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CHOLLET'S PERMANENT CONJECTURE THROUGH ORDER TEN

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Abstract. We prove Chollet's permanent conjecture, $\operatorname{per}(A \circ B) \leq \operatorname{per}(A)\operatorname{per}(B)$, for all $n \times n$ complex Hermitian positive semidefinite matrices A and B of every order $1 \leq n \leq 10$. This advances the unrestricted complex positive semidefinite frontier by four orders beyond the August 2026 record of $n \leq 6$ due to Li, Zhao, and Luo. Our proof combines two new ingredients: (i) a two-parameter double-border spectral induction ($q \to q+2$) that scales the two orthogonal eigendirections of a maximal 2×2 principal block independently and reduces positivity of the resulting tridiagonal border operators on $\operatorname{Sym}^k(\mathbb{C}^2)$ to univariate rational polynomials certified in the Bernstein basis; and (ii) for small off-diagonal entries at $n = 10$, a 3×3 triangle-packing Lieb lower bound paired with Okumura's sharp traceless cubic bound (proved via a two-line sum-of-squares identity) and a cycle-index Frobenius–derangement upper bound. All polynomial inequalities are verified in zero-floating-point exact rational arithmetic, with a substantial core (207 of 480 double-border checks plus all prerequisite, $r = 1$, and $n = 10$ small-entry Bernstein certificates) kernel-verified in Lean 4.27.0. We also report the observed limits of our schedule search at orders $n = 11$ and $n = 12$, where an uncertified intermediate off-diagonal gap remains open.

1. Introduction

For $n \times n$ complex matrices $A = (A_{ij})$ and $B = (B_{ij})$, let $A \circ B = (A_{ij}B_{ij})$ denote their Hadamard (entrywise) product and let

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

denote the permanent of A. In 1982, Chollet [2] conjectured that for all $n \times n$ complex Hermitian positive semidefinite (PSD) matrices A and B,

$$\operatorname{per}(A \circ B) \leq \operatorname{per}(A)\operatorname{per}(B). \tag{1}$$

The problem was studied shortly thereafter by Bapat and Sunder [1] alongside their stronger determinantal-permanental conjecture. Our main result establishes (1) unconditionally through order ten.

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Theorem 1.1. *For every integer $1 \leq n \leq 10$ and all $n \times n$ complex Hermitian positive semidefinite matrices A and B,*

$$\operatorname{per}(A \circ B) \leq \operatorname{per}(A) \operatorname{per}(B).$$

Literature context and relation to prior work. Unrestricted progress on Chollet's conjecture has proceeded order by order:

- Orders $n \leq 3$ were settled by Gregorac and Hentzel [3].
- Order $n = 4$ was proved for real PSD matrices by Hutchinson [4] and for complex Hermitian PSD matrices by Rodtes [11].
- Orders $1 \leq n \leq 6$ were established in August 2026 by Li, Zhao, and Luo [5], who introduced single-border induction ($q \to q+1$) through order six.

Complementary restricted-class results include Sa-nguansin and Rodtes [12] for correlation matrices whose off-diagonal entries lie in specified annuli, Pant and Singh [10] for bipartite-support symmetric Z-matrices and hereditary closure under one-point sums, and Ma [8], who established quantitative bounds strictly for matrices of rank at most four ($n \geq 10$ and order-nine tight frames) while noting that orders 7–9 and unrestricted ranks remained open.

Single-border induction hits an algebraic obstruction at $n = 7$: scaling a single rank-1 border direction by $t = \sqrt{2}$ requires $2^m \leq (m+1)^2$, which fails at $m = 6$ ($64 > 49$), and more generally no positive measure ν on $[0, \infty)$ can satisfy the single-border moment inequalities $k + 1 \leq m_{2k}$ and $m_k \leq k + 1$ ($0 \leq k \leq 6$), since $m_0 \leq 1$ and $m_2 \geq 2$ force $m_6 \geq m_2^3/m_0^2 \geq 8 > 7$ by Hölder's inequality (Remark 2.5).

Overview of the proof. Section 2 records the standard self-conjugate Cauchy–Schwarz reduction to correlation matrices ($A_{ii} = 1$), Loewner monotonicity of the permanent, Lieb's permanental inequality [6], and a self-contained exposition of the single-border induction for $1 \leq n \leq 6$ after Li, Zhao, and Luo [5]. Our new contributions resolve unrestricted orders $n = 7, 8, 9, 10$ via two complementary regimes classified by the maximum off-diagonal modulus $r = \max_{a \neq b} |A_{ab}| \in [0, 1]$ (summarized in Table 1):

(1) *Small off-diagonal entries ($r \leq \varepsilon_n$, Section 3).* For $n = 8$ ($\varepsilon_8 = \frac{7}{20}$) and $n = 9$ ($\varepsilon_9 = \frac{8}{25}$), averaging Lieb's inequality over 2×2 matchings in K_n beats an elementary derangement-counting upper bound on $\operatorname{per}(A \circ \bar{A})$. At $n = 10$, the 2×2 matching threshold saturates near 0.3110, short of our double-border regime at $r = \frac{9}{25} = 0.36$. We bridge this gap by combining a 3×3 triangle-packing Lieb lower bound (using both polynomial and convex Jensen–exponential Taylor bounds) with Okumura's sharp traceless cubic bound [9] (proved via an elementary two-line sum-of-squares identity) and a cycle-index exponential bound, pushing the small-entry regime to $\varepsilon_{10} = \frac{9}{25} = 0.36$ so that it meets our double-border schedule seamlessly.

(2) *Two-parameter double-border induction ($r \in [r_{\mathrm{lo}}, 1]$, Section 4).* Permuting a maximal off-diagonal pair to $(n-1, n)$ and phase-normalizing $A_{n-1,n} = r$, the boundary case $r = 1$ (equal border rows) is closed uniformly for all $7 \le n \le 10$ at pure rational $t = \frac{5}{3}$, $M = \frac{54}{25}$. For $r \in [r_{\mathrm{lo}}, 1)$, we scale the two orthogonal eigendirections of the 2×2 principal block $\left(\begin{smallmatrix} 1 & r \\ r & 1 \end{smallmatrix}\right)$ by two distinct rational functions $(t_1(r), t_2(r))$ with $t_1, t_2 > 1$. The induced operators on $\mathrm{Sym}^k(\mathbb{C}^2)$ decouple into even- and odd-index real symmetric tridiagonal blocks of size at most $\lceil (q+1)/2 \rceil \le 5$. Positive diagonal congruence clears all denominators and square roots, reducing Sylvester's criterion to strictly positive univariate polynomials in $\mathbb{Q}[r]$. At $n = 10$, choosing three piecewise-constant pairs (t_1, t_2) and piecewise-linear multipliers $\mu(r)$ on $[\frac{9}{25}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{7}{10}]$, and $[\frac{7}{10}, 1]$ reduces the maximum 5×5 minor degree from 300 down to 60.

TABLE 1. Summary of regimes closing Chollet's conjecture for all $1 \le n \le 10$.

n	Small-entry regime	Border-induction regime	$r = 1$
1–6	—	Single-border ($q \to q+1$, $t = \sqrt{2}$) [5]	incl.
7	not needed	$5 \to 7$ on $[0, 1]$ (1 quadratic piece)	$t = \frac{5}{3}$, $M = \frac{54}{25}$
8	2×2 Lieb $[0, \frac{7}{20}]$	$6 \to 8$ on $[\frac{3}{10}, 1]$ (overlaps $[0.30, 0.35]$)	$t = \frac{5}{3}$, $M = \frac{54}{25}$
9	2×2 Lieb $[0, \frac{8}{25}]$	$7 \to 9$ on $[\frac{8}{25}, 1]$ (meets at 0.32)	$t = \frac{5}{3}$, $M = \frac{54}{25}$
10	3×3 Lieb + SOS $[0, \frac{9}{25}]$	$8 \to 10$ on $[\frac{9}{25}, 1]$ (3 const.-(t_1, t_2))	$t = \frac{5}{3}$, $M = \frac{54}{25}$

Section 5 details the zero-floating-point exact rational Bernstein certification, the exact scope boundary of the `leanprover/lean4:v4.27.0` kernel verification (`CholletCertificates.lean`), and an independent C++20 empirical floating-point diagnostic across 7,000 complex PSD matrices. Section 6 documents the observed limits of our schedule search and remaining open gaps at $n = 11$ and $n = 12$.

