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# A Conjugacy-Circle Estimate on SU(2) and Strong Banach Property ($T$) for Real Higher Rank Groups of Type $B_2$ and $BC_2$

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November 2026

**Abstract**

We prove that every connected simple real Lie group $G$ of real rank at least two (with arbitrary, possibly infinite center) and every lattice $\Gamma \subset G$ satisfies strong Banach property ($T$) with respect to every class of complex Banach spaces of uniformly nontrivial Rademacher type, as do finite products of higher rank groups and their lattices (Theorem B). The core new analytic ingredient is a uniform spectral-decay estimate for the conjugacy-circle averaging operators $S_\theta$ on SU(2) acting on $L^2(\mathrm{SU}(2); E)$ (Theorem A and Corollary 1.1), supplying the groups of restricted root type $B_2$ and $BC_2$—such as $\mathrm{Sp}_4(\mathbb{R})$, its universal cover $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$, $\mathrm{SO}_0(2,n)$ ($n \geq 3$), and their lattices such as $\mathrm{Sp}_4(\mathbb{Z})$—not directly covered by the $S^2$ latitude estimate. Every external input is a published, refereed result: we derive the base-$A$ digit and bilinear-shift estimates (after Yao) directly from Bourgain's 1982 Hausdorff–Young theorem (Section 2), give a short second proof of the $S^2$ latitude estimate (Theorem C), and write out in full the two-step strong property ($T$) proof for $\mathrm{Sp}_4(\mathbb{R})$, $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$, and all higher rank connected simple real Lie groups (Section 8). The core geometric, fold, freezing, and Littlewood–Paley Hölder engine of Sections 3–5 is formally verified in Lean 4 under explicit structured hypotheses (zero `sorry`, standard axioms only), while Sections 2, 6, 7, and 8 are paper proofs; this manuscript has not yet been refereed.

## 1 Introduction

### 1.1 Setting

For $1 \leq p \leq 2$ and a complex Banach space $E$, let $T_p(E)$ be the least constant $T$ such that

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\|^2\Big)^{1/2} \leq T\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p} \tag{1.1}$$

for all finite families $x_1, \dots, x_n \in E$, where $\varepsilon_j$ are independent random signs (this is [21, Definition 2.1]). For $1 < p \leq 2$ and $1 \leq M < \infty$ let

$$\mathcal{C}_{p,M} = \{E \text{ complex Banach space}: \ T_p(E) \leq M\}.$$

A class $\mathcal{E}$ of complex Banach spaces has *uniformly nontrivial type* ("type $> 1$" in [21]) if $\mathcal{E} \subseteq \mathcal{C}_{p,M}$ for some $1 < p \leq 2$ and $M < \infty$.

On SU(2) we use the probability Haar measure. The left regular representation is given by $(\lambda(g)f)(x) = f(g^{-1}x)$. Each $\lambda(g)$ is an isometry of the Bochner space $L^2(\mathrm{SU}(2); E)$; here $\lambda(g)$

stands for $\lambda(g) \otimes \mathrm{Id}_E$. Put

$$d_\alpha = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \qquad u_\theta = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta} & -1 \\ 1 & e^{-i\theta} \end{pmatrix}, \qquad S_\theta = \frac{1}{2\pi} \int_0^{2\pi} \lambda(d_\alpha u_\theta d_{-\alpha})\, d\alpha.$$

Thus $S_0$ averages $\lambda(g)$ over the "conjugacy circle" $\{g \in \mathrm{SU}(2) : g_{11} = 2^{-1/2}\}$. A direct computation shows $(d_\alpha u_\theta d_{-\alpha})^{-1} = 2^{-1/2} \left(\begin{smallmatrix} e^{-i\theta} & -e^{i\varphi} \\ e^{-i\varphi} & e^{i\theta} \end{smallmatrix}\right)$ with $\varphi = 2\alpha + \pi$, so $S_\theta$ is exactly the operator $S_\theta$ of de la Salle [6, Introduction, before Theorem 1.5], and it is the operator of de Laat and de la Salle [3, Section 3].

The map $\theta \mapsto \lambda(d_\theta)$ is a $2\pi$-periodic isometric representation of the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For $m \in \mathbb{Z}$ put $\Pi_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \lambda(d_\theta)\, d\theta$. For $\psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$ and $R \geq 1$ let

$$P_R = \sum_{m \in \mathbb{Z}} \psi(m/R)\, \Pi_m = \int_{\mathbb{T}} k_R(\theta) \lambda(d_\theta)\, d\theta, \qquad k_R(\theta) = \frac{1}{2\pi} \sum_m \psi(m/R) e^{-im\theta}.$$

Since $k_R$ is the periodization of a dilate of a Schwartz function, $\sup_{R \geq 1} \|k_R\|_{L^1(\mathbb{T})} < \infty$, so $P_R$ is bounded on $L^2(\mathrm{SU}(2); E)$ uniformly in $R$ and $E$. Since $S_0$ commutes with every $\lambda(d_\theta)$, $P_R$ commutes with $S_0$.

### 1.2 Results

**Theorem A** (the SU(2) estimate)**.** *Let $1 < p \leq 2$, $1 \leq M < \infty$ and $\psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Let $A, q$ be a fixed choice of the constants of Corollary 2.6 for the class $\mathcal{C}_{p,M}$ (constructed via Yao's base-$A$ digit method [21] from Bourgain's theorem [2]), put $\varepsilon = \log(1/q)/\log A \in (0, \frac{1}{2})$ (non-effective) and*

$$\eta = \tfrac{1}{8} \min\{1, \varepsilon\} = \tfrac{1}{8}\varepsilon.$$

*There is $C < \infty$, depending only on $p, M, \psi$, such that for every $E \in \mathcal{C}_{p,M}$ and every $R \geq 1$,*

$$\|P_R S_0\|_{L^2(\mathrm{SU}(2);E) \to L^2(\mathrm{SU}(2);E)} \leq C\, R^{-\eta}.$$

**Corollary 1.1** (Hölder form)**.** *For $1 < p \leq 2$ and $1 \leq M < \infty$ there is $C < \infty$ such that for every $E \in \mathcal{C}_{p,M}$ and all $\theta, \theta' \in \mathbb{T}$,*

$$\|S_\theta - S_{\theta'}\|_{L^2(\mathrm{SU}(2);E) \to L^2(\mathrm{SU}(2);E)} \leq C\, d_{\mathbb{T}}(\theta, \theta')^\eta,$$

*with $\eta$ as in Theorem A and $d_{\mathbb{T}}$ the distance on the circle.*

Corollary 1.1 implies the conjugacy-circle condition (1.3) of [6, Theorem 1.5] (and condition (5) of [3]) for the class $\mathcal{C}_{p,M}$. The second condition, (1.4) of [6], is the latitude estimate on the sphere, Theorem C below. Together they give the group theorem, proved in full in Sections 7 and 8.

**Theorem B** (groups and lattices)**.** *Let $G$ be a connected real Lie group whose Lie algebra is simple and has real rank at least two, and let $\Gamma \subset G$ be a lattice. Let $1 < p \leq 2$ and $1 \leq M < \infty$. Then $G$ and $\Gamma$ have de la Salle's property $(*_{\mathcal{C}_{p,M}})$ (Definition 7.1); in particular they have strong Banach property $(T)$ with respect to $\mathcal{C}_{p,M}$, and hence with respect to every class of complex Banach spaces of uniformly nontrivial type. The same holds for the finite products of higher rank groups considered in [6, Theorem 5.11] and their lattices.*

For $\delta \in [-1,1]$ let $T_\delta$ average a function on $S^2 \subset \mathbb{R}^3$ over the circle $\{y : \langle x,y\rangle = \delta\}$ (normalized arc length; $T_{\pm 1} f(x) = f(\pm x)$). The latitude estimate was first proved by Yao [21, Theorem 5.4], with his parabolic partition and his Theorem 4.6. We give a short second proof from Proposition 2.10 (Section 6).

**Theorem C** (the latitude estimate; for $\delta_0 = 0$ and some exponent, Yao [21, Theorem 5.4])**.** *Let $1 < p \le 2$ and $1 \le M < \infty$, and let $\varepsilon$ be as in Theorem A. There is $C < \infty$ such that for every $E \in \mathcal{C}_{p,M}$, every $\delta_0 \in [-\frac{1}{2}, \frac{1}{2}]$ and every $\delta \in [-1,1]$,*

$$\|T_\delta - T_{\delta_0}\|_{L^2(S^2;E)\to L^2(S^2;E)} \le C\,|\delta - \delta_0|^{\varepsilon/2}.$$

The statement (for $\delta_0 = 0$, with some exponent) is Yao's; the proof in Section 6 is new and rests only on Proposition 2.10, which is proved self-containedly in Section 2 directly from Bourgain's theorem [2].

### 1.3 What is new and what is not

Strong property $(T)$ was introduced by Lafforgue [12]. Over non-Archimedean local fields, strong Banach property $(T)$ with respect to every class of uniformly nontrivial type is due to Lafforgue [13] and Liao [15], and de la Salle extended it to all lattices [6, Theorem 1.4]. Over $\mathbb{R}$, strong property $(T)$ for Hilbert spaces was proved by Lafforgue [12] for $\mathrm{SL}_3(\mathbb{R})$ and the groups in his statement that contain it, and by de Laat and de la Salle [3] for all connected simple Lie groups of real rank $\ge 2$, also for their class $\mathcal{E}_{10}$ of Banach spaces and for groups with infinite center; de la Salle [6] reduced the Banach version, for groups and all their lattices, to the compact estimates (1.3)–(1.5) of his Theorem 1.5. He called the real case with all spaces of nontrivial type "the main open problem on the subject" [6, Introduction]; it is recorded as open again in [14]. In [7, Section 2.3] de la Salle conjectured the Hölder bounds for $T_\delta$ and $S_\theta$ for every space of nontrivial type.

Yao's preprint [21] proves the latitude estimate [21, Theorem 5.4] for every class of uniformly nontrivial type, and hence strong Banach property $(T)$ for $\mathrm{SL}_3(\mathbb{R})$ [21, Theorem 1.2]. Combined with [6, Theorem 1.5, second case, $n = 2$], Yao's estimate alone already gives Theorem B for every $G$ whose Lie algebra contains $\mathfrak{sl}_3(\mathbb{R})$, and for its lattices. *That part is not ours.* Every simple real Lie algebra of real rank at least two contains $\mathfrak{sl}_3(\mathbb{R})$ or $\mathfrak{sp}_4(\mathbb{R})$. The remaining algebras, those that contain $\mathfrak{sp}_4(\mathbb{R})$ but no $\mathfrak{sl}_3(\mathbb{R})$, are those of real rank two whose restricted root system is $B_2 = C_2$ or $BC_2$ (Proposition 7.8); they need the circle condition (1.3), which is Corollary 1.1. So the new content of this manuscript is Theorem A, and its consequence: Theorem B for the groups of type $B_2$ and $BC_2$ (all connected groups with these Lie algebras, including those with infinite center) and their lattices. In addition, the short proof of Theorem C in Section 6 gives a second proof of Yao's latitude estimate, and with it of his theorem for $\mathrm{SL}_3(\mathbb{R})$; its point is that in height and azimuth coordinates the latitude operator has no fold (the mixed derivative of the azimuthal shift is $(1-\delta^2)Q^{-3/2} > 0$). Moreover, to eliminate all unrefereed inputs and unwritten sketches, Section 2 gives a self-contained proof of Yao's base-$A$ digit-expansion and bilinear-shift bounds directly from Bourgain's theorem [2, Theorem 1], and Section 8 gives a complete proof of the two-step strong property $(T)$ argument for $\mathrm{Sp}_4(\mathbb{R})$, $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$, and all connected simple real Lie groups of higher rank (filling in [6, Section 3.3], eliminating the auxiliary reference to [5, Proposition 2.8], and repairing minor slips in [3, Lemma 6.5] and [6, Lemma 4.1]).

Several consequences of strong Banach property $(T)$ for uniformly convex spaces are already known by other methods, and *we do not claim them.* Bader, Furman, Gelander and Monod

conjectured that higher rank groups with finite center and their lattices have fixed points for affine isometric actions on superreflexive spaces [1, Conjecture 1.6]; de Laat and de la Salle proved this conjecture in 2023 [4, Theorems A and B], after Oppenheim's work on $\mathrm{SL}_n(\mathbb{Z})$ [19], and also obtained the super-expander consequence [4, Corollary C]. What strong Banach property $(T)$ adds beyond those fixed-point results is: Kazhdan projections and exponential convergence for representations of small exponential growth (not only isometric ones), the full range of nontrivial type (which contains some nonreflexive spaces), and groups with infinite center. Compared with the earlier results on strong property $(T)$ [3, 6], which hold for smaller classes of Banach spaces, the new point of Theorem B is the full range of uniformly nontrivial type.

### 1.4 The idea of the proof of Theorem A

Write a point of $\mathrm{SU}(2) \cong S^3 \subset \mathbb{C}^2$ by a height $y \in [-1, 1]$ and two angles $(t, u) \in \mathbb{T}^2$. Haar measure becomes $\frac{dy}{2}$ times the Haar measure of the torus, $P_R$ acts on the angle $t$ only, and $S_0$ becomes, for each output height $y$, an integral over the input height $x$ of translations of the torus by a vector $\pm\Gamma(x, y) \in \mathbb{R}^2$ (Lemma 3.2). So $S_0$ is a one-variable integral operator whose kernel takes values in the translations of the torus. Its "mixed curvature" is $\partial_x \partial_y \Gamma = \pm(1 - x^2 - y^2)^{-3/2}(\kappa, 1)$ with $\kappa = (x + y - 1)/2$. At an angular frequency $(k, n)$ the bilinear part of the phase is $\propto (n + \kappa k)xy$, and the finite Fourier bilinear-shift estimate (Proposition 2.10, proved in Section 2 from Bourgain's theorem via Yao's digit method) gives decay wherever $n + \kappa k$ is large. The phase degenerates on the fold $n + \kappa(x, y)k = 0$. There we keep the frequency cutoff $\chi\big((n + \kappa k)/B\big)$ in its form as an integral of translations of the torus. Because $\kappa$ depends linearly on the output height $y$, this integral contains a new bilinear translation, of size $R/B$, between the output height and the integration variable of the cutoff; Proposition 2.10 applies to it again (Lemma 4.4). The proof does not use Yao's parabolic partition [21, Lemma 4.4] or his Theorems 4.6 and 5.4.

### 1.5 Organization

Section 2 gives a self-contained proof of uniform finite Fourier decay, the base-$A$ digit-expansion estimates (after [21]), and their directional form on $\mathbb{T}^2$ directly from Bourgain's theorem [2]. Section 3 computes $S_0$ in the torus coordinates. Section 4 proves Theorem A, Section 5 proves Corollary 1.1, and Section 6 proves Theorem C. Section 7 introduces de la Salle's two-step property $(*_{\mathcal{E}})$ and the operator-defined class $\mathcal{D}$, establishes its stability and nontrivial type, classifies the new algebras of type $B_2$ and $BC_2$, and explains the reduction of Theorem B. Section 8 supplies the complete self-contained proof of $(*_{\mathcal{D}})$ for $\mathrm{Sp}_4(\mathbb{R})$, $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$, all connected simple real Lie groups of real rank at least two, finite products, and lattices, completing the proof of Theorem B. Section 9 contains remarks, Section 10 lists the published inputs that are not proved here, and Section 11 records the provenance, the checks, and the exact scope of the accompanying Lean 4 formalization.

**Acknowledgments and AI-assistance disclosure.** This work was directed, orchestrated, and reviewed by the author (Jack Ryan), with AI models used as research, auditing, and formalization assistants during the preparation of the manuscript: Claude (Anthropic), GPT-6.2 Astra and GPT-6.2 Sol (OpenAI), and Gemini 4 Neon (Google) collaborated in finding, writing, auditing, and partially formalizing the proofs. The accompanying Lean 4 project (`https://github.com/jackryan225/strong-banach-property-t`, `lean/`, 5,225 lines, 89 exported theorems, verified by `lake build` and `leanchecker` with zero `sorry` and standard axioms only) machine-checks the core Sections 3–5 engine parameterized over explicit structured hypotheses; Sections 2, 6, 7,

and 8 are paper proofs, and this manuscript has not yet been refereed by a human mathematician. Section 11 records the complete model provenance, scope, and check log.

## 2 Bourgain's finite Fourier estimate and a directional bound

Throughout, $1 < p \le 2$ and $1 \le M < \infty$ are fixed, and constants may depend on $p, M$ and on fixed cutoff functions, never on $E \in \mathcal{C}_{p,M}$, on $R$, or on the other running parameters unless stated otherwise. On the Schwartz space $\mathcal{S}(\mathbb{R})$ we use the Fourier and inverse Fourier transforms

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(s) e^{-is\xi}\, ds, \qquad \check{\varphi}(s) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\xi) e^{is\xi}\, d\xi.$$

For a complex Banach space $E$ and $\Delta > 0$, let $\mathcal{B}_\Delta(E)$ denote the space of all $f \in L^2(\mathbb{R}; E)$ whose distributional Fourier transform is supported in $[-\Delta, \Delta]$, meaning that

$$\int_{\mathbb{R}} f(s)\, \widehat{\varphi}(s)\, ds = 0 \qquad \text{for every } \varphi \in C_c^\infty(\mathbb{R} \setminus [-\Delta, \Delta]).$$

For each such $\varphi$ the map $f \mapsto \int_{\mathbb{R}} f(s)\widehat{\varphi}(s)\, ds$ is a bounded linear operator from $L^2(\mathbb{R}; E)$ to $E$ (of norm at most $\|\widehat{\varphi}\|_{L^2(\mathbb{R})}$), so $\mathcal{B}_\Delta(E)$ is a closed linear subspace of $L^2(\mathbb{R}; E)$.

The base-$A$ digit-expansion method below and the resulting bilinear shift estimates (Propositions 2.5 and 2.7, Corollary 2.6, and Lemma 2.8) are due to J. Yao [21, Lemma 2.3, Section 3, Corollary 3.6, and estimate (10)], adapting the non-Archimedean finite Fourier factorization of Lafforgue [13] and Liao [15]. To keep the present manuscript independent of unrefereed preprints, we give complete, self-contained proofs of all these statements from Bourgain's published Hausdorff–Young theorem [2, Theorem 1].

### 2.1 Stability of type and uniform finite Fourier decay

For an integer $A \ge 2$, equip $G_A = \mathbb{Z}/A\mathbb{Z}$ with normalized counting measure and write $\ell^2_A(E) = L^2(G_A; E)$, with norm $\|f\|_{\ell^2_A(E)} = \big(\frac{1}{A}\sum_{a=0}^{A-1} \|f(a)\|_E^2\big)^{1/2}$. Define the normalized finite Fourier transform $\mathcal{F}_A \otimes \mathrm{Id}_E : \ell^2_A(E) \to \ell^2_A(E)$ by

$$\big((\mathcal{F}_A \otimes \mathrm{Id}_E) f\big)(b) = \frac{1}{A} \sum_{a=0}^{A-1} e^{2\pi i ab/A} f(a) \qquad (b = 0, \dots, A-1),$$

and set $\rho_A(E) = \|\mathcal{F}_A \otimes \mathrm{Id}_E\|_{\ell^2_A(E) \to \ell^2_A(E)}$. For $E = \mathbb{C}$, Plancherel's identity on $G_A$ gives $\rho_A(\mathbb{C}) = A^{-1/2}$.

**Lemma 2.1** (after Yao [21, Lemma 2.3])**.** *Let $E$ be a complex Banach space, let $(\Omega, \mu)$ be a measure space, and let $V$ be a closed subspace of $L^2(\Omega; E)$.*

*(i)* $\rho_A(V) \le \rho_A(E)$ *for every integer $A \ge 2$.*

*(ii)* $T_p(V) \le T_p(E)$ *for every $p \in [1, 2]$; in particular, if $E \in \mathcal{C}_{p,M}$, then $V \in \mathcal{C}_{p,M}$.*

*(iii) For any sequence $(E_k)_{k \ge 1}$ of complex Banach spaces and $Y = \big(\bigoplus_{k=1}^\infty E_k\big)_{\ell^2}$, one has $T_p(Y) \le \sup_{k \ge 1} T_p(E_k)$ for every $p \in [1, 2]$, and $\rho_A(E_k) \le \rho_A(Y)$ for every $k \ge 1$ and every integer $A \ge 2$.*

*Proof.* We first verify (i) and (ii) for $L^2(\Omega;E)$. Let $f_0,\dots,f_{A-1}\in L^2(\Omega;E)$, fix strongly measurable representatives, and put $g_b=\frac{1}{A}\sum_{a=0}^{A-1}e^{2\pi iab/A}f_a\in L^2(\Omega;E)$ for $b=0,\dots,A-1$. For every $\omega\in\Omega$, the definition of $\rho_A(E)$ applied to the vector $(f_a(\omega))_{a=0}^{A-1}\in\ell^2_A(E)$ yields

$$\frac{1}{A}\sum_{b=0}^{A-1}\|g_b(\omega)\|_E^2\le\rho_A(E)^2\,\frac{1}{A}\sum_{a=0}^{A-1}\|f_a(\omega)\|_E^2.$$

Integrating over $\Omega$ and using finite linearity of the integral gives

$$\frac{1}{A}\sum_{b=0}^{A-1}\|g_b\|_{L^2(\Omega;E)}^2\le\rho_A(E)^2\,\frac{1}{A}\sum_{a=0}^{A-1}\|f_a\|_{L^2(\Omega;E)}^2,$$

so $\rho_A(L^2(\Omega;E))\le\rho_A(E)$.

Next let $p\in[1,2]$ with $T_p(E)<\infty$ (otherwise the bound is trivial), and let $f_1,\dots,f_n\in L^2(\Omega;E)$. Since $\mathbb{E}$ is a finite average over $\varepsilon\in\{\pm1\}^n$, finite linearity of the integral and (1.1) applied pointwise in $\omega\in\Omega$ give

$$\mathbb{E}\Big\|\sum_{j=1}^n\varepsilon_jf_j\Big\|_{L^2(\Omega;E)}^2=\int_\Omega\mathbb{E}\Big\|\sum_{j=1}^n\varepsilon_jf_j(\omega)\Big\|_E^2\,d\mu(\omega)\le T_p(E)^2\int_\Omega\Big(\sum_{j=1}^n\|f_j(\omega)\|_E^p\Big)^{2/p}d\mu(\omega).$$

Each function $u_j(\omega)=\|f_j(\omega)\|_E^p$ belongs to $L^{2/p}(\Omega)$ with $\|u_j\|_{L^{2/p}(\Omega)}=\|f_j\|_{L^2(\Omega;E)}^p$. Since $2/p\ge1$, Minkowski's inequality in $L^{2/p}(\Omega)$ gives

$$\Big(\int_\Omega\Big(\sum_{j=1}^n\|f_j(\omega)\|_E^p\Big)^{2/p}d\mu(\omega)\Big)^{p/2}\le\sum_{j=1}^n\|f_j\|_{L^2(\Omega;E)}^p.$$

Raising both sides to the $1/p$ power and combining with the preceding display shows that $T_p(L^2(\Omega;E))\le T_p(E)$. Since a closed subspace $V\subseteq L^2(\Omega;E)$ carries the inherited norm, restricting to elements of $V$ gives both (i) and (ii).

For (iii), put $T=\sup_{k\ge1}T_p(E_k)$; if $T=\infty$ the bound is trivial. Otherwise let $y^{(1)},\dots,y^{(n)}\in Y$ with $y^{(j)}=(y_k^{(j)})_{k\ge1}$. By finite linearity of the sum over $k$ and (1.1) in each $E_k$,

$$\mathbb{E}\Big\|\sum_{j=1}^n\varepsilon_jy^{(j)}\Big\|_Y^2=\sum_{k=1}^\infty\mathbb{E}\Big\|\sum_{j=1}^n\varepsilon_jy_k^{(j)}\Big\|_{E_k}^2\le T^2\sum_{k=1}^\infty\Big(\sum_{j=1}^n\|y_k^{(j)}\|_{E_k}^p\Big)^{2/p}.$$

Applying Minkowski's inequality in $\ell^{2/p}(\mathbb{N})$ as above gives $T_p(Y)\le T$. Finally, each $E_k$ embeds isometrically as a closed subspace of $Y$, so $\rho_A(E_k)\le\rho_A(Y)$. □

We now deduce the uniform finite Fourier decay bound on $\mathcal{C}_{p,M}$ directly from Bourgain's published Hausdorff–Young theorem for $B$-convex Banach spaces.

**Theorem 2.2** (uniform finite Fourier decay, from Bourgain [2, Theorem 1])**.** *Let $1<p\le2$ and $1\le M<\infty$. There exist constants $C_B>0$ and $\eta_0>0$, depending only on $p$ and $M$ (non-effective), such that*

$$\rho_A(E)\le C_B\,A^{-\eta_0}\qquad\text{for every integer } A\ge2 \text{ and every } E\in\mathcal{C}_{p,M}.$$

*Proof.* Suppose to the contrary that for every integer $k \geq 1$ there exist an integer $A_k \geq 2$ and a space $\widetilde{E}_k \in \mathcal{C}_{p,M}$ with $\rho_{A_k}(\widetilde{E}_k) > kA_k^{-1/k}$. Pick $f^{(k)} \in \ell^2_{A_k}(\widetilde{E}_k)$ witnessing $\|(\mathcal{F}_{A_k} \otimes \mathrm{Id}_{\widetilde{E}_k})f^{(k)}\| > kA_k^{-1/k}\|f^{(k)}\|$, and let $E_k \subseteq \widetilde{E}_k$ be the finite-dimensional subspace spanned by $\{f^{(k)}(0), \dots, f^{(k)}(A_k - 1)\}$. Since $E_k \subseteq \widetilde{E}_k$ is a subspace, $T_p(E_k) \leq T_p(\widetilde{E}_k) \leq M$ (Lemma 2.1(ii) on a one-point space), so $E_k \in \mathcal{C}_{p,M}$ and $\rho_{A_k}(E_k) > kA_k^{-1/k}$ (each $E_k$ is isometric to a norm on $\mathbb{C}^{d_k}$ with $d_k = \dim E_k$, so the sequence is chosen from a set).

Let $Y = \big(\bigoplus_{k=1}^{\infty} E_k\big)_{\ell^2}$. By Lemma 2.1(iii), $T_p(Y) \leq M$, so $Y$ has nontrivial Rademacher type $p > 1$, i.e., $Y$ is $B$-convex in the sense of Bourgain [2, p. 255] (equivalently, $Y$ does not contain $\ell^1_n$ uniformly [20]). By Bourgain's theorem [2, Theorem 1, equation (3)], there exist $r \in (1, 2]$ (replacing Bourgain's $\tilde{p} > 1$ by $\min\{\tilde{p}, 2\}$ if necessary) and $K < \infty$, depending only on $Y$, such that for every compact abelian group $G$ (with normalized Haar measure $dg$) and every finitely supported family $(y_\gamma)_{\gamma \in \widehat{G}}$ in $Y$,

$$\Big(\int_G \Big\|\sum_{\gamma \in \widehat{G}} y_\gamma\, \gamma(g)\Big\|_Y^2\, dg\Big)^{1/2} \leq K\Big(\sum_{\gamma \in \widehat{G}} \|y_\gamma\|_Y^r\Big)^{1/r}.$$

Let $r' = r/(r-1) \in [2, \infty)$ be the conjugate exponent. For any integer $A \geq 2$ and any $f \in \ell^2_A(Y)$, applying this inequality on $G = G_A$ with $y_a = f(a)$ $(0 \leq a < A)$ and using Hölder's inequality with $r \leq 2$ on the probability space $G_A$ yields

$$\begin{aligned} A\|(\mathcal{F}_A \otimes \mathrm{Id}_Y)f\|_{\ell^2_A(Y)} &= \Big(\frac{1}{A}\sum_{b=0}^{A-1}\Big\|\sum_{a=0}^{A-1} e^{2\pi i ab/A} f(a)\Big\|_Y^2\Big)^{1/2} \\ &\leq K\Big(\sum_{a=0}^{A-1}\|f(a)\|_Y^r\Big)^{1/r} = K\,A^{1/r}\Big(\frac{1}{A}\sum_{a=0}^{A-1}\|f(a)\|_Y^r\Big)^{1/r} \leq K\,A^{1/r}\|f\|_{\ell^2_A(Y)}. \end{aligned}$$

Dividing by $A$ gives $\rho_A(Y) \leq KA^{-(1-1/r)} = KA^{-1/r'}$ for all integers $A \geq 2$. By Lemma 2.1(iii), for every $k \geq 1$,

$$kA_k^{-1/k} < \rho_{A_k}(E_k) \leq \rho_{A_k}(Y) \leq KA_k^{-1/r'}.$$

Choosing $k \geq \max\{K, r'\}$ gives $k \geq K$ and (since $A_k \geq 2$) $A_k^{-1/k} \geq A_k^{-1/r'}$, a contradiction. $\square$

## 2.2 Exponential Schur multipliers and bandlimited translations

Before turning to the digit expansion, we establish two elementary analytic lemmas. The first replaces the projective tensor product machinery of [21, Section 3.1] by a direct power-series argument.

**Lemma 2.3** (exponential Schur multiplier)**.** *Let $(U, \mu_U)$ and $(V, \mu_V)$ be probability spaces, let $Z$ be a complex Banach space, and let $S \in \mathcal{L}\big(L^2(U; Z), L^2(V; Z)\big)$ be represented by an operator-valued kernel $K : V \times U \to \mathcal{L}(Z)$ in the sense that $\sup_{(v,u)\in V\times U}\|K(v,u)\|_{\mathcal{L}(Z)} \leq M_K < \infty$ and, for every $f \in L^2(U; Z)$ and $\mu_V$-a.e. $v \in V$, the map $u \mapsto K(v,u)f(u)$ is Bochner integrable on $U$ with*

$$(Sf)(v) = \int_U K(v,u)f(u)\, d\mu_U(u).$$

*Let $\alpha : V \to \mathbb{C}$ and $\beta : U \to \mathbb{C}$ be bounded measurable functions, and let $T \in \mathcal{L}(Z)$. Define $(M_\varphi g)(z) = \varphi(z)g(z)$ and $(T^n g)(v) = T^n(g(v))$, with the conventions $\alpha^0 = \beta^0 = 1$ and $T^0 = \mathrm{Id}_Z$.*

*Then the series*

$$\widetilde{S} = \sum_{n=0}^{\infty} \frac{1}{n!} M_{\alpha^n}\, T^n\, S\, M_{\beta^n}$$

*converges absolutely in* $\mathcal{L}\big(L^2(U;Z), L^2(V;Z)\big)$*, satisfies*

$$\|\widetilde{S}\|_{L^2(U;Z)\to L^2(V;Z)} \le \exp\big(\|\alpha\|_\infty \|\beta\|_\infty \|T\|_{\mathcal{L}(Z)}\big)\, \|S\|_{L^2(U;Z)\to L^2(V;Z)},$$

*and is represented in the same sense by the kernel* $\widetilde{K}(v,u) = e^{\alpha(v)\beta(u)^T} K(v,u)$*, which satisfies* $\sup_{(v,u)\in V\times U} \|\widetilde{K}(v,u)\|_{\mathcal{L}(Z)} \le \exp\big(\|\alpha\|_\infty \|\beta\|_\infty \|T\|_{\mathcal{L}(Z)}\big) M_K$*.*

*Proof.* Set $B = \|\alpha\|_\infty \|\beta\|_\infty \|T\|_{\mathcal{L}(Z)}$. For each $n \ge 0$, we have $\|M_{\alpha^n}\| \le \|\alpha\|_\infty^n$, $\|M_{\beta^n}\| \le \|\beta\|_\infty^n$, and $\|T^n\|_{L^2(V;Z)\to L^2(V;Z)} \le \|T\|_{\mathcal{L}(Z)}^n$, whence

$$\Big\| \frac{1}{n!} M_{\alpha^n} T^n S M_{\beta^n} \Big\|_{L^2(U;Z)\to L^2(V;Z)} \le \frac{B^n}{n!}\, \|S\|.$$

Summing over $n \ge 0$ gives absolute convergence in operator norm to $\widetilde{S}$ with $\|\widetilde{S}\| \le e^B \|S\|$. The uniform kernel bound $\|\widetilde{K}(v,u)\|_{\mathcal{L}(Z)} \le e^B M_K$ is immediate from $\|e^{\alpha(v)\beta(u)^T}\|_{\mathcal{L}(Z)} \le e^B$.

Fix $f \in L^2(U;Z)$; since $\mu_U$ is a probability measure, $f \in L^1(U;Z)$. For each $n \ge 0$, $\beta^n f \in L^2(U;Z)$, so there is a $\mu_V$-null set $N_n \subset V$ such that for all $v \in V \setminus N_n$, $u \mapsto K(v,u)\beta(u)^n f(u)$ is Bochner integrable on $U$ and $(SM_{\beta^n} f)(v) = \int_U K(v,u)\beta(u)^n f(u)\, d\mu_U(u)$. Since $T^n \in \mathcal{L}(Z)$ commutes with Bochner integration, the $n$-th term $g_{v,n}(u) = \frac{1}{n!}\alpha(v)^n \beta(u)^n T^n K(v,u) f(u)$ is Bochner integrable in $u$ for all $v \in V \setminus N_n$, with integral $\big(\frac{1}{n!} M_{\alpha^n} T^n S M_{\beta^n} f\big)(v)$. Put $N = \bigcup_{n=0}^\infty N_n$, which is $\mu_V$-null. For every $v \in V \setminus N$ and every $u \in U$, $\|g_{v,n}(u)\|_Z \le \frac{B^n}{n!} M_K \|f(u)\|_Z$, so the series $\sum_{n=0}^\infty g_{v,n}(u)$ converges in $Z$ to $\widetilde{K}(v,u) f(u)$, which is therefore strongly measurable in $u$, and $\sum_{n=0}^\infty \int_U \|g_{v,n}(u)\|_Z\, d\mu_U(u) \le e^B M_K \|f\|_{L^1(U;Z)} < \infty$. By dominated convergence for Bochner integrals, $u \mapsto \widetilde{K}(v,u) f(u)$ is Bochner integrable on $U$ and the partial sums $\widetilde{S}_m = \sum_{n=0}^m \frac{1}{n!} M_{\alpha^n} T^n S M_{\beta^n}$ satisfy $(\widetilde{S}_m f)(v) \to \int_U \widetilde{K}(v,u) f(u)\, d\mu_U(u)$ in $Z$ for every $v \in V \setminus N$. On the other hand $\widetilde{S}_m f \to \widetilde{S} f$ in $L^2(V;Z)$, so along a subsequence $(\widetilde{S}_{m_k} f)(v) \to (\widetilde{S} f)(v)$ for $\mu_V$-a.e. $v \in V$. Comparing the two limits gives the kernel representation. □

**Lemma 2.4** (Bernstein bound and translations on $\mathcal{B}_\Delta(E)$; after Yao [21, Lemma 3.4])**.** *There is an absolute constant* $C_\chi > 0$ *with the following property. For every complex Banach space* $E$ *and every* $\Delta > 0$*, the distributional derivative* $D = d/ds$ *restricts to a bounded linear operator on* $\mathcal{B}_\Delta(E)$ *satisfying* $\|D\|_{\mathcal{L}(\mathcal{B}_\Delta(E))} \le C_\chi \Delta$*. Moreover, for every* $h \in \mathbb{R}$*, the translation* $(\tau_h f)(s) = f(s+h)$ *preserves* $\mathcal{B}_\Delta(E)$ *as an isometry and satisfies*

$$\tau_h|_{\mathcal{B}_\Delta(E)} = e^{hD} = \sum_{n=0}^{\infty} \frac{h^n D^n}{n!},$$

*with convergence in operator norm on* $\mathcal{B}_\Delta(E)$*.*

*Proof.* Fix once and for all $\chi \in C_c^\infty(\mathbb{R})$ equal to 1 on $[-2,2]$, set $\kappa = \check{\chi} \in \mathcal{S}(\mathbb{R})$, and put $C_\chi = \|\kappa'\|_{L^1(\mathbb{R})}$. For $\Delta > 0$, set $\kappa_\Delta(s) = \Delta\kappa(\Delta s)$, so that $\widehat{\kappa_\Delta}(\xi) = \chi(\xi/\Delta)$, $\|\kappa_\Delta\|_{L^1(\mathbb{R})} = \|\kappa\|_{L^1(\mathbb{R})}$, and $\|\kappa_\Delta'\|_{L^1(\mathbb{R})} = \Delta\|\kappa'\|_{L^1(\mathbb{R})} = C_\chi\Delta$. For any $g \in L^1(\mathbb{R})$ and $f \in L^2(\mathbb{R};E)$, vector-valued Young's inequality (Minkowski's integral inequality for the Bochner integral) shows that $(g * f)(t) = \int_{\mathbb{R}} g(s) f(t-s)\, ds$ defines an element of $L^2(\mathbb{R};E)$ with $\|g * f\|_{L^2(\mathbb{R};E)} \le \|g\|_{L^1(\mathbb{R})} \|f\|_{L^2(\mathbb{R};E)}$.

We first show that $\kappa_\Delta * f = f$ in $L^2(\mathbb{R};E)$ for every $f \in \mathcal{B}_\Delta(E)$. Let $\varphi \in C_c^\infty(\mathbb{R})$. Since $\int_{\mathbb{R}}\int_{\mathbb{R}} |\kappa_\Delta(s)|\|f(t-s)\|_E|\widehat{\varphi}(t)|\,ds\,dt \le \|\kappa_\Delta\|_{L^1}\|f\|_{L^2}\|\widehat{\varphi}\|_{L^2} < \infty$, Fubini's theorem gives

$$\int_{\mathbb{R}} (\kappa_\Delta * f)(t)\,\widehat{\varphi}(t)\,dt = \int_{\mathbb{R}} f(u)\Big(\int_{\mathbb{R}} \kappa_\Delta(s)\widehat{\varphi}(u+s)\,ds\Big)\,du.$$

Writing $\widehat{\varphi}(u+s) = \int_{\mathbb{R}} \varphi(\xi)e^{-i(u+s)\xi}\,d\xi$ and applying Fubini's theorem again, the inner integral equals $\int_{\mathbb{R}} \varphi(\xi)\widehat{\kappa_\Delta}(\xi)e^{-iu\xi}\,d\xi = \widehat{\chi(\cdot/\Delta)}\varphi(u)$. Hence

$$\int_{\mathbb{R}} \big((\kappa_\Delta * f)(t) - f(t)\big)\,\widehat{\varphi}(t)\,dt = \int_{\mathbb{R}} f(u)\,(\widehat{\chi(\cdot/\Delta)} - 1)\varphi(u)\,du = 0,$$

because $\chi(\cdot/\Delta) - 1$ vanishes on $[-2\Delta, 2\Delta]$ and thus $(\chi(\cdot/\Delta)-1)\varphi \in C_c^\infty(\mathbb{R}\setminus[-\Delta,\Delta])$. Put $w = \kappa_\Delta * f - f \in L^2(\mathbb{R};E)$. Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and the scalar Fourier transform is an $L^2$-isomorphism, $\{\widehat{\varphi} : \varphi \in C_c^\infty(\mathbb{R})\}$ is dense in $L^2(\mathbb{R})$, whence $\int_{\mathbb{R}} w(t)\psi(t)\,dt = 0$ for every $\psi \in L^2(\mathbb{R})$, and in particular $\int_B w(t)\,dt = 0$ for every bounded Borel set $B \subset \mathbb{R}$. As $w$ is strongly measurable, it is essentially separably valued: after modifying $w$ on a null set, its image lies in a separable closed subspace $E_0 \subseteq E$. Choosing a countable norming sequence $(\Lambda_m)_{m\ge1}$ in the unit ball of $E_0^*$, we have $\int_B \Lambda_m(w(t))\,dt = 0$ for all bounded Borel sets $B$, so $\Lambda_m(w(t)) = 0$ for a.e. $t$ and every $m$; hence $w = 0$ in $L^2(\mathbb{R};E)$, i.e., $\kappa_\Delta * f = f$.

Next, for $f \in \mathcal{B}_\Delta(E)$, define $Df = \kappa'_\Delta * f \in L^2(\mathbb{R};E)$. For every $\psi \in \mathcal{S}(\mathbb{R})$, Fubini's theorem and integration by parts in $s$ yield

$$\begin{aligned}\int_{\mathbb{R}} (\kappa'_\Delta * f)(t)\,\psi(t)\,dt &= \int_{\mathbb{R}} f(u)\Big(\int_{\mathbb{R}} \kappa'_\Delta(s)\psi(u+s)\,ds\Big)\,du \\ &= -\int_{\mathbb{R}} f(u)\Big(\int_{\mathbb{R}} \kappa_\Delta(s)\psi'(u+s)\,ds\Big)\,du \\ &= -\int_{\mathbb{R}} (\kappa_\Delta * f)(t)\,\psi'(t)\,dt = -\int_{\mathbb{R}} f(t)\,\psi'(t)\,dt,\end{aligned}$$

so $Df$ is the distributional derivative of $f$. For $\varphi \in C_c^\infty(\mathbb{R}\setminus[-\Delta,\Delta])$, setting $\psi = \widehat{\varphi}$ gives $\psi'(t) = \widehat{-i\xi\varphi}(t)$ with $-i\xi\varphi \in C_c^\infty(\mathbb{R}\setminus[-\Delta,\Delta])$, so $\int_{\mathbb{R}}(Df)(t)\widehat{\varphi}(t)\,dt = -\int_{\mathbb{R}} f(t)\widehat{-i\xi\varphi}(t)\,dt = 0$; thus $Df \in \mathcal{B}_\Delta(E)$. By Young's inequality, $\|Df\|_{L^2(\mathbb{R};E)} \le \|\kappa'_\Delta\|_{L^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R};E)} = C_\chi\Delta\|f\|_{L^2(\mathbb{R};E)}$, so $D \in \mathcal{L}(\mathcal{B}_\Delta(E))$ with $\|D\| \le C_\chi\Delta$.

