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Linear programming certificates for dimension bounds of hyperbolic Coxeter polytopes

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Abstract

We use linear programming to derive upper bounds for the dimension of hyperbolic Coxeter polytopes from local geometric and arithmetic estimates. Rational certificates give the bounds 57 for finite-volume polytopes, 17 for compact polytopes, and 15 for compact arithmetic polytopes. In the finite-volume case, the certificate is a weight function on incidences between vertices and two-dimensional faces. In the compact cases, the inequalities concern squared norms of orthogonal projections of a fixed tangent vector at a vertex farthest from an interior point. We give a common formulation of these linear programs, prove monotonicity in the dimension threshold, and describe the exact verification of the certificates. The local estimates are taken from the companion geometric proofs.

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1 Introduction

Upper bounds for the dimension of a hyperbolic Coxeter polytope can be proved by comparing local inequalities with a global incidence count or a bound on a vector norm. We formulate these comparisons as linear programs. Their constraints are necessary conditions for a polytope to exist, and their infeasibility proves a dimension bound. The dual linear programs describe the combinations of local inequalities used in the proof.

We recover the following bounds from the local results of the companion paper and supplement [1, 2]:

$$\begin{aligned} &n \le 57 \quad \text{in finite volume,} \\ &n \le 17 \quad \text{in the compact case,} \\ &n \le 15 \quad \text{in the compact arithmetic case.} \end{aligned} \tag{1.1}$$

Guo and Zheng [3] prove the general finite-volume bound $n \le 81$. Our purpose here is to give a uniform linear programming treatment of (1.1).

Throughout the paper, a polytope is convex, finite-sided and full-dimensional. Hyperbolic space has curvature -1. A Coxeter polytope has all dihedral angles of the form π/m, where $m \ge 2$ is an integer. Its reflection group is generated by reflections in the supporting hyperplanes of its facets. Arithmeticity always refers to this group. We write Supplements FV, C and A for Parts I/IV, II/V and III/VI, respectively, of [2]. The local results used below are stated with their hypotheses; their geometric proofs are given in those supplements.

Theorem 1.1 (Linear programming proofs of the dimension bounds)**.** *The local estimates and classifications stated below give the following rational certificates.*

(i) *For every integer $n \geq 58$, there is a nonnegative weight function on vertex–face incidences satisfying the face and vertex inequalities of Proposition 3.3. Consequently the finite-volume incidence problem is infeasible.*

(ii) *The normalized linear program for the squared projection norms of compact polytopes, defined in* (4.3)*, is infeasible on the exhaustive family of configurations in dimensions* $18, \ldots, 29$*. Its dual has a rational solution proving this infeasibility.*

(iii) *The same conclusion holds for compact arithmetic polytopes in dimensions* $16, \ldots, 29$*. At dimensions* 16 *and* 17 *the program uses the arithmetic classifications and inequalities stated below; at dimensions* $18, \ldots, 29$ *it uses the compact configurations and inequalities from part* (ii)*.*

Together with Vinberg's compact bound $n \leq 29$, these certificates imply (1.1)*.*

Theorem 2.3 gives the common linear argument. A hypothetical polytope determines nonnegative variables satisfying homogeneous local constraints and a strict global inequality. Dividing by its positive excess gives a solution of a normalized feasibility problem. A dual inequality contradicts that normalization. Proposition 2.4 proves monotonicity: raising the lower bound on the dimension removes configurations from one fixed family.

Case	Quantities in the linear program	Dual inequalities
Finite volume	The same corner functions summed over faces and over vertices	Lower bounds for face sums and upper bounds for vertex sums
Compact	Squared norms of orthogonal projections of a fixed vector, whose total is less than one	Nonnegative combinations of local inequalities with coefficient at most one on each variable
Compact arithmetic	The same projection norms, subject to additional arithmetic restrictions	The same dual program, with the arithmetic inequalities and excluded configurations

The local estimates and the exhaustive lists of configurations are hypotheses of the linear argument. This includes the reductions leading to the two arithmetic lists. Conditions on the searched dual coefficients have a different role: they ensure that a linear combination proves the desired contradiction. We distinguish these conditions from geometric restrictions on a polytope wherever both occur in the same formulation.

Each local estimate is used over its stated range of ranks, labels and other components. Ordinary dihedral components retain their full integer parameter range. Configurations satisfying these necessary conditions need not be geometrically realizable; every polytope must nevertheless give one of them. All terminating decimals in this paper denote exact rational numbers.

2 Geometric notation and linear programming

2.1 Facet normals and Coxeter diagrams

We use a Lorentz space of signature $(-, +, \ldots, +)$, with hyperbolic space the future unit hyperboloid. The supporting hyperplane of a facet will be called a wall. It has an outward spacelike unit normal u with $\langle u, u \rangle = 1$, and the chamber inequality is $\langle y, u \rangle \leq 0$. For two facets meeting at angle π/m, the corresponding Gram entry is $-\cos(\pi/m)$. At a finite vertex the Gram matrix of the incident facets

is positive definite. The spans of the connected components of its Coxeter diagram are mutually orthogonal and together form the Euclidean tangent space.

A spherical Coxeter diagram has positive definite Gram matrix. A connected affine diagram has positive semidefinite Gram matrix of corank one. We use the standard list of irreducible spherical diagrams:

$$A_r\ (r \geq 1), \quad B_r\ (r \geq 3), \quad D_r\ (r \geq 4), \quad E_6, E_7, E_8, F_4, H_3, H_4, \quad I_2(m)\ (m \geq 4).$$

Here $A_2 = I_2(3)$; the isomorphic diagrams B_2 and D_3 are not listed separately. The *order* of a diagram is its number of nodes, and its *rank* is the rank of its Gram matrix. They agree for a spherical diagram. A connected affine diagram with $r+1$ nodes has Gram rank r. The simplex factor of an ideal vertex link that it determines has dimension r. We use this distinction between order and rank in the vertex-figure formulas below.

All facets and facet normals appearing in a local geometric statement belong to the polytope P, unless explicitly introduced as auxiliary objects. A facet opposite a vertex along a specified edge means a facet containing the other endpoint but not that vertex. Replacing a normal by a projected vector does not produce another facet of P. An auxiliary point asserted to lie in P must satisfy every defining facet inequality. The Coxeter and arithmetic hypotheses are imposed on P; they are not assumed for an arbitrary face.

2.2 A vector at a farthest vertex

Let P be compact. Fix a point $z \in \operatorname{int} P$ and choose a vertex p at maximal distance from z. Put $H = -\langle z, p\rangle = \cosh d(z,p)$. The decomposition

$$z = H(p - x), \qquad x \in p^{\perp} \tag{2.1}$$

defines a vector x in the tangent space at p. Indeed, $x = p - z/H$ is orthogonal to p, and $z^2 = p^2 = -1$ gives

$$\|x\|^2 = 1 - H^{-2} < 1. \tag{2.2}$$

The Lorentz form restricts to a Euclidean inner product on $p^{\perp}$. Let $W_1, \dots, W_k$ be the spans of the connected components of the Coxeter diagram at p, and define

$$e_j = \|\mathrm{proj}_{W_j} x\|^2, \qquad e_j \geq 0, \qquad \sum_{j=1}^{k} e_j = \|x\|^2 < 1. \tag{2.3}$$

Thus e_j is a squared norm, rather than a projection vector. A sum of several e_j is the squared norm of the projection onto the corresponding orthogonal sum of subspaces. Throughout both compact arguments, all these quantities refer to the fixed vector x. Auxiliary points used in a local proof do not supply further terms in (2.3).

2.3 Classes of configurations and local inequalities

Definition 2.1. An *exhaustive family of configuration classes* in dimension n is a finite collection of sets $\mathcal{D}_b$, with $n_b = n$, whose union contains every geometric configuration under consideration. Each class is specified by all the hypotheses of the local results used on it. A class may include a whole parameter range when its inequalities hold uniformly throughout that range.

The classes may overlap. A geometric configuration can be assigned to any class containing it. Some formal configurations in the classes may have no geometric realization; their inclusion preserves the necessary direction of the argument.

A local norm inequality specifies a set S of the orthogonal components and a rational number $c \geq 0$ such that

$$\sum_{j \in S} e_j \geq c \tag{2.4}$$

on a stated configuration class. We call S the *support* of the inequality. A lower bound for the squared norm of a projection onto a subspace of one component also bounds the squared norm on that component, by nonnegativity of the omitted terms. Bounds on overlapping subspaces must still be combined with their repeated coefficients accounted for.

A local theorem may instead show that a specified configuration cannot occur. We call this a *local exclusion*. Its geometric and arithmetic hypotheses form part of the definition of the class. In the homogeneous formulation its scaling variable is set to zero. Such an exclusion is used only on the stated class, whereas exclusion of every class in a given dimension is the conclusion of the LP.

2.4 Homogeneous linear programs

We first state the finite-dimensional alternative underlying the linear certificates.

Lemma 2.2. *Let M be a rational matrix and d a rational column vector of compatible size. Exactly one of the following systems has a solution:*

$$Mu = d, \quad u \geq 0; \tag{2.5}$$

$$M^T y \geq 0, \quad d^T y = -1. \tag{2.6}$$

When either system is feasible, it has a rational solution.

Proof. Let K be the cone generated by the columns of M. It is a closed polyhedral cone. If $d \notin K$, separation gives a vector y with $y^T k \geq 0$ on K and $y^T d < 0$; rescaling gives (2.6). If $d \in K$, its nonnegative column coefficients give (2.5). Both systems cannot be feasible, since they would give $-1 = d^T y = u^T M^T y \geq 0$.

For rationality, a nonempty polyhedron defined over $\mathbb{Q}$ has a rational point: restrict to the affine hull of a minimal face, solve its rational equality system, and approximate a relative interior point by rational points in that affine space. All remaining strict inequalities persist under a sufficiently small approximation. Apply this to the two displayed rational systems. □

Replacing a free variable by a difference of nonnegative variables and introducing slack variables puts a finite linear system in this form. Below we also write its dual inequalities in terms of the geometric quantities they estimate.

Let W be a real vector space and I an index set. To each $i \in I$ assign an integer n_i, a polyhedral cone $C_i \subset \mathbb{R}^{k_i}$ of local variables, a linear balance map $T_i : \mathbb{R}^{k_i} \longrightarrow W$, and a linear functional $d_i : \mathbb{R}^{k_i} \longrightarrow \mathbb{R}$. The label n_i denotes the dimension of the polytope, even when i describes a local configuration. The cone C_i is defined by homogeneous necessary conditions. The map T_i records the terms in the linear identities to be imposed, and d_i their contribution to a strict inequality. Fix these data independently of the lower dimension threshold.

For an integer N consider

$$u_i \in C_i \quad (n_i \geq N), \qquad \sum_{n_i \geq N} T_i u_i = 0, \qquad \sum_{n_i \geq N} d_i(u_i) = 1. \tag{2.7}$$

Only finitely many of the vectors u_i may be nonzero. For infinitely many dimensions, take W to be a direct sum with separate coordinates for each dimension. Incidence identities from different dimensions then remain separate. A finite-sided polytope contributes only finitely many terms.

Theorem 2.3 (Normalization and dual certificate)**.** *Suppose every polytope under consideration of dimension $n \geq N$ determines a finitely supported family of vectors $v_i \in C_i$, zero whenever $n_i \neq n$, with*

$$\sum_i T_i v_i = 0, \qquad D := \sum_i d_i(v_i) > 0.$$

Then (2.7) *has a solution. A linear functional λ on W satisfying*

$$\lambda(T_i u) - d_i(u) \geq 0 \qquad (u \in C_i,\ n_i \geq N) \tag{2.8}$$

proves infeasibility and hence excludes all these polytopes. If I is finite, W is finite-dimensional and all the cones, maps and functionals are rational, infeasibility is equivalent to the existence of such a rational certificate.

Proof. Set $u_i = v_i/D$ to obtain the normalized solution. If a feasible point and a dual certificate both existed, summing (2.8) at that point would give $0 - 1 \geq 0$, a contradiction. For the finite converse, express each rational polyhedral cone by finitely many rational generators and apply Lemma 2.2 to their images $(T_i u, d_i(u))$, with target $(0, 1)$. Its separating functional has the form $(\lambda, -1)$, exactly as in (2.8). □

Proposition 2.4 (Monotonicity in the dimension threshold)**.** *In the fixed formulation* (2.7)*, feasibility at $N' \geq N$ implies feasibility at N. Thus infeasibility at N implies infeasibility at every larger threshold. A certificate at N remains a certificate after restricting to $n_i \geq N'$.*

Proof. Extend a feasible point by zeros on the removed indices. Every cone contains zero, and both sums are unchanged. The same dual inequalities hold after restriction to the remaining indices. □

For an infinite index set only the inequality on finite sums is needed; the converse duality statement is restricted to finite systems. If the configuration classes cover a bounded range of dimensions, a separate theorem must exclude the higher dimensions. Section 7 states this additional information for each application. No passage to a lower-dimensional Coxeter face is involved.

