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SUPERSINGULAR CURVES AND OORT'S CONJECTURE ON NEWTON POLYGONS

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Abstract. Let k be an algebraically closed field of characteristic $p > 0$. Van der Geer asked which genera occur for smooth supersingular curves in a fixed prime characteristic. We show that genera five and six occur for every prime p. More precisely, in each of these two genera the locus of smooth supersingular curves over k has an irreducible component of dimension at least three. We also prove that if C_1 and C_2 are smooth curves over k of positive genera g_1 and g_2 with $g_1 + g_2 \leq 5$, then there is a smooth curve over k of genus $g_1 + g_2$ whose Jacobian has the same Newton polygon as $\mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2)$. This verifies Oort's amalgamation conjecture in total genus at most five.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Throughout, a smooth curve over k is understood to be projective and connected. To each such curve C of genus g one associates a Newton polygon $\mathrm{NP}(C)$. Its $2g$ slopes describe the action of Frobenius on the p-divisible group of the Jacobian $\mathrm{Jac}(C)$. A curve is called *supersingular* when all these slopes are $1/2$. In genus one, this condition is equivalent to the elliptic curve having no nonzero point of order p over k. In higher genus, Jacobians with no nonzero point of order p can have different Newton polygons.

Fix a prime p. Van der Geer asked which genera occur for smooth supersingular curves in characteristic p [19, Question 11.1]. Does every integer $g \geq 1$ occur as the genus of a smooth projective connected curve over $\overline{\mathbb{F}}_p$ whose Jacobian is supersingular? Oort observed that an affirmative answer would follow from his conjecture on the amalgamation of Newton polygons [14, Conjecture 8.5.7 and the discussion in 8.5.8].

In characteristic 2, van der Geer and van der Vlugt constructed, for every $g \geq 1$, a smooth supersingular curve of genus g defined over the prime field $\mathbb{F}_2$ [21, Theorem 3.5]. Oort proved that for every prime p there exists a smooth supersingular curve of genus three over $\overline{\mathbb{F}}_p$ [13]. Kudo, Harashita and Senda proved the corresponding result in genus four [10]. Pries subsequently gave a proof using the dimensions of Newton strata [17, Theorem 1.1]. In genus five, Booher and Pries constructed a smooth supersingular curve for every prime $p \equiv 3 \pmod 4$ as an unramified double cover of a genus-three curve [2, Theorem 1.2]. Their work on abelian covers of the projective line gives further sets of primes in genera five and six [3, Theorem 6.1]. Our first result treats the two genera in every characteristic.

Theorem 1.1. *For every prime p and each $g \in \{5, 6\}$, there is a smooth supersingular curve of genus g over k. The locus of such curves in the moduli space of smooth genus-g curves has an irreducible component of dimension at least three.*

The second result concerns a conjecture of Oort. Given two Newton polygons ξ_1 and ξ_2, write $\xi_1 \oplus \xi_2$ for the polygon obtained by combining the two lists of slopes, with their multiplicities, and arranging them in increasing order. This is the operation on Newton polygons induced by the product of abelian varieties. Oort conjectured that if each ξ_i occurs on the Jacobian of a

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smooth curve over k, then $\xi_1 \oplus \xi_2$ also occurs on the Jacobian of a smooth curve over k [15, Conjecture 8.5.7]. Applying the conjecture repeatedly to a supersingular elliptic curve would give a smooth supersingular curve of every genus.

Theorem 1.2. *Let C_1, C_2 be smooth curves over k of positive genera g_1, g_2 with $g_1 + g_2 \leq 5$. There is a smooth curve C over k of genus $g_1 + g_2$ such that $\mathrm{NP}(C) = \mathrm{NP}(C_1) \oplus \mathrm{NP}(C_2)$.*

To see the difficulty in genus five, recall that the closed Torelli locus also contains Jacobians of reducible curves. For example, take smooth supersingular curves C_2 and C_3 of genera two and three and identify a point of C_2 with a point of C_3. The resulting curve has genus five, and its Jacobian is the product $\mathrm{Jac}(C_2) \times \mathrm{Jac}(C_3)$ with the product polarization. This product is supersingular and lies on the boundary of the Torelli locus. The closed Torelli locus and the supersingular locus have dimensions 12 and 6, respectively, in the 15-dimensional moduli space of principally polarized abelian fivefolds. Thus every irreducible component of their intersection has dimension at least $12 + 6 - 15 = 3$. The products just described already form a family of dimension three. The dimension bound alone therefore gives no reason for the intersection to contain the Jacobian of a smooth curve of genus five. In genus six the same count gives a lower bound of $15 + 9 - 21 = 3$. Here one can glue two smooth supersingular curves of genus three. Each Jacobian varies in two parameters, so their products form a four-dimensional family in the boundary. This family alone already exceeds the lower bound for the intersection.

We begin with two smooth curves whose Jacobians are superspecial. In genus five we take curves of genera two and three, and in genus six we take two curves of genus three. Gluing the curves at a point gives a reducible curve whose Jacobian is the product of the two Jacobians. Write X for this product, equipped with its product polarization. We would like to vary X in a family of supersingular abelian varieties and show that some member of the family is the Jacobian of a smooth curve. To find such a family, we keep track of how we arrive at X. We choose a chain of isogenies from a superspecial abelian variety Y to X. Each step in the chain is a quotient by a finite subgroup scheme, and the polarizations at successive steps are compatible. The chain is called a rigid polarized flag because the composite map from Y to X determines all the intermediate steps. There can, however, be several such maps ending at the same X. This gives us a choice. Once we have chosen the flag, we keep its polarized source Y fixed and allow the successive quotients, and hence the endpoint, to vary. Every endpoint remains supersingular because it is isogenous to Y.

We first explain the role of the flag in the proof of Theorem 1.1. Suppose that p is odd and that C_1 and C_2 are smooth superspecial curves of genera s and t, respectively. We take $(s, t) = (2, 3)$ in genus five and $(s, t) = (3, 3)$ in genus six. Set $X_i = \mathrm{Jac}(C_i)$ and give $X = X_1 \times X_2$ the product polarization. Gluing C_1 and C_2 at a point gives a curve with Jacobian X. Thus X belongs to the closed Torelli locus $\mathcal{T}_g$, where $g = s + t$. We seek a family of supersingular Jacobians through X whose general member is the Jacobian of a smooth curve.

Consider first the case of genus five. We start with two smooth curves, one of genus two and one of genus three, whose Jacobians X_1 and X_2 are superspecial. Gluing the curves at a point gives a reducible curve of genus five with Jacobian $X_1 \times X_2$. We want to find a smooth curve of genus five whose Jacobian is still supersingular. The difficulty is that we can already obtain a three-dimensional family of supersingular Jacobians simply by varying the two factors of the product. Every curve obtained by this construction is still reducible.