Now fix $h \in \mathbb{R}$ and $f \in \mathcal{B}_\Delta(E)$. Clearly $\tau_h$ is an isometry of $L^2(\mathbb{R};E)$. For $\varphi \in C_c^\infty(\mathbb{R}\setminus[-\Delta,\Delta])$, changing variables $u = t + h$ gives

$$\int_{\mathbb{R}} f(t+h)\widehat{\varphi}(t)\,dt = \int_{\mathbb{R}} f(u)\widehat{\varphi}(u-h)\,du = \int_{\mathbb{R}} f(u)\widehat{e^{ih\cdot}\varphi}(u)\,du = 0,$$

so $\tau_h f \in \mathcal{B}_\Delta(E)$. Direct substitution in the convolution integral gives $\tau_h(\kappa_\Delta * f) = (\tau_h\kappa_\Delta) * f$ and $\tau_h(\kappa'_\Delta * f) = (\tau_h\kappa'_\Delta) * f = \kappa'_\Delta * (\tau_h f) = D(\tau_h f)$. By Taylor's theorem with integral remainder, for every $t \in \mathbb{R}$ and $\sigma \ne 0$,

$$\frac{\kappa_\Delta(t+h+\sigma) - \kappa_\Delta(t+h)}{\sigma} - \kappa'_\Delta(t+h) = \sigma\int_0^1 (1-\theta)\,\kappa''_\Delta(t+h+\theta\sigma)\,d\theta.$$

Minkowski's integral inequality and translation invariance of $L^1(\mathbb{R})$ give

$$\Big\|\frac{\tau_{h+\sigma}\kappa_\Delta - \tau_h\kappa_\Delta}{\sigma} - \tau_h\kappa'_\Delta\Big\|_{L^1(\mathbb{R})} \le \frac{|\sigma|}{2}\|\kappa''_\Delta\|_{L^1(\mathbb{R})} \longrightarrow 0 \qquad (\sigma \to 0).$$

By Young's inequality, for every $f \in \mathcal{B}_\Delta(E)$ the map $h \mapsto \tau_h f$ is differentiable from $\mathbb{R}$ into $\mathcal{B}_\Delta(E)$ with $\frac{d}{dh}(\tau_h f) = (\tau_h \kappa'_\Delta) * f = D(\tau_h f)$. Since $D \in \mathcal{L}(\mathcal{B}_\Delta(E))$, the power series $e^{sD} = \sum_{n=0}^{\infty} \frac{s^n D^n}{n!}$ converges in operator norm for every $s \in \mathbb{R}$, and $h \mapsto e^{-hD}$ is differentiable in operator norm with derivative $-e^{-hD}D$. Hence $F(h) = e^{-hD}\tau_h f$ satisfies

$$F'(h) = -e^{-hD}D\tau_h f + e^{-hD}D\tau_h f = 0 \qquad (h \in \mathbb{R}).$$

For every $\Lambda \in \mathcal{B}_\Delta(E)^*$, $\frac{d}{dh}\Lambda(F(h)) = \Lambda(F'(h)) = 0$, so $\Lambda(F(h)) = \Lambda(F(0)) = \Lambda(f)$. By the Hahn–Banach theorem, $e^{-hD}\tau_h f = f$, hence $\tau_h f = e^{hD} f$ for all $h \in \mathbb{R}$ and $f \in \mathcal{B}_\Delta(E)$. □

### 2.3 Base-$A$ digit expansion and bilinear shift estimates

Fix integers $A \geq 2$ and $N \geq 1$, and set $\Lambda_N = 2\pi A^N$. For a complex Banach space $E$, $\Delta > 0$, and $\beta \in [0,1]$, set $Z = \mathcal{B}_\Delta(E)$. By Lemma 2.4, the map $h \mapsto \tau_h|_Z = e^{hD}$ is operator-norm continuous from $\mathbb{R}$ into $\mathcal{L}(Z)$ with $\|\tau_h|_Z\| = 1$. Hence for every $g \in L^2([0,1]; Z)$ and every $y \in [0,1]$, the map $x \mapsto e^{i\Lambda_N xy}\tau_{\beta xy} g(x)$ is strongly measurable and Bochner integrable from $[0,1]$ into $Z$, and

$$(\mathcal{T}_{N,\beta} g)(y) = \int_0^1 e^{i\Lambda_N xy}\, \tau_{\beta xy} g(x)\, dx$$

depends continuously on $y \in [0,1]$ in $Z$ (by dominated convergence), satisfies $\|(\mathcal{T}_{N,\beta} g)(y)\|_Z \leq \|g\|_{L^2([0,1];Z)}$, and hence defines a bounded operator $\mathcal{T}_{N,\beta}$ on $L^2([0,1]; Z)$. Under the canonical Fubini isometry $L^2([0,1]; L^2(\mathbb{R}; E)) \cong L^2([0,1] \times \mathbb{R}; E)$ (verified on finite linear combinations of simple functions $\mathbf{1}_I(x)\mathbf{1}_J(s)e$), $L^2([0,1]; Z)$ identifies with a closed subspace of $L^2([0,1] \times \mathbb{R}; E)$, and testing against $\varphi \in L^2(\mathbb{R})$ shows that

$$(\mathcal{T}_{N,\beta} g)(y,s) = \int_0^1 e^{i\Lambda_N xy} g(x, s + \beta xy)\, dx \qquad \text{for a.e. } (y,s) \in [0,1] \times \mathbb{R}.$$

**Proposition 2.5** (after Yao [21, Proposition 3.5])**.** *Let $E$ be a complex Banach space, let $A \geq 2$ and $N \geq 1$ be integers, and suppose that $\Delta > 0$, $\delta \geq 0$, and $\beta \in [0,1]$ satisfy $\beta\Delta \leq \delta\Lambda_N$. Then every $g \in L^2([0,1]; \mathcal{B}_\Delta(E))$ satisfies*

$$\|\mathcal{T}_{N,\beta} g\|_{L^2([0,1]\times\mathbb{R};E)} \leq C_0\big(C_1\, e^{C_2\delta A}\, \rho_A(E)\big)^N \|g\|_{L^2([0,1]\times\mathbb{R};E)},$$

*where $C_0 = e^{2\pi}$, $C_1 = e^{2\pi}$, and $C_2 = 3\pi C_\chi$ are absolute constants (with $C_\chi$ as in Lemma 2.4).*

*Proof.* Set $Z = \mathcal{B}_\Delta(E)$ and discard the endpoints $\{1\}$ of measure zero. Equip $U = (\mathbb{Z}/A\mathbb{Z})_a^N \times [0,1)_u$ and $V = (\mathbb{Z}/A\mathbb{Z})_b^N \times [0,1)_v$ with the product of normalized counting measure and Lebesgue measure. Identifying each residue class with its representative in $\{0, \dots, A-1\}$, the base-$A$ expansions

$$x(a,u) = \sum_{i=1}^{N} a_i A^{-i} + A^{-N}u, \qquad y(b,v) = \sum_{j=1}^{N} b_j A^{-j} + A^{-N}v \tag{2.1}$$

give measure-preserving bijections $U \to [0,1)$ and $V \to [0,1)$ (indeed $\lfloor A^N x \rfloor = \sum_{i=1}^{N} a_i A^{N-i}$ and $u = A^N x - \lfloor A^N x \rfloor$). Thus the pullback maps $(\mathcal{U}_N g)(a,u) = g(x(a,u))$ and $(\mathcal{V}_N g)(b,v) = g(y(b,v))$ are isometric isomorphisms $L^2([0,1); Z) \to L^2(U; Z)$ and $L^2([0,1); Z) \to L^2(V; Z)$, and for every $F \in L^1([0,1); Z)$ one has $\int_0^1 F(x)\, dx = A^{-N} \sum_{a \in (\mathbb{Z}/A\mathbb{Z})^N} \int_0^1 F(x(a,u))\, du$.

For $j = 1, \dots, N$, set $i_j = N + 1 - j$,

$$m_j(b) = \sum_{k=1}^{j-1} b_k A^{j-1-k} \in \mathbb{Z}_{\geq 0} \quad (m_1 = 0), \qquad \theta_j(b, v) = \sum_{k=j+1}^{N} b_k A^{j-k} + v A^{j-N}.$$

Since $0 \leq b_k \leq A - 1$ and $0 \leq v < 1$, we have $0 \leq \theta_j(b, v) < (A-1) \sum_{k=j+1}^{N} A^{j-k} + A^{j-N} = 1$ (with $\theta_N(b, v) = v$), so also $0 \leq (b_j + \theta_j(b, v))/A < 1$, and

$$A^{j-1} y = m_j + \frac{b_j + \theta_j}{A} \qquad (1 \leq j \leq N).$$

Writing $A^N xy = \sum_{i=1}^{N} a_i A^{N-i} y + uy = \sum_{j=1}^{N} a_{i_j} A^{j-1} y + uy$ therefore yields

$$A^N xy = \sum_{j=1}^{N} a_{i_j} m_j + \sum_{j=1}^{N} \frac{a_{i_j} b_j}{A} + \sum_{j=1}^{N} \theta_j \frac{a_{i_j}}{A} + yu.$$

Put

$$h_0(a, b) = A^{-N} \sum_{j=1}^{N} a_{i_j} m_j(b), \qquad h_1\big((b, v), (a, u)\big) = A^{1-N} \sum_{j=1}^{N} \frac{b_j + \theta_j(b, v)}{A} \frac{a_{i_j}}{A} + A^{-N} y(b, v) u, \tag{2.2}$$

and

$$\Phi\big((b, v), (a, u)\big) = 2\pi \Big( \sum_{j=1}^{N} \theta_j(b, v) \frac{a_{i_j}}{A} + y(b, v) u \Big).$$

Then $xy = h_0 + h_1$, and since $\sum_{j=1}^{N} a_{i_j} m_j \in \mathbb{Z}$, we have

$$e^{i\Lambda_N xy} = \exp\Big( 2\pi i \sum_{j=1}^{N} \frac{a_{i_j} b_j}{A} \Big)\, e^{i\Phi}.$$

We next factor the finite Fourier core. For $0 \leq j \leq N$ define

$$\mathcal{X}_j = L^2\big( (\mathbb{Z}/A\mathbb{Z})_b^j \times (\mathbb{Z}/A\mathbb{Z})_a^{N-j} \times [0, 1)_u ; Z \big),$$

so $\mathcal{X}_0 = L^2(U; Z)$, and let $Y_N = L^2(V; Z)$. For $1 \leq j \leq N$, note that $N - (j-1) = i_j$ and $m_j(b)$ depends only on $b_1, \dots, b_{j-1}$. Define $D_j : \mathcal{X}_{j-1} \to \mathcal{X}_{j-1}$ and $F_j : \mathcal{X}_{j-1} \to \mathcal{X}_j$ by

$$(D_j f)(b_1, \dots, b_{j-1}, a_1, \dots, a_{i_j}, u) = \tau_{\beta a_{i_j} A^{-N} m_j(b)} \big( f(b_1, \dots, b_{j-1}, a_1, \dots, a_{i_j}, u) \big),$$

$$(F_j f)(b_1, \dots, b_j, a_1, \dots, a_{i_j - 1}, u) = \frac{1}{A} \sum_{d=0}^{A-1} e^{2\pi i d b_j / A} f(b_1, \dots, b_{j-1}, a_1, \dots, a_{i_j - 1}, d, u).$$

Since each $\tau_s|_Z$ is an isometry of $Z$, $D_j$ is an isometry of $\mathcal{X}_{j-1}$. For each fixed tuple $(b_1, \dots, b_{j-1}, a_1, \dots, a_{i_j - 1}, u)$, $F_j$ acts as $\mathcal{F}_A \otimes \mathrm{Id}_Z$ in the variable $a_{i_j} \mapsto b_j$; averaging over the remaining discrete variables and integrating over $u \in [0, 1)$ gives $\|F_j\| \leq \rho_A(Z) \leq \rho_A(E)$, where the last step is Lemma 2.1(i) applied to the closed subspace $Z = \mathcal{B}_\Delta(E) \subseteq L^2(\mathbb{R}; E)$. Define $J : \mathcal{X}_N \to Y_N$ by $(Jf)(b, v) = \int_0^1 f(b, u)\, du$; by Cauchy–Schwarz, $\|J\| \leq 1$. Set

$$S_N = J F_N D_N \cdots F_1 D_1 : L^2(U; Z) \longrightarrow L^2(V; Z).$$

Then

$$\|S_N\|_{L^2(U;Z)\to L^2(V;Z)} \le \|J\| \prod_{j=1}^{N} \|F_j\|\|D_j\| \le \rho_A(E)^N. \tag{2.3}$$

Since each $m_k(b)$ depends only on $b_1, \dots, b_{k-1}$ and the translations $\tau_s$ commute and satisfy $\tau_s\tau_t = \tau_{s+t}$, induction on $k = 1, \dots, N$ shows that

$$\begin{aligned}(F_k D_k \cdots F_1 D_1 f)(b_1, \dots, b_k, a_1, \dots, a_{N-k}, u) \\ = \frac{1}{A^k} \sum_{a_{N-k+1},\dots,a_N=0}^{A-1} \exp\Big(2\pi i \sum_{j=1}^{k} \frac{a_{i_j} b_j}{A}\Big)\, \tau_{\beta A^{-N} \sum_{j=1}^{k} a_{i_j} m_j(b)} f(a,u).\end{aligned}$$

Taking $k = N$ and applying $J$ shows that $S_N$ is represented by the kernel

$$K_{S_N}\big((b,v),(a,u)\big) = \exp\Big(2\pi i \sum_{j=1}^{N} \frac{a_{i_j} b_j}{A}\Big)\, \tau_{\beta h_0(a,b)} \in \mathcal{L}(Z),$$

which takes only finitely many values and satisfies $\|K_{S_N}((b,v),(a,u))\|_{\mathcal{L}(Z)} = 1$ everywhere.

We now multiply $K_{S_N}$ by the remaining phase and translation factors by applying Lemma 2.3 sequentially $2(N+1)$ times:

- First, apply Lemma 2.3 $N$ times with $\alpha = \theta_j$, $\beta_U = a_{i_j}/A$, and $T = 2\pi i \,\mathrm{Id}_Z$ $(1 \le j \le N)$, and once with $\alpha = y$, $\beta_U = u$, and $T = 2\pi i \,\mathrm{Id}_Z$. Every scalar function has $L^\infty$ norm at most 1 and $\|T\|_{\mathcal{L}(Z)} = 2\pi$, so these $N+1$ steps multiply the operator norm bound by at most $e^{2\pi(N+1)}$.

- Next, with $D \in \mathcal{L}(Z)$ from Lemma 2.4 ($\|D\|_{\mathcal{L}(Z)} \le C_\chi \Delta$), apply Lemma 2.3 $N$ times with $\alpha = (b_j + \theta_j)/A$, $\beta_U = a_{i_j}/A$, and $T = \beta A^{1-N} D$ $(1 \le j \le N)$, and once with $\alpha = y$, $\beta_U = u$, and $T = \beta A^{-N} D$. Again all scalar factors have $L^\infty$ norm at most 1, so these $N+1$ steps multiply the operator norm bound by at most

$$\exp\Big(\beta(NA^{1-N} + A^{-N})\|D\|_{\mathcal{L}(Z)}\Big) \le \exp\big(C_\chi \beta \Delta (NA^{1-N} + A^{-N})\big).$$

Because all scalar multiples of $\mathrm{Id}_Z$ and all operators of the form $e^{sD}$ commute with one another, the resulting operator $\widetilde{S}_N \in \mathcal{L}\big(L^2(U;Z), L^2(V;Z)\big)$ is represented by the kernel

$$\widetilde{K}\big((b,v),(a,u)\big) = e^{i\Phi}\, e^{\beta h_1 D}\, K_{S_N}\big((b,v),(a,u)\big) = e^{i\Lambda_N xy}\, \tau_{\beta(h_0+h_1)} = e^{i\Lambda_N xy}\, \tau_{\beta xy},$$

where we used $e^{\beta h_1 D}\tau_{\beta h_0} = \tau_{\beta h_1}\tau_{\beta h_0} = \tau_{\beta xy}$ from Lemma 2.4. By the change-of-variables formula for (2.1), for every $g \in L^2([0,1);Z)$ and a.e. $(b,v) \in V$ we have $(\widetilde{S}_N \mathcal{U}_N g)(b,v) = (\mathcal{V}_N \mathcal{T}_{N,\beta} g)(b,v)$, hence $\mathcal{V}_N \mathcal{T}_{N,\beta} \mathcal{U}_N^{-1} = \widetilde{S}_N$.

Using $\beta\Delta \le \delta\Lambda_N = 2\pi\delta A^N$ and the inequality $NA + 1 \le \frac{3}{2} NA$ (valid since $A \ge 2$ and $N \ge 1$), we have

$$C_\chi \beta \Delta (NA^{1-N} + A^{-N}) \le 2\pi C_\chi \delta (NA + 1) \le 3\pi C_\chi \delta A N.$$

Since $\mathcal{U}_N$ and $\mathcal{V}_N$ are isometries, combining (2.3) with the $2(N+1)$ applications of Lemma 2.3 yields

$$\|\mathcal{T}_{N,\beta}\|_{L^2([0,1];Z)\to L^2([0,1];Z)} = \|\widetilde{S}_N\| \le e^{2\pi(N+1)}\, e^{3\pi C_\chi \delta A N}\, \rho_A(E)^N = e^{2\pi}\big(e^{2\pi} e^{3\pi C_\chi \delta A} \rho_A(E)\big)^N. \ \square$$

**Corollary 2.6** (after Yao [21, Corollary 3.6]). *Let $1 < p \le 2$ and $1 \le M < \infty$. There exist an integer $A \ge 2$ and constants $\delta > 0$, $q \in (0,1)$, and $C_* \ge 1$, depending only on $p$ and $M$ (non-effective), such that $\varepsilon = \log(1/q)/\log A$ satisfies $0 < \varepsilon < \frac{1}{2}$ and, for every $E \in \mathcal{C}_{p,M}$, $N \ge 1$, $\beta \in [0,1]$, and $\Delta > 0$ with $\beta\Delta \le \delta\Lambda_N$,*

$$\|\mathcal{T}_{N,\beta} g\|_{L^2([0,1]\times\mathbb{R};E)} \le C_* q^N \|g\|_{L^2([0,1]\times\mathbb{R};E)} \qquad \big(g \in L^2([0,1];\mathcal{B}_\Delta(E))\big).$$

*Proof.* Let $C_0, C_1, C_2$ be the absolute constants of Proposition 2.5 and let $C_B, \eta_0 > 0$ be the constants of Theorem 2.2. Choose an integer $A \ge 2$ large enough that $C_1 C_B A^{-\eta_0} \le \frac{1}{4}$, then choose $\delta > 0$ small enough that $e^{C_2\delta A} \le 2$, and set $q = \frac{1}{2}$ and $C_* = C_0 = e^{2\pi} \ge 1$. For every $E \in \mathcal{C}_{p,M}$, Proposition 2.5 and Theorem 2.2 give $C_1 e^{C_2\delta A}\rho_A(E) \le 2C_1 C_B A^{-\eta_0} \le \frac{1}{2} = q$, proving the operator bound. Moreover, since $T_p(\mathbb{C}) = 1 \le M$ we have $\mathbb{C} \in \mathcal{C}_{p,M}$, so $C_1 A^{-1/2} = C_1\rho_A(\mathbb{C}) \le C_1 C_B A^{-\eta_0} \le \frac{1}{4}$, whence $A \ge 16C_1^2 = 16e^{4\pi} > 4$ and therefore $\varepsilon = \log 2/\log A < \frac{1}{2}$. □

**Proposition 2.7** (after Yao [21, estimate (10) in the proof of Theorem 3.7]). *Let $A, \delta, q, C_*, \varepsilon$ be as in Corollary 2.6. For $r \in \mathbb{R}\setminus\{0\}$, define*

$$(\mathsf{A}_r g)(y,s) = \int_0^1 e^{irxy} g(x, s+xy)\,dx.$$

*For every $a > 0$, every $E \in \mathcal{C}_{p,M}$, every $H \ge 1$, every $r \in \mathbb{R}$ with $|r| \ge aH$, every $\Delta \in (0, \delta|r|]$, and every $g \in L^2([0,1];\mathcal{B}_\Delta(E))$,*

$$\|\mathsf{A}_r g\|_{L^2([0,1]\times\mathbb{R};E)} \le \sqrt{2}\,C_*\Big(\frac{2\pi A}{a}\Big)^{\varepsilon} H^{-\varepsilon}\|g\|_{L^2([0,1]\times\mathbb{R};E)}.$$

*Proof.* All norms below are in $L^2([0,1]\times\mathbb{R};E)$. First, since translations are isometries of $L^2(\mathbb{R};E)$, Minkowski's integral inequality and Cauchy–Schwarz give the trivial bound

$$\|\mathsf{A}_r g\|^2 = \int_0^1 \Big\| \int_0^1 e^{irxy}\tau_{xy} g(x)\,dx\Big\|^2_{L^2(\mathbb{R};E)}\,dy \le \int_0^1 \Big(\int_0^1 \|g(x)\|_{L^2(\mathbb{R};E)}\,dx\Big)^2 dy \le \|g\|^2.$$

Next we reduce $r < 0$ to $r > 0$. If $r = -|r| < 0$, set $\widetilde{g}(x,s) = g(1-x,-s)$. Then $\|\widetilde{g}\| = \|g\|$, and for a.e. $x \in [0,1]$ and every $\varphi \in C_c^\infty(\mathbb{R}\setminus[-\Delta,\Delta])$, putting $\widetilde{\varphi}(\xi) = \varphi(-\xi) \in C_c^\infty(\mathbb{R}\setminus[-\Delta,\Delta])$ gives $\int_{\mathbb{R}} \widetilde{g}(x,s)\widehat{\varphi}(s)\,ds = \int_{\mathbb{R}} g(1-x,u)\widehat{\widetilde{\varphi}}(u)\,du = 0$, so $\widetilde{g} \in L^2([0,1];\mathcal{B}_\Delta(E))$. Substituting $x = 1-u$ yields

$$(\mathsf{A}_{-|r|}g)(y,-s) = \int_0^1 e^{-i|r|(1-u)y} g\big(1-u,-(s-y+uy)\big)\,du = e^{-i|r|y}(\mathsf{A}_{|r|}\widetilde{g})(y,s-y).$$

Since $s \mapsto -s$ and $s \mapsto s-y$ preserve Lebesgue measure on $\mathbb{R}$ and $|e^{-i|r|y}| = 1$, we obtain $\|\mathsf{A}_{-|r|}g\| = \|\mathsf{A}_{|r|}\widetilde{g}\|$.

Assume therefore that $r > 0$. If $r < 2\pi A$, then $aH \le r < 2\pi A$ implies $H < 2\pi A/a$, so $(2\pi A/a)^{\varepsilon} H^{-\varepsilon} > 1$. Since $C_* \ge 1$, the trivial bound gives

$$\|\mathsf{A}_r g\| \le \|g\| \le \sqrt{2}\,C_*\Big(\frac{2\pi A}{a}\Big)^{\varepsilon} H^{-\varepsilon}\|g\|.$$

Now suppose $r \ge 2\pi A = \Lambda_1$, and choose the unique integer $N \ge 1$ such that $\Lambda_N \le r < A\Lambda_N$. Put $\beta = \Lambda_N/r \in (A^{-1}, 1]$ and $\lambda = 1/\beta = r/\Lambda_N \in [1, A)$. Then $\beta\Delta \le \beta\delta r = \delta\Lambda_N$. Substituting $x = \beta u$ gives

$$(\mathsf{A}_r g)(y,s) = \beta\int_0^\lambda e^{i\Lambda_N uy} g(\beta u, s+\beta uy)\,du.$$

Set $m = \lceil \lambda \rceil$, and for $1 \le j \le m$ define $g_j \in L^2([0,1]; \mathcal{B}_\Delta(E))$ by

$$g_j(v,s) = \begin{cases} g\big(\beta(j-1+v), s\big), & j-1+v \le \lambda, \\ 0, & j-1+v > \lambda. \end{cases}$$

Splitting $[0,\lambda]$ into $[j-1,j] \cap [0,\lambda]$ and setting $u = j-1+v$ on the $j$-th piece gives

$$(\mathsf{A}_r g)(y,s) = \beta \sum_{j=1}^{m} e^{i\Lambda_N (j-1)y} \, (\mathcal{T}_{N,\beta} g_j)\big(y, s + \beta(j-1)y\big),$$

while changing variables gives

$$\sum_{j=1}^{m} \|g_j\|^2 = \int_0^{\lambda} \|g(\beta u)\|_{L^2(\mathbb{R};E)}^2 \, du = \lambda \|g\|^2.$$

Since $|e^{i\Lambda_N (j-1)y}| = 1$ and translations in $s$ preserve the norm, the triangle inequality, Corollary 2.6, and Cauchy–Schwarz give

$$\|\mathsf{A}_r g\| \le \beta \sum_{j=1}^{m} \|\mathcal{T}_{N,\beta} g_j\| \le \beta \, C_* q^N \sum_{j=1}^{m} \|g_j\| \le \beta \, C_* q^N \sqrt{m} \Big( \sum_{j=1}^{m} \|g_j\|^2 \Big)^{1/2} = C_* q^N \sqrt{\frac{m}{\lambda}} \, \|g\|.$$

Since $\lambda \ge 1$, we have $m < \lambda + 1 \le 2\lambda$, so $\sqrt{m/\lambda} \le \sqrt{2}$. Finally, $q = A^{-\varepsilon}$ and $r < 2\pi A^{N+1}$ imply $q^N = A^{-N\varepsilon} \le (2\pi A/r)^{\varepsilon} \le (2\pi A/a)^{\varepsilon} H^{-\varepsilon}$, which completes the proof. □

**Lemma 2.8** (frequency-localized bilinear shift bound; after Yao [21, Section 3])**.** *Let $A, \delta, q, \varepsilon$ be as in Corollary 2.6 (so $0 < \varepsilon < \frac{1}{2}$). For $0 < a < b$ there is $C$ depending only on $p, M, a, b$ such that for every $E \in \mathcal{C}_{p,M}$, every $H \ge 1$, and every $G \in L^2([0,1] \times \mathbb{R}; E)$ whose distributional Fourier transform in the second variable is supported, for a.e. $x \in [0,1]$, in $\{aH \le |\xi| \le bH\}$, one has*

$$\Big\| \int_0^1 G(x, s + xy) \, dx \Big\|_{L^2([0,1]_y \times \mathbb{R}_s; E)} \le C \, H^{-\varepsilon} \|G\|_{L^2([0,1]\times\mathbb{R};E)}.$$

*Proof.* Fix $0 < \theta < \min\{\delta, 1/2\}$. Cover the compact annulus $\{a \le |\xi| \le b\}$ by finitely many open intervals $I_\nu = (r_\nu - \theta|r_\nu|, r_\nu + \theta|r_\nu|)$ with $a \le |r_\nu| \le b$, and choose a finite family of functions $\chi_\nu \in C_c^\infty(I_\nu)$ whose sum $\chi = \sum_\nu \chi_\nu$ equals 1 on an open neighbourhood of $\{a \le |\xi| \le b\}$. These choices depend only on $a, b, \theta$. Set $k_\nu = \check{\chi}_\nu \in \mathcal{S}(\mathbb{R})$ and $k_{\nu,H}(s) = H k_\nu(Hs)$, so that $\|k_{\nu,H}\|_{L^1(\mathbb{R})} = \|k_\nu\|_{L^1(\mathbb{R})}$ is independent of $H$. Since convolution with $k_{\nu,H} \in L^1(\mathbb{R})$ is a bounded operator on $L^2(\mathbb{R}; E)$ of norm at most $\|k_\nu\|_{L^1(\mathbb{R})}$ by Young's inequality, applying it fiberwise in $x \in [0,1]$ under the Fubini identification $L^2([0,1] \times \mathbb{R}; E) \cong L^2([0,1]; L^2(\mathbb{R}; E))$ (as above) defines $G_\nu \in L^2([0,1] \times \mathbb{R}; E)$ by $G_\nu(x, \cdot) = k_{\nu,H} * G(x, \cdot)$ for a.e. $x \in [0,1]$, with $\|G_\nu\|_{L^2([0,1]\times\mathbb{R};E)} \le \|k_\nu\|_{L^1(\mathbb{R})} \|G\|_{L^2([0,1]\times\mathbb{R};E)}$.

As in the proof of Lemma 2.4, for every $x \in [0,1]$ at which $G(x,\cdot) \in L^2(\mathbb{R}; E)$ has distributional Fourier transform supported in $\{aH \le |\xi| \le bH\}$ (which holds for a.e. $x$) and every $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \Big( \sum_\nu G_\nu(x,s) - G(x,s) \Big) \widehat{\varphi}(s) \, ds = \int_{\mathbb{R}} G(x,u) \, (\chi(\cdot/H)\widehat{\ } - 1)\varphi(u) \, du = 0,$$

because $\chi(\cdot/H) - 1$ vanishes on a neighbourhood of $\{aH \le |\xi| \le bH\}$. By the same Plancherel density and countable-norming argument as in Lemma 2.4, $G = \sum_\nu G_\nu$ in $L^2([0,1] \times \mathbb{R}; E)$.

For each $\nu$, set $r = Hr_\nu$ (so $aH \le |r| \le bH$) and $\Delta = \theta|r| \le \delta|r|$, and put $g_\nu(x,s) = e^{-irs}G_\nu(x,s)$. Then $\|g_\nu\| = \|G_\nu\|$. For every $x \in [0,1]$ with $G(x,\cdot) \in L^2(\mathbb{R};E)$ and every $\varphi \in C_c^\infty(\mathbb{R}\setminus[-\Delta,\Delta])$, using $e^{-irs}\widehat{\varphi}(s) = \widehat{\varphi(\cdot - r)}(s)$ gives

$$\int_{\mathbb{R}} g_\nu(x,s)\widehat{\varphi}(s)\,ds = \int_{\mathbb{R}} G_\nu(x,s)\widehat{\varphi(\cdot-r)}(s)\,ds = \int_{\mathbb{R}} G(x,u)\,\chi_\nu(\cdot/\widehat{H)\varphi(}\cdot - r)(u)\,du = 0,$$

because $\operatorname{supp}\chi_\nu(\cdot/H) \subseteq (r-\Delta, r+\Delta)$ while $\varphi(\cdot - r)$ vanishes on $[r-\Delta, r+\Delta]$. Thus $g_\nu \in L^2([0,1];\mathcal{B}_\Delta(E))$. Finally, $G_\nu(x,s+xy) = e^{irs}e^{irxy}g_\nu(x,s+xy)$ gives $\int_0^1 G_\nu(x,s+xy)\,dx = e^{irs}(\mathsf{A}_r g_\nu)(y,s)$. Applying Proposition 2.7 to each $\nu$ and summing over the finite index set yields the claim. □

## 2.4 Translations of the torus and the directional estimate

Outside the proofs of Lemma 2.1(ii) and Lemma 2.8 above, the type of $E$ enters Sections 2–6 only through Lemma 2.8 and Lemma 2.1(ii). We now transfer Lemma 2.8 to translations on the torus.

Let $X = L^2(\mathbb{T}^2;E)$ with the probability Haar measure of $\mathbb{T}^2$; by Lemma 2.1(ii), $X \in \mathcal{C}_{p,M}$. For $w \in \mathbb{R}^2$ let $\tau_w$ be the translation $(\tau_w g)(t,u) = g((t,u)+w)$, an isometry of $X$, and for $v \in \mathbb{R}^2$ and $\sigma \in \mathbb{R}$ put $U_v(\sigma) = \tau_{\sigma v}$. Write $e_l(t,u) = e^{i(l_1 t + l_2 u)}$ for $l = (l_1,l_2) \in \mathbb{Z}^2$. A *trigonometric polynomial* in $X$ is a finite sum $g = \sum_l c_l\, e_l$ with $c_l \in E$; its frequencies are the $l$ with $c_l \neq 0$. Such polynomials are dense in $X$ (Fejér means). For $\varphi \in \mathcal{S}(\mathbb{R})$ put $h_\varphi(s) = \frac{1}{2\pi}\widehat{\varphi}(s) \in \mathcal{S}(\mathbb{R})$, so that $\varphi(\sigma) = \int_{\mathbb{R}} h_\varphi(s)e^{is\sigma}\,ds$ by Fourier inversion. For $v \in \mathbb{R}^2$ and $H > 0$, define the Fourier multiplier $\varphi(D_v/H)$ on trigonometric polynomials by $e_l \mapsto \varphi(v\cdot l/H)e_l$.

**Lemma 2.9.** *(i) $\varphi(D_v/H)$ extends uniquely to a bounded operator on $X$, given by*

$$\varphi(D_v/H)g = \int_{\mathbb{R}} h_\varphi(s)\,U_v(s/H)g\,ds,$$

*and satisfies $\|\varphi(D_v/H)\|_{X\to X} \le \|h_\varphi\|_{L^1(\mathbb{R})}$ for every $v \in \mathbb{R}^2$ and $H > 0$.*

*(ii) Let $c > 0$ and $K > 0$. If $g \in X$ is a trigonometric polynomial with frequencies in $\{|l_1|+|l_2| \le cK\}$, then $\|\partial_t g\|_X + \|\partial_u g\|_X \le C_c K\|g\|_X$, and $\|\tau_w g - \tau_{w'}g\|_X \le C_c K|w-w'|\|g\|_X$ for all $w, w' \in \mathbb{R}^2$.*

*Proof.* (i) Since translations act strongly continuously on $X$ (by density of $C(\mathbb{T}^2;E)$) and isometrically, the Bochner integral $g \mapsto \int_{\mathbb{R}} h_\varphi(s)U_v(s/H)g\,ds$ defines a bounded operator on $X$ of norm at most $\|h_\varphi\|_{L^1(\mathbb{R})}$. On each mode $e_l$, $U_v(s/H)e_l = e^{is(v\cdot l)/H}e_l$, so the integral evaluates to $\varphi(v\cdot l/H)e_l$; density of trigonometric polynomials gives the claim.

(ii) Fix $\phi \in C_c^\infty(\mathbb{R})$ equal to 1 on $[-c,c]$ and put $\varphi_1(\sigma) = i\sigma\phi(\sigma)$. For $g = \sum_l c_l e_l$ with $|l_1|+|l_2| \le cK$, we have $\phi(l_1/K) = \phi(l_2/K) = 1$, so $\partial_t g = K\,\varphi_1(D_{(1,0)}/K)g$ and $\partial_u g = K\,\varphi_1(D_{(0,1)}/K)g$; apply (i) with $C_c = 2\|h_{\varphi_1}\|_{L^1(\mathbb{R})}$. For $w, w' \in \mathbb{R}^2$, differentiating the finite sum $r \mapsto \tau_{w'+r(w-w')}g$ yields

$$\tau_w g - \tau_{w'}g = \int_0^1 \tau_{w'+r(w-w')}\big((w_1-w_1')\partial_t g + (w_2-w_2')\partial_u g\big)\,dr,$$

and taking norms gives $\|\tau_w g - \tau_{w'}g\|_X \le |w-w'|(\|\partial_t g\|_X + \|\partial_u g\|_X) \le C_c K|w-w'|\|g\|_X$. □

**Proposition 2.10** (directional form of the bilinear shift estimate)**.** *Let $0 < a_0 < b_0$, and let $\varepsilon$ be as in Corollary 2.6. There is $C$ depending only on $p, M, a_0, b_0$ with the following property. Let $E \in \mathcal{C}_{p,M}$, $v \in \mathbb{R}^2$, $H > 0$, and let $F \in L^2([0,1]; X)$ take values in the trigonometric polynomials with frequencies in a finite set $S \subset \mathbb{Z}^2$ such that $a_0 H \le |v \cdot l| \le b_0 H$ for all $l \in S$. Then*

$$\Big\| \int_0^1 U_v(uw) F(u)\, du \Big\|_{L^2([0,1]_w; X)} \le C(1+H)^{-\varepsilon} \|F\|_{L^2([0,1];X)}.$$

*Proof.* Write $F(u) = \sum_{l \in S} c_l(u) e_l$; each coefficient $c_l(u) = \int_{\mathbb{T}^2} F(u)(\theta_1, \theta_2) e^{-i(l_1\theta_1 + l_2\theta_2)} \frac{d\theta_1\, d\theta_2}{4\pi^2}$ lies in $L^2([0,1]; E)$, so $(u,w) \mapsto U_v(uw)F(u) = \sum_{l \in S} e^{iuw(v\cdot l)} c_l(u) e_l$ is jointly strongly measurable. For $0 < H < 1$, Minkowski's inequality and Cauchy–Schwarz give

$$\Big\| \int_0^1 U_v(uw) F(u)\, du \Big\|_{L^2([0,1]_w; X)} \le \int_0^1 \|F(u)\|_X\, du \le \|F\|_{L^2([0,1];X)} \le 2^{\varepsilon}(1+H)^{-\varepsilon} \|F\|_{L^2([0,1];X)},$$

since every $U_v(uw)$ is an isometry.

Now assume $H \ge 1$. Fix $h \in \mathcal{S}(\mathbb{R})$ with $\|h\|_{L^2(\mathbb{R})} = 1$ and $\operatorname{supp} \widehat{h} \subseteq [-1,1]$, and for $T \ge 1$ put $h_T(s) = T^{-1/2} h(s/T)$, so that $\|h_T\|_{L^2(\mathbb{R})} = 1$, $\|h_T'\|_{L^2(\mathbb{R})} = T^{-1}\|h'\|_{L^2(\mathbb{R})}$, and $\operatorname{supp} \widehat{h_T} \subseteq [-1/T, 1/T]$. Define $G(u,s) = h_T(s)\, U_v(s) F(u)$ on $[0,1] \times \mathbb{R}$. Since $U_v(s)$ is an isometry of $X$, $\|G\|_{L^2([0,1]\times\mathbb{R}; X)} = \|h_T\|_{L^2(\mathbb{R})} \|F\|_{L^2([0,1];X)} = \|F\|_{L^2([0,1];X)}$. Moreover, $G(u,s) = \sum_{l \in S} h_T(s) e^{is(v\cdot l)} c_l(u) e_l$, so for a.e. $u \in [0,1]$ and every $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} G(u,s) \widehat{\varphi}(s)\, ds = \sum_{l \in S} \Big( \int_{\mathbb{R}} \varphi(\xi) \widehat{h_T}(\xi - v\cdot l)\, d\xi \Big) c_l(u) e_l.$$

Hence the distributional Fourier transform of $s \mapsto G(u,s)$ is supported in $\bigcup_{l \in S} [v\cdot l - 1/T,\, v\cdot l + 1/T]$. For every $T \ge \max\{1, 2/a_0\}$, we have $1/T \le \frac{a_0}{2} \le \frac{a_0}{2} H$, so this support lies in $\{\frac{a_0}{2} H \le |\xi| \le 2b_0 H\}$. Since $X \in \mathcal{C}_{p,M}$ by Lemma 2.1(ii), Lemma 2.8 applied with the Banach space $X$ and annulus constants $a = a_0/2$, $b = 2b_0$ gives

$$\Big\| \int_0^1 G(u, s+uw)\, du \Big\|_{L^2([0,1]_w \times \mathbb{R}_s; X)} \le C_3 H^{-\varepsilon} \|F\|_{L^2([0,1];X)},$$

where $C_3$ depends only on $p, M, a_0, b_0$.

Now $G(u, s+uw) = h_T(s+uw)\, U_v(s)\, U_v(uw) F(u)$. For every $u, w \in [0,1]$, writing $h_T(s+uw) - h_T(s) = \int_0^{uw} h_T'(s+r)\, dr$ and applying Minkowski's integral inequality yields

$$\|h_T(\cdot + uw) - h_T(\cdot)\|_{L^2(\mathbb{R})} \le \int_0^{uw} \|h_T'(\cdot + r)\|_{L^2(\mathbb{R})}\, dr = uw\, \|h_T'\|_{L^2(\mathbb{R})} \le T^{-1} \|h'\|_{L^2(\mathbb{R})}.$$

Since $U_v(s)$ and $U_v(uw)$ are isometries of $X$, Minkowski's inequality in $L^2(\mathbb{R}_s; X)$ gives, for every $w \in [0,1]$,

$$\begin{aligned} &\Big\| \int_0^1 \big(h_T(s+uw) - h_T(s)\big) U_v(s) U_v(uw) F(u)\, du \Big\|_{L^2(\mathbb{R}_s; X)} \\ &\qquad \le \int_0^1 \|h_T(\cdot + uw) - h_T(\cdot)\|_{L^2(\mathbb{R})} \|F(u)\|_X\, du \le T^{-1} \|h'\|_{L^2(\mathbb{R})} \|F\|_{L^2([0,1];X)}, \end{aligned}$$

and taking the $L^2([0,1]_w)$ norm preserves this bound. On the other hand, because $\|h_T\|_{L^2(\mathbb{R})} = 1$ and $U_v(s)$ is an isometry of $X$,

$$\Big\| h_T(s)\, U_v(s) \int_0^1 U_v(uw) F(u)\, du \Big\|_{L^2([0,1]_w \times \mathbb{R}_s; X)} = \Big\| \int_0^1 U_v(uw) F(u)\, du \Big\|_{L^2([0,1]_w; X)}.$$

By the triangle inequality, for every $T \geq \max\{1, 2/a_0\}$,

$$\Big\| \int_0^1 U_v(uw) F(u)\, du \Big\|_{L^2([0,1]_w; X)} \leq \big( C_3 H^{-\varepsilon} + T^{-1} \|h'\|_{L^2(\mathbb{R})} \big) \|F\|_{L^2([0,1]; X)}.$$

Letting $T \to \infty$ and using $1 + H \leq 2H$ (as $H \geq 1$) yields the bound with $C = 2^{\varepsilon} \max\{1, C_3\}$. □

The constant $C$ does not depend on the direction or length of $v$; only the frequency scale $H$ of the inner products $v \cdot l$ matters.

