Corollary 5.5 (Uniform regularity and atom singularities)**.** *On an ordinary strictly supercritical interval under the assumptions of Proposition 3.8, every Poisson jump location T_j satisfies*

$$h_X(T_j) = q(T_j) \quad \text{almost surely, simultaneously.} \tag{60}$$

The local uniform Hölder exponent

$$h_X^{\mathrm{unif}}(u) = \lim_{\rho \downarrow 0} \sup\{\eta \geq 0 : X \text{ is uniformly } \eta\text{-Hölder on } (u - \rho, u + \rho)\}$$

equals $q(u)$. The local Young condition alone suffices for both assertions.

Proof. At a jump, $h_B = 0$ and $h_{J_B} = 1$, so the frozen exponent is q. Since $q < 1$ and $\gamma > 0$, the cutoff in (28) is strictly larger than q, so that comparison preserves the exponent. For uniform regularity, work on a compact observation interval I and a local source interval J. The Young condition and $H < 1$ imply $q(t) < 1 - \beta(t) \leq 1 - 1/b_J < \gamma$. Fractional integration of the bounded path B gives a uniform time-increment bound of every order below $\inf_I q$ for $P(t, q)$ at a fixed order in the compact range of q. The order derivative of P is uniformly bounded, and q is γ-Hölder. Corollary 3.6 gives a γ-Hölder remainder. Therefore X is uniformly Hölder of every order below $\inf_I q$ on I. The Poisson locations are dense, and their exponents in (60) give the opposite inequality as I shrinks to u. □

5.1 Exact endpoints and deterministic level sets

For the additive multistable motion, Le Guével and Lévy Véhel [LL20] relate the upper endpoint of the global spectrum to the minimum set of the stability index. For X the relevant deterministic set is a level set of H. To identify its contribution we first restrict the pointwise exponent to a fixed compact set.

Lemma 5.6 (Typical exponents on a deterministic set)**.** *On a strictly supercritical interval, suppose the hypotheses of Proposition 3.8 hold and $3\gamma > 1 + \sup \beta$. For every fixed deterministic compact set F in that interval,*

$$\dim_{\mathrm{H}}\{t \in F : h_X(t) = H(t)\} = \dim_{\mathrm{H}} F \quad \text{almost surely.}$$

We retain the convention $\dim_{\mathrm{H}} \varnothing = -\infty$.

Proof. At each fixed deterministic point, the Poisson approximation exponent is one almost surely, so $h_B(t) = \beta(t)$ almost surely. The fractional-integration sandwich and the all-point identification in Theorem 5.4 therefore give $h_X(t) = H(t)$ almost surely. If $0 < s < \dim_{\mathrm{H}} F$, take a deterministic Frostman measure ν_s on F satisfying $\nu_s(B(t,r)) \le C_s r^s$. Joint measurability follows from path continuity. For exponents below one, the local oscillation bounds can be tested using rational increments and orders. Fubini's theorem therefore gives

$$\nu_s\{t : h_X(t) \neq H(t)\} = 0 \quad \text{almost surely.}$$

The restriction of ν_s to the complementary set retains its Frostman bound and positive total mass, so that set has dimension at least s. Use a countable sequence $s \uparrow \dim_{\mathrm{H}} F$. If F is nonempty and has dimension zero, choose one deterministic point in F to obtain nonemptiness almost surely. The upper bound is inclusion in F. □

Theorem 5.7 (The exact upper endpoint)**.** *Assume the hypotheses of Theorem 5.4, with* $3\gamma > 1 + \beta(u)$*, and assume* $h_X(u) = h = H(u)$ *almost surely. If every neighborhood of* u *contains a point* t *with* $H(t) > h$*, then*

$$d_X(h, u) = 1 \quad \textit{almost surely.}$$

Otherwise $H \le h$ *on a neighborhood of* u*, and*

$$d_X(h, u) = \lim_{\rho \downarrow 0} \dim_{\mathrm{H}}\{t \in (u-\rho, u+\rho) : H(t) = h\} \quad \textit{almost surely.}$$

