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THE ESPERET–HENDREY–LAGOUTTE–MARSELOO–NORIN–STEINER (2025) FLOW RECONFIGURATION PROBLEM: A POSITIVE ANSWER ($k = 40$)

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Abstract. Given an oriented finite multigraph G and an abelian group Γ, a *nowhere-zero Γ-flow* assigns each edge a value in $\Gamma \setminus \{0\}$ with equal incoming and outgoing sums at every vertex. The reconfiguration graph $F(G,\Gamma)$ connects two nowhere-zero Γ-flows when the support of their difference is a cycle; write $M_k(G) = F(G,\mathbb{Z}_k)$, and let $I_k(G)$ be the analogous graph on integer flows F with $1 \le |F(e)| \le k-1$ on every edge. Problem 1.2 of Esperet, Hendrey, Lagoutte, Marseloo, Norin, and Steiner (2025) asked whether a uniform integer $k \ge 5$ independent of G makes $I_k(G)$ connected for every finite bridgeless loopless multigraph G; previously only the graph-dependent bound $k = 36\,|E(G)|(|V(G)|+1)$ was known. We answer Problem 1.2 positively: $I_k(G)$ is connected for every finite bridgeless loopless multigraph G (and $M_k(G)$ is connected with loops allowed) whenever k is a positive multiple of 40, 48, or 66—in particular for $k = 40$. We also prove that $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G and every finite abelian group Γ that is not cyclic of prime-power order and whose order has a divisor d with $4 \le d \le |\Gamma|/10$. In particular, every abelian group of order 40 works, lowering the smallest order of a group (with $|\Gamma| \ge 3$) proved universal in Esperet et al. (2025) from $|\mathbb{F}_2^8| = 256$ to 40. Both results follow from a single master theorem for reconfiguration under a restricted set of steps in a quotient Γ/B, combined with a Markov-chain averaging lemma in direct products and a lifting lemma for proper-divisor steps in $\mathbb{Z}_k$.[1] These results are verified in Lean.[2]

1. Introduction

Imagine a network without single-edge bottlenecks (a bridgeless graph) carrying a non-zero current along every edge so that inflow equals outflow at every junction. Introduced by Tutte [Tu] in 1954 as the planar dual of proper map colouring, *nowhere-zero flows* occupy a central place in graph theory. Tutte's 5-flow conjecture—asking whether integer values in $\{-4,\dots,4\} \setminus \{0\}$ always suffice—has stood open for over seventy years in landmark problem collections including Bondy and Murty's *Unsolved Problems* [BM76, Appendix IV, Problem 50], Jensen and Toft's *Graph Coloring Problems* [JT95, Section 13.1], Zhang's monograph [Zh97], and the Open Problem Garden. Two foundational theorems govern flow *existence*: Tutte [Tu] proved that a graph admits a nowhere-zero integer k-flow if and only if it admits a nowhere-zero $\mathbb{Z}_k$-flow, and more generally that existence of a nowhere-zero Γ-flow depends only on the order $|\Gamma|$ of the finite abelian group Γ; Seymour [Se] proved that every bridgeless multigraph admits a nowhere-zero 6-flow, and hence a nowhere-zero Γ-flow whenever $|\Gamma| \ge 6$, which remains the best general bound.

Rather than asking only whether a valid solution exists, *combinatorial reconfiguration*—surveyed by van den Heuvel [vdH13] and Nishimura [Ni18]—asks whether any two valid solutions can be transformed into each other step by step through valid intermediate states, a question directly tied to whether local Markov-chain random samplers can explore the whole solution space (and, for planar graphs, dual to vertex recolouring). Uniting these two areas, Esperet, Hendrey, Lagoutte, Marseloo,

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[1] Expanded version, with further results [Go1]: https://hexagonmath.org/2611.01084

[2] Theorems 1.1–1.3 are proved in Lean 4 from Lean's three standard axioms alone, and checked with `lake comparator`: https://github.com/david44hello/paper_esperet-2025-flow-reconfiguration/tree/v1.00/lean

Norin, and Steiner [EHLMNS] initiated the systematic study of *nowhere-zero flow reconfiguration* in December 2025, and Esperet highlighted the topic as Problem 14 of the 2026 Barbados Graph Theory Workshop [Co26, pp. 5–6] (where the uniform integer question remained open for all $k \geq 6$). Given a finite multigraph G with a fixed reference orientation and a finite abelian group Γ, the reconfiguration graph $F(G, \Gamma)$ has the nowhere-zero Γ-flows of G as its vertices, with two flows adjacent when the support of their difference is a cycle (a connected 2-regular subgraph); in plain terms, one flow is obtained from the other by adding a constant flow around a single closed circuit while keeping every edge non-zero. We write $M_k(G) = F(G, \mathbb{Z}_k)$ for the modular flow reconfiguration graph and $I_k(G)$ for the analogous graph on integer flows F satisfying $1 \leq |F(e)| \leq k-1$ on every edge (denoted $\mathcal{F}(G, \mathbb{Z}_k)$ and $\mathcal{F}(G, k)$ in [EHLMNS]). The trivial cases $k = 2$ and $|\Gamma| \leq 2$ are always connected, because $F(G, \Gamma)$ then has at most one vertex and any two ± 1-valued integer flows differ on edge-disjoint cycles. For $k, |\Gamma| \geq 3$, however, Esperet et al. showed that reconfiguration behaves very differently from existence: connectivity depends on the group structure ($F(K_4, \mathbb{F}_2^2)$ is connected whereas $F(K_4, \mathbb{Z}_4)$ is disconnected [EHLMNS, Sections 2.4 and 4]), and every group of order 3, 4, or 5 fails on some bridgeless graph (three parallel edges for $\mathbb{Z}_3$, K_4 for $\mathbb{Z}_4$, $K_{3,3}$ for $\mathbb{F}_2^2$ [EHLMNS, Theorem 4.1], and an isolated $\mathbb{Z}_5$-flow on a 20-vertex graph [EHLMNS, Theorem 3.4], so $3 \leq k \leq 5$ fail as well). Moreover, integer flow reconfiguration appears markedly harder than modular reconfiguration.

At the close of their paper [EHLMNS, Section 8], Esperet et al. highlighted two main questions left open for general bridgeless multigraphs (subsequent work of Cranston et al. [CLSWX] proved $|\Gamma| \geq 5.3 \times 10^6$ for 4-edge-connected graphs, leaving the uniform all-bridgeless baseline unchanged):

(1) *Problem 1.2 of* [EHLMNS]*:* Does there exist a single integer $k \geq 5$ independent of G such that $I_k(G)$ is connected for every finite bridgeless loopless multigraph G? They proved only a graph-dependent bound $k = 36\,|E(G)|(|V(G)| + 1)$ [EHLMNS, Theorem 7.1].

(2) *Reducing the universal group order (*[EHLMNS, Problem 1.4 and Section 8]*):* For loopless bridgeless G, they proved connectivity for $\mathbb{F}_2^r$ ($r \geq 8$, smallest order $2^8 = 256$ [EHLMNS, Theorem 6.15]), for $\mathbb{Z}_p \times A$ with prime power $p > 56^8 \cdot 27 + 28 \approx 2.61 \times 10^{15}$ [EHLMNS, Corollary 6.20] (including cyclic prime powers), and for all $|\Gamma| \geq (56^8 \cdot 27 + 28)^{45} + 1 \approx 5.74 \times 10^{693}$ [EHLMNS, Theorem 6.23], asking in Section 8 whether 256 can be reduced.

Table 1 and Figure 1 compare the best prior bounds in [EHLMNS] with our main theorems.

TABLE 1. Main results of this paper compared with the best prior bounds in [EHLMNS] for all finite bridgeless multigraphs G (loopless for $I_k(G)$ and throughout [EHLMNS]).

Best prior result ([EHLMNS])	New result in this paper
Integer flows $I_k(G)$: connected for graph-dependent $k = 36\,|E(G)|(|V(G)| + 1)$ [EHLMNS]; $3 \leq k \leq 5$ fail [EHLMNS]. No uniform $k \geq 5$ was known (Problem 1.2 of [EHLMNS]).	*Uniform modulus for all graphs:* $I_k(G)$ is connected for every positive multiple k of 40, 48, or 66 (Theorem 1.1(b); $M_k(G)$ also connected via divisor steps, loops allowed). In particular, $k = 40$ resolves Problem 1.2 positively.
Abelian group flows $F(G, \Gamma)$: connected for $\mathbb{F}_2^r$ ($r \geq 8$, smallest order 256), $\mathbb{Z}_p \times A$ ($p > 2.61 \times 10^{15}$ prime power), products of ≥ 46 groups, and all $\lvert\Gamma\rvert \geq 5.74 \times 10^{693}$ [EHLMNS]; orders $3, 4, 5$ fail.	*Groups from order 40:* $F(G, \Gamma)$ is connected (loops allowed) for every non-cyclic-prime-power Γ with divisor $4 \leq d \leq \lvert\Gamma\rvert/10$ (Theorem 1.2; every group of order $40, 44, 48, 50$ works; cyclic prime powers $\leq 2.61 \times 10^{15}$ and orders lacking such d stay open).

Main results. We resolve Problem 1.2 of [EHLMNS] positively and sharply lower the smallest known universal group order in [EHLMNS, Section 8], in a single self-contained framework.

Theorem 1.1 (Integer and modular flows)**.** *Let k be a positive multiple of* 40*,* 48*, or* 66*.*

(a) *$M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed).*

(b) *$I_k(G)$ is connected for every finite bridgeless loopless multigraph G.*

In particular, $k = 40$ answers Problem 1.2 of [EHLMNS] *positively.*

Theorem 1.2 (Abelian group flows)**.** *Let Γ be a finite abelian group that is not cyclic of prime-power order, and suppose that $|\Gamma|$ has a divisor d with $4 \le d \le |\Gamma|/10$. Then $F(G,\Gamma)$ is connected for every finite bridgeless multigraph G (loops allowed). In particular, $F(G,\Gamma)$ is connected for every abelian group Γ of order* 40*,* 44*,* 48*, or* 50*.*

Both theorems are direct instances of one general engine. Let Γ be a finite abelian group, $B \le \Gamma$ a subgroup, and $A = \Gamma/B$, with quotient map $x \mapsto \overline{x}$. A *step system* is a subset $\Sigma \subseteq \Gamma \setminus \{0\}$ with $\Sigma = -\Sigma$; we write $\Sigma_B = \Sigma \cap B$ and $\Sigma_A = \{\overline{\lambda} : \lambda \in \Sigma\} \setminus \{0\}$. A *$\Sigma$-move* is an adjacency $f \to f + \lambda\chi_C$ in $F(G,\Gamma)$ with $\lambda \in \Sigma$ (where χ_C is the signed indicator of a directed cycle C, defined in Section 2), and two flows are *Σ-joined* if a sequence of Σ-moves connects them. Consider the following four conditions on (Γ, B, Σ):

(R1) For every set $X \subseteq A$ of at most 6 elements, any two elements of $A \setminus X$ are joined by a walk in $A \setminus X$ with steps in Σ_A.

(R2) For every set $X \subseteq B$ of at most 2 elements, any two elements of $B \setminus X$ are joined by a walk in $B \setminus X$ with steps in Σ_B.

(R3) Any two non-zero elements of Γ are joined by a walk in $\Gamma \setminus \{0\}$ with steps in Σ.

(AV) On every loopless 3-edge-connected multigraph G with $n \ge 2$ vertices and m edges, every nowhere-zero Γ-flow is Σ-joined to a nowhere-zero Γ-flow f whose A-coordinate $\overline{f}$ has at most $m/9$ zeros.

Theorem 1.3 (Master theorem for step systems)**.** *Let Γ be a finite abelian group with a subgroup B and quotient $A = \Gamma/B$ satisfying $|A| \ge 10$ and $|B| \ge 4$, and let Σ be a step system satisfying* (R1)*,* (R2)*,* (R3)*, and* (AV)*. Then on every finite bridgeless multigraph G (loops allowed), every nowhere-zero Γ-flow is Σ-joined to every Γ-flow whose A-coordinate is nowhere zero. In particular, any two nowhere-zero Γ-flows of G are Σ-joined, and $F(G,\Gamma)$ is connected.*

Overview of the proof. The extraction–contraction architecture follows [EHLMNS, Section 6], which finds a dense subgraph on the non-zero support of a flow and lifts moves from the contracted graph using the closure theory of Jaeger, Linial, Payan, and Tarsi [JLPT]. Four new ingredients make the method quantitative and bridge modular flows to integer flows (Figure 2):

(i) *Liftable divisor steps* (Section 2). Call $z \in \mathbb{Z}_k \setminus \{0\}$ *liftable* if every move $f \to f + z\chi_C$ in $M_k(G)$ connects the integer fibres of f and $f + z\chi_C$ in $I_k(G)$. Giving short self-contained proofs of Tutte's lifting theorem and connected fibres (cf. [EHLMNS, Theorem 2.3 and Lemma 5.5]), we show that $\pm q$ is liftable for every proper divisor q of k. Thus any step system Σ of divisor steps that connects $M_k(G)$ automatically connects $I_k(G)$, without resolving whether connectedness of $M_k(G)$ always implies connectedness of $I_k(G)$ (Problem 2.6 of [EHLMNS]).

(ii) *Truncated cubic frames and official cycles* (Section 3). A sparse-zero flow with at most $m/9$ zeros leaves more than $(4/3)n - 2$ non-zero edges, which guarantees a *good subgraph* H: either a cycle of length at most 5, or a subdivision of a loopless 3-edge-connected multigraph H_0 in which every edge becomes a *thread* of at most 3 edges. Expanding each block of H_0 to a truncated cubic 3-edge-connected graph Q_b and taking a well colouring of Q_b from [JLPT, Lemmas 3.2 and 3.3] yields a *frame*: a blue spanning tree partitioned into edge pairs $N_i = \{e_i, f_i\}$ with nested *official cycles* $C_1, \dots, C_s$.