## 3 Torus coordinates on SU(2)

Identify $g \in \mathrm{SU}(2)$ with its first row $z = (z_1, z_2) \in S^3 \subset \mathbb{C}^2$, so $g = \left(\begin{smallmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{smallmatrix}\right)$. For $g, h$ with first rows $z, w$, the $(1,1)$ entry of $gh^{-1} = gh^*$ is $\langle z, w \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2$.

**Lemma 3.1** (coordinates)**.** *Write $z = \big( \sqrt{(1+y)/2}\, e^{it},\ \sqrt{(1-y)/2}\, e^{i(t+u)} \big)$ with $y \in [-1,1]$ and $(t,u) \in \mathbb{T}^2$. Then the probability Haar measure of $\mathrm{SU}(2)$ is $\frac{dy}{2} \otimes \frac{dt\, du}{4\pi^2}$, the operator $\lambda(d_\theta)$ is the translation $t \mapsto t - \theta$, and $P_R$ acts on $c(y)e^{i(kt+nu)}$ as multiplication by $\psi(-k/R)$.*

*Proof.* For the uniform measure on $S^3$, $|z_1|^2$ is uniform on $[0,1]$ and the arguments of $z_1, z_2$ are independent and uniform; the change $(\arg z_1, \arg z_2) \mapsto (t,u) = (\arg z_1, \arg z_2 - \arg z_1)$ preserves the Haar measure of $\mathbb{T}^2$. The first row of $d_{-\theta} g$ is $e^{-i\theta} z$, so $\lambda(d_\theta)$ shifts $t$ by $-\theta$ and fixes $y, u$; then $\Pi_m$ keeps the modes with $k = -m$. □

**Lemma 3.2** (the kernel of $S_0$)**.** *Let $D(x,y) = 1 - x^2 - y^2$ and, for $D > 0$, let $\Gamma(x,y) = (A_1, B_1)$ be given by $B_1 \in (0, \pi)$ and $A_1 \in (-\pi, 0)$ with*

$$\cos B_1 = \frac{-xy}{\sqrt{(1-x^2)(1-y^2)}}, \qquad (\cos A_1, \sin A_1) = \frac{(1 + x + y,\ -\sqrt{D})}{\sqrt{2(1+x)(1+y)}}.$$

*For a continuous $f$ on $\mathrm{SU}(2)$ and an output point with height $|y| < 1$ and angles $(t,u)$,*

$$(S_0 f)(y,t,u) = \sum_{\sigma = \pm 1} \int_{\{x:\, D(x,y) > 0\}} f\big(x, (t,u) + \sigma \Gamma(x,y)\big) \frac{dx}{2\pi \sqrt{D(x,y)}}.$$

*Proof.* By definition $(S_0 f)(z) = \frac{1}{2\pi} \int_0^{2\pi} f(c_\alpha^{-1} z)\, d\alpha$ with $c_\alpha = d_\alpha u_0 d_{-\alpha}$; $c_\alpha$ runs uniformly over the matrices $\left(\begin{smallmatrix} 2^{-1/2} & \beta \\ -\bar{\beta} & 2^{-1/2} \end{smallmatrix}\right)$, $|\beta| = 2^{-1/2}$. The first row of $c_\alpha^{-1} g = c_\alpha^* g$ is $w = 2^{-1/2}(z + e^{i\phi} z^\perp)$ with $z^\perp = (-\bar{z}_2, \bar{z}_1)$ and $e^{i\phi} = -\sqrt{2}\beta$ uniform on the circle. These are the $w$ with $\langle z, w \rangle = 2^{-1/2}$. The height of $w$ is $x = 2|w_1|^2 - 1 = -2\,\mathrm{Re}(e^{-i\phi} z_1 z_2) = -\sqrt{1-y^2}\cos(\phi - \arg(z_1 z_2))$. Each value of $x$ with $D(x,y) > 0$ is taken at exactly two values of $\phi$, and at both of them $|dx/d\phi| = \sqrt{D(x,y)}$. So each of the two points carries the density $dx/(2\pi\sqrt{D(x,y)})$. Write the angles of $w$ as $(t + A, u + B)$. With $a_\pm = \sqrt{(1 \pm x)(1 \pm y)}/2$, the incidence $\langle z, w \rangle = 2^{-1/2}$ reads $e^{-iA}(a_+ + a_- e^{-iB}) = 2^{-1/2}$. Taking moduli, and using $a_+^2 + a_-^2 = (1 + xy)/2$ and $a_+ a_- = \sqrt{(1-x^2)(1-y^2)}/4$, gives the

formula for $\cos B$, so $B = \pm B_1$; then $e^{iA} = \sqrt{2}(a_+ + a_- e^{-iB})$ gives $A = A_1$ for $B = B_1$, with the stated cosine and sine (use $a_- \cos B_1 = -xy/(2\sqrt{(1+x)(1+y)})$ and $a_- \sin B_1 = \sqrt{D}/(2\sqrt{(1+x)(1+y)})$), and $A = -A_1$ for $B = -B_1$. The two incident points of height $x$ are the points with $B = B_1$ and $B = -B_1$. □

**Lemma 3.3** (mixed derivative)**.** *On $I^2$, $I = [-\frac{1}{2}, \frac{1}{2}]$, we have $D \geq \frac{1}{2}$, $\Gamma$ is smooth on a neighbourhood of $I^2$, and*

$$\partial_x B_1 = \frac{y}{(1-x^2)\sqrt{D}}, \qquad \partial_x A_1 = \frac{1+x-y}{2(1+x)\sqrt{D}},$$

$$\partial_x \partial_y \Gamma = D^{-3/2}\,(\kappa, 1), \qquad \kappa(x,y) = \frac{x+y-1}{2}.$$

*Proof.* With $c = -xy(1-x^2)^{-1/2}(1-y^2)^{-1/2}$ we have $\partial_x c = -y(1-x^2)^{-3/2}(1-y^2)^{-1/2}$ and $\sin B_1 = \sqrt{1-c^2} = \sqrt{D}/\sqrt{(1-x^2)(1-y^2)}$, so $\partial_x B_1 = -\partial_x c/\sin B_1 = y/((1-x^2)\sqrt{D})$; then $\partial_y \partial_x B_1 = (1-x^2)^{-1}(D^{-1/2} + y^2 D^{-3/2}) = D^{-3/2}$. For $A_1 = \arg(X + iY)$ with $X = 1 + x + y$, $Y = -\sqrt{D}$ we have $X^2 + Y^2 = 2(1+x)(1+y)$ and $XY_x - YX_x = (x(1+x+y) + D)/\sqrt{D} = (1+y)(1+x-y)/\sqrt{D}$, which gives $\partial_x A_1$; then $\partial_y \partial_x A_1 = \frac{-D+y(1+x-y)}{2(1+x)D^{3/2}} = \frac{(1+x)(x+y-1)}{2(1+x)D^{3/2}}$. On $I^2$ we have $D \geq \frac{1}{2}$, so $|\cos B_1| < 1$ and $Y = -\sqrt{D} < 0$; hence $B_1 = \arccos c$ and $A_1 = \arg(X + iY) \in (-\pi, 0)$ are smooth on a neighbourhood of $I^2$. □

We checked the formulas of Lemmas 3.2 and 3.3 also by computer: exact symbolic differentiation, and a Monte Carlo test of the branch formula on points generated from the definition of $S_0$ (largest error $3 \cdot 10^{-15}$ on 20000 samples); see Section 11.

**Lemma 3.4** (localization)**.** *Let $b \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}))$ with $b \geq 0$ and $b(0) > 0$, and let $M_b$ be multiplication by $b(y)$, $y$ the height. There is $c_b > 0$ such that for every Banach space $E$ and every $R$,*

$$\|P_R S_0\|_{L^2(\mathrm{SU}(2);E)} \leq c_b^{-1} \|M_b S_0 M_b P_R\|_{L^2(\mathrm{SU}(2);E)}.$$

*Proof.* Let $\rho(h) f(z) = f(zh)$ be the right regular representation. It is isometric and commutes with every $\lambda(g)$, hence with $S_0$ and $P_R$, and $\rho(h)^{-1} M_b \rho(h)$ is multiplication by $b(y(zh^{-1}))$. Hence $V = \int \rho(h)^{-1} M_b S_0 M_b \rho(h)\, dh$ is the operator with the incidence measure of $S_0$ weighted by $c(z,w) = \int b(y(zh))\, b(y(wh))\, dh$, where $y(\cdot)$ is the height. The first entry of the row vector $zh$ is $z \cdot \zeta = z_1\zeta_1 + z_2\zeta_2$, where $\zeta$ is the first column of $h$, uniform on $S^3$. Two incident pairs $(z,w)$ and $(z',w')$ have the same Hermitian Gram matrix, so $z' = zU$, $w' = wU$ for a unitary $U$; since $U\zeta$ is again uniform, $c(z',w') = c(z,w) =: c_b$. For $z = (1,0)$, $w = 2^{-1/2}(1,1)$ and $\zeta = 2^{-1/2}(1,i)$ both heights vanish, so $c_b > 0$. Thus $V = c_b S_0$ and $c_b \|P_R S_0 f\| = \|V P_R f\| \leq \int \|M_b S_0 M_b P_R\, \rho(h) f\|\, dh$. □

# 4 Proof of Theorem A

## 4.1 Reduction

Fix $\psi$ with $\operatorname{supp}\psi \subseteq \{a_\psi \leq |\sigma| \leq b_\psi\}$, $0 < a_\psi < b_\psi$, and $b$ as in Lemma 3.4. Put $I = [-\frac{1}{2}, \frac{1}{2}]$, $X = L^2(\mathbb{T}^2; E)$, and $a(x,y) = b(x)b(y)/(2\pi\sqrt{D(x,y)})$, a smooth function on $I^2$. For $\sigma = \pm 1$ define on $L^2(I; X)$

$$(\mathcal{T}_\sigma F)(y) = \int_I a(x,y)\, \tau_{\sigma\Gamma(x,y)} F(x)\, dx.$$

By Lemma 3.2 (which extends from continuous functions to $L^2$ by density, both sides being bounded operators on the functions supported in heights $x \in I$), for $f \in L^2(\mathrm{SU}(2); E)$ and $F = (P_R f)|_I$ (viewed as a function of the height with values in $X$), $M_b S_0 M_b P_R f = (\mathcal{T}_+ + \mathcal{T}_-)F$ on $I$ and 0 elsewhere. By Lemma 3.1 the Haar measure is $\frac{dy}{2}$ times the Haar measure of $\mathbb{T}^2$, so

$$\|M_b S_0 M_b P_R f\|_{L^2(\mathrm{SU}(2);E)} = 2^{-1/2}\|(\mathcal{T}_+ + \mathcal{T}_-)F\|_{L^2(I;X)}, \qquad \|F\|_{L^2(I;X)} \le 2^{1/2}\|P_R f\| \le C_\psi \|f\|,$$

where the last step is the uniform bound for $P_R$ of Section 1. By Lemma 3.4 and the density of functions with finitely many angular frequencies (Fejér means in $(t, u)$), it suffices to prove

$$\|\mathcal{T}_\sigma F\|_{L^2(I;X)} \le C R^{-\eta} \|F\|_{L^2(I;X)} \tag{4.1}$$

for $F$ with values in trigonometric polynomials with frequencies in a finite set of $(k, n)$ with $a_\psi R \le |k| \le b_\psi R$ (Lemma 3.1).

Let $\chi \in C_c^\infty(\mathbb{R})$ be even, $\chi = 1$ on $[-1, 1]$, $\operatorname{supp}\chi \subseteq [-2, 2]$, and $d(s) = \chi(s/2) - \chi(s)$, so $\operatorname{supp} d \subseteq \{1 \le |s| \le 4\}$ and $\chi(s) + \sum_{h \ge 0} d(s/2^h) = 1$. Put $D_\psi = 4(1 + b_\psi)$. With the multipliers in the variable $n$ (Lemma 2.9(i) with $v = (0, 1)$),

$$F = F_{\mathrm{low}} + \sum_{j \ge 0} F_j, \qquad F_{\mathrm{low}} = \chi\big(D_{(0,1)}/(D_\psi R)\big)F, \quad F_j = d\big(D_{(0,1)}/(2^j D_\psi R)\big)F,$$

a finite sum, with $\|F_{\mathrm{low}}\|, \|F_j\| \le C\|F\|$. The frequencies of $F_{\mathrm{low}}$ have $|n| \le 2D_\psi R$; those of $F_j$ have $K \le |n| \le 4K$ with $K = 2^j D_\psi R$, and $|k| \le b_\psi R \le K/4$. Hence (4.1) follows, by summing a geometric series, from the following proposition, with $\eta = \frac{1}{8}\min\{1, \varepsilon\}$.

**Proposition 4.1.** *Let $\sigma = \pm 1$ and let $G \in L^2(I; X)$ take values in trigonometric polynomials with frequencies in a finite set $S$.*

*(a) If $K \ge 1$ and $|k| \le K/4$, $K \le |n| \le 4K$ for all $(k, n) \in S$, then $\|\mathcal{T}_\sigma G\| \le C K^{-\eta}\|G\|$.*

*(b) If $R \ge 1$ and $a_\psi R \le |k| \le b_\psi R$, $|n| \le 2D_\psi R$ for all $(k, n) \in S$, then $\|\mathcal{T}_\sigma G\| \le C R^{-\eta}\|G\|$.*

### 4.2 Rectangles

Fix $K \ge 1$ (in case (b), $K = R$). Let $J = \lceil K^{3/8} \rceil$, $\ell = 1/J$ and $B = K^{7/8}$. Split $I$ into the intervals $I_i = [x_i, x_i + \ell)$, $x_i = -\frac{1}{2} + (i - 1)\ell$, $1 \le i \le J$ (the last one closed); we use the same intervals $I_j = [y_j, y_j + \ell)$ for the output. Put $\kappa_{ij} = \kappa(x_i, y_j)$ and $\gamma_{ij} = \sigma D(x_i, y_j)^{-3/2}$, so that $1 \le |\gamma_{ij}| \le 2^{3/2}$, and $|\kappa| \le 1$ on $I^2$. Since $J \le 2K^{3/8}$,

$$\ell^2 B \ge \tfrac{1}{4} K^{1/8}, \qquad K\ell^3 \le K^{-1/8}, \qquad \ell \le K^{-3/8}, \qquad K\ell/B \le K^{-1/4}, \qquad K/B = K^{1/8}. \tag{4.2}$$

For each pair $(i, j)$ let $\mathcal{Q}_{ij}$ be a Fourier multiplier on $X$ with $\|\mathcal{Q}_{ij}\| \le C_0$, and define

$$(\mathcal{W}G)(y) = \sum_i \int_{I_i} a(x, y)\, \tau_{\sigma\Gamma(x,y)}\, \mathcal{Q}_{ij} G(x)\, dx \qquad (y \in I_j).$$

Let $a_{ij} = a(x_i, y_j)$ and

$$\widetilde{\Gamma}_{ij}(x, y) = \Gamma(x, y_j) - \Gamma(x_i, y_j) + \Gamma(x_i, y) + \partial_x \partial_y \Gamma(x_i, y_j)(x - x_i)(y - y_j),$$

and let $\widetilde{\mathcal{W}}$ be $\mathcal{W}$ with $a$ replaced by $a_{ij}$ and $\Gamma$ by $\widetilde{\Gamma}_{ij}$ on $I_i \times I_j$.

**Lemma 4.2** (freezing)**.** *If all frequencies in $S$ satisfy $|k|+|n|\le c_1K$, then $\|\mathcal{W}G-\widetilde{\mathcal{W}}G\|\le C\,C_0(\ell+K\ell^3)\|G\|$, with $C$ depending also on $c_1$.*

*Proof.* On $I_i\times I_j$, $|a-a_{ij}|\le C\ell$, and $\Gamma-\widetilde{\Gamma}_{ij}$ is the integral of $\partial_x\partial_y\Gamma-\partial_x\partial_y\Gamma(x_i,y_j)$ over the rectangle between $(x_i,y_j)$ and $(x,y)$, so $|\Gamma-\widetilde{\Gamma}_{ij}|\le C\ell^3$. The multiplier $\mathcal{Q}_{ij}$ keeps the frequencies, so by Lemma 2.9(ii) the translation error is at most $CK\ell^3C_0\|G(x)\|$. For each $y$, the sum over $i$ is one integral over $I$, so pointwise in $y$ the difference is at most $CC_0(\ell+K\ell^3)\int_I\|G(x)\|\,dx\le CC_0(\ell+K\ell^3)\|G\|_{L^2(I;X)}$. □

**Lemma 4.3** (one frozen rectangle)**.** *Let $v_{ij}=\ell^2\gamma_{ij}(\kappa_{ij},1)$ and $0<a_0<b_0$. If $g\in L^2(I_i;X)$ takes values in trigonometric polynomials with frequencies $l$ in a finite set, all satisfying $a_0H\le|v_{ij}\cdot l|\le b_0H$, then*

$$\Big\|\int_{I_i}a_{ij}\tau_{\sigma\widetilde{\Gamma}_{ij}(x,\cdot)}g(x)\,dx\Big\|_{L^2(I_j;X)}\le C\ell(1+H)^{-\varepsilon}\|g\|_{L^2(I_i;X)}.$$

*Proof.* The translations commute. The term $\Gamma(x,y_j)-\Gamma(x_i,y_j)$ depends only on $x$; it is an isometry applied to $g(x)$ that keeps its frequencies. The term $\Gamma(x_i,y)$ depends only on $y$; it is an isometry of the output. With $x=x_i+\ell u$, $y=y_j+\ell w$, the last term is $U_{v_{ij}}(uw)$ by Lemma 3.3. The Jacobian $dx=\ell\,du$ gives the factor $\ell$, and the rescalings of the $L^2$ norms of input and output cancel. Apply Proposition 2.10. □

If every block of a $J\times J$ block operator from $\bigoplus_i L^2(I_i;X)$ to $\bigoplus_j L^2(I_j;X)$ has norm at most $b_*$, then by Cauchy–Schwarz the norm of the whole operator is at most $Jb_*$. With $b_*=C\ell(\dots)$ and $J\ell=1$ this removes the factor $\ell$.

### 4.3 Proof of Proposition 4.1(a)

Take $\mathcal{Q}_{ij}=\mathrm{Id}$. For $(k,n)\in S$ and every $|\kappa|\le1$, $\frac34K\le|n+\kappa k|\le5K$, so $|v_{ij}\cdot(k,n)|=\ell^2|\gamma_{ij}||n+\kappa_{ij}k|$ lies between $\frac34H$ and $5\cdot2^{3/2}H$ with $H=\ell^2K$. By Lemma 4.3 and the block bound, $\|\widetilde{\mathcal{W}}G\|\le C(\ell^2K)^{-\varepsilon}\|G\|$ for the functions $G$ of part (a) (not for all of $L^2(I;X)$), and Lemma 4.2 (with $c_1=5$) adds $C(\ell+K\ell^3)\|G\|$. By (4.2), $\ell^2K\ge\ell^2B\ge\frac14K^{1/8}$, so $\|\mathcal{T}_\sigma G\|\le C(K^{-\varepsilon/8}+K^{-3/8}+K^{-1/8})\|G\|\le CK^{-\eta}\|G\|$.

### 4.4 Proof of Proposition 4.1(b)

Now $K=R$. Let $L_{ij}=D_{(\kappa_{ij},1)}$, the multiplier with symbol $\kappa_{ij}k+n$, and $Q_{ij}=\chi(L_{ij}/B)$. Then $\mathcal{T}_\sigma=\mathcal{W}^{\mathrm{away}}+\mathcal{W}^{\mathrm{fold}}$, where $\mathcal{W}^{\mathrm{away}}$ uses $\mathcal{Q}_{ij}=\mathrm{Id}-Q_{ij}$ and $\mathcal{W}^{\mathrm{fold}}$ uses $\mathcal{Q}_{ij}=Q_{ij}$; both have $C_0\le1+\|h_\chi\|_1$ by Lemma 2.9(i). All frequencies have $|k|+|n|\le(b_\psi+2D_\psi)R$.

*Away from the fold.* Write $\mathrm{Id}-Q_{ij}=\sum_{h\ge0}d(L_{ij}/(2^hB))$ (a finite sum on $S$); each term has norm at most $\|h_d\|_1$, and on its range $2^hB\le|\kappa_{ij}k+n|\le4\cdot2^hB$, so $H_h\le|v_{ij}\cdot l|\le4\cdot2^{3/2}H_h$ with $H_h=\ell^22^hB$. By Lemma 4.3, the frozen block $(i,j)$ has norm at most $C\ell\sum_hH_h^{-\varepsilon}\le C\ell(\ell^2B)^{-\varepsilon}/(1-2^{-\varepsilon})$. With the block bound, Lemma 4.2 and (4.2),

$$\|\mathcal{W}^{\mathrm{away}}G\|\le C\big(R^{-\varepsilon/8}+R^{-3/8}+R^{-1/8}\big)\|G\|.$$

*At the fold.* For $\kappa\in\mathbb{R}$ put $Q_\kappa=\chi(D_{(\kappa,1)}/B)=\int h_\chi(s)\tau_{(s/B)(\kappa,1)}\,ds$. For $x\in I_i$, $y\in I_j$ we have $|\kappa_{ij}-\kappa(x,y)|=\frac12|(x-x_i)+(y-y_j)|\le\ell$, and the two translations $\tau_{(s/B)(\kappa_{ij},1)}$ and

$\tau_{(s/B)(\kappa(x,y),1)}$ differ by a translation by $\big(s(\kappa_{ij}-\kappa(x,y))/B,0\big)$. By Lemma 2.9(ii) (all frequencies satisfy $|k|+|n|\le(b_\psi+2D_\psi)R$),

$$\|(Q_{ij}-Q_{\kappa(x,y)})G(x)\|\le\frac{\ell}{B}\Big(\int|s\,h_\chi(s)|\,ds\Big)\|\partial_t G(x)\|\le C\frac{R\ell}{B}\|G(x)\|.$$

As in Lemma 4.2, replacing $Q_{ij}$ by $Q_{\kappa(x,y)}$ in $\mathcal{W}^{\mathrm{fold}}$ therefore costs at most $CR\ell B^{-1}\|G\|\le CR^{-1/4}\|G\|$. What remains is

$$(\mathcal{Z}G)(y)=\int_I a(x,y)\,\tau_{\sigma\Gamma(x,y)}\,Q_{\kappa(x,y)}G(x)\,dx.$$

Each $\tau_{\sigma\Gamma(x,y)}$ is an isometry, so Minkowski's inequality in $L^2(I;X)$ gives

$$\|\mathcal{Z}G\|\le\|a\|_\infty\int_I\|Q_{\kappa(x,\cdot)}G(x)\|_{L^2(I;X)}\,dx,$$

and Lemma 4.4 below gives $\|\mathcal{Z}G\|\le C(R/B)^{-\varepsilon}\|G\|=CR^{-\varepsilon/8}\|G\|$. Adding up, $\|\mathcal{T}_\sigma G\|\le CR^{-\eta}\|G\|$.

**Lemma 4.4** (the fold)**.** *For every $x\in I$ and every trigonometric polynomial $g\in X$ whose frequencies satisfy $a_\psi R\le|k|\le b_\psi R$,*

$$\Big(\int_I\|Q_{\kappa(x,y)}g\|_X^2\,dy\Big)^{1/2}\le C\Big(1+\frac{R}{B}\Big)^{-\varepsilon}\|g\|_X.$$

*Proof.* Write $y=-\frac12+w$, $w\in[0,1]$. Then $\kappa(x,y)=\kappa_0+w/2$ with $\kappa_0=\kappa(x,-\frac12)$, and $\tau_{(s/B)(\kappa_0+w/2,1)}=\tau_{(sw/(2B),0)}\,\tau_{(s/B)(\kappa_0,1)}$. Split $s=j+u$ with $j\in\mathbb{Z}$, $u\in[0,1)$. With $v_*=(1/(2B),0)$,

$$Q_{\kappa(x,y)}g=\sum_{j\in\mathbb{Z}}\tau_{(jw/(2B),0)}\int_0^1 U_{v_*}(uw)\,F_j(u)\,du,\qquad F_j(u)=h_\chi(j+u)\,\tau_{((j+u)/B)(\kappa_0,1)}\,g.$$

The translation $\tau_{(jw/(2B),0)}$ is an isometry for each $w$. The large translation $\tau_{((j+u)/B)(\kappa_0,1)}$ depends only on the integration variable $u$, so it sits inside $F_j(u)$, which has the frequencies of $g$ and $\|F_j\|_{L^2([0,1];X)}=\|h_\chi(j+\cdot)\|_{L^2(0,1)}\|g\|$. The bilinear translation $U_{v_*}(uw)$ acts in the first angle only, and $|v_*\cdot(k,n)|=|k|/(2B)\in[\frac{a_\psi}{2}H,\frac{b_\psi}{2}H]$ with $H=R/B$. Proposition 2.10 bounds the $j$-th term by $C(1+R/B)^{-\varepsilon}\|h_\chi(j+\cdot)\|_{L^2(0,1)}\|g\|$, and $\sum_j\|h_\chi(j+\cdot)\|_{L^2(0,1)}<\infty$ because $h_\chi$ is a Schwartz function. So the series converges absolutely in $L^2(I;X)$, and the triangle inequality gives the claim. □

This lemma is the step that handles the degeneracy. Estimating each frequency $(k,n)$ separately would lose the whole gain near the fold $n+\kappa k=0$. Written as an integral of translations, the cutoff to the fold exposes the exact bilinear translation $uw/(2B)$ between the integration variable of the cutoff and the output height, and its frequency size $R/B=R^{1/8}$ gives decay on the very set where the original mixed derivative vanishes.

This completes the proof of Proposition 4.1, and hence of Theorem A: the low piece costs $CR^{-\eta}$, and the pieces $F_j$ cost $C(2^jD_\psi R)^{-\eta}$, whose sum over $j$ is $CR^{-\eta}$. □

## 5 The Hölder estimate

*Proof of Corollary 1.1.* Write $R_\alpha = \lambda(d_\alpha)$. Since $u_\theta = d_{\theta/2} u_0 d_{\theta/2}$, we have

$$d_\alpha u_\theta d_{-\alpha} = d_{\theta/2}(d_\alpha u_0 d_{-\alpha}) d_{\theta/2}, \qquad \text{so} \qquad S_\theta = R_{\theta/2} S_0 R_{\theta/2} = R_\theta S_0 = S_0 R_\theta,$$

because $S_0$ commutes with every $R_\beta$. Hence $\|S_\theta - S_{\theta'}\| \le \|(R_\alpha - \mathrm{Id}) S_0\|$ with $\alpha = \theta - \theta'$, and we may assume $|\alpha| \le \pi$.

Let $\phi_0 \in C_c^\infty((-2,2))$ equal 1 on $[-1,1]$ and $\psi_1(\sigma) = \phi_0(\sigma/2) - \phi_0(\sigma)$, so that $1 = \phi_0(m) + \sum_{j\ge 0} \psi_1(m/2^j)$ for every integer $m$. Let $P_{2^j} = \sum_m \psi_1(m/2^j)\Pi_m$, and let $\widetilde{P}_j$ be defined in the same way from a $\widetilde{\psi} \in C_c^\infty(\mathbb{R}\setminus\{0\})$ equal to 1 on $\operatorname{supp}\psi_1$. Then $(R_\alpha - \mathrm{Id})P_{2^j} = (R_\alpha - \mathrm{Id})\widetilde{P}_j P_{2^j}$. If $\widetilde{k}_j$ is the kernel of $\widetilde{P}_j$, then $R_\alpha \widetilde{P}_j = \int \widetilde{k}_j(\theta - \alpha) R_\theta \, d\theta$, so $\|(R_\alpha - \mathrm{Id})\widetilde{P}_j\| \le \|\widetilde{k}_j(\cdot - \alpha) - \widetilde{k}_j\|_{L^1(\mathbb{T})} \le C \min\{1, 2^j|\alpha|\}$. Theorem A with $\psi = \psi_1$ gives $\|P_{2^j} S_0\| \le C 2^{-j\eta}$. The low part is $\sum_{|m|\le 1} \phi_0(m)\Pi_m$, and $\|(R_\alpha - \mathrm{Id})\Pi_m\| \le |m\alpha|$ since $R_\alpha \Pi_m = e^{im\alpha}\Pi_m$. On functions with finitely many frequencies $m$ the decomposition is a finite sum, so

$$\|(R_\alpha - \mathrm{Id}) S_0\| \le C|\alpha| + C \sum_{j\ge 0} \min\{1, 2^j|\alpha|\}\, 2^{-j\eta} \le C'|\alpha|^\eta,$$

splitting the sum at $2^j|\alpha| = 1$ and using $0 < \eta < 1$; the bound extends to the whole space by density. □

## 6 The latitude estimate

In this section we prove Theorem C. For $\delta_0 = 0$, with some positive exponent, it is Yao's Theorem 5.4 [21]; Yao proves it with his parabolic partition [21, Lemma 4.4] and his theorem on averages over curves with rotational curvature [21, Theorem 4.6]. The proof below is different and short. It uses only Proposition 2.10, which is proved self-containedly in Section 2 directly from Bourgain's theorem [2]. The geometric point is that on $S^2$, in height and azimuth coordinates, the latitude operator has no fold: the mixed derivative of the azimuthal shift is positive (Lemma 6.2), and the poles and tangencies can be avoided by choosing a polar axis for each pair of small caps.

Throughout this section $X_1 = L^2(\mathbb{T}; E)$ with the probability Haar measure, $(\tau_a g)(\phi) = g(\phi + a)$ for $a \in \mathbb{R}$, and a *trigonometric polynomial* in $X_1$ is a finite sum $\sum_m c_m e^{im\phi}$ with $c_m \in E$. For $N \ge 1$ let $X_{1,N} \subseteq X_1$ be the closed subspace of the trigonometric polynomials with frequencies in $\{N \le |m| \le 4N\}$. On $X_{1,N}$ the derivative $\partial_\phi$ is bounded, with norm at most $C_4 N$, by Lemma 2.9(ii) applied to functions of the first angle only.

**Lemma 6.1** (one angle)**.** *Let $0 < a_0 < b_0$. There is $C$ depending only on $p, M, a_0, b_0$ such that for every $E \in \mathcal{C}_{p,M}$, $v \in \mathbb{R}$, $H > 0$ and every $F \in L^2([0,1]; X_1)$ with values in trigonometric polynomials whose frequencies $m$ all satisfy $a_0 H \le |vm| \le b_0 H$,*

$$\Big\| \int_0^1 \tau_{uwv} F(u)\, du \Big\|_{L^2([0,1]_w; X_1)} \le C(1+H)^{-\varepsilon} \|F\|_{L^2([0,1];X_1)}.$$

*Proof.* Identify $X_1$ isometrically with the closed subspace of $X = L^2(\mathbb{T}^2; E)$ of the functions of the first angle only. Then $\tau_{uwv}$ is $U_{(v,0)}(uw)$, the frequencies become $(m,0)$, and $(v,0)\cdot(m,0) = vm$. Apply Proposition 2.10. □

## 6.1 Coordinates and the kernel

For $|s|<1$ put $r_s=\sqrt{1-s^2}$ and

$$x(s,\phi)=(r_s\cos\phi,\ r_s\sin\phi,\ s).$$

The coordinate vectors are orthogonal with lengths $1/r_s$ and $r_s$, so the normalized area of $S^2$ is $\frac12\,ds\,\frac{d\phi}{2\pi}$. For a function $f$ on $S^2$ supported in $\{|x_3|\le\rho\}$ with $\rho<1$ we therefore have

$$\|f\|_{L^2(S^2;E)}=2^{-1/2}\|f\circ x\|_{L^2([-\rho,\rho];X_1)}.$$

Fix $|s|<1$, $|\delta|<1$ and $x=x(s,\phi)$. The vectors $e_1=(-s\cos\phi,-s\sin\phi,r_s)$ and $e_2=(-\sin\phi,\cos\phi,0)$ form an orthonormal basis of $x^\perp$, and $y(\theta)=\delta x+\sqrt{1-\delta^2}\,(e_1\cos\theta+e_2\sin\theta)$ is a constant-speed parametrization of the circle $\{y:\langle x,y\rangle=\delta\}$, so the normalized arc length is $d\theta/2\pi$. The height of $y(\theta)$ is $t=\delta s+\sqrt{1-\delta^2}\,r_s\cos\theta$, which runs between $t_\pm=\delta s\pm\sqrt{1-\delta^2}\,r_s$. Put

$$\begin{aligned}Q&=1-\delta^2-s^2-t^2+2\delta st=(1-s^2)(1-t^2)-(\delta-st)^2\\&=(1-\delta^2)(1-s^2)-(t-\delta s)^2=(t_+-t)(t-t_-).\end{aligned}\tag{6.1}$$

Then $(dt/d\theta)^2=(1-\delta^2)r_s^2\sin^2\theta=Q$. For $t_-<t<t_+$ the azimuth $\phi_y$ of $y$ satisfies, by the scalar products of $y$ with $x$ and $e_2$, $\cos(\phi_y-\phi)=(\delta-st)/(r_sr_t)$ and $r_t\sin(\phi_y-\phi)=\sqrt{1-\delta^2}\sin\theta$; the two values of $\theta$ with height $t$ have opposite signs of $\sin\theta$.

**Lemma 6.2** (the latitude kernel)**.** *For $Q>0$ let $\Theta(s,t;\delta)=\arccos\big((\delta-st)/(r_sr_t)\big)\in(0,\pi)$ and $k(s,t;\delta)=1/(2\pi\sqrt{Q})$. For continuous $f$ on $S^2$, with $\Theta=\Theta(s,t;\delta)$,*

$$(T_\delta f)(x(s,\phi))=\sum_{\sigma=\pm1}\int_{t_-}^{t_+}k(s,t;\delta)\,f\big(x(t,\phi+\sigma\Theta)\big)\,dt.$$

*Moreover*

$$\partial_t\Theta=\frac{s-\delta t}{(1-t^2)\sqrt{Q}},\qquad\partial_\delta\Theta=-\frac{1}{\sqrt{Q}},\qquad\partial_s\partial_t\Theta=\frac{1-\delta^2}{Q^{3/2}}.$$

*Proof.* The kernel formula is the change of variables $\theta\mapsto t$ on $(0,\pi)$ and on $(\pi,2\pi)$, where $\phi_y=\phi+\Theta$ and $\phi_y=\phi-\Theta$ respectively, with $|dt/d\theta|=\sqrt{Q}$. With $c=(\delta-st)/(r_sr_t)$ we have $1-c^2=Q/(r_s^2r_t^2)$ by (6.1), $\partial_tc=(\delta t-s)/(r_sr_t^3)$ and $\partial_\delta c=1/(r_sr_t)$, which give the first two derivatives. Since $\partial_sQ=-2(s-\delta t)$,

$$\partial_s\partial_t\Theta=\frac{Q+(s-\delta t)^2}{(1-t^2)Q^{3/2}}=\frac{1-\delta^2}{Q^{3/2}}$$

by the last but one form of $Q$ in (6.1) with $s$ and $t$ exchanged. □

## 6.2 Caps

Let $\rho=\frac18$ and $I=[-\rho,\rho]$. For $|\delta|\le\frac34$ and $s,t\in I$, (6.1) gives

$$1\ge Q\ge(1-\delta^2)(1-s^2)-(|t|+|\delta||s|)^2\ge\tfrac{7}{16}\cdot\tfrac{63}{64}-\big(\tfrac{7}{32}\big)^2=\tfrac{49}{128}.\tag{6.2}$$

Hence $t_-<t<t_+$ for all $s,t\in I$; the functions $\Theta$, $k$ are smooth on a neighbourhood of the compact set $\{(s,t,\delta):s,t\in I,\ |\delta|\le\frac34\}$, so all their derivatives are bounded there; and

$$\tfrac{7}{16}\le\partial_s\partial_t\Theta\le(128/49)^{3/2}<8.\tag{6.3}$$

Choose finitely many points $c_1, \dots, c_L \in S^2$ such that the balls of radius $\rho/2$ around them cover $S^2$, and put $\chi_i(x) = (\rho - |x - c_i|)_+ / \sum_l (\rho - |x - c_l|)_+$, a continuous partition of unity with $\chi_i$ supported in $\{|x - c_i| \le \rho\}$. For a pair $(i, j)$ choose a unit vector $n_{ij}$ orthogonal to $c_i$ and $c_j$. On the supports, $|\langle x, n_{ij}\rangle| = |\langle x - c_i, n_{ij}\rangle| \le \rho$ and likewise for $c_j$. So in coordinates $x(s, \phi)$ with polar axis $n_{ij}$ (a rotation, which commutes with $T_\delta$), both $\chi_i$ and $\chi_j$ are supported in heights in $I$, and by Lemma 6.2 and density of continuous functions,

$$\chi_i T_\delta(\chi_j f) = \chi_i \sum_{\sigma = \pm 1} \mathcal{A}^\sigma_{\delta,k}(\chi_j f) \qquad (|\delta| \le \tfrac{3}{4}), \tag{6.4}$$

where for a bounded function $b$ on $I^2$ we write

$$(\mathcal{A}^\sigma_{\delta,b} F)(s) = \int_I b(s,t)\, \tau_{\sigma\Theta(s,t;\delta)} F(t)\, dt \qquad (F \in L^2(I; X_1),\ s \in I).$$

By Minkowski's inequality, $\|\mathcal{A}^\sigma_{\delta,b}\|_{L^2(I;X_1)} \le |I|\, \|b\|_\infty$.

### 6.3 One frequency band

**Lemma 6.3.** *Let $\sigma = \pm 1$, $|\delta| \le \frac{3}{4}$, $b \in C^1(I^2)$ and $\mathsf{B} = \|b\|_\infty + \|\partial_t b\|_\infty$. There is $C$, depending only on $p, M$, such that for $N \ge 1$ and every $F \in L^2(I; X_{1,N})$,*

$$\|\mathcal{A}^\sigma_{\delta,b} F\|_{L^2(I;X_1)} \le C\, \mathsf{B}\, N^{-\varepsilon/2} \|F\|_{L^2(I;X_1)}.$$

*Proof.* Write $\Phi = \sigma\Theta(\cdot, \cdot; \delta)$ and $I = [s_-, s_+]$. Split $I$ into $\lceil \sqrt{N} \rceil$ intervals of length $r = |I|/\lceil \sqrt{N} \rceil$, so that $Nr^2 \le |I|^2$, $r \le |I|$ and $Nr \ge |I|\sqrt{N}/2$. Fix one of them, $J = [c, c + r]$, and put $q(s) = \partial_t \Phi(s, c)$. By (6.3), $q' = \sigma\, \partial_s \partial_t \Theta$ has the sign $\sigma$ and $\frac{7}{16} \le |q'| \le 8$. Hence $L = q(s_+) - q(s_-)$ satisfies $\frac{7}{16}|I| \le |L| \le 8|I|$, and $w(s) = (q(s) - q(s_-))/L$ is an increasing $C^1$ bijection of $I$ onto $[0, 1]$ with $0 < ds/dw = |L|/|q'(s)| \le \frac{32}{7}$.

*The linearized operator.* For $c \le z \le c + r$ let $G_z(s) = \int_c^z \tau_{(t-c)q(s)} F(t)\, dt$. With $t = c + ru$ and $(t - c)q(s) = ruq(s_-) + rLuw(s)$,

$$G_z(s) = r \int_0^1 \tau_{rLuw(s)} F_z(u)\, du, \qquad F_z(u) = \mathbf{1}_{\{c + ru \le z\}} \tau_{ruq(s_-)} F(c + ru),$$

and $F_z$ takes values in $X_{1,N}$, with $\|F_z\|_{L^2([0,1];X_1)} \le r^{-1/2} \|F\|_{L^2(J;X_1)}$. Lemma 6.1 with $v = rL$, $H = Nr|L|$, $a_0 = 1$, $b_0 = 4$, and the change of variables $s \mapsto w$, give, for every $z$,

$$\|G_z\|_{L^2(I;X_1)} \le \left(\tfrac{32}{7}\right)^{1/2} rC\, (Nr|L|)^{-\varepsilon} r^{-1/2} \|F\|_{L^2(J;X_1)} \le C\, r^{1/2} (Nr)^{-\varepsilon} \|F\|_{L^2(J;X_1)}. \tag{6.5}$$

*The remainder.* Let $\mathcal{R}(s, t) = \Phi(s, t) - \Phi(s, c) - (t - c)q(s)$. Then $|\partial_t \mathcal{R}(s, t)| = |\partial_t \Phi(s, t) - \partial_t \Phi(s, c)| \le C|t - c| \le Cr$ on $J$, because $\partial_t^2 \Theta$ is bounded. For $s \in I$ and $t \in J$ let $V_s(t) = b(s, t) \tau_{\mathcal{R}(s,t)}$, an operator on $X_{1,N}$. On $X_{1,N}$ the map $t \mapsto V_s(t)$ is $C^1$ in the operator norm, with $V_s'(t) = \partial_t b\, \tau_{\mathcal{R}} + b\, \partial_t \mathcal{R}\, \tau_{\mathcal{R}} \partial_\phi$, so

$$\|V_s(t)\| \le \mathsf{B}, \qquad \|V_s'(t)\| \le C\mathsf{B}(1 + Nr) \qquad \text{on } X_{1,N}. \tag{6.6}$$

Since $\tau_{\Phi(s,t)} = \tau_{\Phi(s,c)} \tau_{(t-c)q(s)} \tau_{\mathcal{R}(s,t)}$ and $V_s(t) = V_s(c + r) - \int_t^{c+r} V_s'(z)\, dz$, Fubini's theorem gives

$$\mathcal{A}^\sigma_{\delta,b}(\mathbf{1}_J F)(s) = \tau_{\Phi(s,c)} \Big[ V_s(c + r) G_{c+r}(s) - \int_c^{c+r} V_s'(z) G_z(s)\, dz \Big].$$

Here all vectors lie in $X_{1,N}$, on which the operators are bounded. If $F(t)=\sum_m c_m(t)e^{im\phi}$ (with $N\le|m|\le 4N$), then the map $(s,z)\mapsto G_z(s)$ is continuous, so the integrals are well defined. By (6.5), (6.6) and Minkowski's inequality,

$$\begin{aligned}\|\mathcal{A}^\sigma_{\delta,b}(\mathbf{1}_J F)\|_{L^2(I;X_1)} &\le C\mathsf{B}\,(1+r+Nr^2)\,r^{1/2}(Nr)^{-\varepsilon}\|F\|_{L^2(J;X_1)}\\ &\le C\mathsf{B}\,N^{-\varepsilon/2}r^{1/2}\|F\|_{L^2(J;X_1)}.\end{aligned}$$

Finally $\sum_J r^{1/2}\|F\|_{L^2(J;X_1)}\le|I|^{1/2}\|F\|_{L^2(I;X_1)}$ by the Cauchy–Schwarz inequality. □

The step that costs half of the exponent is the choice $r\approx N^{-1/2}$: the variation of the remainder costs $Nr^2$, and keeping it bounded leaves the frequency size $Nr\approx\sqrt{N}$ in (6.5).

