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# The order of excluded minors for surfaces

Alec Shelley

*Department of Applied Physics, Stanford University, Stanford, CA, USA*

**Abstract**

Each closed surface $S$ has a finite list of excluded minors, and a graph embeds in $S$ if and only if none of its minors are on the list. How many vertices can these excluded minors have? Houdaigoui and Kawarabayashi conjectured an almost linear upper bound in the Euler genus $g$ of $S$. For nonorientable surfaces, we prove the conjecture by showing that every excluded minor has $O(g(\log g)^2(\log\log g)^2)$ vertices as $g \to \infty$. For orientable surfaces, we disprove it by exhibiting an excluded minor with $(g+2)^2$ vertices for every even $g \geq 2$.

*Keywords:* excluded minor, graph embedding, surface, face-width, genus
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## 1. Introduction

An *excluded minor* for a surface $S$ is a graph that does not embed in $S$ although every proper minor does. A minor of a graph is a graph obtained by a sequence of vertex deletions, edge deletions, and edge contractions. Consider a loopless graph $G$. If $G$ is planar, the four-color theorem gives a coloring of its vertices with at most four colors in which adjacent vertices receive different colors [1, 2, 24]. By Wagner's formulation of Kuratowski's theorem [18, 33], $G$ can be embedded in the plane without crossings if and only if it has neither $K_5$ nor $K_{3,3}$ as a minor (Figure 1). The five vertices of $K_5$ require five colors, and $K_{3,3}$ is bipartite and requires only two. This suggests that excluding $K_5$ alone suffices for four-colorability, and Wagner's structure theorem shows that this intuition is correct [24, 33]. On the torus even $K_7$ embeds, so seven colors can be necessary [20].

The list of excluded minors is finite for every surface. For the projective plane, Glover and Huneke [11] first proved that finitely many forbidden

subdivisions suffice. Glover, Huneke and Wang [12] exhibited 103 obstructions under subdivision, and Archdeacon [3] proved the list complete. Exactly 35 of these graphs are minor-minimal [4, Section 3]. Archdeacon and Huneke [5] proved finiteness for all nonorientable surfaces, and Robertson and Seymour [25] proved it for every surface. Thomassen [30] gave a short proof. The complete list for the torus remains open. Myrvold and Woodcock tabulate 17 535 excluded minors [21, Table 2], and Ulrigg reports fourteen more [32, Section 1].

For a fixed surface, an explicit upper bound on the number of vertices in each excluded minor reduces the search for the complete list to a finite, though possibly large, set of graphs [8]. Bounds on this number, the *order* of the graph, also enter the analysis of embedding algorithms [17].

For a closed connected surface of Euler characteristic $\chi$, let the *Euler genus* be $2-\chi$. We say the orientable surface $S_\gamma$ of genus $\gamma \geq 0$ has $\gamma$ handles and Euler genus $2\gamma$. The nonorientable surface $N_g$ with $g \geq 1$ crosscaps is the connected sum of $g$ copies of the real projective plane $\mathbb{RP}^2$ and has Euler genus $g$. For a graph $H$, let $e(H)$ be the least Euler genus among surfaces of either type in which $H$ embeds [23].

Seymour [28] proved that an excluded minor for a surface of Euler genus $g$ has at most $2^{2^k}$ vertices, where $k = (3g+9)^9$. Houdaigoui and Kawarabayashi give the bounds $g^{O(\log^3 g)}$ in [13] and $g^{2+o(1)}$ in [14]. Remark 8.1 discusses the scope of their arguments.

### *1.1. Order bounds for surfaces*

An *almost linear* upper bound in $g$ has the form $g^{1+o(1)}$ as $g \to \infty$. Equivalently, for each $\delta > 0$ there is a constant $C_\delta$ such that the order is at most $C_\delta g^{1+\delta}$. Houdaigoui and Kawarabayashi ask whether the maximum order of an excluded minor admits such a bound [14, Section 8]. For nonorientable surfaces, we show that the answer to their question is yes.

**Theorem 1.1.** *There is an absolute constant $C$ such that, for every $g \geq 1$, each excluded minor $G$ for $N_g$ satisfies*

$$|V(G)| \leq C(g+1)\log^2(g+2)\,\log^2\bigl(2+\log(g+2)\bigr).$$

*In particular, for every $\varepsilon > 0$ there is a constant $C_\varepsilon$ such that $|V(G)| \leq C_\varepsilon(g+1)\log^{2+\varepsilon}(g+2)$.*

The disjoint union of $g+1$ copies of $K_{3,3}$ is an excluded minor for $N_g$ with $6(g+1)$ vertices. Theorem 1.1 is therefore sharp up to a factor of $O((\log g)^2(\log\log g)^2)$.

For orientable surfaces the answer to the same question is negative. The examples are the classical projective grids. For $k \geq 3$, form the *projective* $k \times k$ *grid* $P_k$ from the square grid on $\{1,\ldots,k\}^2$ by adding edges between the left and right columns in reverse order. The added edges join $(i,1)$ to $(k+1-i,k)$ for $1 \leq i \leq k$.

**Theorem 1.2.** *Let $\gamma \geq 1$. The projective grid $P_{2\gamma+2}$ is an excluded minor for the orientable surface $S_\gamma$. Consequently, for every even $g \geq 2$, the orientable surface of Euler genus $g$ has an excluded minor with $(g+2)^2$ vertices.*

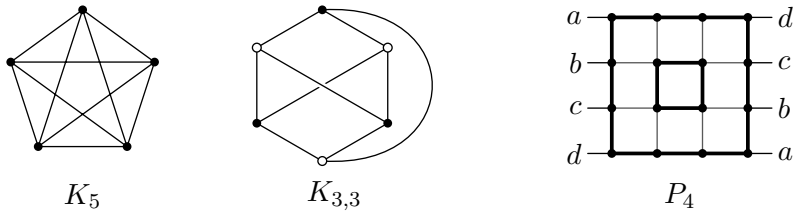


Figure 1: The two excluded minors for the sphere and the torus example $P_4$ from Theorem 1.2. In $P_4$, half-edges with matching letters form one edge. The two bold squares bound nested disks in its projective-plane drawing.

Theorem 1.2 follows from two classical facts. In the usual projective-plane drawing of $P_k$, the least number of points in which a noncontractible simple closed curve meets the graph is $k$. Deleting or contracting any edge lowers this number [15, 22]. The genus formula of Fiedler, Huneke, Richter and Robertson [9] therefore gives orientable genus $\gamma+1$ for $P_{2\gamma+2}$ and at most $\gamma$ after any edge deletion or contraction (see also Fulek and Kynčl [10] for the genus of the grid). Section 13 gives explicit curves and embeddings for these minors.

*1.2. Proof idea for Theorem 1.1*

In a plane graph, even a triangle can enclose arbitrarily many vertices. For a 2-connected excluded minor for $N_g$, we show that a disk piece with $w$ boundary vertices, counted with multiplicity, contains $O((w+1)\log^2(w+2))$ vertices (Lemma 12.1). This reduces the problem to finding a small number of pieces with short boundaries.

For the disk estimate we adapt Houdaigoui and Kawarabayashi's comparison of face boundaries after an edge deletion [14, Proposition 5.5]. By Tutte's characterization of faces in a 3-connected plane graph, the interior faces of the original disk have induced, nonseparating boundary cycles [31, (2.8)]. Many layers around the deleted edge force exponentially many disjoint cycles to cease bounding faces in the new embedding. The Euler genus required by the rest of the graph leaves room for only $O(w+1)$ such cycles, so there are $O(\log(w+2))$ layers (Proposition 11.1).

Short paths to the boundary now give an induction on $w$. For its initial cases we need a bound independent of $g$ when the number of attachments is fixed. Richter's genus formula for graphs meeting in two vertices helps bound the possible arrangements of those attachments [19, Theorem 3.2]. We group the pieces by which arrangements admit a drawing of the piece. Within each group, we label the attachments and apply the labelled form [27, Theorem 1.7] of Robertson and Seymour's graph-minor theorem [26]. This gives only finitely many minor-minimal pieces in each group (Theorem 9.2).

Cutting the surface gives $O(g \log g)$ pieces with $O(\log g)$ boundary vertices each and leaves $O(g \log^2 g)$ other vertices. Summing the disk estimates gives the bound for 2-connected graphs. The block genus formulas of Stahl and Beineke give the general case [29, 19].

*1.3. A vertex bound for embedded graphs*

Theorem 1.3 gives an upper bound on the number of vertices of an embedded graph. Besides the Euler genus of the surface, its inputs include bounds on how many cycles can nest in a disk or run parallel through an annulus, and how many vertices a disk can enclose when its boundary cycle has few attachments to the rest of the graph. A further hypothesis bounds families of noncontractible cycles across the whole surface. For excluded minors satisfying the estimates of Section 8, the theorem gives Corollary 1.4. The proof of Theorem 1.1 uses the cut graph and disk subdivision from this argument with the disk and annulus estimates of Section 12.

Let $S$ be a closed connected surface of Euler genus $h \geq 1$. Let $G$ be a simple 2-connected graph with minimum degree at least 3, embedded in $S$ so that every face is an open disk. A cycle $C$ of $G$ is *contractible* if it bounds a closed disk in $S$. This disk is unique, because $S$ is not the sphere, and we denote it by $\Delta(C)$. An *attachment vertex* of a contractible cycle $C$ is a vertex of $C$ incident with an edge that is not drawn in $\Delta(C)$. A family of cycles is *almost disjoint* if each member has at most one vertex in common

with the union of the other members. A 2-*separation* of $G$ is a pair $(A, B)$ of subgraphs with $A \cup B = G$ that have exactly two common vertices and no common edge, such that $A$ has a vertex outside $B$ and $B$ has at least one edge.

Let $m \geq 1$ and $N \geq 0$ be integers and let $P\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function. We impose the following five hypotheses on the embedding.

(H1) If $(A, B)$ is a 2-separation of $G$ and $B$ is drawn in a closed disk of $S$, then $B$ is a single edge.

(H2) There are no $m + 1$ pairwise disjoint contractible cycles $C_1, \ldots, C_{m+1}$ with $C_i \subset \Delta(C_{i+1})$ for all $i \leq m$.

(H3) There are no $2m + 1$ pairwise disjoint two-sided noncontractible cycles any two of which bound an annulus.

(H4) Every almost disjoint family of noncontractible cycles has at most $N$ members.

(H5) If $C$ is a contractible cycle with at most $a$ attachment vertices, then $\Delta(C)$ contains at most $P(a)$ vertices of $G$.

Hypotheses (H1)–(H4) take their form from the separation and cycle bounds stated by Houdaigoui and Kawarabayashi [14, Sections 4–5]. In (H2) and (H3) we use only the cases of vertex-disjoint cycles. Their disk estimate fixes an attachment bound in terms of the genus [14, Section 6]. In (H5), the disk bound $P(a)$ must hold for each number $a$ of allowed attachments.

**Theorem 1.3.** *There is an absolute constant $c$ with the following property. Let $G$ be a simple 2-connected graph with minimum degree at least 3, embedded in a closed surface of Euler genus $h \geq 1$ so that every face is an open disk. If $G$ satisfies* (H1)–(H5)*, then*

$$|V(G)| \;\leq\; c\,(h\,m^2 + N)\,F(4m + 7), \qquad \text{where } F(s) = 4s \max\{2, P(5s)\}.$$

For fixed $m$ and $P$, the bound is linear in the Euler genus $h$ and the cycle bound $N$ in (H4). The other four hypotheses alone allow arbitrarily large graphs even in the projective plane (Proposition 7.2).

*Proof idea.* By (H2) and (H3) we can cut the surface open into a disk along a graph with $O(hm)$ edges. Short paths to the boundary divide it

into pieces with $O(m)$ boundary vertices each (Lemmas 4.2 and 6.1). When a piece remains a disk after gluing, (H1) and (H5) bound its order. If two boundary copies of one vertex are identified, internally disjoint paths between them through the interior become noncontractible cycles. Across all pieces, (H4) bounds this almost disjoint family, and Menger's theorem bounds the number of vertices needed to separate the copies (Lemma 7.1).

*Application to excluded minors.* When a minimum-Euler-genus embedding of a 2-connected excluded minor satisfies the bounds (B1)–(B5) of Section 8, Theorem 1.3 gives the following corollary.

**Corollary 1.4.** *Let $G$ be a 2-connected excluded minor for a surface of Euler genus $g$. If the embedding of $G$ fixed in [14] has the properties* (B1)–(B5) *of Section 8, then*

$$|V(G)| \;\le\; g\,(\log g)^{O((\log\log g)^3)} \;=\; g^{1+o(1)}.$$

For the orientable surface of even Euler genus $g \ge 2$, Theorem 1.2 explains why this application must be conditional. Its excluded minor $P_{g+2}$ has a drawing in the projective plane with $g/2+1$ concentric square cycles bounding disks one inside the next. Hypothesis (B2) allows only $O(\log g)$ such cycles. For sufficiently large $g$ the violation persists in every embedding of minimum Euler genus (Remark 13.3). Remark 8.1 discusses the remaining hypotheses, and Appendix A treats the Euler genus of subgraphs sharing a vertex.

## 2. Preliminaries

Throughout Sections 2–7, $S$ is a closed connected surface of Euler genus $h \ge 1$, and $G$ is a simple 2-connected graph with minimum degree at least 3, embedded in $S$ so that every face is an open disk. We identify $G$ with its drawing. A closed curve is *contractible* if it is null-homotopic in $S$. A simple closed curve is contractible if and only if it bounds a closed disk, and this disk is unique.

### *2.1. The radial graph*

The *radial graph* $R$ of $G$ is the bipartite multigraph whose nodes are the vertices and the faces of $G$, with one edge joining $v$ and $f$ for each occurrence of the vertex $v$ on the boundary walk of the face $f$. We call the nodes of the first kind *vertex-nodes* and those of the second kind *face-nodes*. We draw $R$ in $S$ by placing each face-node inside its face and each edge inside the

corresponding face. Every face of $R$ is then an open disk bounded by a closed walk of length four, and it contains exactly one edge of $G$, which joins its two vertex-nodes. We call these faces the *quadrilaterals* of $R$.

