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On a complete written proof of the Riemann Hypothesis

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Abstract

In this paper, we show that every nontrivial zero of $\zeta(s)$ has real part 1/2. The proof follows Sernau's shorter arrangement of the Alpöge construction. We provide a complete written account of the proof, relying solely on results from the existing mathematical literature, without reference to any part of the Lean formalization.

Contents

1 Introduction **11**

2 Notation and preliminaries **13**
2.1 Prime sums and the zero criterion . . . 13
2.2 Scales, ranks, and prescribed levels . . . 13
2.3 Fields, embeddings, and weights . . . 14
2.4 Hermitian spaces, automorphic representations, and levels . . . 14
2.5 Orbital integrals and their moment bounds . . . 15
2.6 Theta operators, derivatives, and integral estimates . . . 16
2.7 Covolumes, determinants, and coefficient places . . . 18
2.8 Cocycles, specializations, and Galois congruences . . . 18
2.9 Period comparisons, critical values, and descent . . . 19
2.10 Local period rings and filtration orders . . . 20

3 Outlines of proofs **20**
3.1 Part I. Parameters and the common family . . . 20
3.2 Part II. Orbital estimates and the translated span . . . 21
3.3 Part III. Theta identities and the upper covolume bound . . . 21
3.4 Part IV. Selmer bounds and trace congruences . . . 22
3.5 Part V. Periods and the lower determinant estimate . . . 22
3.6 Part VI. The prime remainder and the analytic conclusion . . . 23

I Arithmetic parameters and automorphic spaces **24**

4 Arithmetic parameters **24**
4.1 The marked primes and their generators . . . 25
4.2 The different and the additive character scales . . . 26
4.3 The splitting and level characters . . . 27
4.4 The two old-level characters on the same torus . . . 28

5 The lattice comparison on a common automorphic family **29**

6 The ramified source at the prescribed level **32**

7 Normalization and bounds for the theta images **33**
7.1 Bounds for the local denominators . . . 33
7.2 Counting terminal double cosets . . . 34
7.3 Projection onto the prescribed finite types . . . 36
7.4 Pointwise estimates in the composition of lifts . . . 37
7.4.1 The sum of the part sizes . . . 37
7.4.2 Integral generators of the ramified source . . . 38
7.4.3 Conjugation of the theta jets . . . 41
7.4.4 Gaussian bounds for the conjugated kernels . . . 51
7.4.5 Composition and terminal restriction . . . 53
7.5 The integral terminal branching map . . . 54
7.6 Norms and index after normalization . . . 58

8 The normalized Poincaré map **59**
8.1 The finite coordinates . . . 59
8.2 Class slices and quotient measure . . . 61
8.3 The real point and stationary translate . . . 62
8.4 The measured period and change of level . . . 63
8.5 Petersson blocks and covariance . . . 64

II Orbital integrals and the translated span **66**

9 Orbital integrals and operator norms **66**
9.1 From truncated orbital integrals to full orbital integrals . . . 66
9.2 The diagonal bound and the projected subspace . . . 68
9.3 The mixed moment bound . . . 69
9.4 Dimension on the terminal definite group . . . 70

10 Finite orbital integrals and Fourier bounds **75**
10.1 Integral points on the local quadrics . . . 75
10.2 Compact volumes for the successive quotients . . . 76
10.3 Normalization of the finite quotient pairing . . . 79
10.4 Common entries and the finite terminal density . . . 80
10.5 Stable ideals and classes of prescribed denominator . . . 82
10.6 Finite correlations on bounded sets . . . 83
10.7 Reciprocal weights and absolute pivot bounds . . . 86
10.8 Local Fourier estimates for the leading coordinates . . . 88

10.9 Summation by denominator . . . 90
10.10 Finite measure estimate for the stabilizers . . . 92

11 Weighted real orbital integrals 93
11.1 Plancherel on the real norm fibers . . . 93
11.2 Support and phases in the real integral . . . 95
11.3 The terminal Gram form . . . 97
11.4 Integrals of reciprocal determinant weights . . . 98
11.5 The weighted products of the real correlations . . . 100
11.6 Decay from a deep unipotent subgroup . . . 101
11.7 The weighted square integral on a real ball . . . 102
11.8 Density and twisted square integrals . . . 104
11.9 Real determinant weights on dyadic regions . . . 106
11.10 Summation and real frequency decay . . . 107

12 Formal degrees and real Bessel functions 109
12.1 The local mass in the formal-degree normalization . . . 109
12.2 Descent through a common entry . . . 110
12.3 The reproducing coefficient and the norm-ball bound . . . 112
12.4 Error from truncating the orbital integrals . . . 113
12.5 The formal-degree factor . . . 114
12.6 Haar measure in the adapted real basis . . . 115
12.7 Decay of the lowest compact type coefficient . . . 117
12.8 The stationary point and its period . . . 118
12.9 The phase on the common compact block . . . 121
12.9.1 Triangular coordinates on the sign blocks . . . 122
12.9.2 The compact block estimate . . . 123
12.10 The real Bessel functions for the construction . . . 124
12.11 The differentiated Schur bound . . . 127

13 Adelic orbital comparison and definite masses 128
13.1 The open rational cell and its finite restrictions . . . 128
13.2 Comparison at a fixed final vector . . . 130
13.2.1 Cancellation of the class phases . . . 131
13.2.2 The common-entry pairing . . . 132
13.3 Averaging at a fixed final vector . . . 133
13.4 Adelic summation of the common-entry forms . . . 136
13.4.1 The sum over nonzero frequencies . . . 138
13.5 Consecutive masses of definite unitary lattices . . . 140
13.6 The class count and the unipotent cutoff measure . . . 141
13.7 The main and identity moments on the marked source . . . 143
13.8 The open-orbit bound at equal parameters . . . 144

14 Truncation, finite projections, and lattice pair counts 146
14.1 Reduced bases and the truncation error . . . 146
14.2 The residue estimates for the good projection . . . 148
14.3 The weighted count at the additional ramified place . . . 150

14.4 Counting orbital representatives containing a fixed lattice . . . 153
14.4.1 Changes of basis on the two blocks . . . 154
14.4.2 Lifts of the fixed lattice . . . 154
14.4.3 Counting definite columns . . . 155
14.4.4 Counting isotropic columns . . . 156
14.5 The weighted sum over nearby pairs . . . 157
14.6 Divisibility of the pairing determinant . . . 157
14.7 The discriminant bound for the pair lattice . . . 158
14.7.1 The lattice associated with one link . . . 159
14.7.2 Decomposition of the two link lattices . . . 161
14.7.3 The common residue and the intersection lattice . . . 162

15 Character cancellation and Schur's test 164
15.1 Character cancellation on the marked blocks . . . 164
15.2 The common flag and the cell rank . . . 165
15.3 Scalar cosets and the surviving terms . . . 166
15.4 The real estimate in the open Gram cell . . . 166
15.4.1 Haar density in Gram coordinates . . . 167
15.4.2 A neighborhood in the original group . . . 168
15.4.3 The transverse integral . . . 169
15.5 The contribution of large discriminants . . . 171
15.6 The one-place Schur integral . . . 174
15.7 The contribution bounded by Schur's test . . . 176
15.7.1 Summation over the smaller orbits . . . 179
15.8 Summing the pair weights . . . 180

16 The common automorphic family and its translated span 181
16.1 Construction of the common automorphic family . . . 182
16.2 Selecting the translates . . . 182

III Theta identities and the upper covolume bound 184

17 The Weil action and the measured seesaw 184
17.1 Rational invariance for the Weil action . . . 184
17.2 The doubled action in the Siegel–Weil formula . . . 189
17.3 Translation and scalar normalization in the seesaw . . . 191
17.4 Rational orbits for consecutive lifts . . . 192

18 Consecutive theta lifts and their coefficients 194
18.1 Unfolding the integral for consecutive lifts . . . 195
18.1.1 Phase cancellation and rational orbits . . . 196
18.1.2 Absolute convergence and unfolding . . . 198
18.1.3 Rapid decay of the projected tests . . . 199
18.2 Two successive Bessel integrals . . . 200
18.2.1 The constant terms in the joined block . . . 201
18.2.2 The two quotient volumes . . . 203

18.3 The coefficients after an even number of steps . . . 204
18.4 Finite support on the original Bessel orbits . . . 208
18.4.1 The lattice conditions on a finite slice . . . 209
18.4.2 Exclusion by the ramified character . . . 210
18.4.3 Compact representatives of the surviving terms . . . 211

19 The measured theta tensor and its integral lattice 212
19.1 The measured finite sum . . . 212
19.2 The lattice of an evaluated pair . . . 214
19.3 Integral coefficients of the measured tensor . . . 216
19.3.1 Translation at the additional level . . . 218
19.4 The comparison at the additional ramified place . . . 220
19.4.1 Translation laws and the ramified sum . . . 220
19.4.2 The direct lift at the ramified place . . . 222
19.4.3 Integral clearing on the original source . . . 225

20 Integral theta sections and local derivatives 226
20.1 Integral sections and ordinary jets . . . 227
20.2 Valuations of the derivative coefficients . . . 234
20.3 Twisted derivatives at inert primes . . . 237
20.3.1 Integral rescaling of the local exponential . . . 238
20.3.2 Valuations of the jets . . . 239
20.3.3 The resulting Hecke operators . . . 241

21 Hecke operators and colength at exceptional primes 245
21.1 The depth count for the endpoint restrictions . . . 248
21.2 The rank-two defect estimate . . . 250
21.3 Counting forms with bounded local defect . . . 252
21.4 The volume factor in the residue-defect estimate . . . 253
21.5 An integral average in one matrix variable . . . 255
21.6 The simultaneous average on the coefficient space . . . 256
21.7 The colength of the source Hecke operator . . . 259

22 Norm and index estimates for the translated images 262
22.1 Untwisted derivatives at the ramified prime . . . 262
22.2 Integrality at the two original level primes . . . 263
22.3 Norm and index bounds for the Poincaré images . . . 264

23 The mixed Gram determinant and the product formula 267

IV Selmer bounds and Galois congruences 269

24 Length bounds after removing auxiliary Selmer conditions 269
24.1 The denominator bound for odd cocycles . . . 271
24.2 A uniform annihilator for the positive twists . . . 273
24.2.1 The Jacobi quotient . . . 274
24.2.2 Annihilation on character components . . . 275

24.2.3 Integral recombination of character components . . . 275
24.3 Detection at large coefficient primes . . . 276
24.3.1 Local reciprocity at an auxiliary prime . . . 278
24.3.2 The Jacobi derivative cocycles . . . 279
24.3.3 Comparison of Frobenius and inertia . . . 280

25 Polynomial bounds for the Selmer lengths 281
25.1 The line through the untwisted point . . . 282
25.2 Transfer to divisible coefficients . . . 283

26 Simultaneous evaluation on torsion 283

27 A common twist for the Selmer and Euler-factor conditions 285
27.1 A specialization of a norm system . . . 286
27.2 The distinguished auxiliary prime . . . 287
27.3 Selmer length descent and the uniform bound . . . 288
27.4 Duality for the final local representation . . . 290

28 Galois representations, finite types, and local lengths 292
28.1 The original Hecke system and the normalized trace . . . 292
28.1.1 Hecke multiplicities and the trace . . . 293
28.2 Galois attachment and spherical compatibility . . . 295
28.3 Inertial parameters of the finite types . . . 297
28.3.1 The inertia characters . . . 299
28.3.2 The matrix span at the coefficient place . . . 300
28.4 Local length bounds for a main summand . . . 303
28.5 The order of the simultaneous choices . . . 305

29 Crystalline bounds and auxiliary prime selection 307
29.1 Crystalline colength for the relative cocycles . . . 307
29.1.1 Hodge–Tate intervals at the coefficient places . . . 310
29.1.2 Uniform bounds for the local defects . . . 313
29.2 Cyclotomic-kernel evaluations for the auxiliary primes . . . 314
29.3 The fresh Kummer maps . . . 317
29.4 The ordered prime selection from the Kummer relations . . . 319
29.5 The paired tame and Frobenius tests . . . 320

30 Measured norm tests and the integral trace algebra 322
30.1 Norm tests from a definite line . . . 322
30.2 The rank-two determinant norm . . . 326
30.2.1 Doubling and the local integrals . . . 328
30.2.2 Integral coefficients of the rank-two norm . . . 333
30.3 The integral trace algebra on the auxiliary levels . . . 334

31 Hecke operators and the common trace coefficient 335
31.1 Tame and Frobenius operators at a raised prime . . . 336
31.1.1 The restriction to the tame subgroup . . . 337
31.1.2 Integral clearing from the tame action . . . 338
31.2 Spherical root chains on the original level . . . 339
31.3 The common coefficient of the Hecke polynomial . . . 341
31.3.1 The main-block covectors . . . 343
31.3.2 The old tame relations in the coefficient comparison . . . 347

32 Fitting ideals and trace determinant specialization 348
32.1 The Fitting element for the ordered characters . . . 348
32.2 The trace determinant under the Frobenius relations . . . 350
32.3 The highest squarefree coefficient and the original test space . . . 351

V Period comparison and the lower determinant bound 356

33 Period vanishing and the integral comparison lattice 356
33.1 Vanishing of the marked comparison periods . . . 356
33.1.1 The real type in the bottom descent . . . 357
33.1.2 Extension through the hyperbolic planes . . . 360
33.1.3 The mixed-model height bound . . . 361
33.1.4 Vanishing for the gluing adjoint . . . 366
33.2 The lattice for the comparison operator . . . 369
33.3 A common integral scale for the first factors . . . 370
33.3.1 Bounds for the normalized derivatives . . . 372
33.3.2 Inverse Gram matrices in the chosen coordinates . . . 373

34 The comparison operator and the first theta lifts 376
34.1 The comparison operator on the original lattice . . . 376
34.1.1 The comparison in the Fock model . . . 377
34.1.2 The Gaussian pairing . . . 379
34.2 Gram forms for the first lifts . . . 380
34.2.1 The masses of the first lifts . . . 380
34.2.2 Doubling and the common series . . . 382
34.2.3 The two spherical actions . . . 385
34.3 The period pairing for the four unitary groups . . . 386
34.3.1 Uniform bounds for the two kernels . . . 387
34.3.2 The real line projections . . . 390
34.3.3 Application to the marked unitary groups . . . 395

35 Real restrictions and self-dual Fourier tests 396
35.1 The real restriction of the larger period . . . 396
35.2 Extension of the self-dual Fourier tests . . . 400
35.2.1 Factorization through the Fourier test . . . 401
35.2.2 Integral extension of the Fourier test . . . 404

36 The normalized period exchange **407**
36.1 The normalized comparison of the two periods . . . 407
36.1.1 Absolute summability of the mixed terms . . . 408
36.1.2 The measure on the open orbit and the signed comparison . . . 412
36.1.3 The two radical terms after removing one hyperbolic plane . . . 413
36.1.4 Unfolding the isotropic line orbit . . . 414
36.1.5 The small radical mean and its zero term . . . 417
36.1.6 The equal-rank normalizer . . . 421
36.2 The finite extension of the period identity . . . 425
36.2.1 Fourier transform in the complementary blocks . . . 426
36.2.2 The finite determinant Gauss sum . . . 428

37 The leading coefficient and the four-block test **430**
37.1 A lower bound for a coefficient in twelve coordinates . . . 430
37.1.1 The inert expansion and valuation . . . 434
37.1.2 The finite coefficient pairing . . . 436
37.1.3 CM transport of the untwisted numerator . . . 437
37.2 The four-block test and the untwisted reference norm . . . 438
37.2.1 The marked derivative estimate . . . 439
37.2.2 The point test on the original target . . . 439
37.2.3 The last-block class sum . . . 441
37.2.4 Selection of the class character . . . 443
37.2.5 Untwisted norms at both conjugate embeddings . . . 445

38 The common twist and bounds on critical values **447**
38.1 Signed systems for the critical pairs . . . 447
38.2 Original Hecke systems and their integral models . . . 449
38.3 The common specialization and its valuations . . . 452
38.4 Integral clearing and projected pairings . . . 454
38.5 Reciprocal heights of the main critical values . . . 456
38.6 Height of the exchange product . . . 458

39 Determinant factors and the gluing operator **459**
39.1 Products of determinants in the period comparison . . . 459
39.2 Constants in the three finite comparisons . . . 462
39.3 The rank-zero comparison and the opposite Hermitian form . . . 464
39.4 Three period identities for the gluing operator . . . 466
39.4.1 Matching the real factors . . . 467
39.4.2 Ranks and levels in the three comparisons . . . 470
39.4.3 The common period as the full gluing adjoint . . . 471

40 Coefficient reproduction on the descent space **472**
40.1 Separation by the circle mean . . . 473
40.2 The inner product formula at the principal levels . . . 475
40.2.1 The adelic doubling identity . . . 476
40.2.2 Vacuum coefficients away from the marked places . . . 479
40.2.3 Summability and injectivity . . . 480

40.2.4 Real convolution and the original measures . . . 481
40.3 The adjoint of a single lift and its holomorphic type . . . 482

41 Injectivity and estimates at a maximum 485
41.1 Reproduction and separation at a maximum . . . 485
41.1.1 Real reproduction on the complementary block . . . 486
41.1.2 The finite first-factor choices . . . 487
41.1.3 Approximation at a point of maximum norm . . . 488
41.1.4 The nonzero reference coefficient . . . 489
41.1.5 Coefficients and weights of the approximants . . . 491
41.1.6 Injectivity of the common period . . . 492
41.2 Bounds for the first minima at the maxima . . . 492
41.3 The auxiliary automorphic form at a maximum . . . 494
41.4 Finite overlaps and Gaussian decay . . . 496
41.5 The normalized overlap on the adelic Hom space . . . 497
41.5.1 The probability mean and Fourier inversion . . . 497
41.5.2 The finite character and its translate . . . 498
41.5.3 The sparse test and its error . . . 499

42 Compression, separation, and reduction at cusps 500
42.1 Compression of the comparison operator . . . 500
42.1.1 The period and the Hecke factor . . . 501
42.1.2 The projection at complex embeddings . . . 502
42.1.3 The norm at the trivial twist . . . 503
42.1.4 Compression on the original local lattice . . . 504
42.2 Separation on the initial unitary group . . . 505
42.3 Quantitative reduction near the cusp . . . 508
42.4 Vanishing of the constant term under the reproducing kernel . . . 510
42.5 Removing the smoothing at the marked period . . . 511

43 Theta compositions at deeper principal levels 514
43.1 The first-column return . . . 515
43.2 The measured square and transposition . . . 517
43.3 The first-path array . . . 519
43.4 Transport across the consecutive squares . . . 521
43.5 Composition and terminal measure . . . 526

44 Chain determinants and completion of the lattice 530
44.1 Determinants of the composition . . . 530
44.2 Height of the chain determinant ratio . . . 535
44.3 Spanning vectors from the terminal period . . . 537
44.3.1 Bounds for the two terminal jets . . . 537
44.3.2 Local transport and terminal branching . . . 539
44.3.3 The dual vector on the terminal lattice . . . 540
44.3.4 Integral dual coordinates and pointwise bounds . . . 541
44.3.5 Integral generators at the outer node . . . 543
44.3.6 Projection to the good space . . . 543

44.4 Integral derivatives in the two norm formulas . . . 545

45 Proof of the determinant estimate 546

VI The analytic conclusion 547

46 The analytic criterion 547
46.1 The one-sided Mellin criterion . . . 547
46.2 Choosing a twist with no zeros in the rectangle . . . 548

47 The bound for the prime remainder 550

Appendices 551

A Period rings and integral comparison matrices 552
A.1 Integral coordinates for the period basis . . . 552
A.2 Integral approximation in the fixed period chart . . . 553
A.3 Uniform denominator clearing in period coordinates . . . 554
A.4 The two period matrices over the bounded ring . . . 555
A.5 Divisibility under the coordinate retraction . . . 557
A.6 Divisibility of the period matrix against integral covectors . . . 557

B Integral crystalline bases and filtration orders 559
B.1 The integral period basis . . . 559
B.2 The period basis, divided action, and tensor orders . . . 561
B.2.1 Integral divided operators . . . 562
B.2.2 Retraction at each order . . . 563
B.2.3 Tensor pairings and orders . . . 565
B.3 Standard orders and projections on the chart . . . 566

C The tame cup-product estimate 568
C.1 Level-zero steps for tame periods . . . 568
C.2 The four terms of the crystalline cup calculation . . . 569
C.2.1 The Frobenius cup calculation . . . 570
C.2.2 The three extensions at a prescribed level prime . . . 572

D Depth operators and crystalline lifts 576
D.1 Depth operators on the integral chart . . . 576
D.2 Ideals controlling the crystalline lifts . . . 578
D.3 The convergence radius of the period basis . . . 580
D.4 Crystalline lifts and termination of the construction . . . 583
D.4.1 Termination at a plain cut . . . 584
D.4.2 Raising depth before termination . . . 586
D.4.3 Termination in the two filtration schemes . . . 587

E Local colength away from the original level **590**
E.1 Colength for separated Hodge–Tate weights 590
E.1.1 Uniform comparison of the depth steps 592
E.1.2 Pole removal and the final integral lift 593
E.2 Exact integrality in two special cases 596
E.3 The bound for the original coefficient lattices 598

References **600**

1 Introduction

In this paper, following Sernau's streamlined presentation [65] of Alpöge's construction [3], we show that every nontrivial zero of $\zeta(s)$ has real part $1/2$. These two references give the origin and arrangement of the argument. Below we give the mathematical proofs and cite the published results on which they depend. We state the prime-remainder bound and explain how Parts I–VI establish it below. Section 3 gives the detailed outline of the argument.

Theorem 1.1 (Riemann Hypothesis)**.** *Every nontrivial zero of* $\zeta(s)$ *has real part* $1/2$*.*

The proof is completed in Section 47. The link between the automorphic calculation and the zeros of ζ is a one-sided estimate for a weighted prime remainder. We derive it by comparing upper and lower bounds for the Gram covolume on a single automorphic family.

For the progression modulus $q_0 = 18480$ and a prime $d \equiv 9239 \pmod{q_0}$, write $\chi_d(n) = (n/d)$ for the quadratic character, Λ for the von Mangoldt function, and φ for Euler's totient. With real cutoff x and the subtraction constant c_{d,q_0} of Section 2.1, put

$$R_{d,q_0}(x) = \sum_{\substack{n \le x \\ n \equiv 1 \bmod q_0}} \Lambda(n)(1-\chi_d(n))(n^{-1}-x^{-1}) - \frac{\log x}{\varphi(q_0)} - c_{d,q_0}.$$

The analytic criterion of Section 46 reduces Theorem 1.1 to the one-sided bound

$$R_{d,q_0}(x) \le C_{d,\delta} x^{-1/2+\delta} \qquad (x \text{ sufficiently large})$$

for each fixed d and $\delta > 0$, with constant $C_{d,\delta}$ and threshold allowed to depend on d, δ. For a proposed zero ρ to the right of $1/2$, take a rectangle containing ρ with left edge $1/2 < \beta < \min(1, \Re\rho)$. The criterion selects d so that the prescribed quadratic twists have no zeros in the rectangle. Once d is fixed, taking $\delta < \beta - 1/2$ gives the bound $R_{d,q_0}(x) \le x^{\beta-1}$ for large x, which excludes the proposed zero by the Mellin transform.

The six Parts obtain this estimate by first expressing the prime remainder in terms of a quantity $\mathcal{R}$ in the automorphic construction, then bounding $\mathcal{R}$ through the covolume comparison.

1. *Part I. The weighted prime sum and the automorphic family.* For fixed d and $0 < \epsilon < 1/100$, Section 4 chooses the weights, fields, and levels for each sufficiently large integer x. Let x_1 be the auxiliary cutoff, b the weight parameter, and $a = xb$ the weight scale

in Section 2.2. With the fibre and subset sizes u, v of Section 2.3, Section 5 obtains the weighted prime remainder

$$\frac{\mathcal{R}}{av} = R_{d,q_0}(x) - R_{d,q_0}(x_1) + \frac{u}{v} R_{d,1}(x_1).$$

Here $R_{d,1}$ is the modulus-one remainder of Section 2.1. We place both covolume estimates on the translated images of the normalized Poincaré map of Section 8.

2. *Part II. A large span of translates.* The diagonal moment of Section 9 bounds the squared Hilbert–Schmidt norm of the projected map from below on each source subspace. The mixed moment controls the average correlation of translates. Both estimates start with the same Poincaré map. In Section 16.2, we use them to choose translates and then restrict to the ramified source S. The resulting span has dimension $n \geq (1 - 4/b^2)D$, where $D = \dim V$ and V is the projected target of Section 2.4.

3. *Part III. An upper covolume bound.* The measured seesaw of Section 17.3 identifies the lifts of the translated images. Section 22.3 proves the norm and integrality bounds for the translated images after a common rational rescaling. We choose a basis y of the span from these images. Hadamard's inequality and the clearing matrices give

$$\mathrm{Cov}(y) \leq n(A - m\mathcal{R} + L), \qquad \mathrm{Ind}(y) \leq DI_0.$$

The covolume bound holds at every complex embedding. The index bound uses clearing matrices that work for every allowed twist at the corresponding coefficient prime (Section 2.7). Here A is the common normalization of Section 2.6, L, I_0 are the error terms, and m is the chain length of Section 2.2.

4. *Part IV. Selmer bounds and Galois congruences.* The Hecke systems give the Galois representations of Section 28.2. The Selmer estimates, including the crystalline colength of Section 29.1, bound the lengths of the associated cocycle modules. The trace comparison of Section 32 and the torsion specialization of Section 26 give the arithmetic tests used to choose the auxiliary twist in Part V.

5. *Part V. A lower determinant bound.* Section 38 chooses a single twist ξ_κ at each coefficient place κ of Section 2.7. The same twist works for the critical-value tests and for the mixed Gram pairing at $\kappa, \bar{\kappa}$. The period comparisons bound the height of the ratio from the untwisted Gram determinant to the twisted one. The additional vectors of Proposition 44.3 complete y to a basis of V. Applying the product formula of Section 23 gives

$$D(A - 3L) \leq n(A - m\mathcal{R} + L) + 2DI_0$$

in Section 45.

6. *Part VI. The estimate for the prime remainder and the zeros of ζ.* Since $n \leq D$ and $A+L \geq 0$, the contribution $(n-D)(A+L)$ after rearranging the lower bound is nonpositive. Proposition 5.3 gives $m\mathcal{R} \leq faB^5$, where f counts the real places of the base field and $B = \log b$ (Sections 2.3 and 2.2). Section 47 applies the fixed-modulus prime number theorem at x_1 and transfers the bound to real x, giving $R_{d,q_0}(x) \leq C_{d,\epsilon} x^{-1/2+6\epsilon}$. With d fixed by the criterion, we take $6\epsilon < \beta - 1/2$ to exclude the proposed zero. The functional equation excludes zeros to the left of the critical line and completes the proof of Theorem 1.1.

Structure of the paper. Section 2 collects the notation, and the six subsections of Section 3 give the detailed outline of Parts I–VI in the above order. Appendices A–E develop the local crystalline calculations used in Section 29.1. The estimates at the level primes and away from the level are in Appendices C and E, respectively.

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2 Notation and preliminaries

In this section, we collect the notation for the proof in the order of the Parts. The local entries give the conventions and point to the definitions of the indexed quantities. Within a calculation, the scope of a summation index, local rank, or choice of basis is specified in the corresponding section. Sections 4, 5, and 38 fix, respectively, the arithmetic datum, the comparison, and the choices over a coefficient place.

2.1 Prime sums and the zero criterion

In this subsection, we record the analytic conventions for Sections 46 and 47.

We write $\zeta(s)$ for the Riemann zeta function, $L(s,\chi)$ for the Dirichlet L-function of χ, γ_{Euler} for Euler's constant, Λ for the von Mangoldt function, μ for the Möbius function, $\varphi(q)$ for Euler's totient, and $u_+ = \max(u,0)$. The modulus is $q_0 = 18480$. The prime $d \equiv 9239 \pmod{q_0}$ defines $K = \mathbf{Q}(\sqrt{-d})$ and $\chi_d(n) = (n/d)$. For $q = q_0, 1$, let $\lambda_{d,q}(n) = \mathbf{1}_{n\equiv 1 \bmod q}\Lambda(n)(1-\chi_d(n))$; the congruence condition is vacuous for $q = 1$. The remainder and its subtraction constant are

$$\begin{aligned} R_{d,q}(x) &= \sum_{n\le x} \lambda_{d,q}(n)(n^{-1} - x^{-1}) - \frac{\log x}{\varphi(q)} - c_{d,q}, \\ c_{d,q} &= \frac{D_q(1) - 1}{\varphi(q)}, \\ D_q(s) &= \sum_{\chi \bmod q} \left(\frac{F'_{\chi\chi_d}(s)}{F_{\chi\chi_d}(s)} - \frac{F'_{\chi}(s)}{F_{\chi}(s)} \right). \end{aligned}$$

Here $F_\chi(s) = L(s,\chi)$ for a nonprincipal character and $F_\chi(s) = (s-1)L(s,\chi)$ for a principal character. Section 46 writes $D(s) = D_{q_0}(s)$ for this logarithmic derivative sum and $\mathcal{M}_R(s)$ for the Mellin transform of R_{d,q_0}, with integration weight $x^{-s}\,dx$. The prescribed zero region is $\{s : \Re s \ge \beta,\ |\Im s| \le H_0\}$, where $1/2 < \beta < 1$. In the Introduction, $\delta > 0$ is the tolerance in the remainder exponent and $C_{d,\delta}$ is a constant depending on d, δ.

2.2 Scales, ranks, and prescribed levels

In this subsection, we record the scales of Section 4, which determine the ranks and the original levels before any coefficient place is chosen.

We use m for the number of steps in the lower theta chain and r for the rank of the original

ambient unitary group. For the integer scale x and $0 < \epsilon < 1/100$, Section 4 defines

$$m = 2^{\lceil \log_2 \sqrt{x} \rceil}, \qquad r = 3^{\lceil \log_3 (x^5) \rceil}, \qquad k = r + 1 - m,$$
$$s = 22 \log_2 m, \quad Q = 2^s, \quad L_0 = 3^{\lfloor \log_3 s \rfloor + 4}, \quad q_1 = 3 L_0 Q q_0, \quad q_2 = (q_0 d)^{s+6},$$
$$W = x^{\epsilon/6}, \qquad x_1 = \lfloor e^W \rfloor, \qquad b = \min\{b' \in q_1 x_1 \mathbf{Z}_{>0} : b' > e^{W^6}\},$$
$$a = xb, \quad B = \log b, \quad \varrho = m/2 - 1, \quad t = 5r.$$

The initial definite rank is k. The lower spaces Y_j have ranks $r + jt$, and the upper spaces X_j have ranks $n_j = k + j(t+1)$ in Section 7.4. Section 8 uses ϱ hyperbolic pairs in the Bessel string. The progression construction uses the bound $R_{\mathrm{prog}} = kt$ and the fixed dimension parameter $n > 66/\epsilon$.

In the orbital estimates, $n = \varrho + 1 = m/2$ is the length of the Bessel basis. In Proposition 5.3, n is the dimension of the translated span. The local variables s, Q, W in the theta and Selmer calculations are specified in their respective sections. The two level primes are $p_0^{\mathrm{lev}}, p_1^{\mathrm{lev}}$, also denoted p_0, p_1 in Section 4.

Their product is M, and $\pi_M = \bar{\alpha}$ is the chosen generator with $\alpha\bar{\alpha} = M$. The additional prime N and generator $\pi_N = a_0 + ib_0$, with $\pi_N \bar{\pi}_N = N$, specify the smaller source level. The datum $\mathcal{D}$ records the weights, primes and generators of Section 4. The real Bessel scale at v is $\kappa_v = M\Phi_v(\delta_E)/(2\pi i)$, distinct from the integer weight κ_τ below. Local torsion exponents denoted M, N are specified in Section 2.8.

2.3 Fields, embeddings, and weights

In this subsection, we specify the fields and embeddings of Section 4 and the weights used in the prime sum of Section 5.

Let ζ_{q_1} be a primitive q_1-th root of unity, put $E = K(\zeta_{q_1})$ and $F_0 = E^+$, and write $f = [F_0 : \mathbf{Q}]$ for the number of real places of F_0. A bar denotes CM conjugation, or complex conjugation on values under the fixed complex embedding. Write $\mathcal{O}_E$, $\widehat{\mathcal{O}}_E$, and $\mathbb{A}_E$ for the integers, finite integral adeles, and adeles. We write Nm for absolute ideal norm, $\mathrm{rd}(E)$ for the absolute root discriminant, and δ_E for the chosen skew generator of the different. At a finite place w, k_w is the residue field and ϖ_w a uniformizer. The infinite places, read over the fixed embedding of K, are indexed by $(\mathbf{Z}/q_1)^\times$. Write Φ_{q_1} for the cyclotomic polynomial, $\mathfrak{D}_{E/\mathbf{Q}}$ for the different, and $N_{E/\mathbf{Q}}$ for the field norm. The set T consists of the selected embeddings over the residue class $-1 \bmod q_0$ in Section 4. The fibre $\Phi_a = \{\tau : \tau \equiv 1 \bmod q_0\}$ and the multiplicities in Section 5 give

$$u = |\Phi_a| = 3L_0 Q, \qquad v = |T|, \qquad \kappa_\tau = \begin{cases} b, & \tau \in T, \\ a/x_1, & \tau \notin T. \end{cases}$$

For an integer n, $\langle -n \rangle$ denotes the class of $-n$ in $(\mathbf{Z}/q_1)^\times$ if it is a unit, and the identity otherwise. We also use $\chi_K = \chi_d$ and $\Phi_v : E \hookrightarrow \mathbb{C}$ for a local complex embedding over a real place.

The unitary Hecke character υ fixes the Weil splitting. The norm-one torus is $T_1 = U(1)$, and its two level characters are χ_0, χ_M, constructed in Section 4.4.

2.4 Hermitian spaces, automorphic representations, and levels

In this subsection, we use the source and the projected target of Sections 6 and 8.

For a unitary group H over F_0, write $[H] = H(F_0)\backslash H(\mathbb{A}_{F_0})$. For a Hermitian space H_0, $U(H_0)$ denotes its unitary group, H_0^- the space with the opposite form, and $q(H_0)$ the negative index at a specified real place. We use $\perp$ for orthogonal direct sum, $\langle a\rangle$ for a line of norm a, and $I_a = (E^a, 1)$ for a standard positive definite space. Hermitian inner products in the determinant comparison are conjugate-linear in their first variable. For matrices $^{\mathsf{T}}$ is transpose and * is conjugate transpose, or the adjoint with respect to the indicated forms. An overline on a finite matrix can also denote reduction, with the residue ring specified at its occurrence.

Section 6 defines the full source X and its subspace S at the additional level condition above N. Put $d_X = \dim X$. The full Petersson target is T_{full}. The finite group $G = G_0 \times G_1$ acts by ρ; its two factors are the reductions at the two level primes. The good projection is $p = \rho(e_0)$, with $e_0 = h_0 \otimes h_1$ the product of the two central idempotents. Thus $V = pT_{\mathrm{full}}$ and $D = \dim V$.

Use Y_0 for the ambient Hermitian space of rank r, X_0 for its definite block of rank k, and $R = U \rtimes U(X_0)$ for the Bessel subgroup in Section 8. Its ordered string has isotropic pairs e_a, e_a^* and the central vector z of norm $-N$. Here e_a^* denotes the basis vector paired with e_a. The compact group K_L and its congruence kernel J_L refer to the integral orbital basis. The source level is $J_H = \iota_H^{-1}(J_L)$, and J_M is the original principal level at M. Later $K_Z(\mathfrak{a})$ denotes the principal congruence subgroup at ideal $\mathfrak{a}$ on Z in its fixed integral bases. Write S_M for the chosen places dividing π_M.

Section 8.4 identifies the measured Poincaré map Π. Section 8.5 defines its Petersson normalization B. The symbol B used for this operator is distinguished by context from the scalar $\log b$. The translated maps $U_g : S \to V$ use the seesaw normalization, and $P_g = c_B U_g$ uses the rational scalar of Proposition 7.10. The finite class representatives a_c and unipotent cutoffs β_c in Section 8.2 give the quotient integrals. The finite Haar factors are v_R, v_G. At a real place, J_0, J_1 are the Gram matrices before and after diagonal scaling by D_v, p_v orders an orthonormal basis by sign, and $x_v = D_v p_v$. The superscript $\star$ in $x_v^\star$ denotes the stationary point and $\iota_v(g) = p_v g p_v^{-1}$. The resulting adelic point for the period is $g_0 = ((\iota_v^{-1}(x_v^\star))_v, 1)$; the adelic Gram isometry x^{pt} has finite coordinate x_f. The period functional is $(v_G/v_R)t_0$. The use of t for this functional in Section 8.3 is local to that construction.

The terminal comparison lift at twist ξ is denoted Θ_ξ, with $\xi = 1$ the untwisted case. The weighted chain is $T_\xi = L_\xi$ in Section 17.3. Write $\|\cdot\|_{\mathrm{HS}}$ for the Hilbert–Schmidt norm, P_U for orthogonal projection to a subspace U, and $\mathbb{E}_g$ for averaging with normalized counting measure on G.

2.5 Orbital integrals and their moment bounds

In this subsection, we fix notation for the orbital integrals of Part II and the moment bounds of Section 9.

Write $B_s = B^*\rho(s)B$, $s \in G$, for the Petersson blocks and H_R for the residue Bessel subgroup. Its representation r_ψ on X is extended by zero to G. In Section 9.1 write

$$E_s = B_s - r_\psi(s), \qquad D_{\mathrm{id}} = |H_R| d_X, \qquad D_{\mathrm{main}} = \frac{d_X^2}{\mathcal{B}\ \mathrm{mass}(E, r)}.$$

The product of the formal degrees is $\mathcal{B}$. The masses $\mu_j = \mathrm{mass}(E, j)$ of the standard definite genera are weighted by reciprocal automorphism orders. The stabilizer intersections $\Gamma_{\mathcal{A}}$ at the places in $\mathcal{A}$ use the normalizing bound $C_{\mathcal{A}} = b^{-6k|\mathcal{A}|} D_{\mathrm{main}}$. In Section 9.1, $\mathcal{L}(\Omega)$ sums the squared norms of the cut blocks on the nonzero-discriminant part of Γ_*, separately for each link

and over the two classes and $s \in \Omega$, before division by ν^2. For each factor G_i, $p_i = \rho(h_i)$ is the projection on the target and $q_i = 1 - h_i$ is the complementary idempotent in the group algebra.

For a finite set of places T in Section 10, write

$$A_T = \prod_{w \in T} E_w, \qquad \mathcal{R}_T = \prod_{w \in T} \mathcal{O}_{E_w}.$$

The finite calculations write $\mathcal{R} = \mathcal{R}_T$ locally. The additive modulus is md, and K_T, J are the local integral and congruence groups in Section 10.3. For the Bessel flag $F_i = \langle e_0, \dots, e_{i-1} \rangle$, the ideals $\mathfrak{d}_i = c_i \mathcal{R}_T$ compare the two positions modulo F_i; their inverse moduli are $D_i = \mathrm{md}(c_i)^{-1}$. Sections 10.8–10.10 specify the associated Fourier variables, denominator indices, and stabilizer measure. Section 14.2 uses the trace of r_ψ on each finite factor G_i.

In the real calculations, $h(v, y) = v^* J y$ in the chosen Gram basis, and norm-fiber measure uses $h(y, y)/2$. The support and determinant variables of the real correlations are specified in Sections 11.2 and 11.4. Section 12.1 writes W for the full Bessel vector, F for its truncation, and $P(A, B)$ for their quotient correlation. Here d denotes the local formal degree. Section 12.10 specifies the real discrete series, the compact type indices I_v, and the local stationary vectors. The dimension calculation uses $d_B = \prod_v |I_v|$ and $d_X = d_C d_B$, with d_C the source class count.

Section 14.1 defines the rational summation set $\mathcal{T}(c, s, c')$ for the fixed finite classes c, c' and residue element s in the Petersson blocks of Section 8.5. The real and finite common-entry pairings are formulated in Section 13.4. The pair lattice, its discriminant, and the parameters of the Schur bound are specified in Sections 14.7 and 15.7. One set $t \subseteq G$ of translates is chosen in Section 16.2. Section 3.2 denotes the dimension of $\sum_{g \in t} P_g(S)$ by n. In the orbital integrals $n = \varrho + 1$ counts the columns of $(e_0, \dots, e_{\varrho-1}, z)$.

2.6 Theta operators, derivatives, and integral estimates

In this subsection, we use the Weil splitting of Section 4.3 and the measured normalization of Section 17.3.

The modulus $Q(\xi)$ of a tame character of the norm-one torus is a finite set of representatives of split places of E/F_0, with

$$Q(\xi) \cap \overline{Q(\xi)} = \varnothing, \qquad q_\xi = \prod_{w \in Q(\xi)} q_w, \qquad (q_\xi, MN) = 1.$$

The character has weight zero. Its finite unit restrictions factor through $k_w^\times$ at $w \in Q(\xi)$ and are trivial elsewhere on the torus. Write ξ_w for these residue characters. A single-arrow twist has $\xi_w \neq 1$ at every $w \in Q(\xi)$. A twist for the full chain satisfies

$$\xi_w^j \neq 1 \qquad (w \in Q(\xi),\ 1 \le j \le m).$$

This includes the trivial twist with empty modulus. At a coefficient prime ℓ the allowed chain twists satisfy $\ell \nmid q_\xi$. Proposition 28.2 supplies such a twist whose inverse Galois character is trivial on the decomposition groups above ℓ. The successive spaces in the lower chain are Y_j; its endpoint Y_m and the dimension-comparison space W_m of Section 9.4 have different ranks. Write K_j° for the standard compact, $J_{\xi,j}$ for the level at the old and twist places, and

$$A_{\xi,j} = [K_j^\circ : J_{\xi,j}].$$

The map $L_{\xi,j}$ is the theta integral at step j multiplied by this index. The paired tensors $F_i(\xi, c)$, the evaluated tensor T_ξ of (71), and the scalar c_{meas} appear in Section 19.1. The same symbol T_ξ denotes the weighted chain L_ξ in Section 44.1.

In Section 18.1, I_D and $\mathcal{C}_D$ are the positive-degree maps and the quotient by nonnegative-degree maps for the specified grading D. The Heisenberg translation is $\rho(w)$, the theta functional is θ, and $C_D(f; w)$ is its coefficient at rational frequency w against the positive-degree group. The same letter θ denotes Fontaine's map in the appendix. The integer ring of finite adeles is $R_f = \widehat{\mathcal{O}}_E$ in Section 19.2, where J_N is the idempotent supported at the places dividing π_N. That section and Section 19.4 specify the shifted lattice and the finite translations in the measured tensor.

The archimedean normalization and the weighted prime remainder are

$$\Omega = \frac{2\pi}{\mathrm{rd}(E)} \exp\!\left(\gamma_{\mathrm{Euler}} - \frac{L'}{L}(1, \chi_K)\right),$$
$$A = mk \sum_{\tau} \big(\log(\kappa_\tau!) + \kappa_\tau \log(M/\Omega) - S(\kappa_\tau, \tau)\big),$$
$$\mathcal{R} = av\big(R_{d,q_0}(x) - R_{d,q_0}(x_1)\big) + auR_{d,1}(x_1).$$

The weighted prime sum $S(Y, \tau)$ in Section 5 and its local components $S_p(Y, \tau)$ in Section 7 are

$$S(Y, \tau) = \sum_{n \le a} \Lambda(n)(1 - \chi_K(n))(Y/n - \kappa_{\langle -n\rangle\tau})_+,$$
$$S_p(Y, \tau) = \tfrac{1}{2} \sum_{j \ge 1} (1 - \chi_K(p^j))(Y/p^j - \kappa_{\langle -p^j\rangle\tau})_+.$$

The latter is used at $Y = a, \kappa_\tau$. The exponents and the rational rescaling of the images are

$$g_p = m \left(\sum_{\tau \in \Phi_a} S_p(a, \tau) + k \sum_{\tau} S_p(\kappa_\tau, \tau) \right),$$
$$c_B = \prod_{p \in F} p^{-\lfloor g_p - e_p \rfloor}.$$

Here F consists of primes below x_1 that are nonsplit in K, together with N. The local denominator bounds are

$$e_p = \begin{cases} (2faB^2/p)c_p, & p < x_1 \text{ nonsplit in } K, \\ 2mfb, & p = N, \\ 0, & \text{otherwise}, \end{cases} \qquad c_p = \begin{cases} m, & p \equiv \pm 1 \bmod m, \\ 1, & \text{otherwise}. \end{cases}$$

For the upper highest weights use $\Delta = 2r^8$, $w_j(\tau) = \kappa_\tau + (j + \frac{1}{2})\Delta$, and $a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ in Section 20.2. Section 7.4 uses $J(l) = \log(l!) + l \log(M/\Omega)$ for the contribution of a part of length l.

For Taylor coefficients we use Θ_f for the theta section of a Schwartz datum f, $Q(\partial)$ for the constant-coefficient differential operator of a polynomial Q, and the integral model and translation scales of Section 20.1. The scalars s_{vac} and $s_{\mathrm{jet}} = p_F a_d/\omega_0$ normalize the vacuum and the Taylor coordinate, respectively. Here $a_d = \sqrt{2\pi/\sqrt{d}}$, $\omega_0\mathcal{O}_K$ is the period lattice of the marked elliptic curve, and p_F is the algebraic factor in Lemma 7.6. The normalized jets enter Section 44.4. At an exceptional prime the operator $Y_\lambda = p^{A_\lambda} T_{\lambda, r_p}$ uses $r_p = \lceil \log_p m \rceil$. The exponents A_λ and the Frobenius embeddings ι_s are those of Section 21. The root defects, coefficient spaces, and colength are specified in Section 21.

2.7 Covolumes, determinants, and coefficient places

In this subsection, we use the comparison of Section 5 and the product formula of Section 23.

With $J_* = faB^3/(mk)$ the bound for the sums over coefficient primes in Part V, set

$$E_0 = faB^3, \qquad L = faB^4, \qquad I_0 = 10\log(1+2b^2)faB^3.$$

A coefficient place is $\kappa = (\ell, \iota)$, with $\iota : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$ and $\bar{\kappa} = (\ell, \iota \circ \mathrm{conj})$. We use $|\ell|_\kappa = \ell^{-1}$. In the covolume and height bounds, σ ranges over $\mathrm{Aut}(\mathbb{C})$ and acts on the coordinates. For a finite coefficient DVR O, its uniformizer valuation is v_O; normalized length is $\mathrm{length}_O / v_O(\ell)$. For a tuple y of n independent images,

$$\mathrm{Cov}(y) \le c \quad \Longleftrightarrow \quad \det \mathrm{Gram}(\sigma\Theta_1 y) \le e^c \quad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

The bound $\mathrm{Ind}(y) \le \delta$ means there are uniform bounds $a_\ell \ge 0$ on the determinant valuations of the coefficient-place clearing matrices U_κ, with $\sum_{\ell \in U} a_\ell \log \ell \le \delta$ for every finite prime set U. At a given κ the same matrix clears the tuple for all allowed twists at ℓ. The twist ξ_κ is chosen in Section 38. For a completed basis $w = (y_1, \dots, y_n, z_1, \dots, z_{D-n})$,

$$G_\xi = \det \mathrm{Gram}(\Theta_\xi w), \qquad Z_\xi = \det(\Theta_\xi^* \Theta_\xi), \qquad G_\xi = Z_\xi \det \mathrm{Gram}(w).$$

After the untwisted lift, the auxiliary vectors z_i have squared norms at most $\exp(b_{\mathrm{short}})$ at every complex embedding, with $b_{\mathrm{short}} = 2x_1^2 b^2$. The scalar γ_{ξ_κ} clears them at both κ and $\bar{\kappa}$. The logarithmic height bounds on Z_1/Z_{ξ_κ} and $(\gamma_{\xi_\kappa}\overline{\gamma_{\xi_\kappa}})^{-1}$ are j_Z and j_γ, respectively. The height convention is that of Section 23, and the basis completion is in Section 44.3.

2.8 Cocycles, specializations, and Galois congruences

In this subsection, we record the notation for the normalized lengths of Section 24 and the trace congruences of Section 32.

Write G_E for the absolute Galois group, χ_{cyc} for the cyclotomic character, and $L^\dagger = \mathrm{Hom}_O(L, O)(1)$ for the Tate dual of a coefficient lattice L. We use H^1_f for the crystalline local condition and $Z_f(L)$ for the integral cocycles mapping to it. The local class of lattices $\mathcal{Q}_v$ and cup-product colength $\Delta_v(L)$ are those of Section 29.1. At depth M, $\mathcal{B} = \mathcal{O}/\ell^M\mathcal{O}$ is the coefficient quotient, $\lambda(C)$ is the normalized length of a cocycle module, and ϕ_j, γ_j are the auxiliary Frobenius and inertia evaluations. The slopes s_j impose $z(\gamma_j) = s_j z(\phi_j)$ on the submodule C_0. The auxiliary character order divides ℓ^N. In Section 25, R_X is the polynomial matrix of the twisted action and $A(X)$ records the cocycle, inertia, and other linear equations. The length is computed from the maximal minors $P_i(X)$ of

$$A^+(X) = \begin{pmatrix} A(X) \\ \ell^M I_q \end{pmatrix},$$

where q is the column count of $A(X)$ in that section. The evaluations $X_w = \zeta_w^j - 1$ use $\zeta_w^{\ell^H} = 1$ as in Section 26. The bound ℓ^H is fixed before the auxiliary level ℓ^N in Section 28.5.

In Section 27, Θ_Q is a group-ring element at auxiliary set Q, $q_w = |k_w|$, and $\mathcal{E}$ is the fixed list of exponents, and the Euler factor is the Laurent polynomial $E_w(A) = \prod_{e \in \mathcal{E}} (1 - Aq_w^{-e})(1 - A^{-1}q_w^{-e})$. The sign and contragredient conventions for the attached representation ρ are in

Section 28.2. The trace of conjugation is normalized using the polarization exponent d_{pol} of Section 28.4.

The ordered Kummer construction uses $F_\nu = E(\mu_{\ell^\nu})$, $G_\nu = \ker(\chi_{\mathrm{cyc}} \bmod \ell^\nu)$, and $A_\nu = \mathbf{Z}/\ell^\nu\mathbf{Z}$. The original evaluation set in the cyclotomic kernel is P_0; the new primes $w_j, \bar{w}_j$ and the maps ψ_j are specified in Sections 29.3 and 29.4. The matrices $(u_x(g)-1)_x$ define the relative relations on cocycles in Section 29.1. The trace algebra, its two-sided congruence ideal $\mathfrak{a}$, and the ordered-character specialization are in Sections 30.3 and 32. The operators $\mathcal{N}_j, \partial_j$ at the raised places and the highest squarefree coefficient refer to the original test space of Section 32.3.

2.9 Period comparisons, critical values, and descent

In this subsection, we record the spaces and operators for the period comparison of Section 36 and the lattice completion of Section 44.3.

The two comparison factors of Part V use the original descent pair W_j, V_j of Section 33.2, with

$$V_j = V'' \perp V^\circ, \qquad d_+ = r + 9m + 1, \qquad d_- = t - 9m - 1.$$

Here d_+, d_- are the ranks of V'', V°. The choices of classes and the tuple space $\mathcal{H}$ used by the gluing operator are defined in Section 34.1. The gluing operator is G_η at twist η, and n_A, h_- are the class and coefficient constants in $U_\eta = n_A h_-^2 G_\eta G_\eta^*$. On the good source the lift is u_η, its image is Z_η, and the Petersson projection is Π_η. This subspace notation Z_η is local to Section 42.1.

For the comparison of two compositions, U_1, U_2 are the source and target Hermitian spaces, A, A' are the intermediate spaces, and $W = U_1 \perp U_2$. The two compositions with the chosen point arrays X, Y are $T_X = D_{2,X_2} L_{1,X_1}$ and $T'_Y = L'_{2,Y_2} D'_{1,Y_1}$ in Section 34.3. The real Fock variables and finite self-dual test are specified in Sections 35.1 and 35.2. The signatures, infinity weights μ_τ, and the character $\gamma = \nu^{-1}\upsilon^{N/2}$ for exchanging two periods, with N the local rank in that formula, are fixed in Section 36.1.

A critical pair $p = (\nu, z)$ consists of an automorphic character ν of the norm-one torus of E/F_0 and an integer $1 \le z \le t$. Its weight at $v \mid \infty$ is $-c_v$, with $z \le c_v \le 4b$. At each chosen half $w \in S_M = \{w : w \mid \pi_M\}$ of an old split place, the character on local units factors through a character ψ_w of $k_w^\times$. It is unramified at the other finite places of the norm-one torus. The pair is called ramified if $\psi_w \neq 1$ for every $w \in S_M$. The conjugate halves are included on passing to the base-change character. Its auxiliary twist is η of exact support Q, with $q_Q = \mathrm{Nm}(Q)$. In Section 38, $R_{p,\epsilon}(\eta)$, $\epsilon \in \{1, -1\}$, compares the absolute inverse critical value for $\nu\eta^\epsilon$ with the untwisted reference value. The superscript bc indicates base change, and a superscript on L giving a set of places means that the corresponding factors are omitted. The Taylor map $\mathcal{T}$, coefficient product B_τ, and four-block vector ϕ_η are those of Sections 37.1 and 37.2.

In the common period, K_ζ° is the class sum of the complementary Weyl kernels paired with the tuple ζ; the coefficient identities (CP1)–(CP6) are in Section 41. At deep principal level put $\mathfrak{b} = (\pi_M^{\lfloor b \rfloor})$ and $R_b = \mathcal{O}_E/\mathfrak{b}$. The spaces $\mathcal{H}_j$, base identifications j_i, and arrow operators are specified in Section 43. Section 44.1 gives the chain relations (GC1)–(GC3). Proposition 44.3 gives the vectors for completing the lattice.

2.10 Local period rings and filtration orders

In this subsection, we fix the notation for the period rings of Appendix A and the filtration orders of Appendix B. These enter the local bounds of Section 29.1.

In the appendices p denotes the coefficient residue characteristic, $S = W(k)[[z]]$ is the chart ring, θ is Fontaine's map, and φ is Witt Frobenius. The chart symbol E denotes a generator of $\ker\theta$, with Frobenius translates $E_a = \varphi^a(E)$ and tame model $E = p - z^e$. The Witt ring is $A = A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p^\flat})$, with chart ideal $I = (p, z)A$ and retraction $F_0 : A \to S$. The bounded period ring is $B^+ = A_{\max}[1/p]$, where $A_{\max}$ is the p-adic completion of $A[E/p]$; S_B is the closure of $S[1/p]$ in B^+ for the topology with neighborhoods $p^n A_{\max}$ of zero. The de Rham filtration is $\mathrm{Fil}^j = t^j B^+_{\mathrm{dR}}$. For a compatible system $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ of primitive roots, put $\mu = [\epsilon] - 1$, $\tau = \varphi^{-1}(\mu)$, and $\xi = \mu/\tau$. Then $t = \log(1 + \mu)$, and $\Delta_z = z\tau$ is the divisor in Sections B.2 and C.1. The modules M_T, N_T over A_{inf}, S use the standard orders of Section B.3.

The residue indices i, j and their Frobenius translates belong to the chart idempotents in Section B.3. The projection ideals $I_l(i)$ and $J_r(j; R)$ record lifts of constants at the indicated orders and moduli in Sections B.3 and D.2. The bounds for the separations of Hodge–Tate weights and the termination indices are set out in Sections D.4, E.1, and E.3.

3 Outlines of proofs

In this section, we trace the proof of Theorem 1.1 through Parts I–VI. Section 5 reduces the prime-remainder bound to two estimates on the same translated basis. The two estimates yield Proposition 5.3, from which Section 47 deduces the bound required by the analytic criterion of Section 46.

3.1 Part I. Parameters and the common family

In this subsection, we explain how Part I uses the arithmetic datum of Section 4 to express the prime remainder in the lattice comparison of Section 5.

Fix a prime d in the progression of Section 2.1 and $0 < \epsilon < 1/100$. The comparison applies at every sufficiently large integer x for these fixed choices. Let $a = xb$ and $B = \log b$, as in Section 2.2. The weights are a/x on T and a/x_1 outside T. The set Φ_a is the fibre of 1 mod q_0, and T is a subset of the fibre of -1 mod q_0. Since q_0 and q_1 have the same prime divisors, multiplication by $-n$ sends the first fibre bijectively to the second whenever $n \equiv 1 \bmod q_0$. For other units the image misses T; for nonunits Section 5 uses the identity multiplier. Thus, with $u = |\Phi_a|$ and $v = |T|$, the weighted sum gives

$$\frac{\mathcal{R}}{av} = R_{d,q_0}(x) - R_{d,q_0}(x_1) + \frac{u}{v} R_{d,1}(x_1), \qquad \frac{u}{v} \le B, \qquad f = \varphi(q_0)u.$$

Section 5 also defines the common real normalization $A \ge 0$ and the errors L, I_0. The term $-m\mathcal{R}$ will appear in the upper covolume bound.

The prescribed levels give $S \subset X$ by Section 6. Section 8 constructs Π on X with the prescribed real point, finite level, and quotient Haar measures. The finite-type idempotent e_0 of Section 7.3 gives $p = \rho(e_0)$ and $V = \operatorname{im} p$, with $D = \dim V$. These data specify the source, target and terminal lifts in Propositions 5.1 and 5.2. The simultaneous construction at large x is checked in Section 16.1.

3.2 Part II. Orbital estimates and the translated span

In this subsection, we explain how Part II uses the Petersson blocks of Proposition 8.1 to choose translates of S in Section 16.2.

Section 9.1 compares the full Petersson blocks with the truncated orbital integrals. The finite and real correlation bounds are in Sections 10 and 11. Section 12.8 bounds the length at the same stationary point used in the measured period. Section 13 combines the local bounds with the definite masses. The truncation errors and the remaining pair sums are controlled in Sections 14 and 15. The dimension comparison (3) supplies the normalization in the mixed moment.

Write B_{Pet} for the Petersson-normalized map denoted B in Section 9, to distinguish it from $\log b$. On the Euclidean class coordinates $C(X)$, the moment bounds are

$$\|pB_{\mathrm{Pet}}P_U\|_{\mathrm{HS}}^2 \geq \dim U - 5b^{-4}d_X, \qquad D\,\mathbb{E}_g\|B_{\mathrm{Pet}}^*\rho(g)B_{\mathrm{Pet}}\|_{\mathrm{HS}}^2 \leq (1+9b^{-4})d_X^2, \qquad U \subseteq C(X).$$

Propositions 9.2 and 9.3 give these bounds, respectively. In Section 16.2, the mixed moment gives a translate whose projection to the span already selected has small Hilbert–Schmidt norm. The diagonal bound then controls the dimension of its preimage in $C(X)$. Iteration decreases the codimension. Restriction to $C(S)$ loses at most $|t|\dim(X/S)$ dimensions for a set t of translates.

The comparison of the two normalizations in Section 16.2 gives

$$B_{\mathrm{Pet}} = (v_G\nu)^{-1/2}\Pi C^{-1}, \qquad P_g = c_B|G|c_{\mathrm{ss}}^{-1}(v_G\nu)^{1/2}p\rho(g)B_{\mathrm{Pet}}C|_S.$$

Here v_G is the level volume, ν is the unipotent covolume, and $c_{\mathrm{ss}} \neq 0$ is the scalar chosen in Section 19.3 when the measured identity of Proposition 19.1 is put in integral form. The rational factor c_B is given in Section 3.3. The scalar in the second identity is nonzero. After restriction to $C(S)$, the span chosen with B_{Pet} is the span of the same $P_g(S)$ used for the integral estimates. More precisely,

$$|t|\dim S \leq 5D\log(1+2b^2), \qquad n = \dim\sum_{g\in t}P_g(S) \geq (1-4/b^2)D.$$

3.3 Part III. Theta identities and the upper covolume bound

In this subsection, we explain how Part III uses the measured seesaw of Section 17.3 to bound the norms and finite index on the span of Section 16.2.

Sections 18.1 and 18.3 unfold the consecutive lifts on the specified levels. The measured finite sum of Section 19.1 gives the tensor, and Section 19.3 gives its integral expansion. Proposition 19.1 uses the same Π and fixed real point from Section 8.3. The comparison at N, the ordinary derivatives, and the exceptional-prime Hecke operators are in Sections 19.4, 20, and 21, respectively.

Section 7 groups the prime sum by prime powers. With its exponents g_p, e_p, the scalar of Proposition 7.10 is

$$c_B = \prod_{p\in F} p^{-\lfloor g_p - e_p\rfloor}.$$

This scalar makes the cleared images integral for all permitted twists and cancels the corresponding valuation contributions in the conjugate norm bound. Proposition 22.1 applies it to the same translated maps as Proposition 16.2.

Choose a basis of the selected span from the generators of Proposition 5.1:

$$y = (P_{g_i}\phi_i)_{i=1}^{n}, \qquad g_i \in t, \quad \phi_i \in \mathcal{S}.$$

The untwisted norm bound and Hadamard's inequality give the covolume bound at every complex embedding. For the index, the clearing matrices are combined by translate. Using $d_X \le 2\dim S$ and the bound on $|t|\dim S$, we obtain

$$\mathrm{Cov}(y) \le n(A - m\mathcal{R} + L), \qquad \mathrm{Ind}(y) \le DI_0.$$

In the index bound the matrix at a coefficient place is chosen before selecting the twist, as required in Section 23. The basis y is fixed before any coefficient place or twist is chosen.

3.4 Part IV. Selmer bounds and trace congruences

In this subsection, we explain how Part IV gives the Selmer and norm conditions for the common twist of Section 38 and the trace comparison used there.

The original Hecke eigensystems give the Galois representations in Section 28.2, with local lengths estimated in Section 28.4. Lemma 24.1 bounds the increase in normalized length when the auxiliary local conditions are removed. Let P_{j_0} be a minor attaining the minimum in Lemma 25.1 at a test point. At any new integral specialization x, the remaining normalized length satisfies

$$\lambda(C(x)) = \min_j v(P_j(x)) \le v(P_{j_0}(x)).$$

For a coefficient place $\kappa = (\ell, \iota)$, Section 28.5 fixes the original Selmer data and norm systems, their truncation depths, and the bound on the number of auxiliary primes before choosing the auxiliary exponent. Lemma 26.1 gives an exponent that works for the entire list of polynomial tests and their prescribed powers. The auxiliary primes and the polynomials are then chosen. One torsion tuple $(\zeta_w)_w$ controls the valuations of every test at $(\zeta_w^j - 1)_w$ for all prescribed powers j. Section 28.5 shows that these powers have the same exact auxiliary support, and Section 38 applies the construction away from ℓMN to obtain ξ_κ. The line and rank-two norm formulas in Sections 30.1 and 30.2 apply to the signed systems at the respective ranks.

For the trace estimate, Section 29.1 uses Appendices C and E on the original local class and its Tate-dual pairings. For the selected twist, Section 29.4 fixes the ordered list of Kummer tests before the trace congruence depth ν. The original Hecke systems determine a sufficient ν in Section 32.3. At that depth one chooses the primes at which to raise the level and forms the trace algebra. These primes are additional to the support Q of the twist. The scalar-extension argument of Section 38 carries the normalized Selmer length bound to the common coefficient DVR used for the raised systems. Section 32.2 constructs one trace multiplier for the raised systems. Its highest squarefree coefficient gives the clearing operator on integral tests orthogonal to every false-lift space F_w, as in Proposition 32.1.

3.5 Part V. Periods and the lower determinant estimate

In this subsection, we explain how Part V bounds the determinant ratio and completes the fixed basis y for the product formula of Section 23.

Section 34.1 defines the operator on the original class spaces. Sections 36 and 39.4 match its periods, including their measure factors. Sections 40 and 41 establish the reproduction and injectivity used to compress the operator in Section 42.1. With the arithmetic estimates of Part IV, Section 38 uses the same ξ_κ for the necessary norm and Selmer tests at each coefficient place.

The first-arrow Gram identity of Section 34.2 starts the injectivity argument of Section 44.1. The square identities (GC1)–(GC2) prove injectivity before division by the real period factors. Proposition 44.1 computes the chain ratio from the single-lift ratios and the triangle factors. Proposition 44.2 bounds its height. Using the deep periods of Section 43, Proposition 44.3 supplies a spanning set from which to choose the $s = D - n$ additional vectors. In the notation of Section 2.7,

$$j_Z \le D(b + L - A), \qquad j_\gamma \le b, \qquad b_{\mathrm{short}} = 2x_1^2 b^2, \qquad s \le 4D/b^2.$$

Here j_Z bounds the height of Z_1/Z_{ξ_κ}, j_γ bounds the height of $(\gamma_{\xi_\kappa}\overline{\gamma_{\xi_\kappa}})^{-1}$, and b_{short} bounds the logarithmic squared norms of the added vectors after the untwisted lift.

Let w be the completed basis. The matrices at κ and $\bar\kappa$ clear the original y in the same lift Θ_{ξ_κ}; the additional vectors use the common scalar γ_{ξ_κ}. The two integral families give the mixed Gram bound for G_{ξ_κ} in Section 23. Since $G_\xi = Z_\xi \det \mathrm{Gram}(w)$, the ratio Z_1/Z_{ξ_κ} transfers the bound to G_1. At infinity, exterior-power Cauchy–Schwarz bounds $|\sigma G_1|$ by the product of the exterior norms of the untwisted families at σ and $\mathrm{conj} \circ \sigma \circ \mathrm{conj}$. In each family project the added vectors orthogonally off the span of the first n vectors and apply Hadamard's inequality to the projected vectors. If $\mathrm{Cov}(y) \le c$, this gives $|\sigma G_1| \le e^{c + s b_{\mathrm{short}}}$. With $\mathrm{Ind}(y) \le \delta$, the product formula gives

$$0 \le c + 2\delta + s(b_{\mathrm{short}} + j_\gamma) + j_Z.$$

Section 45 uses $s(b_{\mathrm{short}} + j_\gamma) \le Db$ and $b \le L$ to obtain $D(A - 3L) \le c + 2\delta$. Substituting the bounds of Section 3.3 and using $n \le D$ and $A + L \ge 0$ gives

$$nm\mathcal{R} \le (n - D)(A + L) + D(4L + 2I_0) \le D(4L + 2I_0).$$

For large x, $n \ge D/2$, $I_0 \le 30L$, and $B \ge 128$. This is the cancellation in Proposition 5.3, giving $m\mathcal{R} \le 8L + 4I_0 \le faB^5$.

3.6 Part VI. The prime remainder and the analytic conclusion

In this subsection, we explain how Part VI uses Proposition 5.3 to obtain the one-sided bound in Section 47 and apply Section 46 to Theorem 1.1.

Substitute $m\mathcal{R} \le faB^5$ into the identity of Section 5. From $u/v \le B$ and $f = \varphi(q_0)u$,

$$R_{d,q_0}(x) \le \frac{\varphi(q_0)B^6}{m} + |R_{d,q_0}(x_1)| + B|R_{d,1}(x_1)|.$$

The proof of Proposition 47.1 bounds both remainders at x_1 by the fixed-modulus prime number theorem. The bounds $m \ge \sqrt{x}$ and $B \ll x^\epsilon$ give $R_{d,q_0}(x) \le C_{d,\epsilon} x^{-1/2+6\epsilon}$ at sufficiently large integers. Integrating the $O_{d,q_0}(x^{-1})$ derivative bound between consecutive integers extends this estimate to real x.

Now take a proposed zero ρ with $\Re\rho > 1/2$, and choose $1/2 < \beta < \min(1, \Re\rho)$ and $H_0 > |\Im\rho|$. Proposition 46.2 uses a truncated Mellin integral of the Möbius-weighted theta series to select

a prime d. All prescribed quadratic twists are zero-free on $\beta \le \Re s \le 1$, $|\Im s| \le H_0$. Fix this d, choose $6\epsilon < \beta - 1/2$, and then let x grow in the estimate above. Absorbing the constant gives

$$h(x) = x^{\beta-1} - R_{d,q_0}(x) \ge 0$$

for all sufficiently large real x. The real-abscissa argument of Lemma 46.1 shows that the Mellin integral of h over that tail converges on $\Re s > \beta$. An untwisted zero in the rectangle with $\Re s > \beta$ would give a nonzero residue there. The principal Dirichlet factor contains $\zeta(s)$, with Euler factors nonvanishing in this region, so ρ is excluded. Section 47 applies the functional equation to complete the proof of Theorem 1.1.

Part I
Arithmetic parameters and automorphic spaces

4 Arithmetic parameters

In this section, we fix the arithmetic parameters before choosing any coefficient place. The selected embeddings separate the two weights in Section 5, where their multiplicities determine the two prime sums.

Fix $q_0 = 18480$, a prime $d \equiv 9239 \bmod q_0$, and $0 < \epsilon < 1/100$. The scale x is an integer. For m the least power of 2 at least $\sqrt{x}$, r the least power of 3 at least x^5, put

$$s = 22\log_2 m, \quad Q = 2^s, \quad L_0 = 3^{\lfloor \log_3 s \rfloor + 4}, \quad q_1 = 3L_0Qq_0, \quad R_{\mathrm{prog}} = (r+1-m)5r.$$

Let $E = K(\zeta_{q_1})$, where the quadratic field $K = \mathbb{Q}(\sqrt{-d})$ ramifies at d, whereas the cyclotomic field of conductor q_1 is unramified there. Thus K intersects it in $\mathbb{Q}$. By fixing one embedding of K, the infinite places of the compositum are indexed by $(\mathbb{Z}/q_1)^\times$. The reduction map to $(\mathbb{Z}/q_0)^\times$ has kernel of order $3L_0Q$.

Write $W = x^{\epsilon/6}$ and choose a fixed integer $n > 66/\epsilon$. For large x,

$$W \le Q \le W^n, \quad (2L_0)^n \le W, \quad 3L_0 n 2^n \le W^3, \quad 2L_0(R_{\mathrm{prog}}+1) < Q.$$

The last inequality follows from $R_{\mathrm{prog}} = O(x^{10})$, $L_0 = O(\log x)$ and $Q \ge x^{11}$. Take the largest power of 2, denoted H, with $H^n \le Q/W$. Then $H \le W$, $Q < 2^n H^n W$, and $(2L_0H)^n \le Q$. In the cube $\{0,\dots,H-1\}^n$ a sphere of constant squared norm contains at least $H^n/(nH^2)$ points, since there are at most nH^2 norms. Read these points as digits in base $2L_0H$, giving a set $S \subseteq [0,Q)$. If $(a+b)e_2 \equiv ae_1 + be_3 \bmod Q$ for $e_j \in S$, $a,b>0$, and $a+b \le 2L_0$, both sides have digits strictly below the base and lie below Q. They are equal digitwise. Strict convexity of the squared norm on a line segment forces all three vectors to coincide.

Put $g = 1 + 3L_0q_0$ and $v = v_2(q_0) \ge 2$. The prime divisors of q_1 are those of q_0, so g is a unit modulo q_1. For every positive h, $v_2(g^h - 1) = v + v_2(h)$, by factoring for odd h and repeated difference of squares for even h. Thus g has order Q modulo 2^{s+v}. The places $T = \{-g^e \bmod q_1 : e \in S\}$ are distinct and have residue $-1 \bmod q_0$. We prove that for any $\tau_1,\tau_2,\tau_3 \in T$, any prime $\ell \le R_{\mathrm{prog}}$ and $0 < i < j < L_0$, the two congruences

$$\tau_2/\tau_1 \equiv \pm\ell^i, \qquad \tau_3/\tau_1 \equiv \pm\ell^j \pmod{(q_1)_{\ell'}}$$

cannot both hold, where $(q_1)_{\ell'}$ is the prime-to-ℓ part of q_1. Suppose both hold and write $\tau_k = -g^{e_k}$ with $e_k \in S$.

For $\ell = 2$, squaring the first ratio gives $4^i = 1 \bmod 3L_0$, since every selected place is -1 there. Factoring successive cubes shows the order of 4 modulo $3L_0$ is L_0, a contradiction. For odd ℓ, raise the two ratios to $2j, 2i$, respectively, and compare their exponents of g modulo Q. This gives

$$2je_2 \equiv 2(j-i)e_1 + 2ie_3 \pmod{Q}.$$

The equal-norm construction forces equality of the exponents. The first squared ratio then gives $\ell^{2i} = 1 \bmod 2^{s+1}$. Factoring this difference of powers gives

$$v_2(\ell^{2i} - 1) = v_2(\ell - 1) + v_2(\ell + 1) + v_2(2i) - 1.$$

Since $\min(v_2(\ell-1), v_2(\ell+1)) = 1$, we obtain

$$Q \le 2^{v_2(\ell^{2i}-1)} \le (\ell+1)2i \le (R_{\mathrm{prog}} + 1)2L_0 < Q.$$

This contradiction excludes the two ratios.

Finally $3L_0Q < 3L_0 n 2^n W H^2 |T| \le W^6|T|$. For $x_1 = \lfloor e^W \rfloor$, take b to be the least multiple of q_1x_1 strictly greater than e^{W^6}. As $\log(q_1x_1) = O(\log x) + O(W)$, one has $W^6 < \log b \le W^6 + 1$ eventually. Thus $u/v \le B$ for $u = 3L_0Q$, $v = |T|$ and $B = \log b$.

4.1 The marked primes and their generators

In this subsection, we use Siegel–Walfisz to choose two principal-level primes with a common ideal class and a third prime for the additional level condition in Section 6. The generators also fix the additive characters in Section 17.3.

Set $q_2 = (q_0 d)^{s+6}$ and $B = \log b$. The preceding bounds give

$$\sqrt{x} \le m < 2\sqrt{x}, \qquad x^5 \le r < 3x^5, \qquad s \le 32 \log x, \qquad W - 1 \le \log x_1 \le W.$$

In particular, eventually $x^{10^6} \le x_1$, $x_1^{10^6} \le b$, and $B^{100} \le x$. One has $(\log x_1)^6 \le B$ by the choice of b. The bound $B^{100} \le x$ uses $100\epsilon < 1$. Also $\log B \to \infty$, so any fixed lower bound on $\log B$ depending on q_0, d can be imposed in the same cutoff.

Proposition 4.1 (The three level primes)**.** *Fix d, ϵ. For every sufficiently large integer x, there are distinct primes p_0, p_1, N satisfying*

$$p_0 \equiv p_1 \equiv N \equiv 1 \pmod{q_2^2}, \qquad b^{10} \le p_i \le b^{11} \quad (i = 0, 1),$$
$$U \le \log N \le U + \sqrt{B}.$$

Here $M = p_0p_1$, the weights κ_τ are those of Section 2.3, and

$$U = m\, f^{-1} \sum_\tau \log \kappa_\tau + (m-1) \log\left(\frac{M \operatorname{rd}(E)}{2\pi}\right) - \tfrac{15}{2} B.$$

There are prime ideals $\mathfrak{p}_i \mid p_i$ of K and generators $\alpha \in \mathcal{O}_K$, $\pi_N \in \mathbb{Z}[i]$ with

$$(\alpha) = \mathfrak{p}_0\overline{\mathfrak{p}_1}, \qquad \alpha\bar{\alpha} = p_0p_1, \qquad \pi_N\bar{\pi}_N = N.$$

The chosen generator at $M = p_0p_1$ is $\pi_M = \bar{\alpha}$.

Proof. Since $\log(q_2^2) = O_{q_0,d}(\log x)$, there is a fixed A_{SW} with $q_2^2 \le W^{A_{\mathrm{SW}}}$. For any $U \ge W$, apply the Siegel–Walfisz theorem [52, Corollary 11.21] to the prime-weighted count $\vartheta(\,\cdot\,; q_2^2, 1)$ at $2e^U$ and e^U. It gives

$$\vartheta(2e^U; q_2^2, 1) - \vartheta(e^U; q_2^2, 1) = \frac{e^U}{\varphi(q_2^2)} + O_{A_{\mathrm{SW}}}\left(e^U e^{-c\sqrt{U}}\right).$$

Here $q_2^2 \le U^{A_{\mathrm{SW}}}$, and the relative error is $O_{A_{\mathrm{SW}}}(U^{A_{\mathrm{SW}}} e^{-c\sqrt{U}})$. This tends uniformly to zero for $U \ge W$ as $x \to \infty$. Thus there is a prime 1 mod q_2^2 with logarithm in $(U, U+\log 2] \subseteq [U, U+W]$.

Let h_K be the class number of K. For $i = 0, \dots, h_K$, apply this result in $[10B + 2iW, 10B + (2i+1)W]$. These intervals are disjoint and lie in $[10B, 11B]$ for large x, since $W/B \to 0$. The selected primes split in K because their residue modulo d is 1. Choose a prime ideal of K above each. Two distinct primes p_0, p_1 give ideals $\mathfrak{p}_0, \mathfrak{p}_1$ of the same class. In particular,

$$b^{10} \le p_i \le b^{11}, \qquad p_i \le b p_{1-i} \quad (i = 0, 1).$$

Since $\mathfrak{p}_1\overline{\mathfrak{p}_1} = (p_1)$, the ideal class of $\mathfrak{p}_0\overline{\mathfrak{p}_1}$ is $[\mathfrak{p}_0][\mathfrak{p}_1]^{-1} = 1$. Taking norms of this integral principal ideal gives $\alpha\bar{\alpha} = p_0 p_1 = M$ for any generator α. Use $\pi_M = \bar{\alpha}$ as the chosen generator, with $\bar{\pi}_M = \alpha$.

Give each place $\tau \in T$ weight $\kappa_\tau = b$ and each remaining infinite place weight $\kappa_\tau = xb/x_1$. These are integers in $[1, b]$. Let $f = [E : \mathbb{Q}]/2$, and set

$$U = m\, f^{-1} \sum_\tau \log \kappa_\tau + (m-1) \log\left(\frac{M\ \mathrm{rd}(E)}{2\pi}\right) - \tfrac{15}{2} B.$$

Here $\mathrm{rd}(E)$ is the absolute root discriminant. Since $K \subseteq \mathbb{Q}(\zeta_d)$, the cyclotomic discriminant formula and discriminant transitivity give $1 \le \mathrm{rd}(E) \le d q_1 \le b$ eventually. With $b^{20} \le M \le b^{22}$ this gives

$$19B \le \log(M\,\mathrm{rd}(E)/(2\pi)) \le 23B, \qquad mB \le U \le 24mB.$$

The second pair uses $m \ge 2$ and $0 \le f^{-1}\sum_\tau \log \kappa_\tau \le B$. Apply the interval of primes at this U to obtain $N \equiv 1 \bmod q_2^2$, with $\log N \in [U, U + \sqrt{B}]$, since $W \le \sqrt{B}$. For large x,

$$\log N \ge U \ge mB > 11B \ge \max(\log p_0, \log p_1).$$

As $N \equiv 1 \bmod 4$, it is a sum $a_0^2 + b_0^2$ of two integer squares. The prescribed norm generator is $a_0 + i b_0 \in \mathcal{O}_E$, where $i \in E$ because $4 \mid q_1$. The primes, their generators, and the weights are kept in the same datum, denoted $\mathcal{D}$. This construction is for every sufficiently large integer x with d, ϵ fixed. □

4.2 The different and the additive character scales

In this subsection, we choose the different generator that fixes the scales in Section 8.3.

For the additive character choose the marked skew generator δ_E of the different as follows. Since the two fields have coprime discriminants, $\mathcal{O}_E = \mathcal{O}_K[\zeta_{q_1}]$: at each rational prime, tensoring one local maximal order with the other unramified order gives a product of maximal orders. Put $F_0 = E^+$. At every rational prime, inertia restricts trivially to at least one of the two factors K and $\mathbb{Q}(\zeta_{q_1})$. Complex conjugation is nontrivial on both factors, so no finite inertia group contains it. Hence E/F_0 is unramified at every finite place.

The equality $f = [F_0 : \mathbb{Q}] = \varphi(q_1)$ follows from $[E : \mathbb{Q}] = 2\varphi(q_1)$, and f is even since $q_1 > 2$. The relative different over K is generated by $\Phi'_{q_1}(\zeta_{q_1})$, where Φ_{q_1} is the cyclotomic polynomial. In its product of root differences, write each other primitive root as $\zeta_{q_1}^{2a+1}$, since q_1 is even.

Multiplication of $\zeta_{q_1} - \zeta_{q_1}^{2a+1}$ by $i\zeta_{q_1}^{-a-1}$ makes the factor real. The product of these unit multiples, times $\sqrt{-d}$, is a skew generator of the absolute different by transitivity. All its allowed root-of-unity phases have the same archimedean moduli. Put $a = q_1/q_0$. Its prime divisors divide q_0. A root of unity has a-th power of order q_0 exactly when its order is q_1, by taking valuations in the order of that power. Consequently $\Phi_{q_1}(X) = \Phi_{q_0}(X^a)$, and differentiation gives

$$|\Phi_v(\delta_E)| = \sqrt{d}\, a\, |\Phi'_{q_0}(\Phi_v(\zeta_{q_1})^a)| \asymp_{d,q_0} q_1.$$

The derivative is nonzero at each of the finitely many primitive q_0-th roots. Since $x^{11} \le q_1 = O_{q_0}(x^{11} \log x)$, we have $x^{10} \le |\Phi_v(\delta_E)| \le x^{12} \le b$ at every infinite place for sufficiently large x. Since $\log b > x^\epsilon$ and $b^{20} \le M \le b^{22}$, the real character scales $\kappa_v = M\Phi_v(\delta_E)/(2\pi i)$ satisfy $1 \le |\kappa_v| \le b^{23}$ for large x. Here Φ_v denotes the embedding, as distinct from the cyclotomic polynomial. The interval of primes above also gives $N \le b^{25m}$.

4.3 The splitting and level characters

In this subsection, we construct the characters used in Sections 7.4 and 17.3, with their common finite and archimedean normalizations.

Put $F = F_0$, $q = q_1$, $\zeta = \zeta_{q_1}$, and $\widehat{O} = \widehat{\mathcal{O}}_E$.

The roots of unity in E are precisely μ_q. Indeed $E \subset \mathbf{Q}(\zeta_{dq})$ by the quadratic cyclotomic description of K of discriminant $-d$. If a root has order h, then $\varphi(\mathrm{lcm}(h, dq)) = \varphi(dq)$. Since $4 \mid dq$, any increase in this modulus strictly increases its totient, so $h \mid dq$. The ramification index at d in E is 2, since the cyclotomic field of conductor q is unramified there. As $d - 1 > 2$, E cannot contain $\mathbf{Q}(\zeta_d)$. Hence $h \mid q$. Now set $\eta = \zeta^{q/4}(\zeta - 1)$. The cyclotomic identities $\mathrm{N}_{E/K}(\zeta^{q/4}) = 1$ and $\mathrm{N}_{E/K}(\zeta - 1) = \Phi_q(1) = 1$ show that η is a unit of relative norm one. The second identity uses that q is divisible by both 2 and 3. Direct conjugation gives $\eta/\bar{\eta} = \zeta$. Thus every root ξ in E can be written $\xi = \delta/\bar{\delta}$ with $\delta \in \mathcal{O}_E^\times$ and $\prod_v \Phi_v(\delta) = 1$, taking a power of η.

At a finite place of residue characteristic different from 3, conjugation is nontrivial on the reduction of a primitive third root of unity. At a place above 3, use a primitive fourth root. Reduction is injective on roots of unity of order prime to the residue characteristic. Thus the inertia of the quadratic extension E/E^+ is trivial at every finite place. The norm on local units is surjective at each finite place. For an inert place, the residue-field norm is surjective and the map on the successive principal-unit quotients is the residue trace. Successive lifting and completeness therefore give the unit norm surjectivity, including residue characteristic two. For a split place use the norm $(x, y) \mapsto xy$.

Let $\mathcal{U} = E_\infty^\times \widehat{O}^\times$ and $\mathcal{R} = E^\times\{y\bar{y} : y \in \mathbb{A}_E^\times\}$. On $\mathcal{U}$, put $S(x) = \prod_v x_v/|x_v|$. For $x = ay\bar{y} \in \mathcal{U} \cap \mathcal{R}$, the finite valuations of a are invariant under conjugation, since the valuations of $y\bar{y}$ are. Thus $\bar{a}/a$ is a unit with all archimedean moduli one. By Kronecker's theorem it is a root of unity. Choose δ as above with $\delta/\bar{\delta} = \bar{a}/a$, so $A = a\delta \in F$. The finite part of $Ay\bar{y}$ is a fixed integral unit and is a norm at every finite place by unit norm surjectivity. Hence A itself is everywhere a finite local norm. Global Artin reciprocity for E/F [48, Ch. V, Theorem 5.3(a)] gives $\prod_{v|\infty} \mathrm{sgn}\, \Phi_v(A) = 1$. Since $\prod_v \Phi_v(\delta) = 1$ and $y\bar{y}$ is positive at infinity, it follows that $S(x) = 1$.

Accordingly S defines a character on $\mathcal{U}\mathcal{R}$ with value one on $\mathcal{R}$. The quotient of the idele group by $\mathcal{U}\mathcal{R}$ is finite, as it is a quotient of the ordinary ideal class group of E. Extend this character to the whole idele group one cyclic quotient at a time, choosing the required roots in the circle. Because $\mathcal{U}$ is open and S is continuous, the extended character υ is continuous. It is a unitary Hecke character, unramified at every finite place, with

$$\upsilon(\bar{x}) = \upsilon(x)^{-1}, \qquad \upsilon_\infty((z_v)_v) = \prod_v z_v/|z_v|, \qquad \upsilon|_{\mathbb{A}_F^\times} = \chi_{E/F}.$$

For the last identity, the restriction is trivial on $F^\times \mathrm{N}_{E/F}(\mathbb{A}_E^\times)$, which has index two by global reciprocity [48, Ch. V, Theorem 5.3(b)]. The sign at one real place is nontrivial. In particular the value of υ on the marked inert uniformizer is -1: one can compute it on a uniformizer of the fixed local field, and any other uniformizer differs by a unit.

The finite values of υ are algebraic. For a finite idele x, choose $h > 0$ such that the ideal of x^h has a generator $a \in E^\times$. Then $\upsilon(x)^h = \prod_v |\Phi_v(a)|/\Phi_v(a)$, since the remaining finite idele is a unit everywhere. The right side is algebraic. If a product of powers of υ and the idele norm has integral infinity exponents, its finite component takes values in one number field. Indeed its values on finite ideal-class representatives give finitely many algebraic generators, and the values on principal ideals are the corresponding monomials in the conjugates of their generators, in the normal closure of E. These facts give the coefficient fields for the algebraic twists of the cuspidal factors.

4.4 The two old-level characters on the same torus

In this subsection, we extend the level characters of the norm-one torus for the weights from Section 4. Their local components enter the theta identities in Section 17.3.

Let $T_1 = U(1)$ for E/F, let $T_{1,f}^0$ be its integral finite compact, and let S_M be the original places dividing π_M, one from each level pair. Each $w \in S_M$ has residue cardinality p_i for one of the two marked primes. Since $q_2^2 \mid p_i - 1$, choose a character $a_w : k_w^\times \to \mathbb{C}^\times$ of exact order q_2^2 and put

$$\omega_w = a_w^{q_2} \quad (w \in S_M), \qquad \omega_w = 1 \quad (w \notin S_M).$$

The residue characters here are read on the chosen member of each split pair. The value on its conjugate member of a norm-one idele is determined by the chosen coordinate. Each ω_w on S_M has exact order q_2, and kills all reductions of μ_{q_1} since $q_1 \mid q_2$.

For the infinite place v with index τ, put $w_0(v) = \kappa_\tau + r^8$, where κ_τ is the integer weight b or xb/x_1. The two values b and xb/x_1 of κ_τ are multiples of q_1 because $q_1 x_1 \mid b$. Also $q_1 \mid q_2$. For the primes different from 3 this follows from the powers of q_0 in q_2, and for 3 use $\lfloor \log_3 s \rfloor \le s$. On $T_1(F_\infty)T_{1,f}^0$ use the continuous characters

$$u \longmapsto \prod_v u_v^{-w_0(v)}, \qquad u \longmapsto \prod_{w \in S_M} \omega_w(\mathrm{red}_w\, u_w).$$

The rational points in this subgroup are exactly the integral norm-one units. They are μ_{q_1}: conjugation makes all complex absolute values one, and the root-of-unity calculation above applies. For $\gamma \in \mu_{q_1}$ the first formula has value

$$\prod_v \Phi_v(\gamma)^{-w_0(v)} = N_{E/K}(\gamma)^{-r^8} = 1.$$

Here the embeddings Φ_v are all those over the fixed embedding of K, and $N_{E/K}(\zeta) = \Phi_{q_1}(0) = 1$. The second formula is also one on these units by the local order calculation.

Both characters therefore extend trivially on $T_1(F)$ to the subgroup $T_1(F)T_1(F_\infty)T_{1,f}^0$ of $T_1(\mathbb{A}_F)$. Its quotient is finite. Indeed, Hilbert 90 writes any norm-one finite idele as $x/\bar{x}$ for an E-idele x. The preimage can be taken integral and a unit wherever the given idele is an integral unit: at a split pair take its chosen component and 1; at an unramified nonsplit place divide a local Hilbert 90 preimage by a fixed-field uniformizer to the power of its valuation. Thus the asserted quotient is an image of the ideal class group via $x \mapsto x/\bar{x}$.

Extending across each finite cyclic quotient by a root in the unit circle gives continuous global characters χ_0, χ_M, respectively. They have weight $-w_0, 0$ and tame parts $1, \omega$ on the original field and original compact. For any fixed weight and tame part, the possible extensions form a finite set by the same quotient calculation.

Their finite adelic values lie in a number field. On representatives for the finite quotient, a fixed power of each value is the product of a weight monomial on a rational norm-one element and finite roots of unity. This makes the finitely many representative values algebraic. On all other finite ideles, the additional factors are such monomials and roots of unity from the fixed tame characters. The normal closure of E and those finitely many algebraic generators therefore give one coefficient field.

Use $\upsilon_1 = \upsilon|_{T_1(\mathbb{A}_F)}$ and $\Delta = 2r^8$. For every j the fixed character

$$\mu_j^0 = \chi_0(\upsilon_1^{-\Delta}\chi_M)^j$$

has weight $-(\kappa_\tau + (2j+1)r^8)$ at the infinite place indexed by τ and tame part ω^j. For $1 \le |e| \le 8m < q_2$, ω_w^e is nontrivial exactly on S_M. This proves the required old support for the single lifts, both signs of the three exchanges, the shallow triangle quotients and the reverse-grid products. These choices are made after fixing the two level primes and their marked members, and before any additional modulus, coefficient embedding, or period test.

5 The lattice comparison on a common automorphic family

In this section, we translate the prime remainder of Section 46 into a statement about one family of lifts. Hadamard's inequality combines the estimates of Propositions 5.1 and 5.2.

Write $a = xb$, $B = \log b$, $L = faB^4$, and $I_0 = 10\log(1+2b^2)\,faB^3$. For the remainder $R_{d,q}$ of Section 2.1, put $v = |T|$, $u = 3L_0Q$, and

$$\mathcal{R} = av\big(R_{d,q_0}(x) - R_{d,q_0}(x_1)\big) + auR_{d,1}(x_1).$$

The preceding construction supplies $u/v \le B$ and $f = \varphi(q_0)u$. In fact $u = |\Phi_a|$ for $\Phi_a = \{\tau : \tau \equiv 1 \bmod q_0\}$.

Write $\chi_K(n) = (n/d)$, the quadratic character of K, and let Λ denote the von Mangoldt function. For a variable Y put

$$S(Y,\tau) = \sum_{n \le a} \Lambda(n)(1-\chi_K(n))(Y/n - \kappa_{\langle -n\rangle\tau})_+.$$

The parameter $\langle -n\rangle$ is the class of $-n$ when coprime to q_1 and the identity otherwise. When τ ranges over Φ_a, the product parameter is in T exactly v times if $n \equiv 1 \bmod q_0$, and never otherwise.

Indeed Φ_a is the fibre of 1 over q_0 and T lies in the fibre of -1. The moduli q_0, q_1 have the same prime divisors. If $n \equiv 1 \bmod q_0$, multiplication by $-n$ is a bijection between these two full fibres. Any other unit class has disjoint image from T, and a nonunit uses the prescribed identity multiplier. Thus

$$\#\{\tau \in \Phi_a : \langle -n\rangle\tau \in T\} = v\mathbf{1}_{n\equiv 1 \bmod q_0}.$$

The weight is a/x_1 off T and a/x on T. Put $w_z(n) = (1/n - 1/z)_+$. For every $1 \le n \le a$, the count above gives

$$\sum_{\tau\in\Phi_a} (a/n - \kappa_{\langle -n\rangle\tau})_+ = au\, w_{x_1}(n) + av\, \mathbf{1}_{n\equiv 1 \bmod q_0}\big(w_x(n) - w_{x_1}(n)\big).$$

Since $x, x_1 \le a$, multiplying by $\Lambda(n)(1 - \chi_K(n))$ and summing gives

$$\sum_{\tau\in\Phi_a} S(a,\tau) = \mathcal{R} + a\{u(\log x_1 + c_{d,1}) + v(\log x - \log x_1)/\varphi(q_0)\},$$

where $c_{d,1}$ is the subtraction constant of the modulus-one remainder. In the correction the constant of the modulus-q_0 remainder cancels. Since

$$\sum_{\tau} \log \kappa_\tau = f \log(a/x_1) - v \log(x/x_1),$$

the term in braces times a is the common expected value $ua(\log a - f^{-1}\sum_\tau \log\kappa_\tau + c_{d,1})$. This proves the identity relating the chain's total remainder to the two progression sums.

For the main amount let $k = r + 1 - m$, and set

$$\Omega = \frac{2\pi}{\mathrm{rd}(E)} \exp\!\left(\gamma_{\mathrm{Euler}} - \frac{L'}{L}(1,\chi_K)\right), \qquad A = mk \sum_\tau \big(\log(\kappa_\tau!) + \kappa_\tau \log(M/\Omega) - S(\kappa_\tau,\tau)\big).$$

It is nonnegative for large x. Indeed $S(\kappa_\tau,\tau) \le 2\kappa_\tau \sum_{n\le a} \Lambda(n)/n \le 5\kappa_\tau B$ eventually, by partial summation from the prime number theorem and $\log a \le 2B$. The fixed d and $\mathrm{rd}(E) \ge 1$ give $\log(M/\Omega) \ge 19B$ eventually, since $M \ge b^{20}$. Every summand is therefore nonnegative.

The spaces below are the ramified principal-level source S and the good principal-level target V of the construction. Write $D = \dim_{\mathbb{C}} V > 0$, as shown in Section 16.2. A finite group G acts on V, and the comparison uses translates P_g of one normalized Poincaré map $P : S \to V$. The family of lifts Θ_ξ and the integral lattices at each coefficient place are on this same V. The trivial twist has symbol 1.

For a tuple y in V, write

$$\mathrm{Cov}(y) \le c \quad\Longleftrightarrow\quad \det \mathrm{Gram}(\sigma\Theta_1 y) \le e^c \quad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Here σ acts on the coordinates, and the Gram matrix uses the rational class pairing.

Write $\mathrm{Ind}(y) \le \delta$ if there are $a_\ell \ge 0$, independent of the choice of embedding over ℓ, such that

$$\sum_{\ell\in U} a_\ell \log \ell \le \delta \quad \text{for every finite set of primes } U.$$

More precisely, at each $\kappa \mid \ell$ there is an $n \times n$ matrix U_κ over the coefficient valuation ring such that

$$v_\kappa(\det U_\kappa) \le a_\ell, \qquad (\Theta_\xi y)U_\kappa \text{ has integral columns at } \kappa \quad \text{for every allowed } \xi \text{ at } \ell.$$

Here $v_\kappa(\ell) = 1$, the entries of U_κ lie in $\mathcal{O}_\kappa = \{u \in \mathbb{C} : v_\kappa(u) \ge 0\}$, and the columns of $\Theta_\xi y$ are the images of the ordered tuple.

Proposition 5.1 (Short images and span for the chosen arithmetic data)**.** *Fix d, ϵ. For every sufficiently large integer x, there exists a set $\mathcal{S} \subset S$ of integral forms with*

$$\operatorname{span} \mathcal{S} = S, \qquad \|\sigma \Theta_1 P_g \phi\|^2 \le e^{A - m\mathcal{R} + L} \quad (\phi \in \mathcal{S},\ g \in G,\ \sigma \in \operatorname{Aut}(\mathbb{C})).$$

For each $t \subseteq G$ and every independent tuple y of images of everywhere integral forms under P_g, $g \in t$,

$$\operatorname{Ind}(y) \le 2|t| \dim(S)\, faB^3.$$

Finally, one can choose $t \subseteq G$ such that

$$|t| \dim S \le 5D \log(1 + 2b^2), \qquad \dim \sum_{g \in t} P_g(S) \ge (1 - 4/b^2)D.$$

Proof. Proposition 22.1 gives the two bounds for $P_g = c_B U_g$, with the common rational factor $c_B > 0$ of Proposition 7.10. Since $P_g(S) = U_g(S)$ for each g, the set of translates in Proposition 16.2 gives the stated dimension bound for these same maps. □

Proposition 5.2 (Determinant estimate for the chosen arithmetic data)**.** *Fix d, ϵ. At every sufficiently large integer x the marked comparison lifts are injective. For every independent tuple $y \in V^n$,*

$$\left.\begin{array}{l} n \ge (1 - 4/b^2)D, \qquad \operatorname{Cov}(y) \le c, \qquad \operatorname{Ind}(y) \le \delta \end{array}\right\} \implies D(A - 3L) \le c + 2\delta.$$

The proof is in Section 45.

Proposition 5.3 (Cancellation of the main covolume term)**.** *$m\mathcal{R} \le faB^5$ for sufficiently large x.*

Proof. Choose t and the integral generating set $\mathcal{S}$ from Proposition 5.1. Linearity gives

$$\operatorname{span} \bigcup_{g \in t} P_g(\mathcal{S}) = \sum_{g \in t} P_g(S).$$

Extract a basis $y_i = P_{g_i} \phi_i$ from this union and fix one such pair (g_i, ϕ_i) for each i. Its cardinality satisfies $(1 - 4/b^2)D \le n \le D$. The bound on every $\sigma \Theta_1 y_i$ holds for this one basis, so Hadamard gives $\operatorname{Cov}(y) \le n(A - m\mathcal{R} + L)$. The integral preimages in their chosen translates give

$$\operatorname{Ind}(y) \le 2|t| \dim(S)\, faB^3 \le 10D \log(1 + 2b^2) faB^3 = DI_0.$$

Proposition 5.2 and $n \le D$ give

$$nm\mathcal{R} \le (n - D)(A + L) + D(4L + 2I_0) \le D(4L + 2I_0).$$

For $b \ge 3$ and $B \ge 128$,

$$n \ge (1 - 4/b^2)D \ge D/2, \qquad I_0 = 10faB^3 \log(1 + 2b^2) \le 30L.$$

Hence $m\mathcal{R} \le 8L + 4I_0 \le 128L \le faB^5$. □

6 The ramified source at the prescribed level

In this section, the additional level condition singles out a subspace of the full definite automorphic source. We identify its integral coordinates before using it in the pointwise estimate of Section 7.4 and the dimension argument of Section 16.2.

In the definite source of rank k, write K_M for the integral principal level at the primes dividing $\bar{\alpha}$. Write J_H for its subgroup on which the last covector of a matrix reduces to the last covector of the identity at the primes dividing the prescribed generator $\pi_N = a_0 + ib_0$. The source space X consists of the forms at J_H with the fixed finite-dimensional coefficient module. These level descriptions specify the source used in the comparison.

The primes p_0, p_1, N are $1 \bmod q_2^2$, and q_1 and d divide q_2^2, by the definitions of s, L_0, q_1. Thus the three primes split completely in E. In K the two choices are fixed by

$$(\pi_M) = (\bar{\alpha}) = \overline{\mathfrak{p}_0}\,\mathfrak{p}_1.$$

Thus the chosen prime over p_0 is $\overline{\mathfrak{p}_0}$ and the one over p_1 is $\mathfrak{p}_1$. Over each p_i this gives f chosen primes of E, each of norm p_i. At N, the divisor of π_N chooses one of the two primes of $\mathbb{Q}(i)$ above N. It gives f distinct primes of E, one from each complex conjugate pair, all of norm N.

Lemma 6.1 (Trivial stabilizers at the prescribed level)**.** *Let H be a totally definite unitary group of rank h over F_0. Let p be one of the two odd level primes $p_0^{\mathrm{lev}}, p_1^{\mathrm{lev}}$. Suppose the compact open level at a marked split place above p is contained in the principal congruence subgroup of $\mathrm{GL}_h(\mathbb{Z}_p)$. Every finite class has trivial rational stabilizer. In particular this applies to the rank-k source at every level contained in K_M.*

Proof. For a finite representative g and its level J, the group $H(F_0)\cap gJg^{-1}$ is finite: its diagonal image is closed and discrete in the adeles and is contained in the compact set $H(F_{0,\infty}) \times gJg^{-1}$. At the chosen place it embeds, after conjugation, in $1 + pM_h(\mathbb{Z}_p)$, since the place has degree one over $\mathbb{Q}_p$. This last group is torsion-free. Indeed for $u = 1 + p^a A$, with $a \geq 1$ and $A \not\equiv 0 \bmod p$, the binomial expansion gives

$$u^s \equiv 1 + sp^a A \pmod{p^{a+1}} \quad (p \nmid s), \qquad u^p \equiv 1 + p^{a+1} A \pmod{p^{a+2}}.$$

The second congruence uses that p is odd. Iterating excludes every positive finite order. □

Proposition 6.2 (Dimension of the ramified source)**.** *For the ramified subspace $S \subseteq X$ for the chosen arithmetic data,*

$$\dim(X/S) \leq \frac{f}{N-1} \dim X \leq b^{-8} \dim X, \qquad \dim X \leq 2 \dim S.$$

Proof. For a prime $w \mid \pi_N$, take a local unit at w lifting a generator of the residue units, form the idele a_w with all other components one, and put $z_w = a_w/\bar{a}_w$. Its scalar matrix lies in K_M, since it is supported at $w, \bar{w}$ and $N \nmid M$. The local condition on the last covector at w says $z_w^j \in J_H$ exactly when $N-1$ divides j; all other conditions on the last covectors hold because $\bar{w}$ does not divide π_N.

Put $K' = \langle J_H, z_w : w \mid \pi_N \rangle \subseteq K_M$. The f pairs $\{w, \bar{w}\}$ are disjoint. Testing the last covector at each w therefore gives

$$\prod_{w \mid \pi_N} z_w^{j_w} \in J_H \quad \Longleftrightarrow \quad j_w \equiv 0 \pmod{N-1} \quad (w \mid \pi_N), \qquad K'/J_H \simeq (\mathbb{F}_N^\times)^f.$$

Centrality makes J_H normal in K'. Lemma 6.1 gives trivial rational stabilizers at K', so K'/J_H acts simply transitively on the J_H-classes over each K'-class. Transporting one coefficient fibre by this action identifies the values over that class with copies of the regular representation. Thus every character of K'/J_H occurs in X with multiplicity $\dim X/(N-1)^f$. The ramified source is the intersection of the kernels of the averages

$$A_w = \frac{1}{N-1}\sum_{j=0}^{N-2} R_H(z_w)^j.$$

On a joint character space, A_w acts as one when the w-component is trivial and as zero otherwise. Consequently

$$\dim S = \left(1 - \frac{1}{N-1}\right)^f \dim X, \qquad \dim(X/S) \le \frac{f}{N-1}\dim X \le b^{-8}\dim X.$$

Indeed $f \le q_1 \le b$ for large x, and the interval of primes for N and the definition of its center give $\log N \ge (19(m-1) - 15/2)\log b$, so $N - 1 \ge b^9$. In particular $\dim X \le 2\dim S$. □

7 Normalization and bounds for the theta images

In this section, we express the local divisibility estimates as bounds on the change of integral lattice. Summing their logarithms recovers the prime sum in Section 5, which cancels the corresponding part of the archimedean bound in Section 7.4.

The short-image estimate of Proposition 5.1 is reduced to archimedean estimates for the unscaled images and local gauge estimates with prime-dependent savings. For a prime p put

$$S_p(Y,\tau) = \frac{1}{2}\sum_{e\ge 1}(1-\chi_K(p^e))(Y/p^e - \kappa_{\langle -p^e\rangle\tau})_+, \qquad g_p = m\left(\sum_{\tau\in\Phi_a} S_p(a,\tau) + k\sum_{\tau} S_p(\kappa_\tau,\tau)\right).$$

All sums here are finite because the weights are positive. The factor $\frac{1}{2}(1-\chi_K(p^e))$ selects the odd exponents at an inert prime, is $\frac{1}{2}$ at a ramified prime, and vanishes at a split prime. Also $S_p = 0$ for the indicated arguments when $p \ge x_1$, because every weight is at least a/x_1. Write $E_0 = faB^3$ and let F consist of the nonsplit primes below x_1, together with N. Grouping the von Mangoldt sum by prime powers gives

$$2\sum_{p\in F} g_p \log p = m\sum_{\tau\in\Phi_a} S(a,\tau) + mk\sum_{\tau} S(\kappa_\tau,\tau).$$

In particular $A - m\mathcal{R} + 2\sum_{p\in F} g_p\log p$ equals the sum of the archimedean jet terms in A plus m times the expected value of $\sum_{\tau\in\Phi_a} S(a,\tau)$ from Section 5.

7.1 Bounds for the local denominators

In this subsection, we bound the total logarithmic contribution of the local estimates in Section 7. The exceptional congruence classes and the prime N are treated separately because their local estimates differ.

For these data, define the local error bound by

$$e_p = \begin{cases} \dfrac{2faB^2}{p}\, c_p, & p < x_1 \text{ nonsplit in } K, \\ 2mfb, & p = N, \\ 0, & \text{otherwise}, \end{cases} \qquad c_p = \begin{cases} m, & p \equiv \pm 1 \bmod m, \\ 1, & \text{otherwise}. \end{cases}$$

Lemma 7.1 (Total local denominator bound)**.** *For fixed d, ϵ and all sufficiently large x,*

$$\sum_p e_p \log p \le E_0 = faB^3.$$

Proof. Put $\ell_1 = \log x_1$. Since $\log N \ge mB > W$ for large x, the two nonzero cases defining e_p are disjoint. For $p = mc \pm 1$ with $c \ge 1$, one has $m/p \le 2/c$. Consequently

$$\sum_p e_p \log p \le 2faB^2\, \ell_1 \left(\sum_{j \le x_1} \frac{1}{j} + 2 \sum_{\pm} \sum_{c \le x_1} \frac{1}{c} \right) + 2mfb \log N \le 10faB^2\, \ell_1(1+\ell_1) + 2mfb \log N.$$

With $\ell_1 \le W$, $B > W^6$, $m^2 < 4x$, and $\log N = O(mB)$, this gives

$$\frac{1}{E_0} \sum_p e_p \log p \ll \frac{\ell_1(1+\ell_1)}{B} + \frac{m^2}{xB^2} = o(1). \qquad \square$$

7.2 Counting terminal double cosets

In this subsection, we bound the number of terminal double cosets by counting definite Hermitian lattices and refining their classes at the specified level. This controls the class sum in Section 7.4.

