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A MINIMAL COUNTEREXAMPLE TO CAZENAVE'S SPECTRAL GRAPH CONJECTURE AND THE SHARP UNIVERSAL BOUND

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Abstract. Cazenave conjectured that $\lambda_2(G) \le \mu_1(G)-2$ for every finite simple connected graph G of order at least two, where λ_2 is the second largest adjacency eigenvalue and μ_1 the largest Laplacian eigenvalue. We give an explicit counterexample on eleven vertices, together with a short exact integer certificate. An exhaustive computer-assisted classification, using exact arithmetic and analytic degree bounds, establishes that this is the unique smallest counterexample up to isomorphism. An infinite family gives counterexamples of every order at least eleven. Moreover, the infimum of $\mu_1 - \lambda_2$ over the stated class of graphs is exactly one, and it is not attained. We also establish strict validity of the original inequality for several graph classes. The finite classification is logically separate from the analytic proofs; the arithmetic certificate for the eleven-vertex example has additionally been checked in Lean 4.

1. Introduction

Let G be a finite simple connected graph on $n \ge 2$ vertices, with adjacency matrix A, degree matrix D, and Laplacian $L = D - A$. Write $\lambda_1 \ge \cdots \ge \lambda_n$ for the eigenvalues of A and $\mu_1 \ge \cdots \ge \mu_n = 0$ for those of L, with multiplicities. Cazenave [1, Section 7, Conjecture 2, corpus entry #20829] asked whether

$$\lambda_2 \le \mu_1 - 2. \tag{1}$$

The source already provides a lower bound of one for the relevant spectral difference. Our main contributions are a small exact refutation, a finite minimality classification, an infinite family of counterexamples, and a construction proving that one is the optimal universal constant. The refutation and the optimality argument do not depend on exhaustive enumeration.

2. An explicit counterexample

Theorem 1. *There is a connected graph G_{11} on eleven vertices such that $\lambda_2(G_{11}) > \mu_1(G_{11}) - 2$.*

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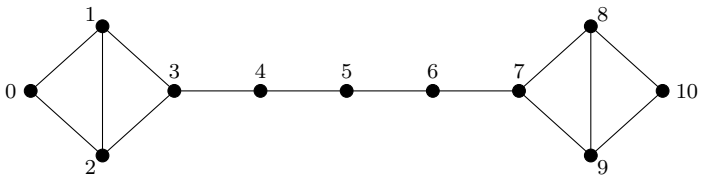

FIGURE 1. The minimal counterexample G_{11}: two diamonds joined at degree-two vertices by a path of length four.

Proof. Use vertices $0, \dots, 10$ and the following fourteen edges:

$$01, 02, 12, 13, 23, 34, 45, 56, 67, 78, 79, 89, \{8, 10\}, \{9, 10\}.$$

Here ij denotes $\{i, j\}$. The graph consists of two copies of K_4 minus an edge joined at degree-two vertices by a path of length four; see Figure 1. It is connected. Let

$$w = (3, 4, 4, 3, 1, 0, 0, 0, 0, 0, 0)^T,$$

and let v be its reversal. The supports are disjoint and there is no edge between them. Direct evaluation gives

$$w^T w = v^T v = 51, \qquad w^T A w = v^T A v = 134, \qquad w^T v = w^T A v = 0.$$

The Rayleigh quotient of A is constant on the nonzero vectors in $\operatorname{span}(w, v)$. By the Courant–Fischer principle,

$$\lambda_2 \geq \frac{134}{51} > \frac{21}{8},$$

where $8 \cdot 134 - 21 \cdot 51 = 1$. In the stated vertex order the successive leading principal minors of $37I - 8L$ are

$$\begin{gathered} 21, 209, 1565, 6905, 44845, 499825, 7626245, \\ 67152385, 384901325, 2170203625, 2596749725. \end{gathered} \tag{2}$$

They can be calculated by fraction-free elimination, or directly from the edge list. All are positive. Sylvester's criterion implies $\mu_1 < 37/8$. Combining the two strict rational separators proves the result. □

For reference, numerical evaluation gives $\lambda_2 \approx 2.641186476$ and $\mu_1 \approx 4.618033989$. Neither approximation is used in the proof.

