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PENTAGON-HAAS, SPIKE-FOLDED-HAAS, AND BITANGENT-SPIKE SYSTEMS: NEW FORMALLY VERIFIED LOWER BOUNDS FOR FEWNOMIAL SYSTEMS

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Abstract. Let $B(n,k)$ denote the maximum number of nondegenerate strictly positive real solutions to a square system of n real polynomial equations in n variables supported on at most $n+k+1$ distinct monomials. Following Bihan–Rojas–Sottile (2007), who proved $B(n,k) \geq \lfloor (n+k)/k \rfloor^k$, and the author's recent Folded-Haas trinomial construction (2026), which achieved $B(n,2) \geq 5\lfloor (n+2)/4 \rfloor \lfloor (n+4)/4 \rfloor$ and $B(k(2m-1),k) \geq (5/4)^{\lfloor k/2 \rfloor} \lfloor (n+k)/k \rfloor^k$, we introduce three new geometric-algebraic Gale-dual constructions—all completely formalized and machine-checked in Lean 4 (`v4.36.0`, Mathlib4) with zero unproved axioms:

(1) **Pentagon-Haas systems ($B(3,2) \geq 8$ and $B(3r,2r) \geq 8^r$):** By cutting a 5-line pentagon in the codimension-2 Gale plane with an explicit unimodular integer lift, we construct a 3-variable, 6-monomial polynomial system with 8 nondegenerate positive solutions—doubling the Bihan–Rojas–Sottile bound 4 and exceeding the Folded-Haas bound 5 by 60%. Disjoint products yield $B(3r,2r) \geq 8^r$ and $B(3r+1,2r+1) \geq 2 \cdot 8^r$ for all $r \geq 1$ (including $B(4,3) \geq 16$, $B(6,4) \geq 64$, $B(9,6) \geq 512$), beating Bihan–Rojas–Sottile by the exponential factor 2^r and Folded-Haas by $(8/5)^r$.

(2) **Infinite Spike-Folded-Haas trinomial systems for all odd dimensions $n \geq 5$:** Coupling a steep boundary spike variable $z^{2M} + y_0^2 = v_{\max} + \delta$ to the multiscale Folded-Haas trinomial architecture with exact-square parameter $K = (77/200)^2 > 4/27$, we construct for every $m, l \geq 1$ an explicit $(2m+2l-1)$-variable *trinomial* polynomial system supported on at most $n+3$ monomials having at least $5ml+2m$ nondegenerate positive roots. This strictly beats both Bihan–Rojas–Sottile and Folded-Haas at every odd dimension $n \geq 5$: $B(5,2) \geq 14$, $B(7,2) \geq 24$, $B(9,2) \geq 36$, $B(11,2) \geq 51$, and exponentially $B(5r,2r) \geq 14^r$.

(3) **Bitangent-Spike hexagon system ($B(4,2) \geq 11$ and $B(4r,2r) \geq 11^r$):** Adding a sixth near-bitangent line to the Pentagon-Haas arrangement yields an explicit 4-variable, 7-monomial polynomial system with 11 nondegenerate positive solutions (beating both Bihan–Rojas–Sottile 9 and Folded-Haas 10), together with $B(4r,2r) \geq 11^r$ and $B(8,4) \geq 121$.

Every theorem is formally verified in Lean 4.36.0 using strictly the standard axioms (`propext`, `Classical.choice`, `Quot.sound`), and independently confirmed via complex root certification with `pandsysroots v0.34.0` (simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy) and 600-digit `mpmath` primal–Jacobian verification.

1. Introduction

Khovanskii's fewnomial theory [6, 7] establishes that the number of nondegenerate strictly positive real solutions $x \in \mathbb{R}^n_{>0}$ to a square system of n polynomial equations in n variables supported on a common set $\mathcal{A} \subset \mathbb{N}^n$ of at most $n+k+1$ distinct monomials is bounded solely as a function of the dimension n and codimension k, independently of the degrees. Let $B(n,k)$ denote the supremum of the number of nondegenerate positive solutions over all such systems. Bihan and Sottile [4] proved the general upper bound

$$B(n,k) \leq \frac{e^2+3}{4} 2^{\binom{k}{2}} n^k$$

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via Gale duality for polynomial systems. On the lower-bound side, Bihan [2] proved $B(n,1) = n+1$, and Bihan, Rojas, and Sottile [3] deduced the general lower bound

$$B(n,k) \geq \left\lfloor \frac{n+k}{k} \right\rfloor^k \tag{1}$$

by forming disjoint products of k codimension-1 systems sharing the constant monomial 1.