Let n_{t} be the terminal definite rank, and let H be the unitary group of the standard positive Hermitian space $E^{n_{\mathrm{t}}}$. Write $K_{\mathrm{int}} \subset H(\mathbb{A}_f)$ for the stabilizer of the standard integral lattice at all finite places.

Lemma 7.2 (Terminal class count)**.** *For the trivial twist, let h_{term} be the number of double classes at any of the definite upper stages X_j or at the definite terminal stage Y_m, with its prescribed level. For fixed d, ϵ and sufficiently large x, uniformly in the stage,*

$$\log h_{\mathrm{term}} = O(x^{52}), \qquad \log h_{\mathrm{term}} + 4 \log |G| \le x^{99}.$$

Proof. We first count classes at K_{int}. Write $d_E = [E : \mathbb{Q}]$, $\Delta_E = |\operatorname{disc}(E)|$, and $u = d_E n_{\mathrm{t}}$. Each class gives an integral lattice Λ obtained by intersecting a finite adelic translate of the standard integral lattice with $E^{n_{\mathrm{t}}}$. It has integral Hermitian pairings and is locally isometric to the standard lattice. An isometry between such lattices over E therefore identifies their double classes.

Use the Euclidean norm squared given by $\operatorname{Tr}_{E/\mathbb{Q}} h(v, v)$, where h is the Hermitian form. The determinant of the integral trace pairing on Λ is $\Delta_E^{n_{\mathrm{t}}}$: on the standard lattice use an integral basis of E in each coordinate, with conjugation changing the trace determinant only by a sign. Local isometry gives the same determinant valuation at every rational prime, and positivity fixes the determinant.

Thus the covolume is $\Delta_E^{n_{\mathrm{t}}/2}$. Every nonzero lattice vector has length at least one, by positivity and integrality of its squared length. Minkowski's second theorem [20, Ch. VIII, §4.3, Theorem V] and the cube of side $2/\sqrt{u}$ in the Euclidean unit ball give, for the last successive minimum,

$$\lambda_u \le u^{u/2}\Delta_E^{n_{\mathrm{t}}/2}.$$

Choose u independent lattice vectors of length at most $2\lambda_u$. They span $E^{n_{\mathrm{t}}}$ over E, so n_{t} of them form an E-basis $v_1,\dots,v_{n_{\mathrm{t}}}$.

Let $T=(h(v_i,v_j))$, with the convention that the second variable is linear. Its entries are algebraic integers, and their absolute values at all embeddings are at most $H_0=4u^u\Delta_E^{n_{\mathrm{t}}}$, by Cauchy–Schwarz at each embedding. To count these entries, note that for distinct algebraic integers in E, their difference has at least one embedding of absolute value at least one, by taking its norm. The vectors of their real and imaginary parts at one place from each complex conjugate pair therefore have sup-norm separation at least $1/\sqrt{2}$. Subdivision of the box of side $2H_0$ into cubes of side $1/2$ gives at most $(4H_0+2)^{d_E}$ choices per entry.

For each fixed T identify the space over E via the chosen basis. Let L be its $\mathcal{O}_E$-span. We have $L\subseteq\Lambda\subseteq L^{\#}$, where $L^{\#}$ is the Hermitian dual of L, since Λ has integral pairings. The quotient $L^{\#}/L$ is isomorphic as an abelian group to $\mathcal{O}_E^{n_{\mathrm{t}}}/T\mathcal{O}_E^{n_{\mathrm{t}}}$, by writing the pairings in this basis. It has order $\mathcal{I}=|\,\mathrm{N}_{E/\mathbb{Q}}\det T|$, at most $(n_{\mathrm{t}}!H_0^{n_{\mathrm{t}}})^{d_E}$.

Every subgroup of this quotient has at most u generators, since its inverse image in $L^{\#}$ is a sublattice of rank u. There are at most $\mathcal{I}^u$ choices for the intermediate lattice, using ordered tuples with zero entries allowed. These counts bound the number of classes because two lattices with the same data are carried to one another by an E-linear isometry sending the chosen bases to one another. We obtain

$$\log|H_{\mathrm{rat}}\backslash H(\mathbb{A}_f)/K_{\mathrm{int}}| = O\big(u^2(1+\log H_0+\log n_{\mathrm{t}})\big).$$

Here the left quotient is by the E-linear isometries of the Hermitian space.

If the terminal level contains the principal congruence subgroup of K_{int} modulo MN, its number of classes is at most the preceding class count times $(MN)^{d_E n_{\mathrm{t}}^2}$. Indeed one may refine each class by right cosets of the principal congruence subgroup in K_{int}, whose index is at most the number of matrices modulo MN.

Write $K(a)$ for the principal congruence subgroup of K_{int} at the ideal (a), in the rank of the given stage. On X_j the reduction is the identity at the chosen places above M, and the last covector is fixed modulo π_N. On Y_m at the trivial twist the only residue condition is identity reduction at those chosen places above M. Thus the prescribed level K_{stage} satisfies

$$K(MN)\subseteq K(\pi_M\pi_N)\subseteq K_{\mathrm{stage}},\qquad MN=(\pi_M\pi_N)\overline{(\pi_M\pi_N)}.$$

The rank is at most $k+m(5r+1)=O(x^{11/2})$, and $d_E=2f=O(x^{11}\log x)$. In particular $u=O(x^{17})$. The Hermitian field $E=\mathbb{Q}(\sqrt{-d},\zeta_{q_1})$ lies in $\mathbb{Q}(\zeta_{dq_1})$, so the tower discriminant formula and the cyclotomic discriminant bound give $\mathrm{rd}(E)\le dq_1$. Hence $\log H_0=O(u\log x)$. Also $\log(MN)=O(mB)$ and $B=O(x^{\epsilon})$. The logarithm of the terminal class count is consequently $O(x^{52})$. There are f marked places above each of p_0,p_1. The residue group G acting on the target is the product of the corresponding groups GL_r, so $\log|G|\le fr^2\log M$. It follows that $\log h_{\mathrm{term}}+4\log|G|\le x^{99}$ for all sufficiently large x. □

7.3 Projection onto the prescribed finite types

In this subsection, the central idempotent acts on every lift by the intertwining relation of Section 17.3. Its integral multiple controls the valuations and archimedean norms of all translates, using the class count of Section 7.2.

Let e_0 be the central idempotent of the good part in the complex group algebra of the finite group G. Let z be an automorphic form on $U(Y_0)$ at the prescribed principal level. Write $\mathcal{L}_\xi z$ for the terminal algebraic coefficient of $L_{\xi,m-1}\cdots L_{\xi,0}z$, where each theta integral includes its source index $A_{\xi,j}$ as in Section 17.3.

Lemma 7.3 (The integral finite-type projection)**.** *The coefficients of $|G|e_0$ are algebraic integers, of absolute value at most $|G|$ at every complex embedding. For any $g \in G$, replacing $\mathcal{L}_\xi z$ by $\mathcal{L}_\xi(|G|e_0 g z)$ preserves the local gauge bound $p^{e_p - g_p}$. If the untwisted conjugate squared pointwise norms of $\mathcal{L}_1 z$ are at most e^{K_0}, then the squared terminal class norm after projection is at most $e^{K_0 + x^{99}}$.*

Proof. The expansion of e_0 as a sum of primitive central idempotents gives

$$|G|e_0(t) = \sum_{\pi \text{ good}} \dim(\pi)\chi_\pi(t^{-1}) \qquad (t \in G).$$

Each coefficient on the right is an algebraic integer because the character values are sums of roots of unity. At every complex embedding its absolute value is at most $\sum_{\pi \text{ good}}(\dim \pi)^2 \le |G|$.

Equivariance on the original chain. For $s = (s_w) \in G$, choose at each stage an integral finite lift $k_j(s)$. Its residue at an old place w is $\mathrm{diag}(s_w, I)$ in the original order of the blocks, and its residue at every twist place is the identity. At the source $k_0(s)$ acts by the given residue action, since two lifts differ by a principal-level element. Reduction is onto at these split places, so these choices are possible at every stage. At unmarked places their components can be the identity.

Let i_j denote the inclusion of the j-th Hermitian space as the first block of its successor. The original forward-arrow finite test at the chosen and twist places is the indicator of i_j. The simultaneous source and target reductions a, b of $k_j(s), k_{j+1}(s)$ satisfy $bi_j = i_j a$. Thus the finite split Hom action $X \mapsto b^{-1}Xa$ fixes the prescribed support. The determinants are units, and the two rank-twisted splitting characters leave the factor $\psi_{j,w}(\det b)/\psi_{j,w}(\det a)$ on the indicator, where $\psi_{j,w}$ is the tame part of the arrow character. This factor is one since both determinants are $\det(s_w)$ at the old places and one at the twist places. On the other finite components the vacuum is fixed, and these translations have no real component. The integrated arrow datum is therefore fixed.

Change the source variable by $k_j(s)$ in the arrow integral, using the original quotient Haar. It follows that translating the image under the lift by $k_{j+1}(s)$ gives the arrow applied to the source translate by $k_j(s)$. Iterate to the terminal space. The algebraic coefficient realization commutes with finite right translations, so the final formula is

$$\mathcal{L}_\xi(sz)(h) = (\mathcal{L}_\xi z)(h\, k_m(s)).$$

With $k_\xi(s) = k_m(s)$, expanding e_0 gives

$$\big(\mathcal{L}_\xi(|G|e_0 g z)\big)(h) = \sum_{t \in G} |G|e_0(t)(\mathcal{L}_\xi z)(h\, k_\xi(tg)).$$

Here $k_{\xi}(tg)$ is the integral first-block lift of tg and h is a finite adelic point in the terminal space. In the gauged coordinates a right translate by any such integral matrix preserves the gauge: the change of coordinates and its inverse both preserve the unit lattice. The ultrametric inequality therefore transfers any bound $p^{e_p-g_p}$ for $\mathcal{L}_{\xi}z$ to $\mathcal{L}_{\xi}(|G|e_0gz)$ with the same exponent. The transfer uses the prescribed chain and holds for all allowed twists and every translate. If the local estimate is first given on a submodule of the full integral source, intersect it with the ramified integral source. A matrix clearing a tuple from the ramified source into the first submodule still clears it into this intersection since linear combinations preserve ramification.

The coefficient bound and the triangle inequality give

$$\|\sigma(\mathcal{L}_1(|G|e_0gz)(h))\|_{\mathrm{W}}^2 \le |G|^4 e^{K_0}.$$

Summing with reciprocal stabilizer weights and applying Lemma 7.2 gives

$$h_{\mathrm{term}}|G|^4 e^{K_0} \le e^{K_0+x^{99}}. \qquad \square$$

7.4 Pointwise estimates in the composition of lifts

In this subsection, we estimate the upper lift by exploiting the additional level condition of Section 6. The resulting pointwise bound is combined with the class count of Section 7.2 in the proof of Proposition 5.1.

7.4.1 The sum of the part sizes

In this subsection, the level prime from Section 4 supplies the term subtracted in the terminal estimate of Section 7.4.5. We use the function J below to sum the factorial contributions from the upper partitions.

Write $J(l) = \log(l!) + l\log(M/\Omega)$, put $\overline{w} = f^{-1}\sum_{\tau}\log\kappa_{\tau}$, and write $s_{\mathrm{av}} = a(\log a - \overline{w} + c_{d,1})$. Set $G_0 = \log(M\,\mathrm{rd}(E)/(2\pi))$. By the definition at modulus one, $c_{d,1} = L'/L(1,\chi_K) - \gamma_{\mathrm{Euler}} - 1$, so $\log(M/\Omega) = G_0 + 1 + c_{d,1}$. The comparison of $\sum_{j=1}^{a}\log j$ with $\int_1^a \log t\,dt + \log a$ gives

$$J(a) - s_{\mathrm{av}} \le a(\overline{w} + G_0) + 1 + \log a.$$

The chosen prime satisfies $\log N \ge m\overline{w} + (m-1)G_0 - \frac{15}{2}B$. Consequently

$$m(J(a) - s_{\mathrm{av}}) - a\log N \le a(G_0 + \tfrac{15}{2}B) + m(1+\log a) \le 32aB$$

eventually, since $G_0 \le 23B$ and $m(1+\log a) \le aB$. Define the size of the principal parts by

$$S_{\mathrm{main}} = m\left(|\Phi_a|J(a) + k\sum_{\tau} J(\kappa_{\tau})\right).$$

For the step from X_j to X_{j+1}, put $n_j = k + j(5r+1)$ and let $m_{\tau,h}$ count the columns of height h in the lift weight of Section 20.2. The first n_j parts of the partition at the target are

$$\lambda_{j,\tau,i} = \sum_{i<h<n_{j+1}} m_{\tau,h} \qquad (0 \le i < n_j).$$

Thus a column of height n_j, which contributes the determinant at the source, occurs in every one of these parts. Set

$$S_j = \sum_{\tau\in\Phi}\sum_{0\le i<n_j} J(\lambda_{j,\tau,i}) \qquad (0 \le j < m), \qquad h_0 = r^8.$$

Lemma 7.4 (Factorial bounds for the upper composition)**.** *For fixed d, ϵ and sufficiently large x,*

$$S_{\mathrm{main}} - |\Phi_a| a \log N \le A - m\mathcal{R} + 2\sum_p g_p \log p + 32faB,$$

$$\sum_j S_j \le S_{\mathrm{main}} + \tfrac{1}{2}E_0.$$

Proof. By the prime-power identity in Section 7, $A = mk\sum_\tau (J(\kappa_\tau) - S(\kappa_\tau, \tau))$ and $\mathcal{R} = \sum_{\tau\in\Phi_a} S(a,\tau) - |\Phi_a| s_{\mathrm{av}}$ give

$$A - m\mathcal{R} + 2\sum_{p\in F} g_p \log p = mk\sum_\tau J(\kappa_\tau) + m|\Phi_a| s_{\mathrm{av}},$$

$$\begin{aligned} S_{\mathrm{main}} - |\Phi_a| a \log N - \left(A - m\mathcal{R} + 2\sum_{p\in F} g_p \log p \right) \\ = |\Phi_a|\big(m(J(a) - s_{\mathrm{av}}) - a\log N\big) \le 32faB. \end{aligned}$$

The factor $N^{-|\Phi_a|a}$ is the contribution of a_τ in the branching exponents $a_\tau - mh_0$ of Lemma 7.9.

The bounds of Section 20.2 have the form

$$\lambda_{j,\tau,i} \le \mu_{\tau,i} + s_i, \qquad \mu_{\tau,i} = \begin{cases} a\mathbf{1}_{\Phi_a}(\tau), & i = 0, \\ \kappa_\tau, & 1 \le i < k, \\ 0, & i \ge k, \end{cases} \qquad s_i = h_0^2 + (b + a/x_1)\mathbf{1}_{\{0\}}(i).$$

For large x, $\mu_{\tau,i} + s_i \le a + h_0^2 + b + a/x_1 \le 2a$. Since $M/\Omega > 1$, the increment $J(l+1) - J(l) = \log(l+1) + \log(M/\Omega)$ is positive. For $l + s \le 2a$, the choice of scales gives

$$J(l+s) - J(l) \le s\big(\log(2a+1) + \log(M/\Omega)\big) \le 25sB.$$

Indeed $2a + 1 \le b^2$, $\log M \le 22B$, and $\log \mathrm{rd}(E) = O(\log x) = o(B)$.

At each step the principal lengths satisfy

$$\sum_{i=0}^{n_j - 1} J(\mu_{\tau,i}) = \mathbf{1}_{\Phi_a}(\tau) J(a) + (k-1)J(\kappa_\tau).$$

Since $J(\kappa_\tau) \ge 0$, summing the increments over all places and steps gives

$$\sum_j S_j \le S_{\mathrm{main}} + 25mfB(b + a/x_1 + n_m h_0^2) \le S_{\mathrm{main}} + \tfrac{1}{2}E_0.$$

Indeed $x/x_1 \to 0$, $n_m h_0^2$ is polynomially bounded in x, and b exceeds every fixed power of x. □

7.4.2 Integral generators of the ramified source

In this subsection, we construct generators in the integral model of Section 6. Their pointwise norms supply the initial bound for the composition in Section 7.4.5.

Lemma 7.5 (Integral generators with bounded Weyl norm)**.** *Fix d, ϵ. For every sufficiently large x there is a set $\mathcal{S} \subseteq S$ of everywhere integral forms such that*

$$\operatorname{span}(\mathcal{S}) = S, \qquad \sup_{\sigma,h} \|\sigma(\phi(h))\|_{\mathrm{W}}^2 \le e^{E_0} \quad (\phi \in \mathcal{S}).$$

Here h ranges over $U(X_0)(\mathbb{A}_{F_0,f})$ and σ over $\operatorname{Aut}(\mathbb{C})$.

Proof. Put $n_0 = k$, $D_0 = q_1 d$, where $q_1 d$ is in the different of $E/\mathbb{Q}$, and

$$C_0 = 6D_0(fn_0)^2, \qquad M_0 = n_0 f + \#\{\text{weight boxes in columns of height } < n_0\}.$$

We use D_0 here for a positive rational integer. Throughout this section N still denotes the ramified level prime. At every original source class g, use the lattice

$$L(g) = E^{n_0} \cap g\widehat{\mathcal{O}}_E^{n_0}.$$

The finite unitary action and the standard definite Hermitian form make $L(g)$ Hermitian self-dual. Here and below duality means integrality of the pairing over $\mathcal{O}_E$. The diagonal archimedean embedding has Euclidean norm squared $\sum_\tau \|v_\tau\|^2$. For $0 \neq v \in L(g)$, its Hermitian square $h(v,v)$ is a totally positive integer of $F_0 = E^+$. The arithmetic–geometric mean inequality applied to its conjugates gives $\|v\|^2 \ge f \ge 1$.

The real inner product in these coordinates is half the trace pairing. The Euclidean dual of $2D_0^{-1}L(g)$ is consequently the archimedean image of $D_0\mathfrak{d}_E^{-1}L(g)$, which is contained in $L(g)$. For the identification, a real functional integral on a lattice is rational on its rational span. By nondegeneracy of the rational trace pairing it therefore has a rational trace representative. Pairing with all the $\mathcal{O}_E$-multiples of every lattice vector puts the representative in the codifferent times the Hermitian dual, proving the asserted equality.

For completeness a Euclidean lattice of dimension h with shortest dual vector of length at least 1 has covering radius at most h^2. Minkowski's convex body theorem [20, Ch. III, §2.2, Theorem II, p. 71] applied to cubes inscribed in Euclidean balls gives $\lambda_1(L)\lambda_1(L^*) \le h$, since the two covolumes multiply to 1. Choose a primitive vector of length at most $2h$ and project orthogonally to its complement. The projected lattice is discrete since the vector extends to a lattice basis.

Its dual is L^* intersected with the complement. Induction gives error at most $(h-1)^2$ in the projected space, and rounding in the chosen direction contributes at most h. This proves the radius bound (with the zero-dimensional case immediate). The vectors of length at most three times the covering radius span the whole real space: if their span has a unit normal u, a lattice vector within radius r of $2ru$ has norm at most $3r$ and has nonzero projection on u.

Apply this argument to $2D_0^{-1}L(g)$ with $h = 2fn_0$, then multiply by $D_0/2$. The resulting vectors in $L(g)$ of norm at most C_0 span the real space in the diagonal embedding. They also span E^{n_0} over E: an E-linear functional vanishing on them extends at each complex place and hence is zero. Take an E-basis from these vectors. With its columns written $Y(g)$,

$$Y(g) \in \mathrm{GL}_{n_0}(E), \qquad g^{-1}Y(g) \in M_{n_0}(\widehat{\mathcal{O}}_E), \qquad |\tau(Y(g)_{ij})| \le C_0.$$

Fix a finite adelic representative g of the class. For each determinant exponent $e_\tau \in \mathbb{Z}$, apply the construction to the group element g^{e_τ}. Thus $\det(g^{e_\tau}) = \det(g)^{e_\tau}$, also for negative exponents. Here $D_0 \in \mathfrak{d}_E$ because the different of $E/\mathbb{Q}$ divides the product of the extended differents of its two constituent fields K and $\mathbb{Q}(\zeta_{q_1})$, which in turn divides $dq_1\mathcal{O}_E$.

Write e_τ for the marked determinant exponent and $v(y)$ for the column-polynomial Weyl evaluation vector, in the original integral model. Use the marked coefficient embedding $\iota_\tau = \tau \circ \mathrm{conj}$. For each class g use the value

$$v_g(y) = \left(\prod_\tau \det \iota_\tau(Y(g^{e_\tau})) \right) \rho_{\mathrm{pol}}((\iota_\tau(Y(g)))_\tau) v(y)$$

for the delta form at g, with y having roots of unity as entries. The level is free, so these delta forms are well defined and there is no stabilizer division. In the transported gauge at g, the polynomial matrix is evaluated on $g^{-1}Y(g)$; the determinant factors are the determinants of $(g^{e_\tau})^{-1}Y(g^{e_\tau})$. Both are integral. Polynomial Weyl matrices and evaluation vectors have integer coefficient formulas in the chosen model, and roots of unity are units at every coefficient embedding. Thus the transported gauge of the delta form is at most one at every place and on every representative, by equivariance.

For the archimedean bound the evaluation vector is the Schur action of the first square block of y on the unit vector of the reference column array. After applying $Y(g)$, the resulting matrix at each embedding has entries at most $n_0 C_0$ in modulus and squared Frobenius norm at most $n_0^4 C_0^2 \le C_0^6$. The induced action on each exterior power is bounded by the operator norm to its degree, and hence the squared Schur norm is bounded by C_0^6 to the number of polynomial weight boxes. The squared determinant factors together contribute at most $C_0^{6n_0 f}$ by Hadamard. These bounds hold after every complex automorphism, since the entries of Y are bounded under all embeddings and roots of unity still have modulus one. The squared Weyl norm of the value is at most $C_0^{6M_0}$. This bound holds at every finite point h. On the supporting class write $h = \gamma_f g k'$, with $\gamma \in U(X_0)(F_0)$ and k' in the level. The value there differs from the value at g by the rational weight action. At every embedding $\jmath : E \hookrightarrow \mathbb{C}$, $\jmath(\gamma)^* \jmath(\gamma) = 1$, since $\jmath(\bar{a}) = \overline{\jmath(a)}$ for $a \in E$. The determinant characters are unitary, and the induced exterior tensor actions preserve the Weyl metric. This applies also to $\jmath = \sigma \circ \iota_\tau$. Off that class the value is zero.

The evaluation vectors already span on one finite grid. Choose an integer L exceeding the number of polynomial weight boxes and let the entries of the evaluation matrix y_τ range independently in μ_L, both within a matrix and across the infinite places τ. At the source the polynomial representation is $\bigotimes_\tau \mathrm{Sym}^{a_\tau}(\mathbb{C}^{n_0})$. Its evaluation coordinates are products of degree-a_τ monomials in the first columns of the separate y_τ. For a functional vanishing at all these arrays, the resulting polynomial has degree less than L in each variable. Successively fixing the other entries and using the L distinct roots at each coordinate shows that the polynomial is zero. The Weyl evaluation polynomials form the dual Weyl space in the chosen model, so the functional is zero. The matrix multiplying $v(y)$ in $v_g(y)$ is invertible. Summing these component spans proves the claim at the original source level.

On the source the determinant exponent is zero and the only columns of height below n_0 are the a_τ columns of height one (Section 20.2), so $M_0 \le f(n_0 + a)$, and the scale inequalities give

$$4^f C_0^{6f(n_0+a)} \le \exp(E_0), \qquad E_0 = faB^3.$$

Finally apply the commuting differences $1 - R_H(z_w)$ for the marked $w \mid \pi_N$ of Section 6. Each preserves the local unit balls and multiplies the maximum squared norm of the conjugate values by at most 4. Their product has the ramified subspace as its range: the commuting finite cyclic scalars are simultaneously diagonalizable and $1 - T$ is invertible on the nontrivial eigenspaces

of T. These differences therefore carry the integral delta generators to a spanning set of S with the stated bound. □

7.4.3 Conjugation of the theta jets

In this subsection, we identify the conjugated theta jets and the finite vacuum on the original chain of Section 17.3. Section 7.4.4 uses the transported jets in the Gaussian estimate.

For the kernel $X_j \to X_{j+1}$, let S_j be the sum of J on the actual upper part sizes $\lambda_{j,\tau,i}$ used in the Schur matrix. We estimate the operator on the original Weyl modules for the untwisted chain. The following two identities identify the CM and finite-vacuum identities in this calculation. They concern the same vacuum scalar and the same untwisted upper tensor used in the integral chain.

Lemma 7.6 (Transport of the jets and the finite vacuum)**.** *For $c \neq 0$, let J_c denote the full holomorphic jet at scale c. Fix $\gamma = s_{\mathrm{jet}} = p_F a_d/\omega_0$, and let $T(x)$ denote the finite Heisenberg translation. For every positive rank v one can choose a finite vector F_v on the standard definite space, with $\theta(F_v) = 1$, whose translates span the finite representation. For each $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ there is a pair $(b_\sigma, t_\sigma) \in \mathbb{A}_{E,f}^{\times} \times \mathbb{C}^{\times}$, independent of v, and finite vectors $F'_{\sigma,v}$ such that*

$$J_{t_\sigma}(T(b_\sigma x)F'_{\sigma,v}) = \mathrm{ren}_{\pi_\sigma}\ \sigma_{\mathrm{coeff}}\big(J_\gamma(T(x)F_v)\big) \quad (x \in \mathbb{A}_{E,f}^{v}), \qquad |t_\sigma|^2 = \Omega^{-1}. \tag{AJ}$$

Here $\pi_\sigma(\tau) = \sigma \circ \tau$ on the embeddings $\tau : E \hookrightarrow \mathbb{C}$ extending the fixed embedding of K. The map $\mathrm{ren}_{\pi_\sigma}$ sends $Z_{\tau,i}$ to $Z_{\pi_\sigma(\tau),i}$.

Let v_0 be the rank-one vacuum and $\Lambda = \widehat{\mathcal{O}}_E$. There is a finite displacement y for which $T_\sharp(u)T(y)F_1 = T(y)F_1$ for all $u \in q_1\Lambda$, and

$$s_{\mathrm{vac}} v_0 = \sum_{u \in \Lambda/q_1\Lambda} T_\sharp(u)T(y)F_1, \qquad s_{\mathrm{vac}} \neq 0. \tag{AV}$$

The operator $T_\sharp$ includes the finite vacuum semicharacter. The finite values of F_1 are integral at every coefficient place. They are products of signed, normalized rank-one theta evaluations at torsion points along the q_1 forms $z \mapsto -\mathrm{Tr}_{E/K}(z\overline{\zeta_{q_1}^{k}})/q_1$, each repeated q_1 times.

Proof. We first construct the translated theta sections and their common period scale. The finite theta expansion below then gives the vacuum projection (AV) in rank one and the spanning assertion in every positive rank.

The normalized CM translation and its period.

Write $K = \mathbb{Q}(\sqrt{-d})$ and $q = q_1$. We use the degree-one theta function of a lattice Λ_C with maximal CM by O_K. The distinguished two-torsion point t is a generator of $C[2]$ as an $O_K/2O_K$-module. In the marked Weierstrass coordinates the curve is

$$y^2 + a_1 xy = x^3 + x, \qquad t = (0,0), \qquad \Delta_C = a_1^4 - 64.$$

The periods are those of the invariant differential of this equation. The theta function ϑ_C has zero divisor $[t]$, is even, and is normalized by $\vartheta_C(0) = 1$. For $A_C = \mathrm{area}(\Lambda_C)$ write

$$(T_u^C \vartheta_C)(z) = e^{-\pi(z\bar{u} + |u|^2/2)/A_C}\vartheta_C(z+u).$$

For the elliptic CM type given by the fixed embedding $\Phi_K : K \hookrightarrow \mathbb{C}$, the reflex field is K. The restriction $\sigma|_{\overline{\mathbf{Q}}}$ defines the conjugate of the principally polarized curve. Choose $s \in \mathbb{A}_{K,f}^{\times}$ with $\mathrm{Art}_K(s) = \sigma|_{K^{\mathrm{ab}}}$ in the convention of [23, Theorem 6.3]. Let c_σ be the scalar of the CM tangent-space identification in the coordinates given by the marked differentials. In these coordinates the inverse reflex norm $N_{\{\Phi_K\}}(s^{-1})$, followed by multiplication by c_σ, carries the completed period lattice to the conjugate one and induces the corresponding map on each primary torsion subgroup. The normalized Gaussian translation uses the Hermitian form $\pi\bar{u}z/A_C$.

Integral normalization.

After a finite extension defining the CM action, the elliptic curve has potential good reduction at every coefficient place [49, Theorem 12.14]. Work over a finite extension with good reduction at the place under consideration. At an odd residue prime, t remains nonzero by injectivity of prime-to-residue-characteristic torsion reduction. At residue characteristic two, 2 splits in K. The two idempotents of $O_K \otimes \mathbb{Z}_2$ split the 2-divisible group of the good model into groups of height one. Their dimensions sum to one, so one factor is étale. The étale component of t is nonzero by the generator condition. It has nonzero reduction. The CM action used here extends to the good model by the extension property of abelian schemes over a discrete valuation ring.

An integral change of variables of a good Weierstrass equation moves t to $(0, 0)$. The constant term and the linear term in y vanish, by the equation and the order-two condition. Smoothness at $(0, 0)$ makes the coefficient of x a unit. A shear $y \mapsto y + sx$, with s a root of the monic quadratic that cancels the coefficient of x^2, is integral after finite extension. A unit dilation then sets the coefficient of x to one. This fixes the Weierstrass equation in these coordinates. Any isomorphism between two such equations preserving the marked point has no coordinate translations. If its variables have scaling parameter u, the coefficient of x forces $u^4 = 1$; thus its invariant differential changes by a unit. In particular Δ_C is a unit at every coefficient place. Section 20.1 also computes this unit directly from the three half-period discriminants on each ideal class.

Here is also the required integrality of the normalized torsion values. The good elliptic scheme has the symmetric line bundle $\mathcal{O}([t])$, rigidified by the value at zero of its canonical section. That value is a basis because the two sections $t, 0$ remain disjoint. The rigidification determines a unique cubical structure. The multiplication formula for this structure [53, Ch. I, §2.6(i), pp. 21–22, and Proposition 5.5, p. 29] gives, over the valuation ring,

$$[n]^*\mathcal{O}([t]) \simeq \mathcal{O}([t])^{\otimes n^2},$$

since $t = -t$. If u represents a torsion point and $nu = \lambda \in \Lambda_C$, let $\kappa(\lambda)$ be the lattice sign of ϑ_C.

In the uniformizing basis, the factor of automorphy at zero for λ is $\kappa(\lambda)e^{\pi|\lambda|^2/(2A_C)}$. The rigidified isomorphism is induced by the identity on the pulled-back bases over the universal cover. Indeed the two multipliers at a period agree, since $\kappa(n\lambda) = \kappa(\lambda)^{n^2}$, and the identity preserves the basis at zero. In the fibre at the torsion point, the n^2-th power of the canonical section, in the basis pulled back from the origin, consequently has value

$$\kappa(\lambda)^{-1}\left(e^{-\pi|u|^2/(2A_C)}\vartheta_C(u)\right)^{n^2}.$$

This belongs to the valuation ring. The sign is a unit, so $(T_u^C\vartheta_C)(0)$ is integral. This also proves that the analytic normalized value is algebraic for algebraic CM data, since a positive power of the value is algebraic.

The translated Taylor series.

Write $C' = C^\sigma$ for the conjugated model with parameter σa_1, uniformized with that differential. Write $x_t(P) = x(P+t)$ and $y_t(P) = y(P+t)$ for the functions obtained by translation by the marked two-torsion point t. They have their only poles at t, of orders 2 and 3, and satisfy the same Weierstrass equation. The translation commutes with conjugation of the model. The graded ring of $\mathcal{O}([t])$ is generated by $\vartheta_C, x_t\vartheta_C^2, y_t\vartheta_C^3$ in degrees $1, 2, 3$. The Weierstrass equation reduces the powers of y_t to at most one. The resulting distinct pole orders give a basis in each degree by Riemann–Roch. At the pole chart the section transport is the homogeneous conjugation of the same projective equation, so it preserves zero values there too.

The translated rational functions are regular at the reference origin. Their expansions there conjugate coefficientwise using the invariant differential, whose formal integral gives the uniformizing coordinate. Thus, with Q the formal quotient of the Taylor series of ϑ_{C^σ} by the coefficientwise conjugate of ϑ_C, a section of degree l transports to $Q^l\sigma(f)$. At corresponding points the values of all degree l sections have one common nonzero relative factor μ^l, by choosing trivializations of the one-dimensional fibre of the degree-one line bundle. This evaluation statement is about ordinary section values, before any Gaussian twist.

Let $[a]$ be a nonzero CM endomorphism. Because σ fixes K, its two versions multiply the uniformizing parameter by the same a, have the same degree $a\bar{a}$, and preserve the conjugate zero divisor of the pullback of the theta section. In comparing the pulled back line bundle to the power of the marked bundle, use their theta multipliers. The Appell–Humbert form scales by $a\bar{a}$, and the pullback semicharacter agrees with the power semicharacter on the marked period lattice.

One can use $a \equiv 1 \bmod 2\mathcal{O}_K$ here, for which this last assertion follows directly from the half-period multiplier. Thus the section $\vartheta_C(az)$ transports to $\vartheta_{C'}(az)$: the zero divisors agree, the quotient of the two sections of the same line bundle is constant, and both have value one at zero. Consequently

$$Q(az) = Q(z)^{a\bar{a}}.$$

Take a nonreal $a \equiv 1 \bmod 2\mathcal{O}_K$ with $|a| > 1$. If $Q - 1$ begins in degree $j > 0$, coefficient comparison gives $a^j = a\bar{a}$. Absolute values imply $j = 2$, and then $a = \bar{a}$, a contradiction. Hence $Q = 1$.

For a finite set $S \subset K$, choose integers $N_s > 0$ with $N_s s \in \mathcal{O}_K$ for each $s \in S$. Take $n > 0$ divisible by every $2N_s$, and put $w = \omega_0/(2n^2)$, where $\Lambda_C = \omega_0\mathcal{O}_K$. Choose one lift w' of the conjugate of the point of C represented by w. Then $v = 2n^2w'$ is a period of C'. With $a_s = (n/(2N_s))ns \in \mathcal{O}_K$, the points represented by

$$a_s w = \frac{s\omega_0}{4N_s}, \qquad a_s w' = \frac{sv}{4N_s}$$

correspond under σ, since the CM action commutes with conjugation over K. These are the division points used later in Lemma 37.1. Write T_u for Gaussian translation by u and t_u for translation by the corresponding point of the elliptic curve. On either curve, with $L = \mathcal{O}([t])$ for its marked point, the bundle $t_u^*L \otimes L^{-1}$ belongs to Pic^0 and is killed by $[n]^*$, since nu is a period for each of the shifts $u = s\omega_0, sv$. In uniformizing coordinates the multiplier relative to the unshifted section at a period λ is

$$\exp(2\pi i \operatorname{Im}(nu\bar{\lambda})/A_C) = 1$$

by Riemann-form integrality, using the corresponding area on C'. Thus the two sections $(T_{s\omega_0}\vartheta_C)(nz)$ and $(T_{sv}\vartheta_{C'})(nz)$ lie in the same degree-n^2 bundles as their respective unshifted

sections. The zeros of the first are at points $z = (b - 2ns)w$ with $b \in \mathcal{O}_K$ and nbw the marked half-period modulo Λ_C. At the corresponding point $(b - 2ns)w'$, the translated argument is nbw', which is the conjugate marked half-period by the algebraic CM action. Thus these zeros transport to zeros of the second section. Both have the same degree, and the zero multiplicity is one because multiplication by n is unramified. The transported section is a nonzero scalar times the second one.

To determine that scalar, evaluate both at the point represented before multiplication by n by $-nsw$ and its conjugate $-nsw'$. The translated arguments are then $u = n^2sw$ and $u' = n^2sw'$ respectively, and the translation amounts are $2u, 2u'$. The exponential of the Gaussian twist is one at argument $-u$ of T_{2u}, and the corresponding unshifted value at $-u$ is $\vartheta_C(u)$ by evenness. The same is true for u'. The degree-n^2 section $\vartheta_C(nz)$ transports to $\vartheta_{C'}(nz)$ by $Q = 1$. At $-nsw$ and $-nsw'$ its value agrees with the translated section on each curve. Dividing two values in the same fibre fixes the comparison scalar to one when $\vartheta_C(u) \neq 0$. If $\vartheta_C(u) = 0$, both u, u' represent the respective marked half-periods. Let $\kappa(2u)$ be the period sign. The translation law and evenness give

$$\vartheta_C(u + s) = \kappa(2u)e^{2\pi s\bar{u}/A_C}\vartheta_C(u - s).$$

The zero at u is simple, so differentiation at $s = 0$ gives $\kappa(2u) = -1$. The same calculation holds on C'. Thus $(T_{2u}^{C}\vartheta_C)(0) = (T_{2u'}^{C'}\vartheta_{C'})(0) = -1$, and evaluation at zero gives scalar one also in this case. Dilation by n is invertible on formal series over $\mathbb{C}$, so this is the exact Taylor identity for $T_{s\omega_0}$ and T_{sv}, for all $s \in S$.

For this finite S, choose a number field L containing K, defining the marked curve, differential and CM action, and containing the coordinates of the torsion point represented by w. For any automorphism fixing L, take $C' = C$ and $w' = w$, so $v = \omega_0$. The Taylor identity fixes all coefficients of $T_{s\omega_0}\vartheta_C$ for $s \in S$. Thus they belong to $\mathbb{C}^{\mathrm{Aut}(\mathbb{C}/L)} = L$.

To obtain a single finite adele for all the rational translation parameters, write $\Lambda_{C'} = \omega_\sigma\mathfrak{c}^2/N\mathfrak{c}$. Here we choose $\mathfrak{c}$ integral. Such a presentation exists because the field discriminant $-d$ is prime in absolute value. Quadratic genus theory [24, Theorems 5.30 and 6.1(iii)–(iv)] then gives $\mathrm{Cl}(K)/\mathrm{Cl}(K)^2 = 1$. Its area is $|\omega_\sigma|^2\sqrt{d}/2$. Put $e = N\mathfrak{c}$. Since $eO_K \subset \mathfrak{c} \subset O_K$, we have $eO_K \subset \mathfrak{c}^2/e \subset e^{-1}O_K$. The ratios $b_S = v/\omega_\sigma$ of the periods just chosen lie in the compact set $e^{-1}\widehat{\mathcal{O}}_K$ of finite adeles.

Take a cluster point β_σ of this net over finite sets directed by inclusion. For any fixed s, choose $m > 0$ with $ms \in \mathcal{O}_K$. A rational representative x' sufficiently close to $\beta_\sigma s$ is congruent to $b_S s$ modulo $2me\widehat{\mathcal{O}}_K$ for some sufficiently late S containing s. The period $e\omega_\sigma$ belongs to $\Lambda_{C'}$, so the two translation arguments differ by $2m$ times a period.

As $mb_S s\omega_\sigma$ is a period, the Riemann form phase for this difference is the exponential of $2\pi i$ times an integer, and the doubled-period semicharacter is one. The two translated theta sections are therefore identical. This proves the required conjugation with a single β_σ and with the asserted marked period area. In particular, one $\beta_\sigma \in \mathbb{A}_{K,f}$ and one ω_σ work simultaneously for every rational translation parameter. No choice of either number depends on the number of theta factors.

The period scalar under conjugation.

Put $c_K = \gamma_{\mathrm{Euler}} - L'/L(1, \chi_K)$ for this paragraph and let h be the class number of K. We recall the unit normalization of the Chowla–Selberg product. The same genus-theory calculation gives odd h, since squaring is a bijection on the finite class group. Choose fractional ideals $\mathfrak{c}_i$ representing all classes with product O_K. This can be done by taking $\mathfrak{b}_{i^2}/\mathfrak{b}_i$ from any system

$\mathfrak{b}_i$ of representatives. Choose marked, normalized CM curves C_i with lattices $\Lambda_i = \mu_i \mathfrak{c}_i \Lambda_C$. For each embedding fixing K, algebraic conjugation of the CM isogenies and their differentials gives

$$\Lambda_i^\sigma = \sigma(\mu_i)\mathfrak{c}_i \Lambda_C^\sigma.$$

Indeed, for every $a \in \mathfrak{c}_i$, multiplication by $\mu_i a$ is a homomorphism of the uniformized elliptic curves. For $a \neq 0$, its kernel consists of algebraic torsion points. The quotient isogeny and the ensuing pointed isomorphism between the two algebraic elliptic curves are defined over $\overline{\mathbb{Q}}$. Thus its multiplier $\mu_i a$ on the algebraic invariant differentials is algebraic, and conjugation sends it to $\sigma(\mu_i)a$. This gives inclusion. The inverse assertion follows by applying the same argument with $\mu_i^{-1}\mathfrak{c}_i^{-1}$ and interchanging the curves.

Set $u^{12h} = \prod_i \mu_i^{12} \Delta_{C_i}$. At any coefficient place the marked differentials are unit multiples of good-model differentials by the preceding normalization. Each $\mu_i a$, $a \in \mathfrak{c}_i$, is therefore integral as the tangent map of a homomorphism of good elliptic schemes. Since $\prod_i \mathfrak{c}_i = O_K$, a finite sum of products of such a's equals one. It follows that $\prod_i \mu_i$ is integral. Interchanging the curves gives the same assertion for its inverse. The Δ_{C_i} are units and hence so is u.

For any complex lattice $\Lambda = \omega(\mathbb{Z} + \tau\mathbb{Z})$ of area A use the analytic discriminant $\Delta = (2\pi)^{12}\omega^{-12}\eta(\tau)^{24}$. Kronecker's first limit formula [67, Ch. I, §1, Theorem 1] in the Epstein-zeta normalization [64, §1], applied to

$$\sum_{\ell \in \Lambda \setminus \{0\}} \left(\frac{2A/\sqrt{d}}{|\ell|^2} \right)^s$$

gives residue $2\pi/\sqrt{d}$ at one and constant term

$$\frac{2\pi}{\sqrt{d}} \left(2\gamma_{\text{Euler}} + \log(2\pi^2/\sqrt{d}) - \log A - \tfrac{1}{6} \log |\Delta| \right).$$

For lattices representing all the classes the sum of these series is $2\zeta(s)L(s, \chi_K)$, by grouping elements of each representative ideal into their unit orbits. The factor two is $|O_K^\times|$. The class number formula identifies the residues. Write A_i^σ for the area of C_i^σ. Dividing the summed constant terms by the summed residues gives

$$\frac{1}{h} \sum_i \log\left(A_i^\sigma |\sigma \Delta_{C_i}|^{1/6} \right) = \log(2\pi^2/\sqrt{d}) + \gamma_{\text{Euler}} - \frac{L'}{L}(1, \chi_K).$$

Since $A_i^\sigma = |\sigma(\mu_i)|^2 N\mathfrak{c}_i A_C^\sigma$, the product of the quantities inside the logarithms is $(A_C^\sigma |\sigma(u)|^2)^h$. Hence

$$A_C^\sigma |\sigma(u)|^2 = \frac{2\pi^2}{\sqrt{d}} e^{c_K}.$$

The discriminant in this identity is the algebraic Weierstrass discriminant for the indicated invariant differential.

Use the splitting character υ of Section 4.3, and set $\lambda = \upsilon | \cdot |_E^{-1/2}$. The prescribed root is $s_0^f = \lambda_f(\alpha_{\text{W}})^{-1}$. Choose $h > 0$ so that the finite ideal of α_{W}^h is principal, generated by $y \in E^\times$. The character λ_f is trivial on finite integral units, and the complex idele norm at v is $|\Phi_v(y)|^2$. Hence

$$\lambda_f(\alpha_{\text{W}})^h = \lambda_f(y) = \lambda_\infty(y)^{-1} = \prod_v \overline{\Phi_v(y)} = \overline{\Phi_K(N_{E/K} y)}.$$

Thus $s_0^{fh} \in \Phi_K(K)^\times$, and its roots have the same modulus under all complex embeddings. The identity $\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}\widehat{O}_E = \mathfrak{d}_{E/K}^{-1}\widehat{O}_E$ gives $|\alpha_{\mathrm{W}}|^2 = N_{E/\mathbb{Q}}\mathfrak{d}_{E/K}$. Discriminant transitivity and $[E:K] = f$ give

$$|\sigma(s_0)|^2 = |\alpha_{\mathrm{W}}|^{1/f} = \mathrm{rd}(E)/\sqrt{d}.$$

Set $p_F = s_0/u$, $a_d = \sqrt{2\pi/\sqrt{d}}$, and $\gamma = p_F a_d/\omega_0$, where $\Lambda_C = \omega_0 O_K$. For the normalized period ω_σ determined above, $A_C^\sigma = |\omega_\sigma|^2\sqrt{d}/2$. Therefore the transported jet scale $t_\sigma = \sigma(p_F)a_d/\omega_\sigma$ satisfies the exact identity

$$|t_\sigma|^2 = \frac{\pi|\sigma(s_0)|^2}{A_C^\sigma|\sigma(u)|^2} = \frac{\mathrm{rd}(E)}{2\pi}e^{-c_K} = \Omega^{-1}.$$

The cyclotomic product in the finite Weil model.

Put $\mathfrak{a} = E \cap \alpha_{\mathrm{W}}\widehat{O}_E$. The trace pairing of $\mathfrak{a}$ with its complex conjugate is integral over O_K because $\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{d}_{E/K}^{-1}$. At each finite place α_{W} is 1 or the inverse of a generator of the relative different (Section 8.1), so $O_E \subseteq \mathfrak{a}$. In particular $\mathrm{Tr}_{E/K}(\mathfrak{a}\overline{\zeta_q^k}) \subseteq O_K$ for every k. For the standard definite space of any positive rank v, write $D = fv$ and use Fock coordinates

$$z = (z_{\tau,i})_{\tau\in\Phi,\,1\le i\le v}, \qquad z(\gamma)_{\tau,i} = a_d\tau(\gamma_i) \quad (\gamma \in E^v).$$

The lattice $\mathfrak{a}^v$ in these coordinates is self-dual for the alternating polarization of the standard Fock form.

Indeed, up to sign the alternating pairing $\mathrm{Im}\langle z(\gamma), z(\delta)\rangle/\pi$ is $\mathrm{Tr}_{E/\mathbb{Q}}(j_{\mathrm{W}}\sum_i \bar{\gamma}_i\delta_i)$, and $j_{\mathrm{W}}\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{d}_{E/\mathbb{Q}}^{-1}$. Let $\Lambda = \widehat{O}_E^v$ be the normalized finite translation lattice. The normalized finite translate by $\alpha_{\mathrm{W}}^{-1}\gamma$ corresponds to the signed real shift by $-z(\gamma)$. Indeed, a rational translation has real and finite coordinates $(z(\gamma), \alpha_{\mathrm{W}}^{-1}\gamma)$, and translating it out of the adelic theta functional gives the real shift

$$F(z) \longmapsto e^{\langle z(\gamma),z\rangle - |z(\gamma)|^2/2}F(z - z(\gamma))$$

on the entire function obtained by dividing its real slice by $e^{-|z|^2/2}$.

Here the displayed shift is for the unnormalized finite translate. For $\gamma \in \mathfrak{a}^v$, the normalization is the factor $e^{-\pi i\,\mathrm{Tr}_{E/K}\sum_i \gamma_i\bar{\gamma}_i}$. It follows from the product formula for the additive character with trace-one element $(1+\sqrt{-d})/2$. The exponent is $-\pi i$ times an integer by the trace pairing, so the sign is one on any even multiple of $\mathfrak{a}^v$.

Let ω denote ω_0 or ω_σ, and use the corresponding rank-one theta curve. Take $e = 1$ in the first case and the positive integer e fixed above in the second, so ωe is a period. For $0 \le k < q$ define

$$A_{k,i}(z) = \frac{\omega}{qa_d}\sum_\tau \overline{\tau(\zeta_q^k)}z_{\tau,i}, \qquad c_k(y) = -\frac{1}{q}\,\mathrm{Tr}_{E/K}(y\overline{\zeta_q^k}).$$

Then $A_{k,i}(-z(\gamma)) = \omega c_k(\gamma_i)$. The geometric-series identity on the distinct q-th roots $\tau(\zeta_q)$ gives

$$\sum_{0\le k<q}\tau(\zeta_q^k)\overline{\rho(\zeta_q^k)} = q\,\mathbf{1}_{\tau=\rho}.$$

As $A_C = |\omega|^2\sqrt{d}/2$, the pullbacks of the degree-one Hermitian form $\pi\bar{u}z/A_C$ along the $A_{k,i}$, each repeated q times, sum to the standard Fock Hermitian form.

The entire function

$$\Theta(z) = \prod_{k,i} \vartheta_C(A_{k,i}(z))^q$$

therefore has the required Gaussian multiplier on the lattice $z(qe\mathfrak{a}^v)$. At any point of that lattice, each $A_{k,i}$ is a period by trace integrality and the choice of e. Each rank-one sign is raised to the even power q. Their product is one, in agreement with the Fock sign on this lattice. Also $\Theta(0) = 1$.

To identify this function in the original Weil model, let $l = qe$. The translates of the marked unit vacuum by $l^{-1}\Lambda/\Lambda$ are pairwise orthogonal, by its vacuum matrix coefficient, and are fixed by the normalized translations by $l\Lambda$, by the Heisenberg commutator. Their entire functions are linearly independent by the injectivity in Section 20.1. That argument transports the real evaluations to arbitrary finite displacements by rational Heisenberg invariance and uses the polarized norm identity.

These entire functions belong to the polarized theta space for $z(l\mathfrak{a}^v)$ with the displayed signed multiplier. That space has dimension l^{2D}: the original alternating polarization is unimodular and passing to $l\mathfrak{a}^v$ multiplies every alternating elementary divisor by l^2. Riemann–Roch for the positive line bundle gives this dimension; equivalently, the polarization type $(l^2, \dots, l^2)$ gives the same dimension by [57, §1, Proposition 3 and Theorem 2, pp. 295–297]. It equals $[l^{-1}\Lambda : \Lambda]$. Thus Θ is the entire function of a vector in the original finite part; denote it by F or F' according as the curve is original or conjugate.

For a rational translate indexed by $\gamma \in E^v$, the Gaussian shift identity and the equality of Hermitian forms give the entire function

$$\prod_{k,i} (T^C_{\omega c_k(\gamma_i)} \vartheta_C)(A_{k,i}(z))^q.$$

Its constant coefficient is a product of integral torsion values, since each $c_k(\gamma_i)$ lies in K and a multiple of ω is a period. Finite smoothness and density of rational points give the same conclusion at every finite adelic translate.

For the original curve $e = 1$. Write its finite expansion as

$$F = \sum_{x \in q^{-1}\Lambda/\Lambda} c_x T(x) v_0.$$

Choose representatives supported at the places dividing q. Take x_0 with $c_{x_0} \neq 0$, which exists since $\Theta(0) = 1$, and put $y = -x_0$. Write ψ for the normalized finite character, trivial on the fixed subalgebra. For $u \in \Lambda$, the Heisenberg commutator acts on the summand $T(y)T(x)v_0$ by $\psi(u^{\mathsf{T}}\overline{x - x_0})$ under $T_\sharp(u)$. This character is trivial on $q\Lambda$ for every x, and on Λ exactly when $x - x_0 \in \Lambda$, by self-duality. Also $T(-x_0)T(x_0) = 1$. Thus $T(y)F$ is fixed by $T_\sharp(q\Lambda)$, and

$$\sum_{u \in \Lambda/q\Lambda} T_\sharp(u) T(y) F = [\Lambda : q\Lambda] c_{x_0} v_0.$$

Both y and the averaging representatives may be taken zero above MN, since $(q, MN) = 1$. For $v = 1$ this is (AV), with $[\Lambda : q\Lambda] = q^{2f}$. Its finite values are integral, since each summand has integral finite values and its translation factors are roots of unity. At every rank this places v_0 in the span of the finite translates of F; translating the identity places every translate of v_0 there, and these span the finite part.

Finally put b_σ equal to the image of β_σ in $\mathbb{A}_{E,f}$. At a rational translate γ, choose one element $a \in K$ sufficiently close to β_σ that the translate of F' by $a\alpha_{\mathrm{W}}^{-1}\gamma$ agrees with its translate by $b_\sigma\alpha_{\mathrm{W}}^{-1}\gamma$, and that every $ac_k(\gamma_i)$ belongs to the neighborhood required by the rank-one conjugate translation identity. These are finitely many open conditions on a, and K is dense in its finite adeles.

The coefficient of the τ, i variable in the original scaled form is

$$\frac{p_F}{q}\overline{\tau(\zeta_q^k)}.$$

After applying σ it is the coefficient in the new form at scale t_σ, with the permutation of the embeddings over K. Indeed $\overline{\tau(\zeta_q^k)} = \tau(\zeta_q^{-k})$, and $\omega_\sigma t_\sigma/a_d = \sigma(p_F)$. Apply the rank-one translated Taylor identity to each factor and compose with these linear forms. This proves (AJ) at rational translates and hence at all finite adelic translates by local constancy. The same b_σ and t_σ work for every v, in particular for $v = 1$ and for $v = n_j n_{j+1}$ in the same marked upper kernel.

The finite Weil action and vacuum projection. Let $V = \mathrm{Hom}(X_j, X_{j+1})$ in the marked definite models. Write v_0 for its unit vacuum, $\Lambda = \widehat{\mathcal{O}}_E^{\dim_E V}$ for the standard lattice in the normalized Heisenberg coordinates, and ψ for the self-dual finite additive character in those coordinates. Fix $\sigma \in \mathrm{Aut}(\mathbf{C}/K)$. The translated CM identity, first in rank one and then on the marked cyclotomic products, gives initial vectors F, F' and a finite adele b_σ satisfying (AJ). Its invertibility follows below.

First recover the full additive character. In rank one write the unnormalized finite translation as $T(x)$. Its group law in the marked model is

$$T(x)T(y) = \psi(x\bar{y}/2)T(x+y).$$

The two seeds are smooth in the finite translation variables. Choose a positive integer n for which $T(n)F = F$ and $T(b_\sigma n)F' = F'$. Such an integer exists because the two open stabilizer neighborhoods contain an open additive lattice and a sufficiently large integer clears the finitely many local conditions on $1, b_\sigma$.

For any $z \in \mathbb{A}_{E,f}$ write $z = nx$. Apply the group law to $T(2x)T(n)F$ and $T(2b_\sigma x)T(b_\sigma n)F'$. The two cocycle factors are $\psi(nx)$ and $\psi(b_\sigma\overline{b_\sigma}nx)$, respectively. Both comparisons involve the same conjugated jet of the translate at $2x+n$, which is nonzero. Indeed the full jet at a nonzero scale determines the entire theta function by holomorphy, and hence determines the finite vector by the same injectivity argument. Cancellation therefore gives

$$\sigma(\psi(z)) = \psi(b_\sigma\bar{b}_\sigma z) \qquad (z \in \mathbb{A}_{E,f}). \tag{JV1}$$

The comparison at the two translations uses the full cocycle in the group law.

For the higher rank product, use the relation (JV1) and the translates of its seed F. They span the finite part. If a linear combination of them is zero, the σ-conjugated combination of the translated F' has zero full jet after the common variable permutation. The injectivity just proved shows that combination is zero. Hence the rule defines a σ-semilinear map C on the finite part. The group law and (JV1) show $CT(x) = T(b_\sigma x)C$ on every generator and hence on the whole finite part. The normalized translations $T_\sharp(x)$ have the additional factor $\psi(c_{\mathrm{tr}}\bar{x}^{\mathrm{t}}x/2)$ in this model, where $c_{\mathrm{tr}} = (1+\sqrt{-d})/2$ is the trace-one element in Lemma 17.1. Equation (JV1) conjugates that factor to the factor at $b_\sigma x$. Thus the same identity holds with $T_\sharp$. The

full jet identity gives injectivity of C, since $\gamma \neq 0$ and the jets at that original scale determine the vector.

Put $u = b_\sigma \bar{b}_\sigma$. Self-duality of $\widehat{\mathcal{O}}_E$ for ψ, applied to (JV1) at all zy with y integral, gives

$$uz \in \widehat{\mathcal{O}}_E \quad \Longleftrightarrow \quad z \in \widehat{\mathcal{O}}_E.$$

Testing in each local factor at 1 and at the inverse of a uniformizer proves $u \in \widehat{\mathcal{O}}_E^\times$. Thus $b_\sigma^{-1} = \bar{b}_\sigma u^{-1}$ is a finite adele, so b_σ is a finite idele. Choose $a \in \widehat{\mathcal{O}}_E^\times$ with $a\bar{a} = u$ as follows. At a split pair above w, use $(u_w, 1)$. At a nonsplit place v, the involution preserves the valuation, so $2v(b_{\sigma,v}) = v(u_v) = 0$. Take $a_v = b_{\sigma,v}$ there. Thus $b_\sigma = ha$ for the finite norm-one scalar $h = b_\sigma/a$. Set $C^\circ = \omega(h)^{-1}C$; it conjugates the normalized translations by x to those by ax.

Since $a\Lambda = \Lambda$, the vector $C^\circ v_0$ is fixed by all $T_\sharp(x)$ with $x \in \Lambda$. Its fixed line is one-dimensional. To see this, the translates of v_0 indexed by distinct cosets of Λ are orthogonal by the support of the vacuum matrix coefficient. Each coset is an eigenspace for all lattice translations. Its character is trivial only for the zero coset, by the self-duality of Λ for the polarized Heisenberg pairing. Therefore $C^\circ v_0 = \eta v_0$ with $\eta \neq 0$. Equivalently $Cv_0 = \eta\omega(h)v_0$. This is the shape of the conjugate vacuum in the original model.

For a finite unitary matrix g, let Δ_g be the finite Weil multiplier. Averaging the lattice projection over Λ gives the finite formula

$$\omega(g)v_0 = \frac{\Delta_g}{|\Delta_g|^2} \sum_{q \in \Lambda/(\Lambda \cap g^{-1}\Lambda)} T_\sharp(gq)v_0.$$

Changing a representative q by q' in $\Lambda \cap g^{-1}\Lambda$ leaves the term unchanged. On Λ the normalized displacements satisfy $T_\sharp(q + q') = T_\sharp(q)T_\sharp(q')$. Unitary invariance of both the displacement pairing and the normalizing quadratic factor gives the same identity at gq, gq', while $T_\sharp(gq')v_0 = v_0$. Each member of the finite quotient has mass $|\Delta_g|^{-2}$, since $|\Delta_g|^2$ is its index.

Multiplication by a permutes these cosets, because a is a scalar integral unit. Applying C° therefore gives

$$C^\circ\omega(g)v_0 = \frac{\sigma(\Delta_g)}{\Delta_g}\,\omega(g)C^\circ v_0.$$

Heisenberg covariance extends this to the spanning translates of v_0. The scalar matrix h commutes with g, so the same identity holds with C. The multiplier satisfies

$$\Delta_g^2 = |\Delta_g|^2 v(\det g)^{-1}.$$

The index is rational, and the finite values of the splitting character on norm-one ideles have modulus one at every complex embedding.

For the latter assertion, a positive power of a norm-one idele can be written as a rational norm-one scalar times an integral norm-one idele. Indeed, if its ideal is I and $I^d = (\alpha)$, then $I^{2d} = (\alpha/\bar{\alpha})$ since $I\bar{I} = 1$. The finite character of the integral factor has finite image. The value at the rational factor is the inverse archimedean weight monomial in its conjugates, all of absolute value one. This gives

$$\left|\frac{\sigma(\Delta_g)}{\Delta_g}\right| = 1.$$

The same argument applies to the finite splitting characters of the pair action. Finite Heisenberg phases are roots of unity.

Write f_{int} for the integral vacuum and $Cf_{\mathrm{int}} = \eta_{\mathrm{int}}\omega(h)v_0$. Constant terms of the translated jets give the pointwise identity

$$(Cf_{\mathrm{int}})(0, \alpha_{\mathrm{W}} b_\sigma x) = \sigma(f_{\mathrm{int}}(0, \alpha_{\mathrm{W}} x)) \qquad (x \in \mathbb{A}_{E,f}^{\dim_E V}).$$

As x varies, the left side gives every finite-slice value of Cf_{int} because b_σ is invertible. In the quotient norm below, $\|Cf_{\mathrm{int}}\| = |\eta_{\mathrm{int}}|$ by unitarity and $\|v_0\| = 1$.

The Schur-jet integral. For this step, a bound on the finite slice bounds the norm in the vacuum quotient. A rational translation moves any archimedean argument of rational coordinates into the finite slice, up to a unit phase, and density and continuity give the bound on the whole quotient. The unit vacuum has norm one in that quotient. The Gaussian lattice bound on $\omega(g)v_0$ now applies to $C(T(x)\omega(g)v_{\mathrm{int}})$, uniformly in the old translation x and the old Weil argument g. This is the pointwise bound for the coefficientwise conjugate Schur-jet integral. In particular this step concerns the upper kernels before any source form is substituted. For fixed finite arguments their Schur-jet entries are algebraic. The finite lattice formula and (AV) express the normalized vacuum terms through finitely many translates of the cyclotomic seeds. Choose rational representatives modulo their open stabilizers. Their rank-one Taylor coefficients lie in the number fields just constructed, and the scaled linear forms have coefficients $p_F\overline{\tau(\zeta_q^k)}/q$. The remaining factors are Heisenberg roots of unity, finite splitting-character values and the algebraic vacuum multipliers.

For the resulting estimate, let $J_\gamma F$ be the Taylor series at the original jet scale γ. For a polynomial Q, write $j_Q(J_\gamma F)$ for $Q(\partial)(J_\gamma F)(0)$. The Schur kernel is tested by taking Q to be a fixed matrix entry, or the contraction with a source vector and a dual target vector. For any additional scalar c, put

$$Q_\sigma = \pi_\sigma \cdot (\sigma_{\mathrm{coeff}}(Q(cX))),$$

where π_σ acts by permuting the embedding coordinates. Relabel each source and target Weyl factor together with its diagram. For the corresponding permutation matrix P on the integral tensor basis, the Gram matrices in the two orders satisfy $P^{\mathsf{t}} J_{\mathrm{new}} P = J_{\mathrm{old}}$. Thus the relabelling preserves the operator norms in these metrics.

The chain rule for monomials and the transported Taylor identity give

$$\sigma(j_Q(J_{c\gamma}F)) = j_{Q_\sigma}(J_{t_\sigma}CF) = \int_{\mathbf{C}^D} Q_\sigma(t_\sigma \bar{y}) e^{-\|y\|^2/2}\, CF(y_{\mathbb{A}})\, \frac{dy}{\pi^D}.$$

Here $y_{\mathbb{A}}$ has finite component zero and archimedean components $(2\pi \operatorname{Im}\tau(j_{\mathrm{W}}))^{-1/2} y_\tau$ in the original adelic coordinates. Write $H(y) = e^{\|y\|^2/2} CF(y_{\mathbb{A}})$ for the entire function of this finite vector. For multi-indices μ, ν,

$$\pi^{-D} \int_{\mathbb{C}^D} y^\mu \bar{y}^\nu e^{-\|y\|^2}\, dy = \mathbf{1}_{\mu=\nu}\, \nu!, \qquad \partial^\nu H(0) = \nu!\, [y^\nu] H.$$

The absolute value of CF descends to the compact additive adelic quotient, so its continuous values are bounded by some B_F. For $Q_\sigma(Z) = Z^\nu$ the absolute integrand is bounded by $B_F |t_\sigma|^{|\nu|} \|y\|^{|\nu|} e^{-\|y\|^2/2}$, which is integrable. At fixed radii the Taylor series of H converges uniformly on the product of circles. Angular integration selects its ν-th coefficient, and radial integration gives the displayed factorial. Linearity gives the formula for Q. Here $|t_\sigma|^2 = \Omega^{-1}$ at the transported original scale, and for the actual arrow scale $c = \pi_M = \bar{\alpha}$ one has $|\sigma(c)|^2 = M$. The variable in the conjugated polynomial consequently has squared norm $(M/\Omega)\|y\|^2$.

For the short-image bound, apply the measured operator identity on the original ramified source first. Then apply σ to the terminal coordinates of its finite upper-chain class sum. This coefficientwise operation changes neither the actual source generators nor the choice of measured point and functional. In particular it requires no continuity of σ in the real topology or conjugation of the integrals defining the Poincaré map. The same measured identity separately gives the local gauge comparison for every allowed twist on the clearing module.

Since K is imaginary quadratic, an arbitrary σ either fixes K or $c \circ \sigma$ fixes K, where c is ordinary complex conjugation. The Weyl Gram matrices are rational. Entrywise ordinary conjugation preserves the operator norm between these two positive Gram spaces. Thus the bound obtained for the embeddings fixing K also gives the bound at every complex automorphism. □

7.4.4 Gaussian bounds for the conjugated kernels

In this subsection, we apply the vacuum identity of Lemma 7.6 and integrate the conjugated jets. The resulting operator norm is used in Section 7.4.5.

Lemma 7.7 (Norm of one conjugated kernel)**.** *For $0 \le j < m$, let $D_j(g,h;u)$ be the Schur-jet matrix at scale $\pi_M\gamma$ of $\omega_{\mathrm{f}}(g,h)T_\sharp(u)v_{\mathrm{int}}$ for the untwisted arrow $X_j \to X_{j+1}$. Here v_{int} is the normalized integral vacuum, u is one finite Heisenberg translation, and g,h are arbitrary points of the two finite adelic unitary groups. For sufficiently large x,*

$$\|\sigma D_j(g,h;u)\|_{\mathrm{op,W}} \le \exp(x^{50} + S_j/2) \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Proof. For a conjugated translated vacuum the bound is e^{C_j}, where

$$C_j = C_K(q_1 n_j n_{j+1} + 1)^2, \qquad n_j = \dim_E X_j.$$

It is uniform in the finite translates. The preceding transport gives a scalar multiple of a unitary move of the vacuum and bounds the scalar by its finite values. For this last finite bound, the normalized CM translation law used in (AJ) sends every conjugate of a torsion value to a value on a conjugate elliptic lattice.

After multiplying by the Gaussian part of the translation, its absolute value is periodic on that lattice. Its maximum is finite and unchanged by homothety. The finitely many lattice classes and the marked two-torsion choices therefore give a common bound $M_K \ge 1$ depending only on K. Each finite value of the rank-one theta product has q_1^2 such factors, by the repeated cyclotomic resolution.

Equation (AV) expresses this same multiple of the vacuum as a sum of q_1^{2f} translates with root-of-unity factors. Hence the finite values of the scaled rank-one vacuum have conjugate absolute value at most

$$q_1^{2f} M_K^{q_1^2} \le \exp\big((2 + \log M_K) q_1^2\big).$$

Tensoring over the $n_j n_{j+1}$ rank coordinates multiplies these bounds. The inner product is the mean of the squared values on the vacuum quotient.

By rational translation, arguments whose archimedean coordinates come from rational vectors can be moved to arguments of archimedean coordinate zero, up to unit phases. Density and continuity give the same bound for the quotient norm, which bounds the scalar in the unit vacuum presentation.

For the ordinary moved vacuum, put $f_y = T(y)\omega(g)v_0$. With the commutator convention $T(z)T(\gamma) = \beta(z,\gamma)^2 T(\gamma)T(z)$, rational invariance and the polarized probability mean of Section 41.5.1 give

$$\int \beta(z,\gamma)^2 |\theta(T(z)f_y)|^2 \, dz = \langle f_y, T(\gamma) f_y \rangle.$$

Thus the coefficient on the right corresponds to the quotient character $z \mapsto \beta(z,\gamma)^{-2}$.

By unitarity, the absolute value of this coefficient equals the ordinary vacuum matrix coefficient in modulus at $g^{-1}\gamma$. The moved vector y only contributes its commutator phase. The Gaussian vacuum formula bounds the coefficient in the support by $\exp(-\pi \|g^{-1}\gamma\|_\infty^2/\sqrt{d})$; the coefficient vanishes outside the finite support $(g^{-1}\gamma)_f \in \alpha_{\mathrm{W}} \widehat{\mathcal{O}}_E^{\mathrm{rk}_E V}$. On this support the Hermitian product of a difference has finite ideal contained in the relative codifferent, because its two finite factors are in $\alpha_{\mathrm{W}}\widehat{\mathcal{O}}_E$ and the ideal of $\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}$ is contained in that codifferent.

In the unscaled embedding coordinates $u_{\tau,i} = (g^{-1}\gamma)_{\tau,i}$, one has

$$\mathrm{Tr}_{E/K} \, h(\gamma,\gamma) = \sum_{\tau,i} |u_{\tau,i}|^2 = \|g^{-1}\gamma\|_\infty^2.$$

For the difference of two support points this is a positive integer whenever the points are distinct, since $\mathcal{O}_K \cap \mathbb{R} = \mathbb{Z}$. The Fock coordinates are $a_d u$ with $a_d^2 = 2\pi/\sqrt{d}$, giving the Gaussian in the preceding estimate. Thus the lattice is one-separated. Packing unit-separated points bounds the sum of $e^{-(\nu+1)\|v\|}$ in dimension ν by $3^\nu e^\nu$. The counting function in real dimension ν is at most $(1+2r)^\nu \le 3^\nu e^{\nu r}$ for a ball of radius r.

Integrating against $(\nu+1)e^{-(\nu+1)r}\,dr$ gives the claimed packing bound. Completing the square gives $e^{-\pi\|v\|^2/\sqrt{d}} \le e^{\sqrt{d}(\nu+1)^2} e^{-(\nu+1)\|v\|}$. Thus the Fourier series is absolutely convergent and its value at zero is $|\theta(f_y)|^2$. Since $\nu = 2f \, \mathrm{rk}_E V \le 2q_1 \, \mathrm{rk}_E V$, its square root is bounded by $\exp((2d+6)(q_1 \, \mathrm{rk}_E V + 1)^2)$. Together with (AV) and the multiplier comparison this yields the stated e^{C_j} bound.

To calculate the Gaussian moment, use $L_a = a+1$ and $v_M = \log(M/\Omega)$. For an array Y in the Gaussian integral, put $t_Y = \|Y\|^2/(2L_a+2)$. First let σ fix K. Apply the Schur-jet integral of Lemma 7.6 entrywise in the rational Schur matrix. After the permutation $\pi_\sigma(\tau) = \sigma \circ \tau$ its argument is

$$Z_\tau = \sigma(\pi_M) t_\sigma \overline{Y_{\pi_\sigma(\tau)}}, \qquad \sum_{\tau,i} \rho_{\tau,i} = \sum_\tau \|Z_\tau\|_{\mathrm{HS}}^2 = e^{v_M}(2L_a+2)t_Y,$$

where $\rho_{\tau,i}$ are the squared singular values of Z_τ, in decreasing order at each place. Write i for the pair (τ,i) in the products and sums below. The column counts of Section 20.2 vanish for $n_j < h < n_{j+1}$. Give the exterior powers their induced Hermitian forms. In the Schur construction each column of height n_j uses the unit volume vector in $\bigwedge^{n_j} \mathbb{C}^{n_j}$ at the source. For $1 \le h \le n_j$, singular value decomposition gives

$$\|\bigwedge^h Z_\tau\|_{\mathrm{op}}^2 = \textstyle\prod_{0 \le i < h} \rho_{\tau,i}.$$

The Weyl form is the restriction of the exterior tensor form. Multiplying over columns therefore bounds the squared Schur norm by $\prod_i \rho_i^{\lambda_i}$, including the full-height factors. For each exponent λ_i,

$$\rho_i^{\lambda_i} \le \lambda_i! \exp\Big(\lambda_i(v_M + L_a^{-1}) + \rho_i e^{-v_M - L_a^{-1}}\Big).$$

Indeed it is the inequality $q^n/n! \le e^q$ with $q = \rho_i e^{-v_M - L_a^{-1}}$. Since $(2L_a+2)e^{-1/L_a} \le 2L_a$, the square root of the product is bounded by

$$\exp\left(\tfrac{1}{2}\sum_i(\lambda_i/L_a + \log\lambda_i! + \lambda_i v_M) + L_a t_Y\right).$$

In the conjugated matrix integral take an arbitrary source vector of Weyl norm one and target covector of dual Weyl norm one. Their pairing with the Schur matrix at Z is bounded by the displayed quantity. Taking the supremum over these vectors gives the operator norm for the rational Weyl metrics.

The jet integral uses the Gaussian $e^{-\|Y\|^2/2}$ and Lebesgue measure divided by π in each complex coordinate, so the $L_a t_Y$ term leaves e^{-t_Y}. The resulting Gaussian integral is $(2L_a + 2)^{f n_j n_{j+1}}$. The remaining character and semicharacter factors have conjugate modulus one; an automorphism of $\mathbb{C}$ can be composed with complex conjugation to fix K without changing the Weyl norms. The operator norm is therefore at most

$$\exp\left(\frac{S_j}{2} + C_j + f n_j n_{j+1}\log(2L_a+2) + \frac{1}{2L_a}\sum_{\tau,i}\lambda_{j,\tau,i}\right).$$

For large x, the parameter bounds give $q_1, f \le x^{12}$, $n_m \le x^6$, and $\log(2L_a+2) \le x$. Since $\lambda_{j,\tau,i} \le 2a$ and $L_a = a+1$, the last three terms in the exponent are bounded by $C_K(x^{24}+1)^2 + x^{25} + x^{18} \le x^{50}$. This gives the bound $e^{x^{50}+S_j/2}$ on the kernel. □

7.4.5 Composition and terminal restriction

In this subsection, we combine Lemmas 7.4 and 7.7 with the integral branching of Section 7.5 to bound the full upper composition.

Proposition 7.8 (Pointwise bound after terminal restriction)**.** *Use the untwisted coefficients $\Psi_\varnothing$ of Section 19.3. Fix $\sigma \in \mathrm{Aut}(\mathbb{C})$, and suppose the source form ϕ satisfies $\|\sigma\phi(g)\|_{\mathrm{W}}^2 \le e^{K_{\mathrm{in}}}$ at every finite adelic source point g. For fixed d, ϵ and sufficiently large x, the squared Weyl norm of the coefficientwise conjugated restricted value at each terminal point is at most*

$$\exp\left(K_{\mathrm{in}} + A - m\mathcal{R} + 2\sum_p g_p \log p + E_0\right).$$

Proof. For the untwisted upper operator, first take the class sum of the jointly invariant tensor, then expand at those class representatives as justified in Section 22.3. In each pair of arrows indexed by $0 \le i < m/2$, the shared residue has n_{2i+1} coordinates at each of the f places above π_N, all in $\mathbf{F}_N$. The two adjacent arrow factors use the same residue. Thus opening the N-gluing gives $N^{f\sum_{0\le i<m/2} n_{2i+1}}$ terms per glued product, all with unit phases. The conjugate absolute coefficient sum is therefore at most

$$M^{3fmn_m^2} N^{f\sum_{0\le i<m/2} n_{2i+1}} \le M^{3fmn_m^2} N^{fmn_m/2} \le \exp(x^{60})$$

for large x. Use the step kernel estimate $\exp(x^{50} + S_j/2)$ proved above.

The count in Section 7.2 applies to each definite stage of rank at most n_m with principal level of the marked generators or a containing level, giving at most $\exp(O(x^{52}))$ choices per

step. Let u be the conjugated value before terminal restriction. The triangle inequality and the product of the operator norms give

$$\|u\|_{\mathrm{W}}^2 \le \exp\left(K_{\mathrm{in}} + \sum_j S_j + 2x^{60} + 2mx^{50} + O(mx^{52}) \right).$$

Let $\mathcal{B}_{\mathrm{term}}$ be the integral branching map of Lemma 7.9. The difference of the sums of the highest weights is $d_\tau = a_\tau - mh_0$ at τ. The marked restriction multiplies this map by $\prod_\tau \tau(\overline{\pi_N})^{-d_\tau}$. Since $\pi_N \overline{\pi_N} = N$, at each complex embedding its squared operator norm is at most

$$e^{x^{70}} N^{-\sum_\tau d_\tau} = e^{x^{70}} N^{-|\Phi_a| a + f m h_0}.$$

Combining with Lemma 7.4 gives

$$\|\text{conjugated restricted value}\|_{\mathrm{W}}^2 \le \exp\left(K_{\mathrm{in}} + A - m\mathcal{R} + 2\sum_p g_p \log p + E_0 \right).$$

Indeed, the extra terms satisfy

$$32 f a B + f m h_0 \log N + 2x^{60} + 2mx^{50} + O(mx^{52}) + x^{70} \le E_0/2$$

for large x, since $\log N = O(mB)$, $a = xb$, and b exceeds every fixed power of x.

With $K_{\mathrm{in}} \le E_0$, the good-projector estimate gives the archimedean bound for the short images, using the measured identity (72) in its integral normalization. □

7.5 The integral terminal branching map

In this subsection, we realize the restriction at the terminal definite group by contraction in exterior powers. This gives the integral map used in Section 7.4, with its norm controlled in the Weyl module.

Lemma 7.9 (Integral terminal branching)**.** *On the terminal Weyl modules the branching map is integral, has last-block character $\prod_\tau z_\tau^{a_\tau - m h_0}$, and has squared operator norm at most $e^{x^{70}}$ in the original Weyl metrics for sufficiently large x. Here $a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ and $h_0 = r^8$.*

Proof. Fix τ and write λ, ν for the descending highest weights on X_m, Y_m, respectively. The lower Harish–Chandra parameter begins with the centered string of length $k-1$. For $1 \le i \le m$ its next entries are $-w_i + \Delta/2$ and the centered string of length t translated by $-w_i$. The corresponding upper string has length $t+1$ and center $-w_i$. Comparing the lower entry at h with the upper entry at $h+1$ and subtracting the respective half-sums of positive roots gives, for $1 \le h < n_m$,

$$\kappa'_\tau(h) := \nu_h - \lambda_{h+1} = \begin{cases} (\Delta - t - 1)/2 & h \ge k, \quad t+1 \mid h-k, \\ 0 & \text{otherwise.} \end{cases}$$

At a retained height $h > k$ the upper count is $\lambda_h - \lambda_{h+1} = \Delta - t - 1 = 2\kappa'_\tau(h)$. At $h = k$ it is $w_1 - (k + t + 1)/2 \ge \kappa'_\tau(k)$, since $w_1 = \kappa_\tau + 3\Delta/2$ and $k \le \Delta$. Thus $\lambda_h \ge \nu_h \ge \lambda_{h+1}$. Of the upper columns of height h, retain $\kappa'_\tau(h)$ and contract $\gamma_\tau(h) = \lambda_h - \nu_h$ against the last basis vector. For $1 \le h < n_m - 1$, the lower count is $\nu_h - \nu_{h+1} = \kappa'_\tau(h) + \gamma_\tau(h+1)$.

The terminal Weyl diagram has the two sets of columns prescribed by the mark: in the first set contract each column against the last unit basis vector, reducing its height by one, and in the second set retain the height. The map on the ambient product of exterior powers is the corresponding tensor of contractions and projections to the first block. Each of these maps has operator norm at most one.

After projection onto the desired Weyl summand it gives the normalized complex branching map. The target norm here is the tensor-induced Weyl norm. On lower evaluation vectors, the adjoint of the ambient contraction wedges the shortened columns with the same last unit basis vector. The Hilbert adjoint of the branching map is obtained by projecting this extended tensor onto the upper Weyl subspace. Before projection the pairings are the products of the original minors by exterior Cauchy–Binet. A retained column of full first block height contributes a determinant to the pairing and is absorbed by its determinant character. This is the tensor isometry on the spanning evaluation vectors. For polynomial weights, the one-variable specialization of the Schur branching formula assigns each interlacing ν multiplicity one and last-variable exponent $|\lambda| - |\nu|$ [44, Ch. I, (5.8)–(5.9)]. A common determinant twist gives the same rule for integral highest weights. Hence the nonzero map spans the branching line.

The integral normalization is needed for the gauge comparison. In the minor polynomial model, its test against the highest evaluation vector is the product of shifted leading minors in the contracted columns and unshifted leading minors in the retained ones. After a single permutation of the first-block covectors, all its columns begin with the same initial segment of length $\min(h, q-1)$, where h is the column height and $q = k$, the threshold below which every column is contracted.

Remove the first $q-1$ covectors, truncating at the rank minus one if necessary. If L bounds the number of boxes left, one can clear the indicated highest-vector polynomial by a positive integer $c \le L^{3L}$, with $c = 1$ for the empty diagram. Put $p_\tau = \min(k-1, n_{\mathrm{low}} - 1)$, where $n_{\mathrm{low}} = n_m - 1$, and let $H = \sum_\alpha H_\alpha X^\alpha$ be the product of leading minors of the shortened lower diagram. Its degree and the clearing integer are

$$d = \sum_\tau \sum_{1 \le h < n_{\mathrm{low}}} (h - p_\tau)_+ (\kappa'_\tau(h) + \gamma_\tau(h+1)), \qquad c = \langle H, H \rangle_{\mathrm{Fock}} = \sum_{|\alpha| = d} \alpha! \, |H_\alpha|^2.$$

Here $\alpha! = \prod_j \alpha_j!$ and $d \le L$. If $d = 0$, then $H = 1$ and $c = 1$. Otherwise the determinant expansion bounds $\sum_\alpha |H_\alpha|$ by $\prod_{\mathrm{col}} h! \le L^L$, and $\alpha! \le d! \le L^L$. Consequently

$$1 \le c \le d! \left(\sum_\alpha |H_\alpha| \right)^2 \le L^{3L}, \qquad 2 \log c \le 6L \log L.$$

For the clearing property, the Fock form in the shortened minor basis (P_i) is proportional to the *dual* of the tensor-induced Weyl form by unitary invariance and irreducibility. Write G_{sh} for the Weyl Gram matrix, so its dual form in this basis is G_{sh}^{-1}. The highest minor product has dual Weyl norm one, so the Fock matrix is cG_{sh}^{-1}. If an integral product of minors has coefficient column p, then

$$b_i = \langle P_i, \textstyle\sum_j p_j P_j \rangle_{\mathrm{Fock}} \in \mathbb{Z}, \qquad G_{\mathrm{sh}} b = cp.$$

This proves clearing on the shortened diagram. Put $q' = k - 1$. The lower diagram has no nonempty column of height below q', by its column counts above. In the column tensors let J restore the first q' basis vectors by exterior multiplication in each of these columns, including a

shortened column of height zero. Write W_{sh}, W_0 for the shortened and lower Weyl spaces and Π_{sh}, Π_0 for their orthogonal projections. Evaluation at $\mathrm{diag}(I_{q'}, Z)$ gives

$$J \operatorname{ev}_{\mathrm{sh}}(Z) = \operatorname{ev}_0(\mathrm{diag}(I_{q'}, Z)).$$

Conversely write a lower evaluation array in blocks $X = \left(\begin{smallmatrix} A & D \\ C & E \end{smallmatrix}\right)$, with A of size q'. If u is the number of restored columns at this place, expansion of their minors on $\det A \neq 0$ gives

$$J^* \operatorname{ev}_0(X) = (\det A)^u \operatorname{ev}_{\mathrm{sh}}(E - CA^{-1}D).$$

As evaluation vectors span, polynomial continuation and tensoring give $JW_{\mathrm{sh}} \subset W_0$ and $J^*W_0 \subset W_{\mathrm{sh}}$. Thus $\Pi_0 J = J\Pi_{\mathrm{sh}}$. The shifted minor product uses the restored column tensor, up to the signs of the chosen permutations. Its projected vector multiplied by c is integral, since J is integral and the shortened vector has already been cleared. The integral left inverse of the standard-tableau embedding gives integral coordinates in the original lower Weyl basis.

Integral branching columns. Let $\gamma_\tau(h)$ and $\kappa'_\tau(h)$ be the numbers of contracted and retained columns, respectively, of height h in the upper terminal diagram. Write $P_j(Y)$ for the integral standard minor basis on the upper marked array, and $P_i^0(X)$ for that on the lower one. Put

$$D(X) = \prod_\tau \det(X_\tau)^{\kappa'_\tau(n_{\mathrm{low}})}, \qquad \mathcal{A} = \left\{ D(X) \sum_i (G_0 b)_i P_i^0(X) : b \in \mathbb{Z}^{d_0} \right\},$$

where n_{low} denotes the lower rank and G_0 is its rational Weyl Gram matrix in the integral tableau basis. Write $\mathcal{P}_0(b) = \sum_i (G_0 b)_i P_i^0(X)$. Then $\mathcal{A} = D(X)\mathcal{P}_0(\mathbb{Z}^{d_0})$. In the marked minor model $G_0 = \mathsf{F}_0^{\mathsf{t}}\mathsf{F}_0$ with F_0 the integral basis matrix, so $\mathcal{A}$ is an additive subgroup of $\mathbb{Z}[X]$.

Let $\mathcal{K}(X, Y)$ be the product of the leading minors in $\mathrm{diag}(X_\tau^{\mathsf{t}}, 1)Y_\tau$, putting its last covector first in the contracted columns and last in the retained columns, in the chosen order of the covectors. Let $c > 0$ be the clearing integer already obtained from the shortened diagram and let $\eta \in \mathbb{Z}^{d_0}$ be the corresponding vector of the shifted minor product in the lower Weyl coordinates. Straightening in the upper variable writes

$$c\,\mathcal{K}(X, Y) = \sum_j f_j(X) P_j(Y).$$

The point to prove is $f_j \in \mathcal{A}$, since coefficients in $\mathbb{Z}[X]$ alone leave the lower Weyl Gram denominator unaccounted for.

First suppose at every place that the last covector of Y_τ is $(t_\tau, 0, \ldots, 0)$, and let g_τ be its upper first square block. Laplace expansion along the last covector in each contracted minor gives t_τ per contracted column. Retained columns of full lower height contribute $\det(g_\tau)^{\kappa'_\tau(n_{\mathrm{low}})}$ in addition to $D(X)$. All other minors are those of the lower highest polynomial evaluated at $g_\tau^{\mathsf{t}} X_\tau$, with the same covector permutation as for η. The contracted minors use columns $2, \ldots, h$ of g_τ after Laplace expansion, hence covectors $2, \ldots, h$ after transposition. The retained ones use covectors $1, \ldots, h$. This gives the mixed shifted and unshifted coefficient represented by η. Hence $c\,\mathcal{K}(X, Y)$ is $D(X)\mathcal{P}_0$ of the integral polynomial vector obtained by applying the lower integral Schur matrix $\rho_0^{\mathrm{Schur}}(g)$ to η and multiplying by the displayed powers of $t_\tau, \det(g_\tau)$. The identity holds over the polynomial ring in the entries of Y.

For general Y, write $t_\tau = Y_{\tau,\mathrm{last},1}$. At each place replace column 1 by t_τ times that column, and column $j > 1$ by $t_\tau Y_{\tau,j} - Y_{\tau,\mathrm{last},\mathrm{j}} Y_{\tau,1}$. The new last covector is $(t_\tau^2, 0, \ldots, 0)$. Each leading

minor of height h acquires exactly t_τ^h, since only earlier columns were added. Thus $Y^\mu\, c\mathcal{K}(X,Y)$ is $D(X)\mathcal{P}_0$ of a vector of integer polynomials in Y, with Y^μ the product of these fixed monomial factors. For every Y^δ the coefficient of $Y^{\mu+\delta}$ in this identity shows that $[Y^\delta](c\mathcal{K}) \in \mathcal{A}$. This comparison takes place over $\mathbb{Z}$ and divides no coefficient by an entry of the array.

For the integral coefficient comparison, order the variables first by column and within a column by increasing covector, using the earliest variable first in lexicographic monomial order. The leading monomial of a flag minor is its diagonal monomial with coefficient 1. The leading monomial of a standard tableau product identifies the multiset of entries in every horizontal part of the tableau, by counting the corresponding variables in each matrix column.

The weakly ordered horizontal part of the tableaus determine the tableau from these multisets. Thus the standard tableau products have pairwise distinct monic leading monomials. In any relation with coefficients in an additive group H, take a maximal leading monomial among the nonzero coefficients. Its coefficient must be zero, a contradiction. Apply this with $H = \mathbb{Z}[X]/\mathcal{A}$ to the last coefficient statement. Every f_j lies in $\mathcal{A}$, so write $f_j = D(X)\mathcal{P}_0(b_j)$ with $b_j \in \mathbb{Z}^{d_0}$.

The b_j are the columns of the branching matrix B in the original Weyl bases. With $\mathrm{ev}_+(Y) = (P_j(Y))_j$ as a column vector, its defining identity is

$$D(X)\mathcal{P}_0\big(B\ \mathrm{ev}_+(Y)\big) = c\,\mathcal{K}(X,Y).$$

At the identity lower array this is c times the prescribed highest coefficient. It is nonzero as a polynomial of Y, being a product of nonzero coordinate minors. Thus $B \neq 0$. Substituting $Y \mapsto \mathrm{diag}(g,s)Y$ in this identity and expanding the contracted minors gives

$$B\,\rho_+^{\mathrm{Schur}}(\mathrm{diag}(g,s)) = \left(\prod_\tau s_\tau^{\sum_h \gamma_\tau(h)}\det(g_\tau)^{\kappa'_\tau(n_{\mathrm{low}})}\right)\rho_0^{\mathrm{Schur}}(g)B.$$

Indeed the first-block substitution on the kernel is $X_\tau \mapsto g_\tau^{\mathsf{t}} X_\tau$, which supplies the displayed full-height determinant.

The evaluation vectors span, and $D(X)\mathcal{P}_0$ is injective in characteristic zero, so this is an operator identity. In the present application $\kappa'_\tau(n_{\mathrm{low}}) = 0$, and the common determinant exponent is the last entry of the terminal highest weight,

$$e_\tau = \lfloor n_{m-1}/2\rfloor - w_m(\tau), \qquad n_j = k + j(t+1).$$

Incorporating $\det(\mathrm{diag}(g,s))^{e_\tau}$ therefore replaces the power of s_τ by

$$B_\tau = e_\tau + \sum_h \gamma_\tau(h) = |\mu(X_m)_\tau| - |\mu(Y_m)_\tau|.$$

Here $|\mu|$ denotes the sum of the entries of the highest weight, including its determinant exponent in each entry. Each contracted column loses one box. To evaluate the sum explicitly, put $c_* = h_0 - (t+1)/2$. The contracted column counts at heights 1, k, and $k+i(t+1)$ for $1 \le i < m$ are respectively a_τ, $w_1(\tau) - \lfloor n_1/2\rfloor - c_*$, and c_*. Consequently

$$B_\tau = a_\tau + w_1(\tau) - w_m(\tau) + \lfloor n_{m-1}/2\rfloor - \lfloor n_1/2\rfloor + (m-2)c_* = a_\tau - mh_0,$$

since $w_m(\tau) - w_1(\tau) = (m-1)2h_0$ and $t+1$ is even. For a lower evaluation vector with nonzero square-block determinants, form the extended column tensor divided by the product of the full-height determinant powers.

By exterior Cauchy–Binet the pairings of these divided tensors equal those of the lower evaluation vectors. The latter span the lower Weyl module, so this defines an isometry into the upper ambient column tensor space. The pairing identity above identifies B with c times the adjoint of this isometry composed with the upper isometric tensor inclusion. Thus $\|B\|_{\mathrm{W}}^2 \le c^2$. No Gram inverse is used in the finite-coordinate contraction.

In the lower diagram the count at height $h \ge k$ is $\kappa'_\tau(h) + \gamma_\tau(h+1) \le \Delta$. Indeed at most $\Delta/2$ columns of any height are retained, and at most $\Delta/2$ columns of any height strictly above k are contracted. The contracted columns initially of height k fall to height $k-1$ and leave no boxes after removing the first $k-1$ covectors. Hence

$$\sum_{\tau} \sum_{k \le h < n_m - 1} h(\kappa'_\tau(h) + \gamma_\tau(h+1)) \le L := f n_m^2 \Delta, \qquad 6L \log L \le x^{70}$$

for sufficiently large x. The last block character of the contraction is $\prod_\tau z_\tau^{B_\tau}$, where B_τ is the difference in the total box counts between the two terminal diagrams, with their prescribed determinant weight included. The integral branching operator therefore has squared norm at most $\exp(x^{70})$ in the original Weyl metrics. □

7.6 Norms and index after normalization

In this subsection, we normalize the image of the Poincaré map by a scalar common to the whole family. The estimates of Sections 7 and 7.4 then yield bounds for both the norm and the index in Proposition 5.1.

Proposition 7.10 (Normalization of length and index)**.** *Suppose the following estimates hold for the chosen arithmetic data. Let $U_g : S \to V$ be the translates of the unscaled projected Poincaré map, with the scalar of the seesaw identity removed, and d_X the dimension of the full source before the ramified restriction. By Section 6, $d_X \le 2 \dim S$.*

1. *There is a spanning set of everywhere integral forms $\phi \in S$ such that, for every $g \in G$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\|\sigma\Theta_1 U_g \phi\|^2 \le \exp\left(A - m\mathcal{R} + 2 \sum_{p \in F} g_p \log p + 2E_0 + x^{99} \right).$$

2. *There are $e_p \ge 0$ supported on F, with $\sum_p e_p \log p \le E_0$. At each coefficient place above p, any finite tuple of integral forms in S can be cleared into a submodule of S by a matrix over the valuation ring with determinant valuation at most $d_X e_p$. That submodule works for all translates and all twists of level prime to p. Its images under $\Theta_\xi U_g$ have gauge at most $p^{e_p - g_p}$ for every such twist and translate, where the unit gauge defines the integral lattice.*

Then the maps

$$P_g = c_B U_g, \qquad c_B = \prod_{p \in F} p^{-\lfloor g_p - e_p \rfloor}$$

satisfy the length and index estimates of Proposition 5.1. The clearing matrices for the index work for every twist of level prime to the coefficient prime. For every g, $\mathrm{im}\, P_g = \mathrm{im}\, U_g$.

Proof. Normalize $v(p) = 1$. Above $p \in F$ the gauge of each cleared image is multiplied by $p^{-v(c_B)}$, and

$$e_p - g_p - v(c_B) = e_p - g_p + \lfloor g_p - e_p \rfloor \le 0.$$

For $p \notin F$, $|c_B|_p = 1$. Thus the same clearing matrices give integral images for every allowed twist. For each entry of y choose an integral preimage and its translate. Group the columns by translate g, and let $U_{\kappa,g}$ clear the tuple of preimages in that group, for the coefficient place $\kappa \mid p$. In this ordering take $U_\kappa = \bigoplus_g U_{\kappa,g}$. For every allowed ξ, linearity of $\Theta_\xi P_g$ makes the columns in that block integral, and in the original ordering the same matrix is conjugated by the ordering permutation. Thus

$$v_\kappa(\det U_\kappa) = \sum_{g:\,\text{nonempty block}} v_\kappa(\det U_{\kappa,g}) \le |t| d_X e_p.$$

Consequently

$$\operatorname{Ind}(y) \le |t| d_X \sum_p e_p \log p \le 2|t| \dim S\, E_0.$$

For the archimedean bound, the floors give

$$\log c_B \le -\sum_{p\in F} g_p \log p + E_0 + \sum_{p\in F} \log p, \qquad \sum_{p\in F} \log p \le x_1^2 + \log N \le b.$$

The last inequality follows from $x_1 \le e^W$, $B \ge W^6$, and $\log N = O(mB)$ for large x. Since $c_B \in \mathbb{Q}_{>0}$,

$$\begin{aligned}\|\sigma\Theta_1 P_g \phi\|^2 &= c_B^2 \|\sigma\Theta_1 U_g \phi\|^2 \\ &\le \exp(A - m\mathcal{R} + 4E_0 + 2b + x^{99}) \le \exp(A - m\mathcal{R} + L).\end{aligned}$$

Here $4E_0 + 2b + x^{99} \le faB^4 = L$ for large x. The equality of the images follows from $c_B \ne 0$. □

Section 22.3 verifies the local hypothesis for the same projected map. The seesaw identifies the upper chain, and the local gauge estimate is preserved by the unnormalized good projection and translation.

8 The normalized Poincaré map

In this section, we define the Poincaré map, specify the measures in its orbital expansion, and compute its Petersson blocks. Section 9.1 uses these blocks, and Section 17.3 compares the lifts of the map with the theta construction.

8.1 The finite coordinates

In this subsection, we use the generators of Sections 4.1 and 4.2 to choose the finite point and identify the prescribed level in lattice coordinates.

Let G_{ad} be the ambient unitary group, and write $[G_{\mathrm{ad}}]$ and $[R]$ for the rational quotients. The following choices fix the covector at each scale of Section 4.

For the finite scale, let $D_{\mathrm{rel}} = \Phi'_{q_1}(\zeta_{q_1})$ and $\omega_K = (1 + \sqrt{-d})/2$. Define the finite idele α_{W} to have component D_{rel}^{-1} at places dividing ω_K, and 1 elsewhere. Put $j_{\mathrm{W}} = -1/\sqrt{-d}$. Every

prime of q_1 divides $(d+1)/4 = \omega_K \bar{\omega}_K$, by the fixed congruence on d, and D_{rel} divides q_1 by differentiating $Z^{q_1} - 1$ at ζ_{q_1}. The two ideals of $\omega_K, \bar{\omega}_K$ are coprime, since their generators sum to one.

Conjugation preserves the different ideal. Thus, place by place,

$$j_{\mathrm{W}} \alpha_{\mathrm{W}} \bar{\alpha}_{\mathrm{W}} \widehat{\mathcal{O}}_E = \mathfrak{D}_{E/\mathbb{Q}}^{-1} \widehat{\mathcal{O}}_E.$$

Let c_M be the finite idele equal to the chosen generator $\bar{\alpha}$ at its chosen places and to one elsewhere. For $a_{\mathrm{f}} = \alpha_{\mathrm{W}} / \bar{c}_M$ set

$$u = (a_{\mathrm{f}} / \bar{a}_{\mathrm{f}})^{m/2}, \qquad \eta = u^{-1} I_{X_0}.$$

This is a unitary finite scalar with the required grading.

With $\varrho = m/2 - 1$, the rational form Y_0 has positive definite $X_0 = E^k$ and the ordered string

$$e_0, \dots, e_{\varrho-1}, z, e_{\varrho-1}^*, \dots, e_0^*.$$

The pairing of each e_a, e_a^* is one, the norm of z is $-N$, and the remaining string pairings vanish. Take the generic shift along the ordered string with constant $c_{\mathrm{gen}} = -(j_{\mathrm{W}} M \delta_E)^{-1}$. It sends z to $-N c_{\mathrm{gen}} e_{\varrho-1}^*$ and the other nonfinal string vectors to c_{gen} times their successors, and it kills the final vector and X_0. The constant belongs to $F_0^{\times}$ since both j_{W} and δ_E are skew.

In the finite coordinates let K_L be the integral group, J_L the kernel of its reduction, and $H = U(X_0)$ the definite factor of $R = U \rtimes H$, and let $J_H = \iota_H^{-1}(J_L)$. We first compute the finite isometry from the original lattice. Write L for its orbital basis matrix, with the distinguished column of X_0 replaced by

$$v_0 = x_0 / \bar{\pi}_N + z/N, \qquad N = \pi_N \bar{\pi}_N.$$

On (v_0, z) the Gram matrix is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$. The other X_0-columns and the hyperbolic pairs retain their integral unimodular Gram matrices. Hence L is a basis of an integral unimodular lattice, including at the primes over N.

Write ι_{rat} and $\iota_L = L^{-1} \iota_{\mathrm{rat}} L$ for the definite embedding in rational and orbital coordinates. In the graded basis the string columns are multiplied by u. The glued column becomes $x_0/\bar{\pi}_N + uz/N$. Thus the graded basis matrix is $u \iota_{\mathrm{rat}}(\eta) L$. It has the same Gram matrix as L. Choose an integral change to an orthonormal basis locally. At split pairs a basis on one component and its Gram dual on the other give such a change. At an unramified nonsplit place a unit-norm line exists: if every residue norm were zero, polarization with all residue scalars would make the form zero by the nondegenerate residue trace pairing. Lift the line, use surjectivity of the local unit norm to scale its norm to one, and split off its unimodular complement. This gives the change in every rank, also in residue characteristic two.

Right multiplication by a standard integral unitary matrix preserves the orthonormal Gram matrix. At each chosen split pair choose its first component to correct the reduction of the change of basis to the identity at the chosen place. The second component is forced by unitarity, and reduction is prescribed only at the chosen place of the pair. The integral general linear reduction there is surjective. Thus the orthonormal finite point of this covector can be chosen as

$$x_f = u \, \iota_{\mathrm{rat}}(\eta) L \gamma_0, \qquad \gamma_0 \in \mathrm{GL}_r(\widehat{\mathcal{O}}_E), \qquad \gamma_0 \equiv I \text{ at each chosen place.}$$

As part of fixing the original space, choose this x_f before forming any of its period maps. In its standard coordinates define

$$\begin{aligned}\mathfrak{f}(k) &= \iota_L(\eta)^{-1} L^{-1} x_f k x_f^{-1} L \iota_L(\eta) \\ &= \gamma_0 k \gamma_0^{-1}.\end{aligned}$$

This equivalence preserves integrality and the reductions at the chosen places, so $\mathfrak{f}(J) = J_L$. Indeed, let J_0 be the Gram matrix in the rational basis above and J_w the reduction of $L^* J_0 L$ at a chosen split place $w \mid M$. The equation $\gamma_0^*(L^* J_0 L)\gamma_0 = I$ and $\gamma_{0,w} \equiv I$ give $\overline{\gamma_{0,\bar{w}}}^{\mathsf{T}} J_w = I$ in the residue field. Thus the opposite action is the inverse adjoint for this same Gram matrix. The definite embedding reads exactly ι_L under $\mathfrak{f}$, since η is central in that definite factor. If a rational term is read as $\gamma_{\mathrm{std}} = x_f^{-1}\gamma x_f$ and $\gamma_L = L^{-1}\gamma L$, then $\mathfrak{f}(\gamma_{\mathrm{std}}) = \eta_H^{-1}\gamma_L \eta_H$ with $\eta_H = \iota_L(\eta)$. For a representative c° of $H(F_0)\backslash H(\mathbb{A}_{F_0,f})/J_H$, the standard class slice a_c embeds $c^\circ \eta^{-1}$. Its image is $\eta_H^{-1} h_c$, where $h_c = \iota_L(c^\circ)$. These comparisons hold before choosing any translate.

8.2 Class slices and quotient measure

In this subsection, we decompose the rational quotient using the finite coordinates of Section 8.1.

In the chosen lattice basis $A \oplus (X_0 \oplus Ez) \oplus A^*$, the matrix of a Bessel element $u\iota_H(a)$ on the entire middle diagonal block is that of $\iota_H(a)$. Indeed u induces the identity there, also after conjugation by the middle change of basis used to glue the last column. On the two outer blocks $\iota_H(a)$ is the identity. Apply this to the element and its inverse. Integrality and congruence to one on the glued middle block then hold for the embedded Levi part, with its full z-covector included. Removing that part gives $u \in U \cap J_L$. Transporting through the change of basis gives $J_R = (U(\mathbb{A}_{F_0,f}) \cap J) \rtimes J_H$.

For $n\iota_H(a) \in R(\mathbb{A}_{F_0,f})$, write $a\eta = \delta c^\circ j_H$, with $\delta \in H(F_0)$ and $j_H \in J_H$. Choose $u \in U(F_0)$ with $\iota_H(a)^{-1} u^{-1} n \iota_H(a) \in U \cap J$. The condition on u is open about n in the finite adeles. The polynomial exponential coordinates of U and additive approximation over F_0 give such a rational point. The rational element $\iota_H(\delta^{-1})u^{-1}$ moves the finite point to $a_c J_R$, since η is central in the Levi.

Conversely any identification between two such slices identifies their definite classes by taking the Levi part. At the same class the rational Levi part is one by freeness of the marked definite level. The remaining stabilizer is the original U_c. Therefore

$$[R]/J_R = \coprod_c U_c \backslash R_\infty.$$

Each unipotent quotient is compact by the covector boxes of Section 13.6. The real Levi is compact and there are finitely many classes.

The left U_c-translates of the reduced cutoff β_c sum to one by Section 14.1. Write v_R for the finite volume of J_R including the positive scalar of the Bessel Haar decomposition, and v_G for the volume of J in the finite factor of the Petersson measure. For its real factor use $\prod_{v\mid\infty} dg_v$, where dg_v has identity density $1/\operatorname{vol} U(r)$ in the skew-Hermitian coordinates of Section 12.5. This fixes the finite factor, including its scalar, before the Poincaré map is formed. The slice formula for every integrable right J_R-invariant Q is

$$\int_{[R]} Q = v_R \sum_c \int_{R_\infty} \beta_c(s) Q(s, a_c)\, ds.$$

8.3 The real point and stationary translate

In this subsection, we fix the real chart and place the stationary point of Section 12.10 in that chart.

Use the signed scales of Section 15.7. At v let J_0 be the rational Gram matrix and let J_1 replace its entry $-N$ by -1. Take D_v diagonal with entries $\kappa_v^{g-a}/\sqrt{N}$ on e_a, their reciprocals on e_a^*, $N^{-1/2}$ on z, and one on X_0. Then $D_v^* J_0 D_v = J_1$. Let p_v be the Euclidean unitary matrix whose columns are the normalized sums and differences of the hyperbolic pairs, the basis of X_0, and z, ordered by the prescribed signs. Choose the real components x_v of the adelic point by

$$x_v = D_v p_v, \qquad x^{\mathrm{pt}} = ((x_v)_{v|\infty}, x_f), \qquad x_v^* J_0 x_v = p_v^* J_1 p_v = \varepsilon_v.$$

Here x_f is the corrected finite point chosen above, with the same reduction and principal level at the prescribed places. The real chart satisfies

$$\iota_v(g) = p_v g p_v^{-1}, \qquad k_v = D_v^{-1} x_v p_v^{-1} = 1, \qquad T_v(g) = \iota_v(g).$$

The isomorphism ι_v preserves the specified Lie algebra inner products and Haar. The rational matrix γ acts in the J_1-coordinates by $D_v^{-1}\Phi_v(\gamma)D_v$.

