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The Avkhadiev–Wirths (2006) generalized Goodman conjecture is true

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Abstract

Let f be meromorphic and one-to-one in the unit disk, with a simple pole at a point $p \in (0,1)$, $f(0) = 0$ and $f'(0) = 1$. Its Taylor coefficients a_n at the origin satisfy the sharp bound $|a_n| \le A_n(p) = (1-p^{2n})/\big((1-p^2)p^{n-1}\big)$. Goodman conjectured this bound, and Jenkins showed that it follows from the Bieberbach conjecture, now de Branges' theorem. Let h be a holomorphic self-map of the disk with $h(0) = 0$. Then $f \circ h$ may have poles, but it is holomorphic near 0. In 2006 Avkhadiev and Wirths conjectured that its Taylor coefficients satisfy the same bound. Their 2009 book named this the generalized Goodman conjecture. It is the meromorphic counterpart of Rogosinski's generalized Bieberbach conjecture, which de Branges' work settled. It was known for $n \le 2$, for $n = 3$ with $p < 0.7$, for $n \ge 4$ with $p < 1/(2n-2)$, and for some classes of f. We prove it for every n and every p. The proof combines three tools: Rogosinski's reduction, by which the nth Taylor coefficient of $f \circ h$ equals the nth coefficient of Ef, for a holomorphic E with $|E| \le 1$ that depends on n; Jenkins' conformal map of the disk slit along $[p,1)$ onto the disk, a map that fixes 0 and has positive coefficients of $z, z^2, z^3, \dots$; and Robertson's inequality for odd univalent functions. For $n \ge 2$, equality holds only when $f(z) = z/\big((1-pz)(1-z/p)\big)$ and h is a rotation. As a consequence, we prove two conjectures of Avkhadiev and Wirths (2008): sharp bounds for the derivatives of a meromorphic map from the disk, or from a convex domain, into a simply connected domain of the sphere.[1] These results are verified in Lean.[2]

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1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ and let $\mathcal{S}$ be the class of functions F that are holomorphic and one-to-one in $\mathbb{D}$ with $F(0) = 0$ and $F'(0) = 1$. Here and below $[w^n]g$ denotes the coefficient of w^n in the Taylor expansion of g at the origin. De Branges proved in 1985 that every $F \in \mathcal{S}$ satisfies $|[w^n]F| \le n$ [11].

Fix $p \in (0,1)$. We write $\mathcal{M}(p)$ for the class of functions f that are meromorphic and one-to-one in $\mathbb{D}$, have a simple pole at p and no other pole, and satisfy $f(0) = 0$, $f'(0) = 1$. Such an f is holomorphic in the disk $|z| < p$, so it has a Taylor expansion $f(z) = z + \sum_{n\ge2} a_n z^n$ there. The sharp bound for these coefficients is

$$|a_n| \le A_n(p) := \frac{1-p^{2n}}{(1-p^2)\,p^{n-1}} = p^{1-n}\sum_{j=0}^{n-1} p^{2j} \qquad (n \ge 1), \tag{1}$$

[1] Expanded version, with further results [13]: https://doi.org/10.5281/zenodo.24570827

[2] Proved in Lean 4 from Lean's three standard axioms and four published theorems, checked with `lake comparator`: https://github.com/david44hello/paper_avkhadiev-wirths-2006-generalized-goodman/tree/v2.70/lean

with equality for $n \geq 2$ only for

$$f_p(z) = \frac{z}{(1-pz)(1-z/p)}. \tag{2}$$

Expanding (2) as a product of two geometric series shows $[z^n]f_p = A_n(p)$. Goodman proved (1) when all the coefficients of f are real, and conjectured it for all $f \in \mathcal{M}(p)$ [14, Theorem 7 and p. 104]. Jenkins showed that it follows from the Bieberbach conjecture [15, pp. 25–27]; see also [7, Section 2.3]. By de Branges' theorem, (1) holds. Its equality statement also holds: Jenkins derived it from the equality case of the Bieberbach conjecture [15, p. 26], which de Branges proved [11, p. 151]. Theorem 1 below, in its simplest case ($h(z) = z$ and no weight), proves this equality statement again; besides de Branges' theorem it uses only Bieberbach's equality case for the second coefficient.

The conjecture. Let h be holomorphic in $\mathbb{D}$ with $|h| < 1$ and $h(0) = 0$; we call such h a *Schwarz function*. For $f \in \mathcal{M}(p)$ the composition $f \circ h$ is again meromorphic in $\mathbb{D}$. It has a pole at every point where h takes the value p, and in general it is not one-to-one. It is, however, holomorphic in the disk $|z| < p$, because $|h(z)| \leq |z|$ there by Schwarz's lemma. So $f \circ h$ has a Taylor expansion at the origin, and its coefficients are the subject of the following conjecture.

