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AN EXPLICIT SPHERICAL D-DESIGN SUBMERSION AND A 2,621-FOLD IMPROVEMENT ON THE ASYMPTOTIC LOWER BOUND FOR POSITIVE MAXWELL EQUILIBRIA

IVAN BESEVIC

Abstract. Let $M^+(n)$ denote the maximum number of isolated equilibrium points of the three-dimensional electrostatic potential $V(x) = \sum_{j=1}^n q_j \|x - a_j\|^{-1}$ generated by n strictly positive point charges $q_j > 0$ at distinct locations $a_j \in \mathbb{R}^3$, and let $M^+_{\mathbb{S}^2}(n) \le M^+(n)$ denote the corresponding maximum when all charges lie on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. In July 2026, Arathoon, Ball, and Kvalheim disproved Maxwell's conjecture ($M^+(5) \ge 24 > 16$) and established the linear lower bound $M^+(n) \ge 10n - 26$ for odd $n \ge 3$; in August 2026, Kintali proved the first superlinear lower bound for positive charges, $M^+(n) \ge (3.596 \times 10^{-4} - o(1))\, n^{3/2}$, which overtakes the linear slope $10n$ only near $n \approx 7.73 \times 10^8$.

We prove the explicit lower bound

$$M^+_{\mathbb{S}^2}(n) \ge \left(\frac{2\sqrt{2}}{3} - o(1)\right) n^{3/2} \approx 0.942809\, n^{3/2},$$

improving the leading asymptotic constant by a factor greater than 2,621 while confining all n strictly positive charges to the unit sphere $\mathbb{S}^2$. Non-asymptotically, for every integer $D \ge 3$ we prove that $n(D) = (\lfloor D/2 \rfloor + 1)(D+1)$ strictly positive charges on $\mathbb{S}^2$ admit at least $N(D) = \frac{1}{3}D(D-1)(D-2)$ pairwise distinct, strictly nondegenerate equilibria inside the open unit ball B^3. In particular, our bound overtakes the $10n$ linear slope already at $D = 21$ ($n = 242$ positive charges: $N(21) = 2{,}660 > 2{,}420$), more than 3×10^6 times earlier than Kintali's crossover.

The proof combines two elementary, closed-form theorems: (1) a *Nested Complex-Plane Harmonic Lattice* achieving $\frac{1}{3}D^3 - O(D^2)$ explicit nondegenerate critical points ($\approx 327.7\times$ denser than Kintali's 12-wave packet) via a unified 4-line closed-form 3×3 Hessian determinant covering both odd and even scales, and (2) an *Explicit Spherical D-Design Submersion Theorem* showing that the differential of the degree-D spherical moment map with respect to $(\delta w, \delta t, \delta\theta)$ on $\mathbb{S}^2$ at the $(\lfloor D/2 \rfloor + 1)(D+1)$-point Gauss–Legendre design has full rank $(D+1)^2 - 1$ for every $D \ge 1$ (using $4\times$ fewer charges than a strength-$2D$ quadrature). The core algebraic, derivative, dual-injectivity, counting, and asymptotic building blocks are formally verified in Lean 4.36.0 (`MaxwellPositive`, zero `sorry`, zero `native_decide`, standard axioms only), while the analytic multipole/submersion/Implicit Function Theorem steps are proved in full on paper. In addition, explicit exact-rational positive realizations on $\mathbb{S}^2$ (satisfying $X_j^2 + Y_j^2 + Z_j^2 = W_j^2$ identically in $\mathbb{Z}$) for $D = 4, 5, 6$ ($n = 15, 18, 28$ charges; $8, 20, 40$ actual 3D Coulomb equilibria inside B^3) are independently certified in pure Rust via `pandsysroots 0.34.0` and 768/1024-bit outward-rounded Pandrosion V31 interval Krawczyk certification.

1. Introduction

In his 1873 *Treatise on Electricity and Magnetism* [6], James Clerk Maxwell asked for the maximum number of isolated critical points (electrostatic equilibria) of the Coulomb potential

$$V(x) = \sum_{j=1}^{n} \frac{q_j}{\|x - a_j\|}, \qquad x \in \mathbb{R}^3 \setminus \{a_1, \dots, a_n\}, \tag{1}$$

generated by n point charges $q_j \in \mathbb{R} \setminus \{0\}$ placed at distinct positions $a_j \in \mathbb{R}^3$. Maxwell conjectured the upper bound $(n-1)^2$ for arbitrary nonzero charges; for strictly positive charges $q_j > 0$, the corresponding maximum $M^+(n)$ long resisted superlinear lower bounds. Even

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the finite-charge finiteness conjecture remains open for $n \geq 4$, while Gabrielov, Novikov, and Shapiro [3] formalized Maxwell's conjecture and Tsai [7] studied the three-charge case.

Recent breakthroughs in 2026 dramatically reshaped both the positive and signed regimes:

- **Disproof of Maxwell's conjecture and linear positive bound:** In July 2026, Arathoon, Ball, and Kvalheim [1] disproved Maxwell's $(n-1)^2$ conjecture already for $n = 5$ strictly positive point charges by constructing $M^+(5) \geq 24 > (5-1)^2 = 16$ nondegenerate equilibria. By iterating their $+20$ axial-pair bifurcation (Proposition 1 of [1]), they established the linear lower bound $M^+(3+2m) \geq 4 + 20m = 10n - 26$ for every odd $n = 3 + 2m \geq 3$ (subsequently sharpened for odd $n \geq 11$ to $M^+(11) \geq 94$ and $M^+(2k+1) \geq 20k - 6$ for $k \geq 5$ in [2]).
- **Superlinear lower bound for positive charges:** In August 2026, Kintali [4] broke the linear barrier for strictly positive charges by proving the first superlinear lower bound,

$$M^+(n) \geq \left(3.596 \times 10^{-4} - o(1)\right) n^{3/2}. \tag{2}$$

- **Cubic lower bound for signed charges:** For *signed* charges $M(n)$, Kintali [5] (August 2026) proved the cubic lower bound $M(n) \geq (\frac{1}{48} - o(1))\, n^3$, showing that $M(n)$ exceeds Maxwell's quadratic threshold $(n-1)^2$ by an unbounded factor as $n \to \infty$.

Kintali's positive-charge bound (2) combines a 12-wave harmonic exponential packet of degree D producing $c_{\text{Kintali}} D^3$ critical points (with effective density $c_{\text{Kintali}} = 2\sqrt{2} \times 3.596 \times 10^{-4} \approx 1.0171 \times 10^{-3}$) with a positive Gauss–Legendre product cubature of a degree-D surface density on $\mathbb{S}^2$ requiring strength $2D$, which uses $(D+1)(2D+1) \approx 2D^2$ positive charges. Because the leading constant is 3.596×10^{-4}, the bound (2) overtakes the $10n$ linear slope only at $n \approx (10/3.596 \times 10^{-4})^2 \approx 7.73 \times 10^8$ charges.

1.1. **Main Results.** In this paper we replace *both* ingredients of the positive-charge construction by completely elementary, closed-form mechanisms, improving the asymptotic constant by more than 2,621× and lowering the threshold to overtake $10n$ by more than six orders of magnitude.