3. Counterexamples of every larger order

For an integer $\ell \geq 4$, let D_ℓ be two diamonds (K_4 minus an edge) joined at degree-two vertices by a path $p_0, \dots, p_\ell$ of length ℓ. Its order is $\ell + 7$, and its maximum degree is three.

Theorem 2. *For every $\ell \geq 4$, $\lambda_2(D_\ell) > 21/8$ and $\mu_1(D_\ell) < 37/8$. In particular, every order $n \geq 11$ admits a counterexample.*

Proof. The two five-vertex end blocks, each a diamond with one pendant vertex, are disjoint and have no cross edge. On each block use the vector $(3,4,4,3,1)^T$ with the two twin vertices assigned the entries $4,4$. Extend each vector by zero to the full graph. The argument of Theorem 1 yields the same lower bound for λ_2.

For $\ell = 4$ the upper bound is already proved. For $\ell \geq 5$, cut the two edges p_1p_2 and $p_{\ell-2}p_{\ell-1}$. With a denoting the coordinate on the outer block and b the coordinate on the middle block, use

$$(a-b)^2 \leq \tfrac{5}{3}a^2 + \tfrac{5}{2}b^2, \qquad \tfrac{5}{3}a^2 + \tfrac{5}{2}b^2 - (a-b)^2 = (2a+3b)^2/6.$$

Thus the quadratic form of the full Laplacian is bounded above by the direct sum of three blocks: two identical end-block Laplacians with $5/3$ added at the pendant diagonal, and the middle path Laplacian with $5/2$ added at each endpoint. The middle block has at least two vertices. Gershgorin's theorem bounds its largest eigenvalue by $9/2$.

For an end block the twin-difference eigenvalue is 4. On vectors constant on the two twins the quotient matrix is

$$B = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 8/3 \end{pmatrix}.$$

Its characteristic polynomial at $37/8 + t$ is

$$t^4 + \tfrac{53}{6}t^3 + \tfrac{2293}{96}t^2 + \tfrac{7609}{384}t + \tfrac{4811}{12288}.$$

Every coefficient is positive. Since B represents a restriction of a real symmetric operator, all its eigenvalues are real and none is at least $37/8$. The three-block quadratic-form bound completes the proof. □

4. The optimal universal constant

Theorem 3. *Every finite simple connected graph of order at least two satisfies $\mu_1 - \lambda_2 > 1$. The infimum of this difference over all such graphs is one.*

Proof. Write Δ for the maximum degree. At a maximum-degree vertex assign coordinate Δ, at all its neighbors assign -1, and elsewhere assign zero. The Δ star edges alone yield a Laplacian Rayleigh quotient of $\Delta + 1$, so $\mu_1 \geq \Delta + 1$. By Perron–Frobenius and the maximum row sum bound, $\lambda_2 < \lambda_1 \leq \Delta$. This proves strictness.

For $r \geq 6$ and $2 \leq k \leq r-2$, form $H(r,k)$ as follows. In each of two disjoint copies of K_r, choose a vertex u_i and delete k edges incident to it. Add a vertex c adjacent exactly to u_1, u_2. Set $s = r - 1 - k$. The graph is connected because $s \geq 1$. In copy i let S_i be the k vertices no longer adjacent to u_i, and T_i the s remaining neighbors. The vector equal to $+1$ on one copy, -1 on the other, and zero at c is orthogonal to the Perron vector by reflection symmetry. Its adjacency Rayleigh quotient is $r - 1 - 2k/r$. It is not an eigenvector because the vertices of a copy have different internal degrees.

