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L. FEJES TÓTH'S STRONG-SOLIDITY CONJECTURE (1979) HOLDS FOR $\{7,3\}$, AND OTHER MISCELLANEA

DAVE GOEL

Abstract. A packing of open circular discs in the sphere, Euclidean plane, or hyperbolic plane is *solid* if no finite subset can be rearranged so as to form, together with the rest, a packing not congruent to the original one; it is *strongly solid* if it remains solid after any one circle is removed. L. Fejes Tóth conjectured that the face-incircles of the regular trihedral tiling $\{p,3\}$ form a strongly solid packing for every $p \geq 6$. A. Bezdek (1979) proved the conjecture for all $p \geq 8$, leaving $p = 6$ (Euclidean) and $p = 7$ (hyperbolic) open. In Part I we prove that the face-incircles of $\{7,3\}$ form a strongly solid packing, leaving the Euclidean hexagonal packing $\{6,3\}$ as the sole remaining open case.[1] Combining this with an exact evaluation of the grade of saturatedness $g_7 = 3$, we obtain the order of solidity $k_7 = 1 = g_7 - 2$, settling L. Fejes Tóth's (1980) order-of-solidity formula at $p = 7$. The proof for $\{7,3\}$ is elementary, self-contained, and computer-free. At $p = 7$, Bezdek's strict hyperbolic distance inequality degenerates to an exact equality reducing to an algebraic identity in $\mathbb{Q}(\cos(\pi/7))$, and concentric-zone runs shorten so his one-by-one joining order deadlocks; we overcome both obstructions with a sharp geometric pinning lemma and a single nine-site closing lemma tailored to the dual triangulation $\{3,7\}$.

In Part II we give self-contained proofs of structural results and obstructions for the remaining open Euclidean case $p = 6$ (all theorems proved by hand), formulated via the set of lattice discs (sites) overlapped by a finite off-lattice packing. We prove that any counterexample must enclose at least one bounded hole of unoccupied lattice triangles, establish an isoperimetric inequality bounding the total number of moved circles quadratically in the number of circles that overlap only three lattice sites, and exhibit an explicit 30-circle D_6-symmetric packing in which every overlapped lattice site is overlapped by at least two circles. Finally, we prove a near-lattice rigidity theorem: for every finite rearranged subset, the Euclidean one-vacancy conjecture holds whenever every moved circle stays within 5/26 of a lattice side from some lattice site (more than seven times Bárány and Dolbilin's 1/40-side threshold), improving to 1/5 of a lattice side when every moved circle overlaps only three lattice sites. The unrestricted Euclidean conjecture for $p = 6$ remains open. Theorem 1.1 and Corollary 1.2 are verified in Lean.[2]

This is the expanded version of the focused version [10]. *Part I (Sections 1–5) reproduces the proof in* [10]*; Part II (Sections 6–7) appears only here, and Section 8 expands the discussion in* [10].

1. Introduction

Imagine packing identical round discs tightly inside a regular tiling of a sphere, the flat plane, or the hyperbolic plane so that three tiles meet at every corner, and then removing a single disc: can any finite group of the remaining discs be lifted and set down again without overlap to form a genuinely new pattern? For nearly half a century, László Fejes Tóth's strong-solidity conjecture has predicted that the answer is no whenever each disc has at least six neighbors. Despite the simplicity of the question, two boundary cases—the flat honeycomb ($p = 6$) and the first hyperbolic case ($p = 7$)—have stood as prominent unsolved problems in discrete geometry since 1979.

Date: First version: November 25, 2026. This version (1.00): November 26, 2026.

[1] This is an expanded version of the following focused version [10]: https://doi.org/10.5281/zenodo.24569387

[2] Theorem 1.1 and Corollary 1.2 are proved in Lean 4 from Lean's three standard axioms alone, and checked with `lake comparator`:
https://github.com/david44hello/paper_fejes-toth-1979-strong-solidity-7-3/tree/v1.00/lean

In the hyperbolic plane $\mathbb{H}^2$, no single canonical notion of global packing density captures the optimality of circle packings. To formulate optimality without density, L. Fejes Tóth [8] introduced *solidity*: a packing of open circular discs (circles) in S^2, $\mathbb{R}^2$, or $\mathbb{H}^2$ is *solid* if no finite subset of the discs can be rearranged (by arbitrary rigid motions, with no continuity or locality restriction) so as to form, together with the remaining discs, a packing not congruent to the original one. For each integer $p \geq 2$, let $\{p, 3\}$ denote the regular trihedral tiling by congruent p-gons meeting three at each vertex (S^2 for $p \leq 5$, $\mathbb{R}^2$ for $p = 6$, and $\mathbb{H}^2$ for $p \geq 7$). By a theorem of Imre [13] (see also [8]), the face-incircles of $\{p, 3\}$ form a solid packing for every $p \geq 2$.

A solid packing is *strongly solid* if it remains solid after any one circle is removed. On S^2, removing one face-incircle of $\{2, 3\}$, $\{3, 3\}$, or $\{4, 3\}$ destroys solidity [2, p. 203], whereas Danzer [5] proved that the eleven circles left after removing one face-incircle of the dodecahedral tiling $\{5, 3\}$ remain uniquely determined up to isometry, so $\{5, 3\}$ is strongly solid. Motivated by this threshold, L. Fejes Tóth conjectured (first published in [2, p. 203] and [9, p. 299]) that the face-incircles of $\{p, 3\}$ form a strongly solid packing for every $p \geq 6$. In 1979, A. Bezdek [2] proved the conjecture for all $p \geq 8$ and remarked that "the problem for $p = 6$ and 7 is still open." Shortly afterward, L. Fejes Tóth [9] defined the *order of solidity* k_p and the *grade of saturatedness* g_p of the face-incircle packing Π_p of $\{p, 3\}$, proved that $k_p = g_p - 2$ for all $p \geq 8$, and wrote that this equality holds for $p = 2, 3, 4, 5$ and "probably also for $p = 6$ and 7," closing with open problems on strong and higher-order solidity [9, pp. 299, 302].

Over the next four decades, the remaining cases were repeatedly singled out: for the Euclidean case $p = 6$, Bárány and Dolbilin [1, p. 107] remarked that "this conjecture is open and the solution seems to be difficult" and Connelly and Dickinson [4, Conjecture 1] placed it at the head of their study of periodic planar disc packings; A. Bezdek, K. Bezdek, and Connelly [3, Remark 2.8.1] highlighted it as "an outstanding conjecture of L. Fejes Tóth" with precisely $p = 6$ and $p = 7$ open; and G. Fejes Tóth, L. Fejes Tóth, and W. Kuperberg [7, Section 3] recorded both $p = 6$ and $p = 7$ as unsolved in their 2022 survey for Chapter 11 of the Springer *Grundlehren* monograph *Lagerungen*.

Our main result settles the hyperbolic case $p = 7$ (Figure 1), leaving the Euclidean hexagonal packing $\{6, 3\}$ as the sole remaining open case.

Theorem 1.1. *The face-incircles of the regular hyperbolic tiling $\{7, 3\}$ form a strongly solid packing.*

Recall from [9] that a packing Π of congruent circles is *solid of order k* ($0 \leq k < |\Pi| - 1$) if removing any k circles leaves a *k-truncation* Π' of Π that admits no finite rearrangement other than one congruent to some k-truncation of Π; its *order of solidity* is the largest such integer k. Its *grade of saturatedness* g is the largest integer such that every circle congruent to those of Π, and distinct from each of them, intersects at least g circles of Π. Since the symmetry group of $\{7, 3\}$ acts transitively on faces, strong solidity is equivalent to solidity of order 1. Combining Theorem 1.1 with an exact evaluation of g_7 resolves Fejes Tóth's 1980 formula at $p = 7$.

Corollary 1.2. *For the face-incircle packing Π_7 of $\{7, 3\}$, the grade of saturatedness is $g_7 = 3$ and the order of solidity is $k_7 = 1 = g_7 - 2$.*

Why $p = 7$ is critical, and overview of the proof. Bezdek's proof for $p \geq 8$ builds an exhaustion $U_1 \subset U_2 \subset \cdots$ of $\Pi_p \setminus \{c_0\}$ by concentric zones around the removed circle c_0, adding one circle at a time while preserving the property of being *good* (admitting no non-trivial rearrangement). At $p = 7$, his proof stalls in two independent ways:

(1) *Metric criticality.* For $\{p, 3\}$ ($p \geq 7$), let r_p be the face-inradius, h_p the altitude of an equilateral triangle of side $2r_p$ in the dual triangulation $\{3, p\}$, and ρ_p^* the critical polar radius at which the angular half-width excluded by a fixed neighbor equals $3\pi/p$ (Section 2). For every $p \geq 8$ one has $\Delta_p := h_p - (r_p + \rho_p^*) < 0$ (Figure 2), and indeed $|h_p - \rho_p^*| < r_p$,

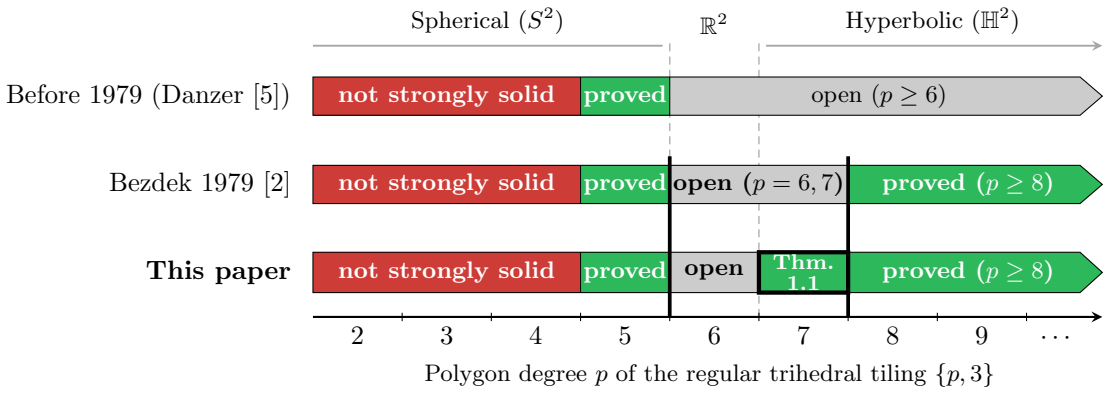

FIGURE 1. Status of L. Fejes Tóth's strong-solidity conjecture for the face-incircles of the regular trihedral tilings $\{p, 3\}$. Red cells fail strong solidity, green cells are proved strongly solid, and grey cells remain open; the black vertical bars bracket the two cases $p \in \{6, 7\}$ left open by A. Bezdek [2] (and likewise left open for $k_p = g_p - 2$ in L. Fejes Tóth [9]). Theorem 1.1 settles strong solidity for $p = 7$ (and Corollary 1.2 settles $k_7 = g_7 - 2$), leaving the Euclidean hexagonal case $p = 6$ as the sole remaining open case.

which yields Bezdek's strict distance bound ($< r_p$) to an edge midpoint [2, p. 206]. At $p = 7$ (dropping the subscript 7), an exact algebraic identity gives $\Delta_7 = 0$, i.e., $h = r + \rho^*$ (Proposition 3.1), weakening the strict distance bound to $\leq r$ with equality at a unique tip point, so Bezdek's Lemma 2 cannot rule out touching configurations directly.

(2) *Combinatorial deadlock.* In Bezdek's zone decomposition of $\{3, p\}$ for $p \geq 8$ [2, p. 207], consecutive sites in Z_m ($m \geq 2$) with two inner neighbors (type 2) are separated by runs of $p - 6$ or $p - 5$ sites with one inner neighbor (type 1), giving runs of length at least 2 (and length $p - 5 \geq 3$ in Z_2), which allows every run to close by Bezdek's Lemmas 1 and 2. At $p = 7$, Lemma 5.1 shows that the runs have length 1 or 2, obstructing the one-by-one joining order around every zone Z_m ($m \geq 2$).

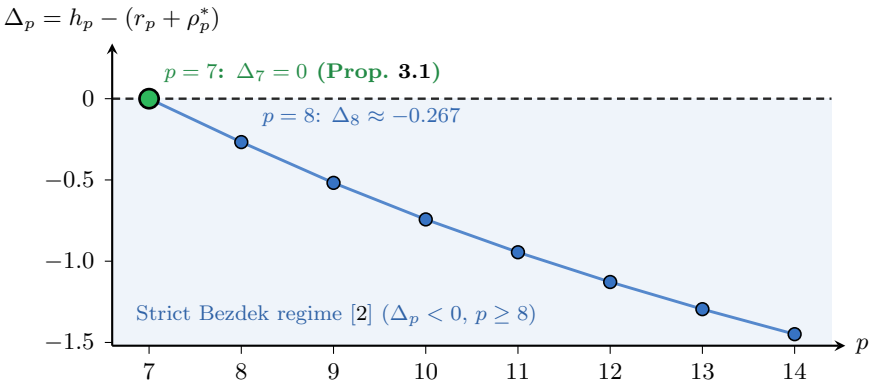

FIGURE 2. The metric gap $\Delta_p := h_p - (r_p + \rho_p^*)$ across hyperbolic regular trihedral tilings $\{p, 3\}$ ($p \geq 7$). On the plotted range $7 \leq p \leq 14$ one has $\rho_p^* \leq h_p$ (with $h_{14} = \rho_{14}^*$), so $\Delta_p = |h_p - \rho_p^*| - r_p$; for every $p \geq 8$, $\Delta_p < 0$ (together with $\rho_p^* < 2r_p < h_p + r_p$ for $p \geq 15$) gives Bezdek's strict distance inequality $|h_p - \rho_p^*| < r_p$. At $p = 7$ the gap vanishes identically ($\Delta_7 = 0$, Proposition 3.1).

We overcome both obstructions simultaneously without any computer assistance. First, we prove $h = r+\rho^*$ via an algebraic identity in $\mathbb{Q}(\cos(\pi/7))$ and give a synthetic two-circles proof of the sharp disc inclusion (Lemma 3.2), pinning any extremal rearranged circle at a unique *tip* point T; a strict open-ball exclusion (Lemma 3.3) then yields a contradiction. Second, we prove a single nine-site closing lemma (Lemma 4.5) and show that the substitution rule $\sigma(1) = 112$, $\sigma(2) = 12$ generating the zones allows Lemma 4.5 to close both types of arcs in every zone Z_m without deadlock.

Organization of this expanded version. Part I (Sections 1–5) gives the complete, computer-free proof of Theorem 1.1 and Corollary 1.2 for $\{7, 3\}$ from the focused version [10]. Part II appears only here: all theorems in Sections 6–7 have self-contained, computer-free proofs (with Proposition 6.6 verified by a routine finite distance check in $\mathbb{Q}(\sqrt{3})$ on five explicit rational generators). Section 6 develops elementary structural theorems and obstructions for $p = 6$, Section 7 proves near-lattice rigidity at displacement $5/26$ of a lattice side ($1/5$ for all-triangular families), and Section 8 expands the discussion of [10].

2. Setting and elementary hyperbolic lemmas

We work in the hyperbolic plane $\mathbb{H}^2$ of curvature -1 with hyperbolic distance $d(\cdot, \cdot)$, open balls $B(x, R) := \{y \in \mathbb{H}^2 : d(x, y) < R\}$, closed balls $\overline{B}(x, R)$, and boundary circles $\partial B(x, R)$. Let $\{7, 3\}$ be the regular heptagonal tiling of $\mathbb{H}^2$ with three heptagons meeting at every vertex, and let $\mathcal{C} := \Pi_7$ be the packing of open face-incircles of $\{7, 3\}$ (Figure 3).