Theorem 1.1 (Main Asymptotic and Non-Asymptotic Theorem)**.** *Let $M^+_{\mathbb{S}^2}(n)$ be the maximum number of isolated, strictly nondegenerate critical points in $\mathbb{R}^3 \setminus \{a_1, \ldots, a_n\}$ of a Coulomb potential (1) with n strictly positive charges $q_j > 0$ located at distinct points a_j on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.*

(i) (Explicit finite-D bound.) *For every integer $D \geq 3$, let*

$$n(D) := \left(\left\lfloor \frac{D}{2} \right\rfloor + 1\right)(D+1), \qquad N(D) := \frac{D(D-1)(D-2)}{3}.$$

Then $M^+_{\mathbb{S}^2}(n(D)) \geq N(D)$. Moreover, all $N(D)$ nondegenerate critical points lie in the open unit ball $B^3 \subset \mathbb{R}^3$.

(ii) (Monotonicity and all-n asymptotic bound.) *For every $n \geq 8 = n(3)$, $M^+_{\mathbb{S}^2}(n) \geq N(D(n))$ where $D(n) = \max\{D \geq 3 : n(D) \leq n\}$. Consequently,*

$$\liminf_{n\to\infty} \frac{M^+_{\mathbb{S}^2}(n)}{n^{3/2}} \geq \frac{2\sqrt{2}}{3} \approx 0.94280904. \tag{3}$$

(iii) (Early crossover beyond $10n$.) *For every $D \geq 21$, we have $N(D) > 10\, n(D)$. In particular, at $D = 21$ ($n = 242$ strictly positive charges on $\mathbb{S}^2$),*

$$M^+_{\mathbb{S}^2}(242) \geq N(21) = 2{,}660 > 2{,}420 = 10 \cdot 242.$$

1.2. **Why the Constant Jumps by 2,621×.** The multiplicative gain over [4] factors cleanly into two independent improvements:

(1) **Harmonic critical-point density (≈ 327.7× gain):** Instead of a 12-wave exponential packet with effective density $c_{\text{Kintali}} \approx 1.0171 \times 10^{-3}\, D^3$, we construct an explicit multiscale polynomial $H^{(D)}(x, y, z)$ by alternating the complex-plane harmonics $\mathrm{Re}(e^{i\pi/4}(x+iy)^d)$ and $\mathrm{Re}(e^{i\pi/4}(y+iz)^d)$. Each scale $d \in \{3, \ldots, D\}$ contributes $(d-2)(d-1)$ explicit, closed-form,

TABLE 1. Comparison of asymptotic and non-asymptotic lower bounds for strictly positive Maxwell equilibria $M^+(n)$.

Result	Support	Lower bound	Ratio to [4]	Beats $10n$ at
ABK (Jul 2026) [1]	$\mathbb{R}^3$	$10n - 26$ (odd $n \geq 3$)	—	—
Besevic (Nov 2026) [2]	$\mathbb{R}^3$	$10n - 16$ (odd $n \geq 11$)	—	—
Kintali (Aug 2026) [4]	$\mathbb{S}^2$	$(3.596 \times 10^{-4} - o(1))n^{3/2}$	$1\times$	$n \approx 7.73 \times 10^8$
This work (Thm. 1.1)	$\mathbb{S}^2$	$(\frac{\mathbf{2\sqrt{2}}}{\mathbf{3}} - \mathbf{o(1)})\mathbf{n^{3/2}}$	$> \mathbf{2{,}621}\times$	$\mathbf{n = 242}$ $(D = 21)$

strictly nondegenerate critical points, summing to $\frac{1}{3}D(D-1)(D-2) \sim \frac{1}{3}D^3$, a density gain of $\frac{1/3}{1.0171\times10^{-3}} \approx 327.73\times$.

(2) **Minimal-strength spherical D-design submersion ($4\times$ fewer charges $\Longrightarrow$ $4^{3/2} = 8\times$ gain):** Discretizing a degree-D surface density on $\mathbb{S}^2$ by quadrature requires a spherical $2D$-design, which costs $(D+1)(2D+1) \approx 2D^2$ charges. We bypass density quadrature completely: viewing the degree-D spherical moments directly as a smooth map $\Phi_D : (\mathbb{R}_{>0} \times (-1,1) \times \mathbb{R}/2\pi\mathbb{Z})^{n(D)} \to \mathbb{R}^{(D+1)^2-1}$ on $\mathbb{S}^2$, we prove that at the Gauss–Legendre spherical D-design of size $(\lfloor D/2 \rfloor + 1)(D+1) = \frac{1}{2}D^2 + O(D)$, the differential $d\Phi_D$ is *surjective for every* $D \geq 1$. Its proof takes 12 lines via exact discrete Fourier decoupling and Hermite double-root counting.

Multiplying the two improvements gives $327.73 \times 8 \approx 2{,}621.83\times$:

$$\frac{2\sqrt{2}/3}{3.596 \times 10^{-4}} \approx 2{,}621.827.$$

1.3. **Verification Scope: Formalized Bricks, Analytic Steps, and Concrete Certificates.** To keep the scope of machine verification completely transparent, we separate what is formally checked by Lean 4, what is proved analytically in the paper, and what is independently certified in Rust:

- **Formally verified in Lean 4.36.0 (`lean/MaxwellPositive/`):** Using Mathlib4 (zero `sorry`, zero `native_decide`, and strictly the three standard axioms `propext`, `Classical.choice`, `Quot.sound`), we formally verify: (a) both odd-scale and even-scale harmonics $H_d^{\mathrm{odd}}, H_d^{\mathrm{even}}$ (exact first and all nine second partial derivatives via `HasDerivAt`, analytic 3D harmonicity $\Delta \equiv 0$, distinctness of all $(d-2)(d-1)$ critical points, vanishing gradient, and closed-form nonzero Hessian determinant for all $d \geq 3$), together with the exact algebraic rescaling decomposition of the nested polynomial $H^{(D)}$; (b) exact mod-$(D+1)$ DFT decoupling, simple and Hermite double-root counting, and dual injectivity of the degree-D spherical harmonic evaluation map for all $D \geq 1$; and (c) the exact counting identity $N(D) = \frac{1}{3}D(D-1)(D-2)$, $n(3) = 8$, the crossover $N(D) > 10\,n(D)$ for all $D \geq 21$, both the subsequence and all-n asymptotic limits to $2\sqrt{2}/3$, and the comparison $> 2{,}621\times$.
- **Proved analytically on paper (not formalized in Lean):** The standard multivariable analysis steps connecting these verified bricks to Theorem 1.1 are proved in full in Sections 2–4: the spherical multipole expansion (15) of the Coulomb potential, translating dual injectivity into surjectivity of $d\Phi_D$ and a smooth local right inverse via the submersion theorem, the uniform C^2 analytic tail estimate for $R_{D+1,\lambda}$, and the Implicit Function Theorem persistence arguments for multiscale nesting (Remark 2.2) and the $\lambda \to 0^+$, $\delta \to 0^+$ perturbations in Theorem 1.1.
- **Independently certified in pure Rust (`rust/`):** We independently certify (a) completeness and interval-certified Morse signatures (c_1, c_2) of both H_d^{odd} and H_d^{even} for $d = 3, 4, 5, 6$ via `pandsysroots 0.34.0` plus 256-bit outward-rounded dyadic interval Hessian determinants, and (b) explicit *exact-rational* positive $\mathbb{S}^2$ Coulomb realizations (rational Pythagorean quadruples $X_j^2 + Y_j^2 + Z_j^2 = W_j^2$ identically in $\mathbb{Z}$) for $(D, n) \in \{(4, 15), (5, 18), (6, 28)\}$ with $8, 20, 40$ actual 3D equilibria certified strictly inside

B^3 via 768/1024-bit outward-rounded dyadic arithmetic, Pandrosion V31 projective order-4 radical enclosures, and 3D interval Krawczyk contraction, with zero `f64/libm` trust in the certification bounds or Morse signs.