The graph $R$ is connected and has minimum degree at least 3. A curve contained in $R$ meets $G$ only in vertices. The *distance* between two nodes is their distance in $R$. Since $R$ is bipartite, a path of $R$ with $j$ vertex-nodes other than its first node has at most $2j+1$ edges, and a cycle of $R$ of length $\ell$ has $\ell/2$ vertex-nodes. Thus a short cycle in $R$ gives a noose through few vertices of $G$. Later we measure distance in $R$ and use cycles of $G$ to separate the resulting distance levels.

*2.2. Normal disks and polygons*

A *noose* is a simple closed curve in $S$ that meets $G$ only in vertices. A *normal disk* is a closed disk in $S$ whose boundary is a noose, and its *weight* is the number of vertices of $G$ on its boundary.

**Definition 2.1.** A *polygon* is a closed disk $\bar{D}$ together with a continuous map $\varphi\colon \bar{D} \to S$ that is injective on the interior of $\bar{D}$, such that $R_D := \varphi^{-1}(R)$ is a plane graph in $\bar{D}$ that contains the boundary of $\bar{D}$ as a cycle and whose remaining faces are mapped homeomorphically onto quadrilaterals of $R$. The nodes of $R_D$ are again called vertex-nodes and face-nodes. A node of $R$ may have several preimages on the boundary of $\bar{D}$, which we call its *occurrences*. The polygon is *$\rho$-thin* if every node of $R_D$ is joined to a node on the boundary of $\bar{D}$ by a path of $R_D$ of length at most $\rho$.

If $K$ is a connected subgraph of $R$ and $D$ is a face of $K$ that is an open disk, then cutting $D$ open along its boundary walk gives a polygon. If $\bar{D}$ is a polygon and $\beta \subseteq \bar{D}$ is a closed disk bounded by a cycle of $R_D$, then $\beta$ is a polygon with the restriction of $\varphi$. Distances in a polygon are measured before the boundary is glued back together. In particular, a path in $R_D$ between two occurrences of the same node becomes a closed walk in $R$. If $\varphi$ is injective on $\beta$, its image is a normal disk, which we identify with $\beta$.

*2.3. Three topological facts*

**Lemma 2.2.** (a) *Let $Y \subsetneq S$ be a compact connected subcomplex such that every closed curve in $Y$ is contractible in $S$. Then $Y$ is contained in an open disk of $S$.*

(b) *A finite graph drawn in the interior of a closed disk that separates a point from the boundary circle contains a cycle that separates them. A finite graph drawn in the interior of a closed annulus that separates the two boundary circles contains a cycle that separates them.*

(c) *Let $u, v$ be two points on the boundary of a closed disk. A finite graph drawn in the disk that contains neither $u$ nor $v$ and separates $u$ from $v$ has a component that meets both arcs into which $u$ and $v$ divide the boundary.*

*Proof.* (a) Let $N$ be a regular neighbourhood of $Y$. Since $N$ retracts onto $Y$, every boundary curve of $N$ is contractible in $S$ and bounds a closed disk. If one such disk contains $N$, its interior contains $Y$. Otherwise the disks are pairwise disjoint and meet $N$ only along their boundary circles. Capping $N$ with these disks gives a closed surface contained in $S$, hence all of $S$. The inclusion $N \hookrightarrow S$ is then surjective on fundamental groups. Its image is trivial by the hypothesis on $Y$, contradicting $h \geq 1$.

Statements (b) and (c) are standard consequences of the Jordan curve theorem. See [20] for background on curves in surfaces. □

## 3. Level cycles and thinness

We grow a union of closed faces of $G$ from a node or a cycle of $R$. Its successive frontiers will give nested cycles of $G$ in a disk and parallel cycles of $G$ in an annulus.

The *region* in which we work is $S$, a polygon, or a closed annulus in $S$ whose boundary curves are cycles of $R$. In a polygon $\bar{D}$ we use $G_D = \varphi^{-1}(G)$ and $R_D$ in place of $G$ and $R$, respectively, and take faces relative to $\bar{D}$. Let $N_0$ consist of a single node or of all nodes of a cycle $\gamma$ of $R$ in the region. A path from $N_0$ to a node $y$ has cost equal to the number of vertex-nodes it visits after its first node. Let $\delta(y)$ be the minimum cost of such a path contained in the region.

For $i \geq 1$, start $Y_i$ with the single node or the whole cycle $\gamma$, as appropriate. Adjoin the intersection of the region with each closed face of $G$ whose face-node $f$ satisfies $\delta(f) \leq i - 1$.

**Lemma 3.1.** (a) *The set $Y_i$ is connected. Every vertex $y$ with $\delta(y) \leq i-1$ lies in the interior of $Y_i$ relative to the region. The frontier of $Y_i$ in the region is a subgraph of $G$ all of whose vertices satisfy $\delta = i$.*

(b) *Suppose that the region is $S$ or a polygon, that $N_0$ is a single node, and that $Y_t$ is contained in an open disk $U$ of $S$, respectively in the interior of the polygon. Then there are pairwise disjoint contractible cycles $C_1, \dots, C_t$ of $G$ with $C_i \subset \Delta(C_{i+1})$ for all $i < t$.*

(c) *Suppose that the region is an annulus $\mathbb{A}$ with boundary cycles $\gamma$ and $\gamma'$, that $N_0$ is the node set of $\gamma$, and that $\delta \geq t+1$ on the nodes of $\gamma'$. Then there are pairwise disjoint cycles $C_1, \dots, C_t$ of $G$ in the interior of $\mathbb{A}$, each of which separates $\gamma$ from $\gamma'$ in $\mathbb{A}$.*

*Proof.* (a) For each face $f$ used to form $Y_i$, a path realising $\delta(f)$ lies in $Y_i$ and joins $f$ to $N_0$. Thus $Y_i$ is connected. If $\delta(y) \leq i-1$, then every face at $y$ has $\delta \leq i-1$, so a neighbourhood of $y$ in the region lies in $Y_i$. A vertex on a closed face $f$ with $\delta(f) \leq i-1$ has $\delta \leq i$. Every point of the frontier lies on $G$. Hence the vertices on the frontier satisfy $\delta = i$, and an edge that meets the frontier lies on it together with its ends.

(b) Since $Y_t$ is compact, there is a closed disk $E$, contained in $U$ or in the polygon, whose interior contains $Y_t$. Fix a point $p$ in the interior of $Y_1$ that is not on $G$. For $i \leq t$, let $W_i$ be the component of $E \setminus Y_i$ that contains the boundary of $E$. Its frontier is a finite subgraph of $G$ that separates $p$ from the boundary of $E$, and by (a) its vertices satisfy $\delta = i$. By Lemma 2.2(b) it contains a cycle $C_i$ that bounds a disk in $E$ containing $p$. The cycles are pairwise disjoint since their vertices have different values of $\delta$. To see their order, observe that $Y_i$ contains both $p$ and $C_i$ and misses $C_{i+1}$. By connectedness, $C_i$ lies on the same side of $C_{i+1}$ as $p$, inside its disk.

(c) We argue as in (b), with $W_i$ the component of $\mathbb{A} \setminus Y_i$ that contains $\gamma'$. Let $i \leq t$ and let $f$ be a face with $\delta(f) \leq i-1$. The open face $f$ does not meet $\gamma'$, since otherwise its node would lie on $\gamma'$. The vertices on the boundary of $f$ have $\delta \leq i \leq t$, so they are not on $\gamma'$ either. Hence $Y_i \cap \gamma' = \emptyset$. The frontier of $W_i$ separates $\gamma$ from $\gamma'$, and Lemma 2.2(b) gives the cycle $C_i$, which lies in the interior of $\mathbb{A}$. □

**Lemma 3.2.** *Assume* (H2).

(a) *For every node $x$ of $R$ there is a cycle of $R$ that is noncontractible, has length at most $4m+6$, and all of whose nodes are within distance $2m+3$ of $x$.*

(b) *Let $K$ be a connected subgraph of $R$ and let $D$ be a face of $K$ that is an open disk. The polygon obtained by cutting $D$ open is $(2m+3)$-thin.*

*Proof.* (a) Apply the definitions above with the region $S$ and $N_0 = \{x\}$, and let $Y = Y_{m+1}$. Every node in $Y$ is joined to $x$ by a path of length at most $2m+3$ inside $Y$. Suppose that every closed curve in $Y$ is contractible. Then $Y \neq S$, so $Y$ lies in an open disk by Lemma 2.2(a), and Lemma 3.1(b) gives $m+1$ cycles that contradict (H2). Hence some closed curve in $Y$ is noncontractible.

The set $Y$ is a union of closed faces of $G$. Let $R_Y$ be the subgraph of $R$ formed by the edges that join a vertex to a face $f$ with $\delta(f) \leq m$. Every closed curve in $Y$ is homotopic in $Y$ to a closed walk of $R_Y$, because an edge $vv'$ of $G$ on the boundary of such a face $f$ is homotopic in the closed face to the path $v, f, v'$. The graph $R_Y$ is connected and has a breadth-first tree rooted at $x$ of depth at most $2m+3$. Its fundamental group is generated by the fundamental closed walks of this tree, and each of them is freely homotopic to a fundamental cycle. A fundamental cycle has length at most $2(2m+3)+1$, hence at most $4m+6$ since $R$ is bipartite. One of the fundamental cycles is therefore noncontractible.

(b) Let $x$ be a node of $R_D$ in the interior of the polygon. Apply the definitions with the polygon as region and $N_0 = \{x\}$. If some node on the boundary of the polygon has $\delta \leq m+1$, then a path of length at most $2m+3$ joins $x$ to it. Suppose that there is no such node, and let $f$ be a face with $\delta(f) \leq m$. The open face $f$ does not meet the boundary of the polygon, since otherwise its node would lie on the boundary. The vertices on the boundary of $f$ have $\delta \leq m+1$, so they are not on the boundary of the polygon. Hence $Y_{m+1}$ lies in the interior of the polygon, and Lemma 3.1(b) contradicts (H2). □

## 4. A short cut graph

Lemma 3.2 supplies short noncontractible cycles near every node of $R$. We first select a family that still reaches the whole graph within distance $6m+7$. Separating its members ensures that any two homotopic members would enclose too many parallel cycles of $G$.

**Lemma 4.1.** *Assume* (H2) *and* (H3). *There are cycles $\gamma_1, \ldots, \gamma_k$ of $R$ with the following properties.*

(i) *Each $\gamma_i$ is noncontractible and has length at most $4m+6$.*

(ii) *Two distinct cycles $\gamma_i$, $\gamma_j$ are at distance at least $4m+4$.*

(iii) *Every node of $R$ is within distance $6m+7$ of some $\gamma_i$.*

(iv) *No two of the cycles are freely homotopic, and $k \le 4h$.*

*Proof.* Choose a maximal family with properties (i) and (ii). Let $x$ be a node at distance more than $6m+7$ from every member. The cycle given by Lemma 3.2(a) for $x$ is at distance more than $4m+4$ from every member, which contradicts maximality. This proves (iii).

Suppose that $\gamma_a$ and $\gamma_b$ are freely homotopic, with $a \neq b$. Two disjoint one-sided simple closed curves are not homotopic, because the intersection form modulo 2 takes the value 1 on the class of a one-sided curve with itself and the value 0 on two disjoint curves. Hence both cycles are two-sided. Two disjoint homotopic noncontractible two-sided simple closed curves bound a closed annulus $\mathbb{A}$ [7, Lemma 2.4]. A path of $R$ from $\gamma_a$ to $\gamma_b$ has at least $4m+4$ edges, so it has at least $2m+2$ vertex-nodes other than its first node. Lemma 3.1(c) with $t = 2m+1$ gives $2m+1$ pairwise disjoint cycles of $G$ in the interior of $\mathbb{A}$, each separating its boundary curves. A simple closed curve in an annulus that separates its boundary circles is isotopic to the core. The cycles are therefore noncontractible and two-sided, and any two of them bound an annulus. This contradicts (H3).

It remains to bound $k$. If $h = 1$, all noncontractible simple closed curves in $S$ are homotopic and $k \le 1$. If $k = 1$ there is nothing to prove. Otherwise cut $S$ along the $k$ curves. Let $k_1$ of them be one-sided and $k_2$ two-sided. The pieces are compact surfaces with $2k_2 + k_1$ boundary circles in total, and their Euler characteristics add up to $2 - h$. No piece is a disk. A piece that is an annulus has a boundary circle that comes from a one-sided curve, since otherwise two of the curves would be homotopic. Thus there are at most $k_1$ such pieces. A closed surface of Euler genus $h$ contains at most $h$ pairwise disjoint one-sided simple closed curves. The cores of the pieces that are Möbius bands, together with the one-sided curves of the family, are such curves, so there are at most $h - k_1$ such pieces and $k_1 \le h$. A piece with Euler characteristic $-c \le -1$ has at most $3c$ boundary circles, and these characteristics add up to $2 - h$. Hence $2k_2 + k_1 \le 3(h-2) + 2k_1 + h$, and $k = k_1 + k_2 \le 4h$. □

**Lemma 4.2.** *Assume* (H2) *and* (H3). *There is a connected subgraph $K^*$ of $R$ with minimum degree at least 2 such that $S \setminus K^*$ is an open disk and*

$$|E(K^*)| \;\le\; (76m + 99)\, h.$$

*Proof.* Let $K_0$ be the union of the cycles of Lemma 4.1, and let $T$ be a breadth-first forest of $R$ rooted at the node set of $K_0$. It spans $R$ and has depth at most $6m + 7$. The graph $K_0 \cup T$ has $k$ components, each with exactly one cycle. Add $k - 1$ edges of $R$ to make it connected, and call the result $H_1$. It is a connected spanning subgraph of $R$ whose cycle space has dimension $k$.

We claim that $H_1$ has at most $k+1$ faces. Consider the boundaries of the faces as elements of the cycle space of $H_1$ over the field with two elements. If the boundaries of a nonempty proper set of faces add up to zero, then no edge has a face of the set on exactly one side. Following the faces around a vertex, and then from a vertex to its neighbours, we find that the set contains all faces, a contradiction. Hence the number of faces is at most $k + 1$.

We use the dual spanning tree of the tree–cotree construction [6] to make each face $\Phi$ of $H_1$ into a disk. Form the dual graph in $\Phi$, with a node for each quadrilateral of $R$ and an edge across each edge of $R$ in $\Phi$. Choose a spanning tree of this dual graph and add to $H_1$ the edges of $R$ whose dual edges are outside the tree. The quadrilaterals remaining in each face are joined along the tree. Thus the resulting graph $H_2$ has the same number $f \leq k+1$ of faces as $H_1$, each an open disk. By Euler's formula $|E(H_2)| = |V(R)| - 2 + h + f$, while $|E(H_1)| = |V(R)| - 1 + k$. Hence at most $h$ edges were added, and the set $Y$ of edges of $H_2$ outside $K_0 \cup T$ has at most $k - 1 + h$ members.

