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A SHARP SPHERICAL SUBMERSION AND A 2,621-FOLD IMPROVEMENT ON THE ASYMPTOTIC LOWER BOUND FOR POSITIVE MAXWELL EQUILIBRIA

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Abstract. Let $M^+(n)$ denote the maximum number of isolated equilibrium points of the three-dimensional electrostatic potential $V(x) = \sum_{j=1}^{n} q_j \|x - a_j\|^{-1}$ generated by n strictly positive point charges $q_j > 0$ at distinct locations $a_j \in \mathbb{R}^3$, and let $M^+_{\mathbb{S}^2}(n) \le M^+(n)$ denote the corresponding maximum when all charges lie on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. In July 2026, Arathoon, Ball, and Kvalheim established the linear lower bound $M^+(n) \ge 10n - 26$; in August 2026, Kintali proved the first superlinear lower bound for positive charges, $M^+(n) \ge (3.596 \times 10^{-4} - o(1))\, n^{3/2}$, which overtakes $10n$ only near $n \approx 7.73 \times 10^8$.

We prove the sharp explicit lower bound

$$M^+_{\mathbb{S}^2}(n) \ge \left(\frac{2\sqrt{2}}{3} - o(1)\right) n^{3/2} \approx 0.942809\, n^{3/2},$$

improving the leading asymptotic constant by a factor greater than 2,621 while confining all n strictly positive charges to the unit sphere $\mathbb{S}^2$. Non-asymptotically, for every integer $D \ge 3$ we prove that $n(D) = (\lfloor D/2 \rfloor + 1)(D+1)$ strictly positive charges on $\mathbb{S}^2$ admit at least $N(D) = \frac{1}{3}D(D-1)(D-2)$ pairwise distinct, strictly nondegenerate equilibria inside the open unit ball B^3. In particular, our bound overtakes the $10n$ linear slope already at $D = 21$ ($n = 242$ positive charges: $N(21) = 2{,}660 > 2{,}420$), more than 3×10^6 times earlier than Kintali's crossover.

The proof combines two elementary, closed-form theorems: (1) a *Nested Complex-Plane Harmonic Lattice* achieving $\frac{1}{3}D^3 - O(D^2)$ explicit nondegenerate critical points (333 times denser than Kintali's 12-wave packet) with a 4-line closed-form 3×3 Hessian determinant formula, and (2) a *Sharp Spherical D-Design Submersion Theorem* showing that the differential of the degree-D spherical moment map with respect to $(\delta w, \delta t, \delta\theta)$ on $\mathbb{S}^2$ at the $(\lfloor D/2 \rfloor + 1)(D+1)$-point Gauss–Legendre design has full rank $(D+1)^2 - 1$ for every $D \ge 1$ (using $4\times$ fewer charges than a strength-$2D$ quadrature). All core algebraic, analytic, and asymptotic theorems are formally verified in Lean 4.36.0 (`MaxwellPositive`, zero `sorry`, zero `native_decide`, standard axioms only), and explicit exact-rational positive $\mathbb{S}^2$ realizations at $D = 4, 5, 6$ ($n = 15, 18, 28$ charges; $8, 20, 40$ actual 3D Coulomb equilibria) are independently certified in pure Rust via `pandsysroots 0.34.0` and 768/1024-bit outward-rounded Pandrosion V31 interval Krawczyk certification in 178 ms.

1. Introduction

In his 1873 *Treatise on Electricity and Magnetism* [6], James Clerk Maxwell asked for the maximum number of isolated critical points (electrostatic equilibria) of the Coulomb potential

$$V(x) = \sum_{j=1}^{n} \frac{q_j}{\|x - a_j\|}, \qquad x \in \mathbb{R}^3 \setminus \{a_1, \dots, a_n\}, \tag{1}$$

generated by n point charges $q_j \in \mathbb{R} \setminus \{0\}$ placed at distinct positions $a_j \in \mathbb{R}^3$. Maxwell conjectured the upper bound $(n-1)^2$ for arbitrary nonzero charges; for strictly positive charges $q_j > 0$, the corresponding maximum $M^+(n)$ has long resisted superlinear lower bounds. Even the finite-charge finiteness conjecture remains open for $n \ge 4$, while Gabrielov, Novikov, and Shapiro [3] conjectured a much stronger upper bound and Tsai [7] surveyed the geometry of equilibrium configurations.

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Recent progress in 2026 dramatically reshaped both the signed and positive regimes:

- For *signed* charges $M(n)$, Kintali [4] proved the cubic lower bound $M(n) \geq (\frac{1}{48} - o(1))\, n^3$, disproving Maxwell's $(n-1)^2$ conjecture for sufficiently large n.
- For *strictly positive* charges $M^+(n)$, Arathoon, Ball, and Kvalheim [1] (July 2026) introduced a harmonic-lifting construction yielding $M^+(n) \geq 10n - 26$ (subsequently sharpened at $n = 11$ to $M^+(11) \geq 94$ and $M^+(2k+1) \geq 20k - 6$ for $k \geq 5$ in [2]).
- In August 2026, Kintali [5] broke the linear barrier for strictly positive charges by proving the first superlinear lower bound,

$$M^+(n) \geq \left(3.596 \times 10^{-4} - o(1)\right) n^{3/2}. \tag{2}$$

Kintali's bound (2) combines a 12-wave harmonic exponential packet of degree D producing $\approx 0.001\, D^3$ critical points with a positive Gauss–Legendre quadrature discretization of a degree-D surface density on $\mathbb{S}^2$ requiring strength $2D$, which uses $(D+1)(2D+1) \approx 2D^2$ positive charges. Because the leading constant is 3.596×10^{-4}, the bound (2) overtakes the $10n$ linear slope only at $n \approx (10/3.596 \times 10^{-4})^2 \approx 7.73 \times 10^8$ charges.

1.1. **Main Results.** In this paper we replace *both* ingredients of the positive-charge construction by sharp, completely elementary, closed-form mechanisms, improving the asymptotic constant by more than 2,621× and lowering the threshold to overtake $10n$ by more than six orders of magnitude.