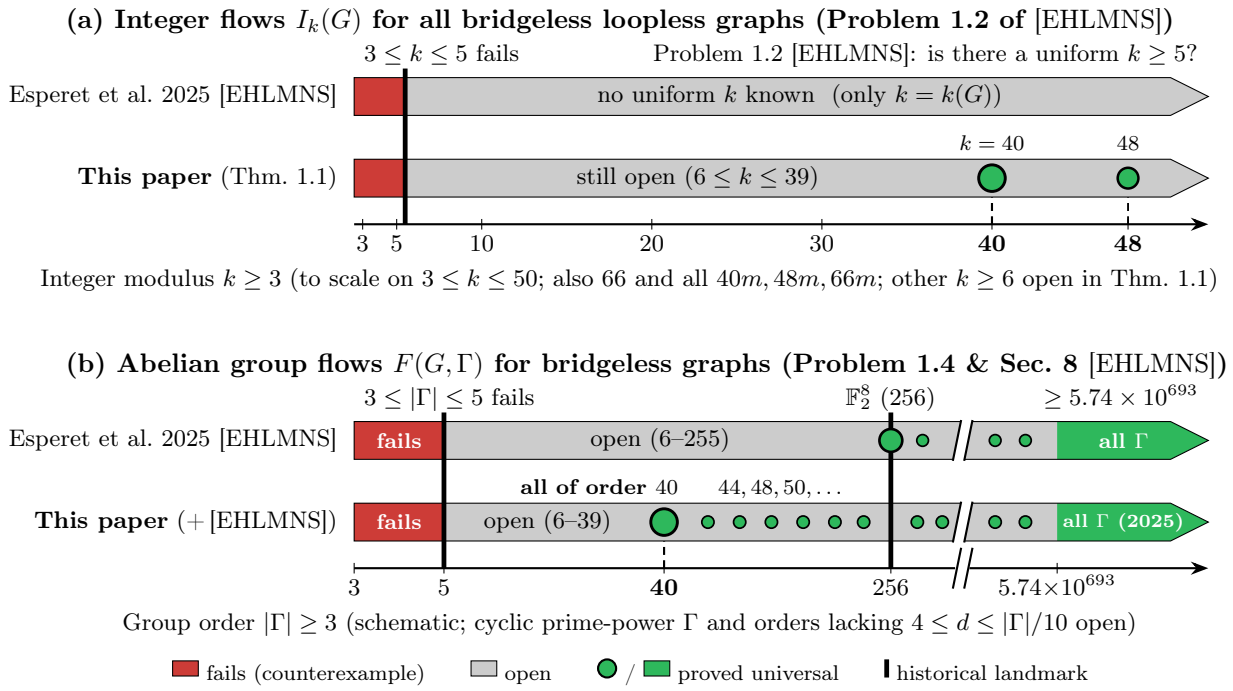


FIGURE 1. What is known about universal flow reconfiguration across all finite bridgeless multigraphs ($k \ge 3$ and $|\Gamma| \ge 3$; for $k = 2$ and $|\Gamma| \le 2$, $I_2(G)$ and $F(G,\Gamma)$ are always connected). In both panels, red marks values ruled out by counterexamples ($3 \le k \le 5$ and every group of order $3, 4, 5$: three parallel edges for $\mathbb{Z}_3$, K_4 for $\mathbb{Z}_4$, $K_{3,3}$ for $\mathbb{F}_2^2$, and a frozen $\mathbb{Z}_5$-flow [EHLMNS, Sections 2.4, 3, and 4]), grey marks open values, green dots and rays mark proved universal values, and thick black vertical bars mark historical benchmarks. (a) Integer flows $I_k(G)$ (loopless G, Problem 1.2 of [EHLMNS]): before this paper, no finite uniform $k \ge 5$ was known (only graph-dependent $k(G)$ [EHLMNS, Theorem 7.1]). Theorem 1.1 proves $k = 40$ and $k = 48$ (green dots), together with $k = 66$ and all positive multiples $40m, 48m, 66m$, pinning the least universal modulus $k \ge 3$ to $6 \le k_{\min} \le 40$; all other moduli $k \ge 6$ not divisible by 40, 48, or 66 remain open in Theorem 1.1. (b) Abelian group flows $F(G,\Gamma)$ (loops allowed in Theorem 1.2; loopless in [EHLMNS]; Problem 1.4 and Section 8 of [EHLMNS]): in [EHLMNS], the smallest universal group of order at least 3 was $\mathbb{F}_2^8$ of order 256 [EHLMNS, Theorem 6.15], alongside $\mathbb{F}_2^r$ ($r \ge 8$), products of at least 46 non-trivial groups, $\mathbb{Z}_p \times A$ for prime powers $p > 56^8 \cdot 27 + 28 \approx 2.61 \times 10^{15}$ [EHLMNS, Corollary 6.20] (including cyclic prime powers), and all groups above $(56^8 \cdot 27 + 28)^{45} + 1 \approx 5.74 \times 10^{693}$ [EHLMNS, Theorem 6.23] (right green ray). Theorem 1.2 lowers the smallest universal order (with $|\Gamma| \ge 3$) relative to [EHLMNS] to 40 (every abelian group of order 40 works) and covers every non-cyclic-prime-power group whose order has a divisor $4 \le d \le |\Gamma|/10$ ($40, 44, 48, 50, \ldots$, continuing with natural density 1, while cyclic prime-power groups up to 2.61×10^{15} remain open in Theorem 1.2).

(iii) *Quantitative avoidance and the threshold* $40 = 10 \times 4$ (Section 4). Because each thread has at most 3 edges, preserving non-zero A-values on H during a reverse-order walk along $C_1, \ldots, C_s$ forbids at most $2 \cdot 3 + 3 = 9$ values at a pair N_i (and deletes at most $3 + 3 = 6$ values of A), while preparing a free thread in B forbids at most 3 values (and deletes at most 2 values of B). This is precisely where $|A| \ge 10$ and $|B| \ge 4$—hence $40 = 10 \times 4$—arise.

(iv) *Product averaging* (Section 5). Whenever $\Gamma = \Gamma_1 \times \Gamma_2$ is a non-trivial direct product and Σ contains generators of both factors, alternating random coordinate updates on the support of the opposite coordinate define a doubly stochastic Markov chain on $\Gamma \setminus \{0\}$ whose cube has strictly positive entries. Hence every edge becomes nearly uniform on $\Gamma \setminus \{0\}$, yielding a flow with fewer than $m/9$ zeros in its A-coordinate whenever $(|B| - 1)/(|\Gamma| - 1) < 1/9$ (automatic for $|A| \geq 9$).

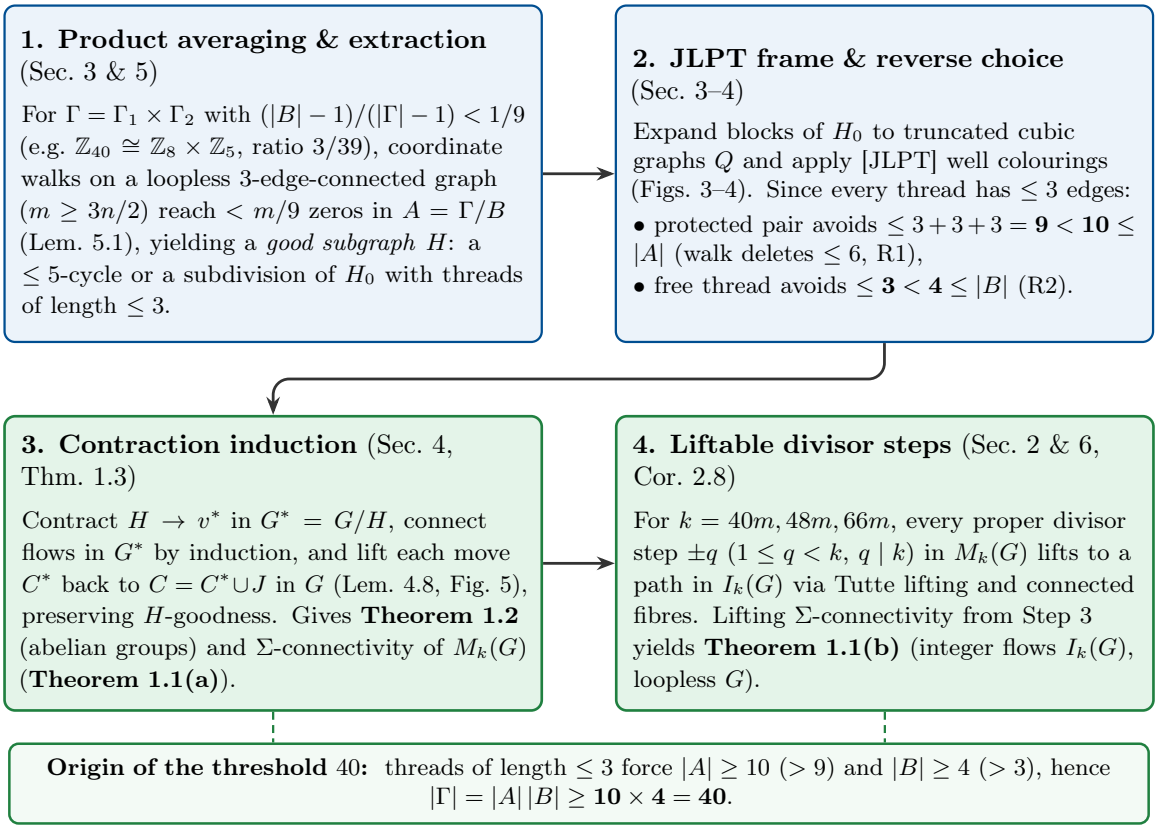


FIGURE 2. Logical architecture of the proof of Theorems 1.1 and 1.2 and the origin of the threshold $40 = 10 \times 4$. Steps 1–3 prove Theorem 1.3, yielding Theorem 1.2 and Σ-connectivity of $M_k(G)$; Step 4 lifts proper-divisor steps from $M_k(G)$ to $I_k(G)$. Only two published theorems are quoted: the well-colouring lemma of Jaeger, Linial, Payan, and Tarsi [JLPT] (in Step 2) and Seymour–Tutte (for a target A-flow in Step 3).

Organization. This focused version is self-contained; further moduli and abelian group bounds are treated in the expanded version [Go1] and companions [Go2, Go3], on which none of the proofs here depends.

Section 2 fixes conventions and proves the liftability of divisor steps. Section 3 develops good subgraphs, truncated cubic expansions, frames, and official cycles. Section 4 proves Theorem 1.3 and its integer consequence. Section 5 proves product averaging and Theorem 1.2. Section 6 proves Theorem 1.1 for all positive multiples of 40, 48, and 66. A brief Discussion section reviews open directions.

Apart from standard facts on finite abelian groups (Chinese remainder theorem, decomposition into cyclic groups of prime-power order, existence of subgroups of every dividing order) and graph blocks, our proofs quote only two published results: the 6-flow theorem of Seymour [Se] and Tutte [Tu]

(Theorem 1.4 below) and the well-colouring lemma of Jaeger, Linial, Payan, and Tarsi [JLPT, Lemmas 3.2 and 3.3] (used in Section 3).

Theorem 1.4 (Seymour [Se]; Tutte [Tu])**.** *Every finite bridgeless multigraph (loops allowed) has a nowhere-zero flow with values in any finite abelian group of order at least* 6*.*

2. Conventions and liftable divisor steps

Graphs, flows, and reconfiguration graphs. All graphs in this paper are finite multigraphs; loops are allowed unless a graph is explicitly called *loopless*. A *cycle* is a connected 2-regular subgraph (a loop has length 1; two parallel edges form a cycle of length 2). An edge lying on no cycle is a *bridge*, and a graph without bridges is *bridgeless* (an edgeless graph is bridgeless). A multigraph is *j-edge-connected* if every partition of its vertex set into two non-empty parts has at least j edges between them.

Fix a reference orientation of G, giving each edge a tail and a head (coinciding for a loop). For a map $x\colon E(G) \to \Gamma$ into an abelian group Γ (finite or infinite), its *boundary* at $v \in V(G)$ is

$$(\partial x)(v) \;=\; \sum_{e\colon \mathrm{head}(e)=v} x(e) \;-\; \sum_{e\colon \mathrm{tail}(e)=v} x(e),$$

where loops cancel. The map x is a *Γ-flow* if $\partial x = 0$, and is *nowhere zero* if $x(e) \neq 0$ for all $e \in E(G)$. Its *support* is $\mathrm{supp}(x) = \{e \in E(G) : x(e) \neq 0\}$. For a directed cycle C, its signed indicator $\chi_C\colon E(G) \to \{-1, 0, 1\}$ takes value $+1$ on edges traversed forward (tail to head), -1 on edges traversed backward, and 0 off C; then $\lambda\chi_C$ is a Γ-flow for every $\lambda \in \Gamma$. Reversing the reference orientation of an edge e and negating all flow values on e preserves flows and adjacency; we change reference orientations freely inside proofs.

For a finite abelian group Γ, the vertices of $F(G, \Gamma)$ are the nowhere-zero Γ-flows of G; two flows are adjacent (a *move*) if their difference is $\lambda\chi_C$ for some cycle C and $\lambda \in \Gamma \setminus \{0\}$, equivalently if the support of their difference is a cycle. We write $M_k(G) = F(G, \mathbb{Z}_k)$. For $k \geq 2$, $I_k(G)$ is the graph on integer flows F of G satisfying $1 \leq |F(e)| \leq k-1$ for every edge e, with two flows adjacent when their difference is $\lambda\chi_C$ for a cycle C and an integer $\lambda \neq 0$. (In Theorem 1.1(b) we state $I_k(G)$ for loopless multigraphs to match [EHLMNS, Problem 1.2], though our lifting lemmas below hold for general multigraphs.) An *integer lift* of a nowhere-zero $\mathbb{Z}_k$-flow f is a vertex $F \in V(I_k(G))$ with $F \equiv f \pmod k$. An edgeless graph has a single empty flow, and an empty reconfiguration graph is regarded as connected.

Cycle decomposition, lifting, and connected fibres. We record the standard cycle decomposition and short self-contained proofs of Tutte's lifting theorem [EHLMNS, Theorem 2.3] and fibre connectivity [EHLMNS, Lemma 5.5] in the exact form needed for divisor moves.

Lemma 2.1 (Cycle decomposition)**.** *Let G be a multigraph.*

(a) *Let A be an abelian group (finite or infinite) and T_0 a spanning forest of G. For each $e \notin E(T_0)$, let C_e be the fundamental cycle of e in T_0, directed forward along e. Then every A-flow h of G has the unique representation $h = \sum_{e \notin E(T_0)} h(e)\chi_{C_e}$. In particular, every A-flow is a sum of cycle flows $x\chi_C$.*

(b) *(Signed cycle decomposition.) Every integer flow h of G is a sum $h = \chi_{O_1} + \cdots + \chi_{O_j}$ $(j \geq 0)$ of directed cycles $O_1, \ldots, O_j$ of G (repetitions allowed) such that $\chi_{O_i}(e)\, h(e) > 0$ for every i and every $e \in E(O_i)$. Consequently, every edge e lies on exactly $|h(e)|$ of the cycles, all cycles lie in $\mathrm{supp}(h)$, and if $h(e) \in \{-1, 0, 1\}$ for all e then $O_1, \ldots, O_j$ are pairwise edge-disjoint.*

Proof. (a) Set $h' = h - \sum_{e \notin E(T_0)} h(e)\chi_{C_e}$. Since C_e contains e with $\chi_{C_e}(e) = 1$ and no other edge outside T_0 (loops included), h' vanishes outside T_0. If $h' \neq 0$, then $\mathrm{supp}(h') \subseteq E(T_0)$ is a non-empty

forest and hence has a leaf, violating $\partial h' = 0$. Uniqueness holds because $\sum_{e \notin E(T_0)} x_e \chi_{C_e}$ evaluates to x_e on $e \notin E(T_0)$.

(b) We induct on $\sum_e |h(e)|$, the case 0 being $j = 0$. Reorient every edge with $h(e) < 0$ so that $h \geq 0$. If some loop has positive value, take O to be that loop directed forward; otherwise conservation ensures that every vertex entered by a positive edge is also left by one, so following positive edges yields a directed cycle O passing all its edges forward. Then $h - \chi_O \geq 0$ has smaller ℓ_1-norm and induction applies. Because all non-zero terms $\chi_{O_i}(e)$ share the sign of $h(e)$ and sum to $h(e)$, the remaining assertions follow immediately. □

Lemma 2.2 (Rounding)**.** *Let $x\colon E(G) \to [0,1]$ have integer boundary $(\partial x)(v) \in \mathbb{Z}$ at every vertex $v \in V(G)$. Then there exists $z\colon E(G) \to \{0,1\}$ with $\partial z = \partial x$ and $z(e) = x(e)$ whenever $x(e) \in \{0,1\}$.*

Proof. Call e *fractional* if $0 < x(e) < 1$. If no edge is fractional, take $z = x$. Otherwise the fractional edges contain a cycle C: a fractional loop is already a cycle, and if no loop is fractional then no vertex meets exactly one fractional edge (since $\partial x(v) \in \mathbb{Z}$ and all non-fractional values are integers), so the fractional subgraph has no vertex of degree 1 and hence contains a cycle C. Direct C and let $\delta > 0$ be minimal such that $x + \delta\chi_C$ hits 0 or 1 on some edge of C. Replacing x by $x + \delta\chi_C$ preserves $[0,1]$-values, the boundary ∂x (since $\partial\chi_C = 0$), and all $\{0,1\}$-edges while strictly decreasing the number of fractional edges. Repeating yields z. □

Corollary 2.3 (Integer lifts)**.** *Let $k \geq 2$ and let f be a $\mathbb{Z}_k$-flow of G (zeros allowed). Then there exists an integer flow F of G with $F \equiv f \pmod k$, $|F(e)| \leq k - 1$ for all e, and $F(e) = 0$ if and only if $f(e) = 0$. In particular, every nowhere-zero $\mathbb{Z}_k$-flow has an integer lift.*

Proof. Let $r(e) \in \{0, \ldots, k-1\}$ represent $f(e)$. Since $\partial r \equiv \partial f = 0 \pmod k$, the map $x = r/k\colon E(G) \to [0,1)$ has integer boundary. Lemma 2.2 supplies $z\colon E(G) \to \{0,1\}$ with $\partial z = \partial x$ and $z(e) = 0$ whenever $r(e) = 0$. Setting $F = r - kz$ gives $\partial F = 0$, $F \equiv f \pmod k$, $F(e) = 0$ when $r(e) = 0$, and $F(e) \in \{r(e), r(e) - k\} \subseteq [-(k-1), -1] \cup [1, k-1]$ when $r(e) \neq 0$. □

Lemma 2.4 (Connected fibres)**.** *Let $k \geq 2$ and let f be a nowhere-zero $\mathbb{Z}_k$-flow of G. Then the integer lifts of f induce a connected subgraph of $I_k(G)$.*

Proof. Let F, F' be integer lifts of f. On each edge e with residue $r(e) \in \{1, \ldots, k-1\}$, both $F(e)$ and $F'(e)$ lie in $\{r(e), r(e) - k\}$, so $h = (F' - F)/k$ is a $\{-1, 0, 1\}$-valued integer flow. By Lemma 2.1(b), $h = \chi_{O_1} + \cdots + \chi_{O_j}$ for pairwise edge-disjoint directed cycles. Setting $F_i = F + k(\chi_{O_1} + \cdots + \chi_{O_i})$ gives a path in $I_k(G)$ from F to F' through lifts of f, since each edge value of F_i equals $F(e)$ or $F'(e)$. □

Liftable divisor steps.