### 6.4 Proof of Theorem C

Put $\beta=\varepsilon/2$; recall $0<\varepsilon<\frac12$ (Corollary 2.6). Let $|\delta|,|\delta_0|\le\frac34$ and $h=|\delta-\delta_0|\le1$. On $I^2$, uniformly in $|\delta|\le\frac34$, the functions

$$k=\frac{1}{2\pi\sqrt{Q}},\qquad \partial_\delta k=\frac{\delta-st}{2\pi Q^{3/2}},\qquad \sigma k\,\partial_\delta\Theta=-\frac{\sigma}{2\pi Q}$$

are smooth with bounded $\mathsf{B}$, by (6.2). For $F$ with values in trigonometric polynomials with frequencies in a finite set, differentiation under the integral gives

$$\partial_\delta\mathcal{A}^\sigma_{\delta,k}F=\mathcal{A}^\sigma_{\delta,\partial_\delta k}F+\mathcal{A}^\sigma_{\delta,\sigma k\partial_\delta\Theta}\,\partial_\phi F.$$

If $F$ takes values in $X_{1,N}$, Lemma 6.3 and $\|\partial_\phi\|\le C_4N$ on $X_{1,N}$ give $\|\partial_\delta\mathcal{A}^\sigma_{\delta,k}F\|\le CN^{1-\beta}\|F\|$; integrating in $\delta$, and using Lemma 6.3 at $\delta$ and at $\delta_0$, we get

$$\|(\mathcal{A}^\sigma_{\delta,k}-\mathcal{A}^\sigma_{\delta_0,k})F\|\le CN^{-\beta}\min\{1,hN\}\|F\|. \tag{6.7}$$

If $F$ takes values in the trigonometric polynomials with frequencies $|m|\le1$, the bound $|I|\|b\|_\infty$ for each term of the derivative gives $\|(\mathcal{A}^\sigma_{\delta,k}-\mathcal{A}^\sigma_{\delta_0,k})F\|\le Ch\|F\|$.

Let $\eta_0\in C_c^\infty((-2,2))$ equal 1 on $[-1,1]$ and $P_K=\eta_0(D_\phi/K)$ on $X_1$ (the multiplier $e^{im\phi}\mapsto\eta_0(m/K)e^{im\phi}$), so that $\|P_K\|\le\|h_{\eta_0}\|_1$ by Lemma 2.9(i). The range of $P_1$ has frequencies $|m|\le1$, and that of $P_{2^{j+1}}-P_{2^j}$ has frequencies $2^j<|m|<4\cdot2^j$, so lies in $X_{1,2^j}$. These multipliers commute with every $\mathcal{A}^\sigma_{\delta,b}$, since translations in $\phi$ do. For $F$ with values in trigonometric polynomials with frequencies in a finite set, $F=P_1F+\sum_{0\le j<j_0}(P_{2^{j+1}}-P_{2^j})F$ for $j_0$ large, and therefore

$$\|(\mathcal{A}^\sigma_{\delta,k}-\mathcal{A}^\sigma_{\delta_0,k})F\|\le C\Big(h+\sum_{j\ge0}2^{-j\beta}\min\{1,h2^j\}\Big)\|F\|\le C'h^\beta\|F\|,$$

splitting the sum at $2^j\approx1/h$ and using $0<\beta<1$, $0<h\le1$. Such $F$ are dense in $L^2(I;X_1)$ (Fejér means in $\phi$), and the operators are bounded, so the bound holds on $L^2(I;X_1)$.

By (6.4), applied in the coordinates of each pair $(i,j)$, and since the factors $2^{\pm1/2}$ between the norms of $L^2(S^2;E)$ and $L^2(I;X_1)$ cancel,

$$\|T_\delta-T_{\delta_0}\|_{L^2(S^2;E)}\le\sum_{i,j}\sum_{\sigma=\pm1}\|\mathcal{A}^\sigma_{\delta,k}-\mathcal{A}^\sigma_{\delta_0,k}\|_{L^2(I;X_1)}\le C\,h^\beta,$$

for $|\delta|,|\delta_0|\le\frac34$ and $h\le1$; the number of pairs depends only on $\rho$. With $\delta_0=0$ this is Theorem C for $|\delta|\le\frac34$; for $|\delta|>\frac34$ use $\|T_\delta-T_0\|\le2\le2(4/3)^\beta|\delta|^\beta$. For the other base points, if $|\delta_0|\le\frac12$ and $|\delta-\delta_0|\le\frac14$ then $|\delta|\le\frac34$ and the bound above applies; if $|\delta-\delta_0|>\frac14$ then $\|T_\delta-T_{\delta_0}\|\le2\le2\cdot4^\beta|\delta-\delta_0|^\beta$. So Theorem C holds with the constant $\max\{C,2\cdot4^\beta\}$. □

*Remark* 6.4 (no decay for $\ell^1$; the exponent). For $E = \ell^1$ and $\delta \neq 0$ the norm of $T_\delta - T_0$ is 2: take a continuous partition of unity $(\psi_j)$ of $S^2$ with supports of diameter less than $|\delta|$ and $f = \sum_j \psi_j e_j$; if $\langle x, y\rangle = \delta$ and $\langle x, y'\rangle = 0$ then $|y - y'| \geq |\delta|$, so $(T_\delta f)(x)$ and $(T_0 f)(x)$ have disjoint supports, nonnegative entries and entry sums one. In the proof, the type of $E$ enters only through Lemma 6.1. For $E = \mathbb{C}$, where Proposition 2.10 holds with the exponent $\frac{1}{2}$ (by Plancherel), the proof gives the exponent $\frac{1}{4}$; with the constants of Theorem A it gives $\varepsilon/2 < \frac{1}{4}$. The sharp exponent is $\frac{1}{2}$ (Legendre polynomials).

# 7 Groups and lattices

## 7.1 De la Salle's criterion

**Definition 7.1** (de la Salle [6, Definition 1.2 and before Theorem 1.4])**.** A *length function* on a locally compact group $G$ is a function $\ell : G \to [0, \infty)$ that is bounded on compact sets, symmetric and subadditive. A *two-step representation* is $(X_0, X_1, X_2, \pi_0, \pi_1)$ with Banach spaces $X_i$ and strongly continuous maps $\pi_i : G \to B(X_i, X_{i+1})$ such that $\pi_1(gg')\pi_0(g'') = \pi_1(g)\pi_0(g'g'')$; then $\pi(gg') = \pi_1(g)\pi_0(g')$ is well defined. For a class $\mathcal{E}$ of Banach spaces, $(G, \ell)$ has property $(*_\mathcal{E})$ if there are $s, t, C > 0$ and probability measures $m_n$ supported in $\{\ell \leq n\}$ such that for every two-step representation with $X_1 \in \mathcal{E}$ and $\|\pi_i(g)\| \leq Le^{s\ell(g)}$ $(i = 0, 1,\ g \in G)$ there is $P \in B(X_0, X_2)$ with $\|\pi(m_n) - P\| \leq CL^2 e^{-tn}$ and $\lim_n \|\pi(\delta_g m_n \delta_{g'}) - \pi(m_n)\| = 0$ for all $g, g' \in G$. $G$ has $(*_\mathcal{E})$ if $(G, \ell)$ has it for every length function $\ell$. (We follow the conventions of [6]: the measures are Borel probability measures, and $n$ runs over the integers for which $\{\ell \leq n\}$ is not empty.)

Property $(*_\mathcal{E})$ implies strong Banach property $(T)$ with respect to $\mathcal{E}$ in the sense of [5, Definition 1.1], [21, Definition 1.1] (take $X_0 = X_1 = X_2$ and $\pi_0 = \pi_1$ a representation) [6, Theorems 1.5 and 5.11].

For $n \geq 2$ and $\delta \in [-1, 1]$ let $T_\delta^{(n)}$ average a function on the sphere $S^n \subset \mathbb{R}^{n+1}$ over $\{y : \langle x, y\rangle = \delta\}$; write $T_\delta = T_\delta^{(2)}$.

**Theorem 7.2** (de la Salle [6, Theorem 1.5])**.** *Let $G$ be a connected simple Lie group with Lie algebra $\mathfrak{g}$, let $\Gamma \subset G$ be a lattice, and let $\mathcal{E}$ be a class of Banach spaces. Then $G$ and $\Gamma$ have $(*_\mathcal{E})$ if one of the following holds.*

1. *$\mathfrak{g}$ contains a subalgebra isomorphic to $\mathfrak{sp}_4$, and there are $\alpha \in (0, 1]$ and $C > 0$ such that for every $X \in \mathcal{E}$: $\|S_\theta - S_{\pi/4}\|_{B(L^2(\mathrm{SU}(2);X))} \leq C|\theta - \frac{\pi}{4}|^{\alpha/4}$ for $\theta \in [0, 2\pi]$ (1.3), and $\|T_\delta - T_0\|_{B(L^2(S^2;X))} \leq C|\delta|^{\alpha/2}$ for $\delta \in [-1, 1]$ (1.4).*
2. *$\mathfrak{g}$ contains a subalgebra isomorphic to $\mathfrak{sl}_{3n-3}$, $n \geq 2$, and there are $\alpha \in (0, 1]$ and $C > 0$ such that for every $X \in \mathcal{E}$: $\|T_\delta^{(n)} - T_0^{(n)}\|_{B(L^2(S^n;X))} \leq C|\delta|^{\alpha/2}$ for $\delta \in [-1, 1]$ (1.5).*

[6, Theorem 5.11] is the version for finite products of higher rank groups. Three features of the proof matter here. (i) For lattices, the class must be stable under $X \mapsto L^2(\Omega, \mu; X)$ [6, Remark 5.5 and before Theorem 5.11]. (ii) In the proof of [6, Theorem 5.4], right invariance is obtained "by symmetry" from the dual maps $\pi(g^{-1})^*$, whose middle space is $X_1^*$. We therefore apply the theorem to a class that is stable under both operations. (iii) The same step gives right invariance for the reflected measures; it gives the statement of [6] when the measures are symmetric. Remark 7.10 shows that this is no restriction.

**Definition 7.3.** For $C \geq 1$, $\eta \in (0,1)$ and $\alpha_1 \in (0, \frac{1}{2}]$ let $\mathcal{D} = \mathcal{D}(C, \eta, \alpha_1)$ be the class of complex Banach spaces $X$ such that, on $L^2(\mathrm{SU}(2); X)$ and $L^2(S^2; X)$ respectively,

$$\|S_\theta - S_{\theta'}\| \leq C\, d_{\mathbb{T}}(\theta, \theta')^\eta \quad (\theta, \theta' \in \mathbb{T}), \qquad \|T_\delta - T_0\| \leq C|\delta|^{\alpha_1} \quad (\delta \in [-1, 1]).$$

**Lemma 7.4.** *(i) For $1 < p \leq 2$ and $1 \leq M < \infty$ there are $C, \eta, \alpha_1$ with $\mathcal{C}_{p,M} \subseteq \mathcal{D}(C, \eta, \alpha_1)$. (ii) Each class $\mathcal{D}(C, \eta, \alpha_1)$ is stable under closed subspaces, under $X \mapsto L^2(\Omega, \mu; X)$ for every measure space, and under $X \mapsto X^*$. (iii) Each class $\mathcal{D}(C, \eta, \alpha_1)$ has uniformly nontrivial type: there are $p_0 \in (1, 2]$ and $M_0 < \infty$ with $\mathcal{D}(C, \eta, \alpha_1) \subseteq \mathcal{C}_{p_0, M_0}$.*

*Proof.* (i) The first bound is Corollary 1.1. The second is Theorem C with $\delta_0 = 0$ and $\alpha_1 = \varepsilon/2 < \frac{1}{4}$. (Yao first proved a bound of this form, with his own exponent, [21, Theorem 5.4]; his $T_\delta$ is the same latitude average.)

(ii) Subspaces: $L^2(K; Y)$ is an invariant closed subspace of $L^2(K; X)$ for $Y \subseteq X$. Bochner $L^2$: the spaces $L^2(K; L^2(\Omega; X))$ and $L^2(\Omega; L^2(K; X))$ are identified isometrically by Fubini (check the norms on finite sums of functions $a(k)b(\omega)x$, which are dense in both), and under this identification the operators act fiberwise, with the same norm bounds. Duals: let $K$ be a compact group with probability Haar measure and $\mu$ a probability measure on $K$. For $f \in L^2(K; X)$ and $g \in L^2(K; X^*)$, Fubini and the invariance of Haar measure give $\int \langle f(x), (\lambda(\mu)g)(x)\rangle\, dx = \int \langle (\lambda(\check{\mu})f)(x), g(x)\rangle\, dx$, where $\check{\mu}$ is the image of $\mu$ under inversion. Since $L^2(K; X^*)$ embeds isometrically into $L^2(K; X)^*$ (check on simple functions, using almost norming vectors on each level set; no Radon–Nikodym property is needed), we get $\|(\lambda(\mu) - \lambda(\nu)) \otimes \mathrm{Id}_{X^*}\| \leq \|(\lambda(\check{\mu}) - \lambda(\check{\nu})) \otimes \mathrm{Id}_X\|$. For the circle measure $\mu_\theta$ of $S_\theta$, $u_\theta^{-1} = d_{\pi/2} u_{-\theta} d_{-\pi/2}$, so $\check{\mu}_\theta = \mu_{-\theta}$, and the bound for all pairs $\theta, \theta'$ passes to $X^*$. The same argument applies to $T_\delta$ on $S^2$: it is its own transpose, because the rotation-invariant probability measure on $\{(x, y) : \langle x, y\rangle = \delta\}$ is symmetric under $(x, y) \mapsto (y, x)$.

(iii) The class $\mathcal{D} = \mathcal{D}(C, \eta, \alpha_1)$ is stable under countable $\ell^2$-direct sums, because

$$L^2\big(K; (\textstyle\bigoplus_k X_k)_{\ell^2}\big) = \big(\textstyle\bigoplus_k L^2(K; X_k)\big)_{\ell^2}$$

and the operators act coordinatewise. Suppose that (iii) fails. Then for every $k \geq 1$ there is $X_k \in \mathcal{D}$ with $T_{1+1/k}(X_k) > k$. Put $Y = (\bigoplus_k X_k)_{\ell^2} \in \mathcal{D}$. If $Y$ had type $p > 1$, then $T_{1+1/k}(X_k) \leq T_{1+1/k}(Y) \leq T_p(Y)$ whenever $1 + 1/k \leq p$, which is false for large $k$. So $Y$ has no nontrivial type, and by the theorem of Pisier [20] and Maurey–Pisier [18] the spaces $\ell^1_n$ are uniformly finitely representable in $Y$: for all $n$ and $\epsilon > 0$ there is a linear map $u$ from $\ell^1_n$ into $Y$ with $\|x\| \leq \|ux\| \leq (1 + \epsilon)\|x\|$. Hence, for $0 < \theta \leq \pi$,

$$\|(S_\theta - S_0) \otimes \mathrm{Id}_{\ell^1_n}\| \leq (1 + \epsilon)\|(S_\theta - S_0) \otimes \mathrm{Id}_Y\| \leq (1 + \epsilon) C \theta^\eta.$$

On the other hand $\sup_n \|(S_\theta - S_0) \otimes \mathrm{Id}_{\ell^1_n}\| = 2$. Indeed $S_\theta - S_0$ is convolution by $\mu = \mu_\theta - \mu_0$, and $\mu_\theta$, $\mu_0$ are probability measures carried by disjoint circles, so $\mu$ has total variation 2. For a finite Borel partition $\mathcal{P}$ of $K = \mathrm{SU}(2)$ let $f_{\mathcal{P}}(x) = (1_A(x))_{A \in \mathcal{P}}$, a unit vector of $L^2(K; \ell^1_{|\mathcal{P}|})$. Then

$$\|(\lambda(\mu) f_{\mathcal{P}})(x)\|_{\ell^1} = \sum_{A \in \mathcal{P}} |\mu(xA^{-1})|,$$

and for partitions of mesh tending to 0 this tends to the total variation of $\mu$ for every $x$ (test against continuous functions of norm at most one, which are uniformly close to functions constant on the cells); dominated convergence gives the claim. For small $\theta$ this contradicts the display. $\square$

**Lemma 7.5** (quotients)**.** *Let $q : \widetilde{G} \to G$ be a continuous surjective homomorphism of locally compact groups, $\ell$ a length function on $G$, and $\mathcal{E}$ a class of Banach spaces. If $(\widetilde{G}, \ell \circ q)$ has $(*_{\mathcal{E}})$, then $(G, \ell)$ has $(*_{\mathcal{E}})$, with the image measures $q_* m_n$.*

*Proof.* $\ell \circ q$ is a length function. A two-step representation of $G$ composed with $q$ is one of $\widetilde{G}$ with the same bounds, $\pi(q_* m_n) = (\pi \circ q)(m_n)$, $q_* m_n$ is carried by $\{\ell \le n\}$ (and if $m_n$ has compact support, then the support of $q_* m_n$ is the compact set $q(\operatorname{supp} m_n) \subseteq \{\ell \le n\}$), and $\delta_g q_* m_n \delta_{g'} = q_*(\delta_{\tilde{g}} m_n \delta_{\tilde{g}'})$ for lifts $\tilde{g}, \tilde{g}'$. □

*Remark* 7.6 (the center of $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$). The center of the universal cover $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ is $\mathbb{Z} \times \mathbb{Z}/2$, not a cyclic group. Indeed the maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$ is $U(2)$, whose universal cover $\mathrm{SU}(2) \times \mathbb{R}$ maps onto it by $(A, t) \mapsto e^{it}A$; the kernel is $\{((-1)^k I, k\pi) : k \in \mathbb{Z}\} \cong \mathbb{Z}$, and the preimage of the center $\{\pm I\}$ of $\mathrm{Sp}_4(\mathbb{R})$ is $\{(\pm I, k\pi) : k \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}/2$, which is the center of $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ because a lift of a central element of a connected group is central. For example $\widetilde{\mathrm{Sp}}_4(\mathbb{R})/\langle(-I, 0)\rangle$ has infinite center, so it is not a finite extension of the adjoint group. The proof of [3, Lemma 6.5] states that the center is cyclic and that quotients by its nontrivial subgroups are finite extensions of the adjoint group; the conclusion of that lemma is still right, because every connected Lie group with Lie algebra $\mathfrak{sp}_4(\mathbb{R})$ is a quotient of $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ by a discrete central subgroup, and Lemma 7.5 transfers $(*_{\mathcal{E}})$ from $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ to every such quotient, whatever its center.

## 7.2 Deduction of Theorem B

Fix $1 < p \le 2$ and $1 \le M < \infty$, and take $C, \eta, \alpha_1$ from Lemma 7.4(i) so that $\mathcal{C}_{p,M} \subseteq \mathcal{D} := \mathcal{D}(C, \eta, \alpha_1)$. Every $X \in \mathcal{D}$ satisfies the compact estimates (1.3), (1.4), and (with $n = 2$) (1.5) of Theorem 7.2 with $\alpha = \min\{1, 4\eta, 2\alpha_1\}$ and constant $\max\{C, 2\}$: indeed, if $|\theta - \frac{\pi}{4}| \le 1$ then

$$\|S_\theta - S_{\pi/4}\| \le C|\theta - \tfrac{\pi}{4}|^\eta \le C|\theta - \tfrac{\pi}{4}|^{\alpha/4},$$

and otherwise the left side is at most $2 \le 2|\theta - \frac{\pi}{4}|^{\alpha/4}$, while $\alpha/2 \le \alpha_1$ gives (1.4) and (1.5). Moreover, by Lemma 7.4(ii)–(iii), $\mathcal{D}$ is stable under closed subspaces, $X \mapsto L^2(\Omega, \mu; X)$, and $X \mapsto X^*$, and has uniformly nontrivial type.

Rather than invoking Theorem 7.2 as a black box—in particular, [6, Section 3.3] leaves the two-step $\mathrm{Sp}_4(\mathbb{R})$ and $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ details to the reader—Section 8 proves $(*_{\mathcal{D}})$ completely from the compact estimates defining $\mathcal{D}$: Theorem 8.11 establishes $(*_{\mathcal{D}})$ (with symmetric measures) for $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$, Theorem 8.15 extends this to every connected simple real Lie group $G$ of real rank at least two (using the closed $\mathfrak{sl}_3(\mathbb{R})$ or $\mathfrak{sp}_4(\mathbb{R})$ subgroup constructed in Proposition 8.14 together with the $\mathrm{SL}_3(\mathbb{R})$ case of [6, Section 3.1]), and Corollary 8.16 deduces $(*_{\mathcal{C}_{p,M}})$ for finite products of higher rank groups and all their lattices $\Gamma$ (using [6, Theorem 1.4, Lemmas 2.6 and 2.8, and Section 5] together with Lemma 7.4(ii)–(iii) and Remark 7.10). Since every class of uniformly nontrivial type is contained in some $\mathcal{C}_{p,M}$, this proves Theorem B. □

We state Theorem B for complex Banach spaces, as [21] does. Real spaces can be reduced to complex ones by complexification (inside $L^2(\mathbb{T}; X)$, which keeps the type constant); we do not write this out.

*Remark* 7.7 (what Theorem B rests on). (1) No unrefereed preprint is used as an input: the finite Fourier bilinear-shift bounds (after [21]) used in Theorems A and C are proved self-containedly in Section 2 directly from Bourgain's theorem [2, Theorem 1]. (2) For groups containing $\mathfrak{sp}_4(\mathbb{R})$ but not $\mathfrak{sl}_3(\mathbb{R})$, the two-step strong property $(*_{\mathcal{D}})$ for the building blocks $\mathrm{Sp}_4(\mathbb{R})$ and

$\widetilde{\mathrm{Sp}}_4(\mathbb{R})$—sketched in [6, Section 3.3] by reference to the one-space arguments of [3]—is proved completely in Section 8 (Theorem 8.11), including self-contained equivariant SU(2) Hölder bounds (Proposition 8.3) and explicit Cartan decompositions in $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ (Propositions 8.6 and 8.7). (3) The descent from $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{R})$ to every connected simple real Lie group $G$ of real rank at least two (including closedness of the rank-2 subgroup, quotients, finite central extensions, and the general length-function domination) is likewise proved completely in Section 8 (Lemma 8.12, Corollary 8.13, Proposition 8.14, and Theorem 8.15), repairing the cyclic-center slip in [3, Lemma 6.5] (Remark 7.6) and the omitted factor in [6, Lemma 4.1]. The remaining published inputs from [6] are the $\mathrm{SL}_3(\mathbb{R})$ case ([6, Section 3.1]), the non-Archimedean case for products ([6, Theorem 1.4]), the product and length-change lemmas ([6, Lemmas 2.6 and 2.8]), and the lattice induction theorem ([6, Section 5], with symmetric measures supplied by Remark 7.10).

### 7.3 Which groups are new

**Proposition 7.8.** *Let $\mathfrak{g}$ be a simple real Lie algebra of real rank two whose restricted root system is of type $B_2 = C_2$ or $BC_2$. Then $\mathfrak{g}$ contains no subalgebra isomorphic to $\mathfrak{sl}_3(\mathbb{R})$. Conversely, every simple real Lie algebra of real rank at least two that is not of this kind contains $\mathfrak{sl}_3(\mathbb{R})$.*

*Proof.* Suppose $\mathfrak{s} \subseteq \mathfrak{g}$ is isomorphic to $\mathfrak{sl}_3(\mathbb{R})$ and let $\mathfrak{a}'$ be the diagonal Cartan subalgebra of $\mathfrak{s}$. Its elements act on $\mathfrak{g}$ (a finite-dimensional representation of $\mathfrak{s}$) diagonalizably with real eigenvalues, so $\mathfrak{a}'$ is an abelian subspace of $\mathfrak{g}$ of real-diagonalizable elements, of dimension $2 =$ the real rank, hence a maximal one, conjugate to the standard one. The roots of $\mathfrak{s}$ are weights of $\mathfrak{a}'$ on $\mathfrak{g}$, so they are restricted roots of $\mathfrak{g}$. The Killing form of $\mathfrak{g}$ restricts to an invariant form on the simple algebra $\mathfrak{s}$, hence to a multiple of the Killing form of $\mathfrak{s}$, and this multiple is positive because both forms are positive definite on $\mathfrak{a}'$. So angles between roots are the same in $\mathfrak{s}$ and in $\mathfrak{g}$, and the restricted root system of $\mathfrak{g}$ contains two roots at an angle of $120^\circ$. The angles between roots of $B_2$ and $BC_2$ ($\pm e_1, \pm e_2, \pm e_1 \pm e_2$ and, for $BC_2$, $\pm 2e_1, \pm 2e_2$) are multiples of $45^\circ$. This is a contradiction.

For the converse, let $\Sigma$ be the restricted root system of $\mathfrak{g}$. If $\Sigma$ is not of type $B_2$ or $BC_2$, then $\Sigma$ contains two roots $\alpha, \beta$ of equal length at an angle of $120^\circ$ such that $\alpha - \beta$, $2\alpha + \beta$ and $\alpha + 2\beta$ are not roots: adjacent simple roots in types $A_n$, $D_n$, $E_n$; two long roots in $G_2$ and $F_4$; and $e_1 - e_2$, $e_2 - e_3$ in $B_n$, $C_n$, $BC_n$ for $n \geq 3$. (The three vectors $\alpha - \beta$, $2\alpha + \beta$, $\alpha + 2\beta$ have squared length $3|\alpha|^2$. Two roots of an irreducible root system have squared lengths in the ratio 3 only in type $G_2$, where the longer one is a long root; so the three vectors are not roots when $\alpha$ is long or the type is not $G_2$.) Let $\vartheta$ be a Cartan involution adapted to $\mathfrak{a}$, and choose nonzero $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$. For $\gamma = \alpha, \beta$ put $E_\gamma = X_\gamma$, $F_\gamma = -\vartheta X_\gamma$ and $H_\gamma = [E_\gamma, F_\gamma]$. The element $[X_\gamma, \vartheta X_\gamma]$ lies in the centralizer of $\mathfrak{a}$ and is mapped to its negative by $\vartheta$, so it lies in $\mathfrak{a}$; it is a negative multiple of the element of $\mathfrak{a}$ dual to $\gamma$ under the Killing form. After scaling $X_\gamma$ by a positive number, $\gamma(H_\gamma) = 2$, so $(E_\gamma, F_\gamma, H_\gamma)$ is an $\mathfrak{sl}_2$-triple, and $\beta(H_\alpha) = \alpha(H_\beta) = -1$ because $\alpha, \beta$ have equal length and make an angle of $120^\circ$. The remaining Serre relations of type $A_2$ hold: $[E_\alpha, F_\beta] \in \mathfrak{g}_{\alpha-\beta} = 0$ and $(\operatorname{ad} E_\alpha)^2 E_\beta \in \mathfrak{g}_{2\alpha+\beta} = 0$, and the same with $\alpha, \beta$ exchanged and with $\vartheta$ applied. So by Serre's presentation theorem [10, Section 18] these elements generate a nonzero homomorphic image of the split algebra $\mathfrak{sl}_3(\mathbb{R})$, which is simple; the image is isomorphic to $\mathfrak{sl}_3(\mathbb{R})$. (Compare [17, Proposition I.1.6.2].) □

From the tables of restricted root systems (for example [9, Chapter X, Table VI], [11, Appendix C]), the algebras of Proposition 7.8 are, for orientation (the proofs above do not use this list): $\mathfrak{so}(2, q)$ for $q \geq 3$ (with $\mathfrak{so}(2, 3) \cong \mathfrak{sp}_4(\mathbb{R})$), $\mathfrak{su}(2, q)$ for $q \geq 2$, $\mathfrak{sp}(2, q)$ for $q \geq 2$,

$\mathfrak{so}^*(8)$, $\mathfrak{so}^*(10)$, $\mathfrak{e}_{6(-14)}$, and $\mathfrak{sp}_4(\mathbb{C})$ viewed as a real Lie algebra. (The list has two repetitions: $\mathfrak{su}(2,2) \cong \mathfrak{so}(2,4)$ and $\mathfrak{so}^*(8) \cong \mathfrak{so}(2,6)$.) The new groups of Theorem B are all connected Lie groups with these Lie algebras, including the covers with infinite center such as $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$, and all their lattices, for example $\mathrm{Sp}_4(\mathbb{Z})$ (see Remark 8.17 for a complete audit of the tables).

*Remark* 7.9 (consequences). The known consequences of strong Banach property $(T)$ should apply to the groups and lattices of Theorem B by the arguments of Lafforgue [12, 13] and [3, Section 7]: fixed points for continuous affine actions on spaces in $\mathcal{C}_{p,M}$ whose linear part has small exponential growth (through the representation on $X \oplus_2 \mathbb{C}$, whose type constant is at most $\max\{M, 1\}$), and the nonembeddability into spaces of $\mathcal{C}_{p,M}$ of the expanders formed by finite quotients of a lattice. We have not written out these arguments, and we state no theorem about them. For isometric actions on uniformly convex spaces and for groups with finite center, the fixed-point statement and the super-expander statement are theorems of de Laat and de la Salle [4]; we claim nothing there.

*Remark* 7.10 (symmetric measures in the lattice step). This remark concerns the step "by symmetry" in the proof of [6, Theorem 5.4]. For a two-step representation $\pi$ of the lattice $\Gamma$, the dual maps $\pi^\vee(\gamma) = \pi(\gamma^{-1})^*$ form a two-step representation with middle space $X_1^*$, and $\pi^\vee(\gamma_1 m) = \pi(\check{m}\,\gamma_1^{-1})^*$ for a finitely supported measure $m$, where $\check{m}$ is the image of $m$ under inversion. For a non-cocompact lattice the measures $m_n^{(0)}$ need not be finitely supported; the identity still holds for them and for their fixed translates, because [6, Lemma 5.6] gives $\sum_\gamma e^{s\ell(\gamma)} m_n^{(0)}(\gamma) < \infty$ in the range of $s$ used there, so that the series defining these operators converge absolutely in operator norm and can be dualized term by term. So left invariance for $\pi^\vee$ gives right invariance for $\check{m}$, and the step is correct as written when the measures $m_n^{(0)}$ of that proof are symmetric. Two facts give this. First, $m_n^{(0)}$ is the image of $\widetilde{m}_n \otimes d\omega$ under $(g, \omega) \mapsto \alpha(g^{-1}, \omega)^{-1} = \alpha(g, g^{-1} \cdot \omega)$, and the action on the fundamental domain preserves the measure; hence the measure built from the reflection of $\widetilde{m}_n$ is the reflection of $m_n^{(0)}$, and $m_n^{(0)}$ is symmetric when $\widetilde{m}_n$ is. Second, symmetric witnesses always exist:

*if $(G, \ell)$ has $(*\mathcal{E})$ with constants $s, t, C$ and measures $m_n$, then it has $(*\mathcal{E})$ with symmetric measures.* Indeed, we may assume $s \le t/3$. Let $\pi$ be a two-step representation as in Definition 7.1, with limit $P$, and let $h \in G$. The maps $\pi_0(\,\cdot\, h)$ and $\pi_1$ form a two-step representation with the constant $Le^{s\ell(h)}$, and its limit is again $P$, by the invariance property of $\pi$. Hence $\|\pi(m_n \delta_h) - \pi(m_n)\| \le 2CL^2 e^{2s\ell(h) - tn}$. Integrating over $h$ with respect to $\check{m}_n$, which is carried by $\{\ell \le n\}$, gives $\|\pi(m_n * \check{m}_n) - \pi(m_n)\| \le 2CL^2 e^{-tn/3}$. The measures $\nu_n = m_n * \check{m}_n$ are symmetric, are carried by $\{\ell \le 2n\}$, and satisfy $\|\pi(\nu_n) - P\| \le 3CL^2 e^{-tn/3}$. Applying this to the two-step representation $\pi_0(\,\cdot\, g')$, $\pi_1(g\,\cdot\,)$, whose limit is also $P$, gives $\|\pi(\delta_g \nu_n \delta_{g'}) - P\| \to 0$ for all $g, g'$. After the change of index $k \mapsto \nu_{\lfloor k/2 \rfloor}$ this is $(*\mathcal{E})$ with symmetric measures. (Here $\nu_n$ is carried by $\{\ell \le 2n\}$. If "supported" is read as the closed support, fix a relatively compact neighbourhood $U$ of the identity and $D = \sup_U \ell$; then the closure of $\{\ell \le 2n\}$ lies in $\{\ell \le 2n + D\}$, and the index change $k \mapsto \nu_{\lfloor (k-D)/2 \rfloor}$ for large $k$, with symmetric probability measures on the finitely many first balls, gives the same conclusion with a larger constant.) The truncation to balls at the end of the proof of [6, Theorem 5.4] keeps the symmetry, because $\ell$ is symmetric.

# 8 Two-step strong property $(T)$ for $\mathrm{Sp}_4(\mathbb{R})$, its universal cover, and all higher rank real groups

The aim of this section is to give a complete, self-contained proof of de la Salle's two-step strong property $(*_{\mathcal{D}})$ for $\mathrm{Sp}_4(\mathbb{R})$ and its universal cover $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ from our two compact estimates (Theorem C and Corollary 1.1), filling in every detail left to the reader in [6, Section 3.3]. We also establish the equivariant $\mathrm{SU}(2)$ Hölder bounds directly by explicit matrix polynomials and Fell absorption (eliminating the auxiliary citation to [5, Proposition 2.8]), repair the cyclic-center slip in [3, Lemma 6.5] and the length-function/exponent details in [6, Lemma 4.1], and deduce Theorem B in full generality.

## 8.1 Realization of $\mathrm{Sp}_4(\mathbb{R})$, root subgroups, and the universal cover $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$

Let $I_2$ denote the $2 \times 2$ identity matrix and set

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \qquad J_0 = -J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

The real symplectic group (denoted $\mathrm{Sp}(2,\mathbb{R})$ in [3]) is

$$G = \mathrm{Sp}_4(\mathbb{R}) = \{g \in \mathrm{GL}_4(\mathbb{R}) : g^T J g = J\},$$

with 10-dimensional simple Lie algebra

$$\begin{aligned} \mathfrak{sp}_4(\mathbb{R}) &= \{X \in M_4(\mathbb{R}) : X^T J + JX = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} : A, B, C \in M_2(\mathbb{R}),\ B^T = B,\ C^T = C \right\}. \end{aligned}$$

The Cartan involution $\vartheta(X) = -X^T$ preserves $\mathfrak{sp}_4(\mathbb{R})$ and yields the Cartan decomposition $\mathfrak{sp}_4(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : A^T = -A,\ B^T = B \right\}, \qquad \mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A^T = A,\ B^T = B \right\}.$$

Define the injective $\mathbb{R}$-algebra homomorphism $\iota : M_2(\mathbb{C}) \to M_4(\mathbb{R})$ by

$$\iota(A + iB) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \qquad (A, B \in M_2(\mathbb{R})).$$

It satisfies $\iota(X^*) = \iota(X)^T$, $\det \iota(X) = |\det X|^2$, and $\iota(iI_2) = J_0 = -J$. Hence $K = G \cap \mathrm{SO}(4) = \iota(\mathrm{U}(2)) \cong \mathrm{U}(2)$ is a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k} = \iota(\mathfrak{u}(2))$. We denote its semisimple compact subgroup and central circle by

$$H = \iota(\mathrm{SU}(2)) \cong \mathrm{SU}(2), \qquad v_t = \iota(e^{it} I_2) = \exp(t J_0) \qquad (t \in \mathbb{R}),$$

so that $K = \{v_t h : t \in \mathbb{R},\ h \in H\}$ and $v_\pi = -I_4$. (Note that $\exp(tJ) = v_{-t}$, so we use $J_0 = -J$ throughout to keep the sign $\exp(tJ_0) = v_t$ consistent.) The center of $G$ is $Z(G) = \{\pm I_4\}$: any central element commutes with all diagonal matrices in $G$, hence is diagonal, and commuting with $\iota(\mathrm{U}(2))$ forces it to be a real scalar matrix $\lambda I_4$ with $\lambda^2 = 1$.

A maximal abelian subspace of $\mathfrak{p}$ is

$$\mathfrak{a} = \big\{ H(\beta,\gamma) = \operatorname{diag}(\beta,\gamma,-\beta,-\gamma) : \beta,\gamma \in \mathbb{R} \big\},$$

so $\operatorname{rank}_{\mathbb{R}}(\mathrm{Sp}_4(\mathbb{R})) = 2$. Put $D(\beta,\gamma) = \exp(H(\beta,\gamma)) = \operatorname{diag}(e^{\beta}, e^{\gamma}, e^{-\beta}, e^{-\gamma})$ and $A = \exp(\mathfrak{a})$. Let $e_1, e_2 \in \mathfrak{a}^*$ be the coordinate functionals $e_1(H(\beta,\gamma)) = \beta$, $e_2(H(\beta,\gamma)) = \gamma$. The restricted root system $\Sigma(\mathfrak{sp}_4(\mathbb{R}), \mathfrak{a})$ is of type $C_2$ $(= B_2)$, with all eight root spaces one-dimensional:

- *Long roots* $\pm 2e_1, \pm 2e_2$: with matrix units $E_{ij}$, $X_{2e_1} = E_{13}$, $X_{-2e_1} = E_{31}$, $H_{2e_1} = H(1,0)$, and $X_{2e_2} = E_{24}$, $X_{-2e_2} = E_{42}$, $H_{2e_2} = H(0,1)$. The corresponding root subgroups $G_{2e_1}, G_{2e_2} \cong \mathrm{SL}_2(\mathbb{R})$ act on $\mathbb{R}e_1 \oplus \mathbb{R}e_3$ and $\mathbb{R}e_2 \oplus \mathbb{R}e_4$ respectively, commute with each other, and intersect $K$ in $\{w_{\theta,0}\}$ and $\{w_{0,\theta'}\}$, where

$$w_{\theta,\theta'} = \iota\big(\operatorname{diag}(e^{i\theta}, e^{i\theta'})\big) = v_{(\theta+\theta')/2}\, \iota(d_{(\theta-\theta')/2}) \qquad (\theta,\theta' \in \mathbb{R}).$$

- *Short roots* $\pm(e_1 - e_2), \pm(e_1 + e_2)$: $X_{e_1-e_2} = E_{12} - E_{43}$, $X_{-(e_1-e_2)} = E_{21} - E_{34}$, $H_{e_1-e_2} = H(1,-1)$, generating $G_{e_1-e_2} = \big\{ \left(\begin{smallmatrix} g & 0 \\ 0 & (g^T)^{-1} \end{smallmatrix}\right) : g \in \mathrm{SL}_2(\mathbb{R}) \big\} \cong \mathrm{SL}_2(\mathbb{R})$ with maximal compact circle $\{\iota(r_\theta)\} \subset H$; and $X_{e_1+e_2} = E_{14} + E_{23}$, $X_{-(e_1+e_2)} = E_{41} + E_{32}$, $H_{e_1+e_2} = H(1,1)$, generating $G_{e_1+e_2} \cong \mathrm{SL}_2(\mathbb{R})$ with maximal compact circle $\{\iota(\cos\theta\, I_2 - i \sin\theta \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))\} \subset H$.

In the adjoint group $\mathrm{PSp}_4(\mathbb{R}) = G/\{\pm I_4\} \cong \mathrm{SO}_0(2,3)$, the short-root subgroups contain $-I_4$ and therefore descend to $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}_0(2,1)$, whereas $w_{\pi,0} = \operatorname{diag}(-1,1,-1,1) \neq \pm I_4$, so the long-root subgroups remain isomorphic to $\mathrm{SL}_2(\mathbb{R})$.

Choosing simple roots $e_1 - e_2$ and $2e_2$ gives the closed positive Weyl chamber $\overline{A^+} = \{D(\beta,\gamma) : \beta \geq \gamma \geq 0\}$ and the Cartan decomposition $G = K\overline{A^+}K$, in which $(\beta,\gamma)$ is uniquely determined by $g \in G$: since $g^{-1} = -Jg^TJ$ is orthogonally similar to $g^T$, the singular values of $g$ are $e^{\beta} \geq e^{\gamma} \geq e^{-\gamma} \geq e^{-\beta}$, and $\ell_G(g) := \log\|g\|_{\mathrm{op}} = \beta$ (Euclidean operator norm on $\mathbb{R}^4$) is a $K$-biinvariant length function on $G$.

We next construct the universal covering group $\widetilde{G} = \widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and the Guichardet–Wigner–Dupont–Rawnsley quasi-morphism $\Phi$ self-containedly (compare [8] and [3, Lemma 2.2]).

**Proposition 8.1** (universal cover, center, quasi-morphism, and length function)**.** *For* $g = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{R})$, *put* $\alpha(g) = \frac{1}{2}\big((A+D)+i(B-C)\big) \in M_2(\mathbb{C})$ *and* $\beta(g) = \frac{1}{2}\big((A-D)+i(B+C)\big) \in M_2(\mathbb{C})$.

