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Three-generated matrix algebras containing a four-regular element

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Abstract

Let A, B, C be pairwise commuting complex $n \times n$ matrices. We prove that $\dim_{\mathbb{C}} \mathbb{C}[A, B, C] \leq n$ whenever this algebra contains an element whose eigenspaces all have dimension at most four. The eigenvalues and Jordan block lengths are arbitrary. The proof reduces to commuting nilpotent operators, studies the top and socle of a minimal counterexample, and uses a canonical-module minor identity to bound an extension with a curvilinear quotient. A change of generators then treats a four-regular element anywhere in the generated algebra. The matrix theorem is verified in Lean.

1 Introduction

The three-matrix analogue of Gerstenhaber's theorem asks whether

$$\dim_{\mathbb{C}} \mathbb{C}[A, B, C] \leq n$$

for every pairwise commuting triple of complex $n \times n$ matrices. Here $\mathbb{C}[A, B, C]$ denotes the unital generated algebra. The problem can also be phrased in terms of the minimum dimension of a faithful module over a three-generated commutative Artinian algebra; see Bergman [1, 2]. The general problem remains open. Luo, Song and Zeng [7] give the general upper bound $2n - 2$ for $n \geq 2$.

For an endomorphism T of a complex vector space, call T *four-regular* if

$$\dim_{\mathbb{C}} \ker(T - \lambda I) \leq 4 \qquad \text{for every } \lambda \in \mathbb{C}.$$

Equivalently, its Jordan form has at most four blocks at each eigenvalue. We prove the following theorem.

*The author conducted this research independently.

Theorem 1. *Let V be a finite-dimensional complex vector space, and let $A, B, C \in \mathrm{End}_{\mathbb{C}}(V)$ commute pairwise. If $\mathbb{C}[A, B, C]$ contains a four-regular element, then*

$$\dim_{\mathbb{C}} \mathbb{C}[A, B, C] \leq \dim_{\mathbb{C}} V.$$

The element may be any polynomial in A, B, C. The bound is sharp: take A to be a single nilpotent Jordan block and $B = C = 0$.

The known three-regular cases and several four-block configurations are treated by Šivic [5, 6]; further module criteria appear in Rajchgot and Satriano [3]. Llopis Vidal [8, Corollary 3.16] treats four nilpotent blocks of size two through the first jet of the commuting-pair scheme. Bergman [1, pp. 6–7] discusses O'Meara's expectation for four-regular algebras. Holbrook and O'Meara [4] describe the relevant block structures and the remaining four-regular boundary. Theorem 1 allows four Jordan blocks of arbitrary lengths at each eigenvalue.

The main step concerns commuting nilpotent endomorphisms t, b, c of a finite-dimensional vector space M over an infinite field k of characteristic zero. We prove that $\dim_k \ker t \leq 4$ implies $\dim_k k[t, b, c] \leq \dim_k M$. A minimal counterexample would have a two-dimensional top and socle. The quotient M/tM then has two possible forms. In one case, a cyclic submodule of maximal dimension has a curvilinear quotient, and a canonical-module minor identity gives the dimension bound. In the other, the annihilator of t embeds in a two-by-two matrix space over $k[b]/(b^2)$; the submodule and quotient bounds imposed by minimality are incompatible. Section 5 passes from this local theorem to Theorem 1.

2 The local nilpotent theorem

Let k be an infinite field of characteristic zero. All lengths in this section are dimensions over k. For a finite-dimensional local k-algebra A, with residue field k and maximal ideal $\mathfrak{m}$, write

$$\operatorname{top} M = M/\mathfrak{m}M, \qquad \operatorname{soc} M = 0 :_M \mathfrak{m}, \qquad \operatorname{type} A = \dim_k \operatorname{soc} A.$$

The action algebra is the image of the unital representation in $\mathrm{End}_k(M)$. Write $D_k N = \mathrm{Hom}_k(N, k)$, with action $(a\varphi)(m) = \varphi(am)$, and put $M[t] = \ker(t : M \to M)$.

Theorem 2 (Local nilpotent bound)**.** *Let t, b, c be commuting nilpotent endomorphisms of a finite-dimensional k-space M. If*

$$\dim_k \ker t \leq 4,$$

then

$$\dim_k k[t, b, c] \leq \dim_k M.$$

We use the ideal-quotient lemma of Section 4 and the curvilinear extension theorem of Section 3. The first applies to quotients of the ideal (X, Y) in $k[X, Y, Z]$. The second applies to a module containing a type-two cyclic submodule with quotient $k[t]/(t^q)$, subject to the stated annihilator containment.

2.1 A minimal counterexample and its socle action

Proof of Theorem 2. Suppose the theorem fails, and choose a counterexample of smallest length n, within the class of all nilpotent commuting triples with first kernel of dimension at most four. Set

$$A = k[t, b, c] \subseteq \mathrm{End}_k(M), \quad \dim_k A = n + e, \quad e \geq 1.$$

The action is faithful. The ideal $\mathfrak{m} = (t, b, c)A$ is nilpotent, and $A/\mathfrak{m} = k$, so A is local. Every proper submodule and every nonzero proper quotient retains the original three operators and is shorter. Submodules retain the kernel bound by inclusion of their t-kernels. For a quotient Q, Q/tQ is a quotient of M/tM, and $\dim \ker(t|Q) = \dim Q/tQ$. Thus the same kernel bound holds for quotients as well. Minimality consequently applies to their image algebras, even when their action is no longer faithful as an A-action.

Put

$$U = \operatorname{top} M, \quad V = \operatorname{soc} M, \quad a = \dim U, \quad s = \dim V, \quad r = \dim M/tM = \dim \ker t.$$

Both a, s are positive, and $a, s \le r \le 4$. Finite duality, with positive action $(\alpha\varphi)(m) = \varphi(\alpha m)$, preserves the annihilator, length and first-kernel dimension, and interchanges a and s. We may therefore arrange $a \le s$.

