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The Avkhadiev–Wirths (2006) generalized Goodman conjecture is true, and other miscellanea

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Abstract

Let f be meromorphic and one-to-one in the unit disk, with a simple pole at a point $p \in (0,1)$, $f(0) = 0$ and $f'(0) = 1$. Its Taylor coefficients a_n at the origin satisfy the sharp bound $|a_n| \le A_n(p) = (1-p^{2n})/\big((1-p^2)p^{n-1}\big)$. Goodman conjectured this bound, and Jenkins showed that it follows from the Bieberbach conjecture, now de Branges' theorem. Let h be a holomorphic self-map of the disk with $h(0) = 0$. Then $f \circ h$ may have poles, but it is holomorphic near 0. In 2006 Avkhadiev and Wirths conjectured that its Taylor coefficients satisfy the same bound. Their 2009 book named this the generalized Goodman conjecture. It is the meromorphic counterpart of Rogosinski's generalized Bieberbach conjecture, which de Branges' work settled. It was known for $n \le 2$, for $n = 3$ with $p < 0.7$, for $n \ge 4$ with $p < 1/(2n-2)$, and for some classes of f. We prove it for every n and every p. The proof combines three tools: Rogosinski's reduction, by which the nth Taylor coefficient of $f \circ h$ equals the nth coefficient of Ef, for a holomorphic E with $|E| \le 1$ that depends on n; Jenkins' conformal map of the disk slit along $[p,1)$ onto the disk, a map that fixes 0 and has positive coefficients of $z, z^2, z^3, \dots$; and Robertson's inequality for odd univalent functions. For $n \ge 2$, equality holds only when $f(z) = z/\big((1-pz)(1-z/p)\big)$ and h is a rotation. As a consequence, we prove two conjectures of Avkhadiev and Wirths (2008): sharp bounds for the derivatives of a meromorphic map from the disk, or from a convex domain, into a simply connected domain of the sphere.[1] We also prove the following. When h has a zero of order at least d at 0 and $n \ge d$, the nth coefficient is at most $A_m(p)$ in modulus, with $m = \lfloor n/d \rfloor$, and this bound is attained exactly when m divides n; for $d = 2$ and $n = 5$ we find the exact maximum. For $n \ge 2$, near-extremal coefficients force f and h near their extremal forms. Finally, a sharp bound for the derivatives of a meromorphic map from any simply connected source domain Ω at a point z_0, in terms of a single number $\tau(\Omega, z_0)$, gives both bounds of 2008 and also covers sources that contain ∞. The numbered results of Sections 1–5 (the focused version) are verified in Lean.[2]

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This is the expanded version of the focused version [16]. Sections 1–5 are that focused version. Sections 6–9 add inner maps with a zero of higher order, a question of Szapiel, stability near the extremal functions, and a sharp derivative bound for an arbitrary source domain.

[1] This is an expanded version of the following focused version [16]: https://doi.org/10.5281/zenodo.24570830

[2] Proved in Lean 4 from Lean's three standard axioms and four published theorems, checked with `lake comparator`: https://github.com/david44hello/paper_avkhadiev-wirths-2006-generalized-goodman/tree/v2.70/lean

1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ and let $\mathcal{S}$ be the class of functions F that are holomorphic and one-to-one in $\mathbb{D}$ with $F(0) = 0$ and $F'(0) = 1$. Here and below $[w^n]g$ denotes the coefficient of w^n in the Taylor expansion of g at the origin. De Branges proved in 1985 that every $F \in \mathcal{S}$ satisfies $|[w^n]F| \le n$ [11].

Fix $p \in (0, 1)$. We write $\mathcal{M}(p)$ for the class of functions f that are meromorphic and one-to-one in $\mathbb{D}$, have a simple pole at p and no other pole, and satisfy $f(0) = 0$, $f'(0) = 1$. Such an f is holomorphic in the disk $|z| < p$, so it has a Taylor expansion $f(z) = z + \sum_{n\ge2} a_n z^n$ there. The sharp bound for these coefficients is

$$|a_n| \le A_n(p) := \frac{1 - p^{2n}}{(1 - p^2)\, p^{n-1}} = p^{1-n} \sum_{j=0}^{n-1} p^{2j} \qquad (n \ge 1), \tag{1}$$

with equality for $n \ge 2$ only for

$$f_p(z) = \frac{z}{(1 - pz)(1 - z/p)}. \tag{2}$$

Expanding (2) as a product of two geometric series shows $[z^n]f_p = A_n(p)$. Goodman proved (1) when all the coefficients of f are real, and conjectured it for all $f \in \mathcal{M}(p)$ [17, Theorem 7 and p. 104]. Jenkins showed that it follows from the Bieberbach conjecture [19, pp. 25–27]; see also [7, Section 2.3]. By de Branges' theorem, (1) holds. Its equality statement also holds: Jenkins derived it from the equality case of the Bieberbach conjecture [19, p. 26], which de Branges proved [11, p. 151]. Theorem 1 below, in its simplest case ($h(z) = z$ and no weight), proves this equality statement again; besides de Branges' theorem it uses only Bieberbach's equality case for the second coefficient.

The conjecture. Let h be holomorphic in $\mathbb{D}$ with $|h| < 1$ and $h(0) = 0$; we call such h a *Schwarz function*. For $f \in \mathcal{M}(p)$ the composition $f \circ h$ is again meromorphic in $\mathbb{D}$. It has a pole at every point where h takes the value p, and in general it is not one-to-one. It is, however, holomorphic in the disk $|z| < p$, because $|h(z)| \le |z|$ there by Schwarz's lemma. So $f \circ h$ has a Taylor expansion at the origin, and its coefficients are the subject of the following conjecture.

Conjecture 1 (Avkhadiev and Wirths [3, 7, 8])**.** *For every $p \in (0, 1)$, every $f \in \mathcal{M}(p)$, every Schwarz function h and every $n \ge 1$,*

$$\big|[z^n](f \circ h)\big| \le A_n(p).$$

Avkhadiev and Wirths stated it in 2006, in a geometric form, as inequality (5) of [3] (Theorem 2 below). In their 2009 monograph *Schwarz–Pick Type Inequalities* [7, Section 8.3, pp. 131–138], they named it the generalized Goodman conjecture and devoted half of the closing open-problems chapter to it, alongside the Krzyż and angle conjectures. Their 2013 paper restates it for coefficients as Conjecture 1 of [8], writing $h(z) = z\omega(z)$ with $|\omega| \le 1$, and credits it to [3]. With $h(z) = z$ it is the inequality (1). It is the meromorphic counterpart of Rogosinski's generalized Bieberbach conjecture: if $F \in \mathcal{S}$ and h is a Schwarz function, then $|[z^n](F \circ h)| \le n$ [26]. The two bounds match, since $A_n(p) \to n$ as $p \to 1$. De Branges' work also settles Rogosinski's conjecture: it proves Robertson's inequality for odd univalent functions [11, pp. 137–138], which implies Rogosinski's conjecture (see [29, p. 172], and Lemmas 4 and 6 below).

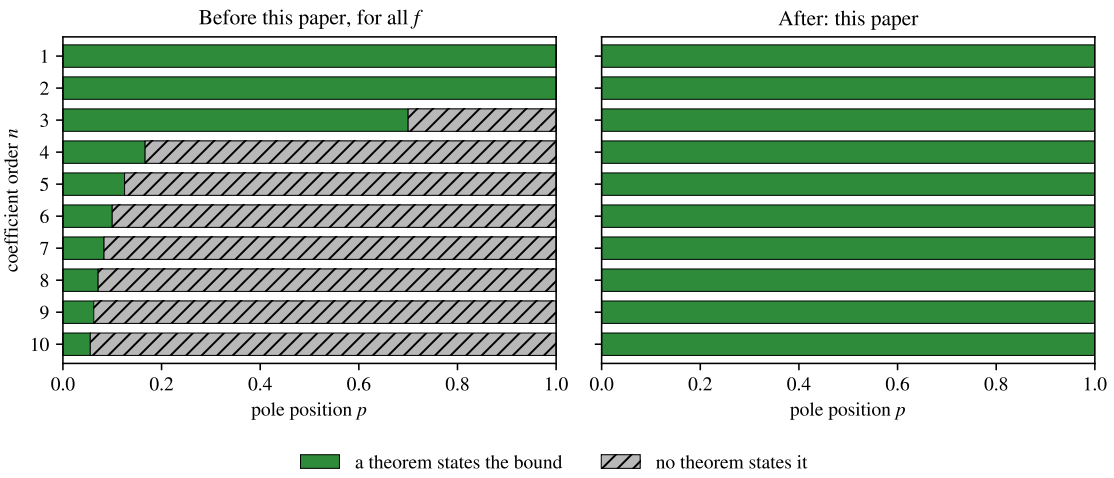

Figure 1: The bound $A_n(p)$ of Conjecture 1 for all $f \in \mathcal{M}(p)$: solid where a theorem states the bound, hatched elsewhere. Left: before this paper ($n = 1$ is Schwarz's lemma; $n = 2$; $n = 3$ with $p < 0.7$; $n \geq 4$ with $p < 1/(2n-2)$ [7, Theorem 8.3]). Right: after (Theorem 1). Rows $n = 1, \ldots, 10$ are drawn; the pattern continues for all n. Results for special classes of f (convex complements [5]; two starlike classes, every n [8]; a subclass, $n \geq 3$ and $p \leq 1/3$ [9]) are not shown.

Earlier results. Avkhadiev and Wirths proved the conjecture for $n = 2$ and every p, for $n = 3$ and $p < 0.7$, and for each $n \geq 4$ and $p < 1/(2n-2)$ [7, Theorem 8.3], and for every n and p when the complement of $f(\mathbb{D})$ is convex [5, Theorem 1] (see also [7, Theorem 8.4]). That concave-domain case has been used to derive Bohr–Rogosinski inequalities for meromorphic subordination (for instance [1, Lemma 4.1]). In [8] they proved the conjecture for every n when the complement of $f(\mathbb{D})$ belongs to one of two classes of starlike sets, and noted that the case of real coefficients was known [8, p. 143]. Bhowmik and Parveen proved it for $n \geq 3$ and $p \leq 1/3$ for a subclass of $\mathcal{M}(p)$ defined by a differential inequality [9]; by contrast, a related λ-deformed coefficient conjecture ($\lambda < 1$) for that subclass was recently disproved for $n = 3$ and large p [20]. Figure 1 compares these results with ours.

Results. We prove the conjecture in the following stronger form, which allows a bounded weight.

Theorem 1. *Let $p \in (0,1)$, $f \in \mathcal{M}(p)$, h a Schwarz function, and e holomorphic in $\mathbb{D}$ with $|e| \leq 1$. Then for every $n \geq 1$*

$$\bigl|[z^n]\, e(z) f(h(z))\bigr| \leq A_n(p).$$

With $e = 1$ this is Conjecture 1. For $e = 1$ and $n \geq 2$, equality holds if and only if $f = f_p$ and $h(z) = \lambda z$ with $|\lambda| = 1$. For $e = 1$ and $n = 1$, equality holds exactly when $h(z) = \lambda z$ with $|\lambda| = 1$.