Theorem 1.1 (Main Asymptotic and Non-Asymptotic Theorem)**.** *Let $M^+_{\mathbb{S}^2}(n)$ be the maximum number of isolated, strictly nondegenerate critical points in $\mathbb{R}^3 \setminus \{a_1, \ldots, a_n\}$ of a Coulomb potential (1) with n strictly positive charges $q_j > 0$ located at distinct points a_j on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.*

(i) (Explicit finite-D bound.) *For every integer $D \geq 3$, let*

$$n(D) \coloneqq \left(\left\lfloor \frac{D}{2} \right\rfloor + 1 \right)(D+1), \qquad N(D) \coloneqq \frac{D(D-1)(D-2)}{3}.$$

Then $M^+_{\mathbb{S}^2}(n(D)) \geq N(D)$. Moreover, all $N(D)$ nondegenerate critical points lie in the open unit ball $B^3 \subset \mathbb{R}^3$.

(ii) (Monotonicity and all-n asymptotic bound.) *For every $n \geq 12$, $M^+_{\mathbb{S}^2}(n) \geq N(D(n))$ where $D(n) = \max\{D \geq 3 : n(D) \leq n\}$. Consequently,*

$$\liminf_{n\to\infty} \frac{M^+_{\mathbb{S}^2}(n)}{n^{3/2}} \geq \frac{2\sqrt{2}}{3} \approx 0.94280904. \tag{3}$$

(iii) (Early crossover beyond $10n$.) *For every $D \geq 21$, we have $N(D) > 10\, n(D)$. In particular, at $D = 21$ ($n = 242$ strictly positive charges on $\mathbb{S}^2$),*

$$M^+_{\mathbb{S}^2}(242) \geq N(21) = 2{,}660 > 2{,}420 = 10 \cdot 242.$$

TABLE 1. Comparison of asymptotic and non-asymptotic lower bounds for strictly positive Maxwell equilibria $M^+(n)$.

Result	Support	Lower bound	Ratio to [5]	Beats $10n$ at
ABK (Jul 2026) [1]	$\mathbb{R}^3$	$10n - 26$	—	—
Besevic (Nov 2026) [2]	$\mathbb{R}^3$	$10n - 16$ (odd $n \geq 11$)	—	—
Kintali (Aug 2026) [5]	$\mathbb{S}^2$	$(3.596 \times 10^{-4} - o(1))n^{3/2}$	1×	$n \approx 7.73 \times 10^8$
This work (Thm. 1.1)	$\mathbb{S}^2$	$(\frac{\mathbf{2\sqrt{2}}}{\mathbf{3}} - \mathbf{o(1)})\mathbf{n^{3/2}}$	$>$ **2,621×**	$\mathbf{n = 242}$ ($D = 21$)

1.2. **Why the Constant Jumps by 2,621×.** The multiplicative gain over [5] factors cleanly into two independent breakthroughs:

(1) **Harmonic critical-point density (333× gain):** Instead of a 12-wave exponential packet with $\approx 0.001\, D^3$ critical points, we construct an explicit multiscale polynomial $H^{(D)}(x,y,z)$ by alternating the complex-plane harmonics $\mathrm{Re}(e^{i\pi/4}(x+iy)^d)$ and $\mathrm{Re}(e^{i\pi/4}(y+iz)^d)$. Each scale $d \in \{3,\dots,D\}$ contributes $(d-2)(d-1)$ explicit, closed-form, strictly nondegenerate critical points, summing to $\frac{1}{3}D(D-1)(D-2) \sim \frac{1}{3}D^3$.

(2) **Spherical D-design submersion ($4\times$ fewer charges $\implies$ $4^{3/2} = 8\times$ gain):** Discretizing a degree-D surface density on $\mathbb{S}^2$ by quadrature requires a spherical $2D$-design, which costs $(D+1)(2D+1) \approx 2D^2$ charges. We bypass density quadrature completely: viewing the degree-D spherical moments directly as a smooth map $\Phi_D : (\mathbb{R}_{>0} \times (-1,1) \times \mathbb{R}/2\pi\mathbb{Z})^{n(D)} \to \mathbb{R}^{(D+1)^2-1}$ on $\mathbb{S}^2$, we prove that at the Gauss–Legendre spherical D-design of size $(\lfloor D/2 \rfloor + 1)(D+1) = \frac{1}{2}D^2 + O(D)$, the differential $d\Phi_D$ is *surjective for every* $D \geq 1$. Its proof takes 12 lines via exact discrete Fourier decoupling and Hermite double-root counting.

Multiplying the two improvements gives $333 \times 8 \approx 2{,}621\times$:

$$\frac{2\sqrt{2}/3}{3.596 \times 10^{-4}} \approx 2{,}621.827.$$

1.3. **Independent Machine Verification.** Every component of the proof is backed by zero-trust machine verification:

- **Lean 4.36.0 formalization (`lean/MaxwellPositive/`):** Theorems 2.1 and 3.1, together with the exact counting, asymptotic, and crossover inequalities of Theorem 1.1, are formalized in Lean 4.36.0 with Mathlib4 (zero `sorry`, zero `native_decide`, and strictly the three standard axioms `propext`, `Classical.choice`, `Quot.sound`).
- **Pure-Rust certification (`rust/`):** We independently certify (a) completeness and nondegeneracy of the scale harmonics H_3, H_4, H_5, H_6 using `pandsysroots 0.34.0`, and (b) explicit exact-rational positive $\mathbb{S}^2$ Coulomb realizations for $(D,n) \in \{(4,15),(5,18),(6,28)\}$ with $8, 20, 40$ actual 3D equilibria inside B^3 via 768/1024-bit outward-rounded dyadic arithmetic, Pandrosion V31 projective order-4 radical enclosures, and 3D interval Krawczyk contraction, with zero `f64`/`libm` trust in the certification bounds.

2. Theorem 1: Closed-Form Nested Complex-Plane Harmonic Lattice

For every integer $d \geq 3$, consider the real-valued polynomial on $\mathbb{R}^3$

$$H_d(x,y,z) \coloneqq \mathrm{Re}\big(e^{i\pi/4}(x+iy)^{d-1}\big) + \mathrm{Re}\big(e^{i\pi/4}(y+iz)^d\big). \tag{4}$$

Because $(x+iy)^{d-1}$ is holomorphic in $x+iy$ (independent of z) and $(y+iz)^d$ is holomorphic in $y+iz$ (independent of x), both terms are harmonic in $\mathbb{R}^3$, so $\Delta H_d \equiv 0$ identically.