Proof. In the first case choose a deterministic sequence $w_n \to u$ with $H(w_n) > h$. For all sufficiently large n, $q(w_n) < h < H(w_n)$ and $3\gamma > 1 + \beta(w_n)$. The interior spectrum at these countably many points gives

$$d_X(h, w_n) = \alpha(w_n)(h - q(w_n)) \longrightarrow 1.$$

Every neighborhood of u contains arbitrarily late w_n, so its exact level-h set has dimension one.

In the second case, the pointwise upper bound $h_X(t) \le H(t) \le h$ shows that, away from u, $E_X(h) \subset \{H = h\}$. Intersect this closed deterministic level set with countably many compact annuli avoiding u. Apply Lemma 5.6 to those compact sets, and use countable stability of Hausdorff dimension. Include the point u, whose exponent is h by assumption, and take a countable decreasing sequence of neighborhoods. This proves the equality. □

Example 5.8 (Every upper endpoint dimension occurs)**.** Fix a constant index $A \in (1, 2)$ and $h \in (1/A, 1)$. For any $d \in [0, 1]$, choose a deterministic compact set F containing a fixed point $u \neq 0$ and having local Hausdorff dimension d at u. For $0 < d < 1$, one may use the two-branch self-similar Cantor set with contraction ratio $2^{-1/d}$. A singleton and an interval give $d = 0$ and $d = 1$, respectively. There is a nonnegative C^∞ function ψ, bounded by one, whose zero set locally is F. For instance, on each bounded gap (a, b) one may set

$$\psi(t) = \exp\left(-\frac{1}{(t-a)(b-t)}\right), \qquad a < t < b,$$

and set $\psi = 0$ on F. Its derivatives tend to zero at the gap endpoints, uniformly for small gaps, so the resulting function is smooth locally. On complementary components meeting the boundary of the observation interval, use a positive smooth function flat at the finite endpoint in F. For a singleton, one may simply use $\psi(t) = \exp(-1/(t-u)^2)$ off u and $\psi(u) = 0$. Choose

$0 < \varepsilon < h - 1/A$ and put $H = h - \varepsilon\psi$. Then H is smooth and strictly supercritical. Lemma 5.6 applied to $\{u\}$ gives $h_X(u) = h$ almost surely, so Theorem 5.7 yields

$$d_X(h, u) = d, \qquad \overline{d}_X(h, u) = 1.$$

Thus even with smooth parameters the exact upper endpoint records the geometry of the deterministic maximum set of H.

Example 5.9 (The exact lower endpoint may be absent)**.** Let $\alpha \equiv A \in (1, 2)$, $1/A < h < 1$, and $u \neq 0$. Choose $H(t) = h + c(t - u)^2$ locally, with $c > 0$ small enough that $H < 1$. Then $q(t) > q(u)$ for $t \neq u$, and the pointwise lower bound gives $h_X(t) \geq q(t) > q(u)$ there. Lemma 5.6 gives $h_X(u) = h$ almost surely at the fixed point u. Hence the exact level $q(u)$ is absent from a small neighborhood, while its window spectrum has value zero.

6 A positive critical contact

At a positive contact, $q(u) = 0$ while $q(t) > 0$ on either side. The exponents at nearby jump locations then approach zero. The value at u has a different exponent, determined by the critical tangent.

