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Both Liouville signs at a fixed even total with a prime or semiprime summand

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Abstract

Let $N = m+n$ be a prescribed large even integer and let m vary in a central interval through primes and products of two distinct primes greater than $x^{2/13}$. After revealing the full z-smooth part of n, every function of that observation predicts the Liouville sign of the remaining cofactor incorrectly on at least one quarter of the population, even with an arbitrary nonnegative weight decreasing under divisibility. Here z may be a sufficiently small fixed power of x. The quarter bound also holds separately on every admissible smooth part of polylogarithmic size. When both factors of the semiprime must exceed $x^{11/40}$, each sign still occupies a positive proportion. The proof uses distribution of the Liouville function in progressions, a linear lower sieve, and the unsigned density of the excluded factor shapes.

Write $\lambda(n) = (-1)^{\Omega(n)}$, where Ω counts prime factors with multiplicity, and let $n_z = \prod_{\ell \le z} \ell^{v_\ell(n)}$. Fix an even N, central summand interval $m \in I \subset [x, 2x]$ of length $Y \asymp x \asymp X$, positive $n = N - m \asymp X$, and $L = \log x$. Let $F(n_z)$ be nonnegative and decreasing in the divisibility order, that is $F(d) \le F(d')$ whenever $d' \mid d$, with n_z the numerical z-smooth part. (Every F decreasing in the size of n_z is of this kind; a divisibility-decreasing function of $n_{z'}$ with $z' < z$, viewed as a function of n_z, is decreasing in the divisibility order but in general not in size.) Let $s(n_z) \in \{-1, 1\}$ be arbitrary. The reference is $\mathbb{E}F = \mathbb{E}\, F(\prod_{\ell \le z} \ell^{v_\ell(N-U_\ell)})$ with independent U_ℓ uniform on $\mathbb{Z}_\ell^\times$. Hence $\Pr_{\mathrm{loc}}(n_z = d) = 0$ unless d is z-smooth and $(d, 2N) = 1$, and then it equals $d^{-1} \prod_{\ell \le z,\ \ell \nmid dN} (1 - 1/(\ell - 1))$. Write

$$\mathcal{P}_\beta = \{m \in I :\ m \text{ prime, or } m = qr \text{ with distinct primes } q, r > x^\beta\}.$$

Theorem 1. *There exists $\delta_0 > 0$ such that for all large X, every such N and $2 \le z \le X^{\delta_0}$,*

$$\sum_{m \in \mathcal{P}_{2/13}} F(n_z) \mathbf{1}_{\lambda(n/n_z) = s(n_z)} \ \ge\ .25 \sum_{m \in \mathcal{P}_{2/13}} F(n_z).$$

The same inequality holds for the opposite prediction. For $\beta = 11/40$ and hence also for $\beta = 1/(1+e)$, with e Euler's number, there is an absolute $c > 0$ such that both inequalities hold on $\mathcal{P}_\beta$ with c in place of .25. These statements hold also for the whole sign $\lambda(n)$, by multiplying the predictor by $\lambda(n_z)$. In particular choose $F = \mathbf{1}_{n_z = 1}$ at any fixed small polynomial cutoff. Then both whole signs occur in full-order counts $\gg Y\mathfrak{S}(N)/(L \log z)$, with m as in the respective assertion; n can be required squarefree.

Here $\mathfrak{S}(N) = 2\prod_{p>2}(1 - (p-1)^{-2}) \prod_{p \mid N,\, p>2} (p-1)/(p-2)$ is the binary singular series. Chen's theorem gives a prime and a two-almost-prime on each sufficiently large even total. Selberg's parity construction gives an obstruction to prescribing the sign of $N - m$ by sieve methods when m must be prime. For even N, J. Guo's archived preprint *The Möbius function at $N - p$* (2026, revised version) gives each sign of $\lambda(N - p)$ on a proportion $\frac{1}{2} + o(1)$ of primes $p < N$, conditional on a zero-free

rectangle for Dirichlet L-functions and on the arguments in Sections 4.2–4.7 and Appendix A of Lü–Wang–Wu (arXiv:2610.20242v1) as stated. The zero-free rectangle would follow from the Riemann hypothesis for Dirichlet L-functions, of which L. Alpöge announced a proof, formalised in Lean, on 20 November 2026 (announcement). The present argument uses neither input, and its predictors may observe the full z-smooth part with z a small power of x. We keep the primes and the indicated semiprimes together and distinguish the two signs of their partners. The sign and the weight can be chosen afresh at every N. See Chen, *Sci. Sinica* 16 (1973), 157–176, and Halberstam–Richert, *Sieve Methods*, p. 239.