Conjecture 1 (Avkhadiev and Wirths [3, 7, 8])**.** *For every $p \in (0,1)$, every $f \in \mathcal{M}(p)$, every Schwarz function h and every $n \geq 1$,*

$$\bigl|[z^n](f \circ h)\bigr| \leq A_n(p).$$

Avkhadiev and Wirths stated it in 2006, in a geometric form, as inequality (5) of [3] (Theorem 2 below). In their 2009 monograph *Schwarz–Pick Type Inequalities* [7, Section 8.3, pp. 131–138], they named it the generalized Goodman conjecture and devoted half of the closing open-problems chapter to it, alongside the Krzyż and angle conjectures. Their 2013 paper restates it for coefficients as Conjecture 1 of [8], writing $h(z) = z\omega(z)$ with $|\omega| \leq 1$, and credits it to [3]. With $h(z) = z$ it is the inequality (1). It is the meromorphic counterpart of Rogosinski's generalized Bieberbach conjecture: if $F \in \mathcal{S}$ and h is a Schwarz function, then $|[z^n](F \circ h)| \leq n$ [21]. The two bounds match, since $A_n(p) \to n$ as $p \to 1$. De Branges' work also settles Rogosinski's conjecture: it proves Robertson's inequality for odd univalent functions [11, pp. 137–138], which implies Rogosinski's conjecture (see [23, p. 172], and Lemmas 4 and 6 below).

Earlier results. Avkhadiev and Wirths proved the conjecture for $n = 2$ and every p, for $n = 3$ and $p < 0.7$, and for each $n \geq 4$ and $p < 1/(2n-2)$ [7, Theorem 8.3], and for every n and p when the complement of $f(\mathbb{D})$ is convex [5, Theorem 1] (see also [7, Theorem 8.4]). That concave-domain case has been used to derive Bohr–Rogosinski inequalities for meromorphic subordination (for instance [1, Lemma 4.1]). In [8] they proved the conjecture for every n when the complement of $f(\mathbb{D})$ belongs to one of two classes of starlike sets, and noted that the case of real coefficients was known [8, p. 143]. Bhowmik and Parveen proved it for $n \geq 3$ and $p \leq 1/3$ for a subclass of $\mathcal{M}(p)$ defined by a differential inequality [9]; by contrast, a related λ-deformed coefficient conjecture ($\lambda < 1$) for that subclass was recently disproved for $n = 3$ and large p [16]. Figure 1 compares these results with ours.

Results. We prove the conjecture in the following stronger form, which allows a bounded weight.

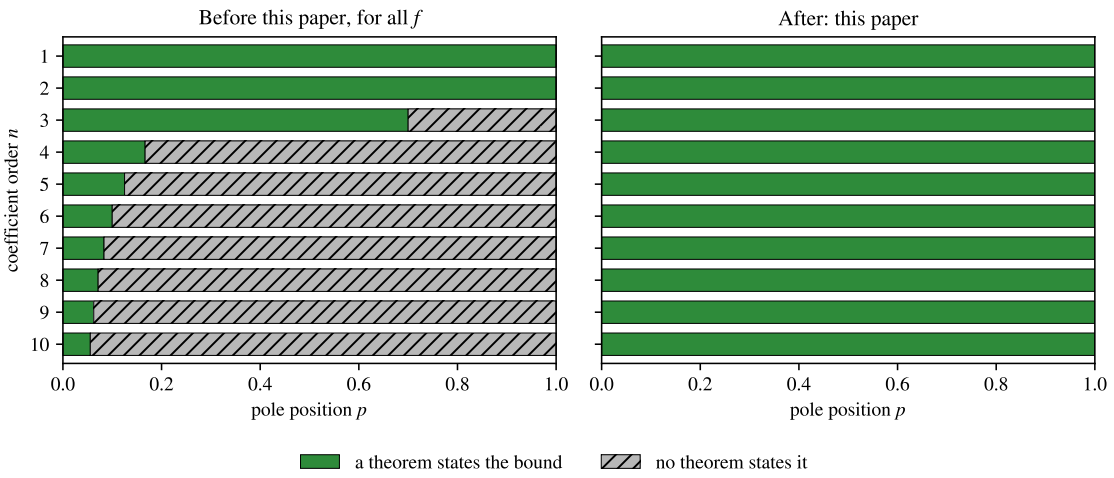

Figure 1: The bound $A_n(p)$ of Conjecture 1 for all $f \in \mathcal{M}(p)$: solid where a theorem states the bound, hatched elsewhere. Left: before this paper ($n = 1$ is Schwarz's lemma; $n = 2$; $n = 3$ with $p < 0.7$; $n \geq 4$ with $p < 1/(2n-2)$ [7, Theorem 8.3]). Right: after (Theorem 1). Rows $n = 1, \ldots, 10$ are drawn; the pattern continues for all n. Results for special classes of f (convex complements [5]; two starlike classes, every n [8]; a subclass, $n \geq 3$ and $p \leq 1/3$ [9]) are not shown.

Theorem 1. *Let $p \in (0,1)$, $f \in \mathcal{M}(p)$, h a Schwarz function, and e holomorphic in $\mathbb{D}$ with $|e| \leq 1$. Then for every $n \geq 1$*

$$\bigl|[z^n]\, e(z) f(h(z))\bigr| \leq A_n(p).$$

With $e = 1$ this is Conjecture 1. For $e = 1$ and $n \geq 2$, equality holds if and only if $f = f_p$ and $h(z) = \lambda z$ with $|\lambda| = 1$. For $e = 1$ and $n = 1$, equality holds exactly when $h(z) = \lambda z$ with $|\lambda| = 1$.