In the language of Robertson [25], who considered holomorphic f, Theorem 1 bounds the coefficients of the functions $e \cdot (f \circ h)$ quasi-subordinate to f.

The original statement of 2006 is geometric. Let $\Pi \subset \widehat{\mathbb{C}}$ be a simply connected domain of the Riemann sphere whose complement has at least two points. For $w_0 \in \Pi \cap \mathbb{C}$ let $R(w_0, \Pi)$ be the conformal radius of Π at w_0, that is, $|\phi'(0)|$ for any conformal map ϕ of $\mathbb{D}$ onto Π with $\phi(0) = w_0$. Let δ_Π be the hyperbolic distance in Π, normalized by curvature -4, so that $\delta_{\mathbb{D}}(0, r) = \operatorname{artanh} r$.

Theorem 2. *Let Π be as above with $\infty \in \Pi$, and let $G : \mathbb{D} \to \Pi$ be holomorphic, as a map into the sphere, with $w_0 = G(0) \neq \infty$. Put $p = \tanh \delta_\Pi(w_0, \infty)$, a number in $(0, 1)$. Then for every $n \geq 1$*

$$\frac{|G^{(n)}(0)|}{n!} \leq R(w_0, \Pi)\, \frac{1 - p^{2n}}{(1 - p^2)\, p^{n-1}} = R(w_0, \Pi)\, A_n(p).$$

For $n \geq 2$ equality holds if and only if G maps $\mathbb{D}$ one-to-one onto Π and the complement of Π is a line segment lying on a line through w_0.

In 2008 Avkhadiev and Wirths extended these estimates to maps defined on other domains, and proved them when the target is a concave domain, that is, when Π is the complement of a compact convex set [6]. For a general simply connected target they stated two conjectures, one for maps on the disk and one for maps on convex domains [6, Conjectures 1 and 2]. Theorem 3 below, a consequence of Theorem 1, proves both.

Again let $\Pi \subset \widehat{\mathbb{C}}$ be a simply connected domain whose complement has at least two points. For a simply connected domain $\Omega \neq \mathbb{C}$, a point $z_0 \in \Omega$ and a meromorphic map $G : \Omega \to \Pi$ with $w_0 = G(z_0)$ finite, put $R_s = R(z_0, \Omega)$ and $R_t = R(w_0, \Pi)$, the conformal radii of the source and of the target. Let $p = \tanh \delta_\Pi(w_0, \infty)$ if $\infty \in \Pi$, and $p = 1$ otherwise; we put $A_k(1) = k$, the limit of $A_k(p)$ as $p \to 1$. For $0 \leq r \leq 1$ define

$$D_n(p, r) = \sum_{k=1}^{n} \binom{n-1}{k-1} r^{n-k} A_k(p) = \frac{(r + 1/p)^{n-1} - p^2 (r + p)^{n-1}}{1 - p^2}, \tag{3}$$

where the second expression, for $0 < p < 1$, follows from $A_k(p) = (p^{1-k} - p^{k+1})/(1 - p^2)$ and the binomial theorem.

Theorem 3. *Let $\Pi \subset \widehat{\mathbb{C}}$ be a simply connected domain whose complement has at least two points, let Ω, z_0, G, R_s, R_t and p be as just defined, and let $n \geq 2$.*

(a) If $\Omega = \mathbb{D}$, then $R_s^n |G^{(n)}(z_0)|/(n!\, R_t) \leq D_n(p, |z_0|)$.

(b) If $\Omega \neq \mathbb{C}$ is convex, then $R_s^n |G^{(n)}(z_0)|/(n!\, R_t) \leq D_n(p, 1)$.

Both bounds are sharp for every p and n. Maps that attain them (Section 5) are built from f_p; for $p = 1$, $f_1(z) = z/(1 - z)^2$ is the Koebe function.

For planar targets ($p = 1$) both bounds were known: (a) is [2, Theorem 2] and (b) is in [4]; for concave Π both are in [6]. Part (a) is [6, Conjecture 1] and part (b) is [6, Conjecture 2]. The new content is the case where $\infty \in \Pi$ and Π is not concave. Part (a) for $n = 2$, for $n = 3$ with $p < 0.7$, and for $n \geq 4$ with $p < 1/(2n - 2)$ also follows from [7, Theorem 8.3] by the argument of Section 5.

The idea of the proof. Three steps reduce Theorem 1 to classical facts. First, the coefficient $[z^n]\, e\, (f \circ h)$ equals $[w^n]\, Ef$ for a holomorphic E with $|E| \leq 1$ (Lemma 4); for $e = 1$ this is due to Rogosinski [26], see also Szapiel [29]. So the inner map disappears and only a bounded factor remains. Second, there is a conformal map q of the slit disk $\mathbb{D} \setminus [p, 1)$ onto $\mathbb{D}$, built from the Joukowski map and one affine map. The coefficient of z^n in q is positive for every $n \geq 1$ (Lemma 5; Jenkins [19, p. 25]). The function $(f \circ q^{-1})/R$, where $R = R(0, \mathbb{D} \setminus [p, 1)) = 4p/(1 + p)^2$ is the conformal radius of $\mathbb{D} \setminus [p, 1)$ at 0, has no pole and lies in $\mathcal{S}$. Third, for $F \in \mathcal{S}$ and $|E| \leq 1$ one has $|[w^k]FE| \leq k$, a consequence of Robertson's inequality (Lemma 6). For $F = (f \circ q^{-1})/R$ and $\Phi = F \cdot (E \circ q^{-1})$ this gives $|[w^k]\Phi| \leq k$, and $Ef = R\, \Phi \circ q$ on $\mathbb{D} \setminus [p, 1)$. Since every power q^k has

nonnegative coefficients, summing these estimates bounds $|[z^n]\, e\,(f \circ h)|$ by the coefficient of z^n in $R \sum_k k\, q^k$, and this series is exactly f_p of (2).

For a general weight e we prove the first step by duality. The third step is classical (Robertson, MacGregor). With $e = 1$ and $h(z) = z$, the second and third steps are Jenkins' proof of (1) [19, pp. 25–26], which assumed the Bieberbach conjecture $|[w^k]F| \le k$, now de Branges' theorem. The point of the proof is that the second and third steps tolerate a bounded factor E, and the first step turns the composition with h into exactly such a factor.

2 Three lemmas

Lemma 4 (transfer)**.** *Let h be a Schwarz function, e holomorphic in $\mathbb{D}$ with $|e| \le 1$, and $n \ge 0$. There is a holomorphic E in $\mathbb{D}$ with $|E| \le 1$ such that*

$$[z^n]\, e(z)\, g(h(z)) = [w^n]\, E(w) g(w)$$

for every function g holomorphic near 0. Moreover $E(0) = e(0) h'(0)^n$.

Proof. Write $g(w) = \sum_{j \ge 0} g_j w^j$ near 0. Since $h^j = O(z^j)$, only $j \le n$ contribute, and $[z^n]\, e\, g(h) = \sum_{j=0}^n g_j \Lambda(w^j)$, where $\Lambda(P) = [z^n]\, e(z) P(h(z))$ for polynomials P. We claim $|\Lambda(P)| \le \|P\|_1$, the norm of P in L^1 of the normalized arc measure σ on the unit circle $\mathbb{T}$. Let U be the Poisson integral of $|P|$ on $\mathbb{T}$. Since $|P|$ is subharmonic, $|P| \le U$ in $\mathbb{D}$, and $U \circ h$ is harmonic. By Cauchy's formula on the circle of radius $r < 1$,

$$r^n |\Lambda(P)| \le \int_{\mathbb{T}} |P(h(r\zeta))|\, d\sigma(\zeta) \le \int_{\mathbb{T}} U(h(r\zeta))\, d\sigma(\zeta) = U(h(0)) = U(0) = \|P\|_1,$$

where the first step uses $|e| \le 1$. Letting $r \to 1$ proves the claim. By the Hahn–Banach theorem and the duality of L^1 and L^∞ there is $v \in L^\infty(\mathbb{T})$, $\|v\|_\infty \le 1$, with $\Lambda(P) = \int_{\mathbb{T}} vP\, d\sigma$ for every polynomial P. As $\Lambda(w^j) = 0$ for $j > n$, the function $w^n v(w)$ has no Fourier coefficients of negative index, so it is the boundary function of a holomorphic E with $|E| \le 1$. For $0 \le j \le n$ the coefficient of w^{n-j} in E is $\int_{\mathbb{T}} w^n v\, \overline{w}^{\,n-j}\, d\sigma = \int_{\mathbb{T}} v\, w^j\, d\sigma = \Lambda(w^j)$. Hence $\sum_{j \le n} g_j \Lambda(w^j) = [w^n]\, Eg$, and $E(0) = \Lambda(w^n) = e(0) h'(0)^n$. □

For $e = 1$ there is an explicit E, which avoids the duality argument. If $n = 0$, then $E = 1$. For $n \ge 1$, put $c_j = [z^n] h^j$ and $Q(w) = \sum_{j=1}^n c_j w^{n-j}$. Then $[z^n]\, g(h) = \sum_j g_j c_j = [w^n]\, Qg$, so $E = Q$ works once we know $|Q| \le 1$. For $|t| = 1$ the function $(1 + th)/(1 - th)$ has positive real part and value 1 at the origin, and its coefficient of z^n is $2 \sum_j c_j t^j$; Carathéodory's inequality bounds it by 2. Taking $t = 1/w$ with $|w| = 1$ gives $|Q(w)| \le 1$ on $\mathbb{T}$, hence in $\mathbb{D}$. This is Rogosinski's argument [26, (1.1.3) and Lemma 2]. Szapiel [29, p. 172, (5)] proves an equivalent bound by the subordination principle and Schwarz's lemma.