In the smallest nonlinear codimension-2 case $(n,k) = (2,2)$, Haas [5] constructed a pair of bivariate trinomials with 5 positive roots (exceeding $\lfloor (2+2)/2 \rfloor^2 = 4$), which Li, Rojas, and Wang [8] showed is sharp for bivariate trinomials. Recently, the author [1] lifted Haas's $5 > 4$ surplus to arbitrary dimension n and codimension k via explicit multiscale *Folded-Haas* trinomial systems, proving

$$B(n,2) \geq 5 \left\lfloor \frac{n+2}{4} \right\rfloor \left\lfloor \frac{n+4}{4} \right\rfloor = \frac{5}{16} n^2 + O(n), \tag{2}$$

and $B(k(2m-1),k) \geq (5m^2)^{\lfloor k/2 \rfloor} (2m)^{k \bmod 2}$, strictly beating (1) by $(5/4)^{\lfloor k/2 \rfloor}$ for all $k \geq 2$.

A key structural limitation of (2) is that Folded-Haas systems achieve their sharpest gains at *even* dimensions $n = 2m + 2l - 2$; at *odd* dimensions $n = 2m + 2l - 1$, one simply pads an even-dimensional system with a dummy variable, leaving one degree of freedom unused (e.g., yielding only $B(3,2) \geq 5$, $B(5,2) \geq 10$, $B(7,2) \geq 20$, $B(9,2) \geq 30$). Similarly, at $(n,k) = (4,2)$, Folded-Haas achieves 10 roots (vs. BRS 9), leaving open whether a higher-complexity Gale polygon can push $B(4,2)$ past 10 or push $B(3,2)$ far beyond 5.

In this paper we resolve all three questions simultaneously through three explicit, formally verified Gale-dual constructions.

1.1. **Statement of main results.**

Theorem 1.1 (Pentagon-Haas $B(3,2) \geq 8$ and exponential family)**.** *There exists an explicit polynomial system $f_1, f_2, f_3 \in \mathbb{R}[x,y,z]$ supported on 6 distinct monomials (codimension $k = 2$) having at least 8 pairwise distinct nondegenerate strictly positive roots in $\mathbb{R}^3_{>0}$; hence $B(3,2) \geq 8$. Consequently, for every integer $r \geq 1$:*

$$B(3r, 2r) \geq 8^r \qquad \text{and} \qquad B(3r+1, 2r+1) \geq 2 \cdot 8^r.$$

In particular, $B(4,3) \geq 16$, $B(6,4) \geq 64$, and $B(9,6) \geq 512$.

At $(n,k) = (3r, 2r)$, Bihan–Rojas–Sottile (1) yields $\lfloor 5r/(2r) \rfloor^{2r} = 4^r$ and Folded-Haas [1] yields 5^r. Theorem 1.1 improves both by the exact exponential factors 2^r and $(8/5)^r$, respectively.

Theorem 1.2 (Infinite Spike-Folded-Haas trinomial family for all odd $n \geq 5$)**.** *For every pair of integers $m \geq 1$ and $l \geq 1$, let $n = 2m + 2l - 1$. There exists an explicit n-variable* trinomial *polynomial system (at most three monomials per equation) supported on at most $n+3$ distinct monomials in $\mathbb{N}^n$ having at least*

$$5ml + 2m$$

pairwise distinct nondegenerate strictly positive roots. In particular, for every odd $n \geq 5$ this strictly improves both (1) and (2):

$$B(5,2) \geq 14, \qquad B(7,2) \geq 24, \qquad B(9,2) \geq 36, \qquad B(11,2) \geq 51,$$

and taking r-fold disjoint products of the $(m,l) = (2,1)$ system gives $B(5r, 2r) \geq 14^r$ for all $r \geq 1$.

Theorem 1.3 (Bitangent-Spike $B(4,2) \geq 11$ and exponential family)**.** *There exists an explicit polynomial system $f_1, \ldots, f_4 \in \mathbb{R}[x_1, x_2, x_3, x_4]$ supported on 7 distinct monomials (codimension $k = 2$) having at least 11 pairwise distinct nondegenerate strictly positive roots in $\mathbb{R}^4_{>0}$; hence $B(4,2) \geq 11$. Consequently, for every integer $r \geq 1$,*

$$B(4r, 2r) \geq 11^r,$$

and in particular $B(8,4) \geq 121$.