Theorem 6.1 (Critical contact)**.** *Let $u \neq 0$, $A = \alpha(u) \in (1, 2)$, $h = 1/A$, and suppose*

$$H(t) = h + c|t - u|^h \quad \text{near } u, \qquad c > 0.$$

Assume $\beta \in C^\gamma$ locally for some

$$\frac{1 + h}{3} < \gamma \leq 1, \tag{61}$$

and suppose that, for some $c_0 > 0$,

$$q(u + s) = H(u + s) - \beta(u + s) \geq c_0|s|^h \tag{62}$$

for all sufficiently small nonzero s. Then X has continuous paths on a sufficiently small neighborhood of u, and

$$r^{-h}\{X(u + r\cdot) - X(u)\} \xrightarrow[M_1]{\text{st}} L_A + c|\cdot|^h D_u. \tag{63}$$

This convergence cannot hold in J_1. Almost surely,

$$h_X(u) = h, \qquad h_X^{\text{unif}}(u) = 0, \qquad E_X(0) \cap (u - \rho, u + \rho) = \varnothing \tag{64}$$

for sufficiently small $\rho > 0$, whereas

$$d_X(s, u) = \begin{cases} As, & 0 < s \leq h, \\ -\infty, & \text{otherwise.} \end{cases} \tag{65}$$

The window spectrum equals As on $0 \leq s \leq h$, including value zero at $s = 0$, and is $-\infty$ elsewhere.

Proof. Since $h > 1/2$, one has $(1+h)/3 > 1-h$. By continuity of α, the local interval can be chosen so that $\gamma > 1 - 1/\sup_I \alpha$. The Hurst profile is $g(v) = c|v|^h$, and (62) is branch (i) of Theorem 4.1. That theorem gives the continuous modification, (63), the failure of J_1 convergence, and $h_X(u) = h$.

On the punctured neighborhood, $H \in C^1$ and $q > 0$. Condition (61) implies both $\gamma > 1/2$ and $3\gamma > 1 + \beta(u)$. Theorem 5.4 therefore gives the full open interior spectrum $d_X(s,u) = As$ for $0 < s < h$ and, on a smaller punctured neighborhood,

$$q(t) + h_B(t) \le h_X(t) \le H(t).$$

In particular $h_X(t) > 0$ there. Together with $h_X(u) = h > 0$, this proves the absence of zero-exponent points. The dense Poisson atom locations have exponents $q(T_j)$, which tend to zero along atoms approaching u. It follows that the local uniform exponent is zero.

Since $H(t) > h$ at every nearby $t \neq u$, Theorem 5.7 gives $d_X(h,u) = 1 = Ah$. For each $s > h$, continuity of H and the upper pointwise bound exclude level s from sufficiently small neighborhoods. Negative levels and level zero are absent. This proves (65). The closed window spectrum follows from Theorem 5.4, using $q(u) = 0$ and $h_X(u) = h$. □

Remark 6.2. The positivity condition (62) follows automatically when $\gamma > h$, since then $\beta(u+s) - \beta(u) = o(|s|^h)$. It also follows when $\beta(t) \le \beta(u)$ near u, independently of the comparison between γ and h. The second possibility permits spatial indices rougher than the prescribed critical Hurst cusp.

Example 6.3 (A singular continuous index)**.** Let C be the usual Cantor distribution function, extended constantly outside $[0,1]$, and take $u = 1/3$,

$$\beta(t) = \frac{3}{5} + \frac{1}{50}\big(C(t) - C(1/3)\big), \qquad \alpha(t) = 1/\beta(t).$$

Its range lies in $[0.59, 0.61]$, so $1 < \alpha < 2$. The function β is singular continuous and has the uniform Hölder bound $\gamma = \log 2/\log 3 > 3/5$. Its variation measure also satisfies $\mu([t-s,t+s]) \le Cs^\gamma$, and hence the Dini condition (29). With $H(t) = 3/5 + c|t-u|^{3/5}$ locally, condition (62) follows from $\gamma > 3/5$. Theorem 6.1 gives pointwise exponent $3/5$, local uniform exponent zero, and exact spectrum $(5/3)s$ for $0 < s \le 3/5$. This index lies outside the absolutely continuous class.