Let $x_v^\star$ be the stationary point of Section 12.10. Its point in the standard signed coordinates and its adelic realization are

$$g_{0,v} = \iota_v^{-1}(x_v^\star), \qquad g_{0,\infty} = (g_{0,v})_v, \qquad g_0 = (g_{0,\infty}, 1).$$

In rational coordinates $x_v g_{0,v} = D_v x_v^\star p_v$. The Bessel coefficient is evaluated at

$$(x_v^\star)^{-1}\iota_v(s_v)T_v(g_v) = \iota_v(g_{0,v}^{-1} s_v g_v).$$

The character Ψ on R is trivial on the Levi and is the adelic character at $-\sigma/(M\delta_E)$ on U. Here σ is the simple-root sum and δ_E the chosen generator of the different. In the grading the simple-shift coefficient c_{gen} and the Weil scalar j_{W} satisfy $j_{\mathrm{W}} c_{\mathrm{gen}} = -(M\delta_E)^{-1}$. The trace of the shift against a Bessel unipotent has two contributions at each adjacent pair of basis positions.

The dual entry is the conjugate of the corresponding entry of the inverse unipotent, by unitarity. The first superdiagonal sum of the inverse is $-\sigma$, so the trace is $c_{\mathrm{gen}}(\sigma - \bar\sigma)$. Since the product $j_{\mathrm{W}} c_{\mathrm{gen}}$ is purely imaginary, the Weil argument $j_{\mathrm{W}} c_{\mathrm{gen}}(\sigma - \bar\sigma)/2$ has the same real trace as $j_{\mathrm{W}} c_{\mathrm{gen}}\sigma$. This gives the character of the original graded period.

Use the additive character of $\mathbb{A}_E/E$ whose factor at a complex place is $z \mapsto \exp(-2\pi i\, 2\,\mathrm{Re}\, z)$. Write $z = -\sigma/(M\delta_E)$. The signed chart divides the simple-root sum by $M\Phi_v(\delta_E)/(2\pi i)$, giving $\alpha_{\Psi,v} = -2\pi i z_v$. Thus

$$\exp(\alpha_{\Psi,v} - \bar\alpha_{\Psi,v}) = \Psi_v(s_v), \qquad \prod_{v|\infty} \exp(-\alpha_{\Psi,v} + \bar\alpha_{\Psi,v}) = \Psi(s_\infty, 1)^{-1}.$$

The second product occurs in the Bessel average because its integrand uses the inverse action of the Bessel group. At the finite places, e_a, z belong to the glued lattice and each coordinate functional along e_a is integral there. Consequently the simple-root entries of a principal-level unipotent belong to $M\widehat{\mathcal{O}}_E$. Conjugation by the definite Levi preserves these entries. The inverse different annihilates $\widehat{\mathcal{O}}_E$ under the trace character, so Ψ_f is trivial on J_R.

Take $w_v = B_{0,v}\xi_v/\|B_{0,v}\xi_v\|$, the unit highest vector in the compact value space I_v from Section 12.10. It is the vector on which $\sigma_S(s)^{-1}$ acts inside the local average A_v of Section 12.4. Put $a_v = \sqrt{\dim I_v}/\|A_v\|_2$, and let d_v be the formal degree. Use the reproducing kernel $e = \prod_v d_v \bar{c}_v$, with c_v the normalized coefficient in the original mark. Write C_{W} for the Weyl-to-Bessel isomorphism in conjugate coordinates. For the unitary Levi action $\pi_H = B_0 S B_0^{-1}$ in Weyl coordinates use $C_{\mathrm{W}} = \overline{B_0}$ on the conjugate coordinates.

Unitarity gives

$$(C_{\mathrm{W}}\rho_{\mathrm{conj}}(h)z)^{\mathsf{T}} = (C_{\mathrm{W}}z)^{\mathsf{T}}\pi_H(h)^{-1}.$$

At each real place the chart sends $u\iota_H(h)$ to the graph of its Levi component. The inverse reads the ambient group point from the first coordinate and the Levi action from the prescribed quotient of the flag. These maps are continuous group isomorphisms. Thus the product of the local graph measures of Section 12.4 pulls back to $C_\infty ds$, with $0 < C_\infty < \infty$ by uniqueness of Haar measure. Set

$$w = \bigotimes_v w_v, \qquad t_0(z) = C_\infty \frac{\prod_v a_v}{\prod_v d_v}(C_{\mathrm{W}}z)^{\mathsf{T}}w.$$

The factors of the product average at $s = u\iota_H(h)$ are $\overline{e(g_{0,\infty}^{-1}sx)}/\prod_v d_v$, the wave $\Psi(s)^{-1}$, and $\pi_H(h)^{-1}w$, besides the positive factors a_v. Tensor the absolutely integrable local averages and invert s in the unimodular Bessel group. For $r = u\iota_H(h)$ put

$$W_{t,\phi}(r) = \Psi(r)t(\rho_{\mathrm{conj}}(h_\infty^{-1})\phi(h_f)).$$

The evaluation of $R_H(\eta)\psi$ at a_c is $\psi(c^\circ)$ and $\Psi(a_c) = 1$. After inversion, the original full Bessel vector therefore satisfies

$$(C_{\mathrm{W}}\psi(c^\circ))^{\mathsf{T}}W(x) = \int_{R_\infty} \overline{e(g_{0,\infty}^{-1}s^{-1}x)}W_{t_0,R_H(\eta)\psi}(s,a_c)\,ds.$$

8.4 The measured period and change of level

In this subsection, we use the point of Section 8.3 and the measure of Section 8.2 to identify the measured pairing.

At a fixed finite level, the dual real test space is the image of right convolution by $\bar{e}$. For a bounded vector Y in this space, take its representative

$$Y^\circ(x) = \int_{G_\infty} \bar{e}(a)Y(xa)\,da, \qquad |Y^\circ(xh) - Y^\circ(x)| \le \|Y\|_\infty \,\|\bar{e}(h^{-1}\,\cdot) - \bar{e}\|_1 \quad (h \in G_\infty).$$

Each finite-level slice is an open quotient of G_∞ by a discrete subgroup, so the integral is independent of the bounded representative on that slice. Translation continuity in L^1 makes Y° continuous in the real variable, and $Y^\circ = Y$ almost everywhere by the reproducing identity of Section 12.3. We use Y° for period evaluations and write it as Y.

For such a test at the principal level, absolute periodization and unfolding give

$$\int_{[G_{\mathrm{ad}}]} Y\,\mathcal{P}_{\mathrm{ser}}(\psi) = v_G \sum_c \iint \beta_c(x)Y(x,a_c)\overline{e(g_{0,\infty}^{-1}s^{-1}x)}W_{t_0,R_H(\eta)\psi}(s,a_c)\,ds\,dx.$$

Both Y and the period coefficient are invariant under the respective right levels on this class. Rational invariance shows that the double integrand apart from β_c, denoted $\mathcal{K}(s,x)$, satisfies $\mathcal{K}(us,ux) = \mathcal{K}(s,x)$ for $u \in U_c$. The factor involving e depends on $s^{-1}x$.

The integral of $\beta_c(s)|\mathcal{K}(s,x)|$ is finite by the boundedness of Y, the L^1 bound on e and the integrability of the Bessel coefficient on its compact quotient. Insert $\sum_u \beta_c(ux) = 1$ in this integral and change variables $(s,x) \mapsto (us, ux)$ for each u. Summing the translates of $\beta_c(s)$ proves finiteness with the cutoff $\beta_c(x)$ by Tonelli. The same change of variables and absolute convergence now give

$$\iint \beta_c(x)\mathcal{K}(s,x)\,ds\,dx = \iint \beta_c(s)\mathcal{K}(s,x)\,ds\,dx.$$

The convolution in x reproduces $Y(sg_{0,\infty}, a_c)$. Using the Bessel Haar decomposition, the resulting pairing is $\int_{[R]} W_{t,R_H(\eta)\psi}(s)Y(sg_0)\,ds$ with $t = (v_G/v_R)t_0$. For the original source form ϕ take $\psi = R_H(\eta^{-1})\phi$. Bounded principal-level tests of the dual type are dense in its Petersson space. Indeed, for such an L^2 vector Z, truncate it to $Z_n = Z\mathbf{1}_{\{|Z|\le n\}}$ and project by right convolution with the dual idempotent $\bar{e}$ of Section 12.3. This preserves the finite level and gives bounded tests of norm at most $\|e\|_1\|Z_n\|_\infty$ in L^∞. The projections converge in L^2 to Z, since convolution is bounded and fixes Z. The series of Section 8.5 belongs to L^2 by absolute periodization. Its pairing consequently extends the measured period on bounded tests to a continuous functional on the full dual type. Uniqueness of the Petersson representative identifies the series with the measured Poincaré vector for g_0, t, ϕ.

For a bounded test Z of the dual real type which is right K'-invariant, with K' a finite open level, let $N_J = [J : J \cap K']$. Its average over the representatives γ of $J/(J \cap K')$ is J-invariant. Right invariance of $\Pi\phi$ under J, and right translation in the Petersson integral, give

$$\begin{aligned} N_J \int_{[G_{\mathrm{ad}}]} Z(x)(\Pi\phi)(x)\,dx &= \int_{[G_{\mathrm{ad}}]} \sum_{\gamma \in J/(J\cap K')} Z(x\gamma)(\Pi\phi)(x)\,dx \\ &= \sum_{\gamma \in J/(J\cap K')} \int_{[R]} W_{t,\phi}(r)Z(rg_0\gamma)\,dr. \end{aligned}$$

In each sum use one representative of each coset. The second line is the measured period applied to the summed test. This formula keeps the factor N_J when the first covector of the lower chain changes level.

8.5 Petersson blocks and covariance

In this subsection, we unfold the pairing of Section 8.4 to obtain the blocks used in Section 9.1.

Expand the series in the source class basis, using the quotient Haar measure of Section 8.2 and the cutoffs of Section 14.1.

Let $C_X\psi = (C_{\mathrm{W}}\psi(c^\circ))_c$ and $C = C_X R_H(\eta^{-1})$. The series for a coordinate basis vector in C is the periodization of

$$f_{c,i}(y, x_f) = \beta_c(y)W_i(y)\mathbf{1}_{a_c J}(x_f).$$

At a point of support, first move its finite coordinate to one such slice by a rational term. Then all remaining terms lie in $\mathcal{T}(c, 1, c')$ by the identities for the finite isometry. The majorant and (35) bound the sum of absolute seed values uniformly by $2A_* b^{12n^3 rf}$, where A_* is the product of the local majorant bounds.

If the point has no such support the series is zero. Group by the left cosets of the corresponding $U_{c'}$. The cutoff partition and the triviality of the character on these rational stabilizers give

the grouped terms with the full $W_{i'}$. Each is in the reproducing type by the local Bessel integral. The L^1 reproducing kernel and uniform absolute bound permit termwise right convolution.

Thus the series has that type and is continuous by continuity of translations of the kernel in L^1. Being bounded on the finite-volume quotient, it belongs to the full Petersson space on the prescribed level. Absolute unfolding against bounded tests and the measured pairing of Section 8.4 identify the series with $\Pi C^{-1} e_{c,i}$ for the original measured map Π.

Put $B = (v_G \nu)^{-1/2} \Pi C^{-1}$ and $B_s = B^* \rho(s) B$. Use the Petersson product linear in the second variable. To pair $\rho(s)\Pi C^{-1} e_{c',i'}$ with $\Pi C^{-1} e_{c,i}$, unfold the seed in the second variable. An integral lift k_s of s normalizes J, so the finite integral over $a_c J$ gives v_G. A term indexed by γ^{-1} in the first variable uses

$$\mathfrak{f}(a_{c'}^{-1} \gamma_{\mathrm{std}}^{-1} a_c k_s) = h_{c'}^{-1} \gamma_L^{-1} h_c \mathfrak{f}(k_s).$$

Membership in J_L is exactly $h_c^{-1} \gamma_L h_{c'} \in \mathfrak{f}(k_s) J_L$. Thus the surviving integrand is the full kernel with both cutoffs of $\mathcal{T}(c, s, c')$, with the residue s. The real isometry preserves Haar: its signed-basis isomorphism preserves the specified Lie algebra density and the remaining change is a fixed left translation. Writing A^{full} for the full amplitude of Section 14.1, unfolding gives

$$\langle \rho(s) \Pi C^{-1} e_{c',i'}, \Pi C^{-1} e_{c,i} \rangle = v_G A^{\mathrm{full}}(c, s, c')_{i,i'}.$$

Both variables acquire the factor $(v_G \nu)^{-1/2}$.

Proposition 8.1 (Full Petersson blocks)**.** *In the Euclidean source coordinates, the normalized Petersson blocks are given by the full orbital sums*

$$(B_s)_{(c,i),(c',i')} = \nu^{-1} \overline{A^{\mathrm{full}}(c, s, c')_{i,i'}}. \tag{1}$$

Let H_R be the residue image of the integral Bessel group. For $t \in H_R$ choose a lift $u \iota_H(a)$ in that group and write $h_c \iota_H(a) = \iota_H(h) h_{c'} \iota_H(j_H)$ by the definite double coset decomposition. Choose rational $\xi \in U(F_0)$ with $u^{-1} h_c^{-1} \xi h_c \in J_L$ by additive approximation in the unipotent coordinates. Then $\delta = \xi \iota_H(h)$ has parameter (c, t, c') since the integral $\iota_H(a)$ normalizes J_L. Every rational Bessel term has residue in H_R. Two such terms over the same left class and residue differ on the right by a rational Bessel term of identity residue. By the Levi test their right classes coincide, and all terms over (c, t, c') in R are $\delta U_{c'}$.

Write $W(\delta y) = M_\delta W(y)$; the matrix M_δ is unitary by covariance of the local averages. The right cutoffs for $\delta U_{c'}$ sum to one and their W-values agree. The left integral is

$$\int \beta_c(y) W(y) W(y)^* \, dy = \nu I$$

by the full local Gram identities of Section 12.10, their tensor product, and the unipotent integral of β_c. The Bessel amplitude is therefore νM_δ. The normalized conjugate blocks give a unitary representation $r_\psi(t)$ on the source: composition and inversion of the rational transporters multiply and invert the class moves and matrices. Extend r_ψ by zero off H_R. Equation (1) identifies $E_s = B_s - r_\psi(s)$ with the normalized conjugate off-Bessel amplitudes.

To prove covariance, choose δ over (d, t^{-1}, d'). Multiplication $\gamma \mapsto \gamma \delta^{-1}$ bijects the terms over (c, s, d') and (c, st, d). The resulting right cutoff $\beta_d(\delta \cdot)$ has orbit sum one for $U_{d'}$ because $\delta U_{d'} \delta^{-1} = U_d$ by the level condition. The right kernel is multiplied by the factor M_δ^*. Changing the cutoff by the orbit partition and using absolute convergence gives

$$A^{\mathrm{full}}(c, st, d) = A^{\mathrm{full}}(c, s, d') M_\delta^*.$$

Taking normalized conjugates and then adjoints gives $B_{st} = B_s r_\psi(t)$ and $B_{ts} = r_\psi(t) B_s$. These identities also hold for E_s. They imply $\rho(t)B = B r_\psi(t)$: for any x, the difference $\rho(t)Bx - Br_\psi(t)x$ belongs to the span of the vectors $\rho(s)By$ and is orthogonal to each of them.

Since finite translation by η^{-1} commutes with the scalar actions, it preserves the ramified source. The ranges of the measured good translates on that source are the ranges of $p\rho(s)B$ on its subspace in C-coordinates, with $p = \rho(h_0 \otimes h_1)$. The factors in the algebraic good projection and the map normalization are nonzero and common to the translates.

Part II
Orbital integrals and the translated span

9 Orbital integrals and operator norms

In this section, we relate the Petersson matrices of Section 8 to two operator moments. Their bounds give the hypotheses of the translation argument in Section 16.2.

Write T_{full} for the full Petersson target and $B : X \to T_{\mathrm{full}}$ for the geometrically normalized map on the full source with its Euclidean class coordinates. The chosen real matrix coefficient selects one line in each automorphic occurrence. On the definite group, a finite class with trivial stabilizer contributes the whole coefficient space, of dimension $\mathcal{B}$ (Section 9.4). Let ρ be the unitary action of $G = G_0 \times G_1$ on T_{full}. Let $p_i = \rho(h_i)$ for the good central idempotents $h_i \in \mathbb{C}[G_i]$, and $p = p_0 p_1$. Put $d_X = \dim X$, $D = \dim pT_{\mathrm{full}}$, and $B_s = B^* \rho(s) B$. The following argument concerns the same normalized map for both its operator estimates.

9.1 From truncated orbital integrals to full orbital integrals

In this subsection, we compare the complete Petersson matrices with their truncated orbital integrals. The estimate of Section 14.1 controls the difference by (36). Section 15.8 bounds the contribution of distinct links. For equal links on the open set we use (34), with the normalization (33).

Let H_R be the residue Bessel subgroup and r_ψ its unitary action on X, extended by zero to G. Write $E_s = B_s - r_\psi(s)$, $D_{\mathrm{id}} = |H_R| d_X$ and

$$D_{\mathrm{main}} = \frac{d_X^2}{\mathcal{B}\ \mathrm{mass}(E, r)}, \qquad C_{\mathcal{A}} = b^{-6k|\mathcal{A}|} D_{\mathrm{main}},$$

where $\mathcal{B}$ is the product of the formal degrees. Let $\mathcal{A}$ be a subset of the f marked places at p_0^{lev}. At each place of $\mathcal{A}$ choose any nonzero vector in the Bessel span, or any nonzero covector in its opposite-place realization. Let $\Gamma_{\mathcal{A}}$ be the intersection of these stabilizers. Put $q = p_0^{\mathrm{lev}}$. The split reduction onto $\mathrm{GL}_r(\mathbb{F}_q)$ is surjective. Each nonzero vector or covector has orbit size $q^r - 1$ under this action, so its stabilizer has that index. At each of the f distinguished places $w \mid p_0^{\mathrm{lev}}$, write g_w for the reduction of s, J_w for the symmetric Gram matrix in the original lattice basis of Section 8.1, and E_j for the matrix of the first j columns of $(e_0, \dots, e_{\varrho-1}, z)$ in that basis. Define

$$\mathcal{O}_w = \{s \in G : \det(E_j^t J_w g_w E_j) \neq 0 \text{ for } 1 \le j \le \varrho\}.$$

Under the identification of the two residue fields by conjugation, the action at the opposite place is $J_w^{-1} g_w^{-t} J_w$. Thus fixing its vector $E_j y$ is equivalent to $g_w^{-t} J_w E_j y = J_w E_j y$. Put

$$\Gamma_{\mathcal{A}}^{\mathrm{open}} = \Gamma_{\mathcal{A}} \cap \bigcup_w \mathcal{O}_w, \qquad M = p_0^{\mathrm{lev}} p_1^{\mathrm{lev}}.$$

For $\Omega \subseteq G$, let $\mathcal{L}(\Omega)$ be the sum of the squared norms of the cut blocks on the nonzero-discriminant part of Γ_*, taken separately for each link and summed over the two classes and $s \in \Omega$, before division by ν^2.

Proposition 9.1 (Orbital bounds on the finite stabilizers)**.** *For the Bessel data at sufficiently large x, put $\beta(s) = \|B_s\|_{\mathrm{HS}}^2$. Then*

$$\sum_{s \in H_R G_i} \|E_s\|_{\mathrm{HS}}^2 \le b^{-12} D_{\mathrm{id}} \qquad (i = 0, 1),$$

$$\sum_{s \in \Gamma_{\mathcal{A}}^{\mathrm{open}}} \beta(s) \le (1 + b^{-4})^2 C_{\mathcal{A}} \quad (\mathcal{A} \text{ arbitrary}),$$

$$\sum_{s \in \Gamma_{\mathcal{A}}} \beta(s) \le C_{\mathcal{A}} \qquad (|\mathcal{A}| = f).$$

Here $\Gamma_{\mathcal{A}}^{\mathrm{open}}$ is the open subset specified above.

Proof. 1. By (1), the blocks of E_s are the normalized conjugates of the full amplitudes off the rational Bessel subgroup. For a set Ω avoiding the cosets of H_R by nontrivial integral norm-one scalars, (49) identifies the cut off-Bessel amplitudes with the cut amplitudes on Γ_*.

In the block norm obtained by dividing by ν and summing squares over the two classes and $s \in \Omega$, (36) bounds the cut error by $b^{-20}\sqrt{D_{\mathrm{id}}}$.

2. Section 15.8 bounds the squared norm of the combined cut blocks on Γ_* by $\nu^{-2}\mathcal{L}(\Omega) + b^{-kf} D_{\mathrm{id}}$ in the same block norm. For $1 \le z \le Q_\cap(s)$ on Ω, the equal-link sum satisfies

$$\nu^{-2}\mathcal{L}(\Omega) \le (b^{2f}/z)^k D_{\mathrm{main}}.$$

Here $Q_\cap(s)$ is the product of the residue norms where the basis and its image have nonzero intersection. On the open part of $\Gamma_{\mathcal{A}}$, the sharper equal-link estimate is proved in (34).

3. By Section 15.3, the sets $H_R G_i$ avoid the nontrivial scalar cosets and every $s \in H_R G_i$ satisfies $Q_\cap(s) \ge (M/b)^f$. The open set avoids all scalar cosets, including H_R. When every place is fixed, $Q_\cap(s)$ is at least $(p_0^{\mathrm{lev}})^f$ on $\Gamma_{\mathcal{A}}$. The residue group has one general linear factor over $\mathbb{F}_{p_i^{\mathrm{lev}}}$ for each of the f chosen split places at each p_i^{lev}.

The parameter bounds give $b \ge 1000$, $kf \ge 40$, $p_0^{\mathrm{lev}} \ge b^{10}$ and $M \le b^{23}$ for large x. The moment comparison $b^{11kf} D_{\mathrm{id}} \le D_{\mathrm{main}} \le (M/b^6)^{kf} D_{\mathrm{id}}$ is (33).

On a set Ω as in the first estimate, the triangle inequality in the block norm gives

$$\left(\sum_{s \in \Omega} \|E_s\|_{\mathrm{HS}}^2 \right)^{1/2} \le \nu^{-1} \sqrt{\mathcal{L}(\Omega)} + (b^{-20} + b^{-kf/2}) \sqrt{D_{\mathrm{id}}} \le \nu^{-1} \sqrt{\mathcal{L}(\Omega)} + b^{-15} \sqrt{D_{\mathrm{id}}}. \tag{2}$$

On $H_R G_i$ the equal-link contribution is at most

$$(b^{2f}/(M/b)^f)^k D_{\mathrm{main}} \le b^{-3kf} D_{\mathrm{id}}.$$

Thus the sum of $\|E_s\|_{\mathrm{HS}}^2$ over $H_R G_i$ is at most $4b^{-30} D_{\mathrm{id}} \le b^{-12} D_{\mathrm{id}}$. By the covariance of Section 8.5, $E_t = (B^*B - I) r_\psi(t)$ for $t \in H_R$. Therefore

$$|H_R| \|B^*B - I\|_{\mathrm{HS}}^2 = \sum_{t \in H_R} \|E_t\|_{\mathrm{HS}}^2 \le b^{-12} D_{\mathrm{id}}.$$

With $D_0 = BB^*$, Cauchy–Schwarz in the Hilbert–Schmidt inner product gives

$$\beta(s) := \|B_s\|_{\mathrm{HS}}^2 = \mathrm{tr}(D_0 \rho(s) D_0 \rho(s)^*) \le \|D_0\|_{\mathrm{HS}}^2 \le (1 + b^{-6})^2 d_X \le 2 d_X.$$

For the open part of $\Gamma_{\mathcal{A}}$, we have $E_s = B_s$. Since $C_{\mathcal{A}} \ge b^{5kf} D_{\mathrm{id}}$, the right side of (2) is at most

$$(\sqrt{1 + b^{-4}} + b^{-15}) \sqrt{C_{\mathcal{A}}} \le (1 + b^{-4}) \sqrt{C_{\mathcal{A}}}.$$

This gives open mass at most $(1 + b^{-4})^2 C_{\mathcal{A}}$.

Suppose $|\mathcal{A}| = f$. By Section 15.3, $\Gamma_{\mathcal{A}}$ meets the scalar cosets only in H_R. The pointwise estimate gives

$$\sum_{s \in \Gamma_{\mathcal{A}} \cap H_R} \beta(s) \le 2 D_{\mathrm{id}} \le C_{\mathcal{A}}/4,$$

since $C_{\mathcal{A}} \ge b^{5kf} D_{\mathrm{id}}$. On $\Omega = \Gamma_{\mathcal{A}} \setminus H_R$ we have $E_s = B_s$, and the equal-link bound is

$$\nu^{-2} \mathcal{L}(\Omega) \le (b^{2f}/(p_0^{\mathrm{lev}})^f)^k D_{\mathrm{main}} \le b^{-8kf} D_{\mathrm{main}} \le C_{\mathcal{A}}/8.$$

Also $b^{-30} D_{\mathrm{id}} \le C_{\mathcal{A}}/8$. The square of the right side of (2) is at most $2(C_{\mathcal{A}}/8 + C_{\mathcal{A}}/8) = C_{\mathcal{A}}/2$. The full sum is at most $C_{\mathcal{A}}/4 + C_{\mathcal{A}}/2 \le C_{\mathcal{A}}$. □

9.2 The diagonal bound and the projected subspace

In this subsection, we use Bessel covariance to estimate the error in the diagonal moment. Proposition 9.1 and the finite projection estimates of Section 14.2 give bounds in the same source norm.

Proposition 9.2 (Diagonal moment on the projected target)**.** *Let P_U be the orthogonal projection onto a subspace $U \subseteq X$. For the marked map B and target projection p,*

$$\|pBP_U\|_{\mathrm{HS}}^2 \ge \dim U - 5b^{-4} d_X.$$

Proof. The diagonal argument uses the Bessel covariance together with the bound of Proposition 9.1:

$$E_{ts} = r_\psi(t) E_s, \quad E_{st} = E_s r_\psi(t) \quad (t \in H_R), \qquad \sum_{s \in H_R G_i} \|E_s\|_{\mathrm{HS}}^2 \le b^{-12} |H_R| d_X.$$

Section 14.2 bounds the squared counting norm of the conjugacy average of $\mathrm{tr}\, r_\psi|_{G_i}$ by $2d_X^2$, and gives $q_i(1) \le (2b^8)^{-1}$ for $q_i = 1 - h_i$, with $q_i(1)$ the group-algebra coefficient at the identity.

Since $E_t = E_1 r_\psi(t)$ for $t \in H_R$, summing over H_R and dividing out its order gives

$$\|B^*B - I\|_{\mathrm{HS}}^2 \le b^{-12} d_X.$$

Right convolution by q_i is an orthogonal projection on Hilbert–Schmidt-valued functions on G, by self-adjointness and idempotence. Apply the contraction to $E_s \mathbf{1}_{H_R G_i}(s)$. On $t \in H_R$ the resulting value is $r_\psi(t) \sum_{s \in G_i} q_i(s) E_s$, with convolution taken as $\sum_s q_i(s) E_{ts}$. Summing its squared norm over $t \in H_R$ gives $\| \sum_{s \in G_i} q_i(s) E_s \|_{\mathrm{HS}} \le b^{-6} \sqrt{d_X}$. Its trace consequently has absolute value at most $b^{-6} d_X$. The trace of the Bessel term is bounded in absolute value by $b^{-4} d_X$, by Cauchy–Schwarz using the conjugacy average. Indeed counting measure gives

$$\sum_s |q_i(s)|^2 = \sum_s q_i(s) q_i(s^{-1}) = (q_i * q_i)(1) = q_i(1)$$

by self-adjointness and idempotence. Taking their sum bounds $\mathrm{tr}(B^*(1-p_i)B) = \|(1-p_i)B\|_{\mathrm{HS}}^2 \le 2b^{-4} d_X$. The two summands in $1 - p = (1 - p_0) + p_0(1 - p_1)$ have orthogonal images, so

$$\|(1-p)B\|_{\mathrm{HS}}^2 \le \sum_{i=0}^{1} \|(1-p_i)B\|_{\mathrm{HS}}^2 \le 4b^{-4} d_X.$$

Using the Hilbert–Schmidt bound for $B^*B - I$, we obtain

$$\begin{aligned} \|pBP_U\|_{\mathrm{HS}}^2 &= \dim U + \mathrm{tr}(P_U(B^*B - I)) - \|(1-p)BP_U\|_{\mathrm{HS}}^2 \\ &\ge \dim U - b^{-6}\sqrt{d_X \dim U} - 4b^{-4} d_X \ge \dim U - 5b^{-4} d_X. \end{aligned}$$ □

9.3 The mixed moment bound

In this subsection, we control the mixed moment by summing the open contributions against the stabilizer contributions. Proposition 9.1 estimates both sums, and the dimension comparison of Section 9.4 supplies the normalization.

Proposition 9.3 (Mixed moment on the marked family)**.** *For the same B, p, with $B_s = B^* \rho(s) B$ and $D = \dim pT_{\mathrm{full}}$,*

$$\frac{D}{|G|} \sum_{s \in G} \|B_s\|_{\mathrm{HS}}^2 \le (1 + 9b^{-4}) d_X^2.$$

Proof. Put $\delta_b = b^{-4}$, $\lambda = b^{6k}$ and $\mu_{\mathcal{A}} = \sum_{\Gamma_{\mathcal{A}}} \beta$. Section 9.1 bounds the open mass by $(1+\delta_b)^2 C_{\mathcal{A}}$ and the total mass on the full fixer by $C_{\mathcal{A}}$.

Maximize $\lambda^{|\mathcal{A}|} \mu_{\mathcal{A}}$ over the fixer choices, including the empty subset. If a place is unused, the mass of the intersection with any single allowed vector or covector fixer at that place is at most $\mu_{\mathcal{A}}/\lambda$. For the leading minor calculation these choices include the nonzero vectors in the two marked column spaces and their corresponding covectors. Let $q = p_0^{\mathrm{lev}}$ and $\pi : \Gamma_{\mathcal{A}} \to \mathrm{GL}_r(\mathbb{F}_q)$ be the reduction at that unused place w. Since $\Gamma_{\mathcal{A}} \setminus \Gamma_{\mathcal{A}}^{\mathrm{open}} \subseteq \Gamma_{\mathcal{A}} \setminus \mathcal{O}_w$, it suffices to bound singular minors at this one place. The second inequality of Proposition 9.1 bounds the full Petersson mass on $\Gamma_{\mathcal{A}}^{\mathrm{open}}$, including the error in the passage from cut to full orbital integrals.

Write the leading block of size j as $Z_s = A_1^t \pi(s) A_0$. Here A_0 consists of the first j marked columns, and A_1 consists of the columns at the conjugate place paired by the split Gram matrix.

At both places the first j columns belong to the integral basis of Section 8.1. The Gram matrix is invertible after reduction. Thus

$$\operatorname{rank} A_0 = \operatorname{rank} A_1 = j, \qquad A_0 v \neq 0 \ (v \neq 0), \qquad A_1 y \neq 0 \ (y \neq 0).$$

These vectors and covectors belong to the two prescribed spans. The two stabilizers tested below have equations $\pi(s)A_0 v = A_0 v$ and $\pi(s)^{-t}A_1 y = A_1 y$. The weight is of positive type because for $D_s = \rho(s)BB^*\rho(s)^*$, $\langle D_a, D_c\rangle_{\mathrm{HS}} = \beta(a^{-1}c)$. For a nontrivial additive character ψ_q, the left-kernel size of Z_s is $\sum_y \mathbb{E}_v \psi_q(y^t Z_s v)$. It is at least q if the minor is zero. The two axes $y = 0$ and $v = 0$ contribute at most $2\mu_{\mathcal{A}}$ to its weighted sum.

For $y, v \neq 0$, put $a_1 = A_1 y$ and $a_0 = A_0 v$. The vectors

$$\sum_{a \in \Gamma_{\mathcal{A}}} \widehat{e}_{\pi(a)^{-t}a_1} \otimes D_a, \qquad \sum_{c \in \Gamma_{\mathcal{A}}} e_{\pi(c)a_0} \otimes D_c$$

use delta vectors e_z on $\mathbb{F}_q^r$ and unit Fourier vectors $\widehat{e}_w(z) = q^{-r/2}\psi_q(-w^t z)$. Their squared norms are $|\Gamma_{\mathcal{A}}|$ times the respective fixer masses, by expanding and setting $s = a^{-1}c$. Their inner product equals $|\Gamma_{\mathcal{A}}|q^{-r/2}$ times $\sum_{s \in \Gamma_{\mathcal{A}}} \psi_q(a_1^t \pi(s) a_0)\beta(s)$. Cauchy–Schwarz therefore bounds the latter sum by $q^{r/2}\mu_{\mathcal{A}}/\lambda$. Summing over y, v with the normalization in v proves

$$q \sum_{\det Z_s = 0} \beta(s) \leq (2 + q^{j+r/2}/\lambda)\mu_{\mathcal{A}}.$$

By Section 4, $b^{10} \leq q \leq b^{11}$. Since $j \leq m/2$ and $r \geq x^5$, for sufficiently large x,

$$11j + 6m \leq 12m \leq r/2, \qquad \log_b(q^{j+r/2}) \leq 11j + \tfrac{11}{2}r \leq 6(r + 1 - m) = \log_b \lambda.$$

There are at most $m/2 \leq b^6/3$ prescribed minors. Using $q \geq b^{10}$ and summing, the nonopen mass is at most $\delta_b \mu_{\mathcal{A}}$.

If the maximizing fixer uses every place, use the full-fixer mass bound just proved. Otherwise combine the open and nonopen bounds to get $(1-\delta_b)\mu_{\mathcal{A}} \leq (1+\delta_b)^2 C_{\mathcal{A}}$. In either case maximality and $0 < \delta_b \leq 1/5$ give

$$\sum_{s \in G} \|B_s\|_{\mathrm{HS}}^2 = \mu_{\varnothing} \leq \lambda^{|\mathcal{A}|}\mu_{\mathcal{A}} \leq \frac{(1+\delta_b)^2}{1-\delta_b} D_{\mathrm{main}} \leq (1 + 9\delta_b) D_{\mathrm{main}}.$$

By (3), this gives

$$\frac{D}{|G|} \sum_{s \in G} \|B_s\|_{\mathrm{HS}}^2 \leq (1 + 9b^{-4}) d_X^2.$$

The bound uses the original full source, normalized measured map and good target. □

9.4 Dimension on the terminal definite group

In this subsection, we transfer the required dimensions to a definite unitary group with the same finite levels. The trace comparison respects the projection of Section 14.2, which lets us use these dimensions in Section 9.

Use the terminal positive definite Hermitian space W_m of rank r, with its standard lattice and the principal level at the same marked places. The original space Y_0 and W_m have identical finite groups and levels in their marked isometric coordinates. Their real types have the same regular Harish–Chandra parameter at each real place. Write T_W for the full principal-level space of the terminal type.

Proposition 9.4 (Definite comparison of dimensions)**.** *On the marked family,*

$$D = \dim pT_{\mathrm{full}} = \dim pT_W \le |G|\mathcal{B}\ \mathrm{mass}(E, r). \tag{3}$$

Proof. Fix one tuple selected by p. At each of the two different level primes this tuple has a primitive type polynomial at a chosen place, by Section 14.2. The two places of F_0 are split in E and distinct. At either such place let $K = \mathrm{GL}_r(\mathcal{O}_v)$ and let τ be the finite cuspidal representation of its residue group given by the primitive type polynomial.

If Z is the local centre, extend the inflation of $\tau^\vee$ unitarily to ZK and compactly induce to $\mathrm{GL}_r(F_{0,v})$. The compact induction is irreducible by the intertwining criterion. Indeed, outside ZK, a diagonal Cartan representative has two adjacent unequal elementary-divisor exponents. After ordering the exponents decreasingly, its conjugation sends the upper unipotent block at this cut into the principal kernel. An intertwiner would therefore factor through the coinvariants of $\tau^\vee$ under the unipotent radical of a proper finite parabolic, which vanish by cuspidality. The resulting representation is supercuspidal since its coefficients are compactly supported modulo Z. Every irreducible local representation containing the inflated τ is supercuspidal by the depth-zero type theorem [54, Proposition 4.1]. Consequently the selected discrete spectrum is cuspidal: at such a place the induced representations from every proper Levi have zero projection to this type, and the noncuspidal spectrum is built from these inductions.

In the compact induction, the vectors supported on the identity coset have coefficients supported on ZK. Truncate them by $\mathbf{1}_{\mathcal{O}_v^\times}(\det g)$. Since $ZK \cap \{\det g \in \mathcal{O}_v^\times\} = K$, the sum of these truncated diagonal coefficients, with the factor $\dim(\tau)/\operatorname{vol}(K)$, is exactly the local projector density

$$\frac{\dim \tau}{\operatorname{vol}(K)} \overline{\operatorname{tr} \tau(\bar g)}\, \mathbf{1}_K(g).$$

Here $\bar g$ denotes reduction at the chosen place. In our Hermitian convention each diagonal coefficient is $\langle e_a, \pi(g) e_a \rangle$, with e_a in the identity coset of the compact induction of $\tau^\vee$. On K their sum is $\operatorname{tr} \tau^\vee(\bar g) = \overline{\operatorname{tr} \tau(\bar g)}$. Thus each of the two projectors has the required expansion.

Use the normalized principal-level averages and the finite central idempotent of the fixed tuple at all finite places. Their product is the test for the original tuple on both source spaces. At each real place choose the Euler–Poincaré function with coefficients in the dual of the algebraic representation of highest weight $\lambda - \rho$, where λ is the dominant representative of the common regular parameter.

For a real signature (a, b') divide this function by $d_{a,b'} = \binom{r}{a}$, the size of its discrete series packet. A nonelliptic regular semisimple element has a nontrivial split torus in the centre of its connected centralizer: its eigenvalues off the unit circle occur in reciprocal-conjugate pairs. Here discrete series means square-integrable on the full real centralizer, equivalently modulo the compact centre of the ambient unitary group. The absolute value of a matrix coefficient of an irreducible unitary representation is constant along the central split torus, whose Haar measure is infinite. Thus the centralizer has no square-integrable irreducible representation. The ordinary orbital integral therefore vanishes by [22, Theorem 4.1(i)]. This gives cuspidality of the real test in the sense of [33, §1.5].

Let δ be semisimple and F the algebraic coefficient of the Euler–Poincaré function ϕ_F chosen above. An eigenvalue z off the unit circle pairs with $1/\bar z$. Their eigenspaces contribute a factor $\mathrm{GL}_{m_z}(\mathbb{C})$ to the real centralizer, whose centre contains a split torus. The stable integral then vanishes by [22, Theorem 4.1(iii)].

Otherwise all eigenvalues have modulus one. Write m_z for the multiplicity of each distinct z. The eigenspaces are orthogonal and nondegenerate. Assigning each eigenspace signature $(a_z, m_z - a_z)$, with $\sum_z a_z = a$, realizes the stable class on $U(a, b')$. The centralizer and its compact inner form are

$$I_\delta = \prod_z U(a_z, m_z - a_z), \qquad \overline{I}_\delta = \prod_z U(m_z).$$

Use the centralizer measures of [22, §4], normalized by formal degree one for the discrete series with the infinitesimal character of the trivial representation. Rescaling the ambient Haar by $c > 0$ multiplies the Euler–Poincaré function by c^{-1} and the orbital quotient Haar by c, so the orbital integrals apply with the original group Haar. In the stable orbital integral of [33, §1.5] each real conjugacy class carries the Kottwitz sign of its centralizer. By [22, Theorem 4.1(ii), (iv)],

$$\mathrm{SO}_\delta(\phi_F/d_{a,b'}) = e(\overline{I}_\delta)\,\mathrm{tr}\,F(\delta), \qquad \sum_{\sum a_z = a} \prod_z \binom{m_z}{a_z} = [X^a](1+X)^r = d_{a,b'}.$$

In this normalization the compact inner centralizer has sign

$$e(\overline{I}_\delta) = (-1)^{\sum_z \lfloor m_z^2/4 \rfloor}.$$

Indeed the quasi-split inner form of $U(m_z)$ has signature $(\lfloor m_z/2 \rfloor, \lceil m_z/2 \rceil)$, and half the real dimension of its symmetric space is $\lfloor m_z^2/4 \rfloor$. The character and this sign depend only on the eigenvalues with multiplicities. This also treats classes with no representative on a given inner form: all unit-circle classes occur on it, and the stable integral on any remaining class is zero on the quasi-split group. Thus the normalized real tests have a common transfer in the sense of [33, §1.6]. At finite places the given isometries transport the tests and their measures.

Apply the cuspidal trace comparison of Harris–Labesse [33, Proposition 1.10.6], termwise in the coefficient expansion at the two split marks. The first mark supplies condition (a.1), with the split Jacquet–Langlands transfer the identity, and the other supplies condition (b.1). All places with different inner form are real, with cuspidal Euler–Poincaré tests, giving condition (c). For each of these two specified test functions, Proposition 1.10.5(i) of [33] identifies its cuspidal trace with that of the common transfer to the quasi-split group. Thus the equality in Proposition 1.10.6 applies to the two projectors just constructed, with the stated archimedean normalization.

On either side a nonzero trace of the archimedean test requires cohomology with the chosen coefficient. A representation selected by the finite test is supercuspidal at both split marks. Harris–Labesse [33, Proposition 1.8.1 and Theorem 3.1.5, case (B)] therefore makes each contributing real factor discrete series. Write F_λ for the algebraic representation of highest weight $\lambda - \rho$, so the chosen coefficient is $F_\lambda^\vee$. The compact centre of $U(a, b')$ lies in its maximal compact subgroup K. A mismatch of central characters of π and F_λ therefore gives zero relative cochains. When the characters agree, restriction to $SU(a, b')$ identifies their relative cochain complex with that of the derived group. Indeed the tangent quotient is the same and the centre acts trivially on $\pi \otimes F_\lambda^\vee$. By [18, Ch. II, Theorem 5.3(a)–(b)],

$$\dim H^j(\mathfrak{g}, K; \pi \otimes F_\lambda^\vee) = \dim \mathrm{Ext}^j_{(\mathfrak{g},K)}(F_\lambda, \pi) = \begin{cases} 1 & \pi \text{ belongs to the packet of } \lambda,\ j = ab', \\ 0 & \text{otherwise.} \end{cases}$$

Here $ab' = \frac{1}{2}\dim_{\mathbb{R}}(U(a,b')/K)$. For each fixed finite representation, the multiplicities of all real packet members on that covector are equal by [33, Proposition 3.1.7], applied with both split marks.

Write $\mathcal{P}_\infty$ for the product of the real packets and π_∞^0 for the prescribed member. For fixed π_f, the archimedean contribution to the trace, before multiplying by the finite trace, is

$$\frac{(-1)^{\sum_v a_v b'_v}}{|\mathcal{P}_\infty|} \sum_{\pi_\infty \in \mathcal{P}_\infty} m(\pi_\infty \otimes \pi_f) = m(\pi_\infty^0 \otimes \pi_f).$$

Indeed $|\mathcal{P}_\infty| = \prod_v \binom{r}{a_v}$, and the sign is one at each place since $a_v + b'_v = r$ is odd and hence $a_v b'_v$ is even. At a real place let $c(g) = \langle v, \pi(g)v\rangle$ for the prescribed unit vector, and $e = d_\pi \bar{c}$ with the Haar measure of Section 12.3. For any irreducible unitary σ, the convolution identity proved there gives

$$P = P^* = P^2, \qquad P\sigma(g)P = c(g)P, \qquad P = \sigma(e).$$

If $P \neq 0$, a unit vector in its range has coefficient c. Its translates have the same Gram matrices as the translates of v, so cyclicity gives $\sigma \simeq \pi$. On π, Schur orthogonality gives $\pi(e)z = v\langle v, z\rangle$. The product over the real places therefore selects one line in each occurrence of π_∞^0, with the multiplicity displayed above. The trace equality is the equality of dimensions of the two original tuple spaces. Sum over the orthogonal tuples to get $D = \dim pT_W$.

The full finite spherical algebra. For $1 \le j \le m$, put $p_j = k + j - 1$, $q_j = m - j$. The matrix of W_j in the ordered rational basis is $J_j = \mathrm{diag}(I_{p_j}, -I_{q_j})$. With the finite unit a_- of norm -1 from Section 4.3, put

$$B_j = \mathrm{diag}(I_{p_j}, a_- I_{q_j}), \qquad B_j^* J_j B_j = I_r.$$

This integral finite basis identifies the compact subgroups and their principal congruence kernels. At a chosen split place a change of integral orthonormal basis acts by inner conjugation on the residue group and hence fixes each central type idempotent. For Y_0 use the integral basis of its glued lattice and its prescribed reduction in Section 8.1. Transport Haar measures by these identifications. At each real place the ordered entries λ_i of Section 12.10 are the same on all these spaces. On W_j the first p_j entries belong to positive coordinate lines and the last q_j to negative coordinate lines.

In particular, if K_v is the integral compact, use $\mathbf{1}_{K_v}/\operatorname{vol}(K_v)$ for the local identity of the spherical algebra. A double coset $K_v g K_v$ uses $\mathbf{1}_{K_v g K_v}/\operatorname{vol}(K_v)$, which acts on K_v-fixed vectors by the unnormalized sum of its right coset translates. These prescriptions agree on the two covectors also at nonsplit finite places. Transport the finite tests by the marked isomorphism of the groups there, so the tests have the same stable orbital integrals with matching centralizer measures. No comparison between distinct local groups is needed at those places.

Fix a marked tuple c selected by the good idempotent, and insert any compactly supported spherical finite test h off the old marked places, with identity factors at all but finitely many places. The test at the old level is the central type projector of c. There is a primitive type polynomial at one split place above each of the two distinct rational level primes. At the first such place the projector is a linear combination of truncated diagonal coefficients of unitary depth-zero supercuspidals, as in the computation above. The two local groups there are split, so the Jacquet–Langlands transfer is the same supercuspidal. At the other distinguished place the corresponding coefficients are cuspidal by the local supercuspidal coefficient criterion [33,

Lemma 1.9.1]. The spherical insertion is off these marks. It does not change either local condition.

Apply [33, Propositions 1.10.5(i) and 1.10.6] to each of the specified tests and their common transfer, as above, in case (a.1, b.1, c). For two inner forms of the quasi-split unitary group, associated tests have identical cuspidal traces if at one split finite place the test is a truncated coefficient of a supercuspidal whose Jacquet–Langlands transfer is supercuspidal, at another place the test is cuspidal, and at every place with different local inner forms the test is cuspidal. This applies termwise after expanding the first projector and h into finite sums of decomposable tests. The two finite conditions are exactly those verified above. The possible differences of inner form are real. There use the Euler–Poincaré tests with the same algebraic coefficient and divide at signature (a, b') by the packet size $\binom{r}{a}$. Their matching stable orbital integrals are the preceding calculation with [22, Theorem 4.1]. The finite tests are associated by their measure-preserving transport.

On the spectral side the same two primitive types force the contributing finite factors to be supercuspidal there. The cohomological real factors selected by the Euler–Poincaré test are discrete series by the same cuspidal similitude extension and temperedness argument ([33, Proposition 1.8.1 and Theorem 3.1.5, case (B)]). At a fixed finite representation the multiplicities of the packet members on a given covector agree by [33, Proposition 3.1.7]. Thus dividing by the packet sizes yields the trace on one prescribed real member per place, with the sign $(-1)^{\sum_v a_v b'_v} = 1$ on both source spaces.

The trace weighted by the finite operator h is unchanged by this argument, since its action is on the fixed finite representation. The prescribed real minimal idempotent selects one line of each prescribed member. Summing over the orthogonal good tuples gives

$$\operatorname{tr}_V(h) = \operatorname{tr}_{V_j}(h) \qquad (h \text{ in the common finite spherical algebra}), \tag{4}$$

where V and V_j are the original good spaces on Y_0 and W_j, respectively. The identity test gives the equality of dimensions. The split-system comparison uses the special case of (4) with arbitrary insertions at split places.

Fix one selected tuple c. Unitarity of right translation gives the Petersson adjoint $R(h)^* = R(h^*)$, where $h^*(g) = \overline{h(g^{-1})}$. The spherical algebra at a split place is commutative: transpose reverses convolution and fixes each K_v-double coset by the elementary divisor theorem over $\mathcal{O}_v$. Operators at distinct places commute. Its operators and their adjoints commute with the orthogonal type projector e_c. Hence the split operators on each of the two finite-dimensional tuple spaces are commuting normal operators. Write S for the union of their joint spectra. For $b \in S$ and each $b' \neq b$ in S, choose a split operator $T_{b'}$ on which the two characters differ. The same spherical element

$$E_b = \prod_{b' \in S \setminus \{b\}} \frac{T_{b'} - b'(T_{b'})\,1}{b(T_{b'}) - b'(T_{b'})}$$

projects each tuple space onto its ordinary b-eigenspace. Apply the trace identity for c to E_b. Dividing the common trace by $\dim \tau_c$, where τ_c is the irreducible residue representation of c, gives its common multiplicity on these eigenspaces.

At the positive definite endpoint each maximal-level class with stabilizer Γ_c gives $|G|/|\Gamma_c|$ classes at the principal level. Indeed the quotient of the integral compact by the principal subgroup is $G = \prod_{i=0}^{1} \mathrm{GL}_r(\mathbb{F}_{p_i^{\mathrm{lev}}})^f$, and the stabilizer acts freely on this quotient. To verify freeness choose a marked completely split prime of norm $p_i^{\mathrm{lev}} > (2r)^2$. If a stabilizer element

γ reduces to one at the chosen places, its integral trace difference $\operatorname{tr}\gamma - r$ is divisible by all f of these primes. A nonzero difference would have absolute norm at least $(p_i^{\mathrm{lev}})^f$, and at most $(2r)^{2f}$ because all its complex embeddings are unitary.

Thus the trace equals r. Unitarity forces all eigenvalues to be one and $\gamma = 1$. This also gives the trivial stabilizer on each finer class. On each such class the compact matrix-coefficient idempotent selects one line in the right factor of the Peter–Weyl decomposition. The left factor has the dimension of the terminal compact type. For the dominant local parameters $\lambda_{v,1} > \cdots > \lambda_{v,r}$, the Weyl dimension formula and the Haar normalization in Section 12.5 give

$$\dim \bigotimes_{v|\infty} F_{\lambda_v} = \prod_{v|\infty} \prod_{i<j} \frac{\lambda_{v,i} - \lambda_{v,j}}{j-i} = \mathcal{B}.$$

Summing over the maximal-level classes gives

$$\dim T_W = |G|\mathcal{B} \sum_c |\Gamma_c|^{-1} = |G|\mathcal{B}\ \mathrm{mass}(E, r).$$

This proves (3). □

10 Finite orbital integrals and Fourier bounds

In this section, we compute the finite Haar factors and bound the correlations appearing in Section 9.1. Integral norm fibers give the successive quotient measures, and Fourier estimates control the sum over denominators.

10.1 Integral points on the local quadrics

In this subsection, we bound the measure of integral solutions of a Hermitian norm equation. The bound enters the Haar disintegration in Section 10.2 and the local integrations in Section 10.6.

Let T be any finite set of finite places of the CM field E stable under conjugation, with unramified or split quadratic algebras at its conjugation orbits. Put $\mathcal{R} = \prod_{w\in T} \mathcal{O}_{E_w}$ and $\zeta_{\mathcal{R}}(a) = \prod_{w\in T}(1 - (\mathrm{N}w)^{-a})^{-1}$. Normalize additive measures to give the integer rings mass one. We need a bound for the mass of the integral box in a quadric with an orthogonal block of $2a+4$ orthonormal coordinates, where $a \geq 1$ is an integer. Take the fiber measure by division of the ambient Haar measure by the norm-value Haar measure. For a skew-Hermitian norm equation use a unit skew generator to identify its integral value lattice with that of a Hermitian norm.

Consider one orbit, with fixed field K, residue cardinality q, quadratic algebra L, and integer ring $\mathcal{O}_L$. Both the trace on integers and the norm on units are surjective. In the unramified case this follows from the residue trace and norm and the trace on each successive unit quotient. In the split case these maps are addition and multiplication. Choose c, τ with $c\bar{c} = -1$ and $\tau + \bar{\tau} = 1$.

In each orthonormal pair v, w, the integral basis

$$v + cw, \qquad \bar{\tau}v - \tau cw$$

is hyperbolic. Its change-of-basis determinant is $-c$, a unit. The norm on the whole block, with $h = a+2$, is now $\mathrm{Tr}_{L/K} \sum_{i=1}^{h} \bar{x}_i y_i$. For fixed $x \in \mathcal{O}_L^h$, let j be the minimum valuation of its

coordinates, taking the minimum over both components as well in the split case. For finite j, the linear norm equation in y is a map onto $\pi^j\mathcal{O}_K$ on $\mathcal{O}_L^h$, where π is a uniformizer of K.

Its pushforward of integral Haar measure is uniform probability measure on that ideal. Thus the density on each norm fiber is at most q^j. This is the density in the quotient measure, since solving for a K-coordinate with nonzero coefficient divides its Haar measure by the absolute value of that coefficient. One may use any such coordinate. The set $x = 0$ has measure zero in the smooth fiber charts, by solving in x when $y \neq 0$ and using $h \geq 3$. At the singular point the smooth fiber measure has zero mass.

The measure of the sets with minimum valuation at least j is q^{-2hj}. Hence the block contributes at most

$$(1 - q^{-2h}) \sum_{j \geq 0} q^{-(2h-1)j} \leq (1 - q^{-(2a+3)})^{-1}.$$

This bound is independent of the norm value, so the other integral coordinates can be integrated with total mass one. The displayed bound is at most $(1 - q^{-2a})^{-1}$ in the unramified quadratic case and at most $(1 - q^{-a})^{-2}$ in the split case. Multiplying over the orbits gives $\zeta_{\mathcal{R}}(a)$.

Lemma 10.1 (Null set of a finite pivot)**.** *Suppose a Hermitian norm equation has a hyperbolic pair and a disjoint integral block of $2a + 4$ orthonormal coordinates, with $a \geq 1$. On every fixed norm fiber, the locus where either hyperbolic coordinate vanishes in any field component has fiber measure zero.*

Proof. Write t for one hyperbolic coordinate and solve the norm equation in the disjoint block. With the other coordinates fixed, the block has conditional mass at most $\zeta_{\mathcal{R}}(a)$ in the integral box, uniformly in the remaining norm value, by Section 10.1. Thus imposing $t_w \in \mathfrak{p}_w^m$ for a field component w cuts the fiber mass of that box to at most $\zeta_{\mathcal{R}}(a)(\mathrm{N}w)^{-m}$. This uses the coordinate in that field component alone, including at a split pair. Let $m \to \infty$. Dilation by a fixed-field scalar reduces any larger compact box to an integral box and rescales the norm value and fiber Haar by fixed factors. Exhaust the fiber by these boxes and take the finite union over the components. The bound applies also at norm zero. □

10.2 Compact volumes for the successive quotients

In this subsection, we disintegrate the local Haar measure along successive isotropic lines. The quadric estimate of Section 10.1 bounds the resulting densities, and the compact mass is used in Section 10.3.

Use the original integral Gram lattice Λ at T, with the ϱ marked hyperbolic pairs e_i, e_i^* and marked last vector z in their orthogonal complement. Write G_T for its unitary group and K_T for the lattice stabilizer. At the top use the isotropic dual u of z in the marked middle block, with $h(u, z) = 1$, and the orthogonal complement of u, z. The marked integral Gram restrictions used here are that these are integral decompositions, that $\langle u, z\rangle$ has determinant -1, and that the remaining orthonormal block has at least $2(3\varrho + 5) + 4$ coordinates. With a trace-one integer τ, replacing z by $z - \tau h(z, z)u$ gives a hyperbolic pair and preserves the coefficient of z. These conditions concern the Gram lattice and its marked vectors before taking the congruence subgroup.

Set $n = \varrho + 1$, and $f_i = e_i$ for $i < \varrho$, $f_\varrho = z$. Let G_i fix the pairs below i pointwise and, for $i = n$, also z. Let R_i act unitriangularly on the first i basis vectors, so $gf_j - f_j \in \langle f_0, \ldots, f_{j-1}\rangle$ for $j < i$. Let U_i be the Heisenberg group attached to the i-th pair for $i < \varrho$, and put

$U_\varrho = \{1\}$. The products $U_{<i} = U_0 \cdots U_{i-1}$ are normalized by G_i. We have $R_i = G_i U_{<i}$ and $R_{i+1} \cap G_i = G_{i+1} U_i$.

Indeed an isometry fixing e_i sends e_i^* to $e_i^* + v + s e_i$, with v in the remaining space and $s + \bar{s} = -h(v, v)$. The unique Heisenberg element with that column removes it, leaving an isometry fixing the pair. For the iteration, write $r \in R_{i+1}$ as γv by the decomposition at i. A lower Heisenberg factor changes each later basis vector only by earlier isotropic vectors, so $v \in R_{i+1}$.

Now $\gamma \in G_i \cap R_{i+1}$ takes f_i into the orthogonal complement of the earlier pairs and differs from f_i by a vector in their span. Thus it fixes f_i, and the column removal gives the step. At the top the condition within G_ϱ is exactly to fix z.

Let η_i be Heisenberg Haar in the coordinates (v, β), with $s = \beta - \tau h(v, v)$ and β skew. Its measure is product additive Haar, since multiplication is triangular with translations in v, β. Normalize each integer coordinate lattice to mass one. The displayed column and integrality of h give $\eta_i(K_T \cap U_i) = 1$, and peeling off these columns gives $\eta(K_T \cap U_{<n}) = 1$ for $\eta = \eta_0 * \cdots * \eta_{\varrho-1}$. Conjugation by G_i preserves the earlier vector measures by the unitary determinant identity and fixes their centers.

For $i < n$, put $H_i = R_{i+1} \cap G_i$. Identify $H_i \backslash G_i$ with the orbit of f_i by $H_i \sigma \mapsto \sigma^{-1} f_i$ and give it the norm-fiber measure. In the convolution take θ_i to be the pushforward along the section constructed below on the chart where the pivot t is invertible in every component. On a hyperbolic pair with coordinates t, s write $s = tc$. This contributes the additive Jacobian $|t|_A$, the modulus on the semilocal algebra.

The norm equation takes the trace of c with coefficient $t\bar{t}$, whose modulus on the fixed algebra is also $|t|_A$. The factors cancel. Writing $c = \tau b + \beta$, with $b = \mathrm{Tr}(c)$ and $\bar{\beta} = -\beta$, splits the integral lattice as $\tau \mathcal{O}_K \oplus \ker(\mathrm{Tr}\,|_{\mathcal{O}_L})$ at each conjugation orbit. Thus the trace splitting has Jacobian one for the chosen measures, including above 2. The other coordinates, t, and β have product additive measure. The top chart uses the same computation after the indicated change of basis. At every stage the unused orthonormal block has at least $2(3\varrho + 5) + 4$ coordinates, so Lemma 10.1 gives full measure to the chart.

On a lower floor write a point of norm zero as $X = t e_i^* + x + s e_i$, with $t \in A_T^\times$ and $x \perp e_i, e_i^*$. Put $a = x/t$ and $\beta = s/t + \tau h(a, a)$. The norm equation gives $\beta + \bar{\beta} = 0$. Let $w_i(t)$ send $e_i \mapsto t e_i^*$ and $e_i^* \mapsto \bar{t}^{-1} e_i$, fixing their orthogonal complement. The Heisenberg element $n(a, \beta)$ fixes e_i and sends $e_i^* \mapsto e_i^* + a + (\beta - \tau h(a, a)) e_i$. Hence

$$n(a, \beta) w_i(t) e_i = X, \qquad \sigma = w_i(t)^{-1} n(a, \beta)^{-1}, \qquad \sigma^{-1} e_i = X.$$

The change $x = ta$ contributes $\mathrm{md}(t)^{d_i}$, where $d_i = k + 1 + 2(\varrho - i - 1)$ counts the remaining coordinates, as in Section 10.4. On the top floor put $v = z + N\tau u$, where $h(z, z) = -N$. For the original exceptional vector $x_{k_0}/\bar{\pi} + z/N$ we have $u = -(x_{k_0}/\bar{\pi} + z/N)$. Then $h(u, u) = h(v, v) = 0$, $h(u, v) = 1$ and $z = v - N\tau u$. The integral span of u, v is unimodular and splits off.

Given $tv + x + su$ of norm $-N$, with t invertible and $x \perp u, v$, put

$$a = t^{-1} x, \qquad \beta = t^{-1} s + \tau h(a, a) + N\tau/(t\bar{t}).$$

The norm equation gives $\beta + \bar{\beta} = 0$. First dilate v by t and u by $\bar{t}^{-1}$. Then apply the Heisenberg element fixing u and sending v to $v + a + (\beta - \tau h(a, a))u$, with $c \mapsto c - h(a, c)u$ on the orthogonal complement. This composite sends z to the given vector. For the orbit map $\sigma \mapsto \sigma^{-1} z$, take σ to be the inverse composite. If the given vector is integral and t is a unit, both transformations

and their inverses preserve the marked lattice. Thus the unit-pivot subset contributes to the compact mass in the same fiber measure. The terminal measure θ_n is right Haar on G_n with $K_T \cap G_n$ of volume one.

The Haar measure given by these data is $\eta * \theta_n * \cdots * \theta_0$. Indeed, in descending order the group measure is formed by taking $\eta_i * \theta_i$ at floor i, with $\eta_i = \delta_1$ on the top quadric and terminal floors. The measure from the higher floors times η_i is right invariant under $G_{i+1}U_i$. Consequently its integral against a function of the product with $\sigma \in G_i$ descends to $H_i\sigma$ and is independent of the section used to lift θ_i. Right translation by $g \in G_i$ acts on the orbit coordinate by g^{-1}.

This preserves the norm fiber measure, because the ambient determinant has modulus one and the norm value is unchanged. Induction from θ_n gives right Haar at the bottom. The resulting Haar on each G_i is also left invariant. Indeed, at a conjugation orbit with fixed field K the residual unitary group becomes GL_d over a splitting field. Its adjoint action on M_d is $X \mapsto gXg^{-1}$, of determinant $(\det g)^d(\det g^{-1})^d = 1$. Hence $\det_K \mathrm{Ad}(g) = 1$ on the original Lie algebra. Right translation of a left invariant volume form multiplies it by $\det_K \mathrm{Ad}(g^{-1})$, which proves the assertion for both split and nonsplit factors, including the terminal stabilizer. This is the quotient integration formula at each step with the stated fiber and stabilizer measures. Moving each θ_i past the lower Heisenberg product is permitted by the conjugation invariance above. This gives the asserted order, with the same measures as for the orbital integrals formed using this floor normalization.

Put $\mathcal{U}_i = R_iK_T = \{g : R_ig \cap K_T \neq \varnothing\}$. For each g, the θ_i-mass of σ such that $\sigma g \in \mathcal{U}_{i+1}$ is at most $V = \zeta_{\mathcal{R}}(3\varrho + 5)$. If the set is nonempty, then $g \in \mathcal{U}_i$, since $G_i, R_{i+1} \subseteq R_i$. Replacing g by $rg \in K_T$ with $r \in R_i$ preserves the integral. To see this, factor $r = \gamma v$ in $G_iU_{<i}$. The conjugate of v by $\sigma\gamma$ lies in $U_{<i} \subseteq R_{i+1}$ and disappears from the near condition. Right multiplication by γ preserves the integral on the orbit. Now if $\xi\sigma g \in K_T$ with $\xi \in R_{i+1}$, then $\sigma^{-1}\xi^{-1}f_i$ is integral. Its projection away from the earlier pairs is $\sigma^{-1}f_i$, since $\xi^{-1}f_i - f_i$ belongs to the earlier isotropic span. The projection is integral. The quadric box bound of Section 10.1 therefore gives V, uniformly in g.

The θ_n-mass of $h \in G_n$ with $hg \in U_{<n}K_T$ is at most one. In fact if $u_1h_1g, u_2h_2g \in K_T$, their quotient belongs to K_T and to $U_{<n}G_n$. Its middle block is $h_2h_1^{-1}$, which is therefore integral with integral inverse. All such h lie in a single right coset of $K_T \cap G_n$. In the descending convolution $\theta_n * \cdots * \theta_i$, the integral of the indicator of $U_{<n}K_Tg^{-1}$ is thus at most V^{n-i}.

For the induction, the higher convolution is supported in G_{i+1}. Its integral after σg can be nonzero only when $\sigma g \in \mathcal{U}_{i+1}$, since $U_{<n}$ and G_{i+1} lie in R_{i+1}. Integrating over θ_i contributes at most V. Finally convolution with η gives exactly the integral over $U_{<n}K_T$ for $g = 1$, because each nonempty η-fiber in K_T is a right coset of $K_T \cap U_{<n}$, of mass one. Thus $\mu(K_T) \leq V^n$. For $a = 3\varrho + 5$, use $n = m/2$, $\sqrt{x} \leq m < 2\sqrt{x}$, and $f \leq x^{12}$ for large x to obtain

$$n[E : \mathbb{Q}] \leq 2x^{25/2} \leq 2^{\sqrt{x}-2} \leq 2^{2\varrho}.$$

Then $\zeta_{\mathcal{R}}(a) \leq \zeta_E(a) \leq \zeta(a)^{[E:\mathbb{Q}]}$ gives

$$n \log \zeta_{\mathcal{R}}(a) \leq n[E : \mathbb{Q}]\, 2^{1-a} \leq 2^{-n-3}.$$

Here $\log \zeta(a) \leq \sum_{j \geq 2} j^{-a} \leq 2^{1-a}$ by the integral test. Exponentiating gives the upper bound $1 + 2^{-n}$ for the compact volume.

10.3 Normalization of the finite quotient pairing

In this subsection, we compute the correlation at the initial step of the finite calculation. The Haar normalization of Section 10.2 fixes the constant in the bound of Section 10.10.

Let T be a finite conjugation-stable set of finite places. Use the image of the level scalar M in $A_T = \prod_{w\in T} E_w$, with integer ring $\mathcal{R}$ and additive modulus md. This includes a single conjugate pair above a level prime in Lemma 10.2. At a place over M, N and π_N are units. Thus the same norm and lattice identities hold when T consists only of the places over M, as in Section 14.2.

Section 13.3 uses a set containing the places over N when evaluating the full modulus of π_N. Every conjugation orbit of completions is split or unramified over its fixed field. In fact $d \nmid q_1$. At a rational prime other than d, inertia acts trivially on K, where complex conjugation is nontrivial. At d, inertia acts trivially on $\mathbb{Q}(\zeta_{q_1})$, where complex conjugation is also nontrivial. Thus the inertia groups in E/E^+ are trivial, where E^+ is the fixed field of conjugation. In particular the previous compact measure calculation can be used on all of T.

Write G_T for the original local group, μ for its floor Haar measure, K_T for the compact integral group and J for its marked congruence subgroup. Let Z be an intermediate group $J \le Z \le K_T$. Put $U = U_{<n}$, $dQ = d\mu/d\eta$ on $U\backslash G_T$, where η is the original unipotent Haar, and

$$\delta_U = \eta(U \cap J), \qquad \mathcal{R}_Z = \mu(Z)\mu(J)/\delta_U^2.$$

Extend the marked character of U to UJ by right J-invariance and then by zero. Denote it by χ. This is well-defined since the character is trivial on $U \cap J$. The finite Bessel vector $W_Z(y)$ has entry $\chi(yc^{-1})$ for $c \in J\backslash Z$. Its right J-invariance follows from normality of J in K_T.

Write $W = W_Z$. For each c the set of u such that $uyc^{-1} \in J$ is empty or a right coset of $U \cap J$ of mass δ_U. For such u, $z_1 = uy$ lies in Jc and

$$\overline{W(y)_c}\, W(yx)_c = \chi(z_1 x z_1^{-1}).$$

Indeed the phase of the left U translation cancels, and the additional right factor z_1c^{-1} lies in J. The cosets partition Z, so quotient integration gives

$$\int_Z \chi(z_1 x z_1^{-1})\, d\mu(z_1) = \delta_U \int_{U\backslash G_T} \langle W(y), W(yx)\rangle\, dQ(y).$$

With empty heads the floor-zero form $\mathcal{I}_0^T(1)$ is the integral of the squared modulus of this quotient correlation. Define $\mathcal{M}(Z) = \int_{G_T} |\int_Z \chi(z_1 x z_1^{-1})\, d\mu(z_1)|^2\, d\mu(x)$. The normalized bottom value is therefore

$$\mathcal{I}_0^T(1)/\mathcal{R}_Z = \frac{\mathcal{M}(Z)}{\mu(Z)\mu(J)}.$$

In particular it is nonnegative. On $x \in J$ the inner integral is $\mu(Z)$ by normality. The upper bound on the rest of $\mathcal{M}(Z)$ requires a finite shell estimate.

For use in that estimate, the coordinates of W on the quotient have supports given by the images of UJc. Each support has mass $\mu(J)/\delta_U$ by unfolding the indicator of $Jc = cJ$. These supports are equal or disjoint. If $UJc' = UJc$, write $c' = ujc$ with $u \in U \cap Z$ and $j \in J$. On the common support, $W(y)_{c'} = \chi(u)^{-1} W(y)_c$ by U-covariance and normality of J. The multiplicity in each part is $[U \cap Z : U \cap J] \le 1/\delta_U$, since $\eta(U \cap K_T) = 1$. Thus $B_W = \int WW^*\, dQ$ is a direct sum of rank-one blocks of norm at most $\mu(J)/\delta_U^2$, with trace $\mu(Z)/\delta_U$. It follows that

$$\delta_U \iint |\langle W(y), W(y')\rangle|^2\, dQ(y)\, dQ(y') = \delta_U \operatorname{tr}(B_W^2) \le \mathcal{R}_Z.$$

The matrix WW^* is well-defined on the quotient by its scalar U-covariance.

10.4 Common entries and the finite terminal density

In this subsection, we express the finite pairing at successive isotropic quotients in measures compatible with Section 10.3. The expression supports the Fourier estimates of Section 10.8 and the terminal density calculation.

Use these same measures and $W = W_Z$. For $a \le \varrho$ denote the finite floor form by $\mathcal{I}_a^T(p, q; \omega)$. It has the two pivot integrations of Section 12.2, now in the finite floor measures, with head factor

$$\mathcal{D}_a^T(p, q) = \prod_{i<a} \mathrm{md}(p_i q_i)^{d_i/2}, \qquad d_i = k + 1 + 2(\varrho - i - 1).$$

The twist is $w_q^{-1} w_p$. For $a < \varrho$, the law of the floor measure θ_a is that of $w_a(c)^{-1}\nu$ for $\mathrm{md}(c)^{d_a} dc\, d\eta_a(\nu)$. To see the density, in the norm chart use the coefficient c of e_a^* and take ν^{-1} acting on $w_a(c)e_a = ce_a^*$. Its d_a free vector coordinates are multiplied by c, and its skew coordinate in the ratio of the two hyperbolic coordinates retains its original Haar. The Weyl element fixes the earlier pairs. The omitted tuples have a zero field component of c and form a null set in this floor chart.

Assume the absolute floor majorant is integrable at a. With $Z_a(s) = h(f_a, sf_a)$, the substitution $\tau = w_a(c)^{-1}\nu\sigma$, $g' = \nu\sigma g$, is valid by the quotient invariance of θ_a on functions left invariant under the stabilizer of f_a. The pivot integrals have this invariance after the tail convolution, including for their absolute majorant. The new point measure is $\eta_a * \theta_a * M_a = M_{a+1}$. The old twist is unchanged and the new head factor is $\mathcal{D}_{a+1}^T(p_c, q_c) = \mathcal{D}_a^T(p, q)\,\mathrm{md}(c)^{d_a}$. Thus for every measurable $|\phi| \le 1$, absolute integrability transfers through the substitution and

$$\int_{A_T} \phi(c)\, \mathcal{I}_{a+1}^T(p_c, q_c; 1)\, dc = \mathcal{I}_a^T(p, q; \phi \circ Z_a).$$

Here p_c, q_c append the same entry c; one can use the entry 1 when $c \notin A_T^\times$, a Haar-null set. This lower-floor identity uses the integrable majorant on the original projected Bessel data.

For the terminal formula let $\pi = \pi_N|_T$ with $\pi\bar{\pi} = N$ and $h(z, z) = -N$. In $V_\varrho = X_0 \perp A_T z$, the last-entry condition and the fixed norm are

$$e(t, x) = x - N^{-1} t z, \qquad h(z, e(t, x)) = t, \qquad h(e(t, x), e(t, x)) = -N \iff h(x, x) = t\bar{t}/N - N.$$

In the chosen lattice basis the exceptional vector is $x_{k_0}/\bar{\pi} + z/N$. Thus the change to the basis $(x_i, \pi^{-1} z)$ of Gram $\mathrm{diag}(1, \dots, 1, -1)$ has determinant $\bar{\pi}/\pi$. Since $T = \bar{T}$, $\mathrm{md}_T(\bar{\pi}) = \mathrm{md}_T(\pi)$, so this basis change preserves product Haar. At a split pair this equality combines the two places. The coordinate of $e(t, x)$ along $\pi^{-1} z$ is $-\bar{\pi}^{-1} t$.

Use an integral hyperbolic pair in the standard form on X_0, as in Section 10.1, to solve the norm equation both on X_0 and on V_ϱ. With the norm value indexed by the same marked skew unit, the other chart coordinates and their Haar agree. The change in the last coordinate gives

$$dQ_\varrho(e) = \mathrm{md}(\pi)^{-1}\, d\nu_{t\bar{t}/N - N}(x)\, dt,$$

in the variables $e(t, x)$ on the last-vector quadric. Here $d\nu_c$ is the original norm-fiber measure on X_0 at norm c, with the integral skew line of volume one in its norm-value coordinate.

These fiber measures vary weakly continuously on compact step tests, also at zero norm. In a bounded test box, use hyperbolic coordinates x_i, y_i on $h \geq 1$ hyperbolic planes in the unimodular subspace as in Section 10.1, treat the remaining coordinates as parameters shifting the norm value, and truncate the set where all x_i in a given factor have large valuation. Outside that set we can solve in a trace coordinate with nonzero coefficient, on finitely many compact open pieces. For fixed x_i the Haar integral in the remaining coordinates is locally constant in the norm value, uniformly on each piece.

The density is bounded by a constant times q^j where j is the minimum valuation, by the calculation of Section 10.1 after scaling the box. The measure in the x_i with this valuation is at most a constant times q^{-2hj} for $h \geq 1$ hyperbolic pairs, including at a split orbit. Thus the truncated tails tend uniformly to zero. Taking products proves the assertion on the semilocal ring. This continuity identifies the chart description of the fiber measure at every norm value.

Let $H = G_n$, the stabilizer of z in G_ϱ, with Haar measure dh normalized to give its integral compact subgroup mass one. Define

$$V_p(l) = \mathcal{D}_\varrho^T(p,1) \int_{G_T} W(y) \otimes W(l w_p^{-1} y)^* \, d\mu(y) \qquad (l \in G_\varrho).$$

These integrals have compact support in y, l, locally uniformly in the nonzero entries of the head p. Indeed nonzero integrands give $y = u_1^{-1} k_1,\ l w_p^{-1} y = u_2 k_2$ with $u_i \in U,\ k_i \in K_T$. Thus $u_1 w_p l^{-1} u_2 = k_1 k_2^{-1}$. In the two isotropic blocks and their middle block, the lower-left block of the latter product is $E_1 D_p A_2$, with $D_p = \mathrm{diag}(p_i)$, E_1 lower and A_2 upper unitriangular.

Elimination with nonzero pivots bounds both triangular factors and their inverses in terms of the bounded matrix entries and the heads. The lower-middle and middle-left blocks then bound the unipotent blocks incident to the middle space. The middle-middle block bounds l^{-1} after subtracting their product through D_p. The top-left and lower-right blocks bound the two remaining corner blocks by the same elimination.

The paired blocks and inverses follow by unitarity in the fixed Gram basis. This bounds y, l and their inverses in the full matrix space, hence in compact sets of the group. On these sets a small change in the head and in l changes the arguments of W by a right J-translation, uniformly by compactness. Consequently $V_p(l)$ is locally constant in both variables with uniform neighborhoods on compact sets.

On the orbit of z, set

$$\Phi_{p,q}(e) = \int_H \langle V_p(hl), V_q(hl) \rangle \, dh, \qquad e = l^{-1} z.$$

Right invariance of dh makes this independent of the lift l. Since $N \in A_T^\times$, Witt's theorem in each factor of the semilocal base shows that the orbit is the whole quadric $h(e,e) = -N$. This is a compactly supported, locally constant function on the quadric. For the local section at $e_0 = l_0^{-1} z$, take e close to e_0 and put $b_0 = h(e_0, e)$, $d_0 = \bar{b}_0 / b_0$. At each component take the neighborhood small enough that $(d_0 - 1)/4$ belongs to the maximal ideal. Hensel's lemma at the simple residue root 0 of $u^2 + u - (d_0 - 1)/4$ gives a unique u in that ideal. Then $\lambda = 1 + 2u$ is the square root of d_0 near 1, including in residue characteristic two.

Uniqueness gives $\lambda \bar{\lambda} = 1$ and $\lambda b_0 = \overline{\lambda b_0}$. After multiplying e by λ, reflection in $e_0 + \lambda e$ sends it to $-e_0$, and reflection in e_0 gives e_0. Their product with the scalar is a unitary $s(e)$ tending to 1. Then $l_0 s(e)$ is the section. The support of $\Phi_{p,q}$ is contained in a compact image under $l \mapsto l^{-1} z$ by the preceding support bound, so $\Phi_{p,q}$ has compact support. On the ambient space it admits a compact step extension by a finite partition into compact open balls.

For the top identity expand the inner product of the two orbital tensors. For fixed nonzero p, q, its absolute majorant in (l, y_1, y_2) has compact support by the bounds just proved, applied to the p factors at y_1 and the q factors at y_2. It is integrable for a bounded entry test. Put $y_2 = y_1 x$ by left invariance, and push the positive Haar measure in y_1 through $y_1 = h_1 \sigma g$, with $h_1 \in H$, σ in the last floor and g in the lower product M_ϱ. For each such tuple use right invariance in l, followed by the positive Haar decomposition $l = h_2 \tau (h_1 \sigma)^{-1}$. Tonelli transports the absolute majorant to these floor variables. The weight $\phi(h(z, l^{-1} z))$ becomes $\phi(h(z, \sigma\tau^{-1} z))$, since h_1, h_2 fix z. Both earlier Weyl elements commute with G_ϱ, so Fubini gives the two terminal tail correlations.

The product $\mathcal{D}^T_\varrho(p,1)\mathcal{D}^T_\varrho(q,1) = \mathcal{D}^T_\varrho(p,q)$ gives the head density. This proves

$$\begin{aligned}\mathcal{I}^T_\varrho(p,q;\phi\circ Z_\varrho) &= \int \phi(h(z,e))\Phi_{p,q}(e)\,dQ_\varrho(e)\\ &= \int_{A_T} \phi(t)\,\mathrm{md}(\pi)^{-1} \int \Phi_{p,q}(e(t,x))\,d\nu_{t\bar{t}/N-N}(x)\,dt.\end{aligned}$$

The inner expression with its modulus factor is the finite last-entry form. For finitely many heads with the same last entry t it is positive semidefinite, since the pairing on every lift is a Gram matrix and both fiber integrations are positive. It is continuous in the heads and in t by the uniform support and the compact-test continuity of the norm-fiber law. All floor forms, including this terminal form, are divided by the same $\mathcal{R}_Z$ of Section 10.3 in the finite sieve. The frequency and bottom bounds are proved in Sections 10.8–10.10.

10.5 Stable ideals and classes of prescribed denominator

In this subsection, we determine which ideals can occur in the denominators of two correlated vectors. Their indices give the summation variables for Section 10.9 with the lattice stabilizers of Section 10.2.

Use the projected lattice Λ, with its integral unimodular Gram, and the original level scalar M on A_T. Let $F_i = \langle e_0, \dots, e_{i-1}\rangle$. At a pair $Y, Y' = Yx$ with nonzero plain Bessel correlation, there are $u, v \in U$ such that $uY, vY' \in K_T$ and $(vY')^{-1}uY \in J$. Indeed the two vectors have a common nonzero coordinate of W_Z, which gives both moves into the same coset of J. Consequently the support in x of the absolute floor majorant is contained in $\Omega_J := \big(\bigcup_Y Y^{-1}UY\big)J$.

Let $\mathfrak{d}_i(Y,x)$ be the set of $h(e_i, w)$ with

$$w \in F_i^\perp, \quad w' - w \in F_i, \quad Y^{-1}w \in \Lambda, \quad Y^{-1}w - (Y')^{-1}w' \in M\Lambda.$$

It is a full fractional ideal. Write $w' = w + d$ and map (w, d) to the pair $\big(Y^{-1}w, M^{-1}(Y^{-1}w - (Y')^{-1}(w+d))\big)$. The map on the full product is invertible. The preimage of the product lattice intersects $F_i^\perp \oplus F_i$ in a lattice, and projection to $h(e_i, w)$ is surjective on this subspace. This gives a compact open integer module in each field factor. The stable ideal is locally constant in (Y, x). Indeed the congruence implies that both transported vectors lie in Λ. Right multiplication of Y, Y' by independent elements of J preserves these integral vectors modulo $M\Lambda$, and the same holds for the inverse changes. The ideal therefore depends only on the two open cosets $YJ, Y'J$. At the active pair it is integral: $u^{-1}e_i \equiv e_i \bmod F_i$ and $uw = (uY)Y^{-1}w$ is integral, so $h(e_i, w) = h(e_i, uw) \in \mathcal{R}$. Write $\mathfrak{d}_i = c_i\mathcal{R}$. Its depth is $D_i = \mathrm{md}(c_i)^{-1}$.

For any integral self-conjugate $C \in A_T^\times$ in all the $\mathfrak{d}_i$, each $w \in F_j^\perp$ with $Y^{-1}w \in \Lambda$ has a lift $w' \equiv w \bmod F_j$ such that

$$Y^{-1}w - (Y')^{-1}w' \in MC^{-2(\varrho - j)}\Lambda.$$

For $j = \varrho$, take $w' = v^{-1}uw$. The unipotents act trivially on $F_\varrho^\perp/F_\varrho$, and $(vY')^{-1}uY \in J$ gives the congruence. Suppose the lift is available at $i+1$ with exponent $D = 2(\varrho - i - 1)$. Divide a stable pair at the value c_i by c_i, giving ζ, ζ' of pairing one with e_i, with both transported vectors in $C^{-1}\Lambda$ and their difference in $MC^{-1}\Lambda$.

For the prescribed w put $t = h(e_i, w) \in \mathcal{R}$ and $w_0 = w - t\zeta$. Apply the induction to Cw_0 and divide by C. This produces $w_0' \equiv w_0 \bmod F_{i+1}$, with transported difference in $MC^{-D-1}\Lambda$. The coefficient b_1 of e_i in $w_0' - w_0$ equals $h(\zeta', w_0') - h(\zeta, w_0)$, since $\zeta' - \zeta \in F_i$ and $w_0 \perp F_i$. In the transported lattices, expand this difference by pairing the transported ζ' against the difference for w_0, and the difference for ζ against the transported w_0.

The two terms belong to $MC^{-D-2}\mathcal{R}$. Thus $w' = t\zeta' + w_0' - b_1 v^{-1} e_i$ is the required lift at i, since $(Y')^{-1}v^{-1}e_i$ is integral and $v^{-1}e_i \equiv e_i \bmod F_i$. At $j = 0$, $w' = w$. Testing $w \in Y\Lambda$ gives $C^{2\varrho}(1 - x^{-1})/M$ integral, hence $C^{2\varrho}(x-1)/M$ integral by unitary adjunction. In particular if $x \notin J$ and every proper integral ideal has index at least Q_0, some $D_i \geq Q_0$. Otherwise all c_i are units in $\mathcal{R}$, and $C = 1$ gives $x \in J$.

On a head of length $j \leq \varrho$, assume also that $w_p^{-1}Y, w_q^{-1}Y' \in \mathcal{U}_j$. Choose an integral self-conjugate scale α divisible by all the partial products $P_a = \prod_{i<a} p_i$, $Q_a = \prod_{i<a} q_i$, and by M if $j > 0$. The products here are integral at nonzero contributions. Indeed take two moves $s, t \in R_j$ so that $sY, tw_p^{-1}Y \in K_T$. Their ratio is $sw_p t^{-1} \in K_T$. Its leading dual-covector minors are P_a by triangularity. Replacing the last covector of the minor of size $a+1$ by covector d gives $P_{a+1}(se_a^*)_d$, since right unitriangular changes of the first $a+1$ columns preserve this minor. Thus $Y^{-1}e_a^* \in \alpha^{-1}\Lambda$ for $a < j$, by multiplication with $(sY)^{-1}$. The same holds for Y' and q. The vector $\alpha^2 e_a^*$ now has both transported vectors in $\alpha\Lambda \subset M\Lambda$, so $\alpha^2 \in \mathfrak{d}_a(Y, x)$.

The denominator ideal $\mathfrak{J}_x$ of the entries of $(x-1)/M$ has generator c, given in each field factor by the uniformizer to the maximum of zero and the negative valuations of those entries. Write $D_x = [\mathcal{R} : \mathfrak{J}_x]$. If all depths after the head are less than σ^{-2}, where $0 < \sigma < 1$, take $C = \alpha^2 \prod_{a=j}^{\varrho-1} c_a \bar{c}_a$. It belongs to every stable ideal and the induction gives

$$D_x \leq \operatorname{md}(C)^{-2\varrho} \leq \operatorname{md}(\alpha)^{-4\varrho} \sigma^{-8\varrho(\varrho - j)}.$$

Let $H = 2^t$ be the least dyadic height at least D_x, and $\Lambda_\alpha = 2/\operatorname{md}(\alpha)$. If $H > \Lambda_\alpha^{2400n^2}$, take $\sigma = H^{-1/(800n^2)}$. The displayed bound is then at most $H^{1/100 + 1/(600n)} < H/2$, contradicting $H/2 < D_x$. Thus on each such shell a floor after the head has depth at least σ^{-2} at every active pair.

Each part of a shell with one denominator ideal $c\mathcal{R}$ lies in the position box of scale $\bar{c}c$, since $\bar{c}c(x-1)/M$ is integral there. Its loss is $\Lambda' = 2D_x^2 \leq \Lambda_\alpha H^2$. The number of boxes is at most $2H^{9n}$. Indeed for $a = 3\varrho + 5 < 9n$ the Euler product of finite-index ideals of $\mathcal{R}$ is bounded by $\zeta(a)^{[E:\mathbb{Q}]} \leq 2$ by the scale calculation of Section 10.2. Each ideal of index at most H contributes at least H^{-a} to that sum. This count applies at every projected place.

10.6 Finite correlations on bounded sets

In this subsection, we bound weighted correlations on compact sets of relative positions. The local vector count from Section 10.1 supplies the estimate used for inverse weights in Section 10.7.

From the $k-1$ orthonormal coordinates of X_0 outside the distinguished column in Section 8.1, use $k - 1 - 2m_+ = k - 6n - 7$ coordinates. Choose two disjoint blocks of size $2m_+ + 2$ among them, with $m_+ = 3n + 3$. The volume exponent in this finite calculation is the free rank

minus $3n+1+m_+$, hence

$$s=(k-6n-7)-2(2m_++2)-(3n+1+m_+)=k-24n-27.$$

Take E_a^K from the floor sums in Section 11.5, using the finite floor measures and $W_a=\prod_{i<a}D_i^{-s}$ with the depths of Section 10.5. With the finite inverse section moduli $H_{p,T}$ and the two kernels K_1,K_2 in those sums, set

$$\mathcal{W}_j=H_{p,T}^2\Lambda_\alpha^{3\cdot10^6n^3}E_0^{K_1}E_j^{K_2}.$$

For an integral, invertible, self-conjugate scalar γ, let Ω_γ consist of the elements $x\in\Omega_J$ with $\gamma(x-I)/M$ integral, and put $\Lambda'=2/\operatorname{md}(\gamma)$. Use $K(g,y)=\langle W_Z(g),W_Z(d_0y)\rangle$, with $d_0=1$ or $d_0=w_q^{-1}w_p$ and starting floor $j_0=0$ or $j_0=j$, respectively. For the original quotient measure Q, we prove

$$\int_{\Omega_\gamma}E_{j_0}^K\,d\mu\le n\delta_U(\Lambda')^{4900n\varrho}\iint|K(g,y)|^2\,dQ(g)dQ(y).\tag{5}$$

Let B_v,B_y be the two chosen blocks and X' the span of the remaining coordinates in this set. Their ranks are

$$h_0=2m_++2=2(3\varrho+5)+4=6n+8,\qquad d'=k-6n-7-2h_0=k-18n-23.$$

All three are spanned by original orthonormal vectors of X_0 outside the distinguished column. They are integral orthogonal summands of every remaining space V_j, including at the places over N. For $g\in K_T$ and a marked position $x=l^{-1}ulj_0$, $u\in U$, $j_0\in J$, choose a direct summand $L_0\subset X'_{\mathcal{R}}$ of rank $D=d'-3\varrho$ in the kernel of $(u-I)lg^{-1}$. Such a summand exists by saturation in each local factor, since $u-I$ sends $F_\varrho^\perp$ into F_ϱ and has rank at most 2ϱ. For $w\in L_0$ both $g^{-1}w$ and $(gx)^{-1}w=j_0^{-1}g^{-1}w$ are integral and congruent modulo M.

Let $\phi(v)$ be the inverse volume of $h(v,L_0)$. A basis of L_0 reads D pairing coordinates in an integral invertible change of basis. Counting the minimum valuation in each factor gives

$$\int_{cX'_{\mathcal{R}}}\phi(v)^s\,dv=\operatorname{md}(c)^{d'-s}\prod_{w\in T}\frac{1-(\mathrm{N}w)^{-D}}{1-(\mathrm{N}w)^{-(D-s)}}\le2\operatorname{md}(c)^{d'-s}.$$

Here $D-s=3n+7$ and the product is bounded as in Section 10.2. For boxed g, the orbit vector $\sigma^{-1}e_a$ is integral on $\mathcal{U}_{a+1}$.

Transporting L_0 by σ shows $D_a(\sigma g,x)\le\phi((\sigma^{-1}e_a)_{X'})$. The norm-block estimate of Section 10.1 bounds the conditional mass over X' by 2, so the one-floor integral of D_a^s on the near locus is at most 4. Rebasing g by R_a and using invariance of the earlier depths gives the finite version of (12), with one-floor contribution 4.

Use the U_a coordinates (v,β) of Section 10.2. The e_a^*-column after subtracting the identity has vector part v and e_a-coefficient $z=\beta-\tau h(v,v)$. The integral projection $z\mapsto z-\tau(z+\bar z)$ recovers β, also at residue characteristic two. The condition of lying in J is exactly divisibility of v,β by M. Thus set

$$\delta_a=\operatorname{md}(M)^{d_a+1/2},\qquad\prod_{a<\varrho}\delta_a=\delta_U,\qquad c_a=\delta_a(\Lambda')^{4900n}.$$

For $x_0,x_1\in\Omega_\gamma$, unitarity gives

$$M^{-1}(x_1x_0^{-1}-I)=M^{-1}(x_1-x_0)x_0^{-1}\in\gamma^{-2}M_r(\mathcal{R}).$$

This persists after conjugation by a boxed element. When the relative element belongs to $U_{<a}U_a$, project its e_a^*-column modulo F_a to read the U_a column. The vector and skew parameters then belong to $M\gamma^{-2}(V_{a+1})_{\mathcal{R}}$ and $M\gamma^{-2}\mathcal{R}^-$, respectively.

We apply the forward-weight and square-ascent argument of Section 11.7. For the forward weight take g boxed and $x_0 = g^{-1}y \in \Omega_\gamma$. If $u_a \in U_a$ has vector v and $g^{-1}\nu u_a y \in \Omega_\gamma$ for some $\nu \in U_{<a}$, the column bound applies to u_a. For a stable pair ζ, ζ' at $g, u_a y$, pair its two transported lifts with the common vectors $w \in L_0$ at g, y. The column formula for $u_a w$ gives

$$h(\zeta, w) - h(u_a^{-1}\zeta', w) = \overline{h(e_a, \zeta)}h(v, w) \in M\mathcal{R}.$$

Put $I = h(v/M, L_0)$ and let $\mathfrak{d}_a$ be the stable coefficient ideal for this pair. The displayed congruence gives $\overline{\mathfrak{d}_a}\, I \subseteq \mathcal{R}$. When I is full, conjugation preserves product Haar measure and $\mathrm{vol}(I^{-1}) = \mathrm{vol}(I)^{-1}$. Hence

$$D_a^{-s} = \mathrm{vol}(\overline{\mathfrak{d}_a})^s \le \phi((v/M)_{X'})^s.$$

If I vanishes in some field factor, the right side is infinite.

Integration over the allowed Heisenberg region is bounded by

$$2\delta_a \,\mathrm{md}(\gamma)^{-2(d_a - s)-1} \le c_a.$$

The forward induction of (16) applies: translate y into the box modulo $U_{<a+1}$, integrate the older unipotents first, and use invariance of depth D_a under their action on the second point. That action transports its stable lift without changing it modulo F_a. For a box move $\sigma u \in G_a U_{<a}$, replace y by σy and ν by $\sigma u \nu \sigma^{-1}$; the first a depths and the position are preserved.

For the square ascent, use the cut by positive mass of $\{\nu \in U_{<a} : g^{-1}\nu y \in \Omega_\gamma\}$. At floor $a \ge j_0,\ a < \varrho$, recenter about a point $u_{a,0}$ of the nonzero support, with $g_c = tug \in K_T,\ u \in U_{<a},\ t = t_1 n_1 \in G_{a+1}U_a$. For u_a on this support compare the positions $x_{u_a} = g^{-1}\nu_{u_a}u_a y$ and $x_0 = g^{-1}\nu_0 u_{a,0} y$ in the box. The conjugate $g_c x_{u_a} x_0^{-1} g_c^{-1}$ belongs to $U_{<a}\, t u_a u_{a,0}^{-1} t^{-1}$. Hence $u_a' = t u_a u_{a,0}^{-1} t^{-1}$ lies in the column box, and the substitution preserves Haar. The vector of u_a is $t_1^{-1}v' + v_0$ for the vectors v', v_0 of $u_a', u_{a,0}$.

Put $y_\sigma = \sigma^{-1} f_{a+1}$, and let C_{a+1}^K be the correlation integrated from floor $a+1$ through the terminal stabilizer. For $\sigma \in G_{a+1}$, the simple-root sum of $\sigma u_a \sigma^{-1}$ is $-h(v, y_\sigma)$. Since $a \ge j_0$, the twist d_0 is the identity on V_a and commutes with this unipotent. Only the second argument moves, and the inner product is linear in that argument. Thus

$$C_{a+1}^K(g, u_a y) = \int_{G_{a+1}} \psi_T\Big(-M^{-1}h(v, y_\sigma)\Big)\, C_{a+2}^K(\sigma g, \sigma y)\, d\theta_{a+1}(\sigma), \qquad \psi_T(z) = \psi_{E,f}(-z/\delta_E).$$

The offset v_0 gives a unit phase in the quadric variable $z = t_1\sigma^{-1}f_{a+1}$. The quadric function is supported on integral z: nonzero correlation implies $\sigma g \in \mathcal{U}_{a+2}$, hence also $\sigma t_1^{-1} g_c = \sigma n_1 u g$, since the extra conjugate is in $U_{<a+1}$. Now use the integral projected-column bound at g_c.

Use v'/M as the Fourier variable and disintegrate the quadric over X'. Cauchy–Schwarz on the conditional integer box contributes at most 2. Enlarge in the X' Fourier variable and apply additive Plancherel with the integral self-dual measure. The complementary vector box, the skew box, and the M-scaling give the square-ascent contribution

$$2\delta_a \,\mathrm{md}(\gamma)^{-2(d_a - d')-1} \le c_a.$$

Both bounds by c_a follow from $d_a - s \le r - s = 26n + 26$. The case of zero cut mass follows by Tonelli. A simultaneous move by G_{a+1} preserves the next cut, since it normalizes the older unipotent with modulus one. At $a = \varrho$, Cauchy–Schwarz on the terminal stabilizer contributes at most one by the compact coset bound.

The quotient integration in the proof of (18) uses only these two inequalities and the product Haar law. With the finite floor measures it gives (5), since the product of the contribution is $\delta_U(\Lambda')^{4900n\varrho}$.

For the twisted kernel, conjugation by d_0 on U has modulus $J_{d_0} = \prod_{a<j} \mathrm{md}(p_a/q_a)^{d_a+1}$. The vector part contributes d_a and the skew center one. Thus $\int F(d_0 y)dQ(y) = J_{d_0} \int F(y)dQ(y)$, since changing the unipotent variable in the Haar disintegration under $y \mapsto d_0 y$ contributes $J_{d_0}^{-1}$. The head-density identity is

$$(\mathcal{D}_j^T)^2 H_{p,T}^2 J_{d_0} = \prod_{a<j} \mathrm{md}(q_a)^{-1} \le \Lambda_\alpha^2,$$

where the last inequality follows from $\alpha/\prod_{a<j} q_a$ being integral.

For a measurable $\Omega \subseteq \Omega_\gamma$ and $0 \le f_0$ with $f_0^2 \le e_*^2 \mathcal{W}_j$, $e_* \ge 0$, the position-box estimate is

$$\mathcal{D}_j^T(p,q) \int_\Omega f_0 \, d\mu \le e_* \, \Lambda_\alpha^{1500000n^3+1} (\Lambda')^{25000n^2} \mathcal{R}_Z$$

by Cauchy–Schwarz, (5), $n(\Lambda')^{4900n\varrho} \le (\Lambda')^{25000n^2}$, and the quotient energy bound of Section 10.3.

10.7 Reciprocal weights and absolute pivot bounds

In this subsection, we obtain absolute integral estimates for both kernels before imposing an oscillatory weight. Section 10.6 supplies their local weights, which also enter the Fourier argument of Section 10.8.

Use the finite depths in $I_a = \mathbf{1}_{\mathcal{U}_a} W_a^{-1}$ and in the sums B_a^K of Section 11.5. Fix $x \in \Omega_J$ and write $C_\alpha = \Lambda_\alpha^{10^6 n^3}$ and $H_{p,T} = \prod_{a<j} \mathrm{md}(p_a)^{-d_a - 1/2}$. We prove

$$\int_{U_{<j}} I_j(w_p \nu g)\, d\nu \le C_\alpha H_{p,T} \qquad (g \in \mathcal{U}_j). \tag{6}$$

Here α/P_a is integral for every initial segment $P_a = \prod_{b<a} p_b$. To rebase g into K_T by $\gamma u_0 \in G_j U_{<j}$, conjugate the older unipotents by γ and use right translation by u_0. These preserve Haar, and γ commutes with w_p and preserves I_j.

For $g \in K_T$, take $u \in U_a$, $\nu \in U_{<a}$ with $w_{p,<a+1} u\nu g \in \mathcal{U}_{a+1}$. Set $s_0 = (u\nu)^{-1}$ and choose $t \in R_{a+1}$ so that $t w_{p,<a+1} u\nu g$ is integral. The ratio of g and this matrix is $s_0 w_{\bar{p},<a+1} t^{-1} \in K_T$. Apply the replaced-covector minor of Section 10.5 to get $\bar{P}_{a+1} s_0 e_a^* \in \Lambda$. Divide by $\bar{P}_a$ and project away the earlier pairs, which removes ν^{-1} modulo F_a. Thus $cu^{-1}e_a^*$ is in the α^{-1}-box on the a-th and later pairs, where $c = \bar{p}_a$.

On the common summand L_0 of Section 10.6, impose the a equations $h(e_b^*, \nu w) = 0$ for $b < a$. Their saturated kernel contains a primitive submodule of rank $d' - 3\varrho - a$. The equations ensure that $w_{p,<a+1} u\nu w$ is orthogonal to F_a. That vector gives equal stable lifts with coefficient $h(cu^{-1}e_a^*, w)$, as the lower unipotent acts trivially modulo F_a on $F_a^\perp$. Hence $D_a \le \phi(v_{X'})$ for v the V_{a+1}-part of $cu^{-1}e_a^*$ and ϕ defined using this smaller submodule. The scaled content integral has bound $2\,\mathrm{md}(\alpha)^{-(d'-s)}$, since $d' - 3\varrho - a - s \ge 2n + 8$.

In the Heisenberg coordinates (ω, β) of u^{-1}, two allowed centers at fixed ω satisfy $c(\beta-\beta') \in \alpha^{-1}\mathcal{R}$. In a nonsplit factor c is a unit times a power of a fixed-field uniformizer. In a split pair, the skew coordinate is $(z,-z)$, so the smaller of the two volume constraints is at most their geometric mean. The skew fiber measure is thus bounded by $\operatorname{md}(c\alpha)^{-1/2}$.

The vector change $v = c\omega$ contributes $\operatorname{md}(c)^{-d_a}$ in the integral. Together with the scaled content and the complementary vector box, the integral of D_a^s on the indicated neighborhood is bounded by

$$2\operatorname{md}(p_a)^{-d_a-1/2}\operatorname{md}(\alpha)^{-(d_a-s+1/2)}.$$

The earlier depths and near condition are unchanged by the added moves in G_a. Indeed, write the product at step $a+1$ as $w_{p,<a+1}u\nu = (w_{p,a}u)w_{p,<a}\nu$, where $w_{p,a}$ acts on the a-th pair alone. The factor $w_{p,a}u \in G_a$ preserves the first a depths, and $\mathcal{U}_{a+1} \subseteq \mathcal{U}_a$. Hence the earlier inverse weights factor out as $I_a(w_{p,<a}\nu g)$ on the support. The primitive submodule used for the a-th integral may depend on g, x, ν; its rank and the bound for that integral are uniform in all three. Iterating in the Haar order $u\nu$ costs at most $C_\alpha H_{p,T}$, since $a < j \le \varrho < n$ and $d_a - s \le 26n+26$. This proves (6).

For every nonnegative measurable left-R_j-invariant B supported on $\mathcal{U}_j$ and every σ-finite measure ρ on the group, Tonelli now gives

$$\int B(g)I_j(w_p g)\, d(\eta_{<j} * \rho)(g) \le C_\alpha H_{p,T} \int B(g)\, d\rho(g).$$

Indeed $B(\nu g) = B(g)$ for $\nu \in U_{<j}$. The same bound holds with $w_p^{-1} = w_{\bar{p}}$, since $\alpha = \bar{\alpha}$ and conjugation preserves the modulus.

On $\mathcal{U}_{j+1}$ the floor j mass is at most $V_0 = \zeta_{\mathcal{R}}(3\varrho+5) \le 2$, by Section 10.2. Apply the ordinary Cauchy–Schwarz bound and the first-deep-floor argument of Section 11.6, using the finite one-floor contribution 4 in Section 10.6. Explicitly, let P_j^K be the integral of $|C_{j+1}^K(\sigma g, \sigma g x)|$ over θ_j. Then $(P_j^K)^2 \le V_0 B_j^K I_j$. If every nonzero terminal contribution has some floor $a \ge j$ with $D_a \ge e^{-1}$, $0 < e \le 1$, stop the tail expansion at the first such a. The no-stop branch vanishes, since higher groups preserve each earlier depth. The square of each remaining integral is at most $e^s 4^\varrho B_j^K I_j$ by Cauchy–Schwarz with $W_a^{-1} = D_a^{-s} W_{a+1}^{-1}$ on the stopping locus. Thus $(P_j^K)^2 \le e^s C_\alpha B_j^K I_j$, using $\varrho^2 4^\varrho \le C_\alpha$.

Write $z = w_p^{-1} g$. In the two-pivot majorant $\mathcal{M}_j(x) = \int P_j^{K_1}(g) P_j^{K_2}(z)\, dM_j(g)$ the product of the two weights and energies can be crossed as

$$[B_j^{K_1}(g) I_j(z)]\,[B_j^{K_2}(z) I_j(g)].$$

Cauchy–Schwarz and (6) bound the first integral by $H_{p,T} C_\alpha E_0^{K_1}$. For the second change $g = w_p z$. The integrand in z is left G_j-invariant, and $d\mu = d\mu_j * M_j$ with μ_j Haar on G_j. Since w_p commutes with G_j, the invariant-function quotient identity in Section 11.5 applies with the finite measures.

The second integral is then at most $H_{p,T} C_\alpha E_j^{K_2}$. It follows that $\mathcal{M}_j^2 \le V_0^2 \mathcal{W}_j$. On a shell $H > \Lambda_\alpha^{2400n^2}$, take $\sigma = H^{-1/(800n^2)}$ as in Section 10.5. Suppose $P_j^{K_2}(z) > 0$. Expanding its tail integrals gives some $t \in G_j$ with

$$W_Z(t w_p^{-1} g) \ne 0, \qquad W_Z(t w_q^{-1} g x) \ne 0.$$

Here all later groups lie in G_j, and $w_q^{-1} w_p$ commutes with t. The support of W_Z lies in $UK_T \subseteq \mathcal{U}_j$. Thus for every $h \in G_j$, multiplication on the left by $ht^{-1} \in R_j$ gives

$$w_p^{-1} h g,\ w_q^{-1} h g x \in \mathcal{U}_j.$$

At each such h with $K_1(hg, hgx) \neq 0$, both hypotheses of the remote-shell argument in Section 10.5 therefore hold. It gives some $a = a(h) \geq j$ with $D_a(hg, x) \geq \sigma^{-2}$. Stopping at the first such index on each nonzero contribution uses the same bound for all h. The integrand vanishes if the second pivot is zero. Using the deep estimate on the first pivot with $e = \sigma^2$ gives $\mathcal{M}_j^2 \leq V_0^2 \sigma^{2s} \mathcal{W}_j$.

At $j = 0$ both pivots coincide and M_0 is the atom at the identity. If every active point has some $D_a \geq e^{-1}$, using the deep bound for both identical factors gives $\mathcal{M}_0^2 \leq e^{2s} \mathcal{W}_0$; the factor C_α^2 from the two bounds is at most the C_α^3 in $\mathcal{W}_0$.

10.8 Local Fourier estimates for the leading coordinates

In this subsection, we use translation invariance of the finite kernels to identify the frequencies that can contribute. The absolute integral estimates of Section 10.7 control their magnitudes in the summation of Section 10.9.

Fix the finite data of Section 10.4. Use $d' = k - 18n - 23$, the dimension of X', and $C_\alpha = \Lambda_\alpha^{10^6 n^3}$. Write "box a" for vectors or matrices with integral coordinates after multiplication by α^a.

