From the Riemann Hypothesis to the Lindelöf Hypothesis:

A Lean 4 Formalization via Two-Disk Interpolation

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Abstract

We give an explicit proof of the classical implication from the Riemann hypothesis to the Lindelöf hypothesis, together with a Lean 4 implementation. A normalized holomorphic logarithm on a zero-free disk, two Borel–Carathéodory estimates and Hadamard interpolation first establish subpower growth on every strictly smaller disk. A second interpolation, applied to the zeta function itself, reaches the closed disk of radius $3/2$ about $2 + it$ without requiring nonvanishing on its boundary. The implementation uses Mathlib and a pinned Abel-continuation estimate from PrimeNumberTheoremAnd (PNT+). The final declaration has precisely the eventual positive-height Lindelöf conclusion with RH as its sole mathematical premise. The author's reported local run compiled 28 modules and reported only the standard axioms `propext`, `Classical.choice` and `Quot.sound` for this declaration. We distinguish this evidence from a fresh rebuild of the exact archived source, which remains outstanding, and provide a statement-level correspondence, dependency inventory and reproducibility protocol. The contribution is a documented formalization of a known conditional theorem, not a proof of RH or of unconditional Lindelöf.

Keywords: Riemann zeta function; Lindelöf hypothesis; Borel–Carathéodory; Hadamard interpolation; Lean 4.
MSC 2020: 11M06, 30A10, 68V20.

1 Introduction and scope

The Riemann hypothesis implies the Lindelöf hypothesis by classical arguments (see Titchmarsh [1]). The objective here is not to establish a new implication, but to expose the analytic and logical interfaces of an explicit proof suitable for a proof assistant. The development uses the existing definition of RH in Mathlib [3]; the formalization of zeta and Dirichlet L-functions by Loeffler and Stoll [2] supplies relevant library context. The outer growth estimate reuses a theorem from PNT+ [4]. The exact source snapshot is separately identified by the Zenodo record [5].

The method avoids a logarithmic Euler product: first estimate a normalized analytic logarithm strictly inside a zero-free disk, then interpolate the zeta function across the limiting radius. This distinction is important because a zero on the critical line is compatible with RH but obstructs extending a logarithm across that zero. We make no claim of priority

among Lean formalizations. A related contemporary preprint reports conditional Lindelöf declarations named `lindelof_of_H` and `lindelof_half`, with Borel–Carathéodory, three-circle and Phragmén–Lindelöf techniques [6]. That report indicates overlap in methods and goals; an exact comparison of the hypotheses, source revisions and kernel audits remains open. Our specific scope is the Riemann zeta function, explicit disk radii, and the direct Mathlib RH premise.

Write ζ for the meromorphic continuation of the Riemann zeta function. Throughout, t is real. RH asserts that the nontrivial zeros of ζ have real part $1/2$.

Theorem 1.1 (Closed-disk subpower estimate)**.** *Assume RH. For every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that for all $t \geq 2$ and all $w \in \mathbb{C}$ with $|w| \leq 3/2$,*

$$|\zeta(2 + it + w)| \leq C_\varepsilon t^\varepsilon. \tag{1}$$

The formal archive proves the eventual version of the critical-line conclusion as a named final declaration. The eventual closed-disk assertion is an intermediate interface in the code. The compact-height extension in Theorem 1.1 and the two-sided formulation below are proved in the exposition; they are not separately audited final declarations in v7.

2 Elementary bounds and zero-free geometry

Set $c_t = 2 + it$ and $f_t(w) = \zeta(c_t + w)$. For $t \geq 2$ and $|w| \leq 7/4$, the real and imaginary parts of $c_t + w$ are both at least $1/4$. In particular the translated disk avoids the pole at 1, so f_t is holomorphic on a neighborhood of the closed disk. For $|w| < 3/2$, the real part is strictly greater than $1/2$. Under RH there are no zeros there: the trivial zeros have negative real part and the nontrivial zeros lie on the critical line.