3 Finite volume: weights on vertex–face incidences

We assign a weight to each incidence of a vertex and a two-dimensional face. Summing these weights first over faces and then over vertices gives the comparison used to bound the dimension. We formulate the required lower bounds at faces and upper bounds at vertices as a linear program. A weight function satisfying these inequalities contradicts the incidence inequality stated below. We also minimize the required vertex bound within a specified family of weight functions.

3.1 Vertex figures and the incidence inequality

Fix an integer $n \geq 58$. A *planar angle*, or *corner*, is an incidence $\kappa = (v, F)$ of a vertex v and a two-dimensional face F. The number of sides of F is denoted by $s(F)$, and

$$D_5(P) = \sum_{\dim F = 2} (5 - s(F)). \tag{3.1}$$

A triangle contributes two, a quadrilateral contributes one, and a face with at least five sides contributes a nonpositive number.

We use the standard vertex-figure description of a finite-volume Coxeter polytope. At a finite vertex the figure is a simplex. At an ideal vertex a horospherical figure is a product

$$L_v = \prod_{i=1}^{k} \Delta^{r_i}, \qquad r_i \geq 1, \qquad \sum_i r_i = n-1. \tag{3.2}$$

Here r_i is the dimension of a simplex factor, not the number of nodes of its affine diagram. That diagram has $r_i + 1$ nodes and Gram rank r_i. These descriptions also apply to nonsimple ideal vertices; see [1, 2].

Put $r = n - 1$. Write the h-polynomial of the figure as

$$h_v(t) = \sum_{j=0}^{r} h_{v,j} t^j, \qquad h_v(t) = \prod_i (1 + t + \cdots + t^{r_i}) \quad \text{at an ideal vertex.} \tag{3.3}$$

At a finite vertex $h_{v,j} = 1$ for every $0 \leq j \leq r$. Define

$$V_v = \sum_{j=0}^{r} h_{v,j}, \qquad G_v = \sum_{j=0}^{r} h_{v,j} \min(j, r-j), \qquad b_n(v) = \frac{5}{2} G_v - \frac{r}{2} V_v. \tag{3.4}$$

The quantity V_v is the number of vertices of the vertex figure. It equals n at a finite vertex and $\prod_i (r_i + 1)$ at an ideal vertex. Every vertex of the figure has degree r, so its number of edges is $E_v = rV_v/2$. These edges correspond to corners at v.

For a finite vertex, the sum in (3.4) gives

$$\beta_n = b_n(v) = \begin{cases} n(n-6)/8, & n \text{ even}, \\ (n-1)(n-5)/8, & n \text{ odd}. \end{cases} \tag{3.5}$$

At an ideal vertex we write

$$K_n(v) = \frac{5 \sum_j h_{v,j} \min(j, n-1-j) - (n-1) V_v}{2}. \tag{3.6}$$

Thus $K_n(v) = b_n(v)$; we use $K_n(v)$ for the bound at an ideal vertex and β_n for the bound at a finite vertex.

We use the following incidence inequality.

Input 3.1 (Incidence excess)**.** Let $P \subset \mathbb{H}^n$ be a finite-volume Coxeter polytope, with the vertex figures described above. The incidence theorem of [2] gives

$$D_5(P) \geq \sum_v b_n(v) + \Delta_n, \qquad \Delta_n \geq 5(n-1) > 0. \tag{3.7}$$

The proof of this inequality, by counting incidences according to height, is given in the supplement. To obtain a contradiction it suffices to prove $D_5(P) \leq \sum_v b_n(v)$; the positive term Δ_n already gives the required strict inequality.

3.2 The incidence linear program and its dual

First fix rational functions on corners, collected in a vector $\Psi(\kappa) \in \mathbb{Q}^d$. These functions will be specified in Section 3.3. Their values at a corner are the same in the face sum and in the vertex sum. A vertex type V consists of a vertex diagram, or of an ideal-link diagram, with the markings needed to evaluate these functions. A small-face type Q consists of a triangle or quadrilateral with compatible markings at its corners. Set

$$A(V) = \sum_{\kappa \text{ at } V} \Psi(\kappa), \qquad A(Q) = \sum_{\kappa \text{ on } Q} \Psi(\kappa), \qquad d(Q) = 5 - s(Q). \tag{3.8}$$

The families of types must include every vertex and every triangle or quadrilateral of a polytope under consideration. We may also include formal types that are not realized by a polytope.

Introduce nonnegative variables x_V for vertex types, y_Q for small-face types, and z_κ for allowed individual corners. The last variables account for corners belonging to faces with at least five sides. The incidence feasibility problem is

$$\sum_V A(V)x_V - \sum_Q A(Q)y_Q - \sum_\kappa \Psi(\kappa)z_\kappa = 0, \tag{3.9}$$

$$\sum_Q d(Q)y_Q - \sum_V b_n(V)x_V = 1. \tag{3.10}$$

For an infinite family of types we require only finitely many variables to be nonzero. This suffices because a polytope has finitely many incidences.

Lemma 3.2. *A finite-volume Coxeter polytope gives a feasible point of* (3.9)–(3.10) *if every vertex, every triangle or quadrilateral, and every corner has one of the specified descriptions.*

Proof. Let f_3 and f_4 be the numbers of triangular and quadrilateral two-faces, and put

$$L = 2f_3 + f_4 - \sum_v b_n(v).$$

Since the other two-faces contribute nonpositively to $D_5(P)$, Input 3.1 gives $L \geq \Delta_n > 0$. Divide each actual vertex count, small-face count and residual-corner count by L. Counting the same corners by vertices and by faces gives (3.9). The definition of L gives (3.10). □

The variables z_κ cannot be omitted: corners on faces with at least five sides still occur in the incidence equalities, although those faces have nonpositive contributions to $D_5(P)$.

Proposition 3.3 (Dual weight inequalities)**.** *Suppose a coefficient vector $\theta \in \mathbb{R}^d$ satisfies*

$$b_n(V) - \theta \cdot A(V) \geq 0 \qquad \textit{for every allowed vertex type,} \tag{3.11}$$

$$\theta \cdot A(Q) - d(Q) \geq 0 \qquad \textit{for every allowed small face,} \tag{3.12}$$

$$\theta \cdot \Psi(\kappa) \geq 0 \qquad \textit{for every allowed corner.} \tag{3.13}$$

Then the incidence feasibility problem is infeasible.

Proof. Use Theorem 2.3, reversing the sign of (3.9). The vertex, small-face and residual-corner columns (T, d) are respectively $(-A(V), -b_n(V))$, $(A(Q), d(Q))$ and $(\Psi(\kappa), 0)$; take $\lambda = \theta$. The three displayed conditions are exactly its certificate inequalities. □

Equivalently, the weight function $w(\kappa) = \theta \cdot \Psi(\kappa)$ has sum at least two on each triangle, at least one on each quadrilateral, and at most $b_n(v)$ at each vertex. On a face with at least five sides its nonnegative sum is at least $5 - s(F)$. Summing over faces and vertices gives $D_5(P) \leq \sum_v b_n(v)$, contrary to Input 3.1. This is weak duality, applied to a finite sum of incidences; no assertion of strong duality for an infinite-dimensional problem is needed.

3.3 The functions defining the corner weights

We now specify the functions used to construct the weights. Write $\phi(\kappa) \in \mathbb{Q}^{26}$ for their values at a corner. Each coordinate of ϕ is fixed; its coefficient in the weight function is a variable of the linear program.

At a finite vertex, the spherical diagram is a disjoint union of connected finite diagrams. Its two marks are the two nodes determining the corner. At an ideal vertex a corner corresponds to an edge of the product (3.2): one simplex factor varies along an edge, and a vertex is selected in every other factor. The varying affine diagram has two marked nodes. Each other factor contributes the finite diagram obtained by deleting the node selected by its vertex. These finite diagrams are included in the corner description.

We number A_r along its path by $0, \dots, r-1$. We number D_r along the spine $0, \dots, r-3$, with leaves $r-2, r-1$ attached to $r-3$. The remaining finite and affine diagrams use the fixed numberings of the rational tables in [2]. Fix one numbering of each vertex diagram and use it at all its corners. Under a diagram isomorphism, transport this numbering together with the markings. The numbering is not chosen separately at each corner.

Set $\phi_0 = 1$. The functions ϕ_1, ϕ_2, ϕ_3 are respectively the prescribed ordinary pair weight for two marks in the same finite component, the affine pair weight in the varying ideal factor, and the ordinary weight for marks in different finite component copies; each is zero outside its stated case. Their fixed rational tables, including exceptional small ranks, are the intrinsic corner tables of [2]. We use their final edition: in particular, the ordinary B_{13} endpoint entry is zero.

The next fifteen coordinate functions are indicators. Each is one when the two marks lie in a copy of the displayed component at the displayed unordered pair, and when the final column holds. Otherwise it is zero. A component in the last column is a connected component of the corner diagram disjoint from the marked component. Put $M_{15} = m_{A_{15}} + m_{D_{15}}$, the number of such finite components.

j	Marked type	Marked pair	Other components
4	A_{13}	$\{0, 12\}$	An A_1 occurs.
5	A_{14}	$\{0, 13\}$	An A_1 occurs.
6,7	A_{15}	$\{0, 13\}, \{1, 14\}$	An A_1 occurs.
8,9	D_{15}	$\{1, 13\}, \{1, 14\}$	An A_1 occurs.
10,11	A_{16}	$\{1, 12\}, \{3, 14\}$	An A_1 occurs.
12,13	D_{16}	$\{0, 14\}, \{0, 15\}$	An A_1 occurs.
14,15	D_{14}	$\{0, 12\}, \{0, 13\}$	An A_1, A_2 or A_3 occurs.
16,17	D_{13}	$\{0, 11\}, \{0, 12\}$	$M_{15} \geq 3$.
18	D_{28}	$\{0, 12\}$	$M_{15} \geq 2$.

In a row with two indices, the two marked pairs correspond in the displayed order. The final seven indicators concern different finite component copies:

j	Marked components and allowed nodes	Other components
19	$A_4 : \{0, 3\}$ and $A_{12} : \{1, 10\}$	An A_1 occurs.
20	$A_2 : \{0, 1\}$ and $A_4 : \{0, 3\}$	An A_1 and an A_{10} occur.
21	$A_2 : \{0, 1\}$ and $D_5 : \{0\}$	An A_1 and an A_9 occur.
22	$A_1 : \{0\}$ and $D_5 : \{0\}$	An A_{11} occurs, or both an A_9 and an A_2 occur.
23	$A_1 : \{0\}$ and $D_5 : \{0\}$	An A_9 and an A_2 occur.
24	$A_{12} : \{0, 11\}$ and $D_{16} : \{0\}$	$M_{15} \geq 2$.
25	Two distinct A_2 copies, with either node in each	An A_{10} and at least two A_1 copies occur.

These functions are indicators of unordered marked pairs. Functions 22 and 23 may both be one; function 25 is zero when both marks belong to the same A_2 component. Every coordinate of ϕ lies in $[0,1]$, but a linear combination of them may have either sign.

There is one additional ideal-corner indicator, denoted χ_B. Write $\Gamma[m]$ for an affine diagram with m nodes and rank $m-1$. The indicator is one precisely when the varying factor is $B[13]$, the marked pair is $\{10,11\}$ or $\{10,12\}$, and the complete residual finite diagram contains at least three components of types A_{15} or D_{15}. It is zero otherwise. Thus $B[13]$ denotes standard type $\widetilde{B}_{12}$, not a finite rank-13 component. Form the residual finite diagram by deleting one selected node from each of the other affine factors. The two marks in the varying factor are kept; deleting them first would define a different indicator.

Let $\epsilon_n = [n = 58]$. Use free vectors $a^{58}, a^{\mathrm{tail}} \in \mathbb{R}^{26}$ and parameters $\lambda, \rho \geq 0$. The weight function to be determined is

$$w(\kappa) = \big(\epsilon_n a^{58} + (1-\epsilon_n) a^{\mathrm{tail}}\big) \cdot \phi(\kappa) + \epsilon_n \lambda \chi_B(\kappa). \tag{3.14}$$

The 53 functions used in the incidence equalities are

$$\Psi(\kappa) = (\epsilon_n \phi(\kappa), (1-\epsilon_n)\phi(\kappa), \epsilon_n \chi_B(\kappa)). \tag{3.15}$$

The auxiliary variable ρ gives a common nonnegative lower bound in the vertex inequalities; there are 54 variables common to all constraints. The normalization at an ideal vertex is specified in Section 3.6. The coefficient vectors are unrestricted in sign. We require the resulting weight to be nonnegative at every corner in the specified classes, rather than require every coefficient to be nonnegative.

