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# A two-pole test of the Liu (2018) conjecture on the norm of the Bergman projection

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**Abstract**

Liu conjectured that the norm of the Bergman projection $P$ on $L^p$ of the upper half-plane equals $\Gamma(2/p)\Gamma(2/p')$, where $1/p + 1/p' = 1$. Near $p = 2$ the conjecture implies the inequality $K(F) = \|(I - P)(F \log |F|)\|^2/\|F\|^2 \le \pi^2/12$ for every zero-free $F$ in the Bergman space $A^2$ satisfying the integrability and pointwise domination hypotheses of Proposition 3.1. Let $S(z) = -\sum_j a_j/(z - w_j)$ be a finite sum of simple poles with $a_j > 0$ and $\operatorname{Im} w_j < 0$ (such an $S$ maps the upper half-plane into itself). Then $K(S^{1+\varepsilon})$ tends to $\pi^2/12$ as $\varepsilon \to 0^+$, with a slope $\sigma(S)$ at $\varepsilon = 0$. For one pole, $\sigma = \pi^2/6 - \frac{3}{2}\zeta(3) < 0$. We prove that $\sigma(S) < 0$ for every $S$ with two poles, so for each such $S$, $K(S^{1+\varepsilon}) < \pi^2/12$ for all sufficiently small $\varepsilon > 0$ (depending on $S$). The proof is computer-assisted. The slope has an explicit finite-part integral formula. Its negativity is proved analytically when the poles are deep compared with their distance apart, and when the masses are very unequal. All other cases are reached by integrating the depth derivative of the finite-part integral down from infinite depth: for depths above a small threshold this derivative is bounded by interval arithmetic, while explicit estimates cover smaller depths. Liu's conjecture itself remains open. The two-pole results (Theorems 1.1 and 3.5) use the computer-checked bounds of Proposition 6.1. These results are verified in Lean.[1]

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## 1 Introduction

We show that for every two-pole Pick function $S(z) = -a_1/(z - w_1) - a_2/(z - w_2)$, with $a_1, a_2 > 0$ and $\operatorname{Im} w_1, \operatorname{Im} w_2 < 0$, there is $\varepsilon_0(S) > 0$ such that $F = S^{1+\varepsilon}$ satisfies (1) below with strict inequality for $0 < \varepsilon < \varepsilon_0(S)$. If Liu's conjecture on the Bergman projection holds for all $p$ near 2, then the inequality (1) of Section 1.2 holds for every $F$ that meets the hypotheses of Proposition 3.1. So for $0 < \varepsilon < \varepsilon_0(S)$ the functions $S^{1+\varepsilon}$ cannot disprove the conjecture by way of Proposition 3.1.

### 1.1 Liu's value and the known bounds

The Bergman projection $P$ of the upper half-plane $\mathbb{U}$ is the orthogonal projection of $L^2(\mathbb{U})$ onto the analytic functions in $L^2(\mathbb{U})$; it extends to a bounded operator on $L^p(\mathbb{U})$ for every $1 < p < \infty$. Its norm there is not known for any $p \ne 2$. Liu proved the two-sided estimate (the case $n = 0$ of [8, Theorem 1], which is stated for the Siegel upper half-space in $\mathbb{C}^{n+1}$; for $n = 0$ it is $\mathbb{U}$)

$$\Gamma(2/p)\,\Gamma(2/p') \;\le\; \|P\|_{L^p \to L^p} \;\le\; \frac{\pi}{\sin(\pi/p)}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

[1] Proved in Lean 4 (three standard axioms, 168-box bounds as one hypothesis), checked with `lake comparator`: https://github.com/david44hello/paper_liu-2018-conjecture-two-pole-test/tree/v2.04/lean

and conjectured that the lower bound is the norm [8]. On the unit disc, Dostanić had conjectured $\|P\|_{L^p\to L^p} = \csc(\pi/p)$ [1]; Liu's lower bound $\Gamma(2/p)\Gamma(2/p')$ on the disc [7] disproved this, and Liu conjectured that this lower bound is the norm on the disc [7] (see also [9]). This is the Bergman-space counterpart of the Gohberg–Krupnik problem for the Riesz projection onto the Hardy space, whose exact $L^p$ norm $\csc(\pi/p)$ was proved by Hollenbeck and Verbitsky [2]. For weighted Bergman projections on the disc, the lower bound of [9] converges to $\csc(\pi/p)$ in the Hardy-space limit $\alpha \to -1$, lending support to Liu's value; however, as noted in [9, Section 5], the plurisubharmonicity argument of [2] does not carry over because $(I-P)f$ is in general not anti-analytic and too irregular. Testing whether multi-pole perturbations can violate the second-order consequence (1) near $p = 2$ therefore offers a concrete check of Liu's conjecture where no proof or counterexample is known.

### 1.2 Near $p = 2$: the quotient $K$

At $p = 2$ the norm is 1. Write $p = 2 + \delta$. Liu's value is $1 + \pi^2\delta^2/24 + O(\delta^3)$, so its $p$-th power is $1 + k_0\delta^2 + O(\delta^3)$ with $k_0 = \pi^2/12$. Below, $\|\cdot\|$ is the norm of $L^2(\mathbb{U}, dx\,dy/\pi)$. Let $F$ be zero-free and analytic in $\mathbb{U}$, with $F$ and $F\log|F|$ in $L^2(\mathbb{U})$, and put $G = (I-P)(F\log|F|)$. Since $P(F-\delta G) = F$, the ratio $\|F\|_p^p/\|F-\delta G\|_p^p$ is at most $\|P\|_{L^p\to L^p}^p$, and its expansion to second order in $\delta$ shows (Proposition 3.1): if Liu's conjecture holds for all $p$ near 2, then

$$K(F) = \frac{\|(I-P)(F\log|F|)\|^2}{\|F\|^2} \;\le\; k_0 = \frac{\pi^2}{12} \tag{1}$$

for every zero-free $F$ that satisfies the two conditions of Proposition 3.1. So one such $F$ with $K(F) > \pi^2/12$ would disprove Liu's conjecture for all $p \neq 2$ close to 2.

### 1.3 The level $\pi^2/12$ and the critical family

For a zero-free $F$ with $F\log F \in A^2$ one has $K(F) = \|(I-P)(F\arg F)\|^2/\|F\|^2$ (Section 2.1). The function $F = -1/z$ is not in $L^2(\mathbb{U})$, and neither is $F\arg F$, so $(I-P)(F\arg F)$ is not defined. Subtracting the analytic function $\frac{\pi}{2}F$ would not change $(I-P)(F\arg F)$ if both were in $L^2$; for $F = -1/z$ this suggests $v = F(\arg F - \pi/2) = -(\pi/2-\theta)/z$ ($z = re^{i\theta}$). This $v$ has the same $\bar\partial$-derivative as $F\arg F$, and it is the far-field form of the function $v_S$ of Section 2.3 (Lemma 2.3). It satisfies $\int|v|^2\,dA = \frac{\pi^2}{12}\int|F|^2\,dA$ on every half-annulus $\{r_1 < |z| < r_2\}$, since $|F|^2 = |z|^{-2}$ and $\int_0^\pi(\pi/2-\theta)^2\,d\theta = \pi^3/12$; for $-1/z$ this ratio is $\pi^2/12$. To construct square-integrable families with this limiting decay, let

$$S(z) = -\sum_{j=1}^{n}\frac{a_j}{z-w_j}, \qquad a_j > 0, \quad \operatorname{Im} w_j < 0,$$

a finite sum of simple poles in the lower half-plane with positive masses $a_j$. Such an $S$ maps $\mathbb{U}$ into $\mathbb{U}$; functions with this property are called Pick (or Nevanlinna) functions. Far away, $S(z) \approx -(a_1+\cdots+a_n)/z$. For $\varepsilon > 0$ the power $S^{1+\varepsilon}$ lies in $L^2(\mathbb{U})$, and $K(S^{1+\varepsilon}) \to \pi^2/12$ as $\varepsilon \to 0^+$. The question is whether $K(S^{1+\varepsilon})$ approaches $\pi^2/12$ from below or from above. We show (Proposition 3.3) that

$$K\big(S^{1+\varepsilon}\big) = \frac{\pi^2}{12} + \sigma(S)\,\varepsilon + o(\varepsilon) \qquad (\varepsilon \to 0^+),$$

where the slope $\sigma(S)$ is given by an explicit finite-part integral. If $\sigma(S) > 0$ for one $S$, then $K(S^{1+\varepsilon}) > \pi^2/12$ for small $\varepsilon$, and Liu's conjecture fails for $p$ near 2 (Proposition 3.1 and Lemma 3.2). For one pole the slope is $\pi^2/6 - \frac{3}{2}\zeta(3) = -0.1581\ldots$ (Corollary 3.4).

## 1.4 The result

**Theorem 1.1.** *Let* $S(z) = -a_1/(z-w_1) - a_2/(z-w_2)$ *with* $a_1, a_2 > 0$ *and* $\operatorname{Im} w_1, \operatorname{Im} w_2 < 0$. *Then* $\sigma(S) < 0$. *Consequently there is* $\varepsilon_0 = \varepsilon_0(S) > 0$ *such that*

$$K\big(S^{1+\varepsilon}\big) < \frac{\pi^2}{12} \qquad \text{for } 0 < \varepsilon < \varepsilon_0.$$

The proof is computer-assisted (Section 6). Table 1 shows what was known about this question before this paper. Theorem 1.1 does not change the known bounds for $\|P\|_{L^p \to L^p}$.

| Best result before this paper | New result |
|---|---|
| *Can $K(S^{1+\varepsilon})$, with $S$ a sum of poles as in Section 1.3, exceed $\pi^2/12$ for small $\varepsilon > 0$? Such a function would disprove Liu's conjecture for $p$ near 2 (Proposition 3.1 and Lemma 3.2).* | |
| Not known. [Liu's lower bound is the limit of the quotients $\|Pf\|_p/\|f\|_p$ of test functions built from one pole [8, proof of Theorem 1; $p \geq 2$].] | No, for two poles: $K(S^{1+\varepsilon}) < \pi^2/12$ for $0 < \varepsilon < \varepsilon_0(S)$, for every two-pole Pick function $S$ (Theorem 1.1; computer-assisted). Three or more poles: open. |

Table 1: The question of Theorem 1.1: what was known before this paper, and the new result.

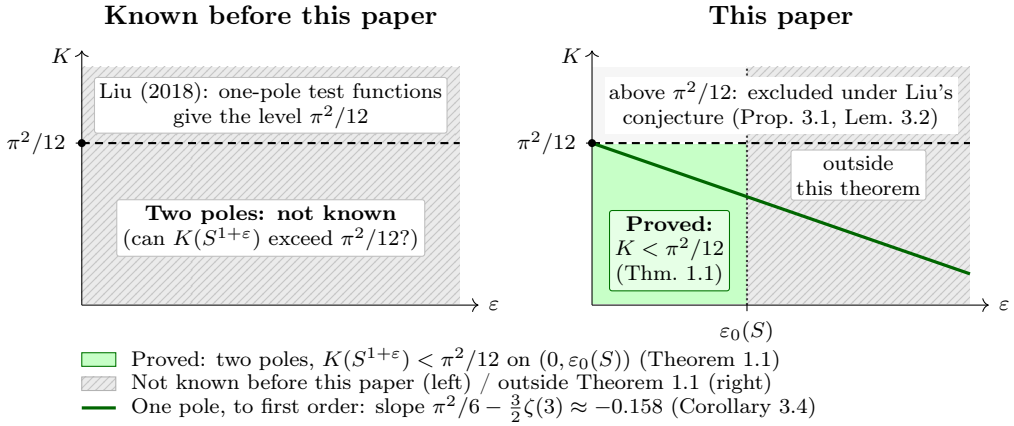


Figure 1: Green: proved in this paper ($K(S^{1+\varepsilon}) < \pi^2/12$ for every two-pole $S$ and $0 < \varepsilon < \varepsilon_0(S)$, Theorem 1.1; $\varepsilon_0(S)$ is schematic and not explicit). Grey (hatched): not known before this paper (left) or outside Theorem 1.1 (right). Along $S^{1+\varepsilon}$, $K \to \pi^2/12$ as $\varepsilon \to 0^+$; Liu's conjecture for $p$ near 2 excludes values above $\pi^2/12$ (Proposition 3.1 and Lemma 3.2). Solid line (right): first-order approximation for one pole (Corollary 3.4). Shaded regions mark what is proved about the question, not attainable values of $K$.

## 1.5 Idea of the proof

The slope is $\sigma(S) = -2\Phi(S)$, where the margin $\Phi(S) = \mathcal{D}(S)/m^2 - \pi^2/12$ is built from a finite-part integral $\mathcal{D}(S)$, the deficit (Section 2), and $m = a_1 + a_2$. So Theorem 1.1 says $\Phi(S) > 0$. Three tools bound it from below.

(a) A *witness identity* (Lemma 4.3): for every witness $W$ (an analytic function on $\mathbb{U}$ that tends to $\pi/2$ at infinity at the rate of Definition 4.1), $\mathcal{D}(S)$ is at least an explicit integral built from $S$ and $W$, with equality for one optimal $W$. For one pole the optimal $W$ is elementary. With it we treat two poles close together compared with their depth (Lemma 4.4); a blend of the

two single-pole witnesses treats two poles of which one has at most $2^{-12}$ times the mass of the other, in any position (Lemma 4.5).

(b) A *flux identity* (Lemma 5.1): if both poles move down together, the derivative of the margin is an integral over the real line of elementary functions. When both poles are deep compared with their separation, $S$ is close to a single pole, and (a) gives a lower bound for the margin there that tends to the one-pole margin $\frac{3}{4}\zeta(3) - \pi^2/12 > 0$. So the margin at any depth is at least this number minus the integral of the flux (Lemma 5.3).

(c) A *computer certificate* (Section 6): after normalisation, a two-pole function has three parameters: the logarithm $s = \log(a_2/a_1)$ of the mass ratio, the angle $\gamma$ of the segment between the poles (Figure 2), and the depth $u$. After Lemma 4.5 it remains to treat $(s, \gamma) \in [-25/3, 25/3] \times [0, \pi/2]$, which is covered by 168 boxes. On each box, a computer program bounds the integral of the flux in interval arithmetic from infinity down to a depth $u_B$ between $e^{-52.5}$ and $e^{-17.5}$, and analytic estimates (Lemmas 5.4 and E.4) handle the rest.

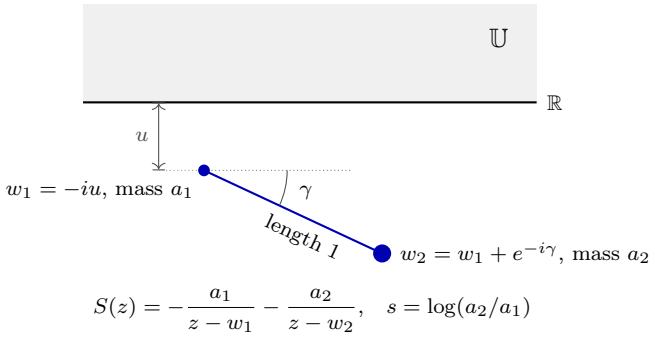


Figure 2: A two-pole Pick function after normalisation (8): the shallower pole at $w_1 = -iu$, the other pole at $w_2 = w_1 + e^{-i\gamma}$, with masses $a_1, a_2$ (dot sizes) and $s = \log(a_2/a_1)$.

Section 8 discusses what stays open: Pick sums with more poles, the inequality (1) for general $F$, and Liu's conjecture itself.

# 2 Setting

## 2.1 The Bergman projection and the quotient $K$

Let $\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane, with the normalised area measure $dA = dx\, dy/\pi$. We write $L^2 = L^2(\mathbb{U}, dA)$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and $A^2$ for the Bergman space of analytic functions in $L^2$. The Bergman projection $P$ is the orthogonal projection of $L^2$ onto $A^2$. We use $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$ and $\Delta = 4\partial\bar\partial$. For $h \in L^2$, the function $(I - P)h$ is the unique solution in $L^2$ of the equation $\bar\partial(\cdot) = \bar\partial h$ (in the sense of distributions) that is orthogonal to $A^2$; among all solutions in $L^2$ it has the least norm.

For $F \in A^2$, $F \not\equiv 0$, with $F \log|F| \in L^2$, put

$$K(F) = \frac{\|(I - P)(F \log|F|)\|^2}{\|F\|^2}. \tag{2}$$

The quotient does not change when $F$ is multiplied by a nonzero constant $c$, because $cF \log|cF|$ differs from $cF \log|F|$ by $c \log|c|\, F \in A^2$. The formula for $K$ in Section 1.3 is elementary: if $F \log F \in A^2$, then $F \log F = F \log|F| + iF \arg F$ is analytic, so $(I - P)(F \log|F|) = -i\,(I - P)(F \arg F)$.

## 2.2 Pick sums

**Definition 2.1.** A *Pick sum* is a function

$$S(z) = -\sum_{j=1}^{n} \frac{a_j}{z - w_j}, \qquad a_j > 0, \quad \operatorname{Im} w_j < 0.$$

Its *total mass* is $m = a_1 + \cdots + a_n$. For $n = 2$ we call $S$ a *two-pole Pick function.*

Since $\operatorname{Im}\bigl(-1/(z-w)\bigr) = \operatorname{Im}(z-w)/|z-w|^2$, we have

$$\operatorname{Im} S(z) = \sum_j a_j \, \frac{\operatorname{Im} z - \operatorname{Im} w_j}{|z - w_j|^2} > 0 \qquad (\operatorname{Im} z \geq 0).$$

So $S$ maps the closed half-plane $\overline{\mathbb{U}}$ into $\mathbb{U}$; in particular $S$ has no zero on $\overline{\mathbb{U}}$, and its argument $\arg S$ is a continuous function on $\overline{\mathbb{U}}$ with values in $(0, \pi)$. At infinity $S(z) = -m/z + O(|z|^{-2})$. For $\varepsilon > 0$ we put

$$g_\varepsilon = S^{1+\varepsilon} = \exp\bigl((1+\varepsilon)\log S\bigr), \qquad \operatorname{Im} \log S = \arg S \in (0, \pi).$$

Then $g_\varepsilon$ is bounded and analytic on $\mathbb{U}$ and $|g_\varepsilon(z)| \sim m^{1+\varepsilon}|z|^{-1-\varepsilon}$ at infinity, so $g_\varepsilon \in A^2$ and $g_\varepsilon \log|g_\varepsilon| \in L^2$; moreover $\|g_\varepsilon\|^2 = m^2/(2\varepsilon) + O(1)$ as $\varepsilon \to 0^+$, since $\int_{\mathbb{U}\cap\{|z|>1\}} |z|^{-2-2\varepsilon}\, dA = 1/(2\varepsilon)$. We call $\varepsilon \mapsto g_\varepsilon$ the *critical family* of $S$.

## 2.3 The function $v_S$ and the deficit

Let $S$ be a Pick sum and put $f_S = S \arg S$. Since $\arg S = \operatorname{Im} \log S$, a short computation gives

$$\bar\partial f_S = S\, \bar\partial \arg S = \frac{i}{2}\, \frac{S}{\overline{S}}\, \overline{S'} =: \rho_S, \qquad |\rho_S| = \tfrac{1}{2}|S'|. \tag{3}$$

The function $f_S$ is not in $L^2$: it decays only like $1/|z|$. But its $\bar\partial$-derivative $\rho_S$ is smooth on $\overline{\mathbb{U}}$ and $O(|z|^{-2})$ at infinity. Let $U_S$ be the bounded solution of

$$-\Delta U_S = 2\rho_S \ \text{ in } \mathbb{U}, \qquad U_S = 0 \ \text{ on } \mathbb{R},$$

and put $v_S = -2\,\partial U_S$. The bounded solution exists and is unique. The difference of two bounded solutions is a bounded harmonic function on $\mathbb{U}$ that vanishes on $\mathbb{R}$, hence 0. For existence, note that $|z - w_j| \geq |z - \operatorname{Re} w_j|$ on $\mathbb{U}$, so $|\rho_S| \leq \frac{1}{2}\sum_j a_j |z - \operatorname{Re} w_j|^{-2}$; and for real $t$ the bounded solution of $-\Delta U = |z-t|^{-2}$ with zero boundary values is $\frac{1}{2}\theta_t(\pi - \theta_t)$, $\theta_t = \arg(z-t) \in (0,\pi)$, which is at most $\pi^2/8$; so the Green potential of $2\rho_S$ converges absolutely and is bounded by $\frac{\pi^2}{8}\sum_j a_j$. Then $\bar\partial v_S = -\frac{1}{2}\Delta U_S = \rho_S = \bar\partial f_S$. Since $\rho_S$ is smooth on a neighbourhood of the closed half-plane, $U_S$ is smooth up to $\mathbb{R}$ (boundary Schauder estimates), so $v_S$ is continuous on $\overline{\mathbb{U}}$. For an analytic $F$ that is zero-free on $\overline{\mathbb{U}}$ and decays fast enough that $F \arg F \in L^2$, the same construction (with $F$ in place of $S$) gives exactly $(I - P)(F \arg F)$ (see Lemma B.1 for the precise statement); so $v_S$ plays the role of "$(I-P)(S \arg S)$" for $S$, although $S \arg S$ is not in $L^2$.

**Lemma 2.2** (Dirichlet estimates)**.** *Let $V$ be smooth on $\overline{\mathbb{U}}$ and zero on $\mathbb{R}$, and write $r = |z|$.*

(a) $|\nabla V(z)| \leq C_0\bigl(r^{-1}\sup|V| + r\sup|\Delta V|\bigr)$*, with the suprema over $\mathbb{U} \cap \{r/4 < |z'| < 2r\}$ and an absolute constant $C_0$.*

(b) *If $V$ is bounded and $|\Delta V| \leq C(1+|z|)^{-3}$, then, as $r \to \infty$, $V = O((1+\log r)/r)$ and $\nabla V = O((1+\log r)/r^2)$, with constants that depend only on $C$.*

*Proof.* (a) If $\operatorname{Im} z \geq r/4$, the interior estimate on the disc of radius $r/8$ about $z$, which lies in $\mathbb{U} \cap \{r/4 < |z'| < 2r\}$, gives the bound. If $\operatorname{Im} z < r/4$, then $z$ lies in the half-disc of radius $r/4$ about

$\operatorname{Re} z$, and the half-disc of radius $r/2$ about $\operatorname{Re} z$ lies in $\{r/4 < |z'| < 2r\}$. Rescale it to unit size. The boundary $W^{2,3}$ estimate for zero Dirichlet data on the inner half-disc (any exponent above 2 will do), together with the embedding of $W^{2,3}$ into $C^1$ in two dimensions, gives the same bound.

(b) The Green potential of a source of size $O((1+|z'|)^{-3})$ on $\mathbb{U}$ is $O((1+\log r)/r)$. Indeed, the Green function is a constant multiple of $k(z,z') = \log\bigl(|z-\bar{z}'|/|z-z'|\bigr)$; split the source into $|z'| < r/2$, where $k(z,z') \le C \operatorname{Im} z'/r$ and the part contributes $O((1+\log r)/r)$; $r/2 \le |z'| \le 2r$, where $\int k(z,z')\,dA(z') = O(r^2)$ over the region and the source is $O(r^{-3})$; and $|z'| > 2r$, where $k(z,z') \le Cr/|z'|$ and the part is $O(1/r)$. This potential is bounded and zero on $\mathbb{R}$, like $V$; so $V$ is the Green potential of $-\Delta V$, by the uniqueness of the bounded solution. Then (a) gives $\nabla V = O\bigl(r^{-1}(1+\log r)/r + r\cdot r^{-3}\bigr) = O((1+\log r)/r^2)$. □

**Lemma 2.3** (Far field)**.** *Let $S$ be a Pick sum of total mass $m$, write $z = re^{i\theta}$, and put $U_0 = \frac{im}{2}\,\theta(\pi-\theta)$. As $r\to\infty$, uniformly in $\theta\in[0,\pi]$,*

$$U_S = U_0 + O\Bigl(\frac{1+\log r}{r}\Bigr), \qquad \nabla(U_S - U_0) = O\Bigl(\frac{1+\log r}{r^2}\Bigr), \qquad v_S = -m\,\frac{\pi/2-\theta}{z} + O\Bigl(\frac{1+\log r}{r^2}\Bigr).$$

*Proof.* From $S = -m/z + O(r^{-2})$ and $S' = m/z^2 + O(r^{-3})$ we get $S/\overline{S} = e^{-2i\theta}(1+O(r^{-1}))$ and hence $\rho_S = \frac{im}{2}r^{-2} + O(r^{-3})$. In polar coordinates $\Delta\bigl(\theta(\pi-\theta)\bigr) = -2r^{-2}$, so $U_0$ solves $-\Delta U_0 = imr^{-2}$ away from the origin and vanishes on $\mathbb{R}$. Let $U_1 = U_S - U_0$, with $U_0$ cut off smoothly near the origin. Then $U_1$ is smooth on $\overline{\mathbb{U}}$, bounded and zero on $\mathbb{R}$, and $|\Delta U_1| \le C(1+|z|)^{-3}$; so Lemma 2.2(b) gives $U_1 = O((1+\log r)/r)$ and $\nabla U_1 = O((1+\log r)/r^2)$. These are the first two bounds. Finally $\partial\theta = 1/(2iz)$, so $-2\partial U_0 = -m(\pi/2-\theta)/z$, and $v_S = -2\partial U_S$ gives the third. □

**Definition 2.4** (Deficit and margin)**.** For a Pick sum $S$ of total mass $m$, the *deficit* of $S$ is

$$\mathcal{D}(S) = \lim_{R\to\infty} \int_{\mathbb{U}\cap\{|z|<R\}} \bigl(k_0|S|^2 - |v_S|^2\bigr)\,dA, \tag{4}$$

and its *margin* is $\Phi(S) = \mathcal{D}(S)/m^2 - k_0$.

**Lemma 2.5.** *The limit* (4) *exists and is finite. The deficit is unchanged when $S(z)$ is replaced by $S(z-x_0)$ with $x_0\in\mathbb{R}$, by $\lambda^{-1}S(z/\lambda)$ with $\lambda>0$, or by $-\overline{S(-\bar{z})}$; and $\mathcal{D}(cS) = c^2\mathcal{D}(S)$ for $c>0$. In particular the margin depends only on the mass fractions $a_j/m$ and on the pole positions up to real translation, dilation and the reflection $w\mapsto -\bar{w}$.*

*Proof.* By Lemma 2.3 the integrand equals $m^2r^{-2}\bigl(k_0 - (\pi/2-\theta)^2\bigr) + O\bigl((1+\log r)r^{-3}\bigr)$ for large $r$. With $dA = r\,dr\,d\theta/\pi$, the main term integrates over the half-annulus $\{R_1 < r < R_2\}$ to $\pi^{-1}m^2\log(R_2/R_1)\int_0^\pi \bigl(k_0 - (\pi/2-\theta)^2\bigr)d\theta$, and the last integral is $\pi k_0 - \pi^3/12 = 0$. The remainder is integrable. So the integrals over $\{|z|<R\}$ form a Cauchy family as $R\to\infty$. The three maps send $S$ to Pick sums with the same masses (the poles move to $w_j + x_0$, $\lambda w_j$ and $-\bar{w}_j$, respectively), and they carry $v_S$ to the corresponding function by uniqueness of the bounded solution; and $v_{cS} = cv_S$. Dilation and reflection preserve $|S|^2dA$ and the half-discs centred at 0. A translation moves the centre of the half-disc; the symmetric difference of the two half-discs of radius $R$ has area $O(R)$, and the integrand is $O(R^{-2})$ there, so the change tends to 0. □

# 3 From the slope to the deficit

## 3.1 Liu's conjecture near $p = 2$

For $1 < p < \infty$ let $p' = p/(p-1)$. Liu conjectured [8] that the norm of $P$ on $L^p(\mathbb{U}, dA)$ equals

$$\lambda_p = \Gamma(2/p)\,\Gamma(2/p') = \frac{\pi x}{\sin \pi x}, \qquad x = \Bigl|1 - \frac{2}{p}\Bigr|; \tag{5}$$

the second form follows, with $y = 2/p$, from $\Gamma(2-y) = (1-y)\Gamma(1-y)$ and $\Gamma(y)\Gamma(1-y) = \pi/\sin \pi y$, since $t \mapsto \pi t/\sin \pi t$ is even. Write $p = 2+\delta$. Then $x = |\delta|/(2+\delta)$ (as $p > 0$), so $\lambda_p = 1 + \pi^2\delta^2/24 + O(\delta^3)$ and

$$\lambda_p^p = 1 + k_0\,\delta^2 + O(\delta^3), \qquad k_0 = \frac{\pi^2}{12}. \tag{6}$$

The next proposition turns (6) into an inequality at $p = 2$. Its proof is in Appendix B.