1 The signed bound for a set of factor shapes

Fix $u > 2$. For m rough through $x^{1/u}$ write down the exponents $\log q/\log x$ of its prime factors with multiplicity. In the limit for j labelled factors these lie on $x_1 + \cdots + x_j = 1$, all $x_i > 1/u$. An unordered selection E of factor shapes is given by symmetric bounded Jordan-measurable sets there and a choice whether to include the point $j = 1$. For definiteness use finitely many affine inequalities with consistent treatment of their zero-measure boundaries, or continuous tests. For $j = 1$ the convention to include the point means including all primes on the central interval. Define

$$b(E) = \mathbf{1}_{\text{prime stratum included}} + \sum_{2\le j\le u} \frac{1}{j!}\int_{E_j} \frac{dx_1\cdots dx_{j-1}}{x_1\cdots x_j}, \qquad U(u) = u\omega(u),$$

where ω is Buchstab's function. Put f for the usual lower linear-sieve function with the standard e^γ normalization. For any fixed $\varepsilon > 0$, choosing $\delta_0 = \delta_0(u, E, \varepsilon) > 0$ sufficiently small, the following holds uniformly in N, F, s, z:

$$\sum_{m\in E} F(n_z) = (b(E) + O(\varepsilon))(Y/L)\mathbb{E}F, \tag{1}$$

$$\sum_{m\in E} F(n_z)\mathbf{1}_{\lambda(n/n_z)=s(n_z)} \ge \left(b(E) - U(u) + \tfrac{1}{2}e^{-\gamma}uf(u/2) - \varepsilon\right)(Y/L)\mathbb{E}F. \tag{2}$$

The constant in the error of (1) can be taken as one after decreasing δ_0. The signed guarantee applies to every allowed shape collection above the indicated unsigned capacity threshold. The ordinary lower sieve is at level just below $x^{1/2}$. The factor $1/2$ multiplying it is the density of the chosen Liouville sign on integer progressions.

Counting on each prescribed total. For (1) fix all but a largest prime factor r of $m = tr$. Every fixed factor exceeds $x^{1/u}$ and $r \gg x^{1/j}$. On primes $\ell \le z$ multiplication by t is invertible. Thus the full small profile of $N - tr$ for a prime r has the same unit law as $N - U$; the arithmetic progression roots are multiplied by t^{-1} at each such prime power. The relative estimate of Lemma 2 applies uniformly at scale $x/t \gg x^{1/u}/O(1)$, with relative error $O_u(e^{-c_u/\delta_0}) + O_A(L^{-A})$. Its small-atom and marked-tail proof uses just the number of roots, their lifts, and Bombieri–Vinogradov on the r variable. No distribution estimate simultaneously through large moduli involving t is needed.

For explicit counting, integrate first over compact exponent boxes in the first $j-1$ factors. If the eliminated exponent stays a fixed distance inside the permitted region and exceeds the others, the r interval cut out by $tr \in I$ is macroscopic at its scale. Its count with weight F is the prime-number-theorem count times $\mathbb{E}F$ with the preceding relative error. The product of harmonic prime sums for the first factors and the factor $1/\log(x/t)$ for the last give the displayed integral by the prime number theorem. Boxes touching a boundary are controlled by the same upper bound on the entire macroscopic interval for r. Their total real measure tends to zero as the exponent mesh shrinks.

A tie for the largest factor has zero measure for $j \ge 2$; choose a labelled sector and divide by the permutation multiplicity. A repeated factor costs $O((x^{1-1/u}+\sqrt{x})\sup_{\text{support}} F)$ globally; here the true observations stay on support since all source factors exceed z. This proves (1), including the prime case, and applies equally to the complement inside the full rough population. Its total integral is $U(u)$. Indeed if $B_j(1/u)$ denotes the full simplex integral for count j, differentiation at a smallest coordinate gives

$$(B_j(1/u))' = B_{j-1}(1/(u-1))/(u-1) \qquad (j \ge 2),$$

with zero when the simplex is empty. Here the derivative is in u, $B_1(1/u) = 1$ for $u > 1$, and the sum on $1 < u \le 2$ is one. This is precisely the defining delay equation for $u\omega(u)$.