2.1. **Constants and dual triangulation.** Let r be the common radius of the circles in $\mathcal{C}$, and let $\alpha := 2\pi/7$. The centers of the circles in $\mathcal{C}$—which we call *sites*—are the vertices of the dual regular triangulation $\{3, 7\}$, whose faces are equilateral hyperbolic triangles of side length $2r$ and vertex angle α, with seven triangles meeting at each site. Distinct sites lie at distance at least $2r$, with equality if and only if they are adjacent in $\{3, 7\}$. From the right triangle formed by a site, a midpoint of a side of $\{3, 7\}$, and a vertex of $\{7, 3\}$ (angles $\pi/7$, $\pi/3$, $\pi/2$), and the half-triangle of $\{3, 7\}$ (angles $\pi/7$, $2\pi/7$, $\pi/2$), standard hyperbolic trigonometry gives

$$\cosh r = \frac{1}{2\sin(\pi/7)}, \qquad \cosh 2r = 2\cosh^2 r - 1 = \frac{\cos\alpha}{1-\cos\alpha}, \qquad \cosh h = \frac{\cosh 2r}{\cosh r}, \tag{1}$$

where h is the altitude of an equilateral triangle of $\{3, 7\}$, and the circumradius R_c of such a triangle (the covering radius of the set of sites) satisfies $\sinh R_c = \sinh r/\sin(\pi/3)$. In a triangle $\triangle uvw$ of $\{3, 7\}$ with side midpoint $M := \mathrm{mid}(u, v)$ and circumcenter O, the right triangle $\triangle uMw$ has hypotenuse $uw = 2r$ exceeding the leg $Mw = h$, while O lies strictly inside the altitude segment Mw and the right triangle $\triangle uMO$ has hypotenuse $uO = R_c$ exceeding the leg $uM = r$; hence

$$r < R_c < h < 2r. \tag{2}$$

Numerically, $2r \approx 1.09055$, $h \approx 0.90380$, and $R_c \approx 0.62067$.

About any site v, we use hyperbolic polar coordinates (ρ, θ) in which the seven neighbors of v in $\{3, 7\}$ sit at $\rho = 2r$ and $\theta = k\alpha$ ($k \in \mathbb{Z}/7\mathbb{Z}$). For $0 < \rho \le 2r$, define

$$\beta(\rho) := \arccos\big(\coth(2r)\tanh(\rho/2)\big). \tag{3}$$

Since $\coth(2r)\tanh r = \cosh(2r)/(\cosh(2r)+1) = \cos\alpha$ by (1), the function β is strictly decreasing on $(0, 2r]$ with $\lim_{\rho\to 0^+}\beta(\rho) = \pi/2$ and $\beta(2r) = \alpha$; in particular, $\beta(\rho) > \alpha$ for all $\rho \in (0, 2r)$. Define the critical radius $\rho^* \in (0, 2r)$ by $\beta(\rho^*) = 3\alpha/2 = 3\pi/7$, namely

$$\rho^* := 2\,\mathrm{artanh}\big(\tanh(2r)\cos(3\pi/7)\big) \approx 0.35853. \tag{4}$$

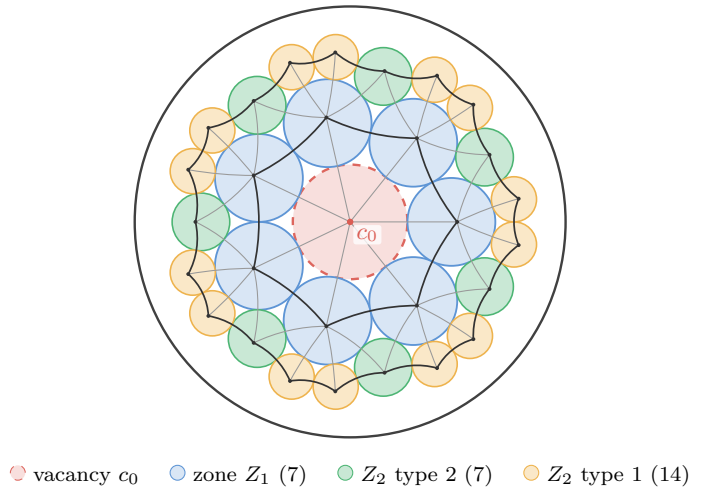

FIGURE 3. The face-incircle packing $\mathcal{C}$ of $\{7,3\}$ and its dual triangulation $\{3,7\}$ in the Poincaré disc model (where congruent hyperbolic circles appear smaller toward the boundary), centered at the removed circle c_0. Zone Z_1 surrounds c_0; zone Z_2 consists of 7 type-2 circles (two neighbors in Z_1) alternating with pairs of type-1 circles (one neighbor in Z_1).

2.2. **Rearrangements, good sets, and neighbor profiles.** Fix one circle $c_0 \in \mathcal{C}$ (the *vacancy*) and set $\mathcal{S} := \mathcal{C} \setminus \{c_0\}$. We identify each circle of $\mathcal{C}$ with its center (site). For a finite subset $s \subset \mathcal{S}$, a *rearrangement* of s is a family X of $|s|$ open circles of radius r which, together with the circles of $\mathcal{S} \setminus s$ (held fixed), forms a packing. A rearrangement X is *trivial* if $(\mathcal{S} \setminus s) \cup X$ is congruent to $\mathcal{S}$.

Lemma 2.1. *A rearrangement X of a finite subset $s \subset \mathcal{S}$ is trivial if and only if every circle of X is inscribed in a face of $\{7,3\}$ (i.e., centered at a site of $\{3,7\}$).*

Proof. If every circle of X is centered at a site, then $(\mathcal{S} \setminus s) \cup X$ consists of all sites of $\{3,7\}$ except one, which is congruent to $\mathcal{S}$ because the symmetry group of $\{7,3\}$ acts transitively on faces. Conversely, suppose $(\mathcal{S} \setminus s) \cup X = \phi(\mathcal{S})$ for an isometry ϕ of $\mathbb{H}^2$. Choose a triangle Δ of $\{3,7\}$ whose three vertices are fixed sites (so they belong to $\mathcal{S} \setminus s \subset \phi(\mathcal{S})$). Then $\phi^{-1}(\Delta)$ is an equilateral triangle of side $2r$ whose vertices are sites of $\{3,7\}$, hence a triangle of $\{3,7\}$ (since distinct sites at distance $2r$ are adjacent). Because the symmetry group of $\{3,7\}$ acts transitively on flags (incident vertex–edge–face triples), ϕ is a symmetry of $\{3,7\}$; thus $\phi(\mathcal{S})$ consists of all sites except $\phi(c_0)$, and every circle of X is centered at a site. □

We say that $s \subset \mathcal{S}$ is *good* if every rearrangement of s is trivial, and that a circle $c \in \mathcal{S} \setminus s$ *can be joined* to a good set s if $s \cup \{c\}$ is good. Relative to a finite set $s \subset \mathcal{S}$ being rearranged, a site of $\{3,7\}$ is *fixed* if it lies in $\mathcal{S} \setminus s$, and *unfixed* if it lies in $s \cup \{c_0\}$ (note that c_0 is always unfixed and carries no fixed circle). A point $P \in \mathbb{H}^2$ is *admissible* if $d(P,f) \geq 2r$ for every fixed site f; every circle of a rearrangement has an admissible center, and distinct circles of a rearrangement have centers at distance at least $2r$. A point P *overlaps* a site q if $d(P,q) < 2r$ (strictly); a circle of a rearrangement whose center overlaps q is called an *overlapper* of q.

Relative to the current set of unfixed sites, we say that a site v has *profile* τ_2 (respectively τ_3, τ_4) if its unfixed neighbors in $\{3,7\}$ are precisely 2 (respectively 3, 4) cyclically consecutive neighbors at angles $0, \alpha$ (respectively $0, \alpha, 2\alpha$; $0, \alpha, 2\alpha, 3\alpha$), with all remaining neighbors of v fixed.

2.3. **Four local geometric lemmas.**

Lemma 2.2 (Polar Lemma)**.** *Let $P \neq v$ have polar coordinates (ρ, θ) about a site v with $0 < \rho \leq 2r$, and let n be a neighbor of v at angle φ (more generally, any point at distance $2r$ from v at angle φ). Then $d(P, n) < 2r$ if and only if $|\theta - \varphi| < \beta(\rho)$ (with angle difference taken in $[0, \pi]$), and equality $d(P, n) = 2r$ holds if and only if $|\theta - \varphi| = \beta(\rho)$.*

Proof. By the hyperbolic law of cosines in $\triangle vPn$,

$$\cosh d(P, n) = \cosh \rho \cosh 2r - \sinh \rho \sinh 2r \cos(\theta - \varphi) < \cosh 2r$$

holds if and only if

$$\cos(\theta - \varphi) > \frac{(\cosh \rho - 1) \cosh 2r}{\sinh \rho \sinh 2r} = \coth(2r) \tanh(\rho/2) = \cos \beta(\rho).$$

Since $\beta(\rho) \in [\alpha, \pi/2)$ and cosine is strictly decreasing on $[0, \pi]$, this is equivalent to $|\theta - \varphi| < \beta(\rho)$, with the same equivalence for equality. □

Lemma 2.3 (Free-Arc Lemma)**.** *Let $P \neq v$ be an admissible point with $d(P, v) = \rho < 2r$, and write $\beta := \beta(\rho)$.*

(a) *If v has zero unfixed neighbors, one unfixed neighbor, or two non-adjacent unfixed neighbors, then no such admissible P exists. If v has profile τ_2, τ_3, or τ_4 (unfixed neighbors at $0, \ldots, (k-1)\alpha$ for $k = 2, 3, 4$), then $\theta \in [\beta - \alpha,\ k\alpha - \beta]$.*

(b) *If v has profile τ_3 with unfixed neighbors a, i, b at $0, \alpha, 2\alpha$, then P always overlaps i; P overlaps a if and only if $\theta < \beta$, and P overlaps b if and only if $\theta > 2\alpha - \beta$. In particular, P overlaps at least one of a and b. If P overlaps only one of a and b, then $\beta \leq 3\alpha/2$, i.e., $\rho \geq \rho^*$.*

(c) *If v has profile τ_4 with unfixed neighbors a, i, j, b at $0, \alpha, 2\alpha, 3\alpha$, then P overlaps a, i, j, b according as $\theta < \beta$, $\theta < \alpha + \beta$, $\theta > 2\alpha - \beta$, $\theta > 3\alpha - \beta$, respectively. If P overlaps neither a nor b, then $\theta \in [\beta,\ 3\alpha - \beta]$ and P overlaps both i and j.*

(d) *The site v itself overlaps no neighbor of v. Consequently, if a site has at least two overlappers in a rearrangement, no circle of that rearrangement is centered at it.*

Proof. By Lemma 2.2, each fixed neighbor at angle $m\alpha$ excludes the open arc $(m\alpha - \beta,\ m\alpha + \beta)$. Since $\beta > \alpha$, the excluded arcs of consecutive fixed neighbors overlap.

(a) If all seven neighbors are fixed, the excluded arcs cover the full circle. If a single unfixed neighbor sits at 0 with fixed neighbors at $\pm\alpha$, the remaining arc is $[\beta - \alpha,\ \alpha - \beta] = \emptyset$ because $\beta > \alpha$; the same holds around each isolated unfixed neighbor when two are non-adjacent. For $k \in \{2, 3, 4\}$ consecutive unfixed neighbors at $0, \ldots, (k-1)\alpha$, the flanking fixed neighbors sit at $-\alpha$ and $k\alpha$, leaving precisely $[\beta - \alpha,\ k\alpha - \beta]$.

(b) For $\theta \in [\beta - \alpha,\ 3\alpha - \beta]$, we have $|\theta - \alpha| \leq 2\alpha - \beta < \beta$ (as $\beta > \alpha$), so P overlaps i by Lemma 2.2. Since $\beta < \pi/2 = 7\alpha/4 < 2\alpha$, any $\theta \in [\beta - \alpha, 3\alpha - \beta]$ satisfies $\theta > -\beta$ and $2\alpha - \theta > -\beta$, hence $|\theta| < \beta \iff \theta < \beta$ and $|2\alpha - \theta| < \beta \iff \theta > 2\alpha - \beta$. Because $2\alpha - \beta < \beta$, at least one inequality holds. If only one holds, then either $\theta \geq \beta$ or $\theta \leq 2\alpha - \beta$; combined with $\theta \in [\beta - \alpha, 3\alpha - \beta]$, this forces $\beta \leq 3\alpha - \beta$ (equivalently $\beta - \alpha \leq 2\alpha - \beta$), i.e., $\beta \leq 3\alpha/2$, hence $\rho \geq \rho^*$.

(c) For $\theta \in [\beta - \alpha,\ 4\alpha - \beta]$, the overlap conditions follow directly from Lemma 2.2 (noting $\beta < \pi/2 < 2\alpha$ and $4\alpha - \beta < \pi$). If P overlaps neither a nor b, then $\theta \in [\beta,\ 3\alpha - \beta]$; since $\beta > \alpha$, we have $2\alpha - \beta < \beta \leq \theta \leq 3\alpha - \beta < \alpha + \beta$, so P overlaps both i and j.

(d) Every neighbor of v lies at distance $2r$ from v, proving the first claim; and because distinct circles of a rearrangement have centers at distance at least $2r$, a circle centered at v excludes every other point of $B(v, 2r)$ from being a center of the rearrangement. □

Lemma 2.4 (Edge Lemma)**.** *Let v, v' be adjacent sites of $\{3, 7\}$ with a common neighbor o that is fixed.*

(i) *Every admissible point P overlapping both v and v' lies in the open half-plane bounded by the geodesic vv' not containing o.*
(ii) *Any two points P, P' overlapping both v and v' and lying in the same open half-plane of vv' satisfy $d(P, P') < 2r$.*

Consequently, at most one circle of a rearrangement overlaps both ends of such an edge (v, v').

Proof. (i) Since P overlaps v', $P \neq v$. Use polar coordinates (ρ, θ) about v with v' at angle 0 and o at $+\alpha$. Overlapping v' gives $|\theta| < \beta(\rho)$, while admissibility against o gives $|\theta - \alpha| \geq \beta(\rho)$. Since $\beta(\rho) \in (\alpha, \pi/2)$, this forces $-\pi/2 < -\beta(\rho) < \theta \leq \alpha - \beta(\rho) < 0$, so P lies strictly on the side of vv' opposite o.

(ii) The geodesic vv' supports the hyperbolic convex hull K of $\{v, v', P, P'\}$, so v and v' are vertices of K. If P' lies in the closed triangle $\triangle vv'P$, then by convexity of $d(P, \cdot)$ along geodesics, $d(P, P') \leq \max(d(P, v), d(P, v')) < 2r$; symmetrically if $P \in \triangle vv'P'$. Otherwise v, v', P, P' are the vertices of a convex quadrilateral with v, v' adjacent, say in cyclic order v, v', Q_1, Q_2 with $\{Q_1, Q_2\} = \{P, P'\}$. The diagonals $[v, Q_1]$ and $[v', Q_2]$ intersect at a point x, so by the triangle inequality,

$$\begin{aligned} 2r + d(P, P') = d(v, v') + d(Q_1, Q_2) &\leq d(v, x) + d(x, v') + d(Q_1, x) + d(x, Q_2) \\ &= d(v, Q_1) + d(v', Q_2) < 4r, \end{aligned}$$

giving $d(P, P') < 2r$. □

Lemma 2.5 (Reuleaux-Triangle Lemma)**.** *For any three mutually adjacent sites u, v, w of $\{3, 7\}$, any two points in the open intersection $B(u, 2r) \cap B(v, 2r) \cap B(w, 2r)$ lie at distance strictly less than $2r$.*

Proof. Let $\mathcal{R} := \overline{B}(u, 2r) \cap \overline{B}(v, 2r) \cap \overline{B}(w, 2r)$. Its boundary consists of three circular arcs: the arc γ_v of $\partial B(v, 2r)$ from u to w, and the symmetric arcs γ_u and γ_w. Pick any $x' \in \mathcal{R}$. If $x' = v$, then $d(x', y) = 2r$ for all $y \in \gamma_v$. Otherwise, because v lies on $\partial B(u, 2r) \cap \partial B(w, 2r)$ and hyperbolic discs are convex, the angle at v between x' and u (and between x' and w) is strictly less than $\pi/2$. In polar angles about v with u at 0 and w at α, the ray vx' therefore lies in $(\alpha - \pi/2, \pi/2)$, while γ_v spans $[0, \alpha]$; in particular, no point of γ_v lies opposite to x' as seen from v. By the hyperbolic law of cosines at v, the distance $d(x', y)$ for $y \in \gamma_v$ increases strictly with the angular separation from x' and thus attains its maximum at an endpoint u or w, where $d(x', u), d(x', w) \leq 2r$. The same holds on γ_u and γ_w. Since $d(x', \cdot)$ is convex along geodesics, its maximum on the convex set $\mathcal{R}$ is attained on $\partial\mathcal{R}$, so $\operatorname{diam} \mathcal{R} \leq 2r$. Finally, for any two distinct points x, y in the open set $B(u, 2r) \cap B(v, 2r) \cap B(w, 2r)$, extending the geodesic segment xy in both directions to $\partial\mathcal{R}$ gives boundary points $x', y' \in \mathcal{R}$ with $d(x, y) < d(x', y') \leq 2r$. □

3. The exact $p = 7$ identity and tip geometry

We now prove the exact algebraic coincidence that distinguishes $\{7, 3\}$ from $\{p, 3\}$ for $p \geq 8$, and extract the sharp tip-disc and strict half-lens bounds.