**Proposition 3.1.** *Let $F \in A^2$ have no zeros in $\mathbb{U}$, put $L = \big|\log|F|\big|$, and assume that for some $\eta_0 > 0$*

$$\int_{\mathbb{U}} |F|^p (1+L)^5\, dA < \infty \quad \text{for } |p-2| \le \eta_0.$$

*Put $G = (I-P)(F\log|F|)$, and assume also that $|G| \le C\,|F|\,(1+L)$ on $\mathbb{U}$ for some $C > 0$. If $\|P\|_{L^{p_n}\to L^{p_n}} \le \lambda_{p_n}$ for a sequence $p_n \to 2$, $p_n \ne 2$, then $K(F) \le k_0$. Equivalently: if $K(F) > k_0$, then $\|P\|_{L^p\to L^p} > \lambda_p$ for every $p \ne 2$ close enough to 2.*

The function $G$ is defined, since the first hypothesis at $p = 2$ gives $F\log|F| \in L^2$: $|F\log|F|| \le |F|(1+L)$.

**Lemma 3.2.** *Let $S$ be a Pick sum and $0 < \varepsilon < 1$. Then $F = S^{1+\varepsilon}$ satisfies the hypotheses of Proposition 3.1, for every $0 < \eta_0 < 2\varepsilon/(1+\varepsilon)$.*

*Proof.* For $\operatorname{Im} z \ge 0$ we have $\operatorname{Im} S(z) > 0$ (Section 2), so $S$ has no zero on $\overline{\mathbb{U}}$, and $zS(z) \to -m$ as $z \to \infty$. Since $S$ is analytic near $\overline{\mathbb{U}}$ and maps $\overline{\mathbb{U}}$ into $\mathbb{U}$, $F$ is analytic near $\overline{\mathbb{U}}$. Hence $c(1+|z|)^{-1-\varepsilon} \le |F| \le C(1+|z|)^{-1-\varepsilon}$ on $\overline{\mathbb{U}}$, and $L \le C(1+\log(1+|z|))$; in particular $F \in A^2$ and $F\log|F| \in L^2$. The integral $\int_{\mathbb{U}} |F|^p(1+L)^5\,dA$ converges when $p(1+\varepsilon) > 2$, that is, for $|p-2| \le \eta_0$ with $\eta_0 < 2\varepsilon/(1+\varepsilon)$. Next, the source $\rho_\varepsilon = \bar\partial(F\log|F|) = \frac12(F/\overline{F})\,\overline{F'}$ is smooth on $\overline{\mathbb{U}}$ and satisfies $|\rho_\varepsilon| = \frac12|F'| \le C(1+|z|)^{-2-\varepsilon}$. For such a source $\rho$ let $\mathcal{C}\rho = -2\,\partial V$, where $V$ is the bounded solution of $-\Delta V = 2\rho$ in $\mathbb{U}$ with $V = 0$ on $\mathbb{R}$ (Appendix B). By Lemma B.1, $G = \mathcal{C}\rho_\varepsilon$ is continuous on $\overline{\mathbb{U}}$ and $O(|z|^{-1-\varepsilon})$, so $|G| \le C(1+|z|)^{-1-\varepsilon}$. With the lower bound on $|F|$, $|G| \le C'|F| \le C'|F|(1+L)$. □

### 3.2 The slope along the critical family

A Pick sum $S$ behaves like $-m/z$ far away, and for $-m/z$ the ratio of $\int|v|^2$ to $\int|F|^2$ over every half-annulus is $k_0$, as in Section 1.3. The following proposition says how $K$ leaves the level $k_0$ along the family; its proof is in Appendix B.

**Proposition 3.3** (Slope formula)**.** *Let $S$ be a Pick sum of total mass $m$. Then, as $\varepsilon \to 0^+$,*

$$K\big(S^{1+\varepsilon}\big) = k_0 + \sigma(S)\,\varepsilon + o(\varepsilon), \qquad \sigma(S) = 2\Big(k_0 - \frac{\mathcal{D}(S)}{m^2}\Big) = -2\,\Phi(S).$$

So $K(S^{1+\varepsilon})$ falls below $k_0$ for all small $\varepsilon > 0$ when the margin $\Phi(S)$ is positive, and rises above $k_0$ when $\Phi(S) < 0$. For one pole the margin is explicit.

**Corollary 3.4** (One pole)**.** *For $S(z) = -a/(z-w)$ with $a > 0$ and $\operatorname{Im} w < 0$,*

$$\mathcal{D}(S) = k_1\,a^2, \qquad k_1 = \tfrac34\zeta(3) = 0.9015426\ldots, \qquad \Phi(S) = k_1 - k_0 = 0.0790756\ldots > 0.$$

*Proof.* By Lemma 2.5 the margin is the same for every single pole, so we may take $S(z) = -1/(z+i)$. Then $S^{1+\varepsilon} = e^{i\pi(1+\varepsilon)}(z+i)^{-1-\varepsilon}$, and $K$ ignores the constant factor. Lemma A.2 in Appendix A gives $K\big((z+i)^{-1-\varepsilon}\big) = k_0 + \big(\pi^2/6 - \frac32\zeta(3)\big)\varepsilon + O(\varepsilon^2)$. Comparing with Proposition 3.3, $-2\Phi(S) = \pi^2/6 - \frac32\zeta(3)$, that is $\Phi(S) = \frac34\zeta(3) - \pi^2/12$. □

### 3.3 The theorem in terms of the deficit

By Proposition 3.3, Theorem 1.1 is equivalent to the following statement, which we prove in the rest of the paper.

**Theorem 3.5.** *Every two-pole Pick function $S$ of total mass $m$ satisfies $\mathcal{D}(S) > k_0\, m^2$, that is, $\Phi(S) > 0$.*

## 4 The witness identity

The deficit is defined through the solution $v_S$ of a boundary value problem. The identity below bounds $\mathcal{D}(S)$ from below by an explicit integral, in which a freely chosen analytic function $W$ replaces the unknown part. The identity is the minimum-norm property of $I - P$, with the divergent norms replaced by the finite part (4).

Write $D_R = \mathbb{U} \cap \{|z| < R\}$ and $z = re^{i\theta}$.

**Definition 4.1.** A *witness* is a function $W$ that is analytic on $\mathbb{U}$, continuous on $\overline{\mathbb{U}}$, and satisfies $W(z) = \pi/2 + O\big((1+\log r)/r\big)$ as $r \to \infty$, uniformly in $\theta$. For a Pick sum $S$ and a witness $W$ put

$$L_S[W] = \lim_{R\to\infty} \int_{D_R} \big(k_0|S|^2 - |S|^2\,|\arg S - W|^2\big)\, dA.$$

The next lemma says that $v_S$ is orthogonal to analytic functions that decay fast enough.

**Lemma 4.2** (Orthogonality)**.** *Let $S$ be a Pick sum, and let $g$ be analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$, with $g = O\big((1+\log r)/r^2\big)$ as $r \to \infty$, uniformly in $\theta$. Then $v_S\bar{g}$ is integrable on $\mathbb{U}$ and*

$$\int_{\mathbb{U}} v_S\, \bar{g}\, dA = 0.$$

*Proof.* By Lemma 2.3, $v_S = O(1/r)$, so $v_S\bar{g} = O((1+\log r)/r^3)$ is integrable on $\mathbb{U}$.

Since $\partial\bar{g} = \overline{\bar{\partial} g} = 0$, we have $v_S\bar{g} = -2\,\partial(U_S\bar{g})$ on $\mathbb{U}$. For $0 < \tau < R$, Green's formula on $D_{R,\tau} = D_R \cap \{\operatorname{Im} z > \tau\}$ gives $\int_{D_{R,\tau}} \partial(U_S\bar{g})\, dx\, dy = \frac{i}{2} \oint_{\partial D_{R,\tau}} U_S\bar{g}\, d\bar{z}$. Let $\tau \to 0^+$: on the segment $\{\operatorname{Im} z = \tau\}$ the function $g$ is bounded and $U_S \to 0$ uniformly, because $U_S$ is continuous on $\overline{\mathbb{U}}$ and vanishes on $\mathbb{R}$. What remains is the integral over the half-circle $|z| = R$, where $|U_S\bar{g}| = O((1+\log R)/R^2)$ ($U_S$ is bounded) along a path of length $\pi R$, so it is $O((1+\log R)/R)$. Since $v_S\bar{g}$ is integrable, letting $R \to \infty$ gives $\int_{\mathbb{U}} v_S\bar{g}\, dA = 0$. □

**Lemma 4.3** (Witness identity)**.** *Let $S$ be a Pick sum and put $W_S^* = (f_S - v_S)/S$. Then $W_S^*$ is a witness, and for every witness $W$ the limit $L_S[W]$ exists and*

$$L_S[W] = \mathcal{D}(S) - \|S\,(W - W_S^*)\|^2.$$

*In particular $\mathcal{D}(S) \geq L_S[W]$, with equality if and only if $W = W_S^*$.*

*Proof.* Since $\bar{\partial}(f_S - v_S) = 0$ and $S$ has no zeros on $\overline{\mathbb{U}}$, $W_S^*$ is analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$. By Lemma 2.3, with $\arg S = \pi - \theta + O(1/r)$,

$$f_S - v_S = -m\frac{\pi-\theta}{z} + m\frac{\pi/2-\theta}{z} + O\Big(\frac{1+\log r}{r^2}\Big) = -\frac{\pi}{2}\,\frac{m}{z} + O\Big(\frac{1+\log r}{r^2}\Big),$$

and dividing by $S = -m/z\,(1+O(1/r))$ gives $W_S^* = \pi/2 + O((1+\log r)/r)$. So $W_S^*$ is a witness.

Let $W$ be a witness and put $h = S(W_S^* - W)$. Then $h$ is analytic on $\mathbb{U}$, continuous on $\overline{\mathbb{U}}$ and $O((1+\log r)/r^2)$, so $h \in A^2$. Since $S(\arg S - W) = f_S - SW = v_S + h$,

$$|S|^2|\arg S - W|^2 = |v_S|^2 + |h|^2 + 2\operatorname{Re}\big(v_S\bar{h}\big).$$

By Lemma 4.2 (with $g = h$), the product $v_S\bar{h}$ is integrable on $\mathbb{U}$ and its integral is 0.

Now integrate over $D_R$ and let $R \to \infty$: the terms $k_0|S|^2 - |v_S|^2$ give $\mathcal{D}(S)$, the term $|h|^2$ gives $\|h\|^2$, and the cross term gives 0. □

The lower bounds on $\mathcal{D}(S)$ in this section and in Appendix C come from Lemma 4.3 with explicit witnesses. For one pole the optimal witness is elementary (Lemma A.3 in Appendix A): for $S(z) = -a/(z - w_0)$ with $w_0 = x_0 - iy_0$,

$$W_S^*(z) = \omega\Big(\frac{z - x_0}{y_0}\Big), \qquad \omega(z) = \frac{\pi}{2} + \frac{\operatorname{Log}(z+i) - \operatorname{Log}(2i)}{z - i}, \tag{7}$$

where Log is the principal logarithm (the singularity at $z = i$ is removable).

## 4.1 Poles close together

When the two poles are close compared with their depth, $S$ is close to the single pole at the mass centre, and the optimal witness of that single pole already gives the bound we need.

**Lemma 4.4** (Merged poles)**.** *Let $S$ be a two-pole Pick function with masses $a_1, a_2$, total mass $m$, mass fractions $\mu_j = a_j/m$, mass centre $w_c = \mu_1 w_1 + \mu_2 w_2$ of depth $u_c = -\operatorname{Im} w_c > 0$ and separation $d = |w_1 - w_2|$. If $d \le u_c/4$, then*

$$\mathcal{D}(S) \;\ge\; k_1\, m^2 - 16.2\, \mu_1\mu_2\, m^2\, \frac{d^2}{u_c(u_c - d)}.$$

*Proof.* Let $S_c(z) = -m/(z - w_c)$ and $\omega_c = W_{S_c}^*$, the optimal witness (7) of this single pole. It is a witness, so $\mathcal{D}(S) \ge L_S[\omega_c]$ by Lemma 4.3. Put $\tilde{z} = z - w_c$ and $e_j = w_j - w_c$, so that $\mu_1 e_1 + \mu_2 e_2 = 0$, $|e_1| = \mu_2 d$, $|e_2| = \mu_1 d$. Then

$$\frac{S}{S_c} = \sum_j \mu_j \frac{\tilde{z}}{\tilde{z} - e_j} = 1 + \eta, \qquad \eta = \sum_j \mu_j \frac{e_j}{\tilde{z} - e_j} = \sum_j \mu_j \frac{e_j^2}{\tilde{z}(\tilde{z} - e_j)},$$

where the second form uses $\sum_j \mu_j e_j = 0$. On $\mathbb{U}$ we have $|\tilde{z}| \ge u_c$ and $|\tilde{z} - e_j| \ge |\tilde{z}| - d \ge (1 - d/u_c)|\tilde{z}|$, so

$$|\eta| \le \frac{(\mu_1\mu_2^2 + \mu_2\mu_1^2)\, d^2}{(1 - d/u_c)|\tilde{z}|^2} = \frac{\mu_1\mu_2\, d^2}{(1 - d/u_c)|\tilde{z}|^2} \le \frac{\mu_1\mu_2\, d^2}{u_c(u_c - d)} \le \frac{1}{48}.$$

Since $|\eta| < 1$, $\arg S = \arg S_c + \arg(1 + \eta)$ and $|\arg(1 + \eta)| \le \arcsin|\eta| \le \frac{\pi}{2}|\eta|$. Put $\iota = \arg S_c - \omega_c$. A direct expansion gives

$$\begin{aligned} &|S|^2\big(k_0 - |\arg S - \omega_c|^2\big) - |S_c|^2\big(k_0 - |\iota|^2\big) \\ &\qquad = |S_c|^2\Big[(2\operatorname{Re}\eta + |\eta|^2)(k_0 - |\iota|^2) - |1 + \eta|^2\big(2\arg(1+\eta)\operatorname{Re}\iota + \arg(1+\eta)^2\big)\Big]. \end{aligned}$$

By Lemma A.3, $|k_0 - |\iota|^2| \le 4.304$ and $|\operatorname{Re}\iota| \le \pi/2 + \log 2 < 2.264$. For $|\eta| \le 1/48$ the absolute value of the bracket is therefore at most $|\eta|\big[8.608 + 4.304|\eta| + (1 + |\eta|)^2(7.113 + 2.468|\eta|)\big] \le 16.17\,|\eta|$. The single-pole part integrates to $L_{S_c}[\omega_c] = \mathcal{D}(S_c) = k_1 m^2$ (equality case of Lemma 4.3 and Corollary 3.4). The difference is absolutely integrable, and $\mathbb{U} \subset \{|\tilde{z}| \ge u_c\}$, so

$$\big|L_S[\omega_c] - k_1 m^2\big| \le 16.17\, \frac{\mu_1\mu_2\, m^2 d^2}{1 - d/u_c} \int_{|\tilde{z}| \ge u_c} |\tilde{z}|^{-4}\, dA = 16.17\, \mu_1\mu_2\, m^2\, \frac{d^2}{u_c(u_c - d)},$$

since $\int_{|\tilde{z}| \ge u_c} |\tilde{z}|^{-4}\, dx\, dy/\pi = u_c^{-2}$. With $\mathcal{D}(S) \ge L_S[\omega_c]$ and $16.17 < 16.2$, this gives the lemma. □

### 4.2 One mass much smaller than the other

Lemma 4.5 below puts no condition on the positions of the poles. Here a blended witness works: it follows each pole's own optimal witness near that pole. The proof, with explicit constants, is in Appendix C.

**Lemma 4.5** (Small mass ratio)**.** *Let $S$ be a two-pole Pick function, and let $m_1 \geq m_2$ be its two masses (the masses $a_1, a_2$ in decreasing order), so that $m = m_1 + m_2$. If $m_2 \leq 2^{-12} m_1$, then for every position of the two poles*

$$\mathcal{D}(S) - k_0\, m^2 \;>\; 0.038\, m_1^2;$$

*in particular $\Phi(S) > 0$.*

## 5 Normalisation and the depth flux

### 5.1 Three coordinates

By Lemma 2.5 the margin of a two-pole Pick function does not change under real translation, dilation, the reflection $w \mapsto -\bar{w}$ of both poles, and multiplication of both masses by the same positive number; it also does not depend on how the two poles are labelled. If $w_1 = w_2$, then $S$ is a single pole and Corollary 3.4 applies. Otherwise label the poles so that $w_1$ is the shallower one ($\operatorname{Im} w_1 \geq \operatorname{Im} w_2$), translate so that $\operatorname{Re} w_1 = 0$, reflect if necessary so that $\operatorname{Re} w_2 \geq \operatorname{Re} w_1$, dilate so that $|w_1 - w_2| = 1$, and scale the masses so that $a_1 + a_2 = 1$. The result is

$$S_{s,\gamma,u}(z) = -\frac{a}{z + iu} - \frac{b}{z + iu - e^{-i\gamma}}, \qquad a = \frac{1}{1+e^s}, \quad b = \frac{e^s}{1+e^s}, \tag{8}$$

with $s = \log(b/a) \in \mathbb{R}$ (the logarithm of the mass ratio), $\gamma \in [0, \pi/2]$ (the angle of the segment from the shallower to the deeper pole, measured downward from the horizontal) and $u > 0$ (the depth of the shallower pole). The value $\gamma = 0$ means equal depths and $\gamma = \pi/2$ means one pole straight below the other. We write $\Phi(s, \gamma, u)$ for the margin of (8), and $\mu = \min(a, b)$ ($= \min(\mu_1, \mu_2)$ in the notation of Lemma 4.4).

Lemma 4.5 covers $|s| \geq 12 \log 2$, and $12 \log 2 = 8.317\ldots < 25/3$. What remains is the compact set of $(s, \gamma)$ with $|s| \leq 25/3$ and $\gamma \in [0, \pi/2]$, at all depths $u > 0$. Lemma 4.4 gives a lower bound for the margin that tends to $k_1 - k_0$ as $u \to \infty$ (Lemma 5.3 below), because the mass centre of (8) has depth $u + b \sin\gamma \geq u$ while the separation is 1. Every finite depth $u > 0$ is then reached by integrating the flux in $u$.

### 5.2 The flux identity

Increasing $u$ moves both poles down by the same amount: $S_{s,\gamma,u+t}(z) = S_{s,\gamma,u}(z + it)$. The derivative of the margin along this motion is a boundary integral.

**Lemma 5.1** (Depth flux)**.** *For fixed $(s, \gamma)$ the function $u \mapsto \Phi(s, \gamma, u)$ is continuously differentiable on $(0, \infty)$, and*

$$\frac{\partial \Phi}{\partial u}(s, \gamma, u) = \varphi(s, \gamma, u) := \int_{\mathbb{R}} \Big( |v_S(x)|^2 - k_0\, |S(x)|^2 \Big) \frac{dx}{\pi}, \qquad S = S_{s,\gamma,u}.$$

*Consequently $\Phi(s, \gamma, u') = \Phi(s, \gamma, u'') - \int_{u'}^{u''} \varphi(s, \gamma, u)\, du$ for $0 < u' < u''$.*

The proof (Appendix D) differentiates the Dirichlet Green function of the moving half-plane $\Omega_t = \{\operatorname{Im} Z > t\}$ in the coordinate $Z = z + it$; the derivative of the Green kernel factors through the Poisson kernel, and this turns the derivative of the (divergent) energy into the boundary integral above.

**Remark 5.2.** The proof of Lemma 5.1 applies unchanged to every Pick sum. For a single pole the deficit does not depend on the depth (Lemma 2.5), so its flux is 0: $\int_{\mathbb{R}} \big(|v_S|^2 - k_0|S|^2\big)\,dx = 0$.

The boundary values $v_S(x)$ have a closed form in elementary functions. For a two-pole Pick function $S$, let $z_0 = (a_1w_2 + a_2w_1)/m$ be its zero, which lies in the lower half-plane. Then $v_S$ is a finite linear combination, with constant coefficients, of the terms

$$\mathcal{T}_{w,c}(z) = \frac{\overline{\operatorname{Log}(z-c)} - \overline{\operatorname{Log}(\bar{w}-c)}}{z-w} - \frac{\operatorname{Log}(z-w) - \operatorname{Log}(\bar{c}-w)}{z-\bar{c}},$$

with $w$ a pole of $S$ and $c$ the zero or a pole of $S$; the exact formula and its proof are in Lemma D.1 (Appendix D).

### 5.3 The anchor at large depth

**Lemma 5.3** (Anchor)**.** *Fix $(s,\gamma)$ with $\gamma \in [0,\pi/2]$, and let $u > 0$. If $\int_u^\infty \varphi(s,\gamma,u')\,du'$ converges, then $\Phi(s,\gamma,u) \ge (k_1 - k_0) - \int_u^\infty \varphi(s,\gamma,u')\,du'$.*

*Proof.* At depth $T \ge 4$ the mass centre has depth $u_c = T + b\sin\gamma \ge T$ and the separation is $d = 1 \le u_c/4$, and $u_c(u_c-1) \ge T(T-1)$. So Lemma 4.4, with $\mu_1\mu_2 \le \frac14$ and $m = 1$, gives $\Phi(s,\gamma,T) \ge (k_1-k_0) - 4.05\,T^{-2}/(1-T^{-1})$. By Lemma 5.1, $\Phi(s,\gamma,u) = \Phi(s,\gamma,T) - \int_u^T \varphi(s,\gamma,u')\,du'$; let $T \to \infty$. □

### 5.4 Small depth

As $u \to 0^+$ the shallower pole approaches the real axis, and the lemma below bounds $|\varphi|$ by $O(1+\log(1/u))$. Section 6 needs only an upper bound on the integral of $\varphi$ near 0, uniform over a box of parameters.

**Lemma 5.4** (Small depth)**.** *For every $s \in \mathbb{R}$, every $\gamma \in [0,\pi/2]$ and $0 < u \le 1/4$,*

$$|\varphi(s,\gamma,u)| \le 2^{34}\mu^{-2}\big(1+\log(1/u)\big), \qquad \mu = \min(a,b).$$

*Hence for $0 < u \le u_0 \le 1/4$, $\int_u^{u_0} |\varphi|\,du' \le \beta(u_0) := 2^{34}\mu^{-2}u_0\big(2+\log(1/u_0)\big)$.*

The proof, in Appendix E, splits the source of the Green problem into a real part and an imaginary part and bounds the boundary trace of each near the two feet (the real points $\operatorname{Re} w_1$ and $\operatorname{Re} w_2$ directly above the poles). For $\sin\gamma > 0$, Lemma E.4 gives a number $c_{\log} = c_{\log}(s,\gamma)$, with $|c_{\log}| \le 64/\sin^4\gamma$, such that $|\varphi - c_{\log}\log(1/u)| \le 4096/(a^2\sin^8\gamma)$ for $0 < u \le \frac14$.

## 6 The computer certificate

This section proves the computer-assisted step of Theorem 1.1. We use the normalisation (8) and its coordinates $(s,\gamma,u)$, and write $\varphi = \varphi(s,\gamma,u)$ for the depth flux of Lemma 5.1. A *box* is a set $B = [s_-,s_+] \times [\gamma_-,\gamma_+]$ of the parameters $(s,\gamma)$. Since $\gamma \le \pi/2$ in (8), we write $(s,\gamma) \in B$ only for the points of $B$ with $\gamma \le \pi/2$.

For a box $B$ and a number $0 < u_B \le 1/4$, let $a_- = 1/(1+e^{s_+})$ and $\mu_- = 1/(1+e^{\max(s_+,-s_-)})$ be the least values of $a$ and $\mu$ on $B$, and put $\beta_B = 2^{34}\mu_-^{-2}u_B\big(2+\log(1/u_B)\big)$ (Lemma 5.4). If $\gamma_- \ge 0.53$, then $\sin\gamma \ge \sin\gamma_-$ on $B$, and $\beta_B$ is instead the smaller of this number and

$$u_B\Big(\frac{64}{\sin^4\gamma_-}\big(1+\log(1/u_B)\big) + \frac{4096}{a_-^2\sin^8\gamma_-}\Big),$$

which bounds $\int_0^{u_B} |\varphi|\,du$ by Lemma E.4. Hence, by Lemma 5.4 (as $\mu \ge \mu_-$ on $B$), or by Lemma E.4 when $\gamma_- \ge 0.53$ (as $a \ge a_-$ and $\sin\gamma \ge \sin\gamma_-$ on $B$, and $\int_0^{u_B} \log(1/u)\,du = u_B(1+\log(1/u_B))$),

$$\sup_{0<u\le u_B} \int_u^{u_B} |\varphi(s,\gamma,u')|\,du' \le \beta_B \qquad \text{for every } (s,\gamma)\in B. \tag{9}$$

**Proposition 6.1.** *For each box $B$ of the cover of Section 6.1 there are numbers $0 < u_B \le 1/4$ and $\underline{\Phi}_B > \beta_B \ge 0$, with $\beta_B$ as above, such that for every $(s,\gamma)\in B$:*

(a) $\Phi(s,\gamma,u) > 0$ *for every* $u \ge u_B$;

(b) $\Phi(s,\gamma,u_B) \ge \underline{\Phi}_B$.

The numbers $u_B$, $\underline{\Phi}_B$ and $\beta_B$ are computed by the program (Section 6.6 gives the smallest value of $\underline{\Phi}_B - \beta_B$).

## 6.1 The cover

By Lemma 4.5 only $|s| \le 25/3$ is needed, and $\gamma \in [0,\pi/2]$ by (8).

The *cover* consists of the 168 boxes of Table 2, in three regions: the corner $0 \le \gamma \le 2^{-20}$, the rows in $\log\gamma$, and the large angles $0.53 \le \gamma \le 1.571$. Inside each region the boxes tile it: in each row the intervals of $s$ share their end points and run from $-25/3$ to $25/3$, and consecutive rows share an edge, as Table 2 shows. The two large-angle rows are cut into 25 intervals of length $2/3$ and 17 intervals of length $50/51$, and each covers $[-25/3, 25/3]$. The rows in $\log\gamma$ are twelve of width 1.1 and one of width 1, and together they run from $-14.83$ to $-0.63$. The regions overlap at their two seams, since $e^{-0.63} > 0.532 > 0.53$ and $e^{-14.83} < 4\cdot 10^{-7} < 2^{-20}$, and $1.571 > \pi/2$. Hence the boxes cover $[-25/3, 25/3]\times[0,\pi/2]$. On the rows the computation uses $\log\gamma$ as its coordinate. The corner boxes contain $\gamma = 0$ (equal depths). Figure 3 shows the cover.

