On Artin L-functions with a cubic complement

Hernandez Juan

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Abstract

We prove a real-zero criterion for the Artin conjecture for representations whose projective image is a Frobenius group with abelian kernel and complement of order three. Nonvanishing of the Dedekind zeta function of the complement fixed field on $(0,1)$ implies cuspidal automorphic matching outside finitely many places and entire continuation of the original Artin L-function. Unconditionally, the exceptional primes of a canonical cyclic descent have summable reciprocal norms.

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1 The statement

Let F be a number field, let $G_F = \mathrm{Gal}(\overline{F}/F)$, and let $\rho : G_F \longrightarrow \mathrm{GL}(V)$ be a continuous representation with finite image. We use the Artin and automorphic local normalizations in which matching unramified parameters have identical Euler polynomials. The Artin conjecture asserts that $L_F(s,\rho)$ is entire when ρ is irreducible and nontrivial [1]. Throughout the paper, an Artin L-function without a superscript contains *all* finite local factors. A completed function additionally contains its archimedean factors and conductor power.

Here the projective image belongs to a classical class of Frobenius groups. Write

$$Q = A \rtimes C, \qquad C = \langle c \rangle \simeq C_3, \qquad |A| = n^2, \tag{1.1}$$

where A is finite abelian and

$$\{a \in A : c(a) = a\} = \{0\}. \tag{1.2}$$

We use additive notation for A. Both nonidentity elements of C act without a nonzero fixed point. In particular the nonzero elements of A occur in orbits of size three, so

$$n^2 \equiv 1 \pmod 3, \qquad 3 \nmid n. \tag{1.3}$$

The coprimality used below is therefore a consequence of the group hypotheses, rather than an additional restriction.

Theorem 1.1. *Let $n > 1$, and let $\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$ be irreducible with finite image. Suppose that its projective image is the group in* (1.1)–(1.2)*. Let K/F be the Galois extension cut out by the projective kernel, and put*

$$E = K^A, \qquad M = K^C.$$

Thus E/F is cyclic cubic and $[M : F] = n^2$. There is a unitary cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, uniquely determined among the descents of the automorphic representation associated with $\rho|_{G_E}$ by $\omega_\pi = \det \rho$, with the following properties.

(i) *Let S be a sufficiently large finite set of finite places, and let $\mathcal{B}$ be the places $v \notin S$ at which the unramified parameters of π_v and ρ differ. Then*

$$\sum_{v \in \mathcal{B}} \frac{1}{\mathrm{N}\, v} < \infty. \tag{1.4}$$

In particular $\mathcal{B}$ has Dirichlet density zero.

(ii) *If*

$$\zeta_M(\sigma) \neq 0 \qquad \text{for every real } 0 < \sigma < 1, \tag{1.5}$$

then $\mathcal{B}$ is finite. Both $L_F(s, \rho)$ and its completed function are entire.

The conclusion in (ii) is about the original irreducible Artin representation, not merely its adjoint. The hypothesis is explicitly conditional. It is not deduced here from a general assertion about Dedekind zeta functions. The generalized tetrahedral family of Lapid, with $A = (\mathbb{F}_q^2, +)$ and $n = q$, is included [9, §5]. The formulation above also permits abelian groups A not supplied with a vector-space structure over a finite field. It asserts a result about a given representation; it does not assert the existence of a representation or a number-field realization for every abstract semidirect product.

The cyclic descent comes from Arthur–Clozel and the determinant normalization in Lapid [2, 9]. The analytic argument uses a positive Euler quotient and Landau's theorem, in the manner of Calegari's work on S_5-extensions [4, Lemmas 15–17]. We give the group-theoretic and local calculations in full, as well as the argument at $s = 0$ and the passage from partial matching to the complete Artin function. Section 8 records a separate classical consequence for primitive solvable adjoints. Appendix A contains the complete source and output of a finite check; none of the proofs depends on that check.

2 The finite group and its endomorphisms

Put $G = \rho(G_F)$, regarded as a finite subgroup of $\mathrm{GL}(V)$. Let $Z = G \cap \mathbb{C}^\times I$, and let N be the inverse image of A in G. Then $G/Z = Q$, $N/Z = A$, $G/N = C$, and $[N, N] \subset Z$. In particular N is nilpotent of class at most two. Averaging a Hermitian form over G makes its action unitary.

Lemma 2.1. *The restriction of V to N is irreducible.*

Proof. Clifford theory writes $V|_N$ as equal-multiplicity copies of the distinct G-conjugates of an irreducible N-module W. Its orbit has size one or three. Size three would imply $3 \mid n$, contrary to (1.3). Thus W is invariant under G.

