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NIM-VALUES OF TWO-MOVE AND SYMMETRIC THREE-MOVE VECTOR SUBTRACTION GAMES, WITH MACHINE-CHECKED PROOFS

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Abstract. In a vector subtraction game a token sits at a point of $\mathbb{N}^n$, a move subtracts one of finitely many fixed vectors, and the player who cannot move loses. For the symmetric three-move game $T(a,b)$ with moves (a,b), (b,a), $(a+b,a+b)$, Larsson, Saha and Yokoo determined the winner when $b \le 2a$; for $b > 2a$ they show pictures of two games and leave the analysis open, and we know of no solution of any such game in the literature. We show that these games are governed by a translation: for an explicit vector w depending on a and b, the Sprague–Grundy value is unchanged under translation by w above the diagonal, and is periodic along the diagonal in a band next to it. A reduction theorem, proved for all $a \le b$, says that both statements hold everywhere once they hold on an explicit finite window, so that they determine the game from a finite table. We prove the two statements (i) for every $m \ge 2$ when $a \le 3$ and $(2m-1)a \le b \le 2ma$; (ii) for every a when $2 \le m \le 14$ and $(2m-1)a \le b \le 2ma$; (iii) for 1792 individual games, of which 862 lie in the complementary ranges $2ma < b < (2m+1)a$, where a single formula for w fits every game tested and the general statement remains a conjecture. We also complete the known cases: the full Sprague–Grundy values of every two-move game in every dimension, by a three-case parity rule, and of $T(a,b)$ for $a \le b \le 2a$. The reduction theorem, the two complete cases and 72 of the individual games are formally verified in Lean 4 with Mathlib. Results (i) and (ii) rest on an SMT solver and the other individual games on a short program; for each claim the paper states which check stands behind it.

1. Introduction

Place a token on a square of the quarter-plane $\mathbb{N}^2$ and fix $1 \le a \le b$. Two players alternately move the token by subtracting (a,b), (b,a) or $(a+b,a+b)$, staying inside the quarter-plane; whoever cannot move loses. This is the symmetric three-move game $T(a,b)$ of Larsson, Saha and Yokoo [7]. They determined its losing positions for $b \le 2a$. For $b > 2a$ they print the computed boards of $T(1,5)$ and $T(2,5)$ and remark that these games "have more complexity"; no rule is given, and we have found no solution of any game $T(a,b)$ with $b > 2a$ in the literature.

Figure 1 shows one of those two boards, now with all Sprague–Grundy values. The picture looks irregular, but the two framed windows are identical. This is the phenomenon the paper is about: *above the diagonal the board repeats under one translation vector w, and next to the diagonal it repeats along the diagonal with period the sum of the two coordinates of w.* Together with the symmetry $x \leftrightarrow y$ these two statements, called (A) and (B) below, reduce the whole infinite board to a finite table.

Main results. The vector w is explicit. Write $u = (a,b)$ and $v = (b,a)$. Every $b \ge a$ satisfies exactly one of $(2m-1)a \le b \le 2ma$ (the *linear ranges*) and $2ma < b < (2m+1)a$ (the *quadratic ranges*) for an integer $m \ge 1$. In the first case $w = 2m\,u + v$; in the second $w = 2(b-ma)\,u + a\,v$, and the translation is valid from a small distance above the diagonal on.

- *A finite check suffices* (Theorem 10, in Lean). For all $a \le b$, statements (A) and (B) hold on the whole board as soon as they hold on an explicit finite window.

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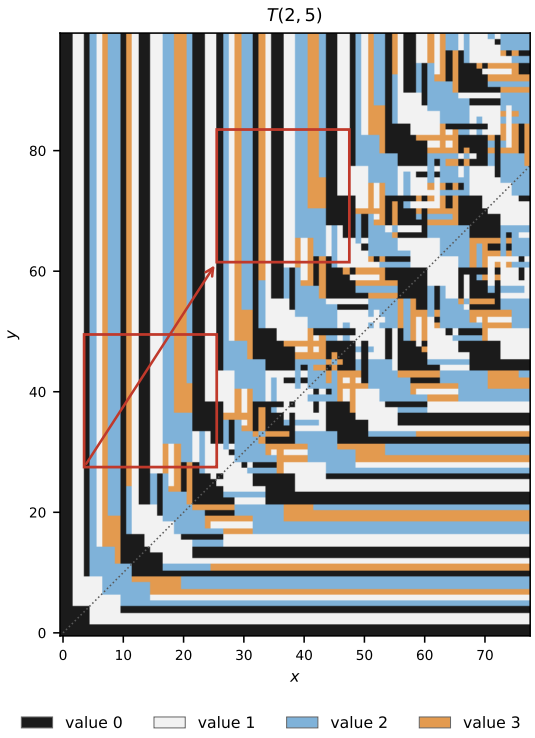

FIGURE 1. Sprague–Grundy values of $T(2,5)$, one of the two games shown without analysis in [7]. The two framed 22×22 windows differ by the translation $w = (22, 34)$ and carry the same values; by Theorem 16 this holds for every position with $y \geq x + 2$. The dotted line is the diagonal.

- *All ratios, small a* (Corollary 22). For $a \leq 3$, (A) and (B) hold in every linear range, for all m. In particular every game $T(1, b)$ is solved, and with it every game in which a divides b.
- *All a, bounded ratio* (Theorem 15). For $2 \leq m \leq 14$, that is for $3a \leq b \leq 28a$ within the linear ranges, (A) and (B) hold for every a.
- *Individual games* (Theorems 13 and 16, Proposition 17). (A) and (B), or their analogues with a margin in the quadratic ranges, hold in 1792 individual games with $b > 2a$, 862 of them in quadratic ranges. One formula for w fits all of them; that it does so for every $b > 2a$ is Conjecture 3 and Conjecture 1.

We also complete the cases that were known at the level of winning and losing: every two-move game in every dimension (Theorem 4) and $T(a, b)$ for $a \leq b \leq 2a$ (Theorem 7), both with full Sprague–Grundy values and both in Lean.

The reduction theorem, the two completed cases and 72 of the individual games are checked by the Lean kernel. The results for small a and for bounded ratio rest on an SMT solver, and the remaining individual games on a short program. Sections 1.4, 8 and 10 say precisely what has to be trusted for each claim, and Section 1.5 what we do and do not know about novelty.

1.1. **The games.** Let $S \subset \mathbb{N}^n \setminus \{0\}$ be finite. In the *vector subtraction game* with move set S, a position is a point $x \in \mathbb{N}^n$, a move replaces x by $x - s$ for some $s \in S$ with $s \leq x$ coordinatewise, and a player with no move loses. These games go back to Golomb [4]; in dimension one they are the classical subtraction games [1]. The *Sprague–Grundy value* (or *nim-value*) $\mathcal{G}(x)$ is defined

by

$$\mathcal{G}(x) = \operatorname{mex}\{\mathcal{G}(x-s) : s \in S,\ s \le x\},$$

where mex of a finite set of non-negative integers is the least non-negative integer not in it [11, 5]. The player to move loses exactly when $\mathcal{G}(x) = 0$ (a *P-position*), and the values, not only the P-positions, are what is needed to play sums of games [1, 10].

We study two families:

- *two-move games* $S = \{u, v\}$ in any dimension, with the *leaper* $u = (a, b)$, $v = (b, a)$ in $\mathbb{N}^2$ as the guiding example;
- the *symmetric additive three-move game* $T(a, b)$ on $\mathbb{N}^2$, with $S = \{u, v, u+v\}$, $u = (a, b)$, $v = (b, a)$, $1 \le a \le b$.

1.2. **What was known.** Larsson, Saha and Yokoo [7] study the *outcomes* (P-positions) of vector subtraction games. For two-move games they give a complete outcome solution, based on the principle that x is a P-position if and only if $x+u+v$ is (their Lemma 2 and Theorem 31). For $T(a, b)$ with $b \le 2a$ they describe the P-positions by update rules along the vectors $(2a+b, 2b+a)$ and $(2b+a, 2a+b)$ (their Proposition 47); the proof is written for $(a, b) = (1, 2)$ and the general case is left to the reader. For $b > 2a$ they show pictures of $T(1, 5)$ and $T(2, 5)$ (their Figures 16 and 18), remark that these "have more complexity" but appear to "settle in periodic patterns", and conjecture that every three-move game has an outcome segmentation (their Conjecture 49). They do not consider nim-values.

Danai, Ellis and Thanatipanonda [2] prove, as their Lemma 3.2, that the nim-values of a two-move vector subtraction game satisfy $\mathcal{G}(x+u+v) = \mathcal{G}(x)$. The rest of their paper concerns subtraction–transfer games, which do not include $T(a, b)$.

In dimension one, the game with moves $\{a, b, a+b\}$ is treated in *Winning Ways* [1], and Larsson and Manabe [6] recently determined its nim-values in what they call the primitive quadratic regime.