## 6.2 The chain in depth

Fix a box $B$. We use break points $u_B = u_0 < u_1 < \cdots < u_N$ as follows.

(i) *The top*: $u_{N-1} = e^{1/2}$ and $u_N = \infty$.

(ii) *Cells*: the intervals $[e^{-3/2}, e^{1/2}], [e^{-7/2}, e^{-3/2}], \dots, [e^{-19/2}, e^{-15/2}]$ (down to $[e^{-27/2}, e^{-23/2}]$ for the 42 boxes with $\gamma \ge 0.53$); so $u_{k+1} = e^2 u_k$ on the cells.

(iii) *Low pieces (small depth $u$, near the real axis)*: below the cells, the intervals $[e^{-1}u_{k+1}, u_{k+1}]$ down to $u_0 = u_B$, where $u_B$ is the first of the numbers $e^{-19/2}, e^{-21/2}, \dots$ (or $e^{-27/2}, e^{-29/2}, \dots$) for which the number $\beta_B$ defined before Proposition 6.1 is at most $1/2000$.

By Lemma 5.1 and Lemma 5.3, for $(s,\gamma)\in B$,

$$\Phi(s,\gamma,u) \;\ge\; k_1 - k_0 - \int_u^{u_N} \varphi(s,\gamma,t)\,dt \qquad (u_B \le u < u_N). \tag{10}$$

The integral converges, as Lemma 5.3 requires: by Lemma 6.2 below, $u^2\varphi(s,\gamma,u)$ is bounded for $u \ge e^{1/2}$, so $\varphi = O(u^{-2})$ as $u\to\infty$.

*Nodes and partners.* We call the two poles and the zero of $S = S_{s,\gamma,u}$ its *nodes*: $w_1 = -iu$, $w_2 = -iu + e^{-i\gamma}$ and $z_0 = -iu + a\,e^{-i\gamma}$, with $a = 1/(1+e^s)$ and $b = 1-a$ as in (8). The interpolation bounds of Section 6.3 need $\varphi$ at complex values of $(s,\gamma,u)$. Each node $c$ has a *partner* $c^{\#}$: $w_1^{\#} = iu$, $w_2^{\#} = iu + e^{i\gamma}$, $z_0^{\#} = iu + a\,e^{i\gamma}$. The partners are analytic in $(s,\gamma,u)$ where $1+e^s \ne 0$ (in particular for $|\operatorname{Im} s| < \pi$), and they equal $\bar w_1$, $\bar w_2$, $\bar z_0$ when the parameters are real; so $c^{\#}$ is the analytic continuation of $\bar c$ in the parameters. For real parameters and real $x$ we have $\overline{\operatorname{Log}(x-c)} = \operatorname{Log}(x-\bar c)$ and $\overline{1/(x-c)} = 1/(x-\bar c)$ for every $c\notin\mathbb{R}$. Hence, by Lemma D.1, the integrand $|v_S(x)|^2 - k_0|S(x)|^2$

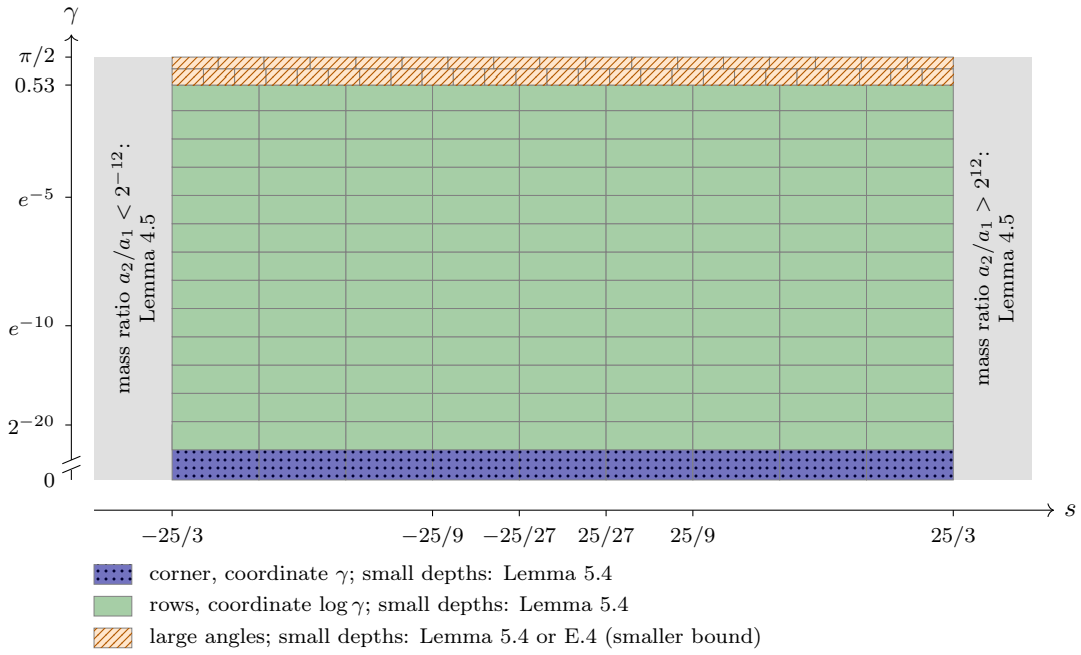


Figure 3: The 168 boxes of the computer step. Horizontal: $s = \log(a_2/a_1)$; vertical: the angle $\gamma$, on a logarithmic scale below 0.53; the bottom band is the corner $0 \le \gamma \le 2^{-20}$. For each box, Proposition 6.1 (computer-assisted) and Lemma 5.4 or E.4 give $\Phi > 0$ at its points with $\gamma \le \pi/2$ and every depth $u > 0$ (Section 7). The colour and pattern show the method: the coordinate in $\gamma$ and the lemma used at small depth. Grey: $|s| > 25/3$, covered by Lemma 4.5.

of the flux (Lemma 5.1) is a finite expression $H(x)$ built from the masses, the factors $1/(x-c)$ and $\mathrm{Log}(x-c)$ with $c$ a node or a partner, and constants $\mathrm{Log}(c'-c)$ with one of $c, c'$ a node and the other a partner. Form $v_S$ and its conjugate using Lemma D.1. Replace conjugate nodes by partners and conjugate masses $\bar{a}, \bar{b}$ by $a, b$; retain the respective constants $i/2$ and $-i/2$. This defines $H(x)$ for complex parameters, the integrand *written with partners*. The logarithm branches are those of Lemma D.2: for $\mathrm{Log}(x-c)$ at a node use $\arg(x-c) \in (-\pi/2, 3\pi/2)$, and at a partner use $\arg(x-c^{\#}) \in (-3\pi/2, \pi/2)$. Each constant logarithm is continued from its half-plane at real parameters, with cut $i[0,\infty)$ for an argument $c - c'^{\#}$ and $-i[0,\infty)$ for $c'^{\#} - c$. A path of integration is *admissible* if it equals $\mathbb{R}$ outside a bounded set and meets none of the vertical rays that run down from the nodes and up from the partners (Lemma D.2); for real parameters $\mathbb{R}$ itself is admissible.

Because the cover extends slightly beyond $\pi/2$, the following lemma also defines the flux for those larger angles.

**Lemma 6.2.** *Let $s$ and $\gamma$ be real, $0 \le \gamma \le \pi$. For $\tilde{u} \ge 0$ let $\varphi_{\tilde{u}}$ be the boundary integral of Lemma 5.1 for the function $-a/(z+i) - b/(z+i-\tilde{u}e^{-i\gamma})$, with $a, b$ as in (8), and let $\varphi(s,\gamma,u)$ be that integral for the function (8) also when $\gamma > \pi/2$. Then $u^2\varphi(s,\gamma,u) = \tilde{u}^{-1}\varphi_{\tilde{u}}$ for $u > 0$ and $\tilde{u} = 1/u$, and $\tilde{u} \mapsto \tilde{u}^{-1}\varphi_{\tilde{u}}$ extends analytically to the disc $|\tilde{u}| < 1$. In particular $u^2\varphi(s,\gamma,u)$ is bounded for $u \ge e^{1/2}$.*

*Proof.* The boundary integral of Lemma 5.1 is homogeneous of degree $-1$ in the configuration: dilation by a factor $\lambda > 0$ divides $S$ and $v_S$ by $\lambda$ (with $x$ replaced by $x/\lambda$) and multiplies $dx$ by $\lambda$. Dilation by $\lambda = \tilde{u} = 1/u$ takes the poles $-iu$ and $-iu + e^{-i\gamma}$ of (8) to $-i$ and $-i + \tilde{u}e^{-i\gamma}$, so $\varphi_{\tilde{u}} = u\,\varphi(s,\gamma,u)$; this is the first claim. The zero of this function is $-i + a\tilde{u}e^{-i\gamma}$. For $|\tilde{u}| < 1$ the offsets $\tilde{u}e^{-i\gamma}$ and $a\tilde{u}e^{-i\gamma}$ of the nodes from $-i$, and those of the partners from $i$, $\tilde{u}e^{i\gamma}$ and $a\tilde{u}e^{i\gamma}$, have modulus less than 1 (as $0 < a < 1$), so the nodes lie in the lower half-plane and the partners in

the upper one. Let $H(x)$ be the integrand of the flux written with partners (above). It contains no difference of two nodes: its denominators are differences of $x$ and a node or a partner, and each of its constant logarithms has one node and one partner in its argument. So no factor of $H$ is singular when the three nodes merge at $\tilde{u} = 0$, and the argument of Appendix D.3 (Fubini and Morera) shows that $\varphi_{\tilde{u}}$ is analytic on $|\tilde{u}| < 1$. At $\tilde{u} = 0$ the masses add up to 1 and the multiplicities, $+1$ for the zero and $-1$ for each pole, add up to $-1$, so $H$ is the integrand of the single pole $-1/(z+i)$, whose flux is 0 (Remark 5.2). Hence $\varphi_0 = 0$, and $\tilde{u}^{-1}\varphi_{\tilde{u}}$ is analytic on $|\tilde{u}| < 1$; in particular it is bounded on $[0, e^{-1/2}]$. □

| Angles | Intervals of $s$ | Boxes |
|---|---|---|
| $0 \le \gamma \le 2^{-20}$ | 9 intervals, end points $\pm\frac{25}{3}, \pm\frac{175}{27}, \pm\frac{125}{27}, \pm\frac{25}{9}, \pm\frac{25}{27}$ | 9 |
| $-14.83 \le \log\gamma \le -0.63$, in 13 rows: $[-14.83, -13.73], \dots, [-2.73, -1.63]$ (width 1.1) and $[-1.63, -0.63]$ | the same 9 intervals in each row | 117 |
| $0.53 \le \gamma \le 1$ | 25 intervals of length 2/3 | 25 |
| $1 \le \gamma \le 1.571$ | 17 intervals of length 50/51 | 17 |
| | | 168 |

Table 2: The cover (Figure 3).

For $u_k \le u \le u_{k+1}$ write $I_k(u) = \int_u^{u_{k+1}} \varphi\, dt$, a function of $(s, \gamma) \in B$ as well.

**Lemma 6.3.** *Suppose that for every $(s, \gamma) \in B$*

$$I_k(u) + \sum_{j>k} I_j(u_j) < k_1 - k_0 \quad \text{for every } k \text{ and every } u \in [u_k, u_{k+1}], \qquad k_1 - k_0 - \sum_{j\ge 0} I_j(u_j) \ge \underline{\Phi}_B. \tag{11}$$

*Then* (a) *and* (b) *of Proposition 6.1 hold on $B$.*

*Proof.* For $u \in [u_k, u_{k+1}]$ the first sum in (11) equals $\int_u^{u_N} \varphi\, dt$, so (10) gives $\Phi(s, \gamma, u) > 0$ for every $u \ge u_B$, since $u_N = \infty$. The second inequality is (10) at $u = u_B$. □

The program verifies (11) with explicit upper bounds for the integrals $I_k$. On the cells and at the top these bounds are polynomials in the parameters plus a constant, and the sum over $j > k$ is kept as one polynomial, so that its maximum over $B$ is taken once rather than as a sum of separate maxima. On a low piece the bound is a constant: the length of the piece times an upper bound for $|\varphi|$ on $B \times [u_k, u_{k+1}]$. For it, $B$ is split into 1, 4 or 16 equal sub-boxes (1, 2 or 4 equal parts in $s$ and in $\gamma$, or $\log\gamma$ on the rows), and the largest sub-box bound is used. In the corner boxes the $\gamma$-range $[0, 2^{-20}]$ of a low piece $[u_k, u_{k+1}]$ is first cut at the points $2^{-20}4^{-j}$ $(j \ge 1)$ that are at least $u_{k+1}$, and each part into equal parts; this keeps the bound sharp and plays no role in its validity. The final check is $\underline{\Phi}_B - \beta_B > 0$, with $\beta_B$ as in Proposition 6.1. Over the 168 boxes, $u_B$ lies between $e^{-52.5}$ and $e^{-17.5}$ and $\beta_B$ is at most 1/2000.

### 6.3 Polynomial bounds on a cell

On a cell $[u_k, u_{k+1}]$ put $u = u_{k+1}e^{\xi_3 - 1}$, $\xi_3 \in [-1, 1]$, write $s$ and $\gamma$ (or $\log\gamma$) as affine images of $\xi_1, \xi_2 \in [-1, 1]$ and set

$$\varphi_\xi(\xi_1, \xi_2, \xi_3) = u\,\varphi\big(s(\xi_1), \gamma(\xi_2), u\big), \qquad \text{so that} \qquad \int_u^{u_{k+1}} \varphi\, dt = \int_{\xi_3}^1 \varphi_\xi\, d\xi_3'.$$

At the top we use the variable $\tilde{u} = 1/u = e^{-1/2}(1-\xi_3)/2$ and set $\varphi_\xi(\xi_1,\xi_2,\xi_3) = \frac{1}{2}e^{-1/2}\,\tilde{u}^{-1}\varphi_{\tilde{u}}$, with $\varphi_{\tilde{u}}$ as in Lemma 6.2; then $\int_{\xi_3}^{1}\varphi_\xi\,d\xi_3' = \int_{u(\xi_3)}^{\infty}\varphi\,dt$.

The program replaces $\varphi_\xi$ by its tensor Chebyshev interpolant $\hat{\varphi}_\xi$ of degrees $(n_1,n_2,n_3)$ at the points of the first kind $\cos\big((2j+1)\pi/(2n+2)\big)$, $0 \le j \le n$, and bounds $|\varphi_\xi - \hat{\varphi}_\xi|$ by the following lemmas. For $\upsilon > 1$ let $E_\upsilon$ be the closed region bounded by the ellipse with foci $\pm 1$ and semi-axes $(\upsilon \pm \upsilon^{-1})/2$. The ellipses used for $\xi_3$ all have $\upsilon = 5/2$; those for $\xi_1$ and $\xi_2$ have $\upsilon$ between $11/5$ and $5/2$ on the cells. At the top, $\upsilon = 3$ for $\xi_1$ on the corner and row boxes and $\upsilon = 5$ on the large boxes, and $\upsilon$ lies between 2 and 5 for $\xi_2$.

**Lemma 6.4.** *Let $\chi$ be analytic on a neighbourhood of $E_\upsilon$ with $|\chi| \le M$ there, and let $\chi_n$ be its polynomial interpolant of degree $n$ at the $n+1$ points of the first kind. Then* $\max_{[-1,1]}|\chi - \chi_n| \le 4M\upsilon^{-n}/(\upsilon-1)$.

*Proof.* The map $\tau \mapsto (\tau+\tau^{-1})/2$ sends the annulus $\upsilon^{-1} \le |\tau| \le \upsilon$ onto $E_\upsilon$, and the circle $|\tau| = \upsilon$ onto its boundary. So $\tilde{\chi}(\tau) = \chi((\tau+\tau^{-1})/2)$ is analytic on a neighbourhood of the annulus, with $\tilde{\chi}(\tau) = \tilde{\chi}(1/\tau)$ and $|\tilde{\chi}| \le M$ on $|\tau| = \upsilon$. Its Laurent series $\tilde{\chi} = \sum_{k\in\mathbb{Z}}\hat{b}_k\tau^k$ has $\hat{b}_{-k} = \hat{b}_k$ and, by Cauchy's estimate on $|\tau| = \upsilon$, $|\hat{b}_k| \le M\upsilon^{-k}$ for $k \ge 0$. Since $T_k(\cos\theta) = \cos k\theta$, putting $\tau = e^{i\theta}$ gives $\chi = \sum_{k\ge 0}\hat{a}_k T_k$ on $[-1,1]$ with $\hat{a}_0 = \hat{b}_0$ and $\hat{a}_k = 2\hat{b}_k$, so $|\hat{a}_k| \le 2M\upsilon^{-k}$ for $k \ge 1$ (see also [10, Theorem 8.1]); the series converges uniformly on $[-1,1]$. At the points $x_j = \cos\theta_j$, $\theta_j = (2j+1)\pi/(2n+2)$, we have $T_k(x_j) = \cos(k\theta_j)$. For $k > n$ write $k = 2(n+1)q \pm k'$ with $0 \le k' \le n+1$; since $(2n+2)q\theta_j = (2j+1)q\pi$, we get $\cos(k\theta_j) = (-1)^q\cos(k'\theta_j)$, and $\cos((n+1)\theta_j) = 0$. So at the interpolation points $T_k$ agrees with $\pm T_{k'}$ for one $k' \le n$, or with 0. Hence $\chi - \chi_n = \sum_{k>n}\hat{a}_k(T_k - \tilde{T}_k)$, where $\tilde{T}_k$ is $\pm T_{k'}$ or 0, and $\max_{[-1,1]}|\chi-\chi_n| \le 2\sum_{k>n}|\hat{a}_k| \le 4M\upsilon^{-n}/(\upsilon-1)$. □

**Lemma 6.5.** *Let $\mathcal{L}_n = 1 + \frac{2}{\pi}\log(n+1)$. For every $n \ge 1$, $\mathcal{L}_n$ is an upper bound for the Lebesgue constant of interpolation at the $n+1$ points of the first kind (classical; see [3, (13)], where $n$ is also the degree). Let $\Pi_1$, $\Pi_2$, $\Pi_3$ denote interpolation in $\xi_1$, $\xi_2$, $\xi_3$. Let $M_3 \ge |\varphi_\xi|$ on ($\xi_3 \in E_{\upsilon_3}$)×($\xi_1,\xi_2 \in [-1,1]$), let $M_2 \ge |\varphi_\xi|$ on ($\xi_2 \in E_{\upsilon_2}$)×($\xi_1 \in [-1,1]$)×($\xi_3$ at the interpolation points), and let $M_1 \ge |\varphi_\xi|$ on ($\xi_1 \in E_{\upsilon_1}$)×($\xi_2 \in [-1,1]$)×($\xi_3$ at the interpolation points), $\varphi_\xi$ being analytic in the complex variable on each of these sets. Then on* $[-1,1]^3$

$$|\varphi_\xi - \hat{\varphi}_\xi| \le \mathrm{Err} := \mathrm{Err}_3 + \mathcal{L}_{n_3}\mathrm{Err}_2 + \mathcal{L}_{n_3}\mathcal{L}_{n_2}\mathrm{Err}_1, \qquad \mathrm{Err}_m = 4M_m\upsilon_m^{-n_m}/(\upsilon_m - 1).$$

*Proof.* $\hat{\varphi}_\xi = \Pi_3\Pi_2\Pi_1\varphi_\xi$ and $\varphi_\xi - \hat{\varphi}_\xi = (\varphi_\xi - \Pi_3\varphi_\xi) + \Pi_3(\varphi_\xi - \Pi_2\varphi_\xi) + \Pi_3\Pi_2(\varphi_\xi - \Pi_1\varphi_\xi)$. The interpolant in $\xi_3$ uses only the values at the interpolation points in $\xi_3$, and its norm is at most $\mathcal{L}_{n_3}$; likewise in $\xi_2$. Apply Lemma 6.4 to each term. □

The interpolation degrees satisfy $1 \le n_1 \le 20$, $1 \le n_2 \le 45$ and $1 \le n_3 \le 46$ (the degrees of each box are recorded in its certificate log; see Code and data). $\mathrm{Err} < 10^{-3}$ on every cell and at the top.

**Lemma 6.6.** *For real values of the other two variables, $\varphi_\xi$ extends analytically in each variable to the region used for it in Lemma 6.5, provided that on that region the nodes of $S$ stay in the lower half-plane and their partners stay in the upper half-plane (proof in Appendix D). At the top cell this holds in the variable $\tilde{u}$ also through $\tilde{u} = 0$, where the nodes merge (Lemma 6.2).*

Analyticity on the interpolation regions follows from Lemma 6.6 where the following separation tests hold, and from Lemma D.2 for the exceptional regions in $s$. The program checks each closed region together with the complex balls that cover its boundary curve (used below for $M_1$, $M_2$, $M_3$). At the top cell it checks that every offset of the proof of Lemma 6.2 has modulus at most $7/8$. The program never evaluates at $\tilde{u} = 0$ ($\xi_3 = 1$). This point is not an interpolation point, and it lies on no boundary curve, since it is a focus of each ellipse $E_\upsilon$. Every ball on which $\tilde{u}^{-1}\varphi_{\tilde{u}}$ is evaluated is

checked to exclude $\tilde{u} = 0$; the bound on the whole region, $\tilde{u} = 0$ included, then follows from the maximum principle. On the other cells the nodes are $w_1$, $w_2$, $z_0$ and their partners are $w_1^{\#}$, $w_2^{\#}$, $z_0^{\#}$, as defined before Lemma 6.2.

- In $\xi_3$: $|\operatorname{Im}\xi_3| < \pi/2$. Then $\operatorname{Re} u > 0$, so each node has imaginary part $-\operatorname{Re} u$ minus a number $\geq 0$, and each partner $\operatorname{Re} u$ plus a number $\geq 0$.
- In $\gamma$ ($s$ and $u$ real, so $0 < a < 1$): $0 < \operatorname{Re}\gamma < \pi$ on the large boxes; on the rows $|\operatorname{Im}\log\gamma| < \pi/2$ and $|\gamma| < \pi$, which give the same. Then $\operatorname{Im} w_2 = -u - e^{\operatorname{Im}\gamma}\sin\operatorname{Re}\gamma < 0$ and $\operatorname{Im} w_2^{\#} = u + e^{-\operatorname{Im}\gamma}\sin\operatorname{Re}\gamma > 0$, and the same holds with the factor $a$ for the zero. On the corner boxes, whose region contains $\gamma = 0$, it checks $\operatorname{Re}\gamma > -\pi/2$ and checks on every cell that its least depth exceeds an upper bound for $e^{|\operatorname{Im}\gamma|}\max(0, -\sin\operatorname{Re}\gamma)$ on the region; then $\operatorname{Im} w_2 \leq -u + e^{|\operatorname{Im}\gamma|}\max(0, -\sin\operatorname{Re}\gamma) < 0$ and $\operatorname{Im} w_2^{\#} \geq u - e^{|\operatorname{Im}\gamma|}\max(0, -\sin\operatorname{Re}\gamma) > 0$, and the same holds with the factor $a$ for the zero.
- In $s$ ($\gamma$ and $u$ real), on the large boxes: $|\operatorname{Im} s| < \gamma_-$ and $\gamma_+ + |\operatorname{Im} s| < \pi$. Since $\gamma_- \leq 1 < \pi/2$, $\operatorname{Re} e^s > 0$, so $1 + e^s \neq 0$ and its argument lies between 0 and $\operatorname{Im} s$. Thus $|\arg a| < \gamma_-$ and $\gamma \pm \arg a \in (0, \pi)$. Hence $\operatorname{Im} z_0 = -u - |a|\sin(\gamma - \arg a) < 0$ and $\operatorname{Im} z_0^{\#} = u + |a|\sin(\gamma + \arg a) > 0$; the poles do not depend on $s$.

On the corner and row boxes the program checks instead $|\operatorname{Im} s| < \pi/2$ on the region in $s$. Their $s$-intervals have half-width $25/27$ and $\upsilon_1 \leq 3$ ($11/5$ or $5/2$ on the cells, 3 on the top cell), so $|\operatorname{Im} s| \leq \frac{25}{27} \cdot \frac{3 - 1/3}{2} = \frac{100}{81} < \frac{\pi}{2}$ on $E_{\upsilon_1}$, so again $1 + e^s \neq 0$. There, at small depth, the zero may leave the lower half-plane (or its partner the upper one), so Lemma 6.6 does not apply; the integral is then taken along an admissible path (the moved path at the end of Section 6.4), and the continuation is given by Lemma D.2, whose part (c) needs $|\operatorname{Im} s| < \pi/2$. On every complex ball at which it evaluates $\varphi_\xi$ by an integral over $\mathbb{R}$, the program checks the hypothesis of Lemma 6.6 itself, in ball arithmetic (every node strictly below $\mathbb{R}$, every partner strictly above). It bounds $|\varphi_\xi|$ on each region by its maximum on the boundary curve (maximum principle, for each fixed real value of the other two variables). For $M_3$ it covers the ellipse curve in $\xi_3$ by small complex balls and covers the real ranges of $\xi_1$ and $\xi_2$ by real balls (intervals), split where needed; $M_2$ and $M_1$ are bounded in the same way.

For every $(\xi_1, \xi_2)$ and $\xi_3$,

$$\int_{\xi_3}^{1} \varphi_\xi \, d\xi_3' \leq \int_{\xi_3}^{1} \hat{\varphi}_\xi \, d\xi_3' + \operatorname{Err}\,(1 - \xi_3),$$

and the integral of $\hat{\varphi}_\xi$ on the right is a polynomial whose coefficients are computed from those of $\hat{\varphi}_\xi$. The full-cell integral ($\xi_3 = -1$) of each earlier cell is kept as a polynomial in $(\xi_1, \xi_2)$; this is possible because the affine maps from $(\xi_1, \xi_2)$ to $(s, \gamma)$ (or $(s, \log\gamma)$) depend only on the box, not on the cell. On cell $k$ the left side of the first inequality in (11) is therefore at most a polynomial in $(\xi_1, \xi_2, \xi_3)$ plus the numbers 2Err of this cell, of the cells above it and of the top. The maximum of a polynomial $\sum c_{l_1 l_2 l_3} T_{l_1}(\xi_1) T_{l_2}(\xi_2) T_{l_3}(\xi_3)$ on $[-1, 1]^3$ is bounded by branch and bound in the angles $\theta_m$ with $\xi_m = \cos\theta_m$: on a product of angle intervals each $T_l(\cos\theta) = \cos(l\theta)$ is enclosed, the pieces whose upper bound lies below the best lower bound found are discarded, and the largest upper bound of the pieces kept bounds the maximum.

### 6.4 Enclosing the flux

Every value of $\varphi$ used by the program is the integral of the closed form of Lemma D.1, over $\mathbb{R}$ or, for the bound $M_1$ of the corner and row boxes, along an admissible path (the moved path, below), computed in ball arithmetic. Any finite subdivision satisfying the stated enclosure tests gives a valid bound. Write $S = a_1 S_{w_1} + a_2 S_{w_2}$, where $a_1 = a$ and $a_2 = b$ are the masses of (8) and

$S_w = -1/(z-w)$ is the single pole of mass 1 at $w$, and put $h_w = |v_{S_w}|^2 - k_0|S_w|^2$. By Remark 5.2, $\int_{\mathbb{R}} h_w\,dx = 0$ for every $w$ below $\mathbb{R}$, and

$$\pi\varphi = \int_{\mathbb{R}} H_\times(x)\,dx, \qquad H_\times = |v_S|^2 - k_0|S|^2 - a_1^2 h_{w_1} - a_2^2 h_{w_2}.$$

The program integrates $H_\times$; this removes exactly the large single-pole terms near the feet. As in Appendix D, $H_\times$ is a function of $x$ that is analytic off the vertical rays that run down from the nodes and up from the partners, so at real parameters it is analytic in the strip $|\operatorname{Im} x| < u_{\min}$, where $u_{\min}$ is the least depth $-\operatorname{Im} c$ of the three nodes $c$. On a ball of complex parameters, $u_{\min}$ is a certified positive lower bound over the ball for $-\operatorname{Im} c$ (on a moved path, for $\operatorname{Re}\big((c^\# - c)/(2i)\big)$, where $c^\#$ is the partner of $c$); both equal the depth at real parameters. Let $x_{\max} = 10^{18}\lceil 1/u_{\min}\rceil$. The tail bound of Lemma 6.7 below uses only $x_{\max} \ge 10^{18}$ and $x_{\max} > 4|c|$ for every node and partner $c$; the program checks the latter.