We therefore keep track of one additional piece of information at $X_1 \times X_2$. We choose a chain of polarized isogenies ending at this product, as in Lemma 3.4. We then hold the source of the chain fixed and allow the isogenies and their endpoint to vary. The reason for choosing a particular chain is the following consequence of Proposition 4.2. If the general endpoint of such a family is a product, its factors must again have dimensions two and three, and *both* must

have a-number at least two. This last assertion is the purpose of the Hermitian construction in Section 3. The chosen chain gives a vector of nonzero norm in the Hermitian space of each factor. Proposition 4.2 shows that if a general factor had a-number one, the corresponding vector would have norm zero.

We can now count the products that remain possible. Supersingular abelian surfaces form a one-dimensional locus, and only finitely many of them have a-number at least two. Supersingular abelian threefolds form a two-dimensional locus, and those with a-number at least two form a locus of dimension at most one. Both bounds follow from Proposition 2.1, since the general point of each component has a-number one. Thus, if the general endpoint were a product, its surface factor could take only finitely many values and its threefold factor could vary in at most one parameter. The family of endpoints would have dimension at most one.

On the other hand, Section 5 shows that every component of the space of flags through our chosen flag, with endpoint in the closed Torelli locus, has dimension at least three. Only finitely many flags with the fixed polarized source can have the same endpoint, so the family of endpoints also has dimension at least three. This contradicts the upper bound of one for a family whose general endpoint is a product. The general endpoint is therefore indecomposable. It is supersingular because it is isogenous to the fixed source, and it is the Jacobian of a smooth curve because it is an indecomposable point of the closed Torelli locus.

In genus six we start with two superspecial Jacobians of dimension three. The lower bound is again three. If the general endpoint were a product, each of its two threefold factors would have a-number at least two. Each could vary in at most one parameter, giving an upper bound of two for the family of products. This proves the same conclusion in genus six. As explained in Section 5, the required starting curves exist for $p \geq 5$ in genus five and for odd p in genus six. For $p = 2$ we use the smooth curves of van der Geer and van der Vlugt [21], and for $g = 5$ and $p = 3$ we use those of Booher and Pries [2, Theorem 1.2]. At any of these smooth Jacobians the intersection dimension bound again gives a family of dimension at least three. This completes the proof of Theorem 1.1.

For the remaining genus-five case in Theorem 1.2 admits a shorter argument, we seek a smooth curve whose Jacobian has three slopes $1/3$, four slopes $1/2$, and three slopes $2/3$. To obtain a starting point with exactly these slopes, we take a genus-two Jacobian with all four slopes equal to $1/2$ and a genus-three Jacobian with slopes $1/3$ and $2/3$, each occurring three times. The slopes of their product are the combined slopes of the two factors.

Here we can work directly in the closed Torelli locus. By Lemma 6.1, the locus with exactly the prescribed polygon has codimension seven in $\mathcal{A}_5$. The closed Torelli locus has dimension twelve, so every component of the intersection through our product has dimension at least $12 - 7 = 5$. Suppose that the general member of one such component were a product. Lemma 4.1 forces it to split into a genus-two Jacobian and a genus-three Jacobian: these are the dimensions of the two indecomposable factors at our chosen point. Both factors have p-rank zero, since the product has no slope zero. The spaces of smooth curves of p-rank zero in genera two and three have dimensions one and three, respectively [1, Theorem 4.1]. Their products can therefore form a family of dimension at most four. This contradicts the lower bound of five. Hence the general member is the Jacobian of a smooth curve with the prescribed slopes. Section 6 combines this case with Theorem 1.1 and the results of Pries [17] to prove Theorem 1.2.

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2. Preliminaries

In this section, we discuss preliminary notions and set up notation. Recall that k is an algebraically closed field of characteristic $p > 0$. Given an abelian variety A, a principal polarization is an isomorphism $A \to A^\vee$ to the dual abelian variety induced by an ample line bundle. The Jacobian of a smooth curve carries a canonical principal polarization. Denote by $\mathcal{M}_g$ the moduli stack of smooth curves of genus $g \geq 2$ over k, and by $\mathcal{A}_g$ the moduli stack of principally polarized abelian varieties of dimension g. The Torelli morphism takes a curve to its polarized Jacobian. Denote by $\mathcal{T}_g$ the closure of its image in $\mathcal{A}_g$. For $g \geq 2$, the moduli stack $\mathcal{M}_g$ has dimension $3g-3$, and $\mathcal{A}_g$ is smooth over k of dimension $g(g+1)/2$ [17, Section 2.3]. By Torelli's theorem, the morphism $\mathcal{M}_g \to \mathcal{A}_g$ has finite geometric fibres. Since $\mathcal{T}_g$ is the closure of its image, it follows that $\dim \mathcal{T}_g = \dim \mathcal{M}_g = 3g-3$.

The points of $\mathcal{T}_g$ are the Jacobians of stable curves of *compact type* [17, Section 2.2]. A nodal curve is of compact type if its dual graph, with a vertex for each irreducible component and an edge for each node, is a tree. Let A/k be an abelian variety of dimension g. Its p-divisible group $A[p^\infty]$ is union $\bigcup_n A[p^n]$. The kernel α_p of Frobenius on the additive group has coordinate ring $k[t]/(t^p)$. The *p-rank* $f(A)$ is defined by $\#A[p](k) = p^{f(A)}$, and the *a-number* is $a(A) = \dim_k \operatorname{Hom}(\alpha_p, A)$.

Write $W = W(k)$ for the Witt ring of k, $K_0 = W[1/p]$ for its fraction field and σ for its Frobenius automorphism. The ring W is a discrete valuation ring with uniformizer p and residue field k, and σ induces $x \mapsto x^p$ on k. Dieudonné theory gives a linear description of the p-power torsion of A. It associates to $A[p^\infty]$ a free $W(k)$-module of rank $2g$, which we denote by $D(A)$, together with two operators, the Frobenius F and the Verschiebung V. For each $n \geq 1$, the quotient $D(A)/p^n D(A)$, with its induced operators, determines the finite group scheme $A[p^n]$ up to isomorphism.

The Newton slopes are those of F on $D[1/p]$, and the multiplicity of a/b is divisible by b. List the $2g$ slopes in nondecreasing order as $\lambda_1, \ldots, \lambda_{2g}$, counting each with its multiplicity. The Newton polygon starts at $(0,0)$. At the i-th step, draw a line segment one unit to the right and λ_i units up. The slopes sum to g and polygon ends at $(2g, g)$. It follows from duality that the multiplicity of a slope λ equals that of $1-\lambda$, and the multiplicity of slope zero equals $f(A)$. We write ss for the supersingular elliptic polygon with slopes $1/2, 1/2$ and ss^j for the sum of j copies.