Definition 2.5 (Liftable step)**.** For $k \geq 3$, an element $z \in \mathbb{Z}_k \setminus \{0\}$ is *liftable* if for every finite loopless multigraph G and every move $f \to f + z\chi_C$ of $M_k(G)$, all integer lifts of f and all integer lifts of $f + z\chi_C$ lie in the same component of $I_k(G)$.

Lemma 2.6 (Step of size 1)**.** *Let $k \geq 2$, let C be a directed cycle of G, and let f and $f + \chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then for every integer lift F of f, the flow $F + \chi_C$ is an integer lift of $f + \chi_C$.*

Proof. Reorient the edges of C so that C passes all its edges forward. On any $e \in E(C)$, the residue $r(e) \in \{1, \ldots, k-1\}$ of $f(e)$ satisfies $r(e) \neq k-1$ (as $f(e)+1 \not\equiv 0 \pmod k$), so $F(e) \in \{r(e), r(e)-k\}$ implies $F(e) + 1 \in [2, k-1] \cup [2-k, -1]$. Outside C nothing changes. □

Lemma 2.7 (Divisor move)**.** *Let $k \geq 3$ and let q be a proper divisor of k ($1 \leq q < k$). Let C be a directed cycle of G, and let f and $f + q\chi_C$ be nowhere-zero $\mathbb{Z}_k$-flows of G. Then there is a path in $I_k(G)$ from an integer lift of f to an integer lift of $f + q\chi_C$.*

Proof. For $q = 1$ this follows from Corollary 2.3 and Lemma 2.6. Suppose $q \geq 2$ and write $k = mq$ with $m \geq 2$. Let w be the reduction of f modulo q, and put $E_0 = \{e \in E(G) : w(e) = 0\}$. By Corollary 2.3 there is an integer flow W of G with $W \equiv w \pmod q$, $|W(e)| \leq q-1$ for all e, and $W(e) = 0$ precisely on E_0. Reorient every edge with $W(e) < 0$ so that $1 \leq W(e) \leq q-1$ on $E(G) \setminus E_0$ and $W(e) = 0$ on E_0; read f and χ_C in this orientation. Call a nowhere-zero $\mathbb{Z}_k$-flow h of G a *state* if $h \equiv w \pmod q$; both f and $f + q\chi_C$ are states.

Claim 1 (*Standard lift of a state*). Every state h admits an integer flow L with $|L(e)| \leq m-1$ for all e such that $F = W + qL$ is an integer lift of h, with $L(e) \neq 0$ and $h(e) \equiv qL(e) \pmod k$ for all $e \in E_0$.

Proof of Claim 1. Since $h - W \equiv 0 \pmod q$, the edgewise quotient $y = (h-W)/q$ is a well-defined $\mathbb{Z}_m$-flow. Corollary 2.3 yields an integer flow L with $L \equiv y \pmod m$, $|L(e)| \leq m-1$, and $L(e) = 0 \iff y(e) = 0$. Put $F = W + qL$; then $\partial F = 0$ and $F \equiv h \pmod k$. If $e \notin E_0$, then $F(e) \equiv W(e) \not\equiv 0 \pmod q$ and $1 - q(m-1) \leq F(e) \leq (q-1) + q(m-1) = k-1$. If $e \in E_0$, then $W(e) = 0$ and $h(e) \neq 0$ force $y(e) \neq 0$, $L(e) \neq 0$, and $1 \leq |F(e)| = q|L(e)| \leq q(m-1) < k$. △

Claim 2 (*One step of size q*). Let O be a directed cycle of G passing every edge of $E(O) \setminus E_0$ forward, and suppose h and $h - q\chi_O$ are states. Then some integer lift of h is adjacent in $I_k(G)$ to some integer lift of $h - q\chi_O$.

Proof of Claim 2. Let L and $F = W + qL$ be given by Claim 1 for h. We verify that $F - q\chi_O \in V(I_k(G))$. It is an integer flow congruent to $h - q\chi_O \pmod k$ and agrees with F off O. On $e \in E(O) \setminus E_0$, we have $\chi_O(e) = 1$, so $F(e) - q = W(e) + q(L(e)-1)$ is congruent to $W(e) \not\equiv 0 \pmod q$ and lies in $[1 - qm,\, (q-1) + q(m-2)] = [1-k,\, k-q-1]$. On $e \in E(O) \cap E_0$, if O traverses e backward we reverse the reference orientation of e and simultaneously replace $h(e)$, $L(e)$, and $F(e)$ by their negatives (since $W(e) = w(e) = 0$, the identity $F = W + qL$, the state condition, and $|L(e)| \leq m-1$ are preserved); now $\chi_O(e) = 1$. By Claim 1, $h(e) \equiv qi \pmod k$ with $1 \leq i \leq m-1$ and $L(e) \in \{i, i-m\}$. Since $(h - q\chi_O)(e) \equiv q(i-1) \not\equiv 0 \pmod k$, we have $i \geq 2$. Thus $F(e) - q = q(L(e)-1) \in \{q(i-1),\, q(i-m-1)\}$ with $1 \leq i-1 \leq m-2$ and $1-m \leq i-m-1 \leq -2$, so $1 \leq |F(e) - q| \leq q(m-1) < k$. Hence $F - q\chi_O$ is an integer lift of $h - q\chi_O$ adjacent to F in $I_k(G)$. △

To finish the proof of Lemma 2.7, we decompose the single step $+q\chi_C$ modulo $k = mq$ into cycles passing every edge outside E_0 forward (matching Claim 2) while hitting each edge of $E_0 \cap E(C)$ only once: consider the integer flow $mW - \chi_C$. On every $e \notin E_0$ we have $mW(e) - \chi_C(e) \geq m - 1 \geq 1$, while on $e \in E_0$ we have $(mW - \chi_C)(e) = -\chi_C(e) \in \{-1, 0, 1\}$. By Lemma 2.1(b), $mW - \chi_C = \chi_{O_1} + \cdots + \chi_{O_j}$ for directed cycles $O_1, \ldots, O_j$ passing every edge in the direction of the sign of $mW - \chi_C$. In particular, every O_i passes its edges outside E_0 forward; edges of $E_0 \setminus E(C)$ lie on no O_i; and each $e \in E_0 \cap E(C)$ lies on a unique O_i, with $\chi_{O_i}(e) = -\chi_C(e)$.

Define $h_0 = f$ and $h_i = h_{i-1} - q\chi_{O_i}$ for $1 \leq i \leq j$. Then $h_j = f - q(mW - \chi_C) \equiv f + q\chi_C \pmod k$. Each h_i is a state: its reduction modulo q is w (so it is non-zero outside E_0), on $E_0 \setminus E(C)$ it equals f, and on $e \in E_0 \cap E(C)$ it equals either $f(e)$ or $f(e) + q\chi_C(e)$, both non-zero. By Claim 2, for each i some integer lift of h_{i-1} is adjacent to some integer lift of h_i, and by Lemma 2.4 all lifts of each h_i lie in one component of $I_k(G)$. Concatenating yields a path in $I_k(G)$ from a lift of f to a lift of $f + q\chi_C$. □

Corollary 2.8 (Liftable divisor steps)**.** *Let $k \geq 3$.*

(a) *Every element $\pm q$ for a proper divisor q of k $(1 \leq q < k)$ is liftable.*

(b) *If $\Sigma \subseteq \mathbb{Z}_k \setminus \{0\}$ is a step system consisting solely of liftable elements and any two nowhere-zero $\mathbb{Z}_k$-flows of a finite loopless multigraph G are Σ-joined, then $I_k(G)$ is connected.*

Proof. (a) Reversing the direction of C turns $f \to f - q\chi_C$ into $f \to f + q\chi_{C'}$, so Lemma 2.7 combined with Corollary 2.3 and Lemma 2.4 shows that $\pm q$ is liftable.

(b) Let $F, F' \in V(I_k(G))$ have reductions f, f' modulo k, which are nowhere-zero $\mathbb{Z}_k$-flows. By hypothesis there is a chain of Σ-moves $f = f_0, f_1, \ldots, f_\nu = f'$. By Corollary 2.3 each f_r has an integer lift; by Definition 2.5 all lifts of f_{r-1} and f_r lie in one component of $I_k(G)$ (and for $\nu = 0$, Lemma 2.4 joins F to F'). Hence F and F' lie in the same component of $I_k(G)$. □

3. Good subgraphs, truncated cubic frames, and official cycles

Good subgraphs and extraction.

Definition 3.1 (Good subgraph)**.** A *good subgraph* of a multigraph is a subgraph H of one of two kinds:

(CYC) a cycle with at most 5 edges and at least 2 vertices;

(SUB) a subdivision of a loopless 3-edge-connected multigraph H_0 on at least two vertices, in which the edges of H_0 are replaced by pairwise internally disjoint paths of at most 3 edges.

The following extraction lemma is [EHLMNS, Lemma 6.1]; we include a short self-contained proof.

Lemma 3.2 (Extraction)**.** *Every loopless multigraph G with $n \geq 3$ vertices and more than $(4/3)n-2$ edges contains a good subgraph.*

Proof. Call a subgraph of G *dense* if it has $n' \geq 3$ vertices and more than $(4/3)n' - 2$ edges. If G has a cycle of at most 5 edges, that cycle has at least 2 vertices (as G is loopless) and is a good subgraph of type (CYC). Assume now that G has girth at least 6 (in particular G is simple), and let H be a dense subgraph of G with the fewest vertices; write $n_H = |V(H)|$ and $m_H = |E(H)|$. Thus $3m_H \geq 4n_H - 5$.

Step 1: $n_H \geq 6$, $m_H > n_H$, and $\delta(H) \geq 2$. Since $m_H > (4/3)n_H - 2 \geq n_H - 1$, H contains a cycle, so $n_H \geq 6$ and $m_H > (4/3)n_H - 2 \geq n_H$. For any vertex $v \in V(H)$, $H - v$ has $n_H - 1 \geq 5 \geq 3$ vertices and is not dense, so $3|E(H-v)| \leq 4(n_H - 1) - 6$, giving $\deg_H(v) = m_H - |E(H-v)| \geq 5/3$, hence $\deg_H(v) \geq 2$.

Step 2: H is connected. If $H = H_1 \sqcup H_2$, each H_i has $\delta(H_i) \geq 2$, hence a cycle and $n_i \geq 6$ vertices; neither is dense, so $m_H = m_1 + m_2 \leq (4/3)(n_1 + n_2) - 4 < (4/3)n_H - 2$, a contradiction.

Step 3: Small edge-cuts. Suppose $V(H) = V_1 \sqcup V_2$ with non-empty parts and at most 2 edges between them; let $H_i = H[V_i]$ have n_i vertices and m_i edges. If both satisfied $m_i \leq (4/3)n_i - 2$, then $m_H \leq (4/3)n_H - 4 + 2$, contradicting density. Thus some H_i satisfies $m_i > (4/3)n_i - 2$. Since $n_i < n_H$, H_i is not dense, forcing $n_i \leq 2$. As H is simple, either $(n_i, m_i) = (1, 0)$ or $(2, 1)$. Because $\delta(H) \geq 2$ and at most 2 edges leave V_i, exactly 2 edges cross the cut and every vertex of V_i has degree 2 in H.

Step 4: Threads and 3-edge-connectivity of H_0. Let $V_0 = \{v \in V(H) : \deg_H(v) \geq 3\}$. Since H is connected with $\delta(H) \geq 2$ and $m_H > n_H$, $V_0 \neq \emptyset$. Decompose H into maximal paths and cycles whose endpoints lie in V_0 and whose interior vertices all have degree 2 (*threads*). If a thread has non-empty interior I, exactly two edges connect I to $V(H) \setminus I \supseteq V_0$, so Step 3 gives $|I| \leq 2$ and the thread has at most 3 edges. No thread can be a cycle (which would have length at most 3, contradicting girth ≥ 6), so every thread is a path with distinct ends in V_0, giving $|V_0| \geq 2$. Replacing each thread by a single edge yields a loopless multigraph H_0 on V_0 of which H is a subdivision with paths of length at most 3. Finally, if $V_0 = V_0' \sqcup V_0''$ had at most 2 edges of H_0 between the parts, assigning each thread's interior vertices to the side of its first endpoint would partition $V(H)$ into two parts, each containing a vertex of degree at least 3, with at most 2 crossing edges—contradicting Step 3. Hence H_0 is 3-edge-connected, so H is of type (SUB). □

Truncated cubic reduction, threads, and flow projection. To apply the well-colouring theorem of [JLPT] (Theorem 3.8 below), we expand each block of H_0 into a truncated cubic 3-edge-connected graph.

Definition 3.3 (Truncated cubic graph)**.** A *truncated cubic* 3*-edge-connected graph* is a graph $Q = Q^+ - v$, where Q^+ is a cubic 3-edge-connected graph on at least four vertices and $v \in V(Q^+)$.

Recall that a *block* of a graph is a maximal connected subgraph without a cut vertex; every edge and every cycle lies in a unique block.

Lemma 3.4 (Truncated cubic reduction)**.** (a) *Let H_0 be a loopless* 3*-edge-connected multigraph on at least two vertices. Every block K of H_0 is loopless and* 3*-edge-connected, has at least two vertices, and has no cut vertex.*

(b) *For every loopless* 3*-edge-connected multigraph K on at least two vertices without a cut vertex, there is a truncated cubic* 3*-edge-connected graph Q_K containing $E(K)$ whose remaining edges $E(Q_K) \setminus E(K)$ form pairwise disjoint cycles, such that contracting each of these cycles to a vertex recovers K.*

Proof. (a) Since H_0 is connected, loopless, and has minimum degree at least 3, each block K is loopless with at least two vertices and no cut vertex. For each $x \in V(K)$, let $V[x]$ be the set of vertices of H_0 reachable from x by a path avoiding $E(K)$. Every vertex of H_0 lies in a unique $V[x]$ (otherwise a path outside $E(K)$ joining distinct vertices of K would enlarge the block K without creating a cut vertex), and every edge outside $E(K)$ has both ends in one $V[x]$. Hence for any bipartition $V(K) = V_1 \sqcup V_2$, the edges of H_0 between $\bigcup_{x\in V_1} V[x]$ and $\bigcup_{x\in V_2} V[x]$ are precisely the edges of K between V_1 and V_2, of which there are at least 3.

(b) Add a new vertex v joined to K by three edges not all incident to the same vertex, forming K^+. Then K^+ is loopless, 3-edge-connected, and has no cut vertex (for $u \in V(K)$, $K - u$ is connected and v has a neighbour in $K - u$); every neighbour of v has degree at least 4 in K^+. Next, replace each vertex $u \in V(K)$ of degree $j \geq 4$ in K^+ by a cycle O_u of length j, attaching its j incident edges bijectively to the j vertices of O_u, yielding a cubic loopless graph K^c with pairwise disjoint cycles O_u.