For completeness, invariance across this cyclic quotient implies extension. Choose $x \in G$ mapping to a generator of G/N, and an intertwiner T on W such that $TW(h)T^{-1} = W(xhx^{-1})$ for $h \in N$. The operator $T^3 W(x^3)^{-1}$ commutes with $W(N)$, and hence is scalar. Also T commutes with $W(x^3)$. Multiplying T by a suitable complex scalar gives $T^3 = W(x^3)$; this defines an extension $\widetilde{W}$ to G. The multiplicity space $\mathrm{Hom}_N(\widetilde{W}, V)$ is then an irreducible ordinary representation of G/N. All irreducible representations of the cyclic group G/N have dimension one. The multiplicity is therefore one. □

The scalar action identifies each commutator with a complex root of unity. Define

$$b : A \times A \longrightarrow \mathbb{C}^\times, \qquad b(a, a') = [x_a, x_{a'}], \tag{2.1}$$

where $x_a, x_{a'}$ are lifts to N, and $[x, y] = xyx^{-1}y^{-1}$. Centrality of the commutators shows that b is well defined, alternating, and multiplicative in each variable. It is invariant under G-conjugation.

Lemma 2.2. *The pairing b is nondegenerate. For one lift x_a of each $a \in A$, with $x_0 = 1$, the matrices $T_a = x_a$ form a basis of* $\mathrm{End}(V)$. *Under conjugation by N, the line $\mathbb{C}T_a$ has character $a' \mapsto b(a', a)$. Moreover* $\mathrm{tr}(T_a) = 0$ *for $a \neq 0$.*

Proof. If $b(a', a) = 1$ for all a', the matrix T_a commutes with N. Lemma 2.1 and Schur's lemma make it scalar; the definition of Z then gives $a = 0$. This proves nondegeneracy. The conjugation assertion follows directly from (2.1). Nondegeneracy makes these $|A| = n^2$ characters distinct. Their nonzero eigenvectors T_a are consequently linearly independent, and their number is $\dim \mathrm{End}(V)$. For $a \neq 0$, choose a' with $b(a', a) \neq 1$. Taking traces in $T_{a'} T_a T_{a'}^{-1} = b(a', a) T_a$ proves the last assertion. □

Proposition 2.3. *As representations of Q,*

$$V \otimes V^\vee \simeq \mathbb{C}[Q/C] = \mathrm{Ind}_C^Q \mathbf{1}. \tag{2.2}$$

Consequently, with every finite local factor included,

$$L_F(s, \rho \otimes \rho^\vee) = \zeta_M(s). \tag{2.3}$$

If $g \in G$ maps to $Q \setminus A$, then

$$|\operatorname{tr} \rho(g)|^2 = 1, \qquad \rho(g)^3 \text{ is scalar.} \tag{2.4}$$

Here $\rho(g)$ denotes the matrix of the already faithful G-representation.

Proof. The scalar subgroup acts trivially on $\mathrm{End}(V)$, so the action factors through Q. Lemma 2.2 says that its restriction to A contains every character of A once. The zero character has line $\mathbb{C}I$, on which all of Q acts trivially. The action of C on nontrivial characters is free: if a character χ is fixed by c, then $\chi((c-1)a) = 1$ for every a; the map $c - 1$ is bijective by (1.2), so $\chi = 1$.

For each orbit of three nontrivial characters, its three weight lines form the irreducible representation $\mathrm{Ind}_A^Q \chi$. Indeed the induced representation has exactly those three distinct A-weights and is irreducible by the usual inner-product form of Frobenius reciprocity. A nonzero map into the three weight lines is therefore an isomorphism. The affine action of $Q = A \rtimes C$ on $A \simeq Q/C$ has precisely the same decomposition after Fourier transform on A. This proves (2.2). Artin induction, including its ramified factors, gives (2.3).

An affine transformation $a' \mapsto a + c^j(a')$, $j = 1, 2$, has a unique fixed point because $1 - c^j$ is bijective. Its permutation character is therefore one. Since the action on V is unitary, the character of $V \otimes V^\vee$ is $|\operatorname{tr} \rho(g)|^2$. Finally $(1 + c + c^2) = 0$ on A: multiply by the bijection $c - 1$ and use $c^3 - 1 = 0$. Thus every element of $Q \setminus A$ has order three, which proves the scalar assertion. □

3 A specified cyclic descent

We first make explicit the finite-group induction used to obtain the automorphic representation over E. Choose a maximal isotropic subgroup $L \subset A$ for b, and let H be its inverse image in N. If $a \in L^\perp$, then $\langle L, a \rangle$ is again isotropic, since b is alternating. Maximality gives $L = L^\perp$. For a perfect pairing on a finite abelian group, $|L|\,|L^\perp| = |A|$: restriction of characters onto L is surjective, with annihilator $L^\perp$. It follows that $|L| = n$.

The group H is abelian because its commutators are the trivial scalars $b(L, L)$. It is normal in N, since $N/Z = A$ is abelian. Choose a one-dimensional constituent θ of $V|_H$. On Z, it equals the faithful scalar character of V. An element $x_a \in N$ stabilizes θ precisely

when $b(a, L) = 1$, equivalently $a \in L^\perp = L$. Thus the inertia group of θ in N is exactly H. Clifford theory (or Frobenius reciprocity and the distinct conjugate characters on H) gives

$$V|_N \simeq \operatorname{Ind}_H^N \theta. \tag{3.1}$$

The induced module is irreducible, has dimension $[N : H] = n$, and contains $V|_N$, which also verifies the isomorphism directly.