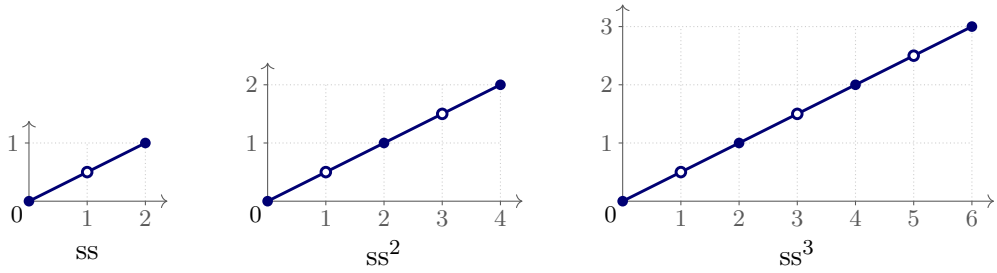

Figure 1. The supersingular Newton polygons ss, ss^2 and ss^3. Each segment between consecutive filled dots is one copy of ss.

The Newton stratum $\mathcal{A}_g[\xi]$ consists of the abelian varieties with polygon ξ. In particular $\mathcal{S}_g = \mathcal{A}_g[\mathrm{ss}^g]$ is the supersingular locus. Over k, a supersingular abelian variety is isogenous to a product of supersingular elliptic curves. It is called *superspecial* if it is isomorphic to such a product as an abelian variety. Equivalently $a(A) = g$, or $FD = VD$ [20, Sections 4–5]. In this case $F^2D = pD$ as lattices.

An isogeny $b : B \to A$ induces an inclusion $b^* : D(A) \to D(B)$ which becomes an isomorphism over K_0. We use this map to view the two modules as lattices in the same vector space. The source B then corresponds to the larger lattice. A principal polarization on A gives $D(A)$ a perfect alternating pairing with $\langle Fx, Fy\rangle = p\,\sigma(\langle x, y\rangle)$. For a lattice L in $D(A)[1/p]$, set $L^t = \{x : \langle x, L\rangle \subset W\}$. Thus $D(A)^t = D(A)$. Let $b : B \to A$ be an isogeny and let $\lambda : A \to A^\vee$ be the given principal polarization. Its pullback to B is the polarization $\lambda_B = b^\vee \circ \lambda \circ b : B \to B^\vee$. Set $L = D(B)$. By contravariance, λ_B induces an injection $D(\lambda_B) : D(B^\vee) \hookrightarrow L$. Under the rational pairing induced by λ, the image of this map consists of the vectors that pair integrally with every element of L. Thus its image is precisely the dual lattice L^t.

Let K_p denote the p-primary part of $\ker(\lambda_B)$. Exactness of contravariant Dieudonné theory gives

$$0 \longrightarrow D(B^\vee) \xrightarrow{D(\lambda_B)} L \longrightarrow D(K_p) \longrightarrow 0.$$

Consequently $D(K_p) \simeq L/L^t$. In particular, when b has p-power degree, this quotient describes the entire kernel of the pullback polarization.

We now describe the flags used in Theorem 1.1. A polarized flag of Li–Oort type is a chain of isogenies

$$Y_{g-1} \longrightarrow Y_{g-2} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0.$$

Every Y_i has dimension g. The source Y_{g-1} is superspecial, the endpoint Y_0 has a principal polarization, and each Y_i carries its pullback. At the i-th arrow the kernel is isomorphic over k to α_p^i. There are also compatibility conditions on the polarizations, see [20, Definition 4.1 and Section 5] for further details.

Write $\rho_i : Y_i \to Y_{i-1}$ for the i-th isogeny, and set $M_i = D(Y_i)$, $D = M_0$ and $H = M_{g-1}$. Pullback by ρ_i identifies M_{i-1} with a sublattice of M_i. Thus the chain of isogenies gives a chain of lattices

$$D = M_0 \subset M_1 \subset \cdots \subset M_{g-1} = H$$

in the common rational Dieudonné module $D[1/p]$. The quotient M_i/M_{i-1} is the Dieudonné module of $\ker(\rho_i)$. Since $\ker(\rho_i) \simeq \alpha_p^i$ and the module of α_p is a one-dimensional k-vector space on which F and V are zero, this quotient has dimension i over k. In particular, both FM_i and VM_i are contained in M_{i-1}.

We take dual lattices with respect to the pairing induced by the principal polarization on Y_0, so $D^t = D$. For an abelian variety Y, write $Y[F^r]$ for the kernel of the r-fold relative Frobenius $Y \to Y^{(p^r)}$. The polarization on the source Y_{g-1} has kernel $Y_{g-1}[F^{g-1}]$, which gives $H^t = F^{g-1}H$. At each stage i, the quotient M_i/M_i^t describes the kernel of the pullback polarization on Y_i. The polarization condition requires that every composition of i operators chosen from F and V act as zero on this quotient. Equivalently, $(F, V)^i M_i \subseteq M_i^t$. For example, when $i = 2$ this reads $F^2M_2 + pM_2 + V^2M_2 \subseteq M_2^t$, since $FV = VF = p$ [20, Definition 4.1 and Section 5]. Recall that $FM_j \subseteq M_{j-1}$ at each step of the flag. Starting at $H = M_{g-1}$ and applying this inclusion $g - 1 - i$ times gives $F^{g-1-i}H \subseteq M_i$. We also have $D \subseteq M_i$, since $D = M_0$ is the first lattice in the chain. Thus every flag satisfies $D + F^{g-1-i}H \subseteq M_i$. The flag is called *rigid* if equality holds for every i [20, Section 5].

This condition also has a description in terms of the isogenies themselves. Let $\rho : Y_{g-1} \to Y_0$ be their composite and set $K = \ker(\rho)$. In a rigid flag, the kernel of the intermediate map $Y_{g-1} \to Y_i$ is precisely $K \cap Y_{g-1}[F^{g-1-i}]$ [9, Definition 3.1]. For instance, when $g = 3$ the

flag is $Y_2 \to Y_1 \to Y_0$. Its only intermediate variety is $Y_1 = Y_2/(K \cap Y_2[F])$, and its lattice is $M_1 = D + FH$.

It follows that fixing the composite isogeny ρ determines a rigid flag up to isomorphism. On fixing the endpoint Y_0, we note that several rigid flags may end at the same Y_0. For a fixed superspecial abelian variety Y with polarization η satisfying $\ker(\eta) = Y[F^{g-1}]$, we write $\mathcal{P}'_\eta$ for the moduli space of rigid flags starting at (Y, η). Its endpoint map sends each flag to Y_0 with its principal polarization.