2. Preliminaries and Self-Contained Base Cases ($1 \le n \le 6$)

Throughout, we write $X \succeq 0$ (respectively $X \succ 0$) to indicate that a Hermitian matrix X is positive semidefinite (respectively positive definite), and we denote by Chollet(n) the assertion that (1) holds for all $n \times n$ complex Hermitian PSD matrices A and B.

Lemma 2.1 (Self-conjugate equivalence and correlation normalization)**.** *Let $n \ge 1$ and let $A, B \in \mathbb{C}^{n \times n}$ be Hermitian PSD.*

(1) $\mathrm{per}(A) \ge 0$, $\mathrm{per}(B) \ge 0$, *and* $\mathrm{per}(A \circ B) \ge 0$.
(2) $|\mathrm{per}(A \circ B)|^2 \le \mathrm{per}(A \circ \bar{A})\ \mathrm{per}(B \circ \bar{B})$.

Consequently, Chollet(n) *holds if and only if*

$$\mathrm{per}(A \circ \bar{A}) \le \mathrm{per}(A)^2 \tag{2}$$

for every $n \times n$ Hermitian PSD correlation matrix A (meaning $A_{ii} = 1$ for all $1 \leq i \leq n$).

Proof. Write $A = U^*U$ with columns $u_1, \ldots, u_n \in \mathbb{C}^n$. On the symmetric tensor power $\mathrm{Sym}^n(\mathbb{C}^n)$ equipped with the inner product $\langle x_1 \vee \cdots \vee x_n,\, y_1 \vee \cdots \vee y_n \rangle = \mathrm{per}(\langle x_i, y_j \rangle)$ (linear in the first slot, conjugate-linear in the second), we have $\mathrm{per}(A) = \|u_1 \vee \cdots \vee u_n\|^2 \geq 0$. By Schur's product theorem, $A \circ B \succeq 0$, proving (1). For (2), set $x_\sigma = \prod_{i=1}^n A_{i,\sigma(i)}$ and $y_\sigma = \prod_{i=1}^n B_{i,\sigma(i)}$; Cauchy–Schwarz on $\mathbb{C}^{n!}$ yields

$$|\,\mathrm{per}(A \circ B)|^2 = \left| \sum_{\sigma \in S_n} x_\sigma y_\sigma \right|^2 \leq \left(\sum_{\sigma \in S_n} |x_\sigma|^2 \right) \left(\sum_{\sigma \in S_n} |y_\sigma|^2 \right) = \mathrm{per}(A \circ \bar{A})\ \mathrm{per}(B \circ \bar{B}).$$

Thus (2) for all PSD A implies (1) for all PSD (A, B), and conversely taking $B = \bar{A}$ recovers (2). If any $A_{ii} = 0$, then row and column i of A vanish and both sides of (2) are 0. Otherwise, replacing A by DAD with $D = \mathrm{diag}(A_{11}^{-1/2}, \ldots, A_{nn}^{-1/2})$ scales both sides of (2) by $\prod_{i=1}^n A_{ii}^{-2}$ and preserves Hermitian PSD. Finally, simultaneous permutation $A \mapsto PAP^T$ and diagonal unitary conjugation $A \mapsto \Theta A \Theta^*$ ($|\Theta_{ii}| = 1$) preserve both $\mathrm{per}(A)$ and $A \circ \bar{A}$. □

Lemma 2.2 (Loewner monotonicity of the permanent)**.** *If $X, Z \in \mathbb{C}^{q \times q}$ are Hermitian PSD, then* $\mathrm{per}(X + Z) \geq \mathrm{per}(X)$.

Proof. Realize X as the Gram matrix of $x_1, \ldots, x_q \in \mathcal{H}_1$ and Z as the Gram matrix of $z_1, \ldots, z_q \in \mathcal{H}_2$ with $\mathcal{H}_1 \perp \mathcal{H}_2$. Then $X + Z$ is the Gram matrix of $x_i \oplus z_i \in \mathcal{H}_1 \oplus \mathcal{H}_2$. Decomposing $\bigvee_{i=1}^q (x_i \oplus z_i)$ into mutually orthogonal sectors $\mathrm{Sym}^{q-k}(\mathcal{H}_1) \otimes \mathrm{Sym}^k(\mathcal{H}_2)$ for $0 \leq k \leq q$ yields

$$\mathrm{per}(X + Z) = \|x_1 \vee \cdots \vee x_q\|^2 + \sum_{k=1}^{q} \|\text{degree-}k \text{ sector}\|^2 \geq \mathrm{per}(X). \quad \square$$

Lemma 2.3 (Lieb's Permanental Inequality [6], Theorem 1)**.** *If $A = \left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{smallmatrix}\right)$ is Hermitian PSD with square diagonal blocks A_{11} and A_{22}, then* $\mathrm{per}(A) \geq \mathrm{per}(A_{11})\,\mathrm{per}(A_{22})$. *By induction, for any partition $\{1, \ldots, n\} = \bigsqcup_{r=1}^m B_r$,* $\mathrm{per}(A) \geq \prod_{r=1}^m \mathrm{per}(A[B_r])$.

The base cases $1 \leq n \leq 6$ were proved via single-border induction by Li, Zhao, and Luo [5]. We include a concise self-contained proof to establish notation for Section 4.

Proposition 2.4 (Orders $1 \leq n \leq 6$, after Li, Zhao, and Luo [5])**.** Chollet(n) *holds for all $1 \leq n \leq 6$.*

Proof. The base $n = 1$ is trivial ($\mathrm{per}([a\bar{a}]) = a^2 = \mathrm{per}([a])^2$). Assume Chollet($q$) for some $1 \leq q \leq 5$, and let $A = \left(\begin{smallmatrix} B & v \\ v^* & 1 \end{smallmatrix}\right)$ be a $(q+1) \times (q+1)$ correlation matrix. By Schur complement, $S := B - vv^* \succeq 0$. Realizing S as the Gram matrix of $w_1, \ldots, w_q \in \mathcal{H}_S$ with a unit vector $e_1 \perp \mathcal{H}_S$, A is

the Gram matrix of $u_i = w_i + v_i e_1$ $(1 \le i \le q)$ and $u_{q+1} = e_1$. Expanding $u_1 \vee \cdots \vee u_q = \sum_{k=0}^{q} \Phi_k \vee (e_1^k/\sqrt{k!})$ with $\Phi_k \in \mathrm{Sym}^{q-k}(\mathcal{H}_S)$ and setting $p_k := \|\Phi_k\|^2 \ge 0$, we have

$$\mathrm{per}(S + tvv^*) = \sum_{k=0}^{q} p_k t^k \quad (t \ge 0), \qquad \mathrm{per}(A) = \sum_{k=0}^{q} (k+1) p_k.$$

Similarly, $A \circ \bar{A} = \left(\begin{smallmatrix} T + ww^T & w \\ w^T & 1 \end{smallmatrix}\right)$, where $w = v \circ \bar{v}$ and $T = (S \circ \bar{S}) + (S \circ \bar{v}v^T) + (\bar{S} \circ vv^*) \succeq 0$ by Schur's product theorem. Writing $\mathrm{per}(T + tww^T) = \sum_{k=0}^{q} \tilde{p}_k t^k$ $(\tilde{p}_k \ge 0)$ and defining $B(\sqrt{2}) := S + \sqrt{2}\, vv^* \succeq 0$, we observe

$$B(\sqrt{2}) \circ \overline{B(\sqrt{2})} = (S \circ \bar{S}) + \sqrt{2}(S \circ \bar{v}v^T + \bar{S} \circ vv^*) + 2ww^T \succeq T + 2ww^T.$$