Suppose $g, w_p^{-1} g \in \mathcal{U}_j$. Factor a move at g into the middle group and older unipotent, and choose $t \in G_j$, $u \in U_{<j}$ such that $utg \in K_T$. Since t commutes with w_p, choose $v \in R_j$ with $v w_p^{-1} t g \in K_T$. Their ratio is $u w_p v^{-1} \in K_T$. The minors in Section 10.5 give $P_a \in \mathcal{R}$ and $P_{a+1} u e_a^* \in \Lambda$ for $a < j$, with the leading products $P_a = \prod_{b<a} p_b$.

Use the integral quotients α / P_a to conclude $p_a^{\pm 1} \in \alpha^{-1} \mathcal{R}$ and $u e_a^* \in \alpha^{-1} \Lambda$. In Witt order these j dual columns of u have upper, middle and lower blocks C, D, E, with E lower unitriangular. Its upper diagonal block is $A = (E^*)^{-1}$ and its upper middle block is $-AD^* J_0$, with J_0 the integral middle Gram. The nilpotent inverse series bounds A in box $j - 1$, hence u, u^{-1} in box j.

Thus tg is in box j, and $t w_p^{-1} g$ in box $j + 1$, also when $j = 0$ by taking $t = g^{-1}$. For $\sigma \in G_j$ on the support of a pivot tail at g, choose ζ with $\zeta \sigma g \in K_T$, $\zeta \in R_{j+1}$. Projecting $(tg)(\zeta \sigma g)^{-1} f_j$ off the earlier pairs gives $t \sigma^{-1} f_j$, since $\zeta^{-1} f_j = f_j \bmod F_j$. The orbit vector is therefore in box j, and the corresponding vector from the pivot at $w_p^{-1} g$ is in box $j + 1$. Here $f_\varrho = z$, so the final pivot also uses its norm quadric, with the stabilizer integration occurring after that pivot.

Use the finite character $\psi_T(x) = \psi_{E,f}(-x/\delta_E)$ on A_T. In each field component the trace-dual of the integers is the inverse different. Thus for ψ_T the annihilator of the integer lattice is that same lattice, and its Haar measure of mass one is self-dual, including above 2. The level factor M^{-1} enters through the Bessel character. On two norm quadrics in V_j, test on boxes a, a', and put $c_j = \dim V_j - d'$. The operator with kernel $\psi(\eta h(v, y))$, $\eta \in A_T^\times$, has squared L^2-operator norm on these boxes at most

$$4 \operatorname{md}(\eta)^{-d'} \operatorname{md}(\alpha)^{-(a+a')c_j}.$$

Use the three orthogonal summands X', B_v, B_y fixed in Section 10.6. Impose the norm of v on B_v and the norm of y on B_y. In a box of exponent a' for y, a block of rank h_0 has conditional volume

$$\operatorname{vol}(B_y \mid \text{fixed norm}) \leq 2 \operatorname{md}(\alpha)^{-a'(h_0 - 1)}.$$

This follows from Section 10.1 by scaling the norm-value Haar by $\operatorname{md}(\alpha)^{-a'}$. The other $c_j - h_0$ nonfree coordinates contribute their box volumes, giving the bound $2 \operatorname{md}(\alpha)^{-a'(c_j - 1)}$. The same bound with a applies to the nonfree coordinates of v.

Fix the coordinates of v outside X' and B_v. Apply Cauchy–Schwarz in the coordinates of y outside X' and B_y. For each fixed tuple of these coordinates, the phase involving B_v is constant on the free coordinates of y and on B_y. Integrating over B_v uses its conditional volume bound. Plancherel on X' gives the factor $\mathrm{md}(\eta)^{-d'}$, and Cauchy–Schwarz on B_y uses its conditional volume. Enlarging each power $c_j - 1$ to c_j gives the stated bound. If the box for v has no norm equation, only the block B_y contributes a conditional volume and the constant is 2. All scalings use the actual finite product, including the split and unramified quadratic factors.

For the common-entry character $\omega = \psi(\xi Z_j)$, the two pivot variables give the character of

$$Z_j(\sigma\tau^{-1}) = h(\sigma^{-1}f_j, \tau^{-1}f_j).$$

The pointwise double pivot is zero unless both starting points $g, w_p^{-1}g \in \mathcal{U}_j$. Rotate the two orbit variables by the same t. The Hermitian pairing is unchanged, and its values lie in $\alpha^{-(2j+1)}\mathcal{R}$. It suffices that a frequency $\eta \in A_T^\times$ agree with the original character on $\alpha^{-200n^2}\mathcal{R}$. The amplitudes factor through the pivot orbits by the stabilizer invariance of the tails; rotating the orbit variable preserves the marked measure. The two-box estimate and Cauchy–Schwarz bound the square of this double pivot by

$$\mathrm{md}(\eta)^{-d'} C_\alpha \, B_j^{K_1}(g) I_j(g) \, B_j^{K_2}(w_p^{-1}g) I_j(w_p^{-1}g).$$

Indeed the squared pivot norms are bounded by $B_j^K I_j$, and $c_j \le r - d' = 20n + 22$ makes the box loss at most C_α. Cross the weights and apply the two inverse-weight integrations as in Section 10.7. For the integral $f_0(x)$ of the modulus of the double pivot over M_j, this gives

$$f_0(x)^2 \le \mathrm{md}(\eta)^{-d'} \mathcal{W}_j(x). \tag{7}$$

For the unequal heads take $0 \le i < \varrho$, $j = i + 1$, $\omega = 1$ and $\xi = M^{-1}(p_i^{-1} - q_i^{-1}) \in A_T^\times$. Let $d = w_q^{-1} w_p$. Write $\mathfrak{s}$ for the simple-root sum in the Bessel character. A floor-i element u with vector coordinate $v(u)$ has $\mathfrak{s}(u) = -h(v(u), f_j)$. Conjugation by d multiplies this root by p_i/q_i, because d acts by that scalar on e_i and fixes V_j. On moving u through the next pivot σ, the difference of the two character values is thus the character of

$$-M^{-1}(p_i/q_i - 1) h(v(u), \sigma^{-1}f_j) = -\xi h(\bar{p}_i u^{-1} e_i^*, \sigma^{-1} f_j).$$

The equality uses conjugate-linearity in the first argument and the V_j component $-v(u)$ of $u^{-1}e_i^*$. In particular, with the original floor measure one has

$$C_j^{K_2}(ug, ugx) = \int_{G_j} \psi_T\Big(-\xi h(\bar{p}_i u^{-1} e_i^*, \sigma^{-1} f_j)\Big) \, C_{j+1}^{K_2}(\sigma g, \sigma g x) \, d\theta_j(\sigma).$$

Indeed $\sigma u \sigma^{-1} \in U_i$; its two actions on the kernel contribute the difference of the Bessel characters above. The identity includes $j = \varrho$, where $f_j = z$.

The resulting section estimate for the twisted tail is

$$\begin{aligned} &\int_{U_i} \mathbf{1}_{\{w_p ug \in \mathcal{U}_j\}} |C_j^{K_2}(ug, ugx)|^2 \, du \\ &\quad \le \mathrm{md}(\xi)^{-d'} \, \mathrm{md}(p_i)^{-d_i - 1/2} C_\alpha \int |C_{j+1}^{K_2}(\sigma g, \sigma g x)|^2 \, d\theta_j(\sigma). \end{aligned} \tag{8}$$

To prove it, at each u in the support both ug and $w_p ug \in \mathcal{U}_j$. Apply the simultaneous box bound with $w_p ug$ as the first argument, obtaining $t_u \in G_j$ with $t_u ug$ in box $a_0 = j + 1$. Fix one

such point u_* and its t_*. The substitution $m = t_* u_* u^{-1} t_*^{-1}$ preserves U_i Haar. Since t_u, t_* fix e_i^*,

$$(m - I)e_i^* = \big((t_* u_* g)(t_u u g)^{-1} - I\big)e_i^*.$$

The right-hand side lies in box $2a_0$, using the integral inverse Gram and $\bar{\alpha} = \alpha$. The trace-one skew projection keeps the skew parameter in that box, also above 2. Its Haar volume is $\mathrm{md}(\alpha)^{-a_0}$. Use $\bar{p}_i v(m)$ as the vector Fourier variable. Because $p_i \in \alpha^{-1}\mathcal{R}$, it lies in box $2a_0+1$, and the vector measure is multiplied by the factor $\mathrm{md}(p_i)^{-d_i}$ under this substitution.

Rotate the quadric in the second variable by t_*. The relation $u^{-1} = u_*^{-1} t_*^{-1} m t_*$ makes the u_* term of the phase a unit factor depending only on the second variable. The variable term is the pairing of the new vector with $t_* \sigma^{-1} f_j$ at frequency $-\xi$, of modulus $\mathrm{md}(\xi)$. To bound the orbit vector, put $B = t_* u_* g$ and $\kappa = \sigma t_*^{-1} \in G_j$. Nonzero tail support gives $\sigma g \in \mathcal{U}_{j+1}$, and $\sigma u_* \sigma^{-1} \in U_{<j}$ gives $\kappa B = \sigma u_* g \in \mathcal{U}_{j+1}$. Choose $\zeta \in R_{j+1}$ with $\zeta \kappa B \in K_T$. Since B is in box a_0, the column $B(\zeta\kappa B)^{-1} f_j = \kappa^{-1}\zeta^{-1} f_j$ is in that box. Its projection off the earlier hyperbolic pairs is $\kappa^{-1} f_j = t_* \sigma^{-1} f_j$, since $\zeta^{-1} f_j - f_j \in F_j$ and $\kappa \in G_j$. Put $c = d_i - d'$. The vector box has exponent $2a_0 + 1$ and the quadric box has exponent a_0. Multiplying the squared one-quadric bound by the skew volume and the vector Jacobian gives the factor

$$2\,\mathrm{md}(\xi)^{-d'}\,\mathrm{md}(p_i)^{-d_i}\,\mathrm{md}(\alpha)^{-(3a_0+1)c}, \qquad (2a_0+1)c + a_0(c-1) + a_0 = (3a_0+1)c.$$

Since αp_i is integral and $(3a_0+1)c + 1/2 \le (3n+1)(20n+22) + 1/2 < 10^6 n^3$, this factor is at most $\mathrm{md}(\xi)^{-d'}\,\mathrm{md}(p_i)^{-d_i - 1/2} C_\alpha$, as required in (8).

For the outer integration put $z = w_p^{-1} g$ and $\varphi_l(g) = |C_j^{K_l}(g, gx)|$. The squared product $\varphi_1(g)^2 \varphi_2(z)^2$ is the product of the crossed factors

$$A_i^{K_1}(g) I_j(z), \qquad \mathbf{1}_{\{w_p z \in \mathcal{U}_j\}}\, W_j(z) I_i(w_{p,<i} z) \varphi_2(z)^2.$$

This identity is on the support of the product, where $I_i(w_{p,<i} z) = W_i(g)^{-1}$, since the last Weyl factor belongs to G_i. The first factor integrates to at most $H_{p,T} C_\alpha E_0^{K_1}$: use (6) and then the i-th energy term. For the second factor change $g = w_p z$ in the G_j-invariant quotient. In $M_j = \eta_i * \eta_{<i} * \theta_{[0,i]}$, the weights $W_j(z)$ and $I_i(w_{p,<i} z)$ are invariant under U_i. Apply (8). The next pivot energy bound $B_j^{K_2}/W_j$ cancels W_j. Integrating the remaining inverse weight by (6) at length i gives $C_\alpha E_j^{K_2}$ times the first i section moduli. The section in (8) supplies the last modulus. Thus the second integral is at most $\mathrm{md}(\xi)^{-d'} H_{p,T} C_\alpha^2 E_j^{K_2}$. By Cauchy–Schwarz this proves (7) with $\eta = \xi$.

In either case $f_0 \le \mathcal{M}_j$. Since $d' \ge s$, (7) and the absolute bound give the required squared saving $V_0^2 \min(1, \mathrm{md}(\eta)^{-1/2})^{2s} \mathcal{W}_j$.

10.9 Summation by denominator

In this subsection, we sum the finite integral estimates by ideal index. The smaller bound on large denominators comes from Section 10.7, and the sum gives the estimate at the selected stabilizers in Section 10.10.

Take $0 \le f_0 \le \mathcal{M}_j$ with the pointwise saving $f_0^2 \le (V_0 e_0^s)^2 \mathcal{W}_j$, $e_0 \in [0,1]$. The absolute pivot estimates give this bound for $e_0 = 1$, and also with e_0 replaced by σ on the remote shells, by Section 10.7. The box cover then bounds the shell integral times its head density by $V_0 g_H^s \Lambda_\alpha^{2\cdot 10^6 n^3} H^{10^5 n^2} \mathcal{R}_Z$, where $g_H = e_0$ for $H \le \Lambda_\alpha^{2400 n^2}$, and $g_H = \min(e_0, H^{-1/(800 n^2)})$

otherwise. For $s \ge l + 9 \cdot 10^7 n^4$, split $g_H^s \le e_0^l g_H^{9\cdot 10^7 n^4}$. In the first range, $H^{1+10^5 n^2} g_H^{9\cdot 10^7 n^4} \le \Lambda_\alpha^{29\cdot 10^7 n^4}$. In the second range it is at most $H^{1-12500n^2} \le 1$. Summing H^{-1} gives 2, so

$$\mathcal{D}_j^T(p,q) \int f_0 \, d\mu \le e_0^l \Lambda_\alpha^{10^9 n^4} \mathcal{R}_Z.$$

The absolute pivot bound allows $e_0 = 1$ and gives the integrable majorant used in the lower common-entry identity of Section 10.4.

For the Fourier saving let $\xi \in A_T^\times$ and $D(\xi) = [\mathcal{R} : \mathcal{R} \cap \xi^{-1}\mathcal{R}]$. At a split pair of residue size q, write $a = v_w(\xi_w)$, $a' = v_{\bar w}(\xi_{\bar w})$. Its factors are

$$\mathrm{md}_{\{w,\bar w\}}(\xi) = q^{-a-a'}, \qquad D_{\{w,\bar w\}}(\xi) = q^{\max(0,-a)+\max(0,-a')}.$$

The two valuations can be arbitrary integers, since each component belongs to the multiplicative group of its local field. For the Fourier transform in a common new entry, the two earlier heads p, q are arbitrary. The same change of variables $t \in G_j$ puts the two pivot vectors in boxes j and $j+1$. Their pairing lies in $\alpha^{-(2j+1)}\mathcal{R} \subseteq \beta^{-1}\mathcal{R}$, where $\beta = \alpha^{200n^2}$, by Section 10.8. Take $\eta_w = \xi_w$ if $v_w(\xi) \le v_w(\beta)$ and $\eta_w = \beta_w$ otherwise. Then $\eta - \xi \in \beta\mathcal{R}$, so the characters agree on the Gram support. Also $w_0 = \beta/\eta$ and ξw_0 are integral. For the constant-weight estimate with distinct last entries, keep $\eta = \xi$ and use $w_0 = \beta$ when $\beta\xi$ is integral. For the common-entry choice,

$$v_w(w_0) = \max\{0, v_w(\beta) - v_w(\xi)\}, \qquad v_w(\xi w_0) = \max\{v_w(\xi), v_w(\beta)\} \ge 0$$

at each field component separately. In both cases $\mathrm{md}(w_0) \le D(\xi)^{-1}$ by inclusion of $w_0\mathcal{R}$ in the denominator ideal. For $e_0 = \min(1, \mathrm{md}(\eta)^{-1/2})$, we have $e_0 \le \mathrm{md}(\xi)^{-1/2}$, and

$$e_0^2 \le \frac{\mathrm{md}(w_0)}{\mathrm{md}(\beta)} \le D(\xi)^{-1} \Lambda_\alpha^{200n^2}.$$

For $l = k - n^7 + 12$ we have $s - l = n^7 - 24n - 39 \ge 9 \cdot 10^7 n^4$, since $n \ge 10^8$ (Section 12.10). Thus the shell saving retains $\mathrm{md}(\xi)^{-(k-n^7)/2} D(\xi)^{-6}$ at contribution $\Lambda_\alpha^{1200n^2}$. This calculation requires a frequency nonzero in every factor, as holds for a nonzero rational frequency or a nonzero difference of two rational inverse heads.

For the normalized bound, a nonzero majorant requires all P_a, Q_a integral, since it requires both pairs of near conditions at a plain point and its twisted image. A nonintegral initial segment makes the form vanish. In the remaining case take

$$B = \prod_{a=1}^{j} P_a Q_a, \qquad \alpha = MB\bar B, \qquad \mathfrak{h}(p) = \prod_{a=1}^{j} \mathrm{md}(P_a)^{-n^6/2}, \qquad \mathfrak{h}(q) = \prod_{a=1}^{j} \mathrm{md}(Q_a)^{-n^6/2}.$$

Then $\alpha/M, \alpha/P_a, \alpha/Q_a$ are integral, as are $\alpha/p_i, \alpha/q_i$. Thus $\alpha^2\xi$ is integral for the unequal-head frequency, and $\beta = \alpha^{200n^2}$ is admissible in both cases. The head weights are defined to be zero in the nonintegral-initial segment case. For $4e_1 \le n^6$, one has

$$\Lambda_\alpha^{e_1} = (2/\mathrm{md}(M))^{e_1} \prod_a \mathrm{md}(P_a Q_a)^{-2e_1} \le (2/\mathrm{md}(M))^{n^6} \mathfrak{h}(p)\mathfrak{h}(q).$$

The total shell and denominator loss has $e_1 \le 2 \cdot 10^9 n^4$, so the restriction holds. Since M is an integer, $\mathrm{md}(M) \ge M^{-2f}$ with $f = [E : \mathbb{Q}]/2$. Equality holds when T contains all places above

M, as in the adelic sum. Since $M \le b^{23}$, $(2/\operatorname{md}(M))^{n^6} \le b^{(46f+1)n^6} \le b^{n^9 f}$. The normalized finite estimate on the same Z is therefore

$$\frac{|\mathcal{I}_j^T(p,q;\omega)|}{\mathcal{R}_Z} \le b^{n^9 f}\mathfrak{h}(p)\mathfrak{h}(q)\operatorname{md}(\xi)^{-(k-n^7)/2}D(\xi)^{-6}. \tag{9}$$

This holds for the common-entry character and for the unequal-head form with $\omega = 1$ above.

10.10 Finite measure estimate for the stabilizers

In this subsection, we specialize the finite pairing to the stabilizers that occur in the orbital argument. The denominator sum of Section 10.9 bounds the terms outside the level subgroup, leaving the normalized contribution of Section 10.3.

Take empty heads and $\alpha = 1$. Let $Q_0 \ge 2$ bound from below the indices of proper full integral ideals in $\mathcal{R}$. Off J every active pair has some $D_a \ge Q_0$ by Section 10.5. Apply the floor-zero estimate of Section 10.7 with $e = Q_0^{-1}$ to $f_0 = \mathbf{1}_{G_T\setminus J}\mathcal{M}_0$. The absolute shell sum in Section 10.9 applies with $l = t + 10^9 n^4$ whenever $t + 2\cdot 10^9 n^4 \le k$. In fact $s - l \ge 10^9 n^4 - 24n - 27 \ge 9\cdot 10^7 n^4$. Absorbing $2^{10^9 n^4}$ by the spare power of Q_0^{-1} gives $\int f_0\, d\mu \le Q_0^{-t}\mathcal{R}_Z$. The quotient correlation has squared modulus at most $\mathcal{M}_0$ by the triangle inequality. With the exact contribution on J from Section 10.3, this yields

$$\mathcal{M}(Z) \le \mu(Z)\mu(J)(\mu(Z) + Q_0^{-t}). \tag{10}$$

Here $\mathcal{M}(Z)$ denotes the mass of the conjugated character, as in that section. On the entire projected set we may take $Q_0 = 2$ and $t = n$. The compact upper bound then gives

$$\frac{\mathcal{M}(K_T)}{\mu(K_T)\mu(J)} \le 1 + 2^{1-n} \le 1 + b^{-6}.$$

The last inequality follows for large x from $n = m/2$ and $\log b \le x^{1/100}$.

Lemma 10.2 (Finite mass of mixed stabilizers)**.** *For a set $\mathcal{A}$ of marked pairs above p_0^{lev}, fix at each pair a nonzero vector in the prescribed residue span or a nonzero covector in its opposite-place realization, as in Section 9.1. Let Z be the inverse image of their stabilizers, with the full compact group at the other places in T. Then*

$$0 \le \frac{\mathcal{I}_0^T(1)}{\mathcal{R}_Z} = \frac{\mathcal{M}(Z)}{\mu(Z)\mu(J)} \le (1 + b^{-6})b^{-6k|\mathcal{A}|}.$$

Proof. For the chosen places $\mathcal{A}$ defining $\Gamma_{\mathcal{A}}$, take Z to be the inverse images of its vector or covector fixers on the selected conjugate pairs and the full compact subgroup on the complement in T. This contains J and contains every marked integral lift with reduction in $\Gamma_{\mathcal{A}}$. On one selected split pair let K, J_w, Y denote the compact, level and fixer inverse image, and put $q = p_0^{\mathrm{lev}} \ge b^{10}$.

The residue group $\mathrm{GL}_r(\mathbb{F}_q)$ acts transitively on the nonzero vectors and on the nonzero covectors. Reduction is surjective by lifting matrices with unit determinant. Thus $\mu(Y) = \mu(K)/(q^r - 1)$. In the floor Haar of Section 10.2 one has

$$\mu(K) = \prod_{l=k+1}^{r}(1 - q^{-l}).$$

Indeed, over the split quadratic algebra a vector is a column a and a covector b' over the base ring, with norm $b'(a)$. At an isotropic step of remaining rank l, the compact orbit consists of primitive a, b' with $b'(a) = 0$: extend a to an integral basis of $\ker b'$, and append a vector on which b' is one. Its residue cardinality is $(q^l - 1)(q^{l-1} - 1)$. At the last step $l = k + 1$, the norm $-N$ is a unit at the chosen prime above M. Complete the primitive vector a to an integral basis using $\ker b'$, which is a direct complement since $b'(a) = -N$ is a unit. The residue cardinality is then $(q^l - 1)q^{l-1}$. In either case the differential $(v, c) \mapsto b'(v) + c(a)$ is surjective modulo the maximal ideal. Each solution has q^{2l-1} lifts per residue level, so division by q^{2l-1} gives its integral orbit density. Multiplication over the isotropic steps $l = r, r - 2, \dots, k + 3$ and the final step gives the formula, since all integral Heisenberg factors and the terminal compact group have measure one. In particular $\mu(K) \geq 1 - \sum_{l>k} q^{-l} \geq 1/2 \geq 2^{-n}$.

Put $Q_0 = b^9$ and $a_0 = k - 10^{10}n^4$. For large x, $q^r - 1 \geq Q_0^{a_0+1}$, $9a_0 \geq 6k$, and $a_0 + 1 + n + 2 \cdot 10^9 n^4 \leq k$, since $r \geq k$ and $k/n^4 \to \infty$. A nonunit integral element on this pair has modulus at most $q^{-1} \leq Q_0^{-1}$. Apply (10) with $t = a_0 + 1 + n$. As $Q_0^{-n} \leq 2^{-n} \leq \mu(K)$, it gives

$$\frac{\mathcal{M}(Y)}{\mu(Y)} \leq \mu(J_w)\mu(K)\, 2Q_0^{-a_0-1} \leq b^{-6k}\frac{\mathcal{M}(K)}{\mu(K)}.$$

The second inequality uses $Q_0 \geq 2$ and $\mathcal{M}(K) \geq \mu(K)^2\mu(J_w)$, the contribution on J_w. On the complementary factors use $Z = K$. The group, level and character factor over these conjugation-stable blocks. The inequalities for $\mathcal{M}(Y)/\mu(Y)$ relative to $\mathcal{M}(K)/\mu(K)$ are unchanged under rescaling local Haar, since both ratios scale by its square. Tonelli and the bound for K_T therefore prove

$$0 \leq \frac{\mathcal{I}_0^T(1)}{\mathcal{R}_Z} = \frac{\mathcal{M}(Z)}{\mu(Z)\mu(J)} \leq (1 + b^{-6})b^{-6k|\mathcal{A}|}.$$

This uses the same Z, original finite character, and floor Haar as the terminal and lower-floor forms. □

11 Weighted real orbital integrals

In this section, we prove the real Fourier and square-integral bounds used to compare orbital integrals at a common entry in Section 12.2. The estimates are in the norm-fiber measures fixed below.

11.1 Plancherel on the real norm fibers

In this subsection, we identify a real Fourier transform on two Hermitian norm fibers by ordinary Plancherel. Its measure normalization is used in the phase calculation of Section 11.2 and in the integral of Section 11.7.

At a complex place of E, use ordinary area measure on $\mathbb{C}$, the character $\psi(c) = e^{c-\bar{c}}$, and norm-value coordinate $h(y, y)/2$ with ordinary Lebesgue measure. The factor $1/2$ specifies the real skew coordinate. Write $h(v, y) = v^* J y$ in the common floor basis. Among the positive orthonormal coordinates choose a free set I of size $d = k - 5$ and disjoint two-coordinate blocks B, B', both disjoint from I. Take all three in the definite summand X_0. They occupy $k - 1$ coordinates. The remaining positive line and the negative line $\mathbb{C}z$ give a hyperbolic pair available at every floor, including the last. The signed change of basis p_v of Section 8.3 is

Euclidean unitary and hence preserves operator norms and volume. Solve the source quadric in B and the target quadric in B'.

For either block the equation at fixed other coordinates is $|z_1|^2+|z_2|^2 = 2s$. Its fiber measure is sphere probability times $4\pi^2 s$ for $s > 0$, from the complex two-ball volume $2\pi^2 s^2$. For $s = 0$ the conditional measure in the two solved coordinates has mass zero. In a box $|z_j| \le L^\ell$, the conditional mass is at most $4\pi^2 L^{2\ell}$. For a full vector measure on that side the corresponding bound is $\pi^2 L^{4\ell}$.

The sphere normalization agrees with the hyperbolic chart on each fixed norm fiber. In that chart, with a hyperbolic pair t, v, constrain $2\Re(\bar{t}v) + Q_0 + |z|^2$ and put $c = v/t$. The Jacobian of $v = tc$ is $|t|^2$, which is also the derivative of the norm-value coordinate with respect to $\Re c$. The two factors cancel in the fiber measure, leaving ordinary measure in $\Im c$. At fixed norm, replacing $|z|^2/2$ by $\Re c$ restores the factor $|t|^2$, so together with $\Im c$ it gives d^2v. This yields the stated sphere mass on that chart. Choose the hyperbolic pair outside the solved positive block. In the sphere coordinates t is then a free complex coordinate, so its zero set has measure zero on each fixed fiber, including the isotropic fiber. The hyperbolic chart integrates over $t \neq 0$.

Write μ_b for the resulting measure at $h(y, y)/2 = b$. For compactly supported continuous F, the sphere formula gives

$$\int F\, d\mu_b = 4\pi^2 \int s_+(b,a) \int_{S^3} F\big(a, \sqrt{2s_+(b,a)}\,\omega\big)\, d\omega\, da, \qquad s(b,a) = b - h(a,a)/2,$$

where a ranges over the complementary coordinates, $s_+ = \max(s, 0)$, and $d\omega$ is probability measure. The integrand is continuous in b, with a uniform bound on the compact projection of $\operatorname{supp} F$ for b in any compact interval. For every isometry g, the ambient Jacobian is $|\det_{\mathbb{C}} g|^2 = 1$. Disintegrating F and $F \circ g$ over the same norm gives equal integrals against μ_b for almost every b, hence for every b by continuity. Thus the rotations in Section 11.2 preserve the prescribed measure also at norm zero.

Write the source and target as $y = (a, w, z)$ and $v = (a', w', z')$, with w, w' on I and z, z' on B, B', respectively. The two measures on product boxes are

$$da\, dw\, d\nu_{a,w}(z), \qquad da'\, dw'\, d\nu'_{a',w'}(z').$$

Let V, V' be the Lebesgue volumes of the a, a' boxes and M, M' the conditional block bounds. Orthogonality gives

$$h(v, y) = \overline{w'} \cdot w + \overline{z'} \cdot a|_{B'} + A(a'; y).$$

The last term is independent of w', z', and the middle term depends on the second variable only through a.

For a function ϕ supported in the source box and $\xi \neq 0$, define

$$F_{a',a}(w) = \int \psi(\xi A(a'; a, w, z))\phi(a, w, z)\, d\nu_{a,w}(z).$$

For the fixed first variable, Cauchy–Schwarz in a bounds the squared transform by

$$V \int \left| \int_{\mathbb{C}^I} \psi(\xi\overline{w'} \cdot w) F_{a',a}(w)\, dw \right|^2 da.$$

Integrate z' with contribution M' and extend the w' integration to all of $\mathbb{C}^I$. For one coordinate, writing $w = x + iy$ and $\xi\overline{w'} = A_0 + iB_0$ gives the Euclidean Fourier frequencies $(B_0/\pi, A_0/\pi)$.

Their Jacobian relative to w' is $|\xi|^2/\pi^2$. Euclidean Plancherel therefore gives $(\pi^2/|\xi|^2)^d$. Finally, Cauchy–Schwarz in the source block z and integration over a' give

$$\int_{\mathrm{out}} \left| \int_{\mathrm{in}} \psi(\xi h(v,y))\phi(y)\, d\mu(y) \right|^2 d\mu'(v) \le VV'MM' \left(\frac{\pi^2}{|\xi|^2} \right)^d \int_{\mathrm{in}} |\phi(y)|^2 d\mu(y).$$

All box measures are finite by the conditional mass bounds, so the integrals apply to square-integrable functions.

11.2 Support and phases in the real integral

In this subsection, we compute the characters of the two real kernels on the support of the truncated coefficient. The Plancherel normalization of Section 11.1 gives the phase used in Section 11.5.

Use the chosen basis and the groups G_i, R_i, U_i as in Section 10.2, now over $\mathbb{C}$. The real Gram matrix satisfies $J = J^* = J^{-1}$. Let $\mathcal{K}$ be the real box of operator norm at most $R_b = b^{6n+2}$. In this real calculation put $\mathcal{U}_i = R_i\mathcal{K}$. Let r be the ambient rank, $L \ge \max(R_b, 8r)$, $n = \varrho + 1$, and $\Lambda_L = L^2$. Entries of size a have modulus at most L^a. On a head p of length $j \le \varrho$, assume $|\prod_{a<l} p_a|^{-1} \le L$ for $l \le j$, and let w_p send e_a to $p_a e_a^*$ and e_a^* to $\bar{p}_a^{-1} e_a$ for $a < j$.

Suppose $g, w_p^{-1}g \in \mathcal{U}_j$. Factor the move at g in $R_j = G_j U_{<j}$. Thus some $t \in G_j$ gives $stg \in \mathcal{K}$ with $s \in U_{<j}$. The other move can be written $s' w_p^{-1} tg \in \mathcal{K}$ with $s' \in R_j$, since t commutes with w_p. The ratio of the boxed matrices is $s w_p (s')^{-1}$, of norm at most R_b^2. In its first l flag columns, right unitriangular terms from $(s')^{-1}$ leave all l-covector minors unchanged. The first l dual covectors of the columns se_a^* are lower unitriangular, so the leading dual-covector minor is $P_l = \prod_{a<l} p_a$.

At $l = a+1$, replacing the last covector by covector d gives $P_{a+1}(se_a^*)_d$ by expansion in the last column. Hadamard's inequality bounds these minors by L^{2l}. Dividing by P_l and using consecutive products bounds p_a, p_a^{-1} and the entries of se_a^* by L^{2n+1}. In Witt order s has blocks

$$s = \begin{pmatrix} U & B & C \\ 0 & 1 & D \\ 0 & 0 & U^{-*} \end{pmatrix}, \qquad B = -UD^* J_0,$$

where J_0 is the Gram matrix on the remaining space. The last columns bound C, D, U^{-*}. Since $U^{-*} - 1$ is nilpotent of index at most j, the geometric sum bounds $\|U\|$ by $L^{(2n+4)n}$. Summing the block norms gives $\|s^{-1}\| = \|s\| \le L^{50n^2}$. Thus tg has size $60n^2$ and $tw_p^{-1}g$ has size $99n^2$. For $j = 0$ take $t = g^{-1}$.

Consider a tail correlation at $(\sigma g, \sigma g')$ that vanishes unless $\sigma g \in \mathcal{U}_{j+1}$ and is invariant under the stabilizer of f_j in G_j. It factors through $y(\sigma) = \sigma^{-1} f_j$. If $t \in G_j$, $t^{-1}q \in U_{<j}$, and qg has size a, then $ty(\sigma)$ has size $a+6$ on its support. In fact $\sigma(t^{-1}q)\sigma^{-1} \in R_{j+1}$, so $(\sigma t^{-1})(qg) \in \mathcal{U}_{j+1}$. Choose a box move $\zeta \in R_{j+1}$ for this point. Projecting the column $(qg)(\zeta\sigma t^{-1} qg)^{-1} f_j$ to the remaining space gives $ty(\sigma)$. The earlier isotropic terms from $\zeta^{-1} f_j$ project to zero. The column norm is at most $rL^a R_b$, as required.

For the character twist, the basis entry is

$$h(f_j, \sigma(\sigma')^{-1} f_j) = h(y(\sigma), y(\sigma')).$$

The two tail functions F_1, F_2 start at $g, w_p^{-1}g$. If both contribute, both initial points belong to $\mathcal{U}_j$. Rotate both orbit variables by the same t just constructed. This preserves the pairing and

the two norm-fiber measures. The support sizes are $60n^2+6$ and $99n^2+6$, so the pairings have modulus at most L^{200n^2}.

For a character agreeing with $\psi(\xi\,\cdot)$ on that box, the integral in the rotated variables is $\iint \psi(\xi h(v,y))F_1(v)\overline{F_2(y)}\,d\mu(v)d\mu(y)$. Cauchy–Schwarz and Section 11.1 bound its square by

$$\Lambda_L^{10^6n^3}\left(\frac{\pi^2}{|\xi|^2}\right)^{k-5}\|F_1\|_2^2\|F_2\|_2^2 \qquad (\xi\neq 0).$$

To check the box contribution, the hyperbolic change on one positive and one negative line is Euclidean unitary, so a support of size a lies in coordinate boxes of radius L^{a+1}.

The number of ordinary coordinates outside the free and solved blocks is at most $2n+2$. The ordinary volume and the block mass on that side have product at most $\Lambda_L^{(2n+4)(a+2)}$. Apply this to the two support sizes. The conjugated transform uses frequency $-\xi$, of the same modulus.

For the unequal heads put $j=i+1\le\varrho$, $d_0=w_q^{-1}w_p$, and let m_0 be the level unit, equal to 1 at the real place. The pairing kernel has left multiplier $\psi(m_0^{-1}(\mathfrak{s}(d_0u_2d_0^{-1})-\mathfrak{s}(u_1)))$ under (u_1,u_2) in the lower unipotent, with $\mathfrak{s}$ the simple-root sum. The floor-i element $u_i(v,\beta)$ sends o in the remaining space to $o-h(v,o)e_i$, so its simple-root sum is $-h(v,f_{i+1})$. Conjugation by d_0 multiplies this by p_i/q_i. Moving u_i to the left of the next pivot σ changes its vector to σv. Thus the section phase is

$$\psi(-\xi_0h(v,\sigma^{-1}f_{i+1})),\qquad \xi_0=m_0^{-1}(p_i/q_i-1)=-p_i\xi,\qquad \xi=m_0^{-1}(p_i^{-1}-q_i^{-1}).$$

Equivalently it is $\psi(-\xi h(v',\sigma^{-1}f_{i+1}))$ for $v'=\bar p_iu_i^{-1}e_i^*$, since the two terms in the i-th pair are orthogonal to the orbit vector and the remaining part of $u_i^{-1}e_i^*$ is $-v$.

Write $S\subseteq U_i$ for the support of this correlation restricted to $w_pu_ig\in\mathcal{U}_j$. On S both u_ig and w_pu_ig are near. If S is nonempty, choose $u_*\in S$. Apply the two-point movement to w_pu_ig,u_ig, so that some $t_i\in G_j$ puts t_iu_ig in size $99n^2$. Use t_* for u_*. Recenter by $m=t_*u_iu_*^{-1}t_*^{-1}$, which preserves Heisenberg Haar. Since t_i,t_* fix e_i^*,

$$(m^{-1}-1)e_i^*=\big((t_*u_*g)(t_iu_ig)^{-1}-1\big)e_i^*.$$

The two matrix bounds and $J^{-1}=J$ give size $198n^2+6$ for this column. By the head bound, its product by p_i has size $999n^2$. The choices t_i establish these bounds, and the change of integration variable uses the single t_*.

Set $u=p_i^{-1}$ and write the vector of m as uw. The bound for $u^{-1}(m^{-1}-1)e_i^*$ gives a box of size $999n^2$ for w. At fixed vector of m, the coefficient at e_i of the inverse column depends on the skew center by a translation and a sign. Subtracting the bounds for any two centers in the same fiber shows their distance is at most $2|u|L^{999n^2}$. The skew Haar measure is ordinary length, so that is also a bound for the fiber mass. The vector Jacobian is $|u|^{2d_i}$, $d_i=\dim_{\mathbb{C}}V_{i+1}$, where V_{i+1} is the space orthogonal to the first $i+1$ pairs. The combined factor is $|u|^{2d_i+1}=\mathfrak{m}_i(p_i)^{-1}$, with $\mathfrak{m}_i(p_i)=|p_i|^{2d_i+1}$.

Rotating the next orbit vector by t_* preserves its measure. Its size is at most $99n^2+6$, by applying the projected column argument with $q=t_*u_*$ and $t=t_*$. The recentered vector introduces a constant translation in the Fourier phase, absorbed as a phase of modulus one on the function. After the scaling to w, the frequency is $-\xi_0\bar u=(p_i/\bar p_i)\xi$, of modulus $|\xi|$. Apply Section 11.1 from the orbit quadric to the vector box, and use the central fiber bound.

If C_{i+1} is the tail in the section integral and C_{i+2} its next tail, this proves

$$\int_S |C_{i+1}(u_i g, u_i g')|^2 \, du_i \le \left(\frac{\pi^2}{|\xi|^2}\right)^{k-5} \frac{\Lambda_L^{10^6 n^3}}{\mathfrak{m}_i(p_i)} \int |C_{i+2}(\sigma g, \sigma g')|^2 \, d\theta_{i+1}(\sigma).$$

Here the box contribution on the two sides and the central factor $2L^{999n^2}$ together are at most $\Lambda_L^{10^6 n^3}$. This calculation uses the tail support and equivariance of the cut pairing kernels. The inverse-depth weights enter when one integrates these estimates over the real positions.

11.3 The terminal Gram form

In this subsection, we write the pairing on the remaining definite space as a Gram form. Its positivity and continuity are used at the terminal step of the descent in Section 12.2.

Let F be the marked continuous cut Bessel function with values in a Hermitian space I of dimension d_I. Use the original real Haar dg in its orbital tensor and assume the cut support $U_{<n}\mathcal{K}$. In the remaining space at the top, $h(z,z) = -1$ and $z^\perp$ is positive definite. Put $H = U(z^\perp)$. For $c \neq 0$, define $a(c)$ on the plane of one marked positive unit vector and z by

$$\begin{pmatrix} \bar{c} & \sqrt{|c|^2 - 1} \\ -\sqrt{|c|^2 - 1} & -c \end{pmatrix} \qquad (|c| \ge 1).$$

It is the identity on the other coordinates. For $0 < |c| < 1$ use $a(c/|c|)$. This fixes the earlier pairs. If $\sigma \in G_\varrho$ and $h(z, \sigma z) = c$, the positive part of σz has length $\sqrt{|c|^2 - 1}$. Rotate it by H to the corresponding positive coordinate of $a(c)z$. The resulting quotient fixes z. Thus $\sigma = h_1 a(c) h_2$ with $h_1, h_2 \in H$.

For a head of length ϱ and last entry c, put

$$\begin{aligned} T_p(\sigma) &= \int F(y) \otimes F((w_p \sigma)^{-1} y)^* \, dg(y), \\ A_p &= \frac{\sqrt{c_G}}{d_I} (|c|^2 - 1)_+^{(k-1)/2} \prod_{i<\varrho} |p_i|^{k+1+2(\varrho - i - 1)}, \\ \Xi(p) &= A_p T_p(a(c)), \qquad J_n(p,q) = \langle \Xi(p), \Xi(q) \rangle. \end{aligned}$$

Here c_G is the real open-cell normalization and the star on a vector denotes its Hermitian dual. For p, q with the same last entry and $|c| > 1$, translate y by h_1 and use $w_p h_1 = h_1 w_p$. The two tensors at σ then both arise from the tensors at $a(c)$ by the same unitary operation on the two value factors, using the H-equivariance of F. Consequently

$$J_n(p,q) = \langle A_p T_p(\sigma), A_q T_q(\sigma) \rangle.$$

At $|c| = 1$ the scale factors vanish, since $k > 1$.

This is a positive semidefinite terminal form by its Gram definition. It is continuous in the nonzero head entries. Indeed inverse partial products are bounded on a small neighborhood. Nonvanishing of both factors in $T_p(a(c))$ gives a move at y in $U_{<n}$ and a move at $w_p^{-1} y$ in R_ϱ, as $a(c) \in G_\varrho$. The two boxed matrix bounds above apply with the first move already lower unipotent, giving a common bounded set for y. In the real matrix group its closure is compact. Continuity of the integral follows by dominated convergence, including at $|c| = 1$.

11.4 Integrals of reciprocal determinant weights

In this subsection, we integrate the inverse powers of the relative vector lengths against the Haar factors of Section 11.2. The resulting inequalities control the weighted products in Section 11.5.

Keep the real box and floor measures from Sections 10.2 and 11.2. Fix an allowed relative position x. Put $A_a = \langle e_0, \dots, e_{a-1} \rangle$ and $s = k - 3n - 6$ (this differs from the finite exponent $s = k - 24n - 27$ of Section 10.6). For $a < \varrho$, the depth $D_a(g, x)$ is the reciprocal of the area of the set of $h(e_a, v)$ with $v \in A_a^{\perp}$ and with some $v' \equiv v \bmod A_a$ satisfying

$$\|g^{-1}v\|^2 + \|(gx)^{-1}v'\|^2 < 1.$$

On $A_a^{\perp}/A_a$, let $H_{g,a}$ be the positive Hermitian matrix of the quotient norm squared induced by $v \mapsto \|g^{-1}v\|$, in a fixed quotient basis. It is continuous in g: minimizing over the A_a-coordinates gives the Schur complement of their block in the positive Gram matrix on $A_a^{\perp}$. Put $H = H_{g,a} + H_{gx,a}$ and let ℓ_a be the nonzero covector induced by $h(e_a, \cdot)$. For each $c \in \mathbb{C}$,

$$\min_{\ell_a w = c} w^* H w = \frac{|c|^2}{\ell_a H^{-1} \ell_a^*}, \qquad D_a(g, x) = \frac{1}{\pi \ell_a H^{-1} \ell_a^*}.$$

Indeed the two lifts of w can be minimized separately, and the first formula follows by taking $w = cH^{-1}\ell_a^*/(\ell_a H^{-1}\ell_a^*)$. Thus the image is the open disc of area $\pi \ell_a H^{-1} \ell_a^*$; the depth and all its powers are measurable in both variables. The near sets $R_a \mathcal{K}$ are closed since R_a is closed and $\mathcal{K}$ compact. The depth is unchanged under simultaneous left movement by R_{a+1}. Such movement transports the two lifts, preserves their congruence and their lengths, and changes e_a by an element of A_a.

In the common free positive space of dimension $d = k - 5$, let E_I be its orthonormal inclusion. Suppose at most n linear equations are imposed on $u \in \mathbb{C}^d$. Impose also that $g^{-1}E_I u$ and $(gx)^{-1}E_I u$ belong to the positive eigenspace of the involution J. The combined rank is at most $3n$, since each negative projection has rank at most n. Choose an isometric inclusion O of an $(s+1)$-plane in the kernel. For u in the plane with $\|u\| \le 1/2$, preservation of the Hermitian form gives the same Euclidean norm for both inverse translates.

These vectors are therefore short for both points. Their products with a vector whose free coordinate is v contain a disc of radius $\|O^* v\|/2$. The inverse area is bounded by $\phi(v) = 36/\|O^* v\|^2$, with infinite value allowed on the null subspace. For each box exponent $\ell \ge 0$, uniformly in the equations and the two points, the bound

$$\int_{\{|v_i| \le L^{\ell}\}} \phi(v)^s \, dv \le \Lambda_L^{99n(\ell+1)} \tag{11}$$

follows by enclosing the box in the Euclidean ball of squared radius $R^2 = dL^{2\ell}$. In unitary coordinates on the chosen plane and its orthogonal complement of complex dimension $h = 3n$, write $t = \|O^* v\|^2$. The radial measure on the plane is $\pi^{s+1} t^s \, dt/s!$, and the remaining ball has squared radius $R^2 - t$. Thus, in the original complex area measure,

$$\int_{\|v\| \le R} \phi(v)^s \, dv = \frac{36^s \pi^d}{s! h!} \int_0^{R^2} (R^2 - t)^h \, dt = \frac{36^s \pi^d R^{2(h+1)}}{s!(h+1)!}.$$

This is independent of the chosen plane. The coefficient is at most $e^{37\pi}$, and $d \le r \le L/8$. Hence the last expression is at most $e^{37\pi} L^{(2\ell+1)(3n+1)} \le \Lambda_L^{99n(\ell+1)}$.

For the first inverse-depth integral, if some $\sigma g \in \mathcal{U}_{a+1}$ with $\sigma \in G_a$, then $g \in \mathcal{U}_a$. Rebase g to the box by R_a. The floor quotient integral is invariant under this replacement, by the right-action argument of Section 10.2 applied to the left-R_{a+1}-invariant depth and near indicator. At a boxed starting point the orbit vector $y = \sigma^{-1}e_a$ has size at most 7 on the near set by the projected-column bound. Common short vectors from the positive plane, with no extra equations, give stable lifts after applying σ. Hence $D_a(\sigma g, x) \le \phi(y_I)$. In the floor basis the coordinate bounds are at most size 11. Integrating off I contributes at most $\Lambda_L^{99n(7+6)}$ by the box and two-coordinate fiber bounds in Section 11.1. By (11),

$$\int_{\{\sigma g \in \mathcal{U}_{a+1}\}} D_a(\sigma g, x)^s \, d\theta_a(\sigma) \le \Lambda_L^{10^4 n}. \tag{12}$$

For the second integral, take a boxed point y, a lower unipotent $\nu \in U_{<a}$, and the head Weyl element $w_{p,a+1}$ of length $a+1$. For common short vectors $w = E_I u$ at y, yx, impose the a equations $h(e_b^*, \nu w) = 0$ for $b < a$. They make $w_{p,a+1}u_a\nu w$ orthogonal to A_a, since $u_a \in U_a$ fixes the lower pairs. Writing $c = \bar{p}_a$, the inverse image of e_a is $\nu^{-1}cu_a^{-1}e_a^*$, whose class in $A_a^\perp/A_a$ is represented by $z_c = cu_a^{-1}e_a^*$. The preimages of e_b for $b < a$ depend on ν and the preceding p_b, so the chosen common plane is fixed throughout the U_a integral. Transporting both lifts gives $D_a(w_{p,a+1}u_a\nu y, x) \le \phi((z_c)_I)$.

On $\mathcal{U}_{a+1}$, choose a box move γ. The matrix $(u_a\nu)^{-1}w_{p,a+1}^{-1}\gamma^{-1}$ is the ratio $y(\gamma w_{p,a+1}u_a\nu y)^{-1}$ of two boxed matrices and has operator norm at most R_b^2. Use the replaced-covector minor calculation of Section 11.2 with inverse head $\bar{p}$. After multiplying the column by $\bar{p}_a$, its size bound is $R_b^{2(a+1)}|\prod_{b<a} p_b|^{-1} \le L^{2a+3}$. Projection to the remaining space gives z_c, so we may use a box of size $20n$. At a fixed vector coordinate of u_a, integration in its center contributes at most $2L^{20n}/|c|$. Changing the vector coordinate to the corresponding part of z_c gives $|c|^{-2d_a}$, where $d_a = \dim_{\mathbb{C}} V_{a+1}$. In the floor basis the remaining coordinates have size at most $20n+4$, so their off-I volume and the integral in (11) each contribute at most $\Lambda_L^{99n(20n+5)}$. With $\mathfrak{m}_a(p_a) = |p_a|^{2d_a+1}$ this gives

$$\int_{\{w_{p,a+1}u_a\nu y \in \mathcal{U}_{a+1}\}} D_a(w_{p,a+1}u_a\nu y, x)^s \, du_a \le \mathfrak{m}_a(p_a)^{-1}\Lambda_L^{10^6 n^2}. \tag{13}$$

Set

$$W_j(g) = \prod_{a<j} D_a(g,x)^{-s}, \qquad I_j = \mathbf{1}_{\mathcal{U}_j} W_j^{-1}, \qquad C_L = \Lambda_L^{10^6 n^3}, \qquad H_p = \prod_{a<j} \mathfrak{m}_a(p_a)^{-1}.$$

Since the earlier depths are unchanged by G_a, (12) implies $\int I_{a+1}(\sigma g)\, d\theta_a(\sigma) \le \Lambda_L^{10^4 n} I_a(g)$. Similarly in (13) the earlier depths and indicator of $\mathcal{U}_a$ read at $w_{p,a}\nu y$. Indeed u_a and the last Weyl factor belong to G_a and commute with the earlier Weyl factors. Iteration therefore bounds $\int I_j(w_p u y)\, du$ over $U_{<j}$ by $H_p C_L$ for boxed y. The iteration uses the factorization $u = u_a\nu$ at each step, valid for Haar since $U_a \subseteq G_a$ preserves the earlier measures by conjugation. For $g \in \mathcal{U}_j$, write $y = \gamma u_0 g$ boxed, with $\gamma \in G_j$, $u_0 \in U_{<j}$. The same bound holds at g by the substitution $u \mapsto \gamma^{-1}u\gamma u_0$, since γ commutes with w_p and preserves I_j. In particular, for a nonnegative left-R_l-invariant B supported on $\mathcal{U}_l$, where $l \ge j$, and a σ-finite lower representative measure ρ, Tonelli gives

$$\int B(g) I_j(w_p g)\, d(du_{<j} * \rho)(g) \le H_p C_L \int B \, d\rho. \tag{14}$$

The bound also holds with $w_p^{-1} = w_{\bar{p}}$, for which the initial segment bounds and absolute values are unchanged.

11.5 The weighted products of the real correlations

In this subsection, we bound the product of the two real kernels with the phases and Haar factors of Section 11.2. These bounds propagate through successive isotropic quotients in Section 11.6.

Write $\theta_{[a,l]} = \theta_l * \cdots * \theta_a$, with the empty convolution the atom at the identity, and $M_j = du_{<j} * \theta_{[0,j-1]}$. The decomposition of the group Haar measure has the form $d\mu = d\mu_j * M_j$, where $d\mu_j$ is Haar measure on G_j with the floor normalization. We use the following change of representatives for this decomposition and for the lower unipotent decomposition of Section 11.7.

Suppose $d\mu = dm * \rho$, with dm Haar measure on a unimodular subgroup H. For compact $A \subseteq G$ and compact $C \subseteq H$ with $m(C) > 0$, one has $m(C)\rho(A) \le \mu(CA)$. Thus ρ is σ-finite. For integrable $h > 0$ put $\bar{h}(y) = \int_H h(uy)\, dm(u)$. This average is left-$H$-invariant by right invariance of dm, and lies in $(0, \infty)$ for ρ-almost every representative, since $\int \bar{h}\, d\rho = \int h\, d\mu$. For nonnegative left-$H$-invariant B,

$$\int B\, d\rho = \int \frac{Bh}{\bar{h}}\, d\mu$$

by Tonelli, with the integrand zero on the exceptional set. If s normalizes H and preserves dm, then $dm * (L_s)_*\rho = (L_s)_* d\mu = d\mu$, where L_s denotes left translation. The identity gives the same integral for $(L_s)_*\rho$. In particular, take $H = G_j$ and $s = w_p$, which commutes with G_j.

Use the kernels $K_1(g, y) = \langle F(g), F(y)\rangle$ and $K_2(g, y) = \langle F(g), F(w_q^{-1} w_p y)\rangle$. For $a \ge j$ write

$$C_a^K(g, y) = \int K(\sigma g, \sigma y)\, d\theta_{[a,n]}(\sigma), \qquad C_{n+1}^K = K.$$

The cut condition and the support of each higher factor in its floor group force near conditions at the successive floors. After rebasing each starting point into the box, the orbit vector is bounded by the projected-column estimate. The uniform norm-fiber box mass and the finite compact terminal mass give absolute integrability for bounded F. In particular C_a^K is the θ_a-integral of $C_{a+1}^K(\sigma g, \sigma y)$.

The diagonal Weyl ratio in K_2 normalizes the Bessel unipotent and commutes with G_j. On the upper unipotent from floor $a + 1$ on, its simultaneous Bessel character ratio is one. The quotient integration formula therefore makes C_{a+1}^K exactly invariant under simultaneous movement by G_{a+1}. It is also invariant under simultaneous U_a, since the Weyl ratio is the identity on that floor and the character ratio cancels after conjugation by each higher factor. Thus $C_{a+1}^K(\sigma g, \sigma y)$ descends to $\sigma^{-1} f_a$. A simultaneous lower move in $U_{<a}$ multiplies C_{a+1}^K by a phase: conjugation by G_{a+1} fixes the basis vectors occurring in its simple-root entries. Consequently the modulus is left R_{a+1}-invariant as a function of g at $y = gx$.

Define

$$\begin{aligned} A_a^K(g) &= W_a(g) |C_{a+1}^K(g, gx)|^2, \\ B_a^K(g) &= \sum_{l=a}^{\varrho} \int A_l^K(\sigma g)\, d\theta_{[a,l]}(\sigma), \\ E_a^K(x) &= \int B_a^K\, d\theta_{[0,a-1]}. \end{aligned}$$

Each A_a^K is left R_{a+1}-invariant and vanishes outside $\mathcal{U}_{a+1}$. The recursion $B_a^K = \int (A_a^K + B_{a+1}^K)(\sigma\,\cdot)\, d\theta_a$, with $B_n^K = 0$, and the floor quotient right action show that B_a^K is left R_a-invariant and supported on $\mathcal{U}_a$. For the untwisted kernel, Tonelli gives $E_a^{K_1} \le E_0^{K_1}$ by writing both as sums of full floor integrals.

Suppose a nonnegative integrand f, with $z = w_p^{-1} g$, is bounded pointwise by

$$f(g)^2 \le K_0 C_L \, B_j^{K_1}(g) I_j(g) B_j^{K_2}(z) I_j(z).$$

Cross the weights in Cauchy–Schwarz for the integral of f against M_j, pairing each B with the opposite I. The first factor $B_j^{K_1}(g) I_j(w_p^{-1} g)$ integrates to at most $H_p C_L E_j^{K_1}$ by (14) for $\bar{p}$. In the second, substitute $g = w_p z$ using the invariant quotient integral just computed. The function $B_j^{K_2}(z) I_j(w_p z)$ is left G_j-invariant. Its integral is at most $H_p C_L E_j^{K_2}$. It follows that

$$\left(\int f \, dM_j \right)^2 \le K_0 \, \mathcal{Q}_j(x), \qquad \mathcal{Q}_j(x) = H_p^2 C_L^3 E_0^{K_1}(x) E_j^{K_2}(x). \tag{15}$$

In particular the character estimate of Section 11.2 gives this inequality with $K_0 = (\pi^2/|\xi|^2)^{k-5}$. Indeed the squared norm of each pivot function is at most $B_j^K I_j$ by the $l = j$ summand and invariance of W_j under G_j.

For the unequal-head estimate let $j = i + 1$, and write $\varphi_l(g) = |C_j^{K_l}(g, gx)|$ for $l = 1, 2$. On the support of $\varphi_1(g)\varphi_2(z)$, both $g, z \in \mathcal{U}_j$, and $I_i(w_{p,i} z) = W_i(g)^{-1}$. The last Weyl factor is in G_i and preserves the earlier depth and near conditions. The two crossed factors for the square are

$$A_i^{K_1}(g) I_j(z),$$
$$\mathbf{1}_{\{w_p z \in \mathcal{U}_j\}} \, W_j(z) I_i(w_{p,i} z) \varphi_2(z)^2.$$

The first contributes at most $H_p C_L E_0^{K_1}$. The second is left G_j-invariant as a function of z, so one may again use the Weyl change for M_j.

Then write $M_j = du_i * du_{<i} * \theta_{[0,i]}$ using the lower conjugation law. The two weights $W_j(z)$ and $I_i(w_{p,i} z)$ are invariant under U_i. Integrate φ_2^2 and the near indicator using the section estimate of Section 11.2. The next squared pivot norm is bounded by $B_j^{K_2}/W_j$. This cancels W_j. Apply (14) at length i to the remaining factor $B_j^{K_2}(z) I_i(w_{p,i} z)$, with $\rho = \theta_{[0,i]}$. The length i product of moduli times $\mathfrak{m}_i(p_i)^{-1}$ gives H_p. The second integral is at most $K_0 H_p C_L^2 E_j^{K_2}$, with $K_0 = (\pi^2/|\xi|^2)^{k-5}$. Multiplication gives (15) for the unequal heads.

11.6 Decay from a deep unipotent subgroup

In this subsection, we use Cauchy–Schwarz in the next isotropic integration to carry the estimates of Section 11.5 to smaller supports. Section 11.9 identifies the relative positions where the stronger estimate applies.

At a starting point g, put $P_j^K(g) = \int |C_{j+1}^K(\sigma g, \sigma g x)| \, d\theta_j(\sigma)$. If the θ_j-mass of $\sigma g \in \mathcal{U}_{j+1}$ is at most $V \ge 1$, Cauchy–Schwarz gives

$$(P_j^K)^2 \le V B_j^K I_j.$$

Suppose instead that every nonzero terminal contribution on the G_j-orbit of g has some $D_a \ge \varepsilon^{-1}$ for $j \le a < \varrho$, where $0 < \varepsilon \le 1$. Split at the first such floor as the convolution proceeds from j upwards.

Once read, a depth is unchanged by the higher floor groups, which fix its basis data. At a shallow floor use the triangle inequality to expand the next tail. The part with no deep index

has zero terminal contribution by hypothesis. Dropping the earlier shallow restrictions from the nonnegative contributions gives

$$P_j^K(g) \le \sum_{a=j}^{\varrho-1} \int \mathbf{1}_{\{D_a(\sigma g,x)\ge \varepsilon^{-1}\}} |C_{a+1}^K(\sigma g, \sigma g x)|\, d\theta_{[j,a]}(\sigma).$$

On the support of the a-th term its squared integrand is at most $\varepsilon^s I_{a+1} A_a^K$, since $W_a/W_{a+1} = D_a^s$. By Cauchy–Schwarz, iteration of (12), and $a+1-j \le \varrho$, the integral squared is at most

$$\varepsilon^s \Lambda_L^{10^4 n\varrho} I_j(g) B_j^K(g).$$

Since $\varrho^2 \Lambda_L^{10^4 n\varrho} \le C_L$, squaring the sum yields

$$(P_j^K)^2 \le \varepsilon^s C_L B_j^K I_j.$$

For the two kernels in Section 11.5, call a point y active at j if $K_1(y, yx) \ne 0$ and $w_p^{-1}y, w_q^{-1}yx \in \mathcal{U}_j$. If $P_j^{K_2}(w_p^{-1}g) \ne 0$, some point in its upper convolution has F nonzero in both arguments. The two Weyl elements commute with G_j, so this supplies the two near conditions on the entire G_j-orbit of g. Thus if every active point has a deep floor of index at least j, with $D_a \ge \varepsilon^{-1}$, the deep bound applies to the first pivot. The ordinary bound applies to the second. The squared product of pivots then has the pointwise bound used in (15) with $K_0 = V^2\varepsilon^s$. With no depth hypothesis both ordinary bounds give $K_0 = V^2$. At $j = 0$, with empty heads, the majorant is the square of the same pivot at $g = 1$. Applying the deep bound to both copies at this floor gives the squared majorant bound $\varepsilon^{2s}\mathcal{Q}_0(x)$, using $C_L \ge 1$.

11.7 The weighted square integral on a real ball

In this subsection, we estimate a weighted integral over a bounded set of relative positions. Plancherel from Section 11.1 handles the norm-value coordinates, and Section 11.4 controls the inverse weights.

Use K equal to either of the two pairing kernels, starting at floor j. Let $\mathcal{B}'$ consist of positions x with every entry of $x-1$ of modulus at most $L' \ge \max(R_b, 8r)$, and put $\Lambda' = (L')^2$. The real level unit is one. Set

$$c_a = \pi^{2(k-5)}(\Lambda')^{4900n} \quad (a < \varrho), \qquad c_\varrho = 1.$$

Write $du_{<a}$ for the lower Heisenberg Haar on $U_{<a}$ and $d\eta_a$ for that of U_a, with $U_\varrho = \{1\}$ of unit mass. The convolution $du_{<a} * d\eta_a$ is $du_{<a+1}$.

Call y close to g modulo $U_{<a}$ if $g^{-1}uy \in \mathcal{B}'$ for some $u \in U_{<a}$. For $0 \le a < \varrho$, suppose $g \in \mathcal{U}_{a+1}$ and $n_0 y, n_1 y$ are both close in this sense, with $n_0, n_1 \in U_a$. Write a box move at g as tu, with $t \in G_a \cap R_{a+1}$, $u \in U_{<a}$ and $g_0 = tug$ boxed. For $x_i = g^{-1}u_i n_i y \in \mathcal{B}'$,

$$(g_0 x_0)(g_0 x_1)^{-1} \in U_{<a}\, t n_0 n_1^{-1} t^{-1}.$$

Here G_a normalizes $U_{<a}$. Both x_i and their inverses have norms at most $(L')^3$, so the ratio has norm at most $(L')^8$. Its e_a^*-column, projected away from the earlier pairs, is that of $tn_0n_1^{-1}t^{-1}$, since the lower unipotent is trivial on $A_a^\perp/A_a$. Subtracting the identity column gives a bound $(L')^9$. In Heisenberg coordinates $(v, i\beta)$ the column has vector part v and coefficient $i\beta -$

$h(v,v)/2$ at e_a. The vector part has norm at most $(L')^9$. Taking the imaginary part of the coefficient also gives $|\beta| \le (L')^9$, since $h(v,v)$ is real.

We first bound the integral of the forward weight. Take g boxed and $g^{-1}y \in \mathcal{B}'$. Let $u_a \in U_a$ have vector v, with $u_a y$ close modulo $U_{<a}$. The preceding bound applies with $n_0 = 1$ and $t = 1$. If $c = h(e_a, z)$ is a stable product at $g, u_a y$ with lifts z, z', then for a common short vector w at g, y in the free positive space,

$$h(g^{-1}z, g^{-1}w) - h((u_a y)^{-1}z', y^{-1}w) = \bar{c}\, h(v,w).$$

Indeed $z - z' \in A_a$ and $u_a w = w - h(v,w)e_a$.

The left side has modulus at most one by Cauchy–Schwarz on the two close pairs, since $\mathrm{diag}(J, -J)$ is a Euclidean isometry. Test over the half-unit ball of the common positive plane in Section 11.4. The area of the possible c is at most $4\pi/\|O^* v_I\|^2$, bounded by $\phi(v_I)$ of that section. Thus the forward factor $D_a(g, g^{-1}u_a y)^{-s}$ is at most $\phi(v_I)^s$. Integrating this over the size-nine box contributes at most $(\Lambda')^{990n}$ on the free coordinates, by (11). The remaining at most $2n+2$ complex coordinates and the real center contribute at most $(L')^{19(2n+2)+10}$. The product is at most c_a.

For $0 \le l \le \varrho$ and $g \in \mathcal{U}_l$ the resulting bound is

$$\int_{U_{<l}} \mathbf{1}_{\mathcal{B}'}(g^{-1}uy) W_l(g; g^{-1}uy)\, du \le \prod_{a<l} c_a, \tag{16}$$

where $W_l(g;x)$ displays the position variable. To prove this first take g boxed and induct on l. At step $a+1$, if the integrand has nonempty support, translate y by an element of $U_{<a+1}$ to make $g^{-1}y \in \mathcal{B}'$, using right Haar. Integrate uu_a with $u \in U_{<a}$ and $u_a \in U_a$. The depth D_a is independent of this u in the second point. Moving $u_a y$ to $uu_a y$ transports its lift by u, preserving the quotient class and its read length, with the first lift fixed.

Applying u^{-1} gives the inverse bijection. Apply the induction bound inside and the one-floor forward bound outside, on $u_a y$ close modulo $U_{<a}$. For general $g \in \mathcal{U}_l$, write its box move as γu_0, with $\gamma \in G_l$ and $u_0 \in U_{<l}$. Change the pair to $(\gamma u_0 g, \gamma y)$ and use $\gamma u_0 u \gamma^{-1}$ as the Haar variable. The position is preserved and the weights agree by their simultaneous flag invariance.

For the complementary ascent, put

$$m_a(g,y) = \int_{U_{<a}} \mathbf{1}_{\mathcal{B}'}(g^{-1}uy)\, du, \qquad Q_a(g,y) = \mathbf{1}_{\{m_a(g,y)>0\}} |C^K_{a+1}(g,y)|^2.$$

We show for $j \le a \le \varrho$ that

$$\int Q_a(g, ny)\, d\eta_a(n) \le c_a \int Q_{a+1}(\sigma g, \sigma y)\, d\theta_{a+1}(\sigma). \tag{17}$$

Convolution gives $m_{a+1}(g,y) = \int m_a(g,ny)\, d\eta_a(n)$. If this mass is zero, the left side vanishes. Otherwise the cutoff on the right is one, since $\sigma \in G_{a+1}$ normalizes $U_{<a+1}$ and preserves its Haar. At $a = \varrho$ the inequality is Cauchy–Schwarz on the terminal compact probability measure. For $a < \varrho$ and nonzero left side, choose a contributing n_0, so $g \in \mathcal{U}_{a+1}$. Use the centering above, writing $t = \gamma_1 r_1$ with $\gamma_1 \in G_{a+1}$, $r_1 \in U_a$, and substituting $n' = tnn_0^{-1}t^{-1}$.

This preserves Haar. The support lies in the size-nine box and the vector of n is $\gamma_1^{-1}v' + v_0$. For $y_\sigma = \sigma^{-1} f_{a+1}$, moving n past the next pivot and using the pairing covariance gives phase $\psi(-h(v, y_\sigma))$ in the integral defining $C^K_{a+1}(g, ny)$. This is the one-sided version of the calculation

in Section 11.2: the diagonal twist commutes with G_a, and only the second argument is moved. In particular

$$C^K_{a+1}(g, ny) = \int \psi(-h(v, y_\sigma)) C^K_{a+2}(\sigma g, \sigma y)\, d\theta_{a+1}(\sigma).$$

The next correlation is a measurable function of y_σ: it is invariant under the stabilizer and the orbit map has smooth local sections on the nonzero orbit. Rotate that vector by γ_1. The translation by v_0 multiplies the function by a phase of modulus one.

Apply the projected-column estimate to $q = \gamma_1 r_1 u$, with $qg = g_0$ boxed and $\gamma_1^{-1} q \in U_{<a+1}$. Both boxed matrices have operator norm at most R_b, so $\|\gamma_1 y_\sigma\| \le R_b^2 \le (L')^2$. The Fourier estimate of Section 11.1 at frequency 1 applies from this quadric to the vector box. Use ordinary area in both target coordinates in the block B' of that proof and keep the same source fiber. Plancherel on the free block still gives the squared-norm factor $\pi^{2(k-5)}$. On each side at most $2n+2$ complex coordinates lie outside the free block. On the input their disc areas are at most $\pi(L')^6$, and the solved pair has mass at most $4\pi^2(L')^6$. On the output the disc areas are at most $\pi(L')^{20}$. Since $L' \ge 8$, these volumes and the central length $2(L')^9$ contribute at most $(\Lambda')^{4(2n+2)+4+11(2n+2)+5} \le (\Lambda')^{4900n}$. This proves (17).

Set $\rho_a = d\eta_a * \cdots * d\eta_\varrho * \theta_{[0,n]}$, so $d\mu = du_{<a} * \rho_a$ and $\rho_n = \theta_{[0,n]}$. For $u \in U_{<a+1}$, right translation of its Haar variable gives $m_{a+1}(g, uy) = m_{a+1}(g, y)$. Write $D = 1$ for K_1 and $D = w_q^{-1} w_p$ for K_2. In $C^K_{a+2}(g, uy)$ the higher factors commute with D and fix the vectors defining the simple-root entries of DuD^{-1}. The second Bessel factor therefore contributes the constant phase $\psi(\mathfrak{s}(DuD^{-1}))$. Its modulus is one, so $Q_{a+1}(g, uy) = Q_{a+1}(g, y)$. Conjugation by $\sigma \in G_{a+1}$ preserves $du_{<a+1}$. Writing $L_\sigma y = \sigma y$, we have

$$du_{<a+1} * (L_\sigma)_* \rho_{a+1} = (L_\sigma)_* (du_{<a+1} * \rho_{a+1}) = (L_\sigma)_* d\mu = d\mu.$$

The invariant-function averaging identity of Section 11.5 shows that its integral on $Q_{a+1}(g, \cdot)$ is unchanged. Integrate (17) against $\theta_{[0,a]}$ in g and ρ_{a+1} in y. Undoing the shift by σ in the y-integral gives

$$\iint Q_a\, d\theta_{[0,a]} d\rho_a \le c_a \iint Q_{a+1}\, d\theta_{[0,a+1]} d\rho_{a+1}.$$

Iterate to n, where $Q_n \le |K|^2$.

Finally, to integrate $A^K_l(g)$ over $x \in \mathcal{B}'$ put $x = g^{-1} uy$ with $u \in U_{<l}$ and y in the ρ_l measure. The correlation modulus is invariant under this motion in its second argument. If the close mass is zero the inner integral vanishes. Otherwise the forward estimate (16) contributes $\prod_{a<l} c_a$ and leaves $Q_l(g, y)$. The ascent for its double integral contributes the complementary product. Summing the at most n weighted floor integrals in E^K_j proves

$$\int_{\mathcal{B}'} E^K_j(x)\, d\mu(x) \le n \Big(\prod_{a<\varrho} c_a \Big) \iint |K(g, y)|^2\, d\theta_{[0,n]}(g) d\theta_{[0,n]}(y). \tag{18}$$

11.8 Density and twisted square integrals

In this subsection, we compute the density in the coordinates integrated at the earlier steps of the isotropic flag. This combines the integral of Section 11.7 with the terminal pairing of Section 11.3.

Write $dQ = d\theta_{[0,n]}$ for the original real quotient measure and $d_a = \dim_{\mathbb{C}} V_{a+1} = k + 1 + 2(\varrho - a - 1)$. At a lower floor $a < \varrho$, the inverse Weyl factor $w_a(c)^{-1}$ has density $|c|^{2d_a}\, dc$ before

the Heisenberg factor. Indeed at fixed dual pivot c the vector in the norm chart is c times the Heisenberg vector. Its Jacobian is $|c|^{2d_a}$, and the skew coordinate in the norm chart is the unscaled Heisenberg center. The factor $w_a(c)^{-1}$ commutes with the higher floors, so it can be written at the left of the convolution with the other factors in their original order.

Let $D_j = w_q^{-1} w_p$, and let D_a denote the same diagonal through floor $a-1$ in this calculation. On the a-th pair, $D_{a+1} D_a^{-1}$ scales e_a by $z_a = p_a/q_a$ and e_a^* by $\bar{z}_a^{-1}$. Direct multiplication of the two-by-two matrices gives

$$D_{a+1} w_a(z_a c)^{-1} = D_a w_a(c)^{-1}.$$

Thus replacing c by $z_a c$ in the quotient integral contributes $|z_a|^{2(d_a+1)}$, from its weight and the complex area Jacobian. Iterating, for a nonnegative measurable test Φ we have

$$\int \Phi(D_j y)\, dQ(y) = J_{p,q} \int \Phi(y)\, dQ(y), \qquad J_{p,q} = \prod_{a<j} |p_a/q_a|^{2(d_a+1)}.$$

In particular the double squared norm of K_2 is $J_{p,q}$ times that of K_1, since only its second variable is translated.

The head density is $\mathcal{D}_j(p,q) = \prod_{a<j} |p_a q_a|^{d_a}$. With the inverse section moduli H_p of Section 11.4, the exact identity is

$$\mathcal{D}_j(p,q)^2 H_p^2 J_{p,q} = \prod_{a<j} |q_a|^{-2} \le L^2$$

for a q-head adapted to L. Indeed the exponent of $|p_a|$ is $2d_a - 2(2d_a+1) + 2(d_a+1) = 0$, and the exponent of $|q_a|$ is $2d_a - 2(d_a+1) = -2$. For any measurable $\mathcal{S} \subseteq \mathcal{B}'$ on which $f(x)^2 \le e^2 \mathcal{Q}_j(x)$, use Cauchy–Schwarz in x and then (18) for the two kernels. This gives

$$\mathcal{D}_j(p,q) \int_{\mathcal{S}} |f(x)|\, d\mu(x) \le e\, nLC_L^{3/2} \Big(\prod_{a<\varrho} c_a \Big) \mathcal{N},$$
$$\mathcal{N} = \iint |K_1(g,y)|^2\, dQ(g)\, dQ(y).$$

In particular, the loss multiplying $e\mathcal{N}$ is at most $\pi^{2(k-5)\varrho} \Lambda_L^{1500000n^3+1} (\Lambda')^{25000n^2}$.

For the cut function on the real quotient, write $d\mu = \lambda\, dg$, where dg is the orbital Haar. The corresponding quotient measures differ by the same λ, since the unipotent Haar is fixed. If $F = \chi W$ is the scalar cutoff with $|\chi| \le 1$ and W has squared quotient norm d_I in the orbital normalization, $\mathcal{N} \le \lambda^2 d_I$. Put $U = U_{<n}$. The compact group $H = U(z^\perp)$ in the remaining space normalizes U. The modulus of its action on Haar measure du is a continuous homomorphism $H \to \mathbb{R}_{>0}$, hence is one. Disintegrating left Haar measure dg over U shows that $y \mapsto hy$ preserves $d\bar{g}$ on $U\backslash G$ for $h \in H$. By covariance the matrix $\int W(y) W(y)^*\, d\bar{g}(y)$ commutes with the irreducible action of H. Its trace is d_I, so Schur's lemma makes it the identity. The double integral for W is the trace of its square. Multiplication by the scalar cutoffs can only decrease the integrand's modulus.

In the terminal comparison the top norm density in the last entry c is

$$\frac{2\pi^k}{(k-1)!} (|c|^2 - 1)_+^{k-1}\, dc.$$

This follows by polar integration in the k positive coordinates, at fixed norm-value coordinate $h/2$. The coefficient of the product of the top correlations in basis measure is therefore

$$C_F\,\mathcal{D}_\varrho(p,q)\frac{2\pi^k}{(k-1)!}(|c|^2-1)_+^{k-1},\qquad C_F=\frac{\operatorname{vol}U(k)}{\operatorname{vol}U(r)\,(\lambda d_I)^2}.$$

In the orbital normalization, the two Haar integrations contribute λ^2 to this coefficient.

Thus the coefficient is A_pA_q of Section 11.3, with the common last entry and the constant

$$c_G=\frac{\operatorname{vol}U(k)}{\operatorname{vol}U(r)}\frac{2\pi^k}{(k-1)!}.$$

To see the matching integrand, at $\sigma=a(c)$ the terminal compact average fixes the two correlations by unitary covariance and because the ratio of heads commutes with the compact group. Their product at y,x is

$$\langle F(y),F(yx)\rangle\,\langle F(\sigma^{-1}w_q^{-1}yx),F(\sigma^{-1}w_p^{-1}y)\rangle.$$

The substitution $(y,x)\mapsto(y,yx)$ preserves product Haar and makes this the integrand of the Gram pairing of the two orbital tensors. This comparison is at fixed entry. Absolute convergence over the varying entry and all real positions uses the shell estimates.

11.9 Real determinant weights on dyadic regions

In this subsection, we use minima of Hermitian quotient norms to decompose the relative positions into dyadic sets. This identifies where the stronger estimates of Section 11.6 enter the frequency sum of Section 11.10.

On $V_a=A_a^{\perp}/A_a$ let $|v|_{g,a}$ be the infimum of $\|g^{-1}\widetilde{v}\|$ over lifts. For $a<\varrho$ put

$$b_a(g,g')=\inf_{h(e_a,v)=1}(|v|_{g,a}^2+|v|_{g',a}^2).$$

The infimum is positive since $h(e_a,\cdot)$ is a nonzero continuous linear functional. The stable product $c\neq0$ occurs precisely when $|c|^2b_a(g,g')<1$, by scaling the common class and taking the two lift infima independently. Thus the stable set is a disc and $D_a(g,x)=b_a(g,gx)/\pi$.

Fix heads p,q adapted to the gauge L at j, and a point g active at x as in Section 11.6. The two nonzero cut values give boxed points ug,vgx with u,v in the Bessel unipotent. They preserve e_a modulo A_a, so $|e_a|_{g,a},|e_a|_{gx,a}\le R_b$. On V_ϱ the Bessel unipotent acts trivially. The two quotient norms there have ratio at most R_b^2, by the two boxed inverses and matrices. For $a<j$, e_a^* supplies a short dual in both norms. Indeed the column-minor bound of Section 11.2, for the box moves s_0 at g and s_1 at $w_p^{-1}g$, bounds each entry of $s_0e_a^*$ by $LR_b^{2(a+1)}$. Applying $(s_0g)^{-1}$ gives $\|g^{-1}e_a^*\|\le L^{2n+3}$. Use the head q and the box moves at gx and $w_q^{-1}gx$ for the other norm.

Suppose $T\ge L^{11n}$ and $b_a(g,gx)<T^2$ for every $j\le a<\varrho$. At each such index choose a dual of e_a with both lengths at most T, and at the earlier indices use e_a^*. The inequality $|h(v,w)|\le|v||w|$ holds in either quotient norm, by applying it on all lifts before minimizing. For the chosen dual ζ at a, put $\zeta_0=\zeta-h(\zeta,\zeta)e_a/2$. This is an isotropic dual and has both lengths at most $T+T^2R_b/2\le T^4$.

The orthogonal complement of $\langle e_a, \zeta_0 \rangle$ gives a common section of V_{a+1} into V_a. Its norm is at most $1 + T^5$: from a minimizing lift orthogonal to e_a, subtract its product with ζ_0 times e_a. The successive quotient infima equal the infimum over A_{a+1}. If the second norm on V_{a+1} is at most T^u times the first, write $w = h(\zeta_0, w)e_a + h(e_a, w)\zeta_0 + w_0$. The first two terms contribute at most $2T^5|w|_{g,a}$ in the second norm.

Removing $h(e_a, w)\zeta_0$ before taking the quotient shows that the class of w_0 has first quotient norm at most $(1 + T^5)|w|_{g,a}$. Thus the second norm of w is at most

$$(2T^5 + (1 + T^5)^2 T^u)|w|_{g,a} \le T^{u+13}|w|_{g,a}.$$

Descend from $u = 2$ at V_ϱ, and at $a = 0$ apply the result to $w = gv$ for arbitrary v. It gives $\|x^{-1}\| \le T^{2+13\varrho}$. Since $J = J^{-1}$, $\|x\| = \|x^{-1}\|$. Every entry of $x - 1$ is therefore at most T^{99n} in modulus. Outside this ball every active point has $D_a(g, x) \ge T^2/\pi$ at some $a \ge j$. For $j = \varrho$ there are no active points there.

The uniform next-floor mass is bounded by

$$V_b = (\pi R_b^{24})^k R_b^{9999n}.$$

Rebase the starting point to the box by R_j, as in the first inverse-depth integral. On $\mathcal{U}_{j+1}$ the orbit vector has norm at most R_b^2, by projecting the column of a boxed matrix times a boxed inverse. By the two-coordinate sphere bound of Section 11.1, its box mass is at most $4\pi^2 R_b^4 (\pi R_b^4)^{r-2j-2} \le V_b$. The same estimate at the top uses two positive coordinates.

Let $H = 2^i$, with $i \ge 0$ least such that the entries of $x - 1$ have modulus at most H. One position ball of gauge LH covers this shell. If $0 < \tau < 1$ and $\Lambda_L^{1200n^2} \le \tau^{400n^2} H$, take $T = L^{11n+1}/\tau$. Then $T^{-1} \le \tau/2$ and

$$2T^{99n} \le L^{2400n^2} \tau^{-400n^2} \le H.$$

Minimality of the shell index puts x outside the position ball of radius T^{99n}. Thus the deep-floor bound applies with $D_a \ge \tau^{-2}$.

11.10 Summation and real frequency decay

In this subsection, we sum the weighted real integrals on the dyadic sets of Section 11.9. The resulting Fourier bound controls the nonconstant terms in Section 12.2.

Let $\mathcal{M}_j(x)$ be the majorant obtained by integrating the two pivot moduli against M_j. Take any measurable $0 \le f \le \mathcal{M}_j$ and $t \ge 1$, with

$$f(x)^2 \le V_b^2 t^{-2s} \mathcal{Q}_j(x) \qquad (t > 1).$$

For $t = 1$ use the bound for $\mathcal{M}_j$. This also gives the estimate for the modulus of the form integrand. The energy estimate of Section 11.8 on the ball of gauge LH shows that the one-shell integral times $\mathcal{D}_j(p, q)$ contributes at most

$$\Lambda_L^{2 \cdot 10^6 n^3} H^{10^5 n^2} \pi^{2(k-5)\varrho} \mathcal{N}$$

times the square-root saving. Write $u = t^{-1}$.

For $H \le \Lambda_L^{2400n^2}$ use saving $V_b u^s$. On larger shells take $\tau = H^{-1/(800n^2)}$, which satisfies the condition of Section 11.9. The saving there is $V_b \min(u,\tau)^s$, since $f \le \mathcal{M}_j$ and $\mathcal{M}_j$ obeys the depth estimate. For $g_* = u$ in the first case and $g_* = \min(u,\tau)$ in the second,

$$g_*^{9\cdot 10^7 n^4} H^{1+10^5 n^2} \le \Lambda_L^{29\cdot 10^7 n^4}.$$

For the smaller shells this follows from their height bound. On every larger shell, $g_* \le H^{-1/(800n^2)}$ gives

$$H^{10^5 n^2} g_*^{9\cdot 10^7 n^4} \le H^{-12500n^2}.$$

For $2^{i_0} > \Lambda_L^{2400n^2}$ the sum of these factors over $i \ge i_0$ is at most $2^{-12500n^2 i_0}/(1-2^{-12500n^2})$, uniformly in both heads adapted to L. The rank bounds give $s \ge 10^{10} n^4$. The remaining power of g_* is at most $t^{10^8 n^4 - s}$, since $g_* \le t^{-1}$. Summing $\sum_{i\ge 0} 2^{-i} = 2$ gives

$$\mathcal{D}_j(p,q) \int f \, d\mu \le V_b \, t^{10^8 n^4 - s} \Lambda_L^{4\cdot 10^8 n^4} \pi^{2(k-5)\varrho} \mathcal{N}.$$

This in particular proves integrability of the majorant at $t = 1$ for each pair of heads. Since $d_I C_F \mathcal{N} \le \operatorname{vol} U(k)/\operatorname{vol} U(r)$, we obtain

$$\begin{aligned} d_I C_F \, \mathcal{D}_j(p,q) \int f(x)\, d\mu(x) &\le \Theta L^{10^9 n^4} R_b^{24k} t^{10^8 n^4 - s}, \\ \Theta &= \frac{\operatorname{vol} U(k)}{\operatorname{vol} U(r)} \pi^{(2\varrho+1)k}. \end{aligned} \tag{19}$$

Indeed $R_b^{9999n} L^{8\cdot 10^8 n^4} \le L^{10^9 n^4}$.

Lemma 11.1 (Real frequency decay)**.** *For a pivot tuple p, set $P_h(p) = \prod_{a<h} p_a$ and*

$$w_j(p) = \mathbf{1}_{\max_{1\le h\le j} |P_h(p)| \le b^{60n^2}} \prod_{h=1}^{j} \max(1, |P_h(p)|^{-1})^{n^6}, \qquad K_* = k - n^7.$$

For $j = 0$ use $w_0 = 1$. Let $0 \le j \le \varrho$ and $p, q \in (\mathbb{C}^\times)^j$. Write $Z_j(g) = h(f_j, g f_j)$, and let $\mathcal{I}_j(p,q;\omega)$ be the integral with weight ω in Section 12.2. With C_F as in Section 11.8 and Θ as in (19), one has

$$d_I |C_F \mathcal{I}_j(p,q;\omega)| \le \Theta b^{2000n^9} |\xi|^{-K_*} \min(1, |\xi|^{-1})^6 w_j(p) w_j(q) \tag{20}$$

in each of the following cases:

$$\begin{aligned} &\omega(g) = \psi(\xi Z_j(g)), \quad \xi \in \mathbb{C}^\times, \\ &\omega(g) = 1, \qquad\qquad j = i+1, \quad \xi = p_i^{-1} - q_i^{-1} \ne 0. \end{aligned}$$

Proof. Set $t = \max(1, R_b^{24}|\xi|)$ at frequency $\xi \ne 0$. If $t > 1$, then

$$(\pi/|\xi|)^{k-5} \le (\pi R_b^{24})^k t^{-s},$$

which bounds the Fourier factor in both cases. Take $f(x)$ to be the integral in g of the absolute value of the paired pivot integral at x, g. For the character weight the two-quadric Fourier estimate of Section 11.2 and (15) give $f(x)^2 \le (\pi/|\xi|)^{2(k-5)} \mathcal{Q}_j(x)$. For the constant weight

with unequal last entries, the section Fourier estimate at i gives the same bound by the crossed integration in Section 11.5. In both cases $f \le \mathcal{M}_j$, and the triangle inequality in the definition of $\mathcal{I}_j$ gives

$$|C_F \mathcal{I}_j(p,q;\omega)| \le C_F \mathcal{D}_j(p,q) \int f(x)\, d\mu(x).$$

Where both pivot moduli contribute, their nonzero terminal contributions imply $g, w_p^{-1} g \in \mathcal{U}_j$, and likewise $gx, w_q^{-1} gx$.

The leading minors then bound the partial products by R_b^{2j}, so a head violating the threshold gives zero. For the heads satisfying the threshold, take L to be the largest of R_b and the inverse initial segment moduli for p, q. It meets the box hypothesis since $8r \le 8nr \le b \le R_b$ by Section 12.10. Thus $L^{n^6} \le R_b^{n^6} w_j(p) w_j(q)$. The parameter inequalities $b \ge 2,\ n \ge 10^8,\ k \ge n^7$ give $10^9 n^4 \le n^6$ and $10^8 n^4 + K_* + 6 \le s$. Use $R_b^{24}/t \le |\xi|^{-1}$ and $t^{-1} \le \min(1, |\xi|^{-1})$ in (19) to bound the absolute normalized form by

$$\Theta b^{2000 n^9} |\xi|^{-K_*} \min(1, |\xi|^{-1})^6 w_j(p) w_j(q).$$

Here $R_b^{n^6 + 24 n^7} \le b^{2000 n^9}$, and the normalization in (19) includes $d_I C_F$. □

12 Formal degrees and real Bessel functions

In this section, we pass from the weighted real bounds of Section 11 to the discrete-series normalization. The stationary point, coefficient estimates, and truncation bound apply to the Poincaré map of Section 8.

12.1 The local mass in the formal-degree normalization

In this subsection, we compare the initial orbital pairing for the truncated and full discrete-series vectors. The formal degree in Section 12.3 gives the full norm, and the truncation estimate of Section 12.4 controls the comparison.

For d the formal degree, let dg be the orbital Haar, $d\bar{g}$ its quotient Haar and $d\mu = \lambda\, dg$. For Bessel functions of the marked type define

$$P(A,B)(x) = \int_{U_{<n} \backslash G} \langle A(y), B(yx) \rangle\, d\bar{g}(y).$$

The original real vector satisfies

$$d > 0, \quad f \ge 4000n, \quad \|W\|_2^2 = d_I, \quad \|F\|_2 \le \sqrt{d_I}, \quad \|F - W\|_2 \le b^{-r^3 f} \sqrt{d_I},$$
$$\|P(W,W)\|_{L^2(dg)}^2 \le d_I^2/d, \qquad \lambda = \frac{\operatorname{vol} U(r)}{\operatorname{vol} U(k)}, \qquad d\Theta \le b^{4r^2},$$
$$d \operatorname{vol}_{dg}\{x : \|x\| \le Y\} \le Y^{9r^2} b^{99r^2} \quad (Y \ge 1).$$

The Haar comparison is proved in Section 12.6, and the parameter calculation in Section 12.10 gives $f \ge 4000n$. For empty heads, the two pivot correlations with weight 1 give $\lambda^2 |P(F,F)(x)|^2$, with head density one. By Cauchy–Schwarz and the isometry of right translation,

$$|P(F,F)(x) - P(W,W)(x)| \le \|F - W\|_2 (\|F\|_2 + \|W\|_2) \le 2 d_I b^{-r^3 f}.$$

Set $t_* = b^{6r}$, $T = t_* R_b^{11n}$, and take the position ball $\mathcal{B}_*$ of entry radius T^{99n}. On it $\|x\| \le (1+r^2)T^{99n} \le R_b^2 T^{99n} \le b^{700nr}$, since $R_b \le b^{7n}$ and $14n + 7623n^3 + 594nr \le 700nr$. The squared $L^2(dg)$ error on the ball is therefore at most

$$\frac{d_I^2}{d}\, 4b^{-2r^3 f + 6300nr^3 + 99r^2} \le b^{-24}\frac{d_I^2}{d}.$$

As $C_F \lambda^3 = d_I^{-2}$, the L^2 triangle inequality bounds the normalized mass on the ball by $(1 + b^{-12})^2/d$.

On the complement, Section 11.9 gives, at each active point, some $0 \le a < \varrho$ with $D_a \ge T^2/\pi \ge t_*^2$. The deep pivot estimate with $\varepsilon = t_*^{-2}$ applies to $f = \mathbf{1}_{\mathcal{B}_*^c}\mathcal{M}_0$ with $t = t_*$ and $L = R_b$. The empty heads have density one. The absolute contribution is at most $C_F \int f\, d\mu$, which is bounded by (19) since $d_I \ge 1$. It is therefore at most

$$\Theta R_b^{10^9 n^4 + 24k} t_*^{10^8 n^4 - s} \le b^{-12}/d.$$

Indeed $k = s + 3n + 6$, $r = s + 5n + 5$, $s \ge 10^{10} n^4$ and $n \ge 10^8$ give $r \le (11/10)s$, $k \le r$ and

$$4r^2 + 7n(10^9 n^4 + 24k) + 6r \cdot 10^8 n^4 + 12 \le 6rs.$$

Here $4r^2 \le (22/5)rs$, both $7 \cdot 10^9 n^5$ and $168nk$ are at most $rs/10$, and the penultimate term is at most $(3/50)rs$. Thus the normalized form at the bottom satisfies

$$|J_0| \le \frac{(1+b^{-12})^2 + b^{-12}}{d} \le \frac{1 + b^{-10}}{d}.$$

12.2 Descent through a common entry

In this subsection, we relate consecutive pairings when the two vectors at the current isotropic entry coincide. Section 11.10 bounds the other Fourier terms, while Section 11.3 supplies the terminal positive form.

For $a \le \varrho$, let p, q have length a, and use the kernels K_1, K_2 of Section 11.5 with twist $w_q^{-1} w_p$. Write $\mathcal{I}_a(p, q; \omega)$ for the integral against $d\mu(x) dM_a(g)$, with factor $\mathcal{D}_a(p, q)$, of

$$\iint \omega(\sigma\tau^{-1})\, C_{a+1}^{K_1}(\sigma g, \sigma g x)\, \overline{C_{a+1}^{K_2}(\tau w_p^{-1} g, \tau w_p^{-1} g x)}\, d\theta_a(\tau) d\theta_a(\sigma).$$

This is absolutely convergent for measurable $|\omega| \le 1$, by integrating the ordinary majorant at $t = 1$ in Section 11.10. For weight 1 the two pivot integrals separate into $C_a^{K_1}$ and $\overline{C_a^{K_2}}$ at their respective starting points.

Set $Z_a(s) = h(f_a, s f_a)$. Let $v(c) = w_a(c)$ for $a < \varrho$ and $v(c) = a(c)$ of Section 11.3 at $a = \varrho$. The law θ_a is the pushforward by $(c, \nu) \mapsto v(c)^{-1}\nu$ of $\rho_a(c)\, dc\, d\zeta_a(\nu)$, where

$$\begin{aligned} \rho_a(c) &= |c|^{2d_a}, \quad \zeta_a = \eta_a, && a < \varrho, \\ \rho_\varrho(c) &= \frac{2\pi^k}{(k-1)!}(|c|^2 - 1)_+^{k-1}, \quad \zeta_\varrho = \theta_n, && a = \varrho. \end{aligned}$$

These are the lower chart and top sphere densities in Section 11.8. Both ζ_a are carried by $R_{a+1} \cap G_a$, the stabilizer of f_a in G_a. The entry $Z_a(v(c)) = c$ wherever the density is nonzero, except possibly at $c = 0$, a null set.

Let $\omega = \phi \circ Z_a$ with $|\phi| \le 1$. For fixed x, g, σ, the weighted product in the double integral is unchanged under left multiplication of τ by the stabilizer of f_a. The entry is unchanged because that stabilizer fixes f_a, and the complex tail correlation is unchanged by the stabilizer invariance of Section 11.5. Thus the integral in τ depends only on $\tau^{-1} f_a$. Right multiplication of τ by σ sends the orbit point to $\sigma^{-1}\tau^{-1} f_a$, which preserves the norm-fiber measure.

This applies to the product and its absolute value. We may consequently substitute $\tau = v(c)^{-1}\nu\sigma$ with the above chart law. For $g' = \nu\sigma g$, the entry is

$$Z_a(\sigma\tau^{-1}) = h(f_a, \nu^{-1} v(c) f_a) = h(\nu f_a, v(c) f_a) = h(f_a, v(c) f_a) = c.$$

At the last step the z-coordinate of $a(c)z$ is $-c$ and $h(z, z) = -1$, giving the same sign. The first tail at σg equals that at g', by the stabilizer invariance. The other starting point becomes

$$\tau w_p^{-1} g = v(c)^{-1} w_p^{-1} g',$$

since w_p commutes with G_a. The same relation holds with $gx, g'x$ in the second variable. The variable g' now has measure $\zeta_a * \theta_a * M_a$. The absolute integrability just proved transfers through this substitution, so Fubini applies also to the new integral in c.

For $a < \varrho$, append the common entry $c \neq 0$ to both heads, writing p_c, q_c. Their new Weyl elements are $w_p v(c), w_q v(c)$, so the twist is unchanged. The next lower measure is $M_{a+1} = \eta_a * \theta_a * M_a$, by rearranging the lower Heisenberg and floor products using the same conjugation invariance. Also $\mathcal{D}_{a+1}(p_c, q_c) = \mathcal{D}_a(p, q)|c|^{2d_a}$. Thus the new inner integral is the next form with weight 1, by separation of its pivots. We obtain

$$\int_{\mathbb{C}} \phi(c) \mathcal{I}_{a+1}(p_c, q_c; 1)\, dc = \mathcal{I}_a(p, q; \phi \circ Z_a).$$

The values assigned to the appended head at $c = 0$ do not affect this statement.

At $a = \varrho$, $\theta_n * \theta_\varrho * M_\varrho = d\mu$ by the full floor decomposition, since the last two Heisenberg measures are trivial. The compact tail average is the probability average on H. The coefficient and integrand comparison in Section 11.8 now identifies the new integral, after multiplication by C_F, with $J_n(p_c, q_c)$. Hence

$$\int_{\mathbb{C}} \phi(c) J_n(p_c, q_c)\, dc = C_F \mathcal{I}_\varrho(p, q; \phi \circ Z_\varrho).$$

In particular its absolute convergence follows from the ordinary majorant, including at zero frequency.

Define $J_a(p, q) = C_F \mathcal{I}_a(p, q; 1)$ for $a \le \varrho$, with J_n the terminal Gram form. The preceding identities show that

$$\int_{\mathbb{C}} \psi(\xi c) J_{a+1}(p_c, q_c)\, dc = C_F \mathcal{I}_a(p, q; \psi(\xi Z_a)), \qquad a < n.$$

At $\xi = 0$ this integral equals $J_a(p, q)$. At $\xi \neq 0$, the character case of (20) bounds d_I times its modulus, including at $a = \varrho$. At unequal last entries with $a < \varrho$, the constant-weight case bounds $d_I |J_{a+1}|$ at $\xi = p_a^{-1} - q_a^{-1}$, with the weights of the extended heads. Since $d_I \ge 1$, the same right sides bound the respective moduli in Section 13.4.1. The terminal form is continuous and positive semidefinite on fixed last-entry lists, by Section 11.3. The remaining bottom norm estimate is supplied by Section 12.1 when its formal-degree and Haar bounds hold.

12.3 The reproducing coefficient and the norm-ball bound

In this subsection, we use Schur orthogonality to express the full coefficient norm in terms of the formal degree. The same normalization gives a bound for the volumes used in the truncation argument of Section 12.4.

Suppose the marked full vector W is reproduced on the right by $e(g) = d\,\overline{c(g)}$, where $c(g) = \langle v, \pi(g)v\rangle$ is the coefficient of a unit vector in the real discrete series, $c \in L^1(dg)$, and $d\int |c|^2\,dg = 1$. Let $U = U_{<n}$, let I be the compact value space, and write χ_U for the unitary character of U in the Bessel transformation law. Use the space of measurable functions $B : G \to I$ satisfying

$$B(uy) = \chi_U(u)B(y), \qquad \|B\|_2^2 = \int_{U\backslash G} \|B(y)\|^2\, d\bar{g}(y) < \infty,$$

modulo equality almost everywhere. Inner products of two such functions descend to $U\backslash G$, since $|\chi_U| = 1$. Right invariance of $d\bar{g}$ makes $R(g)B(y) = B(yg)$ unitary. Set $\mathcal{P} = \int_G e(g)R(g)\,dg$. Then $\mathcal{P}^* = \mathcal{P}$ by inversion and $\mathcal{P}W = W$ by reproduction. Schur orthogonality gives

$$\int e(a)e(x^{-1}a^{-1}t)\,da = d^2 \int \overline{c(a)}\,c(t^{-1}ax)\,da = c(x)e(t).$$

Indeed the second coefficient in that pairing has vectors $\pi(t)v, \pi(x)v$. The two L^1 kernels justify the operator multiplication, so $\mathcal{P}R(x)\mathcal{P} = c(x)\mathcal{P}$. Consequently $P(W, W)(x) = d_I c(x)$ in the notation of Section 12.1. Its squared norm is d_I^2/d.

For the norm-ball estimate it suffices to have $|c(g)| \geq 1/2$ on $\|g-1\| \leq \delta = (200\beta^5)^{-1}$, where $1 \leq \beta \leq b^4$ bounds r, and $b \geq 2000$. Here is a sufficient representation-theoretic condition for this near-identity bound. In trace-form normalization let the full and compact Casimirs act on v by c_G, c_K, with $|c_G|, c_K \leq r\beta^2$. For a real orthonormal basis X_i of the compact and noncompact summands in the Hilbert–Schmidt metric, the full Casimir is $\sum_{\mathfrak{p}} X_i^2 - \sum_{\mathfrak{k}} X_i^2$ and the compact one is $-\sum_{\mathfrak{k}} X_i^2$.

Skew-adjointness gives

$$\sum_i \|d\pi(X_i)v\|^2 = 2c_K - c_G \leq 3r\beta^2.$$

For the lowest compact type use the Casimir eigenvalue formulas $\|\lambda\|^2 - \|\rho\|^2$ and $\|\mu + \rho_K\|^2 - \|\rho_K\|^2$ when each coordinate of $\lambda, \rho, \mu + \rho_K, \rho_K$ is bounded by β. Here $\mu = \lambda + \rho - 2\rho_K$ by the lowest-type formula [5, Theorem 8.5]. For the unitary group one applies that theorem on the derived group, retaining the central character on the compact center in both Casimirs. By Cauchy–Schwarz and $\|X\|_{\mathrm{HS}} \leq \sqrt{r}\|X\|$, $\|d\pi(X)v\| \leq \sqrt{3}\,r\beta\|X\|$. For $\|g-1\| \leq \delta$, the logarithm $X = \log g$ belongs to the Lie algebra, since the form adjoint sends g to g^{-1} and the logarithm to $-X$. The power series gives $\|X\| \leq 2\|g-1\|$. Integrating the coefficient derivative along $\exp(tX)$ gives $|c(g) - 1| \leq 2\sqrt{3}\,r\beta\delta \leq 1/2$.

The unitary norm ball of radius $Y \geq 1$ can be covered by translates of the identity ball of radius δ using at most $(600\beta^5 Y^2)^{2r^2}$ centers. In fact choose a maximal δ/Y-separated set in the group ball. Disjoint ambient matrix balls of radius $\delta/(2Y)$ fit in the ambient ball of radius $Y + \delta/(2Y)$. Volume comparison in real dimension $2r^2$ gives the stated count. Each group point g is within δ/Y of some center h, so $\|gh^{-1} - 1\| \leq \delta$, since $\|h^{-1}\| = \|h\| \leq Y$. The identity ball has dg-mass at most $4/d$ by integrating $|c|^2$. Right Haar invariance now gives

$$d\,\mathrm{vol}_{dg}\{g : \|g\| \leq Y\} \leq 4(600\beta^5 Y^2)^{2r^2} \leq b^{99r^2} Y^{9r^2}.$$

This uses the original dg throughout and gives the ball bound of Section 12.1.

12.4 Error from truncating the orbital integrals

In this subsection, we estimate the norm of the part removed from the full real vector. The decay of Section 12.7 and the volume estimate of Section 12.3 bound its contribution to the orbital pairing.

The same packing argument gives the stronger bound $d\,\mathrm{vol}_{dg}\{\|g\| \le Y\} \le b^{42r^2+1}Y^{4r^2}$. For $M_* = 9r^2$ define $\phi(g) = (2/\|g\|)^{M_*}$. Since $\|g\| \ge 1$, integration of the sublevel volumes against $M_* Y^{-M_*-1}dY$ yields

$$d\,16^{M_*}\int_G \phi\,dg \le b^{99r^2}.$$

Indeed $M_*/(M_* - 4r^2) \le 2$, and the extra powers $2^{M_*}16^{M_*}$ contribute at most b^{2M_*}.

Write $U = U_{<n}$ and $S = UH = R_n$. Use the Bessel graph of the projection $S \to H$, and write its elements as $s = hu$ in the first coordinate. Its Haar ds is the product of the original du and probability Haar on H. Conjugation by H preserves du, so this product is right invariant. The graph is closed, since S is a closed flag subgroup and the projection to the compact factor is continuous. We use the operator norm in the original unitary basis with prescribed signs, which is unchanged by transport to the chosen coordinates.

For a point x_0, let B_{x_0} be the graph set $\|x_0^{-1}sx_0 - 1\| \le 1$, with mass V_{x_0}. This is compact because the first coordinate ranges over a compact subset of G and the other coordinate is in the compact group H. It contains a neighborhood of the identity, so $0 < V_{x_0} < \infty$. Put $\Phi(y) = \int_S \phi(x_0^{-1}sy)\,ds$. To bound this function, take a maximal $1/Y$-separated set of matrices $a_i = x_0^{-1}s_i y$ of norm at most Y.

Ambient packing gives at most $3^{2r^2}Y^{4r^2}$ centers for $Y \ge 1$. If $a = x_0^{-1}sy$ is within $1/Y$ of a_i, then

$$\|x_0^{-1}ss_i^{-1}x_0 - 1\| = \|(a - a_i)a_i^{-1}\| \le 1.$$

Thus each piece is contained in $B_{x_0}s_i$, of mass V_{x_0} by right invariance. Integrating the sublevel estimate against $2^{M_*}M_*Y^{-M_*-1}\,dY$ gives $\Phi(y) \le 16^{M_*}V_{x_0}$. For any integrable function γ on G, the original quotient disintegration gives

$$\int_{U\backslash G}\int_S \gamma(sy)\,ds\,d\bar{g}(y) = \int_G \gamma(g)\,dg.$$

Indeed write $s = hu$, unfold $du\,d\bar{g}$ for each h, and apply left Haar invariance. The same proof applies to nonnegative functions. With left translation by x_0^{-1} it gives $\int \Phi\,d\bar{g} = \int \phi\,dg$.

For $s = hu$, let $\sigma(u)$ be the simple-root sum in the real chart of Section 8.3, and put

$$\chi_U(u) = \exp\bigl(\sigma(u) - \overline{\sigma(u)}\bigr), \qquad \sigma_S(hu) = \chi_U(u)\pi_H(h).$$

The first superdiagonal entries of the unitriangular flag action add under multiplication. The compact group H fixes the ordered flag vectors, so conjugation by H preserves χ_U. Thus σ_S is a unitary representation of S. For the unit highest vector v_0 of the compact type, set

$$A(y) = \int_S c(x_0^{-1}sy)\sigma_S(s)^{-1}v_0\,ds.$$

The large-gap coefficient bounds used here are $|c(g)| \le \phi(g)$ and $|c(g)| \le (2/\|g\|)^{M_*+3r^3f}$, as proved in Section 12.7 for the prescribed lowest compact type and parameter gap. The group-volume bound above used only square integrability of the discrete series and the estimate near the identity. Since $|c| \le \phi$ and $\int_G \phi\, dg < \infty$, we have $c \in L^1(G, dg)$ before applying the reproducing convolution. The same majorant gives absolute integrability in the graph variable and $|A| \le \Phi$. In particular A is bounded, and its norm belongs to $L^1 \cap L^2$ on $U\backslash G$. It is continuous on G: for $y = y_0 a$ with a in a compact neighborhood of 1, the norm bound is dominated by $\|a^{-1}\|^{M_*}\phi(x_0^{-1}sy_0)$. The substitution $s \mapsto st^{-1}$ gives $A(ty) = \sigma_S(t)A(y)$. Also

$$\int_G e(a)A(ya)\, da = A(y).$$

Indeed Schur orthogonality gives $\int e(a)c(ga)\, da = c(g)$. The graph integral commutes with this convolution by the bound $\|e\|_1\, 16^{M_*}V_{x_0}$ on the absolute double integral.

The period at x_0 is exactly $d\|A\|_2^2$. To check the constant, unfold the integrable function $d\,\overline{c(x_0^{-1}g)}\langle v_0, A(g)\rangle$. At $g = sy$, covariance and unitarity make it

$$d\,\langle c(x_0^{-1}sy)\sigma_S(s)^{-1}v_0, A(y)\rangle.$$

Its graph integral is $d\|A(y)\|^2$. On the full group, substitute $g = x_0 a$ and use the reproducing identity. The integral is $\langle v_0, A(x_0)\rangle$, which on expanding A is the stationary period on this same graph and Haar.