Proposition 2.1 (Li–Oort, Oda–Oort)**.** *The following assertions hold.*

(1) Let (A, λ) be a principally polarized supersingular abelian variety of dimension $g \geq 2$. There is a rigid flag

$$Y_{g-1} \longrightarrow Y_{g-2} \longrightarrow \cdots \longrightarrow Y_0 = A$$

whose source Y_{g-1} is superspecial and whose endpoint carries the given polarization λ.

Now fix a polarized source (Y, η) as above. The space $\mathcal{P}'_\eta$ of rigid flags starting at (Y, η) is smooth and irreducible of dimension $\lfloor g^2/4 \rfloor$. Its endpoint map $e_\eta : \mathcal{P}'_\eta \to \mathcal{S}_g$ sends a flag to its principally polarized endpoint. This map is quasi-finite. Every irreducible component of $\mathcal{S}_g$ has dimension $\lfloor g^2/4 \rfloor$ and contains a dense open subset on which $a(A) = 1$.

(2) Let (A, λ) be a principally polarized supersingular abelian variety of dimension $s \geq 2$, and suppose that $a(A) = 1$. There is a superspecial abelian variety B and an isogeny $b : B \to A$ with the following property. Given a superspecial abelian variety C and a homomorphism $u : C \to A$, there is a unique homomorphism $v : C \to B$ such that $u = b \circ v$. This property determines $b : B \to A$ up to unique isomorphism over A. It is called the minimal superspecial isogeny *to A.*

The kernel of the pullback polarization $b^*\lambda$ on B is exactly $B[F^{s-1}]$, the subgroup scheme killed by the $(s-1)$-fold relative Frobenius of B. To express this in terms of lattices, use $b^* : D(A)[1/p] \to D(B)[1/p]$ to identify the two rational Dieudonné modules, and set $L = D(B)$. With respect to the pairing induced by λ, the dual lattice of L is $L^t = F^{s-1}L$.

Proof. For part (1), Li and Oort show that every principally polarized supersingular abelian variety occurs as the endpoint of a rigid flag [11, Propositions 4.1 and 4.3]. The assertions about the flag spaces and their endpoint maps are collected in [9, Theorem 3.3]. The same theorem gives the dimension $\lfloor g^2/4 \rfloor$ of each irreducible component of $\mathcal{S}_g$ and the density of the open locus on which the a-number is one.

For part (2), let $b : B \to A$ be the minimal isogeny from a superspecial abelian variety. Its existence and the stated factorization property are recalled in [20, Section 4]. Since $a(A) = 1$, the kernel of the pullback polarization $b^*\lambda$ is $B[F^{s-1}]$ by [12, Theorems 2.2 and 3.1], as formulated in [16, Theorem 11.5].

It remains to identify the dual lattice L^t, where $L = D(B)$. Set $r = s - 1$ and $K = B[F^r]$. The inclusion $K \hookrightarrow B$ induces a surjection $L \to D(K)$. Since K is the kernel of the polarization $b^*\lambda : B \to B^\vee$, exactness of contravariant Dieudonné theory identifies the kernel of this surjection with the image of $D(B^\vee)$ in L, namely L^t. On the other hand, K is also the kernel of the relative Frobenius $F^r_{B/k} : B \to B^{(p^r)}$. Applying the same argument to this isogeny identifies the kernel of the same surjection with the image of $D(B^{(p^r)})$ in L, namely $F^r L$. Thus $L^t = F^r L = F^{s-1}L$. □

3. A rigid flag ending at a superspecial product

Assume in this section that p is odd. Let D be a Dieudonné lattice of rank $2g$ over $W = W(k)$. We assume that D is superspecial, meaning that $FD = VD$. Since $FV = p$, it follows that

$F^2D = pD$. Equip $D[1/p]$ with the alternating pairing $\langle\,,\,\rangle$ coming from the polarization. We assume that D is self-dual for this pairing.

Definition 3.1. Suppose that p is odd. Let D be a self-dual superspecial Dieudonné lattice of rank $2g$ over $W = W(k)$, with alternating pairing $\langle\,,\,\rangle$. Inside $D[1/p]$, set

$$F^{-1}D = \{z \in D[1/p] : Fz \in D\}.$$

Since $FD \subseteq D$, this lattice contains D. Set $\tau = p^{-1}F^2$. The operator τ preserves both lattices and therefore acts on $F^{-1}D/D$. We define

$$V_D = (F^{-1}D/D)^{\tau=1}.$$

Thus an element of V_D is a class $\bar{z}$ modulo D satisfying $\tau(\bar{z}) = \bar{z}$. A *skeleton representative* of this class is a lift $z \in F^{-1}D$ for which $\tau(z) = z$, or equivalently $F^2z = pz$.

Proposition 3.2. *The space V_D has dimension g over $\mathbb{F}_{p^2}$, and every element of V_D has a skeleton representative. Choose a unit $c \in W(\mathbb{F}_{p^2})$ satisfying $\sigma(c) = -c$. For $\bar{z}, \bar{w} \in V_D$, choose skeleton representatives z, w. Then*

$$h_D(\bar{z}, \bar{w}) = c\,p\langle Fz, w\rangle \pmod{p} \tag{3.1}$$

defines a nondegenerate Hermitian form on V_D with respect to the conjugation $a \mapsto a^p$ of $\mathbb{F}_{p^2}$. If $D = D_1 \perp D_2$ is an orthogonal direct sum of polarized Dieudonné lattices, then $V_D = V_{D_1} \perp V_{D_2}$ for the forms defined using the same choice of c.

Proof. We first describe the vectors in V_D. Since D is superspecial, it has a W-basis $e_1, f_1, \ldots, e_g, f_g$ with $Fe_i = f_i$ and $Ff_i = pe_i$. Thus $F^{-1}D$ has basis $e_i, p^{-1}f_i$ for $1 \le i \le g$. Modulo D, the classes of the e_i vanish, and the classes of $p^{-1}f_1, \ldots, p^{-1}f_g$ form a k-basis of $F^{-1}D/D$. Equivalently, Frobenius identifies this quotient semilinearly with D/FD by sending the class of $p^{-1}f_i$ to the class of e_i.

Each vector $p^{-1}f_i$ is fixed by τ. Since τ acts on scalars by σ^2, its action on the coefficients of a class in $F^{-1}D/D$ is $a \mapsto a^{p^2}$. The class is therefore fixed precisely when all its coefficients belong to $\mathbb{F}_{p^2}$. This proves that $\dim_{\mathbb{F}_{p^2}} V_D = g$. Lifting these coefficients to $W(\mathbb{F}_{p^2})$ gives a representative fixed by τ, as claimed.