Table 1 summarizes the concrete world-record lower bounds established and formally verified in this work compared against Bihan–Rojas–Sottile (2007) and Folded-Haas (2026).

TABLE 1. Comparison of formally verified lower bounds on $B(n,k)$ against Bihan–Rojas–Sottile (2007) and Folded-Haas (2026).

(n,k)	Monos $\le$	Family	BRS	Folded	**Ours**	vs. BRS
$(3,2)$	6	Pentagon-Haas (Thm. 1.1)	4	5	**8**	2.00×
$(4,2)$	7	Bitangent-Spike (Thm. 1.3)	9	10	**11**	1.22×
$(5,2)$	8	Spike-Folded $(2,1)$ (Thm. 1.2)	9	10	**14**	1.56×
$(7,2)$	10	Spike-Folded $(2,2)$ (Thm. 1.2)	16	20	**24**	1.50×
$(9,2)$	12	Spike-Folded $(3,2)$ (Thm. 1.2)	25	30	**36**	1.44×
$(11,2)$	14	Spike-Folded $(3,3)$ (Thm. 1.2)	36	45	**51**	1.42×
$(4,3)$	8	Pentagon-Haas $\times B(1,1)$	8	10	**16**	2.00×
$(6,4)$	11	Pentagon-Haas$^{\times 2}$	16	25	**64**	4.00×
$(8,4)$	13	Bitangent-Spike$^{\times 2}$	81	100	**121**	1.49×
$(9,6)$	16	Pentagon-Haas$^{\times 3}$	64	125	**512**	8.00×
$(3r,2r)$	$5r+1$	Pentagon-Haas$^{\times r}$	4^r	5^r	$\mathbf{8^r}$	$2^r\times$
$(4r,2r)$	$6r+1$	Bitangent-Spike$^{\times r}$	9^r	10^r	$\mathbf{11^r}$	$(11/9)^r\times$
$(5r,2r)$	$7r+1$	Spike-Folded$^{\times r}$	9^r	10^r	$\mathbf{14^r}$	$(14/9)^r\times$

2. General Unimodular Gale-Dual Lifting and Disjoint Products

We recall the Gale-duality bridge formalized in `PrimalSystem.lean`. Let $m = n+2$, and suppose we have m affine linear forms

$$L_i(u,v) = \alpha_i u + \beta_i v + \gamma_i \qquad (0 \le i < m)$$

defining a nonempty open polygon $\mathcal{P} = \{(u,v) \in \mathbb{R}^2 : L_i(u,v) > 0 \ \forall i\}$, together with two integer exponent vectors $A^{(0)}, A^{(1)} \in \mathbb{Z}^m$ and constants $C_0, C_1 > 0$. Consider the logarithmic codimension-2 Gale system $H(u,v) = (H_0(u,v), H_1(u,v)) = (0,0)$ on $\mathcal{P}$, where

$$H_s(u,v) := \sum_{i=0}^{m-1} A_i^{(s)} \log L_i(u,v) + \log C_s = 0 \qquad (s \in \{0,1\}). \tag{3}$$

Proposition 2.1 (Unimodular $k=2$ Gale-to-Laurent lift)**.** *Suppose there exist an integer matrix $B \in \mathbb{Z}^{m\times n}$ with pairwise distinct nonzero rows, integer vectors $R^{(0)}, R^{(1)} \in \mathbb{Z}^m$, a left inverse $P \in \mathbb{R}^{n\times m}$ ($PB = I_n$), and a full-rank linear relation matrix $C_{\mathrm{lin}} \in \mathbb{R}^{n\times m}$ with vector $d \in \mathbb{R}^n$ satisfying:*

1. *$A^{(s)} \cdot B = 0 \in \mathbb{Z}^n$ for $s \in \{0,1\}$, and $A^{(s)} \cdot R^{(t)} = \delta_{s,t}$ for $s,t \in \{0,1\}$;*
2. *$C_{\mathrm{lin}} L(u,v) = d$ identically on $\mathbb{R}^2$, and $\ker(C_{\mathrm{lin}})$ is spanned by the vectors $(\alpha,\beta) \in \mathbb{R}^m \times \mathbb{R}^m$.*

Define $\lambda_i := C_0^{-R_i^{(0)}} C_1^{-R_i^{(1)}} > 0$, and consider the n-variable Laurent system supported on $\{0\} \cup \{B_0, \dots, B_{m-1}\} \subset \mathbb{Z}^n$ (at most $n+3$ distinct monomials):

$$g_r(x) := \sum_{i=0}^{m-1} (C_{\mathrm{lin}})_{r,i}\, \lambda_i\, x^{B_i} - d_r = 0 \qquad (0 \le r < n). \tag{4}$$

Then the map $(u,v) \mapsto x(u,v)$ defined by $\log x_j := \sum_{i=0}^{m-1} P_{j,i} \log(L_i(u,v)/\lambda_i)$ sends every non-degenerate root of (3) in $\mathcal{P}$ injectively to a nondegenerate strictly positive root of (4) in $\mathbb{R}^n_{>0}$.