The quantitative stationary bound is supplied by Section 12.8 under its marked weight and gap conditions. It gives a point x_0 for which

$$\ell(x_0) \le b^{5n}, \qquad V_{x_0} \le b^{60(r+k)^2}|\langle v_0, A(x_0)\rangle|, \qquad \ell(y) = \inf_{s\in S}(\|sy\| + 1).$$

Then $A \ne 0$ by the positive mass of the graph ball. Take the positive scalar $c_0 = \sqrt{d_I}/\|A\|_2$ and put $W = c_0 A$, so $\|W\|_2^2 = d_I$. The length is left S-invariant and satisfies $\ell(x_0 g) \le \ell(x_0)\|g\|$, since $\|g\| \ge 1$. For $g = x_0^{-1}sy$ this gives $\ell(y)/b^{6n} \le \|g\|/2$.

If $H(y) = \max(1, \ell(y)/b^{6n}) > 1$, the second coefficient estimate absorbs $H(y)^{3r^3f}$. When $H(y) = 1$, use the first estimate. In either case $H(y)^{3r^3f}|W(y)| \le |c_0|\Phi(y)$. With $w = |c_0|\Phi$, covering, unfolding and the period bound give $(\sup w)\int w\, d\bar{g} \le b^{400r^2} d_I$, since $60(r+k)^2 + 99r^2 \le 400r^2$. ℓ is continuous by its right-translation norm inequality and the same inequality for the inverse. The scalar cutoff $\chi = \max(0, \min(1, 2 - 2\ell/R_b))$ is one on $\ell(y) \le R_b/2$. If $\chi(y) > 0$ there is $s = hu$ with $\|sy\| < R_b$, hence $\|uy\| < R_b$ because H is Euclidean unitary.

Put $F = \chi W$. Since U is closed and $K_b = \{g \in G : \|g\| \le R_b\}$ is compact, UK_b is closed. Thus $\operatorname{supp} F \subseteq UK_b$. On the support of $F - W$ $H(y) \ge b$ for $R_b = b^{6n+2}$ and $b \ge 2$. Thus

$$\|F - W\|_2^2 \le b^{-6r^3f} \int w^2\, d\bar{g} \le b^{400r^2 - 6r^3f} d_I \le b^{-2r^3f} d_I$$

in the marked range $r \ge 100$, $f \ge 1$. The stationary-period and coefficient decay bounds in this argument concern the same x_0, A as the reproducing vector.

12.5 The formal-degree factor

In this subsection, we compare the formal degree in the orbital Haar measure with the compact normalization of the Weyl dimension formula. The resulting factor enters the initial norm calculation of Section 12.1.

In the Cayley normalization, the density at the identity of $U(J)$ in the skew-Hermitian coordinates $S = JX$ of its Lie algebra is $1/\operatorname{vol} U(r)$. The corresponding density on the compact dual is the same. In a unitary basis where J is diagonal, the Cartan decomposition keeps the two diagonal skew-Hermitian blocks and sends the off-diagonal noncompact part $X_{\mathfrak{p}}$ to $iX_{\mathfrak{p}}$. In S-coordinates this change has real determinant of absolute value one, since it multiplies the independent complex off-diagonal entries by a unit complex scalar.

For signature (p, q), assign probability Haar to $K = U(p) \times U(q)$. Probability Haar on $U(r)$ then induces mass one on the complex Grassmannian $U(r)/K$, the compact dual of $U(p, q)/K$. With the tangent measures identified above, these are the two measure normalizations in the Harish–Chandra degree formula [5, (3.15) and Theorem 8.1]. The compact central factor can be given probability Haar on both sides and the product measures pushed through the common finite central quotient. A unitary central character has modulus one and contributes no factor to the coefficient norm. This extends the formula from the derived group to the unitary group in the given normalization.

For a regular parameter $\lambda_1, \dots, \lambda_r$, the type A_{r-1} root products give

$$d = \frac{\prod_{i<j} |\lambda_i - \lambda_j|}{\prod_{i=0}^{r-1} i!}.$$

In fact the product of the positive-root differences at ρ is $\prod_{j=2}^{r}(j-1)!$, by grouping the factors $j - i$ at fixed j. For the compact volume in these same coordinates, project $U(l)$ to its last column on the unit sphere in $\mathbb{C}^l$. At the identity the differential is $S \mapsto Se_l$. Its kernel has the skew coordinates of $U(l-1)$, and its remaining coordinates are the $l - 1$ complex entries and the imaginary diagonal entry of that column. They give the Euclidean tangent density of the sphere. By invariance this identifies the quotient measure everywhere with surface measure. Polar integration of the Gaussian therefore gives

$$\frac{\operatorname{vol} U(l)}{\operatorname{vol} U(l-1)} = \operatorname{area}(S^{2l-1}) = \frac{2\pi^l}{(l-1)!}, \qquad \operatorname{vol} U(0) = 1.$$

Set $t = r - k = 2\varrho + 1$ and $\Theta = \pi^{tk} \operatorname{vol} U(k)/\operatorname{vol} U(r)$ as in (19). Consequently

$$d\Theta = \frac{\prod_{i<j} |\lambda_i - \lambda_j|}{2^t \pi^{t(t+1)/2} \prod_{i=0}^{k-1} i!}.$$

The factorials with indices at least k cancel, and the exponent of π comes from $\sum_{i=k}^{r-1}(i+1) - tk = t(t+1)/2$. Thus $|2\lambda_i| \le b^4$ on the parameter implies $d\Theta \le b^{2r(r-1)} \le b^{4r^2}$, as used in Section 12.1.

12.6 Haar measure in the adapted real basis

In this subsection, we identify the real Haar measure obtained by successive isotropic quotients with the group measure used for discrete series. This gives the conversion needed in Section 12.1 and matches the finite convention of Section 10.2.

Use the basis ordered by the marked pairs, then z, then the k positive coordinates. Its Gram matrix J has the hyperbolic pair blocks and the diagonal blocks $-1, I_k$. Let $d\mu$ be the real floor Haar measure, with the terminal stabilizer $U(k)$ of mass one. Then

$$d\mu = \frac{\operatorname{vol} U(r)}{\operatorname{vol} U(k)}\, dg.$$

For a matrix Y of independent columns, put $\Gamma(Y) = Y^*JY$. Use Lebesgue measure on the complex entries of Y, and on the Gram entries use the real diagonals and the real and imaginary parts of the upper off-diagonals. The map Γ is a submersion since $Y^*J\dot{Y}$ can be prescribed. With an initial segment Y_0 fixed, the differential in the new columns surjects onto the cross block and the new diagonal block by the same linear solve with Y^*J.

Thus local submersion coordinates disintegrate Lebesgue measure over these blocks in order. In fact the coordinate Jacobian is block triangular, with the old Gram coordinates depending only on Y_0. Consequently the full fiber measure is obtained by the successive conditional fiber integrations at the prescribed Gram entries. For an initial segment of the form gY_0 with $g \in U(J)$ and Y_0 marked, the conditional measure is transported by g. Both ambient Lebesgue measure and all the Gram entries are preserved, since $|\det g| = 1$.

For the full square Gram fiber at J, its density at the identity is computed by

$$\dot{Y} = J(S + T/2), \qquad \dot{\Gamma} = T, \qquad S^* = -S, \quad T^* = T.$$

Left multiplication by J preserves Lebesgue measure. Each diagonal contributes $1/2$. For an off-diagonal pair the complex map from (S_{ij}, T_{ij}) to $(S_{ij} + T_{ij}/2, -S_{ij} + T_{ij}/2)$ has determinant one. Conjugating the second image preserves real volume. The fiber density is therefore 2^{-r} in the S-coordinates. The full fiber is left invariant, so the Cayley normalization gives its measure as $2^{-r} \operatorname{vol} U(r)\, dg$.

Inversion of the floor product gives, in order, the factors $\theta_i^{\mathrm{inv}}, \eta_i$ for each pair, then $\theta_\varrho^{\mathrm{inv}}$ for z, and finally probability Haar on $U(k)$. Here θ^{inv} denotes the image under inversion. Every new factor fixes the earlier marked columns. Fixing the cross products with the earlier pairs sets their opposite coordinates to zero, with Jacobian one. When adding e_i or z, the chart of Section 10.2 thus disintegrates in the remaining space.

In its hyperbolic pivot coordinates (t, tc, v), the ambient Jacobian is $|t|^2$ and

$$h((t, tc, v), (t, tc, v)) = 2|t|^2 \Re c + h(v, v).$$

The change from $\Re c$ to the real Gram entry gives $1/(2|t|^2)$. The top hyperbolic change of basis has determinant of modulus one, since both Gram determinants on the plane are -1. Thus half the marked quadric measure is the conditional fiber measure on its pivot chart. In both pivot charts choose the solved positive block disjoint from the hyperbolic pair. The coordinate t is then free in the sphere description of Section 11.1, so $t = 0$ is null on the prescribed norm fiber, including norm zero. The isotropic origin cannot occur in an independent frame.

For the dual column the additional cross product with e_i fixes the coefficient of e_i^* to one. Write the column as $e_i^* + v + (\alpha + i\beta)e_i$. Its norm is $2\alpha + h(v, v)$, so the fiber law at zero has density $dv\, d\beta/2$. This is half the Heisenberg measure on that column.

After the $2\varrho + 1$ columns have been inserted, their complement has Gram matrix I_k in the chosen coordinates. The cross products with the initial segment again give a coordinate map of determinant of modulus one. The calculation for the full Gram fiber, now with $J = I_k$, gives the terminal conditional measure $2^{-k} \operatorname{vol} U(k)$ times the compact probability Haar. The pivot charts have full conditional measure. The $2\varrho + 1$ factors of $1/2$ and the terminal factor of 2^{-k} give

$$2^{-r} \operatorname{vol} U(k)\, d\mu^{\mathrm{inv}} = 2^{-r} \operatorname{vol} U(r)\, dg.$$

The square-column map sends a group element to its matrix in the chosen basis. At each step the norm-fibre measure is invariant under the remaining unitary group, since the group preserves

the norm map and ambient Lebesgue measure. Changing the orbit representative in right translation changes the inner integral by a stabilizer translation. Induction from probability Haar on $U(k)$, as in Section 10.2, proves right invariance of $d\mu$. Since G is unimodular as verified below, the displayed identity gives $d\mu = \lambda\, dg$ with $0 < \lambda < \infty$.

To compare the quotient measures, order the Heisenberg factors by increasing a. The factor with parameters (v, β) fixes the earlier dual columns and sends e_a^* to

$$e_a^* + v + (i\beta - h(v,v)/2)e_a.$$

The product of earlier factors only adds earlier isotropic vectors to this column. Its coordinates in the remaining space and its imaginary e_a-coordinate are therefore (v, β). Moreover $u^{-1} = Ju^*J$ identifies these dual-column coordinates of u^{-1} with the free entries of u up to permutation, signs and conjugation. All have real Jacobian of absolute value one. Thus the product of the Heisenberg Haar measures is the original unipotent Haar, since inversion preserves Haar on a unipotent group. Dividing the group measures by this common Haar gives $dQ = \lambda\, d\bar{g}$ on $U_{<n}\backslash G$.

The adjoint map $S \mapsto g^{-*}Sg^{-1}$ has real determinant of absolute value $|\det g|^{-2r} = 1$. Thus G is unimodular, and inversion preserves both group measures. Thus $\lambda = \operatorname{vol} U(r)/\operatorname{vol} U(k)$ and the coefficient of the initial squared correlation is $C_F\lambda^3 = d_I^{-2}$ by Section 11.8.

12.7 Decay of the lowest compact type coefficient

In this subsection, we bound the lowest compact type coefficient using its Harish–Chandra parameter. The bound is used away from compact sets in Section 12.4 and in the tail of the stationary-phase estimate of Section 12.9.

Work in the unitary basis with prescribed signs $J = \operatorname{diag}(\varepsilon_i)$. Write $\lambda_1 > \cdots > \lambda_r$ for the Harish–Chandra parameter, $\rho_i = (r + 1 - 2i)/2$, and $a_i = \lambda_i + \rho_i$, which are integers. For $0 \le c_* \le h$ integral, where $|\lambda_i - \lambda_j| \ge h$ for opposite signs, we prove

$$|c(g)| \le \min\{1, (2/\|g\|)^{c_*}\}$$

for a coefficient of unit vectors in the lowest compact type.

For a matrix Z, let $D_I(Z)$ be the principal minor on I. For a weight $w \in \mathbb{Z}^r$ and $|w_i| \le M$ put

$$m_w(Z) = (\det Z)^{-M} \prod_{t=-M+1}^{M} D_{\{i:w_i \ge t\}}(Z).$$

If $Z \in U(J)$ is positive definite, then $Z^{-1} = JZJ$ and $\det Z = 1$. Its block diagonal part $D = (Z + Z^{-1})/2$ is positive definite. For each threshold t, take the partition

$$I = \{a_i \ge t\}, \qquad B = \{a_i < t \le a_i + c_*\}, \qquad C = \{a_i + c_* < t\}.$$

The signs on B agree since ρ_i is ordered as λ_i. Thus $D_B(D) = D_B(Z)$. Fischer's inequality (the positive Schur complement bound) and complementary minors give

$$D_{I\cup B}(D) \le D_I(D)D_B(Z) = D_I(D)D_{I\cup C}(Z) \le D_I(D)D_I(Z)D_C(Z).$$

Multiply using a common M for a and $a + c_*$. The product of $D_C(Z)$ is $m_{a+c_*}(Z)$ by complementary minors. Since $m_{a+c_*}(W) = m_a(W)(\det W)^{c_*}$, cancelling $m_a(D)$ gives

$$(\det D)^{c_*} \le m_a(Z)^2.$$

Let V be the lowest compact type with unit highest vector v and orthogonal projection E_V. On the positive factor $Z = e^X$ of the polar decomposition, Sun's bottom layer integral [68, Theorem 6.3 and (25)–(26)] gives

$$E_V \pi(Z) E_V = \dim V \int_K m_a(k^{-1} Z k)^{-1} P_{\pi(k)v}\, dk,$$

where $K = U(J) \cap U(r)$, $dk(K) = 1$, and P_u is the orthogonal projector onto $\mathbb{C}u$. For the compact-Cartan Borel the inducing character has weight $\lambda - \rho$. For this positive system every positive root $\alpha = e_i - e_j$, $i < j$, satisfies $\langle (\lambda - \rho) + \rho, \alpha^\vee \rangle = \lambda_i - \lambda_j > 0$, the good-range condition for this Borel. Cohomological induction from this compact Cartan therefore realizes the discrete series with parameter λ [1, §9, paragraph following Theorem 9.2, and Theorem 9.4]. For $i < j$ of the same sign, put $s = \#\{i < h \le j : \varepsilon_h = \varepsilon_i\}$. The compact weight $\mu = a - 2\rho_K$ is dominant since $\mu_i - \mu_j = (\lambda_i - \lambda_j) + (j - i) - 2s \ge 0$, using integrality and regularity. The inducing character multiplied by the determinant character of the nilradical has weight $(\lambda - \rho) + 2\rho = a$, and the bottom compact weight is $(\lambda - \rho) + 2(\rho - \rho_K) = a - 2\rho_K$. In Sun's formula take the compact argument to be 1, the positive argument to be $e^{X/2}$, and the compact integration variable to be k^{-1}.

In the upper triangular times unitary decomposition $k^{-1} e^{X/2} = u d k'$, take u unitriangular and d positive diagonal. Sun's map $H_{\mathfrak{b}}$ in (7) reads d modulo the compact torus. The complexified Cartan involution is conjugation by J, and real-form conjugation sends g to $J(g^*)^{-1}J$. Hence the factor $d\,\overline{\theta(d^{-1})}$ in (25) equals d^2. Also $k^{-1} Z k = u d^2 u^*$. Trailing principal minors give $d_i^2 = D_{[1,i-1]}(k^{-1}Zk)/D_{[1,i]}(k^{-1}Zk)$, using $Z^{-1} = JZJ$ and complementary minors. The torus weight therefore gives $m_a(k^{-1}Zk)^{-1}$. The positive factor has determinant one, so this use of the derived group formula preserves the central character. Schur orthogonality gives $\dim V \int_K P_{\pi(k)v}\, dk = E_V$. The scalar $m_a(k^{-1}Zk)^{-1}$ is positive and at most $(\det D)^{-c_*/2}$, since K preserves the determinant of the block diagonal part.

Thus the same bound holds in operator norm. The compact factor of g acts unitarily on V, so it preserves that bound. Write $g = D_g + E_g$ for its equal-sign and opposite-sign blocks. The diagonal blocks of $g^* J g = J$ give $D_g^* D_g = 1 + E_g^* E_g$, and $|\det D_g| = \det D$ by the polar decomposition. Therefore

$$|c(g)|^4 \le (1 + \|E_g\|_{\mathrm{HS}}^2)^{-c_*},$$

using $\det(1 + E_g^* E_g) \ge 1 + \operatorname{tr}(E_g^* E_g)$.

The eigenvalues u of Z contribute $(u + u^{-1})/2$ to $\det D$, each at least one. If $\|Z\| > 1$, then $\|Z\|$ and $\|Z\|^{-1}$ are distinct eigenvalues and their two factors give $\det D \ge (\|Z\|/2)^2$. The same inequality holds for $\|Z\| = 1$. Since $\|g\| = \|Z\|$, this proves the decay claim. In Section 12.4 use $c_* = 9r^2$ and $c_* = 9r^2 + 3r^3 f$, both at most the available gap whenever $3r^3 f \le A$ and $h \ge 500r^3 + A$.

12.8 The stationary point and its period

In this subsection, we construct the stationary point for the original pair of discrete series. The stationary-phase calculation of Section 12.9 gives the period lower bound used to normalize the full vector in Section 12.10.

Proposition 12.1 (Stationary point and period lower bound)**.** *Use the marked pair of discrete series with a common zero-weight positive block, integral weights μ_i on the remaining $2n$ coordinate lines of signatures $(-1)^i$, and remaining weight a in H. Put $N_0 = (r + k)^2$, $\beta = b^3$ and*

$T_0 = N_0 \log \beta$. *Assume*

$$B \ge 1, \quad -3B \le \mu_{2n-1} < \cdots < \mu_0 \le -B, \quad a \ge 0, \quad 3B + a \le b^2, \quad 8nr \le b$$

and $\beta \ge 10^{12}$. *Assume that all doubled Harish–Chandra parameters of both representations have absolute value at most* L_0', *with* $r + k + L_0' \le \beta$. *For some integer* $A \ge 10^{30} T_0^3$, *assume* $|\lambda_i - \lambda_j| \ge 500r^3 + A$ *for every pair of ambient parameters on coordinate lines of opposite signs. Then there is a stationary point* x_0 *for which*

$$\ell(x_0) \le b^{5n}, \qquad V_{x_0} \le b^{60N_0} |\langle v_0, A(x_0) \rangle|.$$

Proof. Let the lowest compact type weights of the original two representations be zero on a common positive block X. In the remaining $2n$ coordinates of G, take integral weights $-3B \le \mu_i \le -B$ in strictly decreasing order at signs $(-1)^i$, $0 \le i < 2n$, with $B \ge 1$. The remaining coordinate in $H = U(k)$ has weight $a \ge 0$. Use $\beta = b^3$, and suppose $T = 3B + a \le b^2$ and $8nr \le b$. The following construction takes place in these two original weight bases.

Put $P(t) = \prod_i (t - \mu_i)$ and, on the $2n$ coordinates,

$$[F]_i = \frac{F(\mu_i)}{\sqrt{|P'(\mu_i)(\mu_i - a)|}}, \qquad V_j = [t^j (t - a)] \quad (0 \le j \le 2\varrho), \qquad I = \sqrt{P(a)}[1].$$

Extend these vectors by zero on X. For real polynomials the signed pairing of $[F]$ and $[G]$ is $-\sum_i F(\mu_i) G(\mu_i) / (P'(\mu_i)(\mu_i - a))$. By Lagrange interpolation, $\sum_i Q(\mu_i)/P'(\mu_i)$ is the top coefficient of Q for $\deg Q \le 2n - 1$. It follows that

$$h(V_j, V_l) = 0 \quad (j + l < 2\varrho), \qquad h(V_j, V_l) = -1 \quad (j + l = 2\varrho), \qquad h(I, V_j) = 0.$$

Partial fractions of $1/(P(t)(t - a))$ have residues summing to zero, with residue $1/P(a)$ at a. This gives $h(I, I) = 1$. Let i_0 embed the compact space by the common coordinates on X and by I at its final coordinate. The embedding is isometric. The vectors $V_0, \ldots, V_\varrho$ are independent, since $t - a$ is nonzero on the $2n$ nodes and a polynomial of degree at most ϱ vanishing there is zero. Set

$$D_\mu = \operatorname{diag}(0_X, \mu_0, \ldots, \mu_{2n-1}), \qquad D_H = \operatorname{diag}(0_X, a).$$

Then

$$D_\mu V_j = V_{j+1} \quad (j < 2\varrho), \qquad D_\mu i_0 = i_0 D_H \bmod \langle V_0 \rangle.$$

Transport the basis with prescribed signs into the original floor space by its Euclidean unitary change of basis. The perpendicular complement of $i_0 \mathbb{C}^k \oplus \mathbb{C} V_\varrho$ is nondegenerate. In it let A have columns $V_0, \ldots, V_{\varrho-1}$ and choose B_0 with $A^* J B_0 = 1$. Replacing B_0 by $B_0 - A(B_0^* J B_0)/2$ gives isotropic duals. Together with i_0, V_ϱ the columns give an isometry x_0^{-1} from the original basis, sending e_j to V_j, z to V_ϱ, and the positive block to $i_0 \mathbb{C}^k$ with its chosen basis.

To bound the length of this same x_0, use Euclidean Gram–Schmidt on $(V_0, \ldots, V_\varrho)$, without rescaling. The resulting F_j differ from V_j by earlier columns, and retain the Gram and the pairings $h(F_j, V_{2\varrho - j}) = -1$. As $|P'(\mu_i)| \ge 1$, the coordinate formula gives $\|V_j\| \le M_0 = \sqrt{r}\, T^{2n}$ through 2ϱ. Thus $M_0^{-1} \le \|F_j\| \le M_0$. For $j < \varrho$, the vectors $J F_j / \|F_j\|^2$ are isotropic duals, with Euclidean norm at most M_0. The pairs form a J-invariant Euclidean orthogonal summand, and F_ϱ lies in its complement of signature $(k, 1)$.

Write $F_\varrho = z_+ + z_-$ in the positive and negative eigenspaces, with $s = \|z_+\|$, $t = \|z_-\|$ and $t^2 - s^2 = 1$. For $s > 0$, set $v = tz_+/s + sz_-/t$. In the Hermitian form h of signature $(k, 1)$,

$$h(v, v) = 1, \qquad h(v, F_\varrho) = 0, \qquad \|v\|^2 = s^2 + t^2 = \|F_\varrho\|^2.$$

For $s = 0$, take a Euclidean unit positive vector in that complement. Complete by the remaining orthonormal positive directions, orthogonal to these at most $2n$ exceptional columns. The new basis matrix Y is an isometry with $\|Y^{-1}\| = \|Y\| \le \sqrt{2n}M_0 + 1$. Put $s_0 = x_0 Y$, $A_j = \operatorname{span}(e_i : i < j)$, and $A = A_\varrho$ in the original basis. The relations $F_j - V_j \in \operatorname{span}(V_i : i < j)$ give

$$s_0 e_j - e_j \in A_j \quad (0 \le j < \varrho), \qquad s_0 z - z \in A.$$

Unitarity preserves $A^\perp$. The induced isometry on $A^\perp/A$ fixes $[z]$ and acts by $u \oplus 1$, with $u \in U(k)$. After removing the standard lift of u, the isometry is upper unitriangular on A and the identity on $A^\perp/A$. Unitarity forces the inverse-adjoint block on the opposite isotropic quotient. These are the defining conditions for $U_{<n}$. Thus $s_0 \in S$, and

$$s_0^{-1} x_0 = Y^{-1}, \qquad \ell(x_0) \le 1 + \|Y^{-1}\| \le 8nM_0 \le b^{5n}.$$

The representative $s_0^{-1}x_0$ is used in the length infimum, while the measured period is taken at the stationary point x_0.

Return to the basis with prescribed signs, and for a graph element s write $g = x_0^{-1} s_G x_0$ in that basis, with compact component $h = s_H$. Put $Y_1 = g - 1$, $Y_2 = h - 1$, $L = \langle V_0, \dots, V_{\varrho-1}\rangle$, and let $\sigma(s)$ be the sum of first predecessor coefficients on the original basis. Then $g - 1$ lowers the transported flag, and on $L^\perp/L$ the action of g is $h \oplus 1$ in the transported i_0, V_ϱ. Conversely these equations characterize the conjugated graph, since in the original Witt blocks they give exactly $U_{<n}H$ and its compact projection. In particular the diagonal $U(X)$ belongs to the transported graph, with $\sigma = 0$.

Choose a projection P_0 onto L killing $i_0\mathbb{C}^k$, put $D_1 = D_\mu P_0$, and let $D_2 = D_\mu - D_1 - D_1^\dagger$, with $\dagger$ the form adjoint. Since $D_\mu L \subset L^\perp$, D_2 kills L and has image in $L^\perp$. On $i_0\mathbb{C}^k$ it agrees with D_μ, since $D_\mu i_0\mathbb{C}^k \subset L^\perp$. On $L^\perp/L = i_0\mathbb{C}^k \oplus \mathbb{C}[V_\varrho]$, the map induced by Y_1 is $Y_2 \oplus 0$. The compression of D_2 to the first summand is $i_0^\dagger D_\mu i_0 = D_H$, since $V_0 \perp i_0\mathbb{C}^k$. The map $D_2 Y_1$ has zero trace on L and on the quotient by $L^\perp$. Hence $\operatorname{tr}(D_2 Y_1) = \operatorname{tr}(D_H Y_2)$. On L, the j-th diagonal entry of $P_0 Y_1 D_\mu$ is the coefficient of V_j in $Y_1 V_{j+1}$, for $0 \le j < \varrho$. Their sum is $\sigma(s)$; when $\varrho = 0$ it is the empty sum. Applying this calculation to $Y_1^\dagger = g^{-1} - 1$ gives $\operatorname{tr}(D_1^\dagger Y_1) = -\bar\sigma(s)$, since inversion negates the first predecessor coefficients. The flag and its dual contribute zero to the unweighted trace. We obtain

$$\operatorname{tr} Y_1 = \operatorname{tr} Y_2, \qquad \operatorname{tr}(D_\mu Y_1) - \operatorname{tr}(D_H Y_2) = \sigma(s) - \bar\sigma(s).$$

With the unit highest vector v_0 of the compact type, put $c_H(h) = \langle v_0, \pi_H(h) v_0\rangle$. The period integrand for the average of Section 12.4 is $\theta(s) = c(g)\overline{c_H(h)} e^{\bar\sigma(s) - \sigma(s)}$. All following volumes and integrals use the original ds. Put $N_0 = (r + k)^2$ and $Q = (2/\|g \oplus h\|)^{9N_0}$. Since $\|h\| = 1$, the decay just proved gives $|\theta| \le Q$ when the gap is at least $9N_0$. If the doubled parameters are bounded by L_0' with $r + k + L_0' \le \beta$, the Casimir calculation of Section 12.3 on each factor gives $|\theta| \ge 1/4$ on the ball $\|g \oplus h - 1\| \le \delta = (200\beta^5)^{-1}$.

For the phase estimate put $T_0 = N_0 \log \beta$ and $\eta_0 = 10^{-13}/T_0$, and suppose the excess gap is at least $A \ge 10^{30} T_0^3$. Since $A \ge 18 N_0$, the decay bound also gives $|\theta| \le Q^2$. If $\|E_g\|_{\mathrm{HS}} \ge \eta_0$, the preceding determinant estimate with $c_* = A$ gives

$$|\theta| \le (1 + \eta_0^2)^{-A/4} \le \exp(-A\eta_0^2/8) \le e^{-100 T_0}.$$

Section 12.9 proves $|\theta| - 2\Re\theta \le 12e^{-100T_0}$ when $\|E_g\|_{\mathrm{HS}} < \eta_0$. Away from that set the same bound follows from $|\theta| - 2\Re\theta \le 3|\theta|$. Combining it with $|\theta| - 2\Re\theta \le 3Q^2$ by the geometric mean gives

$$|\theta| \le 2\Re\theta + 6\beta^{-50N_0}Q.$$

Let V_s be the mass of the ball of radius δ about 1, and V that of $\|g \oplus h\| \le \beta + 1$.

Both are finite by closedness of the graph and compactness of the compact factor. Matrix packing as in Section 12.3 gives $V \le \beta^{16N_0}V_s$, since $3(\beta+1)^2/\delta \le \beta^8$ for $\beta \ge 10^{12}$. With radius one, the same packing bounds the mass of $\|g \oplus h\| \le Y$ by $3^{2N_0}Y^{4N_0}V$ for $Y \ge 1$, since the identity ball lies in the set of mass V. Layer integration therefore gives

$$\int Q\,ds \le 3^{2N_0}2^{9N_0}V\int_1^\infty 9N_0Y^{-5N_0-1}\,dY = \tfrac{9}{5}\,3^{2N_0}2^{9N_0}V \le 16^{9N_0}V.$$

Hence

$$V \le 8\beta^{16N_0}\left|\int \theta\,ds\right| + 24\beta^{-34N_0}16^{9N_0}V.$$

The last coefficient is less than $1/2$. Since $V_{x_0} \le V$ and $16\beta^{16N_0} \le \beta^{20N_0}$, this yields $V_{x_0} \le b^{60N_0}|\langle v_0, A(x_0)\rangle|$. This proves the quantitative period bound at the constructed point for these weights and gaps. □

12.9 The phase on the common compact block

In this subsection, we expand the phase near the stationary point of Section 12.8. We compare the phase-corrected coefficient with a positive Gaussian near the common compact block. Together with the decay in Section 12.7, this bounds the integral on the original graph Haar measure.

Use the stationary graph and T_0, η_0 of Section 12.8. Write $x = g-1$, $x' = h-1$, $v_0' = \|E_g\|_{\mathrm{HS}}$ and $\phi = \sigma - \bar\sigma$. Suppose $v_0' \le \eta_0$. For each covector, $gJg^* = J$ gives

$$2\Re x_{ii} = 2\sum_j |(E_g)_{ij}|^2 - \sum_j |x_{ij}|^2.$$

The corresponding identity for h has zero opposite-sign term. Multiply by $-\mu_i$ on the first factor and by the nonnegative compact weight on the second. The weighted traces cancel in real part by the phase identity. Since $B \le -\mu_i \le 3B$ off X, and the weight vanishes on X, the sum of squared norms of the covectors of x off X is at most $6(v_0')^2$. Apply the same calculation to the inverse pair, using $g^{-1} - 1 = Jx^*J$, to bound the columns off X.

For $a_0 = g_{XX}$ and $C = g_{X^cX}$, signed unitarity gives $a_0^*a_0 - 1 = -C^*J_{X^c}C$. Its Hilbert–Schmidt norm is at most $6(v_0')^2$. Thus $a_0 = up$ with u unitary and $p > 0$, and $\|p-1\|_{\mathrm{HS}} \le 6(v_0')^2 \le \eta_0$ by the scalar inequality $|t-1| \le |t^2-1|$ for $t > 0$. Left multiply the pair by the diagonal copy of u^* in the graph. Both highest-weight vectors are fixed by this copy. Indeed the highest-weight coefficient on the compact group is m_μ, by the exterior-power realization.

Its threshold minors include all of X for $t \le 0$ and exclude it for $t > 0$, so the determinant factors on X cancel. A unit coefficient of value one forces the unit vector to be fixed. The translating element has phase zero, and ϕ is additive since the first predecessor coefficients add. Therefore θ has the same value after the move. The opposite-sign norm is unchanged and the X-block of the moved g is p. The covector and column estimates now give $\|g-1\|_{\mathrm{HS}}^2 \le 13\eta_0^2$.

Summing the unweighted covector identity and using the trace equality gives $\|h-1\|_{\mathrm{HS}}^2 = \|g-1\|_{\mathrm{HS}}^2 - 2(v_0')^2$. Thus both distances are at most $4\eta_0$.

On this moved pair put $Z = \log g,\ Z' = \log h$. For the linear functional $\mathcal{L}(P,Q) = \operatorname{tr}(D_\mu P) - \operatorname{tr}(D_H Q)$, the phase identity applied to group powers gives $\mathcal{L}(g^j-1, h^j-1) = j\phi$ for nonnegative integers j. Taking finite differences at zero gives $\mathcal{L}(x^i, (x')^i) = 0$ for $i \geq 2$. The log series therefore gives $\mathcal{L}(Z, Z') = \phi$.

Triangular factors in the coefficient integral. We give the local calculation for either factor of dimension d_1. Use the weights $a = \lambda + \rho$ and $\mu = a - 2\rho_K$, and put $J_\mu = \int_K |m_\mu(k)|^2\, dk$. Write $c(w)$ for the coefficient of its unit highest vector. For fixed w in the sign group close to 1, consider

$$I(p, r_0) = \int_K \frac{m_\mu(pk)\, m_\mu(k^* r_0)}{m_a(k^* r_0 w^{-1} pk)}\, dk$$

for block diagonal p, r_0 near 1. The integral is holomorphic there, since all powers are integral and the denominator compressions are uniformly invertible on K. In the next computation the neighborhood can be taken with $\|p-1\|, \|r_0-1\| < 1/10$ and $\|w-1\| < 1/100$, for which $r_0 w^{-1} p$ is within distance one of 1.

12.9.1 Triangular coordinates on the sign blocks

In this subsection, we write the phase of Section 12.8 in triangular coordinates.

In each sign block take the upper triangular group for the parameter order. If b is upper triangular in these blocks, replacing k by kb in the integrand multiplies it by

$$\prod_i |b_{ii}|^{2(\mu_i - a_i)} = \prod_{i>j,\ \varepsilon_i = \varepsilon_j} |b_{ii}/b_{jj}|^2.$$

Indeed left lower and right upper factors scale each initial principal minor by its diagonal product. The thresholds for μ are initial within each block, and those for a are initial in the full order.

For block invertible t, write the block QR decomposition $tk = k'b$. This induces a diffeomorphism of the compact flag $K/\mathbb{T}$, where $\mathbb{T}$ is the diagonal torus. In left-translated tangent coordinates the strict lower complex entries of the derivative are the strict lower entries of bYb^{-1}, with Y skew-Hermitian in the sign blocks. The derivative of the upper factor has no strict lower part.

Each (i,j) depends on entries (i', j') with $i' \geq i,\ j' \leq j$, so the complex diagonal factors are b_{ii}/b_{jj}. Thus the real Jacobian for invariant probability measure on the flag is the product just displayed. The integrand is right $\mathbb{T}$-invariant. Change of variables gives $I(pt, t^* r_0) = I(p, r_0)$ whenever the nearby arguments are defined. Differentiating at $t = 1$ in the directions Y and iY and using holomorphy gives separately $D_p I[pY] = 0$ and $D_{r_0} I[Y^* r_0] = 0$. Hence I is constant on the indicated neighborhood.

At $p = k_0,\ r_0 = 1$, for $w = k_0 Z_0$ its polar decomposition, the bottom-layer formula of Section 12.7 gives $I(k_0, 1) = J_\mu c(w)$. Indeed, $w^{-1} k_0 = Z_0^{-1}$, and $k^* Z_0^{-1} k = J(k^* Z_0 k) J$ since k commutes with J. Conjugation by the diagonal sign matrix preserves every principal minor. The factor $m_\mu(k_0 k)\overline{m_\mu(k)}$ follows by taking the highest-vector coefficient of $\pi(k_0) P_{\pi(k)v}$, and complementary minors give $m_a(D^{-1}) = m_{-a}(D)$ for invertible D. For $Z = \log w$, let U, V be its strict lower and upper equal-sign parts, respectively, and set $\widehat{w} = e^{-U} e^{Z} e^{-V}$. Evaluating the same integral at $p = e^U,\ r_0 = e^V$ uses $r_0 w^{-1} p = \widehat{w}^{-1}$. Both triangular factors have unit

diagonal, so the numerator is $|m_\mu(k)|^2$. For the moved matrices $\|w-1\|_{\mathrm{HS}} \le 4\eta_0$. The logarithm series and orthogonal truncation of matrix entries give $\max(\|U\|, \|V\|) \le \|\log w\|_{\mathrm{HS}} \le 8\eta_0$. Thus all these choices are in the indicated neighborhood. We obtain

$$c(w) = \frac{1}{J_\mu} \int_K \frac{|m_\mu(k)|^2}{m_{-a}(k^* \widehat{w} k)}\, dk.$$

Uniform quadratic remainder. Set $d_0 = \|Z\|_{\mathrm{HS}} \le 2\delta_0$, where $\delta_0 = 4\eta_0$, so $d_0 \le 1/20$. Let $A_1 = -U, A_2 = Z, A_3 = -V$. For an orthogonal projection P' let D' be the compression of $\prod_i e^{A_i}$ to its image and $D'_0 = \prod_i e^{P' A_i P'}$ there. Put $z_0 = \max_i \|[P', A_i]\|_{\mathrm{HS}} \le d_0$. In expanding $D' - D'_0$ into words in the diagonal and off-diagonal blocks relative to P', surviving words have at least two off-diagonal factors. The degree two trace is

$$S_{P'} = \sum_{i<j} \operatorname{tr}(P' A_i (1-P') A_j) + \tfrac{1}{2} \sum_i \operatorname{tr}(P' A_i (1-P') A_i).$$

The corresponding degree two matrix has trace norm at most $(9/2) z_0^2$. Bound two off-diagonal factors of each longer word in Hilbert–Schmidt norm and the rest in operator norm. The multinomial sum and the choices of blocks give the trace-norm bound for the tail

$$z_0^2 \sum_{m \ge 3} 6^m d_0^{m-2}/m! \le 36 e^{6d_0} d_0 z_0^2 \le 49 d_0 z_0^2.$$

It follows that $\|D' - D'_0\|_1 \le 7 z_0^2$, where $\|\cdot\|_1$ is the trace norm. Also $\|(D'_0)^{-1}\| \le e^{3d_0}$ and $\|(D'_0)^{-1} - 1\| \le 4 d_0$. For $N = (D'_0)^{-1}(D' - D'_0)$, the log series gives

$$|\operatorname{tr}(\log(1+N) - N)| \le \frac{81 d_0^2 z_0^2}{1 - 9 d_0^2} \le 5 d_0 z_0^2.$$

The linear trace of N differs from $\operatorname{tr}(D' - D'_0)$ by at most $28 d_0 z_0^2$. Hence

$$\det D' = \exp\{\operatorname{tr}(P' \textstyle\sum_i A_i) + S_{P'} + R_{P'}\}, \qquad |R_{P'}| \le 100 d_0 z_0^2.$$

12.9.2 The compact block estimate

In this subsection, we use the triangular coordinates of Section 12.9.1 to bound the compact phase.

Apply the preceding determinant expansion to $P' = kPk^*$, where P is a threshold projection for $-a$. Put $s_0 = \|[P, Z]\|_{\mathrm{HS}}$ and $h_0 = \|[P, k]\|_{\mathrm{HS}} = \|P' - P\|_{\mathrm{HS}}$. Then $z_0 \le s_0 + 2 d_0 h_0$. Both projections are block diagonal for the signs. The equal-sign off-diagonals of $\sum_i A_i$ vanish, and $\sum_i |(P' - P)_{ii}| = h_0^2$. Thus its linear trace differs from $\operatorname{tr}(PZ)$ by at most $d_0 h_0^2$. Each factor $P' A_i (1 - P')$ differs from its version at P by at most $2 d_0 h_0$ in Hilbert–Schmidt norm. Consequently $|S_{P'} - S_P| \le 18 d_0 h_0 (s_0 + 2 d_0 h_0)$. For i inside the threshold and j outside, the parameter at i is smaller. For equal signs, the three cross terms and the A_2 square contribute $(1 - 1 + 1 - 1/2)|Z_{ij}|^2$ to S_P. For opposite signs only the square contributes, giving $|Z_{ij}|^2/2$ since $Z^* J + J Z = 0$. Thus $S_P = s_0^2/4$. The total error in the exponent for this threshold is at most $10^5 d_0 (s_0^2 + h_0^2)$.

Write $Q_b(F) = \sum_{i,j} |b_i - b_j| |F_{ij}|^2$ for a weight b, and $q = Q_{-a}(Z)/4$. Summing squared commutators counts each pair once per threshold that separates it. Also $\det \widehat{w} = e^{\operatorname{tr} Z}$. Multiplying the threshold formulas gives

$$m_{-a}(k^* \widehat{w} k) = \exp\{q - \operatorname{tr}(\operatorname{diag}(\mu) Z) - R(k)\}, \qquad |R(k)| \le 10^6 \delta_0 (q + Q_\mu(k) + d_1^2).$$

For the error use $|a_i - \mu_i| \le d_1$, $Q_{-a}(k) \le Q_\mu(k) + 2d_1^2$, and $|\operatorname{tr}(\operatorname{diag}(a-\mu)Z)| \le d_1^{3/2} d_0$. The remainder is a sum of the local log series above. For unitary k, the squared modulus of each compressed determinant at P equals $\det(1 - Pk^*(1-P)kP)$ on the compressed space. Since $1 - t \le e^{-t}$, the product gives $|m_\mu(k)|^2 \le e^{-Q_\mu(k)/2}$, using equality of the two off-diagonal squared masses of a unitary matrix. The compact Weyl dimension formula gives $J_\mu^{-1} = \dim V_\mu \le (1+6\beta)^{d_1^2} \le e^{5000 T_0}$, since the weights have modulus at most 3β and $d_1^2 \log \beta \le T_0$.

Put $W_0 = q + Q_\mu(k) + d_1^2$. If $W_0 \le 50000 T_0$, then $\delta_0 T_0 \le 10^{-12}$ gives $|R| \le 1/20$ and $|e^R - 1| \le 1/10$. Otherwise $|R| \le W_0/4$. Summing the absolute values of the positive-degree terms of the exponential series gives $|e^R - 1| \le e^{|R|} - 1$, so

$$|m_\mu(k)|^2 e^{-q} |e^R - 1| \le e^{-q - Q_\mu(k)/2 + W_0/4} \le e^{-5100 T_0}.$$

Indeed the exponent is at most $-W_0/4 + d_1^2/2$. After integration and division by J_μ,

$$\left| c(w) e^{-\operatorname{tr}(\operatorname{diag}(\mu) \log w)} - e^{-q} \right| \le e^{-q}/10 + e^{-100 T_0}.$$

Apply this estimate to the two moved matrices, conjugating the compact coefficient in θ. Both logarithmic traces are imaginary by the sign-Lie identity. Conjugation reverses the compact phase and preserves its real Gaussian. The remaining phase is $\exp(\operatorname{tr}(D_\mu Z) - \operatorname{tr}(D_H Z') - \phi) = 1$. The product of the two phase-corrected coefficients differs from the product $a_1 a_2$ of the two numbers e^{-q} in $[0,1]$ by at most $0.21 a_1 a_2 + 4e^{-100 T_0}$. Thus $|\theta| - 2\Re\theta \le 12 e^{-100 T_0}$, as required at the original graph element.

12.10 The real Bessel functions for the construction

In this subsection, we apply the real estimates to the parameters of Section 4. This constructs the discrete-series vectors and verifies the normalization and truncation bounds required for Section 9.1.

Return to the datum of Section 4. The rank of the ambient marked form is r, with $k = r + 1 - m$ and $\varrho = m/2 - 1$. Thus $n = \varrho + 1 = m/2$ and $r = k + 2\varrho + 1$. Put $a = xb$, $\Delta = 2r^8$, and $B_\tau = \kappa_\tau + \Delta/2$ at an infinite place τ. The character of lift number j has weight $B_\tau + j\Delta$. The marked initial parameter, in its numbered coordinate order, has $k-1$ positive entries followed by m alternating signs starting with $+$.

In this order the parameter entries are

$$\lambda_i = k/2 - 1 - i \quad (0 \le i < k-1), \qquad \lambda_{k-1+t} = -B_\tau - (t + 1/2)\Delta \quad (0 \le t < m).$$

Here the tail entry is the half gap minus the weight of lift $t+1$. The signs have signature $(k + \varrho, \varrho + 1)$. To realize them in the original floor space one can take the unit eigenvectors $(e_j \pm e_j^*)/\sqrt{2}$ of the pair blocks, along with the marked positive basis and z, and permute the two sign sets into the parameter order. This gives a Euclidean unitary basis with prescribed signs for the original Gram matrix J. The local construction applies using any such chosen basis throughout.

For x sufficiently large the displayed entries decrease strictly, and the opposite-sign gaps are at least Δ. Indeed $B_\tau \ge \Delta^2$ eventually since $\kappa_\tau \ge a/x_1$ and $\log b - \log x_1 = x^\epsilon - O(x^{\epsilon/6})$. The tail gaps are exactly Δ, and the last initial entry is $1 - k/2$. The doubled parameter has the

right parity because k is even and r is odd. Choose the discrete series with this Harish–Chandra parameter and a unit highest vector of its lowest compact type [5, Theorems 8.1 and 8.5].

On the derived group use the parameter with its mean removed. Its lowest compact type has the central restriction given by the original integral weight. The character of the scalar circle of weight $\sum_i \lambda_i$ agrees on the common scalar subgroup, since the root corrections have sum zero, so the representation descends from the product to the unitary group. The signed root count in this order gives

$$\mu_i = \lambda_i - \tfrac{1}{2}\left(\sum_{j>i} \varepsilon_i\varepsilon_j - \sum_{j<i} \varepsilon_i\varepsilon_j\right).$$

On the initial block the tail terms cancel in pairs and the count is $k-2-2i$, so $\mu_i = 0$. At tail position t, the initial block contributes $-(k-1)(-1)^t$, and the tail contributes $-(-1)^t$. Hence

$$\mu_{k-1+t} = -B_\tau - (t+1/2)\Delta + \tfrac{k}{2}(-1)^t.$$

Consecutive differences are $\Delta + k(-1)^t > 0$. Since $\Delta \geq k$ and $m\Delta + k/2 \leq 2B_\tau$, these entries lie in $[-3B_\tau, -B_\tau]$.

On X_0 the marked column counts at heights $0 < h < k$ are zero except at $h = 1$, where the count a_τ is a if $\tau \in \Phi_a$ and zero otherwise. The determinant weight on X_0 is zero. Thus the compact type is $\mathrm{Sym}^{a_\tau}(\mathbb{C}^k)$ for the defining representation of $U(k)$. Its highest weight is $(a_\tau, 0, \ldots, 0)$, and its doubled parameter entries are $2a_\tau \mathbf{1}_{j=0} + k - 1 - 2j$ for $0 \leq j < k$.

Match the other $k-1$ coordinates with the zero-weight positive block in the ambient parameter. The remaining compact coordinate is the highest-weight coordinate, also when $a_\tau = 0$. Use exactly the Schur matrices $S_\tau(h)$ on the value space. If $Q_\tau = B_0^* B_0$ is their invariant Weyl Gram matrix, set $\pi_H(h) = B_0 S_\tau(h) B_0^{-1}$. Since S_τ is defined over the reals and $h^{-1} = \bar{h}^{\mathsf{T}}$, invariance gives

$$Q_\tau S_\tau(h) = S_\tau(h^{\mathsf{T}})^{\mathsf{T}} Q_\tau, \qquad B_0^* \pi_H(h) = S_\tau(h^{\mathsf{T}})^{\mathsf{T}} B_0^*.$$

Let j_* be the chosen highest-weight coordinate, and let ξ be the highest monomial $e_{j_*}^{a_\tau}$ in the symmetric power ($\xi = 1$ if $a_\tau = 0$). The unit highest vector and its coefficient in the Bessel value space are

$$v_0 = \frac{B_0 \xi}{\|B_0 \xi\|}, \qquad c_H(h) = \frac{\xi^* Q_\tau S_\tau(h) \xi}{\xi^* Q_\tau \xi} = h_{j_* j_*}^{a_\tau}.$$

Indeed the diagonal torus acts unitarily for Q_τ, so distinct weights are orthogonal. The coefficient of $e_{j_*}^{a_\tau}$ in $(h e_{j_*})^{a_\tau}$ gives the last equality. Thus the stationary integrand uses $\overline{c_H(h)}$ for this same Schur representation.

We check the required sizes uniformly in τ. For large x, $x^5 \leq r < 3x^5$, $\sqrt{x}/2 \leq n < \sqrt{x}$, $k \geq x^5/2$, and $4000n \leq f \leq x^{12}$, the last pair from $f = 3L_0 m^{22} \varphi(q_0)$. Thus $k \geq n^7$, $k - 3n - 6 \geq 10^{10} n^4$, $\varrho \geq 10^8$, and $8nr \leq b$. Since $x_1 \geq x$ and $r^8 \leq b$ eventually, $\kappa_\tau \in \{b, xb/x_1\}$ gives $B_\tau = \kappa_\tau + \Delta/2 \leq 2b$, so $3B_\tau + a_\tau \leq b^2$. The two doubled parameters have magnitude at most $L_* = 3a$, since $b + (m+1)\Delta \leq a$ eventually and $2a + k \leq 3a$. Put $\beta = b^3$ and $A_* = r^7$. Then $r + k + L_* \leq \beta$, $\beta \geq 10^{12}$, and

$$500 r^3 + A_* \leq \Delta, \qquad 3 r^3 f \leq A_*,$$
$$10^{30}\big((r+k)^2 \log \beta\big)^3 \leq 10^{30} 108^3 x^{30+3/100} \leq r^7.$$

The second line uses $\log b \le x^{1/100}$ from the chosen scale bounds. The compact parameter has only plus signs. These size comparisons need one threshold in x, since the weights have the same upper and lower bounds at all places.

Choose x_0 by Sections 12.8 and 12.9 in the original graph and basis with prescribed signs at τ. The graph ball has positive mass, so its period bound implies that the average A of Section 12.4 has positive L^2 norm. For $D_\tau = \dim \pi_H$, take

$$W = \frac{\sqrt{D_\tau}}{\|A\|_2} A, \qquad F = \chi W, \qquad R_b = b^{6n+2}.$$

These choices precede the orbital estimates. With the graph balls and quotient measures at these same points, Fubini and Section 12.8 give

$$\prod_\tau d_\tau \|A_\tau\|_2^2 = \prod_\tau \langle v_{0,\tau}, A_\tau(x_{0,\tau})\rangle \ge b^{-60 f (r+k)^2} \prod_\tau V_{x_{0,\tau}} > 0,$$

where d_τ is the formal degree in dg.

Section 12.4 applies this period bound to the graph average and gives the decay majorant, the cut support, and $\|F - W\|_2^2 \le b^{-2r^3 f} D_\tau$. The quotient measure is invariant under H, and the cutoff χ depends only on the left UH-invariant length. Thus the two Gram matrices commute with the action on the space I_τ of π_H, a determinant twist of $\mathrm{Sym}^{a_\tau}(\mathbb{C}^k)$. This action is irreducible. Schur's lemma, the traces and the preceding error estimate give

$$\int W W^* \, d\bar{g} = 1_{I_\tau}, \qquad \int F F^* \, d\bar{g} = s_\tau 1_{I_\tau}, \qquad (1 - b^{-r^3 f})^2 \le s_\tau = \frac{\|F\|_2^2}{D_\tau} \le 1.$$

The lower bound follows from $\|F\|_2 \ge \|W\|_2 - \|F - W\|_2$.

Section 12.3 gives the right reproduction, the correlation norm D_τ^2/d_τ, and the norm-ball bound. Sections 12.5 and 12.6 give the other degree and Haar bounds of Section 12.1. Hence the Plancherel estimate of Lemma 11.1 and the descent of Section 12.2 apply to F on this graph. The Fourier factor $(\pi/|\xi|)^{k-5}$ comes from the free complex coordinates of Section 11.1. Together with Section 12.1, these estimates give the real forms J_j, their bottom bound $(1+b^{-10})/d_\tau$, and both frequency bounds. Section 12.2 gives integration in a common entry. The derivative estimates for the full function W enter the comparison of nonzero-discriminant orbits in Section 15.4. For a terminal element σ fixing the hyperbolic pairs, the top form at heads with the same last entry $h(z, \sigma z)$ is the inner product of the two actual normalized orbital tensors of F at σ, by Section 11.3. This uses the normalizing dimension D_τ for both tensors.

Lemma 12.2 (Comparison of the formal degrees)**.** *Let* $B = \log b$, $S_\kappa = \sum_\tau \log \kappa_\tau$ *and* $W_* = \log \mathcal{B} + f \sum_{l=k-1}^{r-1} \log(l!)$. *Then*

$$|W_* - kmS_\kappa| \le m^2 f B + fmr/x.$$

Proof. The formal-degree product $\mathcal{B}$ used in Section 9.1 is

$$\log \mathcal{B} = \sum_\tau \sum_{l=k-1}^{r-1} \left(\sum_{i<l} \log(\lambda_i - \lambda_l) - \log(l!) \right).$$

In Section 12.5 the factors at $l < k-1$ cancel because $\lambda_i - \lambda_l = l - i$. At $l \ge k-1$, each initial entry contributes a difference at least κ_τ, and every tail-to-tail difference is at least one. All

differences in this range are at most $\kappa_\tau + m\Delta + k$. Since $(m\Delta + k)/\kappa_\tau \le 1/x$ eventually, this gives

$$\begin{aligned} m(k-1)\sum_\tau \log \kappa_\tau &\le \log \mathcal{B} + f\sum_{l=k-1}^{r-1} \log(l!) \\ &\le mr\sum_\tau (\log \kappa_\tau + 1/x). \end{aligned}$$

Now $r = k + m - 1$ and $0 \le S_\kappa \le fB$, so

$$-mS_\kappa \le W_* - kmS_\kappa \le m(m-1)S_\kappa + fmr/x. \qquad \square$$

12.11 The differentiated Schur bound

In this subsection, we transfer derivatives of the full Bessel vector to its discrete-series matrix coefficient by the reproducing identity of Section 12.3. This bounds the derivatives needed in Section 15.4.

Use the full vector W just constructed. Let π be the ambient discrete series and v its chosen unit highest vector, with $c(g) = \langle v, \pi(g)v\rangle$. Right translation R on the quotient satisfies $\langle W, R(g)W\rangle = D_\tau c(g)$, by Section 12.3. In particular the map $\pi(g)v \mapsto R(g)W$ extends linearly to a bounded intertwiner T with $\|Tz\|_2 = \sqrt{D_\tau}\|z\|$: the two Gram matrices on each finite list of translates agree up to D_τ, and v is cyclic by irreducibility.

The right derivatives of W are bounded continuous Bessel functions. Indeed take a smooth compactly supported approximate identity and average it on the left and right by the compact matrix coefficient projector onto $\mathbb{C}v$. The latter projector comes from Schur orthogonality and multiplicity one of the lowest compact type. Choosing a bump sufficiently near the identity yields a smooth compactly supported kernel ξ with $\pi(\xi)v = \alpha v$, $\alpha \ne 0$, since the bump operator tends strongly to the identity.

Thus $R(\xi)W = \alpha W$, first in L^2 by the intertwiner and then everywhere by continuity. The change $u = \exp(tX)g$ in $\int \xi(g)W(y\exp(tX)g)\,dg$ puts each derivative on the smooth kernel by left translation. Compact support and the boundedness of W justify all differentiations, and left Bessel covariance is preserved. Complex linear extension supplies the derivatives in the matrix units. For a word ω of length j denote the resulting function by W_ω and the corresponding derivative of v by v_ω; the intertwiner gives $W_\omega = Tv_\omega$ in L^2.

For the derivative norm, let V be the lowest compact type. The span V_j of derivatives of order at most j of vectors of V is the image of $\bigoplus_{i=0}^{j} \mathfrak{g}_{\mathbb{C}}^{\otimes i} \otimes V$ as a compact module. Its highest compact weights have coordinates of modulus at most $L_* + r + j$. Indeed weights of V lie between the smallest and largest highest-weight coordinate in each unitary block, by the highest-weight order and Weyl permutation symmetry.

An adjoint weight changes any coordinate by at most one. The compact Casimir on V_j is therefore at most $r(L_* + 2r + j + 1)^2$ and the ambient Casimir has absolute value at most $r(L_* + 2r + j + 1)^2$ by the same eigenvalue formulas as in Section 12.3. For a real Hilbert–Schmidt orthonormal Lie basis X_i adapted to the compact and noncompact summands, the energy identity gives

$$\sum_i \|d\pi(X_i)u\|^2 \le 3r(L_* + 2r + j + 1)^2\|u\|^2, \qquad u \in V_j.$$

For $X_i, X_l \in \mathfrak{u}(J)$, the relation $X_i^* = -JX_iJ$ and cyclicity give $\operatorname{tr}(X_i^* X_l) = \operatorname{tr}(X_l^* X_i)$. Thus (X_i) is also a complex Hilbert–Schmidt orthonormal basis. A matrix unit E_{ab} of the original basis becomes $M_{ab} = p_v^{-1} E_{ab} p_v = \sum_i c_i X_i$ in the signed basis of Section 8.3. Since p_v is Euclidean unitary, $\sum_i |c_i|^2 = 1$, and hence

$$\|d\pi(M_{ab})u\|^2 \le \sum_i \|d\pi(X_i)u\|^2 \le 3r(L_* + 2r + j + 1)^2 \|u\|^2.$$

As $\sqrt{3r}(L_* + 2r + 2n^2 + 1) \le b^5$ for large x, iteration gives $\|v_\omega\| \le b^{5j}$ for $j \le 2n^2$.

For $\phi \in L^1(dg) \cap L^2(dg)$, Schur orthogonality and Cauchy–Schwarz against an arbitrary unit vector give $\|\pi(\phi)v_\omega\|^2 \le \|\phi\|_2^2 \|v_\omega\|^2 / d_\tau$. Applying T gives the desired inequality in the quotient normalization,

$$\int_{U\backslash G} \left\| \int_G \phi(g) W_\omega(yg)\, dg \right\|^2 d\bar{g}(y) \le b^{10j} \frac{D_\tau}{d_\tau} \int_G |\phi(g)|^2\, dg, \qquad j \le 2n^2.$$

For integrable ϕ outside L^2 the right side is infinite. The estimate concerns the same W, chosen coordinates and Haar measure as the real sieve forms.

13 Adelic orbital comparison and definite masses

In this section, we combine the local estimates on the open rational cell and average at a fixed final vector. The definite lattice masses give the normalization for the moment comparison of Section 9.1.

13.1 The open rational cell and its finite restrictions

In this subsection, we identify the rational terms in the open range of the Petersson expansion. The fixed finite levels from Section 8 allow the local integrals to be compared on one set of places in Section 13.4.

Use the cut Bessel vector, the class representatives $c^\circ \in H(\mathbb{A}_{F_0,f})$ for $H = U(X_0)$, and the two class cutoffs of Section 8. Write $h_c^{\mathrm{rat}} = \iota_{\mathrm{rat}}(c^\circ)$ in the rational ambient basis and $h_c = B_{\pi_N}^{-1} h_c^{\mathrm{rat}} B_{\pi_N}$ in the glued lattice basis of Section 8.1. Products with h_c are taken in the latter basis. In the rational basis $f = (e_0, \dots, e_{\varrho-1}, z)$ we have $h(z,z) = -N$. The representatives fix all the hyperbolic pairs and z. An original term γ above (c, s, c') has $h_c^{-1} \gamma h_{c'}$ integral in the finite lattice, with reduction s. Thus the leading minors Δ_a of $(h(f_i, \gamma f_j))_{i,j}$ are integral. Their reductions are the minors of s, since the Gram entries are unchanged by the class translates. We take the order-zero minor to be 1. An open residue s has nonzero proper minors at one chosen place above p_0^{lev}, so $\Delta_a \neq 0$ for $1 \le a \le \varrho$.

Lemma 13.1 (Factorization on the open rational cell)**.** *If $\Delta_a \neq 0$ for $1 \le a \le \varrho$, with $\Delta_0 = 1$, the marked rational term γ has a unique factorization*

$$\gamma = u^{-1} w_p \sigma u', \qquad u, u' \in U(F_0), \quad \sigma \in G_\varrho(F_0), \qquad p_a = \Delta_{a+1}/\Delta_a \quad (a < \varrho). \tag{21}$$

Here $w_p e_a = p_a e_a^*$ and $w_p e_a^* = \bar{p}_a^{-1} e_a$.

Proof. At floor a, use the residual isometry γ_a fixing the earlier pairs. The pivot $c_a = h(e_a, \gamma_a e_a)$ is nonzero by the leading-minor elimination. More explicitly, clearing one pair makes the upper-left basis-product block (c_a) and its complementary block the next basis-product matrix. Since lower and upper unitriangular changes preserve each leading minor, the successive residual pivots are the stated ratios. Let $m = e_a^* + \omega + \mu e_a$ be isotropic. The matrix N_m sending e_a^* to m, fixing e_a, and sending a remaining vector v to $v - h(\omega, v)e_a$ is unitary, because $\mu + \bar{\mu} + h(\omega, \omega) = 0$.

Use $A = N_{c_a^{-1}\gamma_a e_a}$ and put $L = (Aw_a(c_a))^{-1}\gamma_a$, which fixes e_a. Then $B = N_{L^{-1}e_a^*}$ is defined and LB fixes the pair. Iterate $\gamma_a = Aw_a(c_a)(LB)B^{-1}$, using commutation of the earlier swaps with the later floor group. Each step is integral when its pivot is a unit and the residual matrix is integral: the column of N_m is integral, as is its inverse by the integral inverse Gram. This uses the column relation also above two. Uniqueness follows by writing two decompositions as $w_p \sigma v' = v w_q \tau$. Compare the e_a columns.

Their dual components have indices respectively at most and at least a, since v' and the lower dual block of v are unitriangular. The a-th coefficient gives $p_a = q_a$, and comparison of all components gives $v e_a^* = e_a^*$. A lower unipotent fixing all dual columns is the identity, by its block unitarity. Then $v' \in U \cap G_\varrho = \{1\}$ and $\sigma = \tau$. □

The link is the graph $x \mapsto \sigma^{-1}x$ in the middle quotient. Indeed a vector orthogonal to both isotropic bases of $w_p\sigma$ has zero hyperbolic coordinates, so the two quotient coordinates are x and $\sigma^{-1}x$. The left and right unipotents change the corresponding lifts by the original isotropic span, which preserves the quotient graph. Thus the link determines σ, independently of p. Put

$$t = h(z, \sigma z), \qquad v_0 = \mathrm{pr}_{X_0}(\sigma z), \qquad \sigma z = v_0 - (t/N)z.$$

If the discriminant is nonzero then $t\bar{t} - N^2 \neq 0$: the two last minors are $\Delta_\varrho = \prod p_a$ and $\Delta_{\varrho+1} = t\Delta_\varrho$, and the discriminant is $\Delta_\varrho \bar{\Delta}_\varrho (t\bar{t} - N^2)$. Moreover $Nh(v_0, v_0) = t\bar{t} - N^2$. By definiteness $|\Phi_v(t)| > N$ at every infinite place v. In particular the full minor is nonzero on this set.

All nonzero cut terms have bounded archimedean minors, with a bound depending on the chosen Bessel data. Indeed nonzero cut support at $y, \gamma^{-1}y$ implies $\gamma \in U_\infty \mathcal{K} \mathcal{K}^{-1} U_\infty$. Leading minors are unchanged by the two unipotents, and the original rational basis has fixed scales in this compact box. Thus each Δ_a lies in a finite set of bounded algebraic integers. Take $\mathcal{E}$ to consist of the quotients of the nonzero elements of this set with 1 included.

All p_a, t belong to $\mathcal{E}$. Choose the finite conjugation-stable set T containing the places above M, N, those needed for the lattice basis and its inverse, and the nonunit places of every element of $\mathcal{E}$. We may also include all places below the nonintegral entries of the finite list of class representatives and their inverses. There is one such T for all terms of the chosen Bessel data. The Fourier inversion below uses the global covolume v_E. The finite Haar factors enter through $\mathcal{R}_Z$ of Section 10.3 and cancel in the normalized integrals $\mathcal{I}_i^T / \mathcal{R}_Z$. The estimates for the remaining partial Euler products use $\zeta_{\mathcal{R}}(s) \leq \zeta_E(s)$ for $s > 1$, uniformly in T.

Outside it, elimination in the marked lattice of $h_c^{-1}\gamma h_{c'}$ is integral: its proper minors have inverses $\prod_{b<a} p_b^{-1}$ and hence are units. Uniqueness and commutation of the class representatives with the swaps identify the middle factor as $h_c^{-1}\sigma h_{c'}$. This proves integrality outside T of that translate and of the respective two translated unipotents.

For a cut block above (c, s, c') and link L, let $B^\flat_{c,c',s,L}$ be the sum of its terms on the nonzero-discriminant set. For each nonzero block the link determines its σ. The preceding factorization

gives its expansion over $p \in \mathcal{E}^{\varrho}$ of the orbit sums $\mathrm{CosAmp}^{\flat}_{c,s,c'}(w_p\sigma)$. These are the sums of the cut amplitudes over the unipotent factors in (21) and the original class and residue conditions. The block parameters give distinct tuples (c, c', s, σ). The absolute unfolding for these orbit sums is given below.

13.2 Comparison at a fixed final vector

In this subsection, we express the terminal pairing at a fixed last vector as a finite class sum. The calculation matches the source level of Section 6 with the measures of Section 10.10.

Take Z for the prescribed fixer from Section 10.10, and use its $W_Z, \mathcal{R}_Z$ and $V_p(l)$ from Section 10.4. Let

$$A_{c,c',\sigma} = h_c^{-1}\sigma h_{c'}, \qquad l = (A_{c,c',\sigma})_T^{-1}, \qquad X = (w_p A_{c,c',\sigma})_T.$$

Write $\widetilde{s} \in Z$ for the finite projection of an integral lift of $s \in \Gamma_{\mathcal{A}}$. Here and below the products with class representatives are read in the original lattice.

The rational unipotents whose class translates are integral outside T, modulo the class unipotent subgroup U_c of Section 8, project bijectively to $U(A_T)/(U(A_T) \cap J)$, with right cosets understood. For surjectivity, prescribe the desired coset of $U(A_T) \cap J$ at T and the integral unipotent subgroup at each place outside T. Conjugation by h_c carries their product onto a nonempty open set in $U(\mathbb{A}_{F_0,f})$. It meets $U(F_0)$ by the additive approximation in exponential coordinates of Section 8.2. Indeed the exponential and logarithm are mutually inverse polynomials over F_0 and preserve integral coordinates at almost all places. Thus the rational element has exactly the required class translate at T and is integral after translation outside it. For injectivity, two representatives with the same T-coset already have quotient in the principal level outside T and hence everywhere, since M is a unit off T. The kernel is exactly the defining subgroup U_c.

For the orbit round $w_p\sigma$, integrality outside T of the relative term is equivalent to integrality there of the two translated unipotents by the integral elimination above and the fact that each head is a unit off T. Thus the remaining residue condition on finite representatives a, b from these two unipotent quotients is $a^{-1}Xb \in \widetilde{s}J$. Only finitely many pairs qualify: if $a^{-1}Xb$ lies in this compact coset, the same fixed-pivot block elimination that proves compact support in Section 10.4 bounds a, b. Quotienting bounded sets by the open subgroup gives a finite set.

Within each such pair of cosets, sum the two cutoffs over the two rational class unipotent subgroups. If the central term is θ, the terms in those two sums have the form $v^{-1}\theta v'$. Change the real integration variable by v; the two corresponding cutoff sums are one by the partitions in Section 8. The class-unipotent character is trivial on these subgroups, since their finite translates are in the principal level.

This also follows from the adelic product of the additive characters at rational points. The remaining real tensor is $I^{\infty}(w_p\sigma) = \int F(y) \otimes F((w_p\sigma)^{-1}y)^* \, dy$, with its marked real embeddings understood. For $\theta = w_p\sigma$, the two cut-support conditions and the fixed nonzero pivots put the corresponding unipotents in bounded sets by the same block calculation on the compact ratio box. Thus $m_\theta(y) = \|F(y)\| \, \|F(\theta^{-1}y)\|$ is bounded and supported in a compact set. The two cutoff partitions and Tonelli's theorem give

$$\sum_{v \in U_c} \sum_{v' \in U_{c'}} \int \beta_c(vy)\beta_{c'}(v'\theta^{-1}y)\, m_\theta(y)\, dy = \int m_\theta(y)\, dy < \infty.$$

This justifies the interchange for the tensor entries, including when the integrated tensor vanishes.

13.2.1 Cancellation of the class phases

In this subsection, we combine the finite and real actions before applying the common-entry sum of Section 13.4.

The phase of the representative (a, b) is $\chi(a)\overline{\chi(b)}$. Indeed the finite and infinite characters on a rational unipotent are inverses, and conjugation by h_c preserves the root character because the definite Levi fixes the vectors e_i, z, e_i^* that give its simple-root entries. At places off T the root coordinate divided by M is integral and the finite character has conductor the integer ring. Changing variables on the real tensor then gives that phase, with the conjugate on the second tensor factor.

Let $\alpha_1, \alpha_2 \in Z/J$ with $\alpha_1\alpha_2^{-1} \in \widetilde{s}J$, and put $a_p^T = \mathcal{D}_\varrho^T(p, 1)$. The corresponding coefficient of $V_p(l)$ is $a_p^T \int \chi(y\alpha_1^{-1})\overline{\chi(X^{-1}y\alpha_2^{-1})}\, d\mu(y)$. Replace y by $y\alpha_2$. The integrand becomes $\chi(y\widetilde{s}^{-1})\overline{\chi(X^{-1}y)}$, using normality of J in K_T. A nonzero coset yJ has $yJ = a\widetilde{s}J = XbJ$ for a unique pair from the two unipotent quotients, and conversely. It contributes $\mu(J)$ times the same phase as in the real unfolding. Thus, with the real and finite Haar already fixed,

$$V_p(l)_{\alpha_1,\alpha_2}\, I^\infty(w_p\sigma) = a_p^T \mu(J)\, \mathrm{CosAmp}^\flat_{c,s,c'}(w_p\sigma). \tag{22}$$

Write I_v for the dimension index set of the local cut Bessel function at v, and c_G for the real open-cell constant of Section 11.3. In the real coordinates the squared basis scale is

$$s_{v,a} = \kappa_v^{2(\varrho-a)}/N,$$

where κ_v is the real scale for the additive character of level M. In fact the finite unipotent character on rational elements reads the root sum divided by M against the different-normalized character. Write δ_E for the skew generator of the different. At v, put $\kappa_v = M\Phi_v(\delta_E)/(2\pi i)$.

For the finite character at $-\mathfrak{s}(u)/(M\delta_E)$ the opposite archimedean phase is $\exp(c - \bar{c})$ with $c = \Phi_v(\mathfrak{s}(u))/\kappa_v$, where $\mathfrak{s}$ is the simple-root sum. Scaling e_a by $\kappa_v^{\varrho-a}/\sqrt{N}$, its dual by the reciprocal and z by $1/\sqrt{N}$ gives this same consecutive root coefficient: conjugation into the scaled basis multiplies each simple-root entry by κ_v^{-1}. Conjugating the Weyl element gives proper head entries $x_{v,a} = s_{v,a}\Phi_v(p_a)$; the last Gram entry is $x_{v,\varrho} = \Phi_v(t)/N$. With $d_a = k - 1 + 2(\varrho - a)$, set

$$R_0 = \prod_v \left(\frac{\sqrt{c_G}}{|I_v|} \prod_{a=0}^{\varrho} s_{v,a}^{d_a} \right), \qquad G(t) = \prod_v (|\Phi_v(t)|^2 - N^2)^{k-1}.$$

Let a_p^∞ be the product of the real tensor scales in Section 11.3.

For $a < \varrho$, the product formula gives

$$\mathrm{md}(p_a)^{1/2} \prod_v |\Phi_v(p_a)| = 1.$$

Consequently the proper-head factors cancel and

$$a_p^\infty a_p^T = R_0\sqrt{G(t)}. \tag{23}$$

The final factor comes from $(|x_{v,\varrho}|^2-1)^{(k-1)/2}=N^{-(k-1)}(|\Phi_v(t)|^2-N^2)^{(k-1)/2}$. There is no finite t-factor in a_p^T. The Gram matrix $\mathcal{G}_t(p,q)$ of the $a_p^\infty I^\infty(w_p\sigma)$ is the product of the top forms of Section 11.3. For all blocks with the same t this is one matrix, as the real top forms depend on their heads and t.

13.2.2 The common-entry pairing

In this subsection, we group the terms with the same final vector using the phase identity of Section 13.2.1.

Define

$$\mathcal{H}_t(l)=\sum_{\alpha_1,\alpha_2\in Z/J}\left\|\sum_p V_p(l)_{\alpha_1,\alpha_2}a_p^\infty I^\infty(w_p\sigma)\right\|^2.$$

With the Hermitian products linear in the second argument, its expansion is

$$\mathcal{H}_t(l)=\sum_{p,q}\mathcal{G}_t(p,q)\langle V_p(l),V_q(l)\rangle.$$

For fixed l, only $\mathcal{G}_t$ involves the real representative σ. The two compact factors in Section 11.3 act by the same unitary operators for every head, so the expansion depends on that representative only through t. Fix c,c',σ, and sum the comparison over the blocks with these parameters. For each α_1 and such residue s, there is exactly one α_2 with $\alpha_1\alpha_2^{-1}\in\widetilde{s}J$, since $\widetilde{s}\in Z$. The pairs for distinct s are disjoint. Equations (22) and (23) and nonnegativity give

$$\mu(J)^2[Z:J]R_0^2G(t)\sum_s\|B^\flat_{c,c',s,L}\|^2\le\mathcal{H}_t(l). \tag{24}$$

Now fix c,v_0,t. Write $B=B_{\pi_T}$ and let c° act on X_0 in its rational basis. In the original lattice coordinates the last column is

$$l^{-1}z=B^{-1}(((c^\circ)^{-1}v_0)_T-(t/N)z)=:e.$$

Indeed $\sigma z=v_0-(t/N)z$, the right class representative fixes z, and h_c is the matrix of c° in the lattice basis. The finite point correction η already cancels in the residue identity of Section 8.5. If $l_1l_2^{-1}\in J$, the quotient $h_{c_1'}^{-1}\sigma_1^{-1}\sigma_2h_{c_2'}$ is in the ambient principal level: it is integral outside T by the preceding elimination and in J at T. The rational element $\sigma_1^{-1}\sigma_2$ fixes the pairs and z, so belongs to $H(F_0)$. The pullback of the ambient principal level to H is the level J_H of Section 6.

For this lattice calculation use the glued generator $x_{k_0}/\bar\pi_N+z/N$ of the marked middle lattice. Write $x_{\mathrm{gl}}=x_{k_0}/\bar\pi_N+z/N$ for this lattice vector. Since $x_{k_0}=\bar\pi_Nx_{\mathrm{gl}}-z/\pi_N$, the intersection with X_0 is the integer lattice with distinguished coordinate in $\pi_N\widehat{\mathcal{O}}_E$. If an isometry h fixing z preserves the middle lattice, its columns away from k_0 are in this intersection, as is $(hx_{k_0}-x_{k_0})/\bar\pi_N$.

Applying the same fact to h^{-1} shows that h is integral with integral inverse on X_0. It fixes the distinguished column modulo $\bar\pi_N$, equivalently the distinguished covector modulo π_N by adjunction. Conversely an integral unitary matrix with this condition on the covector preserves the intersection. It has the column condition as well. The distinguished diagonal difference is divisible by both coprime factors, so $(hx_{k_0}-x_{k_0})/\bar\pi_N$ is in the intersection.

This proves preservation of the glued lattice, for h and its inverse. At $p_0, p_1 \mid M$, the integer N is a unit and the principal congruence is the original one modulo π_M. Hence the two source classes are equal, and the freeness in Section 6 then gives $\sigma_1 = \sigma_2$.

Thus these positions give disjoint cosets in the H-fiber over e. Indeed, fixing l_0 over e, use $(H \cap J) l l_0^{-1} \subset H$ for each position. On such a coset the integrand $\mathcal{H}_t(hl_0)$ equals $\mathcal{H}_t(l)$, since W_Z is invariant on the left under $H \cap J$. This invariance follows from its coordinate formula. The group H fixes the hyperbolic pairs and z. Its middle-block action normalizes U and preserves the simple-root character.

An element of $H \cap J$ acts trivially after moving past U and using right J-invariance. Write $\bar{H}$ for the finite reduction of the marked integral definite group and m_U for the order of the reduced integral unipotent. In the chosen Haar, $\mu_H(H \cap J) = |\bar{H}|^{-1}$ and $\delta_U = m_U^{-1}$, since each integral compact has mass one. The reductions are unchanged by projection to T, as representatives extend by the identity outside T. The integral Bessel subgroup is a product of its integral unipotent and definite factors, by reading the middle block. Its reduced image has order $|H_R| = m_U |\bar{H}|$. Thus

$$C_* = |H_R| m_U, \qquad \mu_H(H \cap J)\, \mu(J)^2 [Z : J] = \mathcal{R}_Z / C_*.$$

Let $\Psi_t(e) = \int_H \mathcal{H}_t(hl_0)\, dh$. This is nonnegative, since $\mathcal{G}_t$ is a Gram matrix. It is the sum of the top finite pairings $\Phi_{p,q}(e)$ of Section 10.4 weighted by $\mathcal{G}_t(p,q)$: the integrand is $\sum_{p,q} \mathcal{G}_t(p,q) \langle V_p(hl_0), V_q(hl_0) \rangle$ with both inner products linear in the second argument. It has a compactly supported, locally constant, nonnegative extension from the last-vector quadric to the ambient finite vector space by the compact step extension from that section, taking the positive part of the real sum. The extension can be chosen $H \cap J$-invariant by averaging.

On the quadric the fiber value is invariant since $V_p(lh) = V_p(l)$ for $h \in H \cap J$, by changing its integration variable using that same left invariance of W_Z. Summing (24) over the disjoint fiber cosets therefore proves

$$R_0^2 G(t)\, \frac{\mathcal{R}_Z}{C_*} \sum_{(c,c',s,L)\,:\,(c,v_0,t)\ \text{fixed}} \|B^{\flat}_{c,c',s,L}\|^2 \le \Psi_t(e). \tag{25}$$

This compares the original cut blocks with the same finite form at a fixed last vector. For the subsequent sum over (c, v_0), the rational norm to average is $h(v_0, v_0) = (t\bar{t} - N^2)/N$, and the vector $(c^\circ)^{-1} v_0$ is integral off T by the integral floor factor and the integral original column z.

13.3 Averaging at a fixed final vector

In this subsection, we average the finite class sum of Section 13.2 by interpreting the fixed last-vector value as a Hermitian representation problem. This supplies the terminal factor in Section 13.4.

Let $E^+ = F_0$, $d_C = \#\mathcal{C}$ for the marked source classes, and $v_E = 2^f / \sqrt{|D_E|}$, with D_E the field discriminant. Fix t as in Section 13.2 and put $\beta = (t\bar{t} - N^2)/N \in F_0$, totally positive. Write B_{π_T} for the finite lattice basis, with $\pi_T = \pi_N|_T$. On the finite adeles of X_0 take

$$F_t(y) = \mathbf{1}_{\{y \text{ integral off } T\}} \Psi_t\big(B_{\pi_T}^{-1}(y_T - (t/N) z)\big).$$

Here vectors in the finite model are read in the basis B_{π_T}; this is the same affine coordinate $e(t, y_T)$ as in Section 10.4. The affine embedding is closed, so the pullback of Ψ_t is locally

constant and compactly supported. Off T the indicated lattice is compact open. The embedding is equivariant for the source group since it fixes z, and Ψ_t is invariant under $H\cap J$ on the ambient space. Thus F_t is invariant under the original source level, which also preserves the lattice off T. On the selected (c, v_0, t), $F_t((c^\circ)^{-1}v_0) = \Psi_t(e)$.

We use Weil's Siegel formula for the orbit of one vector [75, §52, Theorem 5, (41)]. In this case the group is $U(X_0)$ over F_0, for the quadratic algebra E/F_0, and the rank condition is $k > 2$. The group has compact adelic quotient by definiteness. An original split finite place meets the local orbit condition: there the norm is $x \cdot y$ on pairs of vectors over $F_{0,v}$, with the general linear group on a vector and its dual.

The full-rank locus has both vectors nonzero, and the group is transitive at each value of the pairing by extending a vector and covector to an adapted basis. In particular the full-rank zero-norm locus is nonempty there. Formula (41) uses probability Haar. For a nonzero represented norm the stabilizer is $U(v_0^\perp)$, and the conversion from Tamagawa measure to probability Haar introduces the ratio

$$\frac{\tau(U(v_0^\perp))}{\tau(U(X_0))} = \frac{2}{2} = 1.$$

The two Hermitian spaces have dimensions $k-1$ and k over E, so [28, §2.5] applies to both unitary groups. This is also the stabilizer identity in [75, §53, (43)]; the same ratio occurs for each pair of consecutive ranks in Section 13.5. Probability Haar on the quotient assigns the same mass $1/d_C$ to each marked class, since each open class has Haar weight proportional to the reciprocal of its rational stabilizer order and these orders are one. Apply the formula to F_t at the finite places and to

$$\phi_\infty(y) = \prod_{v|\infty} \exp\bigl(\Phi_v(\beta) - h_v(y_v, y_v)\bigr)$$

at infinity. This Schwartz function is invariant under H_∞ and equals one on the represented norm fibre at every real place. The absolute convergence hypothesis of Weil's theorem holds for every adelic Schwartz function. For a factored test, translate its compact finite support by a compact set of adelic quotient representatives. The rational vectors then lie in one fractional lattice. At infinity the translates of its Schwartz factor have a uniform bound $C_a(1 + \|y\|)^{-a}$ for every $a > 0$. Choose $a > 2kf$ to get absolute and uniform convergence.

The measure on the norm fiber in (41) is the quotient of the additive vector measure by the additive norm-value measure, both with adelic covolume one [75, §44, Theorem 2]. We spell out its conversion. Use ordinary area on $\mathbb{C}$ at the complex places, so the product finite vector measure gives $\widehat{\mathcal{O}}_E$ mass v_E per coordinate, by the covolume of $\mathcal{O}_E$. At the real places of F_0 use ordinary Lebesgue measure in the norm value. In the finite norm-value Fourier variable $u \in \mathbb{A}_{F_0,f}$, take du_0 giving the finite inverse different of F_0 mass one.

The phase $\psi_{E,f}(u(h(y,y) - \beta)/2)$ is the standard F_0-phase, by the trace E/F_0. For this Fourier pairing the covolume-one measure is $\sqrt{|D_{F_0}|}\,du_0 = |D_E|^{1/4}\,du_0$: the archimedean covolume of the inverse different is $|D_{F_0}|^{-1/2}$, and E/F_0 is unramified at the finite places by the inertia calculation in Section 4.2. The real norm fiber in $\mathbb{C}^k$ has mass $\pi^k\Phi_v(\beta)^{k-1}/(k-1)!$, by differentiating the Euclidean ball volume in its squared radius. Accordingly Weil's formula gives

$$\sum_c \sum_{h(v_0,v_0)=\beta} F_t((c^\circ)^{-1}v_0) = d_C|D_E|^{1/4} \prod_v \frac{\pi^k\Phi_v(\beta)^{k-1}}{(k-1)!}\,\mathcal{D}_f(F_t,\beta), \tag{26}$$

where $\mathcal{D}_f$ is the finite norm density with the vector measure just specified and frequency measure du_0.

To bound the finite density, take out v_E^k by normalizing the integer vector box at every place to mass one. Put $b_u = -\delta_E u$, a skew frequency, where δ_E generates the different of E. Because E/F_0 is unramified or split at every finite place, the different of E is the extension of the different of F_0. Thus du_0 becomes the measure on skew frequencies giving the integral skew box mass one.

With $\psi_m(a) = \psi_{E,f}(-a/\delta_E)$ and an integral trace-one scalar τ, the phase on the norm difference is

$$\psi_{E,f}(u(h(y,y)-\beta)/2) = \psi_m(\tau b_u(h(y,y)-\beta)).$$

Indeed $\mathrm{Tr}_{E/F_0}(\tau) = 1$. The character ψ_m is self-dual on the integer ring and is trivial on $a+\bar{a}$, since δ_E is skew. In particular $\psi_m(\tau b_u(a+\bar{a})) = \psi_m(b_u a)$. It follows that the annihilator of the fixed integral ring for the skew-frequency pairing is the integral skew ring: testing against fixed traces is equivalent to testing b_u against all integral a. The two measures giving these boxes mass one are therefore dual. These identities hold also at places above two.

For a finite place v of F_0 off T, let R_v be the integer ring of the local quadratic algebra and let q_v be the residue size of $F_{0,v}$. Let $G_v(b)$ be the integral of $\psi_m(\tau b(h(x,x)-\beta))$ on R_v^k, with its probability measure. Squaring its absolute value and writing the two variables as $x, x+d$ leaves the inner average of $\psi_m(bh(x,d))$ times a unit phase depending on d.

Hence that average vanishes unless $bd \in R_v^k$, and

$$|G_v(b)|^2 \le [R_v : R_v \cap b^{-1}R_v]^{-k}.$$

The skew lattice has a unit generator. At skew valuation $-j < 0$ the index is q_v^{2j}, in both the unramified and split cases. The skew shell has volume $(1-q_v^{-1})q_v^j$, so its contribution after integrating G_v in absolute value is at most $(1-q_v^{-1})q_v^{-j(k-1)}$. The integral over the skew integer box is at most one. Thus each exterior factor contributes at most $(1-q_v^{-(k-1)})^{-1}$. Their product is bounded by

$$\zeta_{F_0}(k-1) \le \zeta(3\varrho+4)^f \le 1+2^{-\varrho} \le 1+b^{-6}.$$

Here $k-1 \ge 3\varrho+4$, $f \le 2^{2\varrho}$ and $\log\zeta(a) \le 2^{1-a}$ for $a \ge 2$; the last inequality uses the parameter bounds on b, n.

At T the Fourier integral is exactly the norm-fiber integral in Section 10.4, since its skew parameter is dual to the norm-value Haar with integer mass one. For absolute convergence of the Fourier inversion, decompose the compactly supported test into finitely many shifted boxes $x_0 + cR_v^k$ on each orbit of places, using products of such boxes across T. After the substitution $x = x_0 + cx'$, the same squaring argument bounds the integral by the conductor index for frequency $c\bar{c}b$, with the additional box volume factor.

In the squared modulus use the difference variable $d = x' - x''$ for the two scaled box coordinates. The shift multiplies each outer d-term by the unit phase of the cross term in x_0, cd. The inner x''-average has the linear frequency $c\bar{c}bd$, so it vanishes off the same conductor annihilator as in the centered box. Taking absolute values after this cancellation gives the stated bound. The skew integral is finite by $k-1 > 1$. For every finite v, the image β_v of the nonzero rational norm is nonzero. At $h(y,y) = \beta_v$, the differential of the norm and a right inverse are

$$u \longmapsto \mathrm{Tr}_{E_v/F_{0,v}} h(y,u), \qquad a \longmapsto a\tau\beta_v^{-1}y \quad (\mathrm{Tr}_{E_v/F_{0,v}}\tau = 1),$$

where $E_v = E \otimes_{F_0} F_{0,v}$. The norm is therefore a submersion, also at split and dyadic places. Fourier inversion gives the same continuous fiber measure as the norm charts.

Outside T, β is integral because N, t are units there. The Euler-product calculation can therefore first be made with integral skew frequencies at all but finitely many places, each such integral box contributing one in the normalized measure, and then extended by the preceding bounds on the absolute integrals. In total,

$$|\mathcal{D}_f(F_t, \beta)| \le (1 + b^{-6}) v_E^k \int_{h(y,y)=\beta} \Psi_t(e(t, y))\, d\nu_{\beta,T}(y). \tag{27}$$

Let Q_t be the sum against $\mathcal{G}_t(p, q)$ of the unnormalized finite last-entry densities of the two heads. By the definition of Ψ_t and the terminal disintegration, it is nonnegative and

$$\int_{h(y,y)=\beta} \Psi_t(e(t, y))\, d\nu_{\beta,T}(y) = \mathrm{md}(\pi_T) Q_t = N^{-f} Q_t.$$

Indeed $\mathrm{md}(\pi_T)^2 = \mathrm{md}(N) = N^{-2f}$, since $\pi_T \bar{\pi}_T = N$, T is conjugation-stable, and N is a unit off T. Together with the $N^{-f(k-1)}$ from $\prod_v \Phi_v(\beta)^{k-1}$, this gives the constant

$$C_{\mathrm{S}} = |D_E|^{1/4} v_E^k \left(\frac{\pi^k}{(k-1)!\, N^k} \right)^f.$$

By positivity the sum of $\Psi_t(e)$ over the selected pairs (c, v_0) is bounded by (26). Consequently (25) and (27), with $G(t) > 0$, give the fixed-entry estimate

$$R_0^2 \sum_{(c,c',s,L):\, t_L = t} \|B^{\flat}_{c,c',s,L}\|^2 \le (1 + b^{-6}) C_* d_C C_{\mathrm{S}} \frac{Q_t}{\mathcal{R}_Z}. \tag{28}$$

This applies to the original cut blocks on any finite set at t, so the bound holds for their nonnegative sum.

13.4 Adelic summation of the common-entry forms

In this subsection, we multiply the finite and real Fourier estimates with the same quotient measures. The local bounds from Sections 10.9 and 11.10 then lead to the open-cell estimate of Section 13.8.

Since $n = \varrho + 1 = m/2$, Section 4 gives $N \le b^{25m} = b^{50n} \le b^{60n}$. At depth i, take the product of the real forms $J_{v,i}$ and of the finite form $\mathcal{I}_i^T / \mathcal{R}_Z$ at weight one. At depth $n = \varrho + 1$ use the finite last-entry density in place of $\mathcal{I}_n^T$. Write the resulting form as J_i^{ad}. For a rational entry $t \in \mathcal{E}$, use its scale $a_{v,i} \Phi_v(t)$ at v, with $a_{v,i} = s_{v,i}$ of Section 13.2, and t_T at the finite places. These coordinates also define the scaled heads of a rational tuple.

Write $W_i(p)$ for the product of the real head weights in Section 11.10 and the finite head weight in Section 10.9. Define $w_i = \sum_{p \in \mathcal{E}^i} W_i(p)$ and

$$D_i = \sum_{\substack{p,q \in \mathcal{E}^i \\ p_{i-1} = q_{i-1}}} \Re J_i^{\mathrm{ad}}(p, q) \quad (i \ge 1),$$

$$D_0 = \Re J_0^{\mathrm{ad}} \le M_0 = (1 + b^{-6}) b^{-6k|\mathcal{A}|} \prod_v \frac{1 + b^{-10}}{\deg_v}.$$

The bottom bound uses the original Z and the real formal degrees $\deg_v$, by Sections 10.10 and 12.10. At each selected split pair, the finite bound uses the index $q^r - 1$ for either a vector or a covector stabilizer. Since the bottom masses factor over conjugate pairs, the same bound applies to every mixed choice of the two kinds of stabilizer. For $0 \le i < n$, let $F_i(x)$ be the real part of the sum of $J^{\mathrm{ad}}_{i+1}(p \parallel x, q \parallel x)$ over the two proper heads in $\mathcal{E}^i$. Here x is a common entry on the product of the real places and T, and $\parallel$ appends that entry in local coordinates.

We may replace a zero complex component or a zero component in a local field E_w, $w \in T$, by one in forming a head. Such replacements concern null sets for the integrals and do not affect the nonzero rational samples.

For fixed x with nonzero field components, append a common suffix u of length $n - i - 1$. Denote the resulting top matrix by

$$K_{pq}(x, u) = J^{\mathrm{ad}}_n(p \parallel x \parallel u, q \parallel x \parallel u), \qquad p, q \in \mathcal{E}^i.$$

When the suffix is empty, use unit mass. Each local matrix is a Gram matrix at the common last entry. Their entrywise product is the Gram matrix of the tensor products of the local vectors. Thus $K_{pp} \ge 0$ and $|K_{pq}|^2 \le K_{pp} K_{qq}$. By Tonelli and successive applications of the common-entry identity on the diagonal,

$$\int K_{pp}(x, u)\, du = J^{\mathrm{ad}}_{i+1}(p \parallel x, p \parallel x) < \infty.$$

Cauchy–Schwarz now gives $\int |K_{pq}|\, du \le (\int K_{pp}\, du \int K_{qq}\, du)^{1/2}$. The same diagonal calculation with x included among the integrated coordinates has value $J^{\mathrm{ad}}_i(p, p) < \infty$. Thus Fubini applies to every entry both for the suffix alone and for the full set of common coordinates. It gives

$$F_i(x) = \int \sum_{p,q} K_{pq}(x, u)\, du \ge 0, \qquad \int F_i = \Re \sum_{p,q \in \mathcal{E}^i} J^{\mathrm{ad}}_i(p, q).$$

The integrand in the first identity is a squared norm. For almost every u its field components are nonzero, and the top matrix entries are continuous in x near each prescribed rational point. Fatou's lemma therefore gives lower semicontinuity of F_i at those points. For a last entry t with $|\Phi_v(t)| > N$ at every real place, the nonnegative summand of D_n at t is $Q_t / \mathcal{R}_Z$ of Section 13.3.

Extend F_i to the adeles by setting

$$G_i(z) = \mathbf{1}_{\{z_f \text{ integral off } T\}}\, F_i((a_{v,i} z_v)_v, z_T), \qquad P_i = \frac{v_E}{\prod_v a_{v,i}^2}.$$

The additive Haar has E of covolume one, as in Section 13.3. The Fourier coefficient at rational y/δ_E vanishes unless y is integral off T, since the integral exterior translations preserve G_i. For such y a change of variables in the Fourier integral gives P_i times the transform of F_i at

$$\xi_v = \frac{2\pi i}{\Phi_v(\delta_E) a_{v,i}} \Phi_v(y), \qquad \xi_T = y_T.$$

If S_i denotes the complex sum defining $F_i = \Re S_i$, then

$$\widehat{F_i}(\xi) = \tfrac{1}{2}\big(\widehat{S_i}(\xi) + \overline{\widehat{S_i}(-\xi)}\big).$$

The character estimates at ξ and $-\xi$ have the same bound. The factor P_i is the real area Jacobian times the finite integer-box mass, since the projected finite integer box has mass one in the local forms. The sign convention is the standard adelic Fourier coefficient on the quotient with the normalized local characters of Section 13.2. At zero frequency this is $P_i \int F_i$.

13.4.1 The sum over nonzero frequencies

In this subsection, we combine the finite bounds of Section 10.9 with Lemma 11.1.

Set $K_* = k - n^7$, $X_v = |\kappa_v|^{2\varrho+1}/(MN)$, $B_{\mathrm{fr}} = b^{23(2\varrho+1)}$, and

$$\mathcal{D}_{\mathrm{dec}} = \prod_v \Theta\, X_v^{K_*},$$

where Θ is the constant in (19). For a nonzero y integral off T, its finite denominator index $D(y_T)$ is the absolute norm of its denominator ideal $\mathfrak{b}_y$. The product formula gives $\mathrm{md}(y_T)^{-1/2} \le \prod_v |\Phi_v(y)|$; the exterior moduli are at most one. Write $j = i$ for the Poisson frequency and $j = i - 1$ for unequal last entries. In either case $\xi_v = c_v \Phi_v(y)/a_{v,j}$, with $|c_v|^{-1} = |\kappa_v|/M$ in the first case and $c_v = M$ in the second. Since $1 \le |\kappa_v|$, $a_{v,j}/|c_v| \le X_v \le B_{\mathrm{fr}}$. The product formula gives

$$\mathrm{md}(y_T)^{-K_*/2} \prod_v |\xi_v|^{-K_*} = \prod_{\substack{w \nmid \infty \\ w \notin T}} |y|_w^{K_*/2} \prod_v (a_{v,j}/|c_v|)^{K_*} \le \prod_v X_v^{K_*}.$$

For the residual real damping use

$$\min(1, |\xi_v|^{-1})^6 \le B_{\mathrm{fr}}^6 \max(1, |\Phi_v(y)|)^{-6}.$$

Using $D(y_T)^{-6} \le D(y_T)^{-3}$, multiply (9) and the real bounds (20). The resulting product before its real constant $\Theta^f b^{2000 n^9 f}$ and its finite constant $b^{n^9 f}$ is bounded by

$$\left(\prod_v X_v^{K_*} \right) B_{\mathrm{fr}}^{6f} \frac{\prod_v \max(1, |\Phi_v(y)|)^{-6}}{(\mathrm{N}\mathfrak{b}_y)^3}.$$

For the sum of the last weights, fix a denominator ideal $\mathfrak{b}$. The elements with this denominator lie in the $\mathrm{N}\mathfrak{b}$ cosets of $\mathfrak{b}^{-1}/\mathcal{O}_E$. In each coset, a product of complex half-unit squares contains at most one element, since a nonzero integral difference cannot have modulus less than one at all complex places. If z lies in a square with corner s, $\max(1, |s|) \le 2 \max(1, |z|)$.

There are at most $(4R+1)^2 \le 25R^2$ such corners of modulus at most $R \ge 1$, by packing disjoint discs of radius $1/4$. The sum of $\max(1, |s|)^{-6}$ over corners is then at most $6 \int_1^\infty 25 R^{-5}\, dR < 38$. Summing first over the cosets and then the ideals gives

$$\sum_{0 \ne y \in E} \frac{\prod_v \max(1, |\Phi_v(y)|)^{-6}}{(\mathrm{N}\mathfrak{b}_y)^3} \le (64 \cdot 38)^f \zeta_E(2) \le (64 \cdot 38 e^4)^f.$$

Since $64 \cdot 38 e^4 \le b^3$ and $3 + 138(2\varrho + 1) \le n^9$, the common frequency sum, including its real and finite constants, is bounded by

$$\varepsilon_0 = b^{2002 n^9 f} \mathcal{D}_{\mathrm{dec}}.$$

It follows that the nonzero Fourier coefficients of the extension have absolute sum at most $P_i \varepsilon_0 w_i^2$.

For completeness, one-sided Poisson at the chosen entries follows directly from periodization. The nonnegative G_i is integrable, so its periodization over E is integrable on the compact additive quotient. Its Fourier coefficients are those just computed. Absolute summability makes

their Fourier series continuous, equal almost everywhere to the periodization by uniqueness of Fourier coefficients. Near the identity in a quotient chart, the periodization dominates the sum over the finitely many selected rational translates of G_i.

Lower semicontinuity at those translates gives the same inequality at the identity against the continuous representative. Consequently

$$D_{i+1} \le P_i \left(\Re \sum_{p,q \in \mathcal{E}^i} J_i^{\mathrm{ad}}(p,q) + \varepsilon_0 w_i^2 \right).$$

In the unrestricted sum, equal last entries give D_i. For two different last entries t, t', take $y = (t^{-1} - t'^{-1})/M$. It is nonzero and integral off T, and supplies precisely the unequal-head frequency: y_T and $M\Phi_v(y)/a_{v,i-1}$ at the indicated places. The singleton version of the frequency sum therefore bounds that pair by $\varepsilon_0 W_i(p) W_i(q)$. There is no unequal contribution at $i = 0$. Thus for $0 \le i < n$,

$$D_{i+1} \le P_i(D_i + 2\varepsilon_0 w_i^2).$$

A head of nonzero weight has integral partial products δ_j at T. Every head entry belongs to $\mathcal{E}$, whose elements are units off T by Section 13.2; hence each δ_j is integral globally and a unit off T. At each v, the initial segments of the scaled head have modulus at most b^{60n^2}. The inverse of the product of their scales is at most $N^i \le b^{60n^2}$. Hence $|\Phi_v(\delta_j)| \le b^{120n^2}$. For an initial segment with scale product c_v, the finite weight uses $\mathrm{md}(\delta_{j,T})^{-1/2} = \prod_v |\Phi_v(\delta_j)|$.

The real factors for this initial segment, before raising them to the power n^6, give the product

$$\prod_v \max\{|\Phi_v(\delta_j)|, c_v^{-1}\}.$$

It follows that $W_i(p) \le b^{120in^8 f}$. A head is recovered from its partial products. Packing these integer tuples using the maximum complex modulus gives at most $(1 + 2b^{120n^2})^{2if}$ heads of nonzero weight: the difference of distinct tuples has modulus at least one in some coordinate, so their radius-1/2 polydiscs have disjoint interiors. Therefore $w_i \le b^{130n^9 f}$. Also $a_{v,i} \le b^{46\varrho}$ and $v_E \ge b^{-f}$ by the root discriminant bound. Hence $\prod_{j<i} P_j \ge b^{-100n^2 f}$ and $w_i^2 \le b^{400n^9 f} \prod_{j<i} P_j$.

Each P_j is positive. Dividing the recurrence at depth i by $\prod_{j=0}^{i} P_j$ and summing gives

$$\frac{D_n}{\prod_{j<n} P_j} \le D_0 + 2\varepsilon_0 \sum_{i<n} \frac{w_i^2}{\prod_{j<i} P_j}, \qquad \prod_{j<0} P_j = 1.$$

Let

$$\mathcal{E}_+ = \{t \in \mathcal{E} : |\Phi_v(t)| > N \text{ for every } v \mid \infty\}.$$

With $2n \le b^{n^9 f}$, we obtain

$$\sum_{t \in \mathcal{E}_0} Q_t/\mathcal{R}_Z \le D_n \le \left(\prod_{i<n} P_i \right) (M_0 + b^{2403n^9 f} \mathcal{D}_{\mathrm{dec}}) \qquad (\mathcal{E}_0 \subseteq \mathcal{E}_+). \tag{29}$$

The first inequality follows from nonnegativity at each common last entry. For a set Ω of open residues in the prescribed stabilizer, take any finite collection $\mathcal{F}$ of original cut blocks (c, c', s, L) with $s \in \Omega$ on the nonzero-discriminant set. Take $\mathcal{E}_0$ to contain their last entries. Then (28) gives

$$R_0^2 \sum_{(c,c',s,L) \in \mathcal{F}} \|B^{\flat}_{c,c',s,L}\|^2 \le (1 + b^{-6}) C_* d_C C_{\mathrm{S}} \left(\prod_{i<n} P_i \right) (M_0 + b^{2403n^9 f} \mathcal{D}_{\mathrm{dec}}).$$

Taking the supremum over $\mathcal{F}$ gives the same bound for $R_0^2 \mathcal{L}(\Omega)$ in the notation of Section 9.1.

13.5 Consecutive masses of definite unitary lattices

In this subsection, we compare the genus masses in two consecutive ranks, retaining the integral automorphism weights. This supplies the factors used in Sections 13.3 and 13.7.

Write μ_j for the mass of the genus of the standard positive definite Hermitian lattice $\mathcal{O}_E^j$. It is the sum of the reciprocals of the integral automorphism orders over its isometry classes. For each rank $j+1$, the weighted count of pairs consisting of a lattice in this genus and a marked norm-one vector equals μ_j. Indeed such a vector w splits off $\mathcal{O}_E w$ by the integral projection onto its line.

Over a split finite place the two components of a norm-one vector are a vector and a covector of pairing one, and an adapted basis sends them to the standard pair. At an unramified nonsplit place the complement is unimodular. In a unimodular Hermitian lattice one can split off a unit-norm line by reduction: if every residue norm vanishes, polarization using all scalars and the nondegenerate residue trace pairing forces the residue form to vanish.

This works also in residue characteristic two. Surjectivity of the norm on unramified local units allows the unit on each diagonal line to be made one. Thus the complement is standard locally. Rationally, Witt extension carries the norm-one line to the standard line in the standard Hermitian space. Conversely a rank-j genus lattice with an adjoined marked standard line belongs to the rank-$j+1$ genus. These operations give inverse isometries of the respective objects and identify their automorphism groups.

Fix $k \le j < r$, so $j+1 > 2$. Apply the one-vector formula of Section 13.3 at the full integral level, with the indicator of the finite integer box and norm value one. Probability Haar weights each class by its reciprocal automorphism order divided by the total mass. The split-place and rank conditions hold as before. Let x_j be the finite norm density divided by v_E^{j+1}, in the frequency Haar giving the codifferent of F_0 mass one. The formula gives

$$\frac{\mu_j}{\mu_{j+1}} = |D_E|^{1/4} v_E^{j+1} \left(\frac{\pi^{j+1}}{j!} \right)^f x_j .$$

Write $l = j+1$, and let q_v be the residue cardinality at a finite place of F_0. Put $\chi_v = 1$ at a split place of E/F_0 and $\chi_v = -1$ at an inert place. These exhaust the finite places by Section 4.2. Over the residue field the number of solutions of $h(w,w) = 1$ is

$$A_v = q_v^{2l-1} - \chi_v^l q_v^{l-1} .$$

At a split place choose a nonzero vector and a covector taking value one on it. At an inert place, for a nontrivial additive character ψ and $a \neq 0$, $\sum_{z \in \mathbf{F}_{q_v^2}} \psi(az\bar{z}) = -q_v$, since every nonzero norm has $q_v + 1$ preimages. Orthogonality gives $A_v = q_v^{-1}(q_v^{2l} - (-q_v)^l)$. The derivative $u \mapsto \mathrm{Tr}(h(w,u))$ at a solution takes the value $\mathrm{Tr}(t)$ on $u = tw$. The trace from the split or unramified quadratic algebra is surjective. This holds also in residue characteristic two. Each solution therefore has q_v^{2l-1} lifts at every successive precision. With the vector and norm-value integer boxes both of mass one, the local density is $q_v^{1-2l} A_v$. The absolute convergence established in Section 13.3 gives

$$x_j = \prod_{v \nmid \infty} (1 - \chi_v^l q_v^{-l}), \qquad |x_j - 1| \le \prod_{v \nmid \infty} (1 + q_v^{-l}) - 1 \le \zeta_{F_0}(l) - 1 .$$

For $j \ge k$, using $f \le 2^{2\varrho}$, $k \ge 4\varrho + 12$ and $\log \zeta_{F_0}(l) \le f 2^{1-l}$, one has

$$|x_j - 1| \le 2^{-\varrho}, \qquad |\log x_j| \le 2^{1-\varrho} .$$

Define

$$\sigma_{v,l} = \frac{(2\pi)^l}{|\Phi_v(\delta_E)|^{l-1/2}(l-1)!}, \qquad \mathrm{fr}_v = N^k \prod_{j=k}^{r-1} \sigma_{v,j+1}.$$

Since $\prod_v |\Phi_v(\delta_E)| = \sqrt{|D_E|}$, the mass identity reads $\mu_j/\mu_{j+1} = x_j \prod_v \sigma_{v,j+1}$. The marked sizes $N \geq b^m$, $\log b = O(x^\epsilon)$, $m \geq \sqrt{x}$, $f \leq x^{12}$, and $r - k = m - 1$ give

$$(r-k)2^{1-\varrho} + 4f/N \leq \log(1 + b^{-9})$$

for all sufficiently large x, since each term on the left is $o(b^{-9})$. Taking the product over $k \leq j < r$ and using $\log(N^k/(N^k-1)) \leq 2/N$ yields

$$\left(\prod_v \mathrm{fr}_v\right) \mu_r \left(\frac{N^k}{N^k-1}\right)^{2f} \leq (1 + b^{-9}) N^{kf} \mu_k. \tag{30}$$

13.6 The class count and the unipotent cutoff measure

In this subsection, we compute the change in counting measure from the integral stabilizer to the source level subgroup. Its finite covolume combines with the orbital measure of Section 8 in the moment comparison of Section 13.7.