We next check that (3.1) is independent of the lifts. The unit c may be taken to be the Teichmüller lift of a nonzero element of trace zero in $\mathbb{F}_{p^2}$. Since $F^2D = pD$, we have $F^{-1}D = p^{-1}FD$. Self-duality of D and the identity $\langle Fx, Fy\rangle = p\,\sigma(\langle x, y\rangle)$ then give

$$(FD)^t = p^{-1}FD, \qquad (F^{-1}D)^t = FD.$$

For $z, w \in F^{-1}D$, both Fz and pw belong to D. Hence $p\langle Fz, w\rangle = \langle Fz, pw\rangle$ is integral. Changing w by an element of D changes this expression by an element of pW, since $Fz \in D$. Changing z by $d \in D$ adds $p\langle Fd, w\rangle$, which also belongs to pW because $Fd \in (F^{-1}D)^t$. Thus the residue depends only on $\bar{z}$ and $\bar{w}$.

For skeleton vectors x, y, applying the Frobenius identity twice gives $\sigma^2(\langle x, y\rangle) = \langle x, y\rangle$. Consequently the integral expression in (3.1) belongs to $W(\mathbb{F}_{p^2})$, and its residue lies in $\mathbb{F}_{p^2}$. The form is conjugate-linear in the first variable and linear in the second, since F acts on coefficients by σ. To check Hermitian symmetry, use $F^2z = pz$ to obtain

$$\sigma\big(\langle Fz, w\rangle\big) = p^{-1}\langle F^2z, Fw\rangle = \langle z, Fw\rangle = -\langle Fw, z\rangle.$$

Since $\sigma(c) = -c$, reduction modulo p gives $h_D(\bar{z}, \bar{w})^p = h_D(\bar{w}, \bar{z})$.

Finally, suppose $\bar{z}$ is orthogonal to every element of V_D. Since c is a unit, (3.1) implies that $\langle Fz, w\rangle \in W$ for every skeleton vector $w \in F^{-1}D$. These vectors span $F^{-1}D$ over W, as the basis $e_i, p^{-1}f_i$ above shows. It follows that $Fz \in (F^{-1}D)^t = FD$, and hence $z \in D$. Thus $\bar{z} = 0$,

proving nondegeneracy. For an orthogonal direct sum $D = D_1 \perp D_2$, Frobenius preserves each summand and the cross pairings in (3.1) vanish. This proves the last assertion. □

For a subspace U of V_D, write $U^\perp$ for its orthogonal complement. The following observation will be applied to a factor of a product after specialization.

Lemma 3.3. *Let $D \subset L$ be superspecial lattices of rank $2s$ in the same rational module, with $s \geq 2$. Suppose that $D^t = D$ and $L^t = F^{s-1}L$. Set $Q = L \cap F^{-1}D$ and $U = (Q/D)^{\tau=1}$. Then $U^\perp \subset U$. In particular, every element of $U^\perp$ has norm zero.*

Proof. Let $z \in F^{-1}D$ be a skeleton representative of a class in $U^\perp$. By (3.1), $\langle Fz, w\rangle$ is integral for every skeleton vector $w \in Q$. These vectors span Q over W, since Q is the intersection of two lattices obtained by extending scalars from their skeletons. Hence $Fz \in Q^t$. The dual of an intersection of full lattices is the sum of their duals, as one sees by writing the lattices in an elementary divisor basis. Therefore

$$Fz \in Q^t = L^t + (F^{-1}D)^t = F^{s-1}L + FD.$$

It follows that $z \in F^{s-2}L + D \subset L$. Since $z \in F^{-1}D$, we have $z \in Q$, proving $U^\perp \subset U$. Every vector of $U^\perp$ is now orthogonal to itself. □

Lemma 3.4. *Let X_1 and X_2 be principally polarized superspecial abelian varieties of positive dimension, and give $X = X_1 \times X_2$ the product polarization. Set $D = D(X)$ and $V_i = V_{D(X_i)}$, so that $V_D = V_1 \perp V_2$. For the source lattice H of a rigid polarized flag ending at X, write*

$$T(H) = \big((H \cap F^{-1}D)/D\big)^{\tau=1}.$$

There is a choice of flag for which $T(H) = (n_1 + n_2)^\perp$, where $n_i \in V_i$ and $h_D(n_i, n_i) \neq 0$ for each i. Thus the line perpendicular to $T(H)$ is spanned by a vector whose projection onto each factor has nonzero Hermitian norm.

Proof. Set $g = \dim X$. By Proposition 2.1, there is a rigid polarized flag ending at X. Write its chain of Dieudonné lattices as $D = M_0 \subset \cdots \subset M_{g-1} = H$. We first determine the dimension of $T(H)$. Rigidity gives $M_{g-2} = D + FH$, so the last quotient in the flag is $H/(D + FH)$. This quotient has dimension $g - 1$ over k. Since H/FH has dimension g, the image of D in H/FH has dimension one. It follows that the natural map $D/FD \to H/FH$ has kernel of dimension $g - 1$.

This kernel is $(D \cap FH)/FD$. Frobenius identifies it with the space occurring in the definition of $T(H)$ by the semilinear isomorphism

$$(H \cap F^{-1}D)/D \longrightarrow (D \cap FH)/FD$$

given by $\bar{z} \longmapsto \overline{Fz}$. Both D and H are superspecial, so $\tau = p^{-1}F^2$ preserves both lattices and acts bijectively on the space on the left. Passing to the τ-fixed vectors gives a space of the same dimension over $\mathbb{F}_{p^2}$. Thus $T(H)$ is a hyperplane in V_D. Moreover, the source polarization gives $H^t = F^{g-1}H$. Lemma 3.3, applied with $L = H$ and $s = g$, now shows that the line $T(H)^\perp$ is isotropic. Choose a nonzero vector n on this line. Then $T(H) = n^\perp$ and $h_D(n, n) = 0$.

We next choose the vector that we would like to have in place of n. Each V_i is a nonzero, nondegenerate Hermitian space, so it contains a vector of nonzero norm. Multiplying a vector by $b \in \mathbb{F}_{p^2}^\times$ multiplies its norm by b^{p+1}. Since the norm map from $\mathbb{F}_{p^2}^\times$ to $\mathbb{F}_p^\times$ is surjective, we may choose $n_1 \in V_1$ of norm 1 and $n_2 \in V_2$ of norm -1. The two summands are orthogonal, so $n_1 + n_2$ has norm zero, although each of its components has nonzero norm. The linear map from the line spanned by n to the line spanned by $n_1 + n_2$ which sends n to $n_1 + n_2$ is therefore an isometry. By Witt's extension theorem [18, Theorem 3.4], the isometry between these two lines extends to an isometry u of V_D satisfying $u(n) = n_1 + n_2$. In particular, $uT(H) = (n_1 + n_2)^\perp$.