*(i) For every* $g \in \mathrm{Sp}_4(\mathbb{R})$, $\alpha(g) \in \mathrm{GL}_2(\mathbb{C})$ *and* $\|\alpha(g)^{-1}\beta(g)\|_{\mathrm{op}} < 1$, $\|\overline{\beta(g)}\,\alpha(g)^{-1}\|_{\mathrm{op}} < 1$. *Setting* $\zeta(g) = \det\alpha(g)/|\det\alpha(g)| \in S^1$, *there is a unique smooth 2-cocycle* $\eta : \mathrm{Sp}_4(\mathbb{R}) \times \mathrm{Sp}_4(\mathbb{R}) \to (-\pi,\pi)$ *vanishing at* $(I_4, I_4)$ *such that* $\zeta(g_1g_2) = \zeta(g_1)\zeta(g_2)e^{i\eta(g_1,g_2)}$ *for all* $g_1, g_2 \in \mathrm{Sp}_4(\mathbb{R})$*; moreover* $\eta(g,g^{-1}) = 0$ *and* $\eta(\iota(k),g) = \eta(g,\iota(k)) = 0$ *for all* $k \in \mathrm{U}(2)$ *and* $g \in \mathrm{Sp}_4(\mathbb{R})$.

*(ii) The manifold* $\widetilde{G} = \{(g,\tau) \in \mathrm{Sp}_4(\mathbb{R}) \times \mathbb{R} : e^{i\tau} = \zeta(g)\}$ *with group law* $(g_1,\tau_1)(g_2,\tau_2) = (g_1g_2, \tau_1 + \tau_2 + \eta(g_1,g_2))$ *and inverse* $(g,\tau)^{-1} = (g^{-1}, -\tau)$ *is the universal covering group* $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ *of* $\mathrm{Sp}_4(\mathbb{R})$, *with covering homomorphism* $\sigma(g,\tau) = g$.

*(iii) Denote by* $\widetilde{\exp} : \mathfrak{sp}_4(\mathbb{R}) \to \widetilde{G}$ *the exponential map, and put* $\widetilde{v}_t = \widetilde{\exp}(tJ_0) = (v_t, 2t)$, $\widetilde{D}(\beta,\gamma) = (D(\beta,\gamma),0)$, $\widetilde{\iota}(h) = (\iota(h),0)$ *for* $h \in \mathrm{SU}(2)$, $\widetilde{H} = \widetilde{\iota}(\mathrm{SU}(2))$, *and* $\widetilde{K} = \sigma^{-1}(K)$. *Then* $(t,h) \mapsto \widetilde{v}_t\widetilde{\iota}(h)$ *is a Lie group isomorphism* $\mathbb{R} \times \mathrm{SU}(2) \xrightarrow{\sim} \widetilde{K}$,

$$\ker\sigma = \big\{ \widetilde{v}_{k\pi}\widetilde{\iota}((-1)^k I_2) : k \in \mathbb{Z} \big\} \cong \mathbb{Z},$$
$$Z(\widetilde{G}) = \big\{ \widetilde{v}_{k\pi}\widetilde{\iota}(\varepsilon I_2) : k \in \mathbb{Z},\ \varepsilon \in \{\pm 1\} \big\} \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}).$$

*In particular, $\widetilde{v}_\pi \in Z(\widetilde{G})$ and $\widetilde{\iota}(-I_2) \in Z(\widetilde{G})$ are distinct central elements, and the short-root subgroups of $\widetilde{G}$ are isomorphic to $\mathrm{SL}_2(\mathbb{R})$ while the long-root subgroups are isomorphic to $\widetilde{\mathrm{SL}_2}(\mathbb{R})$.*

*(iv) Every $g \in \widetilde{G}$ admits a Cartan decomposition $g = \widetilde{h}_1 \widetilde{v}_t \widetilde{D}(\beta,\gamma) \widetilde{v}_s \widetilde{h}_2$ with $\widetilde{h}_1, \widetilde{h}_2 \in \widetilde{H}$, unique $\beta \geq \gamma \geq 0$, and $t, s \in \mathbb{R}$ (where $s$ may be chosen in $[0,\pi)$). The smooth map $\Phi : \widetilde{G} \to \mathbb{R}$, $\Phi(g,\tau) = \tau/2$, satisfies*

$$\Phi(\widetilde{h}_1 \widetilde{v}_t \widetilde{D}(\beta,\gamma) \widetilde{v}_s \widetilde{h}_2) = t + s, \qquad \Phi(g^{-1}) = -\Phi(g), \qquad \Phi(\widetilde{h}_1 g \widetilde{h}_2) = \Phi(g),$$

*and*

$$|\Phi(g_1 g_2) - \Phi(g_1) - \Phi(g_2)| < \frac{\pi}{2} \qquad (g_1, g_2 \in \widetilde{G}). \tag{8.1}$$

*(v) The function*

$$\ell_0(g) = \begin{cases} 0 & g \in \widetilde{H}, \\ \log\|\sigma(g)\|_{\mathrm{op}} + |\Phi(g)| + \frac{\pi}{2} & g \notin \widetilde{H} \end{cases}$$

*is an $\widetilde{H}$-biinvariant length function on $\widetilde{G}$, and for every length function $\ell$ on $\widetilde{G}$ there exist $a, b > 0$ such that $\ell \leq a\ell_0 + b$.*

*Proof.* (i) Introduce the unitary matrix $U = 2^{-1/2}\left(\begin{smallmatrix} I_2 & iI_2 \\ I_2 & -iI_2 \end{smallmatrix}\right)$ and $\Sigma = \mathrm{diag}(I_2, -I_2)$. Direct block multiplication gives $UJU^* = -i\Sigma$ and $UgU^* = \left(\begin{smallmatrix} \alpha(g) & \beta(g) \\ \overline{\beta(g)} & \overline{\alpha(g)} \end{smallmatrix}\right)$. Since $g \in \mathrm{Sp}_4(\mathbb{R})$ satisfies both $g^T J g = J$ and $gJg^T = J$, conjugating by $U$ yields $(UgU^*)^*\Sigma(UgU^*) = \Sigma$ and $(UgU^*)\Sigma(UgU^*)^* = \Sigma$. Their $(1,1)$ blocks read

$$\alpha^*\alpha - (\overline{\beta})^*\overline{\beta} = I_2, \qquad \alpha\alpha^* - \beta\beta^* = I_2.$$

Hence $\alpha\alpha^* \geq I_2$, so $\alpha(g)$ is invertible, and

$$(\alpha^{-1}\beta)(\alpha^{-1}\beta)^* = I_2 - \alpha^{-1}(\alpha^{-1})^* < I_2, \qquad (\overline{\beta}\alpha^{-1})^*(\overline{\beta}\alpha^{-1}) = I_2 - (\alpha^{-1})^*\alpha^{-1} < I_2,$$

giving $\|\alpha^{-1}\beta\|_{\mathrm{op}} < 1$ and $\|\overline{\beta}\alpha^{-1}\|_{\mathrm{op}} < 1$. For $g_1, g_2 \in \mathrm{Sp}_4(\mathbb{R})$, the $(1,1)$ block of $U(g_1g_2)U^* = (Ug_1U^*)(Ug_2U^*)$ is

$$\alpha(g_1g_2) = \alpha(g_1)\big(I_2 + M(g_1,g_2)\big)\alpha(g_2), \qquad M(g_1,g_2) = \big(\alpha(g_1)^{-1}\beta(g_1)\big)\big(\overline{\beta(g_2)}\,\alpha(g_2)^{-1}\big).$$

Thus $\|M(g_1,g_2)\|_{\mathrm{op}} < 1$, so both eigenvalues $\lambda_1, \lambda_2$ of $M(g_1,g_2)$ lie in the open unit disk; in particular $\mathrm{Re}(1+\lambda_j) > 0$, so $\arg(1+\lambda_j) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The power series $\log(I_2 + M) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} M^k$ converges locally uniformly with all derivatives on $\mathrm{Sp}_4(\mathbb{R}) \times \mathrm{Sp}_4(\mathbb{R})$ (as $\|M\|_{\mathrm{op}} < 1$ on compact sets), and the smooth function

$$\eta(g_1,g_2) := \mathrm{Im}\,\mathrm{tr}\log(I_2 + M(g_1,g_2)) = \arg(1+\lambda_1) + \arg(1+\lambda_2)$$

takes values in $(-\pi,\pi)$, vanishes at $(I_4, I_4)$, and satisfies $\zeta(g_1g_2) = \zeta(g_1)\zeta(g_2)e^{i\eta(g_1,g_2)}$. The 2-cocycle identity $\eta(g_1,g_2) + \eta(g_1g_2,g_3) = \eta(g_1,g_2g_3) + \eta(g_2,g_3)$ holds because both sides are continuous on the connected space $\mathrm{Sp}_4(\mathbb{R})^3$, vanish at $(I_4,I_4,I_4)$, and have the same exponential $\zeta(g_1g_2g_3)/(\zeta(g_1)\zeta(g_2)\zeta(g_3))$. For $g^{-1} = -Jg^TJ$ we have $Ug^{-1}U^* = \Sigma(UgU^*)^*\Sigma = \left(\begin{smallmatrix} \alpha^* & -\beta^T \\ -\beta^* & \alpha^T \end{smallmatrix}\right)$, so $\overline{\beta(g^{-1})} = -\beta(g)^*$ and $M(g,g^{-1}) = -(\alpha^{-1}\beta)(\alpha^{-1}\beta)^*$ is negative semidefinite Hermitian of

norm $< 1$; hence $I_2 + M(g, g^{-1}) > 0$ has positive determinant and $\eta(g, g^{-1}) = 0$. For $k \in \mathrm{U}(2)$ we have $\alpha(\iota(k)) = k$ and $\beta(\iota(k)) = 0$, so $M(\iota(k), g) = M(g, \iota(k)) = 0$ and $\eta(\iota(k), g) = \eta(g, \iota(k)) = 0$.

(ii) The cocycle identity makes $\widetilde{G}$ a Lie group with identity $(I_4, 0)$ and inverse $(g^{-1}, -\tau)$ (since $\zeta(g^{-1}) = \overline{\zeta(g)}$ and $\eta(g, g^{-1}) = 0$). Because $\mathrm{Sp}_4(\mathbb{R})$ deformation retracts onto $K \cong \mathrm{U}(2)$ via the Iwasawa decomposition, $\pi_1(\mathrm{Sp}_4(\mathbb{R})) \cong \pi_1(\mathrm{U}(2)) \cong \mathbb{Z}$, and $\zeta|_K = \det : \mathrm{U}(2) \to S^1$ induces an isomorphism on $\pi_1$. Thus the $\mathbb{Z}$-cover $\sigma : \widetilde{G} \to \mathrm{Sp}_4(\mathbb{R})$ corresponds to the trivial subgroup of $\pi_1(\mathrm{Sp}_4(\mathbb{R}))$, so $\widetilde{G}$ is connected and simply connected.

(iii) For $k \in \mathrm{U}(2)$ we have $\alpha(\iota(k)) = k$, $\beta(\iota(k)) = 0$, so $\zeta(\iota(k)) = \det k$; for $D(\beta, \gamma)$ we have $\alpha(D(\beta, \gamma)) = \mathrm{diag}(\cosh\beta, \cosh\gamma)$, $\beta(D(\beta, \gamma)) = \mathrm{diag}(\sinh\beta, \sinh\gamma)$, and $\zeta(D(\beta, \gamma)) = 1$. For any $(\beta_1, \gamma_1), (\beta_2, \gamma_2) \in \mathbb{R}^2$,

$$M\big(D(\beta_1, \gamma_1), D(\beta_2, \gamma_2)\big) = \mathrm{diag}(\tanh\beta_1 \tanh\beta_2,\ \tanh\gamma_1 \tanh\gamma_2)$$

is real diagonal with entries in $(-1, 1)$, so $I_2 + M$ is diagonal positive definite, giving $\eta\big(D(\beta_1, \gamma_1), D(\beta_2, \gamma_2)\big) = 0$. Hence $t \mapsto (D(t\beta, t\gamma), 0)$ is a one-parameter subgroup of $\widetilde{G}$ (as are $t \mapsto (v_t, 2t)$ and $t \mapsto (\iota(\exp(tX)), 0)$ for $X \in \mathfrak{su}(2)$, since $\eta$ vanishes on $\iota(\mathrm{U}(2))$). Checking derivatives at $t = 0$ gives $\widetilde{v}_t = (v_t, 2t)$, $\widetilde{D}(\beta, \gamma) = (D(\beta, \gamma), 0)$, and $\widetilde{\iota}(h) = (\iota(h), 0)$. Since $\eta(\iota(k_1), \iota(k_2)) = 0$ for $k_1, k_2 \in \mathrm{U}(2)$, $(t, h) \mapsto \widetilde{v}_t\widetilde{\iota}(h) = (v_t\iota(h), 2t)$ is an isomorphism $\mathbb{R} \times \mathrm{SU}(2) \xrightarrow{\sim} \widetilde{K}$. Now $\sigma(\widetilde{v}_t\widetilde{\iota}(h)) = I_4$ iff $e^{it}h = I_2$, iff $t = k\pi$ and $h = (-1)^k I_2$ ($k \in \mathbb{Z}$), giving $\ker\sigma$. Similarly $\sigma^{-1}(Z(G)) = \sigma^{-1}(\{\pm I_4\}) = \{\widetilde{v}_{k\pi}\widetilde{\iota}(\varepsilon I_2) : k \in \mathbb{Z},\ \varepsilon = \pm 1\} \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$. Every $z \in \sigma^{-1}(Z(G))$ is central in $\widetilde{G}$ because the continuous map $g \mapsto gzg^{-1}z^{-1}$ from connected $\widetilde{G}$ into discrete $\ker\sigma$ maps 1 to 1; conversely $\sigma(Z(\widetilde{G})) \subseteq Z(G)$ by surjectivity of $\sigma$. Finally, the compact generators of $\mathfrak{g}_{e_1 \pm e_2}$ lie in $\iota(\mathfrak{su}(2))$ and exponentiate to circles of period $2\pi$ in $\widetilde{H}$ containing $\widetilde{\iota}(-I_2) \neq 1$, so $\widetilde{G}_{e_1 \pm e_2} \cong \mathrm{SL}_2(\mathbb{R})$, whereas the compact generators of $\mathfrak{g}_{2e_1}, \mathfrak{g}_{2e_2}$ exponentiate to $\widetilde{w}_{\theta,0} = \widetilde{v}_{\theta/2}\widetilde{\iota}(d_{\theta/2})$ and $\widetilde{w}_{0,\theta} = \widetilde{v}_{\theta/2}\widetilde{\iota}(d_{-\theta/2})$, on which $\Phi$ equals $\theta/2$, so $\widetilde{G}_{2e_1}, \widetilde{G}_{2e_2} \cong \widetilde{\mathrm{SL}}_2(\mathbb{R})$.

(iv) Lifting $G = K\overline{A^+}K$ through $\sigma$ yields $\widetilde{G} = \widetilde{K}\,\overline{\widetilde{A}^+}\,\widetilde{K}$; since $\widetilde{v}_\pi \in Z(\widetilde{G})$, shifting $s \mapsto s - k\pi$, $t \mapsto t + k\pi$ places $s \in [0, \pi)$. For $g = v_t\iota(h_1)D(\beta, \gamma)v_s\iota(h_2)$, we have

$$\alpha(g) = e^{it}h_1\,\mathrm{diag}(\cosh\beta, \cosh\gamma)e^{is}h_2,$$

and hence $\zeta(g) = e^{2i(t+s)}$. Thus the continuous maps into $\widetilde{G}$ sending $(h_1, h_2, t, s, \beta, \gamma)$ to $\widetilde{h}_1\widetilde{v}_t\widetilde{D}(\beta, \gamma)\widetilde{v}_s\widetilde{h}_2$ and to $(g, 2(t + s))$ have the same projection in $G$ and agree at the identity, hence coincide everywhere; this gives $\Phi = t + s$. The remaining identities in (iv) follow immediately from (i) and (ii).

(v) Symmetry and $\widetilde{H}$-biinvariance of $\ell_0$ follow from $\|\sigma(g)^{-1}\|_{\mathrm{op}} = \|\sigma(g)\|_{\mathrm{op}}$, $\Phi(g^{-1}) = -\Phi(g)$, and $\Phi(\widetilde{h}_1 g \widetilde{h}_2) = \Phi(g)$. For subadditivity: if $g_1, g_2 \notin \widetilde{H}$ and $g_1 g_2 \notin \widetilde{H}$, then (8.1) gives $|\Phi(g_1 g_2)| + \frac{\pi}{2} < (|\Phi(g_1)| + \frac{\pi}{2}) + (|\Phi(g_2)| + \frac{\pi}{2})$; the remaining cases with $g_1, g_2$ or $g_1 g_2$ in $\widetilde{H}$ are immediate from $\widetilde{H}$-biinvariance. Boundedness on compact sets follows from continuity of $\sigma$ and $\Phi$. Finally, let $\ell$ be any length function on $\widetilde{G}$, and set $C_H = \sup_{\widetilde{H}} \ell$, $C_v = \sup_{|u| \le 1} \ell(\widetilde{v}_u)$, $C_D = \sup_{0 \le r \le 1} \ell(\widetilde{D}(r, 0))$. For $g = \widetilde{h}_1\widetilde{v}_t\widetilde{D}(\beta, \gamma)\widetilde{v}_s\widetilde{h}_2$ with $s \in [0, \pi)$ and $t = \Phi(g) - s$, we have $\widetilde{D}(\beta, \gamma) = \widetilde{D}(\beta, 0)\,\widetilde{\iota}(r_{\pi/2})\widetilde{D}(\gamma, 0)\widetilde{\iota}(r_{-\pi/2})$ (valid in $\widetilde{G}$ by continuity in $\gamma$ from $\gamma = 0$), and $\ell(\widetilde{v}_s) \le C_v\lceil s \rceil \le C_v(\pi + 1)$, so subadditivity gives

$$\ell(g) \le 4C_H + C_D(\beta + 1) + C_D(\gamma + 1) + C_v(|\Phi(g)| + \pi + 1) + C_v(\pi + 1) \le a\ell_0(g) + b$$

for $a = \max\{2C_D, C_v\}$ and $b = 4C_H + 2C_D + C_v(2\pi + 2)$. □

### 8.2 Two-step representations and self-contained equivariant $\mathrm{SU}(2)$ bounds

We work in the setting of Definition 7.1 (two-step representations $(X_0, X_1, X_2, \pi_0, \pi_1)$ and property $(*_{\mathcal{E}})$) and with the class $\mathcal{D} = \mathcal{D}(C, \eta, \alpha_1)$ of Definition 7.3. Recall from Lemma 7.4 that for every $1 < p \le 2$ and $1 \le M < \infty$, there exist $C \ge 1$, $\eta \in (0,1)$ and $\alpha_1 \in (0, \frac{1}{2}]$ (via Corollary 1.1 and Theorem C) such that $\mathcal{C}_{p,M} \subseteq \mathcal{D}(C, \eta, \alpha_1)$, and that $\mathcal{D}$ is stable under closed subspaces, under $X \mapsto L^2(\Omega, \mu; X)$ for every measure space, and under $X \mapsto X^*$.

On $\mathrm{SU}(2)$, recall the matrices $r_\theta = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$, $d_\theta = \left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ and $u_\theta = 2^{-1/2}\left(\begin{smallmatrix} e^{i\theta} & -1 \\ 1 & e^{-i\theta} \end{smallmatrix}\right)$, and put $R = \{r_\theta : \theta \in \mathbb{R}\} \subset \mathrm{SU}(2)$ with probability Haar measure $\lambda_R$. On $L^2(\mathrm{SU}(2); X)$ let $P = \lambda(\lambda_R) \otimes \mathrm{Id}_X$ and

$$\widetilde{T}_\theta = P(\lambda(d_\theta) \otimes \mathrm{Id}_X)P = \lambda(\lambda_R * \delta_{d_\theta} * \lambda_R) \otimes \mathrm{Id}_X, \qquad S_\theta = \lambda(\mu_\theta) \otimes \mathrm{Id}_X,$$

where $\mu_\theta$ is the pushforward of $\frac{d\alpha}{2\pi}$ on $[0, 2\pi]$ under $\alpha \mapsto d_\alpha u_\theta d_{-\alpha}$ (we write $\widetilde{T}_\theta$ for the operator on $L^2(\mathrm{SU}(2); X)$ to distinguish it from the latitude operator $T_\delta$ on $L^2(S^2; X)$).

**Lemma 8.2** (transfer from $S^2$ to $\mathrm{SU}(2)$)**.** *For every complex Banach space $X$ and every $\theta \in \mathbb{R}$,*

$$\|\widetilde{T}_\theta - \widetilde{T}_{\pi/4}\|_{B(L^2(\mathrm{SU}(2);X))} = \|T_{\cos(2\theta)} - T_0\|_{B(L^2(S^2;X))},$$

*and $\widetilde{T}_{-\theta} = \widetilde{T}_\theta$. Consequently, for every $X \in \mathcal{D}(C, \eta, \alpha_1)$ and all $\theta \in \mathbb{R}$, with $s_1 = \alpha_1$, $s_2 = \eta$, $C_1 = 2^{s_1}C$ and $C_2 = C$,*

$$\|\widetilde{T}_\theta - \widetilde{T}_{\pi/4}\|_{B(L^2(\mathrm{SU}(2);X))} \le C_1|\theta - \tfrac{\pi}{4}|^{s_1}, \qquad \|S_{\pm\theta} - S_{\pm\pi/2}\|_{B(L^2(\mathrm{SU}(2);X))} \le C_2|\theta - \tfrac{\pi}{2}|^{s_2}. \quad (8.2)$$

*Proof.* Identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ via the orthonormal basis $e_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$. A direct computation of $ue_ju^*$ for $u = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) \in \mathrm{SU}(2)$ gives the surjective homomorphism $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ with kernel $\{\pm I_2\}$,

$$\mathrm{Ad}\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(a^2+b^2) & \mathrm{Im}(b^2-a^2) & 2\,\mathrm{Im}(ab) \\ \mathrm{Im}(a^2+b^2) & \mathrm{Re}(a^2-b^2) & -2\,\mathrm{Re}(ab) \\ 2\,\mathrm{Im}(a\bar{b}) & 2\,\mathrm{Re}(a\bar{b}) & |a|^2-|b|^2 \end{pmatrix}. \quad (8.3)$$

In particular,

$$\mathrm{Ad}(r_\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha \\ 0 & -\sin 2\alpha & \cos 2\alpha \end{pmatrix}, \qquad \mathrm{Ad}(d_\theta)e_1 = (\cos 2\theta, \sin 2\theta, 0)^T.$$

Hence $\mathrm{Ad}(R)$ is the stabilizer of $e_1 \in S^2$, and $(Uf)(u) := f(\mathrm{Ad}(u)^{-1}e_1)$ defines a linear isometry $U : L^2(S^2; X) \xrightarrow{\sim} \mathrm{Ran}(P) = L^2(R\backslash \mathrm{SU}(2); X)$. For $f \in L^2(S^2; X)$,

$$(\widetilde{T}_\theta Uf)(u) = \int_R f\big(\mathrm{Ad}(u)^{-1}\,\mathrm{Ad}(r)\,\mathrm{Ad}(d_\theta)e_1\big)\,dr = (T_{\cos(2\theta)}f)\big(\mathrm{Ad}(u)^{-1}e_1\big) = (UT_{\cos(2\theta)}f)(u),$$

because $\mathrm{Ad}(R)\,\mathrm{Ad}(d_\theta)e_1$ is the latitude circle $\{z \in S^2 : \langle z, e_1\rangle = \cos(2\theta)\}$ with uniform measure. Since $P$ has norm 1 and $U$ is isometric onto $\mathrm{Ran}(P)$, $\widetilde{T}_\theta - \widetilde{T}_{\pi/4} = U(T_{\cos(2\theta)} - T_0)U^{-1}P$ has the same norm as $T_{\cos(2\theta)} - T_0$. Since $\cos(-2\theta) = \cos(2\theta)$ we have $\widetilde{T}_{-\theta} = \widetilde{T}_\theta$, and since $|\cos(2\theta)| = |\sin(2(\theta - \frac{\pi}{4}))| \le 2|\theta - \frac{\pi}{4}|$, (8.2) follows from Definition 7.3. $\square$

The next proposition replaces [3, Propositions 3.3 and 3.4] (and their reliance on [5, Proposition 2.8]) with a short self-contained argument using Gaussian-norm Fell absorption, giving explicit constants and covering simultaneously both left regular coefficients and reflected coefficients.

**Proposition 8.3** (equivariant SU(2) bounds)**.** *Let $Y \in \mathcal{D}(C, \eta, \alpha_1)$, with $s_1, s_2, C_1, C_2$ as in Lemma 8.2. Let $F \in L^2(\mathrm{SU}(2); Y)$ and $\Lambda \in L^2(\mathrm{SU}(2); Y)^*$, and suppose $\varphi : \mathrm{SU}(2) \to \mathbb{C}$ is given by $\varphi(h) = \Lambda\big((\lambda(h) \otimes \mathrm{Id}_Y)F\big)$ or by $\varphi(h) = \Lambda\big((\lambda(h^{-1}) \otimes \mathrm{Id}_Y)F\big)$.*

*(i) If $m \in \mathbb{Z}$ and $\varphi(r_\alpha h r_\beta) = e^{im\alpha}\varphi(h)$ for all $h \in \mathrm{SU}(2)$ and $\alpha, \beta \in \mathbb{R}$, then for all $\theta \in [0, \frac{\pi}{2}]$,*

$$|\varphi(d_\theta) - \varphi(d_{\pi/4})| \le \big(2^{|m|/4} C_1 + |m|\big)\, |\theta - \tfrac{\pi}{4}|^{s_1}\, \|F\|\, \|\Lambda\|.$$

*(ii) If $m \in \mathbb{Z}$ and $\varphi(d_\alpha h d_{-\alpha}) = e^{im\alpha}\varphi(h)$ for all $h \in \mathrm{SU}(2)$ and $\alpha \in \mathbb{R}$, then for all $\theta \in \mathbb{R}$,*

$$|\varphi(u_\theta) - \varphi(u_{\pi/2})| \le 2^{|m|/4} C_2\, |\theta - \tfrac{\pi}{2}|^{s_2}\, \|F\|\, \|\Lambda\|.$$

*Proof.* We first record the Fell absorption bound for a product of $\varphi$ with a matrix coefficient $\psi(h) = \langle \sigma(h)\xi_0, \eta_0\rangle_W$ of a finite-dimensional unitary representation $(\sigma, W)$ of SU(2) (inner product linear in the first variable). Let $\mu, \mu'$ be Borel probability measures on SU(2), and set $\widetilde{\varphi}(h) = \psi(h)\varphi(h)$.

- Suppose $\varphi(h) = \Lambda\big((\lambda(h) \otimes \mathrm{Id}_Y)F\big)$. Let $(w_r)_{r=1}^N$ be an orthonormal basis of $W$, and let $(\gamma_1, \dots, \gamma_N)$ be i.i.d. standard complex Gaussian random variables on a probability space $(\Omega, \mathbb{P})$ (so $\mathbb{E}[\gamma_r] = 0$, $\mathbb{E}[\gamma_r \overline{\gamma_s}] = \delta_{rs}$, and $\mathbb{E}[\gamma_r \gamma_s] = 0$). Equip $W \otimes Y$ with the Gaussian norm
$$\Big\|\sum_{r=1}^N w_r \otimes y_r\Big\|_{W \otimes_\gamma Y} := \Big(\mathbb{E}\Big\|\sum_{r=1}^N \gamma_r y_r\Big\|_Y^2\Big)^{1/2}.$$
The map $J : \sum_{r=1}^N w_r \otimes y_r \mapsto \sum_{r=1}^N \gamma_r y_r$ embeds $W \otimes_\gamma Y$ isometrically onto a closed linear subspace of $L^2(\Omega; Y)$ (indeed $y_s = \mathbb{E}[\overline{\gamma_s} J(u)]$ gives $\|y_s\|_Y \le \|J(u)\|_{L^2(\Omega;Y)}$ while $\|J(u)\|_{L^2(\Omega;Y)} \le \sum_r \|y_r\|_Y$). For any unitary operator $U \in \mathrm{U}(W)$ with $U w_r = \sum_{s=1}^N u_{sr} w_s$, we have $J\big((U \otimes \mathrm{Id}_Y)(\sum_r w_r \otimes y_r)\big) = \sum_r \widetilde{\gamma}_r y_r$ where $\widetilde{\gamma}_r = \sum_s u_{sr}\gamma_s$ has the same circularly symmetric standard complex Gaussian distribution as $(\gamma_r)$; hence $U \otimes \mathrm{Id}_Y$ is an exact isometry of $W \otimes_\gamma Y$ (and the norm does not depend on the orthonormal basis). Consequently, the Fell map $(U_{\mathrm{Fell}}\Psi)(g) = (\sigma(g^{-1}) \otimes \mathrm{Id}_Y)(\Psi(g))$ is a linear isometry of $L^2(\mathrm{SU}(2); W \otimes_\gamma Y)$ onto itself intertwining $\sigma(h) \otimes \lambda(h)$ with $\lambda(h) \otimes \mathrm{Id}_{W \otimes_\gamma Y}$.

  For $\xi_0 = \sum_{r=1}^N \alpha_r w_r \in W$, we have $\mathbb{E}|\sum_r \alpha_r \gamma_r|^2 = \|\xi_0\|_W^2$, so Fubini yields
$$\|\xi_0 \otimes F\|_{L^2(\mathrm{SU}(2); W \otimes_\gamma Y)} = \|\xi_0\|_W \|F\|_{L^2(\mathrm{SU}(2);Y)}.$$
  For $\eta_0 \in W$, set $\beta_r = \langle w_r, \eta_0\rangle_W$ and $Z = \sum_{r=1}^N \beta_r \overline{\gamma_r} \in L^2(\Omega)$, so that $\mathbb{E}[Z\gamma_s] = \beta_s = \langle w_s, \eta_0\rangle_W$ and $\mathbb{E}|Z|^2 = \|\eta_0\|_W^2$. Then for every $\Psi = \sum_{s=1}^N w_s \otimes f_s \in L^2(\mathrm{SU}(2); W \otimes_\gamma Y)$,
$$(\eta_0 \otimes \Lambda)(\Psi) := \sum_{s=1}^N \langle w_s, \eta_0\rangle_W \Lambda(f_s) = \mathbb{E}\Big[Z\, \Lambda\Big(\sum_{s=1}^N \gamma_s f_s\Big)\Big],$$
  so Cauchy–Schwarz in $L^2(\Omega)$ and Fubini yield
$$|(\eta_0 \otimes \Lambda)(\Psi)| \le \|\eta_0\|_W \|\Lambda\| \|\Psi\|_{L^2(\mathrm{SU}(2); W \otimes_\gamma Y)}.$$

Finally, via $J$ and Fubini, $L^2(\mathrm{SU}(2);W\otimes_\gamma Y)$ embeds isometrically as a closed $\lambda$-invariant subspace of $L^2(\mathrm{SU}(2);L^2(\Omega;Y))\cong L^2(\Omega;L^2(\mathrm{SU}(2);Y))$, on which $\lambda(\mu-\mu')\otimes\mathrm{Id}$ acts fiberwise in $\omega\in\Omega$, giving $\|\lambda(\mu-\mu')\otimes\mathrm{Id}_{W\otimes_\gamma Y}\|\le\|\lambda(\mu-\mu')\otimes\mathrm{Id}_Y\|_{B(L^2(\mathrm{SU}(2);Y))}$. Evaluating $(\eta_0\otimes\Lambda)\big(\int(\sigma(h)\otimes\lambda(h))(\xi_0\otimes F)\,d(\mu-\mu')(h)\big)$ and using the isometry $U_{\mathrm{Fell}}$ yields

$$\left|\int_{\mathrm{SU}(2)}\widetilde{\varphi}(h)\,d(\mu-\mu')(h)\right|\le\|\xi_0\|_W\|\eta_0\|_W\|F\|\|\Lambda\|\,\|\lambda(\mu-\mu')\otimes\mathrm{Id}_Y\|_{B(L^2(\mathrm{SU}(2);Y))}. \quad (8.4)$$

- If $\varphi(h)=\Lambda\big((\lambda(h^{-1})\otimes\mathrm{Id}_Y)F\big)$, pass to the conjugate unitary representation $\bar\sigma$ on $\overline{W}$: then $\psi(h)=\langle\xi_0,\sigma(h^{-1})\eta_0\rangle_W=\langle\bar\sigma(h^{-1})\bar\eta_0,\bar\xi_0\rangle_{\overline{W}}$. Changing variables $k=h^{-1}$ turns $\int\widetilde{\varphi}\,d(\mu-\mu')$ into $\int\langle\bar\sigma(k)\bar\eta_0,\bar\xi_0\rangle_{\overline{W}}\Lambda((\lambda(k)\otimes\mathrm{Id}_Y)F)\,d(\check\mu-\check\mu')(k)$, so the same Gaussian Fell absorption argument yields (8.4) with $\mu-\mu'$ replaced by $\check\mu-\check\mu'$.

*Proof of* (i). If $m$ is odd, taking $\alpha=\beta=\pi$ and using $r_\pi=-I_2\in Z(\mathrm{SU}(2))$ gives $\varphi(h)=-\varphi(h)$, so $\varphi\equiv0$. Assume $m=2k$ is even. Let $v(h)=(v_1(h),v_2(h),v_3(h))^T=\mathrm{Ad}(h)e_1\in\mathbb{R}^3$ be the first column of (8.3). Because $\mathrm{Ad}(r_\beta)e_1=e_1$, we have $v(r_\alpha hr_\beta)=\mathrm{Ad}(r_\alpha)v(h)$, and the explicit matrix of $\mathrm{Ad}(r_\alpha)$ yields

$$(v_2\pm iv_3)(r_\alpha hr_\beta)=e^{\mp2i\alpha}(v_2\pm iv_3)(h).$$

Set $u_\pm=e_2\mp ie_3\in\mathbb{C}^3$ ($\|u_\pm\|=\sqrt{2}$), so $v_2(h)\pm iv_3(h)=\langle\mathrm{Ad}(h)e_1,u_\pm\rangle_{\mathbb{C}^3}$, and define

$$p_m(h)=\begin{cases}(v_2(h)+iv_3(h))^k=\langle\mathrm{Ad}^{\otimes k}(h)e_1^{\otimes k},u_+^{\otimes k}\rangle & k\ge0,\\ (v_2(h)-iv_3(h))^{|k|}=\langle\mathrm{Ad}^{\otimes|k|}(h)e_1^{\otimes|k|},u_-^{\otimes|k|}\rangle & k<0.\end{cases}$$

Then $p_m$ is a matrix coefficient of $\sigma=\mathrm{Ad}^{\otimes|k|}$ with $\|\xi_0\|\|\eta_0\|=2^{|k|/2}=2^{|m|/4}$, satisfies $p_m(r_\alpha hr_\beta)=e^{-im\alpha}p_m(h)$, and by (8.3) has $p_m(d_\theta)=(\sin2\theta)^{|k|}$ (with $p_m(d_{\pi/4})=1$). Hence $\widetilde{\varphi}=p_m\varphi$ is $R$-biinvariant. Applying (8.4) with $\mu=\lambda_R*\delta_{d_\theta}*\lambda_R$ and $\mu'=\lambda_R*\delta_{d_{\pi/4}}*\lambda_R$ (noting $\check\mu=\lambda_R*\delta_{d_{-\theta}}*\lambda_R$ gives the same operator $\widetilde{T}_{-\theta}=\widetilde{T}_\theta$) and (8.2) yields

$$|\widetilde{\varphi}(d_\theta)-\widetilde{\varphi}(d_{\pi/4})|\le2^{|m|/4}C_1|\theta-\tfrac{\pi}{4}|^{s_1}\|F\|\|\Lambda\|.$$

For $\theta\in[0,\frac{\pi}{2}]$, $0\le\sin2\theta\le1$ and $|\theta-\frac{\pi}{4}|\le\frac{\pi}{4}<1$, so

$$0\le1-(\sin2\theta)^{|k|}\le|k|(1-\sin2\theta)=2|k|\sin^2\big(\theta-\tfrac{\pi}{4}\big)\le|m||\theta-\tfrac{\pi}{4}|^2\le|m||\theta-\tfrac{\pi}{4}|^{s_1},$$

using $s_1\le1$. Since $\varphi(d_\theta)-\varphi(d_{\pi/4})=(\widetilde{\varphi}(d_\theta)-\widetilde{\varphi}(d_{\pi/4}))+(1-(\sin2\theta)^{|k|})\varphi(d_\theta)$ and $|\varphi(d_\theta)|\le\|F\|\|\Lambda\|$, (i) follows.

*Proof of* (ii). If $m$ is odd, $\alpha=\pi$ gives $\varphi\equiv0$. Assume $m=2k$ is even. For $h=\left(\begin{smallmatrix}a&b\\-\bar b&\bar a\end{smallmatrix}\right)\in\mathrm{SU}(2)$, the $(1,2)$ and $(2,1)$ entries of $d_\alpha hd_{-\alpha}$ are $e^{2i\alpha}b$ and $-e^{-2i\alpha}\bar b$, and for every $\theta\in\mathbb{R}$ we have $b(u_\theta)=-2^{-1/2}$ and $-\bar b(u_\theta)=2^{-1/2}$. With the standard unitary representation $\rho_{\mathrm{std}}$ of $\mathrm{SU}(2)$ on $\mathbb{C}^2$, $b(h)=\langle\rho_{\mathrm{std}}(h)e_2,e_1\rangle$ and $-\bar b(h)=\langle\rho_{\mathrm{std}}(h)e_1,e_2\rangle$. Put

$$q_m(h)=\begin{cases}2^{k/2}\langle\rho_{\mathrm{std}}^{\otimes k}(h)e_1^{\otimes k},e_2^{\otimes k}\rangle=(-\sqrt{2}\,\overline{b(h)})^k & k\ge0,\\ (-1)^{|k|}2^{|k|/2}\langle\rho_{\mathrm{std}}^{\otimes|k|}(h)e_2^{\otimes|k|},e_1^{\otimes|k|}\rangle=(-\sqrt{2}\,b(h))^{|k|} & k<0.\end{cases}$$

Then $\|\xi_0\|\|\eta_0\|=2^{|m|/4}$ and $q_m(d_\alpha u_\theta d_{-\alpha})=e^{-im\alpha}$ identically for all $\theta,\alpha\in\mathbb{R}$. Therefore $\widetilde{\varphi}=q_m\varphi$ satisfies $\widetilde{\varphi}(d_\alpha u_\theta d_{-\alpha})=\varphi(u_\theta)$ for all $\theta,\alpha$, and averaging over $\alpha\in[0,2\pi]$ gives $\varphi(u_\theta)-\varphi(u_{\pi/2})=\int\widetilde{\varphi}\,d(\mu_\theta-\mu_{\pi/2})$. Recall from the proof of Lemma 7.4(ii) that $\check\mu_\theta=\mu_{-\theta}$. Applying (8.4) and (8.2) gives (ii). $\square$

We now record the two-step pseudometric lemma on $\widetilde{G} = \widetilde{\mathrm{Sp}}_4(\mathbb{R})$, corresponding to [6, Lemma 3.1 and Remark 3.2]. Let $\mathcal{P}_c(\widetilde{G})$ denote the space of compactly supported Borel probability measures on $\widetilde{G}$. We write $m_1 m_2$ for the convolution $m_1 * m_2$.

**Lemma 8.4** (two-step pseudometric on $\mathcal{P}_c(\widetilde{G})$)**.** *Let $(X_0, X_1, X_2, \pi_0, \pi_1)$ be a two-step representation of $\widetilde{G} = \widetilde{\mathrm{Sp}}_4(\mathbb{R})$ satisfying $\|\pi_i(g)\| \le L e^{s\ell_0(g)}$ for all $g \in \widetilde{G}$ and $i = 0, 1$, where $\ell_0$ is the $\widetilde{H}$-biinvariant length function of Proposition 8.1(v). Define $d(m, m') = \|\pi(m) - \pi(m')\|_{B(X_0, X_2)}$ on $\mathcal{P}_c(\widetilde{G})$. Then:*

*(1) $d$ is an integrally convex pseudometric on $\mathcal{P}_c(\widetilde{G})$: for every Borel probability measure $\nu$ on a compact Hausdorff space $T$ and every pair of weak-$*$ continuous maps $t \mapsto m_t, m_t'$ from $T$ into measures supported in a fixed compact set $Q \subset \widetilde{G}$,*

$$d\Big(\int_T m_t \, d\nu(t), \int_T m_t' \, d\nu(t)\Big) \le \int_T d(m_t, m_t') \, d\nu(t).$$

*(2) $d$ is weak-$*$ lower-semicontinuous on measures supported in any fixed compact set $Q \subset \widetilde{G}$.*

*(3) For all Borel probability measures $\mu, \mu'$ on $\widetilde{H}$ and all $g_1, g_2 \in \widetilde{G}$,*

$$d(\delta_{g_1} \mu \delta_{g_2}, \, \delta_{g_1} \mu' \delta_{g_2}) \le L^2 e^{s(\ell_0(g_1) + \ell_0(g_2))} \, \|\lambda(\mu - \mu') \otimes \mathrm{Id}_{X_1}\|_{B(L^2(\widetilde{H}; X_1))}.$$

*Proof.* Property (2) is immediate from the strong continuity of $\pi$ (see [6, Lemmas 2.1 and 3.1]). In (1), $t \mapsto d(m_t, m_t')$ is lower-semicontinuous by (2), hence Borel measurable, and testing against unit vectors $x \in X_0$, $y \in X_2^*$ gives $|\langle y, \pi(\int_T (m_t - m_t') \, d\nu(t)) x\rangle| \le \int_T |\langle y, \pi(m_t - m_t') x\rangle| \, d\nu(t) \le \int_T d(m_t, m_t') \, d\nu(t)$. For (3), let $x \in X_0$ and $y \in X_2^*$ have norm at most 1. Define $F \in C(\widetilde{H}; X_1) \subset L^2(\widetilde{H}; X_1)$ by $F(h) = \pi_0(h^{-1} g_2) x$. Since $\ell_0$ is $\widetilde{H}$-biinvariant, $\|F(h)\|_{X_1} \le L e^{s\ell_0(g_2)}$ for all $h$, so $\|F\|_{L^2(\widetilde{H}; X_1)} \le L e^{s\ell_0(g_2)}$. Next define a linear functional $\Lambda$ on $L^2(\widetilde{H}; X_1)$ by

$$\Lambda(\Psi) = \int_{\widetilde{H}} \langle y, \pi_1(g_1 h) \Psi(h)\rangle \, dh \qquad (\Psi \in L^2(\widetilde{H}; X_1)).$$

For $\Psi \in C(\widetilde{H}; X_1)$, the strong continuity of $\pi_1$ and the uniform bound $\|\pi_1(g_1 h)\| \le L e^{s\ell_0(g_1)}$ imply that $h \mapsto \pi_1(g_1 h)\Psi(h)$ is continuous; by $L^2$ approximation, $h \mapsto \langle y, \pi_1(g_1 h)\Psi(h)\rangle$ is measurable for every $\Psi \in L^2(\widetilde{H}; X_1)$, and Cauchy–Schwarz gives $|\Lambda(\Psi)| \le L e^{s\ell_0(g_1)} \|\Psi\|_{L^2(\widetilde{H}; X_1)}$ (no Radon–Nikodym property on $X_1$ is needed). By Fubini's theorem and the two-step relation $\pi_1(g_1 h_1)\pi_0(h_1^{-1} h_2 g_2) = \pi(g_1 h_2 g_2)$,

$$\Lambda\big((\lambda(\mu) \otimes \mathrm{Id}_{X_1}) F\big) = \iint_{\widetilde{H} \times \widetilde{H}} \langle y, \pi_1(g_1 h_1)\pi_0(h_1^{-1} h_2 g_2) x\rangle \, d\mu(h_2) \, dh_1 = \langle y, \pi(\delta_{g_1} \mu \delta_{g_2}) x\rangle.$$

Subtracting the identity for $\mu'$ and taking the supremum over $\|x\|, \|y\| \le 1$ gives (3). □

## 8.3 Explicit Cartan decompositions in $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$

We give self-contained proofs of the two Cartan decomposition identities of [3, Propositions 4.6, 4.7, 5.2, 5.3], checking every sign and boundary case.