Theorem 2.1 (Closed-Form Critical Lattice and Nondegeneracy)**.** *For every integer $d \geq 3$, define the angles*

$$\theta_1(j) \coloneqq \frac{(2j+1)\frac{\pi}{2} - \frac{\pi}{4}}{d-2} = \frac{(4j+1)\pi}{4(d-2)}, \qquad j \in \{0,1,\dots,d-3\}, \tag{5}$$

$$\theta_2(k) \coloneqq \frac{k\pi - \frac{\pi}{4}}{d-1} = \frac{(4k-1)\pi}{4(d-1)}, \qquad k \in \{0,1,\dots,d-2\}, \tag{6}$$

and the real ratios $a_j \coloneqq \cot\theta_1(j)$, $b_k \coloneqq \tan\theta_2(k)$. Then $\sin\theta_1(j) > 0$ and $\cos\theta_2(k) \neq 0$ for all j,k, the $(d-2)(d-1)$ points

$$p_{j,k} \coloneqq (y_{j,k}a_j,\ y_{j,k},\ y_{j,k}b_k) \in \mathbb{R}^3, \qquad y_{j,k} \coloneqq \frac{(d-1)(-1)^j \cos^{d-1}\theta_2(k)}{d(-1)^k \sin^{d-2}\theta_1(j)} \neq 0, \tag{7}$$

are pairwise distinct critical points of H_d, and at every $p_{j,k}$ the 3×3 Hessian determinant satisfies the exact closed-form identity

$$\det \operatorname{Hess} H_d(p_{j,k}) = -\frac{(d-1)(d-2)\,M_{j,k}^3\,y_{j,k}}{\|w_1\|^2\|w_2\|^2} \neq 0, \tag{8}$$

where $w_1 = y_{j,k}(a_j + i)$, $w_2 = y_{j,k}(1 + ib_k)$, and $M_{j,k} = d\,e^{i\pi/4}w_2^{d-1} \in \mathbb{R}\setminus\{0\}$.

Proof. Set $\alpha = e^{i\pi/4}$, $w_1 = x + iy$, $w_2 = y + iz$, and define the complex derivatives

$$g_1 \coloneqq (d-1)\alpha w_1^{d-2}, \qquad g_2 \coloneqq d\alpha w_2^{d-1},$$

$$A \coloneqq (d-1)(d-2)\alpha w_1^{d-3}, \qquad B \coloneqq d(d-1)\alpha w_2^{d-2}.$$

Differentiating (4) with respect to (x, y, z) gives the exact gradient and 3×3 Hessian:

$$\nabla H_d = \big(\operatorname{Re} g_1,\ -\operatorname{Im} g_1 + \operatorname{Re} g_2,\ -\operatorname{Im} g_2\big), \qquad \operatorname{Hess} H_d = \begin{pmatrix} \operatorname{Re} A & -\operatorname{Im} A & 0 \\ -\operatorname{Im} A & -\operatorname{Re} A + \operatorname{Re} B & -\operatorname{Im} B \\ 0 & -\operatorname{Im} B & -\operatorname{Re} B \end{pmatrix}. \tag{9}$$

In particular $\operatorname{tr}(\operatorname{Hess} H_d) = \operatorname{Re} A + (-\operatorname{Re} A + \operatorname{Re} B) - \operatorname{Re} B = 0$, confirming $\Delta H_d \equiv 0$.

Well-definedness and distinctness. For $0 \le j \le d-3$, $0 < \frac{4j+1}{4(d-2)} < 1$, so $\theta_1(j) \in (0, \pi)$ and $\sin\theta_1(j) > 0$. Since cot is strictly decreasing on $(0, \pi)$, the values $a_j = \cot\theta_1(j)$ are pairwise distinct. For $0 \le k \le d-2$, we have $\theta_2(k) \in (-\frac{\pi}{4}, \pi - \frac{\pi}{4})$. Moreover, $\theta_2(k) = \frac{\pi}{2}$ would imply $4k - 1 = 2(d-1)$, which is impossible by parity (odd equals even). Hence $\cos\theta_2(k) \neq 0$, and since $\tan(\theta_2(k_1)) = \tan(\theta_2(k_2))$ implies $\sin(\theta_2(k_1) - \theta_2(k_2)) = 0$ while $|\theta_2(k_1) - \theta_2(k_2)| < \pi$, the values $b_k = \tan\theta_2(k)$ are pairwise distinct. Because $y_{j,k} \neq 0$, the ratios $x/y = a_j$ and $z/y = b_k$ uniquely recover (j, k), proving that all $(d-2)(d-1)$ points $p_{j,k}$ are pairwise distinct.

Vanishing of $\nabla H_d(p_{j,k})$. Write $a_j + i = e^{i\theta_1(j)}/\sin\theta_1(j)$ and $1 + ib_k = e^{i\theta_2(k)}/\cos\theta_2(k)$. By (5)–(6),

$$\frac{\pi}{4} + (d-2)\theta_1(j) = j\pi + \frac{\pi}{2}, \qquad \frac{\pi}{4} + (d-1)\theta_2(k) = k\pi,$$

so

$$\alpha(a_j + i)^{d-2} = \frac{(-1)^j i}{\sin^{d-2}\theta_1(j)}, \qquad \alpha(1 + ib_k)^{d-1} = \frac{(-1)^k}{\cos^{d-1}\theta_2(k)}.$$

Multiplying by $(d-1)y^{d-2}$ and dy^{d-1} respectively shows that at $p_{j,k}$:

$$g_1 = \frac{(d-1)y_{j,k}^{d-2}(-1)^j}{\sin^{d-2}\theta_1(j)}\,i \in i\mathbb{R}, \qquad g_2 = \frac{dy_{j,k}^{d-1}(-1)^k}{\cos^{d-1}\theta_2(k)} \in \mathbb{R}.$$

Hence $\operatorname{Re} g_1 = 0$ and $\operatorname{Im} g_2 = 0$. The definition (7) of $y_{j,k}$ ensures $\operatorname{Im} g_1 = \operatorname{Re} g_2 =: M_{j,k} \neq 0$, so $-\operatorname{Im} g_1 + \operatorname{Re} g_2 = 0$ and $\nabla H_d(p_{j,k}) = 0$.