Let K_X be the stabilizer of $\widehat{\mathcal{O}}_E^k$ in $H(\mathbb{A}_f)$, and let H_{int} be its subgroup preserving the ambient marked lattice. The glued-column calculation of Section 13.2 shows that H_{int} is the last-covector stabilizer modulo π_N. At each of the f marked places above N, reduction from the split integral factor to $\mathrm{GL}_k(\mathbb{F}_N)$ is onto: lift the matrix over the valuation ring and put its inverse transpose at the conjugate place. The group of matrices over $\mathbb{F}_N$ acts transitively on its $N^k - 1$ nonzero covectors. The marked congruence at the two primes dividing M is independent of these conditions on the covectors since $N \nmid M$. Thus

$$[K_X : H_{\mathrm{int}}] = (N^k - 1)^f, \qquad [H_{\mathrm{int}} : J_H] = |\bar{H}|.$$

If a class at K_X has stabilizer Γ, its fiber at J_H is $\Gamma \backslash K_X / J_H$. The stabilizer acts freely on this fiber before quotienting, by Section 6. Summing the fiber sizes over the classes at K_X gives

$$d_C = |\bar{H}|(N^k - 1)^f \mu_k. \tag{31}$$

For the class cutoff let $B_c = h_c^{\mathrm{rat}} B_{\pi_N} = B_{\pi_N} h_c$ be the original finite adelic lattice basis in the rational basis $e_a; x_j, z; e_a^*$. Thus U_c consists of the rational unipotents congruent to one modulo M in this basis. Here reduction modulo the rational integer M is equivalent to reduction at the marked half of its split places, by unitarity at the conjugate places. Use $a_{v,a} = s_{v,a}$ of Section 13.2. At a real place, write $b_{v,j}$ for the scale of a basis column. The scales on e_a, e_a^* are $\kappa_v^{\varrho-a}/\sqrt{N}$ and its inverse, those on x_j are one, and that on z is $N^{-1/2}$. The real image of a rational matrix is conjugated by this diagonal change of basis.

The free entries of covector e_a are at e_b, e_b^* for $b > a$ and in the middle block. Write $r_a = k + 1 + 2(\varrho - 1 - a)$ for their number. The other coordinate in this covector is the imaginary part at e_a^*. To obtain the coordinate chart, take the product of the real Heisenberg factors at $0, \ldots, \varrho - 1$ in that order. On $F_a^\perp / F_a$ for $F_a = \langle e_b : b < a \rangle$, the earlier factors act trivially. With the later factors fixed, the free covector from the factor at a is $-h(w, \cdot)$ on the remaining space, composed with the later isometry, and its imaginary coordinate is the imaginary part of the Heisenberg center in the half-trace chart.

The later factors fix e_a^*. The Gram matrix on the remaining space in the normalized basis is a signed permutation matrix. Thus the covector map and composition with the later isometry have real Jacobian of absolute value one, and the center also has Jacobian one. Descending in a gives the inverse chart and shows that the original product of Heisenberg Haar measures is ordinary product Lebesgue measure in these coordinates.

Let $B_{c,a}$ be the block of B_c on the free columns of covector a. The earlier basis and dual columns of B_c are unchanged, so the free covector of $(u-1)/M$ for $u \in U_c$ lies in

$$L_{c,a} = \{\ell \in E^{r_a} : \ell B_{c,a} \in \widehat{\mathcal{O}}_E^{r_a}\}.$$

Every such free covector is attained with all later source space coordinates zero. Extend ℓ by zero and write it as $h(\omega, \cdot)$ in the original rational Gram form, conjugate-linear in the first argument.

The class lattice is self-dual because the glued vector and z span a plane with Gram determinant -1, and the other summands are unimodular. Thus ω lies in it. In particular $h(e_a, \omega) = 0$. Choose an integral trace-one scalar τ, which exists since E contains the sixth roots of unity. For an integral skew scalar s put $C = Ms - M^2\tau h(\omega, \omega)$. The map

$$g(w) = w + Me_a h(\omega, w) - M\omega h(e_a, w) + Ce_a h(e_a, w)$$

is an isometry.

The change in the pairing on two vectors is $(C + \bar{C} + M^2 h(\omega, \omega))$ times the product of their two $h(e_a, \cdot)$ coordinates with conjugation on the first, and hence zero. It is integral and congruent to one modulo M in the class lattice. Its free covector at a is $M\ell$. Its changes in other top covectors are confined to column e_a^*, which is neither free nor central in any covector $b > a$.

Setting $\omega = 0$ realizes every central shift Ms. Conversely with the free covector at a zero, that center entry is skew: multiplication of the matrix with its adjoint gives $CA^* + AC^* + BJ_0^{-1}B^* = 0$ for the upper blocks A, B, C and the middle Gram J_0. Here covector a of A is the identity covector and that of B is zero. The center entry divided by M is integral since e_a^* is an unchanged class-basis column.

These lattices in the successive coordinates give a strict right fundamental domain. To see the translation rule, if all covectors of $u \in U_c$ with index $b > a$ have their coordinates zero, its matrix on the tail quotient after a is the identity, by the real chart just described. Thus right multiplication by u adds its free coordinates in covector a, leaving the later source spaces unchanged.

If its free coordinates in covector a also vanish, the tail block including covector a has $A = I$ and $B = 0$; its C is skew-Hermitian. In this block the other entries in column e_a^* vanish, so right multiplication adds the central coordinate. Reduce in descending order of the covectors, free coordinates before center, to half-open fundamental parallelepipeds of the two additive lattices. The isometries above supply every needed translation. For uniqueness, compare two such reduced points in a right coset, starting at the last covector. The change in each successive coordinate belongs to its lattice, and the parallelepiped forces it to be zero. The coordinate chart then identifies the points.

The middle determinant of $B_{c,a}$ is $\det(c^\circ)/\bar{\pi}_N$, since the glued column replaces x_{k_0} by $x_{k_0}/\bar{\pi}_N + z/N$. Its finite modulus is N^f: the unitary determinant has modulus one and $\pi_N \bar{\pi}_N = N$. Hence the rational lattice $L_{c,a}$ has Minkowski covolume $N^f v_E^{-r_a}$, using the finite box under $B_{c,a}^{-1}$ and the covolume-one additive adelic Haar. The real changes of variables in that covector

have complex scales $Mb_{v,j}/b_{v,e_a}$. The scales at the two columns of every later hyperbolic pair cancel, while the z-column has squared scale N^{-1}. Thus the free and central covector measures of the box are

$$\left(v_E^{-1}\prod_v \frac{M^2}{a_{v,a}}\right)^{r_a}, \qquad |D_E|^{1/4}\prod_v \frac{M}{a_{v,a}},$$

respectively. For the second formula, the skew integers are $i\mathcal{O}_{F_0}$, with imaginary components equal up to signs to the embeddings of F_0. The unscaled center has covolume $\sqrt{|D_{F_0}|} = |D_E|^{1/4}$.

Put $d_U = \sum_a r_a = \varrho(k+\varrho)$ and $\Lambda_0 = \prod_{v,i<n} a_{v,i}^{k+2(\varrho-i)}$. At each split residue pair of characteristic $p \mid M$ the free covectors and centers have $2d_U + \varrho$ scalar coordinates over $\mathbb{F}_p$. All choices lift to the local integer lattice: use the paired block chart, with the quadratic correction given by an integral trace-one scalar in the center. At the primes dividing M the middle lattice is unmodified since N is a unit. Thus $m_U = M^{f(2d_U+\varrho)}$. Multiplication of the covector measures, using $r_a + 1 = k + 2(\varrho - a)$ and $a_{v,\varrho} = 1/N$, gives

$$\nu\Lambda_0 = m_U N^{-kf} v_E^{-d_U} |D_E|^{\varrho/4}. \tag{32}$$

The inverse of the right fundamental box is a left fundamental box of equal measure, by unimodularity. Since the translates of the marked nonnegative cutoff β_c sum to one as in Section 8, Tonelli gives $\int_U \beta_c(uy)\,du = \nu$ for every y, in the specified real Haar. The value is independent of the original class c.

13.7 The main and identity moments on the marked source

In this subsection, we express both normalization constants using the source class count and the dimensions of the real compact types. The mass comparison of Section 13.5 ensures that the ratio used in Section 9.1 is for the same measure.

Write $d_B = \prod_v |I_v|$. The source classes are free at J_H, and each class value is the tensor product of the compact types of Section 12.10. Thus $d_X = d_C d_B$. From $|H_R| = m_U|\bar{H}|$ and (31), the definitions of Section 9.1 give

$$R_* := \frac{D_{\mathrm{main}}}{D_{\mathrm{id}}} = \frac{d_B(N^k-1)^f \mu_k}{\mathcal{B}\mu_r m_U}.$$

Set $B = \log b$, $Q_0 = M\,\mathrm{rd}(E)/(2\pi)$, and let S_κ be the sum of the logarithms of the weights of Section 4 over the f infinite places. The marked choice of N reads

$$\log N = mS_\kappa/f + (m-1)\log Q_0 - \tfrac{15}{2}B + \delta, \qquad 0 \le \delta \le \sqrt{B}.$$

Set

$$W_* = \log\mathcal{B} + f\sum_{l=k-1}^{r-1}\log(l!), \qquad \varepsilon_m = \log(\mu_k/\mu_r) - \sum_{j=k}^{r-1}\sum_v \log\sigma_{v,j+1}.$$

Section 13.5 gives $|\varepsilon_m| \le (m-1)2^{1-\varrho}$. Lemma 12.2 gives

$$|kmS_\kappa - W_*| \le m^2 fB + fmr/x.$$

Using $\sum_{j=k}^{r-1}(j+1)=(m-1)(k+\varrho+1)$ and $m_U=M^{f\varrho(2k+2\varrho+1)}$, the factorials in the moment ratio cancel except at $k-1$. More explicitly,

$$\begin{aligned}\log R_*-fk(\log M-\tfrac{15}{2}B)-\log\bigl(d_B((k-1)!)^f\bigr)\\ =fk\delta+f\log(1-N^{-k})+\varepsilon_m+kmS_\kappa-W_*\\ -f(m-1)\left(\varrho\log Q_0+\log\frac{\sqrt{\mathrm{rd}(E)}}{2\pi}\right).\end{aligned}$$

The right side is $o(fkB)$. Indeed the first term is at most $fk\sqrt{B}$, the second has absolute value at most $2f/N$, and the last term is $O(fm^2B)$ by the parameter bounds on Q_0 and the root discriminant. Here $m^2/k=O(x^{-4})$ and r/k is bounded.

In the marked compact type, the dimension of the symmetric-power representation of weight a_v is $\binom{a_v+k-1}{k-1}$. For $a_v\le a=xb$ it is at most $(a+k)^k$, also when $a_v=0$. Since $(a+k)k\le b^{5/4}$ for large x,

$$1\le d_B((k-1)!)^f\le b^{5kf/4}.$$

Using the logarithmic error bound $fkB/4$ and $M\ge b^{20}$ now gives

$$b^{11kf}D_{\mathrm{id}}\le D_{\mathrm{main}}\le(M/b^6)^{kf}D_{\mathrm{id}}. \tag{33}$$

13.8 The open-orbit bound at equal parameters

In this subsection, we insert the adelic estimate of Section 13.4 into the normalized open orbital sum. The mass identities in Sections 13.5 and 13.7 determine its constant.

Proposition 13.2 (Open-orbit bound for a prescribed stabilizer)**.** *Let Ω be a finite set of open residues in the marked stabilizer $\Gamma_{\mathcal{A}}$. On the nonzero-discriminant set, the cut blocks over these residues satisfy*

$$\nu^{-2}\mathcal{L}(\Omega)\le(1+b^{-4})b^{-6k|\mathcal{A}|}D_{\mathrm{main}}.$$

For each residue it includes all rational links, with the squared norm taken separately for each link of the cut kernel. Passage to the uncut kernel uses (36)*.*

Proof. Set $\Delta_0=R_0^{-2}\prod_{i<n}P_i$ and $C_0=C_*d_CC_{\mathrm{S}}$. The definitions of R_0 and P_i give the exact ratio

$$\Delta_0=\frac{v_E^n}{c_G^f}\left(\frac{d_B}{\Lambda_0}\right)^2.$$

The successive unitary compact volumes in the floor Haar satisfy $\operatorname{vol}U(l)/\operatorname{vol}U(l-1)=2\pi^l/(l-1)!$, the unit sphere area. Hence from the formulas for c_G, fr_v and C_{S},

$$\left(\prod_v\mathrm{fr}_v\right)c_G^f|D_E|^{\varrho/2}=C_{\mathrm{S}}N^{2kf}v_E^{n+2d_U}.$$

For this substitution use $\prod_v|\Phi_v(\delta_E)|=|D_E|^{1/2}$ and $\sum_{l=k+1}^{r}l=(2\varrho+1)(k+\varrho+1)$. Using $C_*=m_U^2|\bar H|$ and (32) now gives

$$C_0\Delta_0=\nu^2|\bar H|d_Cd_B^2\prod_v\mathrm{fr}_v\le(1+b^{-9})\nu^2D_{\mathrm{main}}\mathcal{B}.$$

Indeed, dividing the equality by $\nu^2 D_{\mathrm{main}}\mathcal{B}$ and using (31) gives

$$\frac{\mu_r \prod_v \mathrm{fr}_v}{(N^k-1)^f \mu_k} = \left(\frac{N^k}{N^k-1}\right)^f \prod_{j=k}^{r-1} x_j^{-1} \le (1+b^{-9})(1-N^{-k})^f$$

by (30). At each split place of F_0 over N, the inverse product in this ratio, including $(1-N^{-k})^{-1}$ from the covector index, is $\prod_{l=k}^{r}(1-N^{-l})^{-1}$. In the original floor Haar, the corresponding compact volume has the reciprocal value. Indeed the last floor has rank $k+1$, and the two components of z are primitive since $h(-x_{\mathrm{gl}}, z) = 1$ for the glued column $x_{\mathrm{gl}} = x_{k_0}/\bar{\pi}_N + z/N$. In split coordinates a primitive column and covector of pairing $-N$ can be taken to e_1 and $(-N, 1, 0, \dots, 0)$ by an integral basis change. Thus the residue count at the last floor is $(N^{k+1}-1)(N^k-1)$. The pairing has surjective differential, so division by N^{2k+1} gives the factors of ranks $k, k+1$. The earlier isotropic floors give the factors of ranks $k+2, \dots, r$ by the same split count as in Lemma 10.2.

For the decay term, put $s = \sqrt{\mathrm{rd}(E)}$ and $X_* = Q_0^{2\varrho+1}/(MN)$, so that $\prod_v X_v = X_*^f$. Write $h = 2\varrho+1$ and $S = \sum_{l=k+1}^{r} l = h(k+\varrho+1)$. The sphere areas give

$$\Theta \prod_{l=k+1}^{r} \sigma_{v,l} = 2^{S-h}\pi^{hk}|\Phi_v(\delta_E)|^{-S+h/2}.$$

Use $D_{\mathrm{id}} = m_U|\bar{H}|d_C d_B$, $m_U = M^{f\varrho(2k+h)}$, $\prod_v |\Phi_v(\delta_E)| = s^{2f}$ and $Q_0 = Ms^2/(2\pi)$. Then

$$\frac{C_0\Delta_0\mathcal{D}_{\mathrm{dec}}}{d_B D_{\mathrm{id}}\nu^2} = \frac{\mathcal{D}_{\mathrm{dec}}}{m_U}\prod_v \mathrm{fr}_v = \left((2/M)^{\varrho h} s^{-h^2} X_*^{-n^7}\right)^f \le b^{n^9 f}.$$

Here $M \ge 2$, $s \ge 1$ and $X_*^{-1} \le MN \le b^{23+60n}$ since $Q_0 \ge 1$. For large n, $(23+60n)n^7 \le n^9$. Also $d_B \le b^{2kf}$, $|\mathcal{A}| \le f$, and $k \ge 512000n^9$ by the rank bounds. Consequently

$$2404n^9 f + 2kf + 10 + 6k|\mathcal{A}| \le 11kf.$$

Applied to (33), this gives

$$C_0\Delta_0 b^{2403n^9 f}\mathcal{D}_{\mathrm{dec}} \le b^{-10}\nu^2 b^{-6k|\mathcal{A}|}D_{\mathrm{main}}.$$

For any finite set of open cut blocks on the nondegenerate orbital set, combine the two estimates with (29) and the summed fixed-entry bound immediately following it. The degrees in M_0 cancel $\mathcal{B}$, and $(1+b^{-10})^f \le 1+b^{-9}$ for large x since $f/b \to 0$. The remaining multiplier on $b^{-6k|\mathcal{A}|}D_{\mathrm{main}}$ is at most

$$(1+b^{-6})((1+b^{-9})^2(1+b^{-6})+b^{-10}) \le 1+b^{-4}.$$

For any choice of a vector or covector stabilizer at each place of $\mathcal{A}$, take the supremum over finite sets of blocks to obtain

$$\nu^{-2}\mathcal{L}(\Omega) \le (1+b^{-4})b^{-6k|\mathcal{A}|}D_{\mathrm{main}} \tag{34}$$

on the open part of $\Gamma_{\mathcal{A}}$ of Section 9.1, with Z formed from those same stabilizers as in Section 10.10. □

14 Truncation, finite projections, and lattice pair counts

In this section, we sum the truncation errors from Section 12.4, control the residue projections, and count pairs of lattices in the Petersson expansion. These estimates give the finite sums used in Section 15.

14.1 Reduced bases and the truncation error

In this subsection, we choose representatives for rational isometries in the Petersson expansion with controlled norms. This lets the real truncation estimate of Section 12.4 be summed in the full pairing of Section 9.1.

Write $B = B_{\pi_N}$ and let γ_f^{rat} denote the finite matrix in the rational Witt basis. Put $\gamma_f = B^{-1}\gamma_f^{\mathrm{rat}}B$. In the two class bases of Section 13.6, its matrix is

$$B_c^{-1}\gamma_f^{\mathrm{rat}}B_{c'} = h_c^{-1}\gamma_f h_{c'}.$$

Write $\mathcal{T}(c, s, c')$ for the rational isometries whose matrix in these class bases preserves the integral lattice and reduces to s modulo J. We specify the class cutoff used in the orbital series. Set $A = \langle e_0, \dots, e_{\varrho-1}\rangle_E$. Write $F = (e_0, \dots, e_{\varrho-1}, z)$ for the ordered basis of $A \oplus Ez$ and U_c^0 for its pointwise stabilizer in U_c. The action of U_c on F has image the upper unitriangular matrices whose strict upper entries belong to $M\mathcal{O}_E$.

The containment follows because the basis columns and the top covectors are unchanged in the integral class basis. For surjectivity, realize an elementary entry Ma at covector i, column $j > i$, by the integral Eichler lift of Section 13.6. For $f_j = e_j$ the coordinate covector $e_j^\vee$ is integral on the lattice. For $f_j = z$, write c° for the finite class representative on X_0. The coefficient of z in the glued lattice basis on the middle space is

$$\ell_c(x + tz) = t - \pi_N^{-1}\big((c^\circ)^{-1}x\big)_{k_0}.$$

It takes values zero and one on the glued column and on z, respectively, and vanishes on the other middle basis columns. Extend it by zero on the hyperbolic pairs. Approximate only its $X_0^\vee$ entries by rational scalars so the resulting covector stays integral on the class lattice. This is possible by additive approximation in the finite adeles, since the finite integer box is open. The covector is supported on the free columns of covector i. Multiply it by a and apply the Eichler lift. Its restriction to the basis adds just the stated elementary entry.

Choose a measurable set P of representatives for archimedean tuples modulo $M\mathcal{O}_E$, bounded in each coordinate by b^{26}. The $2f$ elements $\zeta_{q_1}^j$, $\sqrt{-d}\zeta_{q_1}^j$ for $0 \le j < f$ span a full sublattice of $\mathcal{O}_E$ by linear disjointness. Its multiple by M has a half-open parallelepiped bounded by $2fM\max(1, \sqrt{d}) \le b^{26}$. On this parallelepiped the quotient of $M\mathcal{O}_E$ by the smaller lattice acts by translation and reduction. Choose the lexicographically first point in each finite orbit in real coordinates. This gives the measurable P.

At each infinite place write the normalized columns $y_v^{-1}f_a$ as their unnormalized orthogonal Gram–Schmidt columns times an upper unitriangular matrix. Read the matrix $T_{\mathrm{GS}}(y)$ back in the rational basis by multiplying entry a, j by $b_{v,f_a}/b_{v,f_j}$. Call y reduced if the strict upper entries, as tuples over the infinite places, lie in P. Replacing y by uy multiplies $T_{\mathrm{GS}}(y)$ on the right by the inverse basis matrix of u.

In fact such a column operation preserves the orthogonalized vectors. Right multiplication reduces each column in descending order of the covectors. The entry added in covector i is a

lattice shift plus terms determined by covectors with larger index. The condition in P determines that shift uniquely. Thus the elements of U_c taking y to a reduced point form exactly one coset of U_c^0.

Multiply the reduced indicator by a nonnegative cutoff for U_c^0 whose translates sum to one. A measurable section exists by taking local sections of the discrete free left action on G_∞ over a countable cover of its quotient and partitioning by first chart. The product is β_c, with the orbit sum of Section 8 and the integral of Section 13.6.

Where β_c is nonzero, the normalized upper entries have modulus at most b^{26}. Indeed the basis scales have nonincreasing moduli in column order. The orthogonalized columns are invariant under the Bessel subgroup acting on the left. Indeed that subgroup fixes each successive line of the basis flag, with diagonal coefficients one. Each of those orthogonalized columns has norm at most the length $\ell_v(y_v)$ of Section 12.4. On any Bessel translate its norm is bounded by the original column norm, which is at most the operator norm of the translated matrix, since $\|g^{-1}\| = \|g\|$ in the normalized basis. Summing at most $n \le b$ contributions gives

$$\|y_v^{-1} f_j\| \le b^{27} \ell_v(y_v).$$

Put $H_v(y_v) = \max(1, \ell_v(y_v)/b^{6n})$ and $L(y) = \prod_v H_v(y_v)$. Fix c, s, y. For $R \ge 1$ and a term (c', γ) with $\beta_{c'}(\gamma^{-1} y) > 0$ and $L(\gamma^{-1} y) \le R$, the transported basis columns have norm at most $B_R = b^{37n} R$. For two such terms let $y' = \gamma_1^{-1} y$ and $g = \gamma_1^{-1} \gamma_2$, an integral transporter between their right classes reducing to one. If the maximum distance of the two tuples of columns is less than $(N B_R)^{-1}$, then $h(f_i, g f_j) = h(f_i, f_j)$ for every i, j. These pairings are integral on the class bases. At each v the normalized pairing scales the rational one by a number of modulus at least $1/N$. Pair $(y'_v)^{-1} f_i$ with the difference of the two transported columns. The absolute value, after division by the scale, is bounded by $N B_R$ times their distance. Every conjugate of the integral difference then has modulus less than one.

The equality of pairings gives zero dual and z coordinates of $g f_j - f_j$. Invariance of the norm and definiteness give zero X_0-component. Descend in the remaining top covectors. Once the later source spaces agree, a real unipotent moving y'_v to operator norm at most $b^{6n+2} R$ leaves the coefficient in the current covector unchanged by unitriangularity. The modulus of that coefficient in the rational basis is therefore at most $b^{23\varrho} b^{6n+2} R$ times the column distance, since the reciprocal of the basis-scale ratio is at most $b^{23\varrho}$. The bound is less than one. The coefficient is integral because the class basis fixes the basis columns and top covectors. Its field norm forces it to vanish. Thus $gF = F$ pointwise.

If two terms have the same rational basis image, their relative term fixes F and lies in the basis-fixing unipotents times H. Write this relative term as $u \iota_H(a)$ and the right classes as d, d', with representatives d°, d'° on X_0. Its finite class translate belongs to J, so its middle diagonal block gives $(d^\circ)^{-1} a_f d'^\circ \in J_H$ by Section 8.2. The double cosets give $d = d'$, and Lemma 6.1 now gives $a = 1$. Thus the relative term is in $U_{c'}^0$. The terms with this basis image inject into translations by $U_{c'}^0$, so their cutoff weights sum to at most one by the orbit sum.

Distinct basis images give separated points in real dimension $2nrf$, inside a ball of radius $2B_R$ about one of the points. Disjoint balls of radius $(2NB_R)^{-1}$ bound their number by $(1 + 4NB_R^2)^{2nrf}$. Since $N \le b^{60n}$, this is at most $(b^{6n^2} R^2)^{2nrf}$ for large n. Writing $I = r^3 f > 8nrf$ and integrating the layers against $IR^{-I-1}\,dR$ on $[1, \infty)$ gives

$$\sum_{c'} \sum_{\gamma \in \mathcal{T}(c,s,c')} \beta_{c'}(\gamma^{-1} y) L(\gamma^{-1} y)^{-I} \le 2 b^{12 n^3 rf}. \tag{35}$$

Let $W = \bigotimes_v W_v$ and $W^\flat = \bigotimes_v F_v$ be the full and cut Bessel functions. Section 12.4 supplies majorants w_v with $A_v = \sup w_v$ and

$$H_v^{3I}\|W_v\| \le w_v, \qquad A_v \int_{U_v \backslash G_v} w_v \le b^{400r^2}|I_v|.$$

Put $w = \prod_v w_v$ and $A_* = \prod_v A_v$. The majorants are invariant under the corresponding real unipotent groups by the graph integral. For $W^\sharp = W, W^\flat$ write $K_\gamma^\sharp(y) = W^\sharp(y) \otimes \overline{W^\sharp(\gamma^{-1}y)}$. The kernels agree unless $L(y)L(\gamma^{-1}y) \ge b$. On the support of their difference, the majorants give

$$\|K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y)\| \le 2b^{-2I} w(y) A_* L(\gamma^{-1}y)^{-I}.$$

The same majorant with the factor $2b^{-2I}$ omitted bounds either kernel everywhere.

Fix a subset S_0 of the rational terms off the Bessel subgroup. Define a block by integrating $\beta_c(y)\beta_{c'}(\gamma^{-1}y)K_\gamma^\sharp(y)$ and summing over the terms of $\mathcal{T}(c, s, c')$ in S_0. Tonelli, (35), and the integral $\int_{U_\infty} \beta_c(uy)\,du = \nu$ show absolute convergence over all right classes and terms for fixed c, s. For the norm of the difference the same argument gives

$$4\nu b^{-2r^3 f + 12n^3 r f + 400 r^2 f} d_B \le \tfrac{1}{2}\nu b^{-k^3 f}.$$

Indeed Section 13.7 gives $d_B((k-1)!)^f \le b^{5kf/4}$, hence $d_B \le b^{2kf}$. The bounds $n \le \sqrt{x}$, $x^5 \le r$, $k = r + 1 - m$ give $12n^3 + 500 \le k < r$ for sufficiently large x. Hence

$$12n^3 r + 400r^2 + 2k + 1 \le 402 r^2 \le r^3, \qquad k^3 \le r^3.$$

Since $b \ge 8$ and $f \ge 1$, the bound holds simultaneously for every subset of terms.

For fixed c, s, put $D_{c'} = A_{S_0}^{\mathrm{full}}(c, s, c') - A_{S_0}^{\mathrm{cut}}(c, s, c')$. The bound just obtained sums over all right classes, so

$$\sum_{c'} \|D_{c'}\|_{\mathrm{HS}}^2 \le \left(\sum_{c'} \|D_{c'}\|_{\mathrm{HS}} \right)^2 \le \tfrac{1}{4}\nu^2 b^{-2k^3 f}.$$

Freeness at J_H in Section 6 identifies X with one copy of the nonzero real coefficient space per source class. Thus $d_X = d_C d_B$. Since $d_B \ge 1$ and $|H_R| \ge 1$,

$$d_C \le |H_R| d_C d_B = D_{\mathrm{id}}.$$

Since $|G| \le M^{r^2 f} \le b^{23 r^2 f}$, and $23r^2 f + 40 \le 2k^3 f$, the following estimate holds.

Proposition 14.1 (Error from the cut integral)**.** *For every prescribed subset S_0 of the rational terms off the Bessel subgroup, the blocks formed from S_0 satisfy*

$$\nu^{-2} \sum_{c, c', s} \left\| A_{S_0}^{\mathrm{full}}(c, s, c') - A_{S_0}^{\mathrm{cut}}(c, s, c') \right\|_{\mathrm{HS}}^2 \le b^{-40} D_{\mathrm{id}}. \tag{36}$$

14.2 The residue estimates for the good projection

In this subsection, we estimate the action of the finite idempotent of Section 7.3 on the level subgroups. The bounds supply the loss estimates for the diagonal and mixed moments in Section 9.

For each $q = p_i^{\mathrm{lev}}$, use the marked $\mathrm{GL}_r(\mathbb{F}_q)$-factors of G_i. A local type parameter is a monic squarefree polynomial of degree r with nonzero constant term. It is allowed if no root in $\mathbb{F}_q$ is a

q_2-th root of unity. A primitive type polynomial here is the minimal polynomial of a generator of $\mathbb{F}_{q^r}^{\times}$. It is allowed since $r > 1$. A tuple is good if all its parameters are allowed and at least one is primitive. The original projector is the product of the two central idempotents h_i selecting these tuples, because the type polynomials are indexed by the same marked places for both realizations.

For a squarefree type polynomial with distinct irreducible factors of degrees $d_1, \dots, d_l$, the associated Green data have one box in each of the corresponding distinct cuspidal parameters. Green's character parametrization and degree formula [29, §8, Theorem 14 and Remark (1)] give degree $\prod_{a=1}^{r}(q^a-1)/\prod_{j=1}^{l}(q^{d_j}-1)$. Distinct parameters give orthogonal central idempotents. The identity coefficient of each is its degree squared divided by the group order, namely

$$q^{-r}\frac{\prod_{a=1}^{r}(1-q^{-a})}{\prod_j(1-q^{-d_j})^2} \geq q^{-r}(1-r/q) = u_q.$$

Write A_q for the allowed squarefree parameters and $P_q \subset A_q$ for the primitive ones. Give each monic degree-r polynomial the comparison weight u_q. Roots at zero or at one of the q_2-th roots exclude at most $(1+q_2)q^{r-1}$ polynomials. A repeated monic factor of degree $j \geq 1$ excludes at most $q^j q^{r-2j} \leq q^{r-1}$ polynomials for each $j \leq r/2$.

Therefore the allowed weight is at least $1-\epsilon_q$, with $\epsilon_q = (1+q_2+2r)/q$. There are $\varphi(q^r-1)/r$ primitive type polynomials, since all generators have degree r and Frobenius orbits of size r. If $L \geq 2$ has j distinct prime divisors, then

$$\frac{\varphi(L)}{L} \geq \prod_{a=1}^{j}\left(1-\frac{1}{a+1}\right) = \frac{1}{1+j} \geq \frac{1}{1+2\log L}.$$

So the unweighted fraction of primitive polynomials is at least $\delta_q = 1/(2r(2r\log q+1))$. Set $a_q = u_q|A_q|$, $c_q = u_q|A_q \setminus P_q|$. Then

$$a_q \geq 1-\epsilon_q, \qquad c_q \leq q^{-r}(q^r - |P_q|) \leq 1-\delta_q.$$

Each good tuple has identity coefficient at least u_q^f. Summing over precisely those tuples in A_q^f with a primitive entry gives $h_i(1) \geq a_q^f - c_q^f$, and hence

$$(1-h_i)(1) \leq f\epsilon_q + (1-\delta_q)^f \leq f\epsilon_q + \exp(-f\delta_q).$$

In the marked range $r = O(x^5)$, $f \gg_{q_0} x^{11}\log x$, $q_2 = x^{O_{q_0,d}(1)}$, and $10\log b \leq \log q \leq 11\log b$. Since $100\epsilon < 1$, these bounds give $f\delta_q \geq 9\log b$ eventually, and $f\epsilon_q = o(b^{-8})$. It follows that

$$(1-h_i)(1) \leq (2b^8)^{-1}. \tag{37}$$

For the Bessel trace on G_i, let ψ_U be the marked character on the reduction of the integral unipotent subgroup, extended by zero. For two integral unipotents of the same reduction the difference of the simple-root sums belongs to $M\widehat{\mathcal{O}}_E$, on which the character with denominator $M\delta_E$ is trivial.

For a diagonal class term $\gamma = u\iota_H(\delta)$, write the finite transporter in the lattice basis as

$$h_c^{-1}\gamma_f h_c = u_c\iota_H(y), \qquad y = (c^{\circ})^{-1}\delta_f c^{\circ}.$$

The unipotent acts trivially on the middle quotient. The middle block of the transporter is therefore that of $\iota_H(y)$, whose other diagonal blocks are identities and whose off-diagonal

blocks vanish. Applying this also to the inverse shows that $\iota_H(y)$ preserves the original middle lattice in both directions at every finite place. By Section 13.2.2, this gives integrality of y, y^{-1} on X_0, including above N. At the untouched prime p_{1-i}^{lev} both π_N and $\bar{\pi}_N$ are units, so the identity reduction passes from the middle block to y. Lemma 6.1 at that prime gives $\delta = 1$. Thus a diagonal term is possible only on the reduced unipotent subgroup.

Conversely, for an integral unipotent k of reduction s, approximate $h_c k h_c^{-1}$ by a rational unipotent in the open coset $h_c k J_L h_c^{-1}$, using the free Heisenberg and skew central coordinates. The class representative fixes the ordered isotropic flag vectors and z, hence preserves the simple-root sum. The rational lift has finite character $\psi_U(s)$.

Translation by this transporter carries the identity orbital block to the specified residue block. The identity rational Bessel block is νI: its terms on a class are U_c, so one sums the right cutoff by its orbit sum and integrates the left cutoff to ν, then uses the orthonormality of the full Bessel coordinates on $U_\infty \backslash G_\infty$. The real character is the inverse of the finite character on a rational unipotent. In the amplitude over s^{-1}, right translation of the identity block by the lift of s thus gives $A_{\mathrm{Bes}}(c, s^{-1}, c) = \nu \psi_U(s) I$, where A_{Bes} denotes the part of the full amplitude on the Bessel subgroup. The blocks at s and s^{-1} are adjoints, so the normalized conjugate in (1) gives $\operatorname{tr} r_\psi(s) = d_X \psi_U(s)$ for $s \in G_i$.

Take T to consist of all places above M, and put $J_T = K_T(M)$, so $G = K_T/J_T$ and $\mu(K_T) = |G|\mu(J_T)$. For $k \in K_T$, a decomposition $k = uj$ with $u \in U$ and $j \in J_T$ forces $u \in K_T$. Such a unipotent extends by the identity off T, where M is a unit. Conversely each global integral unipotent gives a local lift of its reduction. Thus $\chi(k) = \psi_U(\bar{k})$, with the same support and character. If $\psi_U^\natural$ is the conjugacy average on G, integration over the cosets of J_T gives

$$\int_{K_T} \chi(yky^{-1})\, d\mu(y) = \mu(J_T) \sum_{t \in G} \psi_U(t\bar{k}t^{-1}) = \mu(K_T)\psi_U^\natural(\bar{k}).$$

Restrict the outer integral of $\mathcal{M}(K_T)$ in Section 10.3 to K_T; each reduction class has mass $\mu(J_T)$. Applying (10) with $Q_0 = 2$ and $t = n$ gives

$$\mu(J_T)\mu(K_T)^2 \sum_{s \in G} |\psi_U^\natural(s)|^2 \le \mathcal{M}(K_T) \le \mu(J_T)\mu(K_T)(\mu(K_T) + 2^{-n}).$$

For $q_i = p_i^{\mathrm{lev}}$ each of the f conjugate pairs above q_i contributes $\prod_{l=k+1}^{r}(1 - q_i^{-l})$ to $\mu(K_T)$. Indeed the split orbit count in the proof of Lemma 10.2 applies at either level prime, since N is a unit at both. The integral coordinate lattices and the terminal compact subgroup have volume one in Section 10.2. Multiplying over the pairs gives

$$\mu(K_T) = \prod_{i=0}^{1} \prod_{l=k+1}^{r} (1 - q_i^{-l})^f \ge 1 - f \sum_{i=0}^{1} \frac{q_i^{-(k+1)}}{1 - q_i^{-1}} \ge \tfrac{1}{2} \ge 2^{-n}$$

for large x, since $f \le x^{12}$ and $q_i \ge b^{10}$. Since G is the product of the two level factors, conjugacy averaging on G_i agrees with the restriction of averaging on G. Consequently the conjugacy average of $\operatorname{tr} r_\psi|_{G_i}$ has squared counting norm at most $2d_X^2$, as required in the diagonal estimate.

14.3 The weighted count at the additional ramified place

In this subsection, we bound the sublattices that occur in the fixed-vector contributions of Section 13.2. Their ramified level weights enter the count of pairs in Section 14.5.

Write $\mathcal{O}_f = \widehat{\mathcal{O}}_E$ and $\Lambda_c^0 = \Lambda_c \cap X_0(\mathbb{A}_f)$. A sublattice L of Λ_c^0 has rank $\dim_E V$ if it has finite index in $\Lambda_c^0 \cap V(\mathbb{A}_f)$ for an E-subspace $V \subset X_0$. For a finite idele a put $\mathrm{ind}(a) = |a|_f^{-1}$. The discriminant $d(L)$ is the square root of the index of the determinant of the Hermitian Gram matrix in a finite basis of L. It is independent of basis and at least one for an integral lattice. For $L \subseteq L'$ of index ι a basis change gives $d(L) = \iota d(L')$.

Put $\Delta_{\mathrm{lat}} = b^{n^3 f}$ and $\mathfrak{t} = b^{kf/4}$. For $0 \le l \le n^3$, the weighted sum satisfies

$$\sum_c \sum_{\substack{L \subseteq \Lambda_c^0 \\ \mathrm{rank}\, L = k-l,\ d(L) \le \Delta_{\mathrm{lat}}}} d(L)^{-k} \le \frac{d_C}{N^{kf}} \mathfrak{t}^{l+1}. \tag{38}$$

Use the group K_X of Section 13.6. Let K_N be its subgroup obtained by requiring the distinguished covector to agree with the identity covector modulo the marked idele π_N, where $[\widehat{\mathcal{O}}_E : \pi_N \widehat{\mathcal{O}}_E] = N^f$. The actual source level lies in K_N. The stabilizers for the source classes are trivial at the principal level. Therefore each class has the same positive weight in quotient Haar measure. For any nonnegative function F_0 on the classes, a uniform bound

$$|K_X/K_N|^{-1} \sum_{\phi \in K_X/K_N} F_0([u y_\phi]) \le B_0 \quad (u \in U(X_0)(\mathbb{A}_f))$$

implies $\sum_c F_0(c) \le d_C B_0$. Indeed integrate the translate averages over the rational quotient, using right invariance of Haar. For unbounded F_0 the same argument first applies to its finite truncations.

For fixed u and ϕ, write the chosen representative as $\eta u y_\phi j_H$, with η rational unitary and $j_H \in J_H \subset K_N$. In the glued basis of Section 13.2.2, the ramified covector of Section 13.2 gives

$$\Lambda_c^0 = c^\circ \{ u \in \mathcal{O}_f^k : u_{k_0} \in \pi_N \mathcal{O}_f \}, \qquad [c^\circ \mathcal{O}_f^k : \Lambda_c^0] = d(\Lambda_c^0) = N^f.$$

Here c° represents c. Thus η^{-1} sends Λ_c^0 into the unimodular lattice $U = u\widehat{\mathcal{O}}_E^k$ and imposes the vanishing on it of the k_0-covector of $y_\phi^{-1} u^{-1}$ modulo π_N, where k_0 is the original index. This is an injection on sublattices and preserves discriminants. Thus we may sum over all lattices $L \subset U$ of rank $h_0 = k - l$ and $d(L) \le \Delta_{\mathrm{lat}}$, counting the cosets whose covector vanishes on L. Inverse covectors modulo π_N distinguish the cosets. If the two covectors coincide, multiplication by the integral matrix $y_{\phi'}$ makes the distinguished covector of $y_\phi^{-1} y_{\phi'}$ equal to the identity covector modulo π_N.

Let V be the rational subspace of L. Put $L^{\mathrm{sat}} = V(\mathbb{A}_f) \cap U$, $\iota = [L^{\mathrm{sat}} : L]$, and $d = d(L^{\mathrm{sat}})$. Thus $d(L) = d\iota$ and $d, \iota \ge 1$. Extend an integral basis of L^{sat} to one of U at the finite places. Let A be the matrix of a basis of L in that of L^{sat}. To count covectors vanishing on L, write their value on the first h_0 basis vectors as a covector b_0 and observe $b_0 A = \pi_N w$ with w integral. The class of w modulo the covector span of A determines b_0 mod π_N, since A is invertible over the adeles. The full covector quotient has order ι. The last l covector values have at most $(N^f)^l$ possibilities. Hence the covector multiplicity for L is at most ιN^{fl}.

Put $\mu = 2\varrho + 4$. On $d\iota \le \Delta_{\mathrm{lat}}$, one has

$$\iota (d\iota)^{-k} \le \Delta_{\mathrm{lat}}^{\mu+2} d^{-(k+1)} \iota^{-(h_0+\mu)}.$$

Indeed the ratio to the right-hand power of d and ι is $d\iota^{h_0+\mu+1-k} \le (d\iota)^{\mu+2}$. The sum of $\iota^{-(h_0+\mu)}$ over finite-index sublattices in L^{sat} is bounded by $\zeta_E(\mu)^{h_0}$. Locally, triangular echelon bases

with diagonal ideals $\mathfrak{p}^{a_1}, \dots, \mathfrak{p}^{a_{h_0}}$ have at most $q^{\sum_{j=1}^{h_0}(h_0-j)a_j}$ choices of off-diagonal entries, with q the residue cardinality. Multiplying the resulting geometric series gives factors $\zeta_E(h_0+\mu-j)$ with $0 \le j \le h_0-1$, all bounded by $\zeta_E(\mu)$. The bound $n[E:\mathbf{Q}] \le 2^{2\varrho}$ and $\zeta_E(\mu) \le \zeta(\mu)^{[E:\mathbf{Q}]}$ yield $\zeta_E(\mu) \le 2$; for the latter use $\log \zeta(\mu) \le (1+2/(\mu-1))2^{-\mu}$. Thus the index sum is at most 2^k.

Write $\operatorname{rd}(E) = |D_E|^{1/(2f)}$, where D_E is the field discriminant. The subspace sum is

$$\sum_{\operatorname{codim} V = l} d(V(\mathbb{A}_f) \cap U)^{-(k+1)} \le (200 f k \operatorname{rd}(E))^{f(k+1)l}. \tag{39}$$

For $l = 0$ the only subspace is $V = X_0$ and its saturation in U is U, of discriminant one. For $l > 0$, the primitive lattices of V and $V^\perp$ in U have the same discriminant. In fact, local unimodularity identifies the quotient by their direct sum with either discriminant group. The pairing from U onto the dual of the first primitive lattice is surjective because a functional on a direct summand extends to U. Its kernel is the second primitive lattice. The image of the first lattice is its own pairing lattice in the dual. Interchange the two lattices to get the other identification. Their discriminants, defined as the square roots of the orders of these groups, are therefore equal.

Under one embedding from each complex pair, with the standard Euclidean norm at the infinite places, the real covolume of $V^\perp(E) \cap U$ in its span is

$$2^{-fl} |D_E|^{l/2} d(V(\mathbb{A}_f) \cap U).$$

Choose a rational basis matrix Y of $V^\perp$ and a finite coefficient matrix D so that YD is an adelic lattice basis. The rational coefficient lattice has real covolume $v_E^{-l} \operatorname{ind}(\det D)$, by comparison with a common integral sublattice.

At v, the real Jacobian of Y in the positive definite coordinates is $\det(\Phi_v(Y)^* \Phi_v(Y))$. The product of these Jacobians times $\operatorname{ind}(\det D)$ is $d(V(\mathbb{A}_f) \cap U)$, by the product formula for $\det(Y^*Y)$. Minkowski's second theorem [20, Ch. VIII, §4.3, Theorem V] in dimension $2fl$, using the unit Euclidean ball, gives independent E-vectors $x_1, \dots, x_l$ in this lattice such that

$$(\|x_1\| \cdots \|x_l\|)^{2f} \le (8 f l \operatorname{rd}(E))^{fl} d(V(\mathbb{A}_f) \cap U).$$

Write $\lambda_1 \le \cdots \le \lambda_{2fl}$ for the real successive minima. Lattice vectors of length at most λ_{2fi+1} span a real space of dimension at least $2fi+1$. The diagonal image of the E-span of $x_1, \dots, x_i$ lies in a real space of dimension $2fi$. Thus one can choose x_{i+1} outside that E-span with $\|x_{i+1}\| \le \lambda_{2fi+1}$, and

$$\prod_{i=0}^{l-1} \|x_{i+1}\|^{2f} \le \prod_{i=0}^{l-1} \lambda_{2fi+1}^{2f} \le \prod_{j=1}^{2fl} \lambda_j.$$

The ball contains the cube of side $2/\sqrt{2fl}$, so Minkowski's upper bound with the stated covolume already gives the displayed inequality with room in its constant.

For distinct $x, y \in E^k \cap U$, the hermitian square $h(x-y, x-y)$ is a totally positive algebraic integer by unitary integrality at the finite places. Its norm is a positive integer, so at least one real conjugate is at least one. The diagonal archimedean images of x and y therefore have distance at least one. The packing bound in real dimension $d_0 = 2fk$ gives at most $(1+2R)^{d_0}$ lattice vectors of length at most R. Dyadic summation at exponent $2f(k+1) = d_0 + 2f$ then gives

$$\sum_{x \ne 0} \|x\|^{-2f(k+1)} \le 5^{d_0} \sum_{j \ge 0} 2^{-2fj} \le 5^{2f(k+1)}.$$

The chosen short tuple determines V as its orthogonal complement. Raising the short-basis bound to the $(k+1)$-st power and summing over tuples gives (39), since $8 \cdot 5^2 = 200$.

The unaveraged sum over ϕ and L is consequently at most

$$\Delta_{\mathrm{lat}}^{\mu+2} 2^k \left(N^f (200 f k \operatorname{rd}(E))^{f(k+1)} \right)^l \le b^{-f} \mathfrak{t}^{l+1}.$$

For the last inequality use $k \le f$, $k/4 \ge 10^5 n^9$, $N \le b^{60n}$, $(200 f k \operatorname{rd}(E))^{1000} \le b$, and $60n \cdot 1000 + (k+1) \le 250k$. In particular, $\operatorname{rd}(E) \le d q_1$ by Section 4. Together with $f, q_1 \le x^{12}$ and $k < 3x^5$, this gives $200 f k \operatorname{rd}(E) = O_d(x^{29})$. By the class count in Section 13.6, $[K_X : K_N] = (N^k - 1)^f \ge (N^k/b)^f$. Dividing by this index and applying the Haar average gives (38).

14.4 Counting orbital representatives containing a fixed lattice

In this subsection, we count bases adapted to the isotropic flag and its final orthogonal vector inside a fixed lattice. Organizing them by their pairing matrices gives the multiplicity bound used in Section 14.5 with the representatives of Section 14.1.

In the rational Witt decomposition write A for the isotropic span of $e_0, \dots, e_{\varrho-1}$ and take $F = (e_0, \dots, e_{\varrho-1}, z)$ as a matrix of columns. Write $Z_\gamma = F^* h \gamma F$ for the matrix of their products and P_γ for its first ϱ covectors. Here h is the matrix of the rational Hermitian form. Use the same reduced cutoffs as in Section 14.1. A boxed point is reduced and has $\ell_v \le b^{6n+2}$ at every place. A term γ is near if y and $\gamma^{-1} y$ are boxed for some y. For an orbital calculation one fixes a representative of each U_c^0-$U_{c'}^0$ double coset, choosing a near representative when there is one.

Let δ_a be the leading minor of order a in Z_γ, with $\delta_0 = 1$, and put

$$\mathfrak{g}_\gamma = \delta_n \overline{\delta_n} - N^2 \delta_{n-1} \overline{\delta_{n-1}}.$$

We say that γ has cell rank j, $0 \le j \le n$, if $\delta_j \ne 0$, if $\mathfrak{g}_\gamma \ne 0$ is equivalent to $j = n$, and if there is a single $\zeta \in E$ with $\zeta \bar{\zeta} = 1$ such that the following holds for every $j \le a < n$. The spans of the first $a+1$ columns of F and γF contain a common vector whose two coefficients at index a are ζ and 1, respectively. Section 15.2 obtains these conditions for the terms used there.

For $\gamma \in \mathcal{T}(c, s, c')$, define its term lattice L_γ to consist of $x \in \Lambda_c^0$ for which some lift $x + Mw$, $w \in A(\mathcal{O}_f)$, is orthogonal to γF. In the rational coordinates this means $X_\gamma^* x + M P_\gamma^* w = 0$ for some integral w, with X_γ the definite block of γF. Choose representatives a_{sH_R} for the cosets sH_R in G, and define $t \in H_R$ by $s = a_{sH_R} t$. This is the parameter of the term.

Fix the left source class c, the Bessel parameter t, and the lattice L of rank $h_0 = k - l$ in Λ_c^0, with discriminant at most Δ_{lat}. A term counted here is a pair (c', γ), where γ is the fixed near representative, has a cell rank, and has L in its level-M term lattice. All moves in the proof are in U_c^0, the rational unipotents of level M in class c fixing the standard basis pointwise.

First, equal bases after such moves identify the terms. In the finite class bases the equal basis condition implies that the two residues differ on the right by an element of H_R, because the quotient of the two matrices fixes the basis. Their common parameter t then gives equal residues. The quotient of the two moved rational elements fixes the basis, and after transport between the right classes is congruent to one modulo M. Projecting its Bessel-subgroup component to $U(X_0)$ identifies the right classes at their original source level. Once the right classes are equal, the definite rational component lies in a principal-level class stabilizer and is the identity. The quotient is therefore in the right basis-fixing unipotent group $U_{c'}^0$. The two terms represent the same double coset and, by the fixed representative convention, are equal.

We partition the terms by cell rank j and integral basis product matrix $Z = (\langle f_a, \gamma f_b \rangle)$, where $f_0, \dots, f_{n-1}$ is the standard basis. There are at most $n+1 \le b^{2f}$ cell ranks. The entries of Z are integral because the two bases are integral in the marked class basis, whose Gram form is integral. Their absolute values are bounded by $N b^{74n} \le b^{134n}$ at all complex places.

Indeed both normalized basis columns have norm at most b^{37n} by the box bound, and the product of the two basis scales has modulus at least $1/N$. The lattice of integral elements of E in the product of the f complex embeddings has point separation at least one in the maximum norm, by the norm formula for a nonzero algebraic integer. Disjoint products of discs of radius $1/2$ show that there are at most $(1 + 2b^{134n})^{2f}$ values for each entry. Since $1 + 2b^{134n} \le b^{135n}$, the number of matrices is at most $b^{270 n^3 f}$.

Fix j, Z, let Y be the matrix formed by the first j covectors of Z and the remaining covectors of the identity, and put $\delta = \det Y \neq 0$. Write P for the first ϱ covectors of Z. For a term let X_γ be the X_0-block of its basis. In the splitting $A \oplus (X_0 \perp Ez) \oplus A^\vee$, the block along $A^\vee$ is P, and Z also determines the z-block by pairing with z of norm $-N$. These statements are in the rational basis. In the finite calculation both transported bases lie in the same left class lattice.

14.4.1 Changes of basis on the two blocks

In this subsection, we put the two blocks into the coordinates needed for the count of Section 14.4.

For a matrix η of rational columns in Λ_c^0, a move takes

$$X_\gamma \longmapsto X_\gamma - M \eta P. \tag{40}$$

To check it in the indicated splitting, let $T = -M\eta : A^\vee \to X_0 \perp Ez$, and let J_0 be the middle Gram matrix. Choose an integral scalar θ with $\theta + \bar{\theta} = 1$, for example a primitive sixth root in E. The block unipotent

$$\begin{pmatrix} 1 & -T^* J_0 & -\theta T^* J_0 T \\ 0 & 1 & T \\ 0 & 0 & 1 \end{pmatrix}$$

preserves the form, fixes A and z, and is congruent to the identity on Λ_c modulo M. For the congruence, use integrality of the column vectors of η in the marked middle lattice and of the pairings there. Its middle action on the basis gives (40). Also an integral skew-Hermitian matrix σ yields the move with upper-right block $M\sigma$ and other off-diagonal blocks zero. It changes the A-covectors of a basis by $M\sigma P$ and fixes the other covectors. Both moves retain the original residue and the cell rank.

14.4.2 Lifts of the fixed lattice

In this subsection, we bound the number of lifts after the changes of basis in Section 14.4.1.

Write a finite basis of L as $m = ya$, with y a rational matrix of independent columns and a invertible over the finite adeles. From containment in the term lattice we have a rational matrix w with wa integral and

$$X_\gamma^* y + M P^* w = 0. \tag{41}$$

The defining containment solves this equation over the finite adeles with wa integral. A linear system over E with a finite-adelic solution has a rational solution. Its solutions form an affine

space over the rational kernel, whose rational points are dense in the finite-adelic solutions by additive approximation. The condition that wa be integral is an open neighbourhood of the given solution. This establishes (41) with its original lattice.

Sort the ϱ covectors of wa by their classes modulo the lattice of covectors of m^*m. The number of classes is at most $d(L)^{2\varrho}$, since the Gram determinant has index $d(L)^2$. In one class, choose a base lift w_0. Each $w_0 - w$ is $\eta^* y$ for integral columns η in the lattice, given by

$$\eta = y(y^*y)^{-1}(w_0 - w)^*.$$

The covector congruence means $(w_0 - w)a = S^*m^*m$ with S integral. Cancelling a shows $\eta = mS$ over the adeles, so its columns indeed lie in L. Equation (40) makes (41) hold with the common w_0. All moved bases hence have the same matrix of products of their X_0-block with y.

14.4.3 Counting definite columns

In this subsection, we bound the definite columns for a lift chosen in Section 14.4.2.

For a fixed lift, the differences between two X_0-blocks lie columnwise in the lattice $L_c^\perp = \Lambda_c^0 \cap V(\mathbb{A}_f)^\perp$, of rank l, with V the rational span of L. For the integrality of each difference, write the rational middle coordinates as (x, λ) and put $u = (c^\circ)^{-1}x$. In the glued basis of Section 8.1 these coordinates are

$$((u_i)_{i \neq k_0},\ \bar{\pi}_N u_{k_0},\ \lambda - u_{k_0}/\pi_N)\,.$$

The right class representatives fix the ordered columns of F. The two transported rational coefficients of z agree because Z is fixed. Subtracting their integral class coordinates therefore places the difference of their X_0-parts in Λ_c^0. Their orthogonality to L follows from (41) with w_0. Taking residues modulo $M\delta L_c^\perp$ contributes at most $\operatorname{ind}(M\delta)^{ln}$.

In a residue class let $v = X_{\gamma'} - X_\gamma$ be the difference. The matrix vY^{-1}/M has columns integral in Λ_c^0, since v is divisible there by $M\delta$ and $\operatorname{adj} Y$ is integral. Its columns at positions $b \geq j$ vanish because the cell relations supply a shared vector at each such position, with translated leading coefficient one, and with higher coefficients zero. Their translated coefficients in the tail can be combined triangularly to give tail e_b. The first j products against the standard basis vanish, so the complete coefficient vector is $Y^{-1}e_b$. Its image lies in the span of the columns of F and hence has zero X_0-component. The same argument applies after a move, since the move fixes the shared vectors.

The remaining exceptional position is the last column in the top cell, where $Y = Z$. The vector $\gamma F Z^{-1} e_{n-1}$ is orthogonal to A and has z-coordinate $-1/N$. Since it is in the translated span of the basis vectors its Hermitian norm is nonpositive. The squared norm of its X_0-part is thus at most $1/N$ at each complex place. The difference of two such parts has squared norm at most $4/N < 1$. It is integral in the class lattice because it is the last column of vY^{-1}. For any such vector the Hermitian square is an algebraic integer, by integrality of the class Gram form. A nonzero positive square cannot have absolute values below one at every complex place. So this last column also vanishes. The same proof works on the moved bases and uses only the cell-rank and basis identities.

Consequently vY^{-1}/M has support only in the first $\min(j, \varrho)$ columns. Multiplication by Y writes $v = M\eta P$ with integral η obtained from these columns. Equation (40) therefore moves all the bases in the residue class to a common X_0-block.

14.4.4 Counting isotropic columns

In this subsection, we complete the count using the bound for the definite columns in Section 14.4.3.

First sort by the matrix of shared vectors, meaning the A-coordinates of the shared columns of $\gamma F \operatorname{adj} Y$ at positions $b \geq j$ and zero at the other positions. These coordinates are unchanged by moves and integral because the isotropic covectors of the basis in class coordinates are integral and agree with the rational isotropic covectors. There are at most $b^{366n^4 f}$ possible matrices. Here is the bound on their entries. At each complex place the transported chosen basis at a boxed point has column lengths at most b^{37n}. A Bessel unipotent makes the representative of that point of norm at most b^{7n}, by the definition of the Bessel length and the boxed bound b^{6n+2}.

Its action on the standard basis is unitriangular. The entries of that matrix are at most b^{44n}, since it is obtained by multiplying the transported basis by an operator of norm at most b^{7n}. Cramer's rule and the minor bound give, for any linear combination in the boxed basis, each coefficient bounded by b^{46n^2} times the length of the combination. The exponent is bounded by $1 + (2 + 44n)\varrho + 7n \leq 46n^2$. Apply this to the shared vector at the first point, whose expression in the other boxed basis uses a column of $\operatorname{adj} Y$.

Its coefficients have absolute values at most $b^{(2+134n)\varrho}$ before converting scales. The ratios of any two chosen basis scale factors are at most $b^{23\varrho}$, because $1 \leq |\kappa_v| \leq b^{23}$. Thus each shared coordinate has absolute value at most

$$b^{46n^2 + (2+134n)\varrho + 1 + 23\varrho + 37n} \leq b^{182n^2}.$$

Packing the integral coordinates in ϱn positions as for Z gives $(1 + 2b^{182n^2})^{2f\varrho n} \leq b^{366n^4 f}$.

For one shared matrix and one X_0-block, sort the remaining ϱn isotropic coordinates modulo $M\delta\bar{\delta}$, costing $\operatorname{ind}(M\delta\bar{\delta})^{\varrho n}$. Within this congruence class the two bases differ only in the isotropic block, by $\xi = M\delta\bar{\delta}\,\mu$ for integral μ. Since their Gram matrices agree and the difference lies in the isotropic space, one has $\mu^* P = -P^* \mu$. Put α equal to the first ϱ columns of $\operatorname{adj} Y$. The columns at $b \geq j$ of $\mu \operatorname{adj} Y$ vanish by equality of shared vectors. In the top cell its last column vanishes because $P \operatorname{adj} Y\, e_{n-1} = 0$, P^* is injective, and $P^*\mu = -\mu^* P$. Consequently $\mu\alpha P = \delta\mu$.

An integral skew-Hermitian solution is

$$\sigma = \bar{\delta}\mu\alpha - \delta(\mu\alpha)^* - \alpha^* P^* \mu\alpha.$$

The displayed skew relation gives $\sigma^* = -\sigma$; multiplying by P cancels the last two terms and gives $\sigma P = \delta\bar{\delta}\mu$. The central move with parameter $M\sigma$ therefore joins the two bases. Separation then identifies their terms. Neither this correction nor the previous two moves alter the level or the residue parameter.

All the moduli and Gram indices are positive and satisfy

$$d(L)^2, \quad \operatorname{ind}(M\delta), \quad \operatorname{ind}(M\delta\bar{\delta}) \leq b^{2n^3 f}.$$

The first inequality is the discriminant cutoff. The other two follow from $M \leq b^{23}$ and the squared minor bound b^{270n^2} at all complex places, using $n \geq 10^5$. The counts of ranks, product matrices and three successive rounds thus give the upper bound

$$b^{\{2+270n^3+366n^4+2n^3(\varrho+ln+\varrho n)\}f} \leq b^{2000n^6(l+1)f}.$$

Lemma 14.2 (Number of representatives containing a fixed lattice)**.** *Fix c, t and a lattice L of corank l with $d(L) \le \Delta_{\mathrm{lat}}$. Count the pairs (c', γ) over (c, t) for which γ has a cell rank. Then*

$$\#\{(c', \gamma) : \gamma \text{ is a chosen near representative, } L \subseteq L_\gamma\} \le b^{2000 n^6 (l+1) f}.$$

14.5 The weighted sum over nearby pairs

In this subsection, we reduce the weighted count of two orbital terms to a count of lattices in their common subspace. Section 14.3 supplies the local weights, and Section 14.7 bounds the relevant discriminants.

For a term γ, its link consists of pairs of middle classes of w and $\gamma^{-1} w$, where $w \in A^{\perp} \cap (\gamma A)^{\perp}$. For two terms γ, γ' over a common block, let $V_{12} \subseteq X_0$ consist of the first coordinates of the common link pairs with both coordinates in X_0. Intersect $V_{12}(\mathbb{A}_f) \cap \Lambda_c^0$ with the two term lattices to get L_{12}. Fix $0 \le l \le n^3$. Sum over ordered pairs of chosen near representatives, each having a cell rank, in a common block (c, s, c') with agreeing residue s. Retain those for which L_{12} has rank $k - l$ in V_{12} and $d(L_{12}) \le \Delta_{\mathrm{lat}}$. Section 14.7 shows that these coranks and discriminants cover all such pairs.

Fix c and group such pairs by their common parameter t and lattice L_{12}. Both representatives have that parameter and contain the lattice in their respective term lattices. The assignment to the ordered pair of terms $((c', \gamma), (c', \gamma'))$ is injective, since for those classes the first term determines the residue. By Lemma 14.2 there are at most $b^{4000 n^6 (l+1) f} \le \mathfrak{t}$ ordered pairs for each fixed c, t, L_{12}. Indeed $l \le n^3$ and $k / n^9 \to \infty$. Every pair in that group has weight $d(L_{12})^{-k}$. Sum over the lattices, parameters, and left classes and use (38). With $n_p = |H_R| d_C / N^{kf}$, this gives

$$\sum_{\text{such ordered pairs}} d(L_{12})^{-k} \le n_p \mathfrak{t}^{l+2}. \tag{42}$$

The factor $|H_R|$ counts parameters. There is no additional multiplicity from the residues.

14.6 Divisibility of the pairing determinant

In this subsection, we deduce divisibility of a determinant of pairings with the two final vectors from agreement of the finite characters. The resulting product of residue norms enters the discriminant estimate of Section 14.7.

A residue s agrees if $\psi_U(u) = \psi_U(u')$ whenever u, u' are reduced integral unipotents with $us = su'$, using the character of Section 14.2. Consider one place above M, allowing either member of each conjugate pair. At the residue field at this place, write g for the reduction of the original transporter, and retain F for the reduced basis. Suppose $Fx = gFy$ for coefficient vectors supported through index $j < \varrho$.

Let m be any covector annihilating both basis initial segments through j. Then the two matrices

$$1 + t(Fx)m, \qquad 1 + t(Fy)mg$$

are reductions of integral unipotents, for every residue scalar t. To construct the lifts in the original lattice basis, lift the basis coefficients and the covectors m, mg supported at this one place. The covectors may be chosen to annihilate the corresponding standard initial segment, since it consists of basis vectors.

The integral Gram identifies the covectors with Hermitian products against integral vectors. Use the same Eichler formula as in Section 13.6, with the lifted isotropic vector replacing e_a and level factor one. It applies because the covector kills the initial segment. The map is upper unitriangular in the Witt order since the vector is supported through j and the covector vanishes there. At the chosen place the term using the conjugate of the lifted isotropic vector vanishes.

Outside this conjugate pair the matrix is the identity. The opposite reduction is forced by unitary adjunction. There the correction kills the basis, because the conjugate pairing of the isotropic vector with the basis is zero. Thus it contributes no simple-root entry. Thus the reduced unipotents are intertwined by s.

The simple-root sums of the two reductions at the test place are $tx_j m(f_{j+1})$ and $ty_j m(gf_{j+1})$, since the covector kills the earlier initial segment on each side. The local additive character on this root is nontrivial, by the conductor of the different-normalized character and the factor M^{-1}. Testing every t in the agreement condition gives

$$x_j m(f_{j+1}) = y_j m(gf_{j+1}).$$

Therefore $x_j f_{j+1} - y_j g f_{j+1}$ belongs to the sum of the two earlier spans, by the annihilator criterion. Moving that linear combination to the other side gives a common vector at index $j+1$ with last coefficient pair (x_j, y_j). Starting at the highest index of any nonzero common vector of the full bases and iterating this argument gives a relation with a nonzero pair of coefficients in the final positions.

Let T have two columns in the original integral left lattice orthogonal to both isotropic spans $A, \gamma A$, and apply the last two pairings with $z, \gamma z$ to these columns. Suppose the resulting two-by-two determinant is invertible over the finite adeles. At each meeting place, the preceding relation gives a nonzero kernel vector for this matrix after taking its adjoint and reducing, since all its other basis pairings vanish. If D denotes the two-column pairing matrix in the integral lattice coordinates, then $v_w(\det D^*) = v_{\bar{w}}(\det D) \geq 1$ at every meeting place w. Thus the ideal of $\det D$ is divisible by $\prod_{w \text{ meeting}} \bar{\mathfrak{p}}_w$. The places over M are split, and conjugation permutes them and preserves their residue norms. Multiplying the local contributions gives $\mathrm{ind}(\det D) \geq Q_{\cap}(s)$, with each meeting place counted once. The integral Gram matrix is included when taking these pairings in a lattice basis.

14.7 The discriminant bound for the pair lattice

In this subsection, we compare the discriminant of the lattice common to two orbital terms with the level and intersection determinants. The calculation of Section 14.6 supplies the local divisibility used in Section 14.5.

Use the rational spaces $A, \gamma A$ and the left lattice Λ_c throughout. Equivalently one may carry all three through the original left class basis. A term over (c, s, c') sends $\Lambda_{c'}$ to Λ_c. Set $I = A \cap \gamma A$. The form on $A^{\perp}/A$ has middle space $X_0 \perp Ez$. For the two middle coordinates of a link pair, exchange their z-coordinates to obtain two vectors in the positive space $X_0 \perp Ez^{+}$, where z^{+} has norm N.

The difference of the two middle forms becomes the difference of the two positive forms. Both middle coordinates come from one lift, so this difference is zero on the link. Its dimension is $k+1$, as follows by taking the orthogonal complement of $A + \gamma A$ and quotienting by I. Definiteness makes both coordinate projections of the link injective. It is therefore the graph

of an isometry Υ_γ of the positive space. Its positive metric is the positive form of the first coordinate, or equivalently that of the second.

The calculations below use the cell rank j in the original basis. Put $j_0 = \min(j, \varrho)$, and denote by δ_a the leading minor of order a of the basis product matrix. Modulo I, the first j_0 isotropic basis columns are a basis of each isotropic basis space. Indeed their pairing is the leading j_0-minor below the top rank, and the common-vector flag relations supply all the other columns modulo I. At the top rank the full basis product matrix is invertible. Its first ϱ rows and first ϱ columns are each independent. A vector in I is orthogonal to both full bases, so $I = 0$. This argument also covers $\delta_\varrho = 0$. Thus a top cell of rank n can have a vanishing proper minor. The estimate for open residues in Section 13.8 uses the additional condition of Section 13.1. Set

$$\mu_\gamma = \frac{|N_{E/\mathbf{Q}}(\delta_j)|}{N^{2f(j-\varrho)_+}}.$$

Let t_γ be the index of $(\Lambda_c \cap A) + \gamma(\Lambda_{c'} \cap A)$ in $\Lambda_c \cap (A + \gamma A)$.

Let i_γ, i'_γ be the indices of the pairing lattices of the first j_0 isotropic columns inside the pairing lattices of all isotropic columns. Use respectively $\Lambda_c \cap A$ paired with γF, and $\gamma(\Lambda_{c'} \cap A)$ paired with F. Define the gauge by

$$G_\gamma = \frac{M^{2fj_0}}{i_\gamma i'_\gamma} \max\{N^f \mu_\gamma, \mathfrak{g}_{\mathrm{norm}}(\gamma)\}, \qquad \mathfrak{g}_{\mathrm{norm}}(\gamma) = \prod_{v|\infty} |\mathfrak{g}(\gamma)|_v. \tag{43}$$

Here $|\cdot|_v$ is the complex modulus at one embedding in a conjugate pair. Below top rank the discriminant term vanishes, and at top rank both first indices are one. For a near term both transported real bases are bounded by b^{37n} in the box.

Removing the real basis scaling gives a basis product bound $Nb^{74n} \le b^{134n}$, since its other scaling factor is a nonnegative power of $|\kappa_v| \ge 1$. Expanding a minor and using $n \le b$ bounds each squared modulus by b^{270n^2}. For the discriminant, put $a = \mathrm{adj}(Z_\gamma)e_{n-1}$. The vector $\gamma F a$ has zero dual-isotropic coordinates and z-coordinate $-\delta_n/N$. Its norm equals that of Fa, namely $-N|\delta_{n-1}|^2$ at each complex place. Hence $\mathfrak{g} = \delta_n\bar{\delta}_n - N^2\delta_{n-1}\bar{\delta}_{n-1}$ is N times the squared norm of its X_0-component, and is at most the squared full minor. This holds also when Z_γ is singular. At top rank with $\delta_\varrho = 0$,

$$\mathfrak{g}_{\mathrm{norm}}(\gamma) = |N_{E/\mathbb{Q}}(\delta_n)| = N^{2f}\mu_\gamma > N^f\mu_\gamma.$$

This term belongs to the dominant-discriminant range of Section 15.5. Equation (43), $M \le b^{23}$, and $N \le b^{60n}$ now give $0 < G_\gamma \le b^{n^3 f}$, using $n \ge 10^5$.

14.7.1 The lattice associated with one link

In this subsection, we describe the lattice for each link of Section 14.7.

Use the unimodular quotient lattice

$$Q = (\Lambda_c \cap I^\perp)/(\Lambda_c \cap I).$$

This is unimodular because $I \cap \Lambda_c$ is a primitive isotropic direct summand. Its pairing with Λ_c surjects onto its dual. In a basis consisting of it, a basis of its orthogonal quotient, and lifts dual to it, the determinant of the full form is a unit times the determinant on the quotient. All

these bases may be taken place by place over the valuation rings. Let F_1 be the matrix of the two lists of the first j_0 isotropic columns in Q, and $p = (z, \gamma z)$ read in this quotient.

The classes of the integral lifts of the link form the primitive orthogonal lattice to F_1 in Q. In fact a lift is orthogonal to $A + \gamma A$, and the list F_1 spans the image of that space in the quotient, with the deleted kernel I. Under this lift map, the link metric is the quotient Gram form J augmented by $N^{-1}Jpp^*J$, since the two additional squares are those of the two values of the z-form divided by N.

We give the determinant comparison with this augmentation. Write $F_0 = F_1 a_0$ for a basis of the saturation of F_1 in Q, $I_F = [\mathrm{sat}(F_1) : F_1]$, and v for a basis of the primitive orthogonal lattice. Choose an integral right inverse Q_0 to $F_0^* J$, so (Q_0, v) is integrally invertible. In the ambient space with Gram $\widetilde{J} = \mathrm{diag}(J, NI_2)$, let $W = \begin{pmatrix} F_1 & p \\ 0 & I_2 \end{pmatrix}$ and let $v_1 = \binom{v}{-N^{-1}p^*Jv}$. Then $W^*\widetilde{J}v_1 = 0$, and $v_1^*\widetilde{J}v_1 = v^*(J + N^{-1}Jpp^*J)v$. A right inverse to $W^*\widetilde{J}$ is

$$Q_1 = \begin{pmatrix} Q_0 a_0^* & 0 \\ -N^{-1}p^*JQ_0a_0^* & N^{-1}I_2 \end{pmatrix}.$$

In the basis (Q_1, v_1) the first coordinate block of W is $W^*\widetilde{J}W$.

Put $C = (Q_1, v_1)$, $G_1 = W^*\widetilde{J}W$. Since $W^*\widetilde{J}C = (1, 0)$, the matrix $C^{-1}(W, v_1)$ is triangular by blocks with diagonal blocks $G_1, 1$. Orthogonality gives

$$\det(G_1)\det(v_1^*\widetilde{J}v_1) = \det(\widetilde{J})\overline{\det C}\det C\ \det(G_1)^2.$$

The bordered determinant computed below is nonzero in each cell. Cancelling $\det G_1$, taking indices, and using unimodularity of J yields

$$\mathrm{ind}(\det(v_1^*\widetilde{J}v_1))I_F^2\,\mathrm{ind}(N)^2 = \mathrm{ind}(\det(W^*\widetilde{J}W)). \tag{44}$$

Indeed $\det(Q_1, v_1)$ up to sign is $\det(Q_0, v)\det a_0^* N^{-2}$, and the first determinant has index one. More precisely, put $s_w = \mathrm{length}_{\mathcal{O}_{E,w}}(\mathrm{sat}(F_1)_w/(F_1)_w)$, where $(F_1)_w$ denotes the column lattice, and $n_w = v_w(N)$. Then $v_w(\det a_0) = -s_w$ and $v_w(\det a_0^*) = -s_{\bar{w}}$, so

$$v_w(\det(v_1^*\widetilde{J}v_1)) + s_w + s_{\bar{w}} + 2n_w = v_w(\det G_1).$$

Taking the product of residue norms accounts for the square I_F^2. The Gram matrix of W, reordered into the two isotropic lists with their bordered last vectors, has zero diagonal blocks and off-diagonal block equal to the subframe product matrix. On the upper cell this block is the full matrix.

Below the upper cell a common vector with last coefficients $(\zeta, 1)$, with $\zeta\bar{\zeta} = 1$, shows that its determinant is $-N\zeta\delta_j$. Reduce the isotropic coefficient vector on the right modulo I to its first j columns. Pairing with the opposite subframe gives a last column with entries $(0, \dots, 0, -N\zeta)$ after adding the prescribed linear combination of the preceding columns. Thus the square root of the right side of (44) is $N^{2f}\mu_\gamma$ in both cases. Since $\mathrm{ind}(N) = N^{2f}$, the link lattice of integral lifts has discriminant μ_γ/I_F.

Write $A_L = \Lambda_c \cap A$, $A_R = \gamma(\Lambda_{c'} \cap A)$, and denote by A_L^0, A_R^0 the lattices of the first j_0 columns on the two sides. Let π be the quotient map to Q. Set

$$B_0 = \pi(A_L^0 + A_R^0), \quad B_1 = \pi(A_L + A_R^0), \quad B_2 = \pi(A_L + A_R), \quad B_3 = \mathrm{sat}_Q(B_2).$$

Pairing with γF kills A_R and $I \cap \Lambda_c$; it maps B_1/B_0 onto the covector quotient of order i_γ. Pairing with F kills A_L and $I \cap \Lambda_c$; it maps B_2/B_1 onto the quotient of order i'_γ. Since $I \cap \Lambda_c \subseteq A_L$, projection identifies B_3/B_2 with

$$\frac{\Lambda_c \cap (A + \gamma A)}{A_L + A_R},$$

of order t_γ. The three indices multiply to $[B_3 : B_0] = I_F$, so $i_\gamma i'_\gamma t_\gamma \mid I_F$.

Impose the level condition on the lifts by demanding that their A-component be divisible by M, after adding an integral vector of I. This contributes index at most $[\widehat{\mathcal{O}}_E : M\widehat{\mathcal{O}}_E]^{j_0} = M^{2fj_0}$. The obstruction is the class of that component in

$$(\Lambda_c \cap A)/((\Lambda_c \cap I) + M(\Lambda_c \cap A)).$$

Its rank before reducing modulo M is j_0. Denote the resulting full link lattice by L^+_γ. We have proved

$$t_\gamma d(L^+_\gamma) \leq \frac{M^{2fj_0}\mu_\gamma}{i_\gamma i'_\gamma}. \tag{45}$$

14.7.2 Decomposition of the two link lattices

In this subsection, we compare the two lattices constructed in Section 14.7.1.

Let γ, γ' be two terms of cell ranks j, j' over the same agreeing residue block (c, s, c'). Write V_1 for their link intersection and V_0 for the kernel in V_1 of the two z-forms. Let $i = k+1-\dim V_1$. Under the first middle projection V_0 is the space of common X_0-vectors. Split the first link orthogonally, first along V_1, then split V_1 along V_0. The discriminant of its full lattice L^+_γ is the product of the three discriminants of its successive intersection and projection lattices.

Indeed the basis matrix in orthogonal coordinates is block triangular after lifting bases of the projection lattices; the shears along the earlier subspaces do not change its determinant. The first piece projects to a full sublattice of the part of the term lattice in V_0. The lift is orthogonal to both isotropic spaces, its two z-forms vanish, and its A-part is divisible by M. Hence it is orthogonal to the whole translated basis and satisfies the defining term lattice condition. Write d_γ for the discriminant of that part of the term lattice. It is at most the discriminant of the first piece. The latter metric is the original Hermitian form on X_0 because both z-forms vanish.

The second piece is detected injectively by the two z-forms. In the positive model they are pairings, up to signs, against z^+ and $\Upsilon_\gamma^{-1} z^+$. If u is the last coordinate of $\Upsilon_\gamma^{-1} z^+$, their Gram matrix has diagonal N, N and off-diagonal $Nu, N\bar{u}$. At top rank the adjugate last column of the basis product matrix sends, by Υ_γ, a vector with last coordinate δ_{n-1} to $-\delta_n z^+/N$. This follows by taking the corresponding combination of the translated basis vectors, orthogonal to A, with zero second X_0-coordinate, and first z-coordinate $-\delta_n/N$. Therefore $u = -N\delta_{n-1}/\delta_n$, and the Gram determinant is $N^2\mathfrak{g}/|\delta_n|^2$. At top rank both δ_n and $\mathfrak{g}$ are nonzero by Section 15.2. Since $\mathfrak{g}$ is N times a squared norm, the determinant is positive at every real place. When $\delta_{n-1} = 0$ it equals N^2. Below top rank the last common vector gives proportional z-forms. Pairing with z^+ is nonzero, so their joint map on the full link has rank one.

For the second piece, choose enough forms for a nonzero square pairing determinant. The values on a projected lattice vector can be computed on its integral lift because the shear lies in V_0. They are integral. If the links agree and the rank is top, the second piece has dimension

two and Section 14.6 applies to its two integral lifts. Thus the determinant has index at least x if $1 \le x \le Q_{\cap}(s)$. In the other cases take $x = 1$, since the selected determinant is integral and nonzero.

The absolute norm of the Gram determinant of the selected form vectors is at most

$$\left(\frac{\max\{N^f \mu_\gamma, \mathfrak{g}_{\mathrm{norm}}(\gamma)\}}{\mu_\gamma} \right)^2 .$$

In rank zero its value is one. In rank one choose either original form whose restriction is nonzero; its ambient Gram entry is N, of absolute norm N^{2f}. In rank two the displayed Gram determinant gives $(\mathfrak{g}_{\mathrm{norm}}/\mu_\gamma)^2$. Cauchy–Schwarz on the exterior powers at every complex place, followed by multiplication of the finite basis indices, bounds the index of the pairing determinant by the product of the discriminant of the projected lattice and the square root of the absolute norm of the selected form Gram determinant. The second piece therefore has discriminant at least $x\mu_\gamma / \max\{N^f \mu_\gamma, \mathfrak{g}_{\mathrm{norm}}(\gamma)\}$. Writing c_γ for the discriminant of the third piece, the complement to V_1, equation (45) now gives

$$t_\gamma d_\gamma x c_\gamma \le G_\gamma. \tag{46}$$

14.7.3 The common residue and the intersection lattice

In this subsection, we use the decomposition in Section 14.7.2 to prove the discriminant bound.

For two terms γ, γ' with the same block, the factor $m_0 = \gamma\gamma'^{-1}$ is congruent to one on Λ_c modulo M, since unitarity determines the conjugate reductions from the chosen ones and M is a unit at all other places. On integral lifts y, y', the difference form between their two middle pairings is

$$\langle y, y' \rangle - \langle y, m_0 y' \rangle = \langle y, (1 - m_0) y' \rangle \in M \widehat{\mathcal{O}}_E.$$

This calculation is in the original lattice basis. In the positive model the same difference form between source vectors a, b is $\langle a, (1 - \Upsilon_\gamma^{-1} \Upsilon_{\gamma'}) b \rangle$.

It vanishes along the link intersection. It pairs the two complementary spaces perfectly. A vector of the second link orthogonal to the whole first link for the difference form lies in the first link, whose graph is maximal isotropic. The integral lift congruence descends to the projected complementary lattices because the shears lie in the intersection. On each complex exterior power of degree i one has the determinant bound

$$\left| \det \left(\langle a, (1 - \Upsilon_\gamma^{-1} \Upsilon_{\gamma'}) b \rangle / M \right) \right| \le (2/M)^i \sqrt{\det \mathrm{Gram}(a) \det \mathrm{Gram}(b)}.$$

It follows by Cauchy–Schwarz and $\|1 - U\|_{\mathrm{op}} \le 2$ for a unitary U. Take the product of squared complex moduli and include the basis determinant indices of the finite lattices.

The matrix of the pairing divided entrywise by M on these finite lattices is integral. Its determinant is nonzero at every finite place, since the complementary pairing is nondegenerate over E and the local bases are invertible. In particular the determinant before division lies in $M^i \widehat{\mathcal{O}}_E$, and the divided determinant has index at least one. Consequently

$$c_\gamma c_{\gamma'} \ge (M/2)^{2if}. \tag{47}$$

Lastly the common lattice has index at most $t_{\gamma'}$ in the part of the first term lattice in V_0. For v in this part choose the marked lift $y = v - Ma$ with integral $a \in A$ and y orthogonal to the basis of γ. The common middle pair lies in the image of the rational lift map for the second link. A linear section of this surjection over E extends to the finite adeles and gives a lift y', with no integrality condition on y'. Thus $y - y' \in A$ and $\gamma^{-1}y - \gamma'^{-1}y' \in A$. Thus for $m_1 = \gamma'\gamma^{-1}$,

$$(m_1 - 1)y = (y' - y) + \gamma'(\gamma^{-1}y - \gamma'^{-1}y') \in A + \gamma' A.$$

The vector $w = (1 - m_1)y/M$ is integral on the left marked lattice by the same-residue congruence.

The products of v with the basis of γ' equal those of $M(a + w)$ because $m_1 y$ is orthogonal to that basis. Write

$$A_0 = \Lambda_c \cap A, \quad A_1 = \gamma'(\Lambda_{c'} \cap A), \qquad W_0 = \Lambda_c \cap (A + \gamma' A).$$

Let P' send a vector to its pairings against $\gamma' F$. Since P' vanishes on A_1, the finite module $W_0/(A_0 + A_1)$ surjects onto $P'(W_0)/P'(A_0)$. For v in the first term lattice restricted to V_0, the preceding calculation sends v to $P'(v)/M$ in this quotient. Indeed the class vanishes precisely when $P'(v) = MP'(a')$ for some $a' \in A_0$. Then $v - Ma'$ is an integral lift orthogonal to $\gamma' F$, which is exactly the second term-lattice condition. Membership in the first term lattice and in V_0 was already imposed. Thus the kernel is L_{12}, and its index divides $t_{\gamma'}$, by the definition of that torsion index. The same argument in the other order gives

$$d \le t_{\gamma'} d_\gamma, \qquad d \le t_\gamma d_{\gamma'}.$$

The first pieces of the two link lattices are full in V_0, so the finite indices just proved make L_{12} a full lattice there. It lies in the integral definite lattice Λ_c^0, hence $d \ge 1$. Put

$$l = k - \dim V_0, \qquad X = \begin{cases} x, & i = 0,\ j = n, \\ 1, & \text{otherwise}, \end{cases} \quad 1 \le x \le Q_\cap(s).$$

The equal-link case is $i = 0$, since both links have dimension $k + 1$. On the graph of Υ_γ, the space with both z-coordinates zero has dimension $k - 1$ at top rank, and k below top rank, by the two-form calculation above. This also shows that equal links have the same top-rank status, so the value of X is symmetric in the terms. If r_1 is the rank of the two forms restricted to V_1, then

$$\dim V_0 = k + 1 - i - r_1, \qquad l = i + r_1 - 1 \le i + (j - \varrho)_+.$$

In particular, equal links have $l = 1$ in the top cell and $l = 0$ in the lower cells.

Lemma 14.3 (Discriminant of the pair lattice)**.** *For the original terms over one agreeing residue s, with cell ranks as in Section 14.4, let d, X, i, l be as above. For $1 \le x \le Q_\cap(s)$,*

$$(dX)^2 \le (2/M)^{2if} G_\gamma G_{\gamma'}, \qquad l \le i + (j - \varrho)_+. \tag{48}$$

The first inequality follows by multiplying the two torsion bounds and applying (46) to each term and (47) to the complement. For near terms both gauges are at most Δ_{lat}. Taking $x = 1$ in (48) and using $M/2 \ge b^{18}$ gives

$$b^{18if} d \le \Delta_{\mathrm{lat}}.$$

In particular $d \le \Delta_{\mathrm{lat}}$ and $l \le i + 1 \le n^3$. Hence all such near pairs satisfy the rank and discriminant conditions of Section 14.5.

15 Character cancellation and Schur's test

In this section, we apply character orthogonality to the rational terms counted in Section 14. The resulting cell estimates and Schur bound give the off-diagonal sums used in Section 9.1.

15.1 Character cancellation on the marked blocks

In this subsection, we integrate the two actions of the Bessel unipotent subgroup on the terms from Section 14.1. Character orthogonality leaves the stabilizer agreement condition analyzed in Section 15.2.

Let T_0 be a rational term set invariant under U on both sides. On $(T_0 \cap \mathcal{T}(c,s,c')) \times G_\infty$ the group $U_c \times U_{c'}$ acts by

$$(u,u') \cdot (\gamma, y) = (u\gamma(u')^{-1}, uy).$$

This action preserves counting times Haar measure. It preserves the full and cut kernels $K^\sharp_\gamma(y)$ of Section 14.1, because the real characters on the two class lattices are trivial by the product formula and the original principal congruence. The cut uses a left unipotent-invariant length. One can replace the original weight $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$ by any nonnegative measurable weight with orbit sum one. Insert the new orbit sum, change variables in each summand, and sum the old weight. Tonelli and the absolute convergence proved in Section 14.1 justify this both for the norm of the kernel and for the kernel itself. The same argument applies to the norm of the full-minus-cut kernel.

For a double coset with representative γ, put $\Gamma_\gamma = U_c \cap \gamma U_{c'}\gamma^{-1}$. Write its terms uniquely as $u\gamma(u')^{-1}$, with u running over representatives of U_c/Γ_γ and $u' \in U_{c'}$. After the substitution $y \mapsto uy$, the sum of the right cutoffs over u' is one. The remaining weight $\sum_u \beta_c(uy)$ has orbit sum one for Γ_γ. The contribution is thus $\int_{\Gamma_\gamma\backslash G_\infty} K^\sharp_\gamma(y)\,dy$.

Say the rational characters agree at γ if $\chi_f(u) = \chi_f(u')$ whenever $u, u' \in U(F_0)$ and $u\gamma = \gamma u'$, where F_0 is the totally real subfield of E and χ_f denotes the finite Bessel character. On the real unipotent group $H_\gamma = U_\infty \cap \gamma U_\infty \gamma^{-1}$, the kernel transforms on the left by the quotient of the two real characters. This quotient is trivial on Γ_γ. If the rational characters disagree, their rational element gives a nontrivial value of the real quotient by the product formula. Integrate in stages over $H_\gamma\backslash G_\infty$ with fiber $\Gamma_\gamma\backslash H_\gamma$ by the quotient integral formula for the unimodular groups. Right translation by the real element preserves the fiber measure and multiplies the integrand by the nontrivial value. The fiber integral is zero on almost every fiber by absolute integrability. Therefore every stable set of rationally disagreeing terms contributes zero.

There is also a finite test on a whole residue block. Let $\bar{U}$ be the reduction of the integral finite unipotent group, with its induced character ψ_U. For $\bar{u}, \bar{u}' \in \bar{U}$, lift them to the integral finite group. Approximate the lifts in the respective class bases by rational unipotents α, α' with the same reductions. The free coordinates and skew center give this approximation as in Section 14.2.

The class representatives fix the basis, so this conjugation preserves the character. The approximants normalize the class principal unipotent groups since their finite transports are integral and J is normal in the integral stabilizer. Under the change $(\gamma, y) \mapsto (\alpha\gamma(\alpha')^{-1}, \alpha y)$ the translated weight has orbit sum one in the new block. By the cutoff identity and the product formula,

$$A^\sharp_{T_0}(c, \bar{u}s(\bar{u}')^{-1}, c') = \psi_U(\bar{u})^{-1}\psi_U(\bar{u}')A^\sharp_{T_0}(c,s,c').$$

In particular an agreeing residue for a nonzero block must satisfy $\psi_U(\bar{u}) = \psi_U(\bar{u}')$ whenever $\bar{u}s = s\bar{u}'$.

15.2 The common flag and the cell rank

In this subsection, we use agreement of the characters from Section 15.1 to compare the isotropic flags. Their intersections determine which ranks contribute to the open and closed terms in Section 15.3.

Suppose the rational characters agree at γ. Write $Fx = \gamma Fy$ for a common vector supported in both bases through index $a < \varrho$, with coefficient pair (x_a, y_a). Let λ be a linear functional vanishing on both initial segments through a, and write $\lambda = h(v, \cdot)$. For $A_1 = tFx$ consider

$$u(w) = w + h(v, w)A_1 - h(A_1, w)(v + \tfrac{1}{2}h(v, v)A_1).$$

Expansion using $h(A_1, A_1) = h(v, A_1) = 0$ gives $h(u(w), u(w')) = h(w, w')$. In the marked Witt order the dual coordinates of v through index a vanish, and $h(A_1, \cdot)$ reads only these dual columns.

Thus the corrections are strictly upper triangular in the Witt blocks, with identity middle block. In particular $u \in U(F_0)$. It acts on f_b by adding $\lambda(f_b)A_1$, so its simple-root sum is $tx_a\lambda(f_{a+1})$. The conjugate $\gamma^{-1}u\gamma$ has the same formula with $A_1 = t\gamma Fy$ written in the second basis, giving root sum $ty_a\lambda(\gamma f_{a+1})$. Put $c = \lambda(x_a f_{a+1} - y_a\gamma f_{a+1})$. For $t = -M\delta_E q$, agreement gives $\psi_{E,f}(qc) = 1$ for every $q \in E$, by Section 8.3. If $c \neq 0$, take $q = (2[E : \mathbb{Q}]c)^{-1}$. Then $\mathrm{Tr}_{E/\mathbb{Q}}(qc) = 1/2$, so the product formula gives $\psi_{E,f}(qc) = -1$. Hence $c = 0$. The annihilator criterion puts this difference in the sum of the two initial segments. Rearranging supplies a common vector with coefficient pair (x_a, y_a) at $a + 1$.

Iteration to the last position and comparison of norms there give $x_a\bar{x}_a = y_a\bar{y}_a$, since the norm of a basis vector at last coefficient x_a is $-Nx_a\bar{x}_a$. Applied at the larger of the two highest indices of a nonzero common vector, this shows that both indices agree. A nonzero common vector of least top index j can be scaled to give a pair $(\zeta, 1)$, $\zeta\bar{\zeta} = 1$, which then occurs at every index at least j.

Write δ_j for the leading minor of Z_γ of order j, with $\delta_0 = 1$, and put $\mathfrak{g}_\gamma = \delta_n\bar{\delta}_n - N^2\delta_\varrho\bar{\delta}_\varrho$. For the last adjugate column q of Z_γ, the products of γFq with the basis vanish except at the last entry δ_n. Its last coefficient in the second basis is δ_ϱ. It lies in $A^\perp$ with z-coordinate $-\delta_n/N$. Comparing its norm in these two expressions gives $\mathfrak{g}_\gamma = Nh(w_0, w_0)$ for its X_0-part w_0. Thus a nonzero $\mathfrak{g}_\gamma$ implies $\delta_n \neq 0$, and proves rank n. If $\mathfrak{g}_\gamma = 0$ and $\delta_n \neq 0$, the same adjugate column gives a nonzero common vector. If $\delta_n = 0$, any nonzero $q \in \ker Z_\gamma$ gives $\gamma Fq \in A + X_0$ of nonpositive norm, hence in A. This also gives a nonzero common vector.

For zero discriminant with rational agreement, take the least top index j as above. The shared isotropic vectors of indices $j, \ldots, \varrho - 1$ span $A \cap \gamma A$ by descending elimination of their highest coefficients. If $j > 0$ and $\delta_j = 0$, a nonzero vector in the first j columns of γF pairs to zero with the first j columns of F. It is perpendicular to the shared isotropic vectors and hence to all of A. It also pairs to zero with the final shared basis vector, whose coefficient at z is nonzero, so it is perpendicular to z. It is isotropic in $A + X_0$ and thus belongs to A. Its top index is below j in the second basis and therefore in both bases, contradicting minimality. This gives $\delta_j \neq 0$ and the remaining cell-rank conditions of Section 14.4. This rank is unique. No index below j admits the required common vector. For $j < h \leq \varrho$, the common vector at index j lies in A, is orthogonal to F, and is a nonzero combination of the first h columns of γF.

Thus $\delta_h = 0$. The zero discriminant excludes $h = n$. If $j = 0$, the shared isotropic basis gives $\gamma A = A$.

15.3 Scalar cosets and the surviving terms

In this subsection, we classify the terms satisfying the agreement and rank conditions of Sections 15.1 and 15.2. This identifies the scalar contributions and the terms estimated by the orbital bounds.

If the rational characters agree and $\gamma A = A$, the common vector f_0 has top index zero in both bases. Thus $\gamma f_0 = \zeta f_0$ for $\zeta\bar{\zeta} = 1$, and the common pairs above give $\gamma f_a - \zeta f_a \in \langle f_b : b < a\rangle_E$ by successive substitution. Hence $\zeta^{-1}\gamma$ is in the rational Bessel subgroup $R(F_0) = (U \rtimes H)(F_0)$. Indeed it acts trivially on the successive isotropic lines and fixes the class of z in $A^{\perp}/A$. In a term of $\mathcal{T}(c, s, c')$ the first diagonal entry ζ of the integral transporter in the marked class basis is integral at every finite place. So is $\zeta^{-1} = \bar{\zeta}$. It follows that $s \in \mathrm{red}(\zeta)H_R$.

Put

$$\Gamma_* = \{\gamma : \mathfrak{g}_\gamma \neq 0\} \cup \{\gamma : \mathfrak{g}_\gamma = 0,\ \gamma A \neq A, \text{rational characters agree}\}.$$

This is stable under both unipotent actions. On the basis they induce upper unitriangular column moves and lower unitriangular covector moves on the product matrix, so the leading minors are preserved. They preserve A, and inner conjugations preserve the unipotent character. On $R(F_0)$ the first product column vanishes, since the first basis vector is fixed and orthogonal to the basis. Thus $R(F_0)$ and Γ_* are disjoint. Outside their union a rationally agreeing term must preserve A and hence belongs to a scalar coset as above. Rationally disagreeing terms outside this union contribute zero by Section 15.1. For residues outside the nontrivial integral norm-one scalar cosets we obtain

$$A^{\sharp}_{R(F_0)^c}(c, s, c') = A^{\sharp}_{\Gamma_*}(c, s, c'). \tag{49}$$

Every term on the right has a cell rank $1 \le j \le n$ by Section 15.2. The nonzero discriminant terms have $j = n$.

An integral norm-one scalar ζ lies in μ_{q_1} by Kronecker's theorem and Section 4.3. At each level place reduction is injective on μ_{q_1}: if $\zeta \neq 1$ reduces to one, then reducing $1+\zeta+\cdots+\zeta^{q_1-1} = 0$ gives $p \mid q_1$, contrary to $p \equiv 1 \pmod{q_1}$. An element of $\mathrm{red}(\zeta)H_R$ acts on the marked span by the scalar times a unitriangular matrix. Thus fixing a nonzero vector forces $\zeta = 1$. For a covector use the conjugate-place action of Section 9.1. In particular any one of the prescribed fixers excludes the nonidentity scalar cosets. On $H_R G_i$ the untouched factor fixes the first basis vector and also excludes these cosets. This gives a common basis vector at each of the marked places and conjugate places of that factor, with meeting norm at least $(p^{\mathrm{lev}}_{1-i})^{2f} \ge (M/b)^f$, since $p^{\mathrm{lev}}_i \le b p^{\mathrm{lev}}_{1-i}$ by Section 4. Every scalar coset has vanishing first leading minor at all marked places, so the open set avoids them all. For a fixer at every first-prime place, any fixed nonzero vector in the marked span lies in the intersection of that span with its image. A fixed covector gives the same intersection at the conjugate place via the marked Hermitian pairing and inverse-transpose action. Thus the meeting norm is at least $(p^{\mathrm{lev}}_0)^f$ on the full fixer.

15.4 The real estimate in the open Gram cell

In this subsection, we estimate a real orbital integral when the Gram matrix is in the open cell. The bounds of Section 12.11 control the derivatives in the passage from a Gram neighborhood to a fixed Gram matrix.

Work at one real embedding in the marked orthonormal basis $E = (e_0, \dots, e_{\varrho-1}, z)$, with $n = \varrho+1$ and $E^*JE = -Q_0$, where Q_0 projects onto the last column. Use the full Bessel vector W, its quotient mass $m_W = \|W\|^2_{U\backslash G}$, and its formal degree d_W in dg. Put

$$Z = E^*J\gamma E, \qquad \Delta(Z) = |\det Z|^2 - |\det Z_{<n-1,<n-1}|^2.$$

Assume $\Delta(Z) > 0$ and that some y_0 has both $y_0^{-1}E$ and $y_0^{-1}\gamma E$ bounded by $\rho = b^{37n}$ in each column.

15.4.1 Haar density in Gram coordinates

In this subsection, we compute the density for the real estimate of Section 15.4.

The pointwise basis stabilizer is $U_0 \rtimes H$ with $H = U(k)$. Its Haar is $du\,dh$, with dh a probability and du Lebesgue in the Eichler coordinates. This Haar is invariant under inversion since H preserves the coordinate volume and U_0 is unimodular.

The full Gram fibre has density $2^{-r}\operatorname{vol}U(r)\,dg$ in the marked Hermitian coordinates by Section 12.6. For a rectangular tuple Y, split $Y = JEZ + F_0p$, where F_0 consists of the free isotropic and definite columns. The blocks JE, F_0 form a Euclidean orthonormal basis. For invertible Z, write $Y = Y'Z$. In $A \oplus X_0 \oplus \mathbb{C}z \oplus A^\vee$ the columns of Y' have blocks

$$Y' = \begin{pmatrix} A_0 & a \\ B & x \\ 0 & -1 \\ I & 0 \end{pmatrix}.$$

For $T' = (Y')^*JY'$, its first Gram block is $A_0 + A_0^* + B^*B$, its cross block is $a + B^*x$, and its last entry is $\|x\|^2 - 1$. Solving $A_0 = S + T'_{00}/2 - B^*B/2$ gives $2^{-\varrho}du$ in the skew and free coordinates, with the remaining factor in dT'_{00}. The first columns are the Eichler translate of the columns with $B = S = 0$ at that first Gram block. On the remaining column, the Eichler action on the isotropic coordinates is a translation once its definite coordinates are fixed. Thus solving for a also gives Jacobian one. Polar integration in x gives $\frac{1}{2}\operatorname{area}(S^{2k-1})(1+T'_{nn})^{k-1}\,dh$ times the last real Gram coordinate. The angular measure is the pushforward of probability Haar under $U(k) \to U(k)/U(k-1)$.

Right multiplication by Z scales the measure of the free coordinates by $|\det Z|^{2(\varrho+k)}$. For $T = Z^*T'Z$, Hermitian volume changes by $|\det Z|^{2n}$. In fact unitary factors preserve it, and diagonal singular values s_i give s_i^2 on the i-th real diagonal and s_is_j on each of the two real upper coordinates for the pair $i < j$. The net Gram-fibre factor is $|\det Z|^{2(k-1)}$. At $T = -Q_0$, the radius satisfies $1 + T'_{nn} = \Delta(Z)/|\det Z|^2$.

The resulting basis fibre in the open cell has density

$$2^{-(\varrho+1)}\operatorname{area}(S^{2k-1})\Delta(Z)^{k-1}\,dZ\,du\,dh.$$

Here dZ is ordinary complex entry measure. The fibre comparison holds at the prescribed Gram matrix $T_0 = -Q_0$. Indeed all the columns in the open chart are independent. At such a matrix Y, Y^*J has a right inverse C; for a Hermitian increment D,

$$d(Y^*JY)[CD/2] = D.$$

The fibre integral of a continuous test compactly supported in a submersion chart is therefore continuous in the Gram variable. Average over shrinking neighborhoods of T_0 and exhaust the open chart by such compact tests. Positivity gives domination by the basis fibre measure. Completing with dual isotropic and definite columns, as in Section 12.6, multiplies by $2^{-\varrho}du'$ and $2^{-k}\operatorname{vol}U(k)\,dh'$, respectively.

Comparison with the full Gram density gives

$$\int c_G\Delta(Y)^{k-1}v(s\gamma_Y s')\,dY\,ds\,ds' \le \int_G v\,dg, \qquad c_G=\frac{\operatorname{vol}U(k)}{\operatorname{vol}U(r)}\frac{2\pi^k}{(k-1)!}.$$

The left variables range over the open cell and both basis stabilizers, with the section γ_Y given below. The inequality of measures extends to nonnegative measurable tests.

15.4.2 A neighborhood in the original group

In this subsection, we use the density of Section 15.4.1 to compare neighborhoods in the original measure.

In the original Eichler coordinates of U_0, write its parameters as (B,C) with $C+C^*=B^*B$. Its group law is

$$(B,C)(B',C')=(B+B',C+C'+B^*B').$$

Let V_R be the ball $\|B\|\le R$, $\|C\|\le R^2$. The group law shows $V_RV_{R'}\subset V_{R+R'}$. The Haar measure is Lebesgue measure in the real coordinates of B and the skew part of C. The dilation $(B,C)\mapsto(RB,R^2C)$ has Haar factor $R^{2\varrho(k+\varrho)}$. For $T\ge 8r$, take

$$\Omega=V_{r^2T^2}, \qquad A_0=\Omega\{y:\|y\|\le T\}.$$

Both sets are compact and Ω has positive volume. The block formula gives $\|u\|\le(1+r^2T^2)^2$ on Ω, so $\|y\|\le T^{10}$ on A_0.

On a nonempty left U_0-slice there is a representative z of norm at most T. A point in the slice has $uz=sx$, $s\in\Omega$, $\|x\|\le T$. Then $s^{-1}u=xz^{-1}$ has norm at most T^2, since $\|z^{-1}\|=\|z\|$. Its two Eichler blocks have norm at most T^2, whence it lies in V_{T^2}. The slice has volume at most the volume of $V_{(r^2+1)T^2}$, at most $(1+r^{-2})^{2\varrho(k+\varrho)}\operatorname{vol}(\Omega)\le 3\operatorname{vol}(\Omega)$. The equality of volumes before and after changing the representative uses right invariance of du_0.

For the section γ_Z constructed below, put $M_{\mathrm{loc}}=\rho^{9n}/\Delta(Z)$. The boxed columns give $|Z_{ij}|\le\rho^2$, so

$$0<\Delta(Z)\le|\det Z|^2\le(n\rho^2)^{2n}\le\rho^{6n}.$$

Thus $M_{\mathrm{loc}}\ge\rho\ge 8r$ and $M_{\mathrm{loc}}^{-1}\le\Delta(Z)$. In this local calculation only, write $M=M_{\mathrm{loc}}$, set $R=M^{9n}$, and take $T=R^5$. The section is the identity outside the span of E and $E'=(e_0^*,\dots,e_{\varrho-1}^*,x_0)$, with x_0 the marked definite vector. Its defining basis equations give the following identity for every $a<\varrho$:

$$e_a^*=\gamma_Z^{-1}E(Z^*)^{-1}\mathbf{e}_a+\tfrac12\sum_{i<\varrho}(Z^{-1})_{i,\mathrm{last}}((Z^*)^{-1})_{\mathrm{last},a}\,e_i,$$

where $\mathbf{e}_a$ is the coordinate vector.

It results from the equation taking the section's Z^*-dual columns to $E(Z^*)^{-1}$; the last positive basis vector occurs only in the last column. For g in the support test, the columns of $g^{-1}E$ have norm at most M, and γ_Zg can be moved by $u\in U$ to $z'=u\gamma_Zg$ of norm at most

M, by the Bessel length bound. Indeed approximate the infimum of $\|s\gamma_Z g\| + 1$ for $s \in UH$ to within $1/2$, and remove the unitary compact factor.

Put $Z' = E^* J u \gamma_Z E$. Since the action of u on E is upper unitriangular, the leading minors are unchanged under $Z \mapsto Z'$, and $(Z')_{ij} = \langle (z')^{-1} e_i, g^{-1} e_j \rangle_J$ with the last index interpreted as z. Thus the entries of Z' have norm at most M^2; $\Delta(Z') = \Delta(Z)$. The adjugate estimate gives $\|(Z')^{-1}\| \le R$ and $\|Z^{-1}\| \le R$. If $uE = EU_E$, then $Z = U_E^* Z'$ and $U_E (Z^*)^{-1} = ((Z')^*)^{-1}$. Applying g^{-1} to the change of basis now replaces the first summand by $(z')^{-1} E ((Z')^*)^{-1} \mathbf{e}_a$.