Lemma 5 (Jenkins' slit map)**.** *Let $J(z) = z + 1/z$, $B = J(p) = p + 1/p$ and $R = 4/(B + 2)$. There is a conformal map q of $\mathbb{D} \setminus [p, 1)$ onto $\mathbb{D}$ with $q(0) = 0$ and*

$$J(q(z)) + 2 = R\,\big(J(z) + 2\big). \tag{4}$$

The coefficient of z^n in q is positive for every $n \ge 1$, and

$$R \sum_{k \ge 1} k\, q(z)^k = f_p(z) \quad \text{near } 0. \tag{5}$$

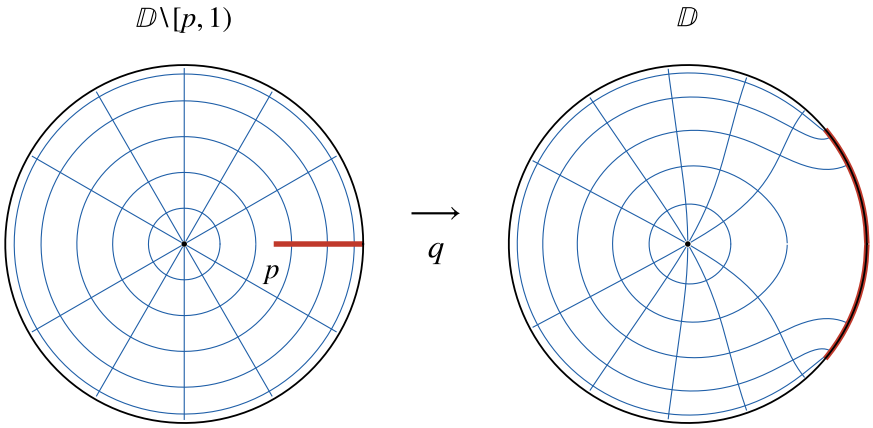

Figure 2: The slit map q of Lemma 5 for $p = 1/2$: circles and radii of the slit disk (left) and their images (right). The two edges of the slit (heavy, left) go to the heavy arc of the unit circle around 1 (right). The coefficient of z^n in q is positive for every $n \geq 1$, so the estimates of Lemma 6 are summed with nonnegative weights (Proposition 7).

Proof. The Joukowski map J sends $\mathbb{D}$ one-to-one onto $\widehat{\mathbb{C}} \setminus [-2, 2]$, with $J(0) = \infty$. On the radius $[p, 1)$ it decreases from B to 2, so it sends $\mathbb{D} \setminus [p, 1)$ onto $\widehat{\mathbb{C}} \setminus [-2, B]$. The affine map $x \mapsto R(x+2)-2$ fixes -2 and ∞ and sends B to 2, so it maps $\widehat{\mathbb{C}} \setminus [-2, B]$ onto $\widehat{\mathbb{C}} \setminus [-2, 2]$. Composing with J^{-1} gives q (Figure 2). It fixes 0: J sends 0 to ∞, the affine map fixes ∞, and J^{-1} sends ∞ to 0. Comparing the poles at 0 in (4) gives $q'(0) = 1/R$; so $R = \chi'(0)$, where $\chi = q^{-1}$, is the conformal radius of $\mathbb{D} \setminus [p, 1)$ at 0.

Since $J(z) + 2 = (1 + z)^2/z$, identity (4) reads $z(1 + q)^2 = Rq(1 + z)^2$. Put $c = 1/R - 1 = (1-p)^2/(4p) > 0$. Then $q(1+z)^2 - z(1+q)^2 = c\,z(1+q)^2$, and the left side factors as $(q-z)(1-zq)$. Hence, near 0,

$$q = z + c\,z\,\frac{(1+q)^2}{1-zq} = z + c\,z\,(1+q)^2 \sum_{m\geq 0} (zq)^m.$$

Write $q = \sum_{j\geq 1} q_j z^j$. Then $q_1 = 1 + c$, and for $n \geq 2$ the coefficient q_n is a polynomial with nonnegative coefficients in $c, q_1, \ldots, q_{n-1}$ that contains the term $2c\,q_{n-1}$. By induction every q_n is positive.

Finally $\sum_k kq^k = q/(1-q)^2 = 1/(J(q)-2)$, and by (4) $J(q) - 2 = R\,(J(z) - B)$, because $4/R = B + 2$. So $R\sum_k kq^k = 1/(J(z) - B) = z/(1 - Bz + z^2) = f_p(z)$. □

Lemma 6 (product estimate)**.** *Let $F \in \mathcal{S}$ and let E be holomorphic in $\mathbb{D}$ with $|E| \leq 1$. Then $|[w^k]\,FE| \leq k$ for every $k \geq 1$.*

Proof. Since $F(w)/w$ has no zeros, $F(w) = wM(w)^2$ with M holomorphic and $M(0) = 1$. The odd function $F_2(z) = zM(z^2)$ satisfies $F_2(z)^2 = F(z^2)$ and is one-to-one: if $F_2(z_1) = F_2(z_2)$ then $z_1^2 = z_2^2$, and $z_2 = -z_1 \neq 0$ would give $F_2(z_1) = -F_2(z_1) = 0$, which is impossible, since F_2 vanishes only at 0. Write $M(w) = \sum_{j\geq 0} b_j w^j$, so b_j is the coefficient of z^{2j+1} in F_2. Robertson's inequality, proved by de Branges [11], states $\sum_{j=0}^{k-1} |b_j|^2 \leq k$. Let $M_k = \sum_{j<k} b_j w^j$. Only these terms of M reach the coefficient of w^{k-1}, so

$$[w^k]\,FE = [w^{k-1}]\,M^2E = [w^{k-1}]\,(M_kE)\,M_k.$$

By the Cauchy–Schwarz inequality for coefficients this is at most $\|M_k E\|_{H^2}\|M_k\|_{H^2} \le \|M_k\|_{H^2}^2 = \sum_{j<k}|b_j|^2 \le k$. □

With $E = 1$ this is Robertson's observation that his inequality implies Bieberbach's [24]; see [11, p. 138]. The form with a bounded factor E is in [15, p. 256], where it is credited to MacGregor [22].

3 Proof of Theorem 1

Proposition 7. *Let $f \in \mathcal{M}(p)$ and let E be holomorphic in $\mathbb{D}$ with $|E| \le 1$. Then for every $n \ge 1$*

$$\bigl|[z^n]\,Ef\bigr| \le A_n(p) - R\,q_n\,(1 - |E(0)|),$$

where R and $q_n = [z^n]q > 0$ are as in Lemma 5.

Proof. Let $\chi = q^{-1}$, which maps $\mathbb{D}$ conformally onto $\mathbb{D} \setminus [p, 1)$, with $\chi(0) = 0$ and $\chi'(0) = R$. Since f has its only pole at $p \notin \chi(\mathbb{D})$, the function $F = (f \circ \chi)/R$ is holomorphic and one-to-one in $\mathbb{D}$, with $F(0) = 0$ and $F'(0) = 1$; so $F \in \mathcal{S}$. The function $\Phi = F \cdot (E \circ \chi)$ satisfies $|[w^k]\Phi| \le k$ by Lemma 6, and $[w^1]\Phi = E(0)$. On $\mathbb{D} \setminus [p, 1)$ we have $Ef = R\,\Phi \circ q$. Here f, E and q are functions of z, while F and Φ are functions of $w = q(z)$. Writing $\Phi(w) = \sum_k d_k w^k$ and using $q(z)^k = O(z^k)$,

$$[z^n]\,Ef = R\sum_{k=1}^{n} d_k\,[z^n]q^k.$$

All the numbers $[z^n]q^k$ are nonnegative by Lemma 5. Hence

$$\bigl|[z^n]\,Ef\bigr| \le R\Bigl(|E(0)|\,q_n + \sum_{k=2}^{n} k\,[z^n]q^k\Bigr) = R\sum_{k=1}^{n} k\,[z^n]q^k - Rq_n(1 - |E(0)|),$$

and $R\sum_{k=1}^{n} k\,[z^n]q^k = [z^n]f_p = A_n(p)$ by (5). □

Proof of Theorem 1. Lemma 4 with $g = f$ gives $[z^n]\,e\,f(h) = [w^n]\,E(w)f(w)$ for some $|E| \le 1$. This is the coefficient bounded in Proposition 7, and that proposition gives the bound.

Now let $e = 1$. Then $E(0) = h'(0)^n$, and by Proposition 7 equality forces $|h'(0)| = 1$, so $h(z) = \lambda z$ with $|\lambda| = 1$ by Schwarz's lemma. Conversely, for such h, $[z^n]f(\lambda z) = \lambda^n a_n$. For $n = 1$ this has modulus $1 = A_1(p)$ for every f. For $n \ge 2$ we must decide when $|a_n| = A_n(p)$. Since $a_n = [z^n]f$, Proposition 7 applies with $E = 1$; then $\Phi = F$, and equality in the proof of Proposition 7 requires $|d_2| = |[w^2]F| = 2$, because $[z^n]q^2 > 0$ for $n \ge 2$. By the equality case of Bieberbach's inequality for the second coefficient [10], $F(w) = w/(1 - \eta w)^2$ with $|\eta| = 1$, so $d_k = k\eta^{k-1}$. Equality in the triangle inequality requires d_1 and d_2 to have the same argument, so $\eta = 1$, F is the Koebe function, and $f = R\,F \circ q = R\,q/(1 - q)^2 = f_p$ by (5). Conversely $f = f_p$ and $h(z) = \lambda z$ with $|\lambda| = 1$ give equality, since $|[z^n]f_p(\lambda z)| = A_n(p)$. □

4 The geometric form

Proof of Theorem 2. Let $\phi : \mathbb{D} \to \Pi$ be a conformal map with $\phi(0) = w_0$; after a rotation of $\mathbb{D}$ the point $\phi^{-1}(\infty)$ lies in $(0, 1)$. Conformal maps preserve hyperbolic distance, so $\delta_{\mathbb{D}}(0, \phi^{-1}(\infty)) = \delta_{\Pi}(w_0, \infty) = \operatorname{artanh} p$, that is, $\phi^{-1}(\infty) = p$. Put $a = \phi'(0)$ and $f = (\phi - w_0)/a$, so $f \in \mathcal{M}(p)$ and $|a| = R(w_0, \Pi)$. The map $h = \phi^{-1} \circ G$ is a Schwarz function; at the poles of G it is holomorphic

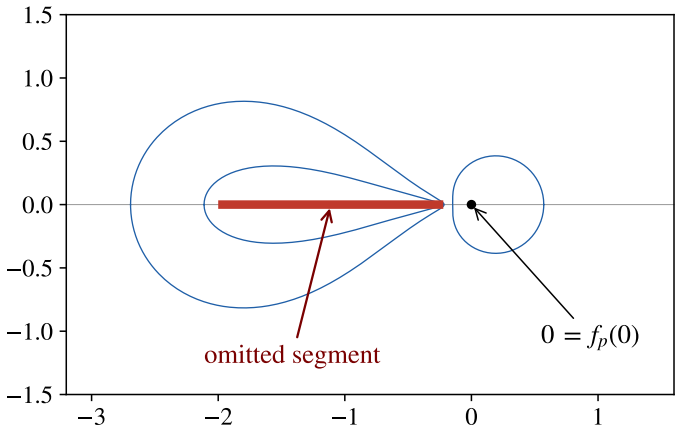

Figure 3: The image $f_p(\mathbb{D})$ of the extremal function, for $p = 1/2$ (the case of equality in Theorems 1 and 2): f_p maps $\mathbb{D}$ onto the sphere minus the heavy segment $[-2, -2/9]$, whose line passes through $f_p(0) = 0$. The curves are images of the circles $|z| = 0.25$, 0.7 and 0.85.

because ϕ^{-1} is holomorphic at ∞ in the spherical coordinate. Near 0 we have $G = w_0 + a\, f \circ h$, so $G^{(n)}(0)/n! = a\,[z^n](f \circ h)$, and Theorem 1 gives the bound.