Example 6.4 (A rougher spatial index with unbounded variation)**.** Fix $u \neq 0$, put $\gamma = 11/20$, and define

$$W(t) = (1 - 2^{-\gamma}) \sum_{n=0}^{\infty} 2^{-n\gamma} \cos(2^n t), \qquad \beta(t) = \frac{3}{5} + \frac{1}{100}\big(W(t-u) - 1\big).$$

Then $W(0) = 1$, $|W| \le 1$, and splitting its series at $2^n|t-s| \asymp 1$ gives $W \in C^\gamma$. Consequently $\beta(u) = 3/5$, $0.58 \le \beta \le 0.60$, and $1 < \alpha = 1/\beta < 2$. At $t_m = 2^{-m}$ all terms in $W(0) - W(t_m)$ are nonnegative, and the term $n = m$ gives

$$W(0) - W(t_m) \ge c\, t_m^\gamma.$$

Thus the pointwise Hölder exponents of β and α at u are exactly $11/20$, strictly below $3/5$.

The function W has unbounded variation on every nonempty open interval. Suppose otherwise and choose a nonnegative smooth function φ compactly supported in such an interval with $\int \varphi > 0$. Then φW would be a compactly supported BV function, so its Fourier transform would satisfy $|\xi\widehat{\varphi W}(\xi)| \le C$. The lacunary expansion, however, gives

$$\widehat{\varphi W}(2^m) = \frac{1-2^{-\gamma}}{2}\, 2^{-m\gamma}\widehat{\varphi}(0) + o(2^{-m\gamma}).$$

For this error estimate, nonresonant frequencies with $n \le m$ are at distance at least 2^{m-1} from 2^m, and those with $n > m$ are at distance at least 2^{n-1}. Rapid decay of $\widehat{\varphi}$ makes their total $o(2^{-m\gamma})$. Multiplication by 2^m contradicts the BV bound. The same unbounded-variation conclusion holds for β and for α, since the reciprocal map is Lipschitz on their compact positive ranges in both directions.

Set $H(t) = 3/5 + c|t-u|^{3/5}$ locally. Since $\beta(t) \le \beta(u)$, one has $q(t) \ge c|t-u|^{3/5}$. Moreover,

$$\frac{11}{20} > \frac{1+3/5}{3} = \frac{8}{15}.$$

Theorem 6.1 therefore gives the full critical M_1 limit and spectrum for a spatial index rougher than H at u and having unbounded variation on every open interval.

7 Further questions

When $1 - 1/\sup\alpha < \gamma < 1/2$, the comparison in Proposition 5.3 gives the spectrum only below $q_0 + \gamma/[A(1-\gamma)]$. Above this cutoff the error and the fractional integral may have the same pointwise exponent. Determining whether cancellation occurs there would also clarify the role of the stronger condition $3\gamma > 1 + \beta(u)$ in the all-point result.

At the boundary $H(u)\alpha(u) = 1$, the proof accommodates both Hölder indices and continuous BV indices whose variation measures satisfy (29). A criterion formulated directly in terms of the variation of the source kernels could cover both cases. Proposition 4.4 shows that such a criterion must also control how H varies immediately after a noise atom if it is to imply M_1 convergence.

For $H(u)\alpha(u) < 1$, the stable L^p limit describes local increments even when the kernel is unbounded at the diagonal. Identifying the singularities in this region requires pointwise regularity estimates for unbounded paths.

A Poisson geometry for a continuous spatial index

This appendix proves Proposition 5.1. Work on compact intervals with $1 < a_I \le \alpha \le b_I < 2$. All probability-one assertions are first made for rational intervals and rational slack parameters, then intersected. The resulting event supports the simultaneous pointwise statements used above.