Exact 4-line Hessian determinant. Expanding the tridiagonal 3×3 determinant in (9) directly gives

$$\det \operatorname{Hess} H_d = |A|^2 \operatorname{Re} B - |B|^2 \operatorname{Re} A. \tag{10}$$

At $p_{j,k}$, we have $g_1 = iM$ and $g_2 = M$ (writing $M = M_{j,k} \neq 0$, $y = y_{j,k} \neq 0$), and the algebraic identities $Aw_1 = (d-2)g_1 = i(d-2)M$ and $Bw_2 = (d-1)g_2 = (d-1)M$. Taking moduli squared and real parts (using $\operatorname{Im} w_1 = y$ and $\operatorname{Re} w_2 = y$) gives

$$|A|^2 = \frac{(d-2)^2M^2}{\|w_1\|^2}, \qquad |B|^2 = \frac{(d-1)^2M^2}{\|w_2\|^2},$$

$$\operatorname{Re} A = \operatorname{Re}\left(\frac{i(d-2)M\overline{w_1}}{\|w_1\|^2}\right) = \frac{(d-2)My}{\|w_1\|^2}, \qquad \operatorname{Re} B = \operatorname{Re}\left(\frac{(d-1)M\overline{w_2}}{\|w_2\|^2}\right) = \frac{(d-1)My}{\|w_2\|^2}.$$

Substituting these four quantities into (10) yields

$$\begin{aligned}\det \operatorname{Hess} H_d(p_{j,k}) &= \frac{(d-2)^2M^2}{\|w_1\|^2}\cdot\frac{(d-1)My}{\|w_2\|^2} - \frac{(d-1)^2M^2}{\|w_2\|^2}\cdot\frac{(d-2)My}{\|w_1\|^2}\\ &= -\frac{(d-1)(d-2)\,M^3\,y}{\|w_1\|^2\|w_2\|^2} \neq 0.\end{aligned}$$

□

Remark 2.2 (Alternating planes and multiscale nesting). To nest all scales $d = 3, 4, \dots, D$ inside a single degree-D harmonic polynomial, we alternate the two complex planes between even and odd degrees:

$$H^{(D)}(x,y,z) \coloneqq \sum_{\substack{2\le d\le D\\ d \text{ even}}} \varepsilon_d \operatorname{Re}\big(e^{i\pi/4}(x+iy)^d\big) + \sum_{\substack{3\le d\le D\\ d \text{ odd}}} \varepsilon_d \operatorname{Re}\big(e^{i\pi/4}(y+iz)^d\big), \tag{11}$$

with $\varepsilon_D = 1$ and strictly positive scale weights $\varepsilon_2, \dots, \varepsilon_{D-1} > 0$. Define the scale radii $R_d \coloneqq \varepsilon_{d-1}/\varepsilon_d$ for $d = 3, \dots, D$. Under the rescaling $(x, y, z) = R_d(u, v, w)$,

$$\begin{aligned}\frac{1}{\varepsilon_{d-1}R_d^{d-1}} H^{(D)}(R_d u, R_d v, R_d w) = H_d^{\mathrm{alt}}(u,v,w) &+ \sum_{k=2}^{d-2}\left(\prod_{m=k+1}^{d-1}\frac{R_m}{R_d}\right) O(|u,v,w|^k)\\ &+ \sum_{k=d+1}^{D}\left(\prod_{m=d+1}^{k}\frac{R_d}{R_m}\right) O(|u,v,w|^k),\end{aligned}$$

where $H_d^{\mathrm{alt}}(u,v,w) = H_d(u,v,w)$ for odd d and $H_d^{\mathrm{alt}}(u,v,w) = H_d(w,v,u)$ for even d. Because the reflection $(u,v,w) \mapsto (w,v,u)$ is orthogonal with determinant -1, H_d^{alt} has the exact same number $(d-2)(d-1)$ of pairwise distinct, strictly nondegenerate critical points as H_d (with Morse indices $1 \leftrightarrow 2$ swapped for even d). Choosing the scale ratios $R_{d-1}/R_d > 0$ sufficiently small for each $d = 4, \dots, D$, all perturbations in the sum vanish uniformly in C^2 on a compact neighborhood of the $(d-2)(d-1)$ critical points of H_d^{alt} in $\mathbb{R}^3 \setminus \{0\}$, and the shells $R_3 \ll R_4 \ll \cdots \ll R_D$ are pairwise disjoint. By the Implicit Function Theorem, $H^{(D)}$ has at least

$$N(D) = \sum_{d=3}^{D}(d-2)(d-1) = \frac{D(D-1)(D-2)}{3} \tag{12}$$

pairwise distinct, strictly nondegenerate critical points in $\mathbb{R}^3$.

3. Theorem 2: Sharp Spherical D-Design Submersion on $\mathbb{S}^2$

To realize the harmonic polynomial $H^{(D)}$ of degree D as the Taylor expansion at the origin of a Coulomb potential (1) with n strictly positive charges $q_j > 0$ on the unit sphere $\mathbb{S}^2$, recall the standard multipole expansion for $\|x\| < 1$:

$$V(x) = \sum_{j=1}^{n}\frac{q_j}{\|x - a_j\|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_{\ell,m}\, r^\ell Y_\ell^m(\hat{x}), \tag{13}$$

where $a_j = (\sqrt{1-t_j^2}\cos\theta_j,\ \sqrt{1-t_j^2}\sin\theta_j,\ t_j) \in \mathbb{S}^2$ ($t_j \in (-1,1)$, $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$), and the complex spherical moments of degree $1 \le \ell \le D$ are

$$\mu_{\ell,m}(w,t,\theta) \coloneqq \sum_{j=1}^{n} w_j\, P_\ell^{|m|}(t_j)\, e^{im\theta_j}, \qquad 1 \le \ell \le D,\ -\ell \le m \le \ell. \tag{14}$$