Because $k + 1 \le 2^k$ and $2^m \le (m+1)^2$ hold for every $0 \le k, m \le 5$, Lemma 2.2 and Chollet(q) yield

$$\begin{aligned} \mathrm{per}(A \circ \bar{A}) &= \sum_{k=0}^{q} (k+1)\tilde{p}_k \le \sum_{k=0}^{q} 2^k \tilde{p}_k = \mathrm{per}(T + 2ww^T) \\ &\le \mathrm{per}\big(B(\sqrt{2}) \circ \overline{B(\sqrt{2})}\big) \le \mathrm{per}\big(B(\sqrt{2})\big)^2 \le \mathrm{per}(A)^2. \qquad \square \end{aligned}$$

Remark 2.5 (Single-border obstruction at $n = 7$). Any single-border moment system for a positive measure ν on $[0, \infty)$ with moments $m_j = \int_0^\infty t^j \, d\nu(t)$ requiring $k + 1 \le m_{2k}$ and $m_k \le k + 1$ for $0 \le k \le q$ fails at $q = 6$ $(n = 7)$: since $m_0 \le 1$ and $m_2 \ge 2$, Hölder's inequality (equivalently, Jensen's inequality for $u \mapsto u^3$ under ν/m_0) forces $m_6 \ge m_2^3/m_0^2 \ge 8$, contradicting $m_6 \le 7$. Likewise, averaging $\mathrm{per}(B(t))^2$ directly via $\mathrm{per}(A \circ \bar{A}) \le \int_1^\infty \mathrm{per}(B(t) \circ \overline{B(t)}) \, d\nu(t) \le \int_1^\infty \mathrm{per}(B(t))^2 \, d\nu(t) \le \mathrm{per}(A)^2$ requires $k + 1 \le m_{2k} \le (k+1)^2$ for $0 \le k \le q$ (testing single-sector weights $p = e_k$), which also fails at $q = 6$ because $m_0 = 1$ and $m_2 \ge 2$ force $m_{12} \ge m_2^6/m_0^5 \ge 64 > 49 = 7^2$.

3. Small Off-Diagonal Entry Regimes $(n = 8, 9, 10)$

For an $n \times n$ correlation matrix A with $\max_{i \ne j} |A_{ij}| \le \varepsilon$, define the nonnegative real symmetric hollow matrix X by $X_{ij} = |A_{ij}|^2 \in [0, \varepsilon^2]$ $(i \ne j)$ and $X_{ii} = 0$, so that $A \circ \bar{A} = I_n + X$. Write $s := \sum_{1 \le i < j \le n} X_{ij}$ and let $d_m = m! \sum_{j=0}^{m} (-1)^j / j!$ denote the number of derangements of m elements. Expanding by support size $m = |\operatorname{supp}(\sigma)|$ gives

$$\mathrm{per}(A \circ \bar{A}) = \mathrm{per}(I_n + X) = 1 + \sum_{m=2}^{n} T_m(X),$$

$$T_m(X) := \sum_{\substack{\sigma \in S_n \\ |\operatorname{supp}(\sigma)| = m}} \prod_{i \in \operatorname{supp}(\sigma)} X_{i,\sigma(i)}.$$

Proposition 3.1 (2×2 matching-averaged Lieb bound for $n = 8, 9$)**.** *Let A be an $n \times n$ correlation matrix with $\max_{i \neq j} |A_{ij}| \leq \varepsilon$, and set $D_n := n - 1$ for n even and $D_n := n$ for n odd. Then*

$$\operatorname{per}(A)^2 \geq \left(1 + \frac{s}{D_n}\right)^2 \geq 1 + \frac{2}{D_n} s \quad \textit{and} \quad \operatorname{per}(A \circ \bar{A}) \leq 1 + C_{aa}(n, \varepsilon)\, s,$$

where

$$C_{aa}(n, \varepsilon) := \frac{2}{n(n-1)} \sum_{m=2}^{n} \binom{n}{m} d_m\, \varepsilon^{2(m-1)}.$$

In particular, $\operatorname{per}(A \circ \bar{A}) \leq \operatorname{per}(A)^2$ *holds for all* 8×8 *correlation matrices with* $\max_{i \neq j} |A_{ij}| \leq \frac{7}{20}$ *and all* 9×9 *correlation matrices with* $\max_{i \neq j} |A_{ij}| \leq \frac{8}{25}$.

Proof. For every matching M of $\lfloor n/2 \rfloor$ disjoint pairs in K_n, Lemma 2.3 yields

$$\operatorname{per}(A) \geq \prod_{\{i,j\} \in M} \operatorname{per}(A[\{i, j\}]) = \prod_{e \in M} (1 + X_e) \geq 1 + \sum_{e \in M} X_e.$$

By S_n-symmetry, a uniformly random matching of size $\lfloor n/2 \rfloor$ contains each edge of K_n with probability $\lfloor n/2 \rfloor / \binom{n}{2} = 1/D_n$. Averaging over M gives $\operatorname{per}(A) \geq 1 + s/D_n$, proving the lower bound.

For the upper bound, for each $\sigma \in S_n$ with $|\operatorname{supp}(\sigma)| = m \geq 2$ and every $r \in \operatorname{supp}(\sigma)$, we have $\prod_{i \in \operatorname{supp}(\sigma)} X_{i,\sigma(i)} \leq \varepsilon^{2(m-1)} X_{r,\sigma(r)}$. Averaging over $r \in \operatorname{supp}(\sigma)$ and summing over all $\binom{n}{m} d_m$ permutations of support size m, each of the $n(n-1)$ directed off-diagonal edges appears $m\binom{n}{m} d_m / (n(n-1))$ times, whence

$$T_m(X) \leq \frac{2\binom{n}{m} d_m}{n(n-1)}\, \varepsilon^{2(m-1)} s \qquad (2 \leq m \leq n). \tag{3}$$

Summing over $m = 2, \ldots, n$ gives $\operatorname{per}(A \circ \bar{A}) \leq 1 + C_{aa}(n, \varepsilon)\, s$. Exact rational evaluation confirms

$$C_{aa}\left(8, \frac{7}{20}\right) = \frac{1658204613950637431}{6553600000000000000} \approx 0.253022 < \frac{2}{7} \approx 0.285714,$$
$$C_{aa}\left(9, \frac{8}{25}\right) = \frac{44355113516753224353344}{209547579288482666015625} \approx 0.211671 < \frac{2}{9} \approx 0.222222. \square$$

At $n = 10$, Proposition 3.1 holds only up to $\varepsilon \approx 0.3110$ (exact threshold ≈ 0.311033), whereas our certified double-border schedule of Section 4 begins at $r = \frac{9}{25} = 0.36$ (with numerical schedule search reaching near 0.356–0.357). To close the gap at $n = 10$, we sharpen both sides: we replace edge-counting at small Frobenius norm by a traceless spectral cycle-index bound incorporating Okumura's sharp cubic inequality [9], and we replace 2×2 matchings by 3×3 triangle packings.

Lemma 3.2 (Traceless symmetric spectral bounds; cubic bound due to Okumura [9])**.** *Let $X \in \mathbb{R}^{n\times n}$ ($n \geq 2$) be real symmetric with $\operatorname{tr}(X) = 0$ and Frobenius norm $N := \|X\|_F = \sqrt{\operatorname{tr}(X^2)}$. Then*

$$\operatorname{tr}(X^L) \leq \left(\frac{n-1}{n}\right)^{(L-2)/2} N^L \quad (L \geq 2), \qquad \textit{and} \qquad \operatorname{tr}(X^3) \leq \frac{n-2}{\sqrt{n(n-1)}}\, N^3.$$

Proof. If $N = 0$ then $X = 0$ and both bounds are trivial. Otherwise let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the eigenvalues of X and set $x_i := \lambda_i/N$, so $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n x_i^2 = 1$. For every i, Cauchy–Schwarz gives

$$x_i^2 = \left(-\sum_{j\neq i} x_j\right)^2 \leq (n-1)\sum_{j\neq i} x_j^2 = (n-1)(1-x_i^2),$$

so $|x_i| \leq \sqrt{(n-1)/n}$. Hence for every $L \geq 2$,

$$\sum_{i=1}^n x_i^L \leq \sum_{i=1}^n |x_i|^{L-2} x_i^2 \leq \left(\frac{n-1}{n}\right)^{(L-2)/2} \sum_{i=1}^n x_i^2 = \left(\frac{n-1}{n}\right)^{(L-2)/2}.$$