Multiplication gives an injective map

$$\operatorname{soc} A \longrightarrow \operatorname{Hom}_k(U, V). \tag{LF.4}$$

Indeed, a socle element kills $\mathfrak{m}M$, has image in V, and is zero if it kills M, by faithfulness. Denote its image by $\mathcal{T}$.

For a nonzero functional $\phi \in U^*$, the inverse image M_ϕ of $\ker \phi$ is a submodule of codimension one. An element annihilating M_ϕ kills $\mathfrak{m}M$; hence its product with $\mathfrak{m}$ acts as zero on M, and faithfulness puts it in $\operatorname{soc} A$. Consequently

$$\operatorname{Ann}_A M_\phi \simeq \{v \in V : v \otimes \phi \in \mathcal{T}\}. \tag{LF.5}$$

Minimality on M_ϕ says that its image algebra has length at most $n - 1$. Therefore the space in (LF.5) has dimension at least $e + 1$.

Similarly, for $0 \ne v \in V$, the line kv is a submodule. If $gM \subseteq kv$, then $\mathfrak{m}gM = 0$, so g is in the algebra socle. Thus

$$\operatorname{Ann}_A(M/kv) \simeq \{\phi \in U^* : v \otimes \phi \in \mathcal{T}\}, \tag{LF.6}$$

and this space also has dimension at least $e + 1$.

We recall the finite linear-algebra estimate needed to combine these two families. If $\Gamma : E \times F \to W$ is bilinear, its values span W, and its slices $\Gamma(e, -)$ and $\Gamma(-, f)$ have ranks at most p, q, respectively, then

$$\dim W \le pq. \tag{LF.7}$$

Here is a proof. Choose a first slice $f = \Gamma(e_0, -)$ of maximum rank $p_0 \le p$, and choose $v_1, \dots, v_{p_0}$ whose images form a basis of $\operatorname{im} f$. The maximum-rank pencil lemma gives $\Gamma(e, \ker f) \subseteq \operatorname{im} f$ for all e. To see that lemma, put f in the block form $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$: a nonzero entry of the kernel-to-cokernel block of another map would give a nonzero coefficient of the pencil parameter in an appropriate one-larger minor of the pencil, contradicting maximality over the infinite field. It follows that every value of Γ lies in $\sum_i \Gamma(E, v_i)$: subtract a linear combination of the v_i to reduce any second argument to $\ker f$, and note that $\operatorname{im} f$ is already in this sum. The sum has dimension at most $p_0 q$, proving (LF.7).

Apply (LF.7) to the quotient map

$$V \times U^* \longrightarrow \operatorname{Hom}_k(U, V)/\mathcal{T}, \qquad (v, \phi) \longmapsto [v \otimes \phi].$$

The rank-one operators span the target. With $d = e + 1$, (LF.5)–(LF.6) give the two slice-rank bounds $a - d$ and $s - d$. In particular

$$d \le a, s, \qquad d(a + s - d) \le \operatorname{type} A \le as. \tag{LF.8}$$

Since $d \geq 2$ and $d \leq a, s$, this also gives

$$2(a + s - 2) \leq \operatorname{type} A. \tag{LF.9}$$

For example, the difference between the left sides of (LF.8) and (LF.9) is $(d-2)(a+s-d-2) \geq 0$.

2.2 Fitting bounds for $H = M/tM$

To distinguish the source from the image algebra, let $S_0 = k[T, B, C] \twoheadrightarrow A$ send the variables to t, b, c, and put $I = \ker(S_0 \to A) = \operatorname{Ann}_{S_0} M$. Let $H = M/tM$, viewed over $R = S_0/(T) = k[B, C]$. In formulas for ideals of R we write b, c for its formal variables as well. Its support is $\mathfrak{n} = (b, c)$. Put

$$\mu = \dim_k A/tA.$$

The ordinary zeroth Fitting ideal supplies

$$\operatorname{Fitt}_0^R(H) \subseteq \overline{I} := (I + (T))/(T), \qquad \operatorname{type} A \leq \mu \leq \dim_k R/\operatorname{Fitt}_0^R(H). \tag{LF.10}$$

Here $R/\overline{I}$ is the quotient algebra A/tA. For completeness, the first containment follows by reducing any free S_0-presentation of M modulo T. Each maximal minor of a relation matrix annihilates all its module generators: multiply the selected square relation matrix by its adjugate. This proves the ordinary Fitting-annihilator containment before reduction and hence after base change. Finally $\operatorname{soc} A \subseteq \operatorname{Ann}_A(t)$, and multiplication by t on A gives $\dim \operatorname{Ann}_A(t) = \dim A/tA = \mu$.

The top of H is U. For top dimension at least three and $\dim_k H \leq 4$, the following list is exhaustive up to a linear change of b, c. The table gives the zeroth Fitting ideals and their colengths.

Top dimension	H	$\operatorname{Fitt}_0(H)$	Colength
4	k^4	$\mathfrak{n}^4$	10
3	k^3	$\mathfrak{n}^3$	6
3	$\omega_Q \oplus k$, $Q = R/\mathfrak{n}^2$	$\mathfrak{n}^3$	6
3	$D \oplus k^2$, $D = R/(c, b^2)$	$(c, b^2)\mathfrak{n}^2$	7

Here $\omega_Q = D_k Q$. To prove exhaustiveness, only the length-four, top-three case needs discussion. Its radical has dimension one and is killed by $\mathfrak{n}$. Multiplication by b, c on the three-dimensional top is therefore a pair of row forms with the same one-dimensional target. If the forms are independent, suitable top bases give the canonical length-three module ω_Q and one simple summand. If they are proportional, a linear coordinate change makes c zero and gives a single length-two b-chain and two simple summands. This is a module decomposition, because all products of two radical operators vanish. The Fitting ideals follow from direct sums and presentations: $\operatorname{Fitt}_0(k) = \mathfrak{n}$, $\operatorname{Fitt}_0(D) = (c, b^2)$, and $\operatorname{Fitt}_0(\omega_Q) = \mathfrak{n}^2$. The last equality is given by the two-by-three presentation with columns $(c, 0), (0, b), (b, -c)$. In the last row of the table the surviving monomials are $1, b, c, b^2, bc, c^2, b^3$, giving colength seven.