Let $n \geq 2$. By Theorem 1, equality holds exactly when $f = f_p$ and h is a rotation, that is, when G maps $\mathbb{D}$ conformally onto $\Pi = w_0 + a\, f_p(\mathbb{D})$. Since $f_p = 1/(J - B)$ and $J(\mathbb{D}) = \widehat{\mathbb{C}} \setminus [-2, 2]$, the complement of $f_p(\mathbb{D})$ is the segment $[-1/(B-2), -1/(B+2)]$, whose line, the real axis, passes through $0 = f_p(0)$ (Figure 3). So the complement of Π is a segment on a line through w_0. Conversely, let the complement of Π be a segment on a line through $w_0 \in \Pi$, with endpoints at distances $0 < \ell_1 < \ell_2$ from w_0. Put

$$p = \frac{\sqrt{\ell_2} - \sqrt{\ell_1}}{\sqrt{\ell_2} + \sqrt{\ell_1}}, \qquad |a| = \frac{4\ell_1\ell_2}{\ell_2 - \ell_1},$$

and choose the argument of a so that $a/|a| = (w_0 - \xi)/\ell_1$, where ξ is the endpoint nearer to w_0; that is, a points from ξ towards w_0. With $B = p + 1/p$ one checks $|a|/(B+2) = \ell_1$ and $|a|/(B-2) = \ell_2$, so $w_0 + a f_p$ maps $\mathbb{D}$ onto Π. Since this map sends p to ∞, this p is $\tanh \delta_\Pi(w_0, \infty)$, and $|a| = R(w_0, \Pi)$. Every conformal map of $\mathbb{D}$ onto Π sending 0 to w_0 is this map composed with a rotation, and attains equality for every n. □

5 The two conjectures of 2008: proof of Theorem 3

The planar case $p = 1$ uses the weighted form of Rogosinski's estimate, $|[z^k]\, E\,(F \circ h)| \leq k$ for $F \in \mathcal{S}$ and $|E| \leq 1$, which follows from Lemmas 4 and 6; for $0 < p < 1$ we use Theorem 1. In both cases we need the following consequence.

Lemma 8. *Let Ω, z_0, G, Π, R_s, R_t and p be as in Theorem 3, let $\phi : \mathbb{D} \to \Omega$ be conformal with $\phi(0) = z_0$ and $\phi'(0) = R_s$, and let $H = (G \circ \phi - w_0)/R_t$. Then $|[z^k]\, EH| \leq A_k(p)$ for every $k \geq 1$ and every holomorphic E with $|E| \leq 1$.*

Proof. Let $\psi : \mathbb{D} \to \Pi$ be conformal with $\psi(0) = w_0$, rotated so that $\psi^{-1}(\infty) = p$ when $\infty \in \Pi$. Then $\psi = w_0 + a\, f$ with $|a| = R_t$ and $f \in \mathcal{M}(p)$, or $f \in \mathcal{S}$ if $\infty \notin \Pi$. The map $h = \psi^{-1} \circ G \circ \phi$ is a Schwarz function and $H = (a/R_t)\, f \circ h$. Theorem 1, or Lemmas 4 and 6 when $p = 1$, give the bound. □

Proof of Theorem 3(a). Take $\phi(z) = (z_0 + z)/(1 + \overline{z_0}z)$, so $R_s = 1 - |z_0|^2$ and $\phi^{-1}(z_0 + R_s x) = x/(1-\overline{z_0}x)$. Hence $G(z_0+R_s x) = w_0+R_t\, H\big(x/(1-\overline{z_0}x)\big)$, and since $[x^n](x/(1-\overline{z_0}x))^k = \binom{n-1}{k-1}\overline{z_0}^{\,n-k}$,

$$\frac{R_s^n G^{(n)}(z_0)}{n!\, R_t} = \sum_{k=1}^{n} \binom{n-1}{k-1} \overline{z_0}^{\,n-k}\, [z^k]H.$$

Lemma 8 with $E = 1$ gives the bound $D_n(p, |z_0|)$. For sharpness take $z_0 = r \geq 0$, $\Pi = f_p(\mathbb{D})$ and $G = f_p \circ \phi^{-1}$; then $R_t = 1$, $H = f_p$, and every term is nonnegative, so equality holds. Rotations of $\mathbb{D}$ give the other z_0. □

Proof of Theorem 3(b). Let ϕ be as in Lemma 8 and $s = (\phi - z_0)/R_s$, a normalized convex map. Then $G(z_0 + R_s x) = w_0 + R_t H(s^{-1}(x))$, and the Lagrange inversion formula gives

$$\frac{R_s^n G^{(n)}(z_0)}{n!\, R_t} = [x^n]\, H(s^{-1}(x)) = \frac{1}{n}\, [z^{n-1}]\, H'(z)\Big(\frac{z}{s(z)}\Big)^n.$$

By the Marx–Strohhäcker theorem [23, 28], $\operatorname{Re}(s(z)/z) > 1/2$; inversion maps this half-plane into the disk $|w - 1| < 1$, so by Schwarz's lemma $z/s(z) = 1 + zv(z)$ with $|v| \leq 1$. Expanding $(1+zv)^n = \sum_j \binom{n}{j} z^j v^j$, the term with v^n does not reach z^{n-1}, and since $[z^{n-1}]\, H' z^j v^j = [z^{k-1}]\, H' v^j$ with $k = n - j \geq 1$, the others are $\binom{n}{j}[z^{k-1}]\, H' v^j$.

For holomorphic $E = \sum \varepsilon_i z^i$ with $|E| \leq 1$, let $\widetilde{E} = \sum_{i<k}(1 - i/k)\varepsilon_i z^i$ be its Fejér mean of order k. It is $\widetilde{E}(z) = \int_{\mathbb{T}} E(z\overline{\zeta})\, K_k(\zeta)\, d\sigma(\zeta)$ with the Fejér kernel $K_k(\zeta) = \sum_{|i|<k}(1 - |i|/k)\zeta^i \geq 0$, which has $\int_{\mathbb{T}} K_k\, d\sigma = 1$; so $|\widetilde{E}| \leq 1$. We need $\widetilde{E}$ because Lemma 8 bounds the coefficients of EH, while the Lagrange inversion formula produces H'. Writing $H = \sum_m \eta_m z^m$ (so $\eta_0 = 0$), both $[z^{k-1}]\, H'E$ and $k\,[z^k]\, H\widetilde{E}$ equal $\sum_{m=1}^{k} m\, \eta_m \varepsilon_{k-m}$, so $|[z^{k-1}]\, H'E| \leq kA_k(p)$ by Lemma 8. We apply this with $E = v^j$ and $k = n - j$. Since $\frac{1}{n}\binom{n}{j}k = \binom{n-1}{k-1}$, summing over j gives the bound $D_n(p, 1)$.

For sharpness take $z_0 = 0$ and $\phi(z) = z/(1 + z)$, which maps $\mathbb{D}$ onto the half-plane $\operatorname{Re} x < 1/2$ with $R_s = 1$, and $G = f_p \circ \phi^{-1}$. Then $R_t = 1$, $\phi^{-1}(x) = x/(1 - x)$, and $[x^n] f_p(x/(1 - x)) = \sum_k \binom{n-1}{k-1} A_k(p) = D_n(p, 1)$. □

6 Inner maps with a zero of higher order

In this section $d \geq 1$ is an integer and h is a Schwarz function with a zero of order at least d at 0. Applying Schwarz's lemma d times gives $h(z) = z^d\, b(z)$ with b holomorphic in $\mathbb{D}$ and $|b| \leq 1$. For $n \geq 0$ we write

$$m = \lfloor n/d \rfloor, \qquad \ell = n - md \in \{0, 1, \ldots, d-1\}.$$

For $d = 1$ every Schwarz function qualifies, $m = n$, and the results below reduce to Theorem 1. Since $h^j = O(z^{dj})$, the function $f \circ h$ can be written near 0 as a series in h, and its coefficient of z^n involves only the first $m + 1$ Taylor coefficients of f. One therefore expects the bound $A_m(p)$ rather than $A_n(p)$. The transfer lemma extends as follows.

Lemma 9. *Let h be a Schwarz function with a zero of order at least d at 0, let e be holomorphic in $\mathbb{D}$ with $|e| \leq 1$, and let $n \geq 0$. There is a holomorphic E in $\mathbb{D}$ with $|E| \leq 1$ such that*

$$[z^n]\, e(z)\, g(h(z)) = [w^m]\, E(w) g(w)$$

for every function g holomorphic near 0, and $E(0) = [z^n]\, e\, h^m$. Conversely, for every holomorphic E with $|E| \leq 1$, the choice $h(z) = z^d$, $e(z) = z^\ell E(z^d)$ gives $[z^n]\, e\, g(h) = [w^m]\, Eg$ for every g.

Proof. The proof of Lemma 4 applies with one change. With $\Lambda(P) = [z^n]\, e(z)P(h(z))$ we again have $|\Lambda(P)| \le \|P\|_1$, and hence $\Lambda(P) = \int_{\mathbb{T}} vP\, d\sigma$ for some v with $\|v\|_\infty \le 1$. Now $\Lambda(w^j) = 0$ for every $j > m$, because $h^j = O(z^{dj})$ and $dj \ge d(m+1) > n$. So the Fourier coefficient of index $-k < 0$ of $w^m v(w)$, which is $\int_{\mathbb{T}} v\, w^{m+k}\, d\sigma = \Lambda(w^{m+k})$, vanishes. Thus $w^m v$ is the boundary function of a holomorphic E with $|E| \le 1$, and for $0 \le j \le m$ the coefficient of w^{m-j} in E is $\Lambda(w^j)$. Writing $g = \sum_j g_j w^j$, we get $[z^n]\, e\, g(h) = \sum_{j \le m} g_j \Lambda(w^j) = [w^m]\, Eg$, and $E(0) = \Lambda(w^m) = [z^n]\, e\, h^m$.

For the converse, $e(z)g(h(z)) = z^\ell\, (Eg)(z^d)$, whose coefficient of $z^n = z^\ell z^{md}$ is $[w^m]\, Eg$. □

So for fixed f the set of values of $[z^n]\, e\, (f \circ h)$, over all such e and h, is exactly the set of values of $[w^m]\, Ef$ over all $|E| \le 1$. In particular it depends on f only through f's first $m+1$ Taylor coefficients.

We also need two facts. The first sharpens the positivity in Lemma 5.