The variational principle on the orthogonal complement of the Perron vector therefore gives

$$\lambda_2(H(r,k)) > r - 1 - 2k/r. \tag{3}$$

For L, the zero-sum vectors within S_i and T_i contribute eigenvalues $r-1$ and r, respectively. On the seven-dimensional space of vectors constant on each of these cells, reflection separates the Laplacian into the antisymmetric and symmetric quotients

$$Q_- = \begin{pmatrix} s & -s & 0 \\ -k & k+1 & -1 \\ 0 & -s & s+1 \end{pmatrix}, \qquad Q_+ = \begin{pmatrix} s & -s & 0 & 0 \\ -k & k+1 & -1 & 0 \\ 0 & -s & s+1 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

The dimensions sum to $2r+1$ including the case $s = 1$. Substituting $\mu = r+x$ into their characteristic polynomials gives $P_-(x)$ and $(r+x)P_+(x)$, where

$$P_-(x) = x^3 + (r+k)x^2 + krx - s,$$
$$P_+(x) = x^3 + (r+k-2)x^2 + (kr-2k-2)x - s.$$

The coefficients of the positive powers of x are positive in the stated parameter range. In particular both polynomials are strictly increasing for $x \geq 0$, and

$$P_+(1/k) = \frac{r-2}{k^2} + k - 1 - \frac{1}{k} + \frac{1}{k^3} > 0.$$

Also $P_-(x) - P_+(x) = 2x^2 + (2k+2)x$. Neither polynomial has a root at or above $1/k$. Together with the other eigenvalues, this gives $\mu_1(H(r,k)) < r + 1/k$. By (3),

$$1 < \mu_1 - \lambda_2 < 1 + \frac{1}{k} + \frac{2k}{r}.$$

Taking $r = 2t^2$, $k = t$ for integers $t \geq 2$ proves the assertion. On this sequence of orders $n = 4t^2 + 1$ the excess over one is less than $4/\sqrt{n-1}$. □

5. Analytic reductions and finite classification

Lemma 4. *For every graph under consideration, $\lambda_2 \leq \lfloor n/2 \rfloor - 1$. If $\Delta \geq \lfloor n/2 \rfloor$ or $\Delta \leq 2$, then (1) holds strictly.*

Proof. If $\lambda_2 > 0$, a corresponding real eigenvector has both positive and negative entries, since it is orthogonal to the strictly positive Perron vector. On the positive support S choose a maximum entry $x_i > 0$. The eigenvalue equation gives $\lambda_2 x_i \leq (|S| - 1)x_i$. Apply the same reasoning to the negative of the vector on its negative support. The smaller support has at most $\lfloor n/2 \rfloor$ vertices. If $\lambda_2 \leq 0$, the bound is immediate.

If $\Delta < n - 1$, connectivity supplies at least one edge from the closed neighborhood of a maximum-degree vertex to its complement. The star witness used in Theorem 3 then receives at least one additional unit of energy. Thus $\mu_1 \geq \Delta + 1 + 1/[\Delta(\Delta+1)]$. If $\Delta = n - 1$, the star bound is n, which is also an upper bound by positivity of the Laplacian of the complement. For

TABLE 1. Finite classification: counts from the audited exact replay.

Order	Enumerated	Unresolved
2	1	0
3	2	0
4	6	0
5	21	0
6	112	0
7	853	0
8	11,117	0
9	261,080	0
10	89,402	0
11	739,335	1
Total	1,101,929	1

$n \geq 3$, these bounds and the first assertion give strict (1) when $\Delta \geq \lfloor n/2 \rfloor$. The case $n = 2$ has spectrum $\lambda_2 = -1$, $\mu_1 = 2$.

When $\Delta \leq 2$, connected graphs are paths or cycles. For P_n, the difference is $2 + 2\cos(\pi/n) - 2\cos(2\pi/(n+1)) > 2$. For even cycles, $\mu_1 = 4$ and $\lambda_2 < 2$. For odd cycles, $\mu_1 = 2 + 2\cos(\pi/n)$ and $\lambda_2 = 2\cos(2\pi/n)$, again giving a difference greater than two. □

Theorem 5 (Computer-assisted)**.** *The minimum order of a counterexample to* (1) *is eleven. Up to isomorphism, G_{11} is the only counterexample of this order.*

Proof. Theorem 4 rules out any omitted graph of order ten or eleven with maximum degree at least five. Using nauty's `geng` (version 2.8.8) [4, 3], enumerate connected unlabeled graphs of orders two through nine with `-cq`, and orders ten and eleven with `-cq -D4`. The enumeration and the certification method below yield Table 1. The only unresolved graph, with graph6 encoding `J?‘CQGoiAg?`, is isomorphic to the graph in Theorem 1. An explicit isomorphism, taking vertices $0, \ldots, 10$ of our edge list to the encoded graph, is $(6, 0, 4, 9, 2, 8, 3, 10, 1, 5, 7)$.