For top dimension two consider the ranks of tensors in the initial relation space. Choose lifts u, v of a top basis. The initial relation space is

$$\mathcal{R}_1 = \ker\bigl[(\mathfrak{n}/\mathfrak{n}^2) \otimes \operatorname{top} H \longrightarrow \mathfrak{n}H/\mathfrak{n}^2 H\bigr]. \tag{LF.11}$$

Its dimension is at least $4 - (\dim H - 2) \geq 2$.

If $\mathcal{R}_1$ contains a rank-two tensor, write its coefficient forms as F_1, G_1; they are independent linear forms in b, c. Their relation in H has error in $\mathfrak{n}^2 H$. Since u, v generate H, subtract $f_2 u + g_2 v$

with $f_2, g_2 \in \mathfrak{n}^2$. Lifting the resulting relation in H to M leaves a t-multiple, which can be written $t(pu + qv)$, since u, v generate M. Hence

$$(F_1 - f_2 - tp)u + (G_1 - g_2 - tq)v = 0. \tag{LF.12}$$

Put $F = F_1 - f_2 - tp$, $G = G_1 - g_2 - tq$ in the algebra A. Their classes together with t span the same cotangent space $\mathfrak{m}/\mathfrak{m}^2$ as t, b, c: the transverse coefficient matrix is nonsingular, f_2, g_2 disappear, and tp, tq only change the coefficient of t. These classes span the cotangent space, whose dimension may be less than three. Nilpotent Nakayama gives $A = k[F, G, t]$. Explicitly, the span assertion implies that the new ideal generates $\mathfrak{m}/\mathfrak{m}^2$, hence generates $\mathfrak{m}$ by iterating modulo its nilpotent powers; splitting coefficients into their scalar and radical parts then shows that the new elements generate the algebra.

Use the new polynomial source $S' = k[X, Y, Z]$ acting by F, G, t. Its ideal $P = (X, Y)$ maps onto M by $X \mapsto v$, $Y \mapsto -u$. This is well defined: the syzygies of X, Y are generated by $(Y, -X)$, whose image is zero by (LF.12). The kernel is an ideal $N \subseteq P$. Thus $M \simeq P/N$, its action algebra is $S'/(N : P) = A$, and the ideal-quotient lemma rules out a counterexample.

It remains to describe $\mathcal{R}_1$ when it has no rank-two tensor. A subspace of two-by-two matrices of rank at most one has dimension at most two and has a common factor. Here is the elementary reason. If two rank-one tensors have independent image vectors and independent functionals, their sum has rank two. Fixing one nonzero tensor, either all images are in its image line, or one independent image forces all functionals into its functional line. Each such space has dimension at most two. In (LF.11) equality must hold. It follows that $\dim \mathfrak{n}H/\mathfrak{n}^2H = 2$; since the whole length is at most four, in fact

$$\dim H = 4, \qquad \mathfrak{n}^2 H = 0.$$

The two common-factor alternatives are exactly

$$H = D^2 \quad \text{or} \quad H = Q \oplus k, \qquad D = R/(c, b^2), \quad Q = R/\mathfrak{n}^2, \tag{LF.13}$$

after a linear change of transverse coordinates in the first case. A common variable factor kills both generators and the other variable identifies the two-dimensional top with the radical, giving D^2. A common generator factor makes one generator simple, while the other has two independent radical images, giving $Q \oplus k$. These basis descriptions prove the module isomorphisms. The rank-two relation case and (LF.13) therefore exhaust the possibilities. In particular, the rank-two relation argument excludes heads of length at most three. In both cases of (LF.13) the Fitting colength is six:

$$\operatorname{Fitt}_0(D^2) = (c, b^2)^2, \qquad \operatorname{Fitt}_0(Q \oplus k) = \mathfrak{n}^3. \tag{LF.14}$$

2.3 Reduction to top and socle dimension two

By (LF.8), $a, s \geq 2$. If $a = 4$, then also $s = 4$, and (LF.9) requires type at least twelve, contrary to the first table row and (LF.10). If $a = 3$, the same bound requires type at least eight, contrary to the remaining table rows. Therefore $a = 2$. The preceding head argument gives $r = 4$, (LF.13), and $\mu \leq 6$. Now $e + 1 \leq a = 2$ forces $e = 1$. Equations (LF.8) and (LF.4) give

$$\operatorname{type} A = 2s, \qquad \operatorname{soc} A \simeq \operatorname{Hom}_k(k^2, \operatorname{soc} M), \qquad s \in \{2, 3\}. \tag{LF.15}$$

The isomorphism is the socle action.

We exclude $s = 3$ by considering a proper submodule or quotient. In this case $\operatorname{type} A = \mu = 6$, so $\operatorname{Ann}_A(t) = \operatorname{soc} A$. If $H = Q \oplus k$, let $P \subset M$ be the inverse image of its simple summand. It has

length $n-3$ and contains tM. Any element annihilating P therefore kills tM; faithfulness gives $gt = 0$, so it lies in the algebra socle. It annihilates P exactly when its full socle matrix kills that one top line. Hence $\dim \operatorname{Ann}_A P = 3$. Its action image has length $n+1-3 = n-2 > \dim P$, a shorter counterexample.

Suppose instead that $H = D^2$. The dual head $D_kM/tD_kM \simeq D_k(M[t])$ has length four and top dimension three. The table leaves $\omega_Q \oplus k$ or $D' \oplus k^2$, where $D' = R/(c', (b')^2)$ for suitable linear coordinates b', c'. In the first case its Fitting ideal is $\mathfrak{n}^3$; this ideal is invariant under linear changes of b, c. Combined with the original head's (c^2, b^2c, b^4), it bounds μ by the colength of (c^2, b^2c, b^3), namely five, a contradiction. In the second case $W = M[t] \simeq D' \oplus k^2$, since the coefficient-of-b' pairing identifies D' with its positive dual. Take its cyclic summand $L = D'$, of length two and socle dimension one. If g annihilates M/L, then $gM \subseteq L \subseteq M[t]$, so $gt = 0$ and $g \in \operatorname{soc} A$. Its image is therefore in $\operatorname{soc} L$. Full socle action gives annihilator dimension exactly two. The quotient has length $n-2$ and image algebra length $n-1$, again contradicting minimality. Applying the argument to the dual module gives the corresponding dual case.