Lemma 2.1 (Center and inner bounds)**.** *For every $t \in \mathbb{R}$, $1/4 \leq |f_t(0)| \leq 2$, and $|f_t(w)| \leq 3$ when $|w| \leq 1/2$.*

Proof. Absolute convergence at real part 2 and the Basel sum give

$$|\zeta(2 + it) - 1| \leq \sum_{n=2}^{\infty} n^{-2} = \frac{\pi^2}{6} - 1 < \frac{3}{4}.$$

The last inequality follows from $\pi < 3.2$. The triangle and reverse triangle inequalities give the center bounds. For the inner disk, $\mathrm{Re}(c_t + w) \geq 3/2$, and the integral test gives

$$|f_t(w)| \leq \sum_{n=1}^{\infty} n^{-3/2} \leq 1 + \int_1^{\infty} x^{-3/2}\, dx = 3.$$

□

The formal center proof uses `basel_tail`; the inner proof uses a convergent majorant $D = \max\{1, \sum_{n\geq 1} n^{-3/2}\}$, without needing the numerical bound $D \leq 3$.

Lemma 2.2 (Outer growth)**.** *For $t \geq 4$ and $|w| \leq 7/4$, $|f_t(w)| \leq 10t$.*

Proof. For $\mathrm{Re}\, s > 1$, partial summation gives

$$\zeta(s) = s \int_1^{\infty} \lfloor x \rfloor x^{-s-1}\, dx = \frac{s}{s-1} - s \int_1^{\infty} \{x\} x^{-s-1}\, dx.$$

The second integral is holomorphic for $\mathrm{Re}\, s > 0$: on any compact subset, the integrands obtained by differentiating a fixed number j of times are dominated by a constant times

$x^{-1-\sigma_0}(1+\log x)^j$, where $\sigma_0 > 0$ bounds the real part below. Dominated differentiation and the identity theorem extend the displayed representation to $\operatorname{Re} s > 0$, $s \neq 1$. Hence, writing $\sigma = \operatorname{Re} s$,

$$|\zeta(s)| \leq 1 + \frac{1}{|s-1|} + \frac{|s|}{\sigma}.$$

For $s = c_t + w$ on the indicated disk, $\sigma \geq 1/4$, $|s-1| \geq t - 7/4 \geq 1$ and $|s| \leq 2 + t + 7/4 \leq 2t$. Thus the right-hand side is at most $2 + 8t \leq 10t$. □

In Lean, the integral derivation is replaced by the PNT+ estimate `norm_riemannZeta_le`. The adapter proves the domain condition $\operatorname{Re} s > 1/10$, $s \neq 1$, and the three inequalities just used; it does not assume an unproved linear-growth axiom.

3 A disk interpolation lemma

Lemma 3.1 (Three circles via three lines)**.** *Let $0 < a < b < R$. Suppose h is holomorphic in $|w| < R$, continuous on $|w| \leq R$, and bounded in modulus by $A > 0$ on $|w| \leq a$ and $B > 0$ on $|w| \leq R$. Then*

$$|h(w)| \leq A^{1-\theta} B^{\theta} \quad (|w| \leq b), \qquad \theta = \frac{\log(b/a)}{\log(R/a)} \in (0,1).$$

Proof. Fix $|w| \leq b$ and set $H(z) = h(w \exp(z - \log b))$ on the vertical closed strip $\log a \leq \operatorname{Re} z \leq \log R$. On the interior the argument of h has modulus strictly less than R; on the two boundaries it has modulus at most a and R, respectively. The function H is continuous on the strip, holomorphic on its interior and uniformly bounded by B. Apply the Hadamard three-lines theorem at $z = \log b$. The argument requires no nonvanishing assumption on h. □

The formal interface `ThreeCircles` assumes differentiability on the closed ball, which is sufficient for these continuity and holomorphy conditions. All functions to which it is applied in this proof are holomorphic on a neighborhood of the corresponding closed ball.