Delete from $H_2$, one at a time, $f - 1$ edges that have two different faces on their two sides. The complement of the graph is then one open disk. Next delete vertices of degree 1 as long as there are any, which keeps the complement an open disk. Let $K^*$ be the result. An edge of $T$ belongs to $K^*$ only if it lies on the path of $T$ from an end of an edge of $Y$ to $K_0$. Therefore

$$|E(K^*)| \leq k(4m+6)+(2(6m+7)+1)(k-1+h) \leq 4h(4m+6)+5h(12m+15).$$

□

Only two properties of the chosen cycles entered the length estimate once the cycles were selected. Their lengths bound the contribution of $K_0$, and the distance to their union bounds the length of each surviving tree path. Section 12 uses this construction with different bounds on both quantities.

**Lemma 4.3.** *Let $S$ have Euler genus $h \geq 1$ and let $R$ be the radial graph of a cellular embedding. Suppose that $k \geq 1$ pairwise vertex-disjoint cycles of $R$ have length at most $a$ each, and every node of $R$ is within distance $d$ of*

*their union. Then $R$ has a connected cut graph of minimum degree at least two with at most*

$$ka + (2d+1)(k-1+h)$$

*edges.*

*Proof.* Use the given cycles as $K_0$ in the proof of Lemma 4.2. The forest has depth at most $d$, the connecting edges number $k-1$, and the construction inside the faces adds at most $h$ edges. After trimming, the forest contributes only paths from the ends of these edges to $K_0$. □

Through Section 7, fix the cut graph $K^*$ of Lemma 4.2 and write $L = |E(K^*)|$. Cutting $S$ along $K^*$ gives a polygon $\bar{D}$ with map $\varphi$. Each edge of $K^*$ occurs twice on the boundary, which therefore has length $2L$. By Lemma 3.2(b) the polygon is $r$-thin for $r = 2m+3$.

**Lemma 4.4.** *Let $u \neq v$ be two occurrences on the boundary of $\bar{D}$ of the same node $a$ of $K^*$. Let $W$ be a simple arc in $\bar{D}$ from $u$ to $v$ whose interior lies in the interior of $\bar{D}$. Then the simple closed curve $\varphi(W)$ is noncontractible.*

*Proof.* Suppose that $\varphi(W)$ bounds a closed disk $\Delta$. The arc $W$ cuts $\bar{D}$ into two closed disks $E$ and $E'$, and $S \setminus (K^* \cup \varphi(W))$ is the disjoint union of the connected sets $\varphi(\operatorname{int} E)$ and $\varphi(\operatorname{int} E')$. Neither the interior nor the exterior of $\Delta$ is contained in $K^*$, so exactly one of the two sets, say $\varphi(\operatorname{int} E)$, lies in $\Delta$, and it equals $\operatorname{int}\Delta \setminus K^*$. Since this set is connected, $K^* \cap \Delta$ contains no cycle. It meets $\varphi(W)$ only in $a$. Hence $K^* \cap \Delta$ is a tree that contains $a$, and no edge of $K^*$ outside $\Delta$ is incident with a vertex of this tree other than $a$. A leaf of the tree other than $a$ would have degree 1 in $K^*$. As $K^*$ has minimum degree at least 2, the tree is the single vertex $a$. Then $\varphi(E) = \Delta$, and the arc of the boundary of $\bar{D}$ on $E$, which contains an edge, is mapped into $\partial\Delta \cap K^* = \{a\}$. This is impossible. □

## 5. Normal disks of small weight

The boundary of a normal disk can pass through the interiors of faces, whereas (H5) applies to a cycle of $G$. To use (H5), we decompose the graph in the disk into blocks and bound the number of attachments of each block's outer cycle.

**Lemma 5.1.** *Assume* (H1) *and* (H5). *A normal disk of weight $s$ contains at most $F(s) = 4s\max\{2, P(5s)\}$ vertices of $G$ in its interior.*

*Proof.* Let $\Delta$ be the disk, let $Z$ be the set of vertices on its boundary, let $X$ be the set of vertices in its interior, and let $H$ be the subgraph of $G$ drawn in $\Delta$. Every edge at a vertex of $X$ belongs to $H$. The graph $G$ is not drawn in $\Delta$, because its faces are disks and $S$ is not the sphere. If $s \leq 1$, some vertex of $G$ lies outside $\Delta$ because $G$ has no loop, and then $X = \emptyset$ because $G$ is 2-connected. Assume $s \geq 2$.

Let $\mathcal{B}$ be the block–cutvertex forest of $H$. We may ignore isolated vertices of $H$, since each belongs to $Z$. We count the leaves and branching nodes of $\mathcal{B}$ first, then the blocks along its paths of degree-two nodes.

*(1)* A block that is a leaf or an isolated node of $\mathcal{B}$ contains a vertex of $Z$ that is not a cutvertex of $H$. Otherwise its only cutvertex would be a cutvertex of $G$, or the block would be a component of $G$. Let $\lambda$ be the number of these blocks. Then $\lambda \leq s$, and $\mathcal{B}$ has fewer than $\lambda$ nodes of degree at least 3.

*(2)* Let $B$ be a block of degree 2 in $\mathcal{B}$, with cutvertices $c$ and $c'$, that contains no vertex of $Z$ other than $c$ and $c'$. Then $c$ and $c'$ separate $B$ from the rest of $G$, and $B$ is drawn in a disk. By (H1), $B$ is the edge $cc'$. We call such a block an *edge block*. Every other block of degree 2 contains a vertex of $Z$ that is not a cutvertex.

*(3)* If two edge blocks share a cutvertex $c$ of degree 2 in $\mathcal{B}$, then $c \in Z$. Otherwise $c$ would have degree 2 in $G$.

*(4)* The nodes of degree 2 of $\mathcal{B}$ form at most $2\lambda$ maximal paths. Along such a path, by (3), consecutive edge blocks are separated by a cutvertex in $Z$ or by a block that is not an edge block. The vertices of $Z$ that are not cutvertices and were found in (1) and (2) are distinct, and they are distinct from the cutvertices in $Z$. Hence the number of edge blocks is at most $s + \lambda$. The total number of blocks is at most $\lambda + \lambda + (s - \lambda) + (s + \lambda) \leq 4s$, and there are fewer cutvertices than blocks.

Consider a block $B$ with at least three vertices. Since $B$ is 2-connected, the face of $B$ containing $S \setminus \Delta$ has a boundary cycle $C_B$. This cycle is contractible. We claim that every vertex and edge in $\Delta(C_B)$ belongs to $B$. A vertex of $Z$ in this disk must lie on $C_B$, and an edge in the disk with both ends in $B$ belongs to $B$. Any component of $H - V(B)$ in a face inside $C_B$ would therefore consist of vertices of $X$. It would meet $B$ by edges to a single vertex, making that vertex a cutvertex of $G$.

An attachment of $C_B$ must now be in $Z$ or be a cutvertex of $H$. There are at most $s + 4s$ such vertices, so (H5) gives $|V(B)| \leq P(5s)$. The other blocks have at most two vertices. Every vertex of $X$ lies in a block, and there

are at most $4s$ blocks. This gives the stated bound. □

## 6. Cells of a thin polygon

In a thin polygon, short paths from interior nodes to the boundary give short cuts. Taking every such cut would give too many pieces. The following construction groups cuts by their boundary endpoints and uses planarity to bound the number of pieces in terms of the boundary length alone.

**Lemma 6.1.** *Let $\bar{D}$ be a $\rho$-thin polygon whose boundary has length $\ell$. There are closed disks in $\bar{D}$, each bounded by a cycle of $R_D$, of the following two kinds.*

(a) *At most $16\ell + 4$ disks, called* loop disks, *each with at most $\rho$ vertex-nodes on its boundary and each meeting the boundary of $\bar{D}$ in at most one node.*

(b) *At most $2\ell$ disks $\beta_{uv}$, called* bigons, *indexed by pairs $\{u, v\}$ of nodes on the boundary of $\bar{D}$. The bigons have pairwise disjoint interiors. Each bigon $\beta_{uv}$ has at most $2\rho+1$ vertex-nodes on its boundary, it meets the boundary of $\bar{D}$ in a subset of $\{u, v\}$, and every node in its interior is joined to $u$ or to $v$ by a path of $R_D$ of length at most $\rho$.*

*Every vertex-node in the interior of $\bar{D}$ lies in a loop disk, or in the interior of a bigon, or belongs to a set of at most $4\ell(\rho + 1)$ further vertex-nodes.*

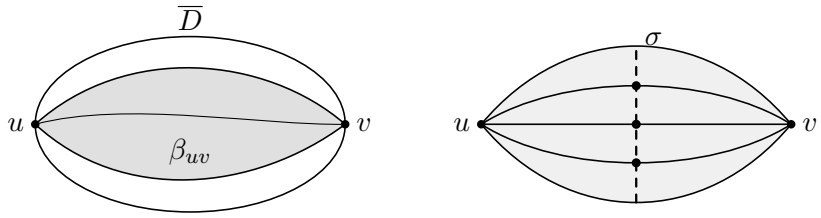


Figure 2: On the left, the two extreme $u$–$v$ chords enclose the shaded bigon $\beta_{uv}$ in the cut disk. On the right, suppose $u$ and $v$ are copies of the same vertex $a$. Each of the three interior paths between them closes to a cycle through $a$ on the surface. The dashed arc cuts the bigon into two disks, each with one copy of $a$.

*Proof.* Let $T_D$ be a breadth-first forest of $R_D$ rooted at the boundary nodes of $\bar{D}$, with $T_u$ the tree at root $u$. The forest has depth at most $\rho$. For each edge $e = xy$ outside the forest and the boundary, form a walk $W_e$ by following the tree path from the root $u$ to $x$, then $e$, then the tree path from $y$ to its root $v$.

If $u \neq v$, then $W_e$ is a path from $u$ to $v$ of length at most $2\rho + 1$ whose inner nodes are in the interior of $\bar{D}$. We call it a *chord* and say that the pair $\{u, v\}$ is *linked*. If $u = v$, then $W_e$ contains a unique cycle $Z_e$. This cycle has length at most $2\rho$ and meets the boundary of $\bar{D}$ in at most the node $u$. We call $e$ a *loop edge* and denote by $\lambda_e$ the closed disk bounded by $Z_e$. Two chords do not cross, because two walks that meet inside a tree of $T_D$ continue together to its root.

Let $\{u, v\}$ be a linked pair. The chords from $u$ to $v$ are linearly ordered. Their union contains two *extreme* chords, and we let $\beta_{uv}$ be the closed disk bounded by the cycle contained in the union of the two extreme chords (Figure 2, left). If there is only one chord from $u$ to $v$, there is no bigon for this pair. A node in the interior of $\beta_{uv}$ belongs to $T_u$ or to $T_v$, since its path in $T_D$ to the boundary leaves $\beta_{uv}$ through an extreme chord and then follows it. This gives the properties in (b), except for the number of bigons. Let $J$ be the union of the boundary of $\bar{D}$ and of all extreme chords.

Contract every tree $T_u$ to its root. The result is a multigraph $M$ drawn in $\bar{D}$ whose vertices all lie on the boundary circle. Its faces in $\bar{D}$ correspond to the quadrilaterals of $R_D$, so each of them is bounded by at most four edges. The linked pairs are the edges of a simple outerplanar graph on the $\ell$ boundary nodes, so there are at most $2\ell$ linked pairs and at most $4\ell$ extreme chords. Each extreme chord has at most $\rho + 1$ vertex-nodes, which accounts for the last set in the statement. The graph $J$ has at most $4\ell + 1$ faces in $\bar{D}$ that are not interiors of bigons.

Fix a face $\Lambda$ of $J$ outside the bigons and consider its image after the contractions. Every chord is extreme or lies in a bigon, so the faces of $M$ within this image meet along loops. A loop at $u$ cuts off a disk meeting the boundary circle only at $u$. All edges inside that disk are again loops at $u$, since every vertex of $M$ lies on the boundary circle.

Exactly one face $\Lambda_0$ of $M$ in this region lies outside all the loop disks. Indeed, a curve from one such face to another that enters the disk of a maximal loop must leave it into the same face. Every maximal loop is an edge of $\Lambda_0$, which has at most four edges.

Now take a node $z$ of $R_D$ in $\Lambda$. Its degree is at least three and a quadrilat-

eral has at most two corners at $z$. At least one incident quadrilateral therefore corresponds to a face other than $\Lambda_0$ and lies inside a maximal loop after contraction. To recover its position in $\bar{D}$, mark a point inside each quadrilateral and contract the trees in disjoint regular neighbourhoods avoiding the marks. Contracting a tree edge shrinks a subpath of $Z_e$ or a branch on one side of it. Consequently $Z_e$ separates the marks in the same way as the corresponding loop of $M$. The closure of the chosen quadrilateral lies in $\lambda_e$, so $\lambda_e$ contains $z$. At most four such disks for each of at most $4\ell + 1$ faces $\Lambda$ suffice. □

**Corollary 6.2.** *Assume* (H1) *and* (H5). *There is an absolute constant $c_0$ such that the following holds. Let $\Delta$ be a normal disk whose boundary is a cycle of $R$ with $w \geq 1$ vertex-nodes, and suppose that $\Delta$ is $\rho$-thin as a polygon. Then $\Delta$ contains at most $c_0\, w\,(\rho + F(2\rho + 1))$ vertices of $G$ in its interior.*

*Proof.* Apply Lemma 6.1 with $\ell = 2w$. Here $\Delta$ is embedded in $S$, so its loop disks and bigons are themselves normal disks. A loop disk contributes at most $\rho$ boundary vertices and $F(\rho)$ interior vertices by Lemma 5.1. For a bigon only its interior remains to be counted. Including the vertices outside these two sets gives at most

$$8w(\rho + 1) + (32w + 4)\big(\rho + F(\rho)\big) + 4w\, F(2\rho + 1). \qquad \square$$

## 7. Proof of Theorem 1.3

Cutting along $K^*$ gave an $r$-thin polygon $\bar{D}$ of boundary length $2L$, where $r = 2m + 3$ and $L = |E(K^*)|$. We apply Lemma 6.1 with $\ell = 2L$ and $\rho = r$, and write $G_D = \varphi^{-1}(G)$. The disk estimate of Section 5 applies to a piece as soon as it embeds back in $S$. The remaining case is a bigon whose two corners are identified at a vertex of $G$.