The game $T(a, b)$ can also be read as a game of king moves on a Young diagram (Remark 3). Gottlieb, Krnc and Muršič [3] study king moves on Young diagrams; they give the nim-values for rectangles (a result of Berlekamp) and for some staircases with square steps, and remark that general staircases may be out of reach. The diagrams arising from $T(a, b)$ are not among the families they treat.

1.3. **Results in detail.** All results below except items (6) and (8)–(11) are proved in Lean (Section 8). Write $c = b - a$.

(1) **Two-move games (Theorem 4).** For every $u, v \in \mathbb{N}^n \setminus \{0\}$ the value $\mathcal{G}(x)$ is given by a three-case parity rule. This adds the values 1 and 2 to the outcome solution of [7]; given the periodicity lemma of [2] the proof is short.

(2) **The leaper (Corollary 5, Theorem 6).** A closed form in floors, and a sharp *bandwidth law*: $\mathcal{G}(x, y)$ equals the value of the one-heap game $\{a, b\}$ at $\min(x, y)$ whenever $|x - y| \ge (b - a)\lceil b/a \rceil$, and this bound cannot be lowered.

(3) $T(a, b)$ **for** $a \le b \le 2a$ **(Theorem 7).** Explicit values in the region where neither $(2a + b, a + 2b)$ nor its mirror image fits; invariance of all values under translation by $(2a + b, a + 2b)$ for $x \le y + 2c$; and period $3(a + b)$ along the diagonal for $|x - y| \le 2c$. The case $\mathcal{G} = 0$ recovers Proposition 47 of [7], now with a proof for all $a \le b \le 2a$.

(4) $T(a, b)$ **for all** $a \le b$ **(Theorems 8–10).** Values depend on a position only through the lattice combinations $aI + bJ$ below each coordinate; values are eventually constant along rows; and a translation statement (A) together with a diagonal-period statement (B) hold everywhere once they hold on a finite window. A variant allows the translation to fail in a strip of fixed width above the diagonal.

(5) **Solved games with $b > 2a$ (Theorem 13).** For $m \in \{2,3,4\}$, $a \le 5$ and $(2m-1)a \le b \le 2ma$, statements (A) and (B) hold with translation vector $(2ma+b, 2mb+a)$ and diagonal period $(2m+1)(a+b)$. Each of these 60 games (33 up to scaling) is thereby determined by a finite table. This includes $T(1,5)$, one of the two games pictured in [7].

(6) **All games with $(2m-1)a \le b \le 2ma$, $2 \le m \le 14$ (Theorem 15).** Statements (A) and (B) hold for every such pair. Its finite hypotheses are verified for all parameters by an SMT solver, not in Lean.

(7) **Solved games in the remaining ranges (Theorem 16).** For 12 coprime pairs with $2ma < b < (2m+1)a$, including $T(2,5)$, the other game pictured in [7], the analogous statements hold with the translation vector $2(b-ma)\,u + a\,v$, at distance at least D above the diagonal for an explicit small D.

(8) **Further single games (Proposition 17).** The statements of item (5) for 870 more coprime pairs, with m up to 100, and those of item (7) for 862 coprime pairs in all (the twelve of item (7) among them); verified by a program, not in Lean.

(9) **All games $T(1,b)$ (Theorem 18).** Statements (A) and (B) for every $b \ge 3$, hence for every m; verified by an SMT solver, not in Lean. By item (4) this determines every game whose ratio b/a is an integer.

(10) **All games $T(2,4m-1)$ (Theorem 20).** Statements (A) and (B) for every $m \ge 2$; verified by an SMT solver, not in Lean.

(11) **All games $T(3,6m-2)$ and $T(3,6m-1)$ (Theorem 21).** Statements (A) and (B) for every $m \ge 2$; SMT solver, not in Lean. With items (9), (10) and (4) this settles every linear range with $a \le 3$ (Corollary 22).

Section 7 states what we could *not* prove: the statements of items (5) and (7) for all a, b in the respective ranges not covered by items (6) and (8)–(11), a local rule describing how values change under the translation below the diagonal, and a classification of the games with $3a < b < 4a$ into five shapes.

1.4. **Status of each claim.** We separate four kinds of claim.

Machine-checked in Lean. Theorems 4, 6, 7, 8, 9, 10 and 12, Corollary 5, Lemma 2 for $T(a,b)$, the existence and uniqueness of the Sprague–Grundy function of each game, and Theorems 13 and 16 for each of the $60+12$ games separately. These depend on the axioms `propext`, `Quot.sound` and, for the two-move games and Theorem 7, `Classical.choice`, and on no others; the 72 single-game theorems use only the first two.

Computer-assisted, not in Lean. Theorems 15, 18, 20 and 21 and Proposition 17. Each combines a Lean theorem (`certifiedD`, `certifiedQ`) with a verification outside Lean: by the SMT solver Z3 for the theorems, by a program using an array library for the proposition.

Proved on paper only. The corollary following Theorem 13 and the corresponding paragraph after Theorem 16 (each game is determined by a finite table); the passage from the pairs listed in Theorems 13 to their multiples and back via Theorem 8; and Remark 3. The proofs printed in Sections 3–5 are summaries of the Lean proofs; where they differ, the Lean proof is the authority.

Checked numerically only. Everything in Section 7, including Conjectures 1–3; the statement in Section 6 that, in the games searched, the translation vectors are exactly the multiples of (2); and the remarks that statement (A) fails just below the diagonal.

1.5. **On novelty.** We searched for the statements above in [7], [2], [6], [3], the papers citing them that we could find, and recent arXiv submissions on subtraction games (November 2026), and did not find them. The search was not exhaustive and no specialist has yet reviewed the claim. Items (1)–(3) are refinements of known outcome results, and item (1) in particular is a

short step from [2]. To our knowledge items (4)–(7) contain the first proved solutions of games $T(a,b)$ with $b > 2a$. A formal proof establishes correctness, not novelty.

2. Preliminaries

For a game with move set S on $\mathbb{N}^n$, a function $G : \mathbb{N}^n \to \mathbb{N}$ is a *Sprague–Grundy function* if for every x: (i) $G(x-s) \neq G(x)$ for every $s \in S$ with $s \leq x$, and (ii) for every $w < G(x)$ there is $s \in S$, $s \leq x$, with $G(x-s) = w$. Since every move decreases the coordinate sum, induction shows that exactly one such function exists; we denote it by $\mathcal{G}$. We use the following two consequences repeatedly.

Lemma 1 (mex test)**.** *If a number m satisfies* (i) *and* (ii) *at x in place of $\mathcal{G}(x)$, then $\mathcal{G}(x) = m$.*

Lemma 2 (equal options)**.** *Let x, x' be positions from which the same moves $s \in S$ are available. If $\mathcal{G}(x-s) = \mathcal{G}(x'-s)$ for each of them, then $\mathcal{G}(x) = \mathcal{G}(x')$.*

For $T(a,b)$ the move set is symmetric under exchanging coordinates, hence $\mathcal{G}(x,y) = \mathcal{G}(y,x)$.

Remark 3 (king moves on a Young diagram). The positions reachable from $P = (x,y)$ in $T(a,b)$ are the points $P - Iu - Jv$ with (I,J) in $R(P) = \{(I,J) \in \mathbb{N}^2 : aI + bJ \leq x,\ bI + aJ \leq y\}$, and a move increases I, or J, or both, by one. So $\mathcal{G}(P)$ is the value, at the corner cell, of the game in which a king steps right, up or diagonally inside the Young diagram $R(P)$, a region cut out by two lines of slopes $-a/b$ and $-b/a$. We do not use this description in the proofs.

Throughout, $u = (a,b)$, $v = (b,a)$, $s = u + v$ and $c = b - a$.

3. Two-move games

Theorem 4. *Let $u, v \in \mathbb{N}^n \setminus \{0\}$ and $x \in \mathbb{N}^n$. Let k be the largest integer with $k(u+v) \leq x$, put $x_0 = x - k(u+v)$, and let m_u (resp. m_v) be the largest integer j with $ju \leq x_0$ (resp. $jv \leq x_0$). Then*

$$\mathcal{G}(x) = \begin{cases} 0 & \text{if } m_u \text{ and } m_v \text{ are both even,} \\ 2 & \text{if one of them is odd and the other is even and positive,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Step 1: $\mathcal{G}(x+u+v) = \mathcal{G}(x)$. This is Lemma 3.2 of [2]; we include the argument, by induction on the coordinate sum of x. Values lie in $\{0,1,2\}$, since there are two moves. Put $z = x + u + v$; its options are $x+u$ and $x+v$, and x is an option of both.

If $\mathcal{G}(x) = 0$, both options of z have positive value, so $\mathcal{G}(z) = 0$.

If $\mathcal{G}(x) = 1$, some option of x has value 0, say $x - u$. Then $x + v - u$ is a position with the option $x - u$, so its value is not 0; hence both options of $x + v$, namely x and $x + v - u$, are non-zero, and $\mathcal{G}(x+v) = 0$. Thus $\mathcal{G}(z) \neq 0$. Also $\mathcal{G}(x+u) \neq 1$ because x is an option of $x + u$. So the options of z have values 0 and an element of $\{0,2\}$, and $\mathcal{G}(z) = 1$.

