hexagon:2611.01162v1

A UNIFORM LOWER BOUND FOR THE DOUBLING RATIO OF $\sum_{k\le n}\tau(2^k-1)$

KEN SASO

ABSTRACT. Let $f(n)=\sum_{k\le n}\tau(2^k-1)$, where τ is the divisor function. Erdős asked whether $f(2n)/f(n)$ has a limit (Erdős Problem #893). Kovač and Luca proved that $\limsup f(2n)/f(n)=\infty$, so no finite limit exists, and that $f(2n)/f(n)\to\infty$ under either of two conjectures on Mersenne-type numbers. We prove that $f(2n)\ge 3f(n)$ for every $n\ge 1$, with equality exactly for $n=1,2,4$ in the range we computed. We isolate the obstruction to going further: the terms $\tau(2^{2^a}-1)$ and the Fermat numbers $F_a=2^{2^a}+1$. Finally, we show that $f(2n)/f(n)\to\infty$ follows from a weak upper bound on $\omega(\Phi_d(2))$ needed only for those d whose odd part has boundedly many divisors; this hypothesis is implied by the second conjecture of Kovač and Luca.

1. Introduction

Let $\tau(m)$ be the number of divisors of m and

$$A(k)=\tau(2^k-1),\qquad f(n)=\sum_{k=1}^{n}A(k).$$

Erdős asked whether $\lim_{n\to\infty}f(2n)/f(n)$ exists [1]. Kovač and Luca [3], building on a heuristic found independently by Cambie (see [1]), proved that

$$\limsup_{n\to\infty}\frac{f(2n)}{f(n)}=\infty,$$

so there is no finite limit, and that the ratio tends to infinity under either of two conjectures [3, Theorem 3]. Whether $f(2n)/f(n)\to\infty$ unconditionally, i.e. whether $\liminf f(2n)/f(n)=\infty$, remains open. Since $2^{2k}-1=(2^k-1)(2^k+1)$ with coprime factors and $\tau(2^k+1)\ge 2$, one has $f(2n)\ge 2f(n)$ trivially; this observation, together with numerical values of the ratio up to $n=138$ and the suggestion that the powers of two form the critical case, appears in a working report on the website *Erdős Problem a Day* [2].

Our main result improves the constant 2 to 3, uniformly in n.

Theorem 1. *For every $n\ge 1$,*

$$f(2n)\ge 3f(n).$$

In particular $\liminf_{n\to\infty}f(2n)/f(n)\ge 3$.

Equality holds for $n=1,2,4$ (we have $f(1),f(2),f(4),f(8)=1,3,9,27$), and for all other $n\le 100$ the inequality is strict (Section 6). The proof uses only the Bang–Zsigmondy theorem and the coprime factorisation $2^{2^v o}-1=(2^o-1)\prod_{i<v}(2^{2^i o}+1)$.

In Section 4 we explain why the same ingredients cannot by themselves give much more: along $n=2^A$ the ratio is controlled by $\tau(F_A)$ and by how much of $f(2^A)$ is carried by the single term $\tau(2^{2^A}-1)$, whose growth is governed by the factorisations of the Fermat numbers

Date: November 27, 2026.

$F_0, \dots, F_{A-1}$. In Section 5 we prove that $f(2n)/f(n) \to \infty$ follows from a hypothesis (Hypothesis 9) that bounds $\omega(\Phi_d(2))$ by $d^{(\log 2-\varepsilon)/\log\log d}$, but only for d whose odd part has a bounded number of divisors (for instance d prime, or d a power of two). This hypothesis is implied by Conjecture 2 of [3], which asks for $\omega(\Phi_d(2)) \le 10 \log d$ for all but finitely many d.

Notation. For $m \ge 1$ write $m = 2^{v_2(m)} o(m)$ with $o(m)$ odd. We write $P_a = A(2^a) = \tau(2^{2^a}-1)$ (so $P_0 = 1$) and $F_a = 2^{2^a}+1$. Logarithms are natural unless written $\log_2$.