Lemma 10. *With q and R as in Lemma 5, write $q = \sum_{k \ge 1} q_k z^k$. Then $q_2 = 2(1-R)/R^2$ and $q_{k+1} > q_k$ for every $k \ge 2$. In particular $R\, q_m \ge 2(1-R)/R$ for every $m \ge 2$.*

Proof. Put $u = (q - z)/(1 + z)$, so $q = z + u + zu$ and $1 + q = (1 + z)(1 + u)$. The identity $z(1 + q)^2 = Rq(1 + z)^2$ from the proof of Lemma 5 becomes $z(1 + u)^2 = R\, (z + u + zu)$, that is,

$$u = z\Big(c + \beta u + \frac{u^2}{R}\Big), \qquad c = \frac{1}{R} - 1, \quad \beta = \frac{2}{R} - 1.$$

Write $u = \sum_{k \ge 1} u_k z^k$. Then $u_1 = c > 0$ and $u_{k+1} = \beta u_k + [z^k]\, u^2/R$. By induction all u_k are positive, and since $\beta > 1$ (because $R < 1$) we get $u_{k+1} > u_k$. For $k \ge 2$ we have $q_k = u_k + u_{k-1}$, so $q_{k+1} - q_k = u_{k+1} - u_{k-1} > 0$. Finally $q_2 = u_2 + u_1 = (\beta + 1)c = 2c/R = 2(1 - R)/R^2$. □

Lemma 11. *Let b be holomorphic in $\mathbb{D}$ with $|b| \le 1$, and let $m \ge 1$ and $\ell \ge 1$ be integers such that m does not divide ℓ. Then*

$$\big|[z^\ell]\, b^m\big| \le C_\ell := \Big(1 - \frac{\ell^\ell}{(\ell + 1)^{\ell+1}}\Big)^{1/2} < 1.$$

Proof. If $b \equiv 0$ there is nothing to prove. Otherwise write $b^m = \sum_j c_j z^j$ and $a = c_\ell$. Every zero of b^m in $\mathbb{D}$ has a multiplicity divisible by m. By Parseval's identity, $\sum_j |c_j|^2 \le 1$. Fix $\rho \in (0, 1)$. On the circle $|z| = \rho$ the Cauchy–Schwarz inequality gives

$$\big|b(z)^m - az^\ell\big| \le \sum_{j \ne \ell} |c_j| \rho^j \le \big(1 - |a|^2\big)^{1/2} \Big(\frac{1}{1 - \rho^2} - \rho^{2\ell}\Big)^{1/2}.$$

Suppose $1 - |a|^2 < (1 - \rho^2)\rho^{2\ell}$. Then the right side is less than $|a|\rho^\ell = |az^\ell|$: with $t = 1 - |a|^2$ this reads $t((1 - \rho^2)^{-1} - \rho^{2\ell}) < (1 - t)\rho^{2\ell}$, that is, $t < (1 - \rho^2)\rho^{2\ell}$. So by Rouché's theorem b^m has exactly ℓ zeros in $|z| < \rho$, counted with multiplicity. Since m does not divide ℓ, this is impossible. Hence $1 - |a|^2 \ge (1 - \rho^2)\rho^{2\ell}$ for every $\rho \in (0, 1)$. The right side is largest at $\rho^2 = \ell/(\ell + 1)$, where it equals $\ell^\ell/(\ell + 1)^{\ell+1}$. □

Theorem 12. *Let $p \in (0, 1)$, $f \in \mathcal{M}(p)$, $d \ge 1$, let h be a Schwarz function with a zero of order at least d at 0, and let e be holomorphic in $\mathbb{D}$ with $|e| \le 1$. For $n < d$ the coefficient $[z^n]\, e\, f(h)$ is 0. For $n \ge d$, with $m = \lfloor n/d \rfloor$, $\ell = n - md$, and R and q_m as in Lemma 5:*

(a) $\big|[z^n]\, e\, f(h)\big| \le A_m(p) - R\, q_m\big(1 - \big|[z^n]\, e\, h^m\big|\big) \le A_m(p)$.

(b) The bound $A_m(p)$ is attained, by $f = f_p$, $h(z) = z^d$ and $e(z) = z^\ell$.

(c) Let $e = 1$. If m divides n, then $|[z^n]f(h)| = A_m(p)$ holds if and only if $h(z) = \lambda z^{n/m}$ with $|\lambda| = 1$ and, when $m \geq 2$, $f = f_p$. If m does not divide n, then

$$\big|[z^n]f(h)\big| \leq A_m(p) - R\, q_m\,(1 - C_\ell) \leq A_m(p) - \frac{2(1-R)}{R}\,(1 - C_\ell),$$

with $C_\ell < 1$ from Lemma 11; so $|[z^n]f(h)|$ stays below $A_m(p)$ by at least an amount that depends only on p and ℓ.

Proof. For $n < d$ we have $m = 0$, and $[w^0]\,Ef = E(0)f(0) = 0$ by Lemma 9. Let $n \geq d$, so $m \geq 1$. By Lemma 9 with $g = f$, $[z^n]\,e\,f(h) = [w^m]\,Ef$ for some $|E| \leq 1$ with $E(0) = [z^n]\,e\,h^m$, and Proposition 7 with m in place of n gives (a). For (b), $z^\ell f_p(z^d)$ has coefficient $[w^m]\,f_p = A_m(p)$ at $z^{\ell+md} = z^n$.

For (c) let $e = 1$. Equality in (a) forces $|[z^n]\,h^m| = 1$, since $q_m > 0$. As $|h^m| \leq 1$, Parseval's identity then forces $h^m = \gamma z^n$ with $|\gamma| = 1$. So m divides n, $h(z)/z^{n/m}$ is holomorphic with constant modulus 1, and $h(z) = \lambda z^{n/m}$ with $|\lambda| = 1$. Then $[z^n]f(h) = \lambda^m a_m$, where $a_m = [z^m]f$. For $m = 1$ this has modulus $1 = A_1(p)$ for every f. For $m \geq 2$, $|a_m| = A_m(p)$ holds only for $f = f_p$, by Theorem 1 with $h(z) = z$. Conversely these pairs attain $A_m(p)$. If m does not divide n, then $m \geq 2$, $\ell \geq 1$ and m does not divide ℓ (because $n = md + \ell$). Writing $h = z^d b$ we get $[z^n]\,h^m = [z^\ell]\,b^m$, which has modulus at most C_ℓ by Lemma 11. Now (a) and Lemma 10 give the bound. □

Parts (b) and (c) show that the weight e matters once $d \geq 2$. For $d = 2$ and $n = 5$, for example, $m = 2$ does not divide 5; for $f = f_p$, $e(z) = z$ and $h(z) = z^2$ the coefficient equals $A_2(p) = p + 1/p$, while for $e = 1$ it is at most $A_2(p) - R\,q_2(1 - \sqrt{3}/2)$, by (c). The exact maximum for $e = 1$ in this case is the following.

Proposition 13. *Let $p \in (0,1)$, $B = p + 1/p$, and let x_* be the unique root in $(1/\sqrt{3}, 1/\sqrt{2})$ of*

$$B\,(1 - 3x^2) + x\,(1 - 2x^2) = 0.$$

For $f \in \mathcal{M}(p)$ and a Schwarz function h with $h(0) = h'(0) = 0$,

$$\big|[z^5]\,f(h)\big| \leq \Psi_* := (1 - x_*^2)(x_*^2 + 2Bx_*).$$

Equality holds if and only if $f = f_p$ and $h(z) = h_*(vz)$ with $|v| = 1$, where $h_*(z) = z^2(x_* + z)/(1 + x_* z)$. The values of $[z^5]\,f(h)$ over all such f and h fill the closed disk of radius Ψ_*, and $\Psi_* < A_2(p)$.

We use a classical inequality of Carlson [12] for bounded functions (see also [21, Lemma 3.2]); we include a short proof of the case we need.

Lemma 14. *If $b = \sum_{j \geq 0} b_j z^j$ is holomorphic in $\mathbb{D}$ with $|b| \leq 1$, then $|b_3| \leq 1 - |b_0|^2 - |b_1|^2$.*

Proof. Let $s = (|b_0|^2 + |b_1|^2)^{1/2}$; by Parseval's identity $s \leq 1$. If $s = 0$ the claim is Cauchy's estimate $|b_3| \leq 1$. Otherwise consider multiplication by b on the Hardy space H^2, an operator of norm at most 1. Take the orthonormal pair $\{1,\ (\overline{b_1} z^2 + \overline{b_0} z^3)/s\}$ of inputs and the orthonormal pair $\{(b_0 + b_1 z)/s,\ z^3\}$ of outputs. The 2×2 matrix of inner products $\langle b\,u_j, v_i\rangle$ between these inputs u_j and outputs v_i (the compression of multiplication by b to these pairs) is $T = \begin{pmatrix} s & 0 \\ b_3 & s \end{pmatrix}$, again of norm at most 1. So $I - T^*T \geq 0$, and its determinant $(1 - s^2)^2 - |b_3|^2$ is nonnegative. □

Proof of Proposition 13. Write $h = z^2 b$ with $|b| \le 1$, $b = \sum b_j z^j$, and $a_2 = [z^2]f$. Only h and $h^2 = z^4 b^2$ reach z^5, so

$$[z^5]\, f(h) = b_3 + 2a_2 b_0 b_1.$$

Put $x = |b_0|$ and $y = |b_1|$. By the Schwarz–Pick lemma $y \le 1 - x^2$; by Lemma 14 $|b_3| \le 1 - x^2 - y^2$; and $|a_2| \le A_2(p) = B$ by (1). Hence $|[z^5]f(h)| \le 1 + Q(x, y)$ with $Q = 2Bxy - x^2 - y^2$, on the region $0 \le x \le 1$, $0 \le y \le 1 - x^2$.

If $Q(x, y) \le 0$ the bound is at most 1. If $Q(x, y) > 0$, then $Q(tx, ty) = t^2 Q(x, y)$ increases with $t \ge 1$, and the ray $t \mapsto (tx, ty)$ meets the curve $y = 1 - x^2$ at some $t \ge 1$ inside the region. So it suffices to bound $1 + Q$ on that curve, where

$$1 + Q = (1 - x^2)(x^2 + 2Bx) =: \Psi(x).$$

Since $\Psi(1/2) = 3/16 + 3B/4 > 1$, the maximum of the bound is $\max_{[0,1]} \Psi$. One computes $\Psi'(x)/2 = B(1 - 3x^2) + x(1 - 2x^2)$. This is positive for $x \le 1/\sqrt{3}$ and negative for $x \ge 1/\sqrt{2}$, and on the interval between, the derivative of $\Psi'(x)/2$, which is $1 - 6Bx - 6x^2$, is negative. So Ψ has the unique maximum point x_*, and $|[z^5]f(h)| \le \Psi(x_*) = \Psi_*$.