For $L = 3$, the sharp bound is Okumura's inequality [9]; we include a self-contained two-line sum-of-squares proof. Set $a := 1/\sqrt{n(n-1)}$ and $b := (n-1)a = \sqrt{(n-1)/n}$. Since $x_i \leq |x_i| \leq b$ for every i, each summand $(b - x_i)(x_i + a)^2 \geq 0$, and summing over $i = 1, \ldots, n$ using $\sum 1 = n$, $\sum x_i = 0$, $\sum x_i^2 = 1$ yields

$$\begin{aligned} 0 \leq \sum_{i=1}^n (b - x_i)(x_i + a)^2 &= \sum_{i=1}^n \bigl(ba^2 + (2ab - a^2)x_i + (b - 2a)x_i^2 - x_i^3\bigr) \\ &= nba^2 + (b - 2a) - \sum_{i=1}^n x_i^3 = \frac{n-2}{\sqrt{n(n-1)}} - \sum_{i=1}^n x_i^3. \end{aligned}$$

□

Lemma 3.3 (Cycle-index exponential bound)**.** *Let $X \in \mathbb{R}_{\geq 0}^{n\times n}$ be real symmetric with zero diagonal and Frobenius norm $N = \|X\|_F$, and let $\rho_2, \ldots, \rho_n \geq 0$ satisfy $\operatorname{tr}(X^L) \leq \rho_L N^L$ for all $2 \leq L \leq n$. Define $P_0 = 1$, $P_1 = 0$, and $P_m = \frac{1}{m}\sum_{L=2}^m \rho_L P_{m-L}$ for $2 \leq m \leq n$. Then*

$$T_m(X) \leq P_m N^m \qquad (2 \leq m \leq n).$$

Proof. Let $W_L(X)$ be the sum of cycle weights over all simple directed L-cycles in K_n (counting each undirected L-cycle of length $L \geq 3$ twice, once per orientation, and each 2-cycle $\{i, j\}$ once). Each simple directed L-cycle appears L times in $\operatorname{tr}(X^L)$ (once per starting vertex), and every remaining closed walk has nonnegative weight since $X_{ij} \geq 0$; thus $W_L(X) \leq \frac{1}{L}\operatorname{tr}(X^L) \leq \frac{\rho_L}{L} N^L$.

Decomposing every $\sigma \in S_n$ with $|\operatorname{supp}(\sigma)| = m$ into its vertex-disjoint directed cycles (say c_L cycles of length L, with $\sum_{L\geq 2} Lc_L = m$) and dropping

vertex-disjointness (each unordered set of c_L disjoint L-cycles occurs $c_L!$ times in $W_L(X)^{c_L}$), we obtain

$$T_m(X) \le \sum_{\sum_{L\ge2} Lc_L = m} \prod_{L=2}^{m} \frac{W_L(X)^{c_L}}{c_L!} = [z^m] \exp\left(\sum_{L=2}^{m} W_L(X)\, z^L \right)$$
$$\le [z^m] \exp\left(\sum_{L=2}^{m} \frac{\rho_L}{L} (Nz)^L \right) = P_m N^m,$$

where the recurrence for P_m follows by differentiating $P(z) = \exp\big(\sum_{L\ge2} \frac{\rho_L}{L} z^L\big)$. □

Theorem 3.4 (Small-entry closure at $n = 10$)**.** *Let A be a 10×10 complex Hermitian PSD correlation matrix with $\max_{i\ne j} |A_{ij}| \le \varepsilon := \frac{9}{25} = 0.36$. Then $\operatorname{per}(A \circ \bar{A}) \le \operatorname{per}(A)^2$.*

Proof. Set $E := \varepsilon^2 = \frac{81}{625}$, $X_{ij} := |A_{ij}|^2 \in [0, E]$ $(i \ne j)$, $X_{ii} := 0$, $s := \sum_{1\le i<j\le 10} X_{ij} \in [0, 45E]$, $N := \|X\|_F = \sqrt{2\sum_{i<j} X_{ij}^2}$, and $u := \sqrt{2Es}$. Since $X_{ij}^2 \le E X_{ij}$, we have $N \le u$, $s = u^2/(2E)$, and $u \le \sqrt{90}\, E < \frac{19}{2} E =: u_{\max} = \frac{1539}{1250}$ (as $19^2 = 361 > 360 = 90 \cdot 2^2$). If $u = 0$ then $A = I_{10}$ and both sides equal 1; assume $u \in (0, u_{\max}]$.

Step 1: 3×3 triangle-packing Lieb lower bounds. For any 3×3 principal submatrix $C = \left(\begin{smallmatrix} 1 & a & b \\ \bar{a} & 1 & c \\ \bar{b} & \bar{c} & 1 \end{smallmatrix}\right)$ of A, we have $|a|, |b|, |c| \le \varepsilon$ and

$$\operatorname{per}(C) = 1 + |a|^2 + |b|^2 + |c|^2 + 2\operatorname{Re}(ac\bar{b}) \ge 1 + |a|^2 + |b|^2 + |c|^2 - 2|a||b||c|.$$

Applying $2xy \le x^2 + y^2$ symmetrically to each pair yields

$$3|a||b||c| = |a|(|b||c|) + |b|(|c||a|) + |c|(|a||b|) \le \varepsilon(|a|^2 + |b|^2 + |c|^2),$$

so setting $c_3 := 1 - \frac{2\varepsilon}{3} = \frac{19}{25}$ and $S_T := |a|^2 + |b|^2 + |c|^2 \in [0, 3E]$, we obtain $\operatorname{per}(C) \ge 1 + c_3 S_T$.

Let $\mathcal{P}$ be the set of all partitions of $\{1, \dots, 10\}$ into three disjoint triangles T_1, T_2, T_3 and one singleton $\{v_0\}$. For every $P = (T_1, T_2, T_3, \{v_0\}) \in \mathcal{P}$, Lemma 2.3 yields

$$\operatorname{per}(A) \ge \prod_{r=1}^{3} \operatorname{per}(A[T_r]) \ge \prod_{r=1}^{3} (1 + c_3 S_{T_r}).$$

Write $S_P := S_{T_1} + S_{T_2} + S_{T_3}$. By S_{10}-symmetry, each of the 45 edges of K_{10} lies in a uniformly random $P \in \mathcal{P}$ with probability $\frac{9}{45} = \frac{1}{5}$, so $\mathbb{E}_{P\in\mathcal{P}}[S_P] = \frac{1}{5} s$. This yields two polynomial lower bounds on $\operatorname{per}(A)^2$:

- *Polynomial bound R_1:* Expanding $\prod_{r=1}^{3}(1 + c_3 S_{T_r}) \ge 1 + c_3 S_P$, averaging over $\mathcal{P}$, and squaring gives

$$\operatorname{per}(A)^2 \ge \left(1 + \frac{c_3}{5} s\right)^2 = \big(1 + a_1 u^2\big)^2 =: R_1(u), \qquad a_1 := \frac{c_3}{10E} = \frac{95}{162}.$$

- *Convex exponential–Taylor bound R_2:* Set $M := 3c_3E = \frac{4617}{15625} \in (0,1)$. By concavity of $y \mapsto \log(1+y)$ on $[0,M]$, $\log(1+y) \geq \frac{\log(1+M)}{M} y$ for all $y \in [0,M]$. Because $M \in (0,1)$, the alternating series $\frac{\log(1+M)}{M} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} M^{k-1}$ has terms strictly decreasing in magnitude, so its sixth partial sum

$$L_6(M) := 1 - \frac{M}{2} + \frac{M^2}{3} - \frac{M^3}{4} + \frac{M^4}{5} - \frac{M^5}{6} \in \left(0,\, \frac{\log(1+M)}{M}\right)$$

is an exact positive rational lower bound ($L_6(M) \approx 0.876060$). Since $c_3 S_{T_r} \in [0,M]$,

$$\operatorname{per}(A) \geq \exp\left(\sum_{r=1}^{3} \log(1 + c_3 S_{T_r})\right) \geq \exp\big(c_3 L_6(M)\, S_P\big).$$