4 First interpolation: a normalized logarithm

Assume RH. The open disk $|w| < 3/2$ is simply connected and f_t is nonzero and holomorphic there for $t \geq 2$. It admits a normalized analytic logarithm g_t satisfying

$$g_t(0) = 0, \qquad \exp g_t(w) = f_t(w)/f_t(0). \tag{2}$$

Indeed, integrate f_t'/f_t from 0 to w; path independence follows from Cauchy's theorem, and differentiation shows that $f_t \exp(-g_t)$ is constant. The Lean construction instead uses a continuous exponential lift G on the simply connected disk. At any fixed x, continuity makes $\operatorname{Im}(G(y) - G(x)) \in (-\pi, \pi)$ near x. The principal logarithm of $f_t(y)/f_t(x)$ then equals $G(y) - G(x)$ locally and is holomorphic, because its argument is near 1. Subtract $G(0)$ to normalize. This does not assert that a pointwise principal logarithm is holomorphic throughout the disk.

Taking moduli in (2) gives $\operatorname{Re} g_t(w) = \log|f_t(w)/f_t(0)|$. By Lemma 2.1, this is at most $\log 12$ on $|w| < 1/2$. We use the normalized Borel–Carathéodory estimate: if $g(0) = 0$ and $\operatorname{Re} g \leq M$ on the open S-disk, with $M > 0$, then for $0 < r < S$,

$$\sup_{|w| \leq r} |g(w)| \leq \frac{2Mr}{S - r}.$$

For clarity, this follows by applying Schwarz's lemma on the S-disk to $g/(2M-g)$, which is holomorphic and bounded by 1 since $\operatorname{Re} g \le M$. Rearrangement gives the stated bound. At $S = 1/2$ and $r = 1/4$ it yields

$$|g_t(w)| \le A = 2\log 12 \quad (|w| \le 1/4). \tag{3}$$

For $0 < R < 3/2$, take $S = (R+3/2)/2$. By Lemmas 2.1 and 2.2, for $t \ge 4$ the real part of g_t is at most $\log(40t)$ on $|w| < S$. Since $\log t \ge 1$, the same estimate yields

$$|g_t(w)| \le B_R \log t \quad (|w| \le R), \qquad B_R = \frac{2R(1+\log 40)}{S-R} > 0. \tag{4}$$

Lemma 4.1 (Interior subpower bound)**.** *Assume RH. For each $0 < r < 3/2$ and $\eta > 0$ there is $T_{r,\eta} \ge 4$ such that $|f_t(w)| \le 2t^\eta$ for every $t \ge T_{r,\eta}$ and $|w| \le r$.*

Proof. Put $b = \max(r, 1)$ and $R = (b+3/2)/2$, so $1/4 < b < R < 3/2$. Apply Lemma 3.1 to g_t using (3) and (4), obtaining

$$|g_t(w)| \le K(\log t)^\theta, \quad \theta = \frac{\log(4b)}{\log(4R)} \in (0,1), \quad K = A^{1-\theta}B_R^\theta.$$

All constants are independent of t, w for the chosen r. For $t \ge \max\{4, \exp((K/\eta)^{1/(1-\theta)})\}$, the right-hand side is at most $\eta \log t$. Using (2) and Lemma 2.1,

$$|f_t(w)| = |f_t(0)| \exp(\operatorname{Re} g_t(w)) \le 2\exp(|g_t(w)|) \le 2t^\eta.$$