2. Theorem 1: Closed-Form Nested Complex-Plane Harmonic Lattice

For any integers $p, q \geq 2$, consider the real-valued polynomial on $\mathbb{R}^3$

$$H_{p,q}(x,y,z) \coloneqq \mathrm{Re}\big(e^{i\pi/4}(x+iy)^p\big) + \mathrm{Re}\big(e^{i\pi/4}(y+iz)^q\big). \tag{4}$$

Because $(x+iy)^p$ is holomorphic in $x+iy$ (independent of z) and $(y+iz)^q$ is holomorphic in $y+iz$ (independent of x), both terms are harmonic in $\mathbb{R}^3$, so $\Delta H_{p,q} \equiv 0$ identically. For every scale degree $d \geq 3$, our alternating multiscale construction uses the two adjacent pairs

$$H_d^{\mathrm{odd}}(x,y,z) \coloneqq H_{d-1,d}(x,y,z), \qquad H_d^{\mathrm{even}}(x,y,z) \coloneqq H_{d,d-1}(x,y,z). \tag{5}$$

Note that H_d^{even} is *not* obtained from H_d^{odd} by an orthogonal reflection $(x,y,z) \mapsto (z,y,x)$ (which conjugates the complex planes and alters the phase $e^{i\pi/4}$); instead, both H_d^{odd} and H_d^{even} are handled simultaneously by the following unified closed-form theorem.

Theorem 2.1 (Unified Closed-Form Critical Lattice and Nondegeneracy)**.** *Let $d \geq 3$ and let $(p,q) \in \{(d-1,d),\,(d,d-1)\}$ (corresponding to H_d^{odd} and H_d^{even} respectively). Define the angles*

$$\theta_1(j) \coloneqq \frac{(2j+1)\frac{\pi}{2} - \frac{\pi}{4}}{p-1} = \frac{(4j+1)\pi}{4(p-1)}, \qquad j \in \{0,1,\ldots,p-2\}, \tag{6}$$

$$\theta_2(k) \coloneqq \frac{k\pi - \frac{\pi}{4}}{q-1} = \frac{(4k-1)\pi}{4(q-1)}, \qquad k \in \{0,1,\ldots,q-2\}, \tag{7}$$

and the real ratios $a_j \coloneqq \cot\theta_1(j)$, $b_k \coloneqq \tan\theta_2(k)$. Then $\sin\theta_1(j) > 0$ and $\cos\theta_2(k) \neq 0$ for all j,k, the $(p-1)(q-1) = (d-2)(d-1)$ points

$$p_{j,k} \coloneqq (y_{j,k}a_j,\ y_{j,k},\ y_{j,k}b_k) \in \mathbb{R}^3, \tag{8}$$

with

$$y_{j,k} \coloneqq \begin{cases} \dfrac{(d-1)(-1)^j\cos^{d-1}\theta_2(k)}{d(-1)^k\sin^{d-2}\theta_1(j)} \neq 0 & \text{if } (p,q) = (d-1,d) \text{ (odd scale } H_d^{\mathrm{odd}}), \\[2ex] \dfrac{(d-1)(-1)^k\sin^{d-1}\theta_1(j)}{d(-1)^j\cos^{d-2}\theta_2(k)} \neq 0 & \text{if } (p,q) = (d,d-1) \text{ (even scale } H_d^{\mathrm{even}}), \end{cases} \tag{9}$$

are pairwise distinct critical points of $H_{p,q}$, and at every $p_{j,k}$ the 3×3 Hessian determinant satisfies the exact closed-form identity

$$\det\mathrm{Hess}\,H_{p,q}(p_{j,k}) = \frac{(p-1)(q-1)(p-q)\,M_{j,k}^3\,y_{j,k}}{\|w_1\|^2\|w_2\|^2} = \mp\frac{(d-1)(d-2)\,M_{j,k}^3\,y_{j,k}}{\|w_1\|^2\|w_2\|^2} \neq 0, \tag{10}$$

with the minus sign for H_d^{odd} and the plus sign for H_d^{even}, where $w_1 = y_{j,k}(a_j+i)$, $w_2 = y_{j,k}(1+ib_k)$, and $M_{j,k} = q\,e^{i\pi/4}w_2^{q-1} \in \mathbb{R}\setminus\{0\}$.

Proof. Set $\alpha = e^{i\pi/4}$, $w_1 = x+iy$, $w_2 = y+iz$, and define the complex derivatives

$$g_1 \coloneqq p\alpha w_1^{p-1}, \qquad g_2 \coloneqq q\alpha w_2^{q-1},$$
$$A \coloneqq p(p-1)\alpha w_1^{p-2}, \qquad B \coloneqq q(q-1)\alpha w_2^{q-2}.$$

Differentiating (4) with respect to (x,y,z) gives the exact gradient and 3×3 Hessian:

$$\nabla H_{p,q} = \big(\mathrm{Re}\,g_1,\ -\mathrm{Im}\,g_1 + \mathrm{Re}\,g_2,\ -\mathrm{Im}\,g_2\big), \quad \mathrm{Hess}\,H_{p,q} = \begin{pmatrix} \mathrm{Re}\,A & -\mathrm{Im}\,A & 0 \\ -\mathrm{Im}\,A & -\mathrm{Re}\,A + \mathrm{Re}\,B & -\mathrm{Im}\,B \\ 0 & -\mathrm{Im}\,B & -\mathrm{Re}\,B \end{pmatrix}. \tag{11}$$

In particular $\mathrm{tr}(\mathrm{Hess}\,H_{p,q}) = \mathrm{Re}\,A + (-\mathrm{Re}\,A + \mathrm{Re}\,B) - \mathrm{Re}\,B = 0$, confirming $\Delta H_{p,q} \equiv 0$.

Well-definedness and distinctness. Since $p,q \geq 2$, both $m_1 \coloneqq p-1 \geq 1$ and $m_2 \coloneqq q-1 \geq 1$. For $0 \leq j \leq m_1 - 1$, $0 < \frac{4j+1}{4m_1} < 1$, so $\theta_1(j) \in (0,\pi)$ and $\sin\theta_1(j) > 0$. Since cot is strictly

decreasing on $(0,\pi)$, the values $a_j = \cot\theta_1(j)$ are pairwise distinct. For $0 \le k \le m_2 - 1$, we have $\theta_2(k) \in (-\frac{\pi}{4}, \pi - \frac{\pi}{4})$. Moreover, $\theta_2(k) = \frac{\pi}{2}$ would imply $4k - 1 = 2m_2$, which is impossible by parity (odd equals even). Hence $\cos\theta_2(k) \neq 0$, and since $\tan(\theta_2(k_1)) = \tan(\theta_2(k_2))$ implies $\sin(\theta_2(k_1) - \theta_2(k_2)) = 0$ while $|\theta_2(k_1) - \theta_2(k_2)| < \pi$, the values $b_k = \tan\theta_2(k)$ are pairwise distinct. Because $y_{j,k} \neq 0$, the ratios $x/y = a_j$ and $z/y = b_k$ uniquely recover (j,k), proving that all $m_1 m_2 = (d-2)(d-1)$ points $p_{j,k}$ are pairwise distinct.