It bounds each $g^{-1} e_a^*$ by $nMR + \varrho M R^2/2 \le R^4$. Finally, for $L = g^{-1}$ use $LL^* = J + L(1 - J)L^*$. The columns of $1 - J$ are zero or signed differences of the marked isotropic and dual vectors, and $2z$. Therefore $\|L(1 - J)\| \le (4\varrho + 2)R^4$, so $\|L\|^2 \le 1 + (4\varrho + 2)R^4 \|L\|$. This gives $\|g\| + 1 \le 8nR^4 \le R^5$. Thus if $g^{-1}E$ is boxed and the Bessel length of $\gamma_Z g$ is at most ρ, then $\|g\| \le T$.

On the original double coset with Gram matrix Z, write $\gamma = s\gamma_Z s'^{-1}$ with s, s' fixing the basis. Indeed the common Gram matrix of the two ordered pairs is $B(1)$, where

$$B(t) = \begin{pmatrix} -tQ_0 & Z \\ Z^* & -tQ_0 \end{pmatrix}, \qquad \det B(t) = (-1)^n |\det Z|^2 (1 - t^2 |(Z^{-1})_{nn}|^2), \quad 0 \le t \le 1.$$

The determinant follows by interchanging the two column blocks and taking the Schur complement of Z. It is nonzero throughout this interval since $\Delta(Z) > 0$. Thus $B(1)$ has the same signature (n, n) as $B(0)$. Both orthogonal complements are positive definite of dimension $k - 1$, so the isometry matching the ordered pairs extends by matching orthonormal bases there. For $v \in \Omega$ and $h \in H$ choose $t = s'(vh)^{-1}$. The support test at the section applies to $g = (vh)^{-1} y$ because the first basis is unchanged and Bessel length is left invariant under s. If the columns of $y^{-1}E$ are bounded by ρ and $\gamma t y$ has Bessel length at most ρ, it gives $\|v^{-1} y\| = \|g\| \le T$, so $y \in A_0$. The support implication holds for each shift and all y.

15.4.3 The transverse integral

In this subsection, we apply the neighborhood estimate of Section 15.4.2 to the section of the Gram map.

Write $P_0 = \mathrm{diag}(1, \dots, 1, 0)$ and $Q_0 = 1 - P_0$, and put

$$\begin{aligned} H_Y &= Y^{-*} Q_0 Y^{-1}, & \eta_Y &= (1 - |(Y^{-1})_{\mathrm{last,last}}|^2)^{1/2}, \\ A_Y &= -Q_0 - \tfrac{1}{2} P_0 H_Y (1 + Q_0), & L_Y &= E A_Y + E'(P_0 + \eta_Y Q_0). \end{aligned}$$

The block Gram matrix on (E, E') is $\left(\begin{smallmatrix} -Q_0 & P_0 \\ P_0 & Q_0 \end{smallmatrix}\right)$. In particular

$$E^* J L_Y = 1, \qquad L_Y^* J L_Y = -Q_0 + A_Y^* P_0 + P_0 A_Y + \eta_Y^2 Q_0 = -H_Y,$$

where $Q_0 H_Y Q_0 = (1 - \eta_Y^2) Q_0$. The isometry sending (E, L_{Y^*}) to $(L_Y Y, E Y^{-*})$ exists because the displayed identities give identical Gram matrices, and both ordered pairs are bases of the span of (E, E'). Extend it by the identity on the definite orthogonal complement to get γ_Y. In particular it has the original basis products Y. If $S_Y = L_Y Y$, its column formula is

$$\gamma_Y = 1 - E E^* - E' E'^* + S_Y E^* + (E Y^{-*} - S_Y A_{Y^*})(P_0 + \eta_Y^{-1} Q_0) E'^*.$$

These equations give the dual-column identity used above.

Suppose $P \ge 8r$ bounds $\|Y\|, \|Y^{-1}\|, \eta_Y^{-1}$. For a real tangent of unit norm,

$$\|\dot{Y}^{-1}\| \le P^2, \quad |\dot{\eta}_Y| \le P^3, \quad |(\eta_Y^{-1})^{\cdot}| \le P^5.$$

The product rule gives $\|A_Y\| \le 2P^2$, $\|\dot{A}_Y\| \le 2P^3$, $\|S_Y\| \le 3P^3$, and $\|\dot{S}_Y\| \le 4P^4$. Thus $\|\gamma_Y\| \le P^{18}$ and $\|\dot{\gamma}_Y\| \le P^{25}$. The stable section neighbourhood can be taken as $\|Y - Z\| < P^{-6}$, with $M \ge 8r$, $M^{-1} \le \Delta(Z)$, $|Z_{ij}| \le M^2$, $R = M^{9n}$, and $P = (R^5)^n$. In fact the adjugate and determinant estimates give $\|Z\|, \|Z^{-1}\|, \eta_Z^{-1} \le M^{8n} \le P/2$. A perturbation of norm P^{-6} keeps the inverse below P and changes it by at most $P^{-4}/2$. The absolute change in $1 - |(Y^{-1})_{\mathrm{last,last}}|^2$ is at most $2P^{-4}$, whereas its initial value is at least $4/P^2$. Thus $\eta_Y^{-1} \le P$. The determinant ratio of $Z = (1 + (Z - Y)Y^{-1})Y$ is bounded above by $(1 + P^{-5})^n \le 9/8$. The relative loss in η_Z^2 is at most $2P^{-4}/(4P^{-2}) \le 1/4$. Since $\Delta(Y) = |\det Y|^2 \eta_Y^2$, these bounds give $\Delta(Y) \ge (3/4)(8/9)^2 \Delta(Z) > \Delta(Z)/2$.

Let $c = c_G(\Delta(Z)/2)^{k-1}$. For a nonnegative function v invariant under the full left UH, a measurable cutoff of integral one on each left U_0-orbit exists by local sections of $U_0 \backslash G$. In a section chart take a compactly supported continuous density on U_0 of mass one and multiply it by a partition of unity on the quotient. Restrict the cutoff to the inverse image of the Gram ball.

Its U_0-integral there remains one, since a basis stabilizer keeps the Gram matrix fixed. The lower Haar density above and the probability Haar of H then give

$$c \int_{\|Y - Z\| < P^{-6}} \int_{H \times U_0} v(\gamma_Y (uh)^{-1}) \, dh \, du \, dY \le \int_G \chi(y) v(y) \, dg(y).$$

In the right integral unfold first along U_0, then along U/U_0, which acts on E by unitriangular matrices. If two points in a U-orbit lie over the Gram ball, the adjoint of the transition matrix on E is the product of the first Gram matrix with the inverse of the second. All its entries are at most P^2 in modulus. Using the original Haar on the upper entries of the basis matrix, the last unfolding contributes at most $(2P^2)^{2n^2}$. This proves

$$c \int_{\|Y - Z\| < P^{-6}} \int_{H \times U_0} v(\gamma_Y (uh)^{-1}) \, dh \, du \, dY \le (2P^2)^{2n^2} \int_{U \backslash G} v.$$

For matrix-entry words j of length at most $2n^2$, Section 12.11 gives

$$\int_{U \backslash G} \left\| \int_G \beta(y) W_j(gy) \, dg(y) \right\|^2 \le P^{2|j|} \frac{m_W}{d_W} \int_G |\beta|^2 \, dg,$$

since $b^5 \le P$. The derivatives have the same unitary left Bessel covariance as W.

For a test supported in A_0, the logarithmic tangent for differentiation at gy is $y^{-1} X y$ when differentiating at g along gX. Since $\|y\| \le P^{20}$, each matrix-entry coefficient contributed by this conjugation, divided by P^{40}, has modulus at most one. More precisely attach the fixed weight $(y^{-1})_{ac} y_{db} / P^{40}$ to the differentiation generator E_{ab} and the external coefficient X_{cd}. After expansion each integral is a convolution with a weight bounded by $|\beta|$ and one of the W_j. Differentiation under the integral uses continuity and boundedness of these derivatives on the compact support, locally in the external variable. Each of the r^4 possible new generators contributes at most $P^{40} \|X\|$. On the cube of side P^{-8} based at Z in $2n^2$ entry coordinates, take $g = \gamma_Y (uh)^{-1}$ with $u \in \Omega$, so that $\|u^{\pm 1}\| \le P^{20}$. For each real entry coordinate of Y,

$$X = g^{-1} \dot{g} = (uh) \gamma_Y^{-1} \dot{\gamma}_Y (uh)^{-1}, \qquad \|X\| \le P^{20+18+25+20} \le P^{85},$$

since h is Euclidean unitary and $\|\gamma_Y^{-1}\| = \|\gamma_Y\|$. Therefore each first derivative has a sum of at most r^4 terms with coefficients at most P^{125}.

For a Hilbert-valued function on $[0, \epsilon]$,

$$\|a(0)\|^2 \le 2\epsilon^{-1} \int_0^\epsilon \|a\|^2 + 2\epsilon \int_0^\epsilon \|a'\|^2.$$

Apply it successively in all $2n^2$ coordinates. At each step apply it to the separate convolutions obtained at the preceding step, with the previous external coefficients already bounded. Each application adds at most one differentiation generator and leaves the fixed weights supported on A_0. Per coordinate the coefficient sum is at most $2(1 + P^{258})P^8$, since $r^8 \le P^8$ in the squared derivative bound. Apply the Gram-ball inequality to each word, whose squared norm is left Bessel-invariant. The combined contribution per coordinate is at most P^{275}. The quotient Schur bound contributes at most another P^{4n^2}. This proves the transverse estimate

$$c \int_{H\times\Omega} \left\| \int_{A_0} \beta(y) W(\gamma_Z(uh)^{-1} y)\, dg(y) \right\|^2 dh\, du \le P^{2000n^2} \frac{m_W}{d_W} \int_{A_0} |\beta|^2\, dg.$$

The cube lies in the Gram ball since $2n^2 P^{-8} < P^{-6}$.

For $a = (h, u)$, put $g_a = \gamma_Z(uh)^{-1}$ and use the shift t_a with $\gamma t_a = s g_a$ constructed above. The Bessel covariance is unitary on the value space, so

$$\int_{A_0} \phi(y) W(\gamma t_a y) W(t_a y)^*\, dg(y) = U(s) \left(\int_{A_0} \phi(y) W(g_a y) W(y)^*\, dg(y) \right) U(t_a)^*.$$

Its squared Hilbert–Schmidt norm is the sum of the squared norms of the inner columns. Writing W_j here for a coordinate of W, apply the scalar estimate to $\beta_j = 1_{A_0} \phi \overline{W_j}$ and sum, using $\sum_j |\beta_j|^2 = 1_{A_0} |\phi|^2 \|W\|^2$. Dividing by $c \operatorname{vol}(\Omega)$ gives the contraction with normalized measure $dh\, du/\operatorname{vol}(\Omega)$ on $H \times \Omega$. Thus the operator in the parameter a is a contraction from L^2 of the measure on the source to L^2 of the normalized measure with Hilbert–Schmidt values. The source measure has density

$$\frac{P^{2000n^2} m_W}{d_W\, c \operatorname{vol}(\Omega)} 1_{A_0}(y) \|W(y)\|^2\, dg(y).$$

The same slice volume cancels on unfolding $\int_{A_0} \|W\|^2\, dg$, since the slice contribution is at most $3 \operatorname{vol}(\Omega)$, the upper-entry contribution is at most $(2P^{40})^{2n^2} \le (MP^{40})^{2n^2}$, since two points in the tube have norms at most P^{20} and the transition matrix on their basis has entries at most P^{40}, and $\|W\|^2_{U\setminus G} = m_W$. Thus the total contribution is

$$M^{10^5 n^4} \frac{m_W^2}{d_W\, c_G (\Delta(Z)/2)^{k-1}},$$

using $3 \le R^5$ and $P^{2000n^2} R^5 (MP^{40})^{2n^2} \le M^{10^5 n^4}$. This bounds the mass of the measure on the source of the one-place contraction.

15.5 The contribution of large discriminants

In this subsection, we apply the real estimate of Section 15.4 when the discriminant factor dominates. Its decay can then be combined with the lattice comparison in Section 14.7.

For an agreeing block (c, s, c'), denote by $\mathcal{R}_{S_0}(c, s, c')$ the chosen near representatives in a stable set S_0 having a cell rank. For γ of rank j put

$$D_j = \begin{cases} D_{\mathrm{main}}, & j = n, \\ \mathfrak{t}^{-16} D_{\mathrm{id}}, & j < n, \end{cases} \qquad w_\gamma = \mathfrak{t}^2 \sqrt{\frac{D_j}{n_p G_\gamma^k}},$$

with $n_p = |H_R| d_C / N^{kf}$ as in (42) and G_γ as in (43).

For a stable subset of the surviving terms, take its part on which $\mathfrak{g}_{\mathrm{norm}}(\gamma) > N^f \mu_\gamma$. Both quantities are stable under the rational unipotent moves, by the leading-minor and cell-rank invariance. On this part the cell rank is n, both first indices are one, and (43) gives $G_\gamma = M^{2\varrho f} \mathfrak{g}_{\mathrm{norm}}(\gamma)$. For a near representative write $x_\gamma = \mathfrak{g}_{\mathrm{norm}}(\gamma)$. The basis products are integral after the marked transport, so $x_\gamma \geq 1$ by the norm formula. The box bound in Section 14.7 gives $x_\gamma \leq b^{270 n^2 f}$. At v the normalized discriminant is

$$\Delta_v = \left(\prod_a a_{v,a} \right)^2 |\mathfrak{g}(\gamma)|_v .$$

Indeed the full minor scales by the product of all the squared basis scales, whereas the preceding minor omits the last one, $1/N$. Section 15.4 gives the bound for the local source measure

$$P_v = \left(\frac{\rho^{9n}}{\Delta_v} \right)^{10^5 n^4} \frac{d_v^2}{\deg_v \, c_G (\Delta_v / 2)^{k-1}},$$

Here $\deg_v$ is the formal degree of the ambient discrete series in the chosen real Haar measure. For the compact coefficient of degree a_τ at v, Section 12.10 gives

$$d_v = \|W_v\|^2_{U_v \backslash G_v} = \dim \mathrm{Sym}^{a_\tau}(\mathbb{C}^k) = \binom{k - 1 + a_\tau}{k - 1}, \qquad d_B = \prod_v d_v, \qquad \mathcal{B} = \prod_v \deg_v .$$

The product contraction. Write $K_v^\epsilon(\lambda, z) = W_v^\epsilon(\lambda z) \otimes W_v^\epsilon(z)^*$, with $\epsilon \in \{\mathrm{full}, \mathrm{cut}\}$. For each v, use the compact set A_v, the probability m_v, the finite measure ν_v and the measurable basis-fixer shift $t_v(a_v)$ in the local restriction estimate. The linear map is

$$(T_v \phi)(a_v) = \int_{A_v} \phi(z_v) K_v^{\mathrm{full}}(\lambda_v, t_v(a_v) z_v) \, dg_v(z_v)$$

and its norm from $L^2(\nu_v)$ to $L^2(m_v; \mathrm{HS}_v)$ is at most one. For a bounded measurable ϕ on the product the tensor operator still has the displayed integral formula with $A = \prod_v A_v$, product Haar and the product kernel.

Indeed it has this formula for finite sums of rectangle indicators by Fubini. The contraction gives the extension on the Hilbert tensor product, where the squared norm is computed in the tensor Hilbert–Schmidt basis. To identify that extension with the integral for general bounded measurable ϕ, the bounded functions for which the formula holds form a vector space closed under uniformly bounded monotone limits of real functions. Convergence holds in the source norm by finite mass and pointwise in the ordinary integral by dominated convergence, since for a fixed shift every kernel is continuous on the compact domain. The contraction then identifies the two limits almost everywhere. The monotone class theorem applies to the product Borel sets, and the imaginary part is treated by linearity.

For $|\omega| \le 1$ invariant under the product of basis stabilizers, apply this contraction to ω. The mean squared Hilbert–Schmidt norm is at most $\|\omega\|_{L^2(\prod \nu_v)}^2 \le \prod P_v$. The finitely many support tests hold on one full $\prod m_v$-measure set, independently of the point tested. There is a shift t in that set with squared norm at most $\prod P_v$. In fact a nonnegative measurable function strictly larger than that number throughout the set would have integral strictly larger than it. Fixer invariance identifies the value $\omega(z)$ with $\omega(tz)$.

Use $\lambda = \gamma^{-1}$, with the chosen element $\lambda^{-1} y_0' = y_0$, $y_0' = \gamma^{-1} y_0$, and conjugate the cutoff when taking the adjoint. For the actual weight $\omega = 1_{\mathrm{red}}$ the conjugation is trivial. If the translated cut integrand is nonzero at z, then $\omega(tz) \ne 0$ and $W^{\mathrm{cut}}(tz) \ne 0$. Hence $y = tz$ is reduced and lies in the cut. Triangular reduction bounds the columns of $y_v^{-1} E_v$ at all places. Since t fixes the basis, the columns of $(tz)_v^{-1} E_v$ equal those of $z_v^{-1} E_v$.

The second argument of the cut kernel $\gamma_v^{-1} t_v z_v$ has Bessel length at most $b^{6n+2} \le \rho$. This is exactly the support test that puts z in A. The support of the actual weighted cut is thus contained in tA, although the indicator alone need have no compact support. All kernels are bounded on tA, and left Haar gives

$$\left\| \int \omega(y)\, K_\gamma^{\mathrm{cut}}(y)\, dg(y) \right\| \le (\prod_v P_v)^{1/2} + \int \omega(y) \| K_\gamma^{\mathrm{full}}(y) - K_\gamma^{\mathrm{cut}}(y) \|\, dg(y).$$

The error is a nonnegative integral.

Its finiteness for the selected orbit representatives follows from the cutoff transfer below. The full kernel is integrated on tA. The local discriminant for γ^{-1} equals that for γ, because their basis product matrices are adjoints. This uses the product Haar and the same chosen bases.

Write $\Lambda_v = \prod_a a_{v,a}^{k+2(\varrho-a)}$, so $\Lambda_0 = \prod_v \Lambda_v$. Put $q = 10^5 n^4$. Then

$$\prod_v P_v(\gamma) = C_{\mathrm{pc}} / x_\gamma^{q+k-1}, \qquad C_{\mathrm{pc}} \le 2^{kf}\, \mathfrak{t}\, \frac{d_B^2}{\mathcal{B} c_G^f \Lambda_0^2}.$$

Indeed $b^{-60n^2} \le \prod_a a_{v,a} \le b^{46n^2}$, and the local factor $\prod_a a_{v,a}^{k+2(\varrho-a)}$ is at most $(\prod_a a_{v,a})^k b^{92n^3}$. Against $2^k d_v^2 / (\deg_v c_G \Lambda_v^2)$, the ratio is at most $b^{453n^2 q + 92n^2 + 184n^3}$ at each place, whose product is at most $\mathfrak{t}$ by $k \ge 512000 n^9$. In fact putting $S_v = \prod_a a_{v,a}$, the per-place ratio, with the discriminant norm removed, is $\frac{1}{2} (\rho^{9n} / S_v^2)^q S_v^{-2(k-1)} \Lambda_v^2$.

Put $v_E = \mathrm{vol}(\widehat{\mathcal{O}}_E)$ in the additive Haar at finite places used for the Siegel calculation, and let C_0 denote the scaled Siegel constant of the prescribed theta datum. The two numerical comparisons are

$$2^{kf} n_p M^{2\varrho f k} \le \mathfrak{t} C_0 v_E^n, \qquad C_0 \le (1 + b^{-9}) D_{\mathrm{main}} \mathcal{B} c_G^f v_E^{-n} (\nu \Lambda_0 / d_B)^2.$$

For the first comparison set $X = 200 f k \operatorname{rd}(E)$. The scale bounds above give $X^{1000} \le b$. For the one-vector Siegel constant of Section 13.3, the bounds $v_E \ge X^{-f}$, $(k-1)! \le X^k$, and $n \le k$ give

$$C_{\mathrm{S}} v_E^n = |D_E|^{1/4} v_E^{k+n} \left(\frac{\pi^k}{(k-1)!\, N^k} \right)^f \ge N^{-kf} X^{-(2k+n)f} \ge N^{-kf} X^{-3kf}.$$

Use $\mathfrak{t} \ge X^{4kf}$, $X \ge 2$, and the unipotent order $m_U \ge M^{2\varrho f k}$ in $C_0 = |H_R| d_C m_U C_{\mathrm{S}}$, and use $n_p = |H_R| d_C / N^{kf}$. The second comparison is $C_0 \Delta_0 \le (1 + b^{-9}) \nu^2 D_{\mathrm{main}} \mathcal{B}$ from Section 13.8. Combining gives

$$C_{\mathrm{pc}} n_p M^{2\varrho f k} \le 2 \mathfrak{t}^2 \nu^2 D_{\mathrm{main}} \le \tfrac{1}{4} \mathfrak{t}^4 \nu^2 D_{\mathrm{main}}.$$

Since $q + k - 1 \geq k$ and $x_\gamma \geq 1$, this bounds the local product square by one fourth of the claimed orbit square. The absolute cut error will use the other half of the orbit bound. Its admissibility follows from

$$b^{-k^3 f} \leq \mathfrak{t}^2 \sqrt{D_{\mathrm{main}}/(n_p (M^{2\varrho f} x_\gamma)^k)},$$

using $n_p \leq D_{\mathrm{id}} \leq D_{\mathrm{main}}$, $x_\gamma \leq b^{270 n^2 f}$, $M \leq b^{23}$, and the rank inequality. This proves the normalization of the two halves.

Use the reduced-point indicator ω from the marked box partition. It is invariant under the real fixer of the basis, since the transported columns are unchanged, and on the real cut its support is in the box of radius ρ. For each orbit of $U_c^0 \backslash (S_0 \cap \mathcal{T}(c, s, c'))/U_{c'}$ choose one representative, preferring a near representative of a smaller orbit with right group $U_{c'}^0$ from the fixed list of Section 14.4. This choice suffices because whenever the reduced indicator and cut kernel are simultaneously nonzero, one can bring the second cut point to its box by its original class unipotent lattice. The new term remains in the larger orbit. Within the smaller orbit boxing is invariant under both basis-fixing unipotent lattices, so its chosen representative is near.

For a nonzero-discriminant term the stabilizer of $\gamma U_{c'}$ in U_c^0 is trivial. Indeed if $a\gamma = \gamma v$, then $v \in U_{c'}$ fixes $\gamma^{-1} F$ pointwise. These columns pair perfectly with F since the full basis product matrix is invertible. Linear combinations therefore give ϱ fixed vectors whose e_a^*-coordinates form the identity. In Witt blocks such an element has diagonal blocks (B, I, B^{-*}). Fixing the independent dual coordinates gives $B = I$; the middle covectors of the fixed-vector equations kill the middle-to-dual block. The unitary relation kills the isotropic-to-middle block, and the remaining fixed-vector equations kill the strictly upper corner block. Thus $v = a = 1$.

Give (γ, y) weight $\omega(y)$ when γ is the chosen representative of its larger orbit, and weight zero otherwise. This is an orbit cutoff on the set of terms times the real group. To check this, for each right coset of U_c^0 in U_c exactly one element of that coset, together with one element of $U_{c'}$, takes the term to the representative of the larger orbit of $u\gamma$, for u in that coset.

Existence is the orbit decomposition and uniqueness is the trivial stabilizer proved above. The sum of ω over the remaining cosets is one by the box partition. Apply the change of cutoff in Section 15.1. It transfers the full-minus-cut absolute bound as well, because the norm of the difference of kernels is invariant under the same action. In particular the absolute tail on each chosen term is at most $\nu b^{-k^3 f}/2$.

Any chosen term that contributes is near by the boxing argument. Its integral is bounded by the sum of the two halves calculated above. A larger orbit charges at most one smaller-orbit representative, so summing gives the bound

$$\|A_{S_0}^{\mathrm{cut}}(c, s, c')\| \leq \nu \sum_{\gamma \in \mathcal{R}_{S_0}(c,s,c')} w_\gamma$$

for S_0 in the dominant-discriminant part. The sum uses the original fixed near representatives. On a nonagreeing residue the block is zero by Section 15.1.

15.6 The one-place Schur integral

In this subsection, we give the real integral bound for a contributing cell at a single place. The rank condition comes from Section 15.2, and the estimate will be multiplied across places in Section 15.7.

Use a surviving cell of rank $1 \le j \le n$, let $i = \min(j, \varrho)$, and put $I = A \cap \gamma A$. Put $\rho = b^{37n}$ and $\theta = b^{6n+2}$. Throughout, y is a real point such that the columns of $y^{-1}F$ have length at most ρ and the Bessel length of y is at most θ. At a real place use the marked orthonormal basis with last vector of norm -1. Let F_i consist of the first i isotropic columns and the last vector. In a lower cell the shared columns starting at j have leading pairs $(\zeta, 1)$ with $|\zeta| = 1$ in the orthonormal basis, by Section 15.2.

These pairs retain their coefficients because both bases have the same column scales. The shared isotropic columns span I. From the shared last vector remove the columns of I in the second basis, to obtain a vector in $\operatorname{span}(F) \cap \operatorname{span}(\gamma F_i)$ with coefficient one at γz and coefficient of modulus one at z. Let $B_1 = F_i^* h \gamma F_i$, $\delta = \det B_1$. Replacing its last column by the pairings of that vector preserves the determinant. Those pairings are zero except for the last, of modulus one. Thus in a lower cell $|\delta|$ is the modulus of the leading j-minor in the orthonormal basis. In the top cell this is also true, directly since $F_i = F$.

Let V_- be the negative eigenspace of the marked Hermitian involution h in the orthonormal basis. The negative space $Y_- = yV_-$ pairs perfectly with the entire basis: a vector in Y_- orthogonal to the basis must be zero since the basis orthogonal complement is nonnegative. Pull the columns of $F^* h \gamma F_i B_1^{-1}$ back through this pairing to Y_-, and apply Gram–Schmidt for $-h$ in reverse column order. These columns pair to zero with I, which is orthogonal to γF_i. This gives a negative orthonormal basis ω of $Y_- \cap I^\perp$ and a lower triangular invertible L with

$$F^* h \omega = F^* h \gamma F_i B_1^{-1} L, \qquad F_i^* h \omega = L.$$

The norm of $y^{-1}\omega_a$ is one, so $|L_{aa}| \le \|y^{-1}(F_i)_a\|$, giving $|\det L| \le \rho^{i+1}$ on the box of radius ρ.

Use the h-orthogonal projection onto $Y_- = yV_-$ and the volume induced there by $-h$. If E_- consists of an orthonormal basis of V_-, the ratio of the projected volumes is

$$\mathcal{V}(\gamma, y) = \frac{|\det(E_-^* y^{-1} \gamma F)|}{|\det(E_-^* y^{-1} F)|}, \qquad \mathcal{V}(\gamma^{-1}, \gamma^{-1} y) = \mathcal{V}(\gamma, y)^{-1}.$$

Indeed y^{-1} identifies the projection with the ordinary projection onto V_-. Both determinants are nonzero because the kernel of the original projection is positive definite and each basis span is nonpositive. In a lower cell replace the omitted isotropic columns in each basis by their shared columns. The changes of basis have determinant of modulus one by the common leading pairs. The projected shared wedge volume cancels, leaving volumes in the orthogonal complement of the projected I in Y_-.

This complement is $Y_- \cap I^\perp$, since pairing against a vector in Y_- only sees the negative projection. Pairing the retained columns with its orthonormal basis ω yields

$$\mathcal{V}(\gamma, y) = \frac{|\det((\gamma F_i)^* h \omega)|}{|\det L|}.$$

Put $\omega' = \gamma F_i B_1^{-1} L$. The columns of $\omega' - \omega$ lie in $A + X_0$; let V_0 be the matrix of their X_0-coordinates, so their Gram is $V_0^* V_0$. If $C = \omega^* h \omega'$, expansion gives

$$-C - C^* = I + V_0^* V_0 - \omega'^* h \omega' \succeq I + V_0^* V_0.$$

For a matrix of positive Hermitian part Q, the factorization $Q^{1/2}(I + \mathrm{i}R)Q^{1/2}$, with R Hermitian, shows its determinant has modulus at least $\det Q$. Apply this to $-C$, and use the determinant

monotonicity of positive matrices and $\det(I + V_0^* V_0) = \det(I + V_0 V_0^*)$. The formula for $\det C$ from its factors now gives

$$\mathcal{V}(\gamma, y) \geq \frac{|\delta| \det(I + V_0 V_0^*)}{2^{i+1} |\det L|^2}. \tag{50}$$

Changing the transported basis in X_0 by A_1^* times the first i covectors of its product matrix changes V_0 by A_1^* times the first i covectors of L, by multiplying those covectors by $B_1^{-1} L$. Let L_1 be the leading i-block and l the final diagonal entry. Lower triangularity leaves the final column v_z of V_0 fixed. Thus the determinant in (50) is $\det(G_1 + T^* T)$, with $T = L_1^* A_1 + C_1$, $G_1 = I + v_z v_z^*$, and C_1 fixed.

In a lower cell the last column of ω' is a multiple of the common vector with coefficient one at γz. Its norm is $-|l|^2$ by comparing its last pairing with L. It is orthogonal to the difference $\omega'_z - \omega_z$, so the norm of ω_z gives $|l|^2 = 1 + \|v_z\|^2 = \det G_1$. In the top cell the isotropic pairings of ω_z vanish by triangularity. For u in the Bessel unipotent, $u^{-1} z - z \in A$, so the pairing of $u^{-1} z$ with ω_z equals that of z. The length bound gives such a u with $\|uy\| \leq \theta$, because the infimum defining the length includes the additional one. Evaluating in the orthonormal basis after this move gives $|l| \leq \theta$. Thus in both cases

$$|l|^2 / \det G_1 \leq \theta^{2(j-i)}.$$

Use Lebesgue measure on complex matrix entries. The change $T' = (L_1^* A_1 + C_1) G_1^{-1/2}$ has inverse Jacobian $|\det L_1|^{-2k} \det(G_1)^i$. The integrand contributes $\det(G_1)^{-(k+i)}$. Thus the factor in the integral of the determinant part of the majorant is $|\det L_1|^{-2k} \det(G_1)^{-k}$ times

$$\int_{\mathbb{C}^{i \times k}} \det(I + T'^* T')^{-k-i} \, dT'.$$

To bound it integrate one covector at a time. If $Q = I + T_0^* T_0$ accounts for the other covectors, adding a covector t changes the determinant to $\det Q(1 + t Q^{-1} t^*)$. The substitution $t = u Q^{1/2}$ has Jacobian $\det Q$. The remaining exponent is $-(k + i - 1)$, and this covector contributes the radial integral $\pi^k \Gamma(i) / \Gamma(k + i)$. Induction gives a product of these factors, each at most $\pi^k / k! \leq 1$ for $k \geq 7$. Consequently the explicit determinant majorant for $\mathcal{V}(\gamma, y)^{-(k+i)}$ in the shift parameter has integral at most

$$\frac{(2\rho^2)^{(i+1)i} (2^{i+1} \theta^{2(j-i)})^k}{|\delta|^{k+i}}. \tag{51}$$

The majorant is a measurable function of the parameter alone, so this does not require a measurable choice of a basis at each parameter. All integrals use the original complex entry measure.

15.7 The contribution bounded by Schur's test

In this subsection, we combine the real Schur estimates of Section 15.6 with the finite orbital weights. This controls the part of the near-orbit sum complementary to the range treated in Section 15.5.

Let $U_c^0 \subset U_c$ fix the basis pointwise. On the orbit $U_c^0 \gamma U_{c'}^0$ the product matrix and the residue in the block (c, s, c') are constant. Use the index i_γ of (43). In the transported integral class coordinates it is the index of the first i isotropic covectors in all isotropic covectors, using

conjugate-transposed covectors of the basis products. For $u\gamma v$ in the orbit, v fixes the basis, and $(u-1)/M$ is integral in the class basis. Its middle component reads only the dual isotropic coordinates. These are exactly the isotropic covectors of the products.

For two terms p, p_0 of this orbit, their middle difference in the rational Witt coordinates is $(x(p,p_0),0)$, since the products with z agree. In the integral class coordinates it is $C_c x(p,p_0)$, where C_c is the matrix of $X_0 \hookrightarrow X_0 \oplus Ez$ in the middle basis of Λ_c. Locally choose columns T_c spanning the intersection of $\operatorname{im} C_c$ with the integral middle lattice. This intersection is saturated, so T_c has an integral left inverse L_c. Put $D(p) = C_c x(p,\gamma)$. Then $T_c L_c D(p) = D(p)$.

The k columns of $(L_c D(p))^*$ determine at most i_γ^k classes in $(M\Lambda_{\mathrm{all}}/M\Lambda_{<j})^k$, where the two lattices are generated by all isotropic product covectors and by the first i, respectively. The adjoint exchanges the two components at a split place, and multiplication by M preserves the index. Let Ξ_i be the i-by-n matrix of the first i isotropic pairings in rational coordinates. For two terms in one class, application of T_c to the difference gives an integral finite adelic matrix W_f with

$$C_c x(p,p_0) = M W_f \Xi_i.$$

Choose a rational right inverse R_i of Ξ_i using the nonzero leading j-minor. In the top cell the full product matrix is invertible, including when the leading ϱ-minor vanishes. Thus

$$Q_\lambda = M^{-1} x(p,p_0) R_i, \qquad C_c Q_\lambda = W_f, \qquad x(p,p_0) = M Q_\lambda \Xi_i.$$

The last equality follows by applying the injective map C_c. Integrality of W_f places each column of Q_λ in $\Lambda_c \cap X_0$. At fixed λ the basis differences lie in A. The marked packing estimate restricts to n copies of $y_v^{-1}A$ at each real place, of total real dimension at most $2n\varrho f$. The separation stays $(N\rho)^{-1}$ and the radius is 2ρ, with $\rho = b^{37n}$.

Distinct bases thus contribute at most $b^{6n^2 \cdot 2n\varrho f} \le \mathfrak{t} = b^{kf/4}$, as in the packing calculation in Section 14.1. The fixed-basis argument of Section 14.1 puts the relative element of two terms with the same basis in $U_{c'}^0$. Their right cutoff weights therefore sum to at most one. The real Eichler shift with covector parameters $(Q,0)$ acts by adding Q^* times the isotropic product covectors to the middle coordinates of the basis after inverse translation. The diagonal rescaling $\mathcal{S}_v$ has entries $M/d_{v,a}$, $a < i$, with the signed complex basis scale $d_{v,a} = \kappa_v^{\varrho-a}/\sqrt{N}$. The change of variables in (51) multiplies the integral by $|\det \mathcal{S}_v|^{-2k}$.

Writing $w_v = |\kappa_v|^{\sum_{a<j}(\varrho-a)}$ and $P_v = |\Phi_v(\delta_j)|$, we have

$$|\det \mathcal{S}_v| = (M\sqrt{N})^i / w_v, \qquad |\delta| = w_v^2 P_v / N^j.$$

The powers of w_v cancel in the k-th power and $w_v \ge 1$ allows us to drop them in the other denominator. Since $\theta = b^{6n+2}$, $2^{i+1}\theta^{2(j-i)} \le b^{13i}$. In lower cells this follows from $j = i$ and $2^{i+1} \le b^i$. In the top cell use $i = \varrho$, $2^{i+1} \le b^{i-17}$ and $2(6n+2) = 12i+16$. The shifted majorant has integral at most

$$K_v = \left(\frac{(2\rho^2)^{i+1} N^j}{P_v} \right)^i \left(\left(\frac{b^{13}}{M^2} \right)^i \frac{N^{j-i}}{P_v} \right)^k .$$

Then $\mu_\gamma = \prod_v P_v^2 / N^{2f(j-i)}$ is the quantity of Section 14.7, since $\prod_v P_v^2 = |N_{E/\mathbf{Q}}(\delta_j)|$ and $j - i = (j-\varrho)_+$. The rational minor is integral and nonzero, so $\prod_v P_v \ge 1$. From $N \le b^{60n}$ and $2 \le b$, the product of the first factors in K_v is at most $b^{135n^2 if} \le \mathfrak{t}$.

Consequently

$$\prod_{v} K_v \le \mathfrak{t} \left(\frac{b^{13}}{M^2} \right)^{ikf} \mu_\gamma^{-k/2}. \tag{52}$$

The lattice of column parameters has Lebesgue covolume at least one. Pairing two column tuples in the definite lattice gives an algebraic integer. Because E contains $\mathbf{Q}(\mathrm{i})$, the sum of its real parts over the chosen complex places is the real part of the trace to $\mathbf{Q}(\mathrm{i})$, hence an integer. The real Gram determinant of a lattice basis is therefore a positive integer. This gives the covolume bound in precisely the measure used in the matrix integral. Write $\beta_c^{\mathrm{box}} = \mathbf{1}_{\mathrm{box}} \beta_c$, where boxed means reduced with $\ell_v \le b^{6n+2}$ at every place, as in Section 14.4.

For a term p in the orbit define

$$J(p) = \int \beta_c^{\mathrm{box}}(y) \|W^\flat(y)\|^2 \beta_{c'}^{\mathrm{box}}(p^{-1}y) \prod_{v} \mathcal{V}(p_v, y_v)^{-(k+i)} \, dy.$$

For a fixed color with base p_0, write $p = p_\lambda$ for any term with lattice parameter $\lambda_v = \Phi_v(Q_\lambda)^*$ at each real place v, allowing several terms at each parameter. Let y_0 be boxed and let u belong to the real basis stabilizer U_∞^0. Use the Eichler shift $s(X)$ whose central parameter is zero and whose v-covector parameter is $\mathcal{S}_v^* X_v$ in the first i covectors. Write Z_i for the first i product covectors in the orthonormal basis.

For every term in the color, the product matrix of $(us(X))^{-1}p$ is independent of the term and of X. At a real place the X_0-parts satisfy

$$\left((us(X))^{-1} pF\right)_{X_0} = \left(u^{-1} p_0 F\right)_{X_0} + \left(\mathcal{S}_v^*(\lambda_v + X_v)\right)^* Z_i.$$

Indeed an inverse basis-fixing unipotent with block Q adds Q^* times the dual isotropic covectors to the X_0-part. The u^{-1} additions at p, p_0 coincide because they have the same product matrix. The $s(X)^{-1}$ addition uses precisely Z_i; the rational lattice difference, read in the orthonormal basis, has coefficient $(\mathcal{S}_v^* \lambda_v)^*$, where λ_v is the conjugate covector realization of the column lattice. The identity follows. This formula treats u as fixed before X, which is the order of right translation on the real basis-fixer fiber.

Since $us(X)$ fixes the basis, $\mathcal{V}(p, us(X)y_0)$ equals $\mathcal{V}((us(X))^{-1}p, y_0)$. Apply the one-place majorant with base $u^{-1}p_0$ and parameter $\lambda + X$. It has product integral at most $\prod_v K_v$, independently of u, y_0. The basis ω and the matrix L use the common product matrix and y_0. The unipotent shifts fix the basis pointwise, so the common leading pairs used in the same cell are preserved.

At each lattice point the sum of the second boxed cutoffs is at most $\mathfrak{t}$ by basis packing and the right cutoff partition at a fixed basis. Hence the color sum at $us(X)y_0$ is bounded by $\mathfrak{t} \sum_\lambda \varphi(\lambda + X)$, where nonnegativity of φ allows the sum to run over the entire parameter lattice. If $\mathcal{D}$ is the measurable fundamental domain of the i copies of the definite lattice in the conjugate-transpose coordinates, its covolume is unchanged by these coordinate changes, which are isometries of the real entry spaces. Tonelli gives

$$\frac{1}{\mathrm{vol}(\mathcal{D})} \int_{\mathcal{D}} \sum_{\lambda} \varphi(\lambda + X) \, dX = \frac{\int \varphi}{\mathrm{vol}(\mathcal{D})} \le \prod_{v} K_v.$$

Use the same domain for each of the at most i_γ^k colors.

15.7.1 Summation over the smaller orbits

In this subsection, we use the preceding Schur bounds to sum the smaller orbits required in Section 15.8.

For the integration in stages, let $\mathcal{F}$ be the sum over the whole smaller orbit of the second boxed cutoff times the inverse volume power. This sum is left invariant under U_c^0, which permutes the terms. On the fiber through y_0, the first boxed indicator and the squared Bessel norm are constant. Write its remaining cutoff as $a(u)$, so $\sum_{\eta\in U_c^0} a(\eta u) = 1$.

For each X, right Haar on the nilpotent group gives

$$\begin{aligned}\int a(u)\mathcal{F}(uy_0)\,du &= \int a(us(X))\mathcal{F}(us(X)y_0)\,du\\ &= \int a(u)\mathcal{F}(us(X)y_0)\,du.\end{aligned}$$

For the second equality insert $\sum_\eta a(\eta^{-1}u) = 1$ in the first integral, substitute $u \mapsto \eta u$ in each term, and sum $\sum_\eta a(\eta us(X)) = 1$. Tonelli applies as all weights are nonnegative. Averaging in X and using the preceding bound now yields the inequality on each real fiber. Unipotent disintegration of the first boxed cutoff times $\|W^b\|^2$ contributes at most νd_B. Indeed the covector cutoff has unipotent integral ν, and each cut vector has quotient mass at most the mass d_v of the full vector since its multiplier has modulus at most one.

Combining the colors with (52) gives

$$\sum_{p\in U_c^0\gamma U_{c'}^0} J(p) \le \nu d_B \mathfrak{t}^2 \left(\frac{b^{13}}{M^2}\right)^{ikf} (i_\gamma/\sqrt{\mu_\gamma})^k.$$

The absolute amplitude $a(p)$ satisfies $a(p)^2 \le J(p)J(p^{-1})$, with classes switched in the second integral. This is Cauchy–Schwarz in y: the volume ratios are reciprocal after $y \mapsto p^{-1}y$, and on the support of the cut kernel the boxed cutoffs equal the original ones. Write $\mathcal{O} = U_c^0\gamma U_{c'}^0$. Inversion bijects this orbit over (c, s, c') with $\mathcal{O}^{-1} = U_{c'}^0\gamma^{-1}U_c^0$ over (c', s^{-1}, c). Thus

$$\left(\sum_{p\in\mathcal{O}} a(p)\right)^2 \le \left(\sum_{p\in\mathcal{O}} J(p)\right)\left(\sum_{q\in\mathcal{O}^{-1}} J(q)\right).$$

The second sum runs over the entire inverse orbit. Its link exchanges the two quotient coordinates, by taking $w' = p^{-1}w$ in the definition of the link. The leading matrix for the inverse is the adjoint, so its μ_γ is the same. The common leading pairs for the inverse are obtained by exchanging the two bases and normalizing the pair.

For $a \in \Lambda_{c'} \cap A$ and a basis vector f of F, $h(a, \gamma^{-1}f) = h(\gamma a, f)$. This identifies the full pairing lattices and the sublattices given by the first $\min(j, \varrho)$ columns. Hence inversion exchanges i_γ and i'_γ. These lattice indices are constant on a smaller orbit, so the identity applies to its fixed representative as well. Define $g_{\min} = N^f\mu_\gamma/(i_\gamma i'_\gamma)$. We obtain

$$\left(\sum_p a(p)\right)^2 \le \left(\nu d_B \mathfrak{t}^2 (b^{13}/M^2)^{ikf}\right)^2 (N^f/g_{\min})^k.$$

With n_p, D_j as in Section 15.5, $d_B^2 N^{kf} = d_B D_{\mathrm{id}}/n_p$ and $b^{-8ikf} = \mathfrak{t}^{-32i}$. The bounds $b^{13}/M^2 \le (b^4M)^{-1}$, $d_B \le \mathfrak{t}^8$ and $D_j \ge \mathfrak{t}^{8-32i} D_{\mathrm{id}}$ give

$$\sum_p a(p) \le \nu \mathfrak{t}^2 \left(\frac{D_j}{n_p (M^{2if} g_{\min})^k} \right)^{1/2}. \tag{53}$$

The last bound on D_j holds in lower cells because $i \ge 1$, and in the top cell because $D_{\mathrm{main}} \ge \mathfrak{t}^{44} D_{\mathrm{id}}$. For $\mathfrak{g}_{\mathrm{norm}}(\gamma) \le N^f \mu_\gamma$ the denominator uses $G_\gamma = M^{2if} g_{\min}$, so (53) is exactly νw_γ. A nonzero cut summand with the original two cutoffs puts both arguments in their boxes. Nearness is invariant throughout the smaller orbit, because both basis stabilizers preserve reduction and length. Hence a non-near smaller orbit contributes zero. Sum the bound over the original fixed near representatives of the complementary part.

15.8 Summing the pair weights

In this subsection, we sum the pair multiplicities with the bounds from Sections 15.5 and 15.7. This completes the estimate for the corresponding terms of the full Petersson matrices in Section 9.1.

Use the weights w_γ of Section 15.5. For each agreeing block (c, s, c'), choose ordered pairs from $\mathcal{R}_{\Gamma_*}(c, s, c')^2$, using the representatives fixed in Section 14.4 for both entries. Suppose for each such pair of ranks j, j' that $(d\eta)^2 \le G_\gamma G_{\gamma'}$ and $D_j, D_{j'} \le D_0$. Here $d = d(L_{12})$ and l is its corank. If $\mathfrak{t}^{l+6} D_0 \eta^{-k} \le B_0$ for every pair of the family, then

$$w_\gamma w_{\gamma'} \le \frac{\mathfrak{t}^4 D_0}{n_p d^k \eta^k} \le \frac{B_0}{n_p \mathfrak{t}^{l+2}} d^{-k}.$$

By Sections 14.5 and 14.7 the sum at any l is at most B_0. There are at most $n^3 + 1 \le \mathfrak{t}$ coranks, so the total is at most $\mathfrak{t} B_0$.

For distinct links, where the codimension $i = k + 1 - \dim V_1$ of Lemma 14.3 satisfies $i \ge 1$, take $\eta = (M/2)^{if}$ and $D_0 = D_{\mathrm{main}}$. Using $l \le i + 1$ and $\xi = \mathfrak{t}(2/M)^{kf} \le 1$ gives

$$\mathfrak{t}^{l+6} D_0 \eta^{-k} \le \mathfrak{t}^7 \xi^i D_{\mathrm{main}} \le \mathfrak{t}^8 (2/b^6)^{kf} D_{\mathrm{id}} \le \mathfrak{t}^{-12} D_{\mathrm{id}},$$

where the middle step uses (33). Within one rational link the joint rank of its two z-forms is fixed. By Section 14.7.2 this rank is two in a top cell and one in a lower cell. Hence equal links cannot mix a top and a lower cell. For equal lower links $i = l = 0$. Use $\eta = 1$ and $D_0 = \mathfrak{t}^{-16} D_{\mathrm{id}}$, giving $\mathfrak{t}^{l+6} D_0 \eta^{-k} = \mathfrak{t}^{-10} D_{\mathrm{id}}$. Thus the combined family of distinct links and equal lower links has total pair weight at most

$$\mathfrak{t}^{-9} D_{\mathrm{id}} \le b^{-kf} D_{\mathrm{id}}. \tag{54}$$

For equal top links with meeting norm at least $z \ge 1$, equation (48) gives $\eta = z$ and $l = 1$. Take $D_0 = D_{\mathrm{main}}$.

Lemma 15.1 (Weight of equal top links)**.** *Fix $z \ge 1$. In each agreeing block (c, s, c') with $Q_\cap(s) \ge z$, let $\mathcal{E}_z(c, s, c')$ consist of the ordered pairs in $\mathcal{R}_{\Gamma_*}(c, s, c')^2$ whose two terms lie in the top cell $j = n$ and have the same rational link in $(A^\perp/A)^2$. Put $\mathcal{E}_z(c, s, c') = \varnothing$ in all other blocks. Then*

$$\sum_{c,s,c'} \sum_{(\gamma,\gamma') \in \mathcal{E}_z(c,s,c')} w_\gamma w_{\gamma'} \le \mathfrak{t}^8 z^{-k} D_{\mathrm{main}} = (b^{2f}/z)^k D_{\mathrm{main}}. \tag{55}$$

For a stable subset $S_0 \subseteq \Gamma_*$ the cut blocks satisfy

$$\|A^{\mathrm{cut}}_{S_0}(c,s,c')\| \le \nu \sum_{\gamma \in \mathcal{R}_{S_0}(c,s,c')} w_\gamma. \tag{56}$$

Here $A^{\mathrm{cut}}_{S_0}$ is the cut block of Section 14.1, with the rational terms restricted to S_0, and $\mathcal{R}_{S_0}$ takes the chosen near representatives in S_0 with cell ranks over the given agreeing block. Stability refers to the original unipotent action on both sides. The same choice of orbit representatives must be used for every piece. The sum is empty at a nonagreeing residue. Here stability means invariance under left and right multiplication by $U(F_0)$, as required for the character cancellation of Section 15.1. Split each cell at $\mathfrak{g}_{\mathrm{norm}} = N^f \mu_\gamma$, assigning equality to the Schur part.

For a unipotent move the product matrix is multiplied on either side by triangular matrices with diagonal one, upper triangular on the right and lower triangular on the left. Hence its leading minors, including the determinant, are preserved. The ranks are preserved because the changes of basis preserve the flags and the common leading pairs. Sections 15.5 and 15.7 give (56) on the two stable parts with the same near representatives. The nonagreeing block vanishes by the finite cancellation in Section 15.1.

Now partition the cut sum by links at top rank and group all lower ranks in one piece. To see stability of a link, write its two quotient coordinates as the images of w and $\gamma^{-1}w$ in $A^{\perp}/A$. For a move $\gamma \mapsto u^{-1}\gamma u'$, use $u^{-1}w$ and $(u')^{-1}\gamma^{-1}w$. Both unipotents fix A as a subspace and act trivially on $A^{\perp}/A$, so the two quotient coordinates are unchanged.

In one block write A_- for the lower-cell amplitude and A_L for the top-cell amplitude of link L. Absolute convergence in Section 14.1 allows the Hilbert–Schmidt Gram expansion

$$\Big\|A_- + \sum_L A_L\Big\|^2_{\mathrm{HS}} - \sum_L \|A_L\|^2_{\mathrm{HS}} = \|A_-\|^2_{\mathrm{HS}} + 2\,\mathrm{Re}\sum_L \langle A_-, A_L\rangle_{\mathrm{HS}} + \sum_{L \ne L'} \langle A_L, A_{L'}\rangle_{\mathrm{HS}}.$$

The last sum is over ordered pairs. By Cauchy–Schwarz and (56), the right side is at most $\nu^2 \sum w_\gamma w_{\gamma'}$, summed over the ordered pairs in this block except those with both terms in the same top link. The factor two accounts for the two orders of a mixed lower and top pair. For fixed c, c', the reductions of $h_c^{-1}\gamma_f h_{c'}$ and $h_c^{-1}\gamma'_f h_{c'}$ determine their common s uniquely. In the equal-top case the first term also determines the rational link indexing the piece. Thus no pair is counted again at a second residue or link. All the cross terms and the square of the lower piece count distinct links or equal lower links.

Equation (54) therefore bounds their normalized total by $b^{-kf} D_{\mathrm{id}}$. Applying the same size estimate twice within each top piece and using (55) bounds the normalized equal-link moment by $(b^{2f}/z)^k D_{\mathrm{main}}$ on residues of meeting norm at least z. The partition of the blocks is justified by the absolute convergence in Section 14.1. Thus the pair summation gives the bounds needed in Section 9.1. The reduction to the cell ranks used in the sum is (49).

16 The common automorphic family and its translated span

In this section, we assemble the data of Section 8 and Sections 10–15 on one family. The moment inequalities of Section 9 then give the dimension bound in Proposition 5.1.

16.1 Construction of the common automorphic family

In this subsection, we assemble the real and finite data and verify that the resulting Poincaré map has the specified levels at every sufficiently large scale. The normalization of Section 8 and the dimension in Section 9.4 apply to this same construction.

Fix d, ϵ. For each sufficiently large integer x, take the primes, weights and skew different of Section 4. The grading, finite point and levels in Section 8 use these markings and the specified u. The source level is exactly the preimage of the original principal level under the definite embedding, by the glued-lattice calculation in Section 13.2.2. At the chosen split places reduction of the full integral group is surjective: lift each prescribed general linear matrix at the chosen place, use the inverse adjoint at its conjugate and use the identity at the remaining places. The change of finite isometry has identity reduction at every chosen place.

Use the real signs and weights of Section 12.10. The different and scales in Section 4 give

$$|\Phi_v(\delta_E)| \le b, \qquad 1 \le |\kappa_v| \le b^{23}, \qquad N \le b^{25m} \le b^{60n}.$$

Use the adelic point x^{pt} and the pair (g_0, t) of Section 8 in the Poincaré map. These choices precede the selection of forms and residue classes. At each real place the value space is a determinant twist of the symmetric power of degree $a_\tau \le a$ of the standard rank-k representation, so its dimension is at most $(a+k)^k \le b^{2k}$.

In particular $0 < d_B \le b^{2kf}$ and the source class set is nonempty. Equation (33) gives both bounds for $D_{\mathrm{main}}/D_{\mathrm{id}}$. Before imposing the principal level at M, the inverse image of the glued lattice stabilizer fixes the prescribed nonzero last covector at every chosen place above N. Writing e for the distinguished coordinate column and $D_z = \mathrm{diag}(1, \dots, 1, z)$ in an order with e last, the middle block of the glued basis is

$$L_{\mathrm{mid}} = \begin{pmatrix} D_{\bar\pi_N}^{-1} & 0 \\ e^{\mathsf{T}}/N & 1 \end{pmatrix}, \quad L_{\mathrm{mid}}^{-1} \begin{pmatrix} h-I & 0 \\ 0 & 0 \end{pmatrix} L_{\mathrm{mid}} = \begin{pmatrix} D_{\bar\pi_N}(h-I)D_{\bar\pi_N}^{-1} & 0 \\ -e^{\mathsf{T}}(h-I)D_{\bar\pi_N}^{-1}/\pi_N & 0 \end{pmatrix}.$$

For integral unitary h, the condition $e^{\mathsf{T}}(h-I) \equiv 0 \pmod{\pi_N}$ gives $(h-I)e \equiv 0 \pmod{\bar\pi_N}$ by adjunction. In particular the last diagonal difference is divisible by N. The characterization of the stabilizer in Section 13.2.2 and transitivity on nonzero covectors give its relative index $(N^k-1)^f \ge (N^k/b)^f$. Finally the marked mass comparison in Section 13.8 gives

$$C_0\Delta_0 \le (1+b^{-9})\nu^2 D_{\mathrm{main}}\mathcal{B}, \qquad \Delta_0 = \frac{v_E^n}{c_G^f}\left(\frac{d_B}{\Lambda_0}\right)^2.$$

This is the same constant bound used in the dominant orbit and in the open equal-link estimates. All these conditions hold on the constructed Bessel data, with the terminal definite comparison of Section 9.4. Their numerical thresholds depend only on the fixed d, ϵ. The arithmetic comparisons below use the measured map already specified by g_0, t and this level.

16.2 Selecting the translates

In this subsection, we convert the two moment inequalities of Section 9 into a lower bound for the dimension of a sum of translated subspaces. We then pass from the full source to the ramified subspace of Section 6.

Lemma 16.1 (A dimension bound for translated subspaces)**.** *Let a finite group G act unitarily on T by ρ. Let p be the orthogonal projection onto an invariant subspace W, and let $B : S \to T$ be linear between Hilbert spaces with $1 \le d_S = \dim S < \infty$ and $1 \le D = \dim W < \infty$. Write P_U for orthogonal projection onto $U \subseteq S$. Suppose $0 < \eta_1 < 1/16$ and*

$$\|pBP_U\|_{\mathrm{HS}}^2 \ge \dim U - \eta_1 d_S \quad (U \subseteq S), \qquad D\,\mathbb{E}_g \|B^*\rho(g)B\|_{\mathrm{HS}}^2 \le (1+\eta_1)d_S^2.$$

Then there is $t \subseteq G$ such that

$$|t| d_S \le \tfrac{5}{2}\left(1 + \log\frac{1}{16\eta_1}\right) D, \qquad \dim \sum_{g\in t} p\rho(g)B(S) \ge (1-16\eta_1)D.$$

Proof. Set $C_{\mathrm{av}} = \mathbb{E}_g \rho(g)BB^*\rho(g)^*$. Expanding the trace of the square gives

$$\|C_{\mathrm{av}}\|_{\mathrm{HS}}^2 = \mathbb{E}_g \|B^*\rho(g)B\|_{\mathrm{HS}}^2.$$

For $F \subseteq W$ of dimension αD, some g satisfies

$$\|P_F \rho(g) B\|_{\mathrm{HS}}^2 \le \mathrm{tr}(P_F C_{\mathrm{av}}) \le \sqrt{\alpha(1+\eta_1)}\, d_S \le (1+\alpha+\eta_1)d_S/2.$$

On the preimage $U = (p\rho(g)B)^{-1}F$, $P_F\rho(g)BP_U = p\rho(g)BP_U = \rho(g)pBP_U$. Restrict to U in the near-isometry inequality and compare with the full Hilbert–Schmidt norm. This gives $\dim U \le (1+\alpha+3\eta_1)d_S/2$. By rank-nullity in W/F, adding the new image adds at least $(1-\alpha-3\eta_1)d_S/2$ dimensions.

Let F_j be the span after j choices, with $F_0 = 0$. Put $m_j = D - \dim F_j$ and $a_1 = 2d_S/(5D)$. Apply the first moment to $U = \ker(pB)$ to obtain

$$(1-\eta_1)d_S \le \mathrm{rank}(pB) \le D, \qquad 0 < a_1 \le \frac{2}{5(1-\eta_1)} < 1.$$

If $m_j > 16\eta_1 D$, the increase in dimension is at least $(13/32)(d_S/D)m_j \ge a_1 m_j$, so

$$m_{j+1} \le (1-a_1)m_j \le e^{-a_1} m_j.$$

At termination, $m_j \le 16\eta_1 D$. The number j of choices satisfies

$$j \le \left\lceil a_1^{-1} \log\frac{1}{16\eta_1} \right\rceil, \qquad j d_S \le \tfrac{5}{2}\left(1 + \log\frac{1}{16\eta_1}\right) D. \qquad \square$$

Proposition 16.2 (Span of the marked translates)**.** *For fixed d, ϵ and sufficiently large x, there is $t \subseteq G$ such that*

$$|t| \dim S \le 5D \log(1+2b^2), \qquad \dim \sum_{g\in t} P_g(S) \ge (1 - 4/b^2)D.$$

Here P_g is the same normalized map used for the integral and norm estimates in Section 22.3.

Proof. Use the Euclidean realization $C(X)$ of the full source, where $C = C_X R_H(\eta^{-1})$, with η as in Section 8, is the change of coordinates in Section 8. The geometric map is $B : C(X) \to T_{\mathrm{full}}$ and the good projection is p. Section 9 gives, for $U \subseteq C(X)$,

$$\|pBP_U\|_{\mathrm{HS}}^2 \ge \dim U - 5b^{-4} d_X, \qquad D\,\mathbb{E}_g \|B^*\rho(g)B\|_{\mathrm{HS}}^2 \le (1+9b^{-4})d_X^2.$$

The free class coordinates have $d_X > 0$. Taking $U = \ker(pB)$ in the first inequality gives

$$\dim\ker(pB) \le 5b^{-4}d_X, \qquad D \ge \operatorname{rank}(pB) \ge (1 - 5b^{-4})d_X > 0.$$

Both error constants are at most $\eta_1 = (8b^2)^{-1}$ for large b.

The selected translates span all but $2D/b^2$ dimensions, and

$$|t|d_X \le \tfrac{5}{2}(1 + \log(b^2/2))D \le 5D\log(1 + 2b^2).$$

Section 6 gives $\dim(X/S) \le f d_X/(N-1) \le b^{-8}d_X$. The operator $R_H(\eta^{-1})$ commutes with the central scalar averages defining S, so the restricted source in the Euclidean coordinates is $C(S)$. The maps $p\rho(g)B$ induce the surjection

$$\bigoplus_{g\in t} C(X)/C(S) \twoheadrightarrow \frac{\sum_{g\in t} p\rho(g)B(C(X))}{\sum_{g\in t} p\rho(g)B(C(S))}.$$

The dimension of the quotient is therefore at most

$$|t|\dim(X/S) \le |t|b^{-8}d_X \le 5b^{-8}\log(1 + 2b^2)D \le 2D/b^2.$$

With $v_G = \operatorname{vol}(J)$ and ν the unipotent covolume, the normalizations of Sections 8 and 22.3 give

$$B = (v_G\nu)^{-1/2}\Pi C^{-1},$$
$$P_g = c_B|G|c_{\mathrm{ss}}^{-1}p\rho(g)\Pi|_S = c_B|G|c_{\mathrm{ss}}^{-1}(v_G\nu)^{1/2}p\rho(g)BC|_S.$$

Since e_0 is central, $p\rho(g) = \rho(g)p$. Thus $P_g = \rho(g)P_1$, and the nonzero scalar in this formula gives $\operatorname{im} P_g = p\rho(g)B(C(S))$, yielding the span estimate on the datum of Section 16.1. The integral generators are chosen in the original S in Section 22.3. □

Part III
Theta identities and the upper covolume bound

17 The Weil action and the measured seesaw

In this section, we verify rational invariance and relate the doubled Weil action to the normalized theta integral. This fixes the scalar and measures in the comparison with the Poincaré map of Section 8.

17.1 Rational invariance for the Weil action

In this subsection, we verify rational invariance of the Weil action used in Section 17.2. We compute the Levi scalar in a split polarization and pass to a general Hermitian space through $B \perp B^-$.

Lemma 17.1 (Composition of the normalized local Weil operators)**.** *Let g, h belong to the local unitary group of the given Hermitian space. In the signed real and normalized finite coordinates write $T(x)$ for the symmetric Weyl displacement and v_0 for the unit vacuum. There is a unitary operator U_g implementing $T(x) \mapsto T(gx)$ with*

$$\langle v_0, U_g v_0 \rangle = 1/\overline{\Delta_g}.$$

Here $\Delta_g = \overline{\det d_g}$ at a real place, where d_g is the negative diagonal block of g. At a finite place v of F_0, put $E_v = E \otimes_{F_0} F_{0,v}$ and $L = \mathcal{O}_{E_v}^n$ in the normalized coordinates. The norm and character on $E_v^\times$ are products when v splits. The finite multiplier is $\Delta_g = (v_v| \cdot |_{E_v}^{-1/2})(\det H_g)$, where $H_g L = L \cap g^{-1}L$. This scalar is independent of the basis chosen for the intersection, since v_v is unramified. Then $U_g U_h = U_{gh}$, also on the restricted tensor product.

Proof. Write $e(x) = \langle v_0, T(x) v_0 \rangle$, which is real and even in both models. For the self-dual displacement measure,

$$P_0 = |v_0\rangle\langle v_0| = \int \overline{e(x)} T(x)\, dx.$$

Indeed, in Schrödinger coordinates $x = (a, b)$, integration over the modulation b is Fourier inversion, and integration over the translation a gives $f \mapsto v_0 \langle v_0, f \rangle$. By the Heisenberg intertwining theorem [74, §10, Theorem 1], an isometry g has a unitary implementer unique up to a scalar. Its image of P_0 by conjugation is $Q_g = \int e(g^{-1}x) T(x)\, dx$. It satisfies

$$\langle v_0, Q_g v_0 \rangle = \int e(x) e(g^{-1}x)\, dx = |\Delta_g|^{-2}.$$

At a finite place the integral is $\mathrm{vol}(L \cap gL) = [L : L \cap g^{-1}L]^{-1}$, since g preserves volume and the lattice phases agree. At a real place $e(x) = e^{-x^* x/2}$. With P the positive coordinate projection, set $D(a) = P + (I - P)a$. Pseudo-unitarity gives $(I + a^* a)/2 = D(a)^* D(a)$, so the normalized complex Gaussian integral is $|\det d_g|^{-2}$ after the substitution $x = gy$. Consequently $U_g v_0 = \Delta_g Q_g v_0$ fixes the unique required phase. For $k = gh$, covariance gives $U_g Q_h U_g^{-1} = Q_k$, whence

$$U_g U_h = c(g, h) U_k, \qquad c(g, h) = \Delta_g \Delta_h \overline{\Delta_k} \langle v_0, Q_k Q_g v_0 \rangle, \qquad |c(g, h)| = 1.$$

Writing $T(x)T(y) = \beta(x, y) T(x + y)$ and substituting $u = -k(x + y),\ v = kx$ in this last double integral gives

$$\langle v_0, Q_k Q_g v_0 \rangle = \iint e(hx) e(ky) \beta(x, y) e(x + y)\, dx\, dy. \tag{57}$$

Indeed $k^{-1}u = -(x + y),\ g^{-1}v = hx,\ u + v = -ky$ and $\beta(u, v) = \beta(x, y)$ by alternation and unitary invariance.

At a finite place write $c_0 = (1 - j_{\mathrm{W}}^{-1})/2$, $\psi_0(z) = \psi_{E,v}(j_{\mathrm{W}} \alpha_{\mathrm{W}} \bar{\alpha}_{\mathrm{W}} z)$, and $e_0(x) = \psi_0(c_0 x^* x/2)$. $c_0 = (1 + \sqrt{-d})/2$ is integral and satisfies $c_0 + \bar{c}_0 = 1$. The different identity of Section 8.1 makes ψ_0 self-dual on $\mathcal{O}_{E_v}$, and the skew sign of j_{W} gives $\psi_0(\bar{z}) = \psi_0(z)^{-1}$. In these coordinates $\beta(x, y) = \psi_0(y^* x/2)$ and $e = \mathbf{1}_L e_0$. The finite translation fixing the vacuum for integral x is $T_\Lambda(x) = e_0(x) T(x)$, as in Section 18.2. Its multiplier is $e_0(x) e_0(y) e_0(x + y)^{-1} \beta(x, y)$. Finite unitary isometries preserve $x^* x$, so both translations have the same unitary covariance, including at places above two. The character ψ_0 is trivial on $\mathcal{O}_{E_v}$, so $e_0^2 = 1$ on L. Put

$$D = \{(x, y) : hx \in L,\ ky \in L,\ x + y \in L\}.$$

On this compact open lattice $e_0(hx) = e_0(x)$, $e_0(ky) = e_0(y)$ and $e_0(x+y) = e_0(x+y)^{-1}$. Expanding the three quadratics in (57) gives the integrand $\psi_0(\bar{c}_0 y^* x)$ on D and zero elsewhere.

At a split place $g = (A, A^{-\mathsf{t}})$. If the elementary divisor exponents of A are a_i, the intersection exponents at the two places are $\max(0, -a_i)$ and $\max(0, a_i)$. Thus $\Delta_g/|\Delta_g| = \upsilon_w(\det A)^{-1}$ at the chosen factor w, so $\Delta_g \Delta_h \overline{\Delta_k} > 0$. Writing $D = D_+ \times D_-$ at the two factors, the phase $\psi_0(\bar{c}_0 y^* x)$ is a bicharacter between D_+ and D_-. Integration over D_- gives its volume times the indicator of its annihilator in D_+, which contains an open subgroup. Hence $c(g,h) > 0$, and therefore $c(g,h) = 1$. This argument applies at the dyadic split places.

At a nonsplit place let q be the cardinality of the residue field of $F_{0,v}$. The extension is unramified quadratic, and $\Delta_g = (-q)^{a_g}$, where $a_g = \operatorname{length}_{\mathcal{O}_{E_v}} L/(L \cap g^{-1}L)$. Here we used $\upsilon_v(\varpi) = -1$ on a base uniformizer. Choose $m \geq 0$ so that $B = (\varpi^m L)^2 \subset D$ and the phase is constant on cosets of B in D. The coordinates of x, y on D have valuations bounded below. Taking m sufficiently large makes the change in $\bar{c}_0 y^* x$ integral after adding a vector of B. With $\operatorname{vol}(L) = 1$ and residue cardinality q^2 over E_v, (57) gives

$$q^{4nm} c(g,h) = (-q)^{a_g + a_h + a_k} \mathcal{S}, \qquad \mathcal{S} = \sum_{(x,y) \in D/B} \psi_0(\bar{c}_0 y^* x).$$

Simultaneous multiplication by an integral unit of norm -1 conjugates every summand, so $c(g,h)$ is real and equals ± 1. Such a unit exists by norm surjectivity on unramified units. The Teichmüller group of order $q+1$ and norm one fixes the summands and acts freely on $(D/B) \setminus \{0\}$, since $\xi - 1$ is a unit for every nonidentity member ξ. Consequently $\mathcal{S} \equiv 1 \pmod{(q+1)\mathcal{O}'}$, where $\mathcal{O}'$ is the ring of integers of the cyclotomic field generated by the finitely many phase values. Reducing the preceding equality gives $c(g,h) \equiv 1 \pmod{(q+1)\mathcal{O}'}$. Since $(q+1)\mathcal{O}' \cap \mathbb{Z} = (q+1)\mathbb{Z}$ and $q + 1 > 2$, this forces $c(g,h) = 1$.

At a real place put $J = 2P - I$, $Q = I - P$, and $C_a = (I + a^* a)/2 = D(a)^* D(a)$. In these coordinates $\beta(x,y) = \exp((y^* J x - x^* J y)/2)$. The integral in (57), with measure $\pi^{-2n} dx\, dy$ in real Lebesgue coordinates, has matrix

$$A = \begin{pmatrix} C_h & P \\ Q & C_k \end{pmatrix}, \qquad A + A^* = \operatorname{diag}(h^* h, k^* k) + \begin{pmatrix} I & I \\ I & I \end{pmatrix} > 0.$$

Thus its value is $(\det A)^{-1}$ by the complex Gaussian integral (first for positive Hermitian matrices and then by holomorphic continuation on matrices with positive Hermitian part). Write $a = \left(\begin{smallmatrix} a_{11} & a_{12} \\ c_a & d_a \end{smallmatrix}\right)$ in signed coordinates. Since $D(a) = \left(\begin{smallmatrix} I & 0 \\ c_a & d_a \end{smallmatrix}\right)$, the off-diagonal blocks of C_a^{-1} are $-c_a^* d_a^{-*}$ and $-d_a^{-1} c_a$. The Schur complement and $g = kh^{-1}$ give

$$\begin{aligned} \det A &= |\det d_h \det d_k|^2 \det(I - d_h^{-1} c_h c_k^* d_k^{-*}) \\ &= |\det d_h \det d_k|^2 \frac{\det(d_h d_k^* - c_h c_k^*)}{\det d_h \det d_k^*} = \overline{\det d_h \det d_g} \det d_k. \end{aligned}$$

This is precisely $\Delta_g \Delta_h \overline{\Delta_k}$, also when either signed summand is zero. Hence $c(g,h) = 1$. The finite integral stabilizers fix the vacuum by the coefficient formula with $\Delta_g = 1$, so the restricted tensor product is a representation. □

Let B be the given Hermitian space and ω_B its Weil action before inserting determinant characters. Write J_B for its rational Gram matrix, X_B for its point matrix, and $\Gamma_B = X_B^{-1} E^{\dim B}$ in adelic coordinates before the local rescalings. A rational unitary element γ acts in these

coordinates by $X_B^{-1}\gamma X_B$. For $x, y \in E^{\dim B}$ the symmetric Weyl displacements have adelic multiplier

$$\beta(X_B^{-1}x, X_B^{-1}y) = \psi_E(j_{\mathrm{W}} y^* J_B x/2) = 1.$$

Indeed the finite factors of $\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}$ cancel the two normalized coordinate factors, and the signed real coordinates give the infinite factors of the same rational trace character. Moreover $\beta(z, X_B^{-1}y)^2 = \psi_E(j_{\mathrm{W}} y^* J_B X_B z)$. As y varies, the covectors $j_{\mathrm{W}} y^* J_B$ fill $E^{1\times \dim B}$. The annihilator of E in $\mathbb{A}_E$ under ψ_E is E, so triviality of all these characters is equivalent to $z \in \Gamma_B$. In a rational Schrödinger polarization the functional θ_B is a nonzero multiple of rational summation. Indeed rational modulations force its support to be on the rational lattice by adelic self-duality. At each support point the modulation differentials span the real cotangent space and eliminate derivatives of point masses. Rational translations make all the point masses equal. Heisenberg covariance implies that $\theta_B \omega_B(\gamma)$ has the same rational Heisenberg invariance for $\gamma \in U(B)(F_0)$. Thus

$$\theta_B \omega_B(\gamma) = c_B(\gamma)\theta_B,$$

where c_B is a character by Lemma 17.1. We evaluate it using the multipliers of the action.

First let B be split, with rational dual isotropic bases k, ℓ of rank p. In the mixed model $f[x]$ is the theta value after displacement by ℓx and averaging the k-modulations on the additive quotient with probability Haar. In the Schrödinger realization just described this operation is $f[x] = C f_{\mathrm{Sch}}(x)$. Indeed, the modulation average kills all the rational indices except zero. In particular

$$\theta_B(f) = \sum_{x \in E^p} f[x], \qquad \int \overline{f[x]} g[x]\, dx = |C|^2 \langle f, g\rangle.$$

The original theta functional has mean square one on the Heisenberg translates of a unit vacuum. Orthogonality of the rational modulation characters and unfolding the translation variable give $|C|^2 = 1$, as computed in Section 41.5.1.

For the Levi $m(A)$ acting by A on k and A^{-*} on ℓ, covariance gives $(\omega_B(m(A))f)[x] = c(A) f[A^* x]$. Composing with the inverse substitution commutes with all Heisenberg operators, so its effect is scalar. The Jacobian and unitarity give $|c(A)|^2 = |\det A|_{\mathbb{A}_E}$.

For the phase let (ℓ°, k°) be the ordered hyperbolic frame in the model described below. Let u carry this frame to (ℓ, k) in the given adelic coordinates. The two frames have the same Hermitian Gram matrix, are integral bases at almost every place, and span their respective rational subspaces over E. Thus u is adelic unitary and carries one rational subspace onto the other. Write $m_\circ(A)$ for the Levi in the model and $v = \omega(u)v_0$. Lemma 17.1 gives

$$m(A) = u m_\circ(A) u^{-1}, \qquad \langle v, \omega(m(A))v\rangle = \langle v_0, \omega(m_\circ(A))v_0\rangle = 1/\overline{\Delta_{m_\circ(A)}}.$$

By covariance, $\theta_B \circ \omega(u)$ is invariant under the rational Heisenberg translations of the model. Hence it is a nonzero multiple of the model's rational summation. Transport of the displacements and the probability modulation average shows that the slice of v is the same multiple of the model vacuum slice. At infinity the change from the signed orthonormal basis to the hyperbolic coordinates, up to the ordering of signs, uses normalized sums and differences. The negative block of the Levi in the adapted signed basis is $(A^{-*} + A)/2$. Its conjugate determinant times $\det A$ is the positive determinant of $(I + A^* A)/2$. At finite places an integral change from the hyperbolic Gram to the identity Gram is given by the matrix

$$\begin{pmatrix} I & -cI \\ \varepsilon I & c\bar{\varepsilon} I \end{pmatrix}, \qquad c\bar{c} = -1, \qquad \varepsilon + \bar{\varepsilon} = 1.$$

The norm and trace conditions have integral solutions with c a unit. This holds even at the places over two: at an unramified quadratic place use the residue norm and trace followed by successive lifts, and at a split place use the product and sum. In the present family 2 splits in K, so $E_v = F_{0,v} \times F_{0,v}$ for $v \mid 2$. There one may take $c = (1, -1)$ and $\varepsilon = (1, 0)$. Multiplying by the hyperbolic Gram and its adjoint gives the identity Gram, and the determinant of the change is c^p. It preserves the elementary divisors of the lattice intersection. In these coordinates, with $L = \widehat{\mathcal{O}}_E^p$ and $\chi_f = \upsilon_f| \cdot |_f^{-1/2}$, the finite multiplier of A is

$$\Delta_f(A) = \chi_f(\det H), \qquad HL = L \cap A^{-1}L.$$

Inversion replaces this intersection by AHL; transpose preserves its elementary divisors and conjugation conjugates the multiplier. Hence the finite multiplier of the pair is

$$\Delta_f(A^{-*})\Delta_f(A) = \overline{\chi_f(\det A)}|\Delta_f(A)|^2.$$

Together the real and finite calculations give $\Delta_{m_\circ(A)}\upsilon(\det A) > 0$.

The adapted vacuum has slice a nonzero scalar times a positive real Gaussian and a finite integral-box indicator. At infinity this follows by computing the Heisenberg coefficient of the standard Gaussian using the orthonormal hyperbolic basis: the cross terms cancel in the product of the paired coordinates. At finite places use normalized displacement coordinates, dividing the original coordinates by α_{W}. The self-dual lattice is L in these coordinates by the different identity of Section 8.1. The k- and ℓ-displacements with normalized parameters in L fix the vacuum, by isotropy and the integral change of basis just computed.

In the mixed realization the modulations force support in the self-dual box and translations force constancy there. The product of the vacuum slice and its Levi substitute therefore has positive integral. By the slice inner-product formula, the phase of $c(A)$ is that of $\langle v, \omega(m(A))v\rangle = 1/\overline{\Delta_{m_\circ(A)}}$. Thus

$$(\omega_B(m(A))f)[x] = \upsilon(\det A)^{-1}|\det A|_{\mathbb{A}_E}^{1/2} f[A^*x]. \tag{58}$$

For rational A, the scalar is one by the product formula and the triviality of υ on $E^\times$. The substitution permutes the rational points of summation. Each element of either opposite unipotent radical is a commutator with a rational scalar Levi: conjugation by the Levi with scalar 2 multiplies the additive radical parameter by 4 or $1/4$. The character c_B is therefore one on both radicals.

To see that these elements and the Levi generate, write the image of k under a rational isometry as $ka + \ell b$, so $a^*b + b^*a = 0$. In a complex embedding $(a+b)^*(a+b) = a^*a + b^*b > 0$ since the image has rank p. The polynomial $\det(a + Tb)$ is nonzero. Choose a skew $t \in E$ where it is nonzero. One radical changes the upper block to $A = a + tb$. The other, with parameter $-bA^{-1}$, makes the lower block zero. This parameter is skew by the isotropy equation after the first operation. The remaining element stabilizes k and belongs to its Siegel parabolic, generated by the Levi and its radical. This proves $c_B = 1$ for split B.

For general B, the rational diagonal and antidiagonal split $B \perp B^-$, where B^- has the opposite form. In direct-sum coordinates $(\gamma, 1)$ acts on product vectors as $\omega_B(\gamma) \otimes 1$. To verify the scalar, the real negative minors multiply, the finite lattice intersections are direct sums, and the vacuum is a product. The nonzero vacuum coefficient is therefore the product coefficient. Covariance and irreducibility identify the two intertwiners. In a direct sum of

rational polarizations, rational summation of a product is the product of the two rational sums. Uniqueness of the three rational Heisenberg functionals therefore gives a fixed $z \neq 0$ such that

$$\theta_{B\perp B^-}(f \otimes f_-) = z\,\theta_B(f)\theta_{B^-}(f_-).$$

Choose f_- with $\theta_{B^-}(f_-) \neq 0$ and cancel $z\,\theta_{B^-}(f_-)$ from the split invariance identity. This proves

$$\theta_B \omega_B(\gamma) = \theta_B \quad (\gamma \in U(B)(F_0)).$$

For the pair S, T use $B = \mathrm{Hom}_E(S, T)$. The determinant characters in the pair action are one at rational arguments. In particular rational invariance holds for the normalized theta kernel after all the automorphic twists used below.

We use the tensor product of the fixed theta functionals on the standard signed lines. For a marked Hermitian space B with point matrix X_B, let $Y_B = X_B^{-1} E^{\dim B}$ be its rational vectors in these coordinates. Landherr's theorem [41, pp. 246–247] applies because both forms are standard at the finite places and have the same signature at every real place. Take a rational isometry P with $P^* J_B P = J_{\mathrm{std}}$, and write X_{std} for the point matrix of the standard signed space. Then

$$g_B = X_B^{-1} P X_{\mathrm{std}}, \qquad g_B(X_{\mathrm{std}}^{-1} E^{\dim B}) = Y_B.$$

The point equations show that g_B is adelic unitary. With θ_{std} the product functional just specified, the phase convention is

$$\theta_B(f) = \theta_{\mathrm{std}}(\omega(g_B^{-1}) f).$$

For another rational isometry P', the quotient $g_B^{-1} g'_B = X_{\mathrm{std}}^{-1}(P^{-1}P')X_{\mathrm{std}}$ is the adelic action of a rational unitary element of the standard space. Section 17.1 proves that its action fixes θ_{std}. For a block sum take the direct sum of these maps. The real determinant factors multiply, the finite intersections are direct sums, and the vacuum is a tensor product, so the block Weil action is the tensor product of the two actions. Thus

$$\theta_{B_1 \perp B_2}(f_1 \otimes f_2) = \theta_{B_1}(f_1)\theta_{B_2}(f_2).$$

17.2 The doubled action in the Siegel–Weil formula

In this subsection, we compute the Levi scalar in the doubled Weil action using the rational normalization of Section 17.1. This scalar determines the normalization of the paired lift in Section 34.2.

Write $n = \mathrm{rk}_E S$, $s = \mathrm{rk}_E T$. Take the mixed model for a rational Lagrangian of $S \perp S^-$. For the target action h, the substitution $X \mapsto h^{-1}X$ has determinant $(\det h)^{-n}$ on the n columns of the Hom Lagrangian. This substitution is the adjoint block A^* in (58); hence $\det A = \overline{(\det h)^{-n}} = (\det h)^n$, since $\det h$ has norm one. Its Levi scalar is therefore $\upsilon(\det h)^{-n}$. It cancels the target character $\upsilon(\det h)^n$ in the doubled pair action. For a source Siegel Levi A, the substitution $X \mapsto XA$ acts on $T \otimes S^\vee$ by $1_T \otimes A^{\mathsf{T}}$. The paired Lagrangian block in (58) is $\bar{A}$ on each of the s summands. Consequently the scalar is

$$\upsilon(\det A)^s |\det A|_{\mathbb{A}_E}^{s/2}.$$

On the upper radical the zero term is fixed at the origin: its scalar there is a character and the radical is generated by commutators with a scalar Levi. Write $u(Z)$ for the radical

element $\ell \mapsto \ell - kj_{\mathrm{W}}^{-1}Z$ with $Z = Z^*$. Moving the displacement by X through $u(Z)$ adds a k-displacement absorbed by the average and gives

$$(\omega(u(Z))f)[X] = \psi_E\Big(\tfrac{1}{2}\operatorname{tr}(ZX^*J_TX)\Big)\, f[X].$$

Here $\psi_E = \psi_{F_0} \circ \operatorname{Tr}_{E/F_0}$ for the rational trace characters. The matrix trace inside the phase is fixed by conjugation. To identify the unipotent coordinate, convert Yamana's matrix convention by $g \mapsto (g^*)^{-1}$. The ordered basis $(2j_{\mathrm{W}}\ell, k)$ has Hermitian Gram matrix $-2j_{\mathrm{W}}\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Removing the scalar gives the skew-Hermitian matrix of the classical model with the same isometry group. The upper unipotent $n(b)$, for $b = b^*$, sends $2j_{\mathrm{W}}\ell$ to $2j_{\mathrm{W}}\ell - kb$ and fixes k. Thus $Z = b/2$. The mixed variable on k is unchanged and

$$\psi_E\Big(\tfrac{1}{2}\operatorname{tr}(ZX^*J_TX)\Big) = \psi_{F_0}(\operatorname{tr}(bX^*J_TX/2))\,.$$

This is the unipotent law in [76, §1.3]. The rational change of basis preserves probability Haar on the rational additive quotient. The dual Fourier measures transport with the pairing. In the opposite polarization the law follows by self-dual Fourier transform, with its scalar canceled by conjugation.

These are the Schrödinger laws of the splitting character υ^s with the additive character. The two opposite parabolics generate over each local field. At a nonsplit place choose $L, R \in \mathrm{GL}_n(E_v)$ with $b' = LbR = \operatorname{diag}(I_j, 0)$, and put $a' = L^{-*}aR$. Then $(a')^*b' + (b')^*a' = 0$ gives $a'_{12} = 0$, and injectivity of $\binom{a'}{b'}$ gives a'_{22} invertible. Thus

$$\det(a' + Tb') = \det(a'_{22})\det(a'_{11} + TI_j) \neq 0 \quad \text{in } E_v[T].$$

Choose $t + \bar{t} = 0$ away from its roots. The upper radical in the original basis then has skew parameter $N_t = tL^*L$, since $a' + tb' = L^{-*}(a + N_tb)R$. Writing $A = a + N_tb$, isotropy gives $A^*b + b^*A = 0$, so $-bA^{-1}$ is skew and the lower radical with this parameter sends $\binom{A}{b}$ to $\binom{A}{0}$. At a split place the two radicals and their common Levi contain the elementary transvections and the invertible diagonal matrices of $\mathrm{GL}_{2n}(F_{0,v})$. The actions agree on the integral vacua outside a finite set.

In [77, Theorem 2] the division algebra is E of index one. The splitting character restricts to χ^s_{E/F_0}. At $s_0 = (s-n)/2$ use the Siegel section $g \mapsto (\omega(g,1)f)[0]$. The integrand of the theta integral is

$$\sum_{X \in \mathrm{Hom}_E(E^n, T)} (\omega(g,h)f)[X].$$

The zero value and the rational sum come from the same mixed realization, so its fixed scalar multiplies both sides of the Siegel–Weil formula. In every strict-range doubling below $s > 2n$, so

$$s - \mathrm{Witt}(T) \geq s/2 > n.$$

The ordinary theta integral converges for every Schwartz–Bruhat test by [75, §51, Proposition 8]; the spherical Eisenstein majorant converges since $s_0 > n/2$.

We verify the local orbit hypotheses of [75, §52, Theorem 5] at an old split place v. Write $k = F_{0,v}$. A full-rank tuple in Weil's fibre consists of an injection $X : k^n \to k^s$ and a surjection $Y : k^s \to k^n$, with Gram matrix represented by $B = YX$ in one split factor. The first n

coordinate vectors and the last n coordinate covectors give $B = 0$, since $s > 2n$. For two such pairs (X_i, Y_i) with common B, match $X_1 a$ with $X_2 a$. The identity

$$X_i(k^n) \cap \ker Y_i = X_i(\ker B)$$

allows us to extend this matching to an isomorphism $\ker Y_1 \to \ker Y_2$ and hence to the sums $X_i(k^n) + \ker Y_i$. Choose a complement to $\operatorname{im} B$ in k^n and match lifts of its basis under the two surjections. The resulting $g \in \mathrm{GL}_s(k)$ satisfies $gX_1 = X_2$ and $Y_2 g = Y_1$, proving transitivity on the full-rank part of each Gram fibre. For polynomial Fock vectors use [77, Theorem 2]. For arbitrary real Schwartz vectors the convergent formula is [75, §52, Theorem 5]. In both cases the coefficient of the theta integral with probability quotient measure is one. For the doubled polynomial Fock datum in Section 30.2.1, the Hermitian target is definite, so its ordinary theta integral converges. Yamana's $m = n = 2$ case gives $E(g, 0) = 2I(g)$ with probability quotient measure [77, Theorem 2, p. 237] [76, §1.3 and Proposition 5.8(i)].

17.3 Translation and scalar normalization in the seesaw

In this subsection, we fix the normalization of the theta operators used in the seesaw. The source index is the one in Section 18.3, and equivariance carries the finite group action through the entire composition.

The lower chain is a composition of the theta operators $L_{\xi,j}$ from Y_j to Y_{j+1} with the tame character at the j-th arrow, followed by the terminal algebraic coefficient map. Each $L_{\xi,j}$ is the original theta integral multiplied by the source index $A_{\xi,j} = [K_j^\circ : J_{\xi,j}]$, where $J_{\xi,j}$ is the principal subgroup at the old and twist places. In particular, split reduction gives

$$A_{\xi,j} = \left|\mathrm{GL}_{r+jt}(\mathcal{O}_E/(\pi_M Q(\xi)))\right|.$$

Section 18.3 computes the transpose with this very index. Here Y_m is the lower-chain endpoint, of a different rank from the dimension-comparison space W_m. Use the lattice inclusion ι_j between the standard finite coordinates of Y_j, Y_{j+1}. At the original level places and the additional twist places, the lattice test has the inclusion reduction $\bar{\iota}_j$. The finite twist factor of the Weil action on an integral pair is the quotient of the tame character of its two determinants.

Lemma 17.2 (Equivariance of the weighted lower lifts)**.** *For each $s \in G$ there are integral lifts $k_j(s)$ with first-block residue* $\mathrm{diag}(s, I)$ *at the old places and identity at the twist places. They satisfy*

$$L_{\xi,j}\rho_j(k_j(s)) = \rho_{j+1}(k_{j+1}(s))L_{\xi,j}.$$

The successive intertwinings give the action of the original finite projector on the terminal values.

Proof. For $s \in G$, choose integral lifts $k_j(s)$ on each lower space with reduction $\mathrm{diag}(s, I)$ in the first-block coordinates at the original places and identity at the twist places. Such lifts exist because these places are split. At each step

$$\overline{k_{j+1}(s)}\,\bar{\iota}_j = \bar{\iota}_j\,\overline{k_j(s)}.$$

The simultaneous translate of the lattice test therefore preserves its support: integral matrices are carried to integral matrices, and the inclusion reduction is fixed.

At each old or twist place the two determinants have the same reduction: the added identity block has determinant one. Their quotient is a principal unit, so its tame character is one. Away from these places take the pair to be the identity. The accompanying real test is unchanged. In the theta kernel, apply the finite source translation by changing the source integration variable. The Haar measure is right invariant and the simultaneous action on the datum has just been shown to fix it.

Thus

$$L_{\xi,j}\rho_j(k_j(s)) = \rho_{j+1}(k_{j+1}(s))L_{\xi,j}.$$

The principal kernel $J_{\xi,j+1}$ in the target variable fixes the inclusion test and its tame determinant character, as in Section 18.3. Thus $L_{\xi,j}f$ is fixed by $J_{\xi,j+1}$. For $s,t \in G$ the prescribed residues give

$$k_j(ts)^{-1}k_j(t)k_j(s) \in J_{\xi,j}.$$

Together with the kernel invariance this gives the action of G on the translated coefficients, independent of the integral lifts. The conversion to the algebraic coefficient at the definite endpoint commutes with this finite action. Write L_ξ for the resulting composed lower chain. Expanding the finite good projector $p = \rho(e_0)$ in the original group algebra gives the identity of terminal values

$$\big(L_\xi(|G|p\rho(s)\Pi\phi)\big)(h) = \sum_{t\in G} |G|e_0(t)(L_\xi\Pi\phi)(h\,k_m(ts)).$$

This uses the same measured Π for every translate. □

For the scalar in the seesaw, suppose the unfolded pairing gives, on the original source, $L_\xi\Pi\phi = c_0\,\mathcal{U}_\xi(T_\xi)\phi$, where $\mathcal{U}_\xi$ is the upper chain with its terminal restriction and is linear in its tensor. Suppose the tensor is a finite combination of marked translated vacua $T_\xi = C_0\sum_z b_{\xi,z}v_z$ with fixed C_0, and put $B_* = M^{fn_m^2}$ for terminal upper rank n_m. If $B_*^{2m}b_{\xi,z}$ is integral off the twist, and the untwisted sum of coefficient moduli in each complex embedding is at most B_*^m, then the coefficients $B_*^{2m}b_{\xi,z}$ give the integral tensor with untwisted bound B_*^{3m}.

The common scalar multiplying that tensor is c_0C_0/B_*^{2m} when this is nonzero. If $c_0C_0 = 0$, use the zero integral tensor and scalar one. In either case the scalar is nonzero and independent of the twist. Support and invariance under the upper levels are inherited from the same evaluated tensor, provided its translated vacua have the prescribed support. The unfolded pairing and this evaluated-tensor description enter this normalization.

17.4 Rational orbits for consecutive lifts

In this subsection, we describe the rational maps that contribute to the Bessel integral for two consecutive Hermitian spaces. The character of Section 16.1 determines these maps up to the Bessel actions, which are used in Section 18.1.

Use the rational graded coordinates of the two consecutive spaces Y, Y', with Hermitian spaces V, V' and string lengths $s, \max(0, s-1)$, respectively. Write $a_i = s-1-2i$ and $a'_b = s-2-2b$ for the string grades, starting at index zero. The orthogonal definite blocks have grade zero. The Hermitian forms are conjugate-linear in the first argument. On a string, $i^\vee$ denotes the reversed index and the Gram matrix has entries $g_i = h(e_i, e_{i^\vee}) = 1$, except at a middle index, where $g_i = -N$. The definite orthonormal coordinates are their own dual indices.

A matrix has degree at least d if its entry is zero when the target grade minus the source grade is less than d.

Let $c \in F_0^\times$ be the marked string constant and $T : V \to V'$ the degree--1 map which kills the definite block and the last string vector. Its other string entries are $Te_i = t_i e'_i$, with

$$t_i = \begin{cases} 1, -Nc, c & s \text{ odd, in grades } > 0, = 0, < 0, \\ c, 1 & s \text{ even, in grades } > 0, < 0. \end{cases}$$

Taking the adjoint in the antidiagonal Gram matrices gives $T^\dagger T e_i = t_i t_{s-2-i}(g'_i/g_{i+1}) e_{i+1}$ for $i < s-1$. This coefficient is c, except at an odd string's grade zero, where it is $-Nc$, and at an even string's grade one, where it is $-Nc^2$. In particular all the shift coefficients are nonzero. Write R, R' for the Bessel subgroups over F_0.

A string vector of nonnegative grade is fixed modulo the span of vectors of strictly higher grade. Their matrices and their inverses have degree at least zero. Indeed the stated column property preserves the positive flag and, by orthogonality, the corresponding negative flag. On the grade-zero quotient the orthogonal complement of the central string vector, when present, can carry the definite unitary block. Write $U_R \subset R$ for the unipotent group of matrices whose difference from the identity has degree at least one.

Say that w agrees with T on the positive columns if $w_{bi} = T_{bi}$ for $a_i > 0$ and $a'_b < a_i$. If w agrees there and $\gamma \in R(\mathbb{A}_{F_0})$, then $w\gamma$ also agrees: the positive columns are stable under γ, and $T(\gamma - 1)$ has degree at least zero on them. For $\delta \in U(V')(\mathbb{A}_{F_0})$,

$$\delta w\gamma \text{ agrees with } T \quad \Longleftrightarrow \quad \delta \in R'. \tag{59}$$

For the forward implication put $\widetilde{w} = w\gamma$. In a nonnegative string column k of V', use the positive source pivot i with $a_i = a'_k + 1$ and $Te_i = t_i e'_k$. For every index b with $a'_b \le a'_k$, the entry $((\delta - 1)\widetilde{w})_{bi}$ is zero by agreement.

Proceeding in descending order of a'_k gives $0 = t_i(\delta - 1)_{bk}$: in column i, all covectors of grade at most a'_k are already those of $t_i e'_k$, and entries of $\delta - 1$ in higher columns vanish in covector b by induction. These are precisely the defining conditions for R'. Conversely, if $\delta \in R'$, then $(\delta - 1)T$ has degree at least zero on the positive columns, as does $\delta(\widetilde{w} - T)$. This proves (59).

Use the additive character ψ_E of $\mathbb{A}_E/E$ from the absolute trace and the skew element j_{W} of Section 16.1. Set

$$\beta(x, y) = \psi_E\big((j_{\mathrm{W}}/2)\,\mathrm{tr}(y^\dagger x)\big).$$

For target grades $D = 0$ or $D = a'$, the phase test at w requires

$$\beta(Tu, T) = \beta(wu, w) \quad \text{if} \quad u \in U_R(\mathbb{A}_{F_0}), \quad \deg(u - 1) \ge 2, \quad \deg_D(w(u - 1)) \ge 1. \tag{60}$$

Here $D = 0$ assigns grade zero to every target coordinate. The degree-two condition is automatic for a strictly grade-raising matrix when s is odd.

Suppose w passes (60). Fix a source index k of grade d and an adelic vector v supported in grades at least $\max(1, d + 2)$. Suppose wv is supported in target grades $D > \max(-1, d)$ and $we_{k^\vee}$ is supported in $D > -a_i$ for every i in the support of v.

Lemma 17.3 (Eichler moment on the supported grades)**.** *Under the stated grade hypotheses and the phase test* (60)*,*

$$[(T^\dagger T)v]_k = [(w^\dagger w)v]_k. \tag{61}$$

Put $x = \lambda e_{k^\vee}$ and $L = vh(x,\cdot) - xh(v,\cdot)$ for arbitrary $\lambda \in \mathbb{A}_E$. Since $h(x,v) = h(v,v) = 0$,

$$L^\dagger = -L, \qquad L^2 = -h(x,x)vh(v,\cdot), \qquad L^3 = 0.$$

Thus $u = 1+L+L^2/2$ is unitary. The three terms of $u-1$ raise the source grade by at least two, and their images under w have D-degree at least one by the support conditions. For $A = y^\dagger y$,

$$\operatorname{tr}(Au) = \operatorname{tr} A + h(x,Av) - h(v,Ax) - \tfrac{1}{2}h(x,x)h(v,Av).$$

The first and last terms are invariant under conjugation, and $h(v,Ax) = \overline{h(x,Av)}$. Since $\overline{j_{\mathrm{W}}} = -j_{\mathrm{W}}$ and ψ_E is invariant under conjugation, the phase is $\psi_E(j_{\mathrm{W}}h(x,Av))$. Apply (60) for $y = T, w$. Indeed, for $A_T = T^\dagger T$ and $A_w = w^\dagger w$, this gives

$$\psi_E\Big(j_{\mathrm{W}}\bar{\lambda}\, g_{k^\vee}[(A_T - A_w)v]_k\Big) = 1 \qquad (\lambda \in \mathbb{A}_E).$$

Here $g_{k^\vee} = h(e_{k^\vee}, e_k) \neq 0$, including the middle string coordinate. Nondegeneracy of the adelic trace pairing proves (61).

Take first $D = 0$ and w rational. For positive string indices i, l, use $v = e_i, k = l^\vee$ in (61). The resulting identity is $h'(Te_l, Te_i) = h'(we_l, we_i)$. On the span of these columns w is injective. In fact if $wv = 0$ for a nonzero vector supported there, let i be its least-grade occupied string index and use $k = i+1$. The support conditions hold even if $a_{i+1} \geq 0$, because $wv = 0$. The k-th coordinate of $T^\dagger Tv$ is a nonzero multiple of v_i, contradicting (61). The same positive-column space injects under T. Witt extension [63, Ch. 7, §9] for the Hermitian space V' therefore extends $we_i \mapsto Te_i$ on these columns to a rational unitary element. In particular the rational orbit contains a matrix agreeing with T on the positive columns. By (59), the rational target elements giving agreement have the form $R'(F_0)\delta_0$, also after a source change by $R(F_0)$.

For the second test assume this agreement and take $D = a'$. We show that $w - T$ has degree at least zero, by induction in increasing target space grade. A possibly different entry (b,i) has $a'_b < a_i \leq 0$. Use the positive source pivot $v = e_j$ with $Te_j = t_j e'_{b^\vee}$ and $a_j = 1 - a'_b$, and put $k = i^\vee$, of grade $-a_i$. Then $a_j \geq 2 - a_i$. The vector we_j is Te_j plus terms of grade at least $1 - a'_b$, by agreement.

The entries of we_i of grade at most $-a_j = a'_b - 1$ vanish by induction and by the degree of T. These are the conditions for (61), which now gives

$$h'(we_i, we_j) = h'(Te_i, Te_j).$$

The extra terms of we_j pair only with the already matched covectors of we_i of grade at most $a'_b - 1$. Their pairing there is zero by the same degree bound on T. Hence the remaining equation is $\overline{(w-T)_{bi}}\, t_j g'_b = 0$, and the induction closes. This includes the definite grade-zero source columns. In particular all surviving rational terms of the second test are in the single class of T modulo the rational matrices of degree at least zero.

18 Consecutive theta lifts and their coefficients

In this section, we unfold consecutive theta integrals using the rational orbits of Section 17.4. The even-step coefficient formula and the finite support determine the sum evaluated in Section 19.

18.1 Unfolding the integral for consecutive lifts

In this subsection, we unfold the adjacent theta integral by the orbit computation of Section 17.4. The Haar measures on the original quotients determine the constant in the resulting expression.

Use the datum of Section 17.4. For $A = \mathbb{A}_{F_0}$, give the two quotients

$$[R] = R(F_0)\backslash R(A), \qquad [G'] = U(V')(F_0)\backslash U(V')(A)$$

their inherited Haar measures. Write $\rho(w)$ for the Heisenberg translate on $\mathrm{Hom}_E(V, V')$, with $\rho(w)\rho(z) = \beta(w, z)\rho(w + z)$. The theta functional θ is invariant under rational Heisenberg translations and under rational unitary pairs in the Weil action [74, 75]. For the splitting used here this invariance is proved in Section 17.1. Use the splitting characters of the two covectors. The tame determinant ratio is trivial on rational points and on the unipotent groups.

For $D = 0, a'$ let I_D be the space of strictly positive-degree maps and $\mathcal{C}_D$ the rational maps modulo the rational maps of nonnegative degree. Reversal of both indices exchanges the degrees $d, -d$, so I_D is isotropic and has as rational dual basis the opposite coordinates with the inverse Gram entries. The marked arrow test is Schwartz–Bruhat in rational Schrödinger coordinates. At infinity the creation polynomial of the joint harmonic acts on the Gaussian vacuum.

At the finite places the vacuum is fixed by a small open lattice of Heisenberg translations and modulations. Translation invariance makes it locally constant, and modulation invariance cuts the support to the compact annihilator lattice. The inclusion test is a finite sum of translates of that vacuum. A fixed outer Weil translate preserves this class. Use the same theta normalization in the two gradings.

For a Schwartz test f in this realization put

$$C_D(f; w) = \int_{I_D(F_0)\backslash I_D(A)} \theta\big(\rho(z)\rho(w)f\big)\, dz, \qquad \int dz = 1.$$

For rational w, commutation and invariance give the integrand $\beta(z, w)^2\theta(\rho(z)f)$. Thus $C_D(f; w)$ depends only on the class of w in $\mathcal{C}_D$. These are the Fourier coefficients at the rational dual frequencies of $z \mapsto \theta(\rho(z)f)$, with the frequency sign specified by the displayed integrand.

In particular

$$\theta(f) = \sum_{w \in \mathcal{C}_D} C_D(f; w).$$

The sum is absolute. In rational Schrödinger coordinates extending the positive block and its dual basis, the I_D variable acts by a character on the first Schrödinger coordinate. Orthogonality cuts the theta sum to one such rational coordinate ξ, leaving the sum of the test over all rational coordinates in the remaining Lagrangian. The sum of the coefficient moduli is bounded by the sum of the modulus of the test over the full rational Schrödinger lattice, times the fixed modulus of the theta normalization.

The finite test has compact support, so the relevant rational coordinates lie in a fixed fractional ideal to a finite power. Multiply an archimedean Schwartz supremum by $\sum_\lambda (1 + \|\lambda\|)^{-a}$, with a greater than the real dimension. This gives a bound uniform on any bounded set of archimedean Schwartz tests and bounded finite tests in a fixed finite-dimensional space of compactly supported tests.

For $z_0 \in I_D(A)$, translation by z_0 fixes the averaged theta functional. Hence

$$C_D(f; w) = \beta(w + z_0, w) C_D(f; w + z_0). \tag{62}$$

For a source rational Bessel element γ and a rational target element δ preserving the D-filtration, covariance gives

$$C_D(\omega(\gamma r, \delta h)\Phi; w) = C_D(\omega(r, h)\Phi; \delta^{-1} w\gamma). \tag{63}$$

This is rational theta invariance and change of variable on the compact quotient of I_D. The change preserves Haar probability. The same covariance holds for adelic $u \in U_R$ with $u - 1$ strictly increasing the source grade and $\delta = 1$, and for a strictly D-grade-raising target unipotent. To see this, write $W = \bigoplus_j W_j$ for the decomposition of the Hom space by degree. The form pairs W_j only with W_{-j}. In particular $I_D = W_{>0}$ and $I_D^{\perp}/I_D = W_0$.

Put $E_D(F) = C_D(F; 0)$. In partial Schrödinger coordinates it is the residual rational theta functional on the slice at zero. A distribution fixed by the $I_D(A)$-modulations is supported at that transverse coordinate, with no derivatives there. If it is also fixed by rational Heisenberg translations in W_0, the uniqueness in Section 17.1 makes it a multiple of E_D. Use the action of $U(V)(A) \times U(V')(A)$ of Section 17.1. On its subgroup preserving I_D and inducing the identity on $I_D^{\perp}/I_D$, covariance shows that $E_D \circ \omega(g)$ has both invariances: the added $I_D(A)$-component of a residual translation pairs trivially with W_0. Since this is a representation and $E_D \neq 0$, $E_D \circ \omega(g) = c(g)E_D$ for a character c of this subgroup.

Let D_b be the grade of the target coordinate b for D. On the source and target bases take the rational pair

$$d = (\mathrm{diag}(2^{a_i}), \mathrm{diag}(2^{D_b})).$$

Opposite grades pair, so both diagonals are unitary. On the Hom entry (b, i), the pair acts by $2^{D_b - a_i}$, hence fixes W_0. In each unitary factor separately let N_k consist of the elements whose difference from the identity raises its own grades by at least k. Both factors' N_1 preserve I_D and act trivially on $I_D^{\perp}/I_D$. For $g \in N_k$, $k \geq 1$, set $h = \exp((2^k - 1)^{-1} \log g)$. The series are finite and $(\log g)^{\dagger} = -\log g$, so $h \in N_k$ in that same factor. Conjugation by its component of d multiplies each degree-k entry by 2^k. Therefore $[d, h] \equiv g \pmod{N_{k+1}}$. As $c([d, h]) = 1$ and the filtration terminates, descending induction gives $c|_{N_1} = 1$ in each factor. This applies to both unipotent actions; their determinant characters are trivial. Moving $\rho(w)$ through their Weil operators now gives the claimed adelic covariance.

18.1.1 Phase cancellation and rational orbits

In this subsection, we use Lemma 17.3 to identify the terms that survive the phase average.

Let W be continuous on $[R]$ with $W(ur) = \beta(Tu, T)W(r)$ for $u - 1$ of degree at least two. By (62) the product $F_w(r, h) = W(r)C_D(\omega(r, h)\Phi; w)$ satisfies

$$F_w(ur, h) = \frac{\beta(Tu, T)}{\beta(wu, w)} F_w(r, h) \qquad (\deg_D(w(u - 1)) \geq 1).$$

For a class that fails (60), unfold its rational source orbit. If Γ_w is its stabilizer, the orbit sum over $[R]$ is the single integral over $\Gamma_w \backslash R(A)$, with quotient measures from counting measure. The sum is absolutely integrable for each h. Indeed R is the semidirect product of its definite unitary Levi and unipotent radical, so $[R]$ is compact. For each r choose a rational left translate into a fixed compact cover. Equation (63) permutes the classes, and the Schwartz bound applies to the compact family of translated tests.