The quotient N/H is finite abelian. Choose a chain

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r = N \tag{3.2}$$

with each quotient H_i/H_{i-1} cyclic of prime order. Every $\operatorname{Ind}_H^{H_i} \theta$ is irreducible: otherwise its induction to N, equal to (3.1), would decompose. Let $\widetilde{K}/F$ be the extension cut out by ρ, with group G. Since $\widetilde{K}^N = E$, the fixed fields in (3.2) form a tower of cyclic prime-degree extensions ending at E.

We use the following classical automorphic results [2, 8, 7] in their stated forms.

(a) Class field theory associates a finite-order Hecke character to a finite one-dimensional Galois character. Cyclic automorphic induction and base change exist for cyclic prime-degree extensions and match unramified induction and restriction of parameters. Induction of a cuspidal representation is cuspidal if its distinct Galois conjugates are inequivalent.

(b) A Galois-invariant cuspidal representation over a cyclic prime-degree extension descends to a cuspidal representation. Its cuspidal descents form a single orbit under twisting by the characters of that extension. Base change carries central characters by norm.

(c) Cuspidal representations of GL_n satisfying equality of their unramified local components outside a finite set are isomorphic (strong multiplicity one).

These are the deep automorphic inputs; no new descent theorem is assumed. Their use for the present family is also described explicitly in [9, Proposition 1 and §5].

Start with the Hecke character corresponding to θ over $\widetilde{K}^H$ and apply cyclic automorphic induction along (3.2). The representation at every stage is cuspidal. To check this last point, if two conjugates of an intermediate cuspidal representation were isomorphic, unramified matching and Chebotarev would identify the corresponding finite Galois representations. Induction to the next group in (3.2) would then be reducible, contradicting the preceding paragraph. We obtain a cuspidal representation Π_E matching $\rho|_{G_E}$ outside a finite set. The finite-order initial character is unitary, and cyclic induction preserves unitarity.

Conjugation by $\mathrm{Gal}(E/F)$ fixes the Galois representation $\rho|_{G_E}$. Matching and strong multiplicity one consequently show that Π_E is invariant. By cyclic descent it has a cuspidal descent π_0 over F. We take this descent unitary; equivalently, its nonunitary real norm twist would remain nonunitary after base change, contrary to the unitarity of Π_E.

Let η be a generator of the order-three group of Hecke characters associated with E/F. The central characters of Π_E and $\rho|_{G_E}$ are equal by their unramified matching and class field theory. Hence $\omega_{\pi_0}/\det\rho = \eta^a$ for some integer a modulo three. Twisting π_0 by η^j changes this ratio to η^{a+jn}. By (1.3) there is exactly one choice of j modulo three giving ratio one. Denote the resulting descent by π. The cyclic descent fiber statement proves its uniqueness with the asserted central character. No matching of π with ρ at inert primes has been assumed.

4 The cubic spectral calculation

Fix $\omega = e^{2\pi i/3}$. For a multiset Y of complex numbers write $p_m(Y) = \sum_{y\in Y} y^m$, and put $L_x(Y) = \prod_{y\in Y}(1-yx)^{-1}$. Superscripts in a multiset below denote multiplicities, not powers.

Lemma 4.1. *Let $n \geq 1$ with $3 \nmid n$, and write $n = 3k+\epsilon$, $\epsilon \in \{1,-1\}$. Put*

$$X = \{1^{[k+\epsilon]}, \omega^{[k]}, (\omega^2)^{[k]}\}.$$

Let Y be a multiset of n cube roots of unity with product one. Set $D = |p_1(Y)|^2$ and $\delta = D-1$. Then

$$\delta \in 3\mathbb{Z}_{\geq 0}, \qquad \delta = 0 \iff Y = X. \tag{4.1}$$

Moreover

$$\frac{L_x(Y\otimes Y^\vee)}{L_x(X\otimes X^\vee)} = \left(\frac{1+x+x^2}{(1-x)^2}\right)^{\delta/3}, \tag{4.2}$$

as an identity of rational functions and of formal power series at zero. Here $Y\otimes Y^\vee$ is the multiset of all ratios y/y'.

Proof. Let (n_0,n_1,n_2) be the multiplicities of $1,\omega,\omega^2$ in Y, and set $a = n_0-n_2$, $b = n_1-n_2$. Since $1+\omega+\omega^2 = 0$,

$$p_1(Y) = a+b\omega, \qquad D = a^2-ab+b^2 = \left(a-\frac{b}{2}\right)^2 + \frac{3b^2}{4}. \tag{4.3}$$

We have $a+b = n-3n_2$, and hence $D \equiv (a+b)^2 \equiv n^2 \equiv 1 \pmod 3$. The nonnegative integer D is thus at least one, giving the first part of (4.1). The determinant condition is $n_1+2n_2 \equiv 0 \pmod 3$, or $b \equiv 0 \pmod 3$.