If $\mathcal{G}(x) = 2$, both $x - u$ and $x - v$ exist and their values are 0 and 1 in some order. By induction $\mathcal{G}(x+v) = \mathcal{G}(x-u)$ and $\mathcal{G}(x+u) = \mathcal{G}(x-v)$, so the options of z have values 0 and 1, and $\mathcal{G}(z) = 2$.

Step 2: the value at x_0. By Step 1, $\mathcal{G}(x) = \mathcal{G}(x_0)$, and $u + v \not\leq x_0$. Consider the positions $x_0 - ju$, $1 \leq j \leq m_u$. From none of them is v available, since $ju + v \leq x_0$ would give $u + v \leq x_0$; and u is available exactly when $j < m_u$. So play from $x_0 - ju$ is forced and lasts $m_u - j$ moves, and $\mathcal{G}(x_0 - ju) = (m_u - j) \bmod 2$. The same holds with u and v exchanged. Hence the options of x_0 have the values $(m_u - 1) \bmod 2$ (present if $m_u \geq 1$) and $(m_v - 1) \bmod 2$ (present if $m_v \geq 1$), and taking the mex gives the three cases. □

The set $\{\mathcal{G} = 0\}$ is the outcome solution of [7]. The statement and proof do not depend on the dimension; in Lean, positions are functions $\iota \to \mathbb{N}$ for an arbitrary finite index type ι.

3.1. **The leaper.** Now let $n = 2$, $u = (a, b)$, $v = (b, a)$ with $1 \le a \le b$. For a position (x, y) put

$$r = \min(x, y) \bmod (a + b), \qquad d = |x - y|, \qquad Y = r + d, \qquad K = \min\big(\lfloor r/a \rfloor, \lfloor Y/b \rfloor\big).$$

Corollary 5 (closed form)**.** *$\mathcal{G}(x, y) = K \bmod 2$ if $r < b$, and $\mathcal{G}(x, y) = 2 - (K \bmod 2)$ if $r \ge b$.*

Proof. By symmetry let $x \le y$. Then $k = \lfloor x/(a + b) \rfloor$ and $x_0 = (r, Y)$, so $m_u = K$ and $m_v = \min(\lfloor r/b \rfloor, \lfloor Y/a \rfloor)$. Since $r < a + b \le 2b$, $\lfloor r/b \rfloor \in \{0, 1\}$. If $r < b$ then $m_v = 0$ and Theorem 4 gives $K \bmod 2$. If $r \ge b$ then $Y \ge r \ge a$, so $m_v = 1$, and $K \ge 1$; Theorem 4 gives 1 for K odd and 2 for K even. □

For $b \le 2a$ this can be rewritten as a sum of two thresholds, $\mathcal{G}(x, y) = [\, r + \min(d, c) \ge b \,] + [\, r + \min((d - c)^+, c) \ge a + b \,]$, where $[\cdot]$ is 1 if the statement holds and 0 otherwise; this form is also verified in Lean, and it fails for every $b > 2a$ we tested.

Let g be the Sprague–Grundy function of the one-heap subtraction game with moves $\{a, b\}$. By Theorem 4 in dimension one, $g(n) = t \bmod 2$ if $\rho < b$ and $g(n) = 2 - (t \bmod 2)$ if $\rho \ge b$, where $\rho = n \bmod (a + b)$ and $t = \lfloor \rho/a \rfloor$.

Theorem 6 (band-width law)**.** *Let $D_0 = (b - a)\lceil b/a \rceil$. If $|x - y| \ge D_0$ then $\mathcal{G}(x, y) = g(\min(x, y))$. If $a < b$, there is a position with $|x - y| = D_0 - 1$ and $\mathcal{G}(x, y) \ne g(\min(x, y))$.*

Proof. Compare Corollary 5 with the formula for g: both use the same case distinction on r, with K in place of $t = \lfloor r/a \rfloor$. So $\mathcal{G}(x, y) = g(\min(x, y))$ if and only if $K \equiv t \pmod 2$, which certainly holds when $\lfloor Y/b \rfloor \ge t$, that is when $r + d \ge tb$. As $r \ge ta$, it suffices that $d \ge t(b - a)$, and $t \le \lfloor (a + b - 1)/a \rfloor = \lceil b/a \rceil$. For sharpness take $x = a\lceil b/a \rceil$, $y = x + D_0 - 1$. Then $x < a + b$, so $r = x$ and $t = \lceil b/a \rceil$, while $Y = tb - 1$ and $K = t - 1$. □

Figure 2 shows the band for $(a, b) = (2, 5)$.

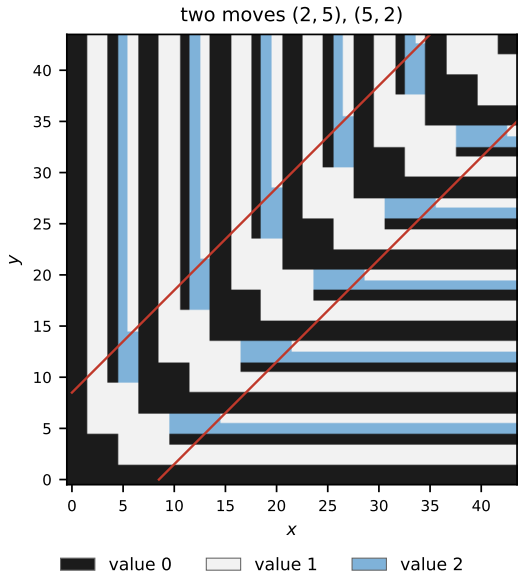

FIGURE 2. Nim-values of the two-move game $\{(2, 5), (5, 2)\}$. Outside the two lines $|x - y| = D_0 = 9$ the value depends only on $\min(x, y)$ (Theorem 6).

4. The three-move game for $a \le b \le 2a$

Let $w_1 = 2u+v = (2a+b,\ a+2b)$ and $w_2 = u+2v$. For a position (x, y) let $L(x, y) \in \{0, 1, 2, 3\}$ be the number of true statements among

(1) $u \le (x, y)$ or $v \le (x, y)$;
(2) $2u \le (x, y)$ or $u + v \le (x, y)$ or $2v \le (x, y)$;
(3) $3u \le (x, y)$ or $3v \le (x, y)$.

Theorem 7. *Let $1 \le a \le b \le 2a$ and let $\mathcal{G}$ be the Sprague–Grundy function of $T(a, b)$.*

(a) (corner values) *If neither $w_1 \le (x, y)$ nor $w_2 \le (x, y)$, then $\mathcal{G}(x, y) = L(x, y)$.*
(b) (translation) *If $x \le y + 2c$, then $\mathcal{G}\big((x, y) + w_1\big) = \mathcal{G}(x, y)$. By symmetry, if $y \le x + 2c$ then $\mathcal{G}\big((x, y) + w_2\big) = \mathcal{G}(x, y)$.*
(c) (diagonal period) *If $|x - y| \le 2c$, then $\mathcal{G}\big(x + 3(a + b),\ y + 3(a + b)\big) = \mathcal{G}(x, y)$.*
(d) (formula above the diagonal) *For $\rho < 2a+b$, $\rho \le Y + 2c$ and $k \ge 0$, $\mathcal{G}\big(\rho+k(2a+b),\ Y + k(a + 2b)\big) = L(\rho, Y)$, and this equals $\lfloor \rho/a \rfloor$ when $Y \ge 3b$.*

In particular (a) *and* (b) *determine every value.*

For $\mathcal{G} = 0$ this is the description of the P-positions in Proposition 47 of [7], with the same vectors w_1, w_2. Part (d) says that far above the diagonal the game behaves like a one-heap game with period $2a + b$ in x.

Figure 3 (left) shows $T(2, 3)$.

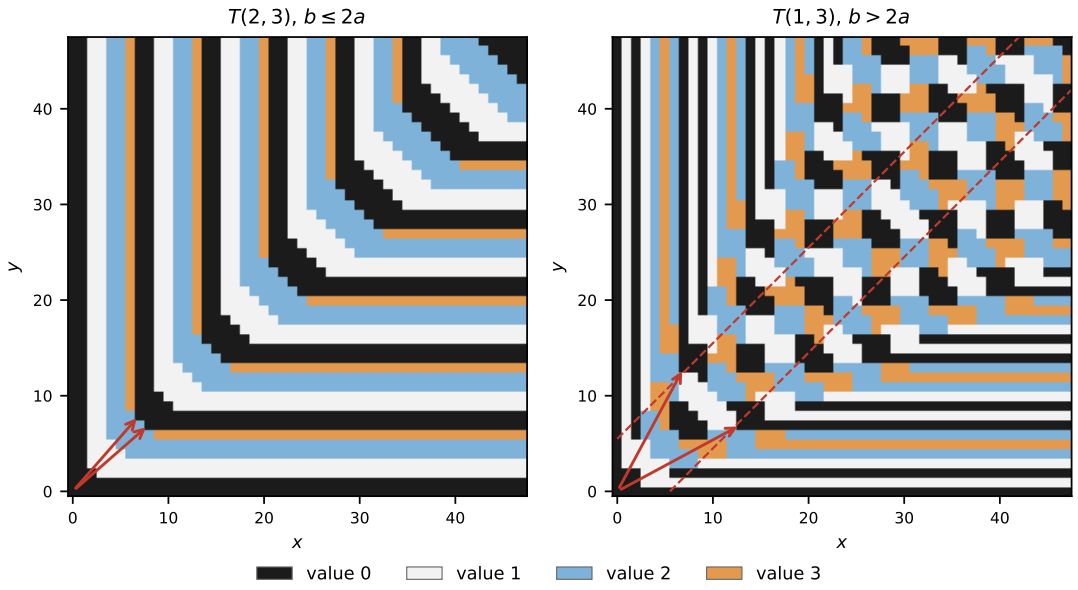

Figure 3. Nim-values of $T(2, 3)$ and $T(1, 3)$. The arrows are the translation vector and its mirror image: $(7, 8)$ on the left (Theorem 7), $(7, 13)$ on the right (Theorem 13). On the right the dashed lines bound the band $|x - y| < 6$, in which the values have period 20 along the diagonal.