2. Tools

Theorem 2 (Bang 1886, Zsigmondy 1892)**.** *For every $d \ge 2$ with $d \ne 6$ there is a prime p with $\operatorname{ord}_p(2) = d$.*

We fix such a prime p_d for each $d \ge 2$, $d \ne 6$. If $\operatorname{ord}_p(2) = d$ then $p \mid 2^m - 1$ if and only if $d \mid m$; in particular the p_d are pairwise distinct.

Lemma 3 (Coprime splitting)**.** *Let o be odd and $v \ge 0$. Then*

$$2^{2^v o} - 1 = (2^o - 1) \prod_{i=0}^{v-1} \bigl(2^{2^i o} + 1\bigr),$$

and the $v+1$ factors on the right are pairwise coprime. Consequently, writing $a_i(o) = \tau(2^{2^i o} + 1)$,

$$A(2^v o) = A(o) \prod_{i=0}^{v-1} a_i(o).$$

Proof. The factorisation follows from $x^2 - 1 = (x-1)(x+1)$ applied repeatedly. If $i < j$ then $2^{2^i o}+1$ divides $2^{2^j o} - 1 = (2^{2^j o}+1) - 2$, so any common divisor of $2^{2^i o}+1$ and $2^{2^j o}+1$ divides 2; both are odd, so they are coprime. The same argument handles $2^o - 1$, which divides every $2^{2^j o} - 1$. □

Lemma 4. *Let $K = 2^a o$ with o odd.*

(i) *$\omega(2^K + 1) \ge \tau(o) - 1$, hence $\tau(2^K + 1) \ge 2^{\tau(o)-1}$.*
(ii) *If $o \ge 3$, then $a_0(o) \ge 3$ and $a_i(o) \ge 4$ for all $i \ge 1$.*

Proof. If $d \mid 2K$ but $d \nmid K$, then p_d divides $2^{2K} - 1 = (2^K - 1)(2^K + 1)$ but not $2^K - 1$, hence $p_d \mid 2^K + 1$.

(i) The numbers $d = 2^{a+1} e$ with $e \mid o$ satisfy $d \mid 2K$, $d \nmid K$, and are pairwise distinct; at most one of them equals 6. This gives $\tau(o) - 1$ distinct primes dividing $2^K + 1$.

(ii) For $i \ge 1$ take $K = 2^i o$ and $d \in \{2^{i+1},\, 2^{i+1} o\}$: both satisfy $d \mid 2K$, $d \nmid K$, both are $\ne 6$, and they are distinct since $o \ge 3$. So $\omega(2^K + 1) \ge 2$ and $a_i(o) \ge 4$. For $i = 0$, $3 \mid 2^o + 1$ as o is odd. If $o \ne 3$, then $d = 2o \ne 6$ gives a prime $p_{2o} \mid 2^o + 1$, and $p_{2o} \ne 3$ because $\operatorname{ord}_3(2) = 2 \ne 2o$; so $a_0(o) \ge 4$. If $o = 3$, then $2^3 + 1 = 9$ and $a_0(3) = 3$. □

Lemma 5. *For $v \ge 1$, $A(3 \cdot 2^v) \ge 2^v P_v$.*

Proof. Let $M = 2^{2^v} - 1$ and $N = 2^{3 \cdot 2^v} - 1$, so $M \mid N$. For $0 \le i \le v$ with $i \ne 1$, the prime $p_{3 \cdot 2^i}$ divides N but not M (as $3 \nmid 2^v$). These are v distinct primes. Write $N = N_1 N_2$, where N_1 is the part of N composed of primes dividing M. Then $M \mid N_1$, so $\tau(N_1) \ge \tau(M) = P_v$, and $\omega(N_2) \ge v$, so $\tau(N_2) \ge 2^v$. Since $\gcd(N_1, N_2) = 1$, $\tau(N) = \tau(N_1)\tau(N_2) \ge 2^v P_v$. □

Finally, by Lemma 3 with $o = 1$, $P_{a+1} = P_a\,\tau(F_a) \geq 2P_a$ for $a \geq 0$. Hence

$$\sum_{a=0}^{b} P_a < 2P_b \qquad (b \geq 0). \tag{1}$$

3. Proof of Theorem 1

For $n < 32$ the inequality is checked numerically (Section 6). Let $n \geq 32$ and $A = \lfloor \log_2 n \rfloor \geq 5$.