In the language of Robertson [20], who considered holomorphic f, Theorem 1 bounds the coefficients of the functions $e \cdot (f \circ h)$ quasi-subordinate to f.

The original statement of 2006 is geometric. Let $\Pi \subset \widehat{\mathbb{C}}$ be a simply connected domain of the Riemann sphere whose complement has at least two points. For $w_0 \in \Pi \cap \mathbb{C}$ let $R(w_0, \Pi)$ be the conformal radius of Π at w_0, that is, $|\phi'(0)|$ for any conformal map ϕ of $\mathbb{D}$ onto Π with $\phi(0) = w_0$. Let δ_Π be the hyperbolic distance in Π, normalized by curvature -4, so that $\delta_{\mathbb{D}}(0, r) = \operatorname{artanh} r$.

Theorem 2. *Let Π be as above with $\infty \in \Pi$, and let $G : \mathbb{D} \to \Pi$ be holomorphic, as a map into the sphere, with $w_0 = G(0) \neq \infty$. Put $p = \tanh \delta_\Pi(w_0, \infty)$, a number in $(0,1)$. Then for every $n \geq 1$*

$$\frac{|G^{(n)}(0)|}{n!} \leq R(w_0, \Pi)\, \frac{1 - p^{2n}}{(1-p^2)\, p^{n-1}} = R(w_0, \Pi)\, A_n(p).$$

For $n \geq 2$ equality holds if and only if G maps $\mathbb{D}$ one-to-one onto Π and the complement of Π is a line segment lying on a line through w_0.

In 2008 Avkhadiev and Wirths extended these estimates to maps defined on other domains, and proved them when the target is a concave domain, that is, when Π is the complement of a compact convex set [6]. For a general simply connected target they stated two conjectures, one for maps on the disk and one for maps on convex domains [6, Conjectures 1 and 2]. Theorem 3 below, a consequence of Theorem 1, proves both.

Again let $\Pi \subset \widehat{\mathbb{C}}$ be a simply connected domain whose complement has at least two points. For a simply connected domain $\Omega \neq \mathbb{C}$, a point $z_0 \in \Omega$ and a meromorphic map $G : \Omega \to \Pi$ with $w_0 = G(z_0)$ finite, put $R_s = R(z_0, \Omega)$ and $R_t = R(w_0, \Pi)$, the conformal radii of the source and of the target. Let $p = \tanh \delta_\Pi(w_0, \infty)$ if $\infty \in \Pi$, and $p = 1$ otherwise; we put $A_k(1) = k$, the limit of $A_k(p)$ as $p \to 1$. For $0 \le r \le 1$ define

$$D_n(p, r) = \sum_{k=1}^{n} \binom{n-1}{k-1} r^{n-k} A_k(p) = \frac{(r + 1/p)^{n-1} - p^2 (r + p)^{n-1}}{1 - p^2}, \tag{3}$$

where the second expression, for $0 < p < 1$, follows from $A_k(p) = (p^{1-k} - p^{k+1})/(1 - p^2)$ and the binomial theorem.

Theorem 3. *Let $\Pi \subset \widehat{\mathbb{C}}$ be a simply connected domain whose complement has at least two points, let Ω, z_0, G, R_s, R_t and p be as just defined, and let $n \ge 2$.*

(a) If $\Omega = \mathbb{D}$, then $R_s^n |G^{(n)}(z_0)|/(n!\, R_t) \le D_n(p, |z_0|)$.

(b) If $\Omega \neq \mathbb{C}$ is convex, then $R_s^n |G^{(n)}(z_0)|/(n!\, R_t) \le D_n(p, 1)$.

Both bounds are sharp for every p and n. Maps that attain them (Section 5) are built from f_p; for $p = 1$, $f_1(z) = z/(1 - z)^2$ is the Koebe function.

For planar targets ($p = 1$) both bounds were known: (a) is [2, Theorem 2] and (b) is in [4]; for concave Π both are in [6]. Part (a) is [6, Conjecture 1] and part (b) is [6, Conjecture 2]. The new content is the case where $\infty \in \Pi$ and Π is not concave. Part (a) for $n = 2$, for $n = 3$ with $p < 0.7$, and for $n \ge 4$ with $p < 1/(2n - 2)$ also follows from [7, Theorem 8.3] by the argument of Section 5.

The idea of the proof. Three steps reduce Theorem 1 to classical facts. First, the coefficient $[z^n]\, e\,(f \circ h)$ equals $[w^n]\, Ef$ for a holomorphic E with $|E| \le 1$ (Lemma 4); for $e = 1$ this is due to Rogosinski [21], see also Szapiel [23]. So the inner map disappears and only a bounded factor remains. Second, there is a conformal map q of the slit disk $\mathbb{D} \setminus [p, 1)$ onto $\mathbb{D}$, built from the Joukowski map and one affine map. The coefficient of z^n in q is positive for every $n \ge 1$ (Lemma 5; Jenkins [15, p. 25]). The function $(f \circ q^{-1})/R$, where $R = R(0, \mathbb{D} \setminus [p, 1)) = 4p/(1 + p)^2$ is the conformal radius of $\mathbb{D} \setminus [p, 1)$ at 0, has no pole and lies in $\mathcal{S}$. Third, for $F \in \mathcal{S}$ and $|E| \le 1$ one has $|[w^k]FE| \le k$, a consequence of Robertson's inequality (Lemma 6). For $F = (f \circ q^{-1})/R$ and $\Phi = F \cdot (E \circ q^{-1})$ this gives $|[w^k]\Phi| \le k$, and $Ef = R\,\Phi \circ q$ on $\mathbb{D} \setminus [p, 1)$. Since every power q^k has nonnegative coefficients, summing these estimates bounds $|[z^n]\, e\,(f \circ h)|$ by the coefficient of z^n in $R \sum_k k\, q^k$, and this series is exactly f_p of (2).