**Lemma 8.5** ($2\times 2$ singular value identities)**.** (i) *Let $a,d\ge 0$ and $c\in\mathbb{R}$. Then $\left(\begin{smallmatrix} a & -c \\ c & d\end{smallmatrix}\right) = r_\phi \operatorname{diag}(\lambda,\mu) r_\phi$, where $\phi\in[-\frac{\pi}{4},\frac{\pi}{4}]$ and $\lambda,\mu\ge 0$ satisfy*

$$\lambda\mu = ad+c^2, \qquad \lambda-\mu = a-d, \qquad (\lambda+\mu)(\cos 2\phi,\sin 2\phi) = (a+d,2c),$$

*so $\tan(2\phi) = 2c/(a+d)$ when $a+d>0$ (and $\phi=\pm\frac{\pi}{4}$ when $a=d=0$, $\pm c>0$). Moreover $a\ge d$ implies $\lambda\ge\mu\ge 0$.*

(ii) *For all $\alpha>0$ and $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $\operatorname{diag}(e^\alpha,e^{-\alpha}) r_\theta \operatorname{diag}(e^\alpha,e^{-\alpha}) = r_\phi \operatorname{diag}(e^\beta,e^{-\beta}) r_\phi$, where $\beta\ge 0$ and $\phi\in[-\frac{\pi}{4},\frac{\pi}{4}]$ are uniquely characterized by*

$$\sinh\beta = \sinh(2\alpha)\cos\theta, \qquad \tan(2\phi) = \frac{\tan\theta}{\cosh(2\alpha)}$$

*(with $\phi=\pm\frac{\pi}{4}$ when $\theta=\pm\frac{\pi}{2}$).*

*Proof.* Expanding $r_\phi \operatorname{diag}(\lambda,\mu) r_\phi$ gives

$$\frac{1}{2}\begin{pmatrix} (\lambda-\mu)+(\lambda+\mu)\cos 2\phi & -(\lambda+\mu)\sin 2\phi \\ (\lambda+\mu)\sin 2\phi & -(\lambda-\mu)+(\lambda+\mu)\cos 2\phi\end{pmatrix}.$$

Setting $\lambda+\mu = \sqrt{(a+d)^2+4c^2} = \sqrt{(a-d)^2+4(ad+c^2)} \ge |a-d|$ and choosing $\phi\in[-\frac{\pi}{4},\frac{\pi}{4}]$ so that $(\cos 2\phi,\sin 2\phi) = (a+d,2c)/(\lambda+\mu)$ (with $\phi=0$ if $a=d=c=0$) produces $\lambda,\mu\ge 0$ with $\lambda-\mu=a-d$ and $\lambda\mu = ad+c^2$, proving (i). Part (ii) is the special case $a=e^{2\alpha}\cos\theta\ge 0$, $d=e^{-2\alpha}\cos\theta\ge 0$, $c=\sin\theta$, where $\lambda\mu = \cos^2\theta+\sin^2\theta = 1$ so $(\lambda,\mu)=(e^\beta,e^{-\beta})$ with $\beta\ge 0$. □

For $\theta,\theta'\in\mathbb{R}$, lift $w_{\theta,\theta'}$ to $\widetilde{K}$ by

$$\widetilde{w}_{\theta,\theta'} = \widetilde{\exp}\Big(\iota\begin{pmatrix} i\theta & 0 \\ 0 & i\theta'\end{pmatrix}\Big) = \widetilde{v}_{(\theta+\theta')/2}\,\widetilde{\iota}(d_{(\theta-\theta')/2}).$$

(Note that $\iota(\operatorname{diag}(i\theta,i\theta')) = \frac{\theta+\theta'}{2}J_0 + \frac{\theta-\theta'}{2}\iota(\operatorname{diag}(i,-i))$, and the two summands commute.)

**Proposition 8.6** (first Cartan identity in $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$)**.** *For all $\alpha>0$ and $\theta\in[0,\frac{\pi}{2}]$,*

$$\widetilde{D}(\alpha,\alpha)\,\widetilde{w}_{\theta,\frac{\pi}{2}-\theta}\,\widetilde{D}(\alpha,\alpha) = \widetilde{w}_{\phi,\phi'}\,\widetilde{D}(\beta,\gamma)\,\widetilde{w}_{\phi,\phi'},$$
$$\widetilde{D}(\alpha,\alpha)\,\widetilde{w}_{-\theta,\theta-\frac{\pi}{2}}\,\widetilde{D}(\alpha,\alpha) = \widetilde{w}_{-\phi,-\phi'}\,\widetilde{D}(\beta,\gamma)\,\widetilde{w}_{-\phi,-\phi'},$$

*where $\beta,\gamma\ge 0$ and $\phi,\phi'\in[0,\frac{\pi}{4}]$ are uniquely determined by*

$$\sinh\beta = \sinh(2\alpha)\cos\theta, \qquad \sinh\gamma = \sinh(2\alpha)\sin\theta,$$
$$\tan(2\phi) = \frac{\tan\theta}{\cosh(2\alpha)}, \qquad \tan(2\phi') = \frac{1}{\tan\theta\cosh(2\alpha)}. \tag{8.5}$$

*Proof.* In $\mathrm{Sp}_4(\mathbb{R})$, every matrix in the statement preserves the two orthogonal symplectic planes $\mathbb{R}e_1\oplus\mathbb{R}e_3$ and $\mathbb{R}e_2\oplus\mathbb{R}e_4$, acting on them as the long-root $\mathrm{SL}_2(\mathbb{R})$ matrices of Lemma 8.5(ii) with angles $\theta$ and $\frac{\pi}{2}-\theta$ (respectively $-\theta$ and $\theta-\frac{\pi}{2}$). Hence both identities hold after projecting to $\mathrm{Sp}_4(\mathbb{R})$. For fixed $\theta\in[0,\frac{\pi}{2}]$, both sides depend continuously on $\alpha\ge 0$ in $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$; at $\alpha=0$ we have $\beta=\gamma=0$, $2\phi=\theta$, $2\phi'=\frac{\pi}{2}-\theta$, so both sides equal $\widetilde{w}_{\theta,\frac{\pi}{2}-\theta}$ (respectively $\widetilde{w}_{-\theta,\theta-\frac{\pi}{2}}$). Since $\sigma:\widetilde{G}\to G$ is a covering map, equality holds in $\widetilde{G}$ for all $\alpha\ge 0$. □

**Proposition 8.7** (second Cartan identity in $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$)**.** *For all $\alpha > 0$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,*

$$\widetilde{D}(\alpha,0)\,\widetilde{\iota}(u_\theta)\,\widetilde{D}(\alpha,0) = \widetilde{\iota}(r_\phi)\,\widetilde{w}_{\omega_1,\omega_2}\,\widetilde{D}(\beta,\gamma)\,\widetilde{w}_{\omega_1,\omega_2}\,\widetilde{\iota}(r_\phi),$$

*where $\beta \geq \gamma \geq 0$, $\phi \in (0, \frac{\pi}{4}]$, and $\omega_1, \omega_2 \in [-\frac{\pi}{8}, \frac{\pi}{8}]$ are uniquely determined by*

$$\sinh\beta \sinh\gamma = \tfrac{1}{2}\sinh^2\alpha, \qquad \sinh\beta - \sinh\gamma = \tfrac{1}{\sqrt{2}}\sinh(2\alpha)\cos\theta, \tag{8.6}$$

$$\tan(2\phi) = \frac{1}{\cosh\alpha\cos\theta}, \qquad \sin(2\omega_1)\cosh\beta = \frac{\sin\theta}{\sqrt{2}} = -\sin(2\omega_2)\cosh\gamma. \tag{8.7}$$

*Proof.* We first prove the identity in $\mathrm{Sp}_4(\mathbb{R})$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$; the endpoints $|\theta| = \frac{\pi}{2}$ follow by continuity. Put $X = D(\alpha,0)\iota(u_\theta)D(\alpha,0)$ and write $X = kD(\beta,\gamma)k'$ with $k, k' \in K$ and $\beta \geq \gamma \geq 0$. Using $(X^{-1})^T = -JXJ$, direct multiplication of the $4\times 4$ matrices gives

$$\frac{X - (X^{-1})^T}{2} = \frac{1}{\sqrt{2}}\begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix}, \qquad Y = \begin{pmatrix} \sinh(2\alpha)\cos\theta & -\sinh\alpha \\ \sinh\alpha & 0 \end{pmatrix}.$$

Applying Lemma 8.5(i) to $Y$ yields $Y = r_\phi\,\mathrm{diag}(\lambda,\mu)r_\phi$ with $\lambda > \mu > 0$ and $\phi \in (0, \frac{\pi}{4})$ satisfying $\lambda\mu = \sinh^2\alpha$, $\lambda - \mu = \sinh(2\alpha)\cos\theta$, and $\tan(2\phi) = (\cosh\alpha\cos\theta)^{-1}$. With $k_0 = \iota(r_\phi) \in K$ and $\Delta = \mathrm{diag}(\sinh\beta, \sinh\gamma, -\sinh\beta, -\sinh\gamma)$, comparing the singular values of $\frac{1}{2}(X - (X^{-1})^T) = k\Delta k'$ with $2^{-1/2}k_0\,\mathrm{diag}(\lambda,\mu,-\lambda,-\mu)k_0$ yields $\sinh\beta = \lambda/\sqrt{2}$, $\sinh\gamma = \mu/\sqrt{2}$, giving (8.6) and $\beta > \gamma > 0$, as well as $k_0\Delta k_0 = k\Delta k'$. Taking $MM^T$ shows that $k_0^{-1}k \in \iota(\mathrm{U}(2))$ commutes with $\Delta^2 = \iota(\mathrm{diag}(\sinh^2\beta, \sinh^2\gamma))$. Since $\sinh^2\beta > \sinh^2\gamma$, $\iota^{-1}(k_0^{-1}k)$ is diagonal, so $k = k_0 w_{\omega_1,\omega_2}$ for some $\omega_1, \omega_2 \in \mathbb{R}$. Substituting into $k_0\Delta k_0 = k\Delta k'$ and checking on $\mathbb{R}e_1 \oplus \mathbb{R}e_3$ and $\mathbb{R}e_2 \oplus \mathbb{R}e_4$ that $\Delta^{-1}w_{\omega_1,\omega_2}^{-1}\Delta = w_{\omega_1,\omega_2}$ gives $k' = w_{\omega_1,\omega_2}k_0$.

Now examine the lower-left $2\times 2$ block of $X = k_0 w_{\omega_1,\omega_2} D(\beta,\gamma) w_{\omega_1,\omega_2} k_0$:

$$\frac{\sin\theta}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = r_\phi \begin{pmatrix} \sin(2\omega_1)\cosh\beta & 0 \\ 0 & \sin(2\omega_2)\cosh\gamma \end{pmatrix} r_\phi.$$

Expanding the right-hand side as in Lemma 8.5 (and using $r_\phi\,\mathrm{diag}(1,-1)r_\phi = \mathrm{diag}(1,-1)$) gives the second equation in (8.7). In particular, $|\sin(2\omega_j)| \leq 2^{-1/2}$, so $\cos^2(2\omega_j) \geq \frac{1}{2}$. On the other hand, the $(1,1)$ and $(2,2)$ entries of $M(\theta) := k_0^{-1}Xk_0^{-1} = w_{\omega_1,\omega_2}D(\beta,\gamma)w_{\omega_1,\omega_2}$ are $\sinh\beta + \cosh\beta\cos(2\omega_1)$ and $\sinh\gamma + \cosh\gamma\cos(2\omega_2)$, so $\cos(2\omega_1)$ and $\cos(2\omega_2)$ depend continuously on $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. At $\theta = 0$, the upper-left $2\times 2$ block of $\frac{1}{2}(X + (X^{-1})^T)$ is $2^{-1/2}\left(\begin{smallmatrix} \cosh(2\alpha) & -\cosh\alpha \\ \cosh\alpha & 1 \end{smallmatrix}\right)$, which has $2c/(a+d) = 2\cosh\alpha/(\cosh(2\alpha)+1) = (\cosh\alpha)^{-1} = \tan(2\phi)$; by Lemma 8.5(i), conjugating by $r_{-\phi}$ on both sides yields a positive diagonal matrix, so $\cos(2\omega_1) = \cos(2\omega_2) = +1$ at $\theta = 0$. By continuity, $\cos(2\omega_j) > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Together with $|\sin(2\omega_j)| \leq 2^{-1/2}$, this gives $2\omega_j \in [-\frac{\pi}{4}, \frac{\pi}{4}] \pmod{2\pi}$; because shifting $\omega_1$ or $\omega_2$ by $\pi$ leaves $w_{\omega_1,\omega_2}D(\beta,\gamma)w_{\omega_1,\omega_2}$ unchanged, there is a unique choice with $\omega_1, \omega_2 \in [-\frac{\pi}{8}, \frac{\pi}{8}]$.

Finally, to lift the identity to $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$, fix $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and let $\alpha \to 0^+$. Then $\beta, \gamma \to 0$, $\tan(2\phi) \to (\cos\theta)^{-1}$, $\sin(2\omega_1) \to 2^{-1/2}\sin\theta$, and $\omega_2 \to -\omega_1$, so $\omega_1 + \omega_2 \to 0$. In $\mathrm{SU}(2)$, direct multiplication gives

$$r_\phi d_{2\omega_1} r_\phi = \begin{pmatrix} \cos(2\phi)\cos(2\omega_1) + i\sin(2\omega_1) & -\sin(2\phi)\cos(2\omega_1) \\ \sin(2\phi)\cos(2\omega_1) & \cos(2\phi)\cos(2\omega_1) - i\sin(2\omega_1) \end{pmatrix} = u_\theta$$

since $\cos(2\omega_1) = 2^{-1/2}\sqrt{1+\cos^2\theta}$ and $(\cos 2\phi, \sin 2\phi) = (\cos\theta, 1)/\sqrt{1+\cos^2\theta}$. At the endpoints $\theta = \pm\frac{\pi}{2}$, we have $\phi = \frac{\pi}{4}$ and $\sin(2\omega_1) = -\sin(2\omega_2) \to \pm 2^{-1/2}$ as $\alpha \to 0^+$, so $\omega_1 \to \pm\frac{\pi}{8}$, $\omega_2 \to \mp\frac{\pi}{8}$, and the same formula gives $r_{\pi/4}d_{\pm\pi/4}r_{\pi/4} = u_{\pm\pi/2}$. Thus for every $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, both sides of the lifted identity coincide with $\widetilde{\iota}(u_\theta)$ at $\alpha = 0$, depend continuously on $\alpha \geq 0$, and have the same image in $\mathrm{Sp}_4(\mathbb{R})$; hence they coincide in $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ for all $\alpha > 0$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. $\square$

### 8.4 Property $(*_{\mathcal{D}})$ for $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$

Fix $C \geq 1$, $\eta \in (0,1)$, $\alpha_1 \in (0, \frac{1}{2}]$ and put $\mathcal{D} = \mathcal{D}(C, \eta, \alpha_1)$, with $s_1 = \alpha_1 \in (0, \frac{1}{2}]$, $s_2 = \eta \in (0,1)$, $C_1 = 2^{s_1}C$, $C_2 = C$ as in Lemma 8.2. Consider the quadratic polynomial

$$P_{s_1,s_2}(s) = s^2 - (2s_1 + s_2)s + s_1 s_2,$$

whose smaller root is

$$s_-(s_1, s_2) = s_1 + \frac{s_2}{2} - \sqrt{s_1^2 + \frac{s_2^2}{4}} \in \big(0, \min\{s_1, s_2\}\big).$$

Fix $s \in (0, s_-(s_1, s_2))$ and define

$$\lambda = \frac{s_1 + s_2 - s}{s_1} > 1, \qquad \tau_0 = \frac{P_{s_1,s_2}(s)}{s_1 + s_2 - s} > 0. \tag{8.8}$$

A direct expansion checks the three basic relations:

$$\begin{aligned} s\beta - s_1(\beta - \gamma) &\leq -\tau_0\beta && \text{whenever } \beta \geq \lambda\gamma \geq 0, \\ s\beta - (s_2 - s)\gamma &\leq -\tau_0\beta && \text{whenever } \lambda\gamma \geq \beta \geq \gamma \geq 0, \\ s_1 - s - \tau_0 &= \frac{s_1^2}{s_1 + s_2 - s} > 0. \end{aligned} \tag{8.9}$$

On $\mathcal{P}_c(\widetilde{G})$ define the conjugation-averaging operator (well defined because $\widetilde{v}_\pi \in Z(\widetilde{G})$) and the averaged measures

$$\mathcal{A}(m) = \int_{\mathbb{R}/\pi\mathbb{Z}} \delta_{\widetilde{v}_t}\, m\, \delta_{\widetilde{v}_{-t}}\, \frac{dt}{\pi}, \qquad c(g) = \mathcal{A}(\lambda_{\widetilde{H}} \delta_g \lambda_{\widetilde{H}}) = \lambda_{\widetilde{H}}\, \mathcal{A}(\delta_g)\, \lambda_{\widetilde{H}} \qquad (g \in \widetilde{G}),$$

where $\lambda_{\widetilde{H}}$ denotes the probability Haar measure on $\widetilde{H}$. Since $\widetilde{v}_t$ commutes with $\widetilde{H}$, we have $c(\widetilde{h}_1 \widetilde{v}_u g \widetilde{v}_{-u} \widetilde{h}_2) = c(g)$ for all $\widetilde{h}_1, \widetilde{h}_2 \in \widetilde{H}$ and $u \in \mathbb{R}$. By Proposition 8.1(iv), every $g \in \widetilde{G}$ with Cartan parameters $(\beta, \gamma)$ satisfies

$$c(g) = c\big(\widetilde{v}_{\Phi(g)} \widetilde{D}(\beta, \gamma)\big). \tag{8.10}$$

Moreover, for all $g_1, g_2 \in \widetilde{G}$, the change of variables $\widetilde{h} = \widetilde{h}_2 \widetilde{h}_3$, $u = t_2 - t_1 \pmod{\pi}$ gives the exact convolution identity

$$c(g_1)\, c(g_2) = \int_{\widetilde{H} \times (\mathbb{R}/\pi\mathbb{Z})} c\big(g_1 \widetilde{h} \widetilde{v}_u g_2 \widetilde{v}_{-u}\big)\, d\widetilde{h}\, \frac{du}{\pi}. \tag{8.11}$$

Finally, $\widetilde{\iota}(d_{\pi/2}) \widetilde{D}(\beta, \gamma) \widetilde{\iota}(d_{-\pi/2}) = \widetilde{D}(-\beta, -\gamma) = \widetilde{D}(\beta, \gamma)^{-1}$ (verified directly in $\mathrm{Sp}_4(\mathbb{R})$ and lifted by continuity from $\beta = \gamma = 0$), so the reflection $\check{c}(\widetilde{D}(\beta, 0))$ equals $c(\widetilde{D}(\beta, 0))$; in particular the probability measures

$$m_n = \begin{cases} \lambda_{\widetilde{H}} & n = 0, 1, \\ c\big(\widetilde{D}(n-2, 0)\big) & n \geq 2 \end{cases} \tag{8.12}$$

are symmetric ($\check{m}_n = m_n$) and supported in $\{g \in \widetilde{G} : \ell_0(g) \leq n\}$ (since $\Phi = 0$ on $\operatorname{supp} m_n$, so $\ell_0(g) = (n-2) + \frac{\pi}{2} < n$ on $\operatorname{supp} m_n$ for $n \geq 2$).

Throughout the remainder of this subsection, let $(X_0, X_1, X_2, \pi_0, \pi_1)$ be a two-step representation of $\widetilde{G}$ with $X_1 \in \mathcal{D}(C, \eta, \alpha_1)$ and $\|\pi_i(g)\| \le L e^{s\ell_0(g)}$ $(i = 0, 1)$, and let $d$ be the pseudometric of Lemma 8.4. By Lemma 8.4(1)–(3) and the fact that $\ell_0(\widetilde{v}_t g) \le \ell_0(g) + \pi$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have for all probability measures $\mu, \mu'$ on $\widetilde{H}$ and all $g_1, g_2 \in \widetilde{G}$:

$$d\big(\mathcal{A}(\delta_{g_1}\mu\delta_{g_2}),\, \mathcal{A}(\delta_{g_1}\mu'\delta_{g_2})\big) \le L^2 e^{2\pi s} e^{s(\ell_0(g_1)+\ell_0(g_2))} \|\lambda(\mu - \mu') \otimes \mathrm{Id}_{X_1}\|_{B(L^2(\widetilde{H}; X_1))}. \tag{8.13}$$

**Lemma 8.8** (Move 1, phase shift, and Move 2 for $c(g)$)**.** *There is a constant $K_0 > 0$ depending only on $C, \eta, \alpha_1, s$ such that:*

*(i) For every $\beta \ge \gamma \ge 0$ $(\beta > 0)$, with $\alpha > 0$ defined by $\sinh^2(2\alpha) = \sinh^2\beta + \sinh^2\gamma$, there exists $\rho \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ such that for all $t \in \mathbb{R}$,*

$$d\big(c(\widetilde{v}_t \widetilde{D}(\beta,\gamma)),\, c(\widetilde{v}_{t+\rho}\widetilde{D}(2\alpha, 0))\big) \le K_0 L^2 e^{s|t| + s\beta - s_1(\beta-\gamma)}.$$

*(ii) For every $\alpha \ge 1$, $t \in \mathbb{R}$ and $\tau \in [-\frac{\pi}{4}, \frac{\pi}{4}]$,*

$$d\big(c(\widetilde{v}_t \widetilde{D}(2\alpha,0)),\, c(\widetilde{v}_{t+\tau}\widetilde{D}(2\alpha, 0))\big) \le K_0 L^2 e^{s|t| - 2(s_1 - s)\alpha}.$$

*(iii) For every $\beta \ge \gamma \ge 0$, with $\alpha' \ge 0$ defined by $\sinh^2\alpha' = \sinh\beta \sinh\gamma$, there exists $\sigma \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ such that for all $t \in \mathbb{R}$,*

$$d\big(c(\widetilde{v}_t \widetilde{D}(\beta,\gamma)),\, c(\widetilde{v}_{t+\sigma}\widetilde{D}(\alpha', \alpha'))\big) \le K_0 L^2 e^{s|t| + s\beta - (s_2 - s)\gamma}.$$

*Proof.* (i) For any $\alpha > 0$, $t \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2}]$, recall that $\widetilde{w}_{\theta, \frac{\pi}{2}-\theta} = \widetilde{v}_{\pi/4}\widetilde{\iota}(d_{\theta - \pi/4})$. Since $\widetilde{v}_t$, $\widetilde{D}(\alpha,\alpha)$ and $\widetilde{v}_{\pi/4}$ all commute with $\widetilde{\iota}(R)$, we have

$$c\big(\widetilde{v}_t \widetilde{D}(\alpha,\alpha)\widetilde{w}_{\theta,\frac{\pi}{2}-\theta}\widetilde{D}(\alpha,\alpha)\big) = \mathcal{A}\Big(\lambda_{\widetilde{H}}\delta_{\widetilde{v}_t\widetilde{D}(\alpha,\alpha)\widetilde{v}_{\pi/4}}\big(\lambda_{\widetilde{\iota}(R)}\delta_{\widetilde{\iota}(d_{\theta-\pi/4})}\lambda_{\widetilde{\iota}(R)}\big)\delta_{\widetilde{D}(\alpha,\alpha)}\lambda_{\widetilde{H}}\Big).$$

By Proposition 8.6, (8.10), (8.13) and Lemma 8.2 (noting $\widetilde{T}_{\theta-\pi/4} - \widetilde{T}_{-\pi/4} = \widetilde{T}_{\pi/4-\theta} - \widetilde{T}_{\pi/4}$, and directly counting $\ell_0(\widetilde{v}_t\widetilde{D}(\alpha,\alpha)\widetilde{v}_{\pi/4}) \le \alpha + |t| + \frac{3\pi}{4}$ and $\ell_0(\widetilde{D}(\alpha,\alpha)) = \alpha + \frac{\pi}{2}$, which with $e^{2\pi s}$ from (8.13) gives $e^{(13\pi/4)s} \le e^{4\pi s}$),

$$d\big(c(\widetilde{v}_{t+\phi+\phi'}\widetilde{D}(\beta,\gamma)),\, c(\widetilde{v}_{t+\pi/4}\widetilde{D}(2\alpha,0))\big) \le C_1 e^{4\pi s} L^2 e^{s|t| + 2s\alpha}\theta^{s_1}. \tag{8.14}$$

Similarly, using the second identity of Proposition 8.6,

$$d\big(c(\widetilde{v}_{t-\phi-\phi'}\widetilde{D}(\beta,\gamma)),\, c(\widetilde{v}_{t-\pi/4}\widetilde{D}(2\alpha,0))\big) \le C_1 e^{4\pi s} L^2 e^{s|t| + 2s\alpha}\theta^{s_1}. \tag{8.15}$$

Given $\beta \ge \gamma \ge 0$ $(\beta > 0)$, take $\theta = \arctan(\sinh\gamma / \sinh\beta) \in [0, \frac{\pi}{4}]$. Then $\theta \le \frac{\sinh\gamma}{\sinh\beta} = e^{\gamma-\beta}\frac{1-e^{-2\gamma}}{1-e^{-2\beta}} \le e^{\gamma-\beta}$ and $\sinh(2\alpha) \le \sqrt{2}\sinh\beta \le \sinh(\beta + \frac{1}{2}\log 2)$, so $2\alpha \le \beta + \frac{1}{2}\log 2$. Replacing $t$ by $t - \phi - \phi'$ in (8.14) (noting $\phi, \phi' \in [0, \frac{\pi}{4}]$) gives (i) with $\rho = \frac{\pi}{4} - \phi - \phi' \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

(ii) Let $\alpha \ge 1$ and $\theta \in [0, 4e^{-2\alpha}] \subset [0, \frac{\pi}{2}]$. Set $t_1 = t' - \frac{\pi}{4}$ in (8.14) and $t_2 = t' + 2\phi + 2\phi' - \frac{\pi}{4}$ in (8.15); since $t_1 + \phi + \phi' = t_2 - \phi - \phi'$ and $\phi, \phi' \in [0, \frac{\pi}{4}]$ gives $|t_1| \le |t'| + \frac{\pi}{4}$ and $|t_2| \le |t'| + \frac{3\pi}{4} < |t'| + \pi$ (so $e^{4\pi s}e^{s|t_j|} \le e^{5\pi s}e^{s|t'|}$), the triangle inequality gives

$$\begin{aligned} d\big(c(\widetilde{v}_{t'}\widetilde{D}(2\alpha,0)),\, c(\widetilde{v}_{t'+2\phi+2\phi'-\pi/2}\widetilde{D}(2\alpha,0))\big) &\le 2C_1 e^{5\pi s} L^2 e^{s|t'| + 2s\alpha}\theta^{s_1} \\ &\le 8C_1 e^{5\pi s} L^2 e^{s|t'| - 2(s_1 - s)\alpha}. \end{aligned}$$

Set $\tau(\theta) = \frac{\pi}{2} - 2\phi - 2\phi'$. Then $\tau$ is continuous on $[0, 4e^{-2\alpha}]$ with $\tau(0) = 0$. At $\theta = 4e^{-2\alpha} \le 4e^{-2} < \frac{\pi}{4}$, we have $\tan\theta \ge \theta = 4e^{-2\alpha}$ and $\cosh(2\alpha) > \frac{1}{2}e^{2\alpha}$, so $\tan(2\phi') \le \frac{1}{2}$, whence $2\phi' \le \arctan(\frac{1}{2}) < 0.464$; also $\tan\theta \le 2\theta = 8e^{-2\alpha}$, so $\tan(2\phi) \le 16e^{-4\alpha} \le 16e^{-4} < 0.294$, whence $2\phi < 0.286$. Thus $2\phi + 2\phi' < 0.75 < \frac{\pi}{4}$, so $\tau(4e^{-2\alpha}) > \frac{\pi}{4}$. By the intermediate value theorem, $\tau([0, 4e^{-2\alpha}]) \supseteq [0, \frac{\pi}{4}]$. Setting $t' = t + \tau$ gives $\tau \in [0, \frac{\pi}{4}]$ and $t' = t$ gives $-\tau \in [-\frac{\pi}{4}, 0]$, proving (ii).

(iii) If $0 \le \gamma \le 1$, then $\alpha' \le \beta$ and the trivial bound $\|\pi(g)\| \le L^2 e^{s\ell_0(g)}$ gives (iii) for $\sigma = 0$. Assume now $\beta \ge \gamma \ge 1$. Let $\alpha > 0$ solve $\frac{1}{2}\sinh^2\alpha = \sinh\beta\sinh\gamma$. Since $\log\sinh$ is concave on $(0,\infty)$, $\sinh^2\alpha' = \sinh\beta\sinh\gamma \le \sinh^2(\frac{\beta+\gamma}{2})$ and $\sinh^2\alpha = 2\sinh\beta\sinh\gamma \le 2\sinh^2(\frac{\beta+\gamma}{2}) \le \sinh^2(\frac{\beta+\gamma}{2} + \frac{1}{2}\log 2)$; on the other hand, $\sinh\beta \ge \frac{1-e^{-2}}{2}e^{\beta}$ and $\sinh\gamma \ge \frac{1-e^{-2}}{2}e^{\gamma}$, while $\sinh\alpha < \frac{1}{2}e^{\alpha}$, giving $e^{2\alpha} > 2(1-e^{-2})^2 e^{\beta+\gamma} > e^{\beta+\gamma}$ and $e^{\alpha'} > (1-e^{-2})e^{(\beta+\gamma)/2} > e^{(\beta+\gamma-1)/2}$. Thus

$$\frac{\beta+\gamma}{2} - \frac{1}{2} < \alpha' \le \frac{\beta+\gamma}{2} < \alpha \le \frac{\beta+\gamma}{2} + \frac{1}{2}\log 2. \tag{8.16}$$

(This corrects the lower bound in [3, (21)].) Moreover, $\sinh(2\alpha) = 2\sinh\alpha\cosh\alpha > 2\sinh^2\alpha = 4\sinh\beta\sinh\gamma$, so

$$0 \le \frac{\sqrt{2}(\sinh\beta - \sinh\gamma)}{\sinh(2\alpha)} < \frac{\sqrt{2}\sinh\beta}{4\sinh\beta\sinh\gamma} = \frac{1}{2\sqrt{2}\sinh\gamma} \le \frac{e^{-\gamma}}{\sqrt{2}(1-e^{-2})} < e^{-\gamma} < 1.$$

Hence there is a unique $\theta \in (0, \frac{\pi}{2}]$ with $\cos\theta = \frac{\sqrt{2}(\sinh\beta - \sinh\gamma)}{\sinh(2\alpha)}$, and $0 \le \frac{\pi}{2} - \theta \le \frac{\pi}{2}\cos\theta < \frac{\pi}{2}e^{-\gamma}$.

Now use $\widetilde{\iota}(d_\psi) = \widetilde{v}_\psi \widetilde{w}_{0,-2\psi}$ and the fact that $\widetilde{w}_{0,-2\psi}$ commutes with $\widetilde{D}(\alpha, 0)$: for all $h \in \mathrm{SU}(2)$ and $\psi \in \mathbb{R}$,

$$\widetilde{D}(\alpha,0)\widetilde{\iota}(d_\psi h d_{-\psi})\widetilde{D}(\alpha,0) = \widetilde{v}_{-\psi}\widetilde{\iota}(d_\psi)\widetilde{D}(\alpha,0)\widetilde{\iota}(h)\widetilde{D}(\alpha,0)\widetilde{\iota}(d_{-\psi})\widetilde{v}_\psi. \tag{8.17}$$

By the invariance of $c$, averaging over $\psi \in [0, 2\pi]$ yields

$$c\big(\widetilde{v}_t\widetilde{D}(\alpha,0)\widetilde{\iota}(u_\theta)\widetilde{D}(\alpha,0)\big) = \mathcal{A}\big(\lambda_{\widetilde{H}}\delta_{\widetilde{v}_t\widetilde{D}(\alpha,0)}\widetilde{\iota}_*(\mu_\theta)\delta_{\widetilde{D}(\alpha,0)}\lambda_{\widetilde{H}}\big).$$

Applying Proposition 8.7 (at $\theta$ and at $\frac{\pi}{2}$, where $\beta = \gamma = \alpha'$ and $\omega_1 + \omega_2 = 0$), (8.13), (8.2) and (8.16) gives

$$d\big(c(\widetilde{v}_{t+\omega_1+\omega_2}\widetilde{D}(\beta,\gamma)),\, c(\widetilde{v}_t\widetilde{D}(\alpha',\alpha'))\big) \le C_2 e^{4\pi s}L^2 e^{s|t|+2s\alpha}|\theta - \tfrac{\pi}{2}|^{s_2} \le K_0' L^2 e^{s|t|+s\beta-(s_2-s)\gamma}.$$

Because $\theta \in (0, \frac{\pi}{2}]$ gives $\omega_1 \in [0, \frac{\pi}{8}]$ and $\omega_2 \in [-\frac{\pi}{8}, 0]$ by Proposition 8.7, we have $\omega_1 + \omega_2 \in [-\frac{\pi}{8}, \frac{\pi}{8}] \subset [-\frac{\pi}{4}, \frac{\pi}{4}]$. Replacing $t$ by $t - \omega_1 - \omega_2$ yields (iii) with $\sigma = -(\omega_1 + \omega_2) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. □

**Proposition 8.9** (convergence of $c(g)$)**.** *There exist a constant $K_1 > 0$ (depending only on $C, \eta, \alpha_1, s$) and an operator $P \in B(X_0, X_2)$ such that for every $g \in \widetilde{G}$ with Cartan parameter $\beta(g) = \log\|\sigma(g)\|_{\mathrm{op}}$,*

$$\|\pi(c(g)) - P\|_{B(X_0,X_2)} \le K_1 L^2 e^{s|\Phi(g)|-\tau_0\beta(g)}. \tag{8.18}$$

*Moreover, for every $\mu \in \mathcal{P}_c(\widetilde{G})$ supported in $\{\ell_0 \le R\}$, there exists ${}_\mu P \in B(X_0, X_2)$ such that $\|\pi(\mu c(g)) - {}_\mu P\| \le K_1 L^2 e^{sR+s|\Phi(g)|-\tau_0\beta(g)}$; the norm difference $(\mu, \mu') \mapsto \|{}_\mu P - {}_{\mu'} P\|$ is weak-$*$ lower-semicontinuous on pairs of measures supported in any fixed compact set; and ${}_{c(g_1)}P = P$ for every $g_1 \in \widetilde{G}$.*

*Proof.* Set $r_0 = 5\lambda + 2$. We first show that for all real $r_1 \le r_2 \le r_1 + 1$ with $r_1 \ge r_0$ and all $t \in \mathbb{R}$,

$$d\big(c(\widetilde{v}_t\widetilde{D}(r_1,0)),\, c(\widetilde{v}_t\widetilde{D}(r_2,0))\big) \le K'L^2 e^{s|t|-\tau_0 r_1}. \tag{8.19}$$

Along the dividing ray $\beta = \lambda\gamma$, the function $\sinh^2(\lambda\gamma) + \sinh^2\gamma$ increases strictly from 0 to $\infty$, so there is a unique point $P_1 = (\beta_1, \gamma_1)$ with $\beta_1 = \lambda\gamma_1$ and $\sinh^2\beta_1 + \sinh^2\gamma_1 = \sinh^2 r_1$. Since $\sinh^2\beta_1 \ge \frac{1}{2}\sinh^2 r_1$, we have $r_1 - \frac{1}{2}\log 2 \le \beta_1 \le r_1$ and $\gamma_1 = \beta_1/\lambda \ge 5$. Next, along the hyperbola $\sinh\beta\sinh\gamma = \sinh\beta_1\sinh\gamma_1 =: A_1$ for $\beta \ge \beta_1$, the quantity $\sinh^2\beta + A_1^2/\sinh^2\beta$ increases strictly from $\sinh^2 r_1$ to $\infty$, so there is a unique point $P_2 = (\beta_2, \gamma_2)$ on this hyperbola with $\beta_2 \ge \beta_1$ and $\sinh^2\beta_2 + \sinh^2\gamma_2 = \sinh^2 r_2$. Then $r_2 - \frac{1}{2}\log 2 \le \beta_2 \le r_2 \le r_1 + 1$ and $\beta_2 \ge \beta_1 = \lambda\gamma_1 \ge \lambda\gamma_2$, so $P_2$ lies in the cone $\{\beta \ge \lambda\gamma\}$. Furthermore, $\sinh\gamma_2 = \frac{\sinh\beta_1}{\sinh\beta_2}\sinh\gamma_1 \ge \frac{\sinh(r_1 - \frac{1}{2}\log 2)}{\sinh(r_1+1)}\sinh\gamma_1 \ge e^{-3}\sinh\gamma_1 \ge \sinh(\gamma_1 - 4)$, so $\gamma_2 \ge \gamma_1 - 4 \ge 1$, and hence

$$s\beta_2 - (s_2 - s)\gamma_2 \le s(r_1+1) - (s_2 - s)(\gamma_1 - 4) \le -\tau_0\beta_1 + s(1 + \tfrac{1}{2}\log 2) + 4(s_2 - s) \le -\tau_0 r_1 + O(1).$$

Now chain the three elementary moves from $(r_1, 0)$ to $P_1$ (Lemma 8.8(i)), from $P_1$ to $P_2$ (two applications of Lemma 8.8(iii) via $(\alpha', \alpha')$), and from $P_2$ to $(r_2, 0)$ (Lemma 8.8(i)): using (8.9), there exists $\psi \in [-\pi, \pi]$ (independent of $t$) such that

$$d\big(c(\widetilde{v}_t\widetilde{D}(r_1,0)),\, c(\widetilde{v}_{t+\psi}\widetilde{D}(r_2,0))\big) \le K''L^2 e^{s|t|-\tau_0 r_1}.$$

Applying Lemma 8.8(ii) at most four times in steps of $\psi/4 \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ at $(r_2, 0)$ (noting $r_2 \ge 2$ and $s_1 - s > \tau_0$ by (8.9)) removes the shift $\psi$ at cost $O(L^2 e^{s|t|-(s_1-s)r_2}) \le O(L^2 e^{s|t|-\tau_0 r_1})$, proving (8.19).

Summing (8.19) along integers $n \ge r \ge r_0$ shows that $(\pi(c(\widetilde{v}_t\widetilde{D}(n,0))))_n$ is Cauchy in $B(X_0, X_2)$ and converges to a limit $P_t \in B(X_0, X_2)$ with $\|\pi(c(\widetilde{v}_t\widetilde{D}(r,0))) - P_t\| \le K'''L^2 e^{s|t|-\tau_0 r}$ for all real $r \ge r_0$ (using (8.19) once from $r$ to $\lceil r \rceil$). By Lemma 8.8(ii), $\|P_t - P_{t+\tau}\| = \lim_{n\to\infty} d(c(\widetilde{v}_t\widetilde{D}(n,0)), c(\widetilde{v}_{t+\tau}\widetilde{D}(n,0))) = 0$ for all $|\tau| \le \frac{\pi}{4}$, so $P_t =: P$ is independent of $t \in \mathbb{R}$.

Now let $(\beta, \gamma)$ be arbitrary with $\beta \ge \gamma \ge 0$, and $t \in \mathbb{R}$. If $\beta < r_0$, (8.18) holds trivially from $\|\pi(c(\widetilde{v}_t\widetilde{D}(\beta,\gamma)))\| \le L^2 e^{s(|t|+r_0+3\pi)}$ and $\|P\| \le K'''L^2 + L^2 e^{s(r_0+3\pi)}$. Assume $\beta \ge r_0$.

- If $\beta \ge \lambda\gamma$, Lemma 8.8(i) and (8.9) compare $c(\widetilde{v}_t\widetilde{D}(\beta,\gamma))$ with $c(\widetilde{v}_{t+\rho}\widetilde{D}(2\alpha, 0))$ ($2\alpha \ge \beta \ge r_0$) up to $K_0 L^2 e^{s|t|-\tau_0\beta}$, and the latter is within $K'''L^2 e^{s(|t|+\pi/4)-\tau_0\beta}$ of $P$.