Proof outline. Parts (c) and (d) follow from (a) and (b): $w_1 + w_2 = 3(a + b)(1, 1)$, and in the band $|x - y| \le 2c$ one of the two translations applies to (x, y) and the other to its translate; (d) is an induction on k.

Parts (a) and (b) are proved together by strong induction on $x + y$, in the form

(R) if neither w_1 nor w_2 fits in (x, y), then $\mathcal{G}(x, y) = L(x, y)$;
(T) if $w_1 \le (x, y)$ and $x \le y + c$, then $\mathcal{G}(x, y) = \mathcal{G}\big((x, y) - w_1\big)$.

Since the two coordinates of w_1 differ by c, statement (T) for all such positions is exactly (b).

Step (R). By Lemma 1 it suffices that L satisfies the mex rule on the region: for each available move the option lies in the region and has a different L, and every smaller value is attained. Both are statements about finitely many linear inequalities in a, b, x, y.

Step (T)*, on or above the diagonal.* Write the position as $Q + w_1$ with $Q = (q_1, q_2)$, $q_1 \le q_2 + c$. If $u + v \le Q$, all three moves are available at Q and at $Q + w_1$, the options of $Q + w_1$ are the translates of the options of Q, each of these satisfies the hypothesis of (T), and Lemma 2 applies. Otherwise Q is near a wall; then $\mathcal{G}(Q) = L(Q)$ by (R), each option of $Q + w_1$ is reduced by (T), its mirror image, or (R) to an explicit value of L, and it remains to check the mex rule for these three values against $L(Q)$, again a finite case analysis.

Step (T)*, below the diagonal* ($y < x \le y + c$). The mirror image of the position lies above the diagonal, so by the previous step $\mathcal{G}(x, y) = \mathcal{G}\big((x, y) - w_2\big)$. If $y \ge 3(a + b)$, applying (T) to $(x, y) - w_2$ and its mirror image to $(x, y) - w_1$ shows that both equal $\mathcal{G}\big((x, y) - w_1 - w_2\big)$. If $y < 3(a + b)$, both $(x, y) - w_1$ and $(x, y) - w_2$ lie in the region of (R), and the claim becomes the identity $L\big((x, y) - w_1\big) = L\big((x, y) - w_2\big)$.

The three case analyses are carried out in Lean by linear integer arithmetic (the `omega` decision procedure); the largest, the existence half of the mex rule in Step (R), is split into 18 cases. We do not reproduce them here. □

5. The three-move game for all $a \le b$: structure and reduction

In this section $1 \le a \le b$ is arbitrary.

Theorem 8 (class transfer)**.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be the Sprague–Grundy functions of $T(a, b)$ and $T(a', b')$. Suppose x, x' satisfy $aI + bJ \le x \iff a'I + b'J \le x'$ for all $I, J \ge 0$, and that y, y' satisfy the same condition. Then $\mathcal{G}(x, y) = \mathcal{G}'(x', y')$.*

Proof. Induction on $x + y$. The cases $(I, J) = (1, 0), (0, 1), (1, 1)$ show that the same moves are available at (x, y) and at (x', y'). The hypothesis is preserved when corresponding combinations are subtracted from x and x', so corresponding options again satisfy it, and Lemma 1 applies. □

In particular $\mathcal{G}(x, y)$ depends on x only through the set $\{(I, J) : aI + bJ \le x\}$, and $\mathcal{G}_{ka,kb}(x, y) = \mathcal{G}_{a,b}(\lfloor x/k \rfloor, \lfloor y/k \rfloor)$ for every $k \ge 1$, so it is enough to study coprime a, b.

Theorem 9 (saturation)**.** *If $y \ge b\lfloor x/a \rfloor$ and $y' \ge b\lfloor x/a \rfloor$, then $\mathcal{G}(x, y) = \mathcal{G}(x, y')$.*

Proof. Induction on x. Under the hypothesis a move is available at (x, y) exactly when its first coordinate fits in x, and likewise at (x, y'). For each of the three moves the options again satisfy the hypothesis, because $b\lfloor (x - a)/a \rfloor = b\lfloor x/a \rfloor - b$, $b\lfloor (x - b)/a \rfloor \le b\lfloor x/a \rfloor - b$ and $b\lfloor (x - a - b)/a \rfloor \le b\lfloor x/a \rfloor - 2b$. Now use Lemma 2. □

Fix integers p, r with

$$p \ge a + b, \qquad r - p \ge 2c, \tag{1}$$

and put $w = (p, r)$, $q = p + r$, $W = r - p$. Consider the statements

(A) $\mathcal{G}(x + p,\ y + r) = \mathcal{G}(x, y)$ for all $y \ge x$;
(B) $\mathcal{G}(x + q,\ y + q) = \mathcal{G}(x, y)$ for all $p \le x \le y < x + W$.

(A) is a translation on and above the diagonal; (B) is a period along the diagonal inside the band $0 \le y - x < W$.

Theorem 10 (reduction to a finite window)**.** *Assume* (1)*. Suppose that*

(H1) (A) *holds for all* $x < a + b$;
(H2) (A) *holds whenever* $a + b \le x < 2p + r + a + b$ *and* $0 \le y - x < c$;
(H3) (B) *holds whenever* $p \le x < 2p + b$.

Then (A) *and* (B) *hold in full.*

Proof. We prove (A) at $P = (x, y)$ and (B) at P by strong induction on the coordinate sum of the larger of the two points involved, $P + w$ respectively $P + (q, q)$. "By induction" refers to instances with a smaller sum.

(A) *at* P, $y \ge x$. If $x < a + b$ this is (H1). Let $x \ge a + b$; then all three moves are available at P and at $P + w$.

If $y - x \ge c$, the options $P - u$, $P - v$, $P - s$ all lie on or above the diagonal. The options of $P + w$ are their translates by w, so by induction they have the same values, and Lemma 2 gives $\mathcal{G}(P + w) = \mathcal{G}(P)$.

If $0 \le y - x < c$ and $x < 2p + r + a + b$, this is (H2).

Let $0 \le y - x < c$ and $x \ge 2p + r + a + b$. Here $P - u$ lies below the diagonal, where (A) is not available. Put $P' = P - (q, q)$. We show $\mathcal{G}(P + w) = \mathcal{G}(P' + w)$; then $\mathcal{G}(P' + w) = \mathcal{G}(P')$ by induction (A), and $\mathcal{G}(P') = \mathcal{G}(P)$ by induction (B), since P' lies in the band with first coordinate at least p. To compare $P + w$ and $P' + w$ we compare options (Lemma 2).

- The u-options $P + w - u$ and $P' + w - u$ differ by (q, q), and for both the difference of coordinates is $(y - x) + W - c \in [W - c, W) \subset [0, W)$. They are equal by induction (B).
- The v-options are $(P - v) + w$ and $(P' - v) + w$, with $P - v$ and $P' - v$ above the diagonal; by induction (A) their values are $\mathcal{G}(P - v)$ and $\mathcal{G}(P' - v)$. These points differ by (q, q) and have coordinate difference $(y - x) + c \in [c, 2c) \subset [0, W)$, so they are equal by induction (B).
- The s-options are treated like the v-options, with coordinate difference $y - x$.

(B) *at* P, $p \le x \le y < x + W$. If $x < 2p + b$ this is (H3). Let $x \ge 2p + b$. We compare the options of $P + (q, q)$ with those of P.

- $P - u$ has coordinate difference $(y - x) - c \in [-c, W - c)$. If it is non-negative, induction (B) applies to $P - u$. If it is negative, the mirror image of $P - u$ has difference in $(0, c] \subset [0, W)$, and induction (B) applies to the mirror image.
- $R = P - v$ has coordinate difference $(y - x) + c \in [c, W + c)$. If it is less than W, induction (B) applies. Otherwise R lies above the band; then $R - w$ has coordinate difference in $[0, c)$ and first coordinate $x - b - p \ge p$, and

  $$\mathcal{G}\big(R + (q, q)\big) = \mathcal{G}\big((R - w) + (q, q) + w\big) = \mathcal{G}\big((R - w) + (q, q)\big) = \mathcal{G}(R - w) = \mathcal{G}(R),$$

  using induction (A), then (B), then (A).
- $P - s$ has the same coordinate difference as P, and induction (B) applies.