For a general weight e we prove the first step by duality. The third step is classical (Robertson, MacGregor). With $e = 1$ and $h(z) = z$, the second and third steps are Jenkins' proof of (1) [15, pp. 25–26], which assumed the Bieberbach conjecture $|[w^k]F| \le k$, now de Branges' theorem. The point of the proof is that the second and third steps tolerate a bounded factor E, and the first step turns the composition with h into exactly such a factor.

2 Three lemmas

Lemma 4 (transfer)**.** *Let h be a Schwarz function, e holomorphic in $\mathbb{D}$ with $|e| \le 1$, and $n \ge 0$. There is a holomorphic E in $\mathbb{D}$ with $|E| \le 1$ such that*

$$[z^n]\, e(z)\, g(h(z)) = [w^n]\, E(w) g(w)$$

for every function g holomorphic near 0. Moreover $E(0) = e(0) h'(0)^n$.

Proof. Write $g(w) = \sum_{j\ge 0} g_j w^j$ near 0. Since $h^j = O(z^j)$, only $j \le n$ contribute, and $[z^n]\, e\, g(h) = \sum_{j=0}^{n} g_j \Lambda(w^j)$, where $\Lambda(P) = [z^n]\, e(z) P(h(z))$ for polynomials P. We claim $|\Lambda(P)| \le \|P\|_1$, the norm of P in L^1 of the normalized arc measure σ on the unit circle $\mathbb{T}$. Let U be the Poisson integral of $|P|$ on $\mathbb{T}$. Since $|P|$ is subharmonic, $|P| \le U$ in $\mathbb{D}$, and $U \circ h$ is harmonic. By Cauchy's formula on the circle of radius $r < 1$,

$$r^n |\Lambda(P)| \le \int_{\mathbb{T}} |P(h(r\zeta))|\, d\sigma(\zeta) \le \int_{\mathbb{T}} U(h(r\zeta))\, d\sigma(\zeta) = U(h(0)) = U(0) = \|P\|_1,$$

where the first step uses $|e| \le 1$. Letting $r \to 1$ proves the claim. By the Hahn–Banach theorem and the duality of L^1 and L^∞ there is $v \in L^\infty(\mathbb{T})$, $\|v\|_\infty \le 1$, with $\Lambda(P) = \int_{\mathbb{T}} vP\, d\sigma$ for every polynomial P. As $\Lambda(w^j) = 0$ for $j > n$, the function $w^n v(w)$ has no Fourier coefficients of negative index, so it is the boundary function of a holomorphic E with $|E| \le 1$. For $0 \le j \le n$ the coefficient of w^{n-j} in E is $\int_{\mathbb{T}} w^n v\, \overline{w}^{\,n-j}\, d\sigma = \int_{\mathbb{T}} v\, w^j\, d\sigma = \Lambda(w^j)$. Hence $\sum_{j\le n} g_j \Lambda(w^j) = [w^n]\, Eg$, and $E(0) = \Lambda(w^n) = e(0) h'(0)^n$. □

For $e = 1$ there is an explicit E, which avoids the duality argument. If $n = 0$, then $E = 1$. For $n \ge 1$, put $c_j = [z^n] h^j$ and $Q(w) = \sum_{j=1}^{n} c_j w^{n-j}$. Then $[z^n]\, g(h) = \sum_j g_j c_j = [w^n]\, Qg$, so $E = Q$ works once we know $|Q| \le 1$. For $|t| = 1$ the function $(1+th)/(1-th)$ has positive real part and value 1 at the origin, and its coefficient of z^n is $2\sum_j c_j t^j$; Carathéodory's inequality bounds it by 2. Taking $t = 1/w$ with $|w| = 1$ gives $|Q(w)| \le 1$ on $\mathbb{T}$, hence in $\mathbb{D}$. This is Rogosinski's argument [21, (1.1.3) and Lemma 2]. Szapiel [23, p. 172, (5)] proves an equivalent bound by the subordination principle and Schwarz's lemma.