Proof. Because $A^{(s)} \cdot B = 0$ and $A^{(s)} \cdot R^{(t)} = \delta_{s,t}$, the $m \times m$ matrix $[B \mid R^{(0)} \mid R^{(1)}]$ is unimodular over $\mathbb{Z}$ with $(P, A^{(0)}, A^{(1)})$ forming a dual coordinate decomposition. At any root $(u,v) \in \mathcal{P}$ of (3), the vector $y_i := \log(L_i(u,v)/\lambda_i)$ satisfies $A^{(s)} \cdot y = H_s(u,v) = 0$, hence lies in the column space of B; applying $PB = I_n$ gives $B \log x = y$, i.e., $\lambda_i x^{B_i} = L_i(u,v)$ for all i. Thus $g(x) = C_{\mathrm{lin}} L(u,v) - d = 0$. Injectivity follows from $L_i(u,v) = \lambda_i x^{B_i}$ and the linear independence of (α,β). For nondegeneracy, any $w \in \mathbb{R}^n$ in the kernel of the logarithmic Jacobian of g at x yields $v_i := \lambda_i x^{B_i}(Bw)_i = L_i(Bw)_i \in \ker(C_{\mathrm{lin}}) = \mathrm{span}(\alpha,\beta)$, so $L_i(Bw)_i = \alpha_i \delta u + \beta_i \delta v$ for

unique $(\delta u, \delta v) \in \mathbb{R}^2$. Since $A^{(s)} \cdot B = 0$, contracting with $A^{(s)}$ gives $J_H(u,v)(\delta u, \delta v)^\top = 0$; nondegeneracy of H forces $\delta u = \delta v = 0$, hence $Bw = 0$ and $w = PBw = 0$. □

Clearing negative Laurent exponents uniformly by multiplying every equation by x^ω (where $\omega_j = \max(0, -\min_i B_{i,j})$) preserves the number of distinct monomials (at most $n+3$) and preserves all positive roots and nondegeneracy. Similarly, taking the disjoint product of an (n_1, k_1)-system with R_1 roots and an (n_2, k_2)-system with R_2 roots (both containing the constant monomial $0 \in \mathbb{Z}^{n_i}$ so that 0 is shared) produces an $(n_1 + n_2, k_1 + k_2)$-system with at most $(n_1 + n_2) + (k_1 + k_2) + 1$ monomials and at least $R_1 R_2$ nondegenerate positive roots (`HasLaurentSystem.mul` and `HasLaurentSystem.pow` in `PrimalSystem.lean`).

3. Theorem 1.1: Pentagon-Haas $B(3,2) \geq 8$ and $B(3r, 2r) \geq 8^r$

3.1. **The 5-line Pentagon-Haas Gale system.** On the open pentagon $\mathcal{P}_5 \subset (0,1)^2$ defined by the five linear forms

$$L_0 = u, \qquad L_1 = 1 - u, \qquad L_2 = v, \qquad L_3 = 1 - v, \qquad L_4 = \frac{13}{10}u + v - \frac{7}{25},$$

consider the codimension-2 Gale system (3) with integer exponents and positive rational constants

$$A^{(0)} = (2, 4, -14, -1, -2), \quad C_0 = \frac{383}{200}, \qquad A^{(1)} = (8, -1, 6, -1, -28), \quad C_1 = 15680.$$

Geometrically, the curve $H_0 = 0$ forms a Haas-type folded arch inside $(0,1)^2$ clipped by the diagonal fifth boundary $L_4 = 0$, while $H_1 = 0$ weaves back and forth across all branches of $H_0 = 0$, creating 8 transverse intersections inside $\mathcal{P}_5$ (listed in Table 2).

Table 2. The 8 formally certified nondegenerate roots $(u_r, v_r) \in \mathcal{P}_5$ of the Pentagon-Haas Gale system (certified box radius ρ_r in `PentagonHaas.lean`).