We have reached

$$a = s = 2, \quad e = 1, \quad \operatorname{type} A = 4, \quad \operatorname{soc} A = \operatorname{Hom}_k(\operatorname{top} M, \operatorname{soc} M). \tag{LF.16}$$

Both the original and the dual heads satisfy (LF.13).

2.4 The case $H = Q \oplus k$

Assume that $H = Q \oplus k$. Choose $u \in M$ so that $\dim_k Au$ is maximal and its coefficient on the Q-generator of H is nonzero. The two requirements are nonempty open conditions on the same affine space: the first is the nonvanishing of a maximal evaluation minor, and the second is the nonvanishing of a linear coordinate. Their intersection has a k-point because k is infinite.

For every u with nonzero top, (LF.16) gives $\operatorname{soc} A\,u = \operatorname{soc} M$. Thus Au contains the whole socle, and

$$R_0 = A/\operatorname{Ann}_A(u) \simeq Au \quad \text{has type two.} \tag{LF.17}$$

Indeed, the socle of the submodule Au is exactly $Au \cap \operatorname{soc} M = \operatorname{soc} M$.

Maximum cyclic length supplies the needed annihilator nesting. Apply the maximum-rank pencil lemma to the evaluation maps $A \to M$, $a \mapsto au$ and $a \mapsto av$, for arbitrary v. It gives

$$\operatorname{Ann}_A(u)M \subseteq Au. \tag{LF.18}$$

Consequently the preimage of $\operatorname{Ann}_A(u)$ in the polynomial source kills M/Au.

The image of u generates a length-three cyclic submodule in the head $Q \oplus k$: its Q-component is a unit. Hence, for $N = M/Au$,

$$\dim N/tN = 1.$$

Nilpotent Nakayama over $k[t]$ now gives an isomorphism $N \simeq k[t]/(t^q)$, with $q \geq 1$. The quotient is nonzero since M has two minimal generators. Operators commuting with the cyclic t-action are multiplication by polynomials in t. Subtract these polynomials from b, c on this same source, making both act as zero on N; their constants are zero by nilpotence. This change preserves the generated algebra and fixes t and Au.

We therefore have precisely the extension and annihilator nesting of Theorem 4, with the type-two parent (LF.17). Its dimension bound contradicts $\dim A = n+1$. If D_kM/tD_kM has this form, apply the argument to D_kM. Duality preserves dimension, the action algebra and the kernel dimension of t. It remains to consider the case in which both quotients have the form D^2.

2.5 The case $H = D^2$ and $M[t] = D^2$

The argument uses the entire ideal $K = \mathrm{Ann}_A(t)$, not just the four-dimensional algebra socle. First, minimality gives

$$\mu \geq r + e = 5. \tag{LF.19}$$

In fact $\mathrm{Ann}_A(tM) = \mathrm{Ann}_A(t)$ by faithfulness. Its dimension is μ. Minimality applied to the submodule tM, of dimension $n - r$, gives $n + e - \mu \leq n - r$. The submodule is proper; the same calculation also covers $tM = 0$. Thus $5 \leq \mu \leq 6$.

The two D^2 quotients may initially have different annihilating linear forms. Their Fitting ideals contain the squares of those forms. If the forms are independent, their squares have quotient colength four, contradicting (LF.19). They therefore align. After changing transverse coordinates and independently normalizing the two chain bases we have module identifications for the same b, c:

$$H = M/tM = D^2, \quad W = M[t] = D^2, \qquad D = k[b]/(b^2), \quad cH = cW = 0. \tag{LF.20}$$

On W, this follows by dualizing the D^2 decomposition for D_kM/tD_kM; the length-two module is self-dual. Choose the bases of H and W independently.

Each element of K kills tM and has image in W. The map

$$K \longrightarrow \mathrm{Hom}_R(H, W) = \mathrm{Mat}_2(D) \tag{LF.21}$$

is injective by faithfulness. In these coordinates (LF.16) says that its socle subspace is exactly $b\,\mathrm{Mat}_2(k)$. Reduction modulo b therefore gives a subspace

$$\overline{K} \subseteq \mathrm{Mat}_2(k), \qquad \dim \overline{K} = \mu - 4 \leq 2, \tag{LF.22}$$

and K is the full inverse image of $\overline{K}$: the entire kernel $b\,\mathrm{Mat}_2(k)$ is contained in the image. Its reduction kernel is exactly the algebra socle, because a map in (LF.21) with zero reduction has image in $bW = \mathrm{soc}\, M$; faithfulness then puts its scalar in $\mathrm{soc}\, A$.

Call a vector in D^2 primitive if its reduction modulo b is nonzero. Write each matrix over D as $Y + bZ$, where $Y, Z \in \mathrm{Mat}_2(k)$. For every primitive $v = v_0 + bv_1 \in W = D^2$, the cyclic submodule $L_v = Dv$ is a submodule, of length two. Its quotient annihilator consists exactly of the elements of K whose maps in (LF.21) have image in L_v. For a fixed reduced matrix $Y = v_0\alpha$, the permitted tail matrices are $Z = v_1\alpha + v_0\beta$, with the row β arbitrary. Thus the tail fiber has dimension two, and

$$\dim \mathrm{Ann}_A(M/L_v) = 2 + \dim\{Y \in \overline{K} : \mathrm{im}\, Y \subseteq kv_0\}. \tag{LF.23}$$

The full inverse-image description in (LF.22) makes this condition sufficient. Minimality on the length-$n - 2$ quotient forces the left side to be at least three. Hence every nonzero target vector v_0 has a nonzero image-line fiber in $\overline{K}$.