Lemma 5 (Jenkins' slit map)**.** *Let $J(z) = z + 1/z$, $B = J(p) = p + 1/p$ and $R = 4/(B+2)$. There is a conformal map q of $\mathbb{D} \setminus [p, 1)$ onto $\mathbb{D}$ with $q(0) = 0$ and*

$$J(q(z)) + 2 = R\,\big(J(z) + 2\big). \tag{4}$$

The coefficient of z^n in q is positive for every $n \ge 1$, and

$$R \sum_{k\ge 1} k\, q(z)^k = f_p(z) \quad \textit{near } 0. \tag{5}$$

Proof. The Joukowski map J sends $\mathbb{D}$ one-to-one onto $\widehat{\mathbb{C}} \setminus [-2, 2]$, with $J(0) = \infty$. On the radius $[p, 1)$ it decreases from B to 2, so it sends $\mathbb{D} \setminus [p, 1)$ onto $\widehat{\mathbb{C}} \setminus [-2, B]$. The affine map $x \mapsto R(x+2) - 2$ fixes -2 and ∞ and sends B to 2, so it maps $\widehat{\mathbb{C}} \setminus [-2, B]$ onto $\widehat{\mathbb{C}} \setminus [-2, 2]$. Composing with J^{-1} gives q (Figure 2). It fixes 0: J sends 0 to ∞, the affine map fixes ∞, and J^{-1} sends ∞ to 0. Comparing the poles at 0 in (4) gives $q'(0) = 1/R$; so $R = \chi'(0)$, where $\chi = q^{-1}$, is the conformal radius of $\mathbb{D} \setminus [p, 1)$ at 0.

Since $J(z) + 2 = (1+z)^2/z$, identity (4) reads $z(1+q)^2 = Rq(1+z)^2$. Put $c = 1/R - 1 = (1-p)^2/(4p) > 0$. Then $q(1+z)^2 - z(1+q)^2 = c\, z(1+q)^2$, and the left side factors as $(q - z)(1 - zq)$. Hence, near 0,

$$q = z + c\, z\, \frac{(1+q)^2}{1 - zq} = z + c\, z\, (1+q)^2 \sum_{m\ge 0} (zq)^m.$$

Write $q = \sum_{j\ge 1} q_j z^j$. Then $q_1 = 1 + c$, and for $n \ge 2$ the coefficient q_n is a polynomial with nonnegative coefficients in $c, q_1, \ldots, q_{n-1}$ that contains the term $2c\, q_{n-1}$. By induction every q_n is positive.

Finally $\sum_k k q^k = q/(1-q)^2 = 1/(J(q) - 2)$, and by (4) $J(q) - 2 = R\,(J(z) - B)$, because $4/R = B + 2$. So $R \sum_k k q^k = 1/(J(z) - B) = z/(1 - Bz + z^2) = f_p(z)$. □

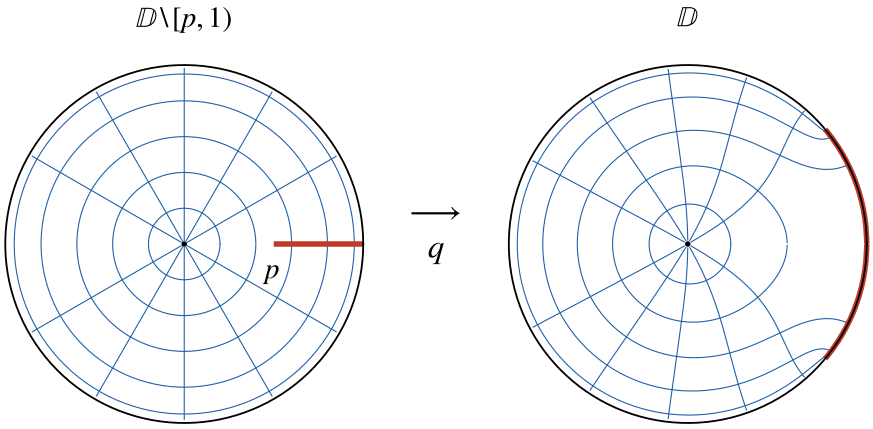

Figure 2: The slit map q of Lemma 5 for $p = 1/2$: circles and radii of the slit disk (left) and their images (right). The two edges of the slit (heavy, left) go to the heavy arc of the unit circle around 1 (right). The coefficient of z^n in q is positive for every $n \geq 1$, so the estimates of Lemma 6 are summed with nonnegative weights (Proposition 7).

Lemma 6 (product estimate)**.** *Let $F \in \mathcal{S}$ and let E be holomorphic in $\mathbb{D}$ with $|E| \leq 1$. Then $|[w^k]\, FE| \leq k$ for every $k \geq 1$.*

Proof. Since $F(w)/w$ has no zeros, $F(w) = wM(w)^2$ with M holomorphic and $M(0) = 1$. The odd function $F_2(z) = zM(z^2)$ satisfies $F_2(z)^2 = F(z^2)$ and is one-to-one: if $F_2(z_1) = F_2(z_2)$ then $z_1^2 = z_2^2$, and $z_2 = -z_1 \neq 0$ would give $F_2(z_1) = -F_2(z_1) = 0$, which is impossible, since F_2 vanishes only at 0. Write $M(w) = \sum_{j\geq 0} b_j w^j$, so b_j is the coefficient of z^{2j+1} in F_2. Robertson's inequality, proved by de Branges [11], states $\sum_{j=0}^{k-1} |b_j|^2 \leq k$. Let $M_k = \sum_{j<k} b_j w^j$. Only these terms of M reach the coefficient of w^{k-1}, so

$$[w^k]\, FE = [w^{k-1}]\, M^2 E = [w^{k-1}]\, (M_k E)\, M_k.$$

By the Cauchy–Schwarz inequality for coefficients this is at most $\|M_k E\|_{H^2} \|M_k\|_{H^2} \leq \|M_k\|_{H^2}^2 = \sum_{j<k} |b_j|^2 \leq k$. □

With $E = 1$ this is Robertson's observation that his inequality implies Bieberbach's [19]; see [11, p. 138]. The form with a bounded factor E is in [12, p. 256], where it is credited to MacGregor [17].