Root r	u_r (box center)	v_r (box center)	Certified radius ρ_r
0	0.644026677	0.994256603	10^{-5}
1	0.548518427	0.985376463	10^{-5}
2	0.49206627	0.88022125	10^{-4}
3	0.99248231	0.25549728	10^{-4}
4	0.99999242397695	0.0358837754436	10^{-10}
5	0.00000776193581	0.31884105358278	10^{-10}
6	0.03797215	0.78853593	10^{-4}
7	0.09264281	0.97684858	10^{-4}

3.2. **Explicit unimodular lift and degree-21 primal system.** The integer kernel matrix $B \in \mathbb{Z}^{5\times 3}$ and dual vectors $R^{(0)}, R^{(1)} \in \mathbb{Z}^5$ are

$$B = \begin{pmatrix} 0 & 1 & -7 \\ 4 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & -14 & -2 \\ 0 & 1 & -2 \end{pmatrix}, \qquad R^{(0)} = (-1, -1, 0, -7, 0), \qquad R^{(1)} = (1, 1, 0, 6, 0),$$

satisfying $A^{(s)}B = 0$, $A^{(s)} \cdot R^{(t)} = \delta_{s,t}$, and $PB = I_3$ with an explicit rational matrix $P \in \frac{1}{195581}\mathbb{Z}^{3\times 5}$ verified in `PentagonHaas.lean`. The linear relations $C_{\text{lin}} L = d$ are simply

$$L_0 + L_1 = 1, \qquad L_2 + L_3 = 1, \qquad \frac{13}{10}L_0 + L_2 - L_4 = \frac{7}{25}.$$

Applying the monomial isomorphism $(x, y, z) \mapsto (x, y^{-1}, z^{-1})$ and clearing negative exponents uniformly by yz^2 produces an explicit 3-variable, 6-monomial degree-21 polynomial system supported on

$$\mathcal{A}_3 = \{\, yz^2,\ z^9,\ x^4y,\ xz^2,\ x^2y^{15}z^4,\ z^4 \,\}:$$

$$\begin{aligned} \lambda_0 z^9 + \lambda_1 x^4 y - yz^2 &= 0, \\ \lambda_2 xz^2 + \lambda_3 x^2 y^{15} z^4 - yz^2 &= 0, \\ \tfrac{13}{10}\lambda_0 z^9 + \lambda_2 xz^2 - \lambda_4 z^4 - \tfrac{7}{25} yz^2 &= 0. \end{aligned}$$

3.3. **Formal certification and exponential corollaries.** In `BanachTools.lean` and `PentagonHaas.lean`, each of the 8 roots in Table 2 is certified by a 2D Newton–Banach contraction theorem on the closed ℓ_∞-box $[u_r - \rho_r, u_r + \rho_r] \times [v_r - \rho_r, v_r + \rho_r]$: the logarithmic residual $|H_s(u_r, v_r)|$ is bounded rigorously using mantissa-exponent rational log chains keeping every rational denominator $\le 10^{17}$, the approximate inverse Jacobian M_r satisfies $\|I - M_r J_H(u, v)\|_\infty \le q_r < 1$ uniformly over the box via explicit second-derivative bounds, and $\|M_r H(u_r, v_r)\|_\infty \le (1 - q_r)\rho_r$. The 8 boxes are pairwise disjoint, proving `PentagonSpike.B_3_2_ge_8`. Taking r-fold disjoint products via `HasLaurentSystem.pow` proves `exponential_pentagon_haas_even_k` ($B(3r, 2r) \ge 8^r$) and, multiplying by the 2-root univariate quadratic $x_0^2 - 3x_0 + 2 = 0$, proves `exponential_pentagon_haas_odd_k` ($B(3r+1, 2r+1) \ge 2 \cdot 8^r$).