If $D = 1$, the last expression in (4.3) implies $|b| \leq 1$. Therefore $b = 0$, and $a = 1$ or -1. The congruence $a+b \equiv n \pmod 3$ selects $a = \epsilon$. Solving gives $n_1 = n_2 = k$ and $n_0 = k+\epsilon$. Conversely these values have determinant one and $D = 1$.

If $3 \mid m$, then $|p_m(Y)|^2 = n^2 = |p_m(X)|^2$. If $3 \nmid m$, conjugation interchanges the two possible traces, so $|p_m(Y)|^2 = D$ and $|p_m(X)|^2 = 1$. The formal logarithm of the left side of (4.2) is consequently

$$\begin{aligned}\sum_{m\geq 1} \frac{|p_m(Y)|^2 - |p_m(X)|^2}{m} x^m &= \delta \sum_{3\nmid m} \frac{x^m}{m} \\ &= \delta\left(-\log(1-x) + \frac{1}{3}\log(1-x^3)\right).\end{aligned}$$

Exponentiating proves (4.2), because $(1-x^3)/(1-x)^3 = (1+x+x^2)/(1-x)^2$. □

The basic factor in this identity has the particularly simple expansion

$$f(x) := \frac{1+x+x^2}{(1-x)^2} = 1 + \sum_{r\geq 1} 3rx^r. \tag{4.4}$$

Indeed $(1-x)^{-2} = \sum_{r\geq 0}(r+1)x^r$; the coefficient is three for $r = 1$, and $(r+1)+r+(r-1) = 3r$ for $r \geq 2$. Thus every factor in (4.2) has nonnegative integer coefficients. Its coefficient of x is δ, and it is identically one exactly when the spectra agree.

5 The global positive quotient

Choose a finite set S of finite places of F containing all places ramified in $\widetilde{K}/F$, all places where π is ramified, and all places below the finite exceptional set in the matching of Π_E with $\rho|_{G_E}$. For a function over F, superscript S means omission of its local factors at $v \in S$. Define

$$\zeta_M^S(s) = \prod_{v \notin S} \prod_{w|v} (1 - (\mathrm{N}\, w)^{-s})^{-1};$$

thus on the right the places are those of M above v.

For $v \notin S$ split in E/F, base change gives equality of the parameters of π_v and $\rho(\mathrm{Frob}_v)$. Now suppose that v is inert in E/F, and put $g = \rho(\mathrm{Frob}_v)$. Proposition 2.3 gives $g^3 = uI$ and $|\operatorname{tr} g|^2 = 1$. If $d = \det g$, then $d^3 = u^n$. There is a unique $t \in \mathbb{C}^\times$ with

$$t^3 = u, \qquad t^n = d. \tag{5.1}$$

To see this, choose one cube root of u. The ratio of its n-th power to d is a cube root of unity; multiplication of the chosen root by a cube root of unity changes that ratio bijectively, since $3 \nmid n$. All the numbers here are roots of unity, so $|t| = 1$.

Unramified cyclic base change cubes the Satake eigenvalues at an inert place. Since Π_E matches $\rho|_{G_E}$, every eigenvalue α of π_v satisfies $\alpha^3 = u$. Their product is d, by the central-character normalization. Divide both the spectrum of g and that of π_v by t. Both resulting multisets consist of cube roots of unity and have product one. The first has squared trace norm one; it is exactly X in Lemma 4.1. Denote the second by Y_v, and write $\delta_v = |p_1(Y_v)|^2 - 1$. For split places set $\delta_v = 0$. We have proved

$$0 \le \delta_v \le n^2 - 1, \quad \delta_v \in 3\mathbb{Z}, \qquad v \in \mathcal{B} \iff \delta_v > 0. \tag{5.2}$$

In particular all unramified parameters of π outside S have modulus one; no general Ramanujan conjecture is being used.

The unramified Rankin–Selberg factor has eigenvalues all pairwise ratios of Satake eigenvalues. The scalar t cancels in these ratios. Lemma 4.1 and (2.3) therefore give, initially for $\operatorname{Re} s > 1$,

$$R(s) := \frac{L_F^S(s, \pi \times \pi^\vee)}{\zeta_M^S(s)} = \prod_{v \notin S} f((\mathrm{N}\, v)^{-s})^{\delta_v/3}. \tag{5.3}$$

This Euler product converges absolutely in that half-plane, since the exponents are bounded and the first nonconstant term of each factor is $O((\mathrm{N}\, v)^{-s})$.

Expanding the product gives an ordinary Dirichlet series

$$R(s) = \sum_{m \ge 1} a_m m^{-s}, \qquad a_m \ge 0. \tag{5.4}$$

One first expands over ideals of F, and then sums the nonnegative coefficients of ideals with the same positive integer norm. There are finitely many ideals of bounded norm, so every a_m is well defined. Equality of norms at different prime ideals causes no cancellation or sign change.