The finite tables used for the dimension-58 calculation are defined only through rank 58. Their validity in higher dimensions is not assumed. The face theorems for arbitrary rank and the vertex estimates below justify the weight function (3.14) for $n \geq 59$.

3.4 Local face inequalities

The local face theorems restrict the marked diagrams and supporting hyperplanes that can occur at a triangle or quadrilateral. They give the following sufficient linear conditions on the weight coefficients.

Input 3.4 (Local face theorems)**.** The corner, triangle and quadrilateral theorems of [2], together with the implications between coefficient conditions established in [4], give a linear system $\mathcal{L}$. It includes the corner and triangle inequalities for both coefficient vectors; the conditions involving χ_B and the finite D_{13} conditions; the conditions for fork diagrams and the additional rank-16 conditions; and all the quadrilateral inequalities, including the A_{13} inequality for each coefficient vector. If the coefficients and auxiliary variables satisfy this system, then (3.12)–(3.13) hold at every corner, triangle and quadrilateral of a polytope in dimension $n \geq 58$. Whenever one of these implications uses equalities implied by the dimension-58 vertex inequalities, the conditions of Section 3.5 are also required. Those equalities are consequences of that linear system, not geometric assumptions.

The unknowns in $\mathcal{L}$ are coefficients of a weight function, not quantities assigned by the polytope. Thus its inequalities are sufficient conditions for the desired weights; they need not hold for every weight function satisfying the face and vertex bounds. For example, the A_{13} face theorem is applied under

$$a_0^{58} + a_4^{58} \geq 1, \qquad a_0^{\mathrm{tail}} + a_4^{\mathrm{tail}} \geq 1. \tag{3.16}$$

We include both inequalities in $\mathcal{L}$.

The hypotheses record the identities of the facets of P, the order of the marks, their embeddings in the numbered diagrams, the selected node deletions at ideal links, and all the facets used to justify an auxiliary construction. An inverse-Gram calculation does not establish these geometric hypotheses. Inequalities from different alternatives are combined only when they apply simultaneously, with each corner counted with its stated multiplicity.

For example, for either $z = a^{58}$ or a^{tail}, the system contains

$$4z_0 + z_3/16 + z_{22} + z_{23} \geq 1, \tag{3.17}$$

$$4z_0 + z_3/2 + z_{21} \geq 1. \tag{3.18}$$

Each inequality is used only for the marked face configurations in its local theorem. The occurrence of a diagram with the same name elsewhere on a face does not suffice. The electronic matrix lists every inequality of $\mathcal{L}$ and identifies its source theorem.

3.5 Finite vertices: a network formulation

At a finite vertex the marks range over all unordered pairs of nodes in a spherical forest. A formal forest has nonnegative integer component multiplicities m_T satisfying

$$\sum_T r_T m_T = 58, \tag{3.19}$$

where r_T is the rank of type T. We use the finite collection of weight types of the rational catalogue. A symbolic dihedral family can be represented by one weight type only when every corner function has the same table throughout the stated parameter range and that range is covered by the local theorem. A finite sample of labels is not used in place of an unbounded family.

Let

$$\Phi(m) = \sum_{\kappa \text{ at the formal vertex}} \phi(\kappa).$$

Every actual finite vertex supplies an allowed formal forest. The converse is not required. We seek the family of inequalities

$$a^{58} \cdot \Phi(m) + \rho \leq 377 \quad \text{for every } m \text{ satisfying (3.19).} \tag{3.20}$$

The right side is the vertex bound $\beta_{58} = 58(58-6)/8$ obtained from the incidence inequality.

We express this family of inequalities by paths in a finite directed acyclic graph. For one component T, let S_T be the sum of its ordinary same-component entries; for different component types T, U, let B_{TU} be the sum of their ordinary cross entries. Using u_j for the jth coordinate vector of $\mathbb{Q}^{26}$ gives

$$\Phi_0 = \binom{58}{2}, \qquad \Phi_1 = \sum_T m_T S_T, \qquad \Phi_2 = 0, \qquad \Phi_3 = \sum_{T<U} m_T m_U B_{TU}. \tag{3.21}$$

Functions 4–18 sum to the component multiplicity times the indicator of the stated condition on the other components. The remaining coordinates follow by counting the unordered mark pairs in the table: for example, $\Phi_{19} = 4 m_{A_4} m_{A_{12}} [m_{A_1} > 0]$ and $\Phi_{25} = 4\binom{m_{A_2}}{2}[m_{A_{10}} > 0, m_{A_1} \geq 2]$. Brackets denote Boolean indicators; the factors four count node choices, not orientations.

Fix the multiplicities $h_H = m_H$ of the types in

$$\mathcal{H} = \{A_9, \dots, A_{17}, B_9, \dots, B_{12}, D_9, \dots, D_{16}, D_{28}\}. \tag{3.22}$$

Also fix $a = m_{A_1}$ and $b = m_{A_2}$, and denote the tuple of these fixed multiplicities by I. Let

$$M_h = 58 - \sum_H r_H h_H, \qquad R_I = M_h - a - 2b.$$

Every nonzero ordinary cross term has exactly one component in $\mathcal{H}$. Once I is fixed, the remaining contribution to $\Phi(m)$ is additive in the other component multiplicities, apart from one indicator for the presence of A_3. More precisely, let $\mathcal{R}$ be the remaining types after removing $\mathcal{H}, A_1, A_2$, and put $\mathcal{S} = \mathcal{R} \setminus \{A_3\}$. Define

$$p_T^h = S_T u_1 + \Big(\sum_H h_H B_{TH}\Big) u_3.$$

For remaining types other than A_4, D_5 use $p_{I,T} = p_T^h$. For these two types, the vector contributions are

$$\begin{aligned} p_{I,A_4} &= p_{A_4}^h + 4h_{A_{12}}[a > 0]u_{19} + 4b[a > 0, h_{A_{10}} > 0]u_{20}, \\ p_{I,D_5} &= p_{D_5}^h + 2b[a > 0, h_{A_9} > 0]u_{21} \\ &\quad + a[h_{A_{11}} > 0 \text{ or } (h_{A_9} > 0, b > 0)]u_{22} + a[h_{A_9} > 0, b > 0]u_{23}. \end{aligned} \tag{3.23}$$

Let K_I contain $\binom{58}{2}u_0$ and all terms of the corner sum involving only the fixed counts h, a, b, including the fixed same-component indicators. Put

$$\gamma_I = h_{D_{14}}[a = b = 0](u_{14} + u_{15}).$$

Then, with $k = m_{A_3}$,

$$\Phi(m) = K_I + kp_{I,A_3} + [k > 0]\gamma_I + \sum_{T \in \mathcal{S}} m_T p_{I,T}. \tag{3.24}$$

By the definitions of the functions, the only indicator not fixed by I is the occurrence of A_3 in the D_{14} conditions. The remaining cross terms are linear in the A_4, D_5 multiplicities; all other terms are fixed by I. This proves the identity.

For each choice (I, k) consistent with total rank 58, an initial edge has vector weight $K_I + kp_{I,A_3} + [k > 0]\gamma_I$ and ends at a vertex labelled by the remaining rank $R_I - 3k$. A subsequent edge selects a type $T \in \mathcal{S}$, decreases the remaining rank by r_T, and has vector weight $p_{I,T}$. Use the rank-feasible initial choices and their reachable remaining ranks. A complete path ends at rank zero. The rank strictly decreases along each subsequent edge, so the graph is acyclic.

Lemma 3.5 (Exact network formulation)**.** *Every formal forest in the specified family is represented by a complete path with vector weight $\Phi(m)$, and every complete path gives component multiplicities of total rank 58. Consequently the maximum scalar path weight is $\max_m a^{58} \cdot \Phi(m)$ for every signed vector a^{58}.*

Proof. The initial choice and subsequent edges select exactly the nonnegative multiplicities in (3.19). Equation (3.24) gives their vector weight. Different orders may represent the same forest, but give the same weight. A unit flow on an acyclic graph is a convex combination of complete paths, so maximizing its linear weight gives the maximum path weight. □

For fixed h, two choices of I may use the same graph of remaining ranks only if their vectors p_{I,A_4}, p_{I,D_5} agree. The other edge weights are already fixed by h. Their initial edges are still kept separately. Equality of scalar weights at one choice of coefficients would not justify identifying these graphs while the coefficients vary.

The finite calculation is divided into subgraphs indexed by S. Within one subgraph, let g index the groups of initial choices with equal remaining edge-weight vectors, as described above. Introduce free potential variables $v_{S,g,\ell}$ at the reachable remaining ranks ℓ, and a free upper bound U_S for the paths starting in S. The linear constraints are

$$\begin{aligned} v_{S,g,0} &= 0, \\ v_{S,g,\ell} - v_{S,g,\ell-r_T} &\geq a^{58} \cdot p_{g,T}, \\ U_S - v_{S,g(I),R_I-3k} &\geq a^{58} \cdot (K_I + k p_{I,A_3} + [k>0]\gamma_I), \\ U_S + \rho &\leq 377. \end{aligned} \tag{3.25}$$

All sets S use the same coefficients a^{58}, ρ and include every remaining type; the potentials are separate variables. Summing along a path gives its required upper bound. Conversely, define each potential to be the maximum scalar weight of a path from that remaining rank to zero. These potentials satisfy the inequalities. Eliminating them therefore gives exactly (3.20), including when some coefficients are negative. All optimization variables are real. The exhaustive construction of the finite graph is supplied with the checked data; the lemma proves the equivalence with the displayed linear program.

3.6 Ideal vertices and estimates for higher dimensions

Let an ideal link have factor ranks $r_1, \ldots, r_k$, total rank $R = n-1$ and $V = \prod_i (r_i+1)$ vertices. For a connected affine factor Γ of rank r, let Q_Γ be the sum of its baseline affine pair weights, given by ϕ_2, and put $q_\Gamma = Q_\Gamma/(r+1)$. Define $C = \sum_i q_{\Gamma_i}$. Denote the sum of the weights at v, before the χ_B term is included, by $C(v)$. For coefficient values (a_0, a_2) it satisfies

$$\frac{C(v)}{V} = \frac{R}{2} a_0 + C a_2 \tag{3.26}$$

This follows because a marked pair in one factor occurs once for each choice of a vertex in every other factor. Only coordinates 0 and 2 of ϕ can be nonzero at ideal corners; χ_B is a separate function.

Let J be the sum of independent random variables, with the i-th variable uniformly distributed on $\{0, \ldots, r_i\}$, and put $Y = 2J - R$. The coefficient distribution of h_v is the distribution of J multiplied by V. Symmetry gives

$$\frac{K_n(v)}{V} = \frac{3R}{4} - \frac{5}{4}\mathbb{E}|Y|, \qquad S_a = \frac{3R}{4} - \frac{5}{4}\mathbb{E}|Y| - \frac{R}{2}a_0 - C a_2. \tag{3.27}$$

The desired inequality is $S_a \geq \rho$. Since these expressions have been divided by V, the corresponding difference between the vertex bound and its weight sum is at least $V\rho$.

Products of total rank 57. For $n = 58$, introduce free factor-bound variables k_r for $1 \leq r \leq 57$. For every connected affine family impose

$$k_r \geq \frac{r}{2} a_0^{58} + q_\Gamma a_2^{58}. \tag{3.28}$$

The family ranges are affine A at ranks 1–57, C at 2–57, B at 3–57, D at 4–57, and the exceptional E_6, E_7, E_8, F_4, G_2. There are 227 individual-family rows. They all remain, even when a_2^{58} is negative.