Lemma A.1 (Poisson approximation input)**.** *Let $\{(T_j, R_j)\}$ be a Poisson point process of intensity $\mathrm{d}t\,\mathrm{d}R/R^2$ on $\mathbb{R}\times(0,1)$. Define*

$$\delta(t) = \sup\{d \ge 1 : |t - T_j| \le R_j^d \text{ for infinitely many } j\}.$$

On one event of probability one, $\delta(t) \geq 1$ for every $t \in \mathbb{R}$, and, for every nondegenerate compact interval I and every continuous $f : I \to (1, \infty)$,

$$\dim_{\mathrm{H}}\{t \in I : \delta(t) = f(t)\} = \sup_{t \in I} \frac{1}{f(t)}.$$

For each fixed deterministic t, $\delta(t) = 1$ almost surely.

Proof. On the unit interval, the balls of radius R_j cover every interior point infinitely often almost surely [BFJS10, Proposition 17]. The localized approximation theorem then gives the exact-target dimension [BS11, Theorem 1.4]. Section 5.4 of [BS11] verifies the required conditions for this Poisson intensity. These conditions concern the realized point system, so the same event works for all continuous targets. To localize, use $(t, R) \mapsto ((t - a)/L, R/L)$ on an interval of length L. The intensity is invariant. At exponent d the transformed power test has the constant factor L^{d-1}, which leaves the supremal exponent unchanged since $R_j \to 0$ along local approximating sequences. At $d = 1$ the covering test is unchanged. Intersect the events for intervals with rational endpoints. For any compact interval, exhaust its interior by rational subintervals and extend the target continuously where needed. The two endpoints have dimension zero. For each fixed t and each $d > 1$, the expected number of sufficiently small balls of radius R^d covering t is finite, since $\int_0^1 R^{d-2}\,\mathrm{d}R < \infty$. Taking rational $d > 1$ and using the covering lower bound proves $\delta(t) = 1$ almost surely. □

A.1 Uniform small-jump oscillations

Let $B^{\leq e}$ be the compensated martingale of marks $|j| \leq e$. For an interval Q of length ℓ, its exponential martingale gives

$$\mathbb{P}\left(\sup_{t \in Q} |B^{\leq e}(t) - B^{\leq e}(\inf Q)| > Kme\right) \leq 2\exp\{-Km + C\ell e^{-b_I}\}. \tag{66}$$

Indeed use the exponential parameter $1/e$, symmetry, and $\cosh(j/e) - 1 \leq C(j/e)^2$ for $|j| \leq e$, so the compensator is bounded by $C\ell e^{-b_I}$. Apply the same estimate to the negative martingale. Choose K large enough to sum over an exponentially growing number of dyadic cells. It follows that, whenever

$$e_m = 2^{-m(1+\varepsilon)}, \qquad \ell_m = 2^{-m\delta}, \qquad \delta > b_I(1 + \varepsilon),$$

all intervals of length at most a fixed multiple of ℓ_m have small-jump oscillation at most Cme_m, eventually almost surely. An arbitrary such interval lies in a bounded number of adjacent cells. The same conclusion applies to any geometric choices of e_m, ℓ_m for which $\ell_m e_m^{-b_I}$ is bounded.

A.2 Normalized Poisson radii and pointwise exponents

For each Poisson mark (T_j, J_j), discard its sign and define

$$R_j = \frac{\alpha(T_j)}{2c(\alpha(T_j))}|J_j|^{\alpha(T_j)}. \tag{67}$$

A change of variables in (4) shows that (T_j, R_j) has intensity $\mathrm{d}t\,\mathrm{d}R/R^2$. Restricting to $0 < R_j < 1$ gives the system of Lemma A.1. Put

$$\delta(t) = \sup\{d \geq 1 : |t - T_j| \leq R_j^d \text{ for infinitely many } j\}.$$

Lemma A.1 gives $\delta(t) \geq 1$ simultaneously at every interior point and supplies the exact continuous-target dimensions.