**Lemma 7.1.** *Assume* (H1)–(H5). *There is an absolute constant $c_1$ such that the number of vertices of $G$ that are not on $K^*$ is at most $c_1\,(L\,m + N)\,F(4m + 7)$.*

*Proof.* Let $\beta \subseteq \bar{D}$ be a closed disk bounded by a cycle of $R_D$. The map $\varphi$ is injective on $\beta$ unless $\beta$ contains two occurrences of the same node. Occurrences of a face-node play no role. If $\beta$ contains an occurrence $u$ of a face-node, remove from $\beta$ a small neighbourhood of $u$ inside the open face. This changes neither the part of $G_D$ in $\beta$ nor the set of vertices on the

boundary of $\beta$. After this modification, the loop disks of Lemma 6.1 are normal disks of weight at most $r$. A bigon $\beta_{uv}$ is a normal disk of weight at most $2r+1$, unless $u$ and $v$ are two occurrences of the same vertex and both lie on the boundary of $\beta_{uv}$. We call a bigon of the latter kind *pinched* (Figure 2, right).

By Lemmas 6.1 and 5.1, the number of vertices in the interior of $\bar{D}$ that are not in the interior of a pinched bigon is at most

$$8L(r+1) + (32L+4)\big(r + F(r)\big) + 4L\,F(2r+1).$$

Fix a pinched bigon $\beta$ whose corners $u$ and $v$ are occurrences of the vertex $a$. In the plane graph $G_\beta$ drawn by $G_D$ in $\beta$, the vertices $u$ and $v$ are distinct and nonadjacent because $G$ has no loop. Write $t_\beta$ for the maximum number of internally disjoint $u$–$v$ paths in $G_\beta$. Menger's theorem gives a separator $\Sigma$ of $t_\beta$ vertices other than $u, v$. We use the separator to cut $\beta$ into normal disks and the paths to bound the cost of these cuts.

Let $A$ be the vertex set of the component of $G_\beta - \Sigma$ that contains $u$, and let $\Xi$ be the union of the closed quadrilaterals of $R_D$ in $\beta$ whose edge of $G_D$ has an end in $A$. All quadrilaterals at $u$ belong to $\Xi$, and no quadrilateral at $v$ does. Consider an edge of $R_D$ that joins a vertex $x$ to a face-node and lies between a quadrilateral in $\Xi$ and a quadrilateral of $\beta$ not in $\Xi$. The edges of $G_D$ in these two quadrilaterals are $xy$ with $y \in A$ and $xy'$ with $x, y' \notin A$. Hence $x$ is a neighbour of $A$ outside $A$, and $x \in \Sigma$. So the frontier of $\Xi$ in $\beta$ is a subgraph of $R_D$ that contains neither $u$ nor $v$, separates $u$ from $v$ in $\beta$, and has all its vertex-nodes in $\Sigma$. By Lemma 2.2(c) it contains a path that joins the two arcs of the boundary of $\beta$ between $u$ and $v$. A shortest such path $\sigma$ has its inner nodes in the interior of $\beta$.

The path $\sigma$ has at most $t_\beta$ vertex-nodes, and it cuts $\beta$ into two closed disks $\beta_u \ni u$ and $\beta_v \ni v$ that are bounded by cycles of $R_D$. Each of them contains one occurrence of $a$ only, so each is a normal disk, bounded by a cycle of $R$ with at most $2r + 1 + t_\beta$ vertex-nodes. Each is $r$-thin because a node in the interior of $\beta$ is joined to $u$ or to $v$ by a path of length at most $r$, and this path either stays in the disk that contains the node or meets its boundary. By Corollary 6.2, the interior of $\beta$ contains at most

$$t_\beta + 2c_0\,(2r+1+t_\beta)\,\big(r + F(2r+1)\big)$$

vertices of $G$.

We now bound $\sum_\beta t_\beta$ over all pinched bigons at once. Delete from $G_\beta$ the vertices on the boundary of $\beta$ other than $u$ and $v$. There are at most $2r+1$ such vertices. The resulting graph has at least $t_\beta - (2r+1)$ internally disjoint paths from $u$ to $v$. The image under $\varphi$ of such a path is a cycle of $G$ through $a$ whose other vertices lie in $\varphi(\operatorname{int}\beta)$, and it is noncontractible by Lemma 4.4. Take these cycles for all pinched bigons. The interiors of the bigons are pairwise disjoint and disjoint from the boundary of $\bar{D}$. Hence each of the cycles has at most one vertex, namely the vertex $a$ of its bigon, in common with the union of the others. The family is almost disjoint, and by (H4)

$$\sum_\beta t_\beta \;\le\; N + 4L(2r+1),$$

since there are at most $4L$ bigons. Adding the bounds, and using $r = 2m+3$ and $F(s) \ge 8s$, we obtain the lemma. □

*Proof of Theorem 1.3.* The graph $K^*$ is bipartite and has minimum degree at least 2, so it has at most $L/2$ vertex-nodes. Every other vertex of $G$ is counted in Lemma 7.1. By Lemma 4.2, $L \le (76m+99)h$. Hence

$$|V(G)| \;\le\; \tfrac{L}{2} + c_1(Lm+N)F(4m+7) \;\le\; c\,(h\,m^2+N)\,F(4m+7). \quad \square$$

In the preceding proof, (H4) bounded the number of paths whose endpoints become the same vertex after gluing. The following family shows why this extra control is needed even when the surface and the other four bounds are fixed.

**Proposition 7.2.** *For $r \ge 3$ let $W_r$ be the graph with a vertex $a$, a cycle $\rho_1\rho_2\cdots\rho_{2r}$ and a cycle $e_1e_2\cdots e_r$, in which $a$ is adjacent to every $\rho_j$ and $e_j$ is adjacent to $\rho_j$ and to $\rho_{j+r}$. It has $3r+1$ vertices and an embedding in the projective plane, with disk faces, that satisfies* (H1), (H2), (H3) *and* (H5) *with $m=2$ and $P(a) = 3a+1$.*

*Proof.* Draw $a$ at the centre of a disk and the cycle $\rho_1\cdots\rho_{2r}$ around it, and glue to this disk a Möbius band whose core is the cycle $e_1\cdots e_r$, so that the faces in the band are the quadrilaterals $\rho_j\rho_{j+1}e_{j+1}e_j$ for $1 \le j \le 2r$, with the indices of $e$ taken modulo $r$. There are $2r$ triangular and $2r$ quadrilateral faces, and $(3r+1) - 7r + 4r = 1$. For each $j \le r$ the cycle $\lambda_j = a\rho_j e_j \rho_{j+r}$ is one-sided.

The graph $W_r$ is 3-connected, so in every 2-separation $(A, B)$ the graph $B$ is an edge, as required by (H1). The projective plane has no two-sided noncontractible simple closed curve, so (H3) holds. For (H2), suppose that $C_1, C_2, C_3$ are pairwise disjoint contractible cycles with $C_1 \subset \Delta(C_2)$ and $C_2 \subset \Delta(C_3)$. A vertex $x$ of $C_1$ lies on some $\lambda_j$. A closed curve drawn next to $\lambda_j$ through its four vertices is a noncontractible noose. It is not contained in $\Delta(C_3)$, so it meets each of $C_2$ and $C_3$ in at least two vertices, and it has at least five vertices, a contradiction.

For (H5), let $C$ be a contractible cycle with $a'$ attachment vertices. A vertex in $\Delta(C)$ that is not an attachment vertex has all its edges, and hence all its neighbours, in $\Delta(C)$. Let $j \leq r$ be such that one of $\rho_j, e_j, \rho_{j+r}$ lies in $\Delta(C)$. If none of the three were an attachment, following the path $\rho_j e_j \rho_{j+r}$ would put all three vertices and both path edges in $\Delta(C)$. The two edges from its ends to $a$ would also lie there. This would put the noncontractible cycle $\lambda_j$ in a disk. Hence one of the three is an attachment vertex. The triples $\{\rho_j, e_j, \rho_{j+r}\}$ are pairwise disjoint, so at most $a'$ of them meet $\Delta(C)$, and $\Delta(C)$ contains at most $3a' + 1$ vertices. □

The cycles $\lambda_1, \ldots, \lambda_r$ pairwise meet in $a$ only, so (H4) holds for $W_r$ only with $N \geq r$.

## 8. A conditional bound for excluded minors

Let $S'$ be a surface of Euler genus $g \geq 1$ and let $G$ be a 2-connected excluded minor for $S'$. It is simple, and it has minimum degree at least 3, because suppressing a vertex of degree 2 gives a proper minor whose embeddings in $S'$ extend to $G$. In [14, Section 4] an embedding $\Pi$ of $G$ in a surface $S$ of minimum Euler genus $h \leq g+2$ is fixed. Since $G$ is not planar, $h \geq 1$, and every face of an embedding of minimum Euler genus of a connected graph is an open disk [20]. Let $q = 9073/9072$ and $m = 2(\lfloor \log_q(3g + 4) \rfloor + 2)$, so that $m = O(\log g)$. The numbering below follows the second arXiv version of [14].

(B1) [14, Lemma 4.8] If $(A, B)$ is a 2-separation of $G$ and $B$ is contained in a disk of $\Pi$, then $B$ is an edge. This is (H1).

(B2) [14, Corollary 5.6] The graph $G$ contains at most $m$ cycles that are $\Pi$-well-nested. For pairwise disjoint cycles this is (H2).

(B3) [14, Proposition 5.11] The graph $G$ contains at most $2m$ cycles that are $\Pi$-well-homotopic. For pairwise disjoint cycles this is (H3).

(B4) [14, Corollary 5.12] A set of almost disjoint $\Pi$-noncontractible cycles of $G$ has at most $4m^2(3g+3)$ members. This is (H4) with $N = 4m^2(3g+3)$.

(B5) Let $C$ be a contractible cycle of $\Pi$ with at most $a$ attachment vertices. Set $\tilde{m}_a = 2(\lfloor \log_q(60a + 180) \rfloor + 2)$ and $\Delta_a = \left(4\sqrt{2a(2\tilde{m}_a+1)^4 \tilde{m}_a^3}\right)^{\tilde{m}_a^2}$. The disk bounded by $C$ contains at most $P_a = a\,\Delta_a(\Delta_a^{2\tilde{m}_a} - 1)/(\Delta_a - 1)$ vertices. This is (H5) with $P(a) = P_a$.

In (B1), containment in a disk of $\Pi$ means containment of the drawing in a closed disk of $S$. Section 6 of [14] takes a particular value of $a$ of order $\log^2 g$. Corollary 1.4 assumes the parameterised statement (B5), since the argument below uses a different value of $a$.

*Proof of Corollary 1.4.* Assume (B1)–(B5). In Theorem 1.3 take $h \le g + 2$, $N = 4m^2(3g + 3)$ and $P(a) = P_a$. The first factor satisfies $hm^2 + N \le 13m^2(g+2)$. The second, $F(4m+7)$, uses (B5) at $a = 5(4m+7) = 20m+35$. We obtain

$$|V(G)| \le 13c\, m^2(g + 2) \cdot 4(4m + 7) \max\{2, P_{20m+35}\}.$$

Here $m = O(\log g)$, and for $a = O(\log g)$ the expression in (B5) gives $P_a = (\log g)^{O((\log\log g)^3)}$. Substitution gives the claimed bound. □

For $N_g$, the passage from 2-connected graphs to general excluded minors is proved at the end of Section 12.

*Remark* 8.1. *Orientability.* Remark 13.3 shows that (B2) fails for orientable $S'$ of sufficiently large Euler genus. In [14], steps that add handles or twisted handles produce an embedding of $G$ of Euler genus at most $g$. To contradict exclusion for $S'$, this must give an embedding in $S'$ itself. For an orientable target, the constructed surface may be nonorientable. At equality for a nonorientable target, an orientable drawing of Euler genus $g$ need not give a drawing in $N_g$.

In particular, if $g(G)$ denotes the minimum Euler genus of $G$, the inequality $g < g(G)$ stated in [14, Section 4] can fail. For the orientable target of Theorem 1.2, the excluded grid $P_{g+2}$ already embeds in the projective plane. For a nonorientable example, $K_7$ has a toroidal embedding and no Klein-bottle embedding [20]. Choose an excluded minor $H$ for the Klein bottle

among its minors. Then $e(H) \leq 2$ by the toroidal embedding and $e(H) \geq 2$ since every projective-plane graph embeds in the Klein bottle. Thus equality can occur for a nonorientable target as well.

*Euler genus classes.* Write $\mathcal{E}_g$ for the class of graphs admitting an embedding of Euler genus at most $g$. For an excluded minor of $\mathcal{E}_g$, any drawing of Euler genus at most $g$ contradicts exclusion. When $g$ is odd, this class is exactly the class of graphs embeddable in $N_g$, since an orientable drawing of smaller Euler genus extends to $N_g$ by adding crosscaps. At even $g$ a graph excluded for $N_g$ can have an orientable drawing of Euler genus exactly $g$. Sections 9–12 allow for this case in the choice and replacement of drawings.

*One-sided cycles.* In [14] two cycles are called Π-homotopic only if they are two-sided. The proof of (B4) there treats a family of homotopic cycles through a common vertex by means of (B3). When Π is nonorientable, a family of one-sided cycles through a common vertex, any two of which bound a disk pinched at that vertex, is not covered by the definitions of [14]. The graphs of Proposition 7.2 contain arbitrarily large families of this kind. Property (B4) for such families requires an extension of (B3) that is not written there.

*Shared vertices.* In the second arXiv version of [14], Lemma 5.2 bounds the sum of two induced Euler genera for certain intersections of subgraphs in a drawing of $G - e$. Disjoint subgraphs satisfy the required inequality by cutting along their regular neighbourhoods. For subgraphs meeting in one vertex, Proposition A.1 gives a drawing of minimum Euler genus in which the proposed upper bound is $g + 6$ and the sum is $2g$, for every $g \geq 7$. This leaves open estimates under further restrictions on the subgraphs or on the drawing. Corollary 1.4 retains its stated hypotheses.