3 Proof of Theorem 1

Proposition 7. *Let $f \in \mathcal{M}(p)$ and let E be holomorphic in $\mathbb{D}$ with $|E| \leq 1$. Then for every $n \geq 1$*

$$\big|[z^n]\, Ef\big| \leq A_n(p) - R\, q_n\, (1 - |E(0)|),$$

where R and $q_n = [z^n]q > 0$ are as in Lemma 5.

Proof. Let $\chi = q^{-1}$, which maps $\mathbb{D}$ conformally onto $\mathbb{D} \setminus [p, 1)$, with $\chi(0) = 0$ and $\chi'(0) = R$. Since f has its only pole at $p \notin \chi(\mathbb{D})$, the function $F = (f \circ \chi)/R$ is holomorphic and one-to-one in $\mathbb{D}$, with $F(0) = 0$ and $F'(0) = 1$; so $F \in \mathcal{S}$. The function $\Phi = F \cdot (E \circ \chi)$ satisfies $|[w^k]\Phi| \leq k$ by

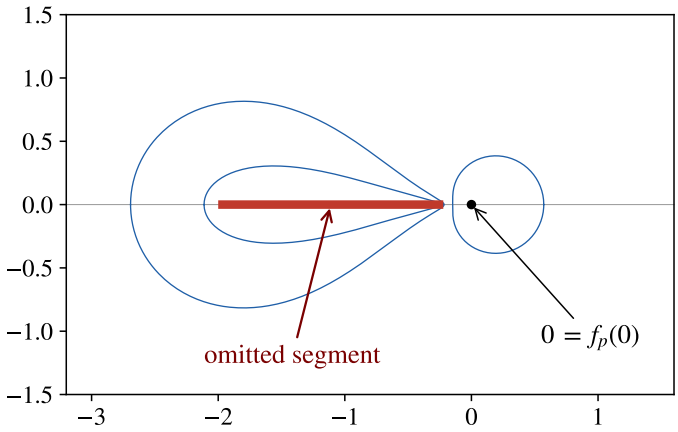

Figure 3: The image $f_p(\mathbb{D})$ of the extremal function, for $p = 1/2$ (the case of equality in Theorems 1 and 2): f_p maps $\mathbb{D}$ onto the sphere minus the heavy segment $[-2, -2/9]$, whose line passes through $f_p(0) = 0$. The curves are images of the circles $|z| = 0.25$, 0.7 and 0.85.

Lemma 6, and $[w^1]\Phi = E(0)$. On $\mathbb{D} \setminus [p, 1)$ we have $Ef = R\,\Phi \circ q$. Here f, E and q are functions of z, while F and Φ are functions of $w = q(z)$. Writing $\Phi(w) = \sum_k d_k w^k$ and using $q(z)^k = O(z^k)$,

$$[z^n]\,Ef = R\sum_{k=1}^{n} d_k\,[z^n]q^k.$$

All the numbers $[z^n]q^k$ are nonnegative by Lemma 5. Hence

$$|[z^n]\,Ef| \le R\Big(|E(0)|\,q_n + \sum_{k=2}^{n} k\,[z^n]q^k\Big) = R\sum_{k=1}^{n} k\,[z^n]q^k - Rq_n(1 - |E(0)|),$$

and $R\sum_{k=1}^{n} k\,[z^n]q^k = [z^n]f_p = A_n(p)$ by (5). □

Proof of Theorem 1. Lemma 4 with $g = f$ gives $[z^n]\,e\,f(h) = [w^n]\,E(w)f(w)$ for some $|E| \le 1$. This is the coefficient bounded in Proposition 7, and that proposition gives the bound.

Now let $e = 1$. Then $E(0) = h'(0)^n$, and by Proposition 7 equality forces $|h'(0)| = 1$, so $h(z) = \lambda z$ with $|\lambda| = 1$ by Schwarz's lemma. Conversely, for such h, $[z^n]f(\lambda z) = \lambda^n a_n$. For $n = 1$ this has modulus $1 = A_1(p)$ for every f. For $n \ge 2$ we must decide when $|a_n| = A_n(p)$. Since $a_n = [z^n]f$, Proposition 7 applies with $E = 1$; then $\Phi = F$, and equality in the proof of Proposition 7 requires $|d_2| = |[w^2]F| = 2$, because $[z^n]q^2 > 0$ for $n \ge 2$. By the equality case of Bieberbach's inequality for the second coefficient [10], $F(w) = w/(1 - \eta w)^2$ with $|\eta| = 1$, so $d_k = k\eta^{k-1}$. Equality in the triangle inequality requires d_1 and d_2 to have the same argument, so $\eta = 1$, F is the Koebe function, and $f = R\,F \circ q = R\,q/(1 - q)^2 = f_p$ by (5). Conversely $f = f_p$ and $h(z) = \lambda z$ with $|\lambda| = 1$ give equality, since $|[z^n]f_p(\lambda z)| = A_n(p)$. □