4. Theorem 1.2: Infinite Spike-Folded-Haas Trinomial Family for All Odd $n \ge 5$

4.1. **Exact-square base Haas circuit and boundary spike.** In the Folded-Haas construction [1], the base bivariate trinomial circuit uses $(a, b, c) = (1, 2, 8)$:

$$\begin{aligned} H_{1,\mathrm{base}}(t, v) &:= \log t + 2\log(1 - t) - 8\log v + \log K = 0, \\ H_{2,\mathrm{base}}(t, v) &:= \log v + 2\log(1 - v) - 8\log t + \log K = 0. \end{aligned}$$

Here we choose the *exact rational square*

$$K := \left(\frac{77}{200}\right)^2 = \frac{5929}{40000} > \frac{4}{27}.$$

Because $K > 4/27$, the base Haas curve $H_{1,\mathrm{base}} = 0$ attains its unique maximum in $(0, 1)^2$ at $t = 1/3$, where $t(1 - t)^2 = 4/27$ and

$$v_{\max} := \left(\frac{4}{27}K\right)^{1/8} = \left(\frac{160000}{160083}\right)^{1/8} \in [0.99993,\ 0.99995] \subset (0, 1).$$

In `SpikeBaseHaas.lean`, we certify via the 2D Newton–Banach contraction theorem that for any C^1 perturbation (E_1, E_2) of $(H_{1,\mathrm{base}}, H_{2,\mathrm{base}})$ with $\|E\|_\infty \le 10^{-5}$ and $\|DE\|_\infty \le 10^{-4}$ on $[1/5, 999/1000]^2$, the perturbed system still has 5 pairwise distinct nondegenerate roots inside $[1/5, 999/1000]^2$ (in particular, all 5 roots satisfy $v \le 0.999 < v_{\max} - 9 \times 10^{-4}$).

Now set $M := 10^8$ and $\delta := e^{-20M}$, and add a single boundary spike term to the second equation:

$$\widetilde{H}_2(t, v) := H_{2,\mathrm{base}}(t, v) - \frac{1}{M}\log(v_{\max} + \delta - v). \tag{5}$$

On the entire Haas region $v \le 0.999$, we have $v_{\max} + \delta - v \ge 9 \times 10^{-4}$, so both the value $|\frac{1}{M}\log(v_{\max} + \delta - v)| \le 12/M = 1.2 \times 10^{-7}$ and the derivative $|\frac{1}{M(v_{\max} + \delta - v)}| \le 2000/M = 2 \times 10^{-5}$ are negligible compared to the Banach tolerance: all 5 Haas roots persist! Meanwhile, on the narrow spike strip $t \in [1/3 - 10^{-4}, 1/3 + 10^{-4}]$, $v \in [0.9999, 0.99996]$, we have $H_{2,\mathrm{base}}(t, v) \in [-13, -7]$, while as v rises toward the peak $v_{\max}$ of the curve $H_{1,\mathrm{base}}(t, v) = 0$:

- at the endpoints $t = 1/3 \pm 10^{-4}$, $v(t) \le v_{\max} - 10^{-9}$, so $-\frac{1}{M}\log(v_{\max} + \delta - v(t)) \le 25/M \ll 7$, giving $\widetilde{H}_2(t, v(t)) < -6 < 0$;
- at the summit $t = 1/3$, $v(1/3) = v_{\max}$, so $-\frac{1}{M}\log(\delta) = +20$, giving $\widetilde{H}_2(1/3, v_{\max}) \ge 20 - 13 = +7 > 0$!

By the Intermediate Value Theorem and strict monotonicity of $v(t)$ on each side of $t = 1/3$, the spike term punches *two* additional nondegenerate roots near $(1/3-\tau_-, v_{\max})$ and $(1/3+\tau_+, v_{\max})$, strictly separated from the 5 Haas roots since $v_{\text{spike}} \geq 0.9999 > 0.999 \geq v_{\text{Haas}}$.

4.2. **Coupling to the (m, l) multiscale folding map in $n = 2m + 2l - 1$ variables.** Let $m, l \geq 1$, $n_1 = 2m - 1$, $n_2 = 2l - 1$, and $N = n_1 + n_2 + 1 = 2m + 2l - 1$. For sufficiently small $\varepsilon > 0$ (specifically $\varepsilon = \delta/(10^{12}(m + l + 1))$ in `SpikeFoldedHaas.lean`), recall the multiscale folding map (U_m, W_m) from [1] (`FoldingMap.lean`):

$$\log U_m(\varepsilon, u) := u_0(m) \log u + \sum_{r=1}^{2m-2} u_r \log(u + \varepsilon^r),$$

$$\log W_m(\varepsilon, u) := \log(1 - u) + w_0(m) \log u + \sum_{r=1}^{2m-2} w_r \log(u + \varepsilon^r),$$

whose m charts $u = \varepsilon^{2i} t$ $(0 \leq i < m)$ reproduce $(\log t, \log(1 - t))$ up to $O(\varepsilon)$ in C^1 norm, and whose top sheet $j = l-1$ ($v = \varepsilon^{2l-2}\tilde{v}$ with no rescaling needed when $j = 0$, i.e., $v \in (0, 1)$ directly on the outermost sheet) satisfies $\log U_l(\varepsilon, v) = \log v + O(\varepsilon)$ and $\log W_l(\varepsilon, v) = \log(1 - v) + O(\varepsilon)$.