In each case the option values agree, and Lemma 2 gives $\mathcal{G}(P + (q, q)) = \mathcal{G}(P)$. □

Remark 11. (H2) and (H3) concern finitely many positions. (H1) concerns the strip $x < a + b$, which is infinite, but by Theorem 9 the values $\mathcal{G}(x, y)$ and $\mathcal{G}(x + p, y + r)$ are constant in y from an explicit bound on, so (H1) is also a finite check.

For the ranges $2ma < b < (2m+1)a$ we need a variant in which the translation is only claimed at distance at least D above the diagonal, and the diagonal period only from $x \ge X_0$ on. For integers $D, X_0 \ge 0$ consider

(A_D) $\mathcal{G}(x + p,\, y + r) = \mathcal{G}(x, y)$ for all $y \ge x + D$;
(B_D) $\mathcal{G}(x + q,\, y + q) = \mathcal{G}(x, y)$ for all $X_0 \le x \le y < x + W + D$.

Theorem 12 (reduction with a margin)**.** *Assume* (1)*. Suppose that* (A_D) *holds for all* $x < a+b$*, and whenever* $a+b \le x < X_0+q+a+b$ *and* $D \le y-x < D+c$*, and that* (B_D) *holds whenever* $X_0 \le x < X_0+p+b$*. Then* (A_D) *and* (B_D) *hold in full.*

Proof. The proof of Theorem 10 goes through with every coordinate difference shifted by D where (A) is used, the band $[0, W)$ replaced by $[0, W+D)$, and the bound $x \ge p$ in (B) replaced by $x \ge X_0$. For (A_D) at P with $D \le y-x < D+c$ and $x \ge X_0+q+a+b$, the u-options of $P+w$ and $P'+w$ have coordinate difference in $[D+W-c, D+W)$, the v-options are translates of points with difference in $[D+c, D+2c)$, and the s-options of points with difference in $[D, D+c)$; all of these lie in $[D, W+D)$, where both (A_D) and (B_D) are available by induction. For (B_D) at P, when $P-v$ lies above the band, $R-w$ has coordinate difference in $[D, D+c)$, and the same chain of equalities applies. Theorem 10 is the case $D=0$, $X_0=p$. □

6. Solved games with $b > 2a$

The intervals $(2m-1)a \le b \le 2ma$ and $2ma < b < (2m+1)a$, $m \ge 1$, partition the range $b \ge a$. We call the first kind *linear* and the second *quadratic*, by analogy with the one-heap game [6] and because of the period lengths found below. Theorem 7 is the linear range $m=1$.

Theorem 13. *Let* $m \in \{2,3,4\}$*,* $1 \le a \le 5$ *and* $(2m-1)a \le b \le 2ma$*. Put* $p = 2ma+b$ *and* $r = 2mb+a$*. Then in* $T(a,b)$*:*

(A) $\mathcal{G}(x+p,\, y+r) = \mathcal{G}(x,y)$ *for all* $y \ge x$*;*
(B) $\mathcal{G}\big(x+(2m+1)(a+b),\, y+(2m+1)(a+b)\big) = \mathcal{G}(x,y)$ *whenever* $p \le x \le y < x+(2m-1)(b-a)$*.*

These are 60 pairs (a,b); 33 of them are coprime, and the others are multiples of coprime pairs in the list, hence equivalent to them by Theorem 8. The vector (p,r) equals $2m\,u+v$, and for $m=1$ it is the vector w_1 of Section 4.

Corollary 14. *Each game in Theorem 13 is determined by a finite table: with* $W = (2m-1)(b-a)$ *and* $q = (2m+1)(a+b)$*, every position can be reduced, without changing its value, to a position* (x,y) *with either* $x < p$ *and* $y \le b\lfloor x/a \rfloor$*, or* $p \le x < p+q$ *and* $x \le y < x+W$*.*

Proof. By symmetry let $x \le y$. If $x \ge p$ and $y-x \ge W$, subtract (p,r) by (A); repeat. Then $x < p$, or $y-x < W$. In the second case, if $x \ge p+q$ subtract (q,q) by (B) and repeat. If $x < p$, Theorem 9 allows us to lower y to $b\lfloor x/a \rfloor$. □

Proof of Theorem 13. For each pair we apply Theorem 10; conditions (1) hold since $r-p = (2m-1)(b-a)$ and $m \ge 2$. The hypotheses are verified by a certificate. A table D of candidate values on a rectangle $[0,X) \times [0,Y)$, with $X \ge 3p+r+a+b$ and Y large enough to contain all positions occurring in (H1)–(H3) after saturation, is produced by a program outside Lean. Lean then checks, by evaluation in its kernel, (i) that D satisfies the mex rule at every cell of the rectangle, which implies $D = \mathcal{G}$ there by induction, and (ii) that D satisfies the equalities (H1)–(H3). The table is not trusted: if it were wrong, check (i) would fail. The rectangle is 38×46 for $T(1,3)$ and 610×961 for $T(5,40)$. □

Figure 3 (right) shows $T(1,3)$. Two differences from the case $b \le 2a$ are visible in the data and explain the form of Theorem 10. First, for $m \ge 2$ the translation (A) fails just below the diagonal in every game we computed, whereas for $m=1$ it extends to $x \le y+2c$. Second, as a consequence, the translation by w and its mirror image no longer overlap, so the diagonal period in the band does not follow from (A) and has to be established separately as (B).

6.1. **All games in the linear ranges with $m \le 14$.** For fixed m the certificate can be made symbolic. Write $\alpha = b-(2m-1)a$ and $\beta = 2ma-b$, so that $a = \alpha+\beta$ and $b = 2m\alpha+(2m-1)\beta$.

Theorem 15. *Let $2 \le m \le 14$. For all $a \ge 1$ and $(2m-1)a \le b \le 2ma$, statements* (A) *and* (B) *of Theorem 13 hold in $T(a,b)$, with $(p,r) = (2ma+b, 2mb+a)$.*

Proof (computer-assisted; not formalized in Lean). Fix m. The interval $[2m-1, 2m]$ of ratios b/a is cut at 13/4, 10/3, 7/2, 11/3 for $m = 2$, and at $2m-1+\frac{1}{3}$, $2m-1+\frac{1}{2}$, $2m-1+\frac{2}{3}$ for $3 \le m \le 14$; the end point $b = 2ma$ is treated as a piece of its own. For each piece we give a *symbolic table*: on each axis a list of breakpoints of the form $i\alpha+j\beta$ with fixed integers $i, j \ge 0$, and one value in $\{0,1,2,3\}$ for each cell between consecutive breakpoints. For every α, β in the piece this defines a function D on the window $[0, 3p+r+a+b) \times [0, Y)$, where $Y = 2p+2r+2b$ for $m \le 3$ and $Y = 6mb+a+1$ for $m \ge 4$. We verify that, for all integers $\alpha, \beta \ge 0$ in the piece, D satisfies the mex rule at every point of the window and the equalities (H1)–(H3); then $D = \mathcal{G}$ on the window and Theorem 10 applies (in Lean, these are exactly the hypotheses of `certifiedD`).

The verification is a finite computation in linear integer arithmetic. The breakpoints and all shifts (a, b, $a+b$, p, r, $p+r$) are linear forms in α, β. For each cell and each shift, the cells that the shifted point can reach, for some admissible α, β, are found by feasibility queries to the SMT solver Z3; for every combination of reachable cells one checks the required property on the table values, and a combination that violates it must be infeasible. For $m = 2$ no combination violates it; for larger m some do, and all of them are infeasible. The breakpoint lists are also checked to be ordered and to start at 0, and the pieces to cover $[2m-1, 2m]$. The tables have about 135×135 cells per piece for $m = 2$ and 2810×2833 for $m = 14$; the check takes 3 seconds for $m = 2$ and about eight minutes for $m = 14$ on a large machine. Changing one table entry makes it fail. Independently, the tables for $m \le 6$ were instantiated for 170 games, including the end points of the pieces, and agree with directly computed values on the whole window. □

The symbolic tables were obtained by fitting the breakpoints of computed games with $7 \le a \le 19$. The proof relies on the correctness of Z3 and of a script of under 200 lines; it is not checked by the Lean kernel. The tables of different pieces are distinct, so for instance the window table of $T(a,b)$ with $3a < b < 4a$ takes exactly five combinatorial forms. The number of cells grows with m, so this method treats one value of m at a time; the cut points, however, follow the same pattern for every $m \ge 3$, which suggests that a description uniform in m exists.