Proposition 3.1 (Exact $p = 7$ identity)**.** *In $\{7, 3\}$, the altitude h, inradius r, and critical radius ρ^* satisfy $h = r + \rho^*$.*

Proof. Set $\lambda := 2\cos(\pi/7) > 0$. Because $\zeta := e^{i\pi/7}$ satisfies $\zeta^6 - \zeta^5 + \zeta^4 - \zeta^3 + \zeta^2 - \zeta + 1 = 0$ and $\lambda = \zeta + \zeta^{-1}$, we have the cubic relation

$$\lambda^3 - \lambda^2 - 2\lambda + 1 = 0, \tag{5}$$

together with $2\cos(2\pi/7) = \lambda^2 - 2$ and $2\cos(3\pi/7) = \lambda^3 - 3\lambda = \lambda^2 - \lambda - 1$. Since $4\sin^2(\pi/7) = 4 - \lambda^2$, equation (1) yields

$$\cosh r = \frac{1}{\sqrt{4 - \lambda^2}}, \qquad \sinh r = \frac{\sqrt{\lambda^2 - 3}}{\sqrt{4 - \lambda^2}}, \qquad \cosh 2r = \frac{\lambda^2 - 2}{4 - \lambda^2}, \qquad \sinh 2r = \frac{2\sqrt{\lambda^2 - 3}}{4 - \lambda^2}.$$

Hence $\cosh h = \cosh(2r)/\cosh r = (\lambda^2-2)/\sqrt{4-\lambda^2}$, and from $(\lambda^2-2)^2-(4-\lambda^2) = \lambda^2(\lambda^2-3)$ we obtain $\sinh h = \lambda\sqrt{\lambda^2-3}/\sqrt{4-\lambda^2} = \lambda \sinh r$. Using (5) ($\lambda^3-\lambda^2 = 2\lambda-1$),

$$\cosh(h-r) = \cosh h\cosh r - \sinh h \sinh r = \frac{(\lambda^2-2)-\lambda(\lambda^2-3)}{4-\lambda^2} = \frac{\lambda-1}{4-\lambda^2},$$
$$\sinh(h-r) = \sinh h\cosh r - \cosh h \sinh r = \frac{(\lambda-(\lambda^2-2))\sqrt{\lambda^2-3}}{4-\lambda^2} = \frac{(2+\lambda-\lambda^2)\sqrt{\lambda^2-3}}{4-\lambda^2}.$$

Since $\cosh(h-r)+1 = (3+\lambda-\lambda^2)/(4-\lambda^2)$, the half-angle identity $\tanh(x/2) = \sinh x/(\cosh x+1)$ gives

$$\tanh\frac{h-r}{2} = \frac{(2+\lambda-\lambda^2)\sqrt{\lambda^2-3}}{3+\lambda-\lambda^2},$$
$$\tanh\frac{\rho^*}{2} = \tanh(2r)\cos\frac{3\pi}{7} = \frac{(\lambda^2-\lambda-1)\sqrt{\lambda^2-3}}{\lambda^2-2}.$$

Cross-multiplying the rational factors gives

$$(2+\lambda-\lambda^2)(\lambda^2-2)-(3+\lambda-\lambda^2)(\lambda^2-\lambda-1) = -(\lambda^3-\lambda^2-2\lambda+1) = 0$$

by (5). Thus $\tanh((h-r)/2) = \tanh(\rho^*/2)$, and injectivity of tanh yields $h-r = \rho^*$, i.e., $h = r+\rho^*$. □

3.1. **Tip points and the synthetic Tip-Disc Lemma.** Let v be a site with four consecutive neighbors n_0, n_1, n_2, n_3 at polar angles $0, \alpha, 2\alpha, 3\alpha$ about v. Let $M := \mathrm{mid}(n_1, n_2)$ be the midpoint of the edge n_1n_2, which sits at polar coordinates $(h,\, 3\alpha/2)$ about v. Define the *tip point*

$$\mathrm{tip}(v;\, n_1, n_2) := (\rho^*,\, 3\alpha/2)$$

in polar coordinates about v, i.e., the point at distance ρ^* from v along the altitude ray vM (Figure 4). By Proposition 3.1, $d(\mathrm{tip}(v; n_1, n_2), M) = h-\rho^* = r$. By Lemma 2.2 with $\beta(\rho^*) = 3\alpha/2$, we have $d(\mathrm{tip}(v; n_1, n_2), n_0) = d(\mathrm{tip}(v; n_1, n_2), n_3) = 2r$. Finally, since $\triangle\, \mathrm{tip}(v; n_1, n_2)\, M\, n_1$ has a right angle at M with both legs of length r, the hyperbolic Pythagorean theorem gives

$$d(\mathrm{tip}(v; n_1, n_2),\, n_1) = d(\mathrm{tip}(v; n_1, n_2),\, n_2) = \mathrm{arccosh}(\cosh^2 r) < 2r \tag{6}$$

(since $\cosh^2 r < 2\cosh^2 r - 1 = \cosh 2r$).

Lemma 3.2 (Tip-Disc Lemma)**.** *Let v have consecutive neighbors n_0, n_1, n_2, n_3 at angles $0, \alpha, 2\alpha, 3\alpha$ and set $M := \mathrm{mid}(n_1, n_2)$ and $T := \mathrm{tip}(v; n_1, n_2)$. Define the polar region*

$$K_2 := \{(\rho, \theta) : 0 < \rho < 2r,\ \theta \in [\beta(\rho),\, 3\alpha - \beta(\rho)]\}$$

about v. Then every point $P \in K_2$ satisfies $d(P, M) \le r$, with equality if and only if $P = T$ (Figure 4). Symmetrically (by reflection across the ray vn_1, or rotation by $-\alpha$), if $0 < \rho < 2r$ and $\theta \in [\beta(\rho)-\alpha,\, 2\alpha-\beta(\rho)]$, then $d(P, \mathrm{mid}(n_0, n_1)) \le r$, with equality if and only if $P = \mathrm{tip}(v; n_0, n_1)$.

Proof. Since $\beta(\rho) > 3\alpha/2$ for $\rho < \rho^*$, the slice of K_2 at radius ρ is empty for $\rho < \rho^*$ and consists of the single point $T = (\rho^*, 3\alpha/2)$ at $\rho = \rho^*$, where $d(T, M) = r$. Fix $\rho \in (\rho^*, 2r)$. By the hyperbolic law of cosines in $\triangle vPM$, the distance $d(P, M)$ is an even, strictly increasing function of $|\theta - 3\alpha/2| \in [0, \pi]$. Hence its maximum over $\theta \in [\beta(\rho),\, 3\alpha - \beta(\rho)]$ is attained at the endpoints, and by symmetry across the line vM it suffices to show that $d(P(\rho), M) < r$ for $P(\rho) := (\rho, \beta(\rho))$ and all $\rho \in (\rho^*, 2r)$.

By Lemma 2.2 (equality case for $0 < \rho \le 2r$), every point $P(\rho)$ lies on $\Gamma := \partial B(n_0, 2r)$, with $P(0) := v$ and $P(2r) = n_1$ also on Γ. Measure the signed central angle $\phi(X) \in (-\pi, \pi]$ of $X \in \Gamma$ about n_0 from the ray $\overrightarrow{n_0 v}$, positive on the n_1-side. Because $0 < \beta(\rho) < \pi/2$, $P(\rho)$ lies on the n_1-side for $\rho \in (0, 2r]$, and the isosceles triangle $\triangle v n_0 P(\rho)$ (legs $2r$, base ρ) satisfies $\sin(\phi(P(\rho))/2) = \sinh(\rho/2)/\sinh(2r)$. Thus $\phi(P(\rho))$ increases strictly from $\phi(v) = 0$ through $\phi(T) \in (0, \alpha)$ to $\phi(n_1) = \alpha = 2\pi/7 < \pi$. In particular, $\Gamma \setminus \{T, n_1\}$ splits into the open arc

$\gamma_1 := \{P(\rho) : \rho \in (\rho^*, 2r)\} = \{X \in \Gamma : \phi(T) < \phi(X) < \alpha\}$ and the complementary open arc γ_2, which contains v ($\phi(v) = 0$).

Now consider $\partial B(M, r)$. Note that $n_0 \neq M$ because $d_0 := d(n_0, M) \geq d(n_0, n_1) - d(n_1, M) = r > 0$. For $X \in \Gamma$ at signed angle $\psi \in (-\pi, \pi]$ about n_0 from the ray $\overrightarrow{n_0 M}$, the hyperbolic law of cosines gives

$$\cosh d(X, M) = \cosh(2r) \cosh d_0 - \sinh(2r) \sinh d_0 \cos \psi.$$

Both $T = P(\rho^*)$ and $n_1 = P(2r)$ lie on $\Gamma \cap \partial B(M, r)$ (as $d(T, M) = d(n_1, M) = r$) and are distinct ($\rho^* < 2r$), so they occur at $\psi = \pm\psi_0$ for a unique $\psi_0 \in (0, \pi)$. Consequently, the two open arcs of $\Gamma \setminus \{T, n_1\}$ are precisely $\{X \in \Gamma : \cos\psi > \cos\psi_0\} = \Gamma \cap B(M, r)$ and $\{X \in \Gamma : \cos\psi < \cos\psi_0\} = \Gamma \setminus \overline{B}(M, r)$. Since $v \in \gamma_2$ satisfies $d(v, M) = h = r + \rho^* > r$, we have $\gamma_2 = \Gamma \setminus \overline{B}(M, r)$ and therefore $\gamma_1 = \Gamma \cap B(M, r)$. That is, $d(P(\rho), M) < r$ for all $\rho \in (\rho^*, 2r)$. □

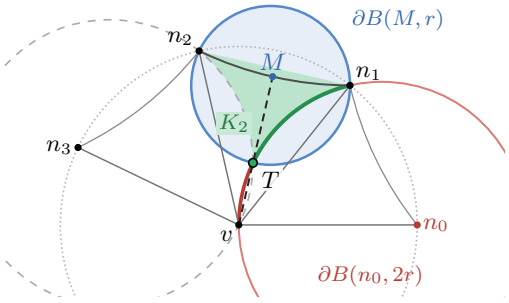

FIGURE 4. Exact Poincaré-disc geometry about a site v at the origin. The boundary circles $\partial B(n_0, 2r)$ (red) and $\partial B(M, r)$ (blue) intersect transversely at $T = \mathrm{tip}(v; n_1, n_2)$ and at n_1. Because v lies on the complementary arc of $\partial B(n_0, 2r)$ (thick red) at distance $h = r + \rho^* > r$ from M, the open arc $\{P(\rho) : \rho \in (\rho^*, 2r)\}$ (thick green) lies strictly inside $B(M, r)$, forcing K_2 (shaded green) into $\overline{B}(M, r)$ with unique boundary contact at T.

Lemma 3.3 (Strict Half-Lens Bound)**.** *Suppose a site v has profile τ_3 with unfixed neighbors a, i, b at $0, \alpha, 2\alpha$ and set $T := \mathrm{tip}(v; i, b)$. Then every admissible point Y overlapping both a and v satisfies $d(Y, T) < 2r$.*

Proof. Since v has profile τ_3, the common neighbor of a and v at angle $-\alpha$ is fixed. By Lemma 2.4(i), Y lies in the open half-lens $H := B(a, 2r) \cap B(v, 2r) \cap H^+$, where H^+ is the open half-plane of av containing i. The closure $\mathrm{cl}(H)$ is convex, bounded by the geodesic segment $[a, v]$, the arc γ_v of $\partial B(v, 2r)$ from a to i, and the arc γ_a of $\partial B(a, 2r)$ from v to i. We evaluate $d(T, \cdot)$ on each boundary piece:

- On $[a, v]$: $d(T, a) = 2r$ and $d(T, v) = \rho^* < 2r$; by convexity of $d(T, \cdot)$ along geodesics, $d(T, x) < 2r$ for every $x \in [a, v] \setminus \{a\}$.
- On γ_v: points lie at polar coordinates $(2r, \varphi)$ about v with $\varphi \in [0, \alpha]$, while T is at $(\rho^*, 3\alpha/2)$. Since $3\alpha/2 - \varphi \in [\alpha/2, 3\alpha/2] \subset (0, \pi)$, the distance $d(T, x)$ increases strictly as φ decreases, reaching $d(T, a) = 2r$ only at $\varphi = 0$ and strictly less elsewhere.
- On γ_a: T lies at distance $2r$ from a, while v and i also lie on $\partial B(a, 2r)$. In the isosceles triangles $\triangle aTv$ (legs $2r$, base $\rho^* < 2r$) and $\triangle aTi$ (legs $2r$, base $d(T, i) < 2r$ by (6)), the apex angles at a are strictly less than $\alpha < \pi/2$ (as each base is $< 2r$, the side of an equilateral triangle of angle α). Hence no point of γ_a lies opposite to T as seen from a; by the law of cosines at a, $d(T, x) \leq \max(d(T, v), d(T, i)) < 2r$ for all $x \in \gamma_a$.

Thus $\mathrm{cl}(H) \subset \overline{B}(T, 2r)$ and touches $\partial B(T, 2r)$ only at $a \notin H$. Extending any geodesic through $Y \in H$ to two boundary points of $\mathrm{cl}(H)$ (which cannot both equal a) and applying convexity of $d(T, \cdot)$ gives $d(Y, T) < 2r$. □

4. Good sets and the joining lemmas

We begin with the bridge and deletion remarks of Bezdek [2, p. 204], made completely explicit for $\{7, 3\}$.

Lemma 4.1 (Bridge and Deletion Remarks)**.** *Let $s \subset \mathcal{S}$ be a finite good set and $c \in \mathcal{S} \setminus s$.*

(a) *In any non-trivial rearrangement X of a finite subset $s' \subset \mathcal{S}$, at least two circles of X are not centered at sites of $\{3, 7\}$. In particular, every singleton $\{c\} \subset \mathcal{S}$ is good.*
(b) *If c cannot be joined to s, then in every non-trivial rearrangement X of $s \cup \{c\}$, at least two circles of X overlap c.*
(c) *Every subset of a good set is good. Call a site $q \in s \cup \{c\}$* qualifying *(for joining c to s) if $(s \cup \{c\}) \setminus \{q\}$ is good. If c cannot be joined to s, then in every non-trivial rearrangement of $s \cup \{c\}$, every qualifying site q has at least two overlappers and carries no circle centered at q.*

Proof. (a) Suppose for contradiction that all circles of X except at most one circle Y are centered at sites. Then all sites of $\{3, 7\}$ are occupied by either fixed circles of $\mathcal{S} \setminus s'$ or inscribed circles of X, except for exactly two empty sites (one from the vacancy c_0 and one from the remaining circle Y). Since the covering radius of $\{3, 7\}$ satisfies $R_c < h < 2r$ by (2), the center of Y lies at distance at most $R_c < 2r$ from some site F. Because Y overlaps no occupied circle, F must be one of the two empty sites. Treating all occupied sites as fixed, F has at most one unoccupied neighbor (the other empty site). By Lemma 2.3(a), no point $P \neq F$ with $d(P, F) < 2r$ avoids overlapping all occupied neighbors of F; hence Y is centered at F, and X is trivial by Lemma 2.1, a contradiction. For $s' = \{c\}$, $|X| = 1 < 2$, so $\{c\}$ is good.