4 The geometric form

Proof of Theorem 2. Let $\phi : \mathbb{D} \to \Pi$ be a conformal map with $\phi(0) = w_0$; after a rotation of $\mathbb{D}$ the point $\phi^{-1}(\infty)$ lies in $(0, 1)$. Conformal maps preserve hyperbolic distance, so $\delta_{\mathbb{D}}(0, \phi^{-1}(\infty)) = \delta_{\Pi}(w_0, \infty) = \operatorname{artanh} p$, that is, $\phi^{-1}(\infty) = p$. Put $a = \phi'(0)$ and $f = (\phi - w_0)/a$, so $f \in \mathcal{M}(p)$ and

$|a| = R(w_0, \Pi)$. The map $h = \phi^{-1} \circ G$ is a Schwarz function; at the poles of G it is holomorphic because ϕ^{-1} is holomorphic at ∞ in the spherical coordinate. Near 0 we have $G = w_0 + a\, f \circ h$, so $G^{(n)}(0)/n! = a\,[z^n](f \circ h)$, and Theorem 1 gives the bound.

Let $n \geq 2$. By Theorem 1, equality holds exactly when $f = f_p$ and h is a rotation, that is, when G maps $\mathbb{D}$ conformally onto $\Pi = w_0 + a\, f_p(\mathbb{D})$. Since $f_p = 1/(J - B)$ and $J(\mathbb{D}) = \widehat{\mathbb{C}} \setminus [-2, 2]$, the complement of $f_p(\mathbb{D})$ is the segment $[-1/(B-2), -1/(B+2)]$, whose line, the real axis, passes through $0 = f_p(0)$ (Figure 3). So the complement of Π is a segment on a line through w_0. Conversely, let the complement of Π be a segment on a line through $w_0 \in \Pi$, with endpoints at distances $0 < \ell_1 < \ell_2$ from w_0. Put

$$p = \frac{\sqrt{\ell_2} - \sqrt{\ell_1}}{\sqrt{\ell_2} + \sqrt{\ell_1}}, \qquad |a| = \frac{4\ell_1\ell_2}{\ell_2 - \ell_1},$$

and choose the argument of a so that $a/|a| = (w_0 - \xi)/\ell_1$, where ξ is the endpoint nearer to w_0; that is, a points from ξ towards w_0. With $B = p + 1/p$ one checks $|a|/(B+2) = \ell_1$ and $|a|/(B-2) = \ell_2$, so $w_0 + a f_p$ maps $\mathbb{D}$ onto Π. Since this map sends p to ∞, this p is $\tanh \delta_\Pi(w_0, \infty)$, and $|a| = R(w_0, \Pi)$. Every conformal map of $\mathbb{D}$ onto Π sending 0 to w_0 is this map composed with a rotation, and attains equality for every n. □

5 The two conjectures of 2008: proof of Theorem 3

The planar case $p = 1$ uses the weighted form of Rogosinski's estimate, $|[z^k]\, E\,(F \circ h)| \leq k$ for $F \in \mathcal{S}$ and $|E| \leq 1$, which follows from Lemmas 4 and 6; for $0 < p < 1$ we use Theorem 1. In both cases we need the following consequence.

Lemma 8. *Let Ω, z_0, G, Π, R_s, R_t and p be as in Theorem 3, let $\phi : \mathbb{D} \to \Omega$ be conformal with $\phi(0) = z_0$ and $\phi'(0) = R_s$, and let $H = (G \circ \phi - w_0)/R_t$. Then $|[z^k]\, EH| \leq A_k(p)$ for every $k \geq 1$ and every holomorphic E with $|E| \leq 1$.*

Proof. Let $\psi : \mathbb{D} \to \Pi$ be conformal with $\psi(0) = w_0$, rotated so that $\psi^{-1}(\infty) = p$ when $\infty \in \Pi$. Then $\psi = w_0 + a\, f$ with $|a| = R_t$ and $f \in \mathcal{M}(p)$, or $f \in \mathcal{S}$ if $\infty \notin \Pi$. The map $h = \psi^{-1} \circ G \circ \phi$ is a Schwarz function and $H = (a/R_t)\, f \circ h$. Theorem 1, or Lemmas 4 and 6 when $p = 1$, give the bound. □

Proof of Theorem 3(a). Take $\phi(z) = (z_0 + z)/(1 + \overline{z_0} z)$, so $R_s = 1 - |z_0|^2$ and $\phi^{-1}(z_0 + R_s x) = x/(1 - \overline{z_0} x)$. Hence $G(z_0 + R_s x) = w_0 + R_t\, H\big(x/(1 - \overline{z_0} x)\big)$, and since $[x^n](x/(1-\overline{z_0}x))^k = \binom{n-1}{k-1}\overline{z_0}^{\,n-k}$,

$$\frac{R_s^n G^{(n)}(z_0)}{n!\, R_t} = \sum_{k=1}^{n} \binom{n-1}{k-1} \overline{z_0}^{\,n-k}\, [z^k] H.$$

Lemma 8 with $E = 1$ gives the bound $D_n(p, |z_0|)$. For sharpness take $z_0 = r \geq 0$, $\Pi = f_p(\mathbb{D})$ and $G = f_p \circ \phi^{-1}$; then $R_t = 1$, $H = f_p$, and every term is nonnegative, so equality holds. Rotations of $\mathbb{D}$ give the other z_0. □