6.2. **The ranges $2ma < b < (2m+1)a$.** In these ranges no vector satisfies (A) on the whole half-plane $y \ge x$ in the games we searched, but a translation exists once a thin strip above the diagonal is excluded. An exhaustive search over all vectors up to about two diagonal periods, in thirteen games, found in each game that the vectors valid from a small distance D on are exactly the multiples of

$$w = 2(b-ma)\,u + a\,v = \big(a(3b-2ma),\ 2b(b-ma)+a^2\big). \tag{2}$$

At the two ends of the range, $b = 2ma$ and $b = (2m+1)a$, this is a times the vector $2m\,u+v$, respectively $2(m+1)\,u+v$, of the neighbouring linear range.

Theorem 16. *For each pair (a,b) in Table 1, with (p,r) given by (2), $q = p+r$, $W = r-p$ and the listed D and X_0, statements* (A_D) *and* (B_D) *hold in $T(a,b)$.*

Proof. Theorem 12, with certificates as in the proof of Theorem 13. □

As before, each of these games is determined by a finite table: positions with $x \ge p$ and $y-x \ge W+D$ are reduced by (A_D), positions in the band with $x \ge X_0+q$ by (B_D), and the strip $x < p$ is handled by Theorem 9. The game $T(2,5)$ is the second game pictured in [7]; its

TABLE 1. The games of Theorem 16.

(a,b)	m	(p,r)	D	X_0	q
$(2,5)$	1	$(22,34)$	2	22	56
$(3,7)$	1	$(45,65)$	6	58	110
$(3,8)$	1	$(54,89)$	3	54	143
$(4,9)$	1	$(76,106)$	12	110	182
$(4,11)$	1	$(100,170)$	4	100	270
$(5,11)$	1	$(115,157)$	20	178	272
$(5,12)$	1	$(130,193)$	10	152	323
$(5,13)$	1	$(145,233)$	10	168	378
$(2,9)$	2	$(38,94)$	2	38	132
$(3,13)$	2	$(81,191)$	6	81	272
$(3,14)$	2	$(90,233)$	3	90	323
$(2,13)$	3	$(54,186)$	2	54	240

values are invariant under translation by $(22,34)$ for $y \geq x+2$, and have period 56 along the diagonal for $x \geq 22$, $0 \leq y-x < 14$.

6.3. **Further single games, verified outside Lean.** For a single game the hypotheses of Theorems 10 and 12 form a finite computation. Besides the Lean certificates above we ran it, for many more games, with a short program that is not checked by Lean.

Proposition 17 (computer-assisted; not formalized in Lean)**.** (i) *Statements* (A) *and* (B) *of Theorem 13, with* $(p,r) = (2ma+b, 2mb+a)$*, hold for the* 870 *coprime pairs with* $(2m-1)a \leq b \leq 2ma$ *and either* $11 \leq m \leq 40$*,* $a \leq 8$*, or* $41 \leq m \leq 100$*,* $a \leq 2$*.*

(ii) *Statements* (A_D) *and* (B_D)*, with* (p,r) *given by* (2)*,* $D = a\big((2m+1)a-b\big)$ *and* $X_0 = p+q$*, hold for the* 862 *coprime pairs with* $2ma < b < (2m+1)a$*,* $m \leq 10$*,* $2 \leq a \leq 25$ *and* $q = p+r \leq 30000$*.*

Proof. For each pair the program `experiments/open_cases.py` computes the values on the window of the reduction theorem by the mex recursion and checks the hypotheses (H1)–(H3), the saturation bounds and the side conditions (1), in exactly the form of the Lean theorem `certifiedQ` (with $D = 0$, $X_0 = p$ in case (i)). All 1732 checks succeed (870 and 862); the largest window has 82600×239002 entries. Theorems 10 and 12 then give the statements. □

In (ii) the parameters D and X_0 are those of Conjecture 3 below, not values fitted game by game; the twelve games of Theorem 16 are among the 862, there with smaller, measured D and X_0. The proposition rests on the correctness of the program and of the array library it uses.

6.4. **The games $T(1,b)$ for all b.** For $a = 1$ every $b \geq 3$ lies in a linear range, with $m = \lceil b/2 \rceil$, and the statements can be proved for all m at once.

Theorem 18 (computer-assisted; not formalized in Lean)**.** *For every* $b \geq 3$*, statements* (A) *and* (B) *of Theorem 13 hold in* $T(1,b)$*. Explicitly:*

- *for even* b*:* $(p,r) = (2b,\, b^2+1)$*, diagonal period* $q = (b+1)^2$*, band width* $W = (b-1)^2$*;*
- *for odd* b*:* $(p,r) = (2b+1,\, b^2+b+1)$*,* $q = (b+1)(b+2)$*,* $W = b(b-1)$*.*

Corollary 19. *For all* $k \geq 1$ *and* $b \geq 3$*, the value of* (x,y) *in* $T(k,kb)$ *is the value of* $(\lfloor x/k \rfloor, \lfloor y/k \rfloor)$ *in* $T(1,b)$ *(Theorem 8). So every game* $T(a,b)$ *whose ratio* b/a *is an integer is determined by a finite table.*

Proof of Theorem 18. For $3 \le b \le 8$ this is Theorem 13. Let $b \ge 8$ be even or $b \ge 9$ odd, and write positions in base b: $x = k_x b + j_x$, $y = k_y b + j_y$ with $0 \le j_x, j_y < b$. The window of Theorem 10 is $[0, X) \times [0, Y)$ with $X = b^2 + 7b + 2$, $Y = 3b^2 + 1$ for even b and $X = b^2 + 8b + 5$, $Y = 3b^2 + b + 1$ for odd b. On it we define a candidate F:

- for $x < p$ (the walls) and for $x \ge p$, $0 \le y - x < W$ (the band), F is given by a decision tree. Each inner node tests a linear inequality, or a parity, in b, the digits k_x, j_x, k_y, j_y, the two digits of x in base $b+1$ and the two digits of $y - x$ in base $b-1$; each leaf carries a value in $\{0, 1, 2, 3\}$. The trees have 21 and 75 leaves for even b, and 23 and 151 leaves for odd b;
- for $x \ge p$ and $y - x \ge W$ we put $F(x, y) = F(x - p, y - r)$; inside the window this is applied at most twice;
- for $y < x$ we put $F(x, y) = F(y, x)$.

The three moves and the translations by (p, r) and (q, q) change the base-b digits by fixed amounts with at most one borrow or carry; for even b, for instance, $p = 2 \cdot b$, $r = b \cdot b + 1$ and $q = (b + 2)\, b + 1$. The digits in base $b \pm 1$ are obtained from those in base b by a case distinction with a bounded number of cases, since $x = k_x(b + 1) + (j_x - k_x)$ and $y - x = (k_y - k_x)(b - 1) + (k_y - k_x + j_y - j_x)$ and all digits are at most $3b + 2$ in the window. Therefore each of the statements "F satisfies the mex rule at every point of the window" and "F satisfies (H1), (H2), (H3)" is a formula of linear integer arithmetic in the five variables b, k_x, j_x, k_y, j_y. The SMT solver Z3 finds no counterexample with $b \ge 8$ even, and none with $b \ge 9$ odd. By induction the first statement gives $F = \mathcal{G}$ on the window. The remaining hypotheses of Theorem 10, conditions (1) and the saturation bounds, are inequalities between polynomials in b (for instance $Y_1 = 2b^2 - 1$) and are checked directly. □

Since F is defined above the band by the translation itself, (H1) and (H2) hold by construction; the content of the verification is the mex rule, in particular at the points above the band, and the diagonal period (H3). The choice of digits is suggested by the game: for even b the diagonal period is $(b+1)^2$ and the band width $(b-1)^2$, and the moves change $y - x$ by multiples of $b - 1$.

The trees were obtained by fitting computed tables for $8 \le b \le 27$; before the solver was run they reproduced the tables for $b \in \{30, 36, 44\}$ and $\{31, 37, 45\}$ without error. As independent checks, the candidate was afterwards compared with the computed table on the whole window for sixteen values of b up to 67 (up to $6.8 \cdot 10^7$ entries each), without a difference; the solver confirms that the digit conversions used in the encoding always have a solution; and changing the value of one leaf makes the solver return a counterexample. The proof relies on Z3, on a script of about 250 lines (`scripts/a1_uniform.py`) and on the tree files; it is not checked by the Lean kernel.

6.5. **The games $T(2, 4m - 1)$ for all m.** The method of Theorem 18 is not limited to $a = 1$. For $a = 2$ the odd values of b in linear ranges are $b = 4m - 1$.

Theorem 20 (computer-assisted; not formalized in Lean)**.** *For every $m \ge 2$, statements* (A) *and* (B) *of Theorem 13 hold in $T(2, b)$ with $b = 4m - 1$. Explicitly*

$$(p, r) = \left(2b + 1,\ \tfrac{b^2+b+4}{2}\right), \qquad q = \tfrac{(b+2)(b+3)}{2}, \qquad W = \tfrac{(b-1)(b-2)}{2}.$$

By Theorem 8 this also determines $T(2k, (4m - 1)k)$ for every $k \ge 1$. For $m \le 14$ the statement is contained in Theorem 15; it is new for $m \ge 15$.