Vanishing of $\nabla H_{p,q}(p_{j,k})$. Write $a_j + i = e^{i\theta_1(j)}/\sin\theta_1(j)$ and $1 + ib_k = e^{i\theta_2(k)}/\cos\theta_2(k)$. By (6)–(7),

$$\frac{\pi}{4} + (p-1)\theta_1(j) = j\pi + \frac{\pi}{2}, \qquad \frac{\pi}{4} + (q-1)\theta_2(k) = k\pi,$$

so

$$\alpha(a_j + i)^{p-1} = \frac{(-1)^j i}{\sin^{p-1}\theta_1(j)}, \qquad \alpha(1 + ib_k)^{q-1} = \frac{(-1)^k}{\cos^{q-1}\theta_2(k)}.$$

Multiplying by py^{p-1} and qy^{q-1} respectively shows that at $p_{j,k}$:

$$g_1 = \frac{p y_{j,k}^{p-1}(-1)^j}{\sin^{p-1}\theta_1(j)}\, i \in i\mathbb{R}, \qquad g_2 = \frac{q y_{j,k}^{q-1}(-1)^k}{\cos^{q-1}\theta_2(k)} \in \mathbb{R}.$$

Hence $\operatorname{Re} g_1 = 0$ and $\operatorname{Im} g_2 = 0$. In both cases $(p,q) = (d-1,d)$ and $(p,q) = (d,d-1)$, the definition (9) of $y_{j,k}$ solves

$$\frac{p y_{j,k}^{p-1}(-1)^j}{\sin^{p-1}\theta_1(j)} = \frac{q y_{j,k}^{q-1}(-1)^k}{\cos^{q-1}\theta_2(k)} =: M_{j,k} \neq 0,$$

so $\operatorname{Im} g_1 = \operatorname{Re} g_2 = M_{j,k}$, giving $-\operatorname{Im} g_1 + \operatorname{Re} g_2 = 0$ and $\nabla H_{p,q}(p_{j,k}) = 0$.

Exact 4-line Hessian determinant. Expanding the tridiagonal 3×3 determinant in (11) directly gives

$$\det \operatorname{Hess} H_{p,q} = |A|^2 \operatorname{Re} B - |B|^2 \operatorname{Re} A. \tag{12}$$

At $p_{j,k}$, we have $g_1 = iM$ and $g_2 = M$ (writing $M = M_{j,k} \neq 0$, $y = y_{j,k} \neq 0$), and the algebraic identities $Aw_1 = (p-1)g_1 = i(p-1)M$ and $Bw_2 = (q-1)g_2 = (q-1)M$. Taking moduli squared and real parts (using $\operatorname{Im} w_1 = y$ and $\operatorname{Re} w_2 = y$) gives

$$|A|^2 = \frac{(p-1)^2 M^2}{\|w_1\|^2}, \qquad |B|^2 = \frac{(q-1)^2 M^2}{\|w_2\|^2},$$

$$\operatorname{Re} A = \operatorname{Re}\left(\frac{i(p-1)M\overline{w_1}}{\|w_1\|^2}\right) = \frac{(p-1)My}{\|w_1\|^2}, \qquad \operatorname{Re} B = \operatorname{Re}\left(\frac{(q-1)M\overline{w_2}}{\|w_2\|^2}\right) = \frac{(q-1)My}{\|w_2\|^2}.$$

Substituting these four quantities into (12) yields

$$\begin{aligned}\det \operatorname{Hess} H_{p,q}(p_{j,k}) &= \frac{(p-1)^2 M^2}{\|w_1\|^2} \cdot \frac{(q-1)My}{\|w_2\|^2} - \frac{(q-1)^2 M^2}{\|w_2\|^2} \cdot \frac{(p-1)My}{\|w_1\|^2} \\ &= \frac{(p-1)(q-1)(p-q)\, M^3\, y}{\|w_1\|^2 \|w_2\|^2} \neq 0.\end{aligned}$$

□

Remark 2.2 (Alternating planes and multiscale nesting). To nest all scales $d = 3, 4, \ldots, D$ inside a single degree-D harmonic polynomial, we alternate the two complex planes between even and odd degrees:

$$H^{(D)}(x,y,z) := \sum_{\substack{2 \le d \le D \\ d \text{ even}}} \varepsilon_d \operatorname{Re}\big(e^{i\pi/4}(x+iy)^d\big) + \sum_{\substack{3 \le d \le D \\ d \text{ odd}}} \varepsilon_d \operatorname{Re}\big(e^{i\pi/4}(y+iz)^d\big), \tag{13}$$

with $\varepsilon_D = 1$ and strictly positive scale weights $\varepsilon_2, \ldots, \varepsilon_{D-1} > 0$. Define the scale radii $R_d \coloneqq \varepsilon_{d-1}/\varepsilon_d$ for $d = 3, \ldots, D$. Under the rescaling $(x, y, z) = R_d(u, v, w)$,

$$\frac{1}{\varepsilon_{d-1} R_d^{d-1}} H^{(D)}(R_d u, R_d v, R_d w) = H_d^{\mathrm{alt}}(u, v, w) + \sum_{k=2}^{d-2} \left(\prod_{m=k+1}^{d-1} \frac{R_m}{R_d} \right) O(|u, v, w|^k) + \sum_{k=d+1}^{D} \left(\prod_{m=d+1}^{k} \frac{R_d}{R_m} \right) O(|u, v, w|^k),$$

where

$$H_d^{\mathrm{alt}}(u, v, w) = \begin{cases} H_d^{\mathrm{odd}}(u, v, w) = \mathrm{Re}\big(e^{i\pi/4}(u + iv)^{d-1}\big) + \mathrm{Re}\big(e^{i\pi/4}(v + iw)^d\big) & (d \text{ odd}), \\ H_d^{\mathrm{even}}(u, v, w) = \mathrm{Re}\big(e^{i\pi/4}(u + iv)^d\big) + \mathrm{Re}\big(e^{i\pi/4}(v + iw)^{d-1}\big) & (d \text{ even}). \end{cases}$$

By Theorem 2.1, in *both* parities H_d^{alt} possesses $(d-2)(d-1)$ explicit, pairwise distinct, strictly nondegenerate critical points in $\mathbb{R}^3 \setminus \{0\}$. Choosing the scale ratios $R_{d-1}/R_d > 0$ sufficiently small for each $d = 4, \ldots, D$, all perturbation terms in the sum vanish uniformly in C^2 on a compact neighborhood of the $(d-2)(d-1)$ critical points of H_d^{alt}, and the shells $R_3 \ll R_4 \ll \cdots \ll R_D$ are pairwise disjoint. By the Implicit Function Theorem, $H^{(D)}$ has at least

$$N(D) = \sum_{d=3}^{D} (d-2)(d-1) = \frac{D(D-1)(D-2)}{3} \tag{14}$$

pairwise distinct, strictly nondegenerate critical points in $\mathbb{R}^3$.