Direct signed lower. Work at $z = X^{\delta_0}$, lifting smaller-cutoff predictors to this one. Keep atoms $d = n_z \le D = X^a$. Under the local law the valuations $v_\ell(N-U)$, $\ell \le z$, are independent; F is coordinatewise decreasing and $\mathbf{1}_{n_z > X^a}$ is coordinatewise increasing, so Harris's inequality gives $\mathbb{E}[F\mathbf{1}_{n_z>X^a}] \le \mathbb{E}F \cdot \Pr_{\text{loc}}(n_z > X^a)$, and Rankin's bound with exponent $1/\log z$ gives $\Pr_{\text{loc}}(n_z > X^a) \ll e^{-a/\delta_0}$. Hence the kept atoms carry local F-mass at least $(1-O(e^{-a/\delta_0}))\mathbb{E}F$. At each atom the base weight $(1+s(d)\lambda(d)\lambda(n))/2$ gives half the integer main on every progression, with arbitrary-log aggregate remainder below level $X^{1/2-b}$, by (3). Impose that exact atom and $P^-(m) > z$ by its small sieve, including the fixed lifts at d. The local factor on integers is $M_z \Pr_{\text{loc}}(n_z = d)$ with $M_z = \prod_{\ell \le z}(1-1/\ell)$. On primes from z through $x^{1/u}$ apply the lower linear sieve at level x^e, $e < 1/2$. Take a small compared with $1/2 - e$ to keep the combined level below $X^{1/2-b}$, for some $b > 0$. For small-range brackets $S^\pm$ and medium-range brackets $T^\pm$, use $S^-T^+ + S^+T^- - S^+T^+$ for the lower bound; the upper brackets are nonnegative and each brackets its exact indicator. The small-range relative gap costs a relative $O(e^{-ca/\delta_0})$ in the normalized medium main. The medium product is $(1+o(1))e^{-\gamma}u/(M_zL)$. We thus have the lower $(\frac{1}{2}e^{-\gamma}uf(eu) - o(1) - O(e^{-ca/\delta_0}))(Y/L)\mathbb{E}F$ on the *whole* rough population for the chosen sign. Choose e close enough to $1/2$, then a, then δ_0. Discarding the sign on the shapes outside E costs at most their actual unsigned mass $(U(u) - b(E) + o_{\delta_0 \downarrow 0}(1))(Y/L)\mathbb{E}F$, by (1). This proves (2). The argument also works when the sign is multiplied by any fixed observed sign at the atom.

2 Two numerical bounds

The elementary bound $\omega(u) < .5653$ holds for $u \ge 3$. On $[2,3]$ the exact expression $(1+\log(u-1))/u$ is at most .568 (differentiate), and $\omega(3) = (1+\log 2)/3 < .564383$. For $3 \le u \le 4$ use $u\omega(u) = 3\omega(3) + \int_3^u \omega(v-1)dv$. Its quotient by u is at most $(3 \cdot .564383 + .568)/4 < .5653$. The recursion preserves this upper bound on every subsequent unit interval. For $4 < u \le 8$, $f(u/2) = 4e^\gamma \log(u/2-1)/u$. Thus the threshold in (2) is at most

$$.5653u - 2\log(u/2-1).$$

At $u = 13/2$ this is < 2.053. The mass of $\mathcal{P}_{2/13}$ is $1 + \log(11/2) > 2.7047$, because the semiprime mass at cutoff β is $\log((1-\beta)/\beta)$. These elementary bounds alone give a fraction strictly larger than .24 before the tolerance.

For a checkable quarter bound, let $M = 8192$, $Q = 10^{12}$, put $B_j = Q$ for $M \le j \le 2M$ and for $j \ge 2M$ set

$$B_{j+1} = B_j + \left\lceil \frac{B_{j+1-M}}{j-M} \right\rceil.$$

Then $U(j/M) \le B_j/Q$ by the delay equation and monotonicity of U. A lower enclosure uses a floor with the previous numerator A_{j-M} and denominator $j+1-M$. The integer recurrence gives