Let $\Upsilon(x) = \log(3|x|/2) + \pi + \Upsilon_{\log}$, where $\Upsilon_{\log}$ is the sum of the moduli of the constant logarithms of $v_S$ and of $\overline{v_S}$ written with partners (Appendix D). At complex parameters each conjugate in $H_\times$ stands for the partner, and the bounds below hold for the partners too.

**Lemma 6.7.** *Suppose that $x_{\max} \ge 10^{18}$ and $x_{\max} > 4|c|$ for every node and partner $c$. Then, on $\mathbb{R}$ and on the moved path below, for $|x| \ge x_{\max}$*

$$|H_\times(x)| \le \big(c_\times \Upsilon(x)^2 + 8k_0|a_1a_2|\big)x^{-2}, \qquad c_\times = 8|a_1a_2| + 32\big(|a_1| + |a_2|\big)^2.$$

*Proof.* For $|x| \ge x_{\max}$ and each node or partner $c$ we have $|x-c| \in [|x|/2, 3|x|/2]$, and every numerator in the terms $\mathcal{T}_{w,c}$ of Lemma D.1 (a logarithm minus a constant logarithm) is at most $\Upsilon(x)$ in modulus. On a moved path a node may lie above $\mathbb{R}$ or a partner below it; then, for $|x| \ge x_{\max}$, $|\arg(x-c)|$ may exceed $\pi$, by at most $\arcsin\frac14$, and the bound still holds: $\log|x-c| \ge \log(x_{\max}/2) > 40$ gives, with Log the branch of Lemma D.2, $|\operatorname{Log}(x-c)| \le \log|x-c| + (\pi + \arcsin\frac14)^2/80 < \log|x-c| + \pi$. Hence $|\mathcal{T}_{w,c}(x)| \le 4\Upsilon(x)/|x|$. Write $v_S = a_1 v_{S_{w_1}} + a_2 v_{S_{w_2}} + v_\times$, where $v_\times$ collects the four terms of Lemma D.1 with $c_k \ne w_j$; their coefficients have moduli $|a_j|/2$, which add up to $|a_1| + |a_2|$. So $|v_{S_{w_j}}| \le 2\Upsilon/|x|$, $|v_\times| \le 4(|a_1| + |a_2|)\Upsilon/|x|$ and $|S_{w_j}| \le 2/|x|$. In the following expansion of $H_\times$, the real parts and squared moduli at complex parameters mean the continued products written with partners, using the convention before Lemma 6.2. The bound follows from $H_\times = 2\operatorname{Re}(a_1a_2 v_{S_{w_1}}\overline{v_{S_{w_2}}}) + 2\operatorname{Re}\big((a_1 v_{S_{w_1}} + a_2 v_{S_{w_2}})\overline{v_\times}\big) + |v_\times|^2 - 2k_0\operatorname{Re}(a_1a_2 S_{w_1}\overline{S_{w_2}})$. The moduli matter for complex parameters; for real ones $a_1 + a_2 = 1$ and $c_\times = 8a_1a_2 + 32$. □

*Values at real parameters.* The program cuts $[-x_{\max}, x_{\max}]$ into finitely many pieces, refined geometrically towards each foot. On each real-line piece the integral is enclosed by the integrator of Arb [4, 5], which combines direct interval enclosures with Gauss–Legendre quadrature and bisection. For a real subinterval $\mathcal{J}$ of length $|\mathcal{J}|$, let $Q_n(\mathcal{J})$ be the $n$-point Gauss–Legendre approximation to $\int_{\mathcal{J}} H_\times\,dx$. If $H_\times$ is analytic on the ellipse with foci at the ends of $\mathcal{J}$ whose semi-axes sum to $\upsilon|\mathcal{J}|/2$, with $\upsilon > 1$ (the region $E_\upsilon$ of Lemma 6.4, moved to $\mathcal{J}$), and $|H_\times| \le M$ there, then

$$\Big|\int_{\mathcal{J}} H_\times(x)\,dx - Q_n(\mathcal{J})\Big| \le \frac{32\,|\mathcal{J}|\,M}{15(\upsilon-1)\upsilon^{2n-1}}.$$

Here $n \ge 1$ is the order of the rule. $H_\times$ is evaluated in ball arithmetic on complex balls; a ball that meets one of the rays gives no finite bound, and the subinterval is then bisected. The result is a ball that contains the integral. Since $\int_{x_{\max}}^\infty \Upsilon(x)^2 x^{-2}dx = \big(\Upsilon(x_{\max})^2 + 2\Upsilon(x_{\max}) + 2\big)/x_{\max}$, Lemma 6.7 bounds the two tails together by $2\big(c_\times(\Upsilon(x_{\max})^2 + 2\Upsilon(x_{\max}) + 2) + 8k_0|a_1a_2|\big)/x_{\max}$, and this number is added to the radius.

*Bounds on parameter balls.* The numbers $M_1, M_2, M_3$ of Lemma 6.5 and the bounds on the low pieces need an upper bound for $|\varphi|$ on a ball of parameters: complex for $M_1, M_2, M_3$, real for a low

piece, where the ball in $u$ is the whole piece. No quadrature rule is used there. The program cuts $[-x_{\max}, x_{\max}]$ into finitely many pieces, refined geometrically towards each foot (the two pieces next to a foot have length $u_{\min}/4$), encloses $H_\times$ on each (piece)×(parameter ball) in ball arithmetic, and uses

$$|\varphi| \le \frac{1}{\pi}\Big( \sum_{\text{pieces}} (\text{length}) \cdot \sup|H_\times| + \text{tail} \Big),$$

with the tail bound of Lemma 6.7 taken over the ball. At the top, where the two poles of the dilated configuration (whose flux is $\varphi_{\tilde{u}}$) are close, a second bound is also computed, and the smaller one is used; it rests on the next lemma. Move the two poles and the zero linearly from their mass centre $w_c = a_1 w_1 + a_2 w_2$ ($t = 0$) to their positions ($t = 1$), and the partners likewise, and let $H_\times(x;t)$ be the integrand $H_\times$ of the moved configuration.

**Lemma 6.8.** *Suppose that for every $t \in [0,1]$ the moved nodes lie below $\mathbb{R}$ and their partners above it. Then $H_\times(x;t)$ is analytic in $t$ on a neighbourhood of $[0,1]$; its Taylor coefficients $H^{[k]}$ at $t = 0$ satisfy $H^{[0]} = 2a_1a_2h_{w_c}$ and $\int_{\mathbb{R}} H^{[0]}\,dx = \int_{\mathbb{R}} H^{[1]}\,dx = 0$, at real and at complex parameters; and $\pi\varphi = \int_{\mathbb{R}}(H_\times - H^{[0]} - H^{[1]})\,dx$, where $H_\times = H_\times(\cdot\,;1)$.*

*Proof.* The factors of $H_\times(x;t)$ are analytic in the nodes and partners as long as the nodes lie below $\mathbb{R}$ and the partners above it (Appendix D.3); by the hypothesis this holds for $t$ on a neighbourhood of $[0,1]$. At $t = 0$ there is one pole of mass $a_1 + a_2 = 1$ at $w_c$, so $H^{[0]} = (1 - a_1^2 - a_2^2)h_{w_c} = 2a_1a_2h_{w_c}$. Since $\sum_j a_j(w_j - w_c) = 0$, $S = S_{w_c} + O(t^2)$; so the sources (3) of $S$ and of $S_{w_c}$ differ by at most a constant (depending on the depth of $w_c$) times $t^2|z - w_c|^{-2}$ on $\mathbb{U}$, and by the domination bound in the proof of Lemma 5.1 (Appendix D) $v_S$ and $v_{S_{w_c}}$ differ by $O(t^2)$ on $\mathbb{R}$ (for complex parameters, by continuation of the formula of Lemma D.1). For real parameters the coefficient of $t$ in $|v_S(x)|^2 - k_0|S(x)|^2$ therefore vanishes for every $x$. It is analytic in the parameters, so it vanishes for complex parameters as well. So $H^{[1]}$ comes only from the two subtracted single-pole terms $a_j^2 h_{w_j}$ and is a derivative of $h_w$ in the position of the pole. Since $\int_{\mathbb{R}} h_w\,dx = 0$ for every $w$ (Remark 5.2), $\int_{\mathbb{R}} H^{[0]}\,dx = \int_{\mathbb{R}} H^{[1]}\,dx = 0$. For complex parameters, treat the partner $w^\#$ of $w$ in $h_w$ as a second variable. The map $(w, w^\#) \mapsto \int_{\mathbb{R}} h_w\,dx$ is holomorphic for $w$ below $\mathbb{R}$ and $w^\#$ above it (Fubini and Morera, as in Appendix D.3), and it vanishes where $w^\# = \bar{w}$ (Remark 5.2). A holomorphic function of $(w, w^\#)$ on a connected domain that vanishes on the totally real set $\{w^\# = \bar{w}\}$ vanishes identically; so $\int_{\mathbb{R}} h_w\,dx$ and its derivatives vanish at all complex parameters (Cauchy's estimate and the bound of that proof allow differentiation under the integral). Hence $\int_{\mathbb{R}} H^{[0]}\,dx = \int_{\mathbb{R}} H^{[1]}\,dx = 0$ also for complex parameters. Finally $\pi\varphi = \int_{\mathbb{R}} H_\times(x;1)\,dx$. □

The program checks, in ball arithmetic, the hypothesis of Lemma 6.8: for every $t \in [0,1]$ the moved nodes stay below $\mathbb{R}$ and their partners above; if this check fails, the program keeps the bound displayed above. By Taylor's formula of order 3 in $t$, on each piece $|H_\times - H^{[0]} - H^{[1]}| \le \sup|H^{[2]}| + \frac{1}{6}\sup_{0\le t\le 1}|\partial_t^3 H_\times|$, and also $|H_\times - H^{[0]} - H^{[1]}| \le \sup|H_\times| + \sup|H^{[0]}| + \sup|H^{[1]}|$; the program takes the smaller of the two on each piece, with the Taylor coefficients and the remainder enclosed in ball arithmetic. The tails are bounded as follows. For $H_\times$ Lemma 6.7 applies. For $H^{[0]} = 2a_1a_2h_{w_c}$ the same argument for one pole of mass 1 (the program checks $x_{\max} > 4|w_c|$ and $x_{\max} > 4|w_c^\#|$) gives $|h_{w_c}(x)| \le 4\big(\Upsilon_0(x)^2 + k_0\big)x^{-2}$, where $\Upsilon_0$ is formed like $\Upsilon$ from the constant logarithms of that pole, so the two tails of $H^{[0]}$ are at most $2|2a_1a_2|\big(4(\Upsilon_0(x_{\max})^2 + 2\Upsilon_0(x_{\max}) + 2) + 4k_0\big)/x_{\max}$. For $H^{[1]}$, the coefficient of $t$ in $H_\times(x;t)$, we use that $H_\times(x;t)$ is analytic in $t$ on the square $|\operatorname{Re} t|, |\operatorname{Im} t| \le \frac{1}{2}$: the program checks that on this square of configurations the nodes stay below $\mathbb{R}$ and the partners above. Then Cauchy's estimate on $|t| = \frac{1}{2}$ gives $|H^{[1]}(x)| \le 2\sup_{|t|=1/2}|H_\times(x;t)|$, and the tail bound for $H_\times$, taken over the same square, bounds the tails of $H^{[1]}$ after multiplication by 2.

*The moved path.* The path of integration is $\mathbb{R}$, with one exception: the bound $M_1$ of a corner or row box, where $s$ is complex. If the program cannot certify on such a ball that the zero lies below $\mathbb{R}$ and its partner above, the program uses a fixed admissible contour valid throughout that parameter ball. Each piece is checked to avoid every cut of Lemma D.2 throughout the ball. Its compact pieces are bounded by their arc lengths times the corresponding suprema of $|H_\times|$; Lemma 6.7 bounds the real tails. Lemma D.2 identifies this contour integral with the continued flux and gives the same integral along any other admissible path.

On the low pieces of the corner and row boxes the parameters are real and the path is $\mathbb{R}$, but near the zero the pieces may follow its foot $a\cos\gamma$, which moves over the sub-box. For numbers $x_\mathrm{l} < x_\mathrm{m} < x_\mathrm{r}$ chosen by the program for each sub-box, with $x_\mathrm{m}$ near the foot of the zero, and each parameter of the sub-box put $x'_\mathrm{l} = x_\mathrm{l} + a\cos\gamma - x_\mathrm{m}$ and $x'_\mathrm{r} = x_\mathrm{r} + a\cos\gamma - x_\mathrm{m}$; then

$$\int_{\mathbb{R}} H_\times\,dx = \Big(\int_{-\infty}^{x_\mathrm{l}} + \int_{x_\mathrm{l}}^{x'_\mathrm{l}} + \int_{x'_\mathrm{r}}^{x_\mathrm{r}} + \int_{x_\mathrm{r}}^{\infty}\Big) H_\times\,dx + \int_{x_\mathrm{l}-x_\mathrm{m}}^{x_\mathrm{r}-x_\mathrm{m}} H_\times(a\cos\gamma + \tilde{x})\,d\tilde{x},$$

the last integral being the one over $[x'_\mathrm{l}, x'_\mathrm{r}]$ after the change of variable $x = a\cos\gamma + \tilde{x}$. Its range of $\tilde{x}$ does not depend on the parameters, and it is cut towards $\tilde{x} = 0$, where the foot of the zero sits for every parameter of the sub-box. The program uses this split only where it certifies $x_\mathrm{l} - |a\cos\gamma - x_\mathrm{m}| > 0$ and $x_\mathrm{r} + |a\cos\gamma - x_\mathrm{m}| < \cos\gamma$ on the whole sub-box, so that the feet 0 and $\cos\gamma$ of the poles lie in the two outer integrals; elsewhere it cuts $\mathbb{R}$ as above.

### 6.5 Arithmetic

All values are enclosed in balls by the Arb library [4], at a precision of 96 bits. Ball arithmetic encloses a real number in an interval stored as a midpoint and a radius, and a complex number in a rectangle formed by two such intervals, one for its real part and one for its imaginary part [5]. The program uses python-flint 0.9.0 [6], the Python interface to Arb. The values of $\varphi_\xi$ at the interpolation points are computed on tight balls that contain these (irrational) points, and their radii are carried into the coefficients of $\hat{\varphi}_\xi$. Every comparison uses certified end points of finite balls. Before each division by, or logarithm of, $x - c$ with $c$ a node or a partner, the program checks that the ball of $x - c$ does not meet the cut of the logarithm. If a test fails on a ball of a corner or row box, the ball is split. On the large-angle boxes a failed test leaves the box without a certificate; none fails. Where the Taylor step of Section 6.4 would divide by a power series in $t$, such as the denominator $x - c(t)$ of a factor $1/(x - c(t))$, whose constant term $x - c(0)$ is not certified to be nonzero, the program keeps the first bound of that section, $\frac{1}{\pi}\big(\sum(\text{length}) \cdot \sup|H_\times| + \text{tail}\big)$.

### 6.6 Proof of Proposition 6.1

For each of the 168 boxes $B$ of Table 2 the program computes the numbers $u_B$, $\underline{\Phi}_B$ and $\beta_B$ and the break points of Section 6.2, and it verifies (11), with the bounds of Sections 6.3 and 6.4 for the integrals $I_k$, and $\underline{\Phi}_B - \beta_B > 0$; all the checks of Sections 6.3–6.5 pass. The smallest value of $\underline{\Phi}_B - \beta_B$ is 0.0285 after rounding down, and it occurs in the central corner box ($|s| \le 25/27$, $0 \le \gamma \le 2^{-20}$). By Lemma 6.3, (a) and (b) hold on $B$, and $u_B \le e^{-17.5} < 1/4$. □

## 7 Proof of Theorem 1.1

By Proposition 3.3 it suffices to prove Theorem 3.5: $\Phi(S) > 0$ for every two-pole Pick function $S$. Let $S$ be one, with masses $a_1, a_2$.

(i) If the two poles coincide, $S$ is a single pole and $\Phi(S) = k_1 - k_0 > 0$ by Corollary 3.4.

(ii) If $\min(a_1, a_2) \le 2^{-12} \max(a_1, a_2)$, then $\Phi(S) > 0$ by Lemma 4.5.

(iii) Otherwise, by Section 5, $\Phi(S) = \Phi(s, \gamma, u)$ for some $|s| < 12 \log 2 < 25/3$, $\gamma \in [0, \pi/2]$ and $u > 0$. The point $(s, \gamma)$ lies in some box $B$ of the cover of Section 6, and Proposition 6.1 gives $\Phi(s, \gamma, u) > 0$ for every $u \ge u_B$. For $0 < u < u_B$, Lemma 5.1 gives

$$\Phi(s, \gamma, u) = \Phi(s, \gamma, u_B) - \int_u^{u_B} \varphi(s, \gamma, u')\, du' \;\ge\; \underline{\Phi}_B - \beta_B > 0,$$

where $\underline{\Phi}_B$ is the certified lower bound at $u_B$ and $\beta_B$ the bound of Lemma 5.4 (or of Lemma E.4) on the box.

In every case $\Phi(S) > 0$. □

# 8 Discussion

Theorem 1.1 proves $\sigma(S) < 0$ and hence $K(S^{1+\varepsilon}) < \pi^2/12$ for small $\varepsilon > 0$ for every two-pole Pick function $S$, ruling out this two-pole family as a near-$p = 2$ counterexample to Liu's conjecture while leaving the known $L^p$ bounds on $\|P\|_{L^p \to L^p}$ unchanged. Going past this two-pole test raises wider questions about larger pole counts, the quotient $K$ on general test functions, and Liu's conjecture itself.

**Many poles.** The closest proved result is Theorem 3.5 for two poles; the next unknown is the same positivity $\Phi(S) > 0$ for Pick sums with $n \ge 3$ poles. Lemmas 4.4 and 4.5 are proved for two poles, and after normalisation an $n$-pole Pick sum has $3n - 3$ real parameters, so a computer cover of the kind used here grows quickly with $n$.

**The quotient $K$.** Even the many-pole statement would give only the first-order behaviour of $K$ along critical families of Pick sums. Proposition 3.1 makes $K(F) \le k_0$ a necessary consequence of Liu's conjecture for every $F$ satisfying its hypotheses; powers of Pick sums form a restricted class, and the inequality for general $F$ remains open.

**Liu's conjecture.** Proposition 3.1 goes in one direction only: Liu's conjecture near $p = 2$ implies $K \le k_0$. It is open whether $K(F) \le k_0$ for all $F$ as in Proposition 3.1 implies $\|P\|_{L^p \to L^p}^p \le 1 + k_0(p-2)^2 + o((p-2)^2)$, and the results of this note give no information on $\|P\|_{L^p \to L^p}$ for any fixed $p \ne 2$.


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# Code and data

The program of Section 6, the list of all boxes, the certificate log of every box, and the Lean 4 files are in the repository named below. Repository:
https://github.com/david44hello/paper_liu-2018-conjecture-two-pole-test/tree/v2.04

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## A Single-pole constants

We use the unit disc $\mathbb{D} = \{|w| < 1\}$, which we also give the normalised area measure $dA = dx\,dy/\pi$ (so $\mathbb{D}$ has measure 1 and $\|w^n\|^2_{L^2(\mathbb{D})} = 1/(n+1)$). For a power series in $w$ we write $[w^m]$ for its coefficient of $w^m$.

### A.1 The Cayley transform

Let $\kappa(w) = i(1+w)/(1-w)$, which maps $\mathbb{D}$ onto $\mathbb{U}$, with inverse $w = (z-i)/(z+i)$. Then

$$\kappa(w) + i = \frac{2i}{1-w}, \qquad \kappa(w) - i = \frac{2iw}{1-w}, \qquad \kappa'(w) = \frac{2i}{(1-w)^2}. \tag{12}$$

For a function $F$ on $\mathbb{U}$ put $(\kappa^* F)(w) = F(\kappa(w))\,\kappa'(w)$. Since $|\kappa'|^2$ is the Jacobian of $\kappa$, we have $\int_{\mathbb{U}} F\,\overline{\widetilde{F}}\,dA = \int_{\mathbb{D}} \kappa^* F\,\overline{\kappa^* \widetilde{F}}\,dA$ whenever $F\overline{\widetilde{F}}$ is integrable. So $\kappa^*$ is a unitary map of $L^2(\mathbb{U})$ onto $L^2(\mathbb{D})$. It maps $A^2(\mathbb{U})$ onto the Bergman space $A^2(\mathbb{D})$, because $\kappa'$ has no zeros. Hence $\kappa^* P = P_{\mathbb{D}} \kappa^*$, where $P_{\mathbb{D}}$ is the Bergman projection of $\mathbb{D}$.

Let $\ell(w) = \operatorname{Log}(1-w) = -\sum_{j\ge 1} w^j/j$, analytic on $\mathbb{D}$. Since $\operatorname{Re}(1-w) > 0$ for $w \in \mathbb{D}$, we have $\arg(1-w) \in (-\pi/2, \pi/2)$, and (12) gives, with principal logarithms,

$$\operatorname{Log}\big(\kappa(w) + i\big) = \operatorname{Log}(2i) - \ell(w). \tag{13}$$

On $\mathbb{D}$ the monomials are orthogonal, and for $n \ge j \ge 0$ the projection of $w^n \bar{w}^j$ is

$$P_{\mathbb{D}}\big(w^n \bar{w}^j\big) = \frac{n-j+1}{n+1}\, w^{n-j}, \tag{14}$$

while $P_{\mathbb{D}}(w^n \bar{w}^j) = 0$ for $n < j$. Indeed $\langle w^n \bar{w}^j, w^m \rangle = \delta_{n,j+m}/(n+1)$ and $\|w^m\|^2 = 1/(m+1)$.

The next lemma identifies the function $v_S$ in Lemmas A.3 and D.1.

**Lemma A.1.** *For $m \ge 0$ let $g_m(z) = (z-i)^m (z+i)^{-m-3}$. Let $V$ be analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$, with $V(z) = O\big((1+\log|z|)/|z|\big)$ as $|z| \to \infty$. If $\int_{\mathbb{U}} V\,\bar{g}_m\,dA = 0$ for every $m \ge 0$, then $V = 0$.*

*Proof.* By (12), $|\kappa(w)| \geq c\,|1-w|^{-1}$ near $w=1$. It follows that $|\kappa^*V(w)| \leq C\big(1+\log(1/|1-w|)\big)|1-w|^{-1}$ there, while $\kappa^*V$ is bounded on the rest of $\mathbb{D}$. Thus $\kappa^*V$ is analytic and integrable on $\mathbb{D}$. Write $\kappa^*V = \sum_n t_n w^n$. Integrating over $\{|w|<r\}$ and letting $r\to 1$ (dominated convergence) gives $\langle \kappa^*V, w^m\rangle = t_m/(m+1)$. Since $\kappa^* g_m = -\frac{1}{4}(1-w)w^m$ and $\int_{\mathbb{U}} V\bar{g}_m\,dA = \int_{\mathbb{D}} \kappa^*V\,\overline{\kappa^* g_m}\,dA$, the hypothesis says $t_m/(m+1) = t_{m+1}/(m+2)$ for every $m$. So $t_m = (m+1)t_0$ and $\kappa^*V = t_0(1-w)^{-2}$. This is compatible with the bound on $\kappa^*V$ near $w=1$ only if $t_0=0$. Thus $\kappa^*V=0$, and $V=0$. □

## A.2 The quotient for one pole

We write $\psi = \Gamma'/\Gamma$ and $\psi_1 = \psi'$, and use the Pochhammer symbol $(x)_n = x(x+1)\cdots(x+n-1)$.