Averaging over $\mathcal{P}$ and applying Jensen's inequality to the convex function exp gives $\operatorname{per}(A) \geq \exp\big(\frac{c_3 L_6(M)}{5} s\big)$. Squaring and truncating $\exp(z) \geq \sum_{j=0}^{6} z^j/j!$ $(z \geq 0)$ yields

$$\operatorname{per}(A)^2 \geq \exp(a_2 u^2) \geq \sum_{j=0}^{6} \frac{(a_2 u^2)^j}{j!} =: R_2(u), \qquad a_2 := \frac{c_3 L_6(M)}{5E}.$$

Step 2: Hybrid upper bound on $\operatorname{per}(A \circ \bar{A})$. By (3) with $n = 10$ and $s = u^2/(2E)$, every $2 \leq m \leq 10$ satisfies

$$T_m(X) \leq b_1(m)\, u^2, \qquad b_1(m) := \frac{\binom{10}{m} d_m}{90}\, E^{m-2} \quad \big(b_1(2) = \tfrac{1}{2}\big).$$

On the other hand, define exact rational constants

$$\rho_2 := 1, \qquad \rho_3 := \frac{17}{20},$$
$$\rho_{2j} := \left(\frac{9}{10}\right)^{j-1} \ (j \geq 2), \qquad \rho_{2j+1} := \left(\frac{9}{10}\right)^{j-1} \frac{19}{20} \ (j \geq 2).$$

Since $17^2 \cdot 90 = 26010 > 25600 = 64 \cdot 20^2$ and $19^2 \cdot 10 = 3610 > 3600 = 9 \cdot 20^2$, we have $\rho_3 > 8/\sqrt{90}$ and $\frac{19}{20} > \sqrt{\frac{9}{10}}$. Hence Lemmas 3.2 and 3.3 together with $N \leq u$ give $T_m(X) \leq P_m u^m$ for all $2 \leq m \leq 10$.

Step 3: Exact rational piecewise Bernstein certification. For any choice $R \in \{R_1, R_2\}$ and subset $F \subseteq \{3, \ldots, 10\}$, we have $\operatorname{per}(A)^2 - \operatorname{per}(A \circ \bar{A}) \geq u^2 D_{R,F}(u)$, where

$$D_{R,F}(u) := \frac{R(u) - 1}{u^2} - b_1(2) - \sum_{m \in \{3,\ldots,10\} \setminus F} b_1(m) - \sum_{m \in F} P_m u^{m-2} \in \mathbb{Q}[u].$$

We partition $[0, u_{\max}]$ into four contiguous subintervals with choices listed in Table 2. On each subinterval $[u_a, u_b]$, the reparameterized polynomial $q(v) = D_{R,F}(u_a + (u_b - u_a)v)$ $(v \in [0,1])$ has all exact rational Bernstein coefficients strictly positive without subdivision (verified independently in `prove_`

`chollet_through_10.py` and kernel-proved in Lean 4.27.0 via `n10_small_entry_bernstein_all_4_pieces`). Hence $D_{R,F}(u) > 0$ on all of $[0, u_{\max}]$, so $\operatorname{per}(A \circ \bar{A}) \le \operatorname{per}(A)^2$. □

TABLE 2. Exact four-piece Bernstein certification of Theorem 3.4 on $[0, \frac{1539}{1250}]$.

Subinterval $[u_a, u_b]$	Lower bound R	Frobenius set F	$\deg D_{R,F}$	Bernstein subintervals
$[0,\ \frac{3}{5}]$	R_1	$\{3,4,5,6,7,8,9,10\}$	8	1
$[\frac{3}{5},\ \frac{4}{5}]$	R_1	$\{3,4,5,6,7\}$	5	1
$[\frac{4}{5},\ 1]$	R_2	$\{3,4,5\}$	10	1
$[1,\ \frac{1539}{1250}]$	R_2	$\emptyset$	10	1

4. Two-Parameter Double-Border Induction ($n = 7, 8, 9, 10$)

Let $q \in \{5,6,7,8\}$ (so $n = q+2 \in \{7,8,9,10\}$), assume Chollet(q), and let A be an $n \times n$ complex Hermitian PSD correlation matrix. By Lemma 2.1, permute a pair (i,j) achieving $r := \max_{a \ne b} |A_{ab}| \in [0,1]$ to positions $(n-1, n)$ and phase-conjugate row and column n so that $A_{n-1,n} = r \in [0,1]$. We treat the boundary case $r = 1$ and the generic case $r \in [r_{\mathrm{lo}}, 1)$ separately.

Theorem 4.1 (Unified $r = 1$ double border for $7 \le n \le 10$)**.** *Let $5 \le q \le 8$, assume* Chollet(q)*, and let A be a $(q+2)\times(q+2)$ Hermitian PSD correlation matrix with $A_{n-1,n} = 1$. Then $\operatorname{per}(A \circ \bar{A}) \le \operatorname{per}(A)^2$.*

Proof. Since $A \succeq 0$ and the 2×2 principal block $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ annihilates $(1,-1)^T$, the last two rows and columns of A coincide:

$$A = \begin{pmatrix} S + vv^* & v & v \\ v^* & 1 & 1 \\ v^* & 1 & 1 \end{pmatrix}, \qquad S \succeq 0 \text{ of order } q.$$

Realizing S as the Gram matrix of $w_1, \dots, w_q \in \mathcal{H}_S$ with unit vector $e_1 \perp \mathcal{H}_S$, we have $u_{n-1} = u_n = e_1$, so wedging with $u_{n-1} \vee u_n = e_1^2$ multiplies the squared norm of the degree-k sector in $\operatorname{Sym}^{q-k}(\mathcal{H}_S) \otimes \operatorname{Sym}^k(\mathbb{C}e_1)$ by $(k+2)!/k! = (k+2)(k+1)$. Thus, with $p_m, \tilde{p}_k \ge 0$ and T, w as in Proposition 2.4,

$$\operatorname{per}(A) = \sum_{m=0}^{q} (m+2)(m+1)p_m, \qquad \operatorname{per}(A \circ \bar{A}) = \sum_{k=0}^{q} (k+2)(k+1)\tilde{p}_k.$$

Set $t := \frac{5}{3}$, $M := \frac{54}{25}$, and $B(t) := S + tvv^* \succeq 0$. Since $t > 1$, $B(t) \circ \overline{B(t)} \succeq T + t^2 w w^T$. Direct exact rational verification confirms that for every $0 \le k, m \le 8$,

$$(k+2)(k+1) \le \frac{54}{25}\left(\frac{5}{3}\right)^{2k} \qquad \text{and} \qquad \frac{54}{25}\left(\frac{5}{3}\right)^{2m} \le \big((m+2)(m+1)\big)^2$$

(equality $6 = 6$ holds at $k = 1$; at $m = 8$, $\frac{54}{25}(\frac{5}{3})^{16} \approx 7656.5610 < 8100 = 90^2$). Applying Lemma 2.2 and Chollet(q) yields

$$\begin{aligned}\operatorname{per}(A \circ \bar{A}) &\le M \sum_{k=0}^{q} t^{2k} \tilde{p}_k = M \operatorname{per}(T + t^2 w w^T) \le M \operatorname{per}\big(B(t) \circ \overline{B(t)}\big) \\ &\le M \operatorname{per}\big(B(t)\big)^2 = \left(\sum_{m=0}^{q} \sqrt{M}\, t^m p_m \right)^2 \le \operatorname{per}(A)^2. \qquad \square\end{aligned}$$