Fibres. For every $k \in [1, n]$ there is exactly one $j \geq 1$ with $2^j k \in (n, 2n]$. Hence $[1, n]$ is the disjoint union of the *fibres*

$$\mathcal{F}(m) = \{m/2,\, m/4,\, \ldots,\, m/2^{v_2(m)}\}, \qquad m \in (n, 2n] \text{ even},$$

and all elements of $\mathcal{F}(m)$ have odd part $o(m)$. Put $S(m) = \sum_{k \in \mathcal{F}(m)} A(k)$.

Step 1: fibres with $o(m) \geq 3$. Let $m = 2^v o$ with $v \geq 1$ and $o \geq 3$, and write $a_i = a_i(o)$. By Lemma 3,

$$\frac{A(m/2^j)}{A(m)} = \prod_{i=v-j}^{v-1} a_i^{-1} \qquad (1 \leq j \leq v).$$

By Lemma 4(ii) this is at most 4^{-j} for $j < v$ and at most $\frac{1}{3}4^{-(v-1)}$ for $j = v$. Therefore

$$\frac{S(m)}{A(m)} \leq \sum_{j=1}^{v-1} 4^{-j} + \frac{1}{3}\,4^{-(v-1)} = \frac{1}{3}, \qquad \text{i.e.} \qquad A(m) \geq 3S(m).$$

Step 2: the fibre with $o(m) = 1$. The only power of two in $(n, 2n]$ is 2^{A+1}; its fibre is $\{1, 2, \ldots, 2^A\}$, with sum $\Phi := \sum_{a=0}^{A} P_a$, and $A(2^{A+1}) = P_{A+1} \geq 2P_A$.

Step 3: the fibre of $3 \cdot 2^v$. There is exactly one $m_3 = 3 \cdot 2^{v_3}$ in $(n, 2n]$. Since $3 \cdot 2^{A-2} < 2^A \leq n < m_3$, we have $v_3 \geq A - 1 \geq 4$. Let $S_3 = S(m_3)$ and $T_3 = A(m_3)$. By Step 1 and Lemma 5, $T_3 \geq \max(3S_3,\, 2^{v_3}P_{v_3})$, so

$$T_3 \geq \tfrac{2}{3} \cdot 3S_3 + \tfrac{1}{3} \cdot 2^{v_3}P_{v_3} \geq 2S_3 + \tfrac{1}{3}\,2^{A-1}P_{A-1},$$

using that P_a is increasing.

Step 4: summing. Let U be the sum of $A(k)$ over $k \leq n$ that are not powers of two, so that $f(n) = U + \Phi$ and $U = \sum S(m)$ over even $m \in (n, 2n]$ with $o(m) \geq 3$. Discarding odd m,

$$\sum_{n < m \leq 2n} A(m) \;\geq\; 3(U - S_3) + T_3 + P_{A+1} \;\geq\; 3U - S_3 + \tfrac{1}{3}\,2^{A-1}P_{A-1} + 2P_A \;\geq\; 2U + 2P_A + \tfrac{1}{3}\,2^{A-1}P_{A-1},$$

since $S_3 \leq U$. On the other hand, by (1), $2\Phi = 2P_A + 2P_{A-1} + 2\sum_{a \leq A-2} P_a < 2P_A + 4P_{A-1}$. As $A \geq 5$ we have $\frac{1}{3}2^{A-1} \geq \frac{16}{3} > 4$, hence $\sum_{n<m\leq 2n} A(m) > 2U + 2\Phi = 2f(n)$, and $f(2n) = f(n) + \sum_{n<m\leq 2n} A(m) > 3f(n)$. □

Remark 6. The proof gives slightly more: for $n \geq 32$ the inequality is strict, and the only fibre that is not tripled "for free" by Step 1 is the dyadic fibre $\{1, 2, \ldots, 2^A\}$. The special fibre of $3 \cdot 2^{v_3}$ is used solely to pay for it.

4. The dyadic obstruction

The proof of Theorem 1 treats every fibre except the dyadic one with a factor 3 or better, and the dyadic fibre is the reason the constant cannot easily be raised. The following elementary inequality makes this precise.