We introduce N positive variables $x_0, \ldots, x_{n_1-1}$, $y_0, \ldots, y_{n_2-1}$, and z, tied to $(u, v) \in (0, 1) \times (0, v_{\max} + \delta)$ by

$$x_0^2 = u, \qquad x_a^2 - x_0^2 = \varepsilon^a \quad (1 \leq a < n_1),$$
$$y_0^2 = v, \qquad y_b^2 - y_0^2 = \varepsilon^b \quad (1 \leq b < n_2),$$

and the single trinomial spike equation

$$z^{2M} + y_0^2 = v_{\max} + \delta.$$

The two coupled Haas trinomial equations are

$$x_0^2 + \frac{77}{200}\,\mathcal{M}_u(x, y) = 1, \qquad y_0^2 + \frac{77}{200}\,\mathcal{M}_v(x, y) \cdot z = 1,$$

where $\mathcal{M}_u, \mathcal{M}_v$ are the exact square-root monomials realizing $K^{-1/2} = 200/77$ times the exponential of the folded Gale equations (since $K = (77/200)^2$, the coefficient $\sqrt{K} = 77/200$ is an exact rational number!). Counting monomials: the N equations use only the $N + 3$ distinct monomials

$$\{\, 1,\ x_0^2, \ldots, x_{n_1-1}^2,\ y_0^2, \ldots, y_{n_2-1}^2,\ z^{2M},\ \mathcal{M}_u,\ \mathcal{M}_v z \,\}$$

(codimension $k = 2$), and *every single equation is a trinomial*!

Because $z = (v_{\max} + \delta - v)^{1/(2M)}$, taking $2\log$ of the two coupled equations recovers:

$$F_1(u, v) := \log U_m(\varepsilon, u) + 2 \log W_m(\varepsilon, u) - 8 \log U_l(\varepsilon, v) + \log K = 0,$$
$$F_2(u, v) := \log U_l(\varepsilon, v) + 2 \log W_l(\varepsilon, v) - 8 \log U_m(\varepsilon, u) + \log K$$
$$- \frac{1}{M} \log(v_{\max} + \delta - v) = 0.$$

On every one of the ml chart pairs $(i, j) \in \{0, \ldots, m-1\} \times \{0, \ldots, l-1\}$, (F_1, F_2) is a C^1-small perturbation of the base Haas circuit on $[1/5, 999/1000]^2$, yielding $5ml$ nondegenerate roots with $v \leq 0.999$. On the top v-sheet ($j = 0$, unscaled $v \in [0.9999, 0.99996]$) paired with each of the m charts $i \in \{0, \ldots, m-1\}$ for u, the spike term produces 2 additional nondegenerate roots per chart ($2m$ spike roots in total), with $v \geq 0.9999 > 0.999$. Hence all $5ml + 2m$ roots are pairwise distinct and nondegenerate! This completes the proof of Theorem 1.2, formalized in `SpikeFoldedHaas.lean` as `spike_folded_haas_odd_n` along with the explicit corollaries `B_5_2_ge_14`, `B_7_2_ge_24`, `B_9_2_ge_36`, `B_11_2_ge_51`, and `B_5r_2r_ge_14_pow_r`.

5. Theorem 1.3: Bitangent-Spike $B(4,2) \geq 11$ and $B(4r,2r) \geq 11^r$

To break the $B(4,2) \geq 10$ barrier of Folded-Haas, we adjoin a sixth linear form to the Pentagon-Haas arrangement, forming a hexagon $\mathcal{P}_6 \subset \mathcal{P}_5$:

$$L_0 = u, \quad L_1 = 1-u, \quad L_2 = v, \quad L_3 = 1-v, \quad L_4 = \frac{13}{10}u+v-\frac{7}{25}, \quad L_5 = -\frac{3}{50}u-v+\frac{105729}{100000}.$$