Now consider $r \in [0,1)$, and partition $A = \left(\begin{smallmatrix} B & V \\ V^* & C \end{smallmatrix}\right)$ with $B \in \mathbb{C}^{q\times q}$, $V = [v_{n-1}, v_n] \in \mathbb{C}^{q\times 2}$, and $C = \left(\begin{smallmatrix} 1 & r \\ r & 1 \end{smallmatrix}\right) \succ 0$. Define the spectral weights

$$\begin{aligned} c &:= \frac{1+r}{2} \in \Big[\frac{1}{2}, 1\Big), & s &:= \frac{1-r}{2} \in \Big(0, \frac{1}{2}\Big], \\ c_2 &:= c^2 + s^2 = \frac{1+r^2}{2}, & s_2 &:= 2cs = \frac{1-r^2}{2} > 0, \end{aligned}$$

and the two rank-1 border projections

$$\begin{aligned} x_+ &:= \frac{v_{n-1} + v_n}{2\sqrt{c}}, & x_- &:= \frac{v_{n-1} - v_n}{2\sqrt{s}}, \\ R_+ &:= x_+ x_+^*, & R_- &:= x_- x_-^*. \end{aligned}$$

Then $v_{n-1} = \sqrt{c}\, x_+ + \sqrt{s}\, x_-$, $v_n = \sqrt{c}\, x_+ - \sqrt{s}\, x_-$, $VC^{-1}V^* = R_+ + R_-$, and the Schur complement $S := B - R_+ - R_- \succeq 0$ is $q \times q$ Hermitian PSD. For any parameters $t_1, t_2 > 1$, define

$$B(t_1, t_2) := S + t_1 R_+ + t_2 R_- \succeq 0.$$

For any $\rho \in [0,1)$ with $c_\rho = \frac{1+\rho}{2}$ and $s_\rho = \frac{1-\rho}{2}$, let $L_{k,\rho} : \operatorname{Sym}^k(\mathbb{C}^2) \to \operatorname{Sym}^{k+2}(\mathbb{C}^2)$ be multiplication by $c_\rho e_1^2 - s_\rho e_2^2$ in the orthonormal basis $\eta_a = e_1^a e_2^{k-a}/\sqrt{a!(k-a)!}$ ($0 \le a \le k$). Since

$$L_{k,\rho}(\eta_a) = c_\rho \sqrt{(a+2)(a+1)}\, \eta_{a+2}^{(k+2)} - s_\rho \sqrt{(k-a+2)(k-a+1)}\, \eta_a^{(k+2)},$$

the Gram operator $G_{k,\rho} := L_{k,\rho}^* L_{k,\rho}$ is real symmetric with nonzero entries only at

$$(G_{k,\rho})_{a,a} = c_\rho^2 (a+2)(a+1) + s_\rho^2 (k-a+2)(k-a+1), \tag{4}$$

$$(G_{k,\rho})_{a,a+2} = (G_{k,\rho})_{a+2,a} = -c_\rho s_\rho \sqrt{(a+2)(a+1)(k-a)(k-a-1)}. \tag{5}$$

For positive scalars $\alpha, \beta > 0$, let $D_k(\alpha,\beta) := \operatorname{diag}(\alpha^{a/2}\beta^{(k-a)/2})_{a=0}^k$ on $\operatorname{Sym}^k(\mathbb{C}^2)$.

Theorem 4.2 (Two-parameter double-border induction)**.** *Let $q \ge 1$, assume* Chollet(q)*, and fix $r \in [0,1)$. Suppose $t_1, t_2 > 1$ and $\mu > 0$ satisfy:*

(a) Loewner positivity: $d_{00} := c_2 t_1^2 - s^2 > 0$, $\mathrm{detD} := c_2 t_1^2 t_2^2 - c^2 t_1^2 - s^2 t_2^2 > 0$, $\mathrm{quad} := c^2 t_2^2 + s^2 t_1^2 - c_2 > 0$, *and with* $w_1 := \mathrm{detD}/\mathrm{quad}$ *and* $w_2 := t_1 t_2$, *the* 2×2 *matrix*

$$\Lambda := \begin{pmatrix} t_1^2 & 0 \\ 0 & t_2^2 \end{pmatrix} - \frac{1}{c_2} \begin{pmatrix} s \\ -c \end{pmatrix} \begin{pmatrix} s & -c \end{pmatrix} - \frac{w_1}{c_2} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} c & s \end{pmatrix}$$

satisfies $\mathrm{tr}(\Lambda) > 0$.

(b) Lower spectral bound: $G_{k,r} - \mu\, D_k(t_1, t_2)^2 \succeq 0$ *for every* $0 \le k \le q$.

(c) Upper spectral bound: $\mu^2 D_k(w_1, w_2)^2 - G_{k,r^2} \succeq 0$ *for every* $0 \le k \le q$.

Then every $(q+2) \times (q+2)$ *Hermitian PSD correlation matrix* A *with* $A_{n-1,n} = r$ *satisfies* $\mathrm{per}(A \circ \bar{A}) \le \mathrm{per}(A)^2$.

Proof. Step 1: Lower bound $\mathrm{per}(A) \ge \mu\, \mathrm{per}(B(t_1, t_2))$. Realize S as the Gram matrix of $w_1, \ldots, w_q \in \mathcal{H}_S$ and choose orthonormal $e_1, e_2 \perp \mathcal{H}_S$. Then A is the Gram matrix of $u_i = w_i + (x_+)_i e_1 + (x_-)_i e_2$ $(1 \le i \le q)$, $u_{n-1} = \sqrt{c}\, e_1 + \sqrt{s}\, e_2$, and $u_n = \sqrt{c}\, e_1 - \sqrt{s}\, e_2$. Since $u_{n-1} \vee u_n = c e_1^2 - s e_2^2$, decomposing $u_1 \vee \cdots \vee u_q = \sum_{k=0}^{q} \Psi_k$ into orthogonal sectors $\Psi_k \in \mathrm{Sym}^{q-k}(\mathcal{H}_S) \otimes \mathrm{Sym}^k(\mathbb{C}^2)$ gives

$$\mathrm{per}\big(B(t_1, t_2)\big) = \sum_{k=0}^{q} \|(I \otimes D_k(t_1, t_2))\Psi_k\|^2, \qquad \mathrm{per}(A) = \sum_{k=0}^{q} \|(I \otimes L_{k,r})\Psi_k\|^2.$$

Condition (b) directly implies $\mathrm{per}(A) \ge \mu\, \mathrm{per}(B(t_1, t_2))$.

Step 2: Exact Schur and Loewner decomposition of $A \circ \bar{A}$. In $A \circ \bar{A}$, the lower-right block is $C \circ \bar{C} = \left(\begin{smallmatrix} 1 & r^2 \\ r^2 & 1 \end{smallmatrix}\right) \succ 0$ with spectral weights (c_2, s_2). Define $a := |x_+|^2$, $b := |x_-|^2 \in \mathbb{R}^q_{\ge 0}$ and $u := x_+ \circ \bar{x}_- \in \mathbb{C}^q$. Since $|v_{n-1}|^2 = ca + sb + 2\sqrt{cs}\,\mathrm{Re}(u)$ and $|v_n|^2 = ca + sb - 2\sqrt{cs}\,\mathrm{Re}(u)$, the border projections of $A \circ \bar{A}$ are $R_2^+ := y_+ y_+^T$ and $R_2^- := y_- y_-^T$, where

$$y_+ := \frac{|v_{n-1}|^2 + |v_n|^2}{2\sqrt{c_2}} = \frac{ca + sb}{\sqrt{c_2}}, \qquad y_- := \frac{|v_{n-1}|^2 - |v_n|^2}{2\sqrt{s_2}} = \sqrt{2}\,\mathrm{Re}(u).$$

Expanding $(S + R_+ + R_-) \circ (\bar{S} + \bar{R}_+ + \bar{R}_-) = (S \circ \bar{S}) + M_{S+} + M_{S-} + aa^T + bb^T + (uu^* + \bar{u}\bar{u}^*)$, where $M_{S\pm} := 2\,\mathrm{Re}(S \circ \bar{R}_\pm) \succeq 0$ by Schur's product theorem, we obtain the exact PSD remainder identities

$$E_+ := aa^T + bb^T - R_2^+ = \frac{1}{c_2}(sa - cb)(sa - cb)^T \succeq 0,$$
$$E_- := (uu^* + \bar{u}\bar{u}^*) - R_2^- = 2\,\mathrm{Im}(u)\,\mathrm{Im}(u)^T \succeq 0.$$

Hence the Schur complement of $C \circ \bar{C}$ in $A \circ \bar{A}$ is $T_0 := (S \circ \bar{S}) + M_{S+} + M_{S-} + E_+ + E_- \succeq 0$.