$$B_{11M/2} = 3088145446039, \qquad B_{13M/2} = 3649640761354.$$

For $r = 9/4, 11/2, 7/4, 29/11$, the first 32 positive terms of $2\sum_{k\ge0} v^{2k+1}/(2k+1)$, $v = (r-1)/(r+1)$, give respective lower bounds $.81092, 1.70474, .55961, .9693$ for $\log r$. At $u = 13/2$ the threshold is therefore less than $3.64965 - 2(.81092) = 2.02781$, strictly less than three quarters of $1 + \log(11/2)$. The positive margin absorbs the sieve tolerance. At $u = 11/2$ the threshold is less than $3.08815 - 2(.55961) = 1.96893$. Primes together with semiprimes whose factors both exceed $x^{11/40}$ have mass $1 + \log(29/11) > 1.9693$. The choice $\beta = 1/(1+e)$ has mass exactly two. Removing large squares on the rough event costs $O(X/z + \sqrt{X})$ at fixed $z = X^\delta$, which proves the last assertion. The integer recurrence and the four rational sums above constitute a reproducible outward certificate. For comparison, Chen's theorem with n prime gives the negative sign, with semiprime factors above $N^{1/10}$ in Chen's proof and above $N^{1/8}$ in Nathanson's proof; Iwaniec's notes, as quoted by Green and Tao for $p+2$, give $p^{3/11}$. These do not reach $x^{2/13}$ in the printed Goldbach form, or $x^{11/40}$ at all, and give no positive sign. Two 2026 preprints announce Goldbach forms in which every prime factor of the almost-prime summand exceeds $N^{0.2665-\varepsilon}$ (M. Ratliff, Zenodo 20377091) or $(N/2)^{0.267}$ (Zenodo 24457310); they would give the negative sign on $\mathcal{P}_{2/13}$, but not on $\mathcal{P}_{11/40}$, and no positive sign.

3 Conditioning on the numerical small part

The quarter bound holds separately on every locally possible atom $n_z = d \le L^J$ for every fixed J. More precisely, there are fixed $a, \delta_0 > 0$ such that, on every prescribed even total and for all $2 \le z \le X^{\delta_0}$, the following holds. Among $d \le X^a$ the atoms failing the quarter bound have total local probability and total actual probability $O_A(L^{-A})$ for every fixed A. Actual probability is uniform over $m \in \mathcal{P}_{2/13}$. A positive-mass atom of zero actual size is included in the exceptional set. The same assertions with a smaller constant hold on $\mathcal{P}_{11/40}$.

Here are the per-atom error estimates behind this assertion. Write $P_d = \Pr_{\mathrm{loc}}(n_z = d)$, $V = Y/L$, $b = b(E)$ and $T = U(u) - \frac{1}{2}e^{-\gamma}uf(u/2)$. Let A_d and $S_{d,\pm}$ be the unsigned and either signed counts in the shape region E, divided by V. For every fixed tolerance $\eta > 0$ choose the exponent boxes and three levels in the preceding proof in terms of η. For all $d \le X^a$ there are nonnegative remainders R_d with

$$|A_d - bP_d| \le \eta P_d + R_d, \quad S_{d,\pm} \ge (b - T - \eta)P_d - R_d, \quad \sum_{d \le X^a} R_d \ll_A L^{-A}.$$

To see uniformity down to $z = 2$, sieve m to $w = X^{\delta_0}$ in the small range. On n specify the atom only through z, ignoring primes in $(z, w]$. The integer small main is now $M_w P_d$. At primes beyond the observed labels it imposes a single forbidden root on m. Every d is retained individually, so no large-atom truncation is needed for this assertion. The small and medium principal sieve gaps are bounded by ηP_d after normalization. Summing absolute progression errors over d costs a fixed divisor power as in (3). The two fixed sign choices cost a factor two. In the unsigned comparison fix the other prime factors at t, apply the atom sieve to the largest prime on its macroscopic interval, and take the sum of absolute progression errors over all d before summing t. The t cost is bounded by the product of reciprocal prime sums over $[x^{1/u}, 2x]$, a bounded quantity for fixed u. Choose a small enough that the small-modulus level is available even on the shortest such prime scale. The finitely many boundary boxes have combined principal upper bound ηP_d. Repeated factors over all

atoms together cost a power saving. These choices give the displayed estimates after subdividing the error allowance.

Take $E = \mathcal{P}_{2/13}$ and choose η strictly inside $b - T - b/4 > 0$. More precisely require $b - T - \eta > (b + \eta)/4$. If an atom fails then $R_d \gg P_d$, hence also $A_d \ll R_d$. Its two total probabilities have the stated bounds. At each admissible atom for even N one has $P_d \gg 1/(d \log(2z))$: every specified positive valuation v at $\ell \nmid N$ contributes exactly ℓ^{-v}, and the unhit allowed odd primes contribute $\prod(1 - 1/(\ell - 1)) \gg 1/\log(2z)$. This proves the individual assertion through L^J. In fact the total of $1/d$ over exceptional admissible $d \le X^a$ also saves every logarithmic power.