We verify that K^c is 3-edge-connected. Suppose $V(K^c) = V_1 \sqcup V_2$ has at most 2 crossing edges. If every cycle O_u lies wholly on one side, contracting the cycles yields a ≤ 2-edge-cut of K^+, impossible. Otherwise some O_u meets both sides, contributing at least 2 crossing edges; thus no other edges cross, so every vertex of O_u in V_i has its unique neighbour outside O_u also in V_i, and every other cycle lies on one side. Contracting all cycles except O_u and deleting O_u disconnects $K^+ - u$, contradicting that K^+ has no cut vertex.

Finally, because every neighbour of v had degree at least 4 in K^+, the three edges at v end at three distinct vertices of K^c, so $|V(K^c)| \geq 4$. Setting $Q_K = K^c - v$ gives a truncated cubic 3-edge-connected graph whose expansion cycles contract back to $K^+ - v = K$. □

For the remainder of Section 3, let H be a subdivision of type (SUB) with underlying 3-edge-connected multigraph H_0, and let A be an abelian group.

Definition 3.5 (Threads and the auxiliary graph)**.** For each edge $t \in E(H_0)$, its replacing path $P_t \subseteq H$ is the *thread* of t, of length $\ell(t) \in \{1, 2, 3\}$. Direct each thread as a path and orient t and all edges of P_t along this direction. Let $K_1, \ldots, K_r$ be the blocks of H_0; for each K_b fix a truncated cubic 3-edge-connected expansion Q_b given by Lemma 3.4 with pairwise disjoint vertex sets, and set $Q = \bigsqcup_{b=1}^{r} Q_b$. Edges of H_0 are *original* edges of Q; edges in $E(Q) \setminus E(H_0)$ are *expansion* edges, with $\ell(t) = 0$. For an A-flow Y of Q, define $\pi(Y)\colon E(H) \to A$ by $\pi(Y)(e) = Y(t)$ for every edge $e \in E(P_t)$ of every original edge t.

Figure 3 illustrates the passage $H \to H_0 = K_b \to Q_b$ on a two-vertex, three-parallel-edge block.

Lemma 3.6 (Flows of H, H_0, and Q)**.** *Let A be an abelian group (finite or infinite).*

(a) *Every cycle of H is a union of threads, and its indicator χ_O is constant ($+1$ or -1) on each thread. Every A-flow of H is constant on each thread and corresponds bijectively to an A-flow of H_0.*

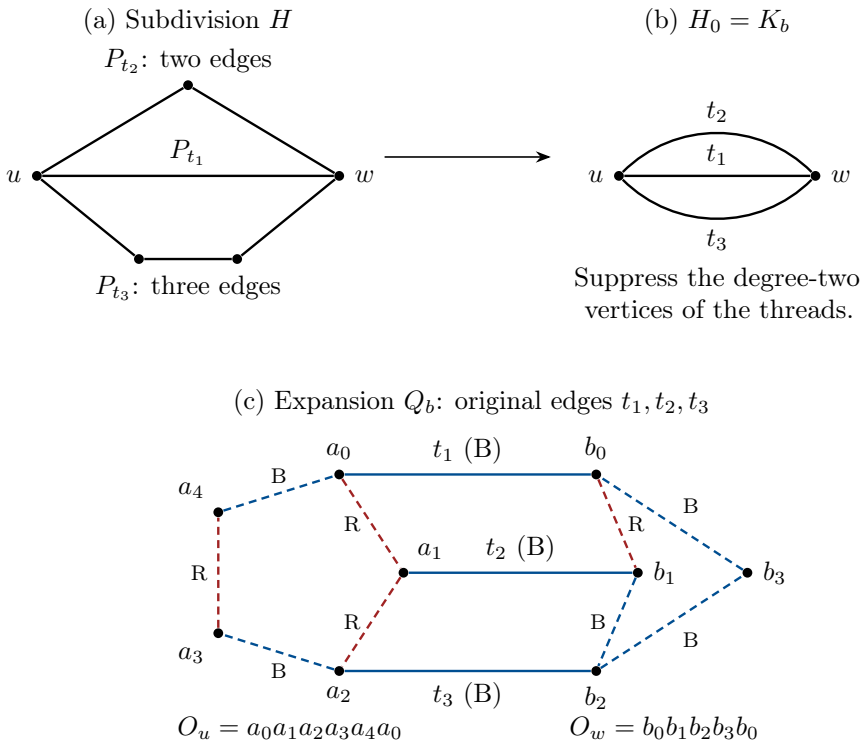


FIGURE 3. Solid edges are original; dashed edges are expansion edges. The letters B and R denote the blue and red edges of the frame, independently of this distinction. In this example all three original edges are blue, whereas expansion edges have both colours. The block K_b has two vertices and three parallel edges. To obtain Q_b by Lemma 3.4, add a vertex v with two edges to u and one to w, expand u and w into cycles of lengths five and four, and delete v. The three resulting degree-two vertices are a_3, a_4, b_3. Contracting O_u and O_w recovers K_b, preserving the identities of t_1, t_2, t_3. The well colouring and flow projection are shown in Figure 4.

(b) *The restriction of an A-flow of H_0 to any block K_b is an A-flow of K_b.*

(c) *For every A-flow Y of Q, its restriction to original edges is an A-flow of H_0 and $\pi(Y)$ is an A-flow of H; conversely, every A-flow of H equals $\pi(Y)$ for some A-flow Y of Q.*

(d) *For any directed cycle C of Q, there are pairwise edge-disjoint directed cycles $O_1, \dots, O_\nu$ of H ($\nu \geq 0$) such that $\pi(x\chi_C) = x(\chi_{O_1} + \cdots + \chi_{O_\nu})$ for all $x \in A$.*

Proof. (a) Interior vertices of threads have degree 2 in H, forcing any cycle or flow of H to be constant along each directed thread; conservation at $V(H_0)$ then agrees in H and H_0. (b) By Lemma 2.1(a), an A-flow of H_0 is a sum of cycle flows, and every cycle lies in one block.

(c) Let $V_x \subseteq V(Q_b)$ be the vertex set of the expansion component contracted to $x \in V(K_b)$. Summing conservation of Y over V_x cancels edges inside V_x and yields conservation at x in K_b; summing over blocks gives conservation in H_0, so $\pi(Y)$ is an A-flow of H by (a). Conversely, given an A-flow of H, let h_0 be the corresponding flow of H_0 (with flow restriction to each K_b by (b)). For each $x \in V(K_b)$, let $\nu(y)$ ($y \in V_x$) be the boundary at y of $h_0|_{K_b}$ extended by 0 on expansion edges; then $\sum_{y \in V_x} \nu(y) = (\partial(h_0|_{K_b}))(x) = 0$. Choosing a spanning tree of expansion edges on V_x and assigning edge values from the leaves inward achieves boundary $-\nu$ on V_x, producing an A-flow Y of Q with $\pi(Y)$ equal to the given flow.

(d) Applying (c) with $A = \mathbb{Z}$ shows that $\pi(\chi_C)$ is a $\{-1, 0, 1\}$-valued integer flow of H; Lemma 2.1(b) writes $\pi(\chi_C) = \chi_{O_1} + \cdots + \chi_{O_\nu}$ with pairwise edge-disjoint cycles, and scaling by $x \in A$ gives the claim. □

Well colourings, frames, and official cycles.

Definition 3.7 (2-constructions and well colourings)**.** A 2-*join* of two vertex-disjoint graphs U' and U'' adds exactly two new edges, each with one end in U' and one in U'' (possibly parallel). A graph is 2-*constructible* if it is a single vertex with no edges, or a 2-join of two 2-constructible graphs. A 2-*construction* fixes a recursive decomposition into *stages* (subgraphs); a stage with more than one vertex is a *node*, joining its two *operands* by its two *edges*.

A *well colouring* of a graph Q_b partitions $E(Q_b)$ into blue and red edges so that: (i) the blue edges form a spanning tree T_b of Q_b, and (ii) the quotient Q_b/red—formed by identifying each connected component of $(V(Q_b), E_{\text{red}})$ to a single vertex while retaining all blue edges—is 2-constructible.

The key structural existence theorem comes directly from Jaeger, Linial, Payan, and Tarsi [JLPT, Lemma 3.3, p. 177] (proved via their constructive generation of truncated cubic 3-edge-connected graphs [JLPT, Lemma 3.2, p. 175]):

Theorem 3.8 (Jaeger, Linial, Payan, and Tarsi [JLPT])**.** *Every truncated cubic* 3*-edge-connected graph admits a well colouring.*

As noted in [JLPT, p. 178] (and proved directly in Lemma 3.9 below), because every 2-constructible graph on j vertices is loopless and has $2(j-1)$ edges, retaining all blue edges in Q_b/red forces no blue edge to have both ends in one red component, yields $2(j-1) = n-1$, and leaves $n-j$ red edges across j components—hence a forest.

We record the elementary combinatorial properties of a well colouring and its 2-construction in one short lemma.

Lemma 3.9 (Counts and stage graphs)**.** *Let Q_b be a truncated cubic* 3*-edge-connected graph on n vertices with a well colouring and a fixed* 2*-construction of Q_b/red. For each stage U, let $V_U \subseteq V(Q_b)$ be the union of the red components belonging to $V(U)$, and let $Q_U = Q_b[V_U]$.*

(a) *Q_b is simple and connected with $3(n-1)/2$ edges; its red edges form a forest, and no blue edge has both ends in one red component.*

(b) *Any two stages of Q_b/red are disjoint or nested; every blue edge belongs to a unique node; and each stage U is an induced subgraph of Q_b/red whose edges are the edges of the nodes contained in U.*

(c) *Q_U is connected, and $E(Q_U)$ consists of the red edges inside V_U together with the edges of all nodes contained in U.*

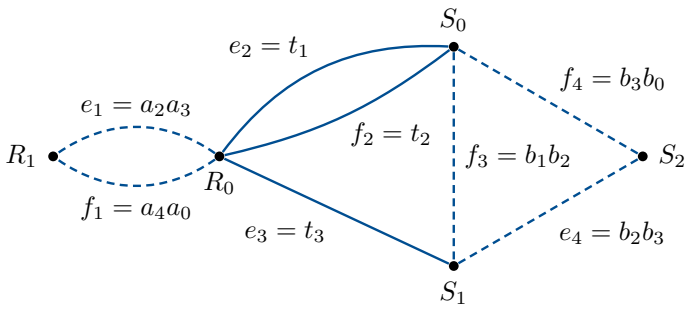
(d) *If a node U joins operands U', U'' by $\{e, f\}$, then $V_U = V_{U'} \sqcup V_{U''}$, each of e, f has one end in $V_{U'}$ and one in $V_{U''}$, and $E(Q_U) = E(Q_{U'}) \cup E(Q_{U''}) \cup \{e, f\}$.*

Proof. (a)–(b) In the cubic 3-edge-connected graph Q_b^+ on $n+1 \geq 4$ vertices, a loop creates a bridge and parallel edges create a 2-edge-cut, so Q_b^+ and $Q_b = Q_b^+ - v$ are simple. Deleting v leaves $3(n+1)/2 - 3 = 3(n-1)/2$ edges and keeps Q_b connected (otherwise some component would receive at most one edge from v). Inducting on the 2-construction of any j-vertex 2-constructible graph U_0 (formed as the 2-join of U_0' and U_0'') shows that U_0 is connected, loopless, and has $2(j-1)$ edges; hence no blue edge of Q_b has both ends in one red component. Moreover, every stage other than U_0 is a stage of U_0' or of U_0'', so any two stages are disjoint or nested and every edge belongs to a unique node. If $U \neq U_0$ is a stage of U_0' and an edge $t \in E(U_0)$ has both ends in $V(U)$, then t is neither one of the two crossing edges of U_0 nor an edge of U_0'', so $t \in E(U_0')$ and hence $t \in E(U)$ by induction; thus U is an induced subgraph of U_0 whose edges are those of the nodes contained in U,

proving (b). Since Q_b/red has $n-1$ blue edges and j red components, $2(j-1) = n-1$, leaving $3(n-1)/2 - (n-1) = (n-1)/2 = n-j$ red edges across j components—hence a forest.

(c)–(d) A red edge stays inside one red component, while a blue edge with both ends in V_U is an edge of Q_b/red between two vertices of U, hence an edge of U by (b). This gives the edge descriptions in (c) and (d); connectedness of Q_U follows by induction on the 2-construction (base case: one red component; node: $Q_{U'}$ and $Q_{U''}$ joined by e). □

A *frame* of H fixes, for every component Q_b of Q, a well colouring (Theorem 3.8), a 2-construction of Q_b/red, and labels $N_i = \{e_i, f_i\}$ for the two edges of each node, with all nodes across all components numbered $1, \dots, s$ in topological order (every node after all nodes contained in its operands). Write U_i for the stage of node i and U_i', U_i'' for its operands. Figure 4 displays the red quotient and 2-construction for the example of Figure 3.



Node	First operand	Second operand
U_1	$\{R_0\}$	$\{R_1\}$
U_2	U_1	$\{S_0\}$
U_3	U_2	$\{S_1\}$
U_4	U_3	$\{S_2\}$

At node i, add exactly the pair $N_i = \{e_i, f_i\}$ between its two operands. Thus $U_4 = Q_b/\text{red}$.

The red components are

$$R_0 = \{a_0, a_1, a_2\}, \quad R_1 = \{a_3, a_4\}, \quad S_0 = \{b_0, b_1\}, \quad S_1 = \{b_2\}, \quad S_2 = \{b_3\}.$$

For any A-flow Y of Q_b, orient t_1, t_2, t_3 from O_u to O_w, and orient each thread from u to w. Then

$$\pi(Y)(e) = Y(t_i) \quad (e \in E(P_{t_i})), \qquad Y(t_1) + Y(t_2) + Y(t_3) = 0.$$

Expansion-edge values disappear under the contraction to K_b; the original-edge values are repeated along their threads in H.

FIGURE 4. The quotient by red components of the graph in Figure 3. Every surviving edge is blue; solid and dashed still distinguish original and expansion edges. This contraction is different from contracting the expansion cycles to recover K_b. The eight blue edges in Q_b form a spanning tree, and the four displayed 2-joins give a 2-construction of its red quotient. They therefore define a frame. The projection π is defined on flows of Q_b, as in Lemma 3.6; the displayed conservation equation constrains $Y(t_1), Y(t_2), Y(t_3)$ to two degrees of freedom.

Definition 3.10 (Protected and free threads; official cycles)**.** Given a frame of H, a thread P_t is *protected* if t is an original blue edge, and *free* if t is an original red edge (called a *free edge*). Let H_P be the union of all protected threads. Let T^e be the subgraph of Q formed by all red edges together with $e_1, \dots, e_s$. For each $1 \le i \le s$, the *official cycle* C_i is the fundamental cycle of f_i in T^e, directed forward along f_i.

Lemma 3.11 (Official cycles and free-edge cycles)**.** *For every frame of H:*

(O1) *For every choice $g = (g_1, \dots, g_s)$ with $g_i \in N_i$, the subgraph T^g formed by all red edges and $g_1, \dots, g_s$ is a spanning tree of every component Q_b and of every stage graph Q_U. In particular, T^e is such a spanning forest of Q.*

(O2) *C_i lies in Q_{U_i} and contains both e_i and f_i.*

(O3) *If $l < i$, then C_l contains neither e_i nor f_i.*

(O4) *If $l > i$, then C_l does not contain f_i.*

(O5) *Every A-flow Y of Q is uniquely a linear combination $Y = \sum_{i=1}^{s} x_i \chi_{C_i}$ with $x_i \in A$.*

(O6) *H_P is connected and contains every vertex of H outside the interiors of free threads; moreover, every free edge ε lies on a cycle D_ε of H containing P_ε and no other free thread, directed forward along P_ε.*

Proof. (O1) Induction on the 2-construction using Lemma 3.9(a),(d): each single-vertex stage is a red tree, and at node i the single edge g_i joins the spanning trees of $Q_{U_i'}$ and $Q_{U_i''}$.