## 9. Planar sides of separations

Fix $g \geq 1$. In Sections 9–12 we work with either of two target classes. For graphs embeddable in $N_g$, a closed connected surface is *admissible* if its Euler genus is less than $g$ or if it is nonorientable of Euler genus $g$. For the class $\mathcal{E}_g$ of graphs of Euler genus at most $g$, every closed connected surface of Euler genus at most $g$ is admissible. In either class, each proper minor of an excluded minor $G$ has a drawing in an admissible surface. Also $e(G) \geq g$, since one can add crosscaps to any drawing of smaller Euler genus to obtain a drawing in $N_g$. For $\mathcal{E}_g$ the inequality is strict.

For a compact surface $R$, cap every boundary circle with a disk and let $h(R)$ be the sum of the Euler genera of the resulting closed components. Applied to a regular neighbourhood of a drawn graph, capping gives the induced cellular embedding of each component. If a planar graph attaches to a graph $B$ at $t$ vertices, the next estimate bounds how much Euler genus remains outside the drawing of $B$.

**Proposition 9.1.** *Let $G = B \cup P$ be connected, with $P$ planar, $E(B) \cap E(P) = \varnothing$, and $t = |V(B) \cap V(P)| \geq 1$. Then*

$$e(B) \geq e(G) - 2(t-1).$$

*Suppose further that $G$ is an excluded minor for the target class, $f \in E(P)$ and $G - f$ is drawn in an admissible surface. Any subgraph of $P - f$ avoiding the $t$ shared vertices has an induced regular neighbourhood of capped Euler genus at most $2(t-1)$.*

*Proof.* Draw $B$ in a connected surface of Euler genus $e(B)$ and draw $P$ in a separate sphere, keeping copies of all common vertices. Join the copies of the first vertex across a connected sum of the surfaces, and contract the joining arc. For each further pair use a separate tube joining small free sectors at the two copies and contract its joining arc. The first operation preserves Euler genus and the others cost two each. This proves the inequality.

In the stated drawing of $G - f$, take disjoint regular neighbourhoods $R_B, R_Q$ of $B$ and of the subgraph $Q$ avoiding the shared vertices. Let $u \leq g$ be the Euler genus of the surface. The sum $h(R_B) + h(R_Q)$ is at most $u$. Indeed, cutting along the boundary circles and capping expresses $u$ as the sum of the capped Euler genera of all the pieces plus twice the cycle rank of their connected incidence graph. As $h(R_B) \geq e(B)$ and $e(G) \geq g \geq u$, the result follows. □

This applies to the planar side of a bigon of Lemma 6.1 whose two boundary nodes are occurrences of the same primal vertex. In its disk model, join the two occurrences of the repeated vertex outside the disk and contract the joining edge. This gives a planar drawing of the abstract side. To apply the Euler genus bound, omit all the attachment vertices, including the repeated vertex.

*9.1. A uniform bound with fixed attachments*

**Theorem 9.2.** *There is a nondecreasing function $p$ with the following property. For every $g \geq 1$, let $G$ be a connected excluded minor for the target class. If $G = B \cup P$ is an edge-disjoint union, $P$ is abstractly planar, and $|V(B) \cap V(P)| = t \geq 1$, then $|V(P)| \leq p(t)$. The bound is independent of $g$ and of the positions of the attachments in a planar drawing of $P$.*

Remove the terminals from the planar side $P$. To reinsert one connected component, we need to know which terminals it can reach from a single complementary component of its exterior. Mark those possible points of access as follows. On the boundary of a regular neighbourhood of an embedded graph $B$, mark each sector between consecutive edge ends at a terminal with that terminal's name. An isolated terminal gives one marked boundary interval. Read the marks on each boundary circle cyclically, suppressing consecutive repetitions of the same name, including repetitions across the start of the word. A circle with one distinct name has reduced word length one.

**Lemma 9.3.** *Let $B$ be drawn on a closed connected surface of Euler genus $u$, and let $a$ of its vertices be distinguished as terminals. Let $D$ be a connected component of the complement of an open regular neighbourhood of $B$, with $b_D$ boundary circles. Put $d = u - e(B)$. Across all circles of $D$, the total length of the reduced words is at most*

$$b_D + 3\binom{a}{2}(2d + 2).$$

*If $B$ has $c$ components, then $h(D) + 2(b_D - 1) \leq d + 2(c - 1)$.*

*Proof.* The last inequality follows by capping the pieces of the surface. If $D_1, \ldots, D_k$ are all the complementary components and $R$ is the regular neighbourhood of $B$, then

$$u = h(R) + \sum_{j=1}^{k} \big(h(D_j) + 2(b_{D_j} - 1)\big) - 2(c - 1), \qquad h(R) \geq e(B).$$

Here $e(B)$ is additive over components, as follows directly from disjoint regular neighbourhoods and connected sums.

To bound the word lengths, fix distinct names $x, y$ and count the $q$ transitions between them in $D$. We may assume $q > 0$. Regard the occurrences

in each reduced word as vertices in cyclic order. The transitions between $x$ and $y$ form a graph of maximum degree two, so a greedy matching selects $r \geq q/3$ with disjoint endpoints. A transition joins a run of consecutive intervals named $x$ to the adjacent run named $y$. The boundary arc from the last interval of the first run to the first interval of the second carries no mark, and the arcs of distinct transitions have disjoint interiors.

These $r$ disjoint transitions allow us to add a copy of $K_{3,r}$ to the drawing. Place a small half-disk in $D$ along each selected arc, with the half-disks pairwise disjoint. In each put a point $u_i$ adjacent to $x$ and $y$ through the two intervals at the ends of its arc. In the remaining connected surface join one new point $z$ to all $u_i$ by disjoint spokes. For example, cut to a disk along arcs avoiding the half-disks and draw the spokes there. Extend the terminal ends through their original vertex sectors. Thus $B \cup K_{3,r}$ embeds on the same surface with intersection exactly $\{x, y\}$ and third vertex $z$ on the side of size three.

Richter's two-terminal formula [23] gives $e(B \cup K_{3,r}) \geq e(B) + e(K_{3,r})$ [19, Theorem 3.2]. That theorem is stated for two connected graphs meeting in $\{x, y\}$. For terminals in the same component of $B$, apply it to that component and $K_{3,r}$, and use additivity over the remaining components. The formula without a terminal edge applies directly if $xy$ is absent. If $xy$ is present, subdivide it once in $B$ at a fresh vertex before applying the formula. Subdivision preserves the Euler genus of both $B$ and its union with the bipartite graph. If $x, y$ belong to different components, the union is obtained by vertex amalgamations at $x$ and $y$, for which Euler genus is additive. Other components are disjoint. The bipartite Euler bound gives $e(K_{3,r}) \geq (r-2)/2$ for $r \geq 2$, and the conclusion $r \leq 2d + 2$ also holds for $r \leq 1$. Summing $q \leq 3(2d + 2)$ over the unordered pairs of names counts all entries in circles of reduced length at least two. Circles with one name account for at most $b_D$ further entries. □

*Proof of Theorem 9.2.* Write $T = V(B) \cap V(P)$ and fix a component $K$ of $P - T$. Its attachments are $T_K = N_G(K)$. Put $a = |T_K|$ and let $L$ consist of $K$, its attachments and all edges with an endpoint in $K$. We first bound $|V(L)|$ in terms of $a$ alone. Connectivity gives $a \geq 1$, and each component of the exterior $B_K = G - V(K)$ contains a neighbour of $K$. Thus $B_K$ has at most $a$ components. Since $L$ is planar, Proposition 9.1 gives

$$0 \leq g - e(B_K) \leq 2(a - 1)$$

since $B_K$ is a proper minor. Consider a drawing, on an admissible surface of Euler genus $u$, of any graph that contains $B_K$. Then $0 \le u - e(B_K) \le g - e(B_K) \le 2(a-1)$. Apply Lemma 9.3 to $B_K$ with the terminals $T_K$, so that $d \le 2(a-1)$ and $c \le a$. A connected complementary component $D$ has $h(D) \le 4(a-1)$ and $b_D \le 2a-1$, and after suppressing successive equal names the total length of its words is at most $(2a-1) + 3\binom{a}{2}(4a-2)$.

Call $D$ together with its named boundary intervals, up to name-preserving homeomorphism, a *marked surface*. Once repetitions are suppressed, the homeomorphism type of $D$, the cyclic words on its boundary circles and the directions in which they are read determine this surface. The bounds on $h(D)$, $b_D$ and the word lengths therefore give a finite list depending only on $a$.

For a graph $J$ with $a$ named terminals, no edges between terminals, and connected nonempty interior $J - T_K$, record for each member of this list whether its interior and all incident terminal edges can be drawn there. The terminal edges are to end in the indicated intervals with matching names, and edges ending at the same name may use different intervals. To see why suppressing successive equal names preserves the test, gather the endpoints of the entire run into one interval in their existing order along a thin boundary collar. Conversely they can all use one of the original intervals. At the vertex of the exterior, distinct edge ends in an interval extend radially to the marked vertex without crossing. There are only finitely many such records.

Give the terminals distinct labels and give all remaining vertices and all edges one common label. These $a+1$ labels form a finite antichain, which is a well-quasi-order. The labelled graph-minor theorem [27, Theorem 1.7] is stated for directed graphs, so orient every edge arbitrarily. It provides, in any infinite sequence of such graphs, two graphs $J'$ and $J$ and disjoint connected subgraphs $\eta(v)$ of $J$, one for each vertex $v$ of $J'$, each containing a vertex with the label of $v$, together with distinct edges of $J$ joining them as the edges of $J'$ require. Since each terminal label occurs exactly once in each graph, $\eta$ of the terminal named $x$ contains the terminal named $x$, and no other $\eta(v)$ contains a terminal. Thus $J'$ is obtained from $J$ by deletions and by contractions that keep the terminals distinct, and we call it a *rooted minor* of $J$. A subgraph $\eta(x)$ of a terminal may contain interior vertices. The graphs of this type are therefore well-quasi-ordered under rooted minors.

For each record, its minimal representatives within this type form an antichain, hence a finite set up to isomorphism. This argument uses only graphs of the specified type and does not require closure under taking minors.

Since the list of records is finite, a single number $s(a)$ bounds the orders of all their minimal representatives. Among the rooted minors of $L$ of this type with the same record as $L$, choose $J$ with $|V(J)| + |E(J)|$ minimum. Then $J$ is a minimal representative and $|V(J)| \le s(a)$.

If $|V(L)| > s(a)$, replacing $L$ by $J$ gives a proper minor $B_K \cup J$ of $G$, because all minor operations take place inside $L$ and keep the terminals distinct. In any admissible drawing of this minor, the connected interior of $J$ lies in one complementary component of $B_K$. Its marked surface is on our list. The common record therefore lets us put $L$ back in this component. The resulting drawing of $G$ is admissible, contradicting exclusion. This proves $|V(L)| \le s(a)$.

It remains to count components. One-terminal components can be inserted at their sole attachment in a small disk and cannot occur. Two components with the same pair of attachment vertices cannot occur either. An $x$–$y$ path in one of them gives a planar drawing of the other with $x, y$ cofacial, since their union is a subgraph of the planar graph $P$. The latter component could then be replaced by the edge $xy$ and restored in a disk along that edge, contradicting minimality. If $xy$ is already an edge of $G$, delete the component instead and restore it along that edge. There are at most $\binom{t}{2}$ two-terminal components.

For the remaining components, contract each component with at least three attachments to one vertex in a planar drawing of $P$, discard terminal–terminal edges, and suppress parallel edges. The resulting simple planar bipartite graph has at least three edges per component vertex. Euler's bound gives at most $2t-4$ such vertices for $t \ge 3$. Adding the $t$ terminals and using the bounds $s(a)$ for $a \le t$ now bounds $|V(P)|$ in terms of $t$. Taking a nondecreasing envelope gives $p$. □

Theorem 9.2 supplies the initial values in two recurrences. Its use of the labelled minor theorem gives an absolute finite bound for each fixed number of attachments, with no numerical constant needed below.

## 10. Comparing disjoint faces in a disk

For the disk and annulus arguments let $G$ be a 2-connected excluded minor for the target class. The graph is simple and has minimum degree at least three. A loop or duplicate edge can be restored locally, and a vertex of

degree two can be restored by subdividing an edge, using a parallel copy if its neighbours were already adjacent.

Fix a cellular reference drawing $\Pi$ of $G$ and an edge $e$. Since $G - e$ is a proper minor, it has an admissible drawing. It is connected, and capping its regular neighbourhood gives a cellular drawing of at most the original Euler genus. If the genus stays equal, the complementary pieces are planar and their incidence graph with the connected neighbourhood is a tree. Each piece then has just one boundary circle. Capping therefore preserves orientability at equality, so the cellular drawing is admissible.

Among all cellular drawings of $G - e$ on admissible surfaces, choose $\Psi$ to maximise the number of simple facial cycles of $\Pi$ that remain facial. We use this same $\Psi$ throughout the comparison for $e$. A simple facial cycle of $\Pi$ that avoids $e$ is *bad* if it ceases to bound a face in $\Psi$. Call a simple closed curve *essential* if it is not null-homotopic. Free homotopy classes here are classes of unoriented curves. Since $G$ is nonplanar, a contractible cycle of $\Pi$ bounds a unique disk.

The comparison of planar and annular drawings follows [14, Lemmas 4.7 and 5.1]. The additional issue in Lemma 10.2 is that replacing one annulus may reverse the relative orientations of its boundary circles. Two disjoint replacements let us preserve the surface type.

**Lemma 10.1.** *Let $C$ be a contractible cycle of $\Pi$ and $F$ a face of $\Pi$ whose closure lies in the open disk bounded by $C$. Then the boundary of $F$ is an induced cycle of $G$, which we also denote by $F$, and $G - V(F)$ is connected. If moreover some cycle of $G$ through $e$ is vertex-disjoint from $F$ and $F$ is bad, then $F$ does not separate the surface of $\Psi$. In particular it is essential there.*

*Proof.* Cap the plane graph in the closed disk bounded by $C$ with a new vertex adjacent to each vertex of $C$ on the other side. The resulting simple plane graph $H$ is 3-connected. To verify 2-connectivity of the disk graph, delete any vertex. The surviving part of $C$ is connected, and a component cut off from it would also be separated from the rest of $G$ by the deleted vertex. In $H$, a two-vertex cut avoiding the cap vertex would isolate an interior component with two attachments. The complementary part of $H$ contains a path between those attachments, so the isolated side is planar with its two attachments cofacial. Replacing that side by an edge in $G$ and restoring it along the edge contradicts minor-minimality. A cut using the cap vertex is already excluded by 2-connectivity of the disk graph.