- If $\gamma \le \beta \le \lambda\gamma$, let $(\beta_*, \gamma_*)$ be the unique point on the dividing ray $\beta_* = \lambda\gamma_*$ with $\sinh\beta_*\sinh\gamma_* = \sinh\beta\sinh\gamma$. Since

  $$\sinh\beta\sinh(\beta/\lambda) \le \sinh\beta\sinh\gamma = \sinh\beta_*\sinh(\beta_*/\lambda),$$

  we have $\beta_* \ge \beta \ge r_0$. Applying Lemma 8.8(iii) twice via $(\alpha', \alpha')$ and using (8.9) compares $c(\widetilde{v}_t\widetilde{D}(\beta,\gamma))$ with $c(\widetilde{v}_{t+\sigma'}\widetilde{D}(\beta_*,\gamma_*))$ ($|\sigma'| \le \frac{\pi}{2}$) up to $O(L^2 e^{s|t|-\tau_0\beta})$; the first bullet then compares the latter with $P$ up to $O(L^2 e^{s|t|-\tau_0\beta_*}) \le O(L^2 e^{s|t|-\tau_0\beta})$.

By (8.10), this proves (8.18).

For $\mu \in \mathcal{P}_c(\widetilde{G})$ supported in $\{\ell_0 \le R\}$, the shifted two-step representation $\pi_0^\mu = \pi_0$, $\pi_1^\mu(g) = \int \pi_1(hg)\,d\mu(h)$ has separate growth bounds $L_0 = L$ and $L_1 = Le^{sR}$. Rescaling $(\pi_0^\mu, \pi_1^\mu) \mapsto (e^{sR/2}\pi_0^\mu, e^{-sR/2}\pi_1^\mu)$ preserves $\pi^\mu = \pi_1^\mu\pi_0^\mu$ while giving both maps the common growth constant $Le^{sR/2}$ (so $(Le^{sR/2})^2 = L^2 e^{sR}$, reflecting the bilinearity of all bounds in $(L_0, L_1)$); applying (8.18) therefore yields ${}_\mu P$ with the stated bound. Weak-$*$ lower-semicontinuity of $(\mu, \mu') \mapsto \|{}_\mu P - {}_{\mu'} P\|$

follows identically to [6, Lemma 3.4]: if $\mu_i \to \mu$ and $\mu_i' \to \mu'$ weak-$*$ in $\{\ell_0 \le R\}$, then for every $n \ge 2$,

$$\|{}_{\mu}P - {}_{\mu'}P\| \le 2K_1L^2e^{sR-\tau_0(n-2)} + d(\mu m_n, \mu' m_n) \le 4K_1L^2e^{sR-\tau_0(n-2)} + \liminf_i \|{}_{\mu_i}P - {}_{\mu_i'}P\|,$$

and letting $n \to \infty$ gives the claim. Finally, for any fixed $g_1 \in \widetilde{G}$, the convolution identity (8.11) gives

$$c(g_1)m_n = \int_{\widetilde{H}\times(\mathbb{R}/\pi\mathbb{Z})} c\big(g_1\widetilde{h}\widetilde{v}_u\widetilde{D}(n-2,0)\widetilde{v}_{-u}\big)\, d\widetilde{h}\,\frac{du}{\pi}.$$

By (8.1) and submultiplicativity of $\|\cdot\|_{\mathrm{op}}$, every element $g' = g_1\widetilde{h}\widetilde{v}_u\widetilde{D}(n-2,0)\widetilde{v}_{-u}$ satisfies $|\Phi(g')| \le |\Phi(g_1)| + \frac{\pi}{2}$ and $\beta(g') \ge (n-2) - \beta(g_1)$. Testing against unit vectors $x \in X_0$, $y \in X_2^*$ and applying Fubini and (8.18) gives

$$\|\pi(c(g_1)m_n) - P\| \le K_1L^2e^{s(|\Phi(g_1)|+\pi/2)-\tau_0(n-2-\beta(g_1))} \xrightarrow{n\to\infty} 0,$$

so ${}_{c(g_1)}P = P$. □

We next prove the two-step decay of all nontrivial $(V,m)$-modes, both in forward and reflected form, without dualizing $X_1$.

**Proposition 8.10** (decay of twisted two-step coefficients)**.** *Let $(\pi_V, V)$ be a finite-dimensional irreducible unitary representation of* SU(2) *and $m \in \mathbb{Z}$ with $(V,m) \ne (\mathbf{1},0)$. There exists a constant $K_2(V,m) > 0$ (depending only on $C, \eta, \alpha_1, s, V, m$) such that for all unit vectors $x \in X_0$, $y \in X_2^*$, $w \in V$ and all $g \in \widetilde{G}$, both $V$-valued functions*

$$C(g) = \int_{\widetilde{H}\times\widetilde{H}\times(\mathbb{R}/\pi\mathbb{Z})} e^{-2imt}\langle y, \pi(h_1\widetilde{v}_t g\widetilde{v}_{-t}h_2)x\rangle\, \pi_V(h_1^{-1})w\, dh_1\, dh_2\, \frac{dt}{\pi},$$
$$\check{C}(g) = \int_{\widetilde{H}\times\widetilde{H}\times(\mathbb{R}/\pi\mathbb{Z})} e^{-2imt}\langle y, \pi\big((h_1\widetilde{v}_t g\widetilde{v}_{-t}h_2)^{-1}\big)x\rangle\, \pi_V(h_1^{-1})w\, dh_1\, dh_2\, \frac{dt}{\pi}$$

*satisfy*

$$\|C(g)\|_V \le K_2(V,m)L^2e^{s|\Phi(g)|-\tau_0\beta(g)}, \qquad \|\check{C}(g)\|_V \le K_2(V,m)L^2e^{s|\Phi(g)|-\tau_0\beta(g)}. \tag{8.20}$$

*Proof.* Let $\widetilde{C}$ denote either $C$ or $\check{C}$. Both satisfy the exact equivariance law

$$\widetilde{C}(\widetilde{h}_1\widetilde{v}_u g\widetilde{v}_{-u}\widetilde{h}_2) = e^{2imu}\pi_V(\widetilde{h}_1)\widetilde{C}(g) \qquad (\widetilde{h}_1, \widetilde{h}_2 \in \widetilde{H},\ u \in \mathbb{R}), \tag{8.21}$$

checked by substituting $h_1' = h_1\widetilde{h}_1$, $h_2' = \widetilde{h}_2h_2$, $t' = t + u \pmod{\pi}$, as well as the a priori bound $\|\widetilde{C}(g)\|_V \le L^2e^{2\pi s}e^{s\ell_0(g)}$ (using $\pi(g) = \pi_1(g)\pi_0(1)$ and $\ell_0(g^{-1}) = \ell_0(g)$).

Put $\Omega = \widetilde{H}\times\widetilde{H}\times(\mathbb{R}/\pi\mathbb{Z})$ with probability measure $d\omega = dh_1\, dh_2\, \frac{dt}{\pi}$, and set $Y = L^2(\Omega; X_1)$. By Lemma 7.4(ii), $Y \in \mathcal{D}(C,\eta,\alpha_1)$. For any $g_1, g_2 \in \widetilde{G}$ and unit vector $v \in V$, consider $h \mapsto \langle\widetilde{C}(g_1\widetilde{\iota}(h)g_2), v\rangle_V$ on SU(2):

- For $\widetilde{C} = C$: factoring $\pi(h_1\widetilde{v}_t g_1\widetilde{\iota}(h)g_2\widetilde{v}_{-t}h_2) = \pi_1(h_1\widetilde{v}_t g_1 k)\pi_0(k^{-1}\widetilde{\iota}(h)g_2\widetilde{v}_{-t}h_2)$ and averaging over $k \in \widetilde{H}$ yields $\langle C(g_1\widetilde{\iota}(h)g_2), v\rangle_V = \Lambda\big((\lambda(h)\otimes \mathrm{Id}_Y)F\big)$, where $F(k,\omega) = \pi_0(k^{-1}g_2\widetilde{v}_{-t}h_2)x$ and $\Lambda(\Psi) = \int_{\widetilde{H}\times\Omega} e^{-2imt}\langle\pi_V(h_1^{-1})w, v\rangle\langle y, \pi_1(h_1\widetilde{v}_t g_1 k)\Psi(k,\omega)\rangle\, dk\, d\omega$ satisfy $\|F\| \le Le^{s(\ell_0(g_2)+\pi)}$ and $\|\Lambda\| \le Le^{s(\ell_0(g_1)+\pi)}$.

- For $\widetilde{C} = \check{C}$: since

$$(h_1\widetilde{v}_t g_1 \widetilde{\iota}(h) g_2 \widetilde{v}_{-t} h_2)^{-1} = h_2^{-1}\widetilde{v}_t g_2^{-1}\widetilde{\iota}(h)^{-1} g_1^{-1}\widetilde{v}_{-t}h_1^{-1},$$

factoring as $\pi_1(h_2^{-1}\widetilde{v}_t g_2^{-1}k)\pi_0(k^{-1}\widetilde{\iota}(h)^{-1}g_1^{-1}\widetilde{v}_{-t}h_1^{-1})$ gives $\langle \check{C}(g_1\widetilde{\iota}(h)g_2), v\rangle_V = \check{\Lambda}\big((\lambda(h^{-1})\otimes \mathrm{Id}_Y)\check{F}\big)$, where $\check{F}(k,\omega) = \pi_0(k^{-1}g_1^{-1}\widetilde{v}_{-t}h_1^{-1})x$ and

$$\check{\Lambda}(\Psi) = \int_{\widetilde{H}\times\Omega} e^{-2imt}\langle \pi_V(h_1^{-1})w, v\rangle \langle y, \pi_1(h_2^{-1}\widetilde{v}_t g_2^{-1}k)\Psi(k,\omega)\rangle\, dk\, d\omega$$

satisfy $\|\check{F}\| \le Le^{s(\ell_0(g_1)+\pi)}$ and $\|\check{\Lambda}\| \le Le^{s(\ell_0(g_2)+\pi)}$.

Thus Proposition 8.3 applies identically to both $C$ and $\check{C}$.

We now establish the analogues of Lemmas 4.14 and 4.15 of [3] for $\widetilde{C}$ on $\widetilde{G}$. First observe two invariance properties of $\widetilde{C}$ from (8.21):

(a) Since $\widetilde{\iota}(r_\psi)$ commutes with $\widetilde{v}_t$ and $\widetilde{D}(\alpha',\alpha')$, every vector $\widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha',\alpha'))$ lies in the subspace $V_1 = \{v \in V : \pi_V(r_\psi)v = v\ \forall \psi \in \mathbb{R}\}$.

(b) By the identity $\widetilde{\iota}(d_\psi)\widetilde{D}(2\alpha,0) = \widetilde{v}_\psi\widetilde{D}(2\alpha,0)\widetilde{v}_{-\psi}\widetilde{\iota}(d_\psi)$ (from (8.17) with $h = I_2$), every vector $\widetilde{C}(\widetilde{v}_t\widetilde{D}(2\alpha,0))$ lies in the subspace $V_2 = \{v \in V : \pi_V(d_\psi)v = e^{2im\psi}v\ \forall \psi \in \mathbb{R}\}$.

Crucially, $V_1 \cap V_2 = \{0\}$: if $V = \mathbf{1}$ then $m \neq 0$ so $V_2 = \{0\}$; if $V \neq \mathbf{1}$ then $\dim V \ge 2$, and any $v \in V_1 \cap V_2$ would span a one-dimensional subspace invariant under both $\{r_\psi\}$ and $\{d_\psi\}$, hence under all of SU(2), contradicting irreducibility. Since $V$ is finite-dimensional, there exists $K(V,m) > 0$ such that

$$\|v_1\|_V + \|v_2\|_V \le K(V,m)\|v_1 - v_2\|_V \qquad (v_1 \in V_1,\ v_2 \in V_2). \tag{8.22}$$

*Equivariant Move 1.* Let $\beta \ge \gamma \ge 0$ ($\beta > 0$), $\sinh^2(2\alpha) = \sinh^2\beta + \sinh^2\gamma$, and $t \in \mathbb{R}$. The function $h \mapsto \widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha,\alpha)\widetilde{v}_{-\pi/4}\widetilde{\iota}(h)\widetilde{D}(\alpha,\alpha))$ satisfies $f(r_\psi h r_{\psi'}) = \pi_V(r_\psi)f(h)$. Diagonalizing the unitary action $\pi_V(r_\psi)$ in an orthonormal basis $e_1,\dots,e_d$ of $V$ with weights $m_1,\dots,m_d \in \mathbb{Z}$ and applying Proposition 8.3(i) to each coordinate $\langle f(h), e_j\rangle_V$ with $\theta = \arctan(\sinh\gamma/\sinh\beta) \in [0,\frac{\pi}{4}]$ gives

$$\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha,\alpha)\widetilde{w}_{-\theta,\theta-\pi/2}\widetilde{D}(\alpha,\alpha)) - \widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha,\alpha)\widetilde{w}_{0,-\pi/2}\widetilde{D}(\alpha,\alpha))\|_V \le C_{V,m}L^2e^{s|t|+s\beta-s_1(\beta-\gamma)}.$$

By Proposition 8.6 and (8.21), the two vectors on the left are $z_1\pi_V(d_1)\widetilde{C}(\widetilde{v}_{t+\rho_1}\widetilde{D}(\beta,\gamma))$ and $z_2\pi_V(d_2)\widetilde{C}(\widetilde{v}_{t+\rho_2}\widetilde{D}(2\alpha,0))$ with $|z_j| = 1$, $d_j \in \{d_\psi\}$, $\rho_1 = -(\phi+\phi') \in [-\frac{\pi}{2},0]$ and $\rho_2 = -\frac{\pi}{4}$. Applying the unitary operator $\bar{z}_1\pi_V(d_1^{-1})$, using property (b) so $\pi_V(d_1^{-1}d_2)$ acts as a scalar of modulus 1 on $\widetilde{C}(\widetilde{v}_{t+\rho_2}\widetilde{D}(2\alpha,0))$, and replacing $t$ by $t-\rho_1$ gives $\rho = \rho_2 - \rho_1 \in [-\frac{\pi}{4},\frac{\pi}{4}]$ and $z \in S^1$ (independent of $t$) such that for all $t \in \mathbb{R}$,

$$\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(\beta,\gamma)) - z\widetilde{C}(\widetilde{v}_{t+\rho}\widetilde{D}(2\alpha,0))\|_V \le C^{(1)}_{V,m}L^2e^{s|t|+s\beta-s_1(\beta-\gamma)}. \tag{8.23}$$

*Equivariant Move 2.* Let $\beta \ge \gamma \ge 1$, $\frac{1}{2}\sinh^2\alpha = \sinh\beta\sinh\gamma = \sinh^2\alpha'$, and $t \in \mathbb{R}$. By (8.17) and (8.21), the function $h \mapsto \widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha,0)\widetilde{\iota}(h)\widetilde{D}(\alpha,0))$ satisfies $f'(d_\psi h d_{-\psi}) = e^{-2im\psi}\pi_V(d_\psi)f'(h)$. Diagonalizing $\pi_V(d_\psi)$ in an orthonormal basis and applying Proposition 8.3(ii) with $\theta \in (0,\frac{\pi}{2}]$ as in Lemma 8.8(iii) gives

$$\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha,0)\widetilde{\iota}(u_\theta)\widetilde{D}(\alpha,0)) - \widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha,0)\widetilde{\iota}(u_{\pi/2})\widetilde{D}(\alpha,0))\|_V \le C'_{V,m}L^2e^{s|t|+s\beta-(s_2-s)\gamma}.$$

By Proposition 8.7 and (8.21),

$$\widetilde{C}\big(\widetilde{v}_t\widetilde{D}(\alpha,0)\widetilde{\iota}(u_\theta)\widetilde{D}(\alpha,0)\big) = e^{-im(\omega_1+\omega_2)}\pi_V(r_\phi)\pi_V(d_{(\omega_1-\omega_2)/2})\widetilde{C}\big(\widetilde{v}_{t+\omega_1+\omega_2}\widetilde{D}(\beta,\gamma)\big).$$

Since $(\omega_1,\omega_2)\mapsto e^{-im(\omega_1+\omega_2)}\pi_V(d_{(\omega_1-\omega_2)/2})$ is Lipschitz on $\mathbb{R}^2$ and (8.7) gives $|\omega_1|+|\omega_2|\le\frac{\pi}{\sqrt{2}}e^{-\gamma}$, replacing this factor by $\mathrm{Id}_V$ incurs an error of at most

$$C''_{V,m}L^2e^{s|t|+s\beta-\gamma} \le C''_{V,m}L^2e^{s|t|+s\beta-(s_2-s)\gamma}$$

(as $s_2-s<s_2\le 1$). At $\theta=\frac{\pi}{2}$ we similarly replace the factor by $\mathrm{Id}_V$ at cost $C''_{V,m}L^2e^{s|t|+(s-1)\alpha'}$, which is also dominated by $e^{s|t|+s\beta-(s_2-s)\gamma}$ because $\gamma-\frac12<\alpha'\le\beta$ by (8.16). Applying $\pi_V(r_{\phi_1}^{-1})$, using property (a) so $\pi_V(r_{\phi_1}^{-1}r_{\phi_2})$ fixes $\widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha',\alpha'))$, and shifting $t\mapsto t-(\omega_1+\omega_2)$ yields $\sigma=-(\omega_1+\omega_2)\in[-\frac{\pi}{4},\frac{\pi}{4}]$ such that for all $t\in\mathbb{R}$,

$$\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(\beta,\gamma))-\widetilde{C}(\widetilde{v}_{t+\sigma}\widetilde{D}(\alpha',\alpha'))\|_V \le C^{(2)}_{V,m}L^2e^{s|t|+s\beta-(s_2-s)\gamma}. \tag{8.24}$$

*Conclusion pointwise in $t\in\mathbb{R}$.* First let $(\beta_0,\gamma_0)$ lie on the dividing ray $\beta_0=\lambda\gamma_0$ with $\gamma_0\ge1$, and let $\sinh^2(2\alpha)=\sinh^2\beta_0+\sinh^2\gamma_0$, $\sinh^2\alpha'=\sinh\beta_0\sinh\gamma_0$. For any $t_0\in\mathbb{R}$, (8.23), (8.24), (8.9) and the triangle inequality give

$$\|\widetilde{C}(\widetilde{v}_{t_0+\sigma}\widetilde{D}(\alpha',\alpha'))-z\widetilde{C}(\widetilde{v}_{t_0+\rho}\widetilde{D}(2\alpha,0))\|_V \le \big(C^{(1)}_{V,m}+C^{(2)}_{V,m}\big)L^2e^{s|t_0|-\tau_0\beta_0}.$$

By (8.22), both terms on the left are individually bounded by $K(V,m)(C^{(1)}_{V,m}+C^{(2)}_{V,m})L^2e^{s|t_0|-\tau_0\beta_0}$. Shifting $t_0$ by $-\sigma$ (respectively $-\rho$), which changes $e^{s|t_0|}$ by at most $e^{s\pi/4}$, yields a constant $C^{(3)}_{V,m}$ such that for all $t\in\mathbb{R}$,

$$\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(\alpha',\alpha'))\|_V+\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(2\alpha,0))\|_V \le C^{(3)}_{V,m}L^2e^{s|t|-\tau_0\beta_0}. \tag{8.25}$$

Now let $(\beta,\gamma)$ be arbitrary with $\beta\ge\gamma\ge0$, and $t\in\mathbb{R}$. If $\beta<\lambda+1$, (8.20) follows from the a priori bound. Assume $\beta\ge\lambda+1$:

- If $\beta\ge\lambda\gamma$, let $(\beta_0,\gamma_0)$ be the unique point on $\beta_0=\lambda\gamma_0$ with $\sinh^2\beta_0+\sinh^2\gamma_0=\sinh^2\beta+\sinh^2\gamma=\sinh^2(2\alpha)$. Then $\beta_0\ge\beta-\frac12\log2>\lambda$, so $\gamma_0\ge1$, and combining (8.23) with (8.25) gives $\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(\beta,\gamma))\|_V\le K'_2L^2e^{s|t|-\tau_0\beta}$.
- If $\gamma\le\beta\le\lambda\gamma$, then $\gamma\ge\beta/\lambda>1$; let $(\beta_0,\gamma_0)$ be the unique point on $\beta_0=\lambda\gamma_0$ with $\sinh\beta_0\sinh\gamma_0=\sinh\beta\sinh\gamma=\sinh^2\alpha'$. Then $\beta_0\ge\beta$ and $\gamma_0=\beta_0/\lambda\ge\beta/\lambda>1$, and combining (8.24) with (8.25) gives $\|\widetilde{C}(\widetilde{v}_t\widetilde{D}(\beta,\gamma))\|_V\le K'_2L^2e^{s|t|-\tau_0\beta}$.

Finally, for general $g=\widetilde{h}_1\widetilde{v}_t\widetilde{D}(\beta,\gamma)\widetilde{v}_{t'}\widetilde{h}_2$, (8.21) shows $\|\widetilde{C}(g)\|_V=\|\widetilde{C}(\widetilde{v}_{\Phi(g)}\widetilde{D}(\beta,\gamma))\|_V$, which proves (8.20). □

**Theorem 8.11** (property $(*_{\mathcal{D}})$ for $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$)**.** *Let $C\ge1$, $\eta\in(0,1)$ and $\alpha_1\in(0,\frac12]$. Then $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$ satisfy property $(*_{\mathcal{D}(C,\eta,\alpha_1)})$ for every length function, with symmetric compactly supported probability measures. Explicitly, on $(\widetilde{\mathrm{Sp}}_4(\mathbb{R}),\ell_0)$, property $(*_{\mathcal{D}})$ holds for every $0<s<s_-(\alpha_1,\eta)$ with decay rate $\tau_0=P_{\alpha_1,\eta}(s)/(\alpha_1+\eta-s)$ and the symmetric measures $m_n$ of* (8.12).

*Proof.* Fix $0 < s < s_-(s_1, s_2)$ and let $(X_0, X_1, X_2, \pi_0, \pi_1)$ be a two-step representation of $\widetilde{G} = \widetilde{\mathrm{Sp}}_4(\mathbb{R})$ with $X_1 \in \mathcal{D}$ and $\|\pi_i(g)\| \le L e^{s\ell_0(g)}$. By Proposition 8.9, there is $P \in B(X_0, X_2)$ such that $\|\pi(m_n) - P\| \le K_1 e^{2\tau_0} L^2 e^{-\tau_0 n}$ for all $n \ge 2$ (and adjusting the constant covers $n = 0, 1$). It remains to prove two-sided invariance: $\lim_{n\to\infty} \|\pi(\delta_{g_1} m_n \delta_{g_2}) - P\| = 0$ for all $g_1, g_2 \in \widetilde{G}$.

For each integer $k \ge 1$, define explicit probability densities on $\widetilde{H} \cong \mathrm{SU}(2)$ and $\mathbb{R}/\pi\mathbb{Z}$ by

$$f_k(h) = a_k^{-1}\Big(\frac{3 + \operatorname{tr} h}{5}\Big)^k \qquad (h \in \mathrm{SU}(2)), \qquad \varphi_k(u) = b_k^{-1}\Big(\frac{3 + 2\cos 2u}{5}\Big)^k \qquad (u \in \mathbb{R}/\pi\mathbb{Z}),$$

normalized so that $\nu_k = f_k\, dh$ is a central probability measure on $\widetilde{H}$ and $\varphi_k(u)\frac{du}{\pi}$ is a probability measure on $\mathbb{R}/\pi\mathbb{Z}$. Since $(3 + \operatorname{tr} h)/5 \in [\frac{1}{5}, 1]$ attains its unique maximum 1 at $h = I_2$, and $(3 + 2\cos 2u)/5 \in [\frac{1}{5}, 1]$ attains its unique maximum 1 at $u = 0 \pmod{\pi}$, we have $\nu_k \to \delta_1$ weak-$*$ on $\widetilde{H}$ and $\varphi_k \frac{du}{\pi} \to \delta_0$ weak-$*$ on $\mathbb{R}/\pi\mathbb{Z}$. Moreover, $f_k$ is a finite linear combination of irreducible characters $\chi_V$ of $\mathrm{SU}(2)$ (as $\mathrm{tr} = \chi_{\mathbb{C}^2}$ and products of characters decompose into finite sums of irreducible characters), with coefficient 1 on $\chi_{\mathbf{1}} \equiv 1$, and $\varphi_k$ is a finite linear combination of $e^{-2imu}$ ($|m| \le k$) with constant term 1.

For any $k \ge 1$ and $g \in \widetilde{G}$, define the probability measure

$$c_k(g) = \int_{\mathbb{R}/\pi\mathbb{Z}} \varphi_k(t)\, \nu_k \delta_{\widetilde{v}_t g \widetilde{v}_{-t}} \lambda_{\widetilde{H}}\, \frac{dt}{\pi}.$$

Expanding $f_k = 1 + \sum_{V \ne \mathbf{1}} b_{k,V} \chi_V$ and $\varphi_k(t) = 1 + \sum_{0<|m|\le k} a_{k,m} e^{-2imt}$ expresses $c_k(g) - c(g)$ as a finite linear combination of the complex measures $\mu_{V,m}(g) = \int_{\mathbb{R}/\pi\mathbb{Z}} e^{-2imt} (\chi_V d\widetilde{h}) \delta_{\widetilde{v}_t g \widetilde{v}_{-t}} \lambda_{\widetilde{H}} \frac{dt}{\pi}$ with $(V, m) \ne (\mathbf{1}, 0)$. Since every irreducible character $\chi_V$ of $\mathrm{SU}(2)$ is real-valued ($\chi_V(h) = \sum_{j=1}^{\dim V} \langle \pi_V(h^{-1}) e_j, e_j \rangle_V$ in an orthonormal basis $e_1, \dots, e_d$ of $V$), we have for all unit vectors $x \in X_0$, $y \in X_2^*$:

$$\langle y, \pi(\mu_{V,m}(g)) x \rangle = \sum_{j=1}^{\dim V} \langle C_{x,y,e_j}(g), e_j \rangle_V, \qquad \langle y, \pi(\check{\mu}_{V,m}(g)) x \rangle = \sum_{j=1}^{\dim V} \langle \check{C}_{x,y,e_j}(g), e_j \rangle_V.$$

Proposition 8.10 therefore yields a constant $K(k) > 0$ such that for all $g \in \widetilde{G}$,

$$\|\pi(c_k(g) - c(g))\| \le K(k) L^2 e^{s|\Phi(g)| - \tau_0 \beta(g)}, \qquad \|\pi(\check{c}_k(g) - \check{c}(g))\| \le K(k) L^2 e^{s|\Phi(g)| - \tau_0 \beta(g)}. \tag{8.26}$$

Now fix $g_1 \in \widetilde{G}$ and define the probability measures $\mu_{k,g_1} = \int_{\mathbb{R}/\pi\mathbb{Z}} \varphi_k(u)\, \nu_k \delta_{\widetilde{v}_u g_1 \widetilde{v}_{-u}} \frac{du}{\pi}$. For every $n \ge 2$, substituting $r = t - u \pmod{\pi}$ as in (8.11) gives

$$\mu_{k,g_1} m_n = \int_{\widetilde{H} \times (\mathbb{R}/\pi\mathbb{Z})} c_k\big(g_1 \widetilde{h} \widetilde{v}_r \widetilde{D}(n-2, 0) \widetilde{v}_{-r}\big)\, d\widetilde{h}\, \frac{dr}{\pi},$$

while $c(g_1) m_n$ is the same integral with $c$ in place of $c_k$. By (8.26) and Fubini (tested against unit vectors $x \in X_0$, $y \in X_2^*$),

$$d(\mu_{k,g_1} m_n, c(g_1) m_n) \le K(k) L^2 e^{s(|\Phi(g_1)| + \pi/2) - \tau_0(n - 2 - \beta(g_1))} \xrightarrow{n\to\infty} 0.$$

Since $\lim_n \pi(c(g_1) m_n) = {}_{c(g_1)}P = P$ by Proposition 8.9, we deduce ${}_{\mu_{k,g_1}}P = P$ for every $k \ge 1$. As $k \to \infty$, the measures $\mu_{k,g_1}$ are all supported in the compact set $\{\widetilde{h} \widetilde{v}_u g_1 \widetilde{v}_{-u} : \widetilde{h} \in \widetilde{H}, u \in [0, \pi]\}$

and converge weak-$*$ to $\delta_{g_1}$. Applying the weak-$*$ lower-semicontinuity of $(\mu,\mu')\mapsto\|{}_\mu P-{}_{\mu'}P\|$ (Proposition 8.9) to $(\mu_{k,g_1},c(g_1))\to(\delta_{g_1},c(g_1))$ gives

$$\|{}_{\delta_{g_1}}P-P\|=\|{}_{\delta_{g_1}}P-{}_{c(g_1)}P\|\le\liminf_k\|{}_{\mu_{k,g_1}}P-{}_{c(g_1)}P\|=0,$$

hence ${}_{\delta_{g_1}}P=P$, i.e. $\lim_n\|\pi(\delta_{g_1}m_n)-P\|=0$ for every $g_1\in\widetilde{G}$.

For right invariance, note that $\check{c}(g)=c(g^{-1})$ and $\check{m}_n=m_n$, and consider the reflected pseudometric $\check{d}(m,m')=d(\check{m},\check{m}')=\|\pi(\check{m})-\pi(\check{m}')\|$, which satisfies $\check{d}(\delta_{g_1}\mu\delta_{g_2},\delta_{g_1}\mu'\delta_{g_2})=d(\delta_{g_2^{-1}}\check{\mu}\delta_{g_1^{-1}},\delta_{g_2^{-1}}\check{\mu}'\delta_{g_1^{-1}})$. For any $\mu\in\mathcal{P}_c(\widetilde{G})$ supported in $\{\ell_0\le R\}$ (so $\operatorname{supp}\check{\mu}\subseteq\{\ell_0\le R\}$), the operators $\pi((\mu m_n)^\vee)=\pi(m_n\check{\mu})$ are governed by the right-shifted two-step representation $\pi_0^{\check{\mu}}(g)=\int\pi_0(gh)\,d\check{\mu}(h)$, $\pi_1^{\check{\mu}}=\pi_1$ (growth bounds $L_0=Le^{sR}$, $L_1=L$). Rescaling $(e^{-sR/2}\pi_0^{\check{\mu}},e^{sR/2}\pi_1^{\check{\mu}})$ and applying Proposition 8.9 directly gives ${}_\mu\check{P}:=\lim_n\pi(m_n\check{\mu})$ with the uniform convergence bound $\|\pi(m_n\check{\mu})-{}_\mu\check{P}\|\le K_1L^2e^{sR-\tau_0(n-2)}$, weak-$*$ lower-semicontinuity of $(\mu,\mu')\mapsto\|{}_\mu\check{P}-{}_{\mu'}\check{P}\|$ on measures supported in a fixed compact set, and ${}_{c(g_2^{-1})}\check{P}=P$ (since $(c(g_2^{-1})m_n)^\vee=\int c\big((g_2^{-1}\widetilde{h}\widetilde{v}_u\widetilde{D}(n-2,0)\widetilde{v}_{-u})^{-1}\big)\,d\widetilde{h}\,\frac{du}{\pi}$ converges to $P$ by (8.18)). Using the second inequality in (8.26), the same argument gives ${}_{\mu_{k,g_2^{-1}}}\check{P}=P$ for all $k\ge1$, whence $\|{}_{\delta_{g_2^{-1}}}\check{P}-P\|\le\liminf_k\|{}_{\mu_{k,g_2^{-1}}}\check{P}-{}_{c(g_2^{-1})}\check{P}\|=0$, i.e. $\lim_n\|\pi(m_n\delta_{g_2})-P\|=0$ for every $g_2\in\widetilde{G}$. Finally, applying right invariance to the left-shifted two-step representation $\pi_1^{(g_1)}(g)=\pi_1(g_1g)$, $\pi_0^{(g_1)}(g)=\pi_0(g)$ (whose limit is $\lim_n\pi(\delta_{g_1}m_n)=P$) yields $\lim_n\|\pi(\delta_{g_1}m_n\delta_{g_2})-P\|=0$ for all $g_1,g_2\in\widetilde{G}$.

Thus $(\widetilde{\mathrm{Sp}}_4(\mathbb{R}),\ell_0)$ satisfies $(*_{\mathcal{D}})$ with symmetric measures. Combining Proposition 8.1(v) and [6, Lemma 2.6], $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ has $(*_{\mathcal{D}})$ for every length function, and Lemma 7.5 transfers this via the covering map $\sigma:\widetilde{\mathrm{Sp}}_4(\mathbb{R})\to\mathrm{Sp}_4(\mathbb{R})$ to $\mathrm{Sp}_4(\mathbb{R})$ with the symmetric pushforward measures $\sigma_*m_n$. □

## 8.5 Descent to all connected higher rank simple Lie groups, products, and lattices

To pass from the building blocks $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{R})$ to all connected simple real Lie groups of real rank at least two, we first record the exact functoriality of $(*_{\mathcal{E}})$ under quotients and finite normal extensions. In contrast to the one-space proof of [3, Lemma 6.4] (which projects onto $N$-invariant vectors in $X$), the two-step version averages $\pi_0$ and $\pi_1$ directly over $N$.

**Lemma 8.12** (quotients and finite normal extensions for two-step representations)**.** *Let $q:\widetilde{G}\to G$ be a continuous surjective homomorphism of locally compact groups with kernel $N=\ker q$, and let $\mathcal{E}$ be any class of Banach spaces.*

*(i) If $\widetilde{G}$ has $(*_{\mathcal{E}})$ for every length function, then so does $G$ (with symmetric measures if $\widetilde{G}$ has symmetric measures).*

*(ii) If $N$ is finite and $G$ has $(*_{\mathcal{E}})$ for every length function, then so does $\widetilde{G}$ (with symmetric measures if $G$ has symmetric measures).*

*Proof.* Part (i) is Lemma 7.5 (since $(q_*m_n)^\vee=q_*(\check{m}_n)$). For (ii), let $\ell$ be a length function on $\widetilde{G}$. Since $N$ is finite, $q$ is proper and $\ell'(h):=\max_{g\in q^{-1}(h)}\ell(g)$ is a length function on $G$: it is bounded on compact sets, symmetric because $q^{-1}(h^{-1})=q^{-1}(h)^{-1}$, and subadditive because any

$z \in q^{-1}(h_1h_2)$ factors as $g_1g_2$ with $g_1 \in q^{-1}(h_1)$ and $g_2 = g_1^{-1}z \in q^{-1}(h_2)$. Let $(\widetilde{m}_n)$ be probability measures supported in $\{\ell' \le n\}$ witnessing $(*_{\mathcal{E}})$ for $(G, \ell')$ with constants $s, t, C$. Define probability measures $m_n$ on $\widetilde{G}$ by $\int_{\widetilde{G}} f\, dm_n = \int_G \big(\frac{1}{|N|}\sum_{z\in N} f(s(h)z)\big)\, d\widetilde{m}_n(h)$, where $s(h) \in q^{-1}(h)$ is any lift (the average over $N$ is independent of $s(h)$). Then $\operatorname{supp} m_n \subseteq q^{-1}(\operatorname{supp} \widetilde{m}_n) \subseteq \{\ell \le n\}$, $m_n = \lambda_N m_n = m_n \lambda_N$ (where $\lambda_N = \frac{1}{|N|}\sum_{z\in N}\delta_z$), and $m_n$ is symmetric whenever $\widetilde{m}_n$ is.

Let $(X_0, X_1, X_2, \pi_0, \pi_1)$ be a two-step representation of $\widetilde{G}$ with $X_1 \in \mathcal{E}$ and $\|\pi_i(g)\| \le Le^{s\ell(g)}$. Define $\bar{\pi}_0 : G \to B(X_0, X_1)$ and $\bar{\pi}_1 : G \to B(X_1, X_2)$ by

$$\bar{\pi}_0(q(g)) = \frac{1}{|N|}\sum_{z\in N} \pi_0(zg), \qquad \bar{\pi}_1(q(g)) = \frac{1}{|N|}\sum_{z\in N} \pi_1(gz).$$

Because $N$ is normal in $\widetilde{G}$, both maps are well defined on $G = \widetilde{G}/N$, strongly continuous, and satisfy $\|\bar{\pi}_i(h)\| \le Le^{s\ell'(h)}$. Moreover, for all $g_1, g_2 \in \widetilde{G}$,

$$\bar{\pi}_1(q(g_1))\bar{\pi}_0(q(g_2)) = \frac{1}{|N|^2}\sum_{z_1,z_2\in N} \pi_1(g_1z_1)\pi_0(z_2g_2) = \frac{1}{|N|}\sum_{z\in N} \pi(g_1zg_2) = \pi(\delta_{g_1}\lambda_N\delta_{g_2}),$$

which by normality of $N$ depends only on $q(g_1g_2)$; hence $(X_0, X_1, X_2, \bar{\pi}_0, \bar{\pi}_1)$ is a two-step representation of $G$ with $\bar{\pi}(\widetilde{m}_n) = \pi(m_n)$ and $\bar{\pi}(\delta_{q(g_1)}\widetilde{m}_n\delta_{q(g_2)}) = \pi(\delta_{g_1}m_n\delta_{g_2})$ for all $g_1, g_2 \in \widetilde{G}$. Thus $(*_{\mathcal{E}})$ for $(G, \ell')$ transfers directly to $(\widetilde{G}, \ell)$. □

**Corollary 8.13** (all connected groups with Lie algebra $\mathfrak{sp}_4(\mathbb{R})$ or $\mathfrak{sl}_3(\mathbb{R})$)**.** *Let $R$ be any connected real Lie group whose Lie algebra is isomorphic to $\mathfrak{sp}_4(\mathbb{R})$ or $\mathfrak{sl}_3(\mathbb{R})$. Then $R$ satisfies $(*_{\mathcal{D}})$ for every length function, with symmetric compactly supported probability measures.*

*Proof.* If $\operatorname{Lie}(R) \cong \mathfrak{sp}_4(\mathbb{R})$, then $R$ is isomorphic to a quotient of $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ by a discrete central subgroup $N \subseteq Z(\widetilde{\mathrm{Sp}}_4(\mathbb{R})) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ (see Remark 7.6; note that this repairs the proof of [3, Lemma 6.5], which treated $Z(\widetilde{\mathrm{Sp}}_4(\mathbb{R}))$ as cyclic and thereby missed the infinite-center quotient $\widetilde{\mathrm{Sp}}_4(\mathbb{R})/\langle\widetilde{\iota}(-I_2)\rangle$). Theorem 8.11 and Lemma 8.12(i) yield $(*_{\mathcal{D}})$ for $R$.

If $\operatorname{Lie}(R) \cong \mathfrak{sl}_3(\mathbb{R})$, then $\mathrm{SL}_3(\mathbb{R})$ has $(*_{\mathcal{D}})$ by [6, Section 3.1 and Remark 3.7] (using our Theorem C, with the symmetric measures $m_n = \lambda_{\mathrm{SO}(3)}\delta_{D(n,0,-n)}\lambda_{\mathrm{SO}(3)}$). Because the maximal compact subgroup $\mathrm{SO}(3)$ of $\mathrm{SL}_3(\mathbb{R})$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$, the universal cover $\widetilde{\mathrm{SL}}_3(\mathbb{R})$ is a 2-fold central extension of $\mathrm{SL}_3(\mathbb{R})$ and is the only other connected group with Lie algebra $\mathfrak{sl}_3(\mathbb{R})$. Lemma 8.12(ii) lifts $(*_{\mathcal{D}})$ from $\mathrm{SL}_3(\mathbb{R})$ to $\widetilde{\mathrm{SL}}_3(\mathbb{R})$. □

**Proposition 8.14** (existence and closedness of the rank-2 subgroup)**.** *Let $G$ be a connected simple real Lie group of real rank at least two, with Lie algebra $\mathfrak{g}$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for a Cartan involution $\vartheta$. Then $\mathfrak{g}$ contains a $\vartheta$-stable Lie subalgebra $\mathfrak{r}$ isomorphic to $\mathfrak{sl}_3(\mathbb{R})$ or $\mathfrak{sp}_4(\mathbb{R})$, and the connected analytic subgroup $R \subset G$ with Lie algebra $\mathfrak{r}$ is closed in $G$.*

*Proof.* Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $\Sigma$ the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. If $\Sigma$ is not of type $B_2 = C_2$ or $BC_2$, the proof of Proposition 7.8 produces a $\vartheta$-stable subalgebra $\mathfrak{r} \cong \mathfrak{sl}_3(\mathbb{R})$ generated by $E_\gamma \in \mathfrak{g}_\gamma$, $F_\gamma = -\vartheta E_\gamma \in \mathfrak{g}_{-\gamma}$, $H_\gamma = [E_\gamma, F_\gamma] \in \mathfrak{a}$ for two roots $\gamma \in \{\alpha, \beta\}$.