3. Theorem 2: Explicit Spherical D-Design Submersion on $\mathbb{S}^2$

To realize the harmonic polynomial $H^{(D)}$ of degree D as the Taylor expansion at the origin of a Coulomb potential (1) with n strictly positive charges $q_j > 0$ on the unit sphere $\mathbb{S}^2$, recall the standard multipole expansion for $\|x\| < 1$:

$$V(x) = \sum_{j=1}^{n} \frac{q_j}{\|x - a_j\|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell,m}\, r^\ell Y_\ell^m(\hat{x}), \tag{15}$$

where $a_j = (\sqrt{1-t_j^2}\cos\theta_j,\ \sqrt{1-t_j^2}\sin\theta_j,\ t_j) \in \mathbb{S}^2$ ($t_j \in (-1,1)$, $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$), and the complex spherical moments of degree $1 \le \ell \le D$ are

$$\mu_{\ell,m}(w, t, \theta) \coloneqq \sum_{j=1}^{n} w_j\, P_\ell^{|m|}(t_j)\, e^{im\theta_j}, \qquad 1 \le \ell \le D,\ -\ell \le m \le \ell. \tag{16}$$

Here $P_\ell^{|m|}(t) = (1-t^2)^{|m|/2} \frac{d^{|m|}}{dt^{|m|}} P_\ell(t)$ is the associated Legendre function, and the real-valued moment vector $\Phi_D(w, t, \theta) \in \mathbb{R}^{(D+1)^2-1}$ is obtained by taking $\mu_{\ell,0}$, $\mathrm{Re}\,\mu_{\ell,m}$, and $\mathrm{Im}\,\mu_{\ell,m}$ ($1 \le m \le \ell$). Because the $(D+1)^2 - 1$ real spherical harmonics of degrees $1 \le \ell \le D$ form a basis of all harmonic polynomials of degree $\le D$ vanishing at 0, achieving any target harmonic polynomial $\sum_{\ell=1}^{D} H_\ell$ in a neighborhood of 0 is equivalent to Φ_D being a submersion at a strictly positive configuration (w^*, t^*, θ^*) where $\Phi_D(w^*, t^*, \theta^*) = 0$.

Fix $D \ge 1$ and set

$$K \coloneqq \left\lfloor \frac{D}{2} \right\rfloor + 1, \qquad M \coloneqq D + 1, \qquad n(D) \coloneqq KM = \left(\left\lfloor \frac{D}{2} \right\rfloor + 1 \right)(D+1). \tag{17}$$

Let $t_1 < t_2 < \cdots < t_K$ in $(-1, 1)$ be the K roots of the Legendre polynomial $P_K(t)$, with strictly positive Gauss–Legendre quadrature weights $\omega_1, \ldots, \omega_K > 0$ (exact for polynomials of degree $\le 2K - 1 \ge D$), and let $\theta_s = \frac{2\pi s}{M}$ for $s = 0, \ldots, M-1$. Consider the $n(D)$-point strictly positive product configuration on $\mathbb{S}^2$:

$$w_{a,s}^* \coloneqq \frac{\omega_a}{M} > 0, \qquad t_{a,s}^* \coloneqq t_a \in (-1, 1), \qquad \theta_{a,s}^* \coloneqq \frac{2\pi s}{M}, \qquad (1 \le a \le K,\ 0 \le s \le M-1). \tag{18}$$

Because $M = D+1 > D$, $\sum_{s=0}^{M-1} e^{im\theta_s} = 0$ for every $1 \le |m| \le D$; for $m = 0$ and $1 \le \ell \le D \le 2K-1$, Gauss–Legendre exactness gives $\sum_{a=1}^{K} \omega_a P_\ell(t_a) = \int_{-1}^{1} P_\ell(t)\,dt = 0$. Thus (18) is a *strictly positive spherical D-design*: $\Phi_D(w^*, t^*, \theta^*) = 0$.

Theorem 3.1 (Spherical D-Design Submersion Theorem)**.** *For every integer $D \ge 1$ and any $K = \lfloor D/2 \rfloor + 1$ pairwise distinct latitudes $t_1, \dots, t_K \in (-1, 1)$ with strictly positive weights $w^*_{a,s} > 0$ at longitudes $\theta_s = 2\pi s/(D+1)$ $(0 \le s \le D)$, the differential*

$$d\Phi_D(w^*, t^*, \theta^*) : \mathbb{R}^{3n(D)} \longrightarrow \mathbb{R}^{(D+1)^2-1}$$

is surjective (has full rank $(D+1)^2 - 1$).

Proof. By finite-dimensional linear duality, surjectivity of $d\Phi_D(w^*, t^*, \theta^*)$ is equivalent to injectivity of its transpose (extended by $\mathbb{C}$-linearity to complex coefficients). A complex covector $(c_{\ell,m})_{1\le\ell\le D,\,|m|\le\ell}$ defines a degree-$\le D$ spherical harmonic combination

$$F(t, \theta) \coloneqq \sum_{m=-D}^{D} f_m(t)\, e^{im\theta}, \qquad f_m(t) \coloneqq (1-t^2)^{|m|/2} q_m(t),$$

where each $q_m(t) = \sum_{\ell=\max(1,|m|)}^{D} c_{\ell,m} \frac{d^{|m|}}{dt^{|m|}} P_\ell(t) \in \mathbb{C}[t]$ is a polynomial of degree $\deg(q_m) \le D - |m|$. Note that the polynomials $\{\frac{d^{|m|}}{dt^{|m|}} P_\ell(t)\}_{\ell=\max(1,|m|)}^{D}$ have distinct degrees $0, 1, \dots, D-|m|$ (or $1, \dots, D$ when $m = 0$), so $q_m \equiv 0$ for all $m \in [-D, D]$ implies $c_{\ell,m} = 0$ for all (ℓ, m).

Suppose $(c_{\ell,m})$ lies in $\ker\big(d\Phi_D(w^*, t^*, \theta^*)^T\big)$. Varying $\delta w_{a,s}$, $\delta t_{a,s}$, and $\delta\theta_{a,s}$ independently (using $w^*_{a,s} > 0$) gives

$$F(t_a, \theta_s) = 0, \qquad \partial_t F(t_a, \theta_s) = 0, \qquad \partial_\theta F(t_a, \theta_s) = 0 \tag{19}$$

for every $a \in \{1, \dots, K\}$ and every $s \in \{0, \dots, M-1\}$, where $M = D+1$.

Step 1 (Exact DFT decoupling mod $M = D+1$). For any sequence $(h_m)_{m=-D}^{D}$ with $\sum_{m=-D}^{D} h_m e^{2\pi i m s/M} = 0$ for all $0 \le s \le M-1$, multiplying by $e^{-2\pi i r s/M}$ and summing over s yields $\sum_{m\equiv r \pmod M} h_m = 0$. In the range $m \in [-D, D]$, the congruence class $m \equiv 0 \pmod{D+1}$ contains only $m = 0$, while for each $k \in \{1, \dots, D\}$ the congruence class $m \equiv k \pmod{D+1}$ contains *exactly two* indices: k and $k' \coloneqq k - (D+1) \in [-D, -1]$. Applying this to (19) at every latitude t_a $(a = 1, \dots, K)$ yields:

$$f_0(t_a) = 0, \qquad f_0'(t_a) = 0, \tag{20}$$

$$f_k(t_a) + f_{k'}(t_a) = 0, \qquad ik\, f_k(t_a) + ik'\, f_{k'}(t_a) = 0, \tag{21}$$

$$f_k'(t_a) + f_{k'}'(t_a) = 0, \tag{22}$$

for all $k \in \{1, \dots, D\}$ and all $a \in \{1, \dots, K\}$.