A face of a 3-connected plane graph is bounded by an induced nonseparating cycle by Tutte's peripheral-cycle characterization [31, (2.8)]. In $H-V(F)$, a path through the cap vertex can be rerouted along the intact cycle $C$. Adjoining the exterior of $C$ thus gives the first assertion.

The specified cycle through $e$ implies that $e$ is not a bridge of $G-V(F)$. Thus $(G-e)-V(F)$ is connected, and $F$ still has no chord. If $F$ separates in the cellular drawing $\Psi$, all the remaining graph lies on one side and the other side is an empty disk. Such an $F$ is facial. □

**Lemma 10.2.** *Let $C$ be a contractible cycle of $\Pi$ such that $e$ is drawn in the open disk bounded by $C$. Consider a family of pairwise vertex-disjoint bad facial cycles whose closed faces in $\Pi$ lie in that open disk. Among the members that are essential in the surface of $\Psi$, each free homotopy class contains at most 12 members.*

*Proof.* A one-sided curve has mod-two self-intersection one, so two disjoint such curves are never freely homotopic. Two disjoint essential two-sided curves in the same class bound an annulus [7, Lemma 2.4]. Within such an annulus each further curve of the family separates the two boundary circles. These curves are ordered by inclusion of the regions between them and one fixed boundary circle. The family thus runs along successive annuli, cyclically if the annuli fill the surface. The exterior graph of $C$, including $C$, is connected, avoids all the curves and lies in one complementary component of their union. With at least 13 curves there is a chain of six consecutive curves $F_1, \dots, F_6$ with all five intervening annuli disjoint from the exterior graph. Indeed at most one gap in a linear chain is occupied by it, and a cyclic chain can be opened there.

Consider the two disjoint subannuli between $F_1, F_3$ and between $F_4, F_6$. In each, the graph carried in the closed subannulus is a subgraph of the reference disk graph. Indeed, its vertices are not in the exterior graph, so in $\Pi$ they lie in the open disk bounded by $C$, and every edge at such a vertex is drawn in that disk. It has a drawing in an abstract annulus with the prescribed boundary cycles by drawing it in the reference sphere and removing the open facial disks of the two boundary cycles. In the target, the two boundary cycles are the only attachments of this subgraph to the retained exterior. Replacing its drawing therefore makes its middle cycle facial.

To check that a previously common face survives, consider its open disk in $\Psi$. The disk misses the two cutting cycles, so it lies wholly inside or wholly outside the target subannulus, even if its boundary meets a cutting cycle. In

the outside case the disk and its incident sectors are retained. In the inside case all its boundary edges belong to the transplanted subgraph and avoid the exterior graph, including $C$. They therefore lie strictly inside $C$ in $\Pi$. There the boundary cycle bounds a unique disk because the reference surface has positive Euler genus. This facial disk is inside $C$, and the two holes removed from the reference drawing are distinct faces whose interiors miss it.

The labelled boundary maps can be glued to the retained surface in either case. Their signs determine whether the replacement reverses the relative orientation of the two boundary circles compared with the old annulus. If either replacement preserves that sign, perform it. If both reverse the sign, perform both. To see that two reversals preserve the surface type, consider the one longer annulus between $F_1$ and $F_6$. In either choice its boundary-to-boundary orientation relation is unchanged, and the longer annulus with the new gluings is homeomorphic to the old one relative to its two outer boundary parametrizations. All gluings preserve Euler characteristic.

If the new drawing is noncellular, cap its regular neighbourhood. Every retained empty facial disk still gives its facial cycle, and the capped drawing is admissible since the uncapped graph draws on the old surface. The middle cycle of each replaced annulus is now facial. This increases the number of common facial cycles, contrary to the choice of $\Psi$. □

**Lemma 10.3.** *Let $Z$ be a graph with $c \geq 1$ components drawn in a closed connected surface of Euler genus $u$, and let $h_Z$ be the sum of its induced capped Euler genera. A family of pairwise disjoint essential simple closed curves avoiding $Z$, with at most* 12 *in any ambient free homotopy class, has size at most $C(u - h_Z + c)$ for an absolute constant $C$. With no $Z$ the bound is $C(u + 1)$.*

*Proof.* Take a regular neighbourhood of $Z$ thin enough to miss the curves, so that each curve lies in a complementary component. In a complementary component of capped Euler genus $k$ with $b$ boundary circles, a disjoint family has $O(k + b)$ noncontractible free homotopy classes, including boundary-parallel classes. For example, attach a once-punctured torus at each boundary. No disk or annulus bounded by curves of the family in the completed surface can contain one of these tori. Essential curves stay essential, and distinct classes stay distinct by the annulus criterion for two-sided curves. Disjoint one-sided curves are already mutually nonhomotopic. Apply the ordinary closed-surface disjoint-curve bound to the resulting surface of

Euler genus $k + 2b$, or cut into pairs of pants directly. A disk contributes zero.

Curves from distinct ambient classes remain in distinct classes in their complementary component. For a nondisk component, $k+b \leq 2(k+2(b-1))$. Summing over the components in the cutting formula of Lemma 9.3, with $h_Z$ the capped genus of the neighbourhood of $Z$, gives $O(u - h_Z + c)$ classes. Each accounts for at most 12 curves. With $Z$ empty, the closed-surface count in the proof of Lemma 4.1 gives $O(u + 1)$ classes and the same multiplicity bound applies. □

## 11. A logarithmic bound on radial depth

Fix a face-node $p$ in the radial graph of the reference drawing $\Pi$. Let $U_j$ be the union of the closed faces of $G$ whose face-nodes are at radial distance at most $2j$ from $p$. In a region cut along radial edges, measure distances and take incidences within that region. We use $U_j$ only when its closed faces avoid the boundary.

Suppose $U_n$ lies in an embedded open disk in the region. For $0 \leq j \leq n$, the frontier facing the boundary of this disk separates $p$ from the boundary and contains a separating cycle $C_j$ of $G$, as in Lemma 3.1. Each vertex on that frontier is incident with both a selected and an unselected face, so its distance from $p$ is exactly $2j + 1$. For $j \geq 1$, the connected set $U_{j-1}$ contains $p$ and avoids the frontier. Consequently $U_{j-1}$ lies inside $C_j$.

**Proposition 11.1.** *There is an absolute constant $c_*$ with the following property. Let $G$ be a 2-connected excluded minor for the target class with a cellular reference drawing $\Pi$, let $p$ be a face-node, and let $n \geq 0$ be such that $U_n$ lies in an open disk that is embedded in the surface of $\Pi$ and, when distances are taken in a cut region, in that region. Then $n \leq c_* \log(g + 2)$. If in addition $G = B \cup P$ is an edge-disjoint union in which $P$ is planar and $1 \leq |V(B) \cap V(P)| \leq t$, and $U_n$ is disjoint from the drawing of $B$ in $\Pi$, then $n \leq c_* \log(t + 2)$.*

We call the two bounds the *global* and the *local* assertion. The local assertion applies to a disk model with two occurrences of one boundary vertex and to a disk neighbourhood inside an annular side, because in each case the open disk containing $U_n$ is embedded injectively in the reference surface.

*Proof.* We may assume $n \geq 24$, since smaller values are absorbed in the constant. Delete an edge $e$ of the face of $p$ and use the drawing $\Psi$ of $G - e$

chosen in Section 10. The closed face of $p$ lies in the open disk bounded by $C_n$. By Lemma 10.1 its boundary is a cycle through $e$. Every face at distance between 10 and $2n-10$ also has its closure in this open disk and is disjoint from the central face. Its boundary is therefore an induced cycle and, if it ceases to be facial, it is nonseparating in $\Psi$ by the same lemma.

Choose a maximal family $\mathcal{M}$ of bad faces in this range with pairwise vertex-disjoint boundaries. Each boundary is essential in $\Psi$. Lemma 10.2 bounds the multiplicity of each homotopy class, so Lemma 10.3 gives $|\mathcal{M}| = O(g+1)$. In the local case $e$ is an edge of $P$, because $U_0$ is disjoint from $B$. Hence $B$ is a subgraph of $G-e$, and the family avoids it. Since $G$ is connected, each component of $B$ contains an attachment. Thus $B$ has at most $t$ components, Proposition 9.1 gives $e(B) \geq g - 2(t-1)$, and Lemma 10.3 with $Z = B$ gives $|\mathcal{M}| \leq C(g - e(B) + t) \leq C(3t-2)$.

For each $j$, let $A_j$ count the faces in $\mathcal{M}$ whose face-nodes have distance at most $2j-10$ from $p$. Fix $12 \leq j \leq n-12$ and let $k_j$ count those whose face-nodes have distance in $[2j-5, 2j+5]$. To prove that $A_j$ grows exponentially with $j$, we will bound this inner count by a constant times the number $k_j$ in the surrounding band (Figure 3).

At a vertex of $C_j$ call the star *good* if all its incident reference faces remain facial in $\Psi$. These faces lie in the distance range used to choose $\mathcal{M}$. From each vertex whose star is not good, draw a radial path to a face-node of $\mathcal{M}$. A path of at most three edges visits an incident bad face, then, if needed, a vertex it shares with a member of the maximal family and that member's face-node. Every chosen face-node lies in the band counted by $k_j$.

The paths ending at one face-node form a connected set of radius at most three. A chain of intersecting sets joins any two nodes in a component of their union. Hence there are at most $k_j$ components, each of diameter at most six times the number of its chosen face-nodes. Take a breadth-first spanning tree of each component. The trees are disjoint, the sum of their diameters is $O(k_j)$, and each stays in the band $[2j-5, 2j+5]$.

We use these trees to modify $C_j$ while keeping $p$ inside it. Contract them in the reference disk bounded by $C_n$, punctured at $p$. Each tree has a disk neighbourhood missing $p$, so the image of $C_j$ still has nonzero winding number. Splitting its closed walk at repeated vertices gives a simple cycle of nonzero winding number, possibly a loop edge. Lift that cycle through each contracted tree it visits, using the unique path between its entry and exit. Since each contracted vertex is visited only once, this gives a simple closed contour $\Gamma$ surrounding $p$. Below, the *inner side* of $\Gamma$ is the side containing $p$.

Along a subpath shared with $C_j$, the inner sides of the two curves can differ.

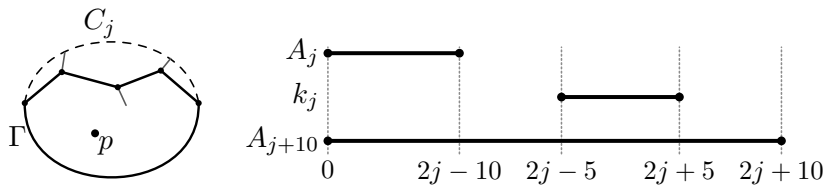


Figure 3: On the left, a portion of the level cycle $C_j$ (dashed) is replaced by a path in a tree to give a contour $\Gamma$ around $p$. On the right, the bars give the ranges of radial distances from $p$ counted by $A_j$, $k_j$ and $A_{j+10}$. The last range contains both of the first two.

At most $k_j$ trees are visited. The sum of the lengths of the lifted paths, counting one additional vertex even for a path of length zero in a visited tree, is $O(k_j)$. If $k_j = 0$, take $\Gamma = C_j$. The other portions are primal subpaths of $C_j$, with good stars at all their interior vertices. The contour remains in the same distance band. Every face counted by $A_j$ lies strictly inside $\Gamma$, since a shortest radial path from $p$ to any of its nodes avoids the band. All reference edges either lie to one side of $\Gamma$ or are primal edges on $\Gamma$, since the latter is drawn in the union of the radial and primal graphs.

Let $Z$ consist of the open edges on the exterior side of $\Gamma$, their endpoints, the primal edges on the contour, and all its primal boundary vertices. Since $e$ lies strictly inside, $\Psi$ induces a drawing of $Z$. We compare the two drawings along the primal portions of the contour. At a good star all incident faces are simple reference faces in the 3-connected planar cap, and no end of $e$ is present. The unordered consecutive pairs of edge ends at the vertex are prescribed by the corners of these retained faces. Each such cycle has a unique facial side at the vertex, since its degree is at least three. The prescribed pairs form a cycle on the edge ends and fix the local rotation up to reversal. Orient the target vertex disk at each good vertex to give the reference rotation. Along an edge with two good ends, the two distinct incident reference faces specify which corners at opposite ends are on the same side of the band. This fixes its signature in the chosen vertex orientations. Hence any reference strip following a primal path of good stars has the same local sides in $\Psi$. The same conclusion holds on a closed good cycle, by the compatibility on its last edge.

If $k_j = 0$, the whole $C_j$ has good stars. All exterior edges then lie on the prescribed exterior side, leaving its inner side free in the restriction to $Z$. This side of $C_j$ gives a facial boundary in the capped neighbourhood of $Z$. Insert the reference interior in that disk. Here $Z$ is connected, since it contains

the boundary cycle and each exterior component meets that cycle. Capping its regular neighbourhood in $\Psi$ gives an admissible surface by the equality argument at the beginning of Section 10. Inserting the interior therefore embeds $G$ on an admissible surface, contrary to exclusion. Thus $k_j \geq 1$.

Partition the primal vertices on $\Gamma$ into $O(k_j)$ consecutive groups. Vertices on the lifted radial paths, including their endpoints, can be taken as singletons. The remaining runs follow a primal path whose stars are good. Split a run once if necessary to ensure that it is a path. Let $q$ be the number of groups. A lifted path in a tree whose component has $m$ chosen centres has length at most $12m$, hence at most $6m + 1$ primal vertices, and there are at most $k_j$ runs before the possible split. Thus $q \leq 9k_j$. All vertices of a run lie in one component of $Z$, and every component meets the contour by connectivity of $G$. Hence $Z$ has at most $q$ components.

Each group now gives one disk along which to attach the reference interior. For a run, work in the capped regular neighbourhood of its component of $Z$. On the inner side of $\Gamma$ we need a disk whose boundary meets the vertices in path order. Take a narrow strip on that side of the path, tapering it at its ends. The retained stars and their agreement along the edges ensure there is no exterior edge in the indicated corners. Its boundary can be drawn through the vertices just inside the path edges. A singleton has a disk in a free sector. The disks can be chosen mutually disjoint since the groups use disjoint vertices and primal subpaths. These statements use only the restricted ribbon at the paths in $Z$.