(O2) By (O1), the path of T^e between the ends of f_i lies inside Q_{U_i}; by Lemma 3.9(d), the only edges of Q_{U_i} between $V_{U_i'}$ and $V_{U_i''}$ are e_i and f_i, so this path must traverse e_i.

(O3) For $l < i$, $C_l \subseteq Q_{U_l}$ by (O2), and by Lemma 3.9(c) and topological numbering every blue edge of Q_{U_l} belongs to a pair N_h with $h \le l < i$.

(O4) Every edge of C_l other than f_l belongs to T^e, which excludes f_i.

(O5) By (O1) and Lemma 3.9(b), T^e is a spanning forest of Q whose non-tree edges are precisely $f_1, \dots, f_s$; apply Lemma 2.1(a).

(O6) In each component Q_b, the blue spanning tree T_b contracts under expansion edges to a connected spanning subgraph of K_b on original blue edges. Since H_0 is connected, replacing original blue edges by their threads shows that H_P is connected and covers all vertices of H_0 and all protected interiors. For a free edge $\varepsilon \in E(K_b)$, its fundamental cycle in T_b contracts to a closed trail in K_b containing ε once and otherwise only original blue edges; extracting a cycle through ε inside this even subgraph and expanding edges to threads yields D_ε. □

Definition 3.12 (List assignments and H-good flows)**.** Fix a frame of H. A *list assignment* Λ assigns to each blue edge t of Q a subset $\Lambda(t) \subseteq A$, with $\Lambda(t) = \emptyset$ on every expansion edge; let $P(\Lambda)$ be the set of A-flows Y of Q satisfying $Y(t) \notin \Lambda(t)$ for all blue edges t.

Let G contain H, and let f be a nowhere-zero Γ-flow of G with A-coordinate $u = \overline{f}$. In case (CYC), f is *H-good* if $u(e) \neq 0$ for all $e \in E(H)$. In case (SUB), f is *H-good* if $u(e) \neq 0$ for every edge e of every protected thread; we then define

$$\Lambda_f(t) = \{\, -u(e) : e \in E(P_t) \,\} \qquad \text{for each original blue edge } t.$$

Note a crucial distinction: the ambient G-flow f need not restrict to a flow of H, because edges of $E(G) \setminus E(H)$ may meet interior vertices of a thread; hence u can take up to $\ell(t) \le 3$ distinct non-zero values along P_t, giving $|\Lambda_f(t)| \le \ell(t) \le 3$. By contrast, for any A-flow Y of Q, the added H-flow $\pi(Y)$ (extended by 0 off $E(H)$) is constant along each directed thread (Lemma 3.6(a)), so $0 \in P(\Lambda_f)$, and $Y \in P(\Lambda_f)$ holds if and only if $u + \pi(Y)$ is non-zero on every edge of every protected thread.

4. The master theorem for step systems

Throughout this section, Γ is a finite abelian group, $B \le \Gamma$ is a subgroup, $A = \Gamma/B$ with quotient map $x \mapsto \overline{x}$, and $\Sigma \subseteq \Gamma \setminus \{0\}$ is a symmetric step system ($\Sigma = -\Sigma$) with $\Sigma_B = \Sigma \cap B$ and $\Sigma_A = \{\overline{\lambda} : \lambda \in \Sigma\} \setminus \{0\}$. For each $\sigma \in \Sigma_A$, fix one lift $\gamma(\sigma) \in \Sigma$ with $\overline{\gamma(\sigma)} = \sigma$. We always assume $|A| \ge 10$ and $|B| \ge 4$, and when $H \subseteq G$ is a subdivision of type (SUB) we orient G on $E(H)$ along the directions of the threads (Definition 3.5).

We use five elementary facts about Γ-flows f with A-coordinate $u = \overline{f}$:

(E1) $f(e) \neq 0$ if and only if $u(e) \neq 0$, or $u(e) = 0$ and $f(e) \in B \setminus \{0\}$.

(E2) Adding $y\chi_C$ with $y \in B$ preserves the A-coordinate u.

(E3) Every flow with values in an abelian group is a sum of cycle flows $x\chi_C$ (Lemma 2.1(a)).

(E4) Every A-flow v of G is the A-coordinate of some Γ-flow of G (write $v = \sum a_r\chi_{C_r}$ by (E3) and lift each a_r to Γ).

(E5) Identifying vertex classes of G and deleting any set of edges with both ends in one class sends every flow of G (by restriction) to a flow of the contracted graph.

Lemma 4.1 (Sparse zeros yield a good subgraph)**.** *Assume* (AV)*. On every loopless 3-edge-connected multigraph G with $n \geq 2$ vertices, every nowhere-zero Γ-flow is Σ-joined to a nowhere-zero Γ-flow f' such that $(V(G), \operatorname{supp}(\overline{f'}))$ contains a good subgraph.*

Proof. Since G is 3-edge-connected, every vertex has degree at least 3, so $m = |E(G)| \geq 3n/2$. Condition (AV) gives a Σ-joined nowhere-zero flow f' whose A-coordinate $u' = \overline{f'}$ satisfies $|\operatorname{supp}(u')| \geq 8m/9 \geq (4/3)n > (4/3)n - 2$. If $n \geq 3$, Lemma 3.2 yields a good subgraph in the loopless multigraph $(V(G), \operatorname{supp}(u'))$. If $n = 2$, then $m \geq 3$ gives $|\operatorname{supp}(u')| \geq 8m/9 > 2$, so two parallel edges form a 2-cycle of type (CYC). □

Reverse choice and the walk lemma. For Lemmas 4.2–4.7, fix a subdivision H of type (SUB) with auxiliary graph Q and a frame with nodes $1, \dots, s$, pairs $N_i = \{e_i, f_i\}$, and official cycles $C_1, \dots, C_s$.

Lemma 4.2 (Reverse choice)**.** *Let Λ be a list assignment on Q.*

(a) *If $|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$ for every node i, then for every A-flow Y_0 of Q there exist $x_1, \dots, x_s \in A$ with $Y_0 + \sum_{i=1}^{s} x_i\chi_{C_i} \in P(\Lambda)$.*

(b) *Let E' be a set of red edges of Q, and for each $\varepsilon \in E'$ let $\Phi(\varepsilon) \subsetneq A$ be a proper subset. Every red edge ε lies on at least one official cycle; write $i(\varepsilon)$ for the least i with $\varepsilon \in E(C_i)$. If*

$$|\Lambda(e_i)| + |\Lambda(f_i)| + \sum_{\varepsilon \in E' :\, i(\varepsilon) = i} |\Phi(\varepsilon)| \;<\; |A| \qquad \text{for every node } i,$$

then there exists $Y \in P(\Lambda)$ such that $Y(\varepsilon) \notin \Phi(\varepsilon)$ for all $\varepsilon \in E'$.

Proof. (a) Choose $x_s, x_{s-1}, \dots, x_1$ in reverse order. Once $x_s, \dots, x_{i+1}$ are fixed, set $Z_i = Y_0 + \sum_{l>i} x_l\chi_{C_l}$. By Lemma 3.11(O2)–(O3), C_i contains e_i and f_i (so $\chi_{C_i}(t) \in \{\pm 1\}$ for $t \in N_i$) while no C_l with $l < i$ meets N_i. Hence the final value $Y(t) = Z_i(t) + x_i\chi_{C_i}(t)$ on $t \in N_i$ depends only on x_i, forbidding $|\Lambda(t)|$ values of x_i. Together e_i and f_i forbid at most $|\Lambda(e_i)| + |\Lambda(f_i)| < |A|$ values, so a valid $x_i \in A$ exists.

(b) For any red edge ε of a component Q_b, let F_ε be its fundamental cycle in the blue spanning tree T_b, directed forward along ε; then χ_{F_ε} is a $\mathbb{Z}$-flow of Q with $\chi_{F_\varepsilon}(\varepsilon) = 1 \neq 0$. By Lemma 3.11(O5) (with $A = \mathbb{Z}$), χ_{F_ε} is a linear combination of $\chi_{C_1}, \dots, \chi_{C_s}$, so ε lies on at least one official cycle and $i(\varepsilon)$ is well defined. Now choose $x_s, \dots, x_1$ in reverse order for $Y = \sum_{i=1}^{s} x_i\chi_{C_i}$, requiring at node i that $Y(t) \notin \Lambda(t)$ for $t \in N_i$ and $Y(\varepsilon) \notin \Phi(\varepsilon)$ for every $\varepsilon \in E'$ with $i(\varepsilon) = i$. Since no C_l with $l < i(\varepsilon)$ contains ε and $\chi_{C_i}(\varepsilon) \in \{\pm 1\}$, each such ε forbids $|\Phi(\varepsilon)|$ values of x_i, leaving at least one valid choice at every node i. □

Lemma 4.3 (Walk lemma)**.** *Assume $|A| \geq 10$ and* (R1)*. Let Λ be a list assignment with $|\Lambda(t)| \leq 3$ for every blue edge t (in particular, $\Lambda = \Lambda_f$ for any H-good flow f). Then any two flows $Y, Y' \in P(\Lambda)$ are joined by a sequence $Y = Y_0, Y_1, \dots, Y_n = Y'$ in $P(\Lambda)$ in which each step $Y_r - Y_{r-1}$ equals $\sigma_r\chi_{C_{l_r}}$ for some $\sigma_r \in \Sigma_A$ and some official cycle C_{l_r}.*

Proof. Call such a sequence *good*. For $1 \leq i \leq s+1$, let Λ_i agree with Λ on $N_i \cup \cdots \cup N_s$ and be empty on $N_1 \cup \cdots \cup N_{i-1}$; thus $\Lambda_1 = \Lambda$ and $P(\Lambda_{s+1})$ consists of all A-flows of Q. We prove by

downward induction on i that any two flows in $P(\Lambda_i)$ are joined by a good sequence in $P(\Lambda_i)$. The strategy at step i is to release the constraints on N_i, take a good sequence $Y_0, \dots, Y_n$ in $P(\Lambda_{i+1})$, and track it inside $P(\Lambda_i)$ by maintaining offsets $Y_r + c_r\chi_{C_i} \in P(\Lambda_i)$ (which leaves $N_{i+1}, \dots, N_s$ untouched) before walking c_n back to 0 at $Y_n = Y'$.

Base $i = s+1$. For any A-flows Y, Y' of Q, Lemma 3.11(O5) writes $Y' - Y = \sum_{l=1}^{s} x_l\chi_{C_l}$. By (R1) (with $X = \emptyset$), Σ_A generates A, so writing each x_l as a sum of elements of Σ_A and adding terms one by one gives a good sequence from Y to Y'.

Inductive step ($i+1 \to i$). Write $\chi = \chi_{C_i}$. For an A-flow Z of Q, set

$$X_i^e(Z) = \{c \in A : Z(e_i) + c\chi(e_i) \in \Lambda(e_i)\}, \qquad X_i^f(Z) = \{c \in A : Z(f_i) + c\chi(f_i) \in \Lambda(f_i)\},$$

and $X_i(Z) = X_i^e(Z) \cup X_i^f(Z)$. Since $\chi(e_i), \chi(f_i) \in \{\pm 1\}$ (O2) and $|\Lambda(t)| \le 3$, we have $|X_i^e(Z)| = |\Lambda(e_i)| \le 3$, $|X_i^f(Z)| = |\Lambda(f_i)| \le 3$, and $|X_i(Z)| \le 6$. Moreover, C_i meets none of $N_{i+1}, \dots, N_s$ (O3), so for every $Z \in P(\Lambda_{i+1})$ and $c \in A$,

$$(*) \qquad Z + c\chi \in P(\Lambda_{i+1}), \qquad \text{and} \qquad Z + c\chi \in P(\Lambda_i) \iff c \notin X_i(Z).$$

Given $Y, Y' \in P(\Lambda_i) \subseteq P(\Lambda_{i+1})$, take a good sequence $Y = Y_0, \dots, Y_n = Y'$ in $P(\Lambda_{i+1})$ with $Y_r - Y_{r-1} = \sigma_r\chi_{C_{l_r}}$. We construct a good sequence in $P(\Lambda_i)$ visiting $Y_r + c_r\chi$ with $c_0 = 0$ and $c_r \notin X_i(Y_r)$ for each r. For $r = 0$, $0 \notin X_i(Y_0)$ by $(*)$. Assuming $Y_{r-1} + c_{r-1}\chi \in P(\Lambda_i)$ is reached, consider three cases:

(i) $l_r = i$. Set $c_r = c_{r-1} - \sigma_r$; then $Y_r + c_r\chi = Y_{r-1} + c_{r-1}\chi \in P(\Lambda_i)$, so no move is needed.

(ii) $l_r < i$. By (O3), C_{l_r} avoids N_i, so $X_i(Y_r) = X_i(Y_{r-1})$ and we simply take $c_r = c_{r-1}$.

(iii) $l_r > i$. By (O4), C_{l_r} does not contain f_i, so $X_i^f(Y_r) = X_i^f(Y_{r-1})$. Therefore

$$|X_i(Y_{r-1}) \cup X_i(Y_r)| \;\le\; |X_i^e(Y_{r-1})| + |X_i^e(Y_r)| + |X_i^f(Y_{r-1})| \;\le\; 3+3+3 \;=\; 9 \;<\; 10 \;\le\; |A|.$$

Choose any $c_r \in A \setminus (X_i(Y_{r-1}) \cup X_i(Y_r))$.

In case (ii), $c_r = c_{r-1} \notin X_i(Y_r)$, so adding $\sigma_r\chi_{C_{l_r}}$ moves directly from $Y_{r-1}+c_{r-1}\chi$ to $Y_r+c_r\chi \in P(\Lambda_i)$. In case (iii), both c_{r-1} and c_r avoid $X_i(Y_{r-1})$, which has size at most 6. By (R1), a walk with steps in Σ_A connects c_{r-1} to c_r inside $A \setminus X_i(Y_{r-1})$; the corresponding χ-steps stay in $P(\Lambda_i)$ by $(*)$ and lead from $Y_{r-1} + c_{r-1}\chi$ to $Y_{r-1} + c_r\chi$. Adding $\sigma_r\chi_{C_{l_r}}$ then lands at $Y_r + c_r\chi \in P(\Lambda_i)$ since $c_r \notin X_i(Y_r)$.

Finally, at $r = n$ both c_n and 0 lie outside $X_i(Y_n)$ (of size at most 6), so (R1) yields a Σ_A-walk in $A \setminus X_i(Y_n)$ from c_n to 0, finishing at $Y_n = Y'$. □

Changing the A-coordinate inside H.

Lemma 4.4 (Preparation in B)**.** *Assume $|B| \ge 4$ and* (R2)*. Let G contain H, let f be an H-good Γ-flow of G with A-coordinate u, and let E° be a set of free edges of H. For each $\varepsilon \in E^\circ$, put $W_\varepsilon = \{-f(e) : e \in E(P_\varepsilon), u(e) = 0\} \subseteq B$, and let $T_\varepsilon \subseteq B$ be a set of at most $\ell(\varepsilon) \le 3$ elements containing W_ε. Then f is Σ-joined, by moves $y\chi_{D_\varepsilon}$ with $y \in \Sigma_B$ and $\varepsilon \in E^\circ$, to an H-good flow f° with A-coordinate u that equals f outside $E(H)$ and on every free thread outside E°, and satisfies $f^\circ = f + y_\varepsilon$ on $E(P_\varepsilon)$ for some $y_\varepsilon \in B \setminus T_\varepsilon$ for every $\varepsilon \in E^\circ$.*

Proof. Here W_ε forbids intermediate zeros along P_ε during the walk in B, while $T_\varepsilon \supseteq W_\varepsilon$ specifies the target values to avoid at the end. Since f is nowhere zero, $0 \notin W_\varepsilon$.