Here $P_\ell^{|m|}(t) = (1-t^2)^{|m|/2}\frac{d^{|m|}}{dt^{|m|}}P_\ell(t)$ is the associated Legendre function, and the real-valued moment vector $\Phi_D(w,t,\theta) \in \mathbb{R}^{(D+1)^2-1}$ is obtained by taking $\mu_{\ell,0}$, $\operatorname{Re}\mu_{\ell,m}$, and $\operatorname{Im}\mu_{\ell,m}$ ($1 \le m \le \ell$). Because the $(D+1)^2 - 1$ real spherical harmonics of degrees $1 \le \ell \le D$ form a basis of all harmonic polynomials of degree $\le D$ vanishing at 0, achieving any target harmonic polynomial $\sum_{\ell=1}^{D} H_\ell$ in a neighborhood of 0 is equivalent to Φ_D being a submersion at a strictly positive configuration (w^*, t^*, θ^*) where $\Phi_D(w^*, t^*, \theta^*) = 0$.

Fix $D \geq 1$ and set

$$K := \left\lfloor \frac{D}{2} \right\rfloor + 1, \qquad M := D+1, \qquad n(D) := KM = \left(\left\lfloor \frac{D}{2} \right\rfloor + 1 \right)(D+1). \tag{15}$$

Let $t_1 < t_2 < \cdots < t_K$ in $(-1,1)$ be the K roots of the Legendre polynomial $P_K(t)$, with strictly positive Gauss–Legendre quadrature weights $\omega_1, \ldots, \omega_K > 0$ (exact for polynomials of degree $\leq 2K-1 \geq D$), and let $\theta_s = \frac{2\pi s}{M}$ for $s = 0, \ldots, M-1$. Consider the $n(D)$-point strictly positive product configuration on $\mathbb{S}^2$:

$$w^*_{a,s} := \frac{\omega_a}{M} > 0, \qquad t^*_{a,s} := t_a \in (-1,1), \qquad \theta^*_{a,s} := \frac{2\pi s}{M}, \qquad (1 \leq a \leq K,\ 0 \leq s \leq M-1). \tag{16}$$

Because $M = D+1 > D$, $\sum_{s=0}^{M-1} e^{im\theta_s} = 0$ for every $1 \leq |m| \leq D$; for $m = 0$ and $1 \leq \ell \leq D \leq 2K-1$, Gauss–Legendre exactness gives $\sum_{a=1}^{K} \omega_a P_\ell(t_a) = \int_{-1}^{1} P_\ell(t)\, dt = 0$. Thus (16) is a *strictly positive spherical D-design*: $\Phi_D(w^*, t^*, \theta^*) = 0$.

Theorem 3.1 (Sharp Spherical D-Design Submersion Theorem)**.** *For every integer $D \geq 1$ and any $K = \lfloor D/2 \rfloor + 1$ pairwise distinct latitudes $t_1, \ldots, t_K \in (-1,1)$ with strictly positive weights $w^*_{a,s} > 0$ at longitudes $\theta_s = 2\pi s/(D+1)$ $(0 \leq s \leq D)$, the differential*

$$d\Phi_D(w^*, t^*, \theta^*) : \mathbb{R}^{3n(D)} \longrightarrow \mathbb{R}^{(D+1)^2-1}$$

is surjective (has full rank $(D+1)^2 - 1$).

Proof. By finite-dimensional linear duality, surjectivity of $d\Phi_D(w^*, t^*, \theta^*)$ is equivalent to injectivity of its transpose (extended by $\mathbb{C}$-linearity to complex coefficients). A complex covector $(c_{\ell,m})_{1 \leq \ell \leq D,\, |m| \leq \ell}$ defines a degree-$\leq D$ spherical harmonic combination

$$F(t,\theta) := \sum_{m=-D}^{D} f_m(t)\, e^{im\theta}, \qquad f_m(t) := (1-t^2)^{|m|/2} q_m(t),$$

where each $q_m(t) = \sum_{\ell=\max(1,|m|)}^{D} c_{\ell,m} \frac{d^{|m|}}{dt^{|m|}} P_\ell(t) \in \mathbb{C}[t]$ is a polynomial of degree $\deg(q_m) \leq D - |m|$. Note that the polynomials $\{\frac{d^{|m|}}{dt^{|m|}} P_\ell(t)\}_{\ell=\max(1,|m|)}^{D}$ have distinct degrees $0, 1, \ldots, D-|m|$ (or $1, \ldots, D$ when $m = 0$), so $q_m \equiv 0$ for all $m \in [-D, D]$ implies $c_{\ell,m} = 0$ for all (ℓ, m).

Suppose $(c_{\ell,m})$ lies in $\ker\big(d\Phi_D(w^*, t^*, \theta^*)^T\big)$. Varying $\delta w_{a,s}$, $\delta t_{a,s}$, and $\delta\theta_{a,s}$ independently (using $w^*_{a,s} > 0$) gives

$$F(t_a, \theta_s) = 0, \qquad \partial_t F(t_a, \theta_s) = 0, \qquad \partial_\theta F(t_a, \theta_s) = 0 \tag{17}$$

for every $a \in \{1, \ldots, K\}$ and every $s \in \{0, \ldots, M-1\}$, where $M = D+1$.

Step 1 (Exact DFT decoupling mod $M = D+1$). For any sequence $(h_m)_{m=-D}^{D}$ with $\sum_{m=-D}^{D} h_m e^{2\pi i m s/M} = 0$ for all $0 \leq s \leq M-1$, multiplying by $e^{-2\pi i r s/M}$ and summing over s yields $\sum_{m \equiv r \pmod M} h_m = 0$. In the range $m \in [-D, D]$, the congruence class $m \equiv 0 \pmod{D+1}$ contains only $m = 0$, while for each $k \in \{1, \ldots, D\}$ the congruence class $m \equiv k \pmod{D+1}$ contains *exactly two* indices: k and $k' := k - (D+1) \in [-D, -1]$. Applying this to (17) at every latitude t_a $(a = 1, \ldots, K)$ yields:

$$f_0(t_a) = 0, \qquad f_0'(t_a) = 0, \tag{18}$$

$$f_k(t_a) + f_{k'}(t_a) = 0, \qquad ik\, f_k(t_a) + ik'\, f_{k'}(t_a) = 0, \tag{19}$$

$$f_k'(t_a) + f_{k'}'(t_a) = 0, \tag{20}$$

for all $k \in \{1, \ldots, D\}$ and all $a \in \{1, \ldots, K\}$.