Write $c$ for the number of components of $Z$ and $h_Z$ for the sum of their capped Euler genera. In the planar drawing of the interior of $\Gamma$, mark matching disks on the exterior side of its boundary, one for each consecutive group in the given order. Remove the interiors of those disks and the corresponding disks in the capped surfaces of $Z$, and identify the matching circles respecting the labelled vertices. Their cyclic orders agree up to reversal, and either gluing orientation is allowed in bounding $e(G)$. Repeated copies of any retained boundary edge can be discarded. The result is connected since every component of $Z$ meets a group. Of the $q$ identifications, $c$ join distinct surfaces, and each remaining identification costs two units of Euler genus. We obtain

$$g \leq e(G) \leq h_Z + 2(q - c), \qquad c \leq q = O(k_j).$$

The faces counted by $A_j$ lie inside $\Gamma$, so their boundary cycles avoid $Z$. If

$u$ is the Euler genus of $\Psi$, then the cutting formula and the last display give

$$u - h_Z + 2(c-1) \le g - h_Z + 2(c-1) \le 2(q-1).$$

Lemma 10.3 gives $A_j \le C' k_j$ for an absolute $C'$. The band counted by $k_j$ starts after the range counted by $A_j$ and ends before the cutoff for $A_{j+10}$. Thus $A_{j+10} \ge A_j + k_j$. Starting at $j = 12$, the first step counts at least one face since $k_j \ge 1$. Each further step of ten with $j \le n - 12$ multiplies the count by at least $1 + 1/C'$. All these faces belong to $\mathcal{M}$. Comparing with $|\mathcal{M}| = O(g+1)$ gives $n = O(\log(g+2))$. On a planar side with at most $t$ attachments, $|\mathcal{M}| = O(t)$ gives $n = O(\log(t+2))$ with the same choice of absolute constant after enlargement. □

## 12. Small disks, annuli and the order bound

In Lemmas 12.1 and 12.2, $G$ is a 2-connected excluded minor for one of the two target classes of Section 9, and $\Pi$ is a cellular embedding of $G$. The constants are absolute, uniformly in $g$ and in the choice of $\Pi$.

A *disk model* is a polygon as in Definition 2.1 in which the primal nodes on the boundary have distinct images, except possibly for two occurrences of one primal vertex. Boundary face-nodes may have repeated images. The *weight* is the number of primal boundary occurrences. All distances in a cut polygon use its own radial graph before identifications. The map is injective on the open disk.

**Lemma 12.1.** *A disk model of weight at most $w$ contains $O((w+1)\log^2(w+2))$ primal vertices, counting boundary occurrences.*

*Proof.* Let $M(w)$ be the supremum of the number of primal occurrences over all such models of weight at most $w$, over both target classes, all $g$, and all cellular reference embeddings of their 2-connected excluded minors. Use zero for an empty supremum. For each model the primal side is abstractly planar. If one boundary vertex is repeated, join its two copies by an edge outside the disk and contract this edge. There are at most $w$ attachments. For a side with at least one attachment, Theorem 9.2 bounds its order independently of $g$, and distinguishing the two boundary copies adds at most one vertex. Connectivity and nonplanarity exclude a nonempty planar side with no attachments. Hence $M(w)$ is finite for each $w$.

Every radial node in the polygon is within $C\log(w+2)$ of its boundary for an absolute $C$. Otherwise start at that node, or at an adjacent face-node

if it is primal. For every $j$ smaller than half its distance to the boundary minus two, $U_j$ and all its incident faces lie in the open polygon and hence in an embedded disk. The local assertion of Proposition 11.1, applied with at most $w$ attachments, then bounds $j$.

Let $\rho$ be the maximum distance to the polygon boundary. Its boundary has length at most $2w$. Lemma 6.1 covers the vertices by $O(w+1)$ subdisks of weight at most $2\rho+1$ and $O((w+1)(\rho+1))$ other vertices, including boundary vertices. A subdisk meets the parent boundary in at most two nodes. Its image therefore has at most the one possible primal identification already present in the parent. Repeated face-node images on the boundary leave the abstract primal side planar. Thus for fixed absolute $A, D$ and $w \geq 2$,

$$M(w) \leq A(w+1)\big(\log(w+2) + M(\lceil D\log(w+2)\rceil)\big).$$

First prove $M(w) \leq C_0(w+1)^2$ by induction. For all large $w$, the smaller argument is strictly below $w$, and

$$A(w+1)(\lceil D\log(w+2)\rceil + 1)^2 \leq (w+1)^2/2,$$
$$A(w+1)\log(w+2) \leq (w+1)^2/2.$$

Since each initial value is finite, one constant $C_0 \geq 1$ covers the initial range. The two inequalities give the induction step. Apply the recurrence once more, now using the quadratic estimate at the argument $\lceil D\log(w+2)\rceil$. This gives $M(w) = O((w+1)\log^2(w+2))$. □

**Lemma 12.2.** *In an embedded annulus with two vertex-disjoint radial boundary cycles of total weight at most $w$, the intrinsic radial distance between the boundary cycles is $O(\log(w+2))$.*

*Proof.* Let $D(w)$ be the supremum of the distances in the statement, over the same graphs and drawings as for $M(w)$. The primal graph in the annulus is abstractly planar and has at most $w$ attachments. Theorem 9.2 bounds its order, and a shortest radial path visits each primal vertex at most once. Its length is at most twice the number of primal vertices plus two. Thus $D(w)$ is finite.

There is a constant $c$ such that any node at distance more than $r = c\log(w+2)$ from both rims is within distance $r$ of an essential radial cycle of length at most $r$ contained in its radius-$r$ neighbourhood in the annulus. To see this, start at an adjacent face-node if necessary. For $j$ a sufficiently large

constant multiple of $\log(w+2)$, but $2j+4 < r$, the closed face neighbourhood $U_j$ avoids both rims. It cannot be contained in an open disk of the annulus by Proposition 11.1 applied with the whole annulus as the planar side.

It follows that $U_j$ contains an essential closed curve of the annulus. To see this, include the annulus as an essential annulus in a torus. If every loop in $U_j$ were contractible in the annulus, it would be contractible in the torus. By Lemma 2.2(a), a small regular neighbourhood of $U_j$ would have an outer boundary bounding a disk in the torus containing $U_j$. That boundary lies in the open annulus. Neither essential rim can lie in its disk, and both are disjoint from the boundary. The disk would thus lie in the annulus, a contradiction.

To obtain a short cycle in the radial graph, deform the essential curve to primal face boundaries and replace each primal edge it uses by the two-edge radial path through an incident selected face. Each replacement is a homotopy within the corresponding face disk. This gives an essential closed walk in the annulus on the radial stars of the selected faces. The union of the radial stars of the included faces is connected and has a spanning tree of depth at most $2j + 1$ from the initial face-node, by the definition of the distances. At least one of its fundamental cycles is essential, and each has length at most $4j + 3$. Enlarge $c$ to absorb these constants.

If a shortest rim-to-rim path has length more than $12r + 10$, take points on it at distance $3\lceil r \rceil$ from the respective rims. The preceding observation supplies two disjoint short core cycles near those points. Put $R_0 = \lceil r \rceil$ and let $d$ be the path length. The distances from the first rim to the two cycles lie in $[2R_0, 4R_0]$ and $[d - 4R_0, d - 2R_0]$, respectively. Each essential cycle separates the rims. Short initial and terminal paths reach the respective cycles without crossing the other one, which gives their order.

The two cycles cut off a middle annulus of total boundary weight at most $2\lceil r \rceil$. One can reach it from either original rim in $O(r)$ steps. Traversing the short cycles to reach the endpoints of a shortest path in the middle annulus also costs $O(r)$. Therefore, for fixed $A, B_0$,

$$D(w) \le A \log(w + 2) + D(\lceil B_0 \log(w + 2) \rceil)$$

with the additive bound alone when the original path was short. For all large $w$, the logarithm of the smaller argument plus two is at most half of $\log(w+2)$. Induction, starting with the finite initial range, gives the claim. □

**Theorem 12.3.** *For either target class, each 2-connected excluded minor satisfies*

$$|V(G)| = O\big((g+1)L^2(\log(L+2))^2\big), \qquad L = \log(g+2).$$

*Proof.* Put $L = \log(g+2)$ and use a cellular reference drawing of minimum Euler genus $h$. We have $1 \le h \le g+2$, since an edge can be added to a drawing of its deletion using one tube. The global assertion in Proposition 11.1, by the radial-star argument of Lemma 12.2, gives an essential radial cycle of length $O(L)$ within distance $O(L)$ of every node. Choose a maximal family of these cycles at mutual distance at least $KL$, for a sufficiently large absolute $K$. Every node is then within $O(L)$ of their union. Two freely homotopic members would be two-sided and cobound an annulus of total weight $O(L)$. Lemma 12.2 would join them in $O(\log(L+2))$ radial steps, fewer than $KL$ after enlarging $K$ to cover bounded values of $g$. Thus the members are disjoint and pairwise nonhomotopic. The disjoint-curve bound in the proof of Lemma 4.1 gives $O(h)$ members.

Apply Lemma 4.3 with $k = O(h)$ and $a, d = O(L)$. The resulting radial cut graph has $O(hL)$ edges. Every node of its cut polygon is within $O(L)$ of the boundary. Indeed, a deeper node, or an adjacent face-node, would give a face neighbourhood in the open polygon contradicting the global assertion of Proposition 11.1.

Lemma 6.1 covers this polygon by $O(hL)$ cells of weight $O(L)$ and $O(hL^2)$ further primal nodes, including boundary occurrences. Each cell meets the polygon boundary in at most two nodes, so it is a disk model of Lemma 12.1. Its contribution is $O(L(\log(L+2))^2)$. Multiplication by the number of cells gives the asserted bound. □

*12.1. Passing from blocks to excluded minors*

*Proof of Theorem 1.1.* For any graph $H$, set $\sigma(H) = 1$ if every drawing of Euler genus $e(H)$ is orientable. Set $\sigma(H) = 0$ otherwise. Under vertex amalgamation of two connected graphs containing cycles, Euler genera add and the indicators multiply by the block formulas of Stahl and Beineke [29] in the form of [19, Theorems 2.2 and 6.2]. A tree attached at one vertex is neutral, since it can be restored in a disk sector. For a bridge joining two graphs, the bridge band connects two distinct regular neighbourhoods, so their capped Euler genera add and their union is orientable precisely when both are orientable. Conversely any two given drawings can be joined by that

band. For disjoint unions the same rules follow by taking connected sums. For the lower bound and the orientability assertion at equality, the cutting formula forces minimum regular neighbourhoods and planar complementary pieces joined along a tree.

An isolated vertex can be restored in a disk. Deleting a bridge preserves $e$ and $\sigma$ by the preceding rules. Contracting an edge in a planar block leaves all blocks of positive Euler genus intact, and every block coming from the contracted planar block is planar. This again preserves $e$ and $\sigma$. These operations preserve embeddability in $N_g$, so an excluded minor has no such vertex or edge. Write $B_1, \dots, B_l$ for its blocks, including all components, and put $b_i = e(B_i) \geq 1$. Then $e(G) = \sum_i b_i \leq g + 2$.

If $e(G) > g$, any elementary minor operation in $B_i$ produces a graph whose Euler genus is at most $g$, so it strictly lowers the Euler genus of $B_i$. Thus $B_i$ is an excluded minor for the class of Euler genus at most $b_i - 1$.

Suppose $e(G) = g$. Exclusion implies $\sigma(G) = 1$. Hence every $B_i$ has $\sigma(B_i) = 1$ and $b_i$ is even. Perform a single deletion or contraction in $B_i$, giving a minor $M$. If $e(M) < b_i$ it embeds in $N_{b_i}$. Otherwise the whole graph after the operation still has Euler genus $g$ and embeds in $N_g$, so its $\sigma$ is zero. All the other block factors have value one, and the same block formula gives $\sigma(M) = 0$. Then $M$ again embeds in $N_{b_i}$. This proves that $B_i$ is an excluded minor for $N_{b_i}$.

We can now sum a bound proportional to each block's Euler genus. Set $L = \log(g + 2)$. Since $b_i \leq g + 2$, Theorem 12.3 gives $|V(B_i)| = O(b_i L^2 (\log(L + 2))^2)$ in both cases, with an absolute factor for the shift of two in the logarithm. When $b_i - 1 = 0$, the two planar obstructions give the same bound. Every vertex belongs to a block, and $\sum_i b_i \leq g + 2$. Summing gives the theorem. □

## 13. Orientable surfaces

For the projective grid $P_k$, the obstruction to embedding in an orientable surface comes from its known genus. Fulek and Kynčl derive the formula below from the projective-plane drawing [10, arXiv:1803.05085, Section 2.2.1]. To prove minimality for even $k$, we will find a curve through $k - 1$ vertices after each edge deletion or contraction.

**Lemma 13.1** ([9, 10])**.** *For $k \geq 4$, the graph $P_k$ has orientable genus $\lfloor k/2 \rfloor$.*

**Lemma 13.2.** *Let $H$ be a graph embedded in the projective plane. Suppose that a noncontractible simple closed curve meets $H$ in exactly $\rho \geq 1$ points, all of them vertices. Then $H$ embeds in the orientable surface of genus $\lfloor \rho/2 \rfloor$.*

*Proof.* Let $v_1, \ldots, v_\rho$ be the vertices on the curve, in order. Cutting the projective plane along the curve gives a closed disk $D$. In $D$ the graph $H$ becomes a plane graph in which each $v_i$ is replaced by two vertices $p_i$ and $q_i$ on the boundary, and the cyclic order on the boundary is $p_1, \ldots, p_\rho, q_1, \ldots, q_\rho$.

For each $i$, attach a band between short boundary intervals around $p_i$ and $q_i$ so that the orientation of $D$ extends over the band. The resulting orientable surface $\Sigma$ has Euler characteristic $1 - \rho$. To count its boundary components, number the $2\rho$ intervals between consecutive attaching intervals by $\mathbb{Z}_{2\rho}$ in cyclic order. Tracing the boundary of $\Sigma$ from interval $x$ takes us across a band and on to interval $x + \rho + 1$. The boundary components are thus the orbits of this translation. There are $\gcd(\rho + 1, 2\rho) = \gcd(\rho + 1, 2)$ of them.