Proof of Theorem 3(b). Let ϕ be as in Lemma 8 and $s = (\phi - z_0)/R_s$, a normalized convex map. Then $G(z_0 + R_s x) = w_0 + R_t H(s^{-1}(x))$, and the Lagrange inversion formula gives

$$\frac{R_s^n G^{(n)}(z_0)}{n!\, R_t} = [x^n]\, H(s^{-1}(x)) = \frac{1}{n}\, [z^{n-1}]\, H'(z) \Big(\frac{z}{s(z)}\Big)^n.$$

By the Marx–Strohhäcker theorem [18, 22], $\mathrm{Re}(s(z)/z) > 1/2$; inversion maps this half-plane into the disk $|w - 1| < 1$, so by Schwarz's lemma $z/s(z) = 1 + z v(z)$ with $|v| \leq 1$. Expanding

$(1+zv)^n = \sum_j \binom{n}{j} z^j v^j$, the term with v^n does not reach z^{n-1}, and since $[z^{n-1}]\, H' z^j v^j = [z^{k-1}]\, H' v^j$ with $k = n - j \geq 1$, the others are $\binom{n}{j}[z^{k-1}]\, H' v^j$.

For holomorphic $E = \sum \varepsilon_i z^i$ with $|E| \leq 1$, let $\widetilde{E} = \sum_{i<k}(1 - i/k)\varepsilon_i z^i$ be its Fejér mean of order k. It is $\widetilde{E}(z) = \int_{\mathbb{T}} E(z\overline{\zeta})\, K_k(\zeta)\, d\sigma(\zeta)$ with the Fejér kernel $K_k(\zeta) = \sum_{|i|<k}(1 - |i|/k)\zeta^i \geq 0$, which has $\int_{\mathbb{T}} K_k\, d\sigma = 1$; so $|\widetilde{E}| \leq 1$. We need $\widetilde{E}$ because Lemma 8 bounds the coefficients of EH, while the Lagrange inversion formula produces H'. Writing $H = \sum_m \eta_m z^m$ (so $\eta_0 = 0$), both $[z^{k-1}]\, H'E$ and $k\,[z^k]\, H\widetilde{E}$ equal $\sum_{m=1}^{k} m\, \eta_m \varepsilon_{k-m}$, so $|[z^{k-1}]\, H'E| \leq k A_k(p)$ by Lemma 8. We apply this with $E = v^j$ and $k = n - j$. Since $\frac{1}{n}\binom{n}{j} k = \binom{n-1}{k-1}$, summing over j gives the bound $D_n(p, 1)$.

For sharpness take $z_0 = 0$ and $\phi(z) = z/(1+z)$, which maps $\mathbb{D}$ onto the half-plane $\operatorname{Re} x < 1/2$ with $R_s = 1$, and $G = f_p \circ \phi^{-1}$. Then $R_t = 1$, $\phi^{-1}(x) = x/(1-x)$, and $[x^n] f_p(x/(1-x)) = \sum_k \binom{n-1}{k-1} A_k(p) = D_n(p, 1)$. □

Discussion

Theorem 1 proves the generalized Goodman conjecture for every n and every p, in a stronger form that allows a bounded weight; Figure 1 shows what was known before. For $e = 1$ and $n \geq 2$ only $f = f_p$ with a rotation h attains the bound. Theorem 3 proves the two conjectures of 2008.

We end with three questions.

(1) *Inner maps with a zero of higher order.* If h has a zero of order at least d at 0 and $n \geq d$, then $|[z^n](f \circ h)| \leq A_m(p)$ with $m = \lfloor n/d \rfloor$, and this bound is attained exactly when m divides n, for example by f_p and $h(z) = z^{n/m}$ (expanded version [13, Theorem 12]); for $d = 2$ and $n = 5$ we find the exact maximum (expanded version [13, Proposition 13]). What is the exact maximum when m does not divide n, beginning with $d = 2$ and $n = 7$?

(2) *Stability.* For $n \geq 2$, if $|[z^n](f \circ h)| \geq A_n(p) - \varepsilon$, then f lies within $C\varepsilon^{1/2}$ of f_p in the chordal metric on the disk $|z| \leq r$, where $r < 1$ and C depends only on n, p and r, and the order $\varepsilon^{1/2}$ cannot be improved (expanded version [13, Theorems 16 and 17]). What are the best constants C?

(3) *Other source domains.* The bound of Theorem 3(a) extends to a meromorphic map from any simply connected source domain Ω whose complement has at least two points, with $|z_0|$ replaced by $\tau(\Omega, z_0) = \sup_{w \in \mathbb{D}} |R_s/(\phi(w) - z_0) - 1/w|$, where ϕ is the conformal map of $\mathbb{D}$ onto Ω with $\phi(0) = z_0$ and $\phi'(0) = R_s > 0$; this number equals $|z_0|$ for the disk and is at most 1 for convex Ω (expanded version [13, Theorem 19]). For $n \geq 2$ we determine the extremal maps when the target contains ∞ (expanded version [13, Theorem 20 and Corollary 21]). Which maps attain the bound for a general source domain when the target does not contain ∞?

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