For $b(z) = (x + z)/(1 + xz)$ one has $b_0 = x$, $b_1 = 1 - x^2$, $b_3 = x^2(1 - x^2)$, so both inequalities above are equalities; and f_p has $a_2 = B$. Hence $f = f_p$, $h = h_*$ give $[z^5]\, f(h) = \Psi(x_*) = \Psi_*$. Replacing $h_*(z)$ by $h_*(vz)$, which is again a Schwarz function with a zero of order at least 2 at 0 for every $|v| \le 1$, multiplies the coefficient by v^5; so the values fill the closed disk of radius Ψ_*.

Conversely, suppose equality holds. Then $Q(x, y) > 0$, so $x, y > 0$, and the argument above forces $y = 1 - x^2$ and $x = x_*$. Equality in $2|a_2|xy \le 2Bxy$ forces $|a_2| = B = A_2(p)$, so $f = f_p$ by Theorem 1, and $a_2 = B > 0$. Equality $|b_1| = 1 - |b_0|^2$ in the Schwarz–Pick lemma forces b to be a disk automorphism, $b(z) = \lambda(x_* + \mu z)/(1 + x_* \mu z)$ with $|\lambda| = |\mu| = 1$. Then $b_3 = \lambda \mu^3 x_*^2 (1 - x_*^2)$ and $b_0 b_1 = \lambda^2 \mu x_* (1 - x_*^2)$, and the two terms of $b_3 + 2B b_0 b_1$ have the same argument only if $\lambda \mu^3 = \lambda^2 \mu$, that is, $\lambda = \mu^2$. Then $h(z) = z^2 b(z) = h_*(\mu z)$.

Finally, $\Psi(x) = x^2(1 - x^2) + 2Bx(1 - x^2)$ with $x^2(1 - x^2) \le \frac{1}{4}$ and $2x(1 - x^2) \le 4/(3\sqrt{3}) < 0.77$ on $[0, 1]$. So $\Psi_* < \frac{1}{4} + 0.77B < B = A_2(p)$, because $B > 2$. □

7 A question of Szapiel

Szapiel asked the following question [29, pp. 174–175, Problem 2]. Let $F(z) = \sum_{j \ge 1} a_j(F) z^j$ be in $\mathcal{S}$, and let $c_1, \dots, c_n$ be complex numbers with

$$\Big| \sum_{j=1}^{n} c_j \zeta^{j-1} \Big| \le 1 \qquad \text{for all } |\zeta| \le 1. \tag{6}$$

Does it follow that $\big|\sum_{j=1}^{n} c_j a_j(F)\big| \le n$ for every $F \in \mathcal{S}$? He observes that the answer is yes for starlike F, and for the closed convex hull of the starlike functions. The answer is yes for all of $\mathcal{S}$, and it follows from Lemma 6. A stronger inequality, which already answers the question, was in print before it was asked. Gruenberg, Rønning and Ruscheweyh [18, Lemma 5.2] state that $\|F * W\|_\infty \le n \|W\|_\infty$ for $F \in \mathcal{S}$ and every polynomial W of degree at most n, where $*$ is the coefficientwise product; they credit it to Sheil-Small [27] and to de Branges [11]. The choice $W(z) = z \sum_j c_j z^{j-1}$ gives Szapiel's inequality at $z = 1$. We record the equality cases and the counterpart for $\mathcal{M}(p)$.

Proposition 15. *Let $c_1, \dots, c_n$ satisfy* (6)*, $n \ge 1$.*

(a) For every $F \in \mathcal{S}$, $\left|\sum_{j=1}^{n} c_j a_j(F)\right| \le n$. For $n \ge 2$, equality holds if and only if $c_j = 0$ for $j < n$, $|c_n| = 1$, and $F(z) = z/(1-\eta z)^2$ with $|\eta| = 1$.

(b) Let $p \in (0,1)$. For every $f \in \mathcal{M}(p)$ with $f = \sum_j a_j z^j$ near 0, $\left|\sum_{j=1}^{n} c_j a_j\right| \le A_n(p)$. For $n \ge 2$, equality holds if and only if $c_j = 0$ for $j < n$, $|c_n| = 1$, and $f = f_p$.

Proof. Put $P(w) = \sum_{j=1}^{n} c_j w^{n-j}$. On the unit circle $|P(w)| = \left|\sum_j c_j \overline{w}^{\,j-1}\right| \le 1$, so $|P| \le 1$ in $\mathbb{D}$, and $\sum_j c_j a_j = [w^n]\, PF$ for every function $F = \sum_j a_j w^j$.

(a) The bound is Lemma 6 with $E = P$. Suppose equality holds, $n \ge 2$. With $F = wM^2$ and M_n as in the proof of Lemma 6, equality forces $\|PM_n\|_{H^2} = \|M_n\|_{H^2}$, that is, $\int_{\mathbb{T}} (1-|P|^2)\,|M_n|^2\, d\sigma = 0$. Since M_n is a polynomial with $M_n(0) = 1$, it vanishes at only finitely many points, so $|P| = 1$ on $\mathbb{T}$. A polynomial of modulus 1 on $\mathbb{T}$ is a unit multiple of a power of w: if its lowest and highest powers w^{i_1} and w^{i_2} were different, $|P|^2$ would have the nonzero Fourier coefficient $c\,\overline{c'}$ at index $i_2 - i_1$, where c, c' are the two extreme coefficients. So $P = \gamma w^{n-k}$ for a single k, that is, $c_k = \gamma$ with $|\gamma| = 1$ and all other $c_j = 0$. Then $n = |a_k(F)| \le k \le n$ by de Branges' theorem [11], so $k = n$ and $|a_n(F)| = n$, which holds only for $F(z) = z/(1-\eta z)^2$, $|\eta| = 1$ [11]. Conversely these choices give equality.

(b) By Proposition 7 with $E = P$, $\left|\sum_j c_j a_j\right| \le A_n(p) - R\, q_n(1 - |c_n|)$, because $P(0) = c_n$. Equality forces $|P(0)| = 1$, so P is the constant c_n and $c_j = 0$ for $j < n$. Then $|a_n| = A_n(p)$, and $f = f_p$ by Theorem 1 with $h(z) = z$. Conversely these choices give equality. □

Part (b) also has a form for an inner map with a zero of higher order. Let d, h, $n \ge d$ and $m = \lfloor n/d \rfloor$ be as in Section 6, let $f \in \mathcal{M}(p)$, and let W be a polynomial of degree at most n. Then the coefficientwise product of $f \circ h$ (a power series near 0) and W is a polynomial, and

$$\|(f \circ h) * W\|_\infty \le A_m(p)\, \|W\|_\infty.$$

Indeed, for $|\zeta| = 1$ and $W \ne 0$ the polynomial $e(z) = z^n W(\zeta/z)/\|W\|_\infty$ satisfies $|e| \le 1$ on $\mathbb{T}$, hence in $\mathbb{D}$, and $[z^n]\, e\,(f \circ h) = ((f \circ h) * W)(\zeta)/\|W\|_\infty$. Theorem 12(a), which allows any weight $|e| \le 1$, gives the bound. The choice $f = f_p$, $h(w) = w^d$ and $W(w) = w^{md}$ attains it.

8 Stability near the extremal functions

By Lemma 4 and Proposition 7, $|[z^n]\, e\, f(h)| = A_n(p)$ for $n \ge 2$ forces $|e(0)| = |h'(0)| = 1$, so e is a unimodular constant and h a rotation; then $f = f_p$ by Theorem 1. Here we show that a coefficient close to $A_n(p)$ forces each of the three functions to be close to its extremal form, with explicit constants. Throughout, $p \in (0,1)$, $B = p + 1/p$, $R = 4/(B+2)$, and q, $\chi = q^{-1}$ are as in Lemma 5 and Proposition 7. For $f \in \mathcal{M}(p)$ we write $a_2 = [z^2]f$; by (1), $|a_2| \le B$, so $B - \operatorname{Re} a_2 \ge 0$, and $B - \operatorname{Re} a_2 = 0$ only for $f = f_p$ (Theorem 1 with $n = 2$ and $h(z) = z$).

Theorem 16. *Let $f \in \mathcal{M}(p)$, $n \ge 2$, and put $\kappa_1 = [z^n]q$ and $\kappa_2 = [z^n]q^2$, both positive by Lemma 5. For every holomorphic E in $\mathbb{D}$ with $|E| \le 1$,*

$$|[z^n]\, Ef| \le A_n(p) - R^2\, \frac{\kappa_1 \kappa_2}{\kappa_1 + 4\kappa_2}\, (B - \operatorname{Re} a_2).$$

By Lemma 4, the same bound holds for $[z^n]\, e\, f(h)$, for every Schwarz function h and every holomorphic e with $|e| \le 1$.

Proof. Let $F = (f \circ \chi)/R \in \mathcal{S}$ as in the proof of Proposition 7, and $\alpha = [w^2]F$. From $J(\chi(w)) + 2 = (J(w) + 2)/R$ (identity (4) at $z = \chi(w)$), comparing the constant terms of the Laurent expansions at 0 gives $\chi(w) = Rw - 2R(1-R)w^2 + O(w^3)$. Hence $\alpha = Ra_2 - 2(1-R)$, and since $4 - 2R = RB$,

$$2 - \operatorname{Re}\alpha = R\,(B - \operatorname{Re} a_2).$$

Let $\Phi = F \cdot (E \circ \chi) = \sum_k d_k w^k$, so $Ef = R\,\Phi \circ q$. Here $d_1 = E(0)$, $d_2 = \alpha E(0) + RE'(0)$, and $|d_k| \le k$ by Lemma 6. With $S_k = [z^n]q^k \ge 0$, as in the proof of Proposition 7,

$$\frac{A_n(p) - |[z^n]\,Ef|}{R} \;\ge\; \kappa_1 + 2\kappa_2 - \bigl|d_1\kappa_1 + d_2\kappa_2\bigr|,$$

because $A_n(p) = R\sum_k kS_k$, $S_k \ge 0$ and $|d_k| \le k$, so the terms with $k \ge 3$ contribute at least 0 to the difference. Put $\rho = |E(0)|$ and $K = \kappa_1 + 2\kappa_2$. By the Schwarz–Pick lemma $|E'(0)| \le 1 - \rho^2$, so

$$\bigl|d_1\kappa_1 + d_2\kappa_2\bigr| \le \rho\,|\kappa_1 + \alpha\kappa_2| + R\kappa_2(1-\rho^2).$$

Bieberbach's inequality $|\alpha| \le 2$ [10] and the identity $K^2 - |\kappa_1 + \alpha\kappa_2|^2 = 2\kappa_1\kappa_2(2 - \operatorname{Re}\alpha) + \kappa_2^2(4 - |\alpha|^2)$ give $K - |\kappa_1 + \alpha\kappa_2| \ge \kappa_1\kappa_2(2 - \operatorname{Re}\alpha)/K$. Hence

$$\frac{A_n(p) - |[z^n]\,Ef|}{R} \;\ge\; \rho\,\frac{\kappa_1\kappa_2}{K}\,(2 - \operatorname{Re}\alpha) + (1-\rho)\bigl\{\kappa_1 + \bigl(2 - R(1+\rho)\bigr)\kappa_2\bigr\}.$$

The factor multiplying ρ and the factor multiplying $1-\rho$ both dominate $\kappa_1\kappa_2(2 - \operatorname{Re}\alpha)/(\kappa_1 + 4\kappa_2)$: the first because $K \le \kappa_1 + 4\kappa_2$; the second because $2 - \operatorname{Re}\alpha \le 4$, $4\kappa_2/(\kappa_1 + 4\kappa_2) \le 1$ and $R(1+\rho) \le 2R < 2$. So the right side is at least $\kappa_1\kappa_2(2 - \operatorname{Re}\alpha)/(\kappa_1 + 4\kappa_2)$, and the identity for $2 - \operatorname{Re}\alpha$ gives the claim. □

The second-coefficient deficit $B - \operatorname{Re} a_2$ controls the distance of f from f_p. We use the chordal distance $d_{\mathrm{ch}}(x,y) = |x-y|/\bigl((1+|x|^2)(1+|y|^2)\bigr)^{1/2}$ on the sphere, and the norm of the Hardy space H^1 for the normalized arc measure σ.