Capping these components gives a closed orientable surface of genus $\lfloor \rho/2 \rfloor$. The band cores are disjoint arcs joining $p_i$ to $q_i$, each meeting $D$ only at its ends. Contracting the cores identifies each pair of copies and gives the required embedding of $H$. □

*Proof of Theorem 1.2.* Let $k = 2\gamma + 2$. By Lemma 13.1, $P_k$ has orientable genus $\gamma + 1$, so it does not embed in the orientable surface of genus $\gamma$. Every proper minor of $P_k$ is a minor of $P_k - e$ or of $P_k/e$ for some edge $e$, because $P_k$ has no isolated vertex. By Lemma 13.2 with $\rho = 2\gamma + 1$, it suffices to find, for every edge $e$ and for each of the two minors, an embedding in the projective plane and a noncontractible simple closed curve that meets the minor in exactly $k - 1$ points, all of them vertices.

We construct the curves in the sphere, the double cover of the projective plane. Let $C_k$ have vertex set $\{1, \ldots, k\} \times \mathbb{Z}_{2k}$. Join each $(i, c)$ to $(i, c+1)$ and, when $i < k$, to $(i + 1, c)$. Embed $C_k$ in the unit sphere by placing $(i, c)$ at a latitude proportional to $k+1-2i$ and at longitude $\pi c/k$. Its faces are the *unit squares*, bounded by the cycles through $(i, c), (i, c+1), (i+1, c+1), (i+1, c)$, and two faces $N$ and $Z$ bounded by the cycles $\{1\} \times \mathbb{Z}_{2k}$ and $\{k\} \times \mathbb{Z}_{2k}$. The map $\tau(i, c) = (k + 1 - i, c + k)$ is the restriction of the antipodal map, which preserves the embedded graph and exchanges $N$ and $Z$. The quotient of the sphere by the antipodal map is the projective plane, and the quotient of $C_k$ is $P_k$. A closed curve in the projective plane is noncontractible if and only if its lifts join a point to its antipode. Hence a simple arc in the sphere from a

point $x$ to $\tau(x)$ that is disjoint from its antipodal image, except at its ends, projects to a noncontractible simple closed curve.

An edge $e$ of $P_k$ has two lifts $\tilde{e}$ and $\tau(\tilde{e})$, and they are disjoint. Deleting both lifts, or contracting both, gives a graph embedded in the sphere that is preserved by $\tau$, and its quotient is an embedding of $P_k - e$, respectively $P_k/e$, in the projective plane. We describe the required curve by an arc in the sphere that passes alternately through vertices and faces of this graph. Up to a rotation of the sphere, $\tilde{e}$ is *vertical*, of the form $(i,c)(i+1,c)$ with $1 \le i \le k-1$, or *horizontal*, of the form $(i,c)(i,c+1)$ with $1 \le i \le k$. The first two constructions below follow a meridian between the polar faces. The other two advance through half the longitudes, starting at a grid vertex and ending at its antipode.

*Contracting a vertical edge.* Take a point $n$ in $N$ and the arc from $n$ through $(1,c),\dots,(k,c)$ to the point $\tau(n)$ of $Z$. It passes through both ends of $\tilde{e}$ consecutively. After the contraction these two ends are one vertex, and the arc, rerouted to pass through this vertex once, passes through $k-1$ vertices.

*Deleting a horizontal edge.* Deleting $\tilde{e} = (i,c)(i,c+1)$ merges the two faces on its sides into a face $F$. If $2 \le i \le k-1$, the vertices $(i-1,c)$ and $(i+1,c)$ are on the boundary of $F$. If $i = 1$, then $F$ includes $N$ and has $(2,c)$ on its boundary. If $i = k$, then $F$ includes $Z$ and has $(k-1,c)$ on its boundary. In each case the arc of the previous paragraph can be rerouted through $F$ so as to avoid $(i,c)$, and it then passes through $k-1$ vertices.

*Contracting a horizontal edge.* Let $\tilde{e} = (i,c)(i,c+1)$ and $x = (i,c)$, so that $\tau(x) = (k+1-i, c+k)$. Choose rows $r_1,\dots,r_k$ with $r_1 = i$, with $r_k = k+1-i$, and with $|r_{j+1} - r_j| \le 1$ for all $j$. This is possible because $|k+1-2i| \le k-1$. The arc through $x, (r_1, c+1), (r_2, c+2), \dots, (r_k, c+k)$ ends at $\tau(x)$, and consecutive vertices on it lie on a common unit square. It passes through $k$ vertices of $P_k$, and the first two of them are the ends of $\tilde{e}$. After the contraction it passes through $k-1$ vertices, as in the first case.

*Deleting a vertical edge.* Let $\tilde{e} = (i,c)(i+1,c)$. Its antipodal image is the edge $(k+1-i, c+k)(k-i, c+k)$. Deleting this edge creates a face $F$ with the vertices $(k-i, c+k\pm1)$ and $(k+1-i, c+k\pm1)$ on its boundary. Let $x = (i, c+1)$, so that $\tau(x) = (k+1-i, c+k+1)$. Choose rows $r_1,\dots,r_{k-1}$ with $r_1 = i$, with $r_{k-1} = k-i$, and with $|r_{j+1} - r_j| \le 1$. This is possible because $|k-2i| \le k-2$. The arc through $(r_1, c+1),\dots,(r_{k-1}, c+k-1)$ and then through $F$ to $\tau(x)$ passes, after projection, through $k-1$ vertices. Figure 4 shows the case $k = 4$.

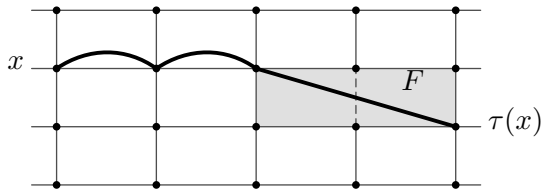


Figure 4: Part of the spherical double cover for $k = 4$, $i = 2$ and $c = 0$. Deleting the dashed lift of the vertical edge merges two squares into the shaded face $F$. The bold arc reaches $\tau(x)$ through $F$. Its endpoints project to the same vertex, so its projection meets the minor in three vertices.

It remains to check that each arc meets its antipodal image only at its ends. In the first two cases take $n$ and $\tau(n)$ to be the poles. Draw the arc close to meridian $c$, making the detour through a merged face on its $c + 1$ side. Continue through both polar caps in the same narrow longitude sector. This sector and its antipodal sector meet only at the poles.

In the last two cases draw the arc with strictly increasing longitude through the square interiors, including the merged pair of squares in the deletion case. In the contraction case use the first edge itself before collapsing it. The open interval of longitudes has length $\pi$ and is disjoint from its antipodal translate. The two contractions identify only the prescribed endpoints of the corresponding portions. Thus each projected curve is simple. □

The grid $P_k$ has $k^2$ vertices and $2k(k-1)+k$ edges. For the torus, Theorem 1.2 gives the excluded minor $P_4$ with 16 vertices and 28 edges. For the orientable surface of genus 2, it gives $P_6$ with 36 vertices and 66 edges. The grid $P_4$ already appears in Juvan's description [16] of the torus obstructions that embed in the projective plane, as discussed by Myrvold and Woodcock [21, Section 4].

*Remark* 13.3. Let $g = 2\gamma$ and $k = g + 2$. In the embedding of $P_k$ in the projective plane described above, the $\gamma + 1$ concentric squares of the planar grid on $\{1, \ldots, k\}^2$ are pairwise disjoint contractible cycles, each contained in the disk bounded by the next. With $m$ as in Section 8, we have $\gamma + 1 > m$ for every $g \geq 517\,320$. Thus (H2) fails in this embedding of minimum Euler genus.

For $g \geq 10^{14}$, (H2) fails in every embedding of $P_k$ with disk faces and Euler genus at most $g + 2$. Such an embedding has at least $k^2 - 2k + 2$ faces of total length $4k^2 - 2k$. Every face has length at least four, since $P_k$ is simple, has minimum degree at least three and has no triangles. The excess of the

total face length over four times the number of faces is at most $6k-8$, so at most $6k-8$ faces have length greater than four. The $k^2-k$ cycles of length four are precisely the images of the unit squares. Comparing this count with the number of quadrilateral faces leaves at most $7k$ nonfacial squares among the $(k-1)^2$ squares of the planar grid.

Put $s=\lfloor\sqrt{k/8}\rfloor$. The planar grid contains $\lfloor(k-1)/s\rfloor^2>7k$ blocks of $s\times s$ squares with disjoint interiors, so some block consists entirely of faces. Its concentric squares give $\lfloor s/2\rfloor$ disjoint nested contractible cycles, more than $m$ in the stated range.

## Appendix A. Induced Euler genera at a shared vertex

In the proof of Proposition 9.1, two disjoint regular neighbourhoods have capped Euler genera whose sum is at most the ambient Euler genus. Here two subgraphs meet in just one vertex, and each carries the full Euler genus of the same drawing of an edge deletion. As before, the induced Euler genus of a connected subgraph is the Euler genus of its capped regular neighbourhood.

**Proposition A.1.** *For every integer $s\geq 7$ there is a simple 2-connected excluded minor $G$ for $N_s$, an edge $f$, and a minimum-Euler-genus drawing of $G-f$ in $N_s$ containing two subgraphs meeting only at one vertex, each of induced Euler genus $s$. For these drawings the sum of the two induced Euler genera exceeds the bound in [14, Lemma 5.2] for one common vertex.*

*Proof.* Fix distinct terminals $x,y$. Let $J=K_5-xy$ and let $G_r$ be the union of $r$ copies of $J$ with corresponding terminals identified and all other vertices distinct. Name the private vertices of copy $i$ by $a_i,b_i,c_i$. Write $H^+=H+xy$. Suppose that the connected graphs $H_1,H_2$ meet exactly in the terminals $x,y$ and neither contains $xy$. For their union $H$, Richter's formula in the form of [19, Theorem 3.2] gives

$$\begin{aligned} e(H) &= \min\{e(H_1)+e(H_2)+2,\ e(H_1^+)+e(H_2^+)\},\\ e(H^+) &= e(H_1^+)+e(H_2^+). \end{aligned}$$

Here $e(J)=0$ and $e(J^+)=1$. Thus $e(G_r^+)=r$ for $r\geq 1$, $e(G_2)=2$, and induction gives $e(G_r)=r$ for $r\geq 2$.

Put $G=G_{s+1}$. It is 2-connected, and $e(G)=s+1$ rules out an embedding in $N_s$. Deleting or contracting an edge in a single part makes that part planar even with $xy$ added, since the resulting graph with the added edge is

a proper minor of $K_5$. Draw the other $s$ augmented parts in projective planes and the modified augmented part in the sphere. In each drawing choose a disk on one side of $xy$ which meets the drawing exactly in this edge, as part of its boundary. Remove the interiors of a pair of such disks and glue the two boundary circles, identifying their marked edges. A fresh narrow disk alongside the common edge permits the next connected sum. Pasting the edges together gives a drawing in $N_s$ after omitting $xy$ if it is absent from the minor. This proves minor-minimality, since there are no isolated vertices. Take $f = a_{s+1}b_{s+1}$ and glue the planar part last. The graph $G - f$ contains $G_s$, so this drawing has minimum Euler genus.

In each of the $s$ intact augmented parts use the following projective-plane drawing. Draw a four-wheel with centre $x$ in a disk, with rim ordered $a_i, y, b_i, c_i$. In the complementary Möbius band draw the diagonals $a_ib_i$ and $yc_i$ as two disjoint spanning arcs of the interval bundle. Their endpoints alternate on its boundary as prescribed. Both triangles $T_i = xa_ib_ix$ and $U_i = xc_iyx$ are one-sided. We can do the preceding connected sums with this choice of drawings. The classes $z_i = [U_i]$ have Gram matrix $I_s$ for the mod-two intersection pairing in $N_s$. Indeed each triangle represents the generator of its projective summand and can be pushed away from the disk used for gluing. The same statement applies to the classes $[T_i]$.

In $G - f$ take $A = \bigcup_{i=1}^{s} T_i$ and let $B$ be the union of the $s+1$ paths $P_i = xc_iy$, including the path in the planar part. These subgraphs meet exactly at $x$. We compute their induced Euler genera using the same intersection pairing in $N_s$. For a regular neighbourhood $R$, the mod-two intersection form on $H_1(R; \mathbb{F}_2)$ has rank $h(R)$, since its radical is generated by boundary classes. Intersections in $R$ agree with those in the ambient surface. For $A$ the cycles $T_i$ form a basis and have Gram matrix $I_s$, so $A$ has induced Euler genus $s$.

In the augmented graph the triangle $P_{s+1} \cup xy$ lies in the disk contributed by the planar part. Its homology class is zero. The cycles $P_i \cup P_{s+1}$ for $1 \leq i \leq s$ form a cycle basis of $B$. Their ambient classes are $z_i$, since adding the triangle $P_{s+1} \cup xy$ to $U_i$ cancels $xy$ over $\mathbb{F}_2$. Their Gram matrix is $I_s$, giving induced Euler genus $s$ for $B$ as well. The asserted inequality would require $2s \leq s + 6$. □

The graphs $G$ in Proposition A.1 have $3s + 5$ vertices. Each restricted Euler genus is at most the ambient Euler genus, so $s = 7$ is the smallest value at which their sum can exceed $s + 6$. The example leaves open estimates for

the particular subgraphs used later in [14] or for a specially chosen drawing. Proposition 9.1 gives one such estimate when all attachments can be omitted from the subgraph being charged.

The following signed rotation system specifies the drawing of $G-f$. Read each list as a cyclic order and index the parts by $1,\dots,s+1$. Around $x$ concatenate $(b_i,c_i,a_i)$ for increasing $i$. Around $y$ concatenate $(a_i,c_i,b_i)$ for decreasing $i$. For a private rim vertex $r_j$ in the cyclic list $(a_i,y,b_i,c_i)$ take rotation $(r_{j+2},r_{j+1},x,r_{j-1})$, with indices modulo four, omitting the deleted edge. The negative edges are $a_ib_i$ and $yc_i$ for $i\le s$.

For $s=7$, the vertex and edge counts $(|V|,|E|)$ are $(26,71)$ for $G-f$, $(15,21)$ for $A$, and $(10,16)$ for $B$. Each capped regular neighbourhood has Euler genus 7 by Proposition A.1. Euler's formula $|F|=2-7-|V|+|E|$ gives 40, 1, and 1 faces, respectively.


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## Declaration of generative AI and AI-assisted technologies in the writing process

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