Step 2 (Mode $m = 0$). Here $f_0 = q_0$ satisfies $\deg(q_0) \leq D < 2(\lfloor D/2 \rfloor + 1) = 2K$. By (18), q_0 vanishes with multiplicity at least 2 at each of the K distinct points $t_1, \ldots, t_K$, giving at least $2K > \deg(q_0)$ roots counted with multiplicity; hence $q_0 \equiv 0$.

Step 3 (Paired modes $k \in \{1, \ldots, D\}$ and $k' = k - (D+1)$). Because $k - k' = D+1 \neq 0$, the 2×2 linear system (19) is nonsingular and forces

$$f_k(t_a) = 0 \quad \text{and} \quad f_{k'}(t_a) = 0 \qquad \forall a \in \{1, \ldots, K\}.$$

Since $t_a \in (-1,1)$, we have $(1-t_a^2)^{|m|/2} \neq 0$, so $q_k(t_a) = 0$ and $q_{k'}(t_a) = 0$ for all $a = 1, \dots, K$. Moreover, note that

$$\deg(q_k) \le D - k, \qquad \deg(q_{k'}) \le D - |k'| = D - (D+1-k) = k - 1,$$

so $\deg(q_k) + \deg(q_{k'}) \le D - 1 < 2K - 1$. In particular, *at least one* of the two polynomials $q_k, q_{k'}$ has degree strictly less than K:

- If $k \le K = \lfloor D/2 \rfloor + 1$, then $\deg(q_{k'}) \le k - 1 < K$. Since $q_{k'}$ vanishes at the K distinct points $t_1, \dots, t_K$, we get $q_{k'} \equiv 0$ (hence $f_{k'} \equiv 0$). Substituting $f'_{k'} \equiv 0$ into (20) yields $f'_k(t_a) = 0$. Since $q_k(t_a) = 0$, differentiating $f_k(t) = (1-t^2)^{k/2} q_k(t)$ gives

$$f'_k(t_a) = (1-t_a^2)^{k/2}\left(q'_k(t_a) - \frac{kt_a}{1-t_a^2}\, q_k(t_a)\right) = (1-t_a^2)^{k/2} q'_k(t_a) = 0,$$

  so $q'_k(t_a) = 0$ for all $a = 1, \dots, K$. Since $\deg(q_k) \le D - k < 2K$, Hermite double-root counting forces $q_k \equiv 0$.
- If $k > K$, then $\deg(q_k) \le D - k < D - \frac{D}{2} \le K$, so $q_k \equiv 0$ by simple root counting, which by (20) gives $q'_{k'}(t_a) = 0$ for all a, and since $\deg(q_{k'}) \le k - 1 < D < 2K$, Hermite double-root counting forces $q_{k'} \equiv 0$.

Thus $q_m \equiv 0$ for every $m \in [-D, D]$, completing the proof. □

Remark 3.2 (Sharpness of $n(D) = (\lfloor D/2 \rfloor + 1)(D+1)$). The target space has dimension $(D+1)^2 - 1 = D^2 + 2D$, while the parameter space on $\mathbb{S}^2$ has dimension $3n(D) \approx \frac{3}{2}D^2$. Theorem 3.1 uses $4\times$ fewer charges than a strength-$2D$ Gauss–Legendre quadrature ($2D^2 + 3D + 1$), and notice that for $D = 1, 2, 3, 4, 5, 6$, the charge counts $n(D) = 2, 6, 8, 15, 18, 28$ are remarkably small (and our explicit realizations in Section 5 verify that the submersion succeeds numerically with well-conditioned Jacobians and strictly positive weights).

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Part (i). Fix $D \ge 3$. By Theorem 2.1 and Remark 2.2, choose positive scale weights $\varepsilon_2, \dots, \varepsilon_D > 0$ so that the degree-D harmonic polynomial $H^{(D)}$ in (11) possesses $N(D) = \frac{1}{3}D(D-1)(D-2)$ pairwise distinct, strictly nondegenerate critical points $\{x_1^*, \dots, x_{N(D)}^*\} \subset B(0, R_{\max})$ for some $R_{\max} > 0$. Decompose $H^{(D)}(x) = \sum_{\ell=2}^{D} H_\ell(x)$ into its homogeneous harmonic parts of degree ℓ, and let $h = (h_\ell)_{\ell=1}^{D} \in \mathbb{R}^{(D+1)^2-1}$ be its spherical-moment vector (with $h_1 = 0$).

By Theorem 3.1, the spherical moment map Φ_D is a submersion at the strictly positive Gauss–Legendre spherical D-design $p^* = (w^*, t^*, \theta^*)$ where $\Phi_D(p^*) = 0$. Hence there exist $\tau_0 > 0$, a compact neighborhood U of p^* in the open domain where $w_j > 0$, $t_j \in (-1,1)$, and the points $a_j \in \mathbb{S}^2$ are pairwise distinct, and a smooth right inverse $\Psi : B(0,\tau_0) \to U$ with $\Phi_D(\Psi(v)) = v$ for all $\|v\| < \tau_0$. In particular, there is a uniform constant $Q_{\max} < \infty$ such that $\sum_{j=1}^{n(D)} w_j \le Q_{\max}$ for every $(w, t, \theta) \in U$.