**Lemma A.2.** *For $1<\alpha<2$ let $F_\alpha(z) = (z+i)^{-\alpha} = \exp\big(-\alpha\,\mathrm{Log}(z+i)\big)$ on $\mathbb{U}$. Then*

$$K(F_\alpha) = \frac{\alpha^2}{4}\Big[2\psi_1(\alpha) - \frac{\Sigma_\alpha}{N_\alpha}\Big], \qquad N_\alpha = \frac{\Gamma(2\alpha-2)}{\Gamma(\alpha)^2}, \qquad \Sigma_\alpha = \sum_{n\geq 0}\Big(\frac{(2-\alpha)_n}{n!}\Big)^2 \frac{\psi_1(n+2)}{n+1}. \tag{15}$$

*As $\varepsilon\to 0^+$,*

$$K\big((z+i)^{-1-\varepsilon}\big) = k_0 + \Big(\frac{\pi^2}{6} - \frac{3}{2}\zeta(3)\Big)\varepsilon + O(\varepsilon^2), \qquad \frac{\pi^2}{6} - \frac{3}{2}\zeta(3) = -0.1581512\ldots < 0.$$

*Proof. Step 1: transfer to the disc.* Put $\tau = 2-\alpha \in (0,1)$ and $e_\tau(w) = (1-w)^{-\tau} = \exp(-\tau\ell(w))$. By (13), $F_\alpha(\kappa(w)) = (2i)^{-\alpha}\exp(\alpha\ell(w))$. We obtain $\kappa^*F_\alpha = 2i(2i)^{-\alpha}e_\tau$ and $\log|F_\alpha(\kappa(w))| = \alpha\,\mathrm{Re}\,\ell(w) - \alpha\log 2$. Therefore

$$\kappa^*\big(F_\alpha \log|F_\alpha|\big) = 2i(2i)^{-\alpha}\Big(\frac{\alpha}{2}\,e_\tau\ell + \frac{\alpha}{2}\,e_\tau\bar{\ell} - \alpha\log 2\; e_\tau\Big).$$

The functions $e_\tau$ and $e_\tau\ell$ are analytic and, since $|e_\tau|^2 = |1-w|^{-2\tau}$ with $2\tau<2$, they lie in $A^2(\mathbb{D})$, and $e_\tau\bar{\ell}\in L^2(\mathbb{D})$. Since $\kappa^*$ is unitary and $\kappa^*P = P_{\mathbb{D}}\kappa^*$,

$$K(F_\alpha) = \frac{\alpha^2}{4}\,\frac{\|(I-P_{\mathbb{D}})(\bar{\ell}e_\tau)\|^2}{\|e_\tau\|^2}. \tag{16}$$

*Step 2: the projection.* Write $e_\tau = \sum_{n\geq 0} c_n w^n$ with $c_n = (\tau)_n/n! > 0$. By (14), $P_{\mathbb{D}}(\bar{\ell}e_\tau) = \sum_m b_m w^m$ with

$$b_m = (m+1)\langle \bar{\ell}e_\tau, w^m\rangle = -(m+1)\sum_{j\geq 1}\frac{c_{m+j}}{j\,(m+j+1)}.$$

(Integrate over $\{|w|<r\}$, where the double series converges absolutely, and let $r\to 1$: the series side converges monotonically, since all its terms have the same sign, and the integral side by dominated convergence.) Euler's integral gives $c_n/(n+1) = \Gamma(n+\tau)/(\Gamma(\tau)\Gamma(n+2)) = \int_0^1 t^{n+\tau-1}(1-t)^{1-\tau}\,dt\,/\,(\Gamma(\tau)\Gamma(2-\tau))$. Inserting this, summing $\sum_{j\geq 1} t^j/j = -\log(1-t)$ under the integral (all terms positive), and using $\int_0^1 t^{x-1}(1-t)^{y-1}\log(1-t)\,dt = B(x,y)\big(\psi(y)-\psi(x+y)\big)$ with $x = m+\tau$, $y = 2-\tau$, we get

$$b_m = c_m\big(\psi(2-\tau) - \psi(m+2)\big). \tag{17}$$

*Step 3: the norms.* By Gauss's summation theorem, for $c > 2\tau$,

$$Z(c) := \sum_{n\geq 0}\frac{(\tau)_n^2}{(c)_n\,n!} = \frac{\Gamma(c)\Gamma(c-2\tau)}{\Gamma(c-\tau)^2}. \tag{18}$$

Since $(2)_n = (n+1)!$, the $n$-th term of $Z(2)$ is $c_n^2/(n+1)$. Summing over $n$, we get $\|e_\tau\|^2 = Z(2) = \Gamma(2-2\tau)/\Gamma(2-\tau)^2 = N_\alpha$. The numbers $c_n^2/((n+1)N_\alpha)$, $n\geq 0$, form a probability distribution;

write $\mathbb{E}$ and Var for mean and variance with respect to it, and put $X_n = \psi(n+2)$, $x_1 = \psi(2-\tau)$, $x_2 = \psi(2-2\tau)$. By (17),

$$\frac{\|P_{\mathbb{D}}(\bar{\ell}e_\tau)\|^2}{N_\alpha} = \sum_m \frac{b_m^2}{(m+1)N_\alpha} = \mathbb{E}\big[(x_1 - X)^2\big] = (x_1 - \mathbb{E}X)^2 + \operatorname{Var} X.$$

The logarithmic derivative in $c$ of the $n$-th term of $Z(c)$ is $\psi(c) - \psi(c+n)$. Differentiating $\log Z$ once and twice termwise (allowed since, on compact subsets of $c > 2\tau$, the terms and their $c$-derivatives are $O(n^{2\tau-c-1}\log^2(n+2))$), and comparing with the derivatives of the right side of (18), at $c = 2$:

$$\mathbb{E}X = 2x_1 - x_2, \qquad \operatorname{Var} X = \psi_1(2-2\tau) - 2\psi_1(2-\tau) + \mathbb{E}\big[\psi_1(n+2)\big].$$

(For the second, $(\log Z)'' = \psi_1(c) - \mathbb{E}_c[\psi_1(c+n)] + \operatorname{Var}_c[\psi(c+n)]$ for the weights of $Z(c)$.) Next, for real $\tau_1, \tau_2$ with $\tau_1 + \tau_2 < 2$, expanding both factors in powers of $w$ and $\bar{w}$ and using Gauss's theorem again,

$$J(\tau_1,\tau_2) := \int_{\mathbb{D}} (1-w)^{-\tau_1}(1-\bar{w})^{-\tau_2}\, dA = \sum_{n\ge 0} \frac{(\tau_1)_n(\tau_2)_n}{n!\,(n+1)!} = \frac{\Gamma(2-\tau_1-\tau_2)}{\Gamma(2-\tau_1)\Gamma(2-\tau_2)}.$$

Since $\partial_{\tau_1}(1-w)^{-\tau_1} = -(1-w)^{-\tau_1}\ell$ and $\partial_{\tau_2}(1-\bar{w})^{-\tau_2} = -(1-\bar{w})^{-\tau_2}\bar{\ell}$, and the differentiated integrands are dominated near $(\tau_1,\tau_2) = (\tau,\tau)$ by an integrable multiple of $|1-w|^{-2\tau'}\big(1+\log^2|1-w|\big)$ for any fixed $\tau' \in (\tau, 1)$, we get $\|\bar{\ell}e_\tau\|^2 = \partial_{\tau_1}\partial_{\tau_2} J(\tau,\tau)$. Differentiating the gamma quotient,

$$\frac{\|\bar{\ell}e_\tau\|^2}{N_\alpha} = \psi_1(2-2\tau) + (x_2 - x_1)^2.$$

By Pythagoras, and since $(x_1 - \mathbb{E}X)^2 = (x_2 - x_1)^2$,

$$\frac{\|(I - P_{\mathbb{D}})(\bar{\ell}e_\tau)\|^2}{N_\alpha} = \psi_1(2-2\tau) - \operatorname{Var} X = 2\psi_1(2-\tau) - \mathbb{E}\big[\psi_1(n+2)\big] = 2\psi_1(\alpha) - \frac{\Sigma_\alpha}{N_\alpha}.$$

With (16) this is (15).

*Step 4: the expansion at $\alpha = 1+\varepsilon$.* Now $\tau = 1-\varepsilon$, and $N_\alpha = \Gamma(2\varepsilon)/\Gamma(1+\varepsilon)^2 = (2\varepsilon)^{-1}\Gamma(1+2\varepsilon)/\Gamma(1+\varepsilon)^2 = (2\varepsilon)^{-1}(1+O(\varepsilon))$. For $0 < \varepsilon \le \frac12$, that is $\tau \in [\frac12, 1]$, we have $0 < c_n \le 1$ and $0 \le \partial_\tau c_n = c_n \sum_{k<n}(\tau+k)^{-1} \le 2 + H_n$, so $0 \le \partial_\tau(c_n^2) \le 2(2+H_n)$, where $H_n$ is the harmonic number. Since $\psi_1(n+2) = \sum_{k\ge n+2} k^{-2} \le (n+1)^{-1}$, the series $\sum_n (2+H_n)\psi_1(n+2)/(n+1)$ converges, and therefore $\Sigma_\alpha = \Sigma + O(\varepsilon)$ with $\Sigma = \sum_{n\ge0}\psi_1(n+2)/(n+1)$. This is an Euler sum:

$$\begin{aligned}\Sigma &= \sum_{n\ge0}\frac{1}{n+1}\sum_{k\ge n+2}\frac{1}{k^2} = \sum_{k\ge2}\frac{H_{k-1}}{k^2} = \frac12\int_0^1 \frac{\log^2(1-x)}{x}\,dx \\ &= \frac12\int_0^1 \frac{\log^2 y}{1-y}\,dy = \frac12\sum_{j\ge0}\frac{2}{(j+1)^3} = \zeta(3),\end{aligned}$$

where the third equality uses $\log^2(1-x) = 2\sum_{k\ge2} H_{k-1}x^k/k$. Hence $\Sigma_\alpha/N_\alpha = 2\zeta(3)\varepsilon + O(\varepsilon^2)$. Also $\psi_1(1+\varepsilon) = \sum_{j\ge1}(j+\varepsilon)^{-2} = \zeta(2) - 2\zeta(3)\varepsilon + O(\varepsilon^2)$. Inserting both into (15),

$$K(F_{1+\varepsilon}) = \frac{(1+\varepsilon)^2}{4}\big[2\zeta(2) - 6\zeta(3)\varepsilon + O(\varepsilon^2)\big] = \frac{\zeta(2)}{2} + \Big(\zeta(2) - \frac32\zeta(3)\Big)\varepsilon + O(\varepsilon^2),$$

and $\zeta(2)/2 = \pi^2/12 = k_0$. The coefficient is negative: $\zeta(2) < \sum_{n\le5} n^{-2} + \int_5^\infty x^{-2}dx < \frac53$, while $\frac32\zeta(3) > \frac32\big(1+\frac18\big) = \frac{27}{16} > \frac53$. □

### A.3 The optimal witness of one pole

**Lemma A.3.** *Let $S(z) = -a/(z-w_0)$ with $a > 0$ and $w_0 = x_0 - iy_0$, $y_0 > 0$. Let $\omega$ be as in (7) and $q = \omega - \pi/2$.*

(a) *$q$ is analytic on $\mathbb{U}$, continuous on $\overline{\mathbb{U}}$, and $\sup_{\overline{\mathbb{U}}}|q| = \log 2$. In particular, $\omega$ is a witness.*
(b) $W_S^*(z) = \omega\big((z-x_0)/y_0\big)$.
(c) *On $\overline{\mathbb{U}}$, $\big|\arg S - W_S^*\big| \le \pi/2 + \log 2 < 2.264$. In particular, $|\operatorname{Re}(\arg S - W_S^*)| < 2.264$.*
(d) *On $\overline{\mathbb{U}}$, $\big|k_0 - |\arg S - W_S^*|^2\big| < 4.304$.*

*Proof.* (a) For $z \in \overline{\mathbb{U}}$ and $0 \le t \le 1$, the number $2i + t(z-i)$ has imaginary part $2 + t(\operatorname{Im} z - 1) \ge 2 - t > 0$. So the path $t \mapsto 2i + t(z-i)$ stays in $\mathbb{U}$, where Log is analytic, and

$$q(z) = \frac{\operatorname{Log}(z+i) - \operatorname{Log}(2i)}{z-i} = \int_0^1 \frac{dt}{2i + t(z-i)}.$$

The integral is analytic in $z \in \mathbb{U}$ and continuous on $\overline{\mathbb{U}}$ (this also shows that $z = i$ is removable, with $q(i) = -i/2$), and $|q(z)| \le \int_0^1 dt/(2-t) = \log 2$. At $z = 0$ the integrand is $-i/(2-t)$. Thus $|q(0)| = \log 2$. Finally, since $q(z) = O\big((1 + \log|z|)/|z|\big)$, the function $\omega = \pi/2 + q$ is a witness.

(b) *Reduction.* Put $\hat z = (z - x_0)/y_0$ and $S_1(\hat z) = -1/(\hat z + i)$. Then $S(z) = (a/y_0)\, S_1(\hat z)$ with $a/y_0 > 0$. It follows that $\arg S(z) = \arg S_1(\hat z)$, $f_S(z) = (a/y_0) f_{S_1}(\hat z)$ and $\rho_S(z) = (a/y_0^2)\rho_{S_1}(\hat z)$. The function $z \mapsto a\, U_{S_1}(\hat z)$ is bounded, vanishes on $\mathbb{R}$ and has $-\Delta = 2\rho_S$. It is therefore $U_S$. Consequently $v_S(z) = (a/y_0) v_{S_1}(\hat z)$ and $W_S^*(z) = W_{S_1}^*(\hat z)$. It suffices to show $W_{S_1}^* = \omega$, that is, the case $S(z) = -1/(z+i)$.

*The pairing on the disc.* Let $S(z) = -1/(z+i)$ and $v_\omega = S(\arg S - \omega)$. By (13), $\arg S(\kappa(w)) = \pi - \arg(\kappa(w) + i) = \pi/2 + \operatorname{Im}\ell(w)$, and by (12), $\omega(\kappa(w)) = \pi/2 - \ell(w)(1-w)/(2iw)$. Since $S(\kappa(w))\kappa'(w) = -1/(1-w)$,

$$\kappa^* v_\omega = -\frac{\arg S \circ \kappa - \omega \circ \kappa}{1-w} = \frac{i}{2}\Big(\frac{\ell}{1-w} + \frac{\ell}{w}\Big) - \frac{i}{2}\,\frac{\bar\ell}{1-w}.$$

By (a), $|\kappa^* v_\omega| \le (\pi/2 + \log 2)\,|1-w|^{-1}$. Since $|1-w|^{-1}$ is integrable on $\mathbb{D}$, the pairings $\langle \kappa^* v_\omega, w^m\rangle = \int_{\mathbb{D}} \kappa^* v_\omega\, \bar w^m\, dA$ exist. The first bracket is analytic, with $[w^m]\,\ell/(1-w) = -H_m$ and $[w^m]\,\ell/w = -1/(m+1)$, and its pairing with $w^m$ is therefore $-H_{m+1}/(m+1)$ (integrate over $\{|w| < r\}$ and let $r \to 1$). For the second, $\bar\ell/(1-w) = -\sum_{j\ge1}\sum_{n\ge0} w^n \bar w^j/j$, and, as in Step 2 of the proof of Lemma A.2,

$$\Big\langle \frac{\bar\ell}{1-w}, w^m \Big\rangle = -\sum_{j\ge1} \frac{1}{j\,(m+j+1)} = -\frac{H_{m+1}}{m+1}.$$

Hence $\langle \kappa^* v_\omega, w^m\rangle = \frac{i}{2}\big(-\frac{H_{m+1}}{m+1}\big) - \frac{i}{2}\big(-\frac{H_{m+1}}{m+1}\big) = 0$ for every $m \ge 0$.

*Uniqueness.* Put $V = v_\omega - v_S = S(W_S^* - \omega)$. It is analytic on $\mathbb{U}$ (both terms have $\bar\partial = \rho_S$), continuous on $\overline{\mathbb{U}}$, and $O(1/r)$ by (a) and Lemma 2.3. Since the functions $g_m$ of Lemma A.1 are analytic near $\overline{\mathbb{U}}$ and $O(r^{-3})$, Lemma 4.2 gives $\int_{\mathbb{U}} v_S\, \bar g_m\, dA = 0$. Also, since $\kappa^* g_m = -\frac14(1-w)w^m$, $\int_{\mathbb{U}} v_\omega\, \bar g_m\, dA = -\frac14\big(\langle \kappa^* v_\omega, w^m\rangle - \langle \kappa^* v_\omega, w^{m+1}\rangle\big) = 0$. So $V = 0$ by Lemma A.1, that is, $W_S^* = \omega$.

(c) By (b), at each point $\arg S - W_S^* = (\arg S - \pi/2) - q$ (with $q$ taken at $(z-x_0)/y_0$), where $\arg S - \pi/2 \in (-\pi/2, \pi/2)$ and $|q| \le \log 2$ by (a). So $|\arg S - W_S^*| \le \pi/2 + \log 2 = 2.26394\ldots$.

(d) By (c), $0 \le |\arg S - W_S^*|^2 \le (\pi/2 + \log 2)^2 = 5.12544\ldots$. Therefore $k_0 - |\arg S - W_S^*|^2 \in [k_0 - 5.12545, k_0] = [-4.30299, 0.82247]$, and its absolute value is below 4.304. □

# B Proofs of Propositions 3.1 and 3.3

## B.1 Proof of Proposition 3.1

We may assume $\eta_0 \le 1$ (the Taylor bound below uses $1 \le p \le 3$). The exponent 5 is what the error terms in (19) need. Let $p = 2 + \delta$ with $0 < |\delta| \le \eta_0$. Since $|G| \le C|F|(1+L)$, $G \in L^p$. On $L^p$, $P$ is the integral operator with the Bergman kernel, which is bounded on $L^p$ [8, Theorem 1]. For $f \in L^2$

and $z \in \mathbb{U}$ the kernel is square-integrable in the other variable, so the integral converges absolutely and is the inner product of $f$ with the reproducing kernel at $z$, that is, the orthogonal projection of $f$ at $z$. Hence the two operators agree on $L^2 \cap L^p$, and $PF = F$, $PG = 0$. So $F = P(F - \delta G)$, and therefore $\|P\|^p_{L^p \to L^p} \geq \|F\|_p^p / \|F - \delta G\|_p^p$, where $\|\cdot\|_p$ is the norm of $L^p(\mathbb{U}, dA)$. We show

$$\|F - \delta G\|_p^p - \|F\|_p^p = -\delta^2 \|G\|^2 + o(\delta^2). \tag{19}$$

Since $\|F\|_p^p \to \|F\|^2$ by dominated convergence (note $|F|^p \leq |F|^{2-\eta_0} + |F|^{2+\eta_0}$), (19) gives $\|F\|_p^p / \|F - \delta G\|_p^p = 1 + K(F)\delta^2 + o(\delta^2)$ as $\delta \to 0$. If $\|P\|_{L^{p_n} \to L^{p_n}} \leq \lambda_{p_n}$ with $p_n = 2 + \delta_n$, then by (6) $1 + K(F)\delta_n^2 + o(\delta_n^2) \leq 1 + k_0 \delta_n^2 + O(\delta_n^3)$, and $\delta_n \to 0$ gives $K(F) \leq k_0$. If $K(F) > k_0$, the same two expansions give $\|F\|_p^p / \|F - \delta G\|_p^p > \lambda_p^p$, hence $\|P\|_{L^p \to L^p} > \lambda_p$, for all small $\delta \neq 0$.

We have $|G/F| \leq C(1 + L)$. Let $E_\delta = \{z : C|\delta|(1 + L) \leq \frac{1}{2}\}$. For $|y| \leq \frac{1}{2}$ and $1 \leq p \leq 3$, $|1 + y|^p = 1 + p \operatorname{Re} y + \frac{p}{2}|y|^2 + \frac{p(p-2)}{2}(\operatorname{Re} y)^2 + R(y)$ with $|R(y)| \leq c|y|^3$ ($c$ absolute). On $E_\delta$ we apply this with $y = -\delta G/F$. Off $E_\delta$ we have $1 < 2C|\delta|(1 + L)$, hence $\big||F - \delta G|^p - |F|^p\big| \leq |F|^p\big((3C|\delta|(1 + L))^3 + 1\big) \leq C'\delta^2 |F|^p (1 + L)^5$ there, and $\int_{\mathbb{U} \setminus E_\delta} |F|^p (1 + L)^5 \, dA \to 0$ by dominated convergence; so the set $\mathbb{U} \setminus E_\delta$ contributes $o(\delta^2)$, and the same bound shows that the integrals below may be taken over all of $\mathbb{U}$ at a cost $o(\delta^2)$. The four terms are:

(i) $-p\delta \operatorname{Re} \int |F|^\delta \bar{F} G \, dA$. With $|F|^\delta = 1 + \delta \log |F| + O\big(\delta^2 L^2 (|F|^{|\delta|} + |F|^{-|\delta|})\big)$ and $\langle G, F\rangle = 0$, $\langle G, F \log |F|\rangle = \langle G, G\rangle$ (because $F \log |F| - G = P(F \log |F|) \in A^2$), this term is $-p\delta^2 \|G\|^2 + O(\delta^3)$.

(ii) $\frac{p}{2}\delta^2 \int |F|^\delta |G|^2 \, dA = \delta^2 \|G\|^2 + O(\delta^3)$.

(iii) $\frac{p(p-2)}{2}\delta^2 \int |F|^p (\operatorname{Re}(G/F))^2 \, dA = O(\delta^3)$, since $p - 2 = \delta$.

(iv) $\int |F|^p |R(-\delta G/F)| \, dA \leq cC^3 |\delta|^3 \int |F|^p (1 + L)^3 \, dA = O(\delta^3)$.

All error terms are dominated by $(|F|^{2-\eta_0} + |F|^{2+\eta_0})(1 + L)^5$. Since $p = 2 + \delta$, the sum of (i) and (ii) is $-\delta^2 \|G\|^2 + O(\delta^3)$. This proves (19). □

## B.2 Proof of Proposition 3.3

Fix a Pick sum $S$ of total mass $m$, fix $\varepsilon_1 \in (0, \frac{1}{2}]$, and write $g = g_\varepsilon = S^{1+\varepsilon}$ for $0 \leq \varepsilon \leq \varepsilon_1$. For a smooth source $\rho$ on $\overline{\mathbb{U}}$ with $|\rho| = O(|z|^{-2})$ let $\mathcal{C}\rho = -2\partial V$, where $V$ is the bounded solution of $-\Delta V = 2\rho$ with $V = 0$ on $\mathbb{R}$; so $\bar\partial \mathcal{C}\rho = \rho$, and $\mathcal{C}\rho_S = v_S$. Let $\mathcal{G}$ be the Dirichlet Green operator of $-\Delta$ on $\mathbb{U}$ with respect to $dx\, dy$, so that $V = 2\mathcal{G}\rho$.

**Lemma B.1.** *Let $\rho$ be a smooth source on $\overline{\mathbb{U}}$ with $|\rho| \leq C(1 + |z|)^{-2-\varepsilon}$ for some $\varepsilon \in (0, 1)$. Then $\mathcal{C}\rho$ is continuous on $\overline{\mathbb{U}}$ and $O(|z|^{-1-\varepsilon})$, so $\mathcal{C}\rho \in L^2$, and $\mathcal{C}\rho = (I - P)h$ for every $h \in L^2$ with $\bar\partial h = \rho$.*

*Proof.* As for $U_S$ in Section 2.3, $V$ is smooth up to $\mathbb{R}$ (boundary Schauder estimates), so $\mathcal{C}\rho$ is continuous on $\overline{\mathbb{U}}$. The Green potential of $|z'|^{-2-\varepsilon}$ is finite (the Green function vanishes to first order at the boundary point 0, so the integrand is $O(|z'|^{-1-\varepsilon})$ near 0, integrable since $\varepsilon < 1$) and homogeneous of degree $-\varepsilon$; it equals $r^{-\varepsilon}\Theta(\theta)$ with $\Theta(\theta) = \varepsilon^{-2}\big(\cos(\varepsilon(\theta - \pi/2))/\cos(\varepsilon\pi/2) - 1\big) \geq 0$, because $\Theta'' + \varepsilon^2\Theta = -1$ and $\Theta(0) = \Theta(\pi) = 0$: the potential vanishes on $\mathbb{R} \setminus \{0\}$, as the Green function does (here $\cos(\varepsilon\pi/2) > 0$ needs $\varepsilon < 1$). Next, $V = 2\mathcal{G}\rho$: the right side is bounded, since $|\rho| \leq C|z'|^{-2}$ and the Green potential of $|z'|^{-2}$ is at most $\pi^2/8$ (Section 2.3), and it vanishes on $\mathbb{R}$; so it is the bounded solution. Since $|\rho| \leq C|z'|^{-2-\varepsilon}$ and the Green function is positive, $|V| \leq 2Cr^{-\varepsilon}\Theta(\theta)$, so $V = O(|z|^{-\varepsilon})$, and Lemma 2.2(a) gives $\nabla V = O(|z|^{-1-\varepsilon})$; since $2 + 2\varepsilon > 2$, $\mathcal{C}\rho \in L^2$. Let $F$ be analytic on $\mathbb{U}$, continuous on $\overline{\mathbb{U}}$ and $O(|z|^{-2})$. As in the proof of Lemma 4.2, $\langle \mathcal{C}\rho, F\rangle$ is a boundary integral of $V\bar{F}$; it vanishes on $\mathbb{R}$, and on the half-circle of radius $R$ it is $O(R^{-1-\varepsilon})$; so $\mathcal{C}\rho \perp F$. The functions $(z + i)^{-n}$, $n \geq 2$, are such $F$, and the Cayley transform of Appendix A maps them to constant multiples of $(1 - w)^{n-2}$, whose span is the polynomials, dense in $A^2$ of the disc. So $\mathcal{C}\rho \perp A^2$. Finally $h - \mathcal{C}\rho$ is in $L^2$ and $\bar\partial(h - \mathcal{C}\rho) = 0$, so it lies in $A^2$, and $\mathcal{C}\rho = (I - P)h$. □

For the sources of Lemma B.1, Green's formula gives $\|\mathcal{C}\rho\|^2 = 4\langle\rho, \mathcal{G}\rho\rangle$: the line contributes nothing because $V = 0$ there, the half-circle term is $O(R^{-2\varepsilon})$ by the rates above, and the cross term $|2\partial V|^2 - |\nabla V|^2$ integrates to a boundary term of the same size (as in Appendix D). The kernel of $\mathcal{G}$ is real, so this energy is unchanged when $\rho$ is replaced by $\bar\rho$.

*Step 1: the two sources.* We have $\bar\partial(g \log|g|) = g\,\bar\partial \log|g| = \frac{1}{2}(g/\bar g)\,\overline{g'}$. With $\rho_0 = \frac{1}{2}(S/\overline{S})\,\overline{S'}$ this is

$$\rho_\varepsilon = (1+\varepsilon)\, S^\varepsilon \rho_0,$$

and $|\rho_\varepsilon| \le C(1+|z|)^{-2-\varepsilon}$; so for $\varepsilon > 0$ Lemma B.1 (with $h = g\log|g|$) gives $K(g) = \|\mathcal{C}\rho_\varepsilon\|^2/\|g\|^2$. Replacing $g/\bar g$ by its value $S/\overline{S}$ at $\varepsilon = 0$ gives the source $b_\varepsilon = \frac{1}{2}(S/\overline{S})\,\overline{g'} = (1+\varepsilon)\,\overline{S}^{\,\varepsilon}\rho_0$. Note $b_0 = \rho_0 = -i\rho_S$ by (3), so $|\mathcal{C}b_0| = |v_S|$.