□

5 Second interpolation: the limiting radius

Proof of Theorem 1.1. Given $\varepsilon > 0$, define

$$L = \log(7/6), \quad d = \min\{1/8, \varepsilon L/8\}, \quad r = 3/2 - 2d.$$

Then $5/4 \le r < 3/2 < 7/4$. The interpolation weight

$$\lambda = \frac{\log((3/2)/r)}{\log((7/4)/r)}$$

lies in $(0,1)$. Its denominator is at least $L > 0$ and its numerator is $\log(1+2d/r) \le 2d/r \le 2d$. Hence $\lambda \le \varepsilon/4$. Take $\eta = \varepsilon/2$ in Lemma 4.1. For $t \ge T_{r,\eta}$, apply Lemma 3.1 to f_t itself with radii $r, 3/2, 7/4$ and the bounds from Lemmas 4.1 and 2.2. For $|w| \le 3/2$ this gives

$$\begin{aligned} |f_t(w)| &\le (2t^{\varepsilon/2})^{1-\lambda}(10t)^\lambda \\ &= 2^{1-\lambda}10^\lambda t^{(\varepsilon/2)(1-\lambda)+\lambda} \le 10t^\varepsilon. \end{aligned}$$

The last inequality follows from $0 < \lambda < 1$, $t \ge 1$ and $(\varepsilon/2)(1-\lambda)+\lambda \le 3\varepsilon/4 \le \varepsilon$. No logarithm is evaluated on the critical circle. On the remaining compact set $[2, T_{r,\eta}] \times \{w : |w| \le 3/2\}$, the function $(t,w) \mapsto |f_t(w)|/t^\varepsilon$ is continuous because the translated points avoid the pole. Its finite maximum, together with 10, gives C_ε in (1). □

Corollary 5.1 (Lindelöf)**.** *RH implies that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|\zeta(1/2+it)| \le C_\varepsilon(1+|t|)^\varepsilon \qquad (t \in \mathbb{R}).$$

Proof. Set $w = -3/2$ in Theorem 1.1. The conjugation identity $\zeta(\overline{s}) = \overline{\zeta(s)}$ follows by analytic continuation from the absolutely convergent Dirichlet series and gives the negative-height estimate. The bounded critical-line segment $|t| \leq 2$ is compact and contains no pole, so increasing the constant completes the proof. □

Remark 5.2. The second interpolation only needs an outer polynomial bound and interior subpower bounds; RH enters through zero-freeness in the logarithmic step. Neither a functional equation nor a zero-density estimate is used in the positive-height argument. The constants here are convenient rather than optimized.

6 Formal theorem and code correspondence

6.1 Precise statement and scope

The final declaration in `ZetaOuterPNT.lean`, in namespace `LindelofDisks`, is (ASCII rendering):

```
theorem lindelof_of_riemann_hypothesis :
    RiemannHypothesis -> CriticalLineLindelof :=
  lindelof_from_RH_and_zeta_bounds
    zeta_center_bounds zeta_inner_bound zeta_outer_linear_pnt
```

The expanded example in `TargetAudit.lean` checks the conclusion against the following statement, rather than relying on the name of the proposition (ASCII rendering):

```
example : RiemannHypothesis -> forall eps : Real, 0 < eps ->
    exists C T : Real, 0 < C /\ 1 <= T /\ forall t : Real,
      T <= t -> norm (riemannZeta ((1 / 2 : Complex)
        + (t : Complex) * Complex.I)) <= C * t ^ eps :=
  lindelof_of_riemann_hypothesis
```

Here C, T depend on ε but not on t. This is the usual eventual positive-height form. The extension to $t \geq 2$ and to negative heights is an ordinary compactness and conjugation argument, as proved above, not a separately checked declaration in v7.

The proposition `CriticalDiskEstimate` in `LindelofCore.lean` has the same quantifiers with an additional universal w in the closed 3/2-disk. The declaration `rh_to_disk_from_components` in `AnalyticObligations.lean` derives it from `SublogFromRH`, `CenterBounds`, `OuterLinear` and `ThreeCircles`. The assembly supplies those components and specializes at $w = -3/2$. For a more direct audit of Theorem 1.1's eventual form, a proposed companion declaration is supplied in the manuscript source bundle (`DiskTargetAudit.lean`); it is not part of the immutable Zenodo v7 release and has not been compiled. This separates an additional test from the reported verification evidence.