Proof. For $m = 2, 3$ this is Theorem 13. Let $m \ge 4$. The window is $[0, X) \times [0, Y)$ with $X = (2m + 7)b + 7$ and $Y = (6m - 1)b + 1$. We proceed as in the proof of Theorem 18, with two changes. First, a low digit j in base b, $b + 2$ or $b - 2$ is split as $j = 2h + e$ with $e \in \{0, 1\}$,

and the trees test h and e in place of j. Second, since p, r, q, W, X, Y now have high digits that are linear in m (for instance $r = 2m \cdot b + 2$), the tests may also compare a high digit with a multiple of m plus a constant. The trees have 52 leaves (walls) and 264 leaves (band). With $b = 4m - 1$ substituted, the mex rule on the window and (H1), (H2), (H3) are formulas of linear integer arithmetic in m and the digits. Z3 finds no counterexample with $m \geq 4$; the mex rule was submitted in 64 cases, according to the parities of four digits, to whether the point lies in the band, and to whether it lies on the walls. □

The trees were fitted on computed tables for $15 \leq b \leq 47$; the band tree needed three rounds in which every entry it got wrong on the tables up to $b = 63$ was added to the training data. Afterwards the candidate agreed with the computed table on the whole window for $b \in \{15, 19, 31, 67, 75, 91\}$ (up to $6.0 \cdot 10^7$ entries). Changing one leaf of either tree makes the solver return a counterexample. The digit forms of the window bounds used in the encoding were compared with the bounds of Theorem 10, and its side conditions were checked, for $4 \leq m \leq 250$. The script is `scripts/fam_uniform.py` (about 290 lines).

For the other family with $a = 2$ and b odd, $b = 4m + 1$ in the quadratic ranges, the same fitting procedure did not produce a correct candidate for the band, so that case remains open (Conjecture 3).

6.6. **The games $T(3, b)$ in the linear ranges, and all linear ranges with $a \leq 3$.** For $a = 3$ the values of b in linear ranges that are not multiples of 3 are $b = 6m - 2$ and $b = 6m - 1$.

Theorem 21 (computer-assisted; not formalized in Lean)**.** *For every $m \geq 2$ and $b \in \{6m - 2, 6m-1\}$, statements* (A) *and* (B) *of Theorem 13 hold in $T(3, b)$, with $(p, r) = (6m+b, 2mb+3)$, $q = p + r$ and $W = (2m - 1)(b - 3)$.*

Proof. For $m = 2, 3$ this is Theorem 13 ($b = 10, 11, 16, 17$). For $m \geq 4$ the proof is that of Theorem 20 with low digits split as $j = 3h+e$, $e \in \{0, 1, 2\}$. The windows are $X = (2m+7)b+12$, $Y = (6m - 1)b + 1$ for $b = 6m - 2$ and $X = (2m + 7)b + 9$, $Y = 6mb + 1$ for $b = 6m - 1$; the trees have 53 and 358 leaves, respectively 29 and 320 leaves. Z3 finds no counterexample to the mex rule (64 cases per family) or to (H1), (H2), (H3). □

The trees were fitted on tables for $b \leq 53$ and refined on tables up to $b = 77$; afterwards the candidates agreed with the computed tables on the whole window for $b \in \{82, 94, 106\}$ and $b \in \{83, 95, 107\}$ (up to $5.3 \cdot 10^7$ entries each). The digit forms of the window bounds and the side conditions were checked for $4 \leq m \leq 167$. The longest single solver case took about twelve hours. Changing one leaf of any of the four trees makes the solver return a counterexample (for the two band trees this test was stopped once counterexamples had been found, before all 32 cases had finished).

Corollary 22. *Statements* (A) *and* (B)*, transported by Theorem 8 where a and b are not coprime, hold for every pair with $a \leq 3$ and $(2m - 1)a \leq b \leq 2ma$, for every $m \geq 2$.*

Proof. For $a = 1$ this is Theorem 18. For $a = 2$, b is either $4m - 1$ (Theorem 20) or even, and then $T(2, b)$ is determined by $T(1, b/2)$. For $a = 3$, b is $6m - 2$ or $6m - 1$ (Theorem 21) or a multiple of 3, and then $T(3, b)$ is determined by $T(1, b/3)$. □

7. Experiments and conjectures

Nothing in this section is proved. All computations use the mex recursion directly; the programs are in the repository.

Conjecture 1. *Statements* (A) *and* (B) *of Theorem 13 hold for all $m \geq 15$ and all $a \geq 4$, $(2m - 1)a \leq b \leq 2ma$.*

Before Theorem 15 was available we tested the statements on 1500×1500 tables for the 105 pairs with $m \in \{2,3,4\}$ and $a \le 7$, without exception. For larger m the evidence is the pattern of Theorem 15 and the 870 games of Proposition 17, which reach $m = 100$. By Theorem 10 the conjecture is a finite statement for each pair, but the window grows with a and b. The method of Theorem 15 should apply to each fixed m, but not to all m at once. For $a \le 3$ the same statement is Corollary 22; whether the digit method works for every fixed a is open.

Conjecture 2 (local rule below the diagonal)**.** *In the setting of Conjecture 1, let* $w = (2ma + b, 2mb + a)$ *and* $W = (2m-1)(b-a)$*, and let* $Q = (x, y)$ *with* $-W \le y - x < 0$*, away from the coordinate axes. Then* $\mathcal{G}(Q + w) = \mathcal{G}(Q)$*, except that*

- $\mathcal{G}(Q + w) = 3$ *if* $\big(\mathcal{G}(Q); \mathcal{G}(Q-u), \mathcal{G}(Q-v), \mathcal{G}(Q-s)\big)$ *is* $(1;2,0,3)$ *or* $(1;3,0,2)$*;*
- $\mathcal{G}(Q + w) = 1$ *if it is* $(3;0,1,2)$ *or* $(3;0,2,1)$*.*

Tested on 16 pairs with $m \in \{2,3,4\}$; a weaker form (the values at Q and $Q + w$ are equal or are $\{1,3\}$, for $-c \le y - x < 0$) was tested on all 105 pairs. We also observed, on 12 games, that in the interior of this region the six values at $Q-2u$, $Q-u-v$, $Q-2v$, $Q-2u-v$, $Q-u-2v$, $Q-2u-2v$ take only 20 of the 4^6 possible combinations. A proof that only these occur would imply Conjectures 1 and 2; we do not have one.

Five shapes. The symbolic tables of Theorem 15 were first found for $m = 2$, as follows. For the 166 coprime pairs with $5 \le a \le 23$ and $3a < b < 4a$, the window table with equal consecutive rows and columns merged took exactly five forms, one for each interval of b/a cut out by $13/4$, $10/3$, $7/2$ and $11/3$, and within an interval every breakpoint was a fixed integer combination of $\alpha = b - 3a$ and $\beta = 4a - b$. Tables fitted on games with $7 \le a \le 19$ reproduced the 33 games with $24 \le a \le 29$ exactly, and the solver then verified them for all parameters. Figure 4 shows two games in the same interval and one in another. Theorem 8 explains why the shape can only change when the order of two combinations $aI + bJ$ changes.

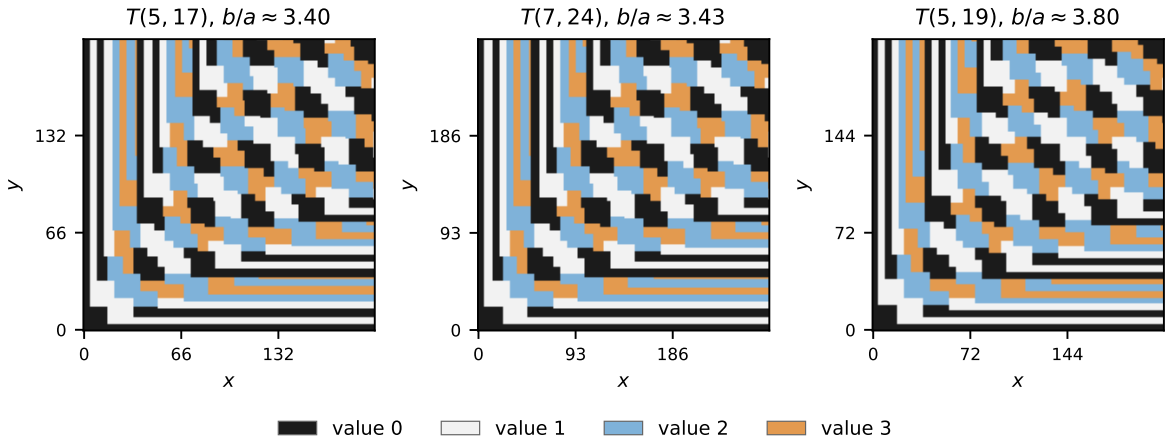

FIGURE 4. Nim-values of three games with $3a < b < 4a$. The first two have b/a in the same interval $(10/3, 7/2)$ and the same combinatorial shape; the third has $b/a \in (11/3, 4)$.