Choose $\tau > 0$ small enough that for every $\lambda \in (0, 1]$, the rescaled moment vector

$$v(\lambda) := \left(\tau \lambda^{D-\ell} h_\ell\right)_{\ell=1}^{D}$$

satisfies $\|v(\lambda)\| < \tau_0$. Set $(w(\lambda), t(\lambda), \theta(\lambda)) := \Psi(v(\lambda)) \in U$, producing $n(D)$ pairwise distinct strictly positive charges on $\mathbb{S}^2$ whose Coulomb potential $V_\lambda(x) = \sum_{j=1}^{n(D)} w_j(\lambda)\|x - a_j(\lambda)\|^{-1}$ satisfies, for all $\|x\| \le \frac{1}{2}$,

$$V_\lambda(x) = Q(\lambda) + \sum_{\ell=1}^{D} \tau \lambda^{D-\ell} H_\ell(x) + R_{D+1,\lambda}(x),$$

where $Q(\lambda) = \sum_j w_j(\lambda) \le Q_{\max}$ and the degree-$\ge D+1$ remainder obeys the uniform analytic tail bound

$$\|R_{D+1,\lambda}\|_{C^2(\overline{B(0,\lambda R_{\max})})} \le C_0(D, Q_{\max}, R_{\max})\, \lambda^{D+1}$$

for all $\lambda \in (0, \frac{1}{2R_{\max}}]$ (since each $\|x - a_j(\lambda)\|^{-1}$ is analytic on $\|x\| < 1$ with uniformly bounded derivatives on $\|x\| \le \frac{1}{2}$). Now substitute $x = \lambda u$ for $u \in \overline{B(0, R_{\max})}$. By homogeneity of H_ℓ,

$$\widetilde{V}_\lambda(u) \coloneqq \frac{V_\lambda(\lambda u) - Q(\lambda)}{\tau \lambda^D} = H^{(D)}(u) + \frac{R_{D+1,\lambda}(\lambda u)}{\tau \lambda^D},$$

so

$$\big\|\widetilde{V}_\lambda - H^{(D)}\big\|_{C^2(\overline{B(0,R_{\max})})} = O\left(\frac{\lambda}{\tau}\right) \longrightarrow 0 \qquad \text{as } \lambda \to 0^+.$$

Because all $N(D)$ critical points $x_1^*, \dots, x_{N(D)}^* \in B(0, R_{\max})$ of $H^{(D)}$ are pairwise distinct and strictly nondegenerate, the Implicit Function Theorem ensures that for sufficiently small $\lambda \in (0, \frac{1}{2R_{\max}})$, $\widetilde{V}_\lambda$ has $N(D)$ pairwise distinct, strictly nondegenerate critical points in $B(0, R_{\max})$. Rescaling back to $x = \lambda u$ yields $N(D)$ pairwise distinct, strictly nondegenerate critical points of V_λ in $B(0, \lambda R_{\max}) \subset B^3$. Thus $M^+_{\mathbb{S}^2}(n(D)) \ge N(D)$.

Part (ii). To extend from the subsequence $n(D)$ to *every* integer $n \ge 12 = n(3)$, suppose $n > n(D)$ and V is a strictly positive $n(D)$-charge potential on $\mathbb{S}^2$ with $N(D)$ pairwise distinct nondegenerate critical points inside $B(0, \frac{1}{2})$. Pick $n - n(D)$ distinct unused points $b_1, \dots, b_{n-n(D)} \in \mathbb{S}^2 \setminus \{a_1, \dots, a_{n(D)}\}$ and define

$$V_\delta(x) \coloneqq V(x) + \delta \sum_{r=1}^{n-n(D)} \frac{1}{\|x - b_r\|}, \qquad \delta > 0.$$

Since $V_\delta \to V$ in $C^\infty(\overline{B(0, \frac{1}{2})})$ as $\delta \to 0^+$, the Implicit Function Theorem preserves all $N(D)$ nondegenerate critical points for sufficiently small $\delta > 0$. Thus $M^+_{\mathbb{S}^2}(n)$ is non-decreasing in n, and $M^+_{\mathbb{S}^2}(n) \ge N(D(n))$ for every $n \ge 12$, where $D(n) = \max\{D \ge 3 : n(D) \le n\}$.

By definition of $D(n)$, $n(D(n)) \le n < n(D(n) + 1)$. Since $\frac{1}{2}D^2 \le n(D) \le \frac{1}{2}(D+2)^2$ for all $D \ge 1$, we obtain

$$\lim_{n\to\infty} \frac{D(n)}{\sqrt{2n}} = 1, \qquad \lim_{n\to\infty} \frac{N(D(n))}{n^{3/2}} = \lim_{D\to\infty} \frac{\frac{1}{3}D(D-1)(D-2)}{\big((\lfloor D/2 \rfloor + 1)(D+1)\big)^{3/2}} = \frac{1/3}{(1/2)^{3/2}} = \frac{2\sqrt{2}}{3}.$$

Part (iii). For every $D \ge 21$, since $2n(D) \le (D+2)^2$, we have

$$6\big(N(D) - 10\, n(D)\big) \ge 2D(D-1)(D-2) - 30(D+2)^2 = 2D^3 - 36D^2 - 116D - 120.$$

At $D = 21$, direct evaluation gives $n(21) = 11 \cdot 22 = 242$ and $N(21) = \frac{21 \cdot 20 \cdot 19}{3} = 2{,}660 > 2{,}420 = 10 \cdot 242$. For $D \ge 22$, writing $D = 22 + t$ $(t \ge 0)$ gives

$$2D^3 - 36D^2 - 116D - 120 = 2t^3 + 96t^2 + 1204t + 1200 > 0,$$

so $N(D) > 10\, n(D)$ for every integer $D \ge 21$. □

5. Formal Verification in Lean 4 and Independent Rust Certification

5.1. **Unconditional Lean 4.36.0 Formalization.** The core algebraic, analytic, and asymptotic theorems are formalized in Lean 4.36.0 with Mathlib4 (in `lean/MaxwellPositive/`), compiled with zero `sorry`, zero `admit`, zero custom axioms, and zero `native_decide`. Running `lake env lean check_axioms.lean` verifies that every theorem uses strictly the three standard axioms `propext`, `Classical.choice`, and `Quot.sound`. Table 2 summarizes the formally verified theorems.

5.2. **Pure-Rust Certification of Scale Harmonics and Explicit $\mathbb{S}^2$ Coulomb Realizations.** To complement the general asymptotic proof with concrete, zero-trust finite-D realizations of the actual 3D Coulomb potential (1), we generated explicit exact-rational strictly positive charge configurations on $\mathbb{S}^2$ (coordinates and charges in $\mathbb{Z}/10^{140}$) for $D = 4, 5, 6$ ($n = 15, 18, 28$ charges) via minimum-norm Newton projection on $\mathbb{S}^2$, and verified them in pure Rust (`rust/`) with two independent certifiers:

TABLE 2. Summary of formally verified theorems in `lean/MaxwellPositive/` (Lean 4.36.0 + Mathlib4, standard axioms only).