It remains to show that this new hyperplane comes from a flag ending at the same X. We first lift u to the lattice D. By [6, Corollaries 2.8 and 2.9], every similitude of the residual Hermitian space lifts to an integral Dieudonné similitude. To apply this result, we compare the residual pairing in that reference with h_D. On our skeleton, Frobenius sends the class of z in V_D to the class of $x = Fz$ modulo FD. If $y = Fw$ as well, then $Fy = F^2w = pw$, and hence

$$c\langle x, Fy\rangle = c\,p\langle Fz, w\rangle \equiv h_D(\bar{z}, \bar{w}) \pmod{p}.$$

The skeleton in [6] uses the convention $F^2x = -px$. To pass from that convention to ours, choose $t \in W(\mathbb{F}_{p^4})$ whose residue lies outside $\mathbb{F}_{p^2}$ and set $a = t - \sigma^2(t)$. Then a is a unit and $\sigma^2(a) = -a$. Thus multiplication by a carries the skeleton defined by $F^2x = -px$ onto the one defined by $F^2x = px$. It also identifies their quotients modulo F, since $\sigma(a)/a$ is a unit in $W(\mathbb{F}_{p^2})$. Under this identification the pairing changes by the factor $c\,a\sigma(a)$, as one sees from

$$c\langle ax, F(ay)\rangle = c\,a\sigma(a)\langle x, Fy\rangle.$$

This factor is fixed by σ, since $\sigma(c) = -c$ and $\sigma^2(a) = -a$. Its reduction therefore belongs to $\mathbb{F}_p^\times$, so the two pairings have the same isometries. Both identifications commute with the action of integral Dieudonné similitudes. The lifting result consequently gives a similitude $\widetilde{u}$ of D inducing u on V_D.

Extend $\widetilde{u}$ to $D[1/p]$ and set $H_i = \widetilde{u}(M_i)$. Since $\widetilde{u}$ preserves D, the new chain still begins with $H_0 = D$. It commutes with F and V, and it scales the polarization pairing by a p-adic unit. In particular, $H_i^t = \widetilde{u}(M_i^t)$. Thus the successive quotients, the polarization conditions and the rigidity formula are all preserved. The hyperplane of the new source lattice is $T(H_{g-1}) = uT(H)$.

We finally realize this chain by isogenies of abelian varieties. Choose m large enough that $p^mD \subseteq p^mH_i \subseteq D$ for every i. By contravariant Dieudonné theory, the quotient D/p^mH_i of D/p^mD corresponds to a subgroup scheme $K_i \subseteq X[p^m]$. The inclusion $H_{i-1} \subseteq H_i$ gives $K_i \subseteq K_{i-1}$. Thus, on setting $Y_i = X/K_i$, we obtain isogenies $Y_i \to Y_{i-1}$. Since $H_0 = D$, we have $K_0 = X[p^m]$, and multiplication by p^m identifies Y_0 with X.

To check that these isogenies give the prescribed lattices, let $q_i : X \to Y_i$ be the quotient map. Since $K_i \subseteq X[p^m]$, there is an isogeny $r_i : Y_i \to X$ such that $r_iq_i = [p^m]_X$. By construction, $q_i^*D(Y_i) = p^mH_i$. It follows from $q_i^*r_i^* = p^m$ that, under the rational identification given by r_i^*, the lattice of Y_i is precisely

$$(r_i^*)^{-1}(D(Y_i)) = p^{-m}q_i^*D(Y_i) = H_i.$$

Give Y_i the pullback of the principal polarization of X along r_i. The isogenies and these polarizations have p-power degree, so their kernels are determined by their Dieudonné modules. The lattice conditions verified above therefore give a rigid polarized flag. Its source is superspecial, since its Dieudonné lattice $H_{g-1} = \widetilde{u}(H)$ is superspecial. This flag ends at X and has $T(H_{g-1}) = (n_1 + n_2)^\perp$, as required. □

4. Specialization of polarized product decompositions

Let $\mathcal{X}$ be a principally polarized abelian scheme over a discrete valuation ring R with fraction field K. We first note how a polarized decomposition behaves under specialization. This observation will also be used for a nonsupersingular Newton polygon in Section 6.

Lemma 4.1. *Let R be the valuation ring of K. Suppose that the geometric special fibre of $\mathcal{X}$ is isomorphic, as a polarized abelian variety, to $X_1 \times X_2$ with its product polarization. Assume that each X_i is indecomposable, meaning that it cannot itself be written as a polarized product of two positive-dimensional factors.*

Suppose that the geometric generic fibre also admits a nontrivial polarized product decomposition, and fix such a decomposition. Then it has exactly two factors. After a finite extension K'/K

with valuation ring R', these factors extend to polarized abelian schemes $\mathcal{X}_1$ and $\mathcal{X}_2$ over R', and the chosen decomposition extends to an isomorphism

$$\mathcal{X}_{R'} \simeq \mathcal{X}_1 \times_{R'} \mathcal{X}_2$$

of polarized abelian schemes. After interchanging the two factors if necessary, the geometric special fibre of $\mathcal{X}_i$ is isomorphic to X_i as a polarized abelian variety for $i = 1, 2$.

Proof. The decomposition of $\mathcal{X}_{\bar{K}}$ into indecomposable polarized factors and the projections onto these factors are defined over a finite extension of K. Replacing R by a valuation ring above it, we may assume that these maps are defined over K.

For each factor, projection followed by inclusion gives an endomorphism e of $\mathcal{X}_K$ satisfying $e^2 = e$. Since the polarization is a product, e is self-adjoint for this polarization. The abelian scheme $\mathcal{X}$ is the Néron model of its generic fibre. Thus e extends uniquely over R [4, Section 1.2, Proposition 8]. Uniqueness preserves the relations among the projectors, including self-adjointness. In particular the map $x \mapsto (ex, (1-e)x)$ identifies $\mathcal{X}$ with $\ker(1-e) \times \ker(e)$, with inverse given by addition. The two factors are abelian schemes, orthogonal for the polarization. The restriction of the polarization to each factor is principal, since their product is principal. Applying this to all the projectors extends the generic polarized decomposition over R.

Each factor has positive relative dimension, so its special fibre accounts for at least one indecomposable factor of the special fibre. The special fibre has precisely two such factors. It follows that the decomposable generic fibre has exactly two factors and that their special fibres are indecomposable. Uniqueness of the polarized decomposition as embedded abelian subvarieties identifies these special fibres with the prescribed X_1, X_2, up to order [5, Corollaire 2]. This also applies when X_1 and X_2 are isomorphic. □

Set $X = X_1 \times X_2$ and suppose that X_1, X_2 are superspecial. Above their product, Lemma 3.4 gives a choice of flag whose normal vector has a component of nonzero norm in each factor. We use that choice to restrict the generic factors in the preceding lemma.