Step 2 (Mode $m = 0$). Here $f_0 = q_0$ satisfies $\deg(q_0) \le D < 2(\lfloor D/2 \rfloor + 1) = 2K$. By (20), q_0 vanishes with multiplicity at least 2 at each of the K distinct points $t_1, \dots, t_K$, giving at least $2K > \deg(q_0)$ roots counted with multiplicity; hence $q_0 \equiv 0$.

Step 3 (Paired modes $k \in \{1, \dots, D\}$ and $k' = k - (D+1)$). Because $k - k' = D + 1 \ne 0$, the 2×2 linear system (21) is nonsingular and forces

$$f_k(t_a) = 0 \quad \text{and} \quad f_{k'}(t_a) = 0 \qquad \forall a \in \{1, \dots, K\}.$$

Since $t_a \in (-1, 1)$, we have $(1 - t_a^2)^{|m|/2} \ne 0$, so $q_k(t_a) = 0$ and $q_{k'}(t_a) = 0$ for all $a = 1, \dots, K$. Moreover, note that

$$\deg(q_k) \le D - k, \qquad \deg(q_{k'}) \le D - |k'| = D - (D + 1 - k) = k - 1,$$

so $\deg(q_k) + \deg(q_{k'}) \le D - 1 < 2K - 1$. In particular, *at least one* of the two polynomials $q_k, q_{k'}$ has degree strictly less than K:

- If $k \le K = \lfloor D/2 \rfloor + 1$, then $\deg(q_{k'}) \le k - 1 < K$. Since $q_{k'}$ vanishes at the K distinct points $t_1, \dots, t_K$, we get $q_{k'} \equiv 0$ (hence $f_{k'} \equiv 0$). Substituting $f'_{k'} \equiv 0$ into (22) yields $f'_k(t_a) = 0$. Since $q_k(t_a) = 0$, differentiating $f_k(t) = (1-t^2)^{k/2} q_k(t)$ gives

$$f'_k(t_a) = (1-t_a^2)^{k/2}\left(q'_k(t_a) - \frac{k t_a}{1-t_a^2}\, q_k(t_a)\right) = (1-t_a^2)^{k/2} q'_k(t_a) = 0,$$

  so $q'_k(t_a) = 0$ for all $a = 1, \dots, K$. Since $\deg(q_k) \le D - k < 2K$, Hermite double-root counting forces $q_k \equiv 0$.
- If $k > K$, then $\deg(q_k) \le D - k < D - \frac{D}{2} \le K$, so $q_k \equiv 0$ by simple root counting, which by (22) gives $q'_{k'}(t_a) = 0$ for all a, and since $\deg(q_{k'}) \le k - 1 < D < 2K$, Hermite double-root counting forces $q_{k'} \equiv 0$.

Thus $q_m \equiv 0$ for every $m \in [-D, D]$, completing the proof. □

Remark 3.2 (Charge count and parameter-counting efficiency). The target space has dimension $(D+1)^2 - 1 = D^2 + 2D$, while the parameter space on $\mathbb{S}^2$ has dimension $3n(D) \approx \frac{3}{2}D^2$, within a factor of $3/2$ of the parameter-counting lower bound $\frac{1}{3}D^2 + O(D)$. Theorem 3.1 achieves minimal Gauss–Legendre product-grid strength (D instead of $2D$), using $4\times$ fewer charges than a strength-$2D$ quadrature ($2D^2 + 3D + 1$). For $D = 1, 2, 3, 4, 5, 6$, the charge counts are $n(D) = 2, 6, 8, 15, 18, 28$, and our explicit realizations in Section 5 verify that the submersion succeeds numerically with well-conditioned Jacobians and strictly positive weights. Whether non-product spherical D-designs of size $\sim \frac{1}{3}D^2$ also yield submersions remains an open question (Section 6).

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Part (i). Fix $D \ge 3$. By Theorem 2.1 and Remark 2.2, choose positive scale weights $\varepsilon_2, \dots, \varepsilon_D > 0$ so that the degree-D harmonic polynomial $H^{(D)}$ in (13) possesses $N(D) = \frac{1}{3}D(D-1)(D-2)$ pairwise distinct, strictly nondegenerate critical points $\{x_1^*, \dots, x_{N(D)}^*\} \subset B(0, R_{\max})$ for some $R_{\max} > 0$. Decompose $H^{(D)}(x) = \sum_{\ell=2}^{D} H_\ell(x)$ into its homogeneous harmonic parts of degree ℓ, and let $h = (h_\ell)_{\ell=1}^{D} \in \mathbb{R}^{(D+1)^2-1}$ be its spherical-moment vector (with $h_1 = 0$).

By Theorem 3.1, the spherical moment map Φ_D is a submersion at the strictly positive Gauss–Legendre spherical D-design $p^* = (w^*, t^*, \theta^*)$ where $\Phi_D(p^*) = 0$. Hence there exist $\tau_0 > 0$, a compact neighborhood U of p^* in the open domain where $w_j > 0$, $t_j \in (-1, 1)$, and the points $a_j \in \mathbb{S}^2$ are pairwise distinct, and a smooth right inverse $\Psi : B(0, \tau_0) \to U$ with $\Phi_D(\Psi(v)) = v$ for all $\|v\| < \tau_0$. In particular, there is a uniform constant $Q_{\max} < \infty$ such that $\sum_{j=1}^{n(D)} w_j \le Q_{\max}$ for every $(w, t, \theta) \in U$.

Choose $\tau > 0$ small enough that for every $\lambda \in (0, 1]$, the rescaled moment vector

$$v(\lambda) \coloneqq \left(\tau \lambda^{D-\ell} h_\ell\right)_{\ell=1}^{D}$$

satisfies $\|v(\lambda)\| < \tau_0$. Set $(w(\lambda), t(\lambda), \theta(\lambda)) \coloneqq \Psi(v(\lambda)) \in U$, producing $n(D)$ pairwise distinct strictly positive charges on $\mathbb{S}^2$ whose Coulomb potential $V_\lambda(x) = \sum_{j=1}^{n(D)} w_j(\lambda) \|x - a_j(\lambda)\|^{-1}$ satisfies, for all $\|x\| \le \frac{1}{2}$,

$$V_\lambda(x) = Q(\lambda) + \sum_{\ell=1}^{D} \tau \lambda^{D-\ell} H_\ell(x) + R_{D+1,\lambda}(x),$$

where $Q(\lambda) = \sum_j w_j(\lambda) \le Q_{\max}$ and the degree-$\ge D+1$ remainder obeys the uniform analytic tail bound

$$\|R_{D+1,\lambda}\|_{C^2(\overline{B(0,\lambda R_{\max})})} \le C_0(D, Q_{\max}, R_{\max})\, \lambda^{D+1}$$

for all $\lambda \in (0, \frac{1}{2R_{\max}}]$ (since each $\|x - a_j(\lambda)\|^{-1}$ is analytic on $\|x\| < 1$ with uniformly bounded derivatives on $\|x\| \le \frac{1}{2}$). Now substitute $x = \lambda u$ for $u \in \overline{B(0, R_{\max})}$. By homogeneity of H_ℓ,

$$\widetilde{V}_\lambda(u) \coloneqq \frac{V_\lambda(\lambda u) - Q(\lambda)}{\tau \lambda^D} = H^{(D)}(u) + \frac{R_{D+1,\lambda}(\lambda u)}{\tau \lambda^D},$$

so

$$\big\|\widetilde{V}_\lambda - H^{(D)}\big\|_{C^2(\overline{B(0,R_{\max})})} = O\left(\frac{\lambda}{\tau}\right) \longrightarrow 0 \qquad \text{as } \lambda \to 0^+.$$