For $z = X^{o(1)}$, outside $o(1)$ of the actual F-weighted population each atom individually has both signs in fractions at least one quarter, uniformly over all decreasing F of positive mass. For $d \le X^a$ the exceptional mass costs $O_A(L^{-A} \sup F)$ and $\sup F = F(1) \le \mathbb{E}F/\Pr_{\mathrm{loc}}(n_z = 1) \ll \log(2z)\,\mathbb{E}F$ for decreasing F. For the rest use the weighted large-divisor tail on the largest-prime variable from the proof of Lemma 2: at each fixed t it tends to zero relative to $\mathbb{E}F$, uniformly on the polynomial scale x/t. Sum over t as above. A single specified atom d of polynomial size can fall in the exceptional set. Ordinary prime distribution in the classes modulo d times the lower-sieve moduli is the outstanding difficulty for such an individual atom.

4 One list of summands for prescribed totals

Fix k even totals N_i sharing the central interval, and write $n_i = N_i - m$. In Theorem 1 the weights and predictor may observe the joint numerical small parts $((n_i)_z)_{i \le k}$. The constant δ_0 may now depend on k. For each specified index i, the quarter bound holds for prediction of $\lambda(n_i/(n_i)_z)$, also on every possible joint atom whose least common multiple is at most L^J. The local law uses a single common unit at each prime for the entire tuple.

Indeed the unsigned box calculation fixes t as before. Multiplication by t permutes the common unit, hence preserves all collisions between observed roots. On the signed side use (3) on n_i, whose progressions specify the joint atom after translating each root. The small sieve has dimension at most $k + 1$, its forbidden unit roots at a prime are determined by the distinct N_j there, and its relative fundamental-lemma error can be made arbitrarily small. At a hit prime the residue class of maximal valuation is fixed up to at most k choices, with at most k prohibited lifts. The combined modulus and its divisor-power multiplicity are as in the joint-profile sieve of the proof of Lemma 2. Beyond the small sieve the only forbidden root is on m, so $b(E)$ and T stay the same. For the joint weighted tail use Lemma 2. On the support, $\sup F \ll_k (\log(2z))^k\,\mathbb{E}F$. Indeed, at $\ell > 2k + 2$ every local vector dominates zero, with local probability at least $1 - k/(\ell - 1)$. At a smaller ℓ fix a unit x realizing a vector (v_i) of finite valuations and choose $1 \le j \le k + 1$ different from all v_i. All units x' with $v_\ell(x' - x) = j$ give vectors coordinatewise at most (v_i) and have total unit probability ℓ^{-j}. Multiplication over the small primes and Mertens' theorem over the remaining ones give the claimed bound, including when some small prime always hits a coordinate. An admissible atom with least common multiple h has mass $\gg_k 1/(h(\log(2z))^{O(k)})$. At its hit primes the product of the exact class probabilities gives at least $1/h$ before excluding the final lifts. The latter and all unhit primes cost at most a product of $(1 - O(k/\ell))$ at large primes and a positive constant at the finitely many small ones. This proves the individual assertion using the total progression-error bound. No joint distribution theorem for several Liouville values enters this calculation.

For example, prescribe any k locally admissible totals, meaning that $\{0, N_1, \dots, N_k\}$ misses a residue class at every prime. On the common event that all n_i are rough through X^δ, with $0 < \delta \le \delta_0(k)$ fixed, take independent uniform samples of $m \in \mathcal{P}_{2/13}$. One can also require every n_i squarefree. The square loss $O_k(X^{1-\delta} + \sqrt{X})$ is negligible even relative to this event, whose mass is $\gg_k (Y/L)(\log X)^{-O(k)}$. Choose the sieve tolerances inside the strict quarter margin before

discarding squares. At any row each sign then has probability at least $1/4$. A list of any integer length $s > \log(2k)/\log(4/3)$ has positive probability of exhibiting both signs at every row, since the failure probability is at most $2k(3/4)^s$. Thus $O(\log(2k))$ common almost-prime summands suffice while all the partners of every chosen summand are polynomially rough and squarefree. Two independent choices differ at an expected number at least $3k/8$ of the rows. In particular some two differ at that many. The signs on different rows may be correlated.

There is also a uniform list when the roughness need only hold for the particular row being represented. Consider up to $O(X)$ even totals sharing one central interval. Sample m uniformly from $\mathcal{P}_{2/13}$ without auxiliary conditions. For each total and sign the probability of a squarefree partner rough through X^δ of that sign is $\gg 1/\log X$ by Theorem 1 and the uniform lower bound for the binary singular series. A union bound now supplies one list of $O((\log X)^2)$ such m covering both signs at every one of these totals. For partners with no roughness restriction the same argument uses $O(\log X)$ summands. On a macroscopic window of totals any list covering even one rough sign requires $\gg_\delta \log X$ summands, since each fixed m has at most $O_\delta(X/\log X)$ rough partners there by the integer upper sieve. This sampling argument follows Erdős's 1954 probabilistic additive complement of the primes. Radomskii's construction for prime complements has the following further consequence for rough partners.