For every primitive $w = w_0 + bw_1 \in H = D^2$, let $P_w \subset M$ be the inverse image of Dw. It has codimension two and contains tM. Its annihilator is in K and has the exact formula

$$\dim \mathrm{Ann}_A(P_w) = 2 + \dim\{Y \in \overline{K} : Yw_0 = 0\}. \tag{LF.24}$$

Indeed, the remaining tail equation is $Zw_0 = -Yw_1$. For every nonzero w_0, evaluation on w_0 from all two-by-two matrices is surjective, with a two-dimensional kernel; all these tails belong to K. Minimality on P_w, also of length $n - 2$, again makes (LF.24) at least three.

The two requirements on the at-most-two-dimensional space $\overline{K}$ are incompatible. Choose nonzero matrices in the image-line fibers of the two coordinate vectors. They have the form $e_1\alpha$ and $e_2\beta$, are

independent, and therefore span $\overline{K}$. A nonzero matrix in the fiber of $e_1 + e_2$ is a linear combination whose two rows coincide; both coefficients are nonzero, so α, β are proportional. Consequently $\overline{K} = k^2 \otimes k\phi$ for a nonzero functional ϕ. For w_0 with $\phi(w_0) \neq 0$, evaluation $\overline{K} \to k^2$, $Y \mapsto Yw_0$, is injective. Its kernel fiber is zero, contradicting (LF.24).

This contradiction proves the theorem. □

3 Canonical annihilator minors

Let k be a field of characteristic zero and put

$$S = k[x_0, x_1, x_2].$$

For a finite-dimensional k-vector space V, write

$$D_k V = \mathrm{Hom}_k(V, k).$$

If V is an S-module, its dual has the positive multiplication action

$$(f\varphi)(v) = \varphi(fv). \tag{CM.1}$$

In particular this is an action by commuting transposes, with no minus sign. All lengths below are dimensions over k. Every symmetric square is the ordinary quotient of the tensor square by $v \otimes w - w \otimes v$.

3.1 The minor identity

Theorem 3 (Canonical annihilator minors)**.** *Let $q : S \twoheadrightarrow R$ be a surjective k-algebra map, where R is finite-dimensional over k. Set $P = D_k R$ and $I = \ker q = \mathrm{Ann}_S P$. Suppose that*

$$0 \longrightarrow E \xrightarrow{\iota} F = Se_1 \oplus Se_2 \xrightarrow{\epsilon} P \longrightarrow 0 \tag{CM.2}$$

is exact. We identify E with $\ker \epsilon \subseteq F$ through ι. Since I annihilates P, the vectors fe_1, fe_2 lie in E for every $f \in I$. Let B be a commutative S-algebra, with its given structure map $\theta : S \to B$, and let $\phi : E \to B$ be any S-linear map. Then, for every $f, g \in I$,

$$\phi(fe_1)\phi(ge_2) - \phi(fe_2)\phi(ge_1) = 0 \quad \text{in } B. \tag{CM.3}$$

We prove the theorem by establishing the symmetric-kernel statement for the two-row presentation (CM.2). Two different interchange signs enter that argument. We fix both explicitly.

The diagonal complex and its interchange

For S-modules M, N, put $W = M \otimes_k N$ and

$$\Delta_i(m \otimes n) = x_i m \otimes n - m \otimes x_i n.$$

The diagonal Koszul complex $C_\bullet(M, N)$ has terms

$$0 \longrightarrow W \xrightarrow{d_3} W^3 \xrightarrow{d_2} W^3 \xrightarrow{d_1} W \longrightarrow 0, \tag{CM.4}$$

where the degree-two coordinate order is $01, 02, 12$, and

$$
\begin{aligned}
d_1(v_0, v_1, v_2) &= \Delta_0 v_0 + \Delta_1 v_1 + \Delta_2 v_2, \\
d_2(w_{01}, w_{02}, w_{12}) &= (-\Delta_1 w_{01} - \Delta_2 w_{02}, \Delta_0 w_{01} - \Delta_2 w_{12}, \Delta_0 w_{02} + \Delta_1 w_{12}), \\
d_3(w) &= (\Delta_2 w, -\Delta_1 w, \Delta_0 w).
\end{aligned} \tag{CM.5}
$$

This complex computes $\mathrm{Tor}_\bullet^S(M, N)$. One construction is the Koszul resolution of the diagonal in $S \otimes_k S$, on the regular sequence $x_i \otimes 1 - 1 \otimes x_i$. Equivalently, the free S-module $S \otimes_k M$ has the three commuting displacement operators $x_i \otimes 1 - 1 \otimes x_i|_M$. A displacement in one variable is injective, since the highest polynomial coefficient of its value equals the highest coefficient of its input. Its cokernel is evaluation at the corresponding operator on M. Commutativity permits repeating this argument in the other two variables. Thus the three displacements form a regular sequence on this free module and their Koszul complex is a free resolution of M. Tensoring it with N, using commutativity of S, gives (CM.4).

When $M = N$, let τ denote ordinary interchange of the two tensor coefficients. Since $\Delta_i \tau = -\tau \Delta_i$, the chain interchange is

$$
J_i = (-1)^i \tau \quad \text{in degree } i. \tag{CM.6}
$$

It is the derived interchange: exchanging the two diagonal variables changes each Koszul generator to its negative. We use $+$ and $-$ for the eigenspaces of this chain interchange on homology.