At nonjump points, we claim

$$h_B(t) = \frac{1}{\alpha(t)\delta(t)}, \tag{68}$$

with value zero if $\delta(t) = \infty$. If $d < \delta(t)$, the corresponding infinitely many jumps and continuity of α give the upper bound $h_B(t) \leq 1/(\alpha(t)d)$. Indeed a pointwise estimate $|B(s) - P(s-t)| \leq C|s-t|^\sigma$ controls both one-sided values at a nearby jump T_j, so $|J_j| \leq 2C|T_j - t|^\sigma$. Let $d \uparrow \delta(t)$ when $\delta(t) > 1$. If $\delta(t) = 1$, use the infinitely many covering balls at $d = 1$.

For the lower bound fix $\eta > \delta(t)$ and a local interval I. Eventually no jump satisfies $|t - T_j| \leq R_j^\eta$. Hence jumps within distance r of t have $|J_j| \leq Cr^{1/(\eta b_I)}$, apart from finitely many locations that disappear when r is small. Apply (66) at $r = 2^{-n}$ and cutoff $e_n = Cr^{1/(\eta b_I)}$. Since $re_n^{-b_I} = O(r^{1-1/\eta})$, the uniform dyadic estimate gives

$$\sup_{|s-t|\leq r} |B(s) - B(t)| \leq C \log(1/r) r^{1/(\eta b_I)}.$$

Let $\eta \downarrow \delta(t)$ through rational values and shrink I to t. This proves (68) under continuity alone. At jumps $h_B = 0$ by definition. At a fixed deterministic point, Lemma A.1 and the absence of a fixed jump give $h_B(t) = \beta(t)$ almost surely.

For a continuous target $\eta > 0$ with $\alpha\eta < 1$, apply Lemma A.1 with the continuous target $f(t) = 1/(\alpha(t)\eta(t)) > 1$. The exact-level dimension is $1/\inf_I f = \sup_I \alpha\eta$, giving (52). The countable Poisson locations have zero noise exponent. Removing them preserves the positive dimension of the target set. The event in Lemma A.1 works for all such targets. To prove (51), take $1 < d < 1/(b_I\kappa)$ for $\kappa > 0$. Apart from the countable jump locations, the sublevel set lies in the limsup of balls of radius R_j^d. For any $D > 1/d$, the expected sum of R_j^{dD} on a bounded source neighborhood is finite because $\int_0^1 R^{dD-2}\,\mathrm{d}R < \infty$. The limsup has dimension at most $1/d$. Let $d \uparrow 1/(b_I\kappa)$, or take arbitrarily large d when $\kappa = 0$.

A.3 A simultaneous upper bound for the primitive

Shrink I when needed so that $1 + b_I < 2a_I$. Choose $a' < a_I$ and $b' > b_I(1+\varepsilon)$, with $b' > a'$ and $(1+b')/a' < 2$, choosing ε sufficiently small first. At scale m let the macrocell length be $R_m = 2^{-ma'}$ and the microcell length be $\ell_m = 2^{-mb'}$. In each microcell consider the event of exactly one mark with size in $[2^{-m}, 2^{1-m}]$ in absolute value, located in its central third, and no other mark larger than $e_m = 2^{-m(1+\varepsilon)}$ in that microcell. Its probability is at least

$$c\ell_m 2^{ma_I} \exp(-C\ell_m e_m^{-b_I}) \geq c'\ell_m 2^{ma_I}.$$

Disjoint microcells have independent events. Therefore the probability that a macrocell has no successful microcell is at most $\exp\{-c2^{m(a_I - a')}\}$. A union over all macrocells and Borel–Cantelli show that every macrocell eventually contains one.

Let τ be its isolated jump time and choose $\rho_m \asymp \ell_m$ so that $[\tau - \rho_m, \tau + \rho_m]$ lies in the microcell. The uniform small-jump estimate gives

$$|J_B(\tau + \rho_m) - 2J_B(\tau) + J_B(\tau - \rho_m)| \geq c2^{-m}\ell_m. \tag{69}$$

This follows by writing the second difference as $\int_0^{\rho_m} (B(\tau + s) - B(\tau - s))\,\mathrm{d}s$. The jump term is $\rho_m J$, and the error is at most $C\rho_m m e_m$.