Theorem 17. *Let $f \in \mathcal{M}(p)$. The function $\Delta = 1/f - 1/f_p$ is holomorphic in $\mathbb{D}$, and*

$$\bigl\|(1-z)\,\Delta(z)\bigr\|_{H^1} \le \frac{4\sqrt{2}}{R}\,\bigl(B - \operatorname{Re} a_2\bigr)^{1/2}.$$

Consequently, for $0 < r < 1$,

$$\sup_{|z|\le r} d_{\mathrm{ch}}\bigl(f(z), f_p(z)\bigr) \le \frac{4\sqrt{2}}{R\,(1-r)^2}\,\bigl(B - \operatorname{Re} a_2\bigr)^{1/2}.$$

The exponent $1/2$ cannot be replaced by a larger number.

Proof. Since f is one-to-one with $f(0) = 0$, $1/f$ is holomorphic in $\mathbb{D}\setminus\{0\}$, with the principal part $1/z$ at 0; the same holds for $1/f_p$. So Δ is holomorphic in $\mathbb{D}$. It lies in H^2: the function $\zeta \mapsto 1/f(1/\zeta) = \zeta - a_2 + \sum_{k\ge1} c_k\zeta^{-k}$ is one-to-one in $|\zeta| > 1$, so $\sum_k k|c_k|^2 \le 1$ by the area theorem [14], and $1/f - 1/z = -a_2 + \sum_k c_k z^k$; the same holds for f_p.

Step 1: the slit disk. Let $F = (f\circ\chi)/R \in \mathcal{S}$ and $\alpha = [w^2]F$ as in the proof of Theorem 16. Let $N = (w/F)^{1/2}$ with $N(0) = 1$, $N = \sum_{j\ge0}\nu_j w^j$, so $\nu_1 = -\alpha/2$. The odd function $z/N(z^2)$ is the function F_2 of Lemma 6, so it lies in $\mathcal{S}$, and $\zeta \mapsto \zeta N(\zeta^{-2}) = \zeta + \sum_{j\ge1}\nu_j\zeta^{1-2j}$ is one-to-one in $|\zeta| > 1$. The area theorem gives $\sum_{j\ge1}(2j-1)|\nu_j|^2 \le 1$. Hence

$$\bigl\|N - (1-w)\bigr\|_{H^2}^2 = 1 + 2\operatorname{Re}\nu_1 + \sum_{j\ge1}|\nu_j|^2 \le 2 - \operatorname{Re}\alpha = R\,\bigl(B - \operatorname{Re} a_2\bigr),$$

and $\|N\|_{H^2}^2 \le 2$. By (5), $f_p(\chi(w)) = Rw/(1-w)^2$, while $f(\chi(w)) = RF(w) = Rw/N(w)^2$. Therefore

$$R\,\Delta(\chi(w)) = \frac{N(w) - (1-w)}{w}\,\bigl(N(w) + 1 - w\bigr),$$

where the first factor is holomorphic because $N(0) = 1$. By the Cauchy–Schwarz inequality on $\mathbb{T}$, $R\,\|\Delta \circ \chi\|_{H^1} \le \bigl(R(B - \operatorname{Re} a_2)\bigr)^{1/2} \cdot 2\sqrt{2}$.

Step 2: back to the disk. We show

$$\|(1-z)\,g\|_{H^1} \le \frac{2}{\sqrt{R}}\,\|g \circ \chi\|_{H^1} \tag{7}$$

for every $g \in H^2$. Since q has real coefficients, $q(\bar z) = \overline{q(z)}$. It maps the segment $(-1, p)$ onto $(-1, 1)$ and, because $q'(0) > 0$, the upper half of $\mathbb{D} \setminus [p, 1)$ onto the upper half-disk. The map q extends continuously to the arc $\{e^{it} : 0 < t < \pi\}$, with values on the upper half of $\mathbb{T}$. For $|z| = 1$ we have $J(z) + 2 = 4\cos^2(t/2)$ when $z = e^{it}$, so (4) gives $q(e^{it}) = e^{i\theta(t)}$ with

$$\cos\frac{\theta(t)}{2} = \sqrt{R}\,\cos\frac{t}{2}, \qquad 0 < t < \pi.$$

So θ increases from $\theta_0 = 2\arccos\sqrt{R}$ to π, and

$$\theta'(t) = \frac{\sqrt{R}\sin(t/2)}{\sin(\theta(t)/2)} \ge \sqrt{R}\sin\frac{t}{2} = \frac{\sqrt{R}}{2}\,|1 - e^{it}|.$$

The lower arc is symmetric. For a polynomial g, the change of variables $\theta = \theta(t)$ on these two arcs, and dropping the arc $|\theta| < \theta_0$, which is the image of the slit, give $\|g \circ \chi\|_{H^1} \ge (\sqrt{R}/2)\|(1-z)g\|_{H^1}$, which is (7). For $g \in H^2$ let g_j be its Taylor sections. Then $(1-z)g_j \to (1-z)g$ in H^2, and $g_j \circ \chi \to g \circ \chi$ in H^2 by Littlewood's subordination theorem [13], since $\chi(0) = 0$. So (7) holds for g.

Steps 1 and 2 with $g = \Delta$ give the H^1 bound. For $|z| \le r$, Cauchy's formula gives $|g(z)| \le \|g\|_{H^1}/(1-|z|)$ for $g \in H^1$; with $g = (1-z)\Delta$ and $|1-z| \ge 1-r$ we get $|\Delta(z)| \le \|(1-z)\Delta\|_{H^1}/(1-r)^2$. Finally, d_{ch} is invariant under $x \mapsto 1/x$ and at most the Euclidean distance, so $d_{\mathrm{ch}}(f(z), f_p(z)) \le |\Delta(z)|$ for $z \ne 0$, and both maps vanish at $z = 0$.

For the exponent, let $\eta = e^{i\lambda}$ and $f_\lambda(z) = z/\bigl((1 - z/p)(1 - \eta p z)\bigr)$. It is in $\mathcal{M}(p)$: its other pole $1/(\eta p)$ lies outside $\mathbb{D}$, and $f_\lambda(z) = f_\lambda(w)$ means $(z-w)(1-\eta z w) = 0$. Here $B - \operatorname{Re} a_2 = p(1-\cos\lambda)$, of order λ^2, while $1/f_\lambda - 1/f_p = (1-\eta)(p-z)$ has size of order $|\lambda|$ at every fixed $z \ne p$ in $\mathbb{D}$; on a fixed disk the chordal distance between f_λ and f_p is therefore at least a positive constant times $|\lambda|$. □

Theorems 16 and 17 together say: if $|[z^n]\,e\,f(h)| \ge A_n(p) - \varepsilon$ for some $n \ge 2$, then $B - \operatorname{Re} a_2 \le \varepsilon\,(\kappa_1 + 4\kappa_2)/(R^2\kappa_1\kappa_2)$, and f is within $C\varepsilon^{1/2}$ of f_p in the chordal metric on $|z| \le r$, where C depends on n, p and r. The same family f_λ, with $e = 1$ and $h(z) = z$, shows that $\varepsilon^{1/2}$ is the right order: $A_n(p) - |[z^n]f_\lambda| \le \sum_{j<n} p^{2j+1-n}(1 - \cos j\lambda)$ is of order λ^2. For the weight and the inner map the dependence is linear.

Proposition 18. *Let $f \in \mathcal{M}(p)$, h a Schwarz function, e holomorphic in $\mathbb{D}$ with $|e| \le 1$, $n \ge 1$, and $\varepsilon = A_n(p) - |[z^n]\,e\,f(h)|$. There are constants u, v of modulus 1 such that for $0 < r < 1$*

$$\sup_{|z|\le r} |e(z) - u| \le \frac{1+r}{1-r}\,\frac{\varepsilon}{R\,q_n}, \qquad \sup_{|z|\le r} |h(z) - vz| \le r\,\frac{1+r}{1-r}\,\frac{\varepsilon}{R\,q_n}.$$

The linear dependence on ε cannot be improved.

Proof. By Lemma 4 and Proposition 7, $\varepsilon \ge R\,q_n(1 - |e(0)|\,|h'(0)|^n)$, and since $|e(0)|, |h'(0)| \le 1$ both $1 - |e(0)|$ and $1 - |h'(0)|$ are at most $1 - |e(0)|\,|h'(0)|^n \le \varepsilon/(Rq_n)$. Let b be holomorphic in

$\mathbb{D}$ with $|b| \le 1$, $b_0 = b(0)$, and let $\xi = b_0/|b_0|$ (any unimodular number if $b_0 = 0$). For $|z| \le r$ the Schwarz–Pick lemma gives $|b(z) - b_0| \le r(1 - |b_0|^2)/(1 - r|b_0|)$, hence

$$|b(z) - \xi| \le (1 - |b_0|)\Big(\frac{r(1 + |b_0|)}{1 - r|b_0|} + 1\Big) \le (1 - |b_0|)\,\frac{1 + r}{1 - r}.$$

Apply this to $b = e$ and to $b = h(z)/z$, for which $b_0 = h'(0)$.