6.2 Dependency and theorem map

The generic interpolation proof uses Mathlib's three-lines theorem `norm_le_interp_of_mem_verticalClosedStrip'`; the outer estimate reuses PNT+ rather than reproducing its Abel-continuation infrastructure. The PNT+ adapter credits Matteo Cipollina and the PNT+ contributors. Old draft comments and a duplicated import in the exact archive are historical artifacts, not additional assumptions. They are retained to preserve the archived bytes and should be cleaned in a future separately versioned release.

Mathematical step	Lean source and declarations
Definitions and geometry	`LindelofCore.lean`: `CriticalDiskEstimate`, `CriticalLineLindelof`, `zeta_ne_zero_on_open_disk`; `DiskGeometry.lean`: `outer_differentiableOn`.
Logarithmic lift	`LogBranch.lean`: `differentiableOn_of_exp_lift`, `normalizedLog_of_RH`, `critical_ball_simplyConnected`.
Borel–Carathéodory	`LogBounds.lean`: `bc_uniform`; `ZetaAssembly.lean`: `small_log_of_zeta`, `large_log_of_zeta`.
Three circles	`ThreeCirclesProof.lean`: `threeCircles_proof`.
Interior subpower	`FirstInterpolation.lean`: `sublog_eventually`, `sublog_of_log_bounds`; `AnalyticObligations.lean`: `interior_of_sublog`.
Critical radius	`Interpolation.lean`: `choose_outer_radius`, `disk_of_interior`.
Concrete estimates	`ZetaSeriesBounds.lean`: `zeta_center_bounds`, `zeta_inner_bound`; `ZetaOuterPNT.lean`: `zeta_outer_linear_pnt`.
Final result and audits	`ZetaOuterPNT.lean`: `lindelof_of_riemann_hypothesis`; `AuditV3.lean`, `CoreTargetAudit.lean`, `TargetAudit.lean`.

Table 1: Correspondence with the unmodified v7 archive.

7 Evidence and reproducibility

7.1 What is established and what remains to be checked

The supplied excerpts in `LOCAL_PORT_AUDIT.txt` and `USER_REPORTED_AUDIT.txt` report successful compilation of 12 core local modules, 14 upstream modules and two final local modules using Lean v4.37.0-rc1. They report:

```
'LindelofDisks.lindelof_of_riemann_hypothesis'
  depends on axioms: [propext, Classical.choice, Quot.sound]
PASS: ZetaOuterPNT
PASS: TargetAudit
```

The three listed axioms are standard Lean axioms. The report does not list `sorryAx`; RH is an explicit premise, not an axiom introduced by this development. These excerpts are author-supplied evidence, not a signed or independently reproduced build log. The exporter checked that compiled objects existed and copied the source bytes; it did not prove that those objects corresponded exactly to the exported sources. A fresh compilation of the deposited snapshot remains necessary to certify the exact source.

In preparing v8, the supplied ZIP was rehashed and all 30 entries in its checksum manifest matched. This is byte-integrity evidence only. The 14 upstream files are represented by recorded hashes, not included in the ZIP; their contents have not been independently retrieved and compiled for this revision. No Lean executable was available in the revision environment. No new axiom audit is claimed.

7.2 Exact release identifier and locked dependencies

Software release: Zenodo record 24546263, version v7, DOI 10.5281/zenodo.24546263. The record and DOI are author-supplied and could not be independently resolved during this revision. The downloadable archive should have the following SHA-256 (lowercase hexadecimal):

```
f1daba8be645b9d93698a3222b7a5e80ef6fb9f82a06b78846a9247e1e3993da
```

The manuscript is v8; the Lean source remains v7. The pins are:

Lean	`leanprover/lean4:v4.37.0-rc1`
Mathlib	`31ce06eaadf4a200c0e1407c5f10953cf606b8f8`
PNT+	`e9906138410f8f76ca66b9bc252d0993f3ee348c`

The archived `lake-manifest.json` additionally pins the transitive Lake packages. Its SHA-256 is