We recall exactly the analytic input used here. For a unitary cuspidal representation π of GL_n, its self Rankin–Selberg function has meromorphic continuation, its incomplete function at any fixed finite set has a simple pole with nonzero residue at $s = 1$, and it has no pole at a real point $0 < s < 1$. These are the classical Rankin–Selberg pole and continuation theorems [8, 7]. The Dedekind function has a simple pole with nonzero residue at one and meromorphic continuation to $\mathbb{C}$. Consequently R is meromorphic on $\mathbb{C}$, and is holomorphic

and nonzero at one. The omitted factors in ζ_M^S have zeros only on $\operatorname{Re} s = 0$; thus (1.5), when assumed, makes R holomorphic at every positive real point.

Already without (1.5), positivity gives

$$\log R(\sigma) = \sum_{v \notin S} \delta_v \sum_{3 \nmid r} \frac{(\mathrm{N}\, v)^{-r\sigma}}{r} \ \geq \sum_{v \notin S} \delta_v (\mathrm{N}\, v)^{-\sigma} \qquad (\sigma > 1). \tag{5.5}$$

The left side has a finite limit as $\sigma \downarrow 1$, because $R(1)$ is positive and nonzero. Monotone convergence and (5.2) imply $\sum_v \delta_v/(\mathrm{N}\, v) < \infty$, proving (1.4). Its Dirichlet density is zero because

$$0 \leq \frac{\sum_{v \in \mathcal{B}} (\mathrm{N}\, v)^{-\sigma}}{\log(1/(\sigma - 1))} \longrightarrow 0 \qquad (\sigma \downarrow 1).$$

This argument asserts Dirichlet density, and does not require an assertion about natural relative prime density.

6 Real nonvanishing and the endpoint at zero

We include the needed positive-coefficient theorem in order to specify the continuation-to-convergence step.

Lemma 6.1 (Landau)**.** *Let $D(s) = \sum_{m \geq 1} a_m m^{-s}$, with $a_m \geq 0$, have finite abscissa of convergence γ. Then its analytic function on $\operatorname{Re} s > \gamma$ has a singularity at the real point γ.*

Proof. Suppose instead that it extends holomorphically to a disk centered at γ of radius $r > 0$. Its continuation together with the right half-plane is holomorphic on their union. Choose $0 < \varepsilon < r/4$ and $s_0 = \gamma + \varepsilon$. The disk centered at s_0 with radius 2ε lies in this union: any of its points to the left of the line $\operatorname{Re} s = \gamma$ lies within $3\varepsilon < r$ of γ. The Taylor series at s_0 therefore converges at $s_0 - t$ for $t = 3\varepsilon/2$.

Termwise differentiation in the half-plane of convergence gives

$$(-1)^j D^{(j)}(s_0) = \sum_{m \geq 1} a_m (\log m)^j m^{-s_0} \geq 0.$$

For each fixed j, convergence of this derivative series follows by comparing with the convergent series at any real point strictly between γ and s_0. Tonelli's theorem for the nonnegative Taylor terms now yields

$$\begin{aligned} D(s_0 - t) &= \sum_{j \geq 0} \frac{t^j}{j!} \sum_{m \geq 1} a_m (\log m)^j m^{-s_0} \\ &= \sum_{m \geq 1} a_m m^{-s_0} \sum_{j \geq 0} \frac{(t \log m)^j}{j!} = \sum_{m \geq 1} a_m m^{-(s_0 - t)} < \infty. \end{aligned}$$

But $s_0 - t = \gamma - \varepsilon/2 < \gamma$, a contradiction. This is the usual positive-coefficient argument [6, Theorem 10]. □

Assume now (1.5). The abscissa of convergence of (5.4) is at most one. If it were a positive real number, Lemma 6.1 would give a positive real singularity of R, contrary to Section 5. It follows that the series converges at every real $\sigma > 0$, also when its abscissa is $-\infty$.

To be explicit about the use of the product after this continuation, its finite Euler subproducts have nonnegative coefficients bounded coefficientwise by those of (5.4). Their

values at $\sigma > 0$ are therefore bounded by the sum of that convergent series. Letting the finite sets of places increase recovers every coefficient and proves convergence of the Euler product to the same sum. Analytic uniqueness identifies this sum with the meromorphic function R in (5.3) wherever it is evaluated on the positive real axis.

If $\mathcal{B}$ were infinite, choose any finite subset $B_0 \subset \mathcal{B}$. For $\sigma > 0$,

$$R(\sigma) \geq \prod_{v \in B_0} f((\mathrm{N}\, v)^{-\sigma})^{\delta_v/3}. \tag{6.1}$$

As $\sigma \downarrow 0$, the expansion $1 - e^{-\sigma \log \mathrm{N}\, v} \sim \sigma \log \mathrm{N}\, v$ gives

$$f((\mathrm{N}\, v)^{-\sigma}) \sim \frac{3}{\sigma^2 (\log \mathrm{N}\, v)^2}. \tag{6.2}$$

The right side of (6.1) is consequently asymptotic to $C_{B_0} \sigma^{-h_{B_0}}$, where

$$C_{B_0} = \prod_{v \in B_0} \left(\frac{3}{(\log \mathrm{N}\, v)^2} \right)^{\delta_v/3} > 0, \qquad h_{B_0} = \frac{2}{3} \sum_{v \in B_0} \delta_v \geq 2|B_0|.$$

On the other hand, meromorphy of R at zero supplies a fixed integer $h \geq 0$ and a constant C such that $|R(\sigma)| \leq C\sigma^{-h}$ for all sufficiently small positive σ. Choosing $|B_0| > h/2$ contradicts the lower bound. Thus $\mathcal{B}$ is finite. This uses the finite order of a meromorphic singularity at zero; Landau's theorem alone would not give the conclusion.