For $f = f_p$, the weight $e = 1 - t$ gives $\varepsilon = tA_n(p)$, and its distance from every unimodular constant is at least t. The inner map $h(z) = (1 - t)z$ gives $\varepsilon = (1 - (1 - t)^n)A_n(p) \le ntA_n(p)$, and on $|z| = r$ its distance from every rotation is at least rt. □

9 A sharp derivative bound for an arbitrary source domain

Theorem 3 treats the disk and convex domains by two different arguments. One number $\tau(\Omega, z_0)$, attached to the source and the base point, covers both cases, and the bound in terms of τ is sharp for every value of τ. Let $\Omega \subset \widehat{\mathbb{C}}$ be a simply connected domain whose complement has at least two points, $z_0 \in \Omega \cap \mathbb{C}$, and let $\phi : \mathbb{D} \to \Omega$ be the conformal map with $\phi(0) = z_0$ and $\phi'(0) = R_s > 0$. Put

$$s = \frac{\phi - z_0}{R_s}, \qquad V(w) = \frac{1}{s(w)} - \frac{1}{w}, \qquad \tau = \tau(\Omega, z_0) = \sup_{w \in \mathbb{D}} |V(w)|.$$

Since $s(w) = w + O(w^2)$, the function V is holomorphic in $\mathbb{D}$ (at a pole of s, $1/s$ vanishes). It is bounded: a closed disk $|x| \le \delta$ lies in $s(\mathbb{D})$ and its preimage under s is a compact subset of $\mathbb{D}$ that contains a disk around 0. Outside the preimage $|1/s| < 1/\delta$ and $|1/w|$ is bounded, and on the preimage V is continuous. So $0 \le \tau < \infty$. For $\Omega = \mathbb{D}$, the map $\phi(w) = (z_0 + w)/(1 + \overline{z_0}w)$ gives $s(w) = w/(1 + \overline{z_0}w)$, so $V \equiv \overline{z_0}$ and $\tau(\mathbb{D}, z_0) = |z_0|$. If $\Omega \ne \mathbb{C}$ is convex, the proof of Theorem 3(b) shows $w/s(w) = 1 + wv(w)$ with $|v| \le 1$, that is, $V = v$ and $\tau \le 1$. If $\infty \in \Omega$, then $V(w_1) = -1/w_1$ at the pole w_1 of s, so $\tau > 1$. We extend (3) to all $\tau \ge 0$ by

$$D_n(p, \tau) = \sum_{k=1}^{n} \binom{n-1}{k-1} \tau^{\,n-k} A_k(p),$$

which for $n \ge 2$ increases strictly with τ.

Theorem 19. *Let Ω, z_0, R_s and $\tau = \tau(\Omega, z_0)$ be as above, let $\Pi \subset \widehat{\mathbb{C}}$ be a simply connected domain whose complement has at least two points, and let $G : \Omega \to \Pi$ be a meromorphic map (holomorphic as a map into the sphere), with $w_0 = G(z_0)$ finite. With $R_t = R(w_0, \Pi)$ and p as in Theorem 3, for every $n \ge 1$*

$$\frac{R_s^n\,|G^{(n)}(z_0)|}{n!\,R_t} \le D_n(p, \tau).$$

For every $p \in (0, 1]$ and every $\tau \ge 0$ there are Ω, z_0, Π and G with these values for which equality holds.

Since $\tau(\mathbb{D}, z_0) = |z_0|$, $\tau \le 1$ for convex Ω, and D_n increases with τ, Theorem 19 contains both parts of Theorem 3. It also covers sources that contain ∞, for which $\tau > 1$.

Proof. Let $H = (G \circ \phi - w_0)/R_t$. Lemma 8 and its proof hold without change for $\Omega \subset \widehat{\mathbb{C}}$, so $|[w^k]\,EH| \le A_k(p)$ for every $k \ge 1$ and every holomorphic E with $|E| \le 1$. The Fejér step in the proof of Theorem 3(b) then gives

$$\big|[w^{k-1}]\,H'E\big| \le k\,A_k(p) \qquad (k \ge 1,\ |E| \le 1). \tag{8}$$

Near $x = 0$ we have $G(z_0 + R_s x) = w_0 + R_t\, H(s^{-1}(x))$, and $w/s(w) = 1 + wV(w)$. If $\tau = 0$, then $s(w) = w$ and the claim is $|[w^n]H| \le A_n(p)$. If $\tau > 0$, the Lagrange inversion formula and the binomial theorem give

$$\frac{R_s^n G^{(n)}(z_0)}{n!\, R_t} = [x^n]\, H(s^{-1}(x)) = \frac{1}{n}\,[w^{n-1}]\, H'(w)\big(1 + wV(w)\big)^n = \sum_{j=0}^{n-1} \frac{1}{n}\binom{n}{j} \tau^j\, [w^{n-1-j}]\, H'\Big(\frac{V}{\tau}\Big)^j.$$

Apply (8) with $E = (V/\tau)^j$ and $k = n - j$, and use $\frac{k}{n}\binom{n}{j} = \binom{n-1}{k-1}$.

For sharpness let $s(w) = w/(1 + \tau w)$, which maps $\mathbb{D}$ onto a disk if $\tau < 1$, a half-plane if $\tau = 1$, and the exterior of a closed disk, with ∞, if $\tau > 1$. Here $V \equiv \tau$. Take $\Omega = s(\mathbb{D})$, $z_0 = 0$, $\phi = s$ (so $R_s = 1$), $\Pi = f_p(\mathbb{D})$ and $G = f_p \circ s^{-1}$ (for $p = 1$, $f_1(z) = z/(1 - z)^2$ is the Koebe function). Then $R_t = 1$ and the target parameter is p. Since $s^{-1}(x) = x/(1 - \tau x)$,

$$[x^n]\, f_p\Big(\frac{x}{1 - \tau x}\Big) = \sum_{k=1}^{n} A_k(p)\, [x^n] \frac{x^k}{(1 - \tau x)^k} = \sum_{k=1}^{n} \binom{n-1}{k-1} \tau^{\,n-k} A_k(p) = D_n(p, \tau). \qquad \square$$

For $0 < p < 1$ the equality cases are the following.

Theorem 20. *In Theorem 19 let $0 < p < 1$, $n \ge 2$ and $\tau > 0$. Equality holds if and only if there are η, β of modulus 1 with*

$$s(w) = \frac{w}{1 + \tau\eta w} \qquad \text{and} \qquad \frac{G(\phi(w)) - w_0}{R_t} = \beta f_p(\eta w).$$

Proof. In the last display of the proof of Theorem 19, the term with index j has modulus at most $\binom{n-1}{k-1}\tau^{\,n-k} A_k(p)$, $k = n - j$. Equality forces equality in every term. The term $j = 0$ is $[w^n]H$, so $|[w^n]H| = A_n(p)$. In the proof of Lemma 8, $H = (a/R_t)\, f \circ h$ with $|a| = R_t$, so Theorem 1 gives $f = f_p$ and a rotation h; that is, $H(w) = \beta f_p(uw)$ with $|\beta| = |u| = 1$. The term $j = n - 1$ is $H'(0)V(0)^{n-1} = \beta u\, V(0)^{n-1}$, so $|V(0)| = \tau = \sup|V|$, and by the maximum principle $V \equiv \tau\eta$ with $|\eta| = 1$; that is, $s(w) = w/(1 + \tau\eta w)$. Then

$$[x^n]\, H(s^{-1}(x)) = \beta \sum_{k=1}^{n} \binom{n-1}{k-1} \tau^{\,n-k} A_k(p)\, u^k \eta^{\,n-k},$$

and all coefficients are positive. The terms $k = 1$ and $k = 2$ have the same argument only if $u = \eta$, and then all terms have the same argument. This proves both directions. $\square$

Corollary 21. *Let $0 < p < 1$ and $n \ge 2$ in Theorem 3.*

(a) For $\Omega = \mathbb{D}$, equality holds if and only if $G(z) = w_0 + R_t\, \beta f_p(\eta\, \phi^{-1}(z))$ with $\phi(w) = (z_0 + w)/(1 + \overline{z_0} w)$, $|\beta| = 1$, and $\eta = \overline{z_0}/|z_0|$ if $z_0 \ne 0$, any $|\eta| = 1$ if $z_0 = 0$.

(b) For convex $\Omega \ne \mathbb{C}$, equality holds if and only if Ω is a half-plane and $G(z) = w_0 + R_t\, \beta f_p(\eta\, \phi^{-1}(z))$ with $\phi(w) = z_0 + R_s\, w/(1 + \eta w)$ and $|\beta| = |\eta| = 1$.

Proof. (a) Here $V \equiv \overline{z_0}$ and $\tau = |z_0|$. For $z_0 \ne 0$ apply Theorem 20; its condition on s holds with $\eta = \overline{z_0}/|z_0|$. For $z_0 = 0$, $D_n(p, 0) = A_n(p)$, and equality means $|[w^n]H| = A_n(p)$, that is, $H(w) = \beta f_p(\eta w)$ as in the previous proof. (b) Here $\tau \le 1$, and since $D_n(p, \tau)$ increases strictly with τ, equality with $D_n(p, 1)$ forces $\tau = 1$. Theorem 20 gives $s(w) = w/(1 + \eta w)$, which maps $\mathbb{D}$ onto the half-plane $\{x : \mathrm{Re}(\eta x) < 1/2\}$, and the form of G. $\square$

Discussion

Theorem 1 proves the generalized Goodman conjecture for every n and every p, in a stronger form that allows a bounded weight; Figure 1 shows what was known before. For $e = 1$ and $n \geq 2$ only $f = f_p$ with a rotation h attains the bound. Theorem 3 proves the two conjectures of 2008. Section 7 corrects the record on one point: the question that Szapiel asked in 1994 [29, pp. 174–175, Problem 2] is answered by an inequality that Gruenberg, Rønning and Ruscheweyh had stated in 1990 [18].

We end with three questions.

(1) *Inner maps with a zero of higher order.* If h has a zero of order at least d at 0 and $n \geq d$, then $|[z^n](f \circ h)| \leq A_m(p)$ with $m = \lfloor n/d \rfloor$, and this bound is attained exactly when m divides n, for example by f_p and $h(z) = z^{n/m}$ (Theorem 12); for $d = 2$ and $n = 5$ we find the exact maximum (Proposition 13). What is the exact maximum when m does not divide n, beginning with $d = 2$ and $n = 7$?

(2) *Stability.* For $n \geq 2$, if $|[z^n](f \circ h)| \geq A_n(p) - \varepsilon$, then f lies within $C\varepsilon^{1/2}$ of f_p in the chordal metric on the disk $|z| \leq r$, where $r < 1$ and C depends only on n, p and r, and the order $\varepsilon^{1/2}$ cannot be improved (Theorems 16 and 17). What are the best constants C?

(3) *Other source domains.* The bound of Theorem 3(a) extends to a meromorphic map from any simply connected source domain Ω whose complement has at least two points, with $|z_0|$ replaced by $\tau(\Omega, z_0) = \sup_{w \in \mathbb{D}} |R_s/(\phi(w) - z_0) - 1/w|$, where ϕ is the conformal map of $\mathbb{D}$ onto Ω with $\phi(0) = z_0$ and $\phi'(0) = R_s > 0$; this number equals $|z_0|$ for the disk and is at most 1 for convex Ω (Theorem 19). For $n \geq 2$ we determine the extremal maps when the target contains ∞ (Theorem 20 and Corollary 21). Which maps attain the bound for a general source domain when the target does not contain ∞?

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