Proposition 4.2. *Suppose that p is odd and that each X_i is indecomposable as a principally polarized abelian variety, of dimension $s_i \geq 2$. Let $y \in \mathcal{P}'_\eta(k)$ be a flag as in Lemma 3.4. Consider a family in $\mathcal{P}'_\eta$ over a discrete valuation ring whose geometric special member is y. If the geometric generic endpoint is decomposable, it has precisely two indecomposable polarized factors. After reordering, their dimensions are s_1, s_2, and each has a-number at least two.*

Proof. Write R for the discrete valuation ring, K for its fraction field and $\mathcal{X}/R$ for the abelian scheme at the end of the flag. By Lemma 4.1, after a finite extension of K the two generic factors extend over R and specialize to X_1, X_2 up to order.

Fix one of the factors over R. Denote it by $\mathcal{A}$, its projection by $\pi : \mathcal{X} \to \mathcal{A}$, and its principal polarization by $\lambda_{\mathcal{A}}$. Its special fibre is some X_i of dimension $s = s_i$. The composite of the arrows of the flag is an isogeny $\rho : Y_R \to \mathcal{X}$ from a fixed superspecial abelian variety Y. Thus $\mathcal{A}_{\bar{K}}$ is supersingular and has positive a-number. Suppose its a-number is one. By Proposition 2.1, there is a minimal superspecial isogeny $b : B \to \mathcal{A}_{\bar{K}}$, and the pullback of $\lambda_{\mathcal{A}}$ has kernel $B[F^{s-1}]$. Since Y is superspecial, the universal property of b gives a homomorphism $h : Y_{\bar{K}} \to B$ such that $bh = \pi\rho$.

We next extend this identity to the special fibre. The superspecial variety B is a product of supersingular elliptic curves over $\bar{K}$. Each of these elliptic curves has a model over k, since its j-invariant is algebraic over $\mathbb{F}_p$. We may therefore choose a model B_0/k and an identification $(B_0)_{\bar{K}} \simeq B$. Homomorphisms between abelian varieties over an algebraically closed field are unchanged by extension to a larger algebraically closed field. In particular $b^*\lambda_{\mathcal{A}}$ is the base change of a polarization η_B on B_0, and $\ker \eta_B = B_0[F^{s-1}]$. After a further finite extension of

K the maps b and h are defined over K. The Néron mapping property now extends them over R, using the constant model $B_{0,R}$. By uniqueness of extension, both $bh = \pi\rho$ and $b^*\lambda_{\mathcal{A}} = \eta_B$ hold over R. On the special fibre the kernel of b is therefore contained in the finite group scheme $\ker \eta_B$. Its source and target have the same dimension, so the specialized map $b : B_0 \to X_i$ is an isogeny.

We now work on the geometric special fibre. Set $D_i = D(X_i)$ and $H = D(Y)$. Using ρ^*, identify the rational Dieudonné modules of X and Y, so that $D_1 \oplus D_2 \subset H$. The identity $h^*b^* = \rho^*\pi^*$ identifies the image of D_i with the displayed i-th summand. Since b^* is an isomorphism after inverting p, the image $L = h^*D(B_0)$ is a superspecial lattice satisfying

$$D_i \subset L \subset H \cap D_i[1/p].$$

Take duals in $D_i[1/p]$ for the pairing given by the principal polarization of X_i. In this identification the pullback by b of that polarization is represented by $L^t \subset L$. Its kernel is $B_0[F^{s-1}]$, as just proved. The lattice description in Proposition 2.1 therefore gives $L^t = F^{s-1}L$.

Set $U = ((L \cap F^{-1}D_i)/D_i)^{\tau=1} \subset V_{D_i}$. Lemma 3.3 shows that $U^\perp \subset U$, so every vector in $U^\perp$ has norm zero. On the other hand, $L \subset H$, so $U \subset T \cap V_{D_i}$, where $T = (n_1 + n_2)^\perp$ is the hyperplane of our flag. Since V_{D_1} and V_{D_2} are orthogonal, $T \cap V_{D_i} = n_i^\perp$. It follows that $n_i \in U^\perp$. But the choice of the flag gives $h_D(n_i, n_i) \neq 0$. This contradiction excludes a-number one for either generic factor. □

5. Supersingular curves of genera five and six

We now apply the specialization result to products of Jacobians. The lower bound for the dimension of their deformations comes from the Torelli locus. Proposition 4.2 gives the upper bound for the deformations whose endpoints are still products.

Proof of Theorem 1.1. We first suppose that there are smooth superspecial curves C_1, C_2 over k of genera $(s,t) = (2,3)$ or $(3,3)$, and that p is odd. Set $g = s+t$. The point $x = \mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2)$, with the product polarization, belongs to $\mathcal{T}_g$. Indeed, it is the polarized Jacobian of the stable curve obtained by identifying a point on C_1 with a point on C_2. Each factor of x is indecomposable. By Lemma 3.4 we may choose a rigid polarized flag y above x to which Proposition 4.2 applies. Fix its polarized source, and write $e_\eta \colon \mathcal{P}'_\eta \to \mathcal{A}_g$ for the endpoint map.

Let Z be an irreducible component of $e_\eta^{-1}(\mathcal{T}_g)$ through y, with its reduced structure. Since every endpoint is supersingular, the general point of Z gives the desired curve as soon as its polarized endpoint is indecomposable. We begin by bounding $\dim Z$ from below. Consider the map $\mathcal{P}'_\eta \times \mathcal{T}_g \to \mathcal{A}_g \times \mathcal{A}_g$ given by e_η and the inclusion of $\mathcal{T}_g$. The inverse image of the diagonal is the fibre product $\mathcal{P}'_\eta \times_{\mathcal{A}_g} \mathcal{T}_g$. Since $\mathcal{A}_g$ is smooth, its diagonal is locally cut out by $\dim \mathcal{A}_g$ equations. One may make this calculation on schemes by choosing a full symplectic level $N \geq 3$ prime to p and a lift of y and x to that level. This finite étale change on the moduli stacks preserves local dimensions. The intersection dimension theorem and Proposition 2.1 now give

$$\dim Z \;\geq\; \left\lfloor \frac{g^2}{4} \right\rfloor + (3g-3) - \frac{g(g+1)}{2}.$$

For $g = 5$ the right side is $6 + 12 - 15 = 3$, and for $g = 6$ it is $9 + 15 - 21 = 3$.

Suppose that the geometric generic endpoint of Z is decomposable. We apply Proposition 4.2 to a family whose generic point is the generic point of Z and whose special point maps to y. For this, take the normalized blow-up of Z at y. The inverse image of y on the blow-up is a nonempty effective Cartier divisor. The local ring at the generic point of a component of its inverse image on the normalization is a discrete valuation ring with fraction field $k(Z)$ and centre y. Pulling back the universal flag gives the required family. Its geometric special fibre is the

chosen flag after extension of the residue field. Proposition 4.2 shows that the two generic factors have dimensions s, t and that each has a-number at least two.