For the cyclic module $R = S/I$, the long exact Tor sequence of $0 \to I \to S \to R \to 0$ gives

$$
\mathrm{Tor}_1^S(R, R) \cong I \otimes_S R \cong I/I^2. \tag{CM.7}
$$

The interchange on this group is negative. To check the sign, choose a free resolution $Q_\bullet \to R$ with $Q_0 = S$. For $f \in I$, choose $a_f \in Q_1$ with $da_f = f$. In the total complex of $Q_\bullet \otimes_S Q_\bullet$, the cycle

$$
a_f \otimes 1 - 1 \otimes a_f
$$

represents the conormal class of f, up to the single global sign coming from the choice of augmentation. Graded interchange negates this cycle. These classes generate (CM.7), without a regular-sequence assumption on I. Consequently

$$
H_1 C_\bullet(R, R)^+ = 0. \tag{CM.8}
$$

Finite duality in dimension three

The evaluation pairing

$$
\langle p \otimes p', r \otimes r' \rangle = p(r) p'(r')
$$

is perfect between $P \otimes_k P$ and $R \otimes_k R$. For the action (CM.1), the displacement operators on these two spaces are adjoint. Combining evaluation with exterior multiplication in k^3 gives a perfect pairing of complementary chain degrees. In the degrees used here it is

$$
\langle w, v \rangle_2 = \langle w_{01}, v_2 \rangle - \langle w_{02}, v_1 \rangle + \langle w_{12}, v_0 \rangle. \tag{CM.9}
$$

The formulas (CM.5) give the adjoint identities

$$
\langle d_3 u, v \rangle_2 = \langle u, d_1 v \rangle_0, \qquad \langle w, d_2 z \rangle_2 = -\langle d_2 w, z \rangle_1, \tag{CM.10}
$$

where the degree-one pairing is also defined by exterior multiplication into the ordered volume element of the Koszul generator space k^3. Thus cycles annihilate boundaries. Perfectness on the finite chain spaces, followed by taking annihilators, shows that the induced pairing

$$H_2C_\bullet(P,P) \ \times \ H_1C_\bullet(R,R) \longrightarrow k \tag{CM.11}$$

is perfect. This pairing follows from finite-dimensional duality of the complexes.

Simultaneously interchanging both tensor coefficients preserves evaluation. However the chain interchanges in degrees two and one have opposite signs. In particular

$$\langle J_2 w, J_1 v\rangle_2 = -\langle w, v\rangle_2. \tag{CM.12}$$

If $J_2[w] = -[w]$, then (CM.8) and $J_1[v] = -[v]$ imply, by (CM.12), that its pairing with every degree-one class is its own negative. Since 2 is invertible, every such pairing is zero. Perfectness of (CM.11) therefore proves

$$\mathrm{Tor}_2^S(P,P)^- = 0. \tag{CM.13}$$

Connecting maps and the symmetric kernel

Flatness of F in (CM.2) gives natural identifications

$$\begin{aligned} \ker[E \otimes_S E \longrightarrow F \otimes_S F] &\cong \mathrm{Tor}_1^S(P,E) \\ &\cong \mathrm{Tor}_2^S(P,P). \end{aligned} \tag{CM.14}$$

For the first line, tensor (CM.2) with E, and then use the injectivity of $F \otimes_S E \to F \otimes_S F$. For the second, tensor the same sequence with P and use the vanishing of positive Tor with F.

Ordinary interchange on the kernel in (CM.14) corresponds to the negative of derived interchange on its right-hand side. Here is a representative check compatible with (CM.5). Choose one k-linear section $\sigma : P \to F$ of ϵ, and define the deviations

$$\delta_i(p) = x_i\sigma(p) - \sigma(x_i p) \in E.$$

For a degree-two cycle $w = (w_{01}, w_{02}, w_{12})$ in $C_\bullet(P,P)$, the composite connecting map is represented, in $E \otimes_S E$, by the image of

$$\Gamma(w) = \sum_{i<j} (\delta_j \otimes \delta_i - \delta_i \otimes \delta_j)(w_{ij}) \quad \text{in } E \otimes_k E. \tag{CM.15}$$

Indeed the first connecting map inserts σ in the first factor and applies the differential. Projection onto E replaces the i-th displacement by δ_i. The second inserts σ in the other factor; its displacement contributes $-\delta_i$. Substitution into (CM.5) gives (CM.15). The resulting class in $E \otimes_S E$ is independent of the section and of the cycle representative: these are the two connecting maps in (CM.14), and a degree-three boundary maps to a balancing relation. The long exact sequences used in (CM.14) show that this formula represents their isomorphism onto the indicated kernel.

Directly from its two terms,

$$\Gamma(\tau w) = -\tau\Gamma(w). \tag{CM.16}$$

Since $J_2 = \tau$, the ordinary symmetric part of the kernel in (CM.14) is the minus part in (CM.13), and is zero. We have proved the following two-row statement:

$$(E \otimes_S E)^{\tau=1} \longrightarrow (F \otimes_S F)^{\tau=1} \quad \text{is injective.} \tag{CM.17}$$

Equivalently, averaging by $(1+\tau)/2$ gives injectivity of the ordinary map $\operatorname{Sym}^2_S E \to \operatorname{Sym}^2_S F$ for (CM.2). Both signs are needed: the three-dimensional pairing reverses the derived parity, while the two connecting maps reverse the relation between that parity and ordinary interchange.

Proof of Theorem 3. The four-term tensor

$$z = (fe_1) \otimes (ge_2) - (fe_2) \otimes (ge_1) + (ge_2) \otimes (fe_1) - (ge_1) \otimes (fe_2) \tag{CM.18}$$

belongs to $E \otimes_S E$ and is fixed by ordinary interchange. Its image in $F \otimes_S F$ is zero: balancing both scalars gives $fg(e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_1 - e_1 \otimes e_2)$. By (CM.17), $z = 0$. Applying $\phi \otimes \phi$ and multiplication in B gives twice the left-hand side of (CM.3). Division by $2 \in k^\times$ proves the identity. □

3.2 Curvilinear extensions

For $q \geq 1$, let $Q_q = k[t]/(t^q)$, and let

$$\tau_q : Q_q \longrightarrow k, \qquad \tau_q(a) = [t^{q-1}]a.$$

The map

$$\kappa_q : Q_q \longrightarrow D_k Q_q, \qquad \kappa_q(a)(b) = \tau_q(ab) \tag{CE.1}$$

is an isomorphism. For example, if the lowest nonzero coefficient of a has degree r, multiplication by t^{q-1-r} detects it under τ_q. Thus the multiplication pairing is nondegenerate, and equality of finite dimensions proves (CE.1). The isomorphism is S-linear for any specified S-algebra structure on Q_q.