7 From cofinite matching to the full function

The following general lemma accounts for finite places and infinity without presupposing their local matching.

Lemma 7.1. *Let $\tau : G_F \to \mathrm{GL}_m(\mathbb{C})$ have finite image, and let Π be a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_F)$, with $m \geq 2$. If their unramified Euler factors agree outside a finite set of places, then both the completed Artin function and $L_F(s, \tau)$ are entire.*

Proof. Write

$$\Lambda(s, \tau) = C_\tau^{s/2} L_\infty(s, \tau) L_F(s, \tau), \qquad \Lambda(s, \Pi) = C_\Pi^{s/2} L_\infty(s, \Pi) L_F(s, \Pi),$$

where the positive constants include the usual conductor and discriminant contributions. Brauer's theorem and the classical Artin functional equation [3] give meromorphic continuation of the first function. The standard GL_m theory gives the same continuation and the functional equation for the second, which is entire because Π is cuspidal and $m \geq 2$ [5]. Choose a finite set T containing every unmatched or ramified finite place. In an initial right half-plane,

$$U(s) := \frac{\Lambda(s, \tau)}{\Lambda(s, \Pi)} = \left(\frac{C_\tau}{C_\Pi} \right)^{s/2} \frac{L_\infty(s, \tau)}{L_\infty(s, \Pi)} \prod_{v \in T} \frac{L_v(s, \tau)}{L_v(s, \Pi)}. \tag{7.1}$$

For each finite place, write $L_v(s, \tau) = P_{\tau,v}((\mathrm{N}\, v)^{-s})^{-1}$ and $L_v(s, \Pi) = P_{\Pi,v}((\mathrm{N}\, v)^{-s})^{-1}$, with both polynomials having constant term one. The eigenvalues of Frobenius on the inertia invariants of τ are roots of unity. Hence all zeros of $P_{\tau,v}$ have modulus one, and

$$\frac{L_v(s, \tau)}{L_v(s, \Pi)} = \frac{P_{\Pi,v}((\mathrm{N}\, v)^{-s})}{P_{\tau,v}((\mathrm{N}\, v)^{-s})}$$

is holomorphic on $\operatorname{Re} s > 0$. No bound on the roots of $P_{\Pi,v}$ is needed.

Put $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. At a real place, the finite-image Artin parameter has gamma factor $\Gamma_{\mathbb{R}}(s)^a\Gamma_{\mathbb{R}}(s+1)^b$; at a complex place it has $\Gamma_{\mathbb{C}}(s)^m$. These factors have no poles in $\operatorname{Re} s > 0$. Every reciprocal automorphic gamma factor is entire, being the reciprocal of a product of shifted gamma functions times nonvanishing exponentials. It follows that the right side of (7.1) is holomorphic on $\operatorname{Re} s > 0$. By meromorphic uniqueness it equals U there.

The same argument applies to the contragredients, since unramified matching passes to them. With $U^\vee(s) = \Lambda(s, \tau^\vee)/\Lambda(s, \Pi^\vee)$, the two completed functional equations give

$$U(s) = \frac{\varepsilon(\tau)}{\varepsilon(\Pi)} U^\vee(1-s). \tag{7.2}$$

Conductor powers have already been included in Λ, so the ratio here is constant and nonzero. The right side is holomorphic on $\operatorname{Re} s < 1$. This half-plane and $\operatorname{Re} s > 0$ cover $\mathbb{C}$. Thus U is entire. Multiplying by the entire $\Lambda(s, \Pi)$ proves entire continuation of $\Lambda(s, \tau)$. Finally $C_\tau^{-s/2}/L_\infty(s, \tau)$ is entire, so the uncompleted Artin function is entire as well. □

Section 6 supplies the hypothesis of Lemma 7.1 for $(\tau, \Pi) = (\rho, \pi)$. Together with Section 5, this completes the proof of Theorem 1.1.

8 A classical consequence for primitive solvable adjoints

There is another unconditional statement whose scope must be kept distinct from Theorem 1.1. A finite-group representation is *primitive* if it is irreducible and is not induced from a representation of a proper subgroup, equivalently if it has no nontrivial system of imprimitivity. Its adjoint here means $\operatorname{Ad}(V) = \operatorname{End}_0(V)$, the trace-zero endomorphisms.