For $j = 2, 3$, denote by B_j the closed subset of $\mathcal{S}_j$ where the a-number is at least two. The general member of each component of $\mathcal{S}_j$ has a-number one, by [9, Theorem 3.3]. Thus $\dim B_2 \leq 0$ and $\dim B_3 \leq 1$, since $\dim \mathcal{S}_2 = 1$ and $\dim \mathcal{S}_3 = 2$. The geometric generic endpoint considered above lies in the image of $B_s \times B_t$ under the product map. Its closure therefore has dimension at most $\dim B_s + \dim B_t$. The map e_η has finite geometric fibres, so the same upper bound holds for $\dim Z$. It is at most one when $g = 5$ and at most two when $g = 6$. Both contradict $\dim Z \geq 3$.

The geometric generic endpoint is therefore indecomposable. Consequently a dense open part of Z maps to smooth supersingular Jacobians. Its image has dimension at least three, by quasi-finiteness of e_η. Since the Torelli map on smooth curves has finite geometric fibres, the supersingular locus in $\mathcal{M}_g$ has a component of dimension at least three.

It remains to check the characteristics in which we can choose C_1 and C_2, and to treat the others. A smooth superspecial curve of genus two exists for every $p \geq 5$ [8, Proposition 3.2 and Theorem 3.3], and a smooth superspecial curve of genus three exists for every odd prime [7, Theorem 1]. This proves the assertion in genus five for $p \geq 5$ and in genus six for every odd p. We may take $C_1 = C_2$ in genus six. In genus five and characteristic three, Booher and Pries give a smooth supersingular curve [2, Theorem 1.2]. In characteristic two, van der Geer and van der Vlugt construct smooth supersingular curves of every positive genus [21, Theorem 2.1]. At each of these smooth Jacobians, apply the intersection dimension theorem directly to $\mathcal{T}_g \cap \mathcal{S}_g$ in $\mathcal{A}_g$. The lower bound is again three. Smooth Jacobians form an open subset of $\mathcal{T}_g$, so this also proves the dimension assertion in the remaining characteristics. □

6. Newton polygon amalgamation in genus at most five

We turn to Theorem 1.2. For $d = 3, 4$, let ν_d be the polygon whose slopes are $1/d$ and $(d-1)/d$, each with multiplicity d. We first consider the genus-five polygon $\xi = \nu_3 \oplus \mathrm{ss}^2$. Its slopes are $1/3, 1/2, 2/3$, with multiplicities $3, 4, 3$, respectively. Pries explains why a dimension estimate over the entire boundary leaves this case open [17, Example 4.6(3)]. We shall make the estimate at a product of Jacobians of genera two and three.

Lemma 6.1. *In every prime characteristic, there exists a smooth curve of genus five with Newton polygon $\nu_3 \oplus \mathrm{ss}^2$.*

Proof. Choose a smooth genus-two curve C_2 of p-rank zero and a smooth genus-three curve C_3 with polygon ν_3. The first exists by [1, Theorem 4.1], and the second by [1, Lemma 5.1]. The polygon of C_2 is ss^2. Indeed, a slope a/b strictly between 0 and $1/2$, in lowest terms, occurs at least b times and so does its complement $1 - a/b$. This requires at least six slopes, whereas C_2 has only four. Glue the two curves at one point and let x be the point of $\mathcal{A}_5$ represented by the polarized product $\mathrm{Jac}(C_2) \times \mathrm{Jac}(C_3)$. Then $x \in \mathcal{T}_5 \cap \mathcal{A}_5[\xi]$, where $\mathcal{A}_5[\xi]$ denotes the stratum with polygon exactly ξ.

Write $\xi(j)$ for the height of the polygon at the integer j. The codimension of the Newton stratum is $\sum_{j=1}^{5} \lceil \xi(j) \rceil$ [17, Section 2.4]. Here the five heights are $1/3, 2/3, 1, 3/2, 2$, and their ceilings sum to seven. Thus the Newton stratum has dimension $15 - 7 = 8$. Since $\mathcal{T}_5$ has dimension 12 and $\mathcal{A}_5$ is smooth of dimension 15, every component Z of the intersection through x has dimension at least $12 + 8 - 15 = 5$.

Suppose that the geometric generic point of Z is decomposable. As in the proof of Theorem 1.1, pass to full level prime to p and take a discrete valuation ring with fraction field $k(Z)$ and centre x. The product at x has two indecomposable factors of dimensions two and three. Lemma 4.1 therefore shows that the generic product also has two indecomposable factors of these dimensions. In particular a product of dimensions one and four cannot specialize to x.

Each of the two generic factors is the Jacobian of a smooth curve. Indeed, the geometric generic point of Z is in $\mathcal{T}_5$, so it is the Jacobian of a stable curve of compact type. Its indecomposable polarized factors are the Jacobians of the components of positive genus. Both have p-rank zero since their product has p-rank zero. The loci of smooth curves of p-rank zero in genera two and three have dimensions $2 \cdot 2 - 3 = 1$ and $2 \cdot 3 - 3 = 3$, respectively [1, Theorem 4.1]. Taking their polarized products gives a locus of dimension at most four. The points of attachment contribute no parameters to the Jacobian. Our assumption would therefore give $\dim Z \leq 4$, contrary to the lower bound of five. The general point of Z is consequently indecomposable and hence is the Jacobian of a smooth curve with polygon ξ. □

Proof of Theorem 1.2. Let $g = g_1 + g_2$, let ζ be the prescribed sum, and write f for its p-rank. If $g - f \leq 2$, the polygon is determined by f. In fact, after removing the f zero slopes and the f unit slopes, at most four slopes remain. The denominator argument above shows that all are $1/2$. Every p-rank occurs on a smooth genus-g curve [1, Theorem 4.1]. This settles genus two and every positive p-rank in genus three. In genus three a sum of p-rank zero comes from an elliptic curve and a genus-two curve of p-rank zero, so the required polygon is ss^3. A smooth supersingular curve of genus three exists in every characteristic [17, proof of Theorem 4.3(1)].

In genus four every symmetric Newton polygon occurs on a smooth curve, and in genus five the same holds for every polygon of positive p-rank [17, Corollaries 4.4 and 4.5]. It remains to consider genus five and $f = 0$. If ζ has a slope a/b strictly between 0 and $1/2$, the summand containing it also contains its complement. These slopes together account for at least b dimensions in that summand. Each summand has dimension at most four, so b is either three or four and $a = 1$. The pair occurs with exactly b copies of each slope, and all remaining slopes are $1/2$. We are thus left with the three possibilities $\nu_4 \oplus \mathrm{ss}$, $\nu_3 \oplus \mathrm{ss}^2$ and ss^5. The first occurs by [17, Corollary 4.1(2)], the second by Lemma 6.1, and the third by Theorem 1.1. □

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