(b) Suppose X is a non-trivial rearrangement of $s \cup \{c\}$. If no circle of X overlaps c, then holding c fixed, any $|s|$ circles of X form a rearrangement of s; since s is good, all $|s|$ of them are centered at sites (Lemma 2.1). Applying this to two different subsets of size $|s|$ (possible when $|s| \geq 1$; the case $s = \emptyset$ is covered by (a)) shows every circle of X is centered at a site, contradicting non-triviality. If exactly one circle $Y \in X$ overlaps c, then $X \setminus \{Y\}$ (with c held fixed) is a rearrangement of s, hence every circle of $X \setminus \{Y\}$ is centered at a site; then Y is the only circle of X possibly not centered at a site, contradicting (a).

(c) Any non-trivial rearrangement of $s' \subset s$, together with the circles of $s \setminus s'$ held fixed, is a non-trivial rearrangement of s. Applying (b) to $s' := (s \cup \{c\}) \setminus \{q\}$ with q in the role of c shows that q has at least two overlappers; by Lemma 2.3(d), no circle is centered at q. □

Remark 4.2 (τ_3 Edge-Count). Suppose a qualifying site q has profile τ_3 with consecutive unfixed neighbors a, i, b in a non-trivial rearrangement X. Since the common neighbor of a, q other than i is fixed, and likewise for q, b, Lemma 2.4 shows that each edge (a, q) and (q, b) carries at most one overlapper of both its endpoints. By Lemma 2.3(b), every overlapper of q overlaps a or b, while by Lemma 4.1(c), q has at least two overlappers. If a single overlapper overlapped both a and b, then by Lemma 2.4 no other circle could overlap (a, q) or (q, b), leaving q with at most one overlapper, a contradiction. Hence q has *exactly two* overlappers: one overlapping (a, q) but not b, and the other overlapping (q, b) but not a.

We now establish the analogues of Bezdek's Lemmas 1 and 2 at $p = 7$, followed by our unified nine-site closing lemma.

Lemma 4.3 (Joining with at most two unfixed neighbors)**.** *Let $s \subset \mathcal{S}$ be good and $c \in \mathcal{S} \setminus s$. If c has at most two unfixed neighbors (relative to $s \cup \{c\}$), then c can be joined to s.*

Proof. Suppose c cannot be joined to s, and take a non-trivial rearrangement X of $s \cup \{c\}$. By Lemma 4.1(b) and Lemma 2.3(d), c has at least two overlappers in X, none centered at c. If c has zero unfixed neighbors, one unfixed neighbor, or two non-adjacent unfixed neighbors, Lemma 2.3(a) rules out any overlapper. If c has two adjacent unfixed neighbors a, i (profile τ_2 at angles $0, \alpha$), then by Lemma 2.3(a) every overlapper of c has $\theta \in [\beta - \alpha,\ 2\alpha - \beta] \subset (-\beta, \beta)$ (since $\beta > \alpha$), hence overlaps a by Lemma 2.2. Because the neighbor of c at $-\alpha$ is a fixed common neighbor of c and a, Lemma 2.4 allows at most one overlapper of (c, a), a contradiction. □

Lemma 4.4 (Profile-τ_3 joining lemma)**.** *Let $s \subset \mathcal{S}$ be good and $c \in \mathcal{S} \setminus s$. Suppose c has profile τ_3 with consecutive unfixed neighbors c_1, c_2, c_3 (at $0, \alpha, 2\alpha$) such that $c_1, c_3 \in s$ (so $c_1, c_3 \neq c_0$). Let c_4 (respectively c_5) be the other common neighbor of c_1, c_2 (respectively c_2, c_3). If all unfixed neighbors of c_1 lie in $\{c_4, c_2, c\}$ and all unfixed neighbors of c_3 lie in $\{c, c_2, c_5\}$, then c can be joined to s.*

Proof. Suppose c cannot be joined to s, and take a non-trivial rearrangement X of $s \cup \{c\}$. Since $c_1, c_3 \in s$, the sets $s \setminus \{c_1\}$ and $s \setminus \{c_3\}$ are good (Lemma 4.1(c)), and relative to $(s \setminus \{c_1\}) \cup \{c\}$ the site c has at most two unfixed neighbors (c_2, c_3), so c can be joined to $s \setminus \{c_1\}$ by Lemma 4.3; thus c_1 qualifies, and symmetrically c_3 qualifies. If c_4 were fixed, then c_1 would have at most two unfixed neighbors and hence at most one overlapper by the proof of Lemma 4.3, contradicting Lemma 4.1(c). Thus c_4 (and symmetrically c_5) is unfixed, so c, c_1, c_3 all have profile τ_3.

Applying Remark 4.2 at c, c_1, and c_3, we obtain overlappers:

- at c: Z on (c_1, c) only and W on (c, c_3) only;
- at c_1: Y_1 on (c_4, c_1) only and Z (the unique circle on (c_1, c)), so Z does not overlap c_4;
- at c_3: W (the unique circle on (c, c_3)) and Y_2 on (c_3, c_5) only, so W does not overlap c_5.

Set $M := \mathrm{mid}(c_2, c)$. In polar coordinates about c_1 (with c_4, c_2, c at $0, \alpha, 2\alpha$), Z overlaps c but not c_4, so $\theta \in [\beta,\ 3\alpha - \beta]$ by Lemma 2.3(a)–(b); Lemma 3.2 gives $d(Z, M) \leq r$, with equality only at $T := \mathrm{tip}(c_1; c_2, c)$. In polar coordinates about c_3 (with c, c_2, c_5 at $0, \alpha, 2\alpha$), W overlaps c but not c_5, so $\theta \in [\beta - \alpha,\ 2\alpha - \beta]$; Lemma 3.2 (mirror form) gives $d(W, M) \leq r$. Since $Z \neq W$,

$$2r \leq d(Z, W) \leq d(Z, M) + d(M, W) \leq 2r,$$

forcing $d(Z, M) = r$ and hence $Z = T$. Now Y_1 is admissible and overlaps both c_4 and c_1; since c_1 has profile τ_3 with neighbors c_4, c_2, c, Lemma 3.3 yields $d(Y_1, Z) = d(Y_1, T) < 2r$. Yet $Y_1 \neq Z$ (as Y_1 overlaps c_4 and Z does not), contradicting that X is a packing. □

Note that in Lemma 4.4, c_2 may be the vacancy c_0 (used to close zone Z_1). We now prove the single nine-site closing lemma that resolves the $p = 7$ combinatorial bottleneck where c_3 has profile τ_4 (four unfixed neighbors).

Lemma 4.5 (Nine-site closing lemma)**.** *Let $s \subset \mathcal{S}$ be good and $c \in \mathcal{S} \setminus s$. Let $c, c_1, \ldots, c_8$ be nine distinct sites of $\{3, 7\}$ such that c_4, c_1, c, c_3, c_5 are five consecutive neighbors of c_2 in cyclic order, c_2, c_3, c_6, c_7, c_8 are five consecutive neighbors of c_5 in cyclic order, and consecutive vertices along the path $c_4, c_1, c, c_3, c_6, c_7, c_8$ are adjacent (see Figure 5). Assume $c_1, c_3, c_6, c_7 \in s$ (so none is c_0), that c_2 and c_5 are unfixed relative to $s \cup \{c\}$, and that relative to $s \cup \{c\}$ the unfixed neighbors of c, c_1, c_3, c_6, and c_7 lie in $\{c_1, c_2, c_3\}$, $\{c_4, c_2, c\}$, $\{c, c_2, c_5, c_6\}$, $\{c_3, c_5, c_7\}$, and $\{c_6, c_5, c_8\}$, respectively. Then c can be joined to s.*

Proof. Suppose c cannot be joined to s, and take a non-trivial rearrangement X of $s \cup \{c\}$. We first list qualifying sites (all belonging to s, using Lemma 4.1(c)):

- c_1 and c_3 qualify by Lemma 4.3 (removing either leaves c with at most two unfixed neighbors).
- c_6 qualifies: in $(s \setminus \{c_6\}) \cup \{c\}$, c has consecutive unfixed neighbors c_1, c_2, c_3 and the unfixed neighbors of c_3 lie in $\{c, c_2, c_5\}$, so c can be joined to $s \setminus \{c_6\}$ by Lemma 4.4.
- c_7 qualifies: first join c to $s \setminus \{c_6, c_7\}$ by Lemma 4.4 (as above), then join c_6 by Lemma 4.3 (its only remaining unfixed neighbors are c_3 and c_5).

If c_4 (respectively c_8) were fixed, then c_1 (respectively c_7) would have at most two unfixed neighbors and hence at most one overlapper by the proof of Lemma 4.3, contradicting Lemma 4.1(c). Thus c_4, c_8 are unfixed, so c, c_1, c_6, c_7 all have profile τ_3, and c_3 has profile τ_4 (with consecutive unfixed neighbors c, c_2, c_5, c_6 at $0, \alpha, 2\alpha, 3\alpha$).

Step 1 (Edge counts). By Remark 4.2 at c_1, c, c_6, c_7, there exist overlappers Y_1 on (c_4, c_1) only, Z on (c_1, c) only, W on (c, c_3) only, E on (c_3, c_6) only, V_1 on (c_6, c_7) only, and V_2 on (c_7, c_8) only. (Equality $W = E$ is not yet ruled out.)

Step 2 (W overlaps c_5). Suppose not. In polar coordinates about c_3 (c, c_2, c_5, c_6 at $0, \alpha, 2\alpha, 3\alpha$), W overlaps c ($\theta < \beta$) and not c_5 ($\theta \le 2\alpha - \beta$), while $\theta \in [\beta - \alpha,\ 4\alpha - \beta]$ by Lemma 2.3(a). Thus $\theta \in [\beta - \alpha,\ 2\alpha - \beta]$, so Lemma 3.2 (mirror form) gives $d(W, M_1) \le r$ where $M_1 := \mathrm{mid}(c, c_2)$. About c_1 (c_4, c_2, c at $0, \alpha, 2\alpha$), Z overlaps c and not c_4, so Lemma 3.2 gives $d(Z, M_1) \le r$ with equality only at $T_1 := \mathrm{tip}(c_1; c_2, c)$. Since $Z \ne W$, $d(Z, W) \ge 2r$ forces $Z = T_1$; Lemma 3.3 at c_1 then gives $d(Y_1, Z) < 2r$ with $Y_1 \ne Z$, a contradiction.

Step 3 (E overlaps c_2). Suppose not. About c_3, E overlaps c_6 ($\theta > 3\alpha - \beta$) and not c_2 ($\theta \ge \alpha + \beta$), so $\theta \in [\alpha + \beta,\ 4\alpha - \beta]$; shifting angles by $-\alpha$, Lemma 3.2 gives $d(E, M_2) \le r$ where $M_2 := \mathrm{mid}(c_5, c_6)$. About c_7 (c_8, c_5, c_6 at $0, \alpha, 2\alpha$), V_1 overlaps c_6 and not c_8, so Lemma 3.2 gives $d(V_1, M_2) \le r$ with equality only at $T_2 := \mathrm{tip}(c_7; c_5, c_6)$. Since $V_1 \ne E$ (as V_1 overlaps c_7 and E does not), $d(V_1, E) \ge 2r$ forces $V_1 = T_2$; Lemma 3.3 at c_7 gives $d(V_2, V_1) < 2r$ with $V_2 \ne V_1$, a contradiction.

Step 4 ($W = E$). By Lemma 2.3(b) at c and Step 2, W overlaps c_2, c_3, c_5. By Lemma 2.3(b) at c_6 and Step 3, E overlaps c_2, c_3, c_5. Since c_2, c_3, c_5 are mutually adjacent, Lemma 2.5 gives $d(W, E) < 2r$ unless $W = E$; hence $W = E$.

Step 5 (Contradiction at c_3). Because c_3 qualifies, it has a second overlapper $G \ne W$ in X. Since $W = E$ is the unique overlapper on (c, c_3) and also on (c_3, c_6), G overlaps neither c nor c_6. By Lemma 2.3(c) at the τ_4 site c_3, G overlaps both c_2 and c_5. Thus G and W both lie in $B(c_2, 2r) \cap B(c_3, 2r) \cap B(c_5, 2r)$, so $d(G, W) < 2r$ by Lemma 2.5, a contradiction. □

5. Zone structure, proof of Theorem 1.1, and proof of Corollary 1.2

Following Bezdek [2, p. 206], define the m-th *zone* Z_m of c_0 inductively by $Z_0 := \{c_0\}$ and $Z_m := \{x \in \mathcal{S} : d_{\{3,7\}}(x, c_0) = m\}$ (the sites at graph distance m from c_0 in $\{3, 7\}$), and set $U_n := Z_1 \cup \cdots \cup Z_n$.

Lemma 5.1 (Zone Substitution Lemma)**.** *For every $m \ge 1$, the zone Z_m forms a simple cycle in $\{3, 7\}$ (the boundary of a convex polygon P_m formed by all triangles of $\{3, 7\}$ whose vertices lie in $Z_0 \cup \cdots \cup Z_m$). Every site $v \in Z_m$ has $i \in \{1, 2\}$ consecutive neighbors in Z_{m-1} (called* type 1 *or* type 2*), two ring neighbors in Z_m, and $5 - i$ consecutive neighbors in Z_{m+1} whose two endpoints are shared with the two ring neighbors of v. Writing w_m for the cyclic word of types (1 or 2) along Z_m, we have $w_1 = 1^7$ and $w_{m+1} = \sigma(w_m)$, where $\sigma(1) = 1\,1\,2$ and $\sigma(2) = 1\,2$.*

Consequently:

(a) *For $m \ge 2$, no two 2's are adjacent in w_m, and maximal runs of 1's have length 1 (a* single S*, arising from a 2 in Z_{m-1}) or length 2 (a* double D*, arising from a 1 in Z_{m-1}).*

(b) *$w_2 = (1\,1\,2)^7$ (all doubles) and $w_3 = (1\,1\,2\,1\,1\,2\,1\,2)^7$ (every arc between consecutive singles has two doubles,* S D D S*).*

(c) *For every $m \ge 4$, the cyclic sequence of arcs between consecutive singles in Z_m corresponds to the letters of w_{m-2}: each 1 gives an* S D D S *arc and each 2 gives an* S D S *arc. In particular, no two* S D S *arcs are adjacent in Z_m.*

Proof. Let V_1 be the union of the seven triangles of $\{3, 7\}$ meeting at c_0, and let V_{m+1} be obtained from V_m by adding all triangles of $\{3, 7\}$ sharing at least one vertex with V_m. By induction on m, we show that V_m is a simple polygon P_m whose boundary vertices are the sites of Z_m in cyclic order, with each vertex $v \in Z_m$ lying in $i + 1 \in \{2, 3\}$ triangles of V_m (corresponding to $i \in \{1, 2\}$ consecutive neighbors in Z_{m-1}). For $m = 1$, V_1 is a regular heptagon of interior

angle $2\alpha = 4\pi/7 < \pi$, and $w_1 = 1^7$. Given the claim for m, every interior angle of P_m equals $(i+1)\alpha \in \{4\pi/7, 6\pi/7\}$, which is strictly less than π; hence P_m is locally convex, and therefore a convex simple closed polygon in $\mathbb{H}^2$.