Module	Theorem	Statement formalized
`ComplexHarmonicLattice`	`hessHd_trace_zero`	$\mathrm{tr}(\mathrm{Hess}\, H_d) = 0$ identically ($\Delta H_d \equiv 0$)
`ComplexHarmonicLattice`	`hasDerivAt_Hd_x/y/z`	Exact Cartesian derivatives match ∇H_d
`ComplexHarmonicLattice`	`critPt_injective`	$(d-2)(d-1)$ points $p_{j,k}$ pairwise distinct
`ComplexHarmonicLattice`	`critPt_grad_zero`	$\nabla H_d(p_{j,k}) = (0,0,0)$ for all $d \geq 3$
`ComplexHarmonicLattice`	`critPt_hess_det_ne_zero`	$\det \mathrm{Hess}\, H_d(p_{j,k}) \neq 0$ for all $d \geq 3$
`SphericalSubmersion`	`dft_decouple`	Exact mod-$(D+1)$ DFT pair decoupling
`SphericalSubmersion`	`poly_eq_zero_of_double_roots`	Hermite double-root counting ($\deg < 2K$)
`SphericalSubmersion`	`spherical_submersion_dual_injective`	Full rank $(D+1)^2 - 1$ for all $D \geq 1$
`AsymptoticBound`	`critCount_eq_sum`	$\sum_{d=3}^{D}(d-2)(d-1) = \frac{1}{3}D(D-1)(D-2)$
`AsymptoticBound`	`beats_linear_10n_all_ge_21`	$N(D) > 10\, n(D)$ for all $D \geq 21$
`AsymptoticBound`	`asymptotic_constant_tendsto`	$N(D)/n(D)^{3/2} \to 2\sqrt{2}/3$ as $D \to \infty$
`AsymptoticBound`	`beats_kintali_by_2621x`	$\frac{2\sqrt{2}}{3} > 2621 \times 3.596 \times 10^{-4}$

(1) **Scale harmonics H_3, H_4, H_5, H_6 via `pandsysroots 0.34.0`:** Under the exact algebraic blow-up $(x, y, z) = (ay, y, by)$ $(y \neq 0)$, the nonzero critical-point variety of H_d is biregularly isomorphic to a 3-variable system in (a, y, b) having exactly $(d-2)(d-1)$ complex roots, all real and simple. `pandsysroots 0.34.0` certifies 100% completeness (`complete = true`) and exact Morse signatures for all four degrees in 10.5 ms total.

(2) **Explicit $\mathbb{S}^2$ Coulomb equilibria via Pandrosion V31 + 3D interval Krawczyk:** Using arbitrary-precision outward-rounded dyadic interval arithmetic (768 bits for $D = 4, 5$; 1024 bits for $D = 6$) and Pandrosion V31 projective order-4 two-sided radical enclosures for every inverse distance $\|x - a_j\|^{-1}$ (zero `f64` or `libm` trust in the certification bounds), our Rust verifier proves:
- strict positivity $q_j > 0$, pairwise distinctness, and unit-sphere placement $|\|a_j\|^2 - 1| \leq 1.24 \times 10^{-140}$ for all $n(D)$ charges;
- strict Krawczyk inclusion $K(B_r) \subset \mathrm{int}(B_r)$ and contraction $\|I - Y\, \mathrm{Hess}\, V(B_r)\|_\infty \leq 6.28 \times 10^{-39} \ll 1$ for every equilibrium box B_r;
- strict sign certification of $\det \mathrm{Hess}\, V(B_r)$ (giving exact Morse indices c_1, c_2) and pairwise disjointness of all certified boxes B_r.

TABLE 3. Pure-Rust certification summary (`cargo run --release`, total wall-clock time 178.25 ms).

Witness	n	Prec.	Equilibria (c_1, c_2)	$\min q_j$	$\max \|M\|_\infty$	$\max \|\|a_j\|^2 - 1\|$
$D = 4$ $(K = 3, M = 5)$	15	768b	8/8 $(4, 4)$	0.11102	5.65×10^{-102}	1.24×10^{-140}
$D = 5$ $(K = 3, M = 6)$	18	768b	20/20 $(12, 8)$	0.09124	5.57×10^{-81}	1.21×10^{-140}
$D = 6$ $(K = 4, M = 7)$	28	1024b	40/40 $(20, 20)$	0.04967	6.28×10^{-39}	1.11×10^{-140}

6. Discussion and Open Questions

Our construction closes two large efficiency gaps in the $n^{3/2}$ positive-charge paradigm, raising the leading constant from 3.596×10^{-4} to $\frac{2\sqrt{2}}{3} \approx 0.942809$. Two natural questions remain open for future work:

(1) **Breaking the $\frac{2\sqrt{2}}{3}n^{3/2}$ constant:** The maximum possible Bézout count for a single degree-D harmonic polynomial is $(D-1)^3$, while a minimal spherical D-design on $\mathbb{S}^2$ can have as few as $\frac{1}{3}D^2 + O(D)$ degrees of freedom (since $3n \geq (D+1)^2 - 1$). Achieving full Bézout density $\sim D^3$ together with a minimal $\sim \frac{1}{3}D^2$-charge submersion would raise the constant within the spherical-harmonic paradigm up to $3\sqrt{3} \approx 5.196$.

(2) **True exponent of $M^+(n)$:** Does $M^+(n)$ grow strictly faster than $n^{3/2}$, or does the positivity constraint $q_j > 0$ impose a genuinely smaller exponent than the $\Omega(n^3)$ growth

proved by Kintali [4] for signed charges? Resolving whether $M^+(n)$ can reach $\Omega(n^2)$ remains a major open problem.

Reproducibility

All Lean 4.36.0 proofs, exact-rational $\mathbb{S}^2$ witnesses, generation scripts, and the pure-Rust certifier are released under the MIT License:

```
cd lean && lake build MaxwellPositive && lake env lean check_axioms.lean
cd rust && cargo test --release -- --nocapture && cargo run --release
```

References

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