If $\Sigma$ is of type $B_2$ or $BC_2$, choose roots $\alpha, \beta \in \Sigma$ forming a base of a $B_2 = C_2$ subsystem: in $B_2 = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$ take $\alpha = e_2$, $\beta = e_1 - e_2$; in $BC_2 = \{\pm e_i, \pm 2e_i, \pm e_1 \pm e_2\}$ take $\alpha = e_1 - e_2$, $\beta = 2e_2$. In both cases, $\alpha - \beta$, $3\alpha + \beta$, and $\alpha + 2\beta$ are not roots in $\Sigma$, and the Killing-form angles give $2\langle\beta, \alpha\rangle/\langle\alpha, \alpha\rangle = -2$ and $2\langle\alpha, \beta\rangle/\langle\beta, \beta\rangle = -1$. Pick nonzero $X_\gamma \in \mathfrak{g}_\gamma$

($\gamma = \alpha, \beta$), put $E_\gamma = X_\gamma$, $F_\gamma = -\vartheta X_\gamma$, $H_\gamma = [E_\gamma, F_\gamma]$. Since $\vartheta(H_\gamma) = -H_\gamma$ and $[H_\gamma, \mathfrak{a}] = 0$, we have $H_\gamma \in \mathfrak{a}$; since $B(H, H_\gamma) = \gamma(H)B(X_\gamma, -\vartheta X_\gamma)$ and $-B(\cdot, \vartheta\cdot)$ is positive definite on $\mathfrak{g}$, scaling $X_\gamma$ by a positive number gives $\gamma(H_\gamma) = 2$, $\beta(H_\alpha) = -2$, and $\alpha(H_\beta) = -1$. Because $\alpha - \beta$, $3\alpha + \beta$, $\alpha + 2\beta \notin \Sigma$, all Chevalley–Serre relations of type $C_2$ hold: $[E_\alpha, F_\beta] = [E_\beta, F_\alpha] = 0$, $(\operatorname{ad} E_\alpha)^3 E_\beta = (\operatorname{ad} E_\beta)^2 E_\alpha = 0$, and their $\vartheta$-images. By Serre's presentation theorem [10, Section 18], the $\vartheta$-stable subalgebra $\mathfrak{r}$ they generate is a nonzero quotient of the simple algebra $\mathfrak{sp}_4(\mathbb{R})$, hence isomorphic to $\mathfrak{sp}_4(\mathbb{R})$.

To prove that $R$ is closed in $G$ (whether $Z(G)$ is finite or infinite), use the global Cartan diffeomorphism $K_G \times \mathfrak{p} \xrightarrow{\sim} G$, $(k, X) \mapsto k\exp(X)$, where $K_G = \exp(\mathfrak{k})$ [11, Theorem 6.31]. Because $\mathfrak{r}$ is $\vartheta$-stable, $\mathfrak{r} = \mathfrak{k}_\mathfrak{r} \oplus \mathfrak{p}_\mathfrak{r}$ with $\mathfrak{k}_\mathfrak{r} \subset \mathfrak{k}$, $\mathfrak{p}_\mathfrak{r} \subset \mathfrak{p}$, and $R = K_R \exp(\mathfrak{p}_\mathfrak{r})$ where $K_R = \exp(\mathfrak{k}_\mathfrak{r}) \subseteq K_G$. Thus $R$ is closed in $G$ iff $K_R$ is closed in $K_G$. Write $\mathfrak{k} = \mathfrak{k}_{\mathrm{ss}} \oplus \mathfrak{z}(\mathfrak{k})$, where $\mathfrak{k}_{\mathrm{ss}} = [\mathfrak{k}, \mathfrak{k}]$ is compact semisimple (so $K_{\mathrm{ss}} = \exp(\mathfrak{k}_{\mathrm{ss}})$ is compact) and $\dim \mathfrak{z}(\mathfrak{k}) \le 1$ (as $\mathfrak{g}$ is simple); hence $K_G = K_{\mathrm{ss}} \cdot Z_0$ with $Z_0 = \exp(\mathfrak{z}(\mathfrak{k}))$ isomorphic to $\{1\}$, $S^1$, or $\mathbb{R}$, and $K_{\mathrm{ss}} \cap Z_0$ finite.

- If $\mathfrak{r} \cong \mathfrak{sl}_3(\mathbb{R})$, then $\mathfrak{k}_\mathfrak{r} \cong \mathfrak{so}(3)$ is semisimple, so $\mathfrak{k}_\mathfrak{r} \subseteq \mathfrak{k}_{\mathrm{ss}}$ and $K_R$ is a compact (hence closed) subgroup of $K_{\mathrm{ss}} \subseteq K_G$.

- If $\mathfrak{r} \cong \mathfrak{sp}_4(\mathbb{R})$, then $\mathfrak{k}_\mathfrak{r} = \mathfrak{h}_\mathfrak{r} \oplus \mathbb{R}J_0$ with $\mathfrak{h}_\mathfrak{r} \cong \mathfrak{su}(2) \subseteq \mathfrak{k}_{\mathrm{ss}}$, so $H_R = \exp(\mathfrak{h}_\mathfrak{r})$ is compact. Moreover, $J_0 = (E_{2e_1} - F_{2e_1}) + (E_{2e_2} - F_{2e_2})$ is the sum of the compact generators of two $\mathfrak{sl}_2$-triples in $\mathfrak{r}$, so $\operatorname{ad}(J_0)$ has eigenvalues in $i\mathbb{Z}$ on $\mathfrak{g}_\mathbb{C}$, whence $\exp(2\pi J_0) \in Z(G)$. Write $J_0 = X_{\mathrm{ss}} + cZ_\mathfrak{k}$ with $X_{\mathrm{ss}} \in \mathfrak{k}_{\mathrm{ss}}$, $Z_\mathfrak{k} \in \mathfrak{z}(\mathfrak{k})$. If $Z_0$ is compact or $c = 0$, then $\exp(\mathbb{R}J_0)$ lies in a compact subgroup $K'$ of $K_G$ ($K' = K_G$ if $Z_0$ is compact, $K' = K_{\mathrm{ss}}$ if $c = 0$); since $Z(G)$ is discrete in $G$ (as $\mathfrak{g}$ is simple), $Z(G) \cap K'$ is discrete in the compact group $K'$ and hence finite, so $\exp(2\pi J_0) \in Z(G) \cap K'$ has finite order $N \ge 1$, whence $\exp(\mathbb{R}J_0) = \exp([0, 2\pi N]J_0)$ is compact and $K_R = H_R \exp(\mathbb{R}J_0)$ is compact. If $Z_0 \cong \mathbb{R}$ and $c \ne 0$, then $\exp(tJ_0) = \exp(tX_{\mathrm{ss}})\exp(tcZ_\mathfrak{k})$ escapes every compact subset of $K_G$ as $|t| \to \infty$, so $\exp(\mathbb{R}J_0)$ is closed in $K_G$, and since $H_R$ is compact, $K_R = H_R \exp(\mathbb{R}J_0)$ is closed in $K_G$.

In every case $K_R$ is closed in $K_G$, so $R$ is closed in $G$. □

**Theorem 8.15** (all connected simple real Lie groups of higher rank)**.** *Let $G$ be a connected simple real Lie group of real rank at least two. Then $G$ satisfies $(*_\mathcal{D})$ for every length function, with symmetric compactly supported probability measures.*

*Proof.* Let $\ell$ be a length function on $G$, and equip $\mathfrak{g}$ with a norm $\|\cdot\|_\mathfrak{g}$. With $\log_+ t := \max\{0, \log t\}$, define a dominating length function

$$\ell'(g) = \ell(g) + \log_+\|\mathrm{Ad}(g)\|_{\mathrm{op}} + \log_+\|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}} \qquad (g \in G).$$

Each summand is nonnegative, symmetric (in the last two combined), bounded on compact sets, and subadditive (since $\|\mathrm{Ad}(gh)\|_{\mathrm{op}} \le \|\mathrm{Ad}(g)\|_{\mathrm{op}}\|\mathrm{Ad}(h)\|_{\mathrm{op}}$), so $\ell'$ is a length function with $\ell \le \ell'$, $\log\|\mathrm{Ad}(g)\|_{\mathrm{op}} \le \ell'(g)$, and $\log\|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}} \le \ell'(g)$ for all $g \in G$ (adapting the $\log\|\mathrm{Ad}(g)\|$ device of [3, proof of Theorem A] in symmetrized $\log_+$ form so that it applies to any length function without the implicit word-length assumption in [6, Lemma 4.1]). By [6, Lemma 2.6], it suffices to prove $(*_\mathcal{D})$ for $(G, \ell')$.

By Proposition 8.14, $G$ contains a closed connected subgroup $R$ with Lie algebra $\mathfrak{r}$ isomorphic to $\mathfrak{sl}_3(\mathbb{R})$ or $\mathfrak{sp}_4(\mathbb{R})$. By Corollary 8.13, $(R, \ell'|_R)$ has $(*_\mathcal{D})$ with constants $s(R), t(R), C(R) > 0$ and symmetric probability measures $m_n$ supported in $\{r \in R : \ell'(r) \le n\}$. Choose a nonzero

ad-diagonalizable element $X \in \mathfrak{a}_{\mathfrak{r}} \subset \mathfrak{r}$, put $a = \exp(X) \in R$, and decompose $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{R}} \mathfrak{g}_\lambda$ into $\mathrm{ad}(X)$-eigenspaces. Since $\mathfrak{g}$ is simple and $X \neq 0$, the subalgebra generated by $\bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda$ is a nonzero ideal of $\mathfrak{g}$, hence equals $\mathfrak{g}$; therefore $G$ is generated by $\bigcup_{\lambda \neq 0} \exp(\mathfrak{g}_\lambda)$.

Fix an integer $c \geq 1$ large enough that $|\lambda| c \geq 2$ for every nonzero eigenvalue $\lambda$ of $\mathrm{ad}(X)$, and choose any $s = s(G)$ with

$$0 < s < \min\left\{ s(R),\ \frac{t(R)}{c\,\ell'(a)},\ \frac{1}{2(1 + c\,\ell'(a))} \right\}$$

(incorporating the factor $c$ in $c\,\ell'(a)$ omitted in [6, Lemma 4.1]). Let $(X_0, X_1, X_2, \pi_0, \pi_1)$ be a two-step representation of $G$ with $X_1 \in \mathcal{D}$ and $\|\pi_i(g)\| \leq L e^{s\ell'(g)}$. For any $\mu, \nu \in \mathcal{P}_c(G)$ supported in $\{\ell' \leq R_1\}$ and $\{\ell' \leq R_2\}$ respectively, the maps $\widetilde{\pi}_0(r) = e^{\frac{s}{2}(R_1 - R_2)} \int \pi_0(rg')\, d\nu(g')$ and $\widetilde{\pi}_1(r) = e^{\frac{s}{2}(R_2 - R_1)} \int \pi_1(gr)\, d\mu(g)$ form a two-step representation of $R$ with norm bound $L e^{\frac{s}{2}(R_1+R_2)} e^{s\ell'(r)}$ and product $\widetilde{\pi}(r) = \pi(\mu \delta_r \nu)$. By $(*_{\mathcal{D}})$ for $(R, \ell'|_R)$, there exists ${}_\mu P_\nu \in B(X_0, X_2)$ such that

$$\|\pi(\mu m_n \nu) - {}_\mu P_\nu\| \leq C(R) L^2 e^{s(R_1+R_2) - t(R)n}, \tag{8.27}$$

and ${}_{\mu\delta_r} P_\nu = {}_\mu P_{\delta_r \nu} = {}_\mu P_\nu$ for all $r \in R$. Moreover, as in [6, Lemma 3.4], $(\mu, \nu, \mu', \nu') \mapsto \|{}_\mu P_\nu - {}_{\mu'} P_{\nu'}\|$ is weak-$*$ lower-semicontinuous on measures supported in any fixed compact set.

We claim that ${}_{\mu\delta_{\exp Y}} P_\nu = {}_\mu P_\nu$ for every $\lambda \neq 0$, $Y \in \mathfrak{g}_\lambda$, and $\mu, \nu \in \mathcal{P}_c(G)$. Replacing $X$ by $-X$ (so $a$ by $a^{-1}$, with $\ell'(a^{-1}) = \ell'(a)$), we may assume $\lambda > 0$. By convolving $\nu$ with a smooth compactly supported approximate identity on $G$ and using the weak-$*$ lower-semicontinuity of the norm difference, we may assume $\nu = \varphi\, dg$ where $\varphi \in C_c^\infty(G)$ ($\varphi \geq 0$, $\int \varphi = 1$) and $dg$ is the bi-invariant Haar measure of the simple (hence unimodular) group $G$. Then there is $C_\varphi < \infty$ such that $\|\delta_{\exp Z} \nu - \nu\|_{\mathrm{TV}} \leq C_\varphi \|Z\|_{\mathfrak{g}}$ for all $Z \in \mathfrak{g}$ with $\|Z\|_{\mathfrak{g}} \leq 1$. Since $a^{cn} \in R$, ${}_{\mu\delta_{\exp Y}} P_\nu = {}_{\mu\delta_{\exp(Y) a^{cn}}} P_\nu$ and ${}_\mu P_\nu = {}_{\mu\delta_{a^{cn}}} P_\nu$. By (8.27),

$$\|{}_{\mu\delta_{\exp Y}} P_\nu - {}_\mu P_\nu\| \leq 2C(R) L^2 e^{s(R_1+R_2+\ell'(\exp Y)) + (sc\ell'(a) - t(R))n} + \|\pi(\alpha_n)\|,$$

where $\alpha_n = \mu\delta_{\exp(Y)a^{cn}} m_n \nu - \mu\delta_{a^{cn}} m_n \nu = \int_{\{\ell'(g) \leq n\}} \mu\delta_{a^{cn} g} \big(\delta_{g^{-1} a^{-cn} \exp(Y) a^{cn} g} \nu - \nu\big)\, dm_n(g)$. Every element in $\mathrm{supp}(\alpha_n)$ has $\ell'$-length at most $R_n := R_1 + R_2 + \ell'(\exp Y) + n(1 + c\ell'(a))$, so (using $\pi(h) = \pi_1(h)\pi_0(1)$) we have $\|\pi(\alpha_n)\| \leq L^2 e^{s(\ell'(1) + R_n)} \|\alpha_n\|_{\mathrm{TV}}$. On the other hand, $g^{-1} a^{-cn} \exp(Y) a^{cn} g = \exp(e^{-\lambda cn} \mathrm{Ad}(g^{-1}) Y)$ satisfies

$$\|e^{-\lambda cn} \mathrm{Ad}(g^{-1}) Y\|_{\mathfrak{g}} \leq e^{(1 - \lambda c)n} \|Y\|_{\mathfrak{g}} \leq e^{-n} \|Y\|_{\mathfrak{g}}$$

for all $g \in \mathrm{supp}\, m_n$. For large $n$ this is at most 1, so $\|\alpha_n\|_{\mathrm{TV}} \leq C_\varphi \|Y\|_{\mathfrak{g}} e^{-n}$ and

$$\|\pi(\alpha_n)\| \leq C_\varphi \|Y\|_{\mathfrak{g}} L^2 e^{s(\ell'(1) + R_1 + R_2 + \ell'(\exp Y))} e^{(s(1 + c\ell'(a)) - 1)n} \xrightarrow{n \to \infty} 0$$

by our choice of $s$. The first term also vanishes as $n \to \infty$ since $sc\ell'(a) < t(R)$, proving ${}_{\mu\delta_{\exp Y}} P_\nu = {}_\mu P_\nu$.

Because $\{g \in G : {}_{\mu\delta_g} P_\nu = {}_\mu P_\nu\ \forall \mu, \nu\}$ is a subgroup containing $\bigcup_{\lambda \neq 0} \exp(\mathfrak{g}_\lambda)$, it equals $G$. Applying the symmetric argument on the right, using

$$\mu\delta_g \delta_{a^{-cn} \exp(Y)} = \big(\mu\delta_{\exp(e^{-\lambda cn} \mathrm{Ad}(g) Y)}\big) \delta_{g a^{-cn}}$$

and smooth $\mu$, yields ${}_{\mu P_{\delta_g \nu}} = {}_\mu P_\nu$ for all $g \in G$. In particular ${}_{\delta_g} P_{\delta_{g'}} = P$ for all $g, g' \in G$, establishing $(*_{\mathcal{D}})$ for $(G, \ell')$ with constants $s(G) = s$, $t(G) = t(R)$, $C(G) = C(R) e^{2s\ell'(1)}$, and the symmetric measures $m_n$. $\square$

**Corollary 8.16** (proof of Theorem B for lattices and products)**.** *Let $G = \prod_{i=1}^{k} G_i$ be a finite product of higher rank simple groups (each $G_i$ either a connected simple real Lie group of real rank at least two, or a connected almost $\mathbf{F}_i$-simple algebraic group of $\mathbf{F}_i$-split rank at least two over a non-Archimedean local field $\mathbf{F}_i$), and let $\Gamma \subset G$ be a lattice. Then for every $1 < p \leq 2$ and $1 \leq M < \infty$, both $G$ and $\Gamma$ satisfy property $(*_{\mathcal{C}_{p,M}})$, and hence strong Banach property $(T)$ with respect to every class of complex Banach spaces of uniformly nontrivial type.*

*Proof.* By Lemma 7.4(i), choose $C, \eta, \alpha_1$ so that $\mathcal{C}_{p,M} \subseteq \mathcal{D} = \mathcal{D}(C, \eta, \alpha_1)$. Every real factor $G_i$ has $(*_{\mathcal{D}})$ with symmetric measures by Theorem 8.15. By Lemma 7.4(iii), $\mathcal{D}$ has uniformly nontrivial Rademacher type, so every non-Archimedean factor $G_i$ has $(*_{\mathcal{D}})$ by [6, Theorem 1.4] (and with symmetric measures by Remark 7.10). By [6, Lemma 2.8], the product $G = \prod_{i=1}^{k} G_i$ has $(*_{\mathcal{D}})$ with symmetric product measures $m_n = \bigotimes_{i=1}^{k} m_n^{(i)}$.

Let $\Gamma \subset G$ be a lattice and $\ell_G$ a word-length function on $G$. By [6, Theorem 5.1 and Lemma 5.2] (building on Lubotzky–Mozes–Raghunathan [16], extended to non-algebraic real groups with infinite center via the bounded 2-cocycle of Guichardet–Wigner [8], and stated in [6] for arbitrary higher rank products including positive-characteristic factors), $\Gamma$ is undistorted in $G$, so by [6, Lemma 2.6] it suffices to prove $(*_{\mathcal{D}})$ for $(\Gamma, \ell_G|_\Gamma)$. By [6, Theorems 5.3 and 5.4, Remark 5.5], using the stability of $\mathcal{D}$ under $X \mapsto L^2(\Omega, \mu; X)$ and $X \mapsto X^*$ (Lemma 7.4(ii)) and the symmetry of the measures $m_n$ (Remark 7.10), $(\Gamma, \ell_G|_\Gamma)$ inherits $(*_{\mathcal{D}})$ from $(G, \ell_G)$. Since $\mathcal{C}_{p,M} \subseteq \mathcal{D}$, both $G$ and $\Gamma$ satisfy $(*_{\mathcal{C}_{p,M}})$. □

*Remark* 8.17 (verification of the classification of $B_2$ and $BC_2$ algebras). We confirm that the list of simple real Lie algebras of real rank two with restricted root system $B_2 = C_2$ or $BC_2$ given after Proposition 7.8 is complete and exact according to [9, Chapter X, Table VI] and [11, Appendix C, Sections 3–4]. Below, root multiplicities are listed in the order $(m_{\text{short}}, m_{\text{long}})$ for $B_2 = C_2$ and in the order $(m_{e_i}, m_{2e_i}, m_{e_i \pm e_j})$ for $BC_2$:

(a) *Complex simple Lie algebras of rank* 2 *viewed over* $\mathbb{R}$*:* only $A_2$ ($\mathfrak{sl}_3(\mathbb{C})$), $B_2 \cong C_2$ ($\mathfrak{sp}_4(\mathbb{C}) \cong \mathfrak{so}_5(\mathbb{C})$, finite center, root multiplicities $2, 2$), and $G_2$ ($\mathfrak{g}_2(\mathbb{C})$). Only $\mathfrak{sp}_4(\mathbb{C})$ has type $B_2/BC_2$.

(b) *Classical non-compact real forms of real rank* 2*:* types AI ($\mathfrak{sl}_3(\mathbb{R})$) and AII ($\mathfrak{su}^*(6) = \mathfrak{sl}_3(\mathbb{H})$) have restricted root system $A_2$. Type BDI gives $\mathfrak{so}(2, q)$ for $q \geq 3$ (type $B_2$, multiplicities $q - 2, 1$; note $\mathfrak{so}(2, 2) \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ is not simple, and $\mathfrak{so}(2, 3) \cong \mathfrak{sp}_4(\mathbb{R})$). Type AIII gives $\mathfrak{su}(2, 2) \cong \mathfrak{so}(2, 4)$ (type $C_2$, multiplicities $2, 1$) and $\mathfrak{su}(2, q)$ for $q \geq 3$ (type $BC_2$, multiplicities $2(q-2), 1, 2$). Type DIII ($\mathfrak{so}^*(2n)$, real rank $\lfloor n/2 \rfloor$) gives $\mathfrak{so}^*(8) \cong \mathfrak{so}(2, 6)$ ($C_2$, multiplicities $4, 1$) and $\mathfrak{so}^*(10)$ ($BC_2$, multiplicities $4, 1, 4$). Type CI gives $\mathfrak{sp}_4(\mathbb{R}) \cong \mathfrak{so}(2, 3)$ ($C_2$, multiplicities $1, 1$). Type CII gives $\mathfrak{sp}(2, 2)$ ($C_2$, multiplicities $4, 3$) and $\mathfrak{sp}(2, q)$ for $q \geq 3$ ($BC_2$, multiplicities $4(q - 2), 3, 4$). Here $\mathfrak{sp}(2, q)$ denotes the quaternionic unitary algebra of signature $(2, q)$ (a real form of $\mathfrak{sp}_{2(2+q)}(\mathbb{C})$), not to be confused with the $4 \times 4$ real symplectic algebra $\mathfrak{sp}_4(\mathbb{R})$.

(c) *Exceptional non-compact real forms:* of the twelve real forms, those of real rank 2 are $\mathfrak{g}_{2(2)}$ ($G_2$), $\mathfrak{e}_{6(-26)}$ ($A_2$), and $\mathfrak{e}_{6(-14)}$ ($BC_2$, multiplicities $8, 1, 6$); note that $\mathfrak{e}_{7(-25)}$ has real rank 3 (type $C_3$, containing $A_2$).

Among these $B_2/BC_2$ algebras, the Hermitian symmetric ones (whose universal covering groups have infinite center, since $\dim \mathfrak{z}(\mathfrak{k}) = 1$) are precisely $\mathfrak{so}(2, q)$ ($q \geq 3$; including $\mathfrak{sp}_4(\mathbb{R}) \cong \mathfrak{so}(2, 3)$ and $\mathfrak{so}^*(8) \cong \mathfrak{so}(2, 6)$), $\mathfrak{su}(2, q)$ ($q \geq 2$), $\mathfrak{so}^*(10)$, and $\mathfrak{e}_{6(-14)}$; the remaining ones ($\mathfrak{sp}(2, q)$ for $q \geq 2$, and $\mathfrak{sp}_4(\mathbb{C})$) have semisimple $\mathfrak{k}$ and finite fundamental group.

# 9 Remarks

*Remark* 9.1 (the scalar case). For $E = \mathbb{C}$ the best exponent in Theorem A is 1/3. By Peter–Weyl, the norm of $P_R S_0$ on $L^2(\mathrm{SU}(2))$ is the supremum over $d \geq 0$ and weights $m \equiv d \pmod 2$, $|m| \leq d$, of $|\psi(m/R) c_{d,m}|$, where $c_{d,m}$ is the diagonal matrix coefficient of $u_0$ at weight $m$ in $\mathrm{Sym}^d \mathbb{C}^2$. Taking the weight-$m$ Fourier coefficient of the character of $d_\theta u_0$ gives, with $N = d+1$,

$$c_{d,m} = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} \, \frac{\sin(N\omega(\theta))}{\sin \omega(\theta)} \, d\theta, \qquad \omega(\theta) = \arccos\Big(\frac{\cos\theta}{\sqrt{2}}\Big), \qquad \omega''(\theta) = \frac{\cos\theta}{(1+\sin^2\theta)^{3/2}}.$$

Sketch of the proof of the rate: the zeros of $\omega''$ are simple, so van der Corput's lemma of orders two and three gives $|c_{d,m}| \leq C N^{-1/3}$ uniformly in $m$; for $m > 0$, $d \equiv m \pmod 2$ and $N = \sqrt{2}\, m + O(1)$, the phase $N\omega(\theta) - m\theta$ has first derivative $O(1)$, second derivative 0 and third derivative of order $N$ at $\theta = \frac{\pi}{2}$, and the rescaling $\theta - \frac{\pi}{2} = N^{-1/3} v$ gives $|c_{d,m}| \geq c N^{-1/3}$. The details are in the blind answer of Section 11 and were checked by its readers; we do not reproduce them. A numerical computation agrees with this rate: for the weights $m = 10, 20, 40, \ldots, 640$ the product $m^{1/3} \sup_d |c_{d,m}|$ lies between 0.75 and 0.85, and the supremum is attained near $d \approx \sqrt{2}\, m$ (script `scripts/hilbert_rate.py`; numerical evidence, not a proof). We do not use the scalar case. The exponent of Theorem A satisfies $\eta = \varepsilon/8 < \frac{1}{16}$, far from $\frac{1}{3}$: the bound $0 < \varepsilon < \frac{1}{2}$ is already proved in Corollary 2.6 (and $\varepsilon \leq \frac{1}{2}$ holds for any choice of constants in Proposition 2.5 and Corollary 2.6, since $\rho_A(\mathbb{C}) = A^{-1/2}$ and already for $E = \mathbb{C}$ the operators $\mathcal{T}_{N,0}$ have norm of order $A^{-N/2}$).

*Remark* 9.2 (no decay for $\ell^1$). On $L^2(\mathrm{SU}(2); \ell^1)$ the norm of convolution by a finite measure $\mu$ is its total variation. The operator $P_R S_0$ is convolution by $\int k_R(\theta) \mu_\theta \, d\theta$, whose total variation is $\int |k_R| \geq \sup_m |\psi(m/R)|$, so for $\psi \neq 0$ it does not tend to 0. In the proof above, the type of $E$ enters only through Lemma 2.8 and Lemma 2.1(ii). Consequently, Lemma 2.8 cannot hold for $L^2(\mathbb{T}^2; \ell^1)$-valued functions: otherwise the arguments of Sections 2–4 would give decay of $P_R S_0$ on $L^2(\mathrm{SU}(2); \ell^1)$, contradicting the total-variation computation above (and indeed $\rho_A(\ell^1) = 1$ for every integer $A \geq 2$).

*Remark* 9.3 (other proofs). Two longer proofs of Theorem A were found earlier in this project, by slope-averaged frequency packets and by a cutoff at the mixed coefficient. Both use in addition Yao's parabolic partition [21, Lemma 4.4] and a dyadic radial regularization of the incidence; they rest on the same mechanism (at the fold, the cutoff to the degenerate direction is an integral of translations whose phase and translation carry the same bilinear form). We do not reproduce them.

*Remark* 9.4 (the exponent). The choice $J = \lceil K^{3/8} \rceil$, $B = K^{7/8}$ is not optimal. The same proof with $J = \lceil K^a \rceil$ and $B = K^{a+1/2}$, $a = (2+\varepsilon)/(6+2\varepsilon)$, gives the five error terms $K^{-\varepsilon(1/2-a)}$ (twice), $K^{1-3a}$, $K^{1/2-2a}$ and $K^{-a}$, and hence Theorem A with $\eta = \varepsilon/(6+2\varepsilon)$. We keep the simpler choice.

# 10 Inputs not proved here

Every external input used in the proofs of Theorems A, B, and C is a published, refereed result; no unrefereed preprint is used as an input (in particular, [21] is cited for attribution of the base-$A$ digit method and historical context, not as an unproved input).

1. Bourgain's Hausdorff–Young theorem for $B$-convex Banach spaces on compact abelian groups [2, Theorem 1, equation (3)] (published): for every Banach space $Y$ of nontrivial Rademacher type, there exist $r \in (1,2]$ and $K < \infty$ such that for every compact abelian group $G$ and every finitely supported family $(y_\gamma)_{\gamma \in \widehat{G}}$ in $Y$,

$$\Big(\int_G \Big\| \sum_{\gamma \in \widehat{G}} y_\gamma \, \gamma(g) \Big\|_Y^2 \, dg\Big)^{1/2} \le K \Big( \sum_{\gamma \in \widehat{G}} \|y_\gamma\|_Y^r \Big)^{1/r};$$

   used in Theorem 2.2.

2. From de la Salle [6] (published):

   - length-function comparison and finite products [6, Lemmas 2.6 and 2.8] (plus strong continuity and weak-$*$ lower-semicontinuity of two-step pseudometrics and limits in [6, Lemmas 2.1, 3.1, and 3.4]);
   - property $(*_{\mathcal{D}})$ for $\mathrm{SL}_3(\mathbb{R})$ from the latitude estimate [6, Section 3.1 and Remark 3.7], used in Corollary 8.13;
   - property $(*_{\mathcal{D}})$ for non-Archimedean higher rank simple algebraic groups [6, Theorem 1.4] (resting on Lafforgue [13] and Liao [15]), used only for non-Archimedean factors in the product case (Corollary 8.16);
   - lattice undistortedness and induction of $(*_{\mathcal{D}})$ to lattices [6, Theorems 5.1, 5.3, 5.4, Lemmas 5.2 and 5.6, and Remark 5.5] (building on [16, 8], including arbitrary higher rank products with positive-characteristic factors), used in Corollary 8.16 and Remark 7.10.

   (By contrast, [6, Section 3.3], [6, Lemma 4.1], [5, Proposition 2.8], and the $\mathrm{Sp}_4$ results of [3] are proved in full in Section 8 and are not used as external inputs.)

3. Standard structure theory of real semisimple Lie groups and root systems (published monographs): the global Cartan diffeomorphism [11, Theorem 6.31] (used in Proposition 8.14); the classification of irreducible root systems and Serre's presentation theorem for split simple Lie algebras [10, Section 18] (used in Propositions 7.8 and 8.14; compare [17, Proposition I.1.6.2] for context); and the tables of restricted root systems [9, Chapter X, Table VI] and [11, Appendix C, Sections 3–4] (used only for the classification list after Proposition 7.8 and Remark 8.17, not in the proofs of Theorems A–C).

4. The theorem of Pisier [20] and Maurey–Pisier [18] (published): a Banach space without nontrivial Rademacher type contains $\ell_n^1$ uniformly. It is used in Lemma 7.4(iii) (for the non-Archimedean factors in the product case of Theorem B) and noted for the equivalence of nontrivial type and $B$-convexity in Theorem 2.2.

# 11 Provenance and verification record

This section is part of the manuscript on purpose: a reader should know how the proofs were found and checked. Unless noted otherwise, dates below are 25 November 2026 (UTC); the two self-contained upgrades (Sections 2 and 8) were written and audited on 26 November 2026 (UTC). The file `CHECKS.md` that accompanies this manuscript lists every check.

*The question.* A second research session of this project, run with GPT-6.2 Sol, isolated Theorem A as the missing estimate and wrote two longer proofs of it (Remark 9.3), two days after Yao's preprint appeared.

*The main proof.* The proof of Theorem A in Sections 2–4 follows a blind proof by GPT-6.2 Astra, which received only the statement, the definitions and the text of Yao's preprint, and did not see the earlier proofs. The call ran twice because of an automatic retry. The text used here is the second answer (written at 09:33 UTC; SHA-256 `a61726e4...b098`). The first answer was overwritten and is lost; it used other coordinates and the same mechanism.

*Checks of the main proof.* (a) Claude Opus 5.5 read the second answer line by line: every step proved, no mathematical error, one surplus citation in the 25 November version (Yao's Lemma 3.4 is not a hypothesis needed to invoke his Corollary 3.6 as a black box, though it is proved and used inside the self-contained proof of Proposition 2.5 in the present version, as Lemma 2.4). (b) Gemini 4 Neon read the same answer line by line as a hostile referee. Its report marks all 24 steps as proved, with no error and no gap, and names the fold lemma as the most delicate step. In a separate conversation, from the bare statement and Sections 2 and 3 of [21] only, Gemini 4 Neon wrote its own proof of Theorem A, by another route (no localization, a two-shift form of Yao's Proposition 3.5) and with a smaller exponent. A Claude Fable 5.2 agent read both answers line by line; the second proof is correct after the repair of one constant. So Theorem A has two independent proofs, from two model families, and each was read line by line by a model of another family. (c) A Claude Opus 5.5 agent wrote this manuscript from the second answer and re-derived every step. A Claude Fable 5.2 agent, which had not written it, then read Sections 2–5 and 7 line by line and re-derived every displayed formula by hand; it found no mathematical error and repaired the wording in several places (among them the proof of the converse in Proposition 7.8). (d) The formulas of Section 3 were checked by exact symbolic differentiation and by Monte Carlo sampling (script `scripts/torus_geometry_check.py`). (e) In the 25 November version, the parts of Yao's preprint then invoked for Lemma 2.8 (Lemma 2.3, Proposition 3.5, Corollary 3.6, estimate (10)) were re-derived line by line by two Claude readers; in the present version, all of these are proved self-containedly in Section 2 directly from [2, Theorem 1]. (f) The manuscript itself went to two referees of other model families. GPT-6.2 Astra checked the parts that are not in the blind proof (Lemma 2.8, Proposition 2.10, Section 5, Section 7, the remarks) and read Sections 3 and 4 again; its verdict was "correct with repairs". Gemini 4 Neon read an earlier text of the whole manuscript as a hostile referee; its verdict was "correct with repairs". Neither report found an error in Sections 2–5. The repairs concern Section 7 and the wording, and all are made in this version: the proof of the converse in Proposition 7.8 (type $G_2$, and a sign in the $\mathfrak{sl}_2$-triples); Lemma 7.4(iii), which the product case of Theorem B needs (argument proposed by Neon); Remark 7.10 on symmetric measures in the lattice step of [6] (argument proposed by Astra); the convention on supports in Lemma 7.5; the constant in (1.3); and three places where a bound was written as an operator norm on the whole space although only the bound on the relevant functions is proved and used. (g) The passages written after these two reports, and Section 6 and the new scope sentences of this version, went to GPT-6.2 Astra for a further referee check (SHA-256 `5204238c...c80f`); a Claude Opus 5.5 agent had read the same passages line by line before. The verdict was "correct with repairs": Section 6, Lemma 7.4(iii), the converse in Proposition 7.8 and the reduction of Section 4 were proved; Remark 7.10 was proved after two additions (absolute convergence of the dual series for non-cocompact lattices, by [6, Lemma 5.6], and the difference between carried measures and closed supports); the attribution of Theorem C to Yao was narrowed to what Yao states (base point 0, some exponent); the lists of inputs were

made consistent; and the exponent for $E = \mathbb{C}$ in Remark 6.4 was qualified. All these repairs are made in this version. A consistency check of the accompanying files by GPT-6.2 Sol led to further changes of wording.

*The latitude estimate.* A Claude Fable 5.2 agent, looking for consequences of the method, observed (and checked by computer algebra) that in height and azimuth coordinates the latitude operator has no fold. GPT-6.2 Astra then wrote the proof of Theorem C that Section 6 follows (SHA-256 `1978791f...55a5`); it received the statement, the directional estimate (Proposition 2.10) as a tool and Sections 2 and 3 of [21], and was told not to read or use Sections 4 and 5 of [21]. That Claude agent read the proof line by line, with every formula checked by computer algebra; a Claude Opus 5.5 agent read it again line by line before it saw any other report. Gemini 4 Neon read it as a hostile referee (SHA-256 `894ab16f...29fd`); its verdict was "proved with repairs", with no mathematical error. Its seven points are made in Section 6: the hypotheses $|\delta| \le \frac{3}{4}$ and $b \in C^1$ in Lemma 6.3; the operators $V_s(t)$ act on the band subspace $X_{1,N}$, where they are bounded; the ranges of the multipliers $P_K$; the density argument; the restriction $h \le 1$; the constant at other base points; and joint measurability. So Theorem C has a proof by one model family, read line by line by two others.

*Checks of the other parts.* The two earlier proofs were checked by GPT-6.2 Astra (a comparison of both proofs, and a hostile line-by-line read of the longer manuscript), by Claude, and the decisive slope-averaging step by Gemini 4 Neon. The deduction of Theorem B from the two compact bounds was checked by GPT-6.2 Astra and by Claude; that check found the error about the center in Remark 7.6. Literature searches (arXiv, citations of [3, 6, 21], registries of AI mathematics, GitHub, MathOverflow), the last one ending at 12:18 UTC, found no earlier proof of Theorem A, of Theorem B for the groups of type $B_2$ and $BC_2$, or of a second proof of the latitude estimate. Such a search cannot be complete.

*Upgrades of 26 November 2026 (eliminating all unrefereed inputs and unwritten sketches).* (1) Section 2 was rewritten to give a complete, self-contained proof of uniform finite Fourier decay (Theorem 2.2), the exponential Schur multiplier and bandlimited Bernstein/translation lemmas (Lemmas 2.3 and 2.4), Yao's base-$A$ digit-expansion bounds (Proposition 2.5 and Corollary 2.6), and the bilinear-shift estimates (Proposition 2.7, Lemma 2.8, and Proposition 2.10) directly from Bourgain's published theorem [2, Theorem 1]. This removes both [21] and `arXiv:1212.4646` as unproved inputs while preserving full attribution to Yao for the base-$A$ digit method. (2) Section 8 was added to give a complete, self-contained proof of de la Salle's two-step strong property $(*_{\mathcal{D}})$ for $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ and $\mathrm{Sp}_4(\mathbb{R})$ (Theorem 8.11), filling in every detail left to the reader in [6, Section 3.3], proving the equivariant SU(2) Hölder bounds directly by matrix polynomials and Fell absorption (Proposition 8.3, eliminating [5, Proposition 2.8]), establishing the descent to all connected simple real Lie groups of real rank at least two with closedness of the rank-2 subgroup and general length functions (Lemma 8.12, Corollary 8.13, Proposition 8.14, Theorem 8.15, and Corollary 8.16), repairing minor slips in [3, Lemma 6.5] and [6, Lemma 4.1], and auditing the classification tables of $B_2$ and $BC_2$ algebras (Remark 8.17). (3) Following initial line-by-line review (which refined the $\log_+$ dominating length function and $\ell'(1)$ factor in Theorem 8.15, the right-shifted two-step representation in Theorem 8.11, and the finite-order compactness step in Proposition 8.14), both new sections underwent full hostile referee audits by Claude Opus 5.5 (`effort: high`) on 26 November 2026. These audits caught two critical issues and 38 minor items, all repaired in the present version: (i) in Theorem 2.2, Bourgain's theorem [2, Theorem 1, equation (3)] was verified directly against the 1982 Pacific Journal of Mathematics text and quoted verbatim with its $L^2$ left-hand side (removing the erroneous $L^{r'}$ left-hand side

and the redundant $\prod G_m$ parenthetical); and (ii) in Proposition 8.3, the non-unitarily-invariant $\ell^2_N(Y)$ norm in the Fell-absorption step was replaced by the unitarily invariant Gaussian norm $W \otimes_\gamma Y \hookrightarrow L^2(\Omega; Y)$, together with all minor repairs to constants, quantifiers, symbol overloads, and citations.

*Lean 4 formalization of the core Sections 3–5 engine (26 November 2026).* The accompanying Lean 4 project in `lean/` (5,225 lines across `BanachT.lean`, `BanachT/Geometry.lean`, `BanachT/ScalesAndFold.lean`, `BanachT/Holder.lean`, and `Check/Axioms.lean`, using `leanprover/lean4:v4.34.1` with `autoImplicit = false`) compiles cleanly with `lake build` (8,928 jobs, 0 errors, 0 warnings, 0 `sorry`) and passes the independent kernel verifier `lake env leanchecker BanachT` (exit code 0). All 89 exported theorems checked in `Check/Axioms.lean` depend only on the three standard Lean axioms `propext`, `Classical.choice`, and `Quot.sound`, with zero occurrences of `sorry`, `admit`, `axiom`, `native_decide`, `unsafe`, `implemented_by`, or `extern`. A fidelity audit by Gemini 4 Neon returned verdict CLEAN. Its exact scope is as follows:

- *Concretely proved from scratch in Lean 4:* all pointwise and differential geometry of Section 3 (`Geometry.lean`: SU(2) matrices and commutation, torus coordinates, incidence branches, Jacobian $\sqrt{D}$, exact `HasDerivAt`/`deriv` first and mixed derivatives $D^{-3/2}$ and $\kappa D^{-3/2}$, $I^2$ bounds, and the double-FTC $O(\ell^3)$ freezing remainder); the full core analytical engine of Section 4 (`ScalesAndFold.lean`: scale and exponent inequalities, geometric sums, $J \times J$ block Cauchy–Schwarz, Bochner $L^2$ Minkowski and Weierstrass $M$-test, `lem_freeze`, `lem_rect`, `lem_fold`, `prop_pieces_a`, `prop_pieces_b`, `theorem_A_T_sigma`, and `theorem_A`); and the circle-commutation and dyadic Littlewood–Paley Hölder deduction of Section 5 (`Holder.lean`: `circleDist`, $S_\theta = R_\theta S_0$, `dyadic_LP_split_tsum`, `corollary_holder`, `theorem_A_implies_holder`, and the end-to-end `bourgain_to_holder`).

- *Parameterized via explicit structured hypotheses (not axioms):* the directional Bourgain input `DirectionalBourgainInput` (Proposition 2.10), the torus/circle Fourier multiplier structures (`AwayFoldDecomp`, `DyadicBandSplit`, `TorusTranslationSpace`, `CircleLittlewoodPaley`), and the localization and Haar-product reduction hypothesis `hred` in `theorem_A` (Lemmas 3.1 and 3.4).

- *Paper proofs only (not yet formalized in Lean 4):* Section 2 (derivation of Proposition 2.10 from [2, Theorem 1]), Section 6 (Theorem C), and Sections 7–8 (Theorem B).

*What has not been done.* No human mathematician has refereed any part of this manuscript; every checker named above is an AI model. Outside the core Sections 3–5 engine formalized in `lean/`, Sections 2, 6, 7, and 8 (as well as the structured interfaces in `lean/`) remain paper proofs only. We do not use any unproved statement from [21] (Sections 2 and 6 provide self-contained proofs from [2]), and we did not check Yao's own proof of his Theorem 5.4. The two-step $\mathrm{Sp}_4(\mathbb{R})$ and $\widetilde{\mathrm{Sp}}_4(\mathbb{R})$ details of [6, Section 3.3] are now written out in full in Section 8.

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