Around each boundary vertex $v \in Z_m$ with $i \in \{1,2\}$ inner neighbors, the cyclic rotation of seven neighbors in $\{3,7\}$ reads: left ring neighbor u, the i inner neighbors in Z_{m-1}, right ring neighbor u', and (by convexity of P_m) $5-i$ outer neighbors outside V_m. Adding the $6-i$ triangles outside P_m incident to v forms a fan whose first and last outer vertices are the third vertices of the outer triangles on vu' and vu, shared with u' and u (hence having at least two neighbors in Z_m). No outer vertex $w \notin V_m$ can be adjacent to two non-consecutive vertices $u_0, u_k \in Z_m$ ($k \geq 2$). Indeed, otherwise the simple $(k+2)$-gon Q bounded by $[w,u_0]$, $[w,u_k]$, and the chain $u_0 - u_1 - \cdots - u_k$ of ∂P_m (enclosing a region exterior to P_m) would have interior angle $(7-(i_j+1))\alpha \geq 4\alpha = 8\pi/7$ at each u_j ($1 \leq j \leq k-1$, where $i_j \in \{1,2\}$ is the inner-neighbor count of u_j) and at least $\alpha = 2\pi/7$ at u_0, u_k, w. Its angle sum would then be at least $(k-1)\frac{8\pi}{7} + 3 \cdot \frac{2\pi}{7} = k\pi + \frac{(k-2)\pi}{7} \geq k\pi$, contradicting the hyperbolic angle-sum bound $< k\pi$ for a $(k+2)$-gon. Hence consecutive fans share only their common outer triangle on the intervening boundary edge of P_m, and non-consecutive fans share no outer vertex (meeting at most at an intervening vertex of Z_m). Consequently, the $3-i$ interior outer vertices of each fan see only v in Z_m (type 1 in Z_{m+1}), the shared end vertices see exactly two neighbors in Z_m (type 2 in Z_{m+1}), and P_{m+1} is again a simple closed polygon whose boundary is Z_{m+1}. Concatenating fans around Z_m gives $\sigma(1) = 112$ and $\sigma(2) = 12$.

Parts (a)–(c) follow immediately from $\sigma(1) = 112$ and $\sigma(2) = 12$: every 2 is followed by at least one 1, every run of 1's has length 1 or 2 according to its parent in w_{m-1}, and the number of doubles between two singles in w_m equals the run length of 1's between their parent 2's in w_{m-1}, which is 1 or 2 according to the grandparent letter (2 or 1) in w_{m-2}. Since w_{m-2} has no adjacent 2's for $m \geq 4$, no two S D S arcs are adjacent in Z_m. □

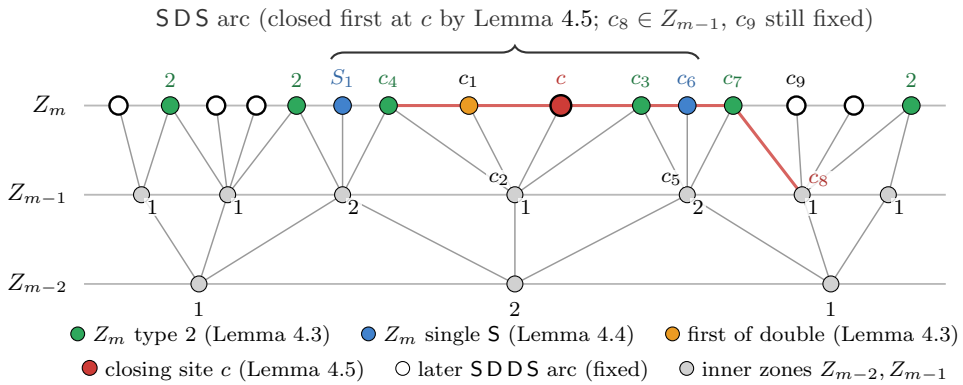

FIGURE 5. Combinatorial structure of three consecutive zones Z_{m-2}, Z_{m-1}, Z_m of $\{3,7\}$ under $\sigma(1) = 112$, $\sigma(2) = 12$ (in the style of Bezdek [2, Fig. 3]). Each 2 in Z_{m-2} produces an isolated S D S arc in Z_m flanked by S D D S arcs. After joining all type-2 vertices in Z_m (green) by Lemma 4.3, every S D S arc is closed first by Lemma 4.5 along the red chain $c_4 - c_1 - c - c_3 - c_6 - c_7 - c_8$ (with $c_8 \in Z_{m-1}$ unfixed and $c_9 \in Z_m$ still fixed); every S D D S arc is then closed by the same Lemma 4.5.

Proof of Theorem 1.1. Every finite subset $s \subset \mathcal{S}$ is contained in $U_N = Z_1 \cup \cdots \cup Z_N$ for some $N \geq 1$. By Lemma 4.1(c), it suffices to prove by induction on $m \geq 1$ that U_m is good.

Zone Z_1. Start with one circle of Z_1, which is good by Lemma 4.1(a). Join the next five circles of Z_1 in cyclic order by Lemma 4.3 (each has two unfixed neighbors: c_0 and one ring neighbor). Join the seventh and last circle c of Z_1 by Lemma 4.4 (with $c_2 = c_0$ and $c_1, c_3 \in Z_1$ its two ring

neighbors; c_1, c_3 lie in the current good set and have unfixed neighbors only in $Z_1 \cup \{c_0\}$). Hence $U_1 = Z_1$ is good.

Zone Z_2 (given U_1 good). By Lemma 5.1(b), $w_2 = (1\,1\,2)^7$. First join all seven type-2 sites of Z_2 by Lemma 4.3 (each has only its two neighbors in Z_1 unfixed). Next, traverse the seven doubles in cyclic order: in each of the first six doubles, join the first site by Lemma 4.3 (one inner neighbor and one adjacent type-2 neighbor unfixed) and the second site by Lemma 4.4 (the type-2 site beyond it still has its other ring neighbor in Z_2, as well as all Z_3 neighbors, fixed). For the seventh and final double, join its first site by Lemma 4.3 and its second site c by Lemma 4.5 (taking $c_4, c_1, c, c_3, c_6, c_7, c_8$ along Z_2 across the seventh double, the intervening type-2 site c_3, and the first double, with $c_2, c_5 \in Z_1$ their two inner neighbors; all nine sites are distinct and $c_1, c_3, c_6, c_7 \in Z_2$ lie in the current good set). Thus U_2 is good.

Zone Z_m for $m \geq 3$ (given U_{m-1} good). First join all type-2 sites of Z_m by Lemma 4.3 (each has only its two inner neighbors in Z_{m-1} unfixed). By Lemma 5.1(b)–(c), the remaining sites of Z_m partition into arcs between singles of types $\mathsf{S\,D\,S}$ (present only for $m \geq 4$, and never adjacent) and $\mathsf{S\,D\,D\,S}$.

(1) *Close every $\mathsf{S\,D\,S}$ arc first* (for $m \geq 4$). Write such an arc as $S_1 - t_1 - d_1 - d_2 - t_2 - S_2$, let $t_3 \in Z_m$ be the type-2 neighbor of S_2 beyond it, and let $c_9 \in Z_m$ be the other ring neighbor of t_3 in Z_m (the first double-site of the adjacent $\mathsf{S\,D\,D\,S}$ arc; Figure 5). Because no two $\mathsf{S\,D\,S}$ arcs are adjacent, c_9 is still fixed (and likewise on the S_1 side). Join S_1 and S_2 by Lemma 4.4 (both of each single's type-2 neighbors still have their other ring neighbors in Z_m, as well as all Z_{m+1} neighbors, fixed), join d_1 by Lemma 4.3, and finally close $c := d_2$ by Lemma 4.5 with $c_4 = t_1$, $c_1 = d_1$, $c_3 = t_2$, $c_6 = S_2$, $c_7 = t_3$, $c_2, c_5 \in Z_{m-1}$ the inner neighbors of (d_1, d_2) and S_2, and $c_8 \in Z_{m-1}$ the second inner neighbor of t_3. Because $c_9 \in Z_m$ and all Z_{m+1} neighbors are fixed, the unfixed neighbors of $c_7 = t_3$ are precisely $\{c_6, c_5, c_8\}$; all nine sites are distinct (six in Z_m and three consecutive sites c_2, c_5, c_8 in Z_{m-1}), and $c_1, c_3, c_6, c_7 \in Z_m$ lie in the current good set.

(2) *Close every $\mathsf{S\,D\,D\,S}$ arc second.* Write such an arc as $S_1 - t_1 - d_1 - d_2 - t_2 - d_3 - d_4 - t_3 - S_2$. Each endpoint single S_1 (and similarly S_2) is either already joined—if the arc on its other side was already closed—or else both arcs meeting at S_1 still have their doubles unjoined, so S_1 can be joined by Lemma 4.4 (both of its type-2 neighbors still have their other ring neighbors in Z_m, as well as all Z_{m+1} neighbors, fixed). Next join d_1 and d_4 by Lemma 4.3, join d_3 by Lemma 4.4 (with d_2 still fixed), and finally close $c := d_2$ by Lemma 4.5 with $c_4 = t_1$, $c_1 = d_1$, $c_3 = t_2$, $c_6 = d_3$, $c_7 = d_4$, $c_8 = t_3$, and $c_2, c_5 \in Z_{m-1}$ the inner neighbors of (d_1, d_2) and (d_3, d_4) (all nine sites are distinct, $c_1, c_3, c_6, c_7 \in Z_m$ lie in the current good set, and no unjoined neighbor beyond t_1 or t_3 is required).

Thus U_m is good for every $m \geq 1$, completing the proof. □

Proof of Corollary 1.2. We first show that $g_7 = 3$. Every point $P \in \mathbb{H}^2$ not at a site of $\{3, 7\}$ lies in some closed triangle $\triangle uvw$ of $\{3, 7\}$. If P lies in the open interior of $\triangle uvw$, then $d(P, u)$, $d(P, v)$, and $d(P, w)$ are all strictly less than $2r$ (since all sides have length $2r$ and distance is strictly convex along geodesics), so the circle of radius r centered at P overlaps at least the three circles at u, v, w. If P lies in the open interior of an edge uv, then $d(P, u)$, $d(P, v)$, and $d(P, w)$ are all strictly less than $2r$ (indeed also $d(P, w') < 2r$ for the reflected vertex w' across uv, giving four overlapped circles). Thus $g_7 \geq 3$.

For the reverse inequality $g_7 \leq 3$, let O be the circumcenter of a triangle $\triangle uvw$ of $\{3, 7\}$. By Proposition 3.1 and (2), $\rho^* = h - r < r$, so

$$\rho^* < r < R_c < h < 2r.$$

Hence the circle at O overlaps u, v, w. Now let $x \notin \{u, v, w\}$ be any other site of $\{3, 7\}$. The three rays $\overrightarrow{Ou}, \overrightarrow{Ov}, \overrightarrow{Ow}$ (at angle $2\pi/3 < \pi$ apart) partition $\mathbb{H}^2$ into three closed convex sectors; without

loss of generality, x lies in the sector Σ_{uv} bounded by $\overrightarrow{Ou}$ and $\overrightarrow{Ov}$. Since $O \in B(u,2r) \cap B(v,2r)$ and $d(x,u), d(x,v) \geq 2r$, the geodesic segment $[O,x] \subset \Sigma_{uv}$ meets $\partial(B(u,2r) \cup B(v,2r))$ at a point $y \in \Sigma_{uv}$; by symmetry between u and v, assume $d(y,u) = 2r \leq d(y,v)$. In polar coordinates about u with v at angle 0 and w at $+\alpha$, the circumcenter O sits at $(R_c, +\alpha/2)$ and Σ_{uv} lies in the closed half-plane on the v-side of the line Ou, so y has angle $\varphi \in [\alpha/2 - \pi,\, \alpha/2] = [-6\pi/7,\, \pi/7]$. Since $d(y,u) = 2r \leq d(y,v)$, Lemma 2.2 at $\rho = 2r$ gives $|\varphi| \geq \alpha = 2\pi/7$, leaving $\varphi \in [-6\pi/7,\, -2\pi/7]$. Hence the angle at u between O and y is $\angle Ouy = \pi/7 - \varphi \in [3\pi/7,\, \pi]$, so $\angle Ouy \geq 3\alpha/2 = \beta(\rho^*) > \beta(R_c)$ (as $R_c > \rho^*$ and β is strictly decreasing). By Lemma 2.2 applied about u (to O at radius $R_c < 2r$ and y on $\partial B(u,2r)$), we obtain $d(O,x) \geq d(O,y) > 2r$. Thus O overlaps only u, v, w, proving $g_7 = 3$.

Since Π_7 is strongly solid (Theorem 1.1) and face-transitive, it is solid of order 1, so $k_7 \geq 1$. Conversely, for any integer $m \geq 2$, choose an m-truncation $\Pi' = \Pi_7 \setminus K$ with $u, v \in K$ and $w \notin K$; moving w to the circumcenter O gives a valid rearrangement $(\Pi' \setminus \{w\}) \cup \{O\}$ of $\{w\}$, since $d(O,x) > 2r$ for all $x \notin \{u,v,w\}$ (cf. [9, p. 301]). If an isometry ϕ of $\mathbb{H}^2$ mapped $(\Pi' \setminus \{w\}) \cup \{O\}$ onto an m-truncation $\Pi'' \subset \Pi_7$, then choosing a triangle Δ of $\{3,7\}$ whose three vertices lie in the cofinite set $\Pi' \setminus \{w\}$ shows—exactly as in the proof of Lemma 2.1—that $\phi(\Delta)$ is a triangle of $\{3,7\}$ and hence ϕ is a symmetry of $\{3,7\}$. Then $\phi(O) \in \Pi'' \subset \Pi_7$ forces $O \in \Pi_7$, contradicting $0 < d(O,u) = R_c < 2r$. Thus Π_7 is not solid of order m for any $m \geq 2$, so $k_7 \leq 1$. Therefore $k_7 = 1 = g_7 - 2$. □

6. Elementary theorems and obstructions for the Euclidean case $p = 6$

We now turn to the single remaining open case of L. Fejes Tóth's conjecture: $p = 6$, the packing of unit open circular discs centered at the spacing-2 triangular lattice

$$\Lambda := \mathbb{Z}\, e_1 + \mathbb{Z}\, e_2 \subset \mathbb{R}^2, \qquad e_1 = (2,0), \quad e_2 = (1,\sqrt{3}),$$

with one disc removed. Let us emphasize at the outset that *the unrestricted Euclidean conjecture for $p = 6$ remains open.* In this section we explain why Bezdek's hyperbolic zone method fails qualitatively at $p = 6$, formulate the equivalent off-lattice kill-set inequality (Proposition 6.1), prove two local packing lemmas and two global structural theorems (Theorems 6.4 and 6.5), and exhibit an explicit 30-point counterexample to a naive inductive heuristic (Proposition 6.6). Section 7 then gives a complete, computer-free proof of near-lattice rigidity up to displacement 5/26 of a lattice side (1/5 for all-triangular families).

6.1. **Why Bezdek's zone method fails at $p = 6$.** At $p = 6$, both pillars of Part I collapse simultaneously:

(1) *Zero metric margin.* In the Euclidean plane (disc radius $r_6 = 1$, vertex angle $\alpha_6 = \pi/3$, triangle altitude $h_6 = \sqrt{3}$), the polar angle excluded by a fixed neighbor at distance 2 is $\beta_6(\rho) = \arccos(\rho/4)$. Solving $\beta_6(\rho_6^*) = 3\alpha_6/2 = \pi/2$ yields $\rho_6^* = 0$, so the metric gap is $\Delta_6 = h_6 - (r_6 + \rho_6^*) = \sqrt{3} - 1 > 0$. Consequently, the strict inequality $d(P,M) < r_p$ of $p \geq 8$ and the sharp pinning $d(P,M) \leq r_7$ of $p = 7$ both fail: admissible overlappers in a τ_3 profile can lie far outside $\overline{B}(M,1)$.

(2) *Adjacent type-2 sites.* In the dual triangulation $\{3,6\}$, zone Z_m ($m \geq 1$) is the boundary of a regular hexagon of side m, containing 6 type-1 corner sites and $6(m-1)$ type-2 side sites. Thus for $m \geq 3$, type-2 sites occur in contiguous runs of length $m - 1$ ($p - 6 = 0$), so joining them leaves chains of adjacent τ_4 profiles that cannot be peeled one circle at a time.

6.2. **The kill-set formulation and local lemmas.** For any finite nonempty set $X \subset \mathbb{R}^2 \setminus \Lambda$, we call X *admissible* if $\|x - y\| \geq 2$ for all distinct $x, y \in X$ (where $\|\cdot\|$ is the Euclidean norm). Define

the *kill set* (the set of overlapped lattice sites) of a point $x \in \mathbb{R}^2$ and of X by

$$K(x) := \{v \in \Lambda : \|x - v\| < 2\}, \qquad K(X) := \bigcup_{x \in X} K(x).$$

Notice the strict inequality: a lattice site at distance exactly 2 from x is *not* killed.