Handle the edges $\varepsilon \in E^\circ$ sequentially, writing f' for the current flow (initially $f' = f$, agreeing with f on every unprocessed free thread). Because the moves on D_ε (Lemma 3.11(O6)) preserve u by (E2) and change no free thread other than P_ε (and nothing outside $E(H)$), each step leaves all other free threads intact. For a single $\varepsilon \in E^\circ$: if $0 \notin T_\varepsilon$, set $y_\varepsilon = 0$ and make no move. If $0 \in T_\varepsilon$, pick $y_\varepsilon \in B \setminus T_\varepsilon$ (possible since $|T_\varepsilon| \le \ell(\varepsilon) \le 3 < 4 \le |B|$). Then $0 \in T_\varepsilon \setminus W_\varepsilon$ implies $|W_\varepsilon| \le |T_\varepsilon| - 1 \le 2$, so (R2) provides a walk $0 = z_0, z_1, \dots, z_\nu = y_\varepsilon$ in $B \setminus W_\varepsilon$ with steps in Σ_B.

Performing $(z_r - z_{r-1})\chi_{D_\varepsilon}$ produces $f' + z_r\chi_{D_\varepsilon}$, which retains A-coordinate u (hence stays non-zero on all protected threads and wherever $u(e) \neq 0$) and has value $f(e) + z_r \neq 0$ on edges of P_ε where $u(e) = 0$ (since $z_r \notin W_\varepsilon$ and D_ε passes P_ε forward). Replacing f' by $f' + y_\varepsilon\chi_{D_\varepsilon}$ leaves $f' = f + y_\varepsilon$ on $E(P_\varepsilon)$ and finishes the step for ε. □

Lemma 4.5 (One step inside H)**.** *Assume $|B| \geq 4$ and* (R2)*. Let G contain H, let f be an H-good Γ-flow of G with A-coordinate u, and let C be a directed cycle of Q and $\sigma \in \Sigma_A$ such that $u' = u + \pi(\sigma\chi_C)$ is non-zero on every edge of every protected thread. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate u' that equals f outside $E(H)$.*

Proof. Put $\lambda = \gamma(\sigma) \in \Sigma$, so $\overline{\lambda} = \sigma$. By Lemma 3.6(d) and (a), $u' = u + \sigma(\chi_{O_1} + \cdots + \chi_{O_\nu})$ on $E(H)$ for pairwise edge-disjoint directed cycles $O_1, \ldots, O_\nu$ of H, each constant (± 1) on every thread it traverses. For each free edge ε, define $\delta_\varepsilon = \lambda \cdot \chi_{O_j}(P_\varepsilon)$ if $P_\varepsilon \subseteq O_j$ and $\delta_\varepsilon = 0$ otherwise; then $\overline{\delta_\varepsilon} = u'(e) - u(e)$ for every $e \in E(P_\varepsilon)$.

Set $T_\varepsilon = W_\varepsilon \cup \{-f(e) - \delta_\varepsilon : e \in E(P_\varepsilon),\, u'(e) = 0\} \subseteq B$ (note $\overline{f(e) + \delta_\varepsilon} = u'(e)$). Each edge $e \in E(P_\varepsilon)$ contributes at most one element to T_ε: if $u(e) = u'(e) = 0$ then $\overline{\delta_\varepsilon} = 0$, which forces P_ε to lie on no O_j (as $\sigma \neq 0$), so $\delta_\varepsilon = 0$ and $-f(e) = -f(e) - \delta_\varepsilon$. Thus $|T_\varepsilon| \leq \ell(\varepsilon) \leq 3$. Applying Lemma 4.4 to the set of all free edges yields an H-good flow f° with A-coordinate u, equal to f outside $E(H)$, such that $f^\circ(e) + \delta_\varepsilon \neq 0$ whenever $e \in E(P_\varepsilon)$ has $u'(e) = 0$.

Now perform the moves $\lambda\chi_{O_1}, \ldots, \lambda\chi_{O_\nu}$ in order. Since the cycles are pairwise edge-disjoint, each intermediate flow takes either its old value $f^\circ(e)$ (nowhere zero) or its new value $f^\circ(e) + \lambda\chi_{O_j}(e)$ (with A-coordinate $u'(e)$, non-zero on protected threads by hypothesis and non-zero on free threads by (E1) when $u'(e) \neq 0$ and by the choice of f° when $u'(e) = 0$). The final flow is H-good, has A-coordinate u', and equals f outside $E(H)$. □

Lemma 4.6 (Changing the A-coordinate inside H)**.** *Assume $|A| \geq 10$, $|B| \geq 4$,* (R1)*, and* (R2)*. Let G contain H, let f be an H-good Γ-flow of G with A-coordinate u, and let $Y \in P(\Lambda_f)$. Then f is Σ-joined, by Σ-moves on cycles of H, to an H-good flow with A-coordinate $u + \pi(Y)$ that equals f outside $E(H)$.*

Proof. Since $0, Y \in P(\Lambda_f)$, Lemma 4.3 gives a sequence $0 = Y_0, Y_1, \ldots, Y_n = Y$ in $P(\Lambda_f)$ with $Y_r - Y_{r-1} = \sigma_r\chi_{C_{l_r}}$ ($\sigma_r \in \Sigma_A$). For each r, $u + \pi(Y_r)$ is non-zero on every protected thread because $Y_r \in P(\Lambda_f)$, so applying Lemma 4.5 iteratively produces H-good flows $f = f_0, f_1, \ldots, f_n$, each equal to f outside $E(H)$, with f_n having A-coordinate $u + \pi(Y)$. □

The path lemma and lifting a contracted move.

Lemma 4.7 (The path)**.** *For any distinct vertices $p, p' \in V(H)$, there is a path J of H from p to p' using at most 3 edges in total on the threads of each pair $N_i = \{e_i, f_i\}$ (J may use arbitrarily many free threads).*

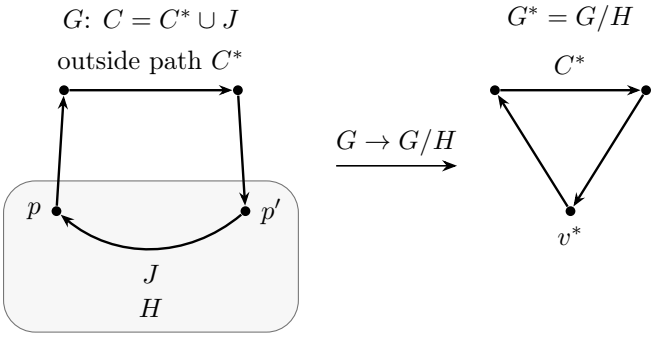
Proof. For any choice $g = (g_1, \ldots, g_s)$ with $g_i \in N_i$, let $H_g \subseteq H$ consist of $V(H_0)$ and the threads of all original edges in T^g. By Lemma 3.11(O1), T^g spans each Q_b, so contracting expansion edges and expanding original edges to threads shows that H_g is connected, covers every vertex of H except interior vertices of threads of blue edges outside $\{g_1, \ldots, g_s\}$, and meets the threads of each pair N_i only along P_{g_i} (at most $\ell(g_i) \leq 3$ edges).

Case 1: p, p' are interior vertices of the same thread P_t. Since $\ell(t) \leq 3$, P_t has at most 2 interior vertices, so the 1-edge subpath J between them uses at most $1 \leq 3$ edge on any pair.

Case 2: Not Case 1, and no pair N_i has p in the interior of one thread and p' in the interior of the other. For each i, at most one edge of N_i has p or p' in its thread interior; pick that edge as g_i (otherwise $g_i = e_i$). Then $p, p' \in V(H_g)$, and any path J from p to p' in H_g uses at most $\ell(g_i) \leq 3$ edges on N_i.

Case 3: p lies in the interior of P_t and p' in the interior of $P_{t'}$, where $N_i = \{t, t'\}$. Let U', U'' be the operands of node i. By Lemma 3.9(d), t and t' each have one end $x_1, x_2 \in V_{U'}$ and one end in

$V_{U''}$, giving each thread one endpoint on the side of U' and one on the side of U''. Let c, c' be the distances along $P_t, P_{t'}$ from p, p' to their endpoints on the side of U'; the distances to the side of U'' are $\ell(t) - c$ and $\ell(t') - c'$. Since $(c + c') + (\ell(t) - c + \ell(t') - c') = \ell(t) + \ell(t')$, one of the two sums is at most $(\ell(t) + \ell(t'))/2 \le \max\{\ell(t), \ell(t')\} \le 3$; by symmetry assume $c + c' \le 3$. By Lemma 3.11(O1), $T^e \cap Q_{U'}$ is a spanning tree of $Q_{U'}$ containing a path R from x_1 to x_2; since $t, t' \notin E(Q_{U'})$, every blue edge of R is an edge e_l with $l \neq i$. Contracting expansion edges and replacing original edges of R by their threads produces a connected subgraph $L \subseteq H$ joining the U'-side endpoints of P_t and $P_{t'}$ while avoiding $P_t, P_{t'}$ and meeting every other pair N_l only on P_{e_l}. Taking a path J from p to p' inside L together with the two U'-side segments of P_t and $P_{t'}$ uses at most $c + c' \le 3$ edges on N_i and at most $\ell(e_l) \le 3$ edges on every other pair N_l. □



Subdivision case	Choice of J	Preparation inside H
$\overline{\lambda} \neq 0$	Lemma 4.7: at most 3 edges in total on each pair's two threads. J may traverse arbitrarily many free threads.	Prepare the protected threads using the enlarged lists and Lemma 4.6, then prepare every free thread met by J using B-moves to obtain a prepared flow f'; then $g = f' + \lambda\chi_C$ is nowhere zero and H-good.
$\overline{\lambda} = 0$	At most two free pieces, each one edge; all other edges lie in H_P. If the endpoints are inside one free thread, take the edge between them.	Use reverse choice and Lemma 4.6 to make the flow's A-coordinate non-zero on every edge of J. The final step has quotient zero and preserves that coordinate.

If H is a cycle with at most five edges, choose an arc J with at most two edges. When $\overline{\lambda} = 0$ no preparation is needed; otherwise a quotient shift along H avoids at most five forbidden values on H and two further values on J, using $|A| \ge 10$ and (R1).

FIGURE 5. Lifting a move in Lemma 4.8, under its stated hypotheses. The picture shows the case in which the edges of C^* form an outside path with distinct ends p, p' in H. The arrows on J complete the direction of C; the preparation changes nothing outside H, lifting the contracted move to an H-good flow g with restriction $g|_{G^*} = g^*$. If those edges already form a cycle in G, including the coincident-end case, the move is lifted directly. A loop at v^* is also covered by the proof, whether its two ends in H coincide or differ. The two subdivision rows concern the quotient of the step λ; the zero-quotient row still requires preparation of the flow's quotient on J.

Lemma 4.8 (Lifting a contracted move)**.** *Assume $|A| \ge 10$, $|B| \ge 4$,* (R1)*, and* (R2)*. Let G be a bridgeless multigraph, let $H \subseteq G$ be a good subgraph, and let $G^* = G/H$ be obtained by identifying $V(H)$ to a single vertex v^* and deleting $E(H)$. If f is an H-good Γ-flow of G with restriction f^* to G^* and $g^* = f^* + \lambda\chi_{C^*}$ is a Σ-move of G^*, then f is Σ-joined to an H-good Γ-flow g of G whose restriction to G^* is g^*.*

Proof. Let $u = \overline{f}$. If the edges of C^* already form a cycle in G (which includes $v^* \notin V(C^*)$ and coincident attachment endpoints in H), set $g = f + \lambda\chi_{C^*}$ directly: g equals g^* on $E(G^*)$ and f on $E(H)$, so it is nowhere zero and H-good.

Otherwise, $E(C^*)$ forms a path in G between two distinct vertices $p, p' \in V(H)$ with all internal vertices outside $V(H)$. For any path $J \subseteq H$ from p to p', $C = C^* \cup J$ is a cycle of G, directed so that $\chi_C = \chi_{C^*}$ on $E(G^*)$ (Figure 5). In each case below we Σ-join f to an H-good flow f' agreeing with f off $E(H)$ and then apply $g = f' + \lambda\chi_C$; on $E(G^*)$ this yields g^*, so we need only verify that g is nowhere zero and H-good on $E(H)$.

Case 1: H is of type (CYC). Direct H and let J be a subpath of H from p to p' with at most 2 edges (since $|E(H)| \le 5$). If $\overline{\lambda} = 0$, set $f' = f$; by (E2), $g = f + \lambda\chi_C$ keeps the same non-zero A-coordinate on $E(H)$, hence is nowhere zero (E1) and H-good. If $\overline{\lambda} \neq 0$, put

$$X = \{c \in A : u + c\chi_H \text{ has a zero on } E(H)\}, \qquad X' = \{c \in A : u + c\chi_H + \overline{\lambda}\chi_C \text{ has a zero on } E(J)\}.$$

Then $|X| \le 5$, $|X'| \le 2$, and $0 \notin X$. Since $|X \cup X'| \le 7 < 10 \le |A|$, pick $c \in A \setminus (X \cup X')$. By (R1) there is a walk $0 = c_0, \dots, c_\nu = c$ in $A \setminus X$ with steps $d_r \in \Sigma_A$. Applying $\gamma(d_r)\chi_H$ keeps $u + c_r\chi_H$ non-zero on $E(H)$ and reaches an H-good flow f' agreeing with f off $E(H)$; then $g = f' + \lambda\chi_C$ has non-zero A-coordinate on $E(H)$ (since $c \notin X \cup X'$).

Case 2: H is of type (SUB) *and $\overline{\lambda} \neq 0$.* Let J be the path of Lemma 4.7, using $j_i \le 3$ edges on the threads of N_i.

(2a) *Protected threads.* For each original blue edge t, set $\Lambda_1(t) = \Lambda_f(t) \cup \{-u(e) - \overline{\lambda}\chi_C(e) : e \in E(P_t) \cap E(J)\}$ (and $\Lambda_1 = \emptyset$ on expansion edges). For every node i, $|\Lambda_1(e_i)| + |\Lambda_1(f_i)| \le \ell(e_i) + \ell(f_i) + j_i \le 3 + 3 + 3 = 9 < 10 \le |A|$. By Lemma 4.2(a) (with $Y_0 = 0$) there exists $Y \in P(\Lambda_1) \subseteq P(\Lambda_f)$. Lemma 4.6 Σ-joins f to an H-good flow $f^{(1)}$ agreeing with f off $E(H)$ with A-coordinate $u_1 = u + \pi(Y)$; then $u_1(e) + \overline{\lambda}\chi_C(e) \neq 0$ for every $e \in E(J)$ on a protected thread.

(2b) *Free threads.* Let E° be the set of free edges ε whose thread meets J. For $\varepsilon \in E^\circ$, define W_ε as in Lemma 4.4 for $f^{(1)}$ and

$$T_\varepsilon = W_\varepsilon \cup \{-f^{(1)}(e) - \lambda\chi_C(e) : e \in E(P_\varepsilon) \cap E(J),\ u_1(e) + \overline{\lambda}\chi_C(e) = 0\} \subseteq B.$$

Since $\overline{\lambda} \neq 0$ and $\chi_C(e) \in \{\pm 1\}$ on $E(J)$, the conditions $u_1(e) = 0$ and $u_1(e) + \overline{\lambda}\chi_C(e) = 0$ are mutually exclusive, so $|T_\varepsilon| \le \ell(\varepsilon) \le 3$. Lemma 4.4 Σ-joins $f^{(1)}$ to an H-good flow $f' = f^{(2)}$ with A-coordinate u_1, agreeing with f off $E(H)$, such that $f^{(2)}(e) + \lambda\chi_C(e) \neq 0$ whenever $e \in E(P_\varepsilon) \cap E(J)$ has $u_1(e) + \overline{\lambda}\chi_C(e) = 0$.

(2c) *Final move.* Set $g = f^{(2)} + \lambda\chi_C$. On $E(H) \setminus E(J)$ nothing changes; on $E(J)$, protected edges have non-zero A-coordinate by (2a), and free edges are non-zero by (E1) and (2b). Thus g is nowhere zero and H-good.