The ideal-vertex calculation uses 354 selected symmetric arrays. For each array the data specify a preceding array and the rank of the factor added to obtain it. For every other factor addition,

the data specify a selected array whose central averages are no greater than the resulting averages. One selected index must work for all coordinates of this comparison. These indices are fixed data; the array averages below are variables of the linear program. Adding a factor acts on an array by uniform convolution followed by subtraction of its factor bound:

$$f'_j = \frac{1}{r+1}\sum_{\ell=0}^{r} f_{j+\ell} - k_r.$$

The initial array is $g_j = (5\min(j, 57-j) - 57)/2$; after the factors have total rank 57 it is a scalar, equal to the normalized vertex bound minus the sum of the factor bounds. After accumulated rank R, put $L = 57 - R$ and let $B_{R,i,a}$ be the average of array i on the central interval $a, \dots, L-a$, for $0 \le a \le \lfloor L/2 \rfloor$. Let I_R be the selected indices at rank R. If a rank-r factor is added to an array $j \in I_{R-r}$, put $A = L - 2a + 1$ and

$$\begin{aligned} \omega_{L,a,r,\ell} &= \frac{A + r - 2\ell}{A(r+1)}, && 0 \le \ell < \min(A, r+1), \\ T^{r,j}_{R,i,a} &= B_{R,i,a} + k_r - \sum_{\ell=0}^{\min(A,r+1)-1} \omega_{L,a,r,\ell} B_{R-r,j,a+\ell}. && (3.29) \end{aligned}$$

These positive coefficients sum to one and express the central average of the convolution as an average of nested central intervals. The rows are:

(i) the 29 initial equalities

$$B_{0,0,a} = \frac{1}{58-2a}\sum_{j=a}^{57-a} \frac{5\min(j, 57-j) - 57}{2}, \qquad 0 \le a \le 28;$$

(ii) $T^{r,j}_{R,i,a} = 0$ for the specified predecessor (r, j) of each selected $i \in I_R$;

(iii) $T^{r,j}_{R,D(R,r,j),a} \le 0$ for every candidate, using its fixed selected comparison array $D(R, r, j)$;

(iv) $\tau \le B_{57,i,0}$ for every final selected index, and $\rho \le \tau$, with τ free.

The index sets and maps are included in the electronic ideal-vertex data of [4]. Together with the connected-family inequalities, this system has 103,265 rows and 5,754 auxiliary variables. Positivity of convolution preserves the comparisons at every factor addition. Hence every product of total rank 57 has the required difference at least $\tau \ge \rho$. The fixed choices of comparison arrays give a sufficient system of linear inequalities; they need not characterize every valid ideal-vertex weight function.

The additional ideal-corner indicator. Keep the preceding additive inequalities and treat χ_B separately. This indicator depends on the other factors as well as the rank-12 factor, so its contribution cannot be included in an unconditional change of k_{12}. If χ_B occurs, the varying $B[13]$ factor has rank 12 and the residual finite rank is 45. Three rank-15 components exhaust that rank. The complete one-node-deletion classification leaves exactly

$$\begin{aligned} \mathrm{I}: &\quad B[13] \times \Gamma_1[16] \times \Gamma_2[16] \times \Gamma_3[16], \\ \mathrm{II}: &\quad B[13] \times D[31] \times \Gamma[16], \qquad \Gamma_i, \Gamma \in \{A, B, D\}. \end{aligned} \tag{3.30}$$

Let $a_\chi = V^{-1}\sum_\kappa \chi_B(\kappa)$. The additional rows are

$$\frac{57}{2}a_0^{58} + Ca_2^{58} + a_\chi\lambda + \rho \le K. \tag{3.31}$$

All thirteen rows have the following compact specification. Put

$$(c_A, c_B, c_D) = (5, 25/4, 411/64), \qquad (d_A, d_B, d_D) = (1, 1/16, 1/4).$$

For every unordered triple $\Gamma_1, \Gamma_2, \Gamma_3 \in \{A, B, D\}$ in type I, and every $\Gamma \in \{A, B, D\}$ in type II, use

	C	a_χ	K
I	$11/2 + \sum_{i=1}^{3} c_{\Gamma_i}$	$(2/13)\prod_{i=1}^{3} d_{\Gamma_i}$	331643/13312
II	$753/62 + c_\Gamma$	$(2/403)d_\Gamma$	16549/806.

(3.32)

Here d_Γ is the proportion of one-node deletions giving the required residual component; the deletion counts are $16, 1, 4$. The values of K are (3.27) for ranks $(12, 15, 15, 15)$ and $(12, 30, 15)$. Thus the specification retains deletion multiplicities, not just the affine factor types.

The remaining rank-58 products in dimension 59. Append all 63 connected-family links of ranks at most 16 for the coefficients a^{tail},

$$k_r \geq \frac{r}{2} a_0^{\mathrm{tail}} + q_\Gamma a_2^{\mathrm{tail}}, \tag{3.33}$$

using the *same* factor bounds k_r as above. Put $k_0 = 0$ and introduce $\mu \geq 0$. The fifteen descent rows are

$$k_r - k_{r-1} + \mu \leq h_r, \qquad 1 \leq r \leq 15, \tag{3.34}$$

where

$$(h_1, \ldots, h_{15}) = \left(\frac{3}{4}, \frac{47}{66}, \frac{89}{132}, \frac{7}{11}, \frac{79}{132}, \frac{37}{66}, \frac{23}{44}, \frac{2213}{4374}, \frac{59}{132}, \frac{9}{22}, \frac{67}{156}, \frac{71}{168}, \frac{5}{12}, \frac{79}{192}, \frac{83}{204}\right).$$

The rank-reduction theorem applies to a rank-58 product whose largest factor rank is between 10 and 16. It provides a factor whose rank can be reduced by one, with the difference between the normalized vertex bound and the sum of factor bounds increasing by at least μ when the factor is restored. The reduced product has rank 57 and satisfies the preceding inequalities. Thus the dimension-59 vertex inequality holds for this class. The same variables k_r must be used in both systems for the comparison to be valid.

All other ideal links in dimensions at least 59. For every other dimension-59 ideal link, and for every ideal link in dimension at least 60, the analytic vertex theorem gives $S_{(0,1)} \geq \varepsilon = 1/5307$. For coefficients a^{tail} the following four affine inequalities suffice:

$$\begin{aligned} a_0^{\mathrm{tail}} &\leq 0, & a_0^{\mathrm{tail}} + a_2^{\mathrm{tail}} &\leq 1, \\ \rho + 29 a_0^{\mathrm{tail}} &\leq \varepsilon, & \rho + 29 a_0^{\mathrm{tail}} + 29 a_2^{\mathrm{tail}} &\leq 29 + \varepsilon. \end{aligned} \tag{3.35}$$

To see the reduction, write $t = 1/n$, $u_0 = a_0^{\mathrm{tail}}$, $u_1 = a_0^{\mathrm{tail}} + a_2^{\mathrm{tail}} - 1$ and

$$P_i(t) = (\varepsilon - \rho)t - \frac{1 - t}{2} u_i.$$

Each P_i is affine, and (3.35) is exactly nonnegativity at the two endpoints of $[0, 1/59]$. The analytic input also gives $0 \leq C \leq R/2$. If $w = 2C/R$, then

$$t(S_a - \rho) = t(S_{(0,1)} - \varepsilon) + (1 - w)P_0(t) + wP_1(t) \geq 0.$$

Thus the four inequalities prove the vertex bound throughout the stated range. This conclusion uses the analytic estimate for $S_{(0,1)}$ in every dimension of that range.

Finite vertices in dimensions at least 59. Let $q \in \mathbb{Q}^{26}$ be the vector displayed in (3.38) below. Introduce $0 \le \xi \le 1$ and impose

$$a_j^{\mathrm{tail}} \le q_j + 2\xi[\, j \in \{16, 17\}\,] \quad (j \ne 2), \qquad \rho \le \frac{1}{4}. \tag{3.36}$$

The analytic finite-vertex estimate asserts that, for every formal finite forest of every rank $n \ge 59$,

$$\beta_n - \big(q + 2\xi(u_{16} + u_{17})\big) \cdot \Phi_n \ge \frac{1}{4}.$$

Since the functions at a finite vertex are nonnegative and function 2 is zero, (3.36) proves its vertex bound. The analytic estimate permits every other component and every rank $n \ge 59$; it is separate from the rank-58 finite calculation. The coordinate inequalities are sufficient conditions on a^{tail}, not a characterization of all valid finite-vertex weight functions.

3.7 The optimal bound within the chosen weight family

Fix a finite family $\mathcal{T}$ of formal forests and a family $\mathcal{C}$ of weight coefficients satisfying the local face and ideal-vertex conditions. Define the maximum finite-vertex weight sum and minimize it over $\mathcal{C}$:

$$U_{\mathcal{T}}(a) = \max_{F \in \mathcal{T}} a \cdot \Phi(F), \qquad \min_{a \in \mathcal{C}} U_{\mathcal{T}}(a). \tag{3.37}$$

Here a is the coefficient vector used at a finite vertex, and $\Phi(F)$ is the sum of its corner-function vectors. The set $\mathcal{C}$ is the projection onto a of the joint system, including all auxiliary-variable conditions. The incidence inequality supplies an independent upper bound for comparison. To prove the dimension bound, one needs a feasible choice whose maximum finite-vertex weight sum does not exceed that upper bound, with the face and ideal-vertex inequalities satisfied by the same coefficients.

In this application $\mathcal{T}$ is the exhaustive family of allowed rank-58 forests and $\beta_{58} = 377$. We give both a feasible weight function and a dual lower bound for this coefficient family. The rank-58 calculation is used only at that rank.

The following rational point uses the coordinate order specified in Section 3.3:

$$\begin{aligned}
(q_0, \dots, q_{12}) &= (0, 1, 1, 1, 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{5}{8}, \tfrac{5}{8}, \tfrac{1}{2}),\\
(q_{13}, \dots, q_{25}) &= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{1}{2}, \tfrac{1}{4}),\\
a^{58} &= q, \qquad a^{\mathrm{tail}} = q + \tfrac{1}{2}(u_{16} + u_{17}),\\
\lambda &= \tfrac{1}{2}, \qquad \rho = 0.
\end{aligned} \tag{3.38}$$

The two D_{13} coefficients are negative in dimension 58 and zero in higher dimensions. Nevertheless, their sum with the other terms in (3.14) is nonnegative at every allowed marked corner.

The auxiliary variables have exact values as well. For the higher-dimensional finite-vertex and rank-reduction inequalities take $\xi = 1/4$ and $\mu = 13/2187$. Take $k_r = \max_{\mathrm{rank}\,\Gamma = r} q_\Gamma$ over all the connected affine families, and assign the $B_{R,i,a}$ using the specified ideal arrays and their defining equalities, with $\tau = 0$. The rational verification checks all these variables and every finite-network potential at the same coefficient values.

For example, the face rows (3.17)–(3.18) take values 17/16 and 1 for both coefficient vectors. The ideal-indicator inequalities have slack at least 939/13312 in type I and 50623/25792 in type II. These examples illustrate the assignment; rational verification checks every local and network inequality at the same coefficients.

There is a useful optimization statement in addition to feasibility. Let $\mathcal{A}$ be the projection onto $(a^{58}, a^{\mathrm{tail}}, \lambda)$ of the complete local coefficient system, including (3.16), with $\rho = 0$. Replace the last network row by $U_S \le U$, for one free common variable U, and minimize U. Eliminating the potentials gives

$$\min_{(a^{58},a^{\mathrm{tail}},\lambda)\in\mathcal{A}} \max_{F} a^{58} \cdot \Phi(F), \tag{3.39}$$

where F ranges over the exhaustive family of allowed rank-58 forests. In this optimization, 377 is no longer imposed as an upper bound.

Proposition 3.6 (Exact restricted minimax)**.** *The optimum in* (3.39) *is 377.*

Proof. The feasible assignment (3.38), including its auxiliary local variables and all network potentials, gives an upper bound of 377. For the converse, write a_0, a_1 for coordinates 0 and 1 of a^{58}. The corner and triangle inequalities give $a_0 \ge 0$ and $3a_0 + 2a_1 \ge 2$. The formal forest

$$F_* = 2D_{17} + D_{24}$$

has total rank 58. The ordinary table gives $S_{D_{17}} = 227/2$ and $S_{D_{24}} = 8 \cdot 24 - 32 - 1 - 4 - 3 - 2 = 150$. All cross-component weights and indicator functions vanish on this forest, so its vector of corner sums is $\Phi(F_*) = 1653u_0 + 377u_1$. Thus one path bound gives $U \ge 1653a_0 + 377a_1$. The exact nonnegative combination

$$U - 377 = (U - 1653a_0 - 377a_1) + \frac{377}{2}(3a_0 + 2a_1 - 2) + \frac{2175}{2}a_0 \tag{3.40}$$

proves the lower bound: the first term is the sum of this forest's network rows and its link to U, and the other terms are multiples of the two local rows. □

The incidence inequality gives the bound 377 independently of this optimization. Feasibility and (3.40) prove that 377 is also the optimum for the chosen weight family and formal forests. The forest F_* need not occur at a polytope vertex. A larger family of weight functions may have a smaller optimum.