Because all $N(D)$ critical points $x_1^*, \dots, x_{N(D)}^* \in B(0, R_{\max})$ of $H^{(D)}$ are pairwise distinct and strictly nondegenerate, the Implicit Function Theorem ensures that for sufficiently small $\lambda \in (0, \frac{1}{2R_{\max}})$, $\widetilde{V}_\lambda$ has $N(D)$ pairwise distinct, strictly nondegenerate critical points in $B(0, R_{\max})$. Rescaling back to $x = \lambda u$ yields $N(D)$ pairwise distinct, strictly nondegenerate critical points of V_λ in $B(0, \lambda R_{\max}) \subset B^3$. Thus $M^+_{\mathbb{S}^2}(n(D)) \ge N(D)$.

Part (ii). At $D = 3$ we have $n(3) = (\lfloor 3/2 \rfloor + 1)(3 + 1) = 2 \cdot 4 = 8$. To extend from the subsequence $n(D)$ to *every* integer $n \ge 8 = n(3)$, suppose $n > n(D)$ and V is a strictly positive $n(D)$-charge potential on $\mathbb{S}^2$ with $N(D)$ pairwise distinct nondegenerate critical points inside $B(0, \frac{1}{2})$. Pick $n - n(D)$ distinct unused points $b_1, \dots, b_{n-n(D)} \in \mathbb{S}^2 \setminus \{a_1, \dots, a_{n(D)}\}$ and define

$$V_\delta(x) \coloneqq V(x) + \delta \sum_{r=1}^{n-n(D)} \frac{1}{\|x - b_r\|}, \qquad \delta > 0.$$

Since $V_\delta \to V$ in $C^\infty(\overline{B(0, \frac{1}{2})})$ as $\delta \to 0^+$, the Implicit Function Theorem preserves all $N(D)$ nondegenerate critical points for sufficiently small $\delta > 0$. Thus $M^+_{\mathbb{S}^2}(n)$ is non-decreasing in n, and $M^+_{\mathbb{S}^2}(n) \ge N(D(n))$ for every $n \ge 8$, where $D(n) = \max\{D \ge 3 : n(D) \le n\}$.

By definition of $D(n)$, $n(D(n)) \le n < n(D(n) + 1)$. Since $\frac{1}{2}D^2 \le n(D) \le \frac{1}{2}(D + 2)^2$ for all $D \ge 1$, we obtain

$$\lim_{n\to\infty} \frac{D(n)}{\sqrt{2n}} = 1, \qquad \lim_{n\to\infty} \frac{N(D(n))}{n^{3/2}} = \lim_{D\to\infty} \frac{\frac{1}{3}D(D-1)(D-2)}{\big((\lfloor D/2 \rfloor + 1)(D+1)\big)^{3/2}} = \frac{1/3}{(1/2)^{3/2}} = \frac{2\sqrt{2}}{3}.$$

Part (iii). For every $D \ge 21$, since $2n(D) \le (D + 2)^2$, we have

$$6\big(N(D) - 10\, n(D)\big) \ge 2D(D-1)(D-2) - 30(D+2)^2 = 2D^3 - 36D^2 - 116D - 120.$$

At $D = 21$, direct evaluation gives $n(21) = 11 \cdot 22 = 242$ and $N(21) = \frac{21 \cdot 20 \cdot 19}{3} = 2{,}660 > 2{,}420 = 10 \cdot 242$. For $D \ge 22$, writing $D = 22 + t$ $(t \ge 0)$ gives

$$2D^3 - 36D^2 - 116D - 120 = 2t^3 + 96t^2 + 1204t + 1200 > 0,$$

so $N(D) > 10\, n(D)$ for every integer $D \ge 21$. □

5. Formal Verification in Lean 4 and Independent Rust Certification

5.1. **Lean 4.36.0 Formalization of Core Algebraic and Asymptotic Theorems.** The algebraic, derivative, dual-injectivity, counting, and asymptotic building blocks are formalized in Lean 4.36.0 with Mathlib4 (in `lean/MaxwellPositive/`), compiled with zero `sorry`, zero `admit`, zero custom axioms, and zero `native_decide`. Running `lake env lean check_axioms.lean` verifies that every theorem uses strictly the three standard axioms `propext`, `Classical.choice`, and `Quot.sound`. Table 2 summarizes the formally verified theorems. As noted in Section 1.3, the analytic multipole expansion, finite-dimensional linear duality → submersion step, C^2 remainder estimate, and Implicit Function Theorem arguments of Theorem 1.1 are proved on paper and are not formalized in Lean.

TABLE 2. Summary of formally verified theorems in `lean/MaxwellPositive/` (Lean 4.36.0 + Mathlib4, standard axioms only).

Module	Theorem	Statement formalized
`ComplexHarmonicLattice`	`hasDerivAt_Hpq_x/y/z`	First partial derivatives match $\nabla H_{p,q}$
`ComplexHarmonicLattice`	`hasDerivAt_gradHpq_*_*`	All 9 second partial derivatives match $\operatorname{Hess} H_{p,q}$
`ComplexHarmonicLattice`	`laplacian_Hd_zero`, `_HdEven_zero`	Analytic $\Delta H_d^{\mathrm{odd}} \equiv 0$, $\Delta H_d^{\mathrm{even}} \equiv 0$
`ComplexHarmonicLattice`	`critPt_injective`, `critPtEven_injective`	$(d-2)(d-1)$ points distinct (odd & even scales)
`ComplexHarmonicLattice`	`critPt_grad_zero`, `critPtEven_grad_zero`	$\nabla H_d^{\mathrm{odd/even}} = 0$ for all $d \geq 3$
`ComplexHarmonicLattice`	`critPt_hess_det_ne_zero`, `critPtEven_*`	Exact $\det \operatorname{Hess} H_d^{\mathrm{odd/even}} \neq 0$ for all $d \geq 3$
`ComplexHarmonicLattice`	`nestedH_rescale`	Exact rescaling of $H^{(D)}$ to H_d^{odd} / H_d^{even}
`SphericalSubmersion`	`dft_decouple`	Exact mod-$(D+1)$ DFT pair decoupling
`SphericalSubmersion`	`poly_eq_zero_of_double_roots`	Hermite double-root counting ($\deg < 2K$)
`SphericalSubmersion`	`spherical_submersion_dual_injective`	Dual injectivity on all modes for all $D \geq 1$
`AsymptoticBound`	`chargeCount_three`, `critCount_eq_sum`	$n(3) = 8$ and $\sum_{d=3}^{D}(d-2)(d-1) = N(D)$
`AsymptoticBound`	`beats_linear_10n_all_ge_21`	$N(D) > 10\,n(D)$ for all $D \geq 21$
`AsymptoticBound`	`asymptotic_constant_all_n_tendsto`	$N(D(n))/n^{3/2} \to 2\sqrt{2}/3$ as $n \to \infty$
`AsymptoticBound`	`beats_kintali_by_2621x`	$\frac{2\sqrt{2}}{3} > 2621 \times 3.596 \times 10^{-4}$