`8a82a39fc967e010492e3851717a454f491e4f9a9b82e7489c2090a6ae489e506`

7.3 Independent rebuild protocol

Download the v7 ZIP and verify its fingerprint before extraction. Install elan, Python 3 and Git. Use a clean working directory, the archived toolchain, and the locked Lake manifest. Resolve the pinned dependencies and obtain the Mathlib cache, without intentionally updating revisions. The archived workflow invokes `lake update`; this may alter the manifest. Do not treat a run against changed pins as a verification of the exact archived dependency snapshot. Check the manifest digest before and after dependency resolution, and compare the Git HEADs of the resolved packages with the recorded revisions (including transitive packages). Stop on any discrepancy. Once the locked packages are available, run from the extracted directory:

```
lake exe cache get
lake env python verify_release.py
```

The archived verifier checks the toolchain, direct Mathlib pin and local source hashes; fetches the pinned PNT+ import closure and compares its 14 recorded hashes; compiles the 28 modules into a new temporary output directory; and checks required axiom reports against the permitted set. It does not by itself independently assert every transitive checkout HEAD or the unchanged manifest hash after a dependency update. A separate preflight helper in the manuscript source bundle addresses those checks; it does not replace the Lean kernel verification.

A successful clean run should produce `lean-check.log`, `axiom-audit.json` and `pnt-observed-hashes.json`, together with the terminal success line for all 28 modules. Preserve these files, the unchanged manifest, the toolchain version, the operating system and the archive fingerprint with the verification report. An optional subsequent audit of the named closed-disk theorem should be performed in the same environment. A nonzero process exit, missing required report, hash mismatch or unexpected axiom must be investigated before claiming a successful rebuild.

8 Conclusion

The proof separates the analytic logarithm inside the zero-free disk from the final interpolation on the critical circle. It yields the classical conditional Lindelöf estimate by explicit inequalities and reusable formal interfaces. The principal named Lean result has only RH as its mathematical premise, according to the exact source statement and reported local audit. The archived project provides a basis for independent reproduction, not a substitute for it. Any future combination with an external proof of RH would require a separate compatibility check, exact statement comparison, and combined axiom audit. No unconditional resolution or claim of priority is made here.

Declarations

Software availability. The original Lean source is linked through Zenodo record 24546263 (DOI 10.5281/zenodo.24546263); see Section 7 for exact version and verification status. Com-

panion manuscript source and optional audit helpers are separate from this release. Upstream contributions retain their original attribution and applicable licenses; the local source license is not specified in the v7 archive.

AI assistance. ChatGPT assisted with formalization development, debugging, source inspection and manuscript revision. The reported local Lean audit is distinguished from an independent rebuild throughout. The author is responsible for verifying the final content, affiliation, attribution and any publication declarations before release.

References

[1] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986, Chapter XIV.

[2] D. Loeffler and M. Stoll, Formalizing zeta and L-functions in Lean, *Annals of Formalized Mathematics* 1 (2025), doi:10.46298/afm.15328; arXiv:2503.00959.

[3] The Mathlib Community, Mathlib4, `https://github.com/leanprover-community/mathlib4`, pinned revision in Section 7.

[4] A. Kontorovich and contributors, PrimeNumberTheoremAnd (PNT+), `https://github.com/AlexKontorovich/PrimeNumberTheoremAnd`, pinned revision in Section 7.

[5] R. Zeraoulia, *From the Riemann Hypothesis to the Lindelöf Hypothesis: A Lean 4 Formalization via Two-Disk Interpolation*, software, v7.0 exact source snapshot, November 2026, Zenodo record 24546263, DOI 10.5281/zenodo.24546263 (author-supplied).

[6] Related preprint, Table 20, row 6 (Corollary 33), reporting the declarations `lindelof_of_H` and `lindelof_half`, Preprints.org, November 2026. Available manuscript extract; bibliographic identity and code not independently established. Included as a provisional related-work pointer rather than a priority claim.