*Step 2: freezing costs $O(\varepsilon)$ in the numerator.* Conjugating $\rho_\varepsilon$ preserves its energy and makes its difference from $b_\varepsilon$ decay as $|z|^{-4}$, up to the bounded power factor. Put $d_\varepsilon = b_\varepsilon - \overline{\rho_\varepsilon} = (1+\varepsilon)\overline{S}^{\,\varepsilon}(\rho_0 - \overline{\rho_0})$. Since $S'/S^2 = 1/m + O(|z|^{-2})$ and $\rho_0 = \frac{1}{2}|S|^2\,\overline{S'/S^2}$, we get $\rho_0 - \overline{\rho_0} = O((1+|z|)^{-4})$ on $\overline{\mathbb{U}}$. The Green potential $\mathcal{G}\tilde\rho$ of a source $\tilde\rho$ with $|\tilde\rho| \le C(1+\log(2+|z|))(1+|z|)^{-4}$ satisfies $|\mathcal{G}\tilde\rho(z)| \le C'\,\mathrm{Im}\, z/(1+|z|^2)$ (split the source into the regions $|z'| < |z|/2$, $|z|/2 \le |z'| \le 2|z|$ and $|z'| > 2|z|$, and use $\int_{\mathbb{R}} \mathcal{G}(z, X+iY)\,dX = \min(\mathrm{Im}\, z, Y)$ on horizontal lines). This applies to $d_\varepsilon$ and to $\partial_\varepsilon d_\varepsilon = d_\varepsilon\big((1+\varepsilon)^{-1} + \log\overline{S}\big)$, uniformly for $0 \le \varepsilon \le \varepsilon_1$: $S$ is bounded on $\overline{\mathbb{U}}$, and $|\log S| \le C(1+\log(2+|z|))$ there, since $S$ has no zero on $\overline{\mathbb{U}}$ and $S \sim -m/z$ at infinity. For $\varepsilon > 0$, polarization gives

$$\tfrac{1}{4}\big(\|\mathcal{C}b_\varepsilon\|^2 - \|\mathcal{C}\rho_\varepsilon\|^2\big) = 2\,\mathrm{Re}\langle\overline{\rho_\varepsilon}, \mathcal{G}d_\varepsilon\rangle + \langle d_\varepsilon, \mathcal{G}d_\varepsilon\rangle.$$

The right-hand side extends to $\varepsilon = 0$, although the two energies on the left diverge separately there: both integrands, with their $\varepsilon$-derivatives, are dominated by $C(1+\log(2+r))(1+r)^{-3}$, which is integrable. So the right side is $C^1$ on $[0, \varepsilon_1]$. At $\varepsilon = 0$ it vanishes: writing $\rho_0 = \rho' + i\rho''$ with $\rho', \rho''$ real, and using that $\mathcal{G}$ maps real functions to real functions, we have $d_0 = 2i\rho''$, $\mathcal{G}d_0 = 2i\,\mathcal{G}\rho''$, and the two integrands are $2\,\mathrm{Re}\big[(\rho' - i\rho'')\overline{2i\,\mathcal{G}\rho''}\big] = -4\rho''\,\mathcal{G}\rho''$ and $4\rho''\,\mathcal{G}\rho''$, which cancel pointwise. Hence $\|\mathcal{C}b_\varepsilon\|^2 - \|\mathcal{C}\rho_\varepsilon\|^2 = O(\varepsilon)$, and since $\|g\|^2 = m^2/(2\varepsilon) + O(1)$,

$$K(g_\varepsilon) = \frac{\|\mathcal{C}b_\varepsilon\|^2}{\|g_\varepsilon\|^2} + O(\varepsilon^2). \tag{20}$$

*Step 3: the far model.* For $0 \le \varepsilon \le \varepsilon_1$ let $y_\varepsilon$ solve $y'' + \varepsilon^2 y = (1+\varepsilon)e^{i\varepsilon\theta}$ on $[0,\pi]$ with $y(0) = y(\pi) = 0$, and put

$$B(\varepsilon) = \frac{1}{\pi}\int_0^\pi \big|\varepsilon y_\varepsilon + i y_\varepsilon'\big|^2 d\theta.$$

In the coordinates $z = e^{t+i\theta}$ $(0 < \theta < \pi)$, and for $g = z^{-1-\varepsilon}$, with $g/\bar g$ replaced by $e^{-2i\theta}$, the function $V = e^{-\varepsilon t}y_\varepsilon(\theta)$ solves the transformed problem: here $S = -1/z$, so $S/\overline{S} = e^{-2i\theta}$ and $b_\varepsilon = \frac{1}{2}e^{-2i\theta}\,\overline{g'} = -\frac{1}{2}(1+\varepsilon)r^{-2-\varepsilon}e^{i\varepsilon\theta}$; since $\Delta = e^{-2t}(\partial_t^2 + \partial_\theta^2)$, the equation $-\Delta V = 2b_\varepsilon$ for $V = e^{-\varepsilon t}y(\theta)$ reads $y'' + \varepsilon^2 y = (1+\varepsilon)e^{i\varepsilon\theta}$, and $V = 0$ on $\mathbb{R}$ reads $y(0) = y(\pi) = 0$. Moreover $z\,\mathcal{C}b = e^{-\varepsilon t}(\varepsilon y_\varepsilon + i y_\varepsilon')$ while $z\,g = e^{-\varepsilon t}e^{-i\varepsilon\theta}$; so on each circle the ratio of $\int|\mathcal{C}b|^2$ to $\int|g|^2$ is $B(\varepsilon)$. Expanding, $y_\varepsilon = y^{[0]} + \varepsilon y^{[1]} + O(\varepsilon^2)$ with $y^{[0]} = \frac{1}{2}\theta(\theta-\pi)$ and $y^{[1]} = \frac{1}{2}\theta(\theta-\pi) + \frac{i}{6}(\theta^3 - \pi^2\theta)$; then $|\varepsilon y_\varepsilon + i y_\varepsilon'|^2 = (y^{[0]\prime})^2(1+2\varepsilon) + O(\varepsilon^2)$ because $\mathrm{Re}\, y^{[1]\prime} = y^{[0]\prime} = \theta - \pi/2$. Therefore

$$B(\varepsilon) = \frac{1}{\pi}\int_0^\pi \Big(\theta - \frac{\pi}{2}\Big)^2 d\theta\,(1+2\varepsilon) + O(\varepsilon^2) = k_0\,(1+2\varepsilon) + O(\varepsilon^2). \tag{21}$$

*Step 4: the frozen numerator.* Let $N(\varepsilon) = \|\mathcal{C}b_\varepsilon\|^2 - k_0\|g_\varepsilon\|^2$, and split $\mathbb{U}$ into $D_R = \mathbb{U}\cap\{|z| < R\}$ $(R \ge 2)$ and its complement $\Omega_R$, and let $V_\varepsilon$ be the bounded potential with $\mathcal{C}b_\varepsilon = -2\partial V_\varepsilon$. On $D_R$, as $\varepsilon \to 0^+$: $|b_\varepsilon - b_0| \le C\varepsilon(1+\log(2+|z|))(1+|z|)^{-2}$, since $S$ is bounded and $|\log S| \le C(1+\log(2+|z|))$ (Step 2), and the Green potential of $(1+\log(2+|z'|))(1+|z'|)^{-2}$ is locally bounded. So $V_\varepsilon \to V_0$

uniformly on compact parts of $\overline{\mathbb{U}}$, and the interior and boundary elliptic estimates, applied to $V_\varepsilon - V_0$, give uniform convergence of the gradients on $\overline{D_{R'}}$ for every $R'$; also $g_\varepsilon \to S$ uniformly on $\overline{D_R}$. Hence $\int_{D_R}(|\mathcal{C}b_\varepsilon|^2 - k_0|g_\varepsilon|^2)\,dA \to \int_{D_R}(|v_S|^2 - k_0|S|^2)\,dA$.

On $\Omega_R$ we compare with the model of Step 3. Since $S = -mz^{-1}(1+O(r^{-1}))$ and $S' = mz^{-2}(1+O(r^{-1}))$, we have $b_\varepsilon = b_\varepsilon^\circ + f_\varepsilon$, where $b_\varepsilon^\circ$ is the model source of Step 3 for $(-m/z)^{1+\varepsilon}$, of modulus $\frac{1}{2}(1+\varepsilon)m^{1+\varepsilon}r^{-2-\varepsilon}$, and $|f_\varepsilon| \le Cr^{-3-\varepsilon} \le Cr^{-3}$ for $r \ge 1$, uniformly in $\varepsilon$ (no logarithm appears). Let $V_\varepsilon^\circ = c_\varepsilon r^{-\varepsilon} y_\varepsilon(\theta)$, with a constant $|c_\varepsilon| \le C$, be the model potential of Step 3 for $b_\varepsilon^\circ$; let $\chi$ be a smooth radial cut-off with $\chi = 0$ for $r \le 1$ and $\chi = 1$ for $r \ge 2$; and put $U_\varepsilon = V_\varepsilon - \chi V_\varepsilon^\circ$. Since $y_\varepsilon(0) = y_\varepsilon(\pi) = 0$, $U_\varepsilon$ is bounded and vanishes on $\mathbb{R}$, and $|\Delta U_\varepsilon| \le Cr^{-3}$ for $r \ge 2$ and $|\Delta U_\varepsilon| \le C$ for $r \le 2$ (this includes the terms in which a derivative falls on $\chi$), uniformly for $0 \le \varepsilon \le \varepsilon_1$. So $U_\varepsilon$ is the Green potential of $-\Delta U_\varepsilon$, and Lemma 2.2(b) gives $U_\varepsilon = O((1+\log r)/r)$ and $\nabla U_\varepsilon = O((1+\log r)/r^2)$, uniformly in $\varepsilon$. Hence on $\Omega_R$ the function $\mathcal{C}b_\varepsilon$ is the model $-2\partial V_\varepsilon^\circ$, of size $r^{-1-\varepsilon}$, up to an error $O((1+\log r)r^{-2})$; and $g_\varepsilon = (-m/z)^{1+\varepsilon}(1+O(r^{-1}))$. Against the model, these errors give integrands $O\big((1+\log r)r^{-3}\big)$ uniformly in $\varepsilon$. So, for $0 \le \varepsilon \le \varepsilon_1$,

$$E_R^\circ(\varepsilon) = \int_{\Omega_R} \Big[\big(|\mathcal{C}b_\varepsilon|^2 - k_0|g_\varepsilon|^2\big) - \big(|2\partial V_\varepsilon^\circ|^2 - k_0|(-m/z)^{1+\varepsilon}|^2\big)\Big]\,dA$$

converges absolutely. At $\varepsilon = 0$ their difference is absolutely integrable, although the individual brackets are not. The function $E_R^\circ$ is continuous on $[0,\varepsilon_1]$: its integrand is dominated by $C(1+\log r)r^{-3}$ uniformly in $\varepsilon$ and converges pointwise. For $\varepsilon > 0$ the model bracket integrates over $\Omega_R$, by Step 3 and $\int_{\Omega_R}|z|^{-2-2\varepsilon}\,dA = R^{-2\varepsilon}/(2\varepsilon)$, to $m^{2+2\varepsilon}\big(B(\varepsilon)-k_0\big)R^{-2\varepsilon}/(2\varepsilon)$; hence

$$\int_{\Omega_R}\big(|\mathcal{C}b_\varepsilon|^2 - k_0|g_\varepsilon|^2\big)dA = m^{2+2\varepsilon}\,\big(B(\varepsilon)-k_0\big)\frac{R^{-2\varepsilon}}{2\varepsilon} + E_R^\circ(\varepsilon).$$

At $\varepsilon = 0$ the model bracket integrates to 0 over every half-annulus ($B(0) = k_0$), so $E_R^\circ(0) = \lim_{R'\to\infty}\int_{R<|z|<R'}(|v_S|^2 - k_0|S|^2)\,dA$, the tail of the finite part (4). By (21) the model term tends to $k_0m^2$ as $\varepsilon \to 0^+$. Letting $\varepsilon \to 0^+$ at fixed $R$, the integral over $D_R$ and $E_R^\circ(0)$ add up to $-\mathcal{D}(S)$ by (4), so

$$\lim_{\varepsilon\to 0^+} N(\varepsilon) = k_0\,m^2 - \mathcal{D}(S). \tag{22}$$

*Step 5: conclusion.* By (20), (22) and $\|g_\varepsilon\|^2 = m^2/(2\varepsilon) + O(1)$,

$$K(g_\varepsilon) = k_0 + \frac{N(\varepsilon)}{\|g_\varepsilon\|^2} + O(\varepsilon^2) = k_0 + \frac{2\varepsilon}{m^2}\big(k_0m^2 - \mathcal{D}(S)\big) + o(\varepsilon).$$

This is Proposition 3.3. □

# C Small-mass constants

In this appendix we prove Lemma 4.5. By Lemma 2.5, $\mathcal{D}(cS) = c^2\mathcal{D}(S)$ and the deficit is invariant under real translation and dilation. Let $w_1$ and $w_2$ be the poles of masses $m_1$ and $m_2$. We may therefore assume $m_1 = 1$ and $w_1 = -i$; then $m_2 \le 2^{-12}$. Put

$$S_{w_1} = -\frac{1}{z+i}, \qquad S_{w_2} = -\frac{1}{z-w_2}, \qquad S = S_{w_1} + m_2S_{w_2}, \qquad \xi = \frac{m_2S_{w_2}}{S_{w_1}} = m_2\,\frac{z+i}{z-w_2}$$

Put $d = |w_1 - w_2|$. If $d = 0$, then $S = (1+m_2)S_{w_1}$ has one pole and Corollary 3.4 gives the claim; so let $d > 0$. Write $r_1 = |z+i|$ and $r_2 = |z-w_2|$ for the distances to the two poles (with $r_1 \ge 1$ on $\overline{\mathbb{U}}$, and $r = |z|$ as before) and $L_{m_2} = \log(2/m_2)$. We use the optimal witnesses $\omega_1 = W^*_{S_{w_1}}$ and $\omega_2 = W^*_{S_{w_2}}$

of Lemma A.3, the functions $v_1 = v_{S_{w_1}} = S_{w_1}\iota$ with $\iota = \arg S_{w_1} - \omega_1$, and $v_2 = S_{w_2}(\arg S_{w_2} - \omega_2)$. By Lemma A.3(c),(d), with $M_\omega = \pi/2 + \log 2 < 2.264$,

$$|v_1| \le M_\omega |S_{w_1}|, \qquad |v_2| \le M_\omega |S_{w_2}|, \qquad \bigl|k_0 - |\iota|^2\bigr| < 4.304. \tag{23}$$

Throughout, $m_2 \le 1/10$ suffices for the region estimates, and only the last step uses $m_2 \le 2^{-12}$.

## C.1 The blended witness

Since $S = S_{w_1}(1+\xi)$ and $\arg S, \arg S_{w_1} \in (0,\pi)$, the difference $\arg S - \arg S_{w_1}$ lies in $(-\pi,\pi)$ and equals $\arg(1+\xi)$. Likewise, $\arg S - \arg S_{w_2} = \arg(1+1/\xi)$. Put

$$\begin{aligned} W_{\mathrm{b}} &= \frac{S_{w_1}\omega_1 + m_2 S_{w_2}\,\omega_2}{S}, \\ g &= S_{w_1}\arg(1+\xi) + m_2 S_{w_2}\arg(1+1/\xi) \\ &= S\arg S - S_{w_1}\arg S_{w_1} - m_2 S_{w_2}\arg S_{w_2}. \end{aligned}$$

$W_{\mathrm{b}}$ is a witness: $S$ has no zero on $\overline{\mathbb{U}}$, and $S_{w_1}\omega_1 + m_2 S_{w_2}\omega_2 = \frac{\pi}{2}S + O(\log r/r^2)$ while $S = -(1+m_2)/z\,(1+O(1/r))$. Pointwise, $S(\arg S - W_{\mathrm{b}}) = v_1 + m_2 v_2 + g$, and $k_0|S|^2 - |v_1+m_2v_2|^2 = (k_0|S_{w_1}|^2 - |v_1|^2) + m_2^2(k_0|S_{w_2}|^2 - |v_2|^2) + 2m_2\,\mathrm{Re}(k_0 S_{w_1}\bar S_{w_2} - v_1\bar v_2)$. Integrate over $D_R$ and let $R\to\infty$. The first two terms give $\mathcal{D}(S_{w_1}) + m_2^2\mathcal{D}(S_{w_2}) = k_1(1+m_2^2)$ (Corollary 3.4). The terms with $g$ converge absolutely, because $g = O(r^{-2})$ and $v_1, v_2 = O(1/r)$. Since $L_S[W_{\mathrm{b}}]$ exists (Lemma 4.3), so does $J = \lim_{R\to\infty}\int_{D_R}\mathrm{Re}(k_0 S_{w_1}\bar S_{w_2} - v_1\bar v_2)\,dA$, and

$$\mathcal{D}(S) \;\ge\; L_S[W_{\mathrm{b}}] = k_1(1+m_2^2) + 2m_2 J - \|g\|^2 - 2\,\mathrm{Re}\langle v_1 + m_2 v_2,\, g\rangle. \tag{24}$$

Put $g_{\mathrm{lin}} = S_{w_1}\,\mathrm{Im}(S_{w_2}/S_{w_1}) + S_{w_2}(\arg S_{w_1} - \arg S_{w_2})$, the limit of $g/m_2$ as $m_2\to 0$, which is $O(r^{-2})$. Then (24) reads $\mathcal{D}(S) \ge k_1(1+m_2^2) + m_2\Psi_1 + \Psi_2$ with

$$\Psi_1 = 2J - 2\,\mathrm{Re}\langle v_1, g_{\mathrm{lin}}\rangle, \qquad \Psi_2 = -2\,\mathrm{Re}\langle v_1, g - m_2 g_{\mathrm{lin}}\rangle - \|g\|^2 - 2m_2\,\mathrm{Re}\langle v_2, g\rangle.$$

## C.2 The size of $g$

We use $|\arg(1+\eta)| \le \arcsin|\eta| \le \frac{\pi}{2}|\eta|$ for $|\eta|\le 1$, $\eta\ne -1$, and $|\arg(1+\eta)| \le \pi$ for every $\eta \ne -1$ (here $1+\xi = S/S_{w_1}$ and $1+1/\xi = S/(m_2 S_{w_2})$ never vanish).

(i) If $|\xi| \le 1$, then $|g| \le \frac{\pi}{2}|S_{w_1}||\xi| + \pi m_2|S_{w_2}| = \frac{3\pi}{2}m_2|S_{w_2}|$. If $|\xi|\ge 1$, then $|g| \le \pi|S_{w_1}| + \frac{\pi}{2}m_2|S_{w_2}|/|\xi| = \frac{3\pi}{2}|S_{w_1}|$. In both cases $|g| \le \frac{3\pi}{2}\min(m_2|S_{w_2}|, |S_{w_1}|)$.

(ii) *Far region* $r_2 \ge 2d$. Write $\xi = m_2 + \xi'$ with $\xi' = m_2(w_2 - w_1)/(z - w_2)$. Then $|\xi'| \le m_2 d/r_2 \le m_2/2$. Put $\eta = \xi'/(1+m_2)$. Since $1+\xi = (1+m_2)(1+\eta)$ and $|\eta| \le m_2 d/r_2$, we get $|\arg(1+\xi)| = |\arg(1+\eta)| \le \frac{\pi}{2}m_2 d/r_2$. Also $\arg(1+1/\xi) = \arg(1+\xi) - \arg\xi$ with $|\arg\xi| = |\arg(1+(w_2-w_1)/(z-w_2))| \le \frac{\pi}{2}d/r_2$. With $|S_{w_1}| \le 1/(r_2-d) \le 2/r_2$, $|S_{w_2}| = 1/r_2$: $|g| \le \frac{\pi}{2}(3+m_2)\,m_2 d/r_2^2$.

(iii) *Disc* $r_2 < r_A = m_2 d/(1-m_2)$. If $r_2 \ge r_A$, then $m_2|z+i| \le m_2(r_2+d) \le r_2$, that is, $|\xi|\le 1$. Thus $\{|\xi|>1\} \subset \{r_2 < r_A\}$. In the disc, $|S_{w_1}| \le 1/(d - r_A) = (1-m_2)/(d(1-2m_2))$.

(iv) *Annulus* $r_A \le r_2 < 2d$: there $\min(m_2|S_{w_2}|, |S_{w_1}|) = m_2/r_2$.

With $dA = dx\,dy/\pi$, the disc $\{r_2 < c\}$ has measure $c^2$, $\int_{r_2<c} r_2^{-1}dA = 2c$, $\int_{c_1<r_2<c_2} r_2^{-2}dA = 2\log(c_2/c_1)$ and $\int_{r_2>c} r_2^{-k}dA = 2c^{2-k}/(k-2)$ for $k>2$. Hence, for $m_2 \le 1/10$,

$$\|g\|^2 \le \tfrac{9\pi^2}{4}\Bigl(\frac{m_2^2}{(1-2m_2)^2} + 2m_2^2 L_{m_2}\Bigr) + \Bigl(\tfrac{\pi}{2}(3+m_2)\Bigr)^2\frac{m_2^2}{4} \le m_2^2\bigl(41 + 44.42\,L_{m_2}\bigr),$$

since $\frac{9\pi^2}{4}\cdot 1.5625 + 5.928 < 41$ and $\frac{9\pi^2}{2} < 44.42$.

## C.3 The first-order term

**Lemma C.1.** $\Psi_1 = 2k_1 + 2\operatorname{Re}\mathcal{Y}$, *where* $\mathcal{Y} = \int_{\mathbb{U}} S_{w_1}\,\overline{(S_{w_2} - S_{w_1})}\,Q\,dA$ *(absolutely convergent) and* $Q = k_0 - |\iota|^2 - i\operatorname{Re}\iota$. *Moreover* $|\Psi_1 - 2k_1| \le 167$ *for every position of* $w_2$.

*Proof. The identity.* The function $S_{w_2}(\omega_1 - \omega_2)$ is analytic on $\mathbb{U}$, continuous on $\overline{\mathbb{U}}$ and $O((1+\log r)/r^2)$, because both witnesses are $\pi/2 + O((1+\log r)/r)$. Substitute $v_1 = S_{w_1}\iota$, $v_2 = S_{w_2}(\arg S_{w_2} - \omega_2)$ and $\bar\iota = \arg S_{w_1} - \bar\omega_1$. A direct expansion, using $|S_{w_1}|^2 \operatorname{Im}(S_{w_2}/S_{w_1}) = -\operatorname{Im}(S_{w_1}\bar S_{w_2})$, gives at every point

$$\operatorname{Re}\big(k_0 S_{w_1}\bar S_{w_2} - v_1\bar v_2\big) - \operatorname{Re}\big(v_1\bar g_{\mathrm{lin}}\big) = \operatorname{Re}\big(S_{w_1}\bar S_{w_2}\,Q\big) - \operatorname{Re}\big(v_1\overline{S_{w_2}(\omega_1-\omega_2)}\big).$$

We integrate over $D_R$ and let $R\to\infty$. Since $\int_{\mathbb{U}} v_1\overline{S_{w_2}(\omega_1-\omega_2)}\,dA = 0$ by Lemma 4.2, we obtain $\Psi_1 = 2\lim_R \int_{D_R}\operatorname{Re}(S_{w_1}\bar S_{w_2}\,Q)\,dA$. Split $S_{w_1}\bar S_{w_2} = |S_{w_1}|^2 + S_{w_1}\overline{(S_{w_2}-S_{w_1})}$. The first part gives $2\lim_R\int_{D_R}|S_{w_1}|^2(k_0-|\iota|^2)\,dA = 2L_{S_{w_1}}[\omega_1] = 2\mathcal{D}(S_{w_1}) = 2k_1$ (Lemma 4.3, equality case). In the second, $|S_{w_2} - S_{w_1}| = d/(r_1r_2)$. The integrand is therefore $O(r^{-3})$, and $\mathcal{Y}$ converges absolutely.

*The bound.* By (23), $|Q| \le c_Q := \sqrt{4.304^2 + 2.264^2} < 4.87$, and $|S_{w_1}\overline{(S_{w_2}-S_{w_1})}| = d/(r_1^2 r_2)$.

Let $d \le 4$. On $\{r_2 > r_1\}$ the integrand is at most $dr_1^{-3}$, and $\int_{r_1\ge 1} r_1^{-3}dA = 2$. On $\{r_2 \le r_1\}$ we have $r_1 \ge \max(r_2, d/2, 1) =: \max(r_2, c_0)$, and therefore $1/(r_1^2r_2) \le \min(c_0^{-2}r_2^{-1}, r_2^{-3})$, whose integral is at most $2c_0^{-1} + 2c_0^{-1}$. Therefore $|\mathcal{Y}| \le c_Q d\,(2 + 4/c_0) \le 16\,c_Q$ (check $d\le 2$ and $2 < d \le 4$ separately).

If $d > 4$, let $\mathcal{N} = \{r_1 < d/2\}$. Outside $\mathcal{N}$, cover $\mathbb{U}$ by $\{r_2 < d/2\}$, $\{r_2 \ge d/2,\ r_1 \le r_2\}$ and $\{r_2 \ge d/2,\ r_2 < r_1\}$. On the first, $r_1 \ge d/2$ and the integral of $d/(r_1^2r_2)$ is at most $d(2/d)^2\cdot 2(d/2) = 4$. On the second it is at most $d\int_{r_1\ge d/2} r_1^{-3}dA = 4$, and on the third at most $d\int_{r_2\ge d/2} r_2^{-3}dA = 4$. Inside $\mathcal{N}$ we have $r_2 > d/2$, and

$$\int_{\mathcal{N}} S_{w_1}\overline{(S_{w_2}-S_{w_1})}\,Q\,dA = \int_{\mathcal{N}} S_{w_1}\bar S_{w_2}\,Q\,dA - \int_{\mathcal{N}}|S_{w_1}|^2 Q\,dA,$$
$$\Big|\int_{\mathcal{N}} S_{w_1}\bar S_{w_2}\,Q\,dA\Big| \le c_Q\frac{2}{d}\int_{r_1<d/2}\frac{dA}{r_1} = 2c_Q,$$

and $\operatorname{Re}\int_{\mathcal{N}}|S_{w_1}|^2 Q\,dA = I(d/2)$, where $I(R') = \int_{\mathbb{U}\cap\{r_1<R'\}}|S_{w_1}|^2(k_0-|\iota|^2)\,dA$. So $|\operatorname{Re}\mathcal{Y}| \le 14c_Q + I^*$ with $I^* = \sup_{R'}|I(R')|$.

*The bound on $I^*$.* In polar coordinates $z = -i + r_1e^{i\chi}$, the circle $r_1 = t > 1$ meets $\mathbb{U}$ in $\chi \in (\chi_0, \pi-\chi_0)$, where $\sin\chi_0 = 1/t$. There $|S_{w_1}|^2 = t^{-2}$, $\arg S_{w_1} = \pi - \chi$ and $\iota = (\pi/2 - \chi) - q$, with $q$ as in Lemma A.3. So $I(R') = \pi^{-1}\int_1^{R'} t^{-1}\int_{\chi_0}^{\pi-\chi_0}(k_0 - |\iota|^2)\,d\chi\,dt$. For $R' \le 20$, $|I(R')| \le 4.304\log 20 < 12.894$ by (23). For $t > 2$ we have $\chi_0 < \pi/6$. Since $\int_0^\pi (k_0 - (\pi/2-\chi)^2)\,d\chi = 0$ and the integrand lies in $[-\pi^2/6, 0]$ for $\chi \le \chi_0$, the angular integral of $k_0 - (\pi/2-\chi)^2$ over $(\chi_0, \pi-\chi_0)$ lies in $[0, \frac{\pi^2}{3}\chi_0] \subset [0, \frac{\pi^3}{6t}]$. The rest of $k_0 - |\iota|^2$ is $2(\pi/2-\chi)\operatorname{Re}q - |q|^2$, of absolute value at most $(\pi + \log 2)|q|$. On the circle, $|q| \le |\operatorname{Log}(z+i) - \operatorname{Log}(2i)|/|z-i| \le (\log t + 5.52)/(t-2)$. Hence the inner integral (over $\chi$) has modulus at most $5.17/t + 12.05(\log t + 5.52)/(t-2)$, and for $R' > 20$

$$|I(R') - I(20)| \le \frac{1}{\pi}\int_{20}^{\infty}\Big(\frac{5.17}{t^2} + 12.05\cdot\frac{20}{18}\cdot\frac{\log t + 5.52}{t^2}\Big)dt < 2.111.$$

So $I^* < 12.894 + 2.111 = 15.005$, and $|\Psi_1 - 2k_1| \le 2\max(16c_Q,\ 14c_Q + I^*) < 2\max(77.92,\ 83.19) < 167$. □

## C.4 The remainder

**Lemma C.2.** *For* $m_2 \le 1/10$ *and every position of* $w_2$, $\Psi_2 \ge -m_2^2(520 + 111L_{m_2})$.

*Proof.* We bound the three terms of $\Psi_2$, using (23) and the regions of the previous subsection.

$\|g\|^2 \le m_2^2(41 + 44.42 L_{m_2})$, shown above.

$2m_2|\langle v_2, g\rangle| \le 2m_2 M_\omega \int |S_{w_2}||g|\, dA$. On the disc, $|g| \le \frac{3\pi}{2}|S_{w_1}|$ and $\int_{r_2<r_A} r_2^{-1} dA = 2r_A$, which gives at most $\frac{3\pi}{2} \cdot 2m_2/(1-2m_2) \le \frac{3\pi}{2} \cdot 2.5\, m_2$. On the annulus, $|g| \le \frac{3\pi}{2} m_2/r_2$, which gives at most $\frac{3\pi}{2} \cdot 2m_2 L_{m_2}$. On the far region, (ii) gives at most $\frac{\pi}{2}(3+m_2)m_2 < 4.87 m_2$. Altogether $2m_2|\langle v_2, g\rangle| \le 2M_\omega\big(\frac{3\pi}{2} \cdot 2.5 + 4.87\big) m_2^2 + 6\pi M_\omega m_2^2 L_{m_2} < m_2^2(75.40 + 42.68 L_{m_2})$.