Now consider the 2×2 matrix $D := \left(\begin{smallmatrix} t_1^2 & 0 \\ 0 & t_2^2 \end{smallmatrix}\right) - \frac{1}{c_2}\left(\begin{smallmatrix} s \\ -c \end{smallmatrix}\right)\begin{pmatrix} s & -c \end{pmatrix} = \frac{1}{c_2}\left(\begin{smallmatrix} d_{00} & cs \\ cs & d_{11} \end{smallmatrix}\right)$, where $d_{11} := c_2 t_2^2 - c^2$. Direct algebra using $c^2 + s^2 = c_2$ yields

$$d_{00} d_{11} - (cs)^2 = c_2\,\mathrm{detD}, \qquad d_{11} c^2 - 2(cs)^2 + d_{00} s^2 = c_2\,\mathrm{quad}.$$

Consequently, $\Lambda = D - \frac{w_1}{c_2}\binom{c}{s}\begin{pmatrix} c & s \end{pmatrix}$ satisfies the exact polynomial identity

$$\det(\Lambda) = \frac{1}{c_2^2}\big(c_2\,\mathrm{detD} - w_1 \cdot c_2\,\mathrm{quad}\big) \equiv 0.$$

Together with $\mathrm{tr}(\Lambda) > 0$ from (a), this forces $\Lambda \succeq 0$. Since $t_1, t_2 > 1$ and $w_2 = t_1 t_2$, we obtain the exact Loewner domination

$$\begin{aligned} &B(t_1,t_2) \circ \overline{B(t_1,t_2)} - (T_0 + w_1 R_2^+ + w_2 R_2^-) \\ &\qquad = (t_1-1)M_{S+} + (t_2-1)M_{S-} + [a\ b]\,\Lambda\,[a\ b]^T + (t_1t_2-1)E_- \succeq 0. \end{aligned}$$

Step 3: Upper bound and inductive closure. Applying the symmetric-tensor decomposition of Step 1 to $A \circ \bar{A}$ (with r replaced by r^2, $S \mapsto T_0$, $R_\pm \mapsto R_2^\pm$) and invoking condition (c), Lemma 2.2, Chollet(q), and Step 1 yields

$$\begin{aligned} \mathrm{per}(A \circ \bar{A}) &\le \mu^2\,\mathrm{per}(T_0 + w_1 R_2^+ + w_2 R_2^-) \le \mu^2\,\mathrm{per}\big(B(t_1,t_2) \circ \overline{B(t_1,t_2)}\big) \\ &\le \mu^2\,\mathrm{per}\big(B(t_1,t_2)\big)^2 \le \mathrm{per}(A)^2. \end{aligned}$$ □

Exact polynomial reduction via tridiagonal Sylvester minors. Because $(G_{k,\rho})_{a,a'}$ vanishes unless $a' \in \{a, a \pm 2\}$, both $G_{k,r} - \mu D_k(t_1,t_2)^2$ and $\mu^2 D_k(w_1,w_2)^2 - G_{k,r^2}$ split into two independent symmetric tridiagonal blocks indexed by even $a \in \{0,2,\dots\}$ and odd $a \in \{1,3,\dots\}$, each of size $m \le \lceil (q+1)/2 \rceil \le 5$. For any symmetric tridiagonal matrix with diagonal entries $d_0,\dots,d_{m-1}$ and squared off-diagonal entries e_j^2 ($0 \le j \le m-2$), the leading principal minors satisfy the three-term recurrence

$$M_0 = 1, \qquad M_1 = d_0, \qquad M_j = d_{j-1}M_{j-1} - e_{j-2}^2 M_{j-2} \quad (2 \le j \le m).$$

Crucially, by (5), the off-diagonal entries enter M_j *only through their squares*, so all square roots disappear identically. In condition (c), because $w_1 = \mathrm{detD}/\mathrm{quad}$ with $\mathrm{quad} > 0$, conjugating by the positive diagonal matrix $\mathrm{diag}(\mathrm{quad}^{a/2})$ replaces the a-th diagonal entry by $\mu^2(\mathrm{detD})^a w_2^{k-a} - (\mathrm{quad})^a (G_{k,r^2})_{a,a}$ and multiplies the $(a, a+2)$ squared off-diagonal entry by $(\mathrm{quad})^{2a+2}$. Thus, for any polynomial schedule $(t_1(r), t_2(r), \mu(r)) \in \mathbb{Q}[r]^3$, every Sylvester minor in conditions (a)–(c) is a pure univariate polynomial in $\mathbb{Q}[r]$.

Proof of Theorem 1.1. Orders $1 \le n \le 6$ follow from Proposition 2.4. For each $n \in \{7,8,9,10\}$, assume Chollet($n-2$) inductively and let A be an $n \times n$ correlation matrix with $r = \max_{a \ne b} |A_{ab}| \in [0,1]$:

- If $r = 1$, Theorem 4.1 closes all four orders $n = 7,8,9,10$.
- If $n = 8$ and $r \le \frac{7}{20}$, or $n = 9$ and $r \le \frac{8}{25}$, Proposition 3.1 closes the inequality. If $n = 10$ and $r \le \frac{9}{25}$, Theorem 3.4 closes the inequality.
- In the remaining range $r \in [r_{\mathrm{lo}}, 1)$ (with $r_{\mathrm{lo}} = 0$ for $n = 7$, $\frac{3}{10}$ for $n = 8$, $\frac{8}{25}$ for $n = 9$, and $\frac{9}{25}$ for $n = 10$), we apply Theorem 4.2 using the explicit rational schedules listed in Table 3. For $n = 7$ at $k = 0$, the exact identities $(1 + r^2) - \mu_7(r) = \frac{1}{2}r^2 \ge 0$ and

$\mu_7(r)^2 - (1 + r^4) = r^2(1 - \frac{3}{4}r^2) \geq 0$ hold on $[0, 1]$. For all remaining sectors and all seven Loewner parameter inequalities in each schedule, every resulting univariate polynomial is certified strictly positive on its interval via exact rational Bernstein basis expansions (with independent monomial re-expansion verification).

Because $[0, \frac{7}{20}] \cup [\frac{3}{10}, 1] = [0, 1]$, $[0, \frac{8}{25}] \cup [\frac{8}{25}, 1] = [0, 1]$, and $[0, \frac{9}{25}] \cup [\frac{9}{25}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{7}{10}] \cup [\frac{7}{10}, 1] = [0, 1]$, all cases are exhausted and Chollet(n) holds for all $1 \leq n \leq 10$. □

TABLE 3. Explicit certified rational schedules for Theorem 4.2 ($n = 7, 8, 9, 10$).

Order / Piece	$[r_{\mathrm{lo}}, r_{\mathrm{hi}}]$	$t_1(r)$	$t_2(r)$	$\mu(r)$	Minors	Deg / sub
$n = 7$	$[0, 1]$	$\frac{71}{50} + \frac{4}{5}r - \frac{9}{20}r^2$	$\frac{71}{50} - \frac{3}{5}r + \frac{11}{50}r^2$	$1 + \frac{1}{2}r^2$	47*	126 / 3
$n = 8$	$[\frac{3}{10}, 1]$	$\frac{69}{50} + \frac{13}{20}r - \frac{7}{25}r^2$	$\frac{7}{5} - \frac{3}{5}r + \frac{11}{50}r^2$	$1 + \frac{3}{5}r^2$	63	184 / 1
$n = 9$	$[\frac{8}{25}, 1]$	$\frac{34}{25} + \frac{29}{50}r - \frac{13}{50}r^2$	$\frac{34}{25} - \frac{14}{25}r + \frac{11}{50}r^2$	$\frac{103}{100} + \frac{11}{20}r^2$	79	224 / 1
$n = 10$ (P1)	$[\frac{9}{25}, \frac{1}{2}]$	$\frac{3}{2}$	$\frac{117}{100}$	$\frac{519}{500} + \frac{1}{4}r$	97	60 / 1
$n = 10$ (P2)	$[\frac{1}{2}, \frac{7}{10}]$	$\frac{159}{100}$	$\frac{11}{10}$	$\frac{9}{10} + \frac{3}{5}r$	97	60 / 1
$n = 10$ (P3)	$[\frac{7}{10}, 1]$	$\frac{161}{100}$	$\frac{103}{100}$	$\frac{21}{20} + \frac{1}{2}r$	97	60 / 4

*Plus exact $k = 0$ identities verified algebraically on $[0, 1]$. Total double-border checks certified: $47 + 63 + 79 + 291 = 480$ (438 Sylvester minors plus 42 scalar Schur/Loewner parameter checks).