Proposition 6.1 (Kill-set equivalence)**.** *The face-incircle packing of* $\{6, 3\}$ *is strongly solid if and only if every admissible set* $X \subset \mathbb{R}^2 \setminus \Lambda$ *satisfies*

$$|K(X)| \geq |X| + 2. \tag{7}$$

Proof. We first note that by Imre's theorem [13] (solidity of the full lattice Λ), every finite modification $Q = (\Lambda \setminus E) \cup X$ ($E \subset \Lambda$ finite, $X \subset \mathbb{R}^2 \setminus \Lambda$ finite, all pairwise distances ≥ 2) satisfies $|E| \geq |X|+1$ whenever $X \neq \emptyset$. Indeed, if $|E| = 0$, then no point of $\mathbb{R}^2$ lies at distance ≥ 2 from all of Λ (the covering radius of Λ is $2/\sqrt{3} < 2$), contradicting $X \neq \emptyset$. If $1 \leq |E| \leq |X|$, deleting $|X| - |E|$ points from X leaves a nonempty set $X' \subseteq X$ of size $|E|$ such that $(\Lambda \setminus E) \cup X'$ is a rearrangement of the finite subset E of Λ with centers off Λ; by solidity of Λ, this rearrangement would have to be congruent to Λ, which is impossible since any isometry ϕ mapping Λ onto $(\Lambda \setminus E) \cup X'$ sends any elementary lattice triangle $\Delta \subset \Lambda \setminus \phi^{-1}(X')$ to an elementary lattice triangle $\phi(\Delta) \subset \Lambda \setminus E \subset \Lambda$ and hence is a symmetry of Λ, contradicting $X' \cap \Lambda = \emptyset$.

Suppose now that (7) fails for some admissible $X \subset \mathbb{R}^2 \setminus \Lambda$, so $|K(X)| \leq |X| + 1$. Applying the preceding paragraph to $E := K(X)$ gives $|K(X)| = |X| + 1$. Translating so that $0 \in K(X)$, the packing $Q := (\Lambda \setminus K(X)) \cup X$ is obtained from $\Lambda \setminus \{0\}$ by rearranging the finite set $K(X) \setminus \{0\}$. Any isometry ϕ mapping $\Lambda \setminus \{0\}$ onto Q sends any elementary lattice triangle $\Delta \subset \Lambda \setminus (\{0\} \cup \phi^{-1}(X))$ (which exists because $\{0\} \cup \phi^{-1}(X)$ is finite) to an equilateral triangle $\phi(\Delta)$ of side 2 whose vertices lie in $Q \setminus X = \Lambda \setminus K(X) \subset \Lambda$. Because distinct sites of Λ lie at distance at least 2 with equality only for adjacent sites, $\phi(\Delta)$ is an elementary triangle of Λ; hence ϕ is a symmetry of Λ, giving $Q = \phi(\Lambda \setminus \{0\}) \subset \Lambda$, contradicting $X \neq \emptyset$ and $X \cap \Lambda = \emptyset$. Thus $\Lambda \setminus \{0\}$ is not solid.

Conversely, suppose $\Lambda \setminus \{0\}$ admits a non-trivial finite rearrangement $Q = (\Lambda \setminus E) \cup X$ with $E \subset \Lambda$ finite, $X \subset \mathbb{R}^2 \setminus \Lambda$ finite, and $|E| = |X| + 1$. If $X = \emptyset$, then $|E| = 1$, so $Q = \Lambda \setminus \{e\}$ is a translate of $\Lambda \setminus \{0\}$, contradicting non-triviality. Hence X is nonempty and admissible, and because Q is 2-separated, every lattice site killed by X must lie in E. Thus $K(X) \subseteq E$, giving $|K(X)| \leq |E| = |X| + 1$, violating (7). □

Lemma 6.2 (Wheel Lemma)**.** *For any admissible set* $X \subset \mathbb{R}^2 \setminus \Lambda$ *and any lattice site* $v \in \Lambda$*, at most* 5 *points of* X *kill* v*, and consecutive rays from* v *to points of* $X \cap B(v, 2)$ *make angles strictly greater than* $\pi/3$*.*

Proof. Let $x, y \in X \cap B(v, 2)$ be distinct, and let $\theta \in [0, \pi]$ be the angle $\angle xvy$. If $\theta \leq \pi/3$, then $\cos\theta \geq 1/2$, so the Euclidean law of cosines in $\triangle vxy$ yields

$$\begin{aligned} \|x - y\|^2 &= \|x - v\|^2 + \|y - v\|^2 - 2\|x - v\|\|y - v\| \cos\theta \\ &\leq \|x - v\|^2 + \|y - v\|^2 - \|x - v\|\|y - v\| \leq \max(\|x - v\|, \|y - v\|)^2 < 4, \end{aligned}$$

contradicting $\|x - y\| \geq 2$. Hence $\theta > \pi/3$. Summing consecutive angles around v gives $k \cdot (\pi/3) < 2\pi$, so $k \leq 5$. □

Lemma 6.3 (Full-type and face-uniqueness lemma)**.** *Let* $X \subset \mathbb{R}^2 \setminus \Lambda$ *be admissible.*

(a) *Every point* $x \in X$ *kills either precisely the* 3 *vertices of one closed elementary lattice triangle* τ *(a* triangular owner*) or precisely the* 4 *vertices of an elementary lozenge formed by two elementary lattice triangles sharing an edge (a* lozenge owner*).*

(b) *No two distinct points of* X *kill all three vertices of the same elementary lattice triangle. In particular, assigning each* $x \in X$ *to a closed elementary triangle* $\tau(x)$ *containing* x *is injective, and the interiors of the full kill tiles (triangles and lozenges) of distinct points of* X *are pairwise disjoint.*

Proof. (a) Pick a closed elementary equilateral triangle $\triangle uvw$ of side 2 containing x, and translate so that $u = 0$, $v = e_1$, $w = e_2$. Because $x \notin \Lambda$, $\|x-u\|, \|x-v\|, \|x-w\| < 2$, so $\{u, v, w\} \subseteq K(x)$. Write any lattice site as $z = ae_1 + be_2$ $(a, b \in \mathbb{Z})$ and $x = \xi e_1 + \eta e_2$ with $\xi, \eta \geq 0$ and $\xi + \eta \leq 1$. Because consecutive parallel lattice lines are spaced $\sqrt{3}$ apart, any site with $a \leq -2$, $b \leq -2$, or $a + b \geq 3$ lies at perpendicular distance at least $2\sqrt{3} > 2$ from $\triangle uvw$ and cannot be killed by x. Among the remaining 15 sites ($a \geq -1$, $b \geq -1$, $a + b \leq 2$), six are the triangle vertices $(0,0), (1,0), (0,1)$ and the three reflected vertices $w' = (1,-1)$, $u' = (1,1)$, $v' = (-1,1)$ across the sides uv, vw, uw. Each of the other nine sites—namely $(-1,0), (0,-1), (-1,-1)$ at u, together with their S_3-symmetric images $(2,0), (2,-1), (3,-1)$ at v and $(0,2), (-1,2), (-1,3)$ at w—satisfies $\|z-p\| \geq 2$ and $\langle y-p, z-p\rangle \leq 0$ for every $y \in \triangle uvw$ at the corresponding vertex $p \in \{u, v, w\}$; since $x \neq p$, this gives $\|x-z\|^2 = \|x-p\|^2 + \|p-z\|^2 - 2\langle x-p, z-p\rangle > 4$. Thus $K(x) \subseteq \{u, v, w, u', v', w'\}$. Finally, any two of u', v', w' lie at distance 4 from each other, so by the triangle inequality x lies within distance < 2 of at most one of them. Hence $K(x)$ is either $\{u, v, w\}$ or $\{u, v, w, z'\}$ for a single reflected vertex $z' \in \{u', v', w'\}$.

(b) For any elementary triangle $\triangle uvw$ of side 2, the Euclidean Reuleaux triangle $\mathcal{R} := \overline{B}(u, 2) \cap \overline{B}(v, 2) \cap \overline{B}(w, 2)$ has diameter 2 (by the exact Euclidean analogue of Lemma 2.5), so any two distinct points in the open intersection $B(u, 2) \cap B(v, 2) \cap B(w, 2)$ lie at distance strictly less than 2. Since X is 2-separated, at most one point of X kills all three vertices of $\triangle uvw$. Because every $x \in \triangle uvw \setminus \Lambda$ kills $\{u, v, w\}$ and every lozenge consists of two elementary triangles whose vertices are all killed by its owner, both assertions follow immediately. □

6.3. **The No-Hole Theorem.** For any admissible $X \subset \mathbb{R}^2 \setminus \Lambda$, choose for each $x \in X$ a closed elementary lattice triangle $\tau(x)$ containing x (if x lies on an open lattice edge, either of the two incident triangles works). By Lemma 6.3(b), τ is injective, so $\mathcal{F} := \{\tau(x) : x \in X\}$ has $|\mathcal{F}| = |X|$. Let $V(\mathcal{F}) \subseteq K(X)$ be the set of vertices of triangles in $\mathcal{F}$, let $U := \bigcup_{\tau \in \mathcal{F}} \tau \subset \mathbb{R}^2$ be the finite planar 2-complex formed by the occupied triangles, let c_U be the number of connected components of U, and let h_U be the number of bounded connected components of $\mathbb{R}^2 \setminus U$ (*holes*). Let Γ be the dual edge-adjacency graph on $\mathcal{F}$ (two triangles adjacent if they share a side), with $e_\Gamma := |E(\Gamma)|$, c_Γ connected components, and cycle rank $\nu := e_\Gamma - |X| + c_\Gamma$.

Theorem 6.4 (No-Hole Theorem)**.** *For every admissible $X \subset \mathbb{R}^2 \setminus \Lambda$ and every choice of containing triangles τ, let* EXTRA $:= K(X) \setminus V(\mathcal{F})$. *Then $\nu \leq h_U$ and*

$$|K(X)| = |X| + c_\Gamma + c_U - \nu - h_U + |\mathrm{EXTRA}|. \tag{8}$$

In particular, if U has no hole ($h_U = 0$), then $|K(X)| \geq |X| + 2$; and every counterexample to (7) *must satisfy $h_U \geq 1$ (for every choice of τ) and $|\mathrm{EXTRA}| \leq 2h_U - 1$.*

Proof. For the planar 2-complex U, Euler's formula gives $|V(\mathcal{F})| - E(U) + |X| = c_U - h_U$, where $E(U) = 3|X| - e_\Gamma$ (each shared triangle edge is counted twice in $3|X|$). Substituting $e_\Gamma = |X| - c_\Gamma + \nu$ gives

$$|V(\mathcal{F})| = 2|X| - e_\Gamma + c_U - h_U = |X| + c_\Gamma - \nu + c_U - h_U,$$

and adding $|\mathrm{EXTRA}| = |K(X)| - |V(\mathcal{F})|$ yields (8).

Now embed Γ in $\mathbb{R}^2$ by placing each vertex at the centroid of its triangle $\tau \in \mathcal{F}$ and each edge as the straight segment joining the centroids of two side-adjacent triangles (crossing only their shared side at its midpoint). This is a plane drawing of Γ inside U, and ν equals its number of bounded faces. Consider any bounded face f of Γ (which may be multiply connected if Γ has nested components). Its outer boundary is a closed walk in the dual honeycomb graph; pick any dual edge on this outer boundary separating f from its exterior, and let $v \in \Lambda$ be the endpoint of the crossed lattice edge lying on the inward side of that outer boundary (facing f). Because the open disc $B(v, 1)$ meets no edge of Γ (midpoints of lattice edges incident to v sit at distance 1 from v, and triangle centroids at $2/\sqrt{3} > 1$), the vertex v lies in f and connects inside $\mathbb{R}^2 \setminus \Gamma$ to every open elementary triangle incident to v.

If all six elementary triangles at v belonged to $\mathcal{F}$, then X would contain six points in $B(v,2)$ (one in each triangle), contradicting Lemma 6.2. Hence at least one open elementary triangle t incident to v does not belong to $\mathcal{F}$. Since $\Gamma \subset U$, the open triangle t lies inside the bounded face f, so the connected component of $\mathbb{R}^2 \setminus U$ containing t cannot cross $\partial f \subset U$; it is therefore a bounded hole of U contained in f. Because distinct bounded faces of Γ are disjoint and each hole is connected and disjoint from Γ, every bounded face contains at least one hole, giving $\nu \le h_U$.

If $h_U = 0$, then $\nu = 0$, and since $c_\Gamma \ge 1$, $c_U \ge 1$, and $|\mathrm{EXTRA}| \ge 0$, equation (8) gives $|K(X)| \ge |X|+2$. If $|K(X)| \le |X|+1$, then $h_U \ge 1$ and $|\mathrm{EXTRA}| = |K(X)|-|X|+\nu+h_U-c_\Gamma-c_U \le 1+2h_U-2 = 2h_U-1$. □

6.4. **The Uniform Triangle-Count Isoperimetric Inequality.** By Lemma 6.3, any admissible set X of size $n = |X|$ partitions into T triangular owners and L lozenge owners ($n = T+L$), whose full kill tiles occupy $F := T+2L = n+L$ pairwise interior-disjoint elementary lattice triangles. Form the planar tile-boundary graph $G(X)$ whose vertices are $K(X)$ and whose edges are the outer boundary edges of all n full tiles (omitting the single internal short diagonal of each lozenge). Let c_G be the number of connected components of $G(X)$ (allowing tiles to touch at vertices), and for each component G_i (viewed in isolation) let h_i be the number of bounded complementary faces of G_i that are not occupied tile interiors; set $H_G := \sum_{i=1}^{c_G} h_i$.

Theorem 6.5 (Uniform Triangle-Count Inequality)**.** *For every admissible set $X \subset \mathbb{R}^2 \setminus \Lambda$ with T triangular owners, L lozenge owners, $n = T+L$, and $F = T+2L$,*

$$|K(X)| - n \ge c_G + \frac{\sqrt{6(F+H_G)} + H_G - T}{2}. \tag{9}$$

Consequently:

(a) *Whenever $T^2 < 6(T+2L)$ (equivalently, $L > T(T-6)/12$), we have $|K(X)| \ge n+2$.*
(b) *Any counterexample to* (7) *satisfies $n \le T^2/12 + T/2$.*

Proof. Rescale Λ by $1/2$ to unit lattice side length, with primitive coordinates $(x,y) \in \mathbb{Z}^2$ along basis vectors meeting at $\pi/3$, so the three lattice-line families have integer indices x, y, and $x+y$.

Consider a single connected component G_i of $G(X)$ containing T_i triangles and L_i lozenges, with $n_i = T_i + L_i$, $F_i = T_i + 2L_i$, V_i vertices, and E_i edges. Each of the n_i tile interiors is a bounded face of G_i. Let p_i be the edge-length of the unbounded exterior boundary walk of G_i, and let $d_1, \dots, d_{h_i}$ be the boundary lengths of the h_i non-tile bounded faces of G_i. Because G_i is a simple graph of lattice edges, every non-tile bounded face has $d_j \ge 3$ and contains at least one unoccupied elementary lattice triangle. By Euler's formula and face-edge incidence,

$$V_i - E_i + (n_i + h_i + 1) = 2, \qquad 2E_i = 3T_i + 4L_i + p_i + \sum_{j=1}^{h_i} d_j.$$

Eliminating E_i and using $d_j - 2 \ge 1$ gives

$$2(V_i - n_i - 1) = p_i - T_i + \sum_{j=1}^{h_i} (d_j - 2) \ge p_i - T_i + h_i. \tag{10}$$

Let a, b, d be the integer widths ($\max - \min$) of the three functionals x, y, and $x+y$ on $V(G_i)$. The closed exterior walk visits a maximizer and a minimizer of each functional (achieved on the convex hull), so the total variation of each functional along the walk is at least twice its width. Because every unit lattice step changes exactly two of $(x, y, x+y)$ by ± 1 and leaves the third unchanged, summing total variations gives $2p_i \ge 2(a+b+d)$, hence $p_i \ge a+b+d$.