$2|\langle v_1, g - m_2 g_{\mathrm{lin}}\rangle| \le 2M_\omega \int |S_{w_1}||g - m_2 g_{\mathrm{lin}}|\, dA$. Here $g - m_2 g_{\mathrm{lin}} = S_{w_1}\big(\arg(1+\xi) - \operatorname{Im}\xi\big) + m_2 S_{w_2} \arg(1+\xi)$. We use $|\operatorname{Log}(1+\eta) - \eta| \le |\eta|^2/(2(1-|\eta|))$ and three regions. On the far region, with $\eta$ as in (ii), $\arg(1+\xi) - \operatorname{Im}\xi = \operatorname{Im}(\operatorname{Log}(1+\eta) - \eta) - m_2 \operatorname{Im}\eta$, of size at most $|\eta|^2/(2(1-|\eta|)) + m_2|\eta| \le m_2^2(d/r_2)(0.53\, d/r_2 + 1) \le 1.27\, m_2^2 d/r_2$ (as $|\eta| \le m_2 d/r_2 \le 1/20$). With $|S_{w_1}| \le 2/r_2$ and $|\arg(1+\xi)| \le \frac{\pi}{2} m_2 d/r_2$ this gives $|g - m_2 g_{\mathrm{lin}}| \le 4.12\, m_2^2 d/r_2^2$, and the far region contributes at most $2M_\omega \cdot 4.12 \cdot 2m_2^2 = 16.48 M_\omega m_2^2$. On $\{|\xi| \le 1/2,\ r_2 < 2d\}$ we have $r_2 \ge 2m_2 d/(1+2m_2)$, $|\arg(1+\xi) - \operatorname{Im}\xi| \le |\xi|^2$ and $|\arg(1+\xi)| \le \frac{\pi}{3}|\xi|$. These bounds give $|g - m_2 g_{\mathrm{lin}}| \le (1 + \frac{\pi}{3}) m_2^2 |S_{w_2}|^2/|S_{w_1}| < 2.57 m_2^2 |S_{w_2}|^2/|S_{w_1}|$, and this region contributes at most $2M_\omega \cdot 2.57 m_2^2 \cdot 2 \log\frac{1+2m_2}{m_2} \le 4 \cdot 2.57\, M_\omega m_2^2 L_{m_2}$. On $\{|\xi| > 1/2\} \subset \{r_2 < r_B\}$, $r_B = 2m_2 d/(1-2m_2) \le 2.5 m_2 d$, we have $|S_{w_1}| \le 1/(d - r_B) \le 1.34/d$, $|g| \le \frac{3\pi}{2}|S_{w_1}|$ and $|m_2 g_{\mathrm{lin}}| \le (1+\pi) m_2/r_2 < 4.15 m_2/r_2$. This region contributes at most $2M_\omega\big(\frac{3\pi}{2} \cdot 1.34^2 \cdot 6.25 + 4.15 \cdot 1.34 \cdot 5\big) m_2^2 < 365.36\, m_2^2$. Altogether $2|\langle v_1, g - m_2 g_{\mathrm{lin}}\rangle| < m_2^2(402.67 + 23.28 L_{m_2})$.

The sum of the three bounds is less than $m_2^2\big((41 + 75.40 + 402.67) + (44.42 + 42.68 + 23.28) L_{m_2}\big) < m_2^2(520 + 111 L_{m_2})$. □

*Proof of Lemma 4.5.* By (24) and Lemmas C.1 and C.2, with $m = 1 + m_2$,

$$\begin{aligned}\mathcal{D}(S) - k_0 m^2 &\ge (k_1 - k_0) - m_2\big(167 - 2(k_1 - k_0)\big) + m_2^2(k_1 - k_0) - m_2^2(520 + 111 L_{m_2}) \\ &\ge (k_1 - k_0) - m_2\big(167 - 2(k_1 - k_0)\big) - m_2^2(520 + 111 L_{m_2}).\end{aligned}$$

The last expression decreases in $m_2$ on $(0, 2^{-12}]$, and at $m_2 = 2^{-12}$ it is at least $0.0790756 - 0.0407329 - 0.0000907 > 0.0382$. Since $m_1 = 1$, this is the claim $\mathcal{D}(S) - k_0 m^2 > 0.038\, m_1^2$. □

# D The flux identity and the closed form of $v_S$

## D.1 Proof of Lemma 5.1

Fix $(s, \gamma)$, put $S_* = S_{s,\gamma,u_*}$ for some $u_* > 0$, and for $t > -u_*$ let $S_t(z) = S_*(z + it)$, so that $S_t = S_{s,\gamma,u_*+t}$. In the coordinate $Z = z + it$ the function $S_t$ on $\mathbb{U}$ becomes $S_*$ on the moving half-plane $\Omega_t = \{\operatorname{Im} Z > t\}$; the source $\rho(Z) = \rho_{S_*}(Z)$ of (3) is *fixed* in these coordinates and only the domain moves. Let $\mathcal{G}_t$ be the Dirichlet Green function of $-\Delta$ on $\Omega_t$, $U_t = \mathcal{G}_t(2\rho)$, and let $\mathcal{P}_y(x) = y/(x^2 + y^2)$ be the Poisson kernel, so that $\mathcal{P}_a * \mathcal{P}_b = \mathcal{P}_{a+b}$ for the convolution with $dx/\pi$.

*The derivative of the Green function.* For $Z, Z' \in \Omega_t$, $\mathcal{G}_t(Z, Z') = \frac{1}{2}\log\big|(Z - \bar{Z}' - 2it)/(Z - Z')\big|$ (with the normalisation of $dA$), and a direct differentiation gives

$$\partial_t \mathcal{G}_t(Z, Z') = -\mathcal{P}_{\operatorname{Im} Z + \operatorname{Im} Z' - 2t}(\operatorname{Re} Z - \operatorname{Re} Z') = -\int_{\mathbb{R}} \mathcal{P}_{\operatorname{Im} Z - t}(\operatorname{Re} Z - x)\, \mathcal{P}_{\operatorname{Im} Z' - t}(\operatorname{Re} Z' - x)\, \frac{dx}{\pi}.$$

The inward normal derivative of $\mathcal{G}_t(\cdot, Z')$ at the boundary point $x + it$ is $\mathcal{P}_{\operatorname{Im} Z' - t}(x - \operatorname{Re} Z')$. Hence $b_t(x) := \partial_y U_t(x + it) = \int_{\Omega_t} \mathcal{P}_{\operatorname{Im} Z' - t}(x - \operatorname{Re} Z')\, 2\rho(Z')\, dA(Z')$, and $\partial_t U_t(Z) = -\int_{\mathbb{R}} \mathcal{P}_{\operatorname{Im} Z - t}(\operatorname{Re} Z - x)\, b_t(x)\, dx/\pi$; there is no term from the moving boundary, because $\mathcal{G}_t$ vanishes there.

*Domination.* $|\rho(Z)| \le \frac{1}{2}\sum_j a_j |Z - w_j|^{-2}$. For the term of $w_j$, at depth $y_j > 0$ below the line $\operatorname{Im} Z = t$, with $x_j = \operatorname{Re} w_j$, the semigroup property gives the exact boundary derivative $a_j \int_0^\infty \mathcal{P}_{2y+y_j}(x - x_j)(y + y_j)^{-1} dy$ (for a source point $Z'$ put $y = \operatorname{Im} Z' - t$; then $|Z' - w_j|^{-2} = \mathcal{P}_{y+y_j}(\operatorname{Re} Z' - x_j)/(y + y_j)$, and $\mathcal{P}_y * \mathcal{P}_{y+y_j} = \mathcal{P}_{2y+y_j}$), which is at most $\min\big(a_j/y_j, \pi a_j/(2|x - x_j|)\big)$.

So $b_t$ and its analogue $b_t^{\mathrm{abs}}$ with $|\rho|$ in place of $\rho$ are in $L^2(\mathbb{R})$, uniformly for $t$ in compact subintervals, and Tonelli gives $\int_{\Omega_t} |\rho||\partial_t U_t|\,dA \le \int_{\mathbb{R}} (b_t^{\mathrm{abs}})^2 dx/\pi < \infty$.

*The energy and the mass.* Let $E_R(t) = \operatorname{Re}\int_{\Omega_t\cap\{|Z|<R\}} \rho\,\overline{U_t}\,dA$. Its derivative has no term from the lower boundary ($U_t = 0$ there), so $E_R'(t) = \operatorname{Re}\int_{\Omega_t\cap\{|Z|<R\}} \rho\,\overline{\partial_t U_t}\,dA$; the rule for differentiating under the integral applies, because $|\partial_t U_t| \le \sup_x |b_t(x)| \le \sum_j a_j/y_j$ up to the moving line. As $R\to\infty$ this tends to the same integral over $\Omega_t$, uniformly for $t\in[t_1,t_2]$ ($-u_* < t_1 < t_2$). Indeed, put $b^*(x) = \sum_j \min\bigl(a_j/(t_1+u_*),\, \pi a_j/(2|x-x_j|)\bigr)$; it is in $L^2(\mathbb{R})$, it does not depend on $t$, and $b_t^{\mathrm{abs}} \le b^*$. The part of the integral over $|Z|\ge R$ is at most $\frac12\int_{\mathbb{R}} b^*\, b_t^{\mathrm{abs},R}\,dx/\pi$, where $b_t^{\mathrm{abs},R} \le b^*$ is $b_t^{\mathrm{abs}}$ with the source cut to $|Z'|\ge R$. For each $x$, $\sup_t b_t^{\mathrm{abs},R}(x) = O(1/R)$, so this tends to 0 uniformly in $t$ by dominated convergence. By the formula for $\partial_t U_t$ and Fubini,

$$\int_{\Omega_t} \rho\,\overline{\partial_t U_t}\,dA = -\int_{\mathbb{R}} \overline{b_t(x)}\Bigl(\int_{\Omega_t} \mathcal{P}_{\operatorname{Im} Z - t}(\operatorname{Re} Z - x)\,\rho(Z)\,dA(Z)\Bigr)\frac{dx}{\pi} = -\frac12\int_{\mathbb{R}} |b_t(x)|^2\,\frac{dx}{\pi},$$

because the inner integral is $b_t(x)/2$ ($\mathcal{P}_y$ is even). So $\lim_{R\to\infty} E_R'(t) = -\frac12\int_{\mathbb{R}} |b_t(x)|^2 dx/\pi$, locally uniformly in $t$; integrating in $t$ first at fixed $R$ and then letting $R\to\infty$ justifies the exchange of limits. The cut mass $M_R(t) = \int_{\Omega_t\cap\{|Z|<R\}} |S_*|^2 dA$ has derivative $-\int_{x^2+t^2<R^2} |S_*(x+it)|^2 dx/\pi$. On $D_{R,t} = \Omega_t\cap\{|Z|<R\}$, Green's formula gives $\pi^{-1}\int_{D_{R,t}} |\nabla U_t|^2 dx\,dy = 2\operatorname{Re}\int_{D_{R,t}} \rho\overline{U_t}\,dA$ plus a term on the arc $|Z| = R$ (the line contributes nothing, as $U_t = 0$ there). We apply Lemma 2.3 to $S_t$ in the original coordinate $z = Z - it$ (so that $U_t(Z) = U_{S_t}(z)$), with $\theta = \arg z$ and $U_0$ as there; on the arc, $\bigl||z| - R\bigr| \le |t|$. So on the arc $U_t = U_0 + O((1+\log R)/R)$ and $\nabla(U_t - U_0) = O((1+\log R)/R^2)$. The normal derivative of $U_0$ on the arc is $O(|t|/R^2)$, since $\nabla U_0$ is orthogonal to $z$ and $Z - z = it$. Hence the normal derivative of $U_t$ is $O((1+\log R)/R^2)$, and the arc term is $O((1+\log R)/R)$. Also $|2\partial U_t|^2 - |\nabla U_t|^2 = -2\operatorname{Im}(\partial_x U_t\,\overline{\partial_y U_t})$ integrates over $D_{R,t}$ to a boundary term, a multiple of $\operatorname{Im}\oint U_t\,d\overline{U_t}$. It is 0 on the line. On the arc, by the same two bounds, $U_t\,d\overline{U_t}$ differs from $U_0\,d\overline{U_0} = \frac{m^2}{4}\theta(\pi-\theta)(\pi-2\theta)\,d\theta$ by $O((1+\log R)/R^2)\,|dZ|$, and $U_0\,d\overline{U_0}$ is real; so this term is $O((1+\log R)/R)$ as well. So the finite part (4) of $k_0|S_t|^2 - |v_{S_t}|^2$ is the finite part of $k_0|S_*|^2\,dA - 2\operatorname{Re}(\rho\overline{U_t})\,dA$ on $\Omega_t$, that is $\mathcal{D}(S_t) = \lim_{R\to\infty}\bigl(k_0 M_R(t) - 2E_R(t)\bigr)$ (as in Lemma 2.5, moving the centre of the cut half-disc costs $o(1)$). Hence

$$\mathcal{D}(S_{t_2}) - \mathcal{D}(S_{t_1}) = \int_{t_1}^{t_2}\int_{\mathbb{R}} \Bigl(|b_t(x)|^2 - k_0|S_*(x+it)|^2\Bigr)\frac{dx}{\pi}\,dt.$$

Finally $U_t = 0$ on the boundary, so $\partial_x U_t = 0$ there and $v_{S_t}(x) = -U_x + iU_y = ib_t(x)$ in the original coordinate; thus $|b_t| = |v_{S_t}|$ on $\mathbb{R}$. The integrand is continuous in $t$ (dominated convergence with the bounds above), so $\mathcal{D}(S_t)$ is $C^1$; dividing by $m^2 = 1$ gives Lemma 5.1. □

## D.2 The closed form of $v_S$

Let $S$ be a Pick sum with distinct poles $w_1,\dots,w_n$ and masses $a_1,\dots,a_n$. Its numerator has degree $n-1$. Since $\operatorname{Im} S > 0$ on $\overline{\mathbb{U}}$, its $n-1$ zeros (with multiplicity) lie in the lower half-plane. List the zeros and the poles as $c_1,\dots,c_{2n-1}$, with $\nu_k = +1$ if $c_k$ is a zero and $\nu_k = -1$ if $c_k$ is a pole, so that $S'/S = \sum_k \nu_k/(z-c_k)$. For $\operatorname{Im} w < 0$ and $\operatorname{Im} c < 0$ put

$$\mathcal{T}_{w,c}(z) = \frac{\overline{\operatorname{Log}(z-c)} - \overline{\operatorname{Log}(\bar w - c)}}{z-w} - \frac{\operatorname{Log}(z-w) - \operatorname{Log}(\bar c - w)}{z-\bar c} \qquad (z\in\overline{\mathbb{U}}). \tag{25}$$

All four logarithms have their arguments in $\mathbb{U}$. They are therefore principal values of analytic functions, and the second quotient has a removable singularity at $z = \bar c \in \mathbb{U}$.

**Lemma D.1.** *For every Pick sum $S$ as above and every $z \in \overline{\mathbb{U}}$,*

$$v_S(z) = \frac{i}{2} \sum_{j=1}^{n} \sum_{k=1}^{2n-1} (-a_j)\, \nu_k\, \mathcal{T}_{w_j,c_k}(z).$$

*Proof.* Call the right side $\tilde{v}$. Each $\mathcal{T}_{w,c}$ is smooth on $\overline{\mathbb{U}}$ and $O((1+\log r)/r)$ at infinity, and $\bar\partial \mathcal{T}_{w,c} = \overline{\partial \operatorname{Log}(z-c)}/(z-w) = 1/\big((z-w)(\bar z - \bar c)\big)$. Hence

$$\bar\partial \tilde{v} = \frac{i}{2}\, S \overline{\sum_k \frac{\nu_k}{z-c_k}} = \frac{i}{2}\, S\, \overline{\Big(\frac{S'}{S}\Big)} = \rho_S.$$

So $V = \tilde{v} - v_S$ is analytic on $\mathbb{U}$. It is continuous on $\overline{\mathbb{U}}$ and $O((1+\log r)/r)$, by Lemma 2.3 for $v_S$. By Lemma 4.2, $\int_{\mathbb{U}} v_S\, \bar g\, dA = 0$ for every $g$ that is analytic near $\overline{\mathbb{U}}$ and $O(r^{-3})$. We show below that the same holds for each $\mathcal{T}_{w,c}$, hence for $\tilde{v}$. Applied to the functions $g_m$ of Lemma A.1, this gives $\int_{\mathbb{U}} V \bar g_m\, dA = 0$ for all $m$. Lemma A.1 then gives $V = 0$.

*Orthogonality of $\mathcal{T}_{w,c}$.* Let $\mathbb{U}_w = \{\operatorname{Im} z > \operatorname{Im} w\}$, a half-plane that contains $\overline{\mathbb{U}}$ and $\bar c$. On $\mathbb{U}_w$ the function $A(z) = \big(\operatorname{Log}(z-w) - \operatorname{Log}(\bar c - w)\big)/(z-\bar c)$ is analytic. Let $\mathcal{A}$ be a primitive of $A$ on $\mathbb{U}_w$, and put

$$\Lambda(z) = \operatorname{Log}(z-w)\Big(\overline{\operatorname{Log}(z-c)} - \overline{\operatorname{Log}(\bar w - c)}\Big) - \mathcal{A}(z) \qquad (z \in \overline{\mathbb{U}}).$$

Since $\partial\overline{\operatorname{Log}(z-c)} = 0$, we have $\partial\Lambda = \mathcal{T}_{w,c}$. Let $g$ be analytic near $\overline{\mathbb{U}}$ and $O(r^{-3})$. Green's formula $\iint \partial(\Lambda \bar g)\, dx\, dy = \frac{i}{2} \oint \Lambda \bar g\, d\bar z$ on $D_R = \mathbb{U} \cap \{|z| < R\}$ gives

$$\int_{\mathbb{U}} \mathcal{T}_{w,c}\, \bar g\, dA = \frac{i}{2\pi} \int_{\mathbb{R}} \Lambda(x)\, \overline{g(x)}\, dx,$$

because $|\Lambda \bar g| = O(\log^2 R/R^3)$ on the half-circle of radius $R$. For real $x$, conjugation commutes with the principal logarithm away from $(-\infty, 0]$. Thus $\overline{\operatorname{Log}(x-c)} = \operatorname{Log}(x - \bar c)$ and $\overline{\operatorname{Log}(\bar w - c)} = \operatorname{Log}(w - \bar c)$, and

$$\Lambda(x) = \operatorname{Log}(x-w)\big(\operatorname{Log}(x-\bar c) - \operatorname{Log}(w-\bar c)\big) - \mathcal{A}(x).$$

We claim that this function of $x \in \mathbb{R}$ extends analytically to the half-plane $L' = \{\operatorname{Im} q < \operatorname{Im} \bar c\}$, which contains the closed lower half-plane, with growth $O(\log^2 |q|)$. Let $\Gamma = \{w - it : t \ge 0\}$ be the vertical ray down from $w$, and let $\lambda$ be the branch of $\log(q-w)$ on $\mathbb{C} \setminus \Gamma$ that equals $\operatorname{Log}(q-w)$ on $\mathbb{U}_w$. Then $A$ and $\mathcal{A}$ continue analytically to $\mathbb{C} \setminus \Gamma$ (with $\lambda$ in place of $\operatorname{Log}(q-w)$, and the point $\bar c$ stays removable; a primitive exists there because $\mathbb{C} \setminus \Gamma$ is simply connected), and $\operatorname{Log}(q - \bar c)$ is analytic on $L'$. So $\tilde\Lambda(q) = \lambda(q)\big(\operatorname{Log}(q-\bar c) - \operatorname{Log}(w-\bar c)\big) - \mathcal{A}(q)$ is analytic on $L' \setminus \Gamma$ and equals $\Lambda$ on $\mathbb{R}$. Across $\Gamma$ the branch $\lambda$ jumps by $2\pi i$. Then $A$ jumps by $2\pi i/(q-\bar c)$, so the jump of $\mathcal{A}$ across $\Gamma$ has derivative $2\pi i/(q-\bar c)$ along $\Gamma$. The jump also tends to 0 as $q \to w$: at $q$ it is the integral of $A$ along the circle about $w$ through $q$, where $|A| = O(\log(1/|q-w|))$, and that circle has length $2\pi|q-w|$. Hence $\mathcal{A}$ is continuous at the tip $w$, and it jumps by $2\pi i \int_w^q dq'/(q'-\bar c) = 2\pi i\big(\operatorname{Log}(q-\bar c) - \operatorname{Log}(w-\bar c)\big)$. The jump of $\tilde\Lambda$ is therefore $2\pi i\big(\operatorname{Log}(q-\bar c) - \operatorname{Log}(w-\bar c)\big) - 2\pi i\big(\operatorname{Log}(q-\bar c) - \operatorname{Log}(w-\bar c)\big) = 0$. So $\tilde\Lambda$ is continuous across $\Gamma$, hence analytic there (Morera). Near $w$ it is bounded, because the factor $\operatorname{Log}(q-\bar c) - \operatorname{Log}(w-\bar c)$ vanishes at $q = w$ and $\mathcal{A}$ is continuous there; so its singularity at $w$ is removable. Since $A = O(\log|q|/|q|)$, $\mathcal{A}$ and $\tilde\Lambda$ are $O(\log^2|q|)$. This proves the claim.

Now $g^*(q) = \overline{g(\bar q)}$ is analytic near the closed lower half-plane and $O(|q|^{-3})$, and $\overline{g(x)} = g^*(x)$ on $\mathbb{R}$. By Cauchy's theorem on the lower half-discs $\{|q| < R,\ \operatorname{Im} q < 0\}$, whose arcs contribute $O(\log^2 R/R^2)$, $\int_{\mathbb{R}} \tilde\Lambda(x) g^*(x)\, dx = 0$. Hence $\int_{\mathbb{U}} \mathcal{T}_{w,c} \bar g\, dA = 0$. □

## D.3 Analytic continuation of the flux in the parameters

Nodes, partners, the integrand $H$ written with partners and admissible paths are defined in Section 6.2, before Lemma 6.2.

*Proof of Lemma 6.6.* On the region of the lemma every node lies in the lower and every partner in the upper half-plane. So for real $x$, each $x - c$ with $c$ a node lies in $\mathbb{U}$, each $x - c^{\#}$ lies in the lower half-plane, and each constant $c' - c$ lies in one fixed open half-plane. All these logarithms are therefore principal values of analytic functions, and no denominator vanishes (the quotient in (25) is a difference quotient, analytic also where its denominator vanishes). Hence $H(x)$ is analytic in each parameter, jointly continuous in $x$ and the parameters, and on compact parts of the region $|H(x)| \le C\big(1 + \log^2(2 + |x|)\big)/(1 + x^2)$, because every factor of $v_S$ and of $\overline{v_S}$ written with partners is $O(\log|x|/|x|)$ and those of $S$ are $O(1/|x|)$. By Fubini and Morera, $\int_{\mathbb{R}} H\,dx$ is analytic in each parameter, and it equals $\pi\varphi$ at real parameters. The same holds for $\varphi_\xi$ of Section 6.3. On a cell $\varphi_\xi = u\varphi$, and $s$, $\gamma$ (or $\log\gamma$) and $u$ are affine or exponential functions of $\xi_1, \xi_2, \xi_3$. At the top cell $\varphi_\xi = \frac{1}{2}e^{-1/2}\tilde{u}^{-1}\varphi_{\tilde{u}}$, where $\varphi_{\tilde{u}}$ is the same kind of integral for the function of Lemma 6.2 and $\tilde{u}$ is affine in $\xi_3$; Lemma 6.2 covers $\tilde{u} = 0$. □

When $s$ is complex and $u$ is small, the zero may leave the lower half-plane (or its partner the upper one), so Lemma 6.6 does not apply on $\mathbb{R}$. The integral is then taken along an admissible path off $\mathbb{R}$; this is justified as follows.

**Lemma D.2.** *Let $\gamma \in [0, \pi/2]$ and $u > 0$ be real. In $H$, use for* $\mathrm{Log}(z - c)$ *the branch whose cut runs vertically down from each node $c$ and up from each partner:* $\arg(z - c) \in (-\pi/2, 3\pi/2)$ *for a node $c$ and* $\arg(z - c^{\#}) \in (-3\pi/2, \pi/2)$ *for a partner $c^{\#}$; at real parameters these agree with the principal value on $\mathbb{R}$. For each constant logarithm use the branch continued from its half-plane at real parameters, with its cut on the half of the imaginary axis outside that half-plane: $i[0,\infty)$ for an argument $c - c'^{\#}$, which lies in the lower half-plane at real parameters, and $-i[0,\infty)$ for $c'^{\#} - c$. Call a path* admissible *if it equals $\mathbb{R}$ outside a bounded set and meets none of these cuts.*

(a) *An admissible path exists, and the integral of $H$ is the same along all of them, as long as no node lies directly above a partner (same real part, imaginary part at least as large).*

(b) *For $|\operatorname{Im} s| < \pi/2$ no node lies directly above a partner.*

(c) *Hence $s \mapsto \frac{1}{\pi}\int H$, along any admissible path, is analytic on the strip $|\operatorname{Im} s| < \pi/2$, and equals $\varphi$ at real $s$.*

*Proof.* (a) The cuts are closed vertical rays: downward ones from the nodes and upward ones from the partners (rays of the same direction from nodes with the same real part are nested). On a vertical line the downward rays cover the heights up to the highest node on it and the upward rays those from the lowest partner on it. If no node lies above a partner, a gap is left, and a path through these gaps exists. The union of the rays and $\infty$ is a tree on the Riemann sphere. Its complement is therefore connected and simply connected. $H$ is analytic on this complement, and $H = O(\log^2|z|/|z|^2)$ uniformly near the ends. Two admissible paths agree outside a bounded set, and by Cauchy's theorem their integrals agree. The constant logarithms are analytic on their domains: each argument has the form $c - c'^{\#}$ or $c'^{\#} - c$ and reaches its cut only if the node $c$ lies directly above the partner $c'^{\#}$, that is, $c - c'^{\#} \in i[0,\infty)$.

(b) Since $\operatorname{Re} e^s > 0$ for $|\operatorname{Im} s| < \pi/2$, the number $a = 1/(1 + e^s)$ lies in the disc $|a - \frac{1}{2}| < \frac{1}{2}$. Thus $\operatorname{Re} a > 0$ and $|a| < 1$. The poles and their partners are as for real parameters. "Node $c$ directly above partner $c'^{\#}$" means $c - c'^{\#} \in i[0,\infty)$. We check the nine pairs, in six cases, with $t \ge 0$ throughout.

- $w_j - w_k^{\#}$ has imaginary part $-2u - \sin\gamma \cdot (0,\ 1 \text{ or } 2) < 0$.
- $z_0 - w_1^{\#} = ae^{-i\gamma} - 2iu \in i[0,\infty)$ forces $a = i(2u + t)e^{i\gamma}$, with $\operatorname{Re} a \le 0$.

- $z_0 - w_2^{\#} \in i[0,\infty)$ forces $a = e^{2i\gamma} + i(2u+t)e^{i\gamma}$, with $|a| = |\cos\gamma + i(\sin\gamma + 2u + t)| \geq 1$.
- $w_1 - z_0^{\#} \in i[0,\infty)$ forces $a = -i(2u+t)e^{-i\gamma}$, with $\operatorname{Re} a \leq 0$.
- $w_2 - z_0^{\#} \in i[0,\infty)$ forces $a = e^{-2i\gamma} - i(2u+t)e^{-i\gamma}$, with $|a| \geq 1$.
- $z_0 - z_0^{\#} = -2i(a\sin\gamma + u) \in i[0,\infty)$ forces $a\sin\gamma = -u - t/2 < 0$. This is impossible at $\gamma = 0$, and for $\gamma > 0$ it gives $\operatorname{Re} a < 0$.

Each case contradicts $\operatorname{Re} a > 0$ or $|a| < 1$.