Lower bound for a list covering polynomially rough partners. Radomskii's large-shift lemma (version 3, Lemma 3.1 and the end of the proof of Theorem 1.1) in fact applies here. Every positive difference in the missing block constructed in his proof has a prime divisor at most $\exp(x/(\log x)^5)$, $x = \frac{1}{3}\log X$. This is below X^δ for any fixed $\delta > 0$ and all large X. Thus any nonnegative list covering every even total in $[X/2, X]$ with even one X^δ-rough partner has size $\gg \log X \log\log X$. Indeed, at a sufficiently small constant times that size his block contains two consecutive integers, hence an even one. This conclusion concerns lists chosen separately at each scale, including arbitrary lists of our permitted source summands. The same proof applies on any prescribed interval of length comparable to X at height $O(X)$: take $x = \log(CX)/3$ for a fixed upper bound on the shift sizes, and use the free subinterval of length $\gg X/(1 + \#\text{list})$ there. The maximum translation $\exp(2x + o(x))$ is smaller than this free interval. The estimate follows from Radomskii's divisibility construction and uses Lemma 3.1 of his 2026 preprint as stated. Its lower bound uses no Liouville input.

5 A prescribed set of smaller semiprime factors

The semiprime selection may prescribe an arbitrary set $\mathcal{Q}_N$ of smaller primes in $[x^{1/u}, \sqrt{x}]$, even after seeing N, F and the predictor. In (1) replace its mass by

$$b_N(\mathcal{Q}_N) = \sum_{q \in \mathcal{Q}_N} \frac{1}{q(1 - (\log q)/L)}.$$

This follows directly from the uniform prime-r comparison at each q. A fixed-factor collar at $\sqrt{x}$ costs $O(1/L)$ in these units, even with arbitrary labels. All other shapes can be evaluated by subtracting this single selected set from the full rough capacity, so (2) is uniform in $\mathcal{Q}_N$. Normalize counts by $(Y/L)\mathbb{E}F$ when positive, and let p_s be the agreement mass from prime m. At $u = 11/2$ every selected set of semiprime columns contributes agreement mass at least

$$b_N(\mathcal{Q}_N) - .96893 - p_s - \varepsilon.$$

For instance, if the chosen sign is absent at all primes, any set of multipliers of harmonic mass at least .969 supplies a positive full-order mass of genuine prime pairs q, r with the chosen partner sign. The statement applies also when agreement at the primes is merely small.

For a numerical consequence take the full band $x^{2/11} < q < \sqrt{x}$ and suppose the agreement mass on genuine primes is $o(1)$ in these units. The harmonic mass of that band is $\log(9/2)+o(1)$. Let $\mathcal{Q}_N$ consist of the multipliers contributing fewer than $0.1(Y/L)\mathbb{E}F/[q(1-\log q/L)]$ agreements apiece. The last inequality gives $0.9b_N(\mathcal{Q}_N) \le .96893 + o(1) + \varepsilon$. Thus a complementary set of harmonic mass greater than .42 consists entirely of multipliers that individually meet the stated lower bound, on choosing the tolerance small. This is a conditional conclusion on each fixed total with a deficient genuine prime slot.

What an improved distribution level would give. If (3) held to every level X^{d-b} for a fixed $d \le 1$, the threshold in (2) would be $U(u) - \frac{1}{2}e^{-\gamma}uf(du)$. All other steps use ordinary prime distribution at the scale of the largest factor. At $d = 1$ and $2 < u \le 3$ this threshold is exactly one, since $U(u) = 1 + \log(u-1)$ and $e^{-\gamma}uf(u)/2 = \log(u-1)$. Thus an almost absent sign among prime inputs would force almost the entire opposite-source semiprime sector, on each chosen set of smaller-prime labels, to carry that sign. Precisely, its disagreement mass would be $\le p_s + \varepsilon$ in the preceding normalization. The cutoff on the observed partner is sufficiently small in terms of the accuracy. This is a conditional implication of full level for the reflected Liouville progression sums. It still allows a perfect correspondence between the source factor parity and the partner sign. Related asymptotic-sieve prime/semiprime alternatives under Elliott–Halberstam precede this observation; see Debouzy as discussed by Evans, §1.1.