Proposition 8.1. *Let $\tau : G_F \to \mathrm{GL}(V)$ be primitive, irreducible, and have finite solvable image G. There exist subgroups $J_i \subset G$ and nontrivial one-dimensional characters χ_i of J_i such that*

$$\operatorname{Ad}(V) \simeq \bigoplus_i \operatorname{Ind}_{J_i}^G \chi_i. \tag{8.1}$$

In particular the full Artin function $L_F(s, \operatorname{Ad}\tau)$ is entire. There is no parity restriction on $\dim V$.

We use one classical structural theorem, in precisely this form: a primitive irreducible representation of a finite solvable group is, after inflation to a finite central extension, a tensor product of strongly irreducible representations [12, Proposition 3.4]. Here "strongly irreducible" means that restriction to each normal subgroup is either scalar or irreducible. The theorem is a finite-group statement; it requires no realization of the central extension as a Galois group. To check its usual quasiprimitivity hypothesis, Clifford theory shows that a nonhomogeneous restriction to a normal subgroup would induce the representation from a proper inertia subgroup, contrary to primitivity. Thus every such restriction is homogeneous. A finite central extension of a solvable group remains solvable, so the solvability needed for the factors is preserved as well.

Proof of Proposition 8.1. First let a finite solvable group B act faithfully and strongly irreducibly on W, with $d = \dim W > 1$. Set $Z_B = Z(B)$. By faithfulness and Schur's lemma, Z_B consists exactly of the scalar elements. Choose $D \triangleleft B$ with D/Z_B a minimal nontrivial normal subgroup of B/Z_B. Solvability makes D/Z_B elementary abelian. The restriction

of W to D is not scalar, since $D \not\subset Z_B$, so strong irreducibility makes it irreducible. Any element of $Z(D)$ then acts as a scalar on W, giving $Z(D) = Z_B$.

The faithful scalar character λ of Z_B defines the alternating pairing

$$(xZ_B, yZ_B) \longmapsto \lambda([x,y]) \quad \text{on } D/Z_B.$$

It is nondegenerate: an element in its radical commutes with all of D, because λ is faithful, and hence lies in $Z(D) = Z_B$. As in Lemma 2.2, the character ϑ of $W|_D$ vanishes outside Z_B. Character orthogonality therefore gives

$$1 = \frac{1}{|D|}\sum_{x\in D} |\vartheta(x)|^2 = \frac{|Z_B|d^2}{|D|}, \qquad |D : Z_B| = d^2.$$

The matrices of one representative from each coset in D/Z_B thus give d^2 distinct conjugation weight lines and a basis of $\mathrm{End}(W)$. Normality of D makes B permute these lines, possibly with scalar multipliers. One line is $\mathbb{C}I$; every other basis matrix has trace zero.

Now apply the quoted structural theorem to V. It supplies a finite central extension $p : \widetilde{G} \to G$ and strongly irreducible modules $W_1, \dots, W_r$ with $p^*V \simeq W_1 \otimes \cdots \otimes W_r$. For each nonlinear factor apply the preceding argument to its faithful image; a linear factor contributes only its identity endomorphism. The tensor products of the resulting basis matrices give a basis of $\mathrm{End}(V) \simeq \bigotimes_i \mathrm{End}(W_i)$ whose lines are permuted by $\widetilde{G}$. The identity tensor is I_V. Every other tensor has a factor of trace zero and consequently has trace zero. Since $\ker p$ is central, it acts by scalars on every W_i, and trivially on $\mathrm{End}(W_i)$. The line action therefore descends to G.

For each G-orbit of nonidentity lines, choose one line, let J_i be its stabilizer, and let χ_i be its scalar action. The orbit span is $\mathrm{Ind}_{J_i}^{G} \chi_i$: the induced-module map sends a coset translate of a chosen nonzero line vector to its translate in $\mathrm{End}(V)$. Distinct cosets give distinct independent basis lines, proving bijectivity. If χ_i were trivial, Frobenius reciprocity would give a G-fixed vector in this trace-zero orbit span. But $\mathrm{End}_G(V) = \mathbb{C}I_V$, and a nonzero scalar matrix in characteristic zero does not have trace zero. Thus every χ_i is nontrivial, which proves (8.1).

Let L/F be the extension cut out by τ, and put $F_i = L^{J_i}$. Full Artin induction gives

$$L_F(s, \mathrm{Ad}\,\tau) = \prod_i L_{F_i}(s, \chi_i).$$

Each factor is the entire Hecke L-function of a nontrivial finite-order character. This proves the assertion. In degree one the adjoint is zero and the empty product is one. □

This proposition is a consequence of classical structure theory, not a proof of the original Artin conjecture for all primitive solvable representations. Nor does the proof claim that every irreducible constituent of the adjoint is itself monomial: the positive induced decomposition (8.1) is the precise statement needed. Wilde's stronger permutation-character conclusion for $V \otimes V^\vee$ has an odd-degree hypothesis [12, Theorem C and the subsequent even-degree discussion]; the monomial line argument above does not.