Case 3: H is of type (SUB) *and $\overline{\lambda} = 0$ (so $\lambda \in \Sigma_B$).* Since every thread has length at most 3, any interior vertex of a thread is adjacent to an endpoint of that thread, and two interior vertices of the same thread are adjacent. If p, p' are interior vertices of the same free thread, take J to be the single edge between them; otherwise step from p (and from p') by one edge to an adjacent endpoint q (respectively q') in H_P if it lies in a free interior (setting $q = p$ or $q' = p'$ otherwise), and join q to q' inside the connected subgraph H_P (Lemma 3.11(O6)). Thus J meets at most two free threads, each in a single edge e_ε. Let E' be this set of at most 2 free edges, and put $\Phi(\varepsilon) = \{-u(e_\varepsilon)\}$ for $\varepsilon \in E'$. For every node i, $|\Lambda_f(e_i)| + |\Lambda_f(f_i)| + \sum_{\varepsilon \in E' :\, i(\varepsilon) = i} |\Phi(\varepsilon)| \le 3 + 3 + 2 = 8 < 10 \le |A|$. Lemma 4.2(b) yields $Y \in P(\Lambda_f)$ with $Y(\varepsilon) \neq -u(e_\varepsilon)$ for all $\varepsilon \in E'$. Lemma 4.6 Σ-joins f to an H-good flow f' agreeing with f off $E(H)$ whose A-coordinate $u' = u + \pi(Y)$ is non-zero on every edge of J (both protected edges and the free edges e_ε). Since $\overline{\lambda} = 0$, $g = f' + \lambda\chi_C$ preserves u' (E2), hence is nowhere zero (E1) and H-good. □

Proof of the master theorem.

Proof of Theorem 1.3. Call a Γ-flow h a *target* if its A-coordinate $\overline{h}$ is nowhere zero (so h is nowhere zero by (E1)). Since $|A| \geq 10 \geq 6$, Theorem 1.4 gives every bridgeless multigraph G a nowhere-zero A-flow, which lifts to a target h of G by (E4). Because cycles lie inside connected components and $\Sigma = -\Sigma$, proving that on every connected bridgeless multigraph G every nowhere-zero Γ-flow f is Σ-joined to every target h implies both claims of Theorem 1.3.

We induct on $n = |V(G)|$, treating loopless graphs on n vertices before graphs with loops. For $n = 1$ and loopless G, $E(G) = \emptyset$ and there is nothing to prove. Otherwise G falls into one of three cases.

Case (a): G has a loop. Delete all loops to obtain G^-, which is connected, bridgeless, and loopless on n vertices. By induction (the loopless case on n vertices), $f|_{G^-}$ is Σ-joined to $h|_{G^-}$ in G^-, which lifts directly to G leaving loops fixed. For each loop e of G, both $f(e)$ and $h(e)$ lie in $\Gamma \setminus \{0\}$; by (R3), a walk in $\Gamma \setminus \{0\}$ with steps in Σ joins $f(e)$ to $h(e)$, and the corresponding moves on the 1-cycle e transform the flow to h.

Case (b): G is loopless and has a 2-edge-cut $\{e_1, e_2\}$. Contract e_1 (identifying its ends and deleting e_1) to obtain a connected bridgeless multigraph G/e_1 on $n-1$ vertices (possibly with loops). By (E5) and vertex conservation, restriction to $E(G) \setminus \{e_1\}$ gives a bijection between flows of G and flows of G/e_1; moreover, conservation across the cut forces $g(e_1) = \pm g(e_2)$ for every flow g of G, so restriction preserves nowhere-zero flows and targets. Every cycle C' of G/e_1 lifts to a cycle C of G (either C' itself or $C' \cup \{e_1\}$) with $\chi_C = \chi_{C'}$ on $E(G/e_1)$, lifting every Σ-move of G/e_1 to a Σ-move of G. Induction on G/e_1 finishes Case (b).

Case (c): G is loopless and 3-edge-connected, $n \geq 2$. By Lemma 4.1, f is Σ-joined to a nowhere-zero flow f' whose A-coordinate $u' = \overline{f'}$ contains a good subgraph H in its support; in particular f' is H-good, and $|V(H)| \geq 2$. Thus $G^* = G/H$ is connected, bridgeless, and has fewer than n vertices (possibly with loops). By induction, $f'|_{G^*}$ is Σ-joined in G^* to $h|_{G^*}$ through a sequence $g_0, \ldots, g_\nu = h|_{G^*}$. Applying Lemma 4.8 ν times Σ-joins f' in G to an H-good flow f_ν agreeing with h outside $E(H)$; write $u_\nu = \overline{f_\nu}$.

If H is of type (CYC), then $\overline{h} - u_\nu = c\chi_H$ for some $c \in A$. The set $X = \{c' \in A : u_\nu + c'\chi_H$ has a zero on $E(H)\}$ has size at most 5 and avoids 0 and c. By (R1), a Σ_A-walk in $A \setminus X$ from 0 to c gives Σ-moves $\gamma(d_r)\chi_H$ reaching a flow f'' with $\overline{f''} = \overline{h}$ agreeing with h off $E(H)$. Then $h - f'' = y\chi_H$ with $y \in B$; by (R2) (with $X = \emptyset$), y is a sum of elements of Σ_B, and applying those moves on H keeps the A-coordinate equal to $\overline{h}$ (nowhere zero) and reaches h.

If H is of type (SUB), then $\overline{h} - u_\nu$ is an A-flow of H, so Lemma 3.6(c) yields an A-flow Y of Q with $\pi(Y) = \overline{h} - u_\nu$. Since $u_\nu + \pi(Y) = \overline{h}$ is nowhere zero, $Y \in P(\Lambda_{f_\nu})$. Lemma 4.6 Σ-joins f_ν to a flow f'' agreeing with h off $E(H)$ with $\overline{f''} = \overline{h}$. Then $h - f''$ is a B-flow of H, which by (E3) and (R2) ($X = \emptyset$) is a sum of cycle flows $y\chi_O$ ($y \in \Sigma_B$, $O \subseteq H$). Adding these one by one keeps the A-coordinate equal to $\overline{h}$ (hence nowhere zero by (E1)–(E2)) and arrives at h. □

Combining Theorem 1.3 with Corollary 2.8(b) immediately yields the integer form: if $\Gamma = \mathbb{Z}_k$ ($k = ab$ with $a \geq 10$, $b \geq 4$), $B = a\mathbb{Z}_k$, $A = \mathbb{Z}_a$, and Σ is a step system of liftable elements satisfying (R1), (R2), (R3), and (AV), then $M_k(G)$ is connected for every finite bridgeless multigraph G (loops allowed) and $I_k(G)$ is connected for every finite bridgeless loopless multigraph G.

5. Product averaging and the abelian group theorem

Lemma 5.1 (Averaging in a product)**.** *Let $\Gamma = \Gamma_1 \times \Gamma_2$ be a finite abelian group with both factors non-trivial, let $B \le \Gamma$ be a subgroup with $A = \Gamma/B$ satisfying*

$$\frac{|B|-1}{|\Gamma|-1} < \frac{1}{9} \qquad \text{(in particular, whenever } |A| \ge 9\text{)},$$

and let $\Sigma \subseteq \Gamma \setminus \{0\}$ be a step system containing a generating set of $\Gamma_1 \times 0$ and a generating set of $0 \times \Gamma_2$. Then on every multigraph G with m edges, every nowhere-zero Γ-flow is Σ-joined to a nowhere-zero Γ-flow whose A-coordinate has at most $m/9$ zeros. In particular, (AV) *holds.*

Proof. If $m = 0$ there is nothing to prove; assume $m \ge 1$ and put $g = |\Gamma|$. Write elements of Γ as $x = (x_{(1)}, x_{(2)})$ and Γ-flows as $\varphi = (\varphi_{(1)}, \varphi_{(2)})$. Since $\Sigma = -\Sigma$, every element of $\Gamma_1 \times 0$ is a finite sum of elements in $\Sigma_1 = \Sigma \cap (\Gamma_1 \times 0)$, and similarly for $\Sigma_2 = \Sigma \cap (0 \times \Gamma_2)$. For a nowhere-zero Γ-flow φ, let $G\langle\varphi_{(2)}\rangle = (V(G), \mathrm{supp}(\varphi_{(2)}))$ and $G\langle\varphi_{(1)}\rangle = (V(G), \mathrm{supp}(\varphi_{(1)}))$.

Step 1: Random coordinate rounds. A *type*-1 *round* replaces φ by $(\varphi_{(1)} + \xi, \varphi_{(2)})$, where ξ is chosen uniformly at random from the finite group of Γ_1-flows of $G\langle\varphi_{(2)}\rangle$ (extended by 0 off $\mathrm{supp}(\varphi_{(2)})$); a *type*-2 *round* updates $\varphi_{(2)}$ uniformly by a Γ_2-flow of $G\langle\varphi_{(1)}\rangle$. Every outcome is nowhere zero (on $\mathrm{supp}(\varphi_{(2)})$ the second coordinate stays non-zero; outside it nothing changes) and is Σ-joined to φ: decomposing $\xi = \sum_{r=1}^{\nu} z_r \chi_{C_r}$ in $G\langle\varphi_{(2)}\rangle$ (Lemma 2.1(a)) and writing each $(z_r, 0)$ as a sum of elements of Σ_1 yields a sequence of Σ-moves keeping the second coordinate fixed.

Step 2: Single-edge transition kernel. Define matrices Π_1, Π_2 indexed by $\Gamma \setminus \{0\}$ by

$$\Pi_1(x,y) = 1/|\Gamma_1| \text{ if } x_{(2)} = y_{(2)} \neq 0, \qquad \Pi_1(x,x) = 1 \text{ if } x_{(2)} = 0, \qquad \Pi_1(x,y) = 0 \text{ otherwise},$$
$$\Pi_2(x,y) = 1/|\Gamma_2| \text{ if } x_{(1)} = y_{(1)} \neq 0, \qquad \Pi_2(x,x) = 1 \text{ if } x_{(1)} = 0, \qquad \Pi_2(x,y) = 0 \text{ otherwise}.$$

We claim that if φ' is the result of a random type-i round applied to a fixed nowhere-zero flow φ, then $\Pr[\varphi'(e) = y] = \Pi_i(\varphi(e), y)$ for every $e \in E(G)$ and $y \in \Gamma \setminus \{0\}$. Take $i = 1$ (by symmetry). If $e \notin \mathrm{supp}(\varphi_{(2)})$, then $\varphi_{(2)}(e) = 0$ and $\xi(e) = 0$, matching Π_1. If $e \in \mathrm{supp}(\varphi_{(2)})$, then e cannot be a bridge of $G\langle\varphi_{(2)}\rangle$: otherwise summing conservation of $\varphi_{(2)}$ over one side of the bridge would force $\varphi_{(2)}(e) = 0$. Hence e lies on a cycle C of $G\langle\varphi_{(2)}\rangle$, so $z\chi_C$ takes every value $z \in \Gamma_1$ on e; thus the evaluation homomorphism $\xi \mapsto \xi(e)$ maps onto Γ_1 with equisized cosets, making $\varphi_{(1)}(e) + \xi(e)$ uniform on Γ_1 while keeping $\varphi_{(2)}(e)$ fixed.

Step 3: Finite-horizon chain. Fix an arbitrary integer $R \ge 0$ and a starting nowhere-zero flow $f = f^{(0)}$. Form a sequence $f^{(0)}, f^{(1)}, \ldots, f^{(R)}$ by applying at step r ($0 \le r < R$) a random round of type 1 (for even r) or type 2 (for odd r), drawn uniformly from the flow group of the current support subgraph conditional on $f^{(0)}, \ldots, f^{(r)}$. The admissible R-round histories form a finite probability space in which every realizable sequence has positive probability and produces a nowhere-zero Γ-flow $f^{(R)}$ that is Σ-joined to f. For any fixed edge e, let μ_r be the marginal distribution of $f^{(r)}(e)$ on $\Gamma \setminus \{0\}$ (as a row vector). Conditioning on $f^{(r)} = \varphi$ and summing over φ via Step 2 gives $\mu_{r+1} = \mu_r \Pi_1$ for even r and $\mu_{r+1} = \mu_r \Pi_2$ for odd r; thus $\mu_{2h} = \mu_0 \Pi^h$ for $\Pi = \Pi_1 \Pi_2$ and $2h \le R$.

Step 4: Positivity of Π^3 and contraction to uniform. Both Π_1 and Π_2 are symmetric and stochastic, so Π is doubly stochastic and fixes the uniform vector $\eta = (1/(g-1), \ldots, 1/(g-1))$. For any $x, y \in \Gamma \setminus \{0\}$: (i) $\Pi(x,x) > 0$; (ii) if $x_{(2)} \neq 0$ and $y_{(1)} \neq 0$, then $\Pi(x,y) > 0$ via $(y_{(1)}, x_{(2)})$; (iii) if $x_{(2)} = 0$, then $\Pi(x, (x_{(1)}, z)) > 0$ for every $z \in \Gamma_2$; (iv) if $y_{(1)} = 0$, then $\Pi((z, y_{(2)}), y) > 0$ for every $z \in \Gamma_1$. Since Γ_1 and Γ_2 are non-trivial, for any $x, y \in \Gamma \setminus \{0\}$ there is x' with $x'_{(2)} \neq 0$ and $\Pi(x, x') > 0$ (take $x' = x$ or apply (iii)). If $y_{(1)} \neq 0$, then $\Pi^3(x,y) \ge \Pi(x,x')\Pi(x',y)\Pi(y,y) > 0$ by (ii) and (i); if $y_{(1)} = 0$, pick $z \in \Gamma_1 \setminus \{0\}$ and set $x'' = (z, y_{(2)})$, giving $\Pi^3(x,y) \ge \Pi(x,x')\Pi(x',x'')\Pi(x'',y) > 0$ by (ii) and (iv).

Every positive entry of Π_i is at least $1/|\Gamma_i|$, so every entry of Π^3 is at least $1/(|\Gamma_1||\Gamma_2|)^3 = 1/g^3$. Putting $\delta = (g-1)/g^3 \in (0,1)$ and writing Π_{u} for the all-$1/(g-1)$ matrix, both Π^3 and Π_{u} are

doubly stochastic, so $\Pi^3 = \delta\Pi_{\mathrm{u}} + (1-\delta)\Pi_{\mathrm{r}}$ for a doubly stochastic matrix Π_{r} (in particular, $\eta\Pi_{\mathrm{r}} = \eta$). Hence for every probability row vector μ, $\mu\Pi_{\mathrm{u}} = \eta$ yields $\mu\Pi^3 - \eta = (1-\delta)(\mu - \eta)\Pi_{\mathrm{r}}$; since every row-stochastic matrix contracts the ℓ_1-norm on row vectors, $\|\mu\Pi^3 - \eta\|_1 \le (1-\delta)\|\mu - \eta\|_1$ and inductively $\|\mu_0\Pi^{3j} - \eta\|_1 \le 2(1-\delta)^j$ for every $j \ge 0$.

Step 5: Expected number of zeros. Fix $j \ge 0$ large enough that $2(1-\delta)^j < 1/9 - (|B|-1)/(|\Gamma|-1)$, and run Step 3 with $R = 6j$. Because $f^{(6j)}$ is nowhere zero, $\overline{f^{(6j)}(e)} = 0$ holds if and only if $f^{(6j)}(e) \in B \setminus \{0\}$. Therefore, for every edge e,

$$\Pr\big[\overline{f^{(6j)}(e)} = 0\big] \;=\; \sum_{y \in B\setminus\{0\}} \mu_{6j}(y) \;\le\; \frac{|B|-1}{|\Gamma|-1} + \|\mu_{6j} - \eta\|_1 \;\le\; \frac{|B|-1}{|\Gamma|-1} + 2(1-\delta)^j \;<\; \frac{1}{9}.$$

By linearity of expectation (no independence across edges is needed), the expected number of zeros of $\overline{f^{(6j)}}$ is strictly less than $m/9$. Hence some realizable $6j$-round outcome $f^{(6j)}$ is nowhere zero, Σ-joined to f, and has fewer than $m/9$ zeros in its A-coordinate. □

Remark 5.2 (Internal direct sums and the index bound). Since Lemma 5.1 is invariant under group isomorphism, it applies whenever Γ has two non-trivial subgroups Γ_1, Γ_2 with $\Gamma_1 \cap \Gamma_2 = \{0\}$ and $\Gamma_1 + \Gamma_2 = \Gamma$ (for instance $\Gamma_1 = c'\mathbb{Z}_k$ and $\Gamma_2 = c\mathbb{Z}_k$ when $k = cc'$ with $\gcd(c, c') = 1$, $c, c' \ge 2$) such that Σ contains generators of both. Moreover, if $|A| = a \ge 9$ and $|B| = b$, then $(b-1)/(ab-1) < 1/a \le 1/9$ automatically (since $a > 1 \implies a(b-1) < ab - 1$).