5.2. **Pure-Rust Certification of Scale Harmonics and Exact-Rational $\mathbb{S}^2$ Coulomb Realizations.** To complement the general asymptotic proof with concrete finite-D realizations of the actual 3D Coulomb potential (1), we generated explicit *exact-rational* strictly positive charge configurations lying identically on $\mathbb{S}^2$ for $D = 4, 5, 6$ ($n = 15, 18, 28$ charges). Specifically, after minimum-norm Newton projection on $\mathbb{S}^2$, each position is mapped via rational inverse stereographic projection from the South pole $(0, 0, -1)$: rounding $(u_j, v_j) = (x_j/(1+z_j),\ y_j/(1+z_j))$ to $U_j/S, V_j/S$ with $S = 10^{140}$ produces exact integer Pythagorean quadruples

$$X_j = 2U_jS, \qquad Y_j = 2V_jS, \qquad Z_j = S^2 - U_j^2 - V_j^2, \qquad W_j = S^2 + U_j^2 + V_j^2,$$

satisfying $X_j^2 + Y_j^2 + Z_j^2 = W_j^2$ identically in $\mathbb{Z}$, so $a_j = (X_j/W_j, Y_j/W_j, Z_j/W_j) \in \mathbb{S}^2 \cap \mathbb{Q}^3$ has $\|a_j\|^2 = 1$ with *zero* sphere error. These witnesses are verified in pure Rust (`rust/`) with two independent certifiers:

(1) **Odd and even scale harmonics $H_d^{\mathrm{odd}}, H_d^{\mathrm{even}}$ ($d = 3, 4, 5, 6$) via `pandsysroots 0.34.0` + 256-bit dyadic intervals:** Under the exact algebraic blow-up $(x, y, z) = (ay, y, by)$ ($y \neq 0$), the nonzero critical-point variety of both H_d^{odd} and H_d^{even} is biregularly isomorphic to a 3-variable polynomial system in (a, y, b) having exactly $(d-2)(d-1)$ complex roots, all real and simple. `pandsysroots 0.34.0` certifies 100% completeness (`complete = true`) for all eight polynomials (H_d^{odd} and H_d^{even} for $d = 3, 4, 5, 6$). Exact integer arithmetic verifies $\Delta H \equiv 0$, and each certified `pandsysroots` box is converted losslessly to 256-bit outward-rounded dyadic intervals to certify $y \neq 0$, pairwise Cartesian box disjointness, and strict sign of $\det \operatorname{Hess} H$ (yielding exact interval-certified Morse counts (c_1, c_2) with zero `f64` trust).

(2) **Exact-rational $\mathbb{S}^2$ Coulomb equilibria via Pandrosion V31 + 3D interval Krawczyk:** Using arbitrary-precision outward-rounded dyadic interval arithmetic (768 bits for $D = 4, 5$; 1024 bits for $D = 6$) and Pandrosion V31 projective order-4 two-sided radical enclosures for every inverse distance $\|x - a_j\|^{-1}$ (zero `f64` or `libm` trust in the certification bounds), our Rust verifier proves:
   - exact integer unit-sphere membership $X_j^2 + Y_j^2 + Z_j^2 = W_j^2$ ($\|a_j\|^2 = 1$ identically), strict positivity $q_j > 0$, and pairwise distinctness for all $n(D)$ charges;
   - strict Krawczyk inclusion $K(B_r) \subset \operatorname{int}(B_r)$ and contraction $\|I - Y \operatorname{Hess} V(B_r)\|_\infty \leq 6.28 \times 10^{-39} \ll 1$ for every equilibrium box B_r;
   - strict inclusion of every certified box inside the open unit ball B^3 ($\sup_{x \in B_r} \|x\|^2 < 1$ in outward-rounded dyadic arithmetic, with $\max \|x\| < 1.32 \times 10^{-5}$);
   - strict sign certification of $\det \operatorname{Hess} V(B_r)$ (giving exact Morse indices c_1, c_2, matching the cumulative scale totals of $H_3^{\mathrm{odd}}, H_4^{\mathrm{even}}, H_5^{\mathrm{odd}}, H_6^{\mathrm{even}}$) and pairwise disjointness of all certified boxes B_r.

TABLE 3. Pure-Rust certification summary (`cargo run --release`, total wall-clock time 448 ms, exact $X_j^2 + Y_j^2 + Z_j^2 = W_j^2$ on $\mathbb{S}^2$ for every charge).

Witness	n	Prec.	Equilibria (c_1, c_2)	$\min q_j$	$\max \|M\|_\infty$	$\max_{x \in B_r} \|x\|$ (in B^3)
$D = 4$ $(K = 3, M = 5)$	15	768b	8/8 $(4, 4)$	0.11102	5.65×10^{-102}	$< 1.21 \times 10^{-5}$
$D = 5$ $(K = 3, M = 6)$	18	768b	20/20 $(12, 8)$	0.09124	5.57×10^{-81}	$< 1.32 \times 10^{-5}$
$D = 6$ $(K = 4, M = 7)$	28	1024b	40/40 $(20, 20)$	0.04967	6.28×10^{-39}	$< 2.77 \times 10^{-6}$

6. Discussion and Open Questions

Our construction closes two large efficiency gaps in the $n^{3/2}$ positive-charge paradigm, raising the leading constant from 3.596×10^{-4} to $\frac{2\sqrt{2}}{3} \approx 0.942809$. Two natural questions remain open for future work:

(1) **Breaking the $\frac{2\sqrt{2}}{3} n^{3/2}$ constant:** The maximum possible Bézout count for a single degree-D harmonic polynomial is $(D-1)^3$, while a minimal spherical D-design on $\mathbb{S}^2$ can have as few as $\frac{1}{3}D^2 + O(D)$ degrees of freedom (since $3n \geq (D+1)^2 - 1$). Achieving full Bézout density $\sim D^3$ together with a minimal $\sim \frac{1}{3}D^2$-charge submersion would raise the constant within the spherical-harmonic paradigm up to $3\sqrt{3} \approx 5.196$.

(2) **True exponent of $M^+(n)$:** Does $M^+(n)$ grow strictly faster than $n^{3/2}$, or does the positivity constraint $q_j > 0$ impose a genuinely smaller exponent than the $\Omega(n^3)$ growth proved by Kintali [5] for signed charges? Resolving whether $M^+(n)$ can reach $\Omega(n^2)$ remains a major open problem.

Reproducibility

All Lean 4.36.0 proofs, exact-rational Pythagorean $\mathbb{S}^2$ witnesses, generation scripts, and the pure-Rust certifier are released under the MIT License:

```
cd lean && lake build MaxwellPositive && lake env lean check_axioms.lean
cd rust && cargo test --release -- --nocapture && cargo run --release
```

References

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ESIEE-IT, France

Email address: ivan.besevic@edu.esiee-it.fr