For any point t in the macrocell, all three evaluation points in (69) are within CR_m of t. A pointwise Hölder bound of order $\zeta \in (1,2)$ at t has an affine polynomial, annihilated by this second difference. It would bound the left side by CR_m^ζ. This is impossible if $a'\zeta > 1+b'$. Hence $h_{J_B}(t) \le (1+b')/a'$. Shrink rational intervals to each t and let the slack parameters decrease to zero. This proves $h_{J_B}(t) \le 1 + 1/\alpha(t)$ simultaneously. The limiting upper bound is less than two, so exponents $1 < \zeta < 2$ suffice. At a jump, directly $h_{J_B} = 1$.

A.4 Pairs of jumps and the noncusp exception

Fix $\delta > b_I(1+\varepsilon)$, $e_m = 2^{-m(1+\varepsilon)}$, and $\ell_m = 2^{-m\delta}$. Cover by grid intervals Q of length comparable to ℓ_m, and let $3Q$ have the same center and three times the length. Define $F_{\delta,\varepsilon}$ as the limsup of the unions of $3Q$ for which the number of marks in $3Q$ with $|j| > e_m$ is at least two. Counting in the enlarged cell captures pairs across grid boundaries. A cell has mean mark count at most $C\ell_m e_m^{-b_I}$, so the expected number of such cells is at most $C\ell_m e_m^{-2b_I}$. Their expected d-dimensional covering sum is bounded by

$$C \sum_m 2^{-m\{\delta(1+d) - 2b_I(1+\varepsilon)\}}.$$

It follows that

$$\dim_{\mathrm{H}} F_{\delta,\varepsilon} \le \max\left(0, \frac{2b_I(1+\varepsilon)}{\delta} - 1\right), \tag{70}$$

and the set is empty if the expression before the maximum is negative.

We show that outside this paired-jump set, points with $h_B(t) < 1/\delta$ are cusps. We first use a physical-jump observation. For $\xi > b_I$, if t eventually avoids all balls $|t - T_j| \le |J_j|^\xi$, the largest jump within radius r is at most $r^{1/\xi}$. Equation (66) at that cutoff proves $h_B(t) \ge 1/\xi$. Consequently, if $h_B(t) = h_* < 1/b_I$, then for every $b_I < \xi < 1/h_*$ infinitely many jumps satisfy that inequality.

Take $t \notin F_{\delta,\varepsilon}$ with $h_* < 1/\delta$. Choose $\delta < \delta' < \xi < 1/h_*$. Along the approximating jumps of size $|J_j| \asymp 2^{-m}$, the distance from t is much smaller than $\rho_m = 2^{-m\delta'}$. These jumps are the only marks above e_m in the larger window of radius comparable to $2^{-m\delta}$. The small-jump estimate on the ρ_m scale applies since $\delta' > b_I(1+\varepsilon)$. The argument of (69) now gives a second difference of magnitude at least $c2^{-m}\rho_m$, with its three arguments within $C\rho_m$ of t. Thus $h_{J_B}(t) \le 1 + 1/\delta'$. Let $\delta' \uparrow 1/h_*$. Ordinary integration gives the reverse inequality $h_{J_B}(t) \ge h_* + 1$, so equality holds. If $h_* = 0$, take arbitrarily large δ'. At jumps the primitive has exponent one.

For $\kappa < 1/b_I$, choose $\varepsilon > 0$ small and $b_I(1+\varepsilon) < \delta < 1/\kappa$. The preceding argument puts $\mathcal{O}_\kappa(I)$ inside $F_{\delta,\varepsilon}$. Let $\delta \uparrow 1/\kappa$ and $\varepsilon \downarrow 0$ in (70). This proves the exceptional-set bound and its emptiness assertion. For $\kappa = 0$ use arbitrarily large δ. Together with the preceding subsections this completes the proof of Proposition 5.1.

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