9 Scope of the criterion

For the finite-field family, a useful order-three transformation is

$$c = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q), \qquad c^3 = I, \quad \det(c - I) = 3.$$

When $3 \nmid q$, this action is fixed-point-free on nonzero vectors of $\mathbb{F}_q^2$. Given an irreducible degree-q representation with this projective image, Theorem 1.1 applies with $[M : F] = q^2$. This is a recognizable classical family with unbounded possible degree, not a condition formulated in terms of an unknown local matching set.

The case $q = 2$ is tetrahedral and its automorphy is classical [10, 11]. Some other instances are already monomial and hence have holomorphic Artin functions by Hecke theory. For example, in the finite-field Heisenberg setting where the commutator pairing comes from the standard alternating form, $q \equiv 1 \pmod 3$ gives a C-invariant isotropic $\mathbb{F}_q$-line. More generally, whenever A has a C-invariant maximal isotropic subgroup L, let H be its preimage as in Section 3, and let J be the preimage of $L \rtimes C$ in G. The characters of H extending the scalar character form a set of size $|H/Z| = n$, permuted by $J/H = C_3$. Since $3 \nmid n$, it has a fixed character θ. The cyclic-extension argument of Lemma 2.1 extends θ to J. Nondegeneracy of b makes N/H act freely and transitively on these n characters. Thus this θ is conjugate to the character chosen in Section 3, and $\operatorname{Ind}_H^N \theta \simeq V|_N$. The induction to G of the chosen extension of θ to J restricts irreducibly to N, namely as $\operatorname{Ind}_H^N \theta$, and is therefore irreducible. Its difference from ρ is a character of G/N, so ρ too is induced from a linear character of J. This proves the stated monomiality. Such an inducing subgroup need not be subnormal; this observation about Hecke holomorphy is not by itself a general automorphic induction theorem.

The nonsplit finite-field action $q \equiv 2 \pmod 3$, $q > 2$, is therefore a natural place to apply the conditional criterion. In characteristic two one must specify an actual finite-image representation with the stated projective image. The elementary upper-unitriangular Heisenberg model does not automatically supply every required order-three action. Our projective hypotheses avoid any implicit use of such a construction.

The generalized Riemann hypothesis for ζ_M, together with $\zeta_M(1/2) \neq 0$, is a sufficient condition for (1.5). The generalized Riemann hypothesis alone does not exclude a zero at the central point. Finally, neither Theorem 1.1 nor Proposition 8.1 is a three-dimensional solvable Artin theorem: the former has $3 \nmid n$, and the latter concerns the adjoint function.

A Complete finite computation

Lemma 4.1 was proved for every degree by (4.3) and its congruences. The following independent finite check enumerates every nonnegative multiplicity triple of sum $n \le 1000$, excluding multiples of three. It checks the positivity congruence for all triples and the precise equality case for every determinant-one triple. Python's integers have arbitrary precision. No external data or third-party package is used. Run the displayed source with Python 3 without the `-O` option, so that assertions are active.

```
"""Exhaustive exact-integer check of the cubic spectral lemma.

Run with Python 3 (without -O). No third-party packages or data are used.
This finite check is not a substitute for the proof for all degrees.
"""
import json

LIMIT = 1000
triples = determinant_one = equality_cases = 0
for n in range(1, LIMIT + 1):
    if n % 3 == 0:
        continue
    epsilon = 1 if n % 3 == 1 else -1
    k = (n - epsilon) // 3
    target = (k + epsilon, k, k)
    equalities_in_degree = 0
    for n2 in range(n + 1):
        for n1 in range(n - n2 + 1):
            n0 = n - n1 - n2
            a, b = n0 - n2, n1 - n2
            D = a * a - a * b + b * b
            triples += 1
            assert D >= 1 and (D - 1) % 3 == 0
            if (n1 + 2 * n2) % 3 != 0:
                continue
            determinant_one += 1
            assert (D == 1) == ((n0, n1, n2) == target)
            if D == 1:
                equalities_in_degree += 1
                equality_cases += 1
    assert equalities_in_degree == 1

assert equality_cases == LIMIT - LIMIT // 3
print(json.dumps({
    "maximum_degree": LIMIT,
    "multiplicity_triples_checked": triples,
    "determinant_one_triples_checked": determinant_one,
    "norm_one_determinant_one_cases": equality_cases,
    "expected_one_per_degree": LIMIT - LIMIT // 3,
    "arithmetic": "Python arbitrary-precision integers",
    "status": "PASS"
}, indent=2))
```

The complete output of the run used for this manuscript is:

```
{
  "maximum_degree": 1000,
  "multiplicity_triples_checked": 111779112,
  "determinant_one_triples_checked": 37259704,
  "norm_one_determinant_one_cases": 667,
  "expected_one_per_degree": 667,
  "arithmetic": "Python arbitrary-precision integers",
  "status": "PASS"
}
```

There are $1000-\lfloor 1000/3 \rfloor = 667$ eligible degrees and exactly one determinant-one equality case in each. The computation checks the finite normalization and equality cases only. It

certifies neither unbounded-degree statements nor global holomorphy; those conclusions follow from the proofs in the text.

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