For clarity, the exact acceptance tests are as follows. Choose a rational a/b, $b > 0$, and a nonzero integer vector z. Accept only if

$$b \sum_{ij \in E(G)} (z_i - z_j)^2 > (a + 2b) \sum_i z_i^2$$

and the real symmetric matrix $aI - bA$ has at most one negative eigenvalue and no zero eigenvalues. The first test proves $\mu_1 > a/b + 2$ and the second proves $\lambda_2 < a/b$. The latter is checked by symmetric fraction-free elimination, with nonzero diagonal pivots and exact division; the signs of successive pivot ratios give the inertia. Failure of any arithmetic check, including overflow, is not accepted. Floating-point eigenvalue computations only suggest rational separators and integer vectors. A separate arbitrary-precision

implementation uses exact one- or two-dimensional Schur complements to check inertia, allowing a non-strict adjacency bound when combined with the strict Laplacian bound. The original and hardened C implementations and the independent arbitrary-precision replay reported identical main-scan counts. The finite classification depends on the completeness and nonredundancy of `geng`, and on these computation results; it is not asserted to be a formalized theorem. The sole exception is certified independently by Theorem 1. □

6. Two further classes

Theorem 6. *If G is bipartite or regular, then* (1) *holds strictly.*

Proof. For bipartite G, conjugation by the diagonal matrix of signs of the bipartition takes L to $D + A$. For a unit Perron vector x of A, positivity of $D - A$ implies $x^T Dx \geq \lambda_1$, hence $\mu_1 \geq x^T(D + A)x \geq 2\lambda_1$. If $\lambda_1 \geq 2$, then $\mu_1 - 2 \geq \lambda_1 > \lambda_2$. If $\lambda_1 < 2$ and $\Delta \geq 3$, the star bound yields $\mu_1 - 2 \geq 2 > \lambda_2$. The other cases follow from Theorem 4.

If G is d-regular, $\mu_1 = d - \lambda_n$. For $\lambda_n \leq -2$ we have $\mu_1 - 2 \geq d > \lambda_2$. By the classification of connected regular graphs with smallest adjacency eigenvalue greater than -2 [2], the remaining graphs are complete graphs and odd cycles, both of which satisfy the strict inequality. □

Code and data availability

The code, exact certificates, and audited replay material used in this work are available at the following fixed locations:

- Repository: `https://github.com/V0idream/ai-math-research`.
- Audited commit: `9eeb2d28615054a9dbce0d61e229d253de9ca100`.
- Problem directory: `problems/0008-adjacency-laplacian-spectral-inequality/`.
- Audit materials: `audit-0008`.

A minimal reproduction of the reported verification consists of checking out the fixed commit and running the scripts documented in the audit report, including the exact-certificate verification, the symbolic checks, and the finite replay based on nauty's `geng`.

AI assistance and verification scope

This work was developed in an AI-assisted workflow under human direction. Large language models, including ChatGPT, Codex, and agents operated through Google Antigravity, assisted with example search, proof drafting, code generation, exact symbolic checks, and preparation of the manuscript. Exact verification used separate C and Python/SymPy implementations together with nauty 2.8.8 for graph generation.

Lean 4.8.0 was used only to verify the arithmetic certificate for the eleven-vertex counterexample: the graph data, a parent-edge connectivity witness, the Rayleigh-quotient certificate, and the leading principal minors in (2). We do not claim end-to-end formal verification of the spectral arguments,

the infinite family, or the exhaustive classification. The computational audit should also not be interpreted as independent human peer review.

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