Remark 3.7 (Choice of the weight family). Vinberg [5, §6.12, p. 91] gives a related E_7 obstruction for a restricted family of nonnegative corner weights in the compact case. His requirement that weights vanish when the marks lie in different connected components is absent from the family in Section 3.3, which allows cross-component and contextual contributions. Such obstructions concern the specified diagram and weight classes and do not preclude stronger certificates using additional geometric compatibility.

In the system with upper bound 377, $\max_F a^{58} \cdot \Phi(F) + \rho \le 377$ and $\rho \ge 0$. The same inequalities force $a_0 = 0$, $a_1 = 1$ and $\rho = 0$. Consequently a requirement $\rho > 0$ would exclude this valid solution. The strict incidence inequality already gives the contradiction when $\rho = 0$.

3.8 The finite-volume dimension bound

The estimates used in each dimension are summarized below. In every case the same weight function is used for the face and vertex sums.

Class	Estimate used
Finite vertices, $n = 58$	The finite directed graph and all its potential inequalities.
Finite vertices, $n \geq 59$	The analytic finite-vertex estimate and coordinate inequalities (3.36).
Ideal vertices, $n = 58$, no corner with $\chi_B = 1$	The system for rank-57 products and the connected-family inequalities.
Ideal vertices, $n = 58$, some corner with $\chi_B = 1$	The exhaustive classification of node deletions (3.30) and all thirteen rows (3.31).
Ideal vertices, $n = 59$, largest factor rank 10–16	The same factor bounds and the rank-reduction inequalities.
Other ideal vertices at 59, and all $n \geq 60$	The analytic ideal estimate and the four inequalities (3.35).
All corners and small faces, $n \geq 58$	The exhaustive local geometric classification and theorems in Input 3.4.

Theorem 3.8 (Finite-volume bound)**.** *Assume the local geometric estimates, the exhaustive diagram classifications and the incidence inequality stated above. The verified rational solution excludes finite-volume hyperbolic Coxeter polytopes in every integer dimension $n \geq 58$.*

Proof. Suppose such a polytope exists. The last row of the table gives nonnegative corner weights whose sums are at least two on each triangle and at least one on each quadrilateral. The preceding rows give

$$\sum_{\kappa \text{ at } v} w(\kappa) \leq b_n(v)$$

at every finite or ideal vertex. The dual inequalities contradict the feasible point supplied by Lemma 3.2. □

4 Linear programs for squared projection norms

The two compact cases use the same specialization of Theorem 2.3. For each configuration class b, let e_b be the vector of squared projection norms defined in (2.3), grouping orthogonal components when convenient. Write all the local inequalities valid on this class as

$$A_b e_b \geq c_b, \qquad e_b \geq 0. \tag{4.1}$$

The matrices A_b and vectors c_b are rational and nonnegative, and every row of A_b has a positive entry. In our applications A_b has entries 0 and 1: a row specifies the components in the sum it estimates. Allowing general nonnegative coefficients gives the same argument.

Suppose more generally that the geometric upper bound is

$$h_b^T e_b < B_b, \qquad h_b > 0, \quad B_b > 0. \tag{4.2}$$

Here h_b and B_b are fixed rational data. For the compact cases $h_b = \mathbf{1}$ and $B_b = 1$, by (2.3). Variables belonging to different configuration classes are kept separate. Within one class, every inequality concerning a given orthogonal component uses the same coordinate of e_b.

4.1 Normalization and duality

For each class introduce nonnegative variables τ_b, E_b and δ_b, and impose

$$\begin{gathered} A_b E_b \geq c_b \tau_b, \qquad \delta_b = B_b \tau_b - h_b^T E_b \geq 0, \\ \sum_b \delta_b = 1. \end{gathered} \tag{4.3}$$

The scalar τ_b is a scaling coefficient, E_b is the scaled norm vector, and δ_b is its difference from the scaled upper bound. For a class ruled out by a local geometric theorem, add

$$\tau_b = 0. \tag{4.4}$$

This also forces $E_b = 0$ and $\delta_b = 0$, since $h_b > 0$. The normalization by $\sum_b \delta_b$ expresses the strict inequality as a closed linear feasibility problem, without choosing a fixed positive error tolerance.

Theorem 4.1 (Normalization and linear programming duality)**.** *For a finite exhaustive family of configuration classes, the following assertions hold.*

(i) *System* (4.3)–(4.4) *is feasible if and only if some class not locally excluded admits a vector* e_b *satisfying* (4.1) *and* (4.2)*. In particular, a polytope in any of the specified dimensions determines a feasible point.*

(ii) *Put*

$$m_b = \min\{h_b^T e : e \geq 0,\ A_b e \geq c_b\}. \tag{4.5}$$

Then

$$m_b = \max\{c_b^T \alpha : \alpha \geq 0,\ A_b^T \alpha \leq h_b\}, \tag{4.6}$$

and both optima are attained at rational points. The normalized system is infeasible if and only if $m_b \geq B_b$ *for every class not locally excluded.*

(iii) *Therefore, nonnegative rational multipliers satisfying*

$$A_b^T \alpha_b \leq h_b, \qquad c_b^T \alpha_b \geq B_b \tag{4.7}$$

for every such class prove infeasibility. Conversely, if the normalized system is infeasible, these multipliers exist.

Proof. Given a vector with $s = B_b - h_b^T e_b > 0$, take $\tau_b = 1/s$, $E_b = e_b/s$, $\delta_b = 1$, and set all variables on other classes to zero. Conversely, a normalized feasible point has some $\delta_b > 0$. Then $\tau_b > 0$, and division by τ_b gives the local inequalities and

$$h_b^T (E_b/\tau_b) = B_b - \delta_b/\tau_b < B_b.$$

Every polytope belongs to a class of the exhaustive family, so the first construction applies to its squared projection norms. The scaling coefficients need no upper bound: a small positive value of s may require a large τ_b.

For the minimum, a sufficiently large multiple of $\mathbf{1}$ satisfies all local inequalities. Since $h_b > 0$, every nonempty finite objective sublevel is compact, so the minimum is attained. To establish duality, omit the subscript and consider the polyhedral cone

$$K = \{(Ae - u, h^T e + t) : e, u, t \geq 0\}.$$

It contains (c, m) but no $(c, m - \varepsilon)$ for $\varepsilon > 0$. A nonzero supporting functional $(-\alpha, \beta)$, nonnegative on K and zero at (c, m), gives

$$\alpha \geq 0, \quad \beta \geq 0, \quad A^T \alpha \leq \beta h, \quad \alpha^T c = \beta m.$$

If $\beta = 0$, the positive entry in each nonnegative row of A forces $\alpha = 0$, a contradiction. Dividing by β gives a dual optimizer. The reverse inequality follows by multiplying $Ae \geq c$ by any feasible

dual vector. A vertex of the compact primal optimal face is a rational vertex of the feasible polyhedron. The dual optimal face is then a nonempty rational polyhedron and has a rational point by Lemma 2.2.

Finally, multiplying the scaled inequalities by the dual multipliers gives

$$h_b^T E_b \geq \alpha_b^T A_b E_b \geq (c_b^T \alpha_b)\tau_b \geq B_b \tau_b.$$

Hence every δ_b is zero, contradicting their normalization. In Theorem 2.3, take C_b to be the local cone defined by (4.3), with zero scaling coefficient on excluded classes, $T_b = 0$ and $d_b = \delta_b$. The displayed inequality gives its dual condition $-d_b \geq 0$. Part (ii) proves the converse in part (iii). □

4.2 The dual inequalities in the compact cases

For $h_b = \mathbf{1}$, $B_b = 1$, the computed dual solution satisfies

$$\alpha_b \geq 0, \qquad A_b^T \alpha_b \leq \mathbf{1}, \qquad c_b^T \alpha_b \geq \eta, \qquad \eta \geq 1. \tag{4.8}$$

For a row supported on S_ℓ, the coefficient of e_{bj} in the chosen combination is $\sum_{\ell : j \in S_\ell} \alpha_{b\ell}$. The inequality $A_b^T \alpha_b \leq \mathbf{1}$ requires this coefficient to be at most one for each component. Nonnegativity of the unused terms then bounds the combination by the total squared norm.

Example. Suppose $e_I + e_A \geq a$ and $e_I + e_B \geq b$. Adding both inequalities with coefficient one would give coefficient two on e_I. Coefficients $1/2, 1/2$ give instead

$$e_I + \tfrac{1}{2} e_A + \tfrac{1}{2} e_B \geq \tfrac{1}{2}(a + b).$$

The left side is at most $e_I + e_A + e_B$, since all three variables are nonnegative. If the right side is at least one, this contradicts the strict upper bound on the total squared norm.

For a fixed dimension-labelled family, define

$$\Gamma_N = \min_{\substack{b \text{ not locally excluded} \\ n_b \geq N}} m_b, \qquad \min \varnothing = +\infty. \tag{4.9}$$

The normalized LP is infeasible if and only if $\Gamma_N \geq 1$. Increasing N removes classes, so Γ_N cannot decrease. This is the specialization of Proposition 2.4. The equivalence concerns the specified linear inequalities; a feasible vector need not be realized by a geometric configuration.

The computation imposes (4.8) and maximizes the sum of the local dual objective values. The resulting rational solution is checked exactly. Only its feasibility is needed for the proof. The compact and compact arithmetic cases use different configuration classes and local inequalities in this same program.

5 The compact case

We apply the local estimates of Supplement C of [2] to the squared norms of the orthogonal projections of x. These estimates hold under the hypotheses stated below.

Fix the farthest vertex and the vector x defined in (2.1). For a connected spherical component C of the Coxeter diagram at that vertex, let $\langle C \rangle$ denote the span of its facet normals, and put

$$e_C = \|\mathrm{proj}_{\langle C \rangle}\, x\|^2.$$

The spans of the components are mutually orthogonal and together form the tangent space at the vertex. Hence

$$e_C \geq 0, \qquad \sum_C e_C = \|x\|^2 < 1. \tag{5.1}$$

For disjoint collections $\mathcal{S}$ of components, write $e_{\mathcal{S}} = \sum_{C \in \mathcal{S}} e_C$. All such quantities in this section refer to the same vector x. An estimate said to allow arbitrary other components imposes no further condition on the remaining spherical components at this vertex; the opposite facets are still facets of the same polytope.

Vinberg's theorem [5] excludes compact Coxeter polytopes in dimensions at least 30. We therefore consider each dimension $18 \leq n \leq 29$ directly.

5.1 Small components and exceptional wall configurations

We use the usual notation for irreducible spherical Coxeter diagrams. The rank-two diagrams are $I_2(m)$, where $m \geq 3$ is an integer; A_2 denotes $I_2(3)$. The isomorphic types B_2 and D_3 are not listed separately. A component is *ordinary* if it does not belong to one of the exceptional configurations defined below.

Each exceptional configuration consists of components of the diagram at the chosen vertex together with a specified opposite facet. Thus O_7, for example, requires more than the occurrence of a bond labelled seven. Diagram labels always describe pairs of facets of the polytope; projected auxiliary vectors are not assigned such labels.

Proposition 5.1 (Local estimates for ordinary components)**.** *For each ordinary component C in Table 1, e_C is strictly greater than the listed bound. These estimates, proved in Supplement C, allow arbitrary other finite components.*

Table 1: The ordinary component lower bounds. Each decimal in this paper is an exact rational number.

Component	Lower bound	Component	Lower bound
A_1	1/10	A_2	319/2000
$I_2(m),\ m \geq 4$	1/7	A_3	1/4
B_3, H_3	2/7	A_4	2/7
B_4, D_4, F_4	1/3	H_4	11/36
A_5	3/8	D_5	5/14
A_6, B_5, B_6, D_6	3/7	B_7, D_7	1/2
B_8, D_8	4/7	E_6	9/20
E_7	5/9	E_8	2/3

The table includes the improved estimate for B_3, H_3 from Supplement C. The dihedral estimate holds for every integer $m \geq 4$.

We next define the exceptional configurations. Let p be the unit timelike vector representing the chosen vertex and let u be the unit normal of an incident A_1 facet. An opposite facet whose spatial normal lies in this rank-one component has, in the short case, a unit normal of the form

$$c_I = \sqrt{h}\, p - \sqrt{1+h}\, u, \qquad 0 < h < \frac{1}{2}. \tag{5.2}$$

We denote this configuration by I. Configuration J involves two A_1 components and one opposite facet whose spatial normal lies in their direct sum and is orthogonal to every other component. That

facet is separated from one of the two incident facets (the corresponding diagram edge is dotted), and it meets the other at angle $\pi/3$. The square of the coefficient of p in its unit normal lies in $(1/3, 3/8)$. Both A_1 components occur together in this configuration.