Translating G_i so that $V(G_i) \subset [0,a] \times [0,b]$ in (x,y)-coordinates, let $\sigma_1 \ge 0$ and $a+b-\sigma_2$ ($\sigma_2 \ge 0$) be the minimum and maximum values of $x+y$ on $V(G_i)$, so $d = a+b-\sigma_1-\sigma_2$ and $\sigma_1, \sigma_2 \le \min(a,b)$. Every bounded face of G_i lies inside the hexagon obtained by cutting off the

two corner right triangles $\{x+y<\sigma_1\}$ and $\{x+y>a+b-\sigma_2\}$ (which have disjoint interiors and coordinate areas $\sigma_1^2/2$ and $\sigma_2^2/2$) from the rectangle $[0,a]\times[0,b]$ of area ab. Since each elementary lattice triangle has coordinate area $1/2$ and the F_i+h_i occupied and hole triangles are interior-disjoint,

$$\begin{aligned} F_i+h_i &\le 2ab-\sigma_1^2-\sigma_2^2 \le 2ab-\frac{(\sigma_1+\sigma_2)^2}{2} = 2ab-\frac{(a+b-d)^2}{2} \\ &= ab+ad+bd-\frac{a^2+b^2+d^2}{2} = \frac{(a+b+d)^2}{6} - \frac{2}{3}\big(a^2+b^2+d^2-ab-ad-bd\big) \\ &\le \frac{(a+b+d)^2}{6} \le \frac{p_i^2}{6}, \end{aligned}$$

using $\sigma_1^2+\sigma_2^2 \ge (\sigma_1+\sigma_2)^2/2$ and $a^2+b^2+d^2 \ge ab+ad+bd$. Thus $p_i \ge \sqrt{6(F_i+h_i)}$.

Since distinct components of $G(X)$ have disjoint vertex sets, summing (10) over $i=1,\dots,c_G$ and applying $\sum_i \sqrt{6(F_i+h_i)} \ge \sqrt{6(F+H_G)}$ yields (9).

For (a), if $T^2<6F$, then $\sqrt{6(F+H_G)}+H_G-T>0$ and $c_G\ge 1$, so $|K(X)|-n>1$; integrality gives $|K(X)|\ge n+2$. For (b), a violation $|K(X)|\le n+1$ forces $T\ge\sqrt{6F}=\sqrt{6(2n-T)}$, i.e., $n\le T^2/12+T/2$. □

6.5. **Refutation of universal private kills.** A tempting inductive strategy for (7) is to conjecture that every admissible set $X\subset\mathbb{R}^2\setminus\Lambda$ contains at least one *private killed site*—a site $v\in K(X)$ killed by exactly one point $x\in X$—which would permit deleting x to drop $|X|$ by 1 and $|K(X)|$ by at least 1. We show that this general private-kill property fails, even with generous margins.

Proposition 6.6 (Counterexample to universal private kills)**.** *There exists an admissible* 30*-point set $X_{30}\subset\mathbb{R}^2\setminus\Lambda$ with full hexagonal symmetry D_6 whose kill set $K(X_{30})$ consists of* 54 *lattice sites (surplus $|K(X_{30})|-|X_{30}|=24$), every one of which is killed by at least two points of X_{30} (*48 *sites with* 2 *killers and* 6 *sites with* 3 *killers).*

Proof. With $\Lambda=\mathbb{Z}(2,0)+\mathbb{Z}(1,\sqrt{3})$, fix the center site $Z:=(5,5\sqrt{3})\in\Lambda$. On the bottom side of the hexagonal annulus around Z, define five rational points:

$$p_0=(1.479767,\,0.513396),\qquad p_1=(3.292220,\,1.464194),\qquad p_2=(5,\,0.331489),$$

together with the reflections $q_1=(6.707780,\,1.464194)$ and $q_0=(8.520233,\,0.513396)$ across the vertical line $x=5$. Let X_{30} be the orbit of these five points under rotations by multiples of $\pi/3$ about Z.

All squared distances belong to $\mathbb{Q}(\sqrt{3})$ and verify directly:

- $\min_{x\ne y\in X_{30}}\|x-y\|\approx 2.046705>2$ (admissible, with no point on Λ);
- the killed set $K(X_{30})$ is precisely the union of the hexagonal rings at graph distances 4 and 5 from Z (24+30 = 54 sites), separating a fixed 37-site central island (graph distance ≤ 3 from Z) from the fixed exterior (distance ≥ 6), with $\min_{x\in X_{30},\,v\in\Lambda\setminus K(X_{30})}\|x-v\|\approx 2.063540>2$;
- every site $v\in K(X_{30})$ has at least two killers in X_{30}, with second-nearest distance at most $2-0.047131\approx 1.952869<2$. □

Geometrically, X_{30} forms a D_6-symmetric ring of 24 lozenge owners (the orbits of p_0,p_1,q_1,q_0) and 6 triangular owners (the orbit of p_2) wrapping around a fixed 37-site lattice island: locally every killed site is covered at least twice, yet the global surplus (24) remains far above 2. Thus any proof of (7) for $p=6$ cannot rely on private kills alone without using the low-surplus constraint $|K(X)|\le|X|+1$.

7. Near-lattice rigidity for $\{6,3\}$ up to 5/26 of a lattice side

Bárány and Dolbilin [1] proved that the Euclidean hexagonal circle packing minus one circle is *uniformly stable*: no non-trivial finite rearrangement can move every circle by less than 1/40 of a disc diameter (i.e., 1/40 of a lattice side; they also remarked on p. 108 that their method extends to 1/16 and breaks down at 1/12). In this section we give a complete, self-contained hand proof improving this displacement threshold by more than a factor of seven—to 5/26 of a lattice side in general ($100/13 \approx 7.69$ times 1/40), and to 1/5 of a lattice side when every moved circle overlaps only three lattice sites.

For convenience in this section, we rescale Λ by 1/2 to the *unit-side* triangular lattice $\Lambda_1 := \frac{1}{2}\Lambda = \mathbb{Z}(1,0) + \mathbb{Z}(\frac{1}{2}, \frac{\sqrt{3}}{2})$, so discs have diameter 1 (radius 1/2), admissible sets $X \subset \mathbb{R}^2 \setminus \Lambda_1$ have pairwise distances ≥ 1, and $K_1(X) := \{v \in \Lambda_1 : \exists\, x \in X,\ \|x - v\| < 1\}$. On the side-2 scale of Section 6, the thresholds 5/26 and 1/5 correspond to 5/13 and 2/5, respectively.

Theorem 7.1 (Near-lattice rigidity for $\{6,3\}$)**.** *Let $X \subset \mathbb{R}^2 \setminus \Lambda_1$ be any finite nonempty 1-separated set.*

(a) *If every point of X lies within Euclidean distance 5/26 of some lattice site in Λ_1, then $|K_1(X)| \geq |X| + 2$.*
(b) *If every point of X kills precisely the 3 vertices of an elementary triangle of Λ_1 and lies within Euclidean distance 1/5 of some lattice site in Λ_1, then $|K_1(X)| \geq |X| + 2$.*

The proof rests on three elementary lemmas: a concave lower bound on triangle area in squared side lengths, orientation propagation under a bond cap below $\sqrt{2}$, and a localized ring-rigidity theorem charged along eighteen disjoint outward lattice rays.

Lemma 7.2 (Concave triangle-area charge)**.** *Suppose (a, κ) is equal to $(29/200, 155/169)$ or to $(119/800, 556/625)$. Every triple of squared side lengths $(q_1, q_2, q_3) \in [1, 1+\kappa]^3$ realizes a nondegenerate Euclidean triangle whose area $\Delta(q_1, q_2, q_3)$ satisfies*

$$\Delta(q_1, q_2, q_3) \geq \frac{\sqrt{3}}{4} + \frac{a}{2}\big((q_1 - 1) + (q_2 - 1) + (q_3 - 1)\big), \tag{11}$$

with equality at $(1, 1, 1)$.

Proof. Since $1 + \kappa < 2$, every side length lies in $[1, \sqrt{2})$, so the sum of any two sides is at least $2 > \sqrt{2}$ and every point of $[1, 1+\kappa]^3$ defines a nondegenerate triangle. By Heron's formula, $16\Delta(q)^2 = H(q) := 2(q_1q_2 + q_2q_3 + q_3q_1) - (q_1^2 + q_2^2 + q_3^2)$, which decomposes as

$$H(q) = L(q)^2 - \|N(q)\|^2, \qquad L(q) := \frac{q_1 + q_2 + q_3}{\sqrt{3}}, \quad N(q) := \Big(q_1 - q_2,\, \frac{q_1 + q_2 - 2q_3}{\sqrt{3}}\Big) \in \mathbb{R}^2.$$

For any two vectors q, q' with $H(q), H(q') > 0$ and $L(q), L(q') > 0$, the Cauchy–Schwarz inequality in $\mathbb{R}^3$ yields

$$\sqrt{H(q)}\sqrt{H(q')} + \langle N(q), N(q')\rangle \leq \sqrt{H(q) + \|N(q)\|^2}\sqrt{H(q') + \|N(q')\|^2} = L(q)L(q').$$

Multiplying by 2 and adding $H(q) + H(q')$ yields

$$\big(\sqrt{H(q)} + \sqrt{H(q')}\big)^2 \leq L(q+q')^2 - \|N(q+q')\|^2 = H(q+q'),$$

and taking square roots gives $\sqrt{H(q)} + \sqrt{H(q')} \leq \sqrt{H(q+q')}$. Because $\sqrt{H}$ is positively homogeneous of degree 1, it is concave on the cone $\{H > 0, L > 0\}$, and subtracting the affine right-hand side of (11) preserves concavity. Hence it suffices to check (11) at the eight corners of $[1, 1+\kappa]^3$.

At $(1,1,1)$ both sides equal $\sqrt{3}/4$. For $k \in \{1,2,3\}$ squared sides equal to $1+\kappa$ and $3-k$ equal to 1, we have $H = 3 + 2\kappa - \kappa^2$ $(k=1)$, $H = 3 + 4\kappa$ $(k=2)$, and $H = 3(1+\kappa)^2$ $(k=3)$. Multiplying (11) by 4 and squaring where needed reduces the three corner tests to

$$(1 + 4a^2)\kappa \leq 2 - 4\sqrt{3}\,a \quad (k=1), \qquad 8\sqrt{3}\,a + 16a^2\kappa \leq 4 \quad (k=2), \qquad 6a \leq \sqrt{3} \quad (k=3).$$

Using the rational upper bound $\sqrt{3} < 26/15$ (since $26^2 = 676 > 675 = 3 \cdot 15^2$):

- For $(a, \kappa) = (29/200,\ 155/169)$: $(1+4a^2)\kappa + 4(26/15)a = \frac{1{,}084{,}100}{1{,}000{,}000} \cdot \frac{155}{169} + \frac{3{,}016}{3{,}000} = \frac{1{,}680{,}355}{1{,}690{,}000} + \frac{377}{375} < 0.9943 + 1.0054 < 2$, while $8(26/15)a + 16a^2\kappa < 2.011 + 0.309 < 4$ and $36a^2 = 30{,}276/40{,}000 < 3$.
- For $(a, \kappa) = (119/800,\ 556/625)$: $(1+4a^2)\kappa + 4(26/15)a = \frac{696{,}644}{640{,}000} \cdot \frac{556}{625} + \frac{12{,}376}{12{,}000} < 0.96834 + 1.03134 < 2$, while $8(26/15)a + 16a^2\kappa < 2.063 + 0.315 < 4$ and $36a^2 < 3$. □

Lemma 7.3 (Orientation propagation)**.** *Let $\psi\colon \Lambda_1 \setminus \{0\} \to \mathbb{R}^2$ be a finitely supported perturbation ($\psi(v) = v$ outside a finite set) such that $\|\psi(v) - \psi(w)\| \geq 1$ for all distinct $v, w \in \Lambda_1 \setminus \{0\}$, and $\|\psi(v) - \psi(w)\| < \sqrt{2}$ for every originally adjacent pair $v, w \in \Lambda_1 \setminus \{0\}$ ($\|v - w\| = 1$). Then every* intact *elementary triangle (an elementary triangle of Λ_1 not incident to 0) retains its positive orientation under ψ.*

Proof. Because every intact side has image length in $[1, \sqrt{2})$, no intact elementary triangle maps to a degenerate triangle. Suppose two intact elementary triangles sharing an edge vw map to triangles on the same side of the segment $AB := [\psi(v), \psi(w)]$, with opposite vertices $C, D \in \mathbb{R}^2$. All six pairwise distances among A, B, C, D are at least 1, and the five distances other than possibly $\|C - D\|$ are strictly less than $\sqrt{2}$.

If A, B, C, D are the vertices of a strictly convex quadrilateral, then AB is a hull edge (since C, D lie on the same side of AB). At least one interior angle of this quadrilateral is at least $\pi/2$; its spanning diagonal connects two vertices separated by two sides of lengths ≥ 1, so by the law of cosines that diagonal has length at least $\sqrt{1+1} = \sqrt{2}$. Yet every diagonal joins an endpoint of AB to C or D, hence has length $< \sqrt{2}$, a contradiction.

Otherwise one of C, D lies in the closed triangle formed by the other three points (say $D \in \triangle ABC$, since C and D lie strictly on the same side of the line AB, so neither A nor B can lie in the triangle formed by the other three points). If D lies in the open interior of $\triangle ABC$, then some two of the three rays from D to A, B, C form an angle at least $2\pi/3$, forcing the opposite side of $\triangle ABC$ to have length at least $\sqrt{1+1+1} = \sqrt{3} > \sqrt{2}$; if D lies on a side of $\triangle ABC$, the two subsegments have length at least 1 each, forcing that side to have length at least $2 > \sqrt{2}$. Yet all three sides of $\triangle ABC$ have length $< \sqrt{2}$, a contradiction.

Thus adjacent intact elementary triangles have consistent orientation under ψ. Because the side-adjacency graph of intact elementary triangles of $\Lambda_1 \setminus \{0\}$ is connected and ψ is the identity outside a bounded set, every intact elementary triangle has positive orientation. □

Lemma 7.4 (Localized ring rigidity)**.** *Take $(a, \kappa, R) = (29/200, 155/169, 5/26)$ or $(a, \kappa, R) = (119/800, 556/625, 1/5)$. Suppose $\psi\colon \Lambda_1 \setminus \{0\} \to \mathbb{R}^2$ is a finitely supported perturbation satisfying:*

(i) *$\|\psi(v) - \psi(w)\| \geq 1$ for all distinct $v, w \in \Lambda_1 \setminus \{0\}$;*
(ii) *$\|\psi(v) - \psi(w)\|^2 \leq 1 + \kappa$ for every originally adjacent pair $v, w \in \Lambda_1 \setminus \{0\}$;*
(iii) *$\|\psi(w_j) - w_j\| \leq R$ for each of the six unit neighbors $w_j = (\cos(j\pi/3), \sin(j\pi/3))$ ($j \in \mathbb{Z}/6\mathbb{Z}$) of the vacancy 0.*

Then $\psi(v) = v$ for all $v \in \Lambda_1 \setminus \{0\}$.