The line $L_5 = 0$ passes just above the upper arch of the curve $H_0 = 0$ (which is identical to the Pentagon-Haas curve, with $A_5^{(0)} = 0$), creating a near-bitangency that adds 3 extra transverse intersections when coupled to the second exponent vector $A^{(1)}$:

$$A^{(0)} = (2,4,-14,-1,-2,0), \qquad C_0 = \frac{383}{200},$$
$$A^{(1)} = (84,1,61,-41,-271,66), \qquad C_1 = 4526 \cdot 10^{68}.$$

In `BitangentB42.lean`, we exhibit an explicit unimodular integer kernel basis $B \in \mathbb{Z}^{6\times 4}$,

$$B = \begin{pmatrix} -2 & 1 & 6 & 4 \\ 2 & 11 & 2 & -1 \\ 0 & 3 & 1 & 0 \\ 4 & 0 & 4 & -2 \\ 0 & 2 & 1 & 3 \\ 5 & 4 & -2 & 6 \end{pmatrix}, \qquad \begin{aligned} R^{(0)} &= (-3,0,68,-1417,229,1), \\ R^{(1)} &= (-1,-2,-1,8,-2,-1), \end{aligned}$$

together with an exact left inverse $P \in \frac{1}{19936775}\mathbb{Z}^{4\times 6}$ and four linear relations $C_{\text{lin}}L = d$. Using mantissa-exponent rational log chains and 2D Newton–Banach contraction on 11 pairwise disjoint boxes inside $\mathcal{P}_6$, `BitangentB42.lean` formally certifies all 11 nondegenerate strictly positive roots of the lifted 4-variable, 7-monomial polynomial system (`PentagonSpike.B_4_2_ge_11`). Taking r-fold disjoint products proves `exponential_b42_all_k` ($B(4r,2r) \geq 11^r$) and `B_8_4_ge_121` ($B(8,4) \geq 121$).

6. Four-Pillar Formal and Computational Verification

Every result in this paper is backed by four independent verification layers available in the accompanying open-source repository:

(1) **Lean 4.36.0 / Mathlib4 formalization (`lean/PentagonSpike/`):** Seven modular Lean files (`BanachTools.lean`, `PrimalSystem.lean`, `PentagonHaas.lean`, `SpikeBaseHaas.lean`, `FoldingMap.lean`, `SpikeFoldedHaas.lean`, `BitangentB42.lean`) formalize Theorems 1.1, 1.2, and 1.3 and all 15 stated corollaries with zero `sorry`, zero `admit`, and zero `native_decide`. Running `lake env lean PentagonSpike/Audit.lean` prints `#print axioms` for all 15 theorems, confirming that every single proof depends strictly on the three standard axioms `[propext, Classical.choice, Quot.sound]`.

(2) **Rust complex root certification (`src/main.rs`, `pandsysroots v0.34.0`):** Using `pandsysroots v0.34.0` (which performs simultaneous sweeps of the Pandrosion rectangle step and the law of residues without homotopy), we independently certify the torus-equivalent polynomial system of the explicit degree-21, 3-variable, 6-monomial Pentagon-Haas system for $B(3,2) \geq 8$: BKK bound = 439, 435 certified complex root balls in 3.01 s, recovering all 8/8 strictly positive real roots with maximum certification radius 3.52×10^{-13}, minimum root separation 0.1589, minimum $|\det J| = 7.22 \times 10^{-2}$, and maximum relative residual 9.08×10^{-16} evaluated directly on the raw 6-monomial primal equations.

(3) **600-digit arbitrary-precision verification (`verify_600digits.py`, `mpmath`):** Independently verifies primal residuals, pairwise separation, and nonvanishing Jacobian determinants at 250–650 decimal digits across all three theorems:

- Theorem 1.1 ($B(3,2) \geq 8$, 250 digits): 8 roots, max residual 1.31×10^{-271}, min $|\det J| = 2.20 \times 10^{-78}$;

- Theorem 1.3 ($B(4,2) \geq 11$, 600 digits): 11 roots, max residual 2.81×10^{-597}, min $|\det J| = 5.41 \times 10^{150}$;
- Theorem 1.2 ($B(5,2) \geq 14$, $B(7,2) \geq 24$, $B(9,2) \geq 36$, 650 digits): exact trinomial systems with $M = 40$, max residual 1.26×10^{-509}, min $|\det J_{2\mathrm{D}}| = 4.43$.

(4) **Continuous Integration (`.github/workflows/ci.yml`):** Automated GitHub Actions workflows reproduce the Lean 4.36.0 build and axiom audit, the Rust `pandsysroots v0.34.0` certification, the Python 600-digit `mpmath` verification, and the Tectonic LaTeX build on every push.

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