The two rank-two exceptional configurations are

$$N_7 = (\text{incident } A_2,\ \text{isolated opposite contact } 7),$$
$$O_7 = (\text{incident } I_2(7),\ \text{isolated opposite contact } 3).$$

Here the contact label records the angle between the opposite facet and an incident facet in the indicated component. Isolation means that the spatial part of the opposite normal is orthogonal to every other component. In particular, the definition specifies an opposite facet as well as the incident rank-two diagram.

Proposition 5.2 (Exceptional configurations)**.** *At most one of the exceptional configurations I, J, N_7, O_7 occurs at the chosen vertex. Supplement C gives the following bounds on the sum of e_C over the indicated incident components:*

configuration	*incident components*	*lower bound*
I	A_1	0
J	$A_1 + A_1$	17/50
N_7	A_2	9/100
O_7	$I_2(7)$	9/100.

The positive bounds are strict. The ordinary-component bounds are not also imposed on components belonging to an exceptional configuration.

In sums of diagrams, each of the symbols I, J, N_7, O_7 stands for its indicated incident components, together with the stated conditions on an opposite facet. Thus the incident components form the vertex diagram; the additional facet is not included in its rank.

The assertion that these configurations cannot occur together is part of the local geometric theorem. It includes the cases in which two candidate opposite facets would involve a common incident component. It does not follow from the list of spherical diagram types alone.

Supplement C also gives estimates involving several components. If T_j is an ordinary irreducible component of rank j, then

$$e_{I+A_1} > 1/5, \qquad e_{I+2A_1} > 3/10, \tag{5.3}$$
$$e_{I+T_2} > 21/100, \qquad e_{I+T_3} > 2/7. \tag{5.4}$$

Each left side is the sum over the components displayed in its subscript. If several of these inequalities are used together, the coefficient of e_I must satisfy the same dual inequality in Theorem 4.1.

If N_7 or O_7 occurs, the squared projection norm for every other ordinary rank-two component is greater than $1/5$. For an ordinary A_1 component outside the exceptional configuration, it is greater than $1/5$ in the N_7 case and greater than $2/15$ in the O_7 case. These bounds concern only the additional component. Their hypotheses, and those of the preceding inequalities, are stated in the ordinary-component appendix of Supplement C.

5.2 Large components

The A-component theorems give

$$e_{A_r} > \frac{1}{2} \quad (r \geq 7), \qquad e_{A_r} > \frac{4}{7} \quad (r \geq 8), \qquad e_{A_r} > \frac{5}{7} \quad (r \geq 13). \tag{5.5}$$

All three estimates allow arbitrary other finite components. Table 2 gives the additional estimates at specified ranks. Each is a bound for e_C of the fixed vector x, including when its proof uses auxiliary points or adjacent vertices.

Table 2: Additional A-component bounds, allowing arbitrary other finite components.

Rank	Lower bound
12	13/20
15	413/500
16	43/50
17	6/7
18	91/100
19	563/625
20	.92274

For B_r, D_r components with $r \geq 9$, one has

$$e_{B_r}, e_{D_r} > \frac{9}{14}. \tag{5.6}$$

The next estimate concerns an initial sequence of coordinates in a B_r or D_r component, in the coordinate convention of Supplement C.

Proposition 5.3 (An estimate for initial coordinates)**.** *Let $6 \leq M \leq 16$. Suppose the component is B_r with $r \geq M+5$, or D_r with $r \geq M+6$. Put*

$$f_i = \sqrt{i} - \frac{61}{100\sqrt{2}}, \quad E_M = \frac{f_5^2}{5} + \sum_{i=5}^{M-1} (\sqrt{i+1} - \sqrt{i})^2, \quad H_M = \left\lfloor \frac{M-5}{2} \right\rfloor.$$

The sum of squares of the first $M+1$ coordinates of the projection of x to this component is greater than

$$E_M - H_M/3200 + 3969/800000. \tag{5.7}$$

The estimate allows arbitrary other finite components.

The first two terms give the earlier estimate on the same coordinates. The last term improves that estimate under the same hypotheses; it is not a contribution from another subspace. The program uses the rational lower bounds $P_M + 3969/800000$, where

M	7	8	9	10	11
P_M	.7349975	.7683275	.797425	.823735	.8472225
M	12	13	14	15	16
P_M	.8689525	.88864	.90715	.9240775	.9401975

Supplement C proves $P_M < E_M - H_M/3200$. The finite linear program uses $7 \leq M \leq 16$. If both the earlier and improved estimates are included for the same coordinates, the earlier inequality is redundant.

Three additional estimates apply to the following full vertex diagrams. They are stated separately because the preceding rank conditions do not cover the relevant terminal coordinates:

(i) If the full vertex diagram is $B_{16} + T_2$ or $D_{16} + T_2$ in dimension 18, with T_2 ordinary, the squared projection norm on the large component is greater than .8689525.

(ii) If the full vertex diagram is $B_{17} + A_1$ in dimension 18, with ordinary nonshort A_1, the squared projection norm on B_{17} is greater than .90715.

(iii) If the full vertex diagram is $D_{19} + A_1$ in dimension 20, with ordinary nonshort A_1, the extension through the terminal D_4 subdiagram and the improved initial-coordinate estimate give $e_{D_{19}} > .91211125$.

These local results are proved in Supplement C at the stated ranks. They are used as component estimates, independently of the eventual dimension exclusions. In particular, the estimate requiring ordinary T_2 is not applied to N_7 or O_7, and the ordinary A_1 estimates are not applied to I.

5.3 An exhaustive family of configurations

Call a component *large* if it is of type A_r, $r \geq 7$, or B_r, D_r, $r \geq 9$. Every large component has squared projection norm greater than one half by (5.5) and (5.6). Table 1 shows that every other ordinary component contributes at least its rank divided by fourteen.

Lemma 5.4 (A bound for the remaining components)**.** *Let $\mathcal{S}$ be a collection of components, none large, of total rank S. Whenever an exceptional configuration occurs, include all of its incident components in the collection. Then*

$$e_{\mathcal{S}} \geq \frac{S-1}{14}. \tag{5.8}$$

If no exceptional configuration occurs, or the configuration is J, the stronger bound $S/14$ holds. The possible loss is $1/14$ in case I and $37/700$ in cases N_7 and O_7.

Proof. Sum the rank-over-fourteen estimates on the ordinary components. Configuration I replaces a rank-one contribution by a nonnegative quantity, so decreases the sum by at most $1/14$. The two incident components of J have total rank two, and the bound $17/50$ exceeds $2/14$. Each rank-two configuration N_7 or O_7 contributes at least $9/100$; the difference from $2/14$ is $37/700 < 1/14$. Proposition 5.2 permits at most one exceptional configuration. □

Fix $18 \leq n \leq 29$. We construct an exhaustive family of configuration classes as follows. A class may specify only component ranks and norm bounds when the individual diagram types are not needed.

1. If no component is large, introduce one variable for the sum of all component contributions and impose $e \geq (n-1)/14$.

2. If at least two components are large, select two distinct large components. Introduce a separate variable for each, with lower bound $1/2$, and a third variable for the sum over the remaining components. The exact ranks of the selected components need not be specified: their orthogonality and common lower bound suffice.

3. If exactly one component is large, of type T and rank r, put $S = n - r$. Apply any local exclusion whose hypotheses hold. Otherwise, if $S \geq 8$, use $e_T \geq 1/2$ and $e_{\mathcal{S}} \geq (S-1)/14$.

4. If $0 < S < 8$, let $d(T, r)$ be the strongest applicable component bound in Section 5.2 that allows arbitrary other finite components. Estimates requiring a particular additional component are not used to define $d(T, r)$. If

$$d(T, r) + (S-1)/14 \geq 1, \tag{5.9}$$

use the class specified by these two bounds on the large component and the sum over the remaining components.

5. In every remaining case, list every spherical forest of rank S with no large component, first without an exceptional configuration and then with each exceptional configuration that is allowed on that forest. Include all ordinary, large-component, joint and additional-component estimates that apply to this class, and the local exclusions of Section 5.4 with their stated hypotheses. For $S = 0$, the full vertex diagram consists of the one large component.

The classes described by sums of components are included in the linear program. Their displayed inequalities, each with multiplier one, already imply a total contribution at least one; this includes $S = 8$ in step three. The test (5.9) determines when individual remaining components need to be listed. It changes neither the hypotheses nor the validity of any local estimate.

Proposition 5.5 (Exhaustiveness of the compact configurations)**.** *Let a compact Coxeter polytope of dimension $18 \le n \le 29$ and a farthest vertex be given. Its vertex configuration either satisfies the hypotheses of one of the stated local exclusions or belongs to one of the classes just constructed. The squared projection norms of x satisfy every inequality assigned to that class.*

Proof. The spherical diagram has no large component, at least two large components, or exactly one. The first two cases give steps one and two. In the third case, its type and rank determine S. Lemma 5.4 and the applicable component estimate justify steps three and four when their numerical conditions hold. Otherwise step five includes the remaining component multiset and its exceptional configuration, if any. Components of equal type can be relabelled, together with their facets, to place a marked component first. This relabelling does not change any other parameter: each ordinary dihedral component keeps its full integer range, and an O_7 mark fixes the label seven only on its incident component. Thus every inequality and local exclusion is applied with its required component and facet hypotheses. Orthogonality gives (5.1) for the fixed vector x. □

For the explicit enumeration in step five, use the following ordered list of component types through rank seven:

rank	types
1	A_1
2	$A_2 = I_2(3),\ I_2(m)\ (m \ge 4)$
3	A_3, B_3, H_3
4	A_4, B_4, D_4, F_4, H_4
5	A_5, B_5, D_5
6	A_6, B_6, D_6, E_6
7	B_7, D_7, E_7.

Choose types in nondecreasing order until their ranks sum to S. Then mark no exceptional configuration, one A_1 as I, two A_1 components as J, one A_2 as N_7, or one dihedral component as O_7. Ordering and permutation of equal components avoid repetitions. All dihedral parameters remain universally quantified. This gives the 829 classes used below. Their exhaustiveness follows from Proposition 5.5, independently of this count.

5.4 Local exclusions

Table 3 states the configuration classes excluded by local theorems of Supplement C. Their scaling coefficient is zero in the normalized linear program; they are not assigned an artificial lower bound on a component norm. Except for C1, each displayed forest is the full Coxeter diagram at the chosen vertex, and the statements are used only for these full diagrams. Every Coxeter label and parameter for separated facets ranges over the full set in the cited local theorem.

Table 3: Local exclusions in the compact case, proved in Supplement C.

Code	Full vertex diagram or class	Local theorem
C1	Any A_r, $21 \le r \le 29$; B_r, $23 \le r \le 29$; or D_r, $24 \le r \le 29$, with arbitrary other finite components	Upper bound on the rank of a component
C2	$B_r + I$, $r \ge 20$, or $D_r + I$, $r \ge 21$	The classical-component theorem in the presence of I
C3	$A_{16} + N_7$ or $A_{16} + O_7$	The endpoint theorems for N_7 and O_7 in dimension 18
C4	$A_{17} + A_1$ or $A_{17} + I$, with ordinary A_1	The theorem for a rank-one complement in dimension 18
C5	$B_{16} + N_7/O_7$ or $D_{16} + N_7/O_7$	The rank-sixteen theorems for N_7 and O_7, including both signs at the D fork
C6	$B_{17} + I$	The theorem for a short isolated opposite facet associated with the A_1 component
C7	$D_{17} + A_1$ or $D_{17} + I$, with ordinary A_1	The separate theorems for an ordinary A_1 and for I
C8	A_{18}	The endpoint exclusion for the indicated full diagram
C9	B_{18}	The terminal-facet theorem, including coincident and distinct opposite facets
C10	D_{18}	The endpoint and fork theorem, including intersecting, parallel and separated opposite facets
C11	$A_{17} + N_7/O_7$ or $A_{18} + I$	The separate local theorems in dimension nineteen
C12	$D_{17} + T_2$, $B_{17} + N_7/O_7$, or $B_{18}/D_{18} + A_1$, with ordinary T_2, A_1	The classical-component extension theorem in dimension nineteen
C13	$D_{17} + N_7/O_7$, $B_{18} + I$, or $D_{18} + I$	The theorems for N_7, O_7 and I in dimension nineteen
C14	A_{19}, B_{19} or D_{19}	The respective full-diagram theorems for types A, B and D
C15	$A_{19} + I$	The theorem for A_{19} with configuration I
C16	$B_{18}/D_{18} + N_7/O_7$ or $B_{19}/D_{19} + I$	The classical-component extension theorem in dimension twenty
C17	A_{20}, B_{20} or D_{20}	The respective full-diagram theorems in dimension twenty
C18	$A_{19} + N_7/O_7$ or $A_{20} + I$	The A-component theorems for N_7, O_7 and I
C19	$D_{19} + N_7/O_7$ or $D_{20} + I$	The D-component extension theorem in dimension twenty-one
C20	B_{21} or D_{21}	The respective full-diagram theorems in dimension twenty-one

Code	Full vertex diagram or class	Local theorem
C21	B_{22}, D_{22} or D_{23}	The terminal-facet theorems for the indicated full diagrams

For example, C10 includes every terminal-facet and fork alternative for the full diagram D_{18}. Auxiliary vertices used in a local proof do not contribute additional terms to (5.1).