Proof. Since $1 + \kappa < 2$, Lemma 7.3 guarantees that every intact elementary triangle has positive orientation under ψ. Write $u_j := \psi(w_j) - w_j$ ($j \in \mathbb{Z}/6\mathbb{Z}$, indices cyclic modulo 6), and define

$$S := \sum_{j=0}^{5} \langle w_j, u_j \rangle, \qquad Q := \frac{1}{2} \sum_{j=0}^{5} \det(u_j, u_{j+1}),$$

$$D := \sum_{j=0}^{5} \|u_{j+1} - u_j\|^2, \qquad \Lambda_{\text{ring}} := \sum_{j=0}^{5} \big(\|\psi(w_{j+1}) - \psi(w_j)\|^2 - 1 \big).$$

Because $w_{j-1}+w_{j+1}=w_j$, expanding $\|\psi(w_{j+1})-\psi(w_j)\|^2-1$ as $\|u_{j+1}-u_j\|^2+2\langle w_{j+1}-w_j, u_{j+1}-u_j\rangle$ and summing over j gives

$$\Lambda_{\text{ring}} = D + 2\sum_{j=0}^{5}\langle 2w_j - w_{j-1} - w_{j+1}, u_j\rangle = D + 2S \geq 0 \tag{12}$$

(since every ring edge has length at least 1). Similarly, the signed area of the image hexagon $(\psi(w_0),\dots,\psi(w_5))$ is

$$\frac{1}{2}\sum_{j=0}^{5}\det(w_j+u_j,\, w_{j+1}+u_{j+1}) = \frac{3\sqrt{3}}{2} + \frac{1}{2}\sum_{j=0}^{5}\det(w_{j-1}-w_{j+1},\, u_j) + Q = \frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{2}S + Q,$$

because rotating w_j counterclockwise by $\pi/2$ maps w_j to $\frac{1}{\sqrt{3}}(w_{j+1}-w_{j-1})$, so $\det(w_{j-1}-w_{j+1},u_j) = \sqrt{3}\langle w_j, u_j\rangle$. Hence the signed area deficit of the six-sided hole is

$$dA := \frac{3\sqrt{3}}{2} - \text{Area}_{\text{signed}}\big(\psi(w_0),\dots,\psi(w_5)\big) = -\frac{\sqrt{3}}{2}S - Q.$$

Choose a large regular lattice hexagon Ω containing the six-sided hole and the finite support of $\psi-\text{id}$ strictly in its interior (so $\psi=\text{id}$ on $\partial\Omega$ and every stretched ray edge belongs to two intact triangles inside Ω). Summing the positive areas of all intact elementary triangles inside Ω via oriented edge determinants cancels all internal intact edges and leaves $\text{Area}(\Omega) - \text{Area}_{\text{signed}}(\psi(w_0),\dots,\psi(w_5))$; subtracting the original total area $\text{Area}(\Omega) - \frac{3\sqrt{3}}{2}$ shows that the sum of area excesses $\Delta(\psi(\tau)) - \frac{\sqrt{3}}{4}$ over all intact elementary triangles equals dA exactly.

For each $j \in \mathbb{Z}/6\mathbb{Z}$, consider the three outward lattice rays starting at w_j in unit directions $e \in \{w_{j-1}, w_j, w_{j+1}\}$. These 18 rays are pairwise edge-disjoint and use no ring edge (any two rays lying on the same lattice line run in opposite directions away from a ring edge or across the vacancy). Along a single ray $v_0 = w_j$, $v_{m+1} = v_m + e$ ($m \geq 0$), writing $\psi(v_m) = v_m + \delta_m$ (with $\delta_0 = u_j$ and $\delta_m = 0$ for large m),

$$\sum_{m\geq 0}\big(\|\psi(v_{m+1})-\psi(v_m)\|^2 - 1\big) = \sum_{m\geq 0}\big(\|\delta_{m+1}-\delta_m\|^2 + 2\langle e, \delta_{m+1}-\delta_m\rangle\big) \geq -2\langle e, u_j\rangle,$$

and the left-hand side is also ≥ 0 (since every bond has length ≥ 1). Set

$$L_{\text{ray}} := \sum_{j=0}^{5}\sum_{e\in\{w_{j-1},w_j,w_{j+1}\}} \max\big(-\langle e, u_j\rangle,\, 0\big).$$

Because each ray edge belongs to two intact elementary triangles while each ring edge belongs to one, summing Lemma 7.2 over all intact triangles and dropping only nonnegative bond excesses off the 18 rays and ring yields

$$M := dA - 2aL_{\text{ray}} - \frac{a}{2}\Lambda_{\text{ring}} \geq 0. \tag{13}$$

Set $z := \frac{\sqrt{3}}{4} - a > 0$. Substituting $dA = -\frac{\sqrt{3}}{2}S - Q$ and $2S = \Lambda_{\text{ring}} - D$ from (12) gives

$$M = -2a(S + L_{\text{ray}}) + (zD - Q) - \Big(z + \frac{a}{2}\Big)\Lambda_{\text{ring}}.$$

We bound the first two terms separately:

(1) At each j, since $w_{j-1} + w_j + w_{j+1} = 2w_j$, we have

$$\langle w_j, u_j\rangle + \sum_{e\in\{w_{j-1},w_j,w_{j+1}\}} \max(-\langle e, u_j\rangle, 0) = \frac{1}{2}\sum_{e\in\{w_{j-1},w_j,w_{j+1}\}} |\langle e, u_j\rangle| \geq \frac{\sqrt{3}}{2}\|u_j\|,$$

because by D_6 symmetry a unit vector at angle $\theta \in [0, \pi/6]$ to the nearest lattice direction has $|\cos\theta| + |\cos(\pi/3-\theta)| + |\cos(\pi/3+\theta)| = 2\cos\theta \geq \sqrt{3}$. Summing over j gives $S + L_{\text{ray}} \geq \frac{\sqrt{3}}{2}\sum_{j=0}^{5} \|u_j\|$.

(2) Identifying $\mathbb{R}^2 \cong \mathbb{C}$ and expanding $u_j = \frac{1}{\sqrt{6}}\sum_{k=0}^{5} \hat{u}_k e^{ijk\pi/3}$ via the discrete Fourier transform, we obtain

$$D = \sum_{k=0}^{5} 2\big(1 - \cos\tfrac{k\pi}{3}\big)|\hat{u}_k|^2, \qquad Q = \frac{1}{2}\operatorname{Im}\sum_{j=0}^{5} \overline{u_j} u_{j+1} = \frac{1}{2}\sum_{k=0}^{5} \sin\tfrac{k\pi}{3}\,|\hat{u}_k|^2.$$

Thus

$$zD - Q = \sum_{k=0}^{5} \Big(2z\big(1 - \cos\tfrac{k\pi}{3}\big) - \frac{1}{2}\sin\tfrac{k\pi}{3}\Big)|\hat{u}_k|^2.$$

Evaluating the six eigenvalues for $k = 0, 1, 2, 3, 4, 5$ gives 0, $z - \frac{\sqrt{3}}{4}$, $3z - \frac{\sqrt{3}}{4}$, $4z$, $3z + \frac{\sqrt{3}}{4}$, and $z + \frac{\sqrt{3}}{4}$. Since $0 < z < \frac{\sqrt{3}}{4}$, the maximum eigenvalue is $3z + \frac{\sqrt{3}}{4} = \sqrt{3} - 3a$ (at $k = 4$), so $zD - Q \leq (\sqrt{3} - 3a)\sum_{j=0}^{5} \|u_j\|^2$.

Combining both bounds with (13) and $\Lambda_{\text{ring}} \geq 0$ yields

$$0 \leq M \leq \sum_{j=0}^{5} \|u_j\|\big((\sqrt{3} - 3a)\|u_j\| - \sqrt{3}\,a\big) - \Big(z + \frac{a}{2}\Big)\Lambda_{\text{ring}}.$$

For both parameter pairs $(a, R) = (29/200,\ 5/26)$ and $(119/800,\ 1/5)$, the inequality $(\sqrt{3} - 3a)R < \sqrt{3}\,a$ is equivalent to $(R - a)\sqrt{3} < 3aR$, which holds strictly using $\sqrt{3} < 26/15$:

- At $(29/200, 5/26)$: $(R - a)(26/15) = \frac{123}{2600} \cdot \frac{26}{15} = \frac{41}{500} = 0.082 < \frac{435}{5200} \approx 0.08365 = 3aR$.
- At $(119/800, 1/5)$: $(R - a)(26/15) = \frac{41}{800} \cdot \frac{26}{15} = \frac{533}{6000} \approx 0.08883 < \frac{357}{4000} = 0.08925 = 3aR$.

Since $\|u_j\| \leq R$ for every j, every term in the sum is strictly negative unless $\|u_j\| = 0$. Therefore $u_j = 0$ for all $j = 0, \ldots, 5$.

It follows that $dA = 0$, so every intact elementary triangle has zero area excess and therefore, by Lemma 7.2, every intact edge retains length exactly 1. Starting from any fixed triangle outside the support of $\psi - \mathrm{id}$, across any fixed unit edge there is a unique positively oriented equilateral triangle; propagating across the connected side-adjacency graph of intact triangles of $\Lambda_1 \setminus \{0\}$ gives $\psi(v) = v$ for all $v \in \Lambda_1 \setminus \{0\}$. □

Proof of Theorem 7.1. Suppose for contradiction that an admissible set $X \subset \mathbb{R}^2 \setminus \Lambda_1$ satisfies $|K_1(X)| \leq |X| + 1$ under the displacement hypothesis of (a) or (b), with $R = 5/26$ in (a) and $R = 1/5$ in (b). Because $2R < 1$, each $x \in X$ has a unique nearest lattice site $v(x) \in \Lambda_1$ (with $\|x - v(x)\| \leq R$), and distinct points of X have distinct nearest sites (since $\|x - y\| \geq 1 > 2R$). Clearly $v(x) \in K_1(x)$ for every $x \in X$. Since $|K_1(X)| \leq |X| + 1$, at most one site in $K_1(X)$ does not belong to $v(X)$; translate Λ_1 so that 0 is this extra site (if one exists) or any site outside $K_1(X)$ otherwise. Define the finitely supported perturbation $\psi\colon \Lambda_1 \setminus \{0\} \to \mathbb{R}^2$ by setting $\psi(v(x)) := x$ for each $x \in X$ and $\psi(v) := v$ for all $v \in \Lambda_1 \setminus (\{0\} \cup v(X))$. Because every fixed site $v \notin \{0\} \cup v(X)$ lies outside $K_1(X)$, it sits at distance ≥ 1 from every $x \in X$, so ψ satisfies hypothesis (i) of Lemma 7.4; since $\|\psi(v) - v\| \leq R$ everywhere, hypothesis (iii) holds as well.

To verify hypothesis (ii) (the squared bond-length cap $1 + \kappa$) in (a), every originally adjacent pair v, w satisfies $\|\psi(v) - \psi(w)\| \leq 1 + 2(5/26) = 18/13$, so $\|\psi(v) - \psi(w)\|^2 \leq 324/169 = 1 + 155/169$. Lemma 7.4 forces $\psi = \mathrm{id}$, contradicting $X \cap \Lambda_1 = \emptyset$.

For (b), take any $x \in X$ with displacement $\delta := x - v(x)$ of length $\rho := \|\delta\| \in (0, 1/5]$. Because $K_1(x)$ is an elementary triangle containing $v(x)$, its other two vertices are $v(x) + e$ and $v(x) + e'$ for two adjacent unit lattice vectors e, e' at angle $\pi/3$ (every lattice site not adjacent to $v(x)$ lies at distance at least $\sqrt{3} - 1/5 > 1$ from x). Let $b_{\parallel} := (e + e')/\sqrt{3}$ be the unit vector along their bisector,

and let $e_\perp := e' - e$ be the unit lattice vector orthogonal to $b_\parallel$. Since $v(x) \pm e_\perp \notin K_1(x)$, we have $\|\delta \mp e_\perp\|^2 \geq 1$, giving $|\langle \delta, e_\perp \rangle| \leq \rho^2/2$. Moreover, $v(x) + e, v(x) + e' \in K_1(x)$ implies $\|\delta - e\|^2 < 1$ and $\|\delta - e'\|^2 < 1$, hence $\langle \delta, e \rangle > \rho^2/2 > 0$ and $\langle \delta, e' \rangle > 0$, so $\langle \delta, b_\parallel \rangle > 0$. Thus the angle θ between δ and $b_\parallel$ satisfies $\sin\theta = |\langle \delta, e_\perp \rangle|/\rho \leq \rho/2 \leq 1/10$. Because every unit lattice direction makes angle at least $\pi/6$ with $b_\parallel$, the projection of δ onto any unit lattice vector e_0 satisfies

$$|\langle e_0, \delta \rangle| \leq \rho \cos\Big(\frac{\pi}{6} - \arcsin\frac{1}{10}\Big) \leq \frac{1}{5}\Big(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{99}}{10} + \frac{1}{2} \cdot \frac{1}{10}\Big) = \frac{\sqrt{297}+1}{100}.$$

This bound holds trivially at fixed sites ($\delta = 0$) as well. Therefore, for any originally adjacent pair v, w ($w - v = e_0$), setting $\Delta\delta := (\psi(w) - w) - (\psi(v) - v)$ gives $\|\Delta\delta\| \leq 2/5$ and $|\langle e_0, \Delta\delta \rangle| \leq (\sqrt{297}+1)/50$, hence

$$\|\psi(w) - \psi(v)\|^2 - 1 = 2\langle e_0, \Delta\delta \rangle + \|\Delta\delta\|^2 \leq \frac{\sqrt{297}+1}{25} + \frac{4}{25} = \frac{\sqrt{297}+5}{25} < \frac{556}{625},$$

since $\sqrt{297} < 431/25 = 17.24$ ($431^2 = 185{,}761 > 185{,}625 = 297 \cdot 625$). Applying Lemma 7.4 with (119/800, 556/625, 1/5) forces $\psi = \mathrm{id}$, again contradicting $X \cap \Lambda_1 = \emptyset$. □

8. Discussion

Summary and a note on the historical record. Theorem 1.1 and Corollary 1.2 settle the $p = 7$ case of L. Fejes Tóth's strong-solidity conjecture and of his order-of-solidity formula $k_p = g_p - 2$, reducing both problems to the single remaining case $p = 6$: the face-incircle packing of the Euclidean hexagonal tiling $\{6, 3\}$ (the densest triangular-lattice circle packing) with one circle removed. Part II establishes elementary structural obstructions for $p = 6$—including the No-Hole Theorem, the Uniform Triangle-Count Inequality, the 30-circle packing X_{30} refuting universal private kills, and near-lattice rigidity up to 5/26 of a lattice side (1/5 for all-triangular families).

We also clarify an attribution in the literature. Bárány and Dolbilin [1, p. 107] cite only L. Fejes Tóth's 1968 paper [8] for the one-vacancy conjecture, while Connelly and Dickinson [4, §1.3] cite both [8] and [9] without citing Bezdek [2]. Inspection of [8] shows that it introduces solidity and proves solidity of $\{p, 3\}$ without discussing the removal of a circle; the strong-solidity conjecture for $\{p, 3\}$ ($p \geq 6$) first appears in print in A. Bezdek [2, p. 203] and L. Fejes Tóth [9, p. 299].

Open directions. Three natural problems stand directly beyond our result:

(1) *The Euclidean case $p = 6$.* Earlier partial results include uniform stability under 1/40 of a diameter [1] (re-proved by rigidity methods in [3, Theorem 2.8.1]), strong solidity of the hexagonal tiling itself under translations [12], and Connelly and Dickinson's reduction to a torus density-gap conjecture [4]. Theorem 7.1 extends uniform stability to 5/26 of a lattice side (1/5 for all-triangular families), while Theorems 6.4 and 6.5 show that any counterexample must enclose a hole and satisfy $n \leq T^2/12 + T/2$; settling the full unrestricted conjecture requires a global planar discharging or area-excess mechanism that handles large displacements.

(2) *Exact value and asymptotics of g_p for $p \geq 8$.* While Corollary 1.2 establishes $g_7 = 3$ and L. Fejes Tóth [9] proved $k_p = g_p - 2$ for all $p \geq 8$ together with a heuristic lower bound $g_p \gtrsim c\sqrt{p}$ for a constant $c > 0$, the exact value and rigorous asymptotic order of g_p as $p \to \infty$ remain open.

(3) *Trihedral Archimedean tilings and higher-order solidity in $\mathbb{R}^2$.* Plain solidity has been established for many trihedral Archimedean circle packings [6, 11], but their strong solidity and orders of solidity remain largely unexplored; in the Euclidean plane, L. Fejes Tóth's Problem 2 [9, p. 302]—whether there exists any packing of congruent convex discs that is solid of order 2—likewise remains open.

Acknowledgments

This work was done overwhelmingly by AI models, steered by the author in a continuous critical dialogue, variously with Google's Gemini, Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

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