One cannot use only the one-point progression and unsigned capacities here to force either sign separately on the prime-only population. For the unconditional, unfiltered signed test take a comparison sign on the m-variable equal to $\lambda(m)$ or its negative. It satisfies the very same zero-mean level estimates on all residue classes used for that test, by (3) and reflection of progressions. Its sign is constant on all genuine primes and opposite on all genuine semiprimes. This identifies the missing use of the actual dependence on $N-m$. It is not a counterexample for the true Liouville function on the partner. It is an instance of Selberg's parity example, in which the sets $\{n : \lambda(n) = \mp 1\}$ attain the linear-sieve functions F and f (Halberstam–Richert, *Sieve Methods*, p. 239).

A Distribution estimates used in the proof

Write $L = \log X$. For fixed $b > 0$, $A, C > 0$ and $C' > 0$ we use

$$\sum_{q \le X^{1/2-b}} \tau(q)^C \max_{a \bmod q} \max_{y \le C'X} \left| \sum_{n \le y,\ n \equiv a\ (q)} \lambda(n) \right| \ll_{b,C,A,C'} XL^{-A}. \tag{3}$$

For reduced classes apply the Bombieri–Vinogradov discrepancy theorem for multiplicative functions of the Granville–Shao class, which contains every 1-bounded completely multiplicative function, to λ. Its Dirichlet inverse is μ^2, and its restriction to primes is minus one. For λ the Siegel–Walfisz bound follows by applying the classical zero-free region and Siegel's theorem to $L(2s,\chi^2)/L(s,\chi)$. The implied constants are ineffective. For $\mu^2(n) = \sum_{d^2|n} \mu(d)$ it follows by counting in reduced classes after the square-divisor expansion (or from $L(s,\chi)/L(2s,\chi^2)$). For the principal average, removing the primes dividing q amounts to convolution with the indicator of the squarefree divisors of q. For $d \mid \operatorname{rad} q$ with $d \le \sqrt{y}$ apply the prime number theorem for λ at y/d. The larger d contribute reciprocal mass at most $y^{-1/4}\prod_{p|q}(1+p^{-1/2}) = y^{-1/4}X^{o(1)}$ at polynomial y. Sum the principal averages after division by $\varphi(q)$. For a nonreduced class write $g = (a,q)$. Complete multiplicativity

reduces the sum to a reduced class at scale X/g and modulus $q/g \le (X/g)^{1/2-b/2}$. Summing over g costs a fixed logarithmic power in the main bound. For the maximum in y use a grid with mesh equal to the ambient scale divided by a fixed large power of L, with the trivial progression bound between successive points. Divisor weights are absorbed by Cauchy–Schwarz using the trivial progression bound and mean divisor moments. This proves (3). The reduced-class input is Granville–Shao, Theorem 1.1(b).

Lemma 2. *Fix k, $a > 0$ sufficiently small, and $A > 0$. Let r be a prime in an interval of length comparable to its scale R, with $R \ge X^\eta$ for fixed $\eta > 0$. Let t be invertible at every prime through $z = R^{1/u}$, with u sufficiently large. Observe the valuations of the k numbers $N_i - tr$ through z. Their reference law uses a common Haar unit at each prime, independent between primes. For every nonnegative bounded coordinatewise decreasing H and $|G| \le H$, the difference of the actual and local expectations of G is*

$$\ll_{k,a,A,\eta} \left(e^{-c_{k,a}u} + (\log R)^{-A}\right)\mathbb{E}_{\mathrm{loc}}H.$$

The same estimate bounds the actual or local expectation of H restricted to the least common multiple of the observed smooth parts exceeding R^a.

Proof. Take $D = R^a$ with $4a < 1/3$. Let h be the least common multiple of the numerical smooth parts. On $h \le D$ specify the exact vector of valuations at all primes dividing h. At $p^e \parallel h$ the class of $tr \bmod p^e$ is one of at most k roots. It fixes the smaller valuations. In the fundamental-lemma sieve below, impose the at most k excluded final lifts modulo p^{e+1} as a local sieve condition at p. On each feasible class its density is at most $\min(k, p-1)/p$. This gives uniform relative error even when h has many prime factors. Outside h, sieve away every nonzero root $N_i \bmod p$ up to z with upper and lower fundamental-lemma weights of level D. The dimension is at most k by Mertens' theorem. Saturated classes have local and actual mass zero. Exact lift exclusions factor in the principal term, giving relative error $O_k(e^{-c_{k,a}u})$ per feasible atom. A final modulus divides a product hde' with $d \mid \operatorname{rad} h$, $e' \le D$, so is at most D^3. All choices giving that modulus number at most a fixed power of its divisor function, counting at most k roots at each prime. Multiplication by t^{-1} permutes the same unit classes. Ordinary Bombieri–Vinogradov for r in its interval therefore bounds the sum over atoms of the absolute normalized distribution errors by $O_{A,k}(\log R)^{-A}$, with a stronger log saving before absorbing divisor weights.