Proof of Theorem 1.2. Since Γ is finite abelian and not cyclic of prime-power order (and $|\Gamma| \ge 10d \ge 40$), its decomposition into cyclic groups of prime-power order has at least two non-trivial factors, so $\Gamma \cong \Gamma_1 \times \Gamma_2$ with both factors non-trivial. Let $B \le \Gamma$ be a subgroup of order d, and set $A = \Gamma/B$; then $|B| = d \ge 4$ and $|A| = |\Gamma|/d \ge 10$.

Take the full step system $\Sigma = \Gamma \setminus \{0\}$. Then $\Sigma_B = B \setminus \{0\}$ and $\Sigma_A = A \setminus \{0\}$, so (R1), (R2), and (R3) hold immediately by single steps. Since Σ contains all non-zero elements of both direct factors and $|A| \ge 10 \ge 9$, Lemma 5.1 gives (AV). Theorem 1.3 then proves that $F(G, \Gamma)$ is connected for every finite bridgeless multigraph G. For $40 \le n \le 50$, a divisor d with $4 \le d \le n/10$ exists precisely for $n \in \{40, 44, 48, 50\}$ ($d = 4$ for $40, 44, 48$; $d = 5$ for 50), and none of these four integers is a prime power, so every abelian group of order 40, 44, 48, or 50 is covered. □

6. The moduli 40, 48, and 66 and their multiples

Two short elementary lemmas verify (R1) in $\mathbb{Z}_a$ and (R2) in $\mathbb{Z}_b$ for divisor step sets.

Lemma 6.1 (At most two non-steps in $\mathbb{Z}_a$)**.** *Let $a \ge 10$ and let $D \subseteq \mathbb{Z}_a \setminus \{0\}$ satisfy $D = -D$ and $|D| \ge a - 3$. Then for every set $X \subseteq \mathbb{Z}_a$ with $|X| \le 6$, any two elements of $\mathbb{Z}_a \setminus X$ are joined by a walk in $\mathbb{Z}_a \setminus X$ with steps in D.*

Proof. Let $D^{\mathrm{c}} = (\mathbb{Z}_a \setminus \{0\}) \setminus D$; then $|D^{\mathrm{c}}| \le 2$ and $D^{\mathrm{c}} = -D^{\mathrm{c}}$. Suppose toward a contradiction that $\mathbb{Z}_a \setminus X$ splits into two non-empty sets U and V with no step of D between them. Then $v - u \in D^{\mathrm{c}}$ for all $u \in U$ and $v \in V$, forcing $|V| \le |D^{\mathrm{c}}| \le 2$ and $|U| \le |D^{\mathrm{c}}| \le 2$. Yet $|U| + |V| = |\mathbb{Z}_a \setminus X| \ge a - 6 \ge 4$. Hence $a = 10$ and $|U| = |V| = |D^{\mathrm{c}}| = 2$. In $\mathbb{Z}_{10}$, the unique non-zero self-inverse element is 5, so $D^{\mathrm{c}} = -D^{\mathrm{c}}$ of size 2 must have the form $D^{\mathrm{c}} = \{d_0, -d_0\}$ with $d_0 \ne -d_0$. Writing $U = \{u_1, u_2\}$ gives $V = u_1 + D^{\mathrm{c}} = u_2 + D^{\mathrm{c}}$. Since $u_2 + d_0 \in u_1 + D^{\mathrm{c}}$ differs from $u_1 + d_0$, we get $u_2 + d_0 = u_1 - d_0$, and similarly $u_2 - d_0 = u_1 + d_0$. Subtracting yields $4d_0 = 0$ in $\mathbb{Z}_{10}$, so $5 \mid 2d_0$, hence $5 \mid d_0$, so $2d_0 = 0$ in $\mathbb{Z}_{10}$, contradicting $d_0 \ne -d_0$. □

Lemma 6.2 ($\mathbb{Z}_b$ minus two elements)**.** *Let $b \ge 4$, let d be a proper divisor of b ($1 < d < b$), and let $X \subseteq \mathbb{Z}_b$ have $|X| \le 2$. Then any two elements of $\mathbb{Z}_b \setminus X$ are joined by a walk in $\mathbb{Z}_b \setminus X$ with steps in $\{\pm 1, \pm d\}$.*

Proof. The steps ± 1 form a b-cycle on $\mathbb{Z}_b$. If $|X| \le 1$, removing X leaves a cycle or path, which is connected using ± 1 alone. If $X = \{x, x'\}$ with $x \ne x'$, removing X leaves two arcs (if one is empty we are done). By translating and swapping x, x' if needed, write $x = 0$, $x' = n + 1$, and the two arcs as $I = \{1, \dots, n\}$ and $I' = \{n+2, \dots, b-1\}$ with $1 \le |I| \le |I'|$. Then $n \le (b-2)/2$, and since $d \mid b$ with $d < b$ we have $d \le b/2$. Consequently,

$$n + 2 \;\le\; n + d \;\le\; \frac{b-2}{2} + \frac{b}{2} \;=\; b - 1,$$

so $n + d \in I'$, and the single step $+d$ bridges $n \in I$ to $n + d \in I'$. □

Proof of Theorem 1.1. Write $k = k_0 m$ with $k_0 \in \{40, 48, 66\}$ and $m \ge 1$. Choose (a, b) according to Table 2, so $k = ab$ with $a \ge 10$ and $b = b_0 m \ge 4$ even. Set $\Gamma = \mathbb{Z}_k$, $B = a\mathbb{Z}_k \cong \mathbb{Z}_b$, $A = \Gamma/B \cong \mathbb{Z}_a$, and take the step system of all proper divisors with signs:

$$\Sigma \;=\; \{\, \pm q \in \mathbb{Z}_k : q \mid k,\ 1 \le q < k \,\}.$$

Then $\Sigma \subseteq \mathbb{Z}_k \setminus \{0\}$ and $\Sigma = -\Sigma$. We verify every hypothesis of Theorem 1.3 and Corollary 2.8:

TABLE 2. Parameters for Theorem 1.1 ($k = k_0 m$, $m \ge 1$): base divisor steps in $\mathbb{Z}_{k_0}$, omitted non-zero residues $D^{\mathrm{c}} \subseteq \mathbb{Z}_a$ for $m = 1$, and averaging bound.

k_0	a	$b = b_0 m$	Base divisors $q \mid k_0$ $(1 \le q < k_0)$	Missing mod a	$(b-1)/(ab-1)$
40	10	$4m$	$1, 2, 4, 5, 8, 10, 20$	± 3	$< 1/10 < 1/9$
48	12	$4m$	$1, 2, 3, 4, 6, 8, 12, 16, 24$	± 5	$< 1/12 < 1/9$
66	11	$6m$	$1, 2, 3, 6, 11, 22, 33$	± 4	$< 1/11 < 1/9$

(1) *Liftability.* Every element of Σ is liftable by Corollary 2.8(a).
(2) *Condition* (R1). Every proper divisor q_0 of k_0 satisfies $1 \le q_0 < k_0 \le k$ and $q_0 \mid k$, so $\pm q_0 \in \Sigma$. Reducing the base divisors in Table 2 modulo a already yields every non-zero element of $\mathbb{Z}_a$ except the pair D^{c} consisting of ± 3 (for $k_0 = 40$, $a = 10$), ± 5 (for $k_0 = 48$, $a = 12$), or ± 4 (for $k_0 = 66$, $a = 11$). Hence $\Sigma_A \supseteq \mathbb{Z}_a \setminus (\{0\} \cup D^{\mathrm{c}})$, so $|\Sigma_A| \ge a - 3$ (for $m > 1$, Σ_A may be strictly larger), and Lemma 6.1 gives (R1).
(3) *Condition* (R2). Because $b = b_0 m \ge 4$ is even, $d = 2$ is a proper divisor of b ($1 < 2 < b$) and $1 \le a < 2a < 4a \le ab = k$, so a and $2a$ are proper divisors of k. Under the isomorphism $B = a\mathbb{Z}_k \xrightarrow{\sim} \mathbb{Z}_b$ ($aj \mapsto j$), the subset $\{\pm a, \pm 2a\} \subseteq \Sigma_B$ corresponds to $\{\pm 1, \pm 2\} \subseteq \mathbb{Z}_b$, so Lemma 6.2 gives (R2).
(4) *Condition* (R3). Since $1 \in \Sigma$, the walk $x, x+1, \dots, x'$ joins any $1 \le x < x' \le k-1$ inside $\mathbb{Z}_k \setminus \{0\}$.
(5) *Condition* (AV). Let $c = 2^{v_2(k)}$ be the largest power of 2 dividing k, and put $c' = k/c$. Since k_0 is divisible by both 2 and an odd prime (5 for 40; 3 for 48, 66), we have $c \ge 2$, $c' \ge 3$ odd, $\gcd(c, c') = 1$, and $cc' = k$. Thus c and c' are both proper divisors of k, hence both lie in Σ and generate the complementary internal direct factors $\Gamma_1 = c'\mathbb{Z}_k$ (order c) and $\Gamma_2 = c\mathbb{Z}_k$ (order c'). Since $|A| = a \ge 10 \ge 9$, Remark 5.2 and Lemma 5.1 establish (AV).

Since $|A| = a \ge 10$ and $|B| = b \ge 4$, Theorem 1.3 proves that any two nowhere-zero $\mathbb{Z}_k$-flows of a finite bridgeless multigraph G (loops allowed) are Σ-joined, giving (a). For loopless G, Corollary 2.8(b) then gives (b). □

Remark 6.3 (Why 40 is the exact threshold of this method). The bound $k = 40 = 10 \times 4$ matches the exact avoidance counts of Section 4: $|A| \ge 10$ (exceeding $3 + 3 + 3 = 9$) and $|B| \ge 4$ (exceeding 3). For $k = 44$, the unique factorization with $a \ge 10$ and $b \ge 4$ is $44 = 11 \cdot 4$; its proper divisors $1, 2, 4, 11, 22$ project onto only six residues $\{\pm 1, \pm 2, \pm 4\} \subseteq \mathbb{Z}_{11}$, so deleting the six neighbours of 0 disconnects $\mathbb{Z}_{11}$ and (R1) fails. Thus Theorem 1.1 does not cover $I_{44}(G)$, whereas Theorem 1.2

covers $M_{44}(G) = F(G, \mathbb{Z}_{44})$—making $k = 44$ a concrete open instance of Problem 2.6 of [EHLMNS]. Similarly, cyclic prime-power groups $\mathbb{Z}_{p^r}$ lack any non-trivial direct-product split for Lemma 5.1.

Discussion

Theorems 1.1 and 1.2 resolve Problem 1.2 of [EHLMNS] positively (with $k = 40$ and all positive multiples of 40, 48, and 66) and lower the smallest order of a known universal abelian group (for $|\Gamma| \geq 3$) relative to [EHLMNS] from 256 to 40. Further moduli and abelian group bounds appear in [Go1, Go2, Go3], and Eulerian graphs are settled for all $k \geq 4$ in [Go4].

Three main directions remain open.

(1) *The least universal integer modulus.* Theorem 1.1 proves $k = 40$ suffices for all bridgeless loopless multigraphs, while $k = 5$ fails [EHLMNS, Theorem 3.4]; whether $I_k(G)$ is connected for all bridgeless loopless G when $k = 6$—or for any integer modulus $6 \leq k \leq 39$—remains open.

(2) *Further integer moduli and Problem 2.6 of* [EHLMNS]. All moduli $k \geq 6$ not divisible by 40, 48, or 66 remain open in Theorem 1.1, and even combined with [Go1] the moduli 41–47 and all primes remain open for $I_k(G)$ over all graphs. For $k = 44$, Theorem 1.2 proves $M_{44}(G)$ is connected while $I_{44}(G)$ is open, giving an explicit test case for Problem 2.6 of [EHLMNS] (restated as Conjecture A in [CLSWX]).

(3) *Small abelian groups and cyclic prime powers.* Theorem 1.2 lowers the smallest order of a known universal abelian group (with $|\Gamma| \geq 3$) relative to [EHLMNS] to 40, and [Go2] reaches order 32 for groups of exponent dividing 4 (including $\mathbb{F}_2^5$), with partial results for $\mathbb{F}_2^3$ and $\mathbb{F}_2^4$; all groups of order 6–39 remain open in Theorem 1.2 (as do cyclic prime-power groups up to 2.61×10^{15} [EHLMNS, Corollary 6.20], treated from order 3024 onward in [Go3]), and even with [Go2] every group of order 6–31—in particular $\mathbb{Z}_6$, $\mathbb{F}_2^3$, and $\mathbb{F}_2^4$—remains open for general bridgeless graphs.

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References

[BM76] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, 1976 (Appendix IV: Unsolved Problems, Problem 50).

[CLSWX] D. W. Cranston, J. Li, B. Su, Z. Wang, and N. Xu, *Reconfiguration of nowhere-zero flows*, arXiv:2606.24685v1, June 23, 2026.

[Co26] J. Codsi (collector), *Open Problems for the 2026 Barbados Graph Theory Workshop*, Bellairs Research Institute, January 2026 (Problem 14, from L. Esperet, pp. 5–6), https://web.math.princeton.edu/~pds/barbados26/problems2.pdf

[EHLMNS] L. Esperet, K. Hendrey, A. Lagoutte, M. Marseloo, S. Norin, and R. Steiner, *Nowhere-zero flow reconfiguration*, arXiv:2512.17342v4, December 2025 (v4: July 3, 2026).

[Go1] D. Goel, *The Esperet–Hendrey–Lagoutte–Marseloo–Norin–Steiner (2025) flow reconfiguration problem has a positive answer* (expanded version), preprint, October 2026 (revised November 25, 2026), https://hexagonmath.org/2611.01084 and https://doi.org/10.5281/zenodo.23824039

[Go2] D. Goel, *Flow reconfiguration with one modulus for all graphs, Cereceda's conjecture for bipartite 2-degenerate graphs, and other miscellanea*, preprint, October 23, 2026, https://hexagonmath.org/2611.00520 and https://doi.org/10.5281/zenodo.23603237

[Go3] D. Goel, *Nowhere-zero flow reconfiguration in every finite abelian group of order at least 3024*, preprint, October 23, 2026, https://doi.org/10.5281/zenodo.23605837

[Go4] D. Goel, *The Cranston–Li–Su–Wang–Xu (2026) conjecture is true*, preprint, October 23, 2026, arXiv:2610.40863 (October 29, 2026) and https://doi.org/10.5281/zenodo.23601787

[JLPT] F. Jaeger, N. Linial, C. Payan, and M. Tarsi, *Group connectivity of graphs—a nonhomogeneous analogue of nowhere-zero flow properties*, J. Combin. Theory Ser. B 56 (1992), 165–182, doi:10.1016/0095-8956(92)90016-Q

[JT95] T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley-Interscience, New York, 1995 (Chapter 13, Section 13.1).

[Ni18] N. Nishimura, *Introduction to reconfiguration*, Algorithms 11 (2018), no. 4, Art. 52.

[Se] P. D. Seymour, *Nowhere-zero* 6*-flows*, J. Combin. Theory Ser. B 30 (1981), 130–135.

[Tu] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6 (1954), 80–91.

[vdH13] J. van den Heuvel, *The complexity of change*, in *Surveys in Combinatorics 2013*, London Math. Soc. Lecture Note Ser. 409, Cambridge Univ. Press, 2013, pp. 127–160.

[Zh97] C.-Q. Zhang, *Integer Flows and Cycle Covers of Graphs*, Monographs and Textbooks in Pure and Applied Mathematics 205, Marcel Dekker, New York, 1997.