(c) Near a given $s_0$, fix an admissible path for $s_0$. It has positive distance from the rays at $s_0$: both are closed, the rays lie on finitely many vertical lines, and the path equals $\mathbb{R}$ outside a bounded set. Only $z_0$ and its partner move with $s$, and they move continuously; so for $s$ close to $s_0$ the path stays admissible. Along it $H$ is analytic in $s$, with tails dominated uniformly. The integral is therefore analytic in $s$. By (a) these local functions agree, and at real $s$ the real axis is admissible and the integral is $\pi\varphi$. □

# E The bound near $u = 0$

Throughout, $(s,\gamma)$ is fixed, $a = 1/(1+e^s)$ is the mass of the shallower pole, $b = 1-a$ and $\mu = \min(a,b) \leq \frac{1}{2}$. Put $S_0(Z) = -a/Z - b/(Z - e^{-i\gamma})$, so that $S_{s,\gamma,u}(z) = S_0(z+iu)$ by (8). We write $Z = X + iY$, use $dx/\pi$ on lines and $dA = dX\,dY/\pi$ on areas, and $\|\cdot\|_r$ for the norm of $L^r(\mathbb{R}, dx/\pi)$, and $\langle f, h\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f\,\bar{h}\,dx/\pi$. The Poisson kernel $\mathcal{P}_y(x) = y/(x^2+y^2)$ has $\|\mathcal{P}_y\|_1 = 1$, and $\|\partial_y \mathcal{P}_y\|_1 = \|\partial_x \mathcal{P}_y\|_1 = 2/(\pi y) \leq 1/y$.

## E.1 The boundary function and the flux

Put $\hat{\rho} = (S_0/\overline{S_0})\,\overline{S_0'} = -2i\rho_{S_0}$ (see (3)). For one pole $-a/(Z-w)$ this is $a/|Z-w|^2 > 0$. For our $S_0$ write $\hat{\rho} = \hat{\rho}_1 + \hat{\rho}_2 + \hat{\rho}_\times$, with $\hat{\rho}_1 = a/|Z|^2$ and $\hat{\rho}_2 = b/|Z - e^{-i\gamma}|^2$. For a function $h$ on $\{Y > u\}$ let $T_u h(x) = \int_u^\infty (\mathcal{P}_{Y-u} * h_Y)(x)\,dY$, where $h_Y(X) = h(X+iY)$.

**Lemma E.1.** *For $u > 0$ let $\mathcal{V}_j = T_u\hat{\rho}_j$, $c_u = T_u \operatorname{Re}\hat{\rho}_\times$ and $q_u = T_u \operatorname{Im}\hat{\rho}_\times$. Then $|v_S| = |\mathcal{V}_1 + \mathcal{V}_2 + c_u + iq_u|$ on $\mathbb{R}$ for $S = S_{s,\gamma,u}$, and*

$$\varphi(s,\gamma,u) = \int_{\mathbb{R}} \big(2\mathcal{V}_1\mathcal{V}_2 + 2(\mathcal{V}_1+\mathcal{V}_2)c_u + c_u^2 + q_u^2\big)\frac{dx}{\pi} - 2k_0 \operatorname{Re}\langle aS_{w_1}, bS_{w_2}\rangle_{\mathbb{R}}, \tag{26}$$

*where $S_w = -1/(z-w)$ (Section 6.4) and $2\operatorname{Re}\langle aS_{w_1}, bS_{w_2}\rangle_{\mathbb{R}} = 4ab\,(\sin\gamma+2u)/\big(\cos^2\gamma + (\sin\gamma+2u)^2\big)$. For the single pole $a_jS_{w_j}$ alone the cross terms vanish, so $\mathcal{V}_j = |v_{a_jS_{w_j}}|$ on $\mathbb{R}$; with its foot $x_1 = 0$, $x_2 = \cos\gamma$ and its depth $y_1 = u$, $y_2 = \sin\gamma + u$,*

$$\mathcal{V}_j(x) = a_j \int_{y_j}^\infty \frac{t\,dt}{(t+y_j)\big((x-x_j)^2+t^2\big)}, \qquad 0 \leq \mathcal{V}_j(x) \leq \min\Big(\frac{a_j}{y_j}, \frac{\pi a_j}{2|x-x_j|}\Big), \tag{27}$$

*where $a_1 = a$, $a_2 = b$.*

*Proof.* In the coordinate $Z = z + iu$ the half-plane $\mathbb{U}$ is $\{Y > u\}$. Apply the boundary-derivative formula of Appendix D on $\{Y > u\}$ with fixed source $2\rho_{S_0}$. It shows that on $\mathbb{R}$ the function $v_S$ equals $i$ times the integral of $\mathcal{P}_{Y-u}(x-X)\,2\rho_{S_0}$ over $\{Y > u\}$; since $2\rho_{S_0} = i\hat{\rho}$, this gives $|v_S| = |T_u\hat{\rho}|$, and $T_u$ is linear. The formula and the bound (27) for one pole are those of Appendix D (with $t = 2y + y_j$). By Remark 5.2, $\int_{\mathbb{R}} \big(\mathcal{V}_j^2 - k_0 a_j^2 |x-w_j|^{-2}\big)dx/\pi = 0$. Now $|v_S|^2 - \mathcal{V}_1^2 - \mathcal{V}_2^2$ is the integrand of (26), and $|S|^2 = |aS_{w_1}|^2 + |bS_{w_2}|^2 + 2\operatorname{Re}\big(aS_{w_1}\overline{bS_{w_2}}\big)$, where $2\operatorname{Re}\langle aS_{w_1}, bS_{w_2}\rangle_{\mathbb{R}}$ has the value in (26) by residues ($\overline{w_2} - w_1 = \cos\gamma + i(\sin\gamma + 2u)$). Subtract the two zero single-pole fluxes from Lemma 5.1. □

## E.2 The cross source

**Lemma E.2.** *Let $r_1 = |Z|$, $r_2 = |Z - e^{-i\gamma}|$, $r_* = |Z - ae^{-i\gamma}|$ and $Y_\gamma = X\sin\gamma + Y\cos\gamma$. Then*

$$\hat\rho_\times = \frac{2iab\,e^{i\gamma}Y_\gamma}{r_1^2 r_2^2(\bar Z - ae^{i\gamma})}, \qquad \operatorname{Re}\hat\rho_\times = -\frac{2ab\,Y_\gamma^2}{r_1^2 r_2^2 r_*^2} \le 0, \qquad |\hat\rho_\times| = \frac{2ab\,|Y_\gamma|}{r_1^2 r_2^2 r_*},$$

*and $|Y_\gamma| \le \min(r_1, r_2, r_*)$.*

*Proof.* $S_0 = -(Z - ae^{-i\gamma})/(Z(Z - e^{-i\gamma}))$, and partial fractions give $\hat\rho_\times = \dfrac{ab\,e^{i\gamma}}{\bar Z - ae^{i\gamma}}\Big(\dfrac{1}{Z(\bar Z - e^{i\gamma})} - \dfrac{1}{(Z - e^{-i\gamma})\bar Z}\Big)$; the bracket is $(Ze^{i\gamma} - e^{-i\gamma}\bar Z)/(r_1^2 r_2^2)$ and $Ze^{i\gamma} - e^{-i\gamma}\bar Z = 2iY_\gamma$. Multiplying numerator and denominator by $Z - ae^{-i\gamma}$ gives $\operatorname{Re}\hat\rho_\times$, since $\operatorname{Im}\big(e^{i\gamma}(Z - ae^{-i\gamma})\big) = Y_\gamma$. Finally $Y_\gamma = \operatorname{Im}(Ze^{i\gamma}) = \operatorname{Im}\big((Z - e^{-i\gamma})e^{i\gamma}\big) = \operatorname{Im}\big((Z - ae^{-i\gamma})e^{i\gamma}\big)$ and $|e^{i\gamma}| = 1$. □

On $\{Y > 0\}$ we have $r_2 \ge |Z - \cos\gamma|$ and $r_2 \ge \sin\gamma$, $r_* \ge a\sin\gamma$. Two sets of pointwise bounds follow from Lemma E.2, each by cancelling one factor with $|Y_\gamma| \le r_j$.

*(i) Bounds for $\cos\gamma \ge \frac12$, uniform in $\sin\gamma$.* Let $r_0 = \mu\cos\gamma/8 \in [\mu/16, 1/16]$. On $\{|Z| < r_0\}$ we have $r_2 \ge \cos\gamma/2$ and $r_* \ge a - r_0 \ge a\cos\gamma/2$; on $\{|Z - \cos\gamma| < r_0\}$, $r_1 \ge \cos\gamma/2$ and $r_* \ge b\cos\gamma/2$; off both discs, $r_1, r_2 \ge r_0$ and $\max(r_1, r_2) \ge \frac12$; for $|Z| \ge 4$, $r_2, r_* \ge |Z|/2$. With $x_1 = 0$, $x_2 = \cos\gamma$, $\cos\gamma \ge \frac12$ and $\mu \le a, b \le 1$ this gives:

- on $\{|Z| < r_0\}$: cancel $Y_\gamma^2 \le r_1^2$, so $|\operatorname{Re}\hat\rho_\times| \le 2ab/(r_2^2 r_*^2) \le 32b/(a\cos^4\gamma) \le 512/\mu$; cancel $|Y_\gamma| \le r_1$, so $|\hat\rho_\times| \le 2ab/(r_1 r_2^2 r_*) \le 16b/(r_1\cos^3\gamma) \le 128/|Z|$;
- on $\{|Z - \cos\gamma| < r_0\}$: the same with $r_2$ cancelled, $|\operatorname{Re}\hat\rho_\times| \le 2ab/(r_1^2 r_*^2) \le 32a/(b\cos^4\gamma) \le 512/\mu$ and $|\hat\rho_\times| \le 2ab/(r_1^2 r_2 r_*) \le 16a/(r_2\cos^3\gamma) \le 128/|Z - \cos\gamma|$ (as $r_2 \ge |Z - \cos\gamma|$);
- off both discs and $|Z| < 4$ (the *bulk set*): cancel $|Y_\gamma| \le r_*$; as $r_1 r_2 \ge r_0/2$ and $ab \le \mu$, $|\hat\rho_\times| \le 2ab/(r_1^2 r_2^2) \le 8ab/r_0^2 = 512ab/(\mu^2\cos^2\gamma) \le 2048/\mu$;
- for $|Z| \ge 4$: cancel $|Y_\gamma| \le r_*$, so $|\hat\rho_\times| \le 2ab/(r_1^2 r_2^2) \le 8ab/|Z|^4 \le 32/|Z|^4$.

In each case $|\operatorname{Re}\hat\rho_\times| \le |\hat\rho_\times|$ as well.

*(ii) Bounds for fixed $\sin\gamma > 0$.* Let $C_2 = 8/(a\sin^4\gamma) \ge 8$. On $\{Y > 0\}$ we use $r_2 \ge \sin\gamma$ and $r_* \ge a\sin\gamma$. Cancelling $Y_\gamma^2 \le r_1^2$ gives $|\operatorname{Re}\hat\rho_\times| \le 2ab/(r_2^2 r_*^2) \le 2b/(a\sin^4\gamma)$; cancelling $|Y_\gamma| \le r_1$ gives $|\hat\rho_\times| \le 2ab/(r_1 r_2^2 r_*) \le 2b/(r_1\sin^3\gamma)$; and for $|Z| \ge 2$, where $r_2, r_* \ge |Z| - 1 \ge |Z|/2$, cancelling $|Y_\gamma| \le r_*$ gives $|\hat\rho_\times| \le 8ab/|Z|^4 \le 4/|Z|^4$.

**Lemma E.3.** (a) *If $\cos\gamma \ge \frac12$ and $C_1 = 2^{15}/\mu$, then for $0 < u \le \frac14$: $\|c_u\|_1 \le C_1$, $\|c_u\|_\infty \le 5C_1$ and $\|q_u\|_2 \le C_1$.* (b) *If $\sin\gamma > 0$, then for $0 \le u \le \frac14$: $\|c_u\|_1 \le C_2$, $\|c_u\|_\infty \le 3C_2$ and $\|q_u\|_2 \le M_q := 16/\sin^3\gamma$, with $q_0 \in L^2(\mathbb{R})$ at $u = 0$. Moreover $|c_0(x) - c_0(0)| \le 4C_2|x|\big(1 + \log(1/|x|)\big)$ for $0 < |x| \le \frac14$, and $\|c_u - c_0\|_\infty \le 4C_2\,u\big(1 + \log(1/u)\big)$ for $0 < u \le \frac14$.*

*Proof.* By Tonelli and $\|\mathcal{P}_y\|_1 = 1$, $\|c_u\|_1 \le \int_{Y>0} |\operatorname{Re}\hat\rho_\times|\,dA$ and $\|c_u\|_\infty \le \int_0^\infty \sup_X |\operatorname{Re}\hat\rho_\times(X + iY)|\,dY$; by Minkowski and Young, $\|q_u\|_2 \le \int_0^\infty \|(\operatorname{Im}\hat\rho_\times)_Y\|_2\,dY$.

(a) The half-disc $\{|Z| < 4, Y > 0\}$ has $dA$-area 8 and $\int_{|Z|>4} 32|Z|^{-4}dA = 1$, so $\int |\operatorname{Re}\hat\rho_\times|\,dA \le 8 \cdot 2048/\mu + 1 \le C_1$, and $\int_0^\infty \sup_X |\operatorname{Re}\hat\rho_\times|\,dY \le 4 \cdot 2048/\mu + \int_4^\infty 32Y^{-4}dY \le 5C_1$. On a horizontal line $\|1/|\cdot + iY - x_j|\|_2 = Y^{-1/2}$, the bulk set has length at most 8, and $32|Z|^{-4} \le 32/(1 + X^2)$ with $\|(1+X^2)^{-1}\|_2^2 = \frac12$; so $\|(\operatorname{Im}\hat\rho_\times)_Y\|_2 \le 256Y^{-1/2} + 4096/\mu + 32$ for $Y < 4$, and $\|(\operatorname{Im}\hat\rho_\times)_Y\|_2 \le 32Y^{-7/2}$ for $Y \ge 4$ (as $\int(1+X^2)^{-4}dx/\pi = \frac{5}{16}$). Integrating, $\|q_u\|_2 \le 1024 + 16384/\mu + 128 + \frac25 < 16961/\mu < C_1$, using $\mu \le \frac12$.

(b) As in (a), now with the bounds (ii). The half-disc $\{|Z| < 2, Y > 0\}$ has $dA$-area 2 and $\int_{|Z|>2} 4|Z|^{-4}dA = \frac12$, so $\int |\operatorname{Re}\hat\rho_\times|\,dA \le 2 \cdot 2b/(a\sin^4\gamma) + \frac12 \le C_2$. The bound $2b/(a\sin^4\gamma) \le C_2$ for $Y < 2$ and $4Y^{-4}$ for $Y \ge 2$ give $\int_0^\infty \sup_X |\operatorname{Re}\hat\rho_\times|\,dY \le 2C_2 + \frac16 \le 3C_2$. On a horizontal line

$\|1/|\cdot+iY|\|_2 = Y^{-1/2}$, so $\|(\operatorname{Im}\hat\rho_\times)_Y\|_2 \le C_3 Y^{-1/2}+2$ for $Y<2$, with $C_3 = 2b/\sin^3\gamma$; here the 2 bounds the part of the line where $|Z|\ge 2$, since there $|Z|^4 \ge 2(1+X^2)$, so $4|Z|^{-4} \le 2/(1+X^2)$, whose norm is $\sqrt{2}$. Also $\|(\operatorname{Im}\hat\rho_\times)_Y\|_2 \le 4Y^{-7/2}$ for $Y\ge 2$; integrating, $\|q_u\|_2 \le 2\sqrt{2}\,C_3+4+1 \le 3C_3+5 \le 11/\sin^3\gamma \le M_q$. For $c_u - c_0$ split the $Y$-integral at $u$, $2u$ and 2: the four parts are at most $C_2u$, $2C_2u$, $\int_{2u}^2 2C_2u\,Y^{-1}dY = 2C_2u\log(1/u)$ (since $\|\mathcal{P}_{Y-u}-\mathcal{P}_Y\|_1 \le u/(Y-u) \le 2u/Y$) and $\int_2^\infty (2u/Y)\,4Y^{-4}dY = u/8$; their sum is at most $4C_2u(1+\log(1/u))$. For $c_0(x)-c_0(0)$ split the $Y$-integral at $|x|$ and 2: the three parts are at most $\int_0^{|x|} 2C_2\,dY = 2C_2|x|$ (each of the two terms is at most $C_2$), $\int_{|x|}^2 C_2|x|\,Y^{-1}dY = C_2|x|\log(2/|x|)$ (since $\|\mathcal{P}_Y(\cdot-x)-\mathcal{P}_Y\|_1 \le |x|/Y$) and $\int_2^\infty (|x|/Y)\,4Y^{-4}dY = |x|/16$; as $C_2\ge 8$, their sum is at most $C_2|x|(3+\log(1/|x|)) \le 4C_2|x|(1+\log(1/|x|))$. □

## E.3 Proof of Lemma 5.4

Every $\gamma\in[0,\pi/2]$ has $\cos\gamma\ge\frac12$ or $\sin\gamma\ge\frac12$.

*Case* $\cos\gamma\ge\frac12$ (this includes $\gamma=0$). The poles 0 and $e^{-i\gamma}$ of $S_0$ stay at distance 1, so $\gamma\to0$ together with $u\to0$ does not merge them; it moves them towards the two boundary points 0 and 1, and the bounds below are uniform in that limit. Let $\mathcal{I}_j=\{|x-x_j|<r_0\}$ (disjoint, as $r_0<\cos\gamma/2$) and $\mathcal{I}_0=\mathbb{R}\setminus(\mathcal{I}_1\cup\mathcal{I}_2)$. By (27): on $\mathcal{I}_j$ the other trace satisfies $\mathcal{V}_{\text{other}} \le \pi a_{\text{other}}/\cos\gamma < 7$; on $\mathcal{I}_0$, $\mathcal{V}:=\mathcal{V}_1+\mathcal{V}_2<2/r_0$; and

$$\int_{\mathcal{I}_j}\mathcal{V}_j\,\frac{dx}{\pi} \le a_j\big(1+\log(1/u)\big), \tag{28}$$

since, if $y_j\le r_0$, the part $|x-x_j|<y_j$ gives at most $2a_j/\pi$ and the rest at most $a_j\log(r_0/y_j)\le a_j\log(1/u)$, while if $y_j>r_0$ the whole integral is below $a_j$.

*Upper bound.* Drop the last term $-2k_0\operatorname{Re}\langle aS_{w_1},bS_{w_2}\rangle_{\mathbb{R}}\le0$ of (26). On $\mathcal{I}_1\cup\mathcal{I}_2$, $\int 2\mathcal{V}_1\mathcal{V}_2 \le 14(1+\log(1/u))$; and since on $\mathcal{I}_j$ we have $2\mathcal{V}|c_u| \le 2\mathcal{V}_j\|c_u\|_\infty + 14|c_u|$ (as $\mathcal{V}_{\text{other}}<7$), (28) and Lemma E.3(a) give $\int 2\mathcal{V}|c_u| \le 10C_1(1+\log(1/u))+14C_1 \le 30C_1(1+\log(1/u))$. On $\mathcal{I}_0$, $\int 2\mathcal{V}_1\mathcal{V}_2 \le \int_{\mathcal{I}_0}(\mathcal{V}_1^2+\mathcal{V}_2^2) \le \sum_j \pi a_j^2/(2r_0) < 32/\mu$ and $\int 2\mathcal{V}|c_u| \le (4/r_0)\|c_u\|_1 \le 64C_1/\mu$. Finally $\|c_u\|_2^2 \le \|c_u\|_\infty\|c_u\|_1 \le 5C_1^2$ and $\|q_u\|_2^2 \le C_1^2$. With $C_1=2^{15}/\mu$ and $\mu\le\frac12$, $\varphi \le (14+30C_1)\big(1+\log(1/u)\big)+32/\mu+64C_1/\mu+6C_1^2 \le 2^{34}\mu^{-2}\big(1+\log(1/u)\big)$.

*Lower bound.* Drop $q_u^2\ge0$. On $\mathcal{I}_j$ write $(\mathcal{V}+c_u)^2-\mathcal{V}_1^2-\mathcal{V}_2^2 = 2\mathcal{V}_j(\mathcal{V}_{\text{other}}+c_u)+(\mathcal{V}_{\text{other}}+c_u)^2-\mathcal{V}_{\text{other}}^2$; since $|\mathcal{V}_{\text{other}}+c_u|\le 6C_1$, this is at least $-12C_1\mathcal{V}_j-\mathcal{V}_{\text{other}}^2$, whose integral over $\mathcal{I}_1\cup\mathcal{I}_2$ is at least $-12C_1(1+\log(1/u))-16$. On $\mathcal{I}_0$ the same expression equals $2\mathcal{V}_1\mathcal{V}_2+2\mathcal{V}c_u+c_u^2 \ge -2\mathcal{V}|c_u|$, with integral at least $-64C_1/\mu$. The last term of (26) is $-2k_0\operatorname{Re}\langle aS_{w_1},bS_{w_2}\rangle_{\mathbb{R}} \ge -k_0\cdot 2ab/\cos\gamma \ge -1$, as $\cos^2\gamma+(\sin\gamma+2u)^2 \ge 2(\sin\gamma+2u)\cos\gamma$. Hence $\varphi \ge -(12C_1+64C_1/\mu+17)(1+\log(1/u)) \ge -2^{22}\mu^{-2}(1+\log(1/u))$.

*Case* $\sin\gamma\ge\frac12$. By Lemma E.4, $|\varphi| \le |c_{\log}|\log(1/u)+4096/(a^2\sin^8\gamma) \le 2^{10}\log(1/u)+2^{20}a^{-2} \le 2^{21}\mu^{-2}(1+\log(1/u))$.

In both cases $|\varphi|\le 2^{34}\mu^{-2}(1+\log(1/u))$, and $\int_0^{u_0}(1+\log(1/u))\,du = u_0(2+\log(1/u_0))$ gives $\beta(u_0)$. □

## E.4 A bound with a logarithmic term ($\sin\gamma>0$)

**Lemma E.4.** *Let* $\sin\gamma>0$. *There is a real number* $c_{\log}=c_{\log}(s,\gamma)$ *with* $|c_{\log}|\le 64/\sin^4\gamma$ *such that for* $0<u\le\frac14$

$$\big|\varphi(s,\gamma,u)-c_{\log}\log(1/u)\big| \le \frac{4096}{a^2\sin^8\gamma}.$$

*Proof.* Let $\Xi_u = \mathcal{V}_2 + c_u$ and $\Xi_* = \Xi_0(0)$, where $\Xi_0$ is the same function at $u = 0$ (the trace of pole 2 at its depth $\sin\gamma$, plus $c_0$). Put $c_{\log} = 2a\Xi_*$. By (27), $\mathcal{V}_2(x) = b\int_0^\infty \mathcal{P}_{2y+\sin\gamma+u}(x-\cos\gamma)\,(y+\sin\gamma+u)^{-1}dy$; since $|\mathcal{P}_y| \le 1/y$ and $|\partial_y\mathcal{P}_y|, |\partial_x\mathcal{P}_y| \le 1/y^2$, differentiating under the integral gives $|\partial_u\mathcal{V}_2|, |\partial_x\mathcal{V}_2| \le 2b\int_0^\infty (y+\sin\gamma+u)^{-3}dy \le b/\sin^2\gamma \le C_2$. With Lemma E.3(b) this gives

$$|\Xi_*| \le 4C_2, \qquad \|\Xi_u\|_\infty \le 4C_2, \qquad \|\Xi_u - \Xi_0\|_\infty \le 5C_2\,u\big(1+\log(1/u)\big) \quad (0 < u \le \tfrac14),$$
$$|\Xi_0(x) - \Xi_*| \le 5C_2|x|\big(1+\log(1/|x|)\big) \quad (0 < |x| \le \tfrac14);$$

in particular $|c_{\log}| \le 8aC_2 = 64/\sin^4\gamma$. The single-pole flux identity gives $\|\mathcal{V}_2\|_2^2 = k_0b^2/(\sin\gamma+u) \le 1/\sin\gamma$. Lemma E.3(b) gives $\|c_u\|_2^2 \le \|c_u\|_\infty\|c_u\|_1 \le 3C_2^2$; the triangle inequality then gives $\|\Xi_u\|_2 \le 3C_2$.

By (26), $\varphi = J_u + \int(2\mathcal{V}_2c_u + c_u^2 + q_u^2)\,dx/\pi - 2k_0\,\mathrm{Re}\langle aS_{w_1}, bS_{w_2}\rangle_{\mathbb{R}}$ with $J_u = 2\int \mathcal{V}_1\Xi_u\,dx/\pi$. Since $\cos^2\gamma + (\sin\gamma+2u)^2 = 1 + 4u\sin\gamma + 4u^2 \ge 1$, we have $2\,\mathrm{Re}\langle aS_{w_1}, bS_{w_2}\rangle_{\mathbb{R}} \le 4ab(1+2u) \le \frac32$, so $0 \le 2k_0\,\mathrm{Re}\langle aS_{w_1}, bS_{w_2}\rangle_{\mathbb{R}} \le 2$.

Let $\mathcal{A}_u = \int_{|x|<1/4}\mathcal{V}_1\,dx/\pi$. The part $|x| < u$ gives at most $2a/\pi$; on $u < |x| < \frac14$, $0 \le \pi a/(2|x|) - \mathcal{V}_1(x) \le (au/x^2)\big(\frac32 + \log(|x|/u)\big)$, whose integral is at most $5a/\pi$; and $\int_{u<|x|<1/4}\pi a/(2|x|)\,dx/\pi = a(\log(1/u) - \log 4)$. Hence $|\mathcal{A}_u - a\log(1/u)| \le 4a$, and also $\mathcal{A}_u \le a(1+\log(1/u))$.

The bound on $\pi a/(2|x|) - \mathcal{V}_1$ holds. Indeed $\pi/(2|x|) = \int_0^\infty dt/(x^2+t^2)$, so by (27) $\pi a/(2|x|) - \mathcal{V}_1(x) = a\int_0^u \frac{dt}{x^2+t^2} + au\int_u^\infty \frac{dt}{(t+u)(x^2+t^2)} \ge 0$; the first integral is at most $u/x^2$, and the second is at most $x^{-2}\log(|x|/u)$ over $[u,|x|]$ and at most $\int_{|x|}^\infty t^{-3}\,dt = 1/(2x^2)$ over $[|x|,\infty)$.

Now split $J_u - c_{\log}\log(1/u)$. Outside $|x| < \frac14$: $\|\mathcal{V}_1\mathbf{1}_{|x|>1/4}\|_2 \le 3a$, so this part is at most $18C_2$. Inside: $\big|2\int\mathcal{V}_1(\Xi_0 - \Xi_*)\big| \le 5aC_2\int_{|x|<1/4}(1+\log(1/|x|))\,dx \le 12C_2$; $\big|2\int\mathcal{V}_1(\Xi_u - \Xi_0)\big| \le 10aC_2u(1+\log(1/u))^2 \le 20C_2$, as $u(1+\log(1/u))^2 < 2$ for $0 < u \le \frac14$; and $|2\Xi_*\mathcal{A}_u - c_{\log}\log(1/u)| = 2|\Xi_*|\,|\mathcal{A}_u - a\log(1/u)| \le 32C_2$. So $|J_u - c_{\log}\log(1/u)| \le 100C_2$.

Finally, since $2|\mathcal{V}_2c_u| \le \mathcal{V}_2^2 + c_u^2$, $\int(|2\mathcal{V}_2c_u| + c_u^2 + q_u^2)\,dx/\pi \le \|\mathcal{V}_2\|_2^2 + 2\|c_u\|_2^2 + \|q_u\|_2^2 \le 7C_2^2 + M_q^2$. Since $C_2 \ge 8$,

$$|\varphi - c_{\log}\log(1/u)| \le 100C_2 + 7C_2^2 + M_q^2 + 2 \le 20C_2^2 + M_q^2 \le \frac{1280+256}{a^2\sin^8\gamma} < \frac{4096}{a^2\sin^8\gamma}. \qquad \square$$