Proposition 7. *For $A \geq 1$ let $Q_A = \sum A(m)$, the sum over $m \leq 2^{A+1}$ with $2^A \nmid m$. Then*

$$\frac{f(2^{A+1})}{f(2^A)} \leq 1 + \tau(F_A) + \frac{Q_A}{P_A}.$$

Proof. The only $m \leq 2^{A+1}$ with $2^A \mid m$ are 2^A and 2^{A+1}, contributing $P_A + P_{A+1} = P_A(1 + \tau(F_A))$. Hence $f(2^{A+1}) = Q_A + P_A(1 + \tau(F_A))$, and $f(2^A) \geq P_A$. □

Two features make the right-hand side hard to control. First, $\tau(F_A) = 2$ whenever F_A is prime, and it is not known whether F_A is composite for all large A. Second, $P_A = P_{A-1}\tau(F_{A-1})$, while every prime factor of F_{A-1} has multiplicative order exactly 2^A and therefore divides none of the numbers $2^m - 1$ with $2^A \nmid m$, i.e. none of the terms of Q_A. So the size of Q_A/P_A depends on how many divisors F_{A-1} has compared with Mersenne numbers of unrelated index. Lemma 4 and its relatives give lower bounds for these quantities but no upper bound for $\tau(F_{A-1})$ better than the trivial one coming from the fact that prime factors of F_{A-1} are $\equiv 1 \pmod{2^A}$. Consequently:

Corollary 8. *If $\liminf_{n\to\infty} f(2n)/f(n) = \infty$, then $\tau(F_A) + Q_A/P_A \to \infty$ as $A \to \infty$. In particular, if there were infinitely many A with F_A prime and $Q_A = o(P_A)$, then $\liminf f(2n)/f(n) = 3$.*

The second statement follows from Proposition 7 and Theorem 1. Heuristically Q_A is much larger than P_A (the highly composite indices $m \leq 2^{A+1}$ dominate f; see Section 6), so we expect $\liminf = \infty$; but proving $Q_A/P_A \to \infty$ unconditionally requires an upper bound on the number of prime factors of Fermat numbers that is far beyond current knowledge. In this sense the constant 3 marks the limit of arguments that, like ours, use only lower bounds of Zsigmondy type. (This last sentence is a heuristic assessment, not a theorem.) The relevance of the powers of two was also pointed out in [2], where it is shown that $\liminf f(2n)/f(n) \geq 4$ would follow from $\sum_{2^j \leq n} \tau(2^{2^j} - 1) = o(f(n))$.

5. A weak sufficient condition for divergence

Hypothesis 9. There is $\varepsilon > 0$ such that for every $C \geq 1$ there is $d_0(C)$ with

$$\omega(\Phi_d(2)) \leq d^{(\log 2 - \varepsilon)/\log\log d}$$

for all $d \geq d_0(C)$ with $\tau(o(d)) \leq C$.

The hypothesis concerns only d of the shape $2^v \cdot o$ with $\tau(o)$ bounded, e.g. primes p (Mersenne numbers $2^p - 1$), powers of two (Fermat numbers), and $2p$ (the numbers $(2^p+1)/3$). The known bound is $\omega(\Phi_d(2)) \ll \varphi(d)/\log d$, and Hypothesis 9 asks only for $d^{o(1)}$-type savings of a specific shape. It is implied by [3, Conjecture 2], which asserts $\omega(\Phi_d(2)) \leq 10 \log d$ for all $d \geq 2$ with finitely many exceptions.

Theorem 10. *Under Hypothesis 9, $f(2n)/f(n) \to \infty$.*

Proof. Since $2^{2k} - 1 = (2^k - 1)(2^k + 1)$ with coprime factors,

$$f(2n) \geq \sum_{k \leq n} A(2k) = \sum_{k \leq n} A(k)\,\tau(2^k + 1). \tag{2}$$

Fix $C \geq 1$ and let $B = B_C(n) = \{k \leq n : \tau(o(k)) \leq C\}$. For $k \notin B$, Lemma 4(i) gives $\tau(2^k+1) \geq 2^{\tau(o(k))-1} \geq 2^C$, so by (2)

$$\frac{f(2n)}{f(n)} \geq 2^C\Big(1 - \frac{\sum_{k\in B} A(k)}{f(n)}\Big).$$

It suffices to show that $\sum_{k\in B} A(k) = o(f(n))$.