5.5 The dual certificate

The construction gives 829 configuration classes, of which 151 are ruled out by local exclusions. For each remaining class the dual has one nonnegative multiplier for each applicable norm inequality. The coefficient of every component contribution is at most one. The rational solution satisfies Theorem 4.1 simultaneously for all classes. In particular, inequalities involving I use the same variable e_I.

Theorem 5.6 (The compact dimension bound)**.** *The local estimates of Sections 5.1, 5.2 and 5.4, together with spherical classification, make the normalized linear program infeasible in dimensions* $18, \dots, 29$*. Together with Vinberg's upper bound* 29*, they imply* $n \leq 17$ *for compact hyperbolic Coxeter polytopes.*

Proof. By Proposition 5.5, a hypothetical polytope belongs to one of the listed classes. A locally excluded class is impossible. For any other class, Theorem 4.1 and the rational dual solution imply that the total squared projection norm is at least one, contrary to (5.1). The exact verification checks all 3157 dual variables and 2224 inequalities; the records are given in the reproduction appendix. Vinberg's theorem excludes $n \geq 30$. □

The configuration list is obtained from spherical classification and the stated component estimates. It does not assume the earlier list of eleven forests at rank eighteen.

6 The compact arithmetic case

For a compact arithmetic Coxeter polytope, the reflection group is an arithmetic lattice. We apply the additional local results of Supplement A of [2], keeping their hypotheses on the defining field and the normal lines of facets. The norm inequality remains (5.1). The preceding compact classes apply in dimensions $18, \dots, 29$; we add arithmetic classes at ranks 16 and 17.

6.1 Arithmetic hypotheses and local results

In the quadratic-form model used in Supplement A, K is a totally real number field, q is a quadratic form over K, and B is its associated symmetric bilinear form. At one distinguished real embedding $\sigma_0 : K \hookrightarrow \mathbb{R}$, the form q has Lorentzian signature and defines the hyperbolic space containing the polytope. At every other real embedding σ, the conjugate form q^σ is positive definite.

Choose vectors α_i on the facet-normal lines over this same field; these are the root vectors used below. The associated Cartan entries are

$$C_{ij} = \frac{2B(\alpha_i, \alpha_j)}{B(\alpha_i, \alpha_i)}. \tag{6.1}$$

The local arithmetic theorems require the cyclic products

$$C_{i_1 i_2} C_{i_2 i_3} \cdots C_{i_\ell i_1} \tag{6.2}$$

to belong to the ring of integers $\mathcal{O}_K$. Replacing α_i by $t_i\alpha_i$, with $t_i \in K^\times$, multiplies C_{ij} by t_j/t_i; these factors cancel in each cyclic product. The field reductions used below are those proved in Supplement A with these hypotheses.

At an embedding $\sigma \neq \sigma_0$, linearly independent conjugate normal vectors have a positive-definite Gram matrix. The Gram matrix of all facet normals may be redundant, and is then only positive semidefinite. Normalizing each vector to unit length may introduce square roots. The resulting field of real cosine entries therefore does not replace the common field K or the square classes of the root norms. The norm class of α_i is the class of $B(\alpha_i, \alpha_i)$ in $K^\times/(K^\times)^2$; rescaling by $t_i \in K^\times$ multiplies this norm by t_i^2 and leaves its class unchanged.

Applications of Bugaenko's theorem [6] use the specified invariant lattice L and the proved inclusion $\Gamma(P) \subseteq \mathrm{Ref}(L)$. Here $\Gamma(P)$ is the reflection group of the polytope, and $\mathrm{Ref}(L)$ is the full reflection group of that lattice. Equality of defining fields alone does not establish the inclusion.

At ranks 16 and 17, Supplement A gives necessary lists of configurations after its local arithmetic reductions. We take those lists as proved local results. The linear program combines their norm inequalities and local exclusions; it does not derive the preceding reductions.

Input 6.1 (Local arithmetic results)**.** The following results of Supplement A are used with all of their stated hypotheses.

(i) The common totally real field, facet-normal lines, integral Cartan cycles, root-norm square classes and positivity at every real embedding other than the distinguished one are as required by each local theorem.

(ii) The fourteen rank-seventeen families and nineteen rank-sixteen families in Tables 5 and 6 are exhaustive necessary lists, including all exceptional configurations, other components and field alternatives occurring in their proofs.

(iii) The estimates in Section 6.2 and Tables 5–6 hold for the indicated components of the full vertex diagrams. Each component bound includes every contribution used in its local proof.

(iv) The local exclusions of Table 7 hold for the stated facet configurations, including boundary values and the identification of each opposite facet associated with an A_1 component.

(v) The four finite classifications in Table 4 are exhaustive on their stated parameter sets. The geometric reductions prove that the relevant facet configurations belong to those sets.

(vi) Each application of the golden-lattice theorem uses the lattice and full reflection-group inclusion specified above.

These statements contain arithmetic information in addition to positive semidefiniteness of real Gram matrices.

In case (ii), the fields are $\mathbb{Q}(2\cos(2\pi/7))$ and $\mathbb{Q}(2\cos(2\pi/9))$. The required finite contact has Coxeter label seven or eighteen, respectively. Thus an endpoint opposite wall meets the corresponding removed incident wall at angle $\pi/7$ or $\pi/18$, respectively. These are the two cases of the A_{17} endpoint reduction in Supplement A.

The exclusion of the full diagram D_{16} is proved separately by a direct argument over dyadic completions. It does not use one of the four finite classifications in Table 4. It is distinct from the exclusion of $D_{16} + I$ at rank seventeen.

Table 4: Four finite classifications used in Supplement A. An algebraic solution need not correspond to a polytope.

No.	Classified parameter set	Conclusion used in the local proof
(i)	The quadratic-field parameter set for full diagram A_{17}	No solutions
(ii)	The two cubic-field parameter sets for full diagram A_{17}, each with the required endpoint contact at a finite Coxeter angle	No solutions in either set
(iii)	The common quadratic parameter set obtained by separate reductions for full diagrams B_{17} and D_{17}	Two possible pairs of parameters; each is then subject to the appropriate B- or D-component norm estimate
(iv)	The quadratic parameter set obtained from full diagram B_{16}	One possible pair of parameters, to which the estimate along the specified edge is applied

6.2 Notation and local norm inequalities

The quantities e_C continue to denote squared norms of orthogonal projections of the fixed vector x. Write T_2 for an ordinary irreducible $I_2(m)$, $m \geq 3$, T_3 for A_3, B_3 or H_3, and T_4 for one of A_4, B_4, D_4, F_4, H_4. Let $\mathcal{K}$ denote either exceptional configuration N_7 or O_7; they are kept as separate alternatives. An unqualified A_1 is ordinary and nonshort. The configurations I, J, N_7, O_7 are those of Section 5.1.

Set

$$\mathcal{R}_5 = \{A_5, D_5, A_4 + A_1, H_4 + A_1, T_3 + T_2, 2T_2 + A_1\},$$
$$\mathcal{R}'_5 = \{A_5, D_5, A_4 + A_1, H_4 + A_1, A_3 + T_2, 2T_2 + A_1\}.$$

Every member contributes at least $5/14$ by Table 1. The restriction to A_3 in $\mathcal{R}'_5$ is part of the rank-sixteen classification.

In addition to the ordinary estimates of Table 1, Supplement A gives

$$e_{I+T_2} > 21/100, \qquad e_{I+T_3} > 2/7, \qquad e_{I+T_4} > 23/70, \tag{6.3}$$
$$e_{I+A_1} > 5051/25000 = .20204. \tag{6.4}$$

It also gives $e_{T_2} > 1/5$ for an ordinary component outside N_7 or O_7, and $e_{A_1} > 2/15$ for an ordinary component outside O_7.

Tables 5 and 6 state the remaining local bounds on the full vertex diagrams in their middle columns. Every listed dihedral parameter and field alternative ranges over the set specified in the local theorem. Separate summands refer to estimates on disjoint component sets, unless a joint estimate is explicitly named. An estimate assigned to the large component bounds that component alone, even when its proof uses other facets. For example, $16/19$ for A_{14} with an ordinary T_2 and .91286445198 with N_7 or O_7 are different statements. The ordinary T_2 estimate $3/19$ follows from the arithmetic dihedral theorem when $m \geq 4$, and from $319/2000 > 3/19$ when $m = 3$. For the indicated B/D components with N_7 or O_7, the bounds .91859 and .93859 already include the correction .0487 from the opposite facets. That correction is part of the bound on the same component, rather than a separate summand. We display the inequalities used by the certificate below. Additional valid inequalities in the electronic model may be assigned multiplier zero.

6.3 The rank-seventeen configurations

Table 5 gives the fourteen families from Supplement A. Braces and semicolons separate alternatives, all of which are included. The local exclusions X1–X13 are stated in Table 7.

Table 5: The rank-seventeen families and the inequalities used in the dual solution.

No.	Full vertex diagram	Norm bounds with sum at least one, or local exclusion
1	$A_{11} + I + (A_5$ or $D_5)$	$13/20 + 5/14 = 141/140$; the nonnegative contribution from I is omitted
2	$A_{11} + \mathcal{K} + U_4$, $U_4 \in \{A_4, B_4, D_4, F_4\}$	$13/20 + 9/100 + 2/7$
3	$A_{12} + R$, $R \in \mathcal{R}_5$	$5/7 + 5/14$
4	A_{12} plus $I + T_4$; $I + T_3 + A_1$; or $I + 2T_2$	5/7 plus, respectively, the joint 23/70; the joint 2/7 and 1/10; or the joint 21/100 and 1/7
5	A_{14} plus T_3; $T_2 + A_1$; or $I + T_2$	4/5 plus, respectively, 1/4; $1/7 + 1/10$; or the joint 21/100
6	A_{15} plus T_2; $2A_1$; $I + A_1$; N_7; or O_7	T_2: X1. The next two give $.8348 + .2$ and $.8348 + .20204$. Either exceptional configuration gives $.9191408 + .09$
7	A_{16} plus ordinary A_1 or configuration I	X2 for both rank-one alternatives
8	A_{17}	X3, using classifications (i), (ii) and the reductions to their parameter sets
9	$B_{15} + N_7$ or $B_{15} + O_7$	$.93859 + .09 = 1.02859$
10	$B_{16} + I$	X4
11	B_{17}	$e_{B_{17}} \geq 1.00184$, after the separate reduction and norm estimates for both parameter pairs
12	$D_{15} + N_7$ or $D_{15} + O_7$	$.91859 + .09 = 1.00859$
13	D_{16} plus ordinary A_1 or configuration I	X5 for ordinary A_1; X6 for I
14	D_{17}	$e_{D_{17}} \geq 1.00184$, after its reduction for the D fork and its norm estimates for both parameter pairs

The fourteen families expand to 39 parametrized configuration classes. Some may not be geometrically realizable. Repeated symbols such as $2T_2$ denote components with independent parameters. For $I + 2T_2$, apply the joint estimate to I and one dihedral component, and the ordinary estimate to the other dihedral component. Thus e_I occurs with coefficient one.

6.4 The rank-sixteen configurations

Table 6 gives the nineteen rank-sixteen families, including the restricted family $\mathcal{R}'_5$.

Table 6: The rank-sixteen families and the inequalities used in the dual solution.