Theorem 4 (Curvilinear extension bound)**.** *Let $R = S/I$ be a nonzero finite local k-algebra with residue field k, and suppose its socle has dimension exactly two. Give Q_q, $q \geq 1$, a quotient S-algebra structure with annihilator J. Suppose the finite S-module M fits into an exact sequence*

$$0 \longrightarrow R \xrightarrow{i} M \xrightarrow{p} Q_q \longrightarrow 0, \qquad I \subseteq J. \tag{CE.2}$$

Then its image algebra satisfies

$$\dim_k S/\operatorname{Ann}_S M \leq \dim_k M. \tag{CE.3}$$

Proof. The canonical module $P = D_k R$ has exactly two minimal generators: its top is dual to the socle of R. Nakayama therefore gives a surjection $\epsilon : S^2 \twoheadrightarrow P$. Use the presentation (CM.2).

Put $X = D_k M$. Dualizing (CE.2) and using the isomorphism (CE.1) gives the exact sequence

$$0 \longrightarrow Q_q \xrightarrow{j = p^* \kappa_q} X \xrightarrow{d = i^*} P \longrightarrow 0. \tag{CE.4}$$

Choose $x_1, x_2 \in X$ with $d(x_i) = \epsilon(e_i)$, and let $\ell : S^2 \to X$ send e_i to x_i. For $e \in E$, $d\ell(e) = 0$. Consequently there is a unique S-linear map $\phi : E \to Q_q$ satisfying

$$j\phi(e) = \ell(e). \tag{CE.5}$$

Surjectivity of ϵ and exactness of (CE.4) imply

$$X = j(Q_q) + \ell(S^2). \tag{CE.6}$$

Let $L = \operatorname{Ann}_S X$. The quotient $X \twoheadrightarrow P$ implies $L \subseteq I$. If $f \in I$, then $fX \subseteq j(Q_q)$, and the nesting $I \subseteq J$ says that f kills this kernel. Define

$$e_\phi : I \longrightarrow Q_q^2, \qquad e_\phi(f) = (\phi(fe_1), \phi(fe_2)). \tag{CE.7}$$

By (CE.5), $fx_i = j\phi(fe_i)$. Thus $e_\phi(f) = 0$ implies that f kills both chosen lifts. It also kills $j(Q_q)$, so (CE.6) gives $fX = 0$. The converse follows by evaluating on the two lifts and using injectivity of j. We have proved the exact identity

$$\ker e_\phi = L, \qquad \operatorname{im} e_\phi \cong I/L. \tag{CE.8}$$

Write $V_\phi = \operatorname{im} e_\phi \subset Q_q^2$. Theorem 3 applied to the same ϵ and the same ϕ says that all determinant pairings of its vectors vanish. Consider the scalar alternating form

$$\Omega((a,b),(c,d)) = \tau_q(ad - bc). \tag{CE.9}$$

It is nondegenerate: pairing (a,b) with every $(0,d)$ and every $(c,0)$ detects a and b, respectively, by (CE.1). The subspace V_ϕ is isotropic. In a $2q$-dimensional space with a nondegenerate alternating form,

$$\dim_k V_\phi^\perp = 2q - \dim_k V_\phi.$$

Since $V_\phi \subseteq V_\phi^\perp$, it follows that

$$\dim_k V_\phi \le q. \tag{CE.10}$$

Finally, (CE.8) and the exact sequence $0 \to I/L \to S/L \to R \to 0$ give

$$\dim_k S/L = \dim_k R + \dim_k V_\phi \le \dim_k R + q = \dim_k M. \tag{CE.11}$$

Finite duality separates vectors, so (CM.1) gives $\operatorname{Ann}_S D_k M = \operatorname{Ann}_S M$. This proves (CE.3). The map $S \to \operatorname{End}_k(M)$ has exactly this kernel, so S/L is its image algebra. □

The homological background is the Koszul resolution of the diagonal, the long exact Tor sequence and graded tensor interchange; see [9, Tags 062D, 00LY and 0FNG].

4 A two-generated ideal quotient

Let k be any field. All algebras in this section are commutative and unital, and lengths mean dimensions over k.

Lemma 5 (Ideal-quotient bound)**.** *Let $S = k[x,y,z]$, let $P = (x,y)$, and let N be an ideal of S. Set*

$$M = (P+N)/N \simeq P/(P \cap N).$$

Suppose that M has finite length and that the actions of x and y on M are nilpotent. Then

$$\ell(S/(N:P)) \le \ell(M). \tag{SIQ}$$

The action of z may be arbitrary. If $N \subseteq P$, the right side is $\ell(P/N)$.

Proof. Put $A = S/(N : P)$. The equality $\operatorname{Ann}_S M = (N : P)$ makes the action of A on M faithful, so A is finite dimensional. Write $\bar{x}, \bar{y}, \bar{z}$ for the images of the coordinates in A, put $I = (\bar{x}, \bar{y})A$, and set

$$W = \{m \in M : xm = ym = 0\}, \qquad \kappa = \dim_k W.$$

The map induced by $S/N \longrightarrow S/(N : P)$ restricts to a map $M \longrightarrow A$ whose image is exactly I. Its kernel consists of the classes of $p \in P$ for which $pP \subseteq N$, hence is exactly W. Consequently

$$\ell(M) = \ell(I) + \kappa. \tag{SIQ.1}$$

We claim that $\ell(A/I) \leq \kappa$. The case $M = 0$ is immediate, so assume $M \neq 0$. Let $B = k[\bar{x}, \bar{y}] \subseteq A$. Since the action is faithful and $\bar{x}, \bar{y}$ are commuting nilpotents, B is a finite local algebra with nilpotent maximal ideal $(\bar{x}, \bar{y})B$ and residue field k. Write $DM = \operatorname{Hom}_k(M, k)$, with its positive dual B-action. The annihilator pairing identifies

$$DM/(\bar{x}, \bar{y})DM \simeq DW.$$

Finite Nakayama therefore gives κ generators of DM over B. Multiplication by $\bar{z}$ is B-linear. The determinant trick applied to these generators gives a monic polynomial

$$p(t) \in B[t], \qquad \deg p \leq \kappa,$$

which annihilates this endomorphism of DM. Positive finite duality preserves the annihilator, so faithfulness implies $p(\bar{z}) = 0$ in A.