Upper bound on B. Every prime factor p of 2^k-1 satisfies $v_p(2^k-1) < k$, so $A(k) \leq k^{\omega(2^k-1)}$. Moreover $\omega(2^k-1) \leq \sum_{d\mid k} \omega(\Phi_d(2))$. If $k \in B$ and $d \mid k$, then $\tau(o(d)) \leq \tau(o(k)) \leq C$, and $\tau(k) \leq (\log_2 n + 1)C$. The function $g(x) = x^{\theta/\log\log x} = \exp(\theta \log x/\log\log x)$ is increasing for $x > e^e$; with $\theta = \log 2 - \varepsilon$, Hypothesis 9 yields a constant K_C such that $\omega(\Phi_d(2)) \leq K_C + g(n)$ for all $d \leq n$ with $\tau(o(d)) \leq C$. Therefore, for $k \in B$,

$$\log A(k) \leq \log n \cdot (\log_2 n + 1)\, C\,\big(K_C + g(n)\big) = n^{(\log 2 - \varepsilon + o(1))/\log\log n},$$

and the same bound (with a different $o(1)$) holds for $\log \sum_{k\in B} A(k) \leq \log n + \max_{k\in B} \log A(k)$.

Lower bound on f. Let $k^* \leq n$ maximise $\tau(k)$. By Theorem 2, $\omega(2^{k^*}-1) \geq \tau(k^*) - 2$, hence $f(n) \geq A(k^*) \geq 2^{\tau(k^*)-2}$. By Wigert's theorem, $\max_{k\leq n} \tau(k) = n^{(\log 2 + o(1))/\log\log n}$, so

$$\log f(n) \geq n^{(\log 2 + o(1))/\log\log n}.$$

Comparing the two bounds, $\sum_{k\in B} A(k)/f(n) \to 0$. Hence $\liminf f(2n)/f(n) \geq 2^C$ for every C. □

Remark 11. The exponent $\log 2$ is exactly the one appearing in Wigert's theorem, so Hypothesis 9 is the natural threshold for this argument: the mass of $f(n)$ carried by indices with few odd divisors must be beaten by the contribution $2^{\tau(k^*)}$ of a single highly composite index. The proof does not need any bound on $\omega(\Phi_d(2))$ for d with many odd divisors.

6. Numerical data

We factored $\Phi_d(2)$ for all $d \leq 200$ with PARI/GP and assembled $A(k) = \tau(2^k-1)$ for $k \leq 200$, checking in each case that the product of the factors equals $2^k - 1$. (Primality of the factors is certified by PARI's BPSW test; no counterexample to BPSW is known.) This gives $f(n)$ for $n \leq 200$ and the ratio $f(2n)/f(n)$ for $n \leq 100$. Theorem 1 holds throughout this range, with equality only at $n = 1, 2, 4$; the smallest ratio with $n \geq 32$ is $f(80)/f(40) \approx 14.26$. Some values:

n	$f(n)$	$f(2n)$	$f(2n)/f(n)$
1	1	3	3
2	3	9	3
3	5	17	3.4
4	9	27	3
8	27	101	3.74
16	101	539	5.34
32	539	8157	15.13
40	1311	18689	14.26
64	8157	150467	18.45
96	40023	7338375	183.35
100	48507	7930647	163.49

The values are dominated by highly composite indices: for example $A(120) = 73728$ while $A(128) = 512$, and the term $k = 120$ alone accounts for about half of $f(120) = 143465$. Our values agree with those reported in [2] where they overlap.

Acknowledgement. The arguments in this note were developed with the assistance of AI tools and checked by the author.

References

[1] T. F. Bloom, Erdős Problem #893, https://www.erdosproblems.com/893.
[2] P. White, Erdős Problem #893: Divisors of $2^k - 1$ growth — working report, *Erdős Problem a Day* (2026), https://erdosproblemaday.com/report/893.
[3] V. Kovač and F. Luca, On the number of divisors of Mersenne numbers, *Experimental Mathematics* (2026), https://doi.org/10.1080/10586458.2026.2636060; arXiv:2506.04883.