No.	Full vertex diagram	Norm bounds with sum at least one, or local exclusion
1	$A_{10} + I + (A_5 \text{ or } D_5)$	$404/625 + 5/14$
2	$A_{10} + \mathcal{K} + U_4$, $U_4 \in \{A_4, B_4, D_4, F_4\}$	$404/625 + 9/100 + 2/7$
3	$A_{11} + R$, $R \in \mathcal{R}_5'$	$13/20 + 5/14 = 141/140$
4	$A_{11} + I + T_4$	The joint $A_{11} + I$ bound $18/25$, plus the disjoint T_4 bound $2/7$, giving $176/175$
5	$A_{11} + O_7 + A_3$	$13/20 + 9/100 + 13/50 = 1$
6	A_{12} plus A_4; H_4; or $2T_2$	$5/7 + 2/7 = 1$
7	$A_{12} + I + T_3$	$5/7$ plus the joint $I + T_3$ bound $2/7$
8	$A_{12} + \mathcal{K} + T_2$	$5/7 + 9/100 + 1/5$, using the stronger ordinary-component bound in the presence of N_7 or O_7
9	A_{13} plus A_3; $T_2 + A_1$; $I + T_2$; or $O_7 + A_1$	Respectively: $53/70 + 1/4$; $53/70 + 1/7 + 1/10 = 1$; X7; or $233/300 + 9/100 + 2/15 = 1$
10	A_{14} plus T_2; $2A_1$; $I + A_1$; N_7; or O_7	Respectively: $16/19 + 3/19 = 1$; $4/5 + .2 = 1$; $4/5 + .20204$; or $.91286445198 + .09$ with either N_7 or O_7
11	A_{15} plus ordinary A_1 or configuration I	X8 for every isolated and nonisolated rank-one alternative
12	A_{16}	X9
13	$B_{14} + N_7$ or $B_{14} + O_7$	$.91859 + .09 = 1.00859$
14	B_{15} plus ordinary A_1 or configuration I	X10 for ordinary A_1; X11 for I
15	B_{16}	$e_{B_{16}} \geq 1.00640742675$, after its reduction to the parameter set in (iv) and the edge estimate for the single solution
16	D_{13} plus $I + T_2$ or $O_7 + A_1$	.82436 plus the bound .21 on the remaining components; joint on $I + T_2$, and coarsened from $.09 + 2/15$ on $O_7 + A_1$
17	D_{14} plus ordinary $I_2(m)$, $m \geq 4$; N_7; or O_7	Ordinary: $.84816 + 3/19$. Either exceptional configuration: $.91859 + .09$
18	D_{15} plus ordinary A_1 or configuration I	X10 for ordinary A_1; X12 for I
19	D_{16}	X13, the quadratic-field reduction and dyadic theorem

Every displayed sum is at least one and therefore contradicts (5.1). No uniform positive difference from one is needed. For example, family 5 uses three estimates on disjoint component sets whose bounds sum to one. Family 4 uses the joint estimate on $A_{11} + I$:

$$e_{A_{11}} + e_I \geq 18/25, \qquad e_{T_4} \geq 2/7.$$

The component sets in these two inequalities are disjoint. The bound $18/25$ applies to the sum on $A_{11} + I$; it cannot be assigned to A_{11} alone or added to another estimate with a second full coefficient on e_I.

The nineteen families give 49 parametrized configuration classes. Together the two lists contain 88 alternatives: 72 are treated by norm inequalities, and 16 by local exclusions. The dihedral parameters retain their full, possibly unbounded ranges.

6.5 Local arithmetic exclusions

Table 7 states every local exclusion used in Tables 5 and 6. Each is a local theorem of Supplement A with the stated arithmetic and geometric hypotheses. It sets the scaling coefficient of that configuration class to zero in the normalized linear program, rather than an additional norm estimate.

Table 7: Local arithmetic exclusions. The indicated diagram is the full vertex diagram.

Code	Full vertex diagram	Local theorem
X1	$A_{15} + T_2$, rank 17	The ordinary-dihedral theorem, including all defining-field and small-norm alternatives.
X2	$A_{16} + A_1$ or $A_{16} + I$, rank 17	The theorem for opposite facets associated with the A_1 component, including isolated and nonisolated cases.
X3	A_{17}, rank 17	The endpoint and field reductions to the finite parameter sets in (i) and (ii).
X4	$B_{16} + I$, rank 17	The theorem for a short isolated opposite facet, for both quadratic and cubic fields.
X5	$D_{16} + A_1$, rank 17, with ordinary A_1	The ordinary rank-one theorem, including both alternatives at the fork.
X6	$D_{16} + I$, rank 17	The theorem for this full diagram over fields of degrees two, three and four, including both fork signs and every dyadic case.
X7	$A_{13} + I + T_2$, rank 16	The endpoint theorem involving the remaining components, with every ordinary dihedral label.
X8	$A_{15} + A_1$ or $A_{15} + I$, rank 16	The rank-one opposite-facet theorem, including every height boundary.
X9	A_{16}, rank 16	The endpoint and field theorem for the full diagram, including its finite-contact cases and ratios of Lorentz time coordinates.
X10	$B_{15} + A_1$ or $D_{15} + A_1$, rank 16, with ordinary A_1	The respective nonisolated reductions and ordinary isolated theorem, including $h = 1/2$.
X11	$B_{15} + I$, rank 16	The isolated-short theorem for $0 < h < 1/2$.
X12	$D_{15} + I$, rank 16	The isolated-short theorem for $0 < h < 1/2$, including both signed forks.
X13	D_{16}, rank 16	The quadratic-field reduction and the dyadic theorem, including the odd, inert, ramified and split cases.

For each rank-one theorem, fix the opposite facet associated with the indicated A_1 component. If its spatial normal has a nonzero projection to the large component, the nonisolated case applies. Otherwise the facet is isolated, and the parameter given by the square of the coefficient of p in its unit normal satisfies $0 < h < 1/2$ or $h \geq 1/2$. The equality case is included in the ordinary theorem. Thus these alternatives concern facets of the polytope, not arbitrary formal extensions of its diagram.

For the full diagrams B_{17} and D_{17}, the local reductions separately place the parameters in class (iii) of Table 4. The component estimates are then applied to both possible parameter pairs. For B_{16}, the reduction to class (iv) and the estimate along its specified edge treat the single possible pair. These finite classifications do not give empty parameter sets. The full diagram D_{16} is instead excluded by the dyadic theorem X13.

6.6 The dual certificate for the arithmetic case

The arithmetic tables give an explicit dual solution. For every alternative treated by norm inequalities, assign multiplier one to the displayed inequalities on disjoint component sets and zero to any additional inequalities. Each component variable then has total coefficient at most one, while the sum of the lower bounds is at least one. Alternatives X1–X13 have zero scaling coefficient and therefore zero normalized deficit, by their local exclusion theorems.

The computation uses the same linear-program construction for all 88 arithmetic alternatives and the 829 compact classes in dimensions 18–29. There are 917 classes in total, of which 167 are locally excluded. The checked dual has 3324 variables and 2488 inequalities. The reproduction appendix records these linear verifications separately from the local geometric proofs.

Theorem 6.2 (The compact arithmetic dimension bound)**.** *Assume the local results of Input 6.1, the compact local results of Section 5, and Vinberg's compact upper bound* 29*. The combined normalized linear program for dimensions* $16, \ldots, 29$ *is infeasible. Consequently a compact arithmetic hyperbolic Coxeter polytope satisfies* $n \leq 15$*.*

Proof. At ranks 16 and 17, every arithmetic facet configuration belongs to one of the respective necessary lists. A locally excluded class is impossible. On each remaining class, the dual inequalities force the sum of squared projection norms to be at least one, contrary to (5.1). In dimensions 18–29, the compact classes and Theorem 5.6 apply without change. Vinberg's theorem excludes dimensions 30 and above. □

7 Exclusion in all higher dimensions

Proposition 2.4 applies to each of the three fixed families of configurations. The following table summarizes the additional estimates needed to cover the full dimension range.

Case	Finite calculation	Higher dimensions
Finite volume	Finite and ideal vertices at $n = 58$	The analytic face and vertex estimates, together with the stated reduction of ideal links, apply at every $n \geq 59$
Compact	Exhaustive classification at $18 \leq n \leq 29$	Vinberg's theorem gives $n \leq 29$
Compact arithmetic	Arithmetic classifications at $n = 16, 17$, and the compact classes at $18 \leq n \leq 29$	The same compact theorem gives $n \leq 29$

Vinberg's bound excludes compact polytopes in dimensions $n \geq 30$ [5, 7]. The arithmetic program uses the compact classes and inequalities at dimensions 18–29 unchanged. The general finite-volume case instead uses the analytic estimates proved for every higher dimension. Thus the certificates exclude all dimensions at least 58, 18 and 16, respectively. This conclusion requires the stated higher-dimensional estimates; infeasibility of an isolated exact-dimension problem would not by itself imply it.

A lower threshold would require an exhaustive family of classes and valid local inequalities in the newly included dimensions. The monotonicity proposition does not assert that an arbitrary face of a Coxeter polytope is again a Coxeter polytope.

8 Exact verification of the linear certificates

For the compact cases, the matrix records the support, rational lower bound and applicability class of each local inequality. The dual programs for all classes are solved together. A numerical solver provides multipliers, which are converted to rational numbers and checked against every matrix row and variable bound by exact arithmetic. A separate run regenerates the configuration lists and both the primal and dual matrices, compares the resulting files byte for byte, and checks the rational solution again.

Problem	Classes	Local exclusions	Primal variables/rows	Dual variables/rows
Compact	829	151	3471/4876	3157/2224
Arithmetic and compact classes	917	167	3869/5147	3324/2488

In finite volume the local matrix has 5 810 variables and 103 606 rows. The rank-58 forest inequalities are represented by the complete collection of finite networks. The local matrix and every network use the same weight coefficients. Their exact feasible assignment gives the upper bound 377, and the eight-row dual combination gives the matching lower bound. These checks used no numerical optimizer and kept the network inequalities in separate matrices. The subsequent verification repeated the local inequalities, the completeness checks joining the networks, and the lower-bound certificate, using the previously checked networks.

An inequality may be redundant on a class. For example, adding a positive correction to the initial-segment norm estimate gives a stronger inequality on the same support. Either may remain in the matrix. Conversely, a zero multiplier in one dual solution proves only that this solution does not use the corresponding inequality; it does not prove redundancy of that inequality.

9 Further constraints

Additional necessary conditions may restrict a configuration class or add inequalities to its cone. Additional functions of a corner may enlarge the class of weight functions available in finite volume. These modifications preserve the argument when every polytope still determines a feasible point and the configuration data remain fixed as the dimension threshold varies.

Some local results could instead be expressed by semidefinite constraints on Gram matrices. Such a formulation would have to preserve the geometric applicability of every constraint. In the arithmetic case it would also have to account for integral Cartan cycles, conjugate forms at the other real embeddings, and the specified reflection-group inclusion in a lattice. Real Gram positivity alone does not imply those arithmetic conditions.

A Sources and reproducibility

The geometric results are given in the dimension-bound article and its supplement [1, 2]. The finite-volume weight function also uses the comparisons between its two dimension ranges and the analytic estimates described in the research notes. The repository [4] provides the reading guide `notes/lp/UNIFORM_LP_PARADIGM.md` and the result summary `notes/lp/existing_proof_lp_recovery_results_20261118.md`. The principal source files and verification records are listed below. The repository's reproduction guide distinguishes the publicly available files from the larger research data not yet distributed with the source checkout.

Record	Path
Finite-volume verification	`audit/joint58_complete_lp_execution_20261110.json`
Subsequent verification	`audit/lp_rerun_report_execution_20261115.json`
Compact configuration generator	`scripts/compact17_proof_recovery_energy_catalogue_20261118.py`
Arithmetic configuration generator	`scripts/existing_proof_arithmetic_energy_domains_20261118.py`
Common LP construction	`scripts/run_existing_proof_energy_lp_20261118.py`
Exact matrix checker	`scripts/finite_volume_lp_matrix_20261106.py`
Compact matrices and certificates	`results/existing_proof_compact17_energy_master01_20261118/`
Arithmetic matrices and certificates	`results/existing_proof_arithmetic15_energy_master01_20261118/`
Independent verification review	`audit/existing_proof_energy_lp_execution_independent_review_20261118.json`
Verification summary	`audit/existing_proof_lp_recovery_completion_root_acceptance_20261118.json`

Each compact output directory contains the configuration list, normalized primal problem, dual problem, rational solution, and construction and verification reports. The execution records include the commands, source hashes, resource limits and measurements. The mathematical applicability of each local inequality is checked in the source review; the rational checker verifies its linear combination. File hashes identify the exact versions used.

Reproduction uses the repository's serial monitored runner. The `export-solve` stage constructs the problem and finds a dual solution; a separate `verify` stage regenerates both matrices and checks the rational certificate. For arithmetic, both the arithmetic and compact configuration generators are supplied. Each run requires fresh output and monitor paths. The execution reports give the exact commands and resource limits.

The resulting certificates prove the linear consequences of the stated local theorems and classifications. The geometric proofs, field reductions, four finite arithmetic classifications and lattice theorems retain the hypotheses stated in the text. Their earlier verification records are cited separately from the new global LP checks.

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