Modulo I, every element of B is a scalar. The image of p is thus a monic polynomial in $k[t]$ of the same degree, annihilating the image of $\bar{z}$ in A/I. Since A/I is generated by that one image, division by the monic relation gives $\ell(A/I) \leq \kappa$. Combining this inequality with (SIQ.1) yields

$$\ell(A) = \ell(I) + \ell(A/I) \leq \ell(I) + \kappa = \ell(M).$$

□

Corollary 6. *Let M be an S-module that is finite-dimensional over k. Suppose that M is generated by u, v, that $xu + yv = 0$, and that x, y act nilpotently. Then $\ell(S/\operatorname{Ann}_S M) \leq \ell(M)$.*

Proof. The presentation of the polynomial ideal (x, y) has one relation $(-y, x)$. The assignments $x \mapsto v$, $y \mapsto -u$ respect that relation and define a surjection $(x, y) \twoheadrightarrow M$. Its kernel is an ideal $N \subseteq (x, y)$, so Lemma 5 applies. □

5 Changing generators and passing to the global algebra

We first record an elementary rank-specialization lemma.

Lemma 7. *Let k be infinite, let M be finite-dimensional, and let $T, U \in \operatorname{End}_k(M)$. For every nonzero $g \in k[X]$, there is a scalar $\lambda \in k$ such that*

$$g(\lambda) \neq 0, \qquad \dim_k \ker(T + \lambda U) \leq \dim_k \ker T.$$

Proof. Choose a nonzero minor of T of order $\operatorname{rank} T$. The corresponding minor of $T + XU$ is a polynomial nonzero at zero. Its product with g is nonzero, so it does not vanish on all of k. At a point where the product is nonzero, the rank is at least that of T. If $T = 0$, use the empty minor. □

Lemma 8. *Let k be infinite of characteristic zero. Let R be a finite-dimensional commutative augmented k-algebra with nilpotent augmentation ideal $\mathfrak{m}$, and let M be a finite faithful R-module. Suppose $a, b, c \in \mathfrak{m}$ generate R as a k-algebra. If an element $s \in \mathfrak{m}$ acts on M with kernel dimension at most four, then*

$$\dim_k R \le \dim_k M.$$

Proof. The classes of a, b, c span $\mathfrak{m}/\mathfrak{m}^2$. Write

$$s \equiv \alpha a + \beta b + \gamma c \pmod{\mathfrak{m}^2}.$$

For $\lambda \neq -\alpha$, the classes of $s + \lambda a, b, c$ still span this quotient. Nilpotence of $\mathfrak{m}$ then gives

$$k[s + \lambda a, b, c] = R.$$

Indeed, iterating the spanning relation through the powers of $\mathfrak{m}$ expresses every element of R as a polynomial in the three elements. Only spanning of the cotangent space is used.

Apply Lemma 7 to the actions of s, a, with $g(X) = X + \alpha$. The resulting generators $s + \lambda a, b, c$ are commuting nilpotents, and the first has kernel dimension at most four. Theorem 2 bounds their image algebra by $\dim_k M$. Faithfulness identifies that image with R. □

Proof of Theorem 1. The zero-dimensional case is immediate. Decompose V into the nonzero simultaneous generalized eigenspaces of A, B, C:

$$V = \bigoplus_i V_i.$$

Let $R = \mathbb{C}[A, B, C]$ and let R_i be its image on V_i. Subtracting the three joint eigenvalues gives commuting nilpotent generators of R_i. Thus R_i is a local augmented complex algebra with nilpotent augmentation ideal and residue field $\mathbb{C}$.

Choose a four-regular $T \in R$, and write T_i for its restriction to V_i. If $\epsilon_i : R_i \to \mathbb{C}$ is the augmentation, put

$$\tau_i = \epsilon_i(T_i), \qquad s_i = T_i - \tau_i I.$$

The element s_i lies in the augmentation ideal, and inclusion of V_i in V induces an injection

$$\ker(s_i \mid V_i) \hookrightarrow \ker(T - \tau_i I \mid V).$$

The space on the right has dimension at most four. Lemma 8 therefore gives $\dim_{\mathbb{C}} R_i \le \dim_{\mathbb{C}} V_i$. Restriction to the summands injects R into $\prod_i R_i$, so

$$\dim_{\mathbb{C}} R \le \sum_i \dim_{\mathbb{C}} R_i \le \sum_i \dim_{\mathbb{C}} V_i = \dim_{\mathbb{C}} V.$$

□

6 Formal verification

The Lean companion proves the quantified complex-matrix statement of Theorem 1. It includes the local nilpotent theorem, the canonical minor identity, the ideal-quotient argument and the changes of generators and invariant summands used above.

The matrix theorem is recorded as

`Gerstenhaber.FourBlocks.AlgebraElement.`
`complex_matrix_contained_four_regular_bound`

The companion's statement map records the declarations for the intermediate results. The toolchain is Lean 4.36.0-rc2 with mathlib revision

`07977d2921637684e9ecdf70fc1927570233637d`.

Running `python3 tools/verify.py` checks the source hashes, compiles the proof, re-elaborates the final source files and audits the axioms. The dependency audit reports the standard axioms

`propext`, `Classical.choice`, `Quot.sound`.

AI assistance

GPT-series agents assisted with mathematical exploration, proof development, Lean formalization and manuscript preparation. The author is responsible for the mathematical statements and the released proof companion.

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