Conjecture 3 (the remaining ranges)**.** *Let* $m \ge 1$*,* $2ma < b < (2m+1)a$ *and* $\gcd(a,b) = 1$*, and let* (p,r) *be given by* (2)*,* $q = p + r$*,* $W = r - p$*. Then* (A_D) *and* (B_D) *hold with* $D = a\big((2m+1)a - b\big)$ *and* $X_0 = p + q$*.*

By Proposition 17 the conjecture holds for the 862 coprime pairs with $a \le 25$, $m \le 10$ and $q \le 30000$; it was first tested for the 95 of them with $a \le 12$, $m \le 4$ and $q \le 3000$, on tables of side $3q + 2p$. The smallest admissible D equals $a((2m+1)a - b)$ in 48 of these games and is smaller in the others; the smallest admissible X_0 is p in 61 games and never exceeds $p + 0.35\,q$. Here $q = (a+b)\big(2b - (2m-1)a\big)$ grows quadratically with the moves, in contrast to $(2m+1)(a+b)$ in the linear ranges; a similar contrast is known for the one-heap game [6]. Together with Conjecture 1, this would give a translation vector for every $b > 2a$: $2m\,u + v$ in the linear ranges and (2) in the others, the two agreeing up to the factor a where the ranges meet. One consistency check is available: far above the diagonal, Theorem 9 shows that $\mathcal{G}(x, y)$ is the value of the one-heap game $\{a, b, a+b\}$ at x, so the first coordinate of any translation vector must be a multiple of the period of that game. For the 140 coprime pairs with $a \le 9$ and $2a < b \le 7a$ we computed that period, and it equals the first coordinate p of our vector in every case ($2ma + b$ in the linear ranges, $a(3b - 2ma)$ in the others); for $2a < b < 3a$ this agrees with the heap period $a(3b - 2a)$ of [6], and for $(a, b) = (2, 5)$ with the period 22 noted in [7]. We do not know why the margin D appears.

8. Formalization

The theorems of Sections 3–6 that are not marked "computer-assisted" (that is, all except Theorems 15, 18, 20, 21, Proposition 17 and Corollary 22) are verified in Lean 4 [9] (version 4.36.0) with Mathlib [8] (v4.36.0). The development contains no `sorry`, no additional axioms and no `native_decide`; the theorems depend only on Lean's standard axioms. In each case the statement is about an arbitrary function satisfying the Sprague–Grundy property, and existence and uniqueness of such a function are proved separately, so the results are not vacuous. We did not use a library of combinatorial game theory; the definitions are self-contained and short.

The certificates of Theorems 13 and 16 store the table as base-4 digits of natural numbers and are checked by kernel evaluation; the largest takes about nine minutes and 80 GB of memory. Table 2 lists the Lean names. The Lean sources, the symbolic tables of Theorem 15 and the scripts that reproduce every number in Section 7 are in the repository

`https://github.com/ChakrawarShubham/leaper-subtraction-games-lean`.

The Lean sources referred to in this paper are the directory `lean/src`, whose git tree hash is `6477707cb7d2b04197bdfd15f96d519b51dd3857`. An independent replay of all 104 modules of the development with `leanchecker`, the stand-alone kernel checker distributed with Lean, passed on 24 November 2026 (Mathlib was taken from its compiled files, not replayed).

TABLE 2. Lean names of the main statements.

Statement	Lean name
Theorem 4, $n = 2$	`TwoMove.twoMove_sg`
Theorem 4, any dimension	`VecTwoMove.vecTwoMove_sg`
Corollary 5	`Leaper.leaper_sg_general`, `leaper_sg_closed_form`
Theorem 6	`Leaper.band_outside`, `band_sharp`
Theorem 7	`ThreeMove.threeMove_main`, `value_small`, `translation`, `diag_period`, `upper_formula`, `far_value`
Theorem 8	`ThreeMove.class_transfer`
Theorem 9	`ThreeMove.sat`
Theorem 10	`ThreeMove.reduce`, `certifiedD`
Theorem 13	`ThreeMove.three_`a`_`b, one per game
Theorem 12	`ThreeMove.reduceD`, `certifiedQ`
Theorem 16	`ThreeMove.threeQ_`a`_`b, one per game

9. WHAT A READER MUST TRUST

To trust the theorems of the first kind in Section 1.4, a reader needs to check that the Lean statements say what this paper says, and nothing else of ours. For the three-move game the statements use two definitions (file `ThreeDefs.lean`; Unicode symbols are rendered as mathematics):

```
def Move (a b x y x' y' : ℕ) : Prop :=
  (a ≤ x ∧ b ≤ y ∧ x' = x - a ∧ y' = y - b) ∨
  (b ≤ x ∧ a ≤ y ∧ x' = x - b ∧ y' = y - a) ∨
  (a + b ≤ x ∧ a + b ≤ y ∧ x' = x - (a + b) ∧ y' = y - (a + b))

def IsSG (a b : ℕ) (G : ℕ → ℕ → ℕ) : Prop :=
  ∀ x y, (∀ x' y', Move a b x y x' y' → G x' y' ≠ G x y) ∧
         (∀ w, w < G x y → ∃ x' y', Move a b x y x' y' ∧ G x' y' = w)
```

The statements for the two games pictured in [7] are, verbatim up to line breaks:

```
theorem three_1_5 {G : ℕ → ℕ → ℕ} (hG : IsSG 1 5 G) :
    (∀ x y, x ≤ y → G (x + 11) (y + 31) = G x y) ∧
    (∀ x y, 11 ≤ x → x ≤ y → y < x + 20 → G (x + 42) (y + 42) = G x y)

theorem threeQ_2_5 {G : ℕ → ℕ → ℕ} (hG : IsSG 2 5 G) :
    (∀ x y, x + 2 ≤ y → G (x + 22) (y + 34) = G x y) ∧
    (∀ x y, 22 ≤ x → x ≤ y → y < x + 12 + 2 → G (x + 56) (y + 56) = G x y)
```

The other 70 statements have the same form with other constants, and the general theorems are stated in the same vocabulary; subtraction is truncated subtraction on $\mathbb{N}$, which is why each move carries its inequalities. That a function with property `IsSG` exists and is unique is `ThreeMove.sg_isSG` and `ThreeMove.IsSG.unique`.

Behind the statements one trusts the Lean kernel and the three axioms named in Section 1.4. The certificates are checked by evaluation inside the kernel (`decide +kernel`), so the kernel's built-in arithmetic on natural numbers is part of what is trusted; compiled code is not (`native_decide` is not used). The tables, the programs that produced them, Mathlib's proofs and the AI system that wrote the proofs are not trusted: they are checked.

For Theorem 15 one must trust more: the solver Z3, the checking script `check_symbolic.py` (under 200 lines), and the correspondence, which is not machine-checked, between the conditions

that the script verifies and the hypotheses of `certifiedD`. For Proposition 17 one must trust the program `open_cases.py` (under 100 lines), the Python interpreter and the NumPy library. For Theorem 18 one must trust Z3, the script `a1_uniform.py` and its encoding of the window bounds in digits, which was compared with the bounds of the theorem numerically for $b < 400$. The same holds for Theorems 20 and 21 with the script `fam_uniform.py`, where the comparison was made for $m \leq 250$ and $m \leq 167$ respectively.

10. Limitations

- *No human expert has checked the work.* The Lean theorems are checked by the kernel. The rest, including the claim of novelty, the choice of what to state, and the paper-only arguments listed in Section 1.4, has not been reviewed by a specialist in combinatorial games.
- *The general problem is open.* For $b > 2a$ we prove the two statements for thirteen values of m in the linear ranges and, for $a \leq 3$, for all m; otherwise only for single games. For $a \geq 4$ we have no proof that is uniform in m, and in the ranges $2ma < b < (2m+1)a$ no result beyond single games and their multiples; the formula (2), the margin D and the start X_0 of the band are fitted to data.
- *Theorem 15 is weaker than the others.* It is not checked by the Lean kernel. The independent comparison of its tables with computed games covers $m \leq 6$ only; for $7 \leq m \leq 14$ the tables were checked by the solver alone.
- *The proofs verify, they do not explain.* Theorems 13, 15 and 16 are established by certificates. They show that the pattern holds and give no reason for the vector $2m\,u+v$ or for (2).
- *Values, not strategies in closed form.* For $b > 2a$ a game is "solved" in the sense that every value is found by reduction to a finite table (up to 610×961 entries); we give no closed formula for the table.
- *Cost.* The largest Lean certificate needs about 80 GB of memory. The 72 certificates are checked one file at a time; only four of them are part of the default build of the library.
- *A small area.* These games were introduced recently and few people work on them. The result should be read as a contribution to that topic, and as a test of how far computer search and formal proof reach in it.

Use of AI and acknowledgements

The computational exploration, the formulation of the conjectures, the Lean proofs and the first draft of this text were produced by an AI system (Claude, Anthropic) working under the author's direction. Correctness of the theorems rests on the Lean kernel and, for the results marked computer-assisted, on the solver and the programs named in Section 9, not on the AI system or on the author's reading of the proofs. The literature search was also carried out with the system and is limited as described in Section 1.5.

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