For a local vector v at a prime $p \le 2k+2$ choose a unit x producing it and an integer $1 \le j \le k+1$ distinct from every v_i. The units x' with $v_p(x'-x) = j$ have total probability p^{-j} and their vectors lie below v. For $p > 2k+2$ the zero vector has probability at least $1-k/(p-1)$. Thus the local supremum of a decreasing test, with other primes fixed, is at most a bounded multiple of its local mean. At the large primes this factor can be $(1-k/(p-1))^{-1}$. Iteration gives $\sup_{\mathrm{supp}} H \ll_k (\log(2z))^k \mathbb{E}_{\mathrm{loc}}H$. More generally, for any independent local factors $w_p \ge 1$,

$$\mathbb{E}_{\mathrm{loc}}[H \prod_p w_p] \le \mathbb{E}_{\mathrm{loc}}H \prod_p (1 + C_k \mathbb{E}_p(w_p - 1)).$$

Indeed integrate one prime at a time; integration preserves decrease in the remaining coordinates. For $w_p = p^{\max_i v_i/\log z}$ the product is bounded uniformly for large z. The local tail of the maximum is at most $k/\varphi(p^j)$ at level j, and $p^{1/\log z} - 1 \ll \log p/\log z$. This proves the reference tail bound by Rankin's inequality.

For the arithmetic tail put $u_j = 2^j u$, $z_j = R^{1/u_j}$, and write h_j for the least common multiple through z_j. Stop at the first $u_J \ge C_A \log\log R$. If necessary enlarge the initial cutoff to $R^{1/(C_A \log\log R)}$ and ignore the extra coordinates; the resulting terminal error can be made $O((\log R)^{-A})$. Let H_j be H with the coordinates above z_j set to zero. Then $H \le H_j$ and

$\mathbb{E}_{\mathrm{loc}} H_j \ll_k (u_j/u)^{O(k)} \mathbb{E}_{\mathrm{loc}} H$, by requiring the omitted prime band to have all its vectors zero. All primes in it exceed $2k+2$ for large R.

Write $s_j = 1/\log z_j$ and define a nonnegative multiplicative divisor weight by $c_j(p^e) = p^{es_j} - p^{(e-1)s_j}$. For $h_j > D$ there is a divisor in $(D/z_j, D]$ by stopping a product of prime factors at the last step below D. Hence $\sum_{d \mid h_j,\, d \le D} c_j(d) \ge e^{au_j - 1}$. On $h_{j+1} \le D < h_j$ use this mark together with $H_{j+1}\mathbf{1}_{h_{j+1} \le D}$. Prime powers in d below the smaller cutoff are read from the atom. For each additional $p^e \parallel d$ prescribe one of the at most k classes N_i mod p^e. The combined modulus is at most D^4. Its multiplicity is divisor-bounded, and $c_j(d) \le e^{au_J} \le (\log R)^{O_A(1)}$. The upper sieve on atoms just proved therefore bounds the marked arithmetic expectation by a constant times $\mathbb{E}_{\mathrm{loc}}(H_{j+1} h_j^{s_j})$ plus $O_{A,k}((\log R)^{-A} \sup_{\mathrm{supp}} H)$. Apply the one-prime bound to that local mean. The resulting j-th arithmetic tail is $\ll e^{-au_j}(u_{j+1}/u)^{O(k)} \mathbb{E}_{\mathrm{loc}} H$ up to the logarithmic remainder. At $h_J > D$ the unconditioned mark gives $\ll (e^{-au_J} + (\log R)^{-A}) \sup_{\mathrm{supp}} H$. The sum of all these bounds proves the lemma, using stronger logarithmic savings at the start. It also applies if finitely many extra factors of $\log R$ enter the progression multiplicities. □

For the signed integer sieve used in the main argument, specify the same bounded atom with its final lifts imposed inside the sieve, and require the variable m to be a unit up to z. At each prime the main probability is $(1 - 1/p)$ times the common-unit probability of the specified observation. The small sieve now has at most $k+1$ roots. Its moduli are at most D^3. The medium linear sieve adds a factor at most x^e, so (3) bounds the signed progression remainders after taking a sufficiently small. The main coefficient of the sign selector on every progression is one half. The same divisor-power count handles summation over all atoms, even when their signs have been chosen independently.