5. Exact Certification, Lean 4 Scope, and Empirical Audit

All computational components of the proof are archived as standalone, zero-dependency verification artifacts:

(1) *Zero-floating-point exact rational certifier* (`prove_chollet_through_10.py`). Using Python 3 standard-library `fractions.Fraction` exclusively, the script certifies Proposition 2.4, Theorem 4.1, Proposition 3.1, all four piecewise polynomials of Theorem 3.4, and all 480 double-border checks (438 Sylvester minors plus 42 scalar Schur/Loewner parameter checks) of Table 3 in 6.00 s. On every Bernstein certificate $b_0, \ldots, b_d$, the script independently re-expands $\sum_{j=0}^{d} b_j \binom{d}{j} u^j (1-u)^{d-j}$ back into the monomial basis and asserts exact rational equality. All 480 checks (438 minors plus 42 scalar checks) and all 4 small-entry polynomials were also independently re-certified without tridiagonal shortcuts via full $m \times m$ cofactor determinants, pure-integer Möbius–Descartes sign-change checks, and exact rational Sturm sequences on all degree ≤ 20 minors.

(2) *Lean 4.27.0 kernel verification* (`CholletCertificates.lean`) *and exact scope boundary.* Written in pure Lean 4 (`Init` only, zero Mathlib imports, zero `sorry`, zero `axiom`, zero `native_decide`), `CholletCertificates.lean` compiles cleanly under `leanprover/lean4:v4.27.0` (exit 0, 0 errors, 0 warnings, ≈ 24 s). Across all 49 named theorems in the file (including private helper lemmas), axiom

inspection confirms 23 use zero axioms (including `single_border_n1_to_6`, `r1_edge_case_n7_to_10`, `lieb_matching_n8`, and `lieb_matching_n9`) and 26 depend only on `[propext]`. Using zero-GCD pure-integer homogenized polynomials (`IPoly`) and tail-recursive $O(k)$ binomials (`combFast`), the Lean kernel verifies:

- the standalone structural soundness lemma `bernsteinSum_pos` (strictly positive cleared-denominator Bernstein coefficients imply strict positivity at every rational barycentric point in $[0,1]$);
- all prerequisite and small-entry certificates: $1 \le n \le 6$ single-border, $r = 1$ ($7 \le n \le 10$), $n = 8, 9$ Lieb bounds, all rational constants and exact divisibility for $n = 10$, all four $n = 10$ small-entry Bernstein pieces, and $n = 7$, $k = 0$;
- **207 of 480** exact rational/integer double-border checks via `by decide`: all 42 Schur-complement and Loewner scalar parameter checks ($7 \le n \le 10$), all 135 Lower Sylvester minors for $n = 10$ ($0 \le k \le 8$), and 30 Upper Sylvester minors for $n = 10$ ($0 \le k \le 3$).

The remaining **273 of 480** double-border minors ($n = 10$ Upper $4 \le k \le 8$, 105 minors of degree ≤ 60; $n = 7$ L/U $1 \le k \le 5$, 40 minors of degree ≤ 126; $n = 8$ L/U $0 \le k \le 6$, 56 minors of degree ≤ 184; $n = 9$ L/U $0 \le k \le 7$, 72 minors of degree ≤ 224) are certified in zero-floating-point Python and Möbius–Descartes/Sturm audits and encoded in Lean as the unproved executable Boolean `verifyAllDoubleBorderThrough10 : Bool`. Crucially, neither the standalone rational lemma `bernsteinSum_pos` nor `verifyAllDoubleBorderThrough10` is formally linked inside Lean to real-interval positivity of `IPoly.certifyPositive` or to complex matrix permanents; the analytical matrix-permanent reductions reside in Sections 2–4.

(3) *Independent C++20 empirical floating-point diagnostic* (`audit_chollet_through_10.cpp`). Separate from the exact proof, every intermediate matrix (S, T_0, $B \circ \bar{B} - M_{\mathrm{mid}}$), spectral bound, Lieb bound (R_1, R_2), traceless Frobenius bound ($\mathrm{tr}(X^L) \le \rho_L N^L$), and forced $r = 1$ case was independently evaluated in double precision via Glynn's permanent formula across 7,000 complex Hermitian PSD matrices (3,000 at $n = 7$, 2,000 at $n = 8$, 1,200 at $n = 9$, and 800 at $n = 10$) spanning ranks $2, \ldots, n$, returning exit status 0 with zero violations (and nonzero exit status upon any violation).

6. Observed Schedule-Search Limits at Orders $n = 11$ and $n = 12$

To probe where the present two-regime proof architecture encounters an intermediate off-diagonal gap, Table 4 records the exact 2×2 matching Lieb threshold alongside the observed reach of our initial numerical search over block-Lieb packings ($[3,3,3,2]$ at $n = 11$, $[3,3,3,3]$ at $n = 12$, paired

with Lemmas 3.2–3.3) and two-parameter double-border schedules for $n = 7, \ldots, 12$.

TABLE 4. Observed reach of our initial schedule search vs. small-entry regimes ($n = 7, \ldots, 12$).

n	2×2 Lieb ε	Block-Lieb + SOS	DB r (search)	Seam status
7	0.3804	—	0.0000	Closed ($[0, 1]$ certified)
8	0.3608	—	0.1930	Closed (certified seam $[0.30, 0.35]$)
9	0.3239	—	0.2948	Closed (certified seam at 0.32)
10	0.3110	0.3612	0.3573	Closed (certified seam at 0.36)
11	0.2863	0.3445	0.4181	Open (refined gap $\subseteq (0.345, 0.417)$)
12	0.2771	0.3326	0.4585	Open (refined DB reaches 0.457)

We emphasize that the block-Lieb and double-border figures in Table 4 reflect the observed limits of our particular numerical schedule search rather than proved lower bounds over all admissible two-parameter double-border schedules or block-Lieb refinements. Indeed, further rational schedule refinement improves slightly on every entry for $8 \leq n \leq 12$: valid rational double-border schedules extend down to $r = 0.192$, 0.294, 0.356, 0.417, and 0.457 for $n = 8, 9, 10, 11, 12$, respectively, and $[3, 3, 3, 2]$ triangle-packing reaches $\varepsilon = \frac{69}{200} = 0.345$ at $n = 11$. While these refinements narrow the uncertified intermediate gap at $n = 11$ to a subset of $(0.345, 0.417)$ (width at most 0.072), an appreciable gap persists at both $n = 11$ and $n = 12$ even though the single-rank $r = 1$ edge case of Theorem 4.1 still closes at $n = 11$ ($t = 1.56$) and $n = 12$ ($t = 1.52$). Extending this framework to $n \geq 11$ naturally points toward anisotropic three-parameter triple-border induction ($q \to q + 3$, bordering by a 3×3 principal block) or sharper multi-cycle permanental upper bounds.

Data Availability and Methodology

All exact rational verification scripts, certificate logs, empirical C++20 diagnostic code, and the Lean 4.27.0 file `CholletCertificates.lean` are publicly available at `https://github.com/coracle-us/conjectures` (package `chollet-permanent-through-10`). This proof was developed using AI-assisted mathematical discovery combined with independent zero-trust computational and Lean 4 kernel verification following the methodology of Loader, Oppenheim, and Osborne [7].

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