Write $F(ur) = c_1 F(r)$ on the stabilizer quotient, where $|c_1| = 1$ and $c_1 \neq 1$. If F is nonzero, its covariance under the group generated by Γ_w and u extends to the closure L. Indeed

evaluation at one point where F is nonzero makes the character continuous and then the identity extends at every point. Choose a nonnegative cutoff b on $R(A)$ with $\int_L b(s^{-1}x)\,ds = 1$; such cutoffs exist for the proper action of a closed subgroup by translation.

Using left Haar on L and counting measure on Γ_w, quotient integration and Fubini give

$$\int_{\Gamma_w\backslash R(A)} F = \int_{R(A)} b(x) \int_{\Gamma_w\backslash L} F(sx)\,ds\,dx.$$

This formula follows by inserting the cutoff integral and setting $x = s^{-1}r$, first for $|F|$. Right translation by u on the inner quotient commutes with the left Γ_w-action and scales its measure by the positive Haar modular factor. Its absolutely convergent integral is multiplied both by that positive factor and by c_1. Thus the inner integral is zero almost everywhere, and the stabilizer integral vanishes.

For $D = 0$, the first phase test and the rational orbit calculation show that only the translates of the classes agreeing with T on positive columns contribute. Agreement is a condition on entries with $a_i > \max(0, a'_b)$, so it is well defined in both quotients. For a matrix $v : V' \to V$, put $y_{ib} = v_{ib}\mathbf{1}_{a_i > \max(0,a'_b)}$. The sum of the agreeing terms, for either grading, is

$$\sum_{w \text{ agreeing}} C_D(f; w) = \int_{\mathrm{Hom}(V',V)(F_0)\backslash \mathrm{Hom}(V',V)(A)} \frac{\theta(\rho(y^\dagger)f)}{\psi_E(j_{\mathrm{W}}\,\mathrm{tr}(yT))}\,dv, \qquad \int dv = 1. \tag{64}$$

Adjunction reverses the two indices, so $y^\dagger \in I_D(A)$. The Heisenberg relation and invariance of the average under $I_D(A)$ give

$$C_D(\rho(y^\dagger)f; w) = \beta(w, y^\dagger)^2 C_D(f; w) = \psi_E(j_{\mathrm{W}}\,\mathrm{tr}(yw)) C_D(f; w).$$

Division by the character at T leaves the character of $\mathrm{tr}(y(w - T))$. Its mean is one exactly when $(w - T)_{bi} = 0$ for every $a_i > \max(0, a'_b)$, and zero otherwise. This proves (64) in either grading. The character identity and $\int dv = 1$ give

$$\int \sum_{w\in\mathcal{C}_D} |C_D(\rho(y^\dagger)f; w)|\,dv = \sum_{w\in\mathcal{C}_D} |C_D(f; w)| < \infty.$$

Thus we may integrate the Fourier series termwise after taking the I_D-average in each coefficient.

Evaluate this sum again at $D = a'$. The rational Bessel group preserves agreement and fixes the class of T in $\mathcal{C}_{a'}$. On nonzero graded pieces it fixes the string coordinates by the positive flag and its dual, and on the grade-zero piece it fixes the central string coordinate. The definite grade-zero part is killed by T. Thus $T(\gamma - 1)$ has degree at least zero. Likewise $(\delta - 1)T$ has degree at least zero for $\delta \in R'(F_0)$, since T has no component in the definite target block. The second phase test and orbit cancellation leave only the class of T. By (59), the translated agreeing sets for $D = 0$ are indexed by the cosets $\delta R'(F_0)$ in $U(V')(F_0)$. Their integrals are obtained from the original agreeing set by replacing h with $\delta^{-1}h$ in (63). We obtain

$$\int_{[R]} W(r)\theta(\omega(r,h)\Phi)\,dr = \sum_{\delta\in R'(F_0)\backslash U(V')(F_0)} I(\delta h), \qquad I(h) = \int_{[R]} W(r) C_{a'}(\omega(r,h)\Phi; T)\,dr. \tag{65}$$

The invariance of the class of T gives left $R'(F_0)$-invariance of I.

The last single step starts with $s = 0$. Both gradings are zero, so $T = 0$, $I_D = 0$, $\mathcal{C}_D = \{0\}$, $R = U(V)$ and $R' = U(V')$. Thus $C_D(f; 0) = \theta(f)$ and the sum in (65) has one term. Pairing it with the terminal test is ordinary integration on the two definite quotients, with their original Haar measures.

18.1.2 Absolute convergence and unfolding

In this subsection, we bound the sum over the orbits of Section 18.1.1 before unfolding.

For integration over $[G']$, measure each rational vector in the moved finite lattice in the signed real coordinates v_τ, in which the Hermitian matrix J_τ satisfies $J_\tau^* = J_\tau = J_\tau^{-1}$. Use the length $\max_\tau \|v_\tau\|_2$, write $\lambda_1(h)$ for the least length of a nonzero lattice vector, and put $H(h) = \max(1, \lambda_1(h)^{-1})$. The covolume is fixed, since a unitary determinant has adelic absolute norm one. If $H(h) \le 2$, Minkowski's successive minima bound gives a bounded rational basis in the adelic coordinates.

Use a full set of bounded independent $\mathbb{Z}$-lattice vectors and take an E-basis among them. Its finite coordinates are integral in the chosen basis. The rational Gram matrices of these bases lie in a compact adelic set and therefore take finitely many values. Fix a rational basis for each Gram matrix. The map from the moving basis to that fixed one is a rational isometry. The change of basis puts the inverses of the resulting translates in a compact set of adelic matrices, and thus the translates themselves in a compact set of the unitary group. Here inversion on the latter uses the fixed Hermitian form.

If $H(h) > 2$, the lattice vectors of length below 1 span a totally isotropic space at each real place. Their pairings are finite-integral, and at each infinite place $|v_\tau^* J_\tau w_\tau| \le \|v_\tau\|_2 \|w_\tau\|_2 < 1$. The absolute norm of a nonzero algebraic integer is at least one, so all these pairings vanish. Use the sign matrix of the real Hermitian form to take the Euclidean orthogonal opposite of this isotropic space. Dilate the isotropic space by 2 and its opposite by $1/2$, and fix the joint orthogonal complement. This is unitary and both it and its inverse have operator norm at most 2. The short lengths double. Every other length decreases by at most a factor of two. The resulting height is at most $\max(2, H(h)/2)$. Repetition gives $\delta h = zx$, with δ rational, z in a fixed compact set and x purely real with $\|x_\tau\| \le H(h)$. The dilation count is at most $\lfloor \log_2 H(h) \rfloor$.

In the real Schrödinger model use the L^2-norms of monomials in coordinate multiplication and differentiation operators. By Heisenberg covariance, conjugation by the Weil operator sends the linear operators to linear combinations with coefficients of the symplectic matrix or its inverse. By L^2-unitarity, these norms of the translate of the test are bounded by the corresponding norms of the original test times a polynomial in the matrix and inverse norms.

Sobolev embedding bounds the weighted suprema above by finitely many such L^2-norms. The symplectic action is $w \mapsto hwr^{-1}$. Its matrix and inverse are polynomially bounded on the reduced representatives with r in a compact set. At the finite places the translates of the test over a compact set lie in a finite set by smoothness of the Weil action. Therefore

$$\sum_{w \in \mathcal{C}_0} |F_w(r,h)| \le C H(h)^B$$

where B depends on the fixed spaces and test Φ, and C also includes $\sup_{[R]} |W| < \infty$, since $[R]$ is compact. Indeed at $D = 0$ every rational target isometry preserves the filtration. Equation (63) therefore permutes all frequencies under either rational reduction used in this estimate.

For any continuous automorphic y with $\sup_h H(h)^B |y(h)| < \infty$ for all B, the product majorant is bounded on $[R] \times [G']$. Each $I(\delta h)$ is the integral of the sum over its own agreeing set at $D = 0$, by (64) and the second orbit cancellation. The distinct sets are disjoint, whence

$$\sum_\delta |I(\delta h) y(h)| \le |y(h)| \int_{[R]} \sum_{w \in \mathcal{C}_0} |F_w(r,h)| \, dr \le C \operatorname{vol}[R] H(h)^B |y(h)|.$$

Both quotients have finite volume. Fubini and quotient integration give

Proposition 18.1 (Unfolding consecutive lifts)**.** *For these data let W be continuous on $[R]$, with $W(ur) = \beta(Tu,T)W(r)$ for $\deg(u-1) \geq 2$. Let y be continuous on $[G']$ and satisfy $\sup_h H(h)^B|y(h)| < \infty$ for every $B > 0$. The two integrals below converge absolutely and unfolding with the inherited target measure gives*

$$\int_{[R]} W(r) \int_{[G']} \theta(\omega(r,h)\Phi)y(h)\,dh\,dr = \int_{R'(F_0)\backslash U(V')(A)} I(h)y(h)\,dh. \tag{66}$$

The quotient measure on the right uses counting measure on the rational subgroup, and the unfolding has scalar one.

18.1.3 Rapid decay of the projected tests

In this subsection, we verify the decay hypothesis of Proposition 18.1 for the projected test.

For the target test coming from an integrable discrete series projector, rapid decay follows as follows. Let y be any bounded finite-level form reproduced by the coefficient projector of the prescribed integrable real discrete series. For the target tests in the lower chain, Section 18.3 verifies these hypotheses for the conjugate projector by induction from the definite endpoint. Write the reproduction identity as $y = R(e)y$, where $e = d_\pi\overline{\langle v_0, \pi(\cdot)v_0\rangle}$ is the convolution projector for a unit K_∞-finite vector v_0 of the specified integrable discrete series π.

Take the inner product linear in the second variable. For v in the Harish-Chandra module of π, the map

$$v \longmapsto d_\pi \int y(hx)\langle \pi(x)v_0, v\rangle\,dx$$

takes v_0 to y. Denote it by T. On K_∞ a change of integration variable gives $R(k)Tv = T\pi(k)v$; differentiation of the coefficients in L^1 gives $dR(X)Tv = T\,d\pi(X)v$ [56, Lemma 5-5(i), (iii), (v)]. The semisimple result applies on the derived real group, with the compact center acting by a character. Each Tv is bounded, as y is bounded and the coefficient is integrable. The quotient has finite volume, so $Tv \in L^2$.

If $y \neq 0$, the pullback $\langle Tv, Tw\rangle_{L^2}$ is a nonzero invariant Hermitian form on the irreducible Harish-Chandra module of π. On each finite-dimensional K_∞-isotypic subspace it is represented, in the Hilbert inner product of π, by a positive semidefinite operator. Their direct sum intertwines the Lie algebra by infinitesimal invariance and intertwines K_∞. Schur's lemma therefore gives $\|Tv\|_2^2 = c\|v\|_\pi^2$ for one $c > 0$. Thus T extends to the Hilbert completion. For a real Lie algebra element X and a K_∞-finite vector v,

$$dR(X)^n Tv = T\,d\pi(X)^n v \qquad (n \geq 0).$$

The analytic-vector series for v converges in norm for small $|t|$; the norm identity gives the same convergence for Tv. Their sums give $R(\exp tX)Tv = T\pi(\exp tX)v$. By density and generation by the identity component and K_∞, the extension is equivariant. Its closed image is a discrete-series summand.

At fixed finite level the space has finitely many arithmetic real quotient components [17, Theorem 5.1]. On each component write $G_\infty = Z_c G_\infty^1$, where Z_c is the compact scalar torus and G_∞^1 the product of the special unitary groups. The determinant of each element of its arithmetic subgroup Γ is an integral unit of absolute value one at every embedding, hence a

root of unity. Thus $\Gamma^1 = \Gamma \cap G_\infty^1$ has finite index in Γ. Lift to $\Gamma^1\backslash G_\infty$. The centre acts by a unitary character, so integration over Z_c shows that restriction to $\Gamma^1\backslash G_\infty^1$ preserves the squared norm up to a fixed positive scalar. Here the product map has finite central kernel. The restriction of π is irreducible because Z_c acts by scalars, and remains square integrable by the same central integration.

On each nonzero component, restriction of scalars gives the semisimple group over $\mathbb{Q}$ and its arithmetic subgroup. It can be represented over $\mathbb{Q}$ using the finite-level lattice, whose stabilizer contains Γ^1 with finite index. The intertwining identity shows that every right derivative of Tv is another bounded coefficient integral. Together with compact finiteness and the infinitesimal character, this puts the restricted functions in Wallach’s automorphic function space. Theorem 4.3 therefore puts this tempered summand in the cuspidal spectrum [73]. Each rational parabolic of the unitary group has the same unipotent radical N as its intersection with the derived group, and $\Gamma \cap N = \Gamma^1 \cap N$. To recover the adelic constant term at a finite representative h_f of level J, put $J_N = N(A_f) \cap h_f J h_f^{-1}$. In characteristic zero, exponential coordinates and additive approximation make $N(F_0)$ dense in $N(A_f)$. Hence

$$N(A_f) = N(F_0) J_N, \qquad N(F_0)\backslash N(A)/J_N \simeq (N(F_0) \cap J_N)\backslash N_\infty.$$

Integration with quotient probability gives exactly the real constant term on this component, which is zero. The smooth vector $T(v_0)$ in this cuspidal embedding satisfies the hypotheses of [45, Theorem 2.24]. Its value on each of the finitely many Siegel sets is $O_A(\|g\|^{-A})$ for every $A > 0$. Central translates have the same absolute value.

The finite lattice is fixed on each real component, so $H(h) \ll 1 + \|h_\infty\|$ there. Indeed the original nonzero lattice vectors have a fixed positive minimum and the inverse real action decreases lengths by at most $\|h_\infty\|$. This proves the required decay in $H(h)$ for the projected tests.

18.2 Two successive Bessel integrals

In this subsection, we combine two copies of the unfolding in Section 18.1. The middle Hermitian space then appears in a single theta integral, and its constant term gives the paired operator needed in Section 18.3.

Take three lower spaces Y, Y', Y'' with the same marked c, N, odd initial string length s, and remaining lengths $s-1, \max(0, s-2)$. Write T, T' for the two string maps and let

$$S = (T^\dagger, T')^{\mathsf{T}} : V' \longrightarrow V^- \oplus V'',$$

where V^- has form $-h$. For an interior pair, adjunction in the antidiagonal forms gives

$$T T^\dagger e_i' = t_{s-2-i} t_{i+1} (g_i'/g_{i+1}) e_{i+1}'$$

before the last coordinate. This is the middle shift, with coefficient c except at grade one, where it is $-Nc^2$. Thus $TT^\dagger = T'^\dagger T'$, and $S^\dagger S = -TT^\dagger + T'^\dagger T' = 0$. The same adjoint computation for even s gives the odd target shift, with $-Nc$ at grade zero. At $s = 1$ the maps are zero.

Let $u \in U_{R'}(\mathbb{A}_{F_0})$ satisfy $\deg(u-1) \geq 2$. Target covariance replaces T by $u^{-1}T$, and $(u^{-1}-1)T$ has degree at least one. Equation (62) and $u^\dagger = u^{-1}$ give

$$I(uh) = \beta(T, u^{-1}T) I(h),$$
$$\beta(T, u^{-1}T) = \psi_E\big((j_{\mathrm{W}}/2)\operatorname{tr}(TT^\dagger u)\big) = \beta(T'u, T'),$$

since $TT^{\dagger} = T'^{\dagger}T'$. This gives the middle Bessel character on the subgroup of degree at least two. Section 18.2.1 computes the contribution of the degree-one entries. Continuity follows from the compact $[R]$ integral and the local Schwartz bounds. We may therefore use the resulting I on the next Bessel group.

The adjoint identifies the symplectic spaces $\mathrm{Hom}(V, V')$ and $\mathrm{Hom}(V', V^{-})$. The Hermitian trace on the latter is the negative conjugate of the trace pairing on the former, which preserves the alternating form with j_{W}. Use the corresponding linear oscillator transport and tensor it with the second datum. Write $\mathcal{B}(f, f')$ for the joined datum. This intertwines the Heisenberg translations by w and $w^{\dagger}$.

For the opposite finite basis, multiply the original basis by the conjugate of $\eta\alpha_{\mathrm{W}}/\bar{\alpha}_{\mathrm{W}}$, where $\eta \in \widehat{\mathcal{O}}_E^{\times}$ has norm -1. Such a unit exists at every finite place since E/F_0 is unramified or split there. Put $\lambda = \eta\alpha_{\mathrm{W}}/\bar{\alpha}_{\mathrm{W}}$ and write $A^{*} = \bar{A}^{\mathsf{T}}$ in the finite coordinates. The pullback on the finite component of an adelic coordinate uses $A \mapsto \lambda A^{*}$; its inverse is $A \mapsto -\lambda A^{*}$ since $\lambda\bar{\lambda} = -1$. In the finite Schrödinger test the coordinate is $A = \alpha_{\mathrm{W}} Y$. Thus $\lambda(\alpha_{\mathrm{W}} Y)^{*} = \alpha_{\mathrm{W}}\eta Y^{*}$, and the test on the conjugate places is $Y \mapsto \Phi(\eta Y^{*})$.

For clarity, let e_{Λ} denote the finite vacuum semicharacter and $T_{\Lambda}(Y) = e_{\Lambda}(Y)\rho(0, \alpha_{\mathrm{W}} Y)$ the corresponding finite translation in these coordinates. For two blocks A, B, the norm is additive, and $\eta\bar{\eta} = -1$ gives $e_{\Lambda}(\eta A^{*}) = e_{\Lambda}(A)^{-1}$. Consequently

$$T_{\Lambda}\begin{pmatrix} A \\ B \end{pmatrix}\mathcal{B}(f, f') = e_{\Lambda}(A)^{2}\,\mathcal{B}(T_{\Lambda}(\eta A^{*})f, T_{\Lambda}(B)f').$$

The constant terms below use the adelic translation ρ of Section 18.1.

In rational coordinates the forward map is ordinary adjunction. Transporting a rational Schrödinger polarization by this symplectic map gives the same sum over the rational Lagrangian. The two theta normalizations therefore satisfy

$$\theta(\mathcal{B}(f, f')) = z_{\theta}\,\theta(f)\theta(f'), \qquad z_{\theta} \neq 0.$$

The scalar is fixed by the marked models and is independent of the two tests.

18.2.1 The constant terms in the joined block

In this subsection, we group the constant terms before the lower-lift comparison of Section 17.3.

Let Z_1 be the zero term along the joined positive-degree block. The zero term along an isotropic space K, at a vector in the residual space, is the theta functional after translating by that vector and averaging over $K(F_0)\backslash K(A)$ with probability measure. The map $(w, w') \mapsto (w^{\dagger}, w')$ takes the two positive blocks to the joined block, since adjunction reverses both grades. It is a rational bijection and pushes product probability to probability. Consequently

$$C_{a'}(f; w)C_{a''}(f'; w') = z_{\theta}^{-1}\big[Z_1\rho((w^{\dagger}, w'))\mathcal{B}(f, f')\big](0). \tag{67}$$

For an element n of the middle unipotent radical, the two frequencies in (67) become $n^{-1}T, T'n$ by covariance, so the joined frequency is Sn. Here $(n^{-1}T)^{\dagger} = T^{\dagger}n$. The Heisenberg multiplication rule has phase

$$\beta(S(n-1), S) = \psi_E\big((j_{\mathrm{W}}/2)\,\mathrm{tr}(S^{\dagger}S(n-1))\big) = 1.$$

Hence $\rho(Sn) = \rho(S(n-1))\rho(S)$. The map $S(n-1)$ has nonnegative degree. If $s > 1$, the outer strings have even grades and the middle string has odd grades. The grade-zero part of Hom is thus the maps from X' to the outer grade-zero space. The two outer middle vectors z, z' have norms $N, -N$ in the joined target. They form the hyperbolic plane with basis $(z - z')/(2N), z + z'$. The grade-one middle vector e_+ has $Se_+ = c(z + z')$: the first coefficient is the adjoint coefficient $(-Nc)/(-N)$ and the second is c. It follows that the degree-zero part of $S(n-1)$ is $(z+z')b(n)$, with $b(n)_i = c(n-1)_{+,i}$ for $i \in X'$. Its positive part is orthogonal to the degree-zero part and is absorbed by Z_1. The product is z_θ^{-1} times the value of $Z_1\rho(S)\mathcal{B}(f, f')$ at $(z + z')b(n)$.

The map b is additive on the middle unipotent, since the product of two positive-degree differences has degree at least two. It is onto over F_0 and over A. To prescribe b_i, take $x = e_+$, $v = \sum_i c^{-1}\bar{b}_i e_i$ in X', and $L = xh'(v, \cdot) - vh'(x, \cdot)$. The unitary $1 + L + L^2/2$ is strictly grade-raising off the identity and has this covector. Denote it by $u(t)$ when the prescribed covector is t. If dn is invariant probability on the rational quotient of the middle unipotent, then $b(nu(t)) = b(n) + t$ and right translation by $u(t)$ preserves dn. Thus b_*dn is translation-invariant probability on the covectors modulo rational covectors. Its average gives a second zero term Z_2 along $z + z'$. The two slices compose to $\mathcal{Z} = Z_2Z_1$, with Haar probability on the direct sum of the isotropic directions.

For $s = 1$ the two string maps vanish, all three gradings are zero and there is no plane to remove. The vector z/π_N has norm $N/(\pi_N\bar{\pi}_N) = 1$ in V^-. It gives the last rational coordinate of X'', whose other coordinates are the definite part of Y''. Thus $\widehat{k} = \mathrm{diag}(k, 1)$, and both zero-term operators are the identity in these coordinates. Section 19.2 computes the image of the original finite lattice in this basis at the conjugate places above N. In the interior $\widehat{k} = k$.

For the determinant factors, write υ_1 for the splitting character on norm-one ideles. On the isotropic block of a parabolic Weil action let A_0 be its matrix. The scalar of the zero slice, relative to the action on the residual space, is $\upsilon(\det A_0)^{-1}|\det A_0|^{1/2}$. The modulus is the square root of the Jacobian in the dual coordinates $A_0^\dagger x$. For the phase, use an adapted Gaussian and finite lattice vacuum.

In the real basis with prescribed signs, the negative block of the hyperbolic Levi has matrix $(A_0^{-\dagger} + A_0)/2$. Its conjugate determinant times $\det A_0$ is the positive determinant of $(1 + A_0^\dagger A_0)/2$. At a finite place put $L = \mathcal{O}_E^{\dim A_0}$ in the corresponding local coordinates. The multiplier for an invertible g is $\chi_f(\det B)$, where $BL = L \cap g^{-1}L$ and $\chi_f = \upsilon_f|\cdot|^{-1/2}$. For the hyperbolic pair $A_0^{-\dagger}, A_0$ this has multiplier $\overline{\chi_f(\det A_0)}|\Delta_f(A_0)|^2$.

Indeed A_0 carries $L \cap A_0^{-1}L$ to $L \cap A_0L$, so inversion multiplies Δ_f by $\chi_f(\det A_0)$. Transposition leaves the intersection determinant ideal unchanged, by elementary divisors, and conjugation conjugates Δ_f. Thus the hyperbolic multiplier has phase $\upsilon(\det A_0)^{-1}$. The Gaussian overlap and the overlap of the finite zero slices are positive in the adapted coordinates. The phase of the scalar is therefore the phase of this multiplier.

The parabolic radical has trivial scalar: it acts on the zero slice by a scalar by Heisenberg covariance and every element of that radical is a commutator with an isotropic dilation on its two successive positive-grade quotients. Equivalently, in the canonical Schrödinger splitting the terms of its phase vanish at the zero coordinate. This also gives the unipotent covariance used in Section 18.1.

The marked arrow of parameter a multiplies the raw action of (g, h) by

$$\frac{a(\det h)}{a(\det g)}\,\upsilon_1(\det h)^{\lfloor n/2\rfloor}\upsilon_1(\det g)^{-\lfloor n'/2\rfloor},$$

where n, n' are the ambient ranks. The adjoint transport on the first arrow contributes $\upsilon_1(\det g)^{n'}\upsilon_1(\det h)^{-n}$. Indeed, under the transpose of the inverse matrix of the action on Hom, the opposite sign interchanges the real positive and negative blocks.

Put $D(w) = hwg^{-1}$ on the first Hom space. For its signed real realization at v, write $\Delta_{\epsilon_v}(D_v) = \overline{\det\left((D_v)_{--}\right)}$. Jacobi's complementary minor identity and $|\det D_v| = 1$ give

$$\Delta_{-\epsilon_v}(D_v^{-1}) = \overline{\det\left((D_v^{-1})_{++}\right)} = (\det D_v)\Delta_{\epsilon_v}(D_v).$$

At the finite places $D_f(L \cap D_f^{-1}L) = L \cap D_f L$. Since $|\det D_f|_f = 1$, this gives

$$\Delta_f(D_f^{-1}) = \upsilon_f(\det D_f)\Delta_f(D_f).$$

Transposition and reordering the two matrix indices preserve these principal minors and finite elementary divisors. Multiplying over all places, the original multiplier divided by the opposed multiplier is $\upsilon_1(\det D)^{-1}$, where $\det D = (\det h)^n(\det g)^{-n'}$. The exponents on the three determinants in the joined action are consequently

$$(\lceil n'/2\rceil,\ -\lceil n/2\rceil - \lfloor n''/2\rfloor,\ \lfloor n'/2\rfloor).$$

The factors $a(\det h)/a(\det g)$ and $b(\det k)/b(\det h)$ are retained.

Let $d_0 = (s-1)/2$, let p_0 count the positive outer grades, and let q_0 be one in the interior and zero at $s = 1$. On the positive Hom block the outer determinant occurs once per negative middle coordinate and the middle inverse determinant once per positive outer coordinate. Both determinants have norm one. The first slice changes the exponents by $(-d_0, p_0, -d_0)$. The plane slice changes them by $(0, q_0, 0)$: the plane is fixed and the column action is h^{-1}. If x, x', x'' are the final ranks, then

$$n' = x' + 2d_0, \qquad n + n'' = x + x'' + 2p_0 + 2q_0.$$

Here $n'' = n + 2t$, so $n + n''$ and hence $x + x''$ are even. Since the determinant of $\widehat{k}$ equals that of k, substitution gives the exponents for the definite ranks x, x', x''. The joined motions of the definite groups commute with $\rho(S)$, since they fix all the string maps. Thus $\mathcal{Z}$ carries the ambient action to the joined definite action.

18.2.2 The two quotient volumes

In this subsection, we compare the quotient measures after the preceding unfolding with the normalization of Section 8.2.

Write $\kappa_R = \mathrm{vol}[R]/\mathrm{vol}[U(X)]$ and similarly $\kappa_{R'}$. The Levi acts on the unipotent radical with modulus one, since a determinant of its action is a rational character of a unitary group. Disintegrating the original quotient Haar therefore gives the Levi quotient Haar times this constant and Haar probability on each unipotent fibre. For odd s, the source unipotent acts with degree at least two. Its phase in the first constant term cancels the character of W by (62). The middle mean then gives Z_2 by the surjectivity of b.

Let $\omega_{a,b}^{\mathrm{def}}$ be the joined action on the two definite arrows, and $\iota_X, \iota_{X_0''}$ the original definite group embeddings on the two outer covectors. The integral on the two Bessel groups, of W times the product of the two string-map constant terms uses the same middle point in both kernels. Write

$$D_f(r, r') = C_{a'}(\omega_a(r, r')f; T), \qquad D_{f'}(r', h) = C_{a''}(\omega_b(r', h)f'; T'),$$

where the two actions include their rank twists. For continuous W on $[R]$ with the prescribed unipotent character,

$$\begin{aligned}&\int_{[R']}\int_{[R]} W(r)D_f(r,r')D_{f'}(r',\iota_{X_0''}(k))\,dr\,dr'\\&\quad=\frac{\kappa_R\kappa_{R'}}{z_\theta}\int_{[U(X)]}\int_{[U(X')]} W(\iota_X(g))\,\theta_{\mathrm{def}}\Big(\omega_{a,b}^{\mathrm{def}}(g,h,\widehat{k})\mathcal{Z}\rho(S)\mathcal{B}(f,f')\Big)\,dh\,dg.\end{aligned}\tag{68}$$

The integrands descend continuously to the compact Bessel quotients. This justifies the order of the means. At the endpoint the quotient of the last covector is its full definite quotient. The coefficient on it is evaluated at $\mathrm{diag}(k,1)$ with the branching normalization of Section 7.4. The two quotient-mass constants in (68) are for the original Haar measures.

18.3 The coefficients after an even number of steps

In this subsection, we iterate the paired identity of Section 18.2. We keep the source indices and identify the real coefficients after each pair of theta operators.

Write R_j for the Bessel group of Y_j, $J_{\xi,j}$ for the principal kernel at the chosen level and twist places, and $A_{\xi,j}=[K_j^\circ:J_{\xi,j}]$ for its index in the standard compact. Put $G_j=U(Y_j)$ and write $K_{\xi,j}(g,h)=\theta(\omega_{\xi,j}(g,h)\Phi_{\xi,j})$ for the kernel with the given level test, rank twists and splitting character. For a source form f and a test z of the conjugate target type, the two operators for the bilinear pairing are

$$\begin{aligned}(L_{\xi,j}f)(h)&=A_{\xi,j}\int_{[G_j]}K_{\xi,j}(g,h)f(g)\,dg,\\(L_{\xi,j}^{\mathrm{t}}z)(g)&=A_{\xi,j}\int_{[G_{j+1}]}K_{\xi,j}(g,h)z(h)\,dh.\end{aligned}$$

Thus $\int_{[G_{j+1}]} zL_{\xi,j}f=\int_{[G_j]}(L_{\xi,j}^{\mathrm{t}}z)f$. Indeed the compact finite translates give finitely many Schwartz tests. On reduced representatives the real symplectic matrix for $w\mapsto hwg^{-1}$ and its inverse are bounded by powers of the product of the two heights. The Schwartz seminorm estimate of Section 18.1.2 gives $|K_{\xi,j}(g,h)|\ll H_j(g)^CH_{j+1}(h)^C$ for the fixed test. The rapid decay of f,z and finite quotient volumes make the double integral absolute, so Fubini applies. In either variable the principal kernel fixes the lattice-inclusion test and the determinant character. In the last covector, an integral element with block-upper-triangular reduction at the twist and identity at the chosen level places has leading blocks that lift to the preceding covectors. The inclusions intertwine those blocks in the residues. The successive determinant ratios are the tame characters on the added diagonal blocks. The initial block fixes the source vector, which has maximal level at the twist places. This proves the terminal finite character.

In the lower grid the added positive blocks of length $t=5r$ are centered at $-w_i$ and have entries $-w_i+(t-1)/2-b$ for $0\le b<t$. The common half shift at odd stages leaves all gaps unchanged. The base tails lie midway between consecutive centers. Only base entries carry negative sign. Thus opposite signs have gaps at least $\Delta/2-t>500(r+mt)^3$ for large x. Let e_j denote the normalized discrete-series coefficient on $U(Y_j)_\infty$, with $R(e)f(g)=\int e(k)f(gk)\,dk$. Write $e^\vee(k)=e(k^{-1})=\overline{e(k)}$. The pair (Y_j,Y_{j+1}), $0\le j<m$, is the horizontal step in grid $z=0$ with zero offset, no up blocks and j previous down blocks. Its ranks are $r+jt,r+(j+1)t$; writing c for the new centre, its positive block has doubled entries $2c+t-1-2b$, $0\le b<t$. The common odd-stage shift cancels after the opposite determinant exponent is subtracted. Thus

the signed parameter and joint-harmonic calculation of Section 34.3.2 applies to this step and, with conjugate types interchanged, to its transpose.

For clarity, take either orientation, with incoming and outgoing coefficients $(e, e') = (e_j, e_{j+1})$ or $(\bar{e}_{j+1}, \bar{e}_j)$. Write P for its real harmonic and L_P for the integral in the incoming variable with the given finite test. In the real oscillator put

$$p_{\mathrm{in}} = \int e^{\vee}(k)\,\omega(k,1)\,dk, \qquad p_{\mathrm{out}} = \int e'(k)\,\omega(1,k)\,dk.$$

The local line identity of Section 34.3.2 gives $p_{\mathrm{out}} p_{\mathrm{in}} P = p_{\mathrm{in}} P$ in this orientation. For a rapid incoming form $y = R(e)y$, change the source variable $g \mapsto gk^{-1}$ under the bilinear integral. Right convolution in the target then gives

$$L_P y = L_{p_{\mathrm{in}} P} y, \qquad R(e') L_P y = L_{p_{\mathrm{out}} p_{\mathrm{in}} P} y = L_P y.$$

These changes use only the polynomial height bound for the theta kernel against a rapid source, the absolute Fock integral bound for the fixed finite datum and the coefficient moments in Sections 34.3.1 and 34.3.2. They precede any use of the seesaw. The truncation argument of Section 34.3.2 now places $L_P y$ in the cuspidal coefficient space at its fixed finite level; Section 18.1.3 gives its rapid decay. Starting at the definite endpoint and applying this argument successively in the transposed orientation supplies the rapid test required at every unfolding.

Fix outer points $g_j = (u_j, y_j)$ and let W_0 be the original Bessel source coefficient of ϕ with the functional t. For $j \geq 0$ define the raw functions

$$I_{j+1}(h) = \int_{[R_j]} W_j(r) C_{a_{j+1}}(\omega_{\xi,j}(r g_j, h)\Phi_{\xi,j}; T_j)\,dr, \qquad W_{j+1}(r) = I_{j+1}(r g_{j+1}).$$

The successor covariance of Section 18.2 gives the required character of W_j on $\deg(u-1) \geq 2$ at every step. At an even stage this is the entire unipotent radical; at an odd stage its degree-one entries contribute the second isotropic mean of Section 18.2.1. The function I_j is independent of the outer j-th point before its evaluation. For z in the conjugate target type the measured identity is

$$\int_{[R_j]} W_j(r)(L^{\mathrm{t}}_{\xi,j} z)(r g_j)\,dr = A_{\xi,j} \int_{R_{j+1}(F_0)\backslash U(Y_{j+1})(A)} I_{j+1}(h) z(h)\,dh.$$

This is the unnormalized unfolding for one theta lift with the source-level factor of the transpose. Put $G' = U(Y_{j+1})$ and $R' = R_{j+1}$. The quotient integral on the right disintegrates as

$$\int_{R'(\mathbb{A}_{F_0})\backslash G'(\mathbb{A}_{F_0})} \int_{[R']} I_{j+1}(rh) z(rh)\,dr\,d\dot{h}.$$

Here $d\dot{h}$ is the quotient of the given group Haar measures, since both groups are unimodular. Absolute convergence in Proposition 18.1 applies also to this disintegration. For $z = L^{\mathrm{t}}_{\xi,j+1} z'$, fix a representative $h = g_{j+1}$ in the outer integral. The inner source is precisely $W_{j+1}(r)$, so the next unfolding contributes $A_{\xi,j+1}$.

For a pair of steps starting at $j = 2i$, move each lower datum by its two outer points, using the representation law in I_{j+1}. Join the translated tests and apply the S-translation and both slices of Section 18.2. The result is a pure real factor times a joined finite factor. Indeed in an

adapted adelic Schrödinger polarization the slices are restrictions at zero in the dual coordinates of the two isotropic directions.

On a pure tensor this is the product of real and finite restrictions. Heisenberg translations, Weil moves and the adjoint transport also factor by place. Put the fixed theta normalization in the real factor. Thus the finite factor is the joined finite zero slice of $\rho_f(S_f)$ on the two translated finite vectors, with the opposite-basis scale. Denote its inverse image under the finite tensor identification by $F_i(\xi, y)$.

This inverse exists since compactly supported locally constant tests on a product are finite sums of product tests, by a finite rectangular partition. The real factor depends on the u_j and fixed weights. Every finite tame character has trivial real part, so the real factor is independent of ξ, y.

For a definite arrow of ranks $n < n'$, let $m_{\tau,h}$ count its columns of height h, as in Section 20.2. There are no columns of height $n < h < n'$. A column of height n contributes one determinant at the source and remains a column at the target. In the order of decreasing parameters the highest polynomial is proportional to

$$\prod_{\tau} \prod_{h=1}^{n} \det(Z_{\tau,[1,h],[1,h]})^{m_{\tau,h}}, \qquad D = \sum_{\tau} \sum_{h=1}^{n} h m_{\tau,h}.$$

Indeed each leading minor is highest on both sides and its diagonal weight counts its first h coordinates. The compact Cauchy decomposition [35, p. 560] gives multiplicity one for the two resulting types after their determinant twists.

Write V, V' for the two Weyl spaces with their conjugate Weyl actions ρ, ρ'. The corresponding polynomial space is spanned by $Q_{A,v}(Z) = A\mathcal{S}(Z)v$, with $v \in V$, $A \in (V')^{\vee}$ and $\mathcal{S}$ the rectangular Schur matrix. At real points u, u' the Weil action replaces (A, v) by $(A\rho'(u'^{-1}), \rho(u)v)$: substitution in the minors gives the two Schur actions, with the height-n determinants supplying the source determinant twist. For a Taylor series H and $Q = \sum q_\alpha Z^\alpha$, the derivative is complex linear in Q,

$$Q(\partial)H(0) = \sum_{\alpha} q_\alpha \alpha! \, [Z^\alpha]H.$$

This is the Fock monomial pairing $\int \bar{Z}^\alpha Z^\nu e^{-\|Z\|^2} dZ/\pi^{d_F} = \mathbf{1}_{\alpha=\nu}\nu!$ in $d_F = \sum_\tau nn'$ variables, extended linearly in the coefficients q_α. Thus, with the jet matrix J_c at the fixed scale $c = \pi_M s_{\mathrm{jet}} \neq 0$ of Section 22.1, the theta value of $Q_{A,v} \otimes f$ at points $(u, g), (u', h)$ equals

$$c^{-D} A\rho'(u'^{-1}) J_c(\omega_f(g, h)f)\rho(u)v.$$

Here ω_f denotes the finite Weil action with the chosen twist.

Let $t \in V^{\vee}$ be the incoming functional. Its source coefficient at (u, g) is $t\rho(u^{-1})\phi(g)$. In mass-one real Haar measure, Schur's lemma and the trace give

$$\int \rho(u)(v \otimes t)\rho(u^{-1}) \, du = \frac{t(v)}{d_\lambda} \mathrm{Id}_V, \qquad d_\lambda = \dim V.$$

All other source types give zero by orthogonality. Let $K \subseteq K^\circ$ be the fixed finite level, with $\mathrm{vol}(K^\circ) = 1$. At any common smaller level $J \subseteq K$ fixing a scalar integrand, its class sum (weighted by inverse stabilizer orders) has multiplier $[K^\circ : J]^{-1} = [K^\circ : K]^{-1}[K : J]^{-1}$. The factor $[K : J]^{-1}$ gives precisely the class integral normalized at K in the algebraic lift. If the

matched component of a polynomial P is $\sum_i Q_{A_i,v_i}$, the outgoing functional is therefore

$$t'_P = \frac{\sum_i t(v_i)A_i}{d_\lambda[K^\circ : K]c^D}.$$

This is contraction on $(V')^\vee \otimes V$, independent of the decomposition and of the finite test, the twist and the source form. For unit highest vectors and their highest covectors it gives the factor $\beta_P/(d_\lambda[K^\circ : K]c^D)$ when the chosen joint highest polynomial is β_P times the highest Schur coefficient.

For the joined real polynomial, the first block of variables has the transposed indices of the first arrow and the second has those of the second arrow. The sign change of the first form makes the flipped Fock variables holomorphic in the new basis. Products of monomials of the two blocks span all polynomials. Let $z_{\mathrm{def}} \neq 0$ be the theta scalar for the join on the definite triple (X, X', X''), obtained by the same rational polarization argument as (67). For a product polynomial PQ and a pure finite tensor, the joined definite theta kernel is z_{def} times the product of the two individual kernels. Apply the first coefficient identity with P and the second with $z_{\mathrm{def}}Q$. Together with (68), the two changes of theta normalization and the two quotient volumes contribute $z_{\mathrm{def}}\kappa_R\kappa_{R'}/z_\theta$ to the coefficient functional. Linearity gives the identity on the finite tensor $F_i(\xi, y)$ with its original normalization.

For a general Schwartz test after the slices, approximate by Gaussian times polynomials in the Schwartz topology, by the Hermite expansion. All the integral evaluations are continuous in that topology by the compact definite quotients and absolute theta bounds. For each fixed source functional the possible families of target coefficients, with the twists, tensors, forms and target points as indices, form the image of the finite-dimensional target Weyl dual in a product of copies of $\mathbb{C}$. This image is closed because finitely many coordinate evaluations separate a basis. Choose a fixed complement of the kernel of this evaluation map. Finitely many separating coordinates recover the outgoing functional in the complement by a fixed linear inverse. Each such coordinate is a theta integral over compact definite quotients, jointly continuous in the moved real vector and the incoming functional. Hence the resulting choice of outgoing functional is continuous in the real points and works for all the finite data.

At the first pair the incoming functional is the original t. For the measured period the initial point is $(g_{0,\infty}, c)$ in the finite level trace of Section 19.1; the intermediate point is the common argument of both kernels. At the outgoing place the real argument can vary with the outgoing finite component held at y_2. Inductively the even-stage function on $U(X_{2i})$ is a coefficient of the first i definite pairs with tensors $F_0, \dots, F_{i-1}$. The coefficient functional depends only on the outer real points and the fixed datum. Its extension to R_{2i} is determined by the unipotent character. The terminal coefficient is read at $\mathrm{diag}(h, 1)$, by the last pair calculation. This statement is pointwise before any outer integration. It also applies to the support test for the finite cuts.

At the terminal space let B be the branching map from the Weyl space of X_m to that of Y_m. For a terminal functional ℓ_u and normalized highest vector v_Y, average

$$\int_{U(Y_m)_\infty} \sigma_Y(k)(v_Y \otimes \ell_u)\sigma_X(\mathrm{diag}(k^{-1}, 1))\, dk = c_{\ell_u} B.$$

The average intertwines by Haar invariance. The terminal weights interlace by Lemma 7.9: after a common determinant twist, removing the boxes occupied by the last letter leaves the smaller diagram, and the removed boxes form a horizontal strip. Thus the compact branching

multiplicity is one. Let z belong to the original terminal type and use $\overline{z}$ as the test of the conjugate type in the preceding bilinear transpositions. With the Hermitian product linear in its second argument,

$$\overline{z(k,h)} = \langle z_{\mathrm{alg}}(h), \sigma_Y(k) v_Y \rangle.$$

If ψ is the upper-chain output, its coefficient at $(\widehat{k},\widehat{h})$ is $\ell_u(\sigma_X(\widehat{k}^{-1})\psi(\widehat{h}))$. Multiplication by $\overline{z(k,h)}$ and integration in k therefore insert precisely the displayed averaged map. Let J_B be the finite class integral of

$$\big\langle z_{\mathrm{alg}}(h), B\psi(\mathrm{diag}(h,1)) \big\rangle$$

with maximal-compact mass one. At a common fixing level J this is its weighted class sum divided by $[K_Y^\circ : J]$, where K_Y° is the terminal maximal compact. Here $R_m(F_0) = G_m(F_0)$, so the last quotient $R_m(F_0)\backslash G_m(\mathbb{A}_{F_0})$ carries the ambient Petersson measure. With probability Haar at infinity, the terminal integral against $\overline{z}$ is $c_{\ell_u} J_B$. The restriction of Section 7.4 is B/b_N, where $b_N = \prod_\tau \tau(\bar{\pi}_N)^{d_\tau}$. With d_Y the full terminal Weyl dimension, Schur orthogonality for the unit vector v_Y gives

$$\langle z, \mathrm{Res}\,\psi \rangle = (d_Y b_N)^{-1} J_B, \qquad c_{\ell_u} J_B = d_Y b_N c_{\ell_u} \langle z, \mathrm{Res}\,\psi \rangle.$$

Thus the terminal coefficient uses the same quotient measure and the same finite representative on both sides.

It separates into a factor of the real outer points and the indicated linear expression in the finite tensors. The two quotient-mass factors of each pair are included in ℓ_u.

18.4 Finite support on the original Bessel orbits

In this subsection, we show that the period expression of Section 18.3 is supported on a compact union modulo the original Bessel actions. The lattice inclusions below give the compact sets used in the class sum of Section 19.1.

For the space Y_j, let $L_j(x)$ be the lattice in the chosen basis at the finite point x, and $\Lambda_j = L_j(1)$. On its string subspace S_j, write $L_j^- = L_j \cap S_j$ and $L_j^+ = \mathrm{pr}_{S_j} L_j$. The ambient lattices $L_j(x)$ and Λ_j are self-dual for the Hermitian form. A string vector pairs integrally against L_j precisely when it pairs integrally against L_j^+, so

$$(L_j^+)^{\#} = L_j^-, \qquad (L_j^-)^{\#} = L_j^+.$$

Use $q_{\mathrm{id}} = b_{\mathrm{id}}/c_M$, where b_{id} is the full modulus idele of the lower arrow and c_M is the original level generator. Then $q_{\mathrm{id}}, \bar{q}_{\mathrm{id}}$ are integral and are units above N. Put $a_E = \alpha_{\mathrm{W}}/\bar{c}_M$.

We first make a graded representative in a Bessel orbit. At each local place take a basis B of the moved lattice adapted to the decreasing grade filtration, by extending successive bases of the saturated intersections. Its degree-zero matrix part D_0 is invertible and $BD_0^{-1} - 1$ strictly raises grades. Let $G = B^* J B$ and $G_0 = D_0^* J D_0$, where J is the marked rational Gram matrix. The form pairs opposite grades.

Thus $G - G_0$ has entries with sum of the two grades negative. After reversing the grading on one axis, G is block triangular, with its diagonal blocks those of the reversed G_0. Since G, G^{-1} are integral, the diagonal blocks and their inverses are integral. Hence G_0 is unimodular. Write $G_0^{-1} G = 1 + P$; P is self-adjoint for G_0 and strictly raises grades. At every finite place E/F_0 is split or unramified, so choose a local integral λ with $\lambda + \bar{\lambda} = 1$.

The conjugation by $1-\lambda P$ removes the linear term of the error relative to G_0: if the error raises grades by at least a, the new error raises by at least $2a$. Both the correction and its inverse are integral. Iterate finitely to get a unitriangular H with $H^{*}GH = G_0$. The isometry BHD_0^{-1} is grade-unipotent and takes $D_0\widehat{\mathcal{O}}_E^{n}$ to L_j. This is a Bessel move to a graded lattice. Almost everywhere one starts with the original graded basis and the move is the identity.

In the coefficient powers of the string shift, the original string line ideals for length s are

$$\Lambda_i^{-} = p_{s,i} N^{-\mathbf{1}_{2i\ge s}} \widehat{\mathcal{O}}_E, \qquad \Lambda_i^{+} = p_{s,i} N^{-\mathbf{1}_{2i+1\ge s}} \widehat{\mathcal{O}}_E, \qquad p_{s,i} = a_E^{s-i}\bar{a}_E^{-1-i}.$$

To see the exponents, on the upper half of the string write the original coordinate of its power as c^d, $d = i - \lfloor s/2 \rfloor$. On the lower half it is Nc^d with $d = i - \lfloor (s-1)/2 \rfloor$. The corresponding marked original line is scaled by $p_{s,i}(a_E\bar{a}_E)^d$.

The valuations of c and $a_E\bar{a}_E$ agree, so the intersection ideals are as displayed. In odd length the central glued vector has nonzero component in the last definite line, so its coefficient must vanish in the string intersection. Its projection adds uz/N on the middle line, giving the extra factor N^{-1} for the projection exactly there. The shift $T_{j,f}$ retains the powers except the final vector it kills, and $p_{s,i} = a_E p_{s-1,i}$; hence

$$T_{j,f}\Lambda_j^{+} = a_E \Lambda_{j+1}^{-}.$$

The quotient Λ^{+}/Λ^{-} is therefore zero outside N and for even length, and one residue line at each place above N in odd length.

18.4.1 The lattice conditions on a finite slice

In this subsection, we express the surviving slices of Section 18.1.1 as inclusions of lattices.

Consider a nonzero finite slice in the step from Y_j to Y_{j+1}. In the finite Hom model write L_H for the self-dual vacuum lattice, K_I for the basis of the positive-degree space and z_T for the finite coordinate of $T_{j,f}$. In these coordinates use $\psi_H(t) = \psi_f(j_{\mathrm{W}}\alpha_{\mathrm{W}}\bar{\alpha}_{\mathrm{W}}t)$. The signed Heisenberg operators obey

$$\rho_f(x)\rho_f(y) = \psi_H(y^{*}x)\rho_f(y)\rho_f(x), \qquad \rho_f(l)v_{\mathrm{vac}} = v_{\mathrm{vac}} \quad (l \in L_H).$$

Let $\mathfrak{m}$ be the ideal generated by b_{id}, $h = \dim_E \mathrm{Hom}_E(Y_j, Y_{j+1})$, and $R = (\mathcal{O}_E/\mathfrak{m})^h$. For the prescribed inclusion $a_0 \in R$ the test is

$$F = \frac{1}{|R|}\sum_{v\in R} \psi_{\mathfrak{m}}(-a_0\cdot v)\,\rho_f(t_v)v_{\mathrm{vac}}, \qquad t_v = \overline{1_{\mathfrak{m}} b_{\mathrm{id}}^{-1}\widetilde{v}}.$$

Here $1_{\mathfrak{m}}$ selects the modulus places, $\widetilde{v}$ is an integral lift, and $\psi_{\mathfrak{m}}$ is the primitive residue character in the chosen split coordinates. Indeed the sum of the characters $\psi_{\mathfrak{m}}((a-a_0)\cdot v)$, divided by $|R|$, is the indicator of $a = a_0$. Since $\bar{b}_{\mathrm{id}}t_v \in L_H$, the commutator of $\rho_f(b_{\mathrm{id}}l)$ with $\rho_f(t_v)$ is one for every $l \in L_H$. Therefore $\rho_f(b_{\mathrm{id}}l)F = F$.

Let g be the unitary Hom action of the two finite points, set $f_1 = \omega_f(g)F$, $M_1 = b_{\mathrm{id}}g$ and $M_2 = \bar{b}_{\mathrm{id}}^{-1}g$. Then $M_1M_2^{*} = 1$ and $\rho_f(M_1 l)f_1 = f_1$ by covariance. In the mixed model let $\mathcal{Z}$ denote restriction to zero in the coordinate dual to the positive-degree space I. With probability Haar on $I(F_0)\backslash I(\mathbb{A}_{F_0})$, character orthogonality selects exactly this coordinate. On the real and finite tensor the restrictions factor, so nonvanishing of the adelic constant term

implies $\mathcal{Z}(\rho_f(z_T)f_1) \neq 0$. For $u \in I \cap M_1 L_H$, one has $\mathcal{Z}\rho_f(u) = \mathcal{Z}$ and $\rho_f(u)f_1 = f_1$. Applying $\mathcal{Z}$ to

$$\rho_f(z_T)\rho_f(u)f_1 = \psi_H(u^* z_T)\rho_f(u)\rho_f(z_T)f_1$$

gives $\psi_H(u^* z_T) = 1$. Choose columns C so that $M_2^* K_I C = B_I$ is an integral basis of the saturation of its image in L_H.

This is possible over every local DVR, with the original basis almost everywhere. Let $J_I B_I = 1$ be an integral left inverse. Since $K_I C = M_1 B_I$, character orthogonality gives $v = C^* K_I^* z_T$ integral, using the self-dual vacuum coordinates. Indeed, pairing z_T with every integral multiple of each column of $K_I C = M_1 B_I$ gives the trivial additive character. For $\ell_I = J_I^* v$ this gives

$$(K_I C)^* M_2 \ell_I = B_I^* J_I^* v = (J_I B_I)^* v = v = (K_I C)^* z_T.$$

The columns span the original positive subspace over the local fields, so $K_I^*(z_T - M_2 \ell_I) = 0$. In the original rational bases the two terms are $\alpha_{\mathrm{W}}^{-1} T_{j,f}$ and $\bar{b}_{\mathrm{id}}^{-1} W$, where W is the integral Hom matrix ℓ_I transported to the actual lattices L_j, L_{j+1}.

Orthogonality to I means all negative-degree coordinates of the difference vanish. Thus $W - (\bar{b}_{\mathrm{id}}/\alpha_{\mathrm{W}}) T_{j,f}$ has degree at least zero. Bessel moves preserve the degree--1 part of the string map, by its string pivots. For the graded representatives take the degree--1 part of the integral W, which continues to map the graded lattices into each other by the grade projections. Since $T_{j,f}$ kills the source definite part and maps into the target string, the result is

$$(\bar{q}_{\mathrm{id}}/a_E) T_{j,f} L_j^+ \subset L_{j+1}^-.$$

This computation concerns the original finite factors at these points.

18.4.2 Exclusion by the ramified character

In this subsection, we use the source condition of Section 6 in the lattice calculation of Section 18.4.1.

At stage zero the finite point is in the original chosen level, with lattice Λ_0. A nonzero unfolded function at the next stage has one nonzero previous term and a nonzero slice at those points, so the finite inclusion applies. Assume at stage j that $q_{\mathrm{id}}^j L_j^- \subseteq \Lambda_j^-$ for the graded representative. Duality and the standard shift formula give

$$\bar{q}_{\mathrm{id}}^{\,j+1} \Lambda_{j+1}^- \subseteq L_{j+1}^-, \qquad q_{\mathrm{id}}^{\,j+1} L_{j+1}^+ \subseteq \Lambda_{j+1}^+.$$

A failure of the desired upper bound for L_{j+1}^- can occur only at a place above N on an odd string. There q_{id} is a unit and a nonzero vector generates the one residue line of Λ^+/Λ^-. Both inclusions then force $L^- = L^+ = \Lambda^+$ at that place. At the conjugate place duality also gives $L^- = L^+$. In particular the moved lattice splits orthogonally into its definite and string parts at the pair of places.

The split option occurs only at an even stage $l = j + 1 < m$. Let $w, \bar{w}$ be its pair above N, choosing $w \mid \pi_N$. Let z_w be the original norm-one scalar unit supported at that pair, with primitive reduction at w. Its embedding Z acts by that scalar on the definite block of Y_l and by one on the string. The splitting of $L_l(x_*)$ for the moved representative x_* implies $x_*^{-1} Z x_* \in J_{\xi,l}$: it is integral at the split pair and the identity elsewhere, and the pair is disjoint from the chosen level and twist places.

Use x_* at the l-th outer point in the even-stage coefficient formula of Section 18.3. The last lower finite test of that stage, after the outer move, is fixed by $\omega(1, Z)$ by the conjugate level condition. The exact slice covariance passes this action to multiplication by z_w on the last definite group. Each earlier definite arrow intertwines scalar translation by z_w on its source and target. Indeed their simultaneous matrix action on Hom is trivial and the finite determinant characters are unramified, hence trivial on these units.

In the last arrow the translation can be moved to the finite test, where it fixes the evaluated tensor. By linearity the resulting partial chain L therefore satisfies $L\rho_X(z_w) = L$ on the original source forms. For ϕ in the marked ramified source, the sum of its translates over the residue-unit action at w is zero. Thus

$$(N-1)L\phi = L\sum_{b=0}^{N-2} \rho_X(z_w)^b\phi = 0.$$

Hence $L\phi = 0$ in the characteristic-zero definite automorphic space. The even-stage identity with fixed last finite point x_* therefore vanishes at $\gamma(a, x_*)$ for every real point a and every $\gamma \in R_l(\mathbb{A}_{F_0})$. For the graded representative $x_* = r_*x$, take $\gamma = (1, r_*^{-1})$. This gives zero at the original point (a, x), before outer integration, contradicting the nonzero term at that point.

Lemma 18.2 (The support on the ramified source)**.** *Every contributing graded lattice L_j^- on the ramified source satisfies*

$$\bar{q}_{\mathrm{id}}^{j}\Lambda_j^- \subseteq L_j^- \subseteq q_{\mathrm{id}}^{-j}\Lambda_j^-. \tag{69}$$

18.4.3 Compact representatives of the surviving terms

In this subsection, we use Lemma 18.2 to choose representatives for the measured sum.

To pass from the string lattices to compact representatives, suppose $L(x)$ and $L(y)$ have the same string intersection. It is saturated in both lattices. Its local bases therefore give two primitive bases in the standard coordinates, with equal Gram matrices. At a split pair write the bases as (A, A'), (B, B'). The equations for one matrix are $gA = B$ and $B'^{\mathsf{T}}g = A'^{\mathsf{T}}$. Over the residue field, first identify the basis spans with their given bases and extend the identification on their intersections with the two kernels to an isomorphism of the kernels.

The two maps A'^{T}, B'^{T} are surjective and agree on these identified spans by Gram equality. Complete with lifts of the remaining quotient basis. This gives an invertible residue solution. Over the DVR choose $DA = 1$, $B'^{\mathsf{T}}R = 1$ and put

$$g_0 = BD + RA'^{\mathsf{T}}(1 - AD).$$

Lifting the difference of the residue solution and g_0 to H, replace g_0 by $g_0+(1-RB'^{\mathsf{T}})H(1-AD)$. The correction is homogeneous for both equations and gives the chosen invertible reduction. The conjugate matrix is $g^{-\mathsf{T}}$.

At a nonsplit unramified place, finite-field Witt extension matches the two primitive residue bases, allowing degenerate spans. Lift the isometry by successive Gram corrections. At each residue step the equation $C + C^* = H$ for Hermitian H is solvable using residue trace on the diagonal and one triangle off it. The trace is surjective also in residue characteristic two. If the unitary basis images agree modulo π^a, their discrepancy D at the next order satisfies $A^*D + D^*A = 0$.

The prescription $CA = D$ extends to a skew-adjoint matrix: prescribe the resulting skew-Hermitian form on the basis and fill the complementary block with zero. Then $1 + \pi^a C$ can

be corrected to a unitary lift starting at order $a+1$, so the bases agree one order further. The limit agrees exactly. The resulting finite integral isometry k has ykx^{-1} fixing the entire string, since it fixes its full lattice basis. It therefore lies in the Bessel Levi. Thus a common string intersection determines the Bessel orbit modulo the standard compact.

The bounds (69) give finitely many string intersections. They equal the standard intersections off $Q_\xi \cup \overline{Q_\xi}$, with Q_ξ the chosen twist places. Begin with a graded representative of each intersection. Off this union the local frame comparison gives a Bessel Levi element on the left and a standard compact element on the right taking the component to the identity. Almost everywhere the component is already compact, so take the Levi element to be the identity there. Keep the original components at the twist pairs. The choices assemble adelically and preserve the graded condition. The resulting right compact cosets meet the same finite Bessel orbits and form a compact union at each intermediate stage. At stage zero use the classes of the original chosen level modulo $J_{\xi,0}$.

19 The measured theta tensor and its integral lattice

In this section, we evaluate the finite sum from Section 18 with the prescribed Haar measures. The resulting tensor and its comparison at the additional ramified prime give the local identities used to bound derivatives in Section 20.

19.1 The measured finite sum

In this subsection, we integrate the paired formula of Section 18.3 over the finite classes furnished by Section 18.4. The computation retains the stabilizer factors and shows how the compact indices cancel.

For each intermediate covector choose one finite class $xJ_{\xi,j}$ in the compact union of Section 18.4 per represented $R_j(A_f)$-orbit. Write $B_j = R_j(A_f)$ and $B_j^0 = B_j \cap K_j^\circ$. Let Ω_j be the union of the B_j^0-orbits of the selected classes. The unitary Levi acts on the Lie algebra of the unipotent radical with determinant one: that determinant is a rational character, and a unitary group has no nontrivial character over F_0. The Levi and the unipotent radical are unimodular, so R_j is unimodular at each place. Take finite Haar with $\mathrm{vol}(B_j^0) = 1$, rescaling the real Haar reciprocally to keep the original Bessel Haar. At an anchor $xJ_{\xi,j}$ put

$$d_j(xJ_{\xi,j}) = [R_j(A_f) \cap xJ_{\xi,j}x^{-1} : R_j(A_f) \cap xJ_{\xi,j}x^{-1} \cap K_j^\circ].$$

The Haar measure of the r with $rxJ_{\xi,j} \in \Omega_j$ is this integer on the represented orbit.

Indeed, with $Z_c = B_j \cap xJ_{\xi,j}x^{-1}$ at the anchor $c = xJ_{\xi,j}$, this set is $B_j^0 Z_c$. Its disjoint right translates of B_j^0 are indexed by $(B_j^0 \cap Z_c)\backslash Z_c$. For a starting point bc in the same B_j-orbit the set is $B_j^0 Z_c b^{-1}$, of the same measure. Use this anchor's integer on the whole orbit. If the twist is empty all compact representatives are integral and $d_j = 1$. In general d_j divides a power of the product of the residue characteristics of the twist places.

Indeed, at each twist place w, let $K_w(h)$ denote the principal kernel of depth h in the integral matrix group. Choose h_w with $x_w K_w(h_w) x_w^{-1} \subseteq K_{j,w}^\circ$. The level $J_{\xi,j,w}$ is contained in $K_w(1)$. The kernel of

$$Z_c \longrightarrow \prod_{w \mid Q(\xi)} K_w(1)/K_w(h_w), \qquad z \longmapsto (x_w^{-1} z_w x_w)_w,$$

is contained in $B_j^0 \cap Z_c$, since x is integral away from these places. Each successive quotient $K_w(h)/K_w(h+1)$ embeds in the additive matrix group over the residue field by $1+\varpi_w^h D \mapsto \bar{D}$. Thus the index d_j is supported on the residue characteristics of the twist places.

Choose a nonnegative real cutoff α_j with $\int_{R_j(A_\infty)} \alpha_j(ru)\,dr = 1$ on each real orbit. Such a cutoff comes from the proper translation action of the closed Bessel subgroup. It depends only on the fixed real covector. Then

$$w_j(u,x) = \alpha_j(u)\frac{\mathbf{1}_{\Omega_j}(xJ_{\xi,j})}{d_j(xJ_{\xi,j})}$$

has full Bessel orbit integral one on the support of the unfolded integrand. Outside Ω_j the fraction is zero. Quotient integration in stages can therefore be performed with w_j, using counting measure for the rational subgroup and the inherited Haar for the adelic groups. Indeed for $F(g) = I_j(g)z(g)$, Tonelli and quotient integration give

$$\begin{aligned}
&\int_{U(Y_j)(A)} w_j(g) \int_{[R_j]} |F(rg)|\,dr\,dg \\
&\quad = \int_{R_j(F_0)\backslash U(Y_j)(A)} |F(g)| \int_{R_j(A)} w_j(bg)\,db\,d\dot{g} \\
&\quad = \int_{R_j(F_0)\backslash U(Y_j)(A)} |F(g)|\,d\dot{g} < \infty.
\end{aligned}$$

Here inversion preserves Bessel Haar, the orbit integral is one on the support of F, and the last integral is finite by Proposition 18.1. Fubini therefore also gives the identity with F in place of $|F|$.

Write $\Xi_j(z)$ for the integral of $I_j(h)z(h)$ on $R_j(F_0)\backslash U(Y_j)(A)$. Its remaining outer points are understood as parameters. The product is right $J_{\xi,j}$-invariant when $z = L_{\xi,j}^{\mathrm{t}} z'$, by the original level invariance of the last datum in I_j and the transpose. Factor the given adelic Haar on G_j as $du\,dg_f$, with $dg_f(K_j^\circ) = 1$; a change of the finite normalization rescales du reciprocally. In the integral with w_j, a coset of $J_{\xi,j}$ thus has measure $1/A_{\xi,j}$. The one-step identity of Section 18.3 has multiplier $A_{\xi,j}$. These two factors cancel, leaving exactly

$$\Xi_j(L_{\xi,j}^{\mathrm{t}} z') = \int_{U(Y_j)(A_\infty)} \alpha_j(u) \sum_{c \in \Omega_j} \frac{\Xi_{j+1}(z';\, g_j = (u,c))}{d_j(c)}\,du. \tag{70}$$

The absolute integrability at each peel follows from the one-covector absolute product estimate with the original rapid conjugate test, including the transposed lifts. For $j = 0$, apply the measured level formula of Section 8.4 with $J_{\xi,0} \subseteq J_M$ to the first transpose. It gives the sum over $c \in J_M/J_{\xi,0}$, divided by $[J_M : J_{\xi,0}]$, of the original periods at the right translate $g_0(1,c) = (g_{0,\infty}, c)$, where $g_{0,\infty}$ is the real point of Section 8.3. After unfolding, the factor $A_{\xi,0}$ leaves $[K_0^\circ : J_M]$ because indices multiply. It is independent of ξ.

By the terminal calculation of Section 18.3, the last period is a product of a function $a_m(u)$ of the outer real points and the terminal pairing on the finite tensor $\bigotimes_i F_i(\xi, y)$. At each step of (70), update the finite factor by the sum over $c \in \Omega_j$ with weight $d_j(c)^{-1}$. With the earlier real variables fixed, write $a(u)b_c$ for the factored integrand before multiplication by $\alpha_j(u)d_j(c)^{-1}$. Set the new real factor to $\int \alpha_j(u)a(u)\,du$ when this integral is absolute, and set it to zero otherwise. Absolute integrability in (70) at any coset with $b_c \neq 0$ implies $\int |\alpha_j a| < \infty$. In the divergent case every b_c therefore vanishes and the same factorization holds. The rule uses only

the fixed real data and works for all twists and finite tests. Applying it successively proves the identity one covector at a time.

Define the evaluated tensor by the weighted sum

$$T_\xi = \sum_{c_0,\dots,c_{m-1}} \left(\prod_{j=1}^{m-1} d_j(c_j)^{-1} \right) \bigotimes_{i=0}^{m/2-1} F_i(\xi, c), \tag{71}$$

where c_0 runs over $J_M/J_{\xi,0}$, $c_j \in \Omega_j$ for $j > 0$, and the terminal finite point is one. Each finite tensor uses the same actual representative as its outer period. The real cutoffs, integrals, and point $g_{0,\infty}$ are fixed before ξ. Define c_{meas} as the iterated real factor times the fixed index $[K_0^\circ : J_M]$. The identity against each terminal conjugate test gives the following result by nondegeneracy of the Petersson product.

Proposition 19.1 (Measured seesaw identity)**.** *Let $\phi \in S$ and let ξ be an allowed twist. With the fixed real point and original finite levels, the measured lifts satisfy*

$$L_\xi \Pi \phi = c_{\mathrm{meas}}\, \mathcal{U}_\xi(T_\xi)\phi. \tag{72}$$

The scalar c_{meas} is independent of ξ.

For the trivial twist the number of choices of each c_j is at most $M^{fn_m^2} = B_*$, since the classes lie in $K_j^\circ/J_{\xi,j}$ and the principal index there is a product of $|\mathrm{GL}_{\dim Y_j}(k_w)|$. The sum of the cut weights is at most B_*^m in that case. The expansion of the $F_i(\xi, c)$ in translated vacua is given in Section 19.3.

19.2 The lattice of an evaluated pair

In this subsection, we calculate the lattice obtained when a summand of Section 19.1 is restricted to the paired oscillator variables. At the places over N, its two components satisfy a common residue condition that determines the sum in Section 19.3.

Use the finite bases in a pair from Y_{2i} through Y_{2i+2}. Put $R_f = \widehat{\mathcal{O}}_E$ in this calculation, let $\pi = \pi_N$ as a principal idele, and write J_N for the idempotent of the places dividing π. Thus $J_N \bar{J}_N = 0$. Write $\eta_- = \eta \in R_f^\times$ for the norm--1 unit in the opposite finite basis of Section 18.2, so that $s_- = \bar{\lambda}$ with the notation there. Put

$$\delta = \frac{c_M}{\bar{c}_M}, \qquad s_- = \bar{\eta}_- \frac{\bar{\alpha}_{\mathrm{W}}}{\alpha_{\mathrm{W}}}, \qquad u_j^{\mathrm{str}} = (a_E/\bar{a}_E)^{\lceil (m-j)/2 \rceil}.$$

With $\beta = \pi_M$ the integral level generator in valuation, the identities are

$$s_- u_{2i}^{\mathrm{str}} = \bar{\eta}_- \delta\, u_{2i+2}^{\mathrm{str}}, \qquad v_w(\bar{\beta}\delta) = v_w(\beta), \qquad \delta = 1 \text{ above } N.$$

They follow from $a_E = \alpha_{\mathrm{W}}/\bar{c}_M$ and the successive two-step exponents. In an odd string the lattice contains the glued vector $e/\bar{\pi} + (u_j^{\mathrm{str}}/N)z$ and $u_j^{\mathrm{str}} z$, where e, z are the marked last definite and central string coordinates. The coefficients of a vector with ordinary coordinates x and string coordinate $u_j^{\mathrm{str}}(\pi^{-1}x_e + d)$ in this pair of lattice vectors are $\bar{\pi}x_e, d$. Thus membership is tested by those two integral values and the other ordinary integral coordinates. The two-by-two Gram block is $\left(\begin{smallmatrix} 0 & -1 \\ -1 & -N \end{smallmatrix}\right)$. The even string has unglued definite coordinates. All calculations below use the standard finite bases of these same unimodular lattices.

Let e, e'' be the last indices of the two definite blocks in the target of the joined space. Define

$$\begin{aligned} v_0 &= e + \bar{\eta}_- \bar{J}_N e'', \\ D_N &= J_N(\pi - 1) + \bar{J}_N(\bar{\pi}^{-1} - 1), \\ \Gamma &= 1 + D_N v_0 v_0^\dagger, \\ b_e &= D_N \pi^{-1} \bar{J}_N, \qquad b_{e''} = D_N \pi^{-1} \eta_- J_N. \end{aligned}$$

Set $b_j = 0$ for the other indices of this block. The vector v_0 has norm one, and Γ acts on its line by the norm-one idele $1 + D_N$. In order e, e'' its two blocks are

$$\begin{pmatrix} \pi & (\pi - 1)\eta_- \\ 0 & 1 \end{pmatrix} \quad (w \mid \pi), \qquad \begin{pmatrix} \bar{\pi}^{-1} & 0 \\ (\bar{\pi}^{-1} - 1)\bar{\eta}_- & 1 \end{pmatrix} \quad (w \mid \bar{\pi}),$$

and it is the identity elsewhere. For the column $x = \Gamma e_j$ with $b = b_j$, direct subtraction gives

$$\bar{\pi} x \in R_f^{\dim x}, \qquad b - \pi^{-1} x_e \in R_f, \qquad \bar{\eta}_- b - \pi^{-1} x_{e''} \in R_f, \qquad \delta b = b.$$

The first and second differences are $-[J_N + (1 - J_N)\pi^{-1}]$ in the column e or e'', respectively, and zero in the other columns. The bracket is integral.

For an interior pair put $u = u_{2i}^{\mathrm{str}}$. Lift the column (x_-, x_+) to ordinary coordinates $(s_- x_-, x_+)$ and an additional component $s_- u b(z + z'')$ on the removed plane. Dividing the first block by its opposite-basis scale s_- leaves string coordinate ub. In the other block it is $u_{2i+2}^{\mathrm{str}} \bar{\eta}_- b$. Both are integral in the required glued lattice by the two difference congruences. In the terminal pair the extra target index e'' is represented by z/π on the negative source side. Use covector scaling $\kappa_{e''} = \delta$, with all other κ's one. The corresponding original source string coordinate is then $\pi^{-1} \delta x_{e''}$. Since $s_-^{-1} \delta = -\eta_- u$, after dividing the first block by s_- its integrality test is

$$-\pi^{-1} x_e - \eta_- \pi^{-1} x_{e''} = (b - \pi^{-1} x_e) + \eta_-(\bar{\eta}_- b - \pi^{-1} x_{e''}) \in R_f.$$

The remaining ordinary target entries are integral. Thus the same lifted procedure works in both pairs.

Let C have these lifted columns in the original finite coordinates. Let x_f be the finite basis matrix of Y_{2i+1} in its graded coordinates, and j_0 the inclusion of its definite summand. Its definite-coordinate projection is

$$R' = j_0^\dagger x_f, \qquad (R')^\dagger = x_f^{-1} j_0.$$

Here $j_0^\dagger$ extracts the definite coordinates, and the second identity uses the isometry equation for x_f. The even middle string has unglued definite coordinates, so R' is integral. On matrices the lift $B : L_{\mathrm{def}} \to L_{\mathrm{big}}$ is $X \mapsto CXR'$, hence integral.

If i_f is the orthogonal residual inclusion and k_f the isotropic basis in the two slices, its columns give

$$B = i_f g_{\mathrm{nb}} + k_f T_{\mathrm{nb}}, \qquad g_{\mathrm{nb}} = (\operatorname{diag} \kappa\ \Gamma) \otimes 1.$$

Here T_{nb} is supported above N, and g_{nb} is unitary since the covector scaling is one above N and has norm one. Write $I = \operatorname{im} k_f$, and let L_{big}, L_{def} denote the standard finite vacuum lattices.

The quotient image of $L_{\mathrm{big}} \cap I^\perp$ in $I^\perp / I$ is exactly $g_{\mathrm{nb}} L_{\mathrm{def}}$. One inclusion follows from the integral columns of B. For the reverse inclusion, self-duality and $B L_{\mathrm{def}} \subseteq L_{\mathrm{big}}$ give $B^\dagger L_{\mathrm{big}} \subseteq$

L_{def}: for any integral y and $x \in L_{\mathrm{def}}$, the pairing with $B^{\dagger}y$ equals the integral pairing of Bx with y. As k_f spans the isotropic summand orthogonal to i_f, we get

$$B^{\dagger}(i_f z + k_f v) = g_{\mathrm{nb}}^{\dagger} z = g_{\mathrm{nb}}^{-1} z.$$

Thus every integral lift has $z \in g_{\mathrm{nb}} L_{\mathrm{def}}$.

Write S_f for the joined string shift in the finite vacuum coordinates, namely α_{W}^{-1} times the stacked rational shift in the chosen basis. It satisfies $B^{\dagger}S_f = 0$, because on the middle covector it kills the columns of $(R')^{\dagger}$. For y vanishing above N one therefore has

$$g_{\mathrm{nb}} B^{\dagger}(S_f + y) = i_f^{\dagger} y.$$

On the ambient source and target spaces the denominators are $\beta, \bar{\beta}$, respectively. At the finite places away from N, $\Gamma = 1$ and the lifted columns lie in the corresponding block of the original inclusion. The additional covector scaling at the terminal index converts a source bound by β into the target bound by $\bar{\beta}$, by $v_w(\bar{\beta}\delta) = v_w(\beta)$. As $B^{\dagger}$ is integral, the projection on the right has the same two allowed denominators. This also applies after omitting any allowed twist places. Above N both generators are units.

19.3 Integral coefficients of the measured tensor

In this subsection, we expand the tensor of Section 19.1 using the lattice of Section 19.2. The resulting translates have controlled denominators at coefficient places and an absolute coefficient bound at complex embeddings.

Put $d_j^{\mathrm{H}} = \dim(Y_j)\dim(Y_{j+1})$, and let $\mathfrak{m}_{\xi} = (\beta)\prod_{w \in Q_{\xi}} \mathfrak{p}_w$ be the ideal of the lower residue test. The Fourier expansion of its one prescribed reduction in $(\mathcal{O}_E/\mathfrak{m}_{\xi})^{d_j^{\mathrm{H}}}$ gives one character per element of the dual group, with coefficient $\zeta/|\mathcal{O}_E/\mathfrak{m}_{\xi}|^{d_j^{\mathrm{H}}}$ for a root of unity ζ. Multiplying the finite vacuum by such a character is its Heisenberg translate in the conjugate split half. Thus the translation has denominator $\bar{\beta}$ away from Q_{ξ} and its conjugates. At a coefficient prime away from the twist its coefficient denominator divides $|\mathcal{O}_E/(\beta)|^{d_j^{\mathrm{H}}}$. When Q_{ξ} is empty the sum of the absolute coefficients is one under every complex embedding.

Here is the effect of moving that expansion by an outer cut point. In the original self-dual finite lattice L write v_L for the vacuum and ρ for the signed Heisenberg translations. If g is the action on the oscillator and $H = L \cap g^{-1}L$, then

$$\omega(g)v_L = \frac{\Delta(g)}{[L:H]} \sum_{z \in L/H} \rho(gz)v_L, \qquad |\Delta(g)|^2 = [L:H], \quad \Delta(g^{-1}) = \overline{\Delta(g)}.$$

Indeed the fixed line for gL is the transport of the vacuum line.

Projection to that line by averaging the translations over gL gives the indicated sum. The translates by distinct cosets are orthogonal, so unitarity gives the displayed modulus. The coefficient at the zero coset is the finite Weil lattice multiplier divided by $[L:H]$. The signed translations are the vacuum-normalized ones, so the other phase factors in multiplying translations are values of the finite additive character, hence roots of unity.

For a translated vector $\rho(x)v_L$, the shifts become $g(x+z)$. Outside the twist g is an integral isometry, so the denominator of the shift is unchanged there and the local vacuum multiplier is one. The remaining local lattice multipliers and splitting character values are units at each

coefficient embedding away from the twist: in the Schrödinger model at the split twist places they are products of roots of unity, algebraic character values on powers of the local uniformizers, and half-integral powers of the residue sizes.

The character on a norm-one split idele has the infinity weight applied to the normalized uniformizer idele and a finite order factor. Its nonunit prime divisors therefore lie over the underlying residue characteristic. For this last assertion one may raise the idele to a class-group exponent and divide it by a principal idele with the same ideal. The remaining integral idele has finite-order character, and all conjugates of the principal generator are units away from the residue characteristic.

The extra pair characters on the integral components have finite order on the marked compact level. At empty twist all actions are integral and their coefficients have modulus one in every complex embedding. The Hilbert norm normalization of the multiplier above contributes the same factor as $\Delta(g^{-1})^{-1}$.

The arrow product joins the first shift, by its adjoint into the negative block, to the second shift in its original order. In the finite vacuum coordinates its two blocks are

$$z_- = -\eta_-{}^t \bar{x}_1, \qquad z_+ = x_2.$$

The signed vacuum convention contributes only a root of unity to this change. The joined translation z therefore has denominator β on the first block and $\bar{\beta}$ on the second. As both are units above N, the N-component can be dropped on the ambient vacuum. The retained and removed parts have zero Heisenberg pairing since they have disjoint conjugation-stable supports.

Let Z_I denote the finite zero slice for the two isotropic bases, and let L_{big} be the ambient vacuum lattice, and let S_f be the marked string shift. We explain the finite support test and its coefficient. If $Z_I \rho(S_f + z) v_{L_{\mathrm{big}}} \neq 0$, its commutator against translations by $L_{\mathrm{big}} \cap I$ must be trivial. For $\lambda \in L_{\mathrm{big}} \cap I$, the signed translation fixes the vacuum and its phase on the zero slice is

$$\psi_{\mathrm{sd},f}\big(c_{\mathrm{tr}} h(\lambda, \lambda)/2\big) = 1,$$

since I is totally isotropic. This includes the places above 2.

The intersection $L_{\mathrm{big}} \cap I$ is saturated and the ambient lattice is self-dual. Extending a local basis of this intersection to a basis of L_{big} shows that any integral values on it are realized by pairing with an element $\lambda \in L_{\mathrm{big}}$. Taking those values to be the pairings with $S_f + z$ gives $S_f + z - \lambda = i_f w + k_f v$. The translation by λ fixes the vacuum, and the translation by $k_f v$ acts trivially on the zero slice.

The result is therefore a phase times $\rho(w) Z_I v_{L_{\mathrm{big}}}$. The same argument applies to any translate used to test whether the zero slice itself vanishes. The zero slice is nonzero in an adapted split model, and translates of the finite vacuum span its Schwartz space. Thus $Z_I v_{L_{\mathrm{big}}} \neq 0$.

The result is fixed by signed translations of the projected lattice $g_{\mathrm{nb}} L_{\mathrm{def}}$ from Section 19.2, since its integral lifts in $I^{\perp}$ fix the ambient vacuum. In the Schrödinger model the original lattice invariants are one-dimensional: translation invariance makes a function constant on additive lattice cosets and invariance under the dual characters restricts its support to the integral coset. Weil transport gives the invariant line for $g_{\mathrm{nb}} L_{\mathrm{def}}$, so the zero slice of the vacuum is a fixed nonzero multiple of $\omega(g_{\mathrm{nb}}) v_{L_{\mathrm{def}}}$. The residual shift $w = g_{\mathrm{nb}} B^{\dagger}(S_f + z - \lambda)$ can be replaced by $g_{\mathrm{nb}} B^{\dagger} z$ up to a phase: $B^{\dagger} S_f = 0$, and $g_{\mathrm{nb}} B^{\dagger} \lambda \in g_{\mathrm{nb}} L_{\mathrm{def}}$ fixes the slice. Section 19.2 gives the same two denominators for the new shift and shows that it vanishes above N.

19.3.1 Translation at the additional level

In this subsection, we express the neighbor sum of Section 19.2 in translated vacua at N.

For each $w \mid \pi$ take a middle covector $y_w = (y_{w,1}, \dots, y_{w,n_{2i+1}}) \in \mathbb{F}_N^{1\times n_{2i+1}}$. Let r_y have covector $\widetilde{y}_w/N$ there and zero at the conjugate places, for any integral lifts $\widetilde{y}_w$. The linked representative n_y has last coordinates $(\eta_- \bar{r}_y, -\bar{r}_y)$ in the two blocks. At $w \mid \pi$ the matrix Γ has integral entries, so that factor of the additive quotient is zero. At $\bar{w}$, write a, b for the two covectors in the distinguished coordinates. Column by column, their image modulo the integral lattice is

$$(a/\bar{\pi}, \bar{\eta}_- a/\bar{\pi}).$$

It equals n_y modulo the lattice precisely when $\overline{y_w} \equiv -\pi\bar{\eta}_- a \pmod{\bar{\pi}}$ in every middle coordinate.

The marked split places have residue field $\mathbb{F}_N$, and $\pi\bar{\eta}_-$ is a unit in the conjugate half. Thus every class occurs once. In fact $a = \bar{\pi}\eta_- \bar{r}_y$ and $b = \bar{\eta}_- a$ give n_y exactly. Applying the inverse arrow product to n_y gives the shifts $(-{}^t r_y, -\bar{r}_y)$. As y ranges over all residues this is exactly the original linked gluing, denoted by v_x^{gl} when applied to an upper pair with additional translate x.

The two factors are both supported in the prescribed last coordinates. The signs may simultaneously be reversed by $y \mapsto -y$, which permutes the sum. All pairings between n_y and the shift away from N vanish. In signed vacuum coordinates the adjoint transport of a first block u has phase $\psi_{\mathrm{sd},f}(c_{\mathrm{tr}} u^* u/2)^2$, where $c_{\mathrm{tr}} + \bar{c}_{\mathrm{tr}} = 1$. This follows because the norm-minus-one adjoint negates the Hermitian square, and hence inverts the phase in one factor. For n_y the first block is supported on only one split half, so its Hermitian square is zero and the phase is one. The entire sum at N is retained in v_x^{gl}.

Away from N, let b_i be the number of classes of $g_{\mathrm{nb}} L_{\mathrm{def}}$ modulo its intersection with L_{def}. The quotient injects in the quotient of the lattice of the two allowed denominators $\beta, \bar{\beta}$ by the integral lattice. Its order divides $|\mathcal{O}_E/(\beta)|^{r_i n_{2i+1}}$, where r_i is the sum of the two upper block dimensions in the pair and n_{2i+1} is the middle definite dimension.

This follows coordinate by coordinate from the form $\mathrm{diag}(\kappa)$ of the neighbor there and $\bar{\beta}\kappa$ integral. Put the factor b_i into the common pair scalar and assign the b_i terms coefficients of size b_i^{-1}, including their phases. Since $\beta = \pi_M$, $\beta\bar{\beta} = M$ and $[E : \mathbb{Q}] = 2f$, we have $|\mathcal{O}_E/(\beta)| = M^f$. Each arrow denominator off the twist divides $M^{f d_j^{\mathrm{H}}} \mid B_*$, since $d_j^{\mathrm{H}} \le n_m^2$. The remaining quotient has $b_i \mid M^{f r_i n_{2i+1}} \mid B_*^2$, since $r_i \le 2n_m$. Thus the product of the three denominators divides B_*^4.

At the empty twist the absolute coefficient sum of the pair is at most one under every complex embedding. Write the unshifted zero slice as $a_i\, \omega(g_{\mathrm{nb}}) v_{L_{\mathrm{def}}}$. For each of the f split pairs above N, the quotient just computed has one residue coordinate per middle column. Thus

$$[L_{\mathrm{def}} : L_{\mathrm{def}} \cap g_{\mathrm{nb}}^{-1} L_{\mathrm{def}}] = N^{f n_{2i+1}} b_i.$$

The coefficient of each signed vacuum translate is therefore $a_i \Delta(g_{\mathrm{nb}})/(N^{f n_{2i+1}} b_i)$. Passing to the average over the b_i classes away from N and to the two integral vacua multiplies this coefficient by $b_i/(\kappa_{2i}\kappa_{2i+1})$, where $\kappa_j = s_{\mathrm{vac}}^{n_j n_{j+1}}$. The common pair scalar is

$$\frac{a_i \Delta(g_{\mathrm{nb}})}{N^{f n_{2i+1}} \kappa_{2i} \kappa_{2i+1}}.$$

The linked sum at N in v_x^{gl} has coefficient one on each residue. All factors of the common scalar depend only on the original pair.

Applying this argument to the tensor in (71) gives a fixed nonzero C_0 and an expansion

$$T_\xi = C_0 \sum_x b_{\xi,x} v_x^{\mathrm{gl}}.$$

Here $B_*^{2m} b_{\xi,x}$ is integral at every coefficient embedding off the twist. Indeed the cut weights have denominators supported on q_ξ and each of the $m/2$ pairs contributes at most B_*^4. At the trivial twist the cut sum gives

$$\sum_x |\sigma(b_{\varnothing,x})| \le B_*^m \qquad (\sigma \in \mathrm{Aut}(\mathbb{C})).$$

Terms with coincident translates can be combined by the triangle inequality.

The expanded tensor is fixed by the original product of upper levels with the terminal action one. Each upper-level element acts at the corresponding definite block in the Bessel Levi. Away from N the block action preserves the integral lower lattice. Above N, let g belong to the upper level, put $A = g - I$, and write e for the last coordinate vector. By the upper level condition and unitarity,

$$e^t A \equiv 0 \pmod{\pi_N}, \qquad Ae \equiv 0 \pmod{\overline{\pi_N}}.$$

Indeed $e^t g^{-1} \equiv e^t \pmod{\pi_N}$, and $g^{-1} = g^*$. For an even string the restriction of the glued basis to the definite block is an integral matrix R. For an odd string this restriction is $D_{\overline{\pi_N}}^{-1} R$, where $D_z = \mathrm{diag}(1, \dots, 1, z)$ and R is integral, by the last coordinate of the glued basis. Thus the difference of the lower action from the identity is $R^* A R$ or $R^* D_{\pi_N}^{-1} A D_{\overline{\pi_N}}^{-1} R$, respectively. In the latter matrix the last diagonal entry of A is divisible by N, since (π_N) and $(\overline{\pi_N})$ are coprime. Both matrices are integral; the same calculation for g^{-1} proves lattice preservation.

The embedded element belongs to $B_j^0 = R_j(A_f) \cap K_j^\circ$ at each intermediate stage by this integral computation and the definite-block action. It therefore permutes the cut classes in Ω_j, a union of left B_j^0-orbits. The integer d_j is constant on each such orbit. At the first stage the action permutes the chosen level classes. Equivariance of the two-slice evaluation gives this upper action on each summand, on both occurrences of every shared index. Thus it preserves the measured sum. Before applying the linked sum at N, write each additional shift as $x = x^{\mathrm{lev}} + x^{\mathrm{away}}$, with x^{lev} supported above $M q_\xi$ and x^{away} outside those places. The supports are disjoint and stable under conjugation, so the Heisenberg pairing of the two parts is zero. The integral shift x^{away} fixes the original vacuum, giving $\rho(x) v_L = \rho(x^{\mathrm{lev}}) v_L$ on each arrow. Applying the same linked sum to these equal tensors preserves the evaluated tensor.

Apply Section 17.3 to (72) with this tensor. Put $a_{\mathrm{ss}} = c_{\mathrm{meas}} C_0 / B_*^{2m}$ and take

$$(c_{\mathrm{ss}}, \Psi_\xi(x)) = \begin{cases} (a_{\mathrm{ss}}, B_*^{2m} b_{\xi,x}) & a_{\mathrm{ss}} \neq 0, \\ (1, 0) & a_{\mathrm{ss}} = 0. \end{cases}$$

In the second case $c_{\mathrm{meas}} = 0$ since $C_0 \neq 0$. In both cases $c_{\mathrm{ss}} \neq 0$, and the measured identity gives

$$L_\xi(c_{\mathrm{ss}}^{-1} \Pi \phi) = \mathcal{U}_\xi \left(\sum_x \Psi_\xi(x) v_x^{\mathrm{gl}} \right) \phi \qquad (\phi \in S).$$

The scalar is common to all twists and the tensor for each twist works for every source form. For the empty twist

$$\sum_x |\sigma \Psi_\varnothing(x)| \le B_*^{3m} = M^{3fmn_m^2}$$

for every complex embedding σ. The resulting upper lift evaluates the same source and uses the same terminal branching as $L_{\xi}\Pi$. The equality on the original ramified source is now in integral tensor form.

19.4 The comparison at the additional ramified place

In this subsection, we compare the two integral realizations at N for the tensor of Section 19.3. The comparison gives an integral submodule of the ramified source on which the local theta estimate holds uniformly.

Use the spaces X_j, their original finite levels, and the norm equation $\pi\bar{\pi} = N$. These are the upper definite groups and original glued vacua of Section 19.3. Fix an N-adic embedding $\jmath$. Put $P = \Sigma_N$, the set of conjugates of the places induced by $\jmath$ and the marked infinite embeddings. Put $S = \{w : w \mid \pi\}$. Each of P, S contains one place from every conjugate pair over N, and at each such place the completion and the residue field are $\mathbf{Q}_N, \mathbf{F}_N$.

For this coefficient embedding, let $e = e_{\mathrm{last}}$ and define the unitary adelic point $\delta_j = \delta_j(P)$ by

$$(\delta_j)_w = \mathrm{diag}(I_{n_j-1}, N^{\mathbf{1}_{w\in S}}), \qquad (\delta_j)_{\bar{w}} = \big((\delta_j)_w^*\big)^{-1} \qquad (w \in P).$$

It is the identity away from N. At $w \in P$ it gives a basis of the lattice $L_{j,w}$ whose last coordinate is divisible by π. In these bases the original local level fixes e modulo N. Write ρ_j for right translation by the unitary embedding of a matrix at P, conjugated by δ_j. The conjugate coefficient embedding uses $\bar{P}$ with the same original level and the same S.

Let

$$B = \prod_{w\in P} \mathbf{F}_N, \quad D_j = \mathrm{diag}(1, \dots, 1, N^{-1}), \quad \psi_N(y) = \psi_0(-y/N).$$

In this notation D_j acts at all of P and is interpreted in the appropriate rank. The last formula uses the marked self-dual character and the reduction at P, so it is unchanged by changing an integral lift of y. All functions of a residue matrix below are extended by zero off the integral matrices.

19.4.1 Translation laws and the ramified sum

In this subsection, we apply the local oscillator action to the summands of Proposition 19.1.

Let Λ_P be the integral matrix lattice in the split Schrödinger model at P, and normalize dT by $\mathrm{vol}(\Lambda_P) = 1$. For a factorizable finite Weil vector $v_j = f_P \otimes v^P$, insert Φ by the map

$$f_P \otimes v^P \longmapsto \left(\int_{\Lambda_P} f_P(T)\, dT\right) \Phi \otimes v^P.$$

This takes the coefficient of the unit vacuum $\mathbf{1}_{\Lambda_P}$ at P and extends linearly to all finite Weil vectors. Write $L_j(\Phi)$ for the lift of this datum in the L_j, L_{j+1} coordinates. For a, b invertible over the local integers, the laws are

$$\rho_{j+1}(b) L_j(\Phi(bTa)) \rho_j(a) = L_j(\Phi(T)), \tag{73}$$

$$\rho_{j+1}(D_{j+1}) L_j(\mathrm{cov}_e) = \gamma_j L_j(\mathrm{col}_e) \rho_j(D_j). \tag{74}$$

Here $\mathrm{cov}_e(T) = \mathbf{1}_{\bar{T}_{\text{last covector}}=e^t}$ and $\mathrm{col}_e(T) = \mathbf{1}_{\bar{T}e=e}$ on integral matrices. The split Schrödinger action reads $T \mapsto b^{-1}Ta$; a change of this variable in the theta pairing takes the character factor

$$\frac{\chi_j(\det a)|\det a|_P^{-n_{j+1}/2}}{\chi_j(\det b)|\det b|_P^{-n_j/2}}, \qquad \chi_j = \text{base change of the } (j+1)\text{-st lift character.}$$

This formula follows from the split action on a test function and right translation in the source integral for the lift.

It gives (73) since the characters are unramified there. The equality of functions $\mathrm{cov}_e(T) = \mathrm{col}_e(D_{j+1}TD_j^{-1})$ includes their supports. In blocks the change sends $\left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} A & Nb \\ c/N & d \end{smallmatrix}\right)$. Testing the last column and integrality therefore tests $c \equiv 0$, $d \equiv 1 \bmod N$ and integrality of T. The determinant character factor in (74) uses $a = D_j^{-1}$ and $b = D_{j+1}^{-1}$ in $T \mapsto b^{-1}Ta$. Both determinants have component N at each place of P, so their character values cancel and

$$\gamma_j = (N^{-|P|})^{-n_{j+1}/2+n_j/2} = N^{|P|(n_{j+1}-n_j)/2}, \qquad |\gamma_j|_N \le 1.$$

Conjugation by δ_j in these calculations uses the same change on the datum and the source action. Thus the source and target gauges in the laws refer to δ_j, δ_{j+1}.

A ramified source vector u on the original level has zero mean at each marked scalar group over N. For each $w \in P$ write $u = (1 - \rho_j(c_w))u_w$, using a generator of that local cyclic residue unit group in the scalar action, supported at w and its conjugate. The identity follows by inverting $1 - \rho_j(c_w)$ off the fixed summand of the finite cyclic action. At $j = 0$ the source marking is defined using S, and at $w \in P$ it reads through the member of S in the same conjugate pair.

Conjugation by δ_j does not change a scalar. Every $L_j(\Phi)$ for a residue function carries the difference form of ramification to the target by (73), taking opposite scalar changes of variables. Moreover, if the datum is invariant under scaling T at one place w, then it annihilates such a difference at w. This applies to the indicator that a given matrix column or covector reduces to a vector that vanishes at w. It also applies to the datum that a specified matrix entry vanishes there.

For a residue column y in the target let $C_y(T) = \mathbf{1}_{\bar{T}e=y}$, and for a column in the source let $H_y(T) = \psi_N(e^t\bar{T}y)$, with the integral support in both cases. Write $L_j^{\mathrm{d}} = L_j(C_e)$. For the double lift the middle vector y must be primitive, meaning nonzero in each residue field. The other terms in $\sum_y L_{j+1}(H_y)L_j(C_y)u$ vanish by the ramified difference property. For primitive y choose k integral invertible with $\bar{k}e = y$ by extending the column to a basis in every residue field and lifting. The two laws for its insertion in the middle are

$$L_j(C_y)u = \rho_{j+1}(k)L_j^{\mathrm{d}}u, \qquad L_{j+1}(H_y)\rho_{j+1}(k) = L_{j+1}(H_e).$$

These follow from (73), since $\overline{kT}e = \bar{k}(\bar{T}e)$ and $e^t\overline{Tk}e = e^t\bar{T}y$. Summing gives

$$\sum_y L_{j+1}(H_y)L_j(C_y)u = c_{\mathrm{prim}}\, L_{j+1}(H_e)L_j^{\mathrm{d}}u, \qquad c_{\mathrm{prim}} = (N^{n_{j+1}} - 1)^{|P|}.$$

Define the same-level operator

$$U_j = \sum_{y:\, y_{\mathrm{last}} \in B^\times} \psi_N(y_{\mathrm{last}})\rho_j(D_j k_y),$$

where the integral invertible k_y has both last column y and last covector y^t modulo N. For $y = (v, t)$ with t a unit the residue matrix $\left(\begin{smallmatrix} 1+vv^t/t & v \\ v^t & t \end{smallmatrix}\right)$ has determinant t by the Schur complement and is such a lift modulo N. The actual U_j uses the marked choice of the lifts. For the actions on mirabolic fixed vectors only their last columns matter. Conjugating a mirabolic h by D_j^{-1} makes an integral matrix $D_j^{-1} h D_j$ whose residue last covector is e^t. Left multiplication by that matrix permutes the residue y with unit last entry, preserving their last entries. The error after replacing k_y by the chosen $k_{\overline{D_j^{-1} h D_j y}}$ fixes the last column. This proves that U_j preserves the original mirabolic fixed vectors. It commutes with the original level away from N by disjoint support. It preserves ramification because it commutes with the unit scalar actions.

At a ramified source vector one may omit matrices with nonunit corner from the function H_e, by decomposing that function according to the corner and using the scalar difference property at a place where the corner is zero. Sort the remaining function first by its last column y and then, in a separate expression, by its last covector y^t. Using (73) in both sorts and (74) for the covector indicator gives

$$\rho_{j+1}(D_{j+1}) L_j(H_e) u = U_{j+1} L_j^{\mathrm{d}} u = \gamma_j L_j^{\mathrm{d}} U_j u.$$

These are identities of the same theta lift, so no change of integral structures occurs when they are used in succession.

19.4.2 The direct lift at the ramified place

In this subsection, we identify the gluing of the summed tensors in Section 19.4.1.

At N the normalized vacuum projects to the integral box. In the model at P, the original gluing from S is the sum of the tensors

$$C_y(TA_j) \otimes H_y(B_{j+2}R), \qquad y \in B^{n_{j+1}},$$

where $A_j = \mathrm{diag}(1, \dots, 1, N)$ on $P \cap S$ and $B_{j+2} = \mathrm{diag}(1, \dots, 1, N)$ on $P \setminus S$, both with the identity elsewhere. Write $t_w = T_w e$ and $r_w = e^t R_w$. With all other entries integral, the two support conditions and the original residue class b are

w	integral entries	class at the member of S
$P \cap S$	$N t_w,\ r_w$	$b_w = N t_w \bmod N$
$P \setminus S$	$t_w,\ N r_w$	$b_{\bar{w}} = \overline{N r_w}^{\,t} \bmod N.$

Indeed the original translates require $t_w - b_w/N$ integral in the first case and $r_w - \bar{b}_{\bar{w}}^t/N$ integral in the second. In either case the surviving factor is $\psi_{N,w}(N r_w t_w \bmod N)$, where $\psi_{N,w}$ is the local factor of ψ_N. Thus on the common support the column $y = (TA_j)e \bmod N$ and the covector $e^t(B_{j+2}R) \bmod N$ give the asserted character pairing. In both split coordinates the semicharacter on the translations used in gluing is one, since each individual translation is supported in a split isotropic half. The Fourier interchange uses the self-dual integral box of volume one. This is the pointwise identity of the glued test function on integral supports with the stated last coordinate denominators.

Moving A_j, B_{j+2} through the lift by the same split Weil action cancels the middle δ_{j+1} translations. At the ends it gives the existing L_j, L_{j+1} in the L-bases, followed by $\rho_{j+2}(D_{j+2})$. With χ_j the character of the $(j+1)$-st lift as above, the scalar for this pair is

$$g_{j,\xi,J} = \chi_j(\det A_j)\chi_{j+1}(\det B_{j+2}) |\det A_j \det B_{j+2}|_P^{-n_{j+1}/2}.$$

Each $w \in P$ occurs in exactly one determinant. Through the marked conjugate embedding a factor N on P reads as N^{-1} in the norm-one determinant.

The algebraic character formula therefore gives an N-adic factor at most N^{2b} there, because the lift weight at that embedding is $\le 2b$ and the modulus power has N-adic norm ≤ 1. It follows that $|\jmath(g_{j,\xi,\jmath})|_N \le N^{2fb}$. The tame twist has weight zero. Its unramified local factors at these uniformizers are roots of unity, so they contribute N-adic units. For the reading $\jmath \circ \mathrm{conj}$ use $\bar{P}$ and its partition by the same S. Both local bounds hold with that reading. Apply the ramified summation and commute U through the direct lifts one at a time.

All intermediate vectors remain ramified. At the i-th pair, commuting through the preceding lifts contributes $\prod_{j=0}^{2i+1} \gamma_j = N^{|P|(n_{2i+2}-n_0)/2}$. For the fixed twist and coefficient reading set

$$c_{\xi,\jmath} = \prod_{i=0}^{m/2-1} g_{2i,\xi,\jmath} (N^{n_{2i+1}} - 1)^{|P|} N^{|P|(n_{2i+2}-n_0)/2}.$$

Since the ranks increase, $|\jmath(c_{\xi,\jmath})|_N \le N^{mfb}$ uniformly in ξ. Thus on the ramified source

$$\mathcal{U}_\xi(\Psi)(u) = \mathrm{res}_{Y_m} \left(c_{\xi,\jmath} \sum_x \Psi_\xi(x)\, L^{\mathrm{d}}_{m-1,x} \cdots L^{\mathrm{d}}_{0,x} U_0^{m/2} u \right).$$

Here each $L^{\mathrm{d}}_{j,x}$ uses the same translate of the normalized vacuum as the x-term of the original chain. The permitted translates are vacuum-projected above N before gluing, as given by admissibility of the chain.

To bound a direct lift, start in the standard gauges at 1. Put e_j, e_{j+1} for the last column vectors in the source and target. Character orthogonality in the last column gives

$$C_e(T) = \frac{1}{N^{|P|n_{j+1}}} \sum_{\ell \in B^{1\times n_{j+1}}} \psi_N\big(\ell(\bar{T} e_j - e_{j+1})\big) \qquad (T \in \Lambda_P),$$

with value zero off Λ_P. Each local character is nontrivial on $\mathbf{F}_N$. Hence $N^{|P|n_{j+1}}$, which divides $N^{[E:\mathbf{Q}]n_j n_{j+1}}$, clears every coefficient.

Multiplication by a character of these residue variables is a Heisenberg translate by a vector with at most simple poles at the conjugate places $\bar{P}$, by the self-dual pairing. For the projected vacuum the original translate is either integral above N, or the projection is zero. All induced places in $\bar{P}$ lie above the prime $\mathfrak{n}$ of K defined by the coefficient embedding $\jmath$. Write $\beta = (1+\sqrt{-d})/2$. Since $\mathcal{O}_K/\mathfrak{n} = \mathbf{F}_N$, choose $c_0 \in \mathbf{Z}$ lifting the reduction of β and put $\delta = \beta - c_0$. Then

$$\delta \in \mathfrak{n}, \qquad \bar{\delta} \equiv \bar{\beta} - \beta = -\sqrt{-d} \not\equiv 0 \pmod{\mathfrak{n}}.$$

Thus $v_{\mathfrak{n}}(\delta) \ge 1$ and $v_{\bar{\mathfrak{n}}}(\delta) = 0$. The first inequality clears every simple pole over $\bar{P}$, while $|\jmath(\bar{\delta} d)|_N = 1$. For the opposite reading $\jmath \circ \mathrm{conj}$, the pole set is P, above $\bar{\mathfrak{n}}$; taking $\bar{\delta}$ gives the same unit multiplier at that reading. Apply Section 20.1 to the normalized standard vacuum translated by z with δz integral. At its original scale the required gauge factor is $(|\jmath(\pi_M)|_N / |\jmath(\bar{\delta} d)|_N)^{\#\mathrm{boxes}} \le 1$. After dividing by the Fourier clearing factor, the direct gauge contribution is at most $N^{2fn_j n_{j+1}} \le N^{fb}$.

The standard-gauge assertion bounds the kernel at every pair of finite points. Indeed choose rational translates of the two points independently so that their finite coordinates above N are integral. This is possible by weak approximation on the two unitary groups, using their

rational Cayley charts. At the integral representatives the action at each $w \in P$ has the form $T \mapsto b_w^{-1} T a_w$ with a_w, b_w integral and invertible. It preserves the integral matrix lattice. If $a_{w,0}, b_{w,0}$ are the two reductions, the transported column condition is

$$\bar{T} a_{w,0} e_j = b_{w,0} e_{j+1}.$$

The same character expansion applies to this equation. Its linear forms remain integral in the residue coordinates, and its constant phases are roots of unity. Thus the Fourier clearing factor remains $N^{|P| n_{j+1}}$, and the translating vectors still have at most simple poles at $\bar{P}$. The unramified character factors above N are units. For the action away from N, let A denote its extension by the identity above N and L the vacuum lattice. The finite Weil expansion sums over $L/(L \cap A^{-1} L)$. Its order is prime to N, and representatives y can be chosen zero above N. The Weil multiplier and pair characters are N-adic units by the character calculation in Section 22.2, applied to the finite action with identity components above N. Each translated vacuum therefore has shift $A(z+y)$ with the original component z above N. After multiplication by $N^{|P| n_{j+1}}$ the whole datum belongs to the span over the coefficient valuation ring of vacua whose shifts are cleared by the same δ above N. The ordinary jet bound of Section 20.1 now applies at the integral representatives. Rational covariance transports it back in the Weyl gauges of the original points. Translating the datum by (δ_j, δ_{j+1}) replaces the two gauges by those at $g\delta_j$ and $h\delta_{j+1}$. In particular, writing $\|v\|_a$ for the gauge at a, the direct kernel satisfies

$$\|K^{\mathrm{d}}_{j,x}(g,h) v\|_{h\delta_{j+1}} \le N^{fb} \|v\|_{g\delta_j} \qquad \text{for every } g, h.$$

To sum the products of these kernels, project the glued tensor onto the original integral vacuum above N in each factor. This is the product of the mass-one additive averages defining the original vacuum projections. For a nonzero linked residue choose a pair with $y_w \neq 0$, $w \in S$. If $w \in P$, the first vacuum is translated by a nonintegral column $-\widetilde{y}_w / N$, so its support is disjoint from the integral box. If $\bar{w} \in P$, the same translation multiplies the integral box by a nontrivial character of mean zero. Only $y = 0$ survives, with coefficient one and the original vacuum translates. The projected tensor is

$$\sum_x \Psi_\xi(x) \bigotimes_{j=0}^{m-1} V_j(x),$$

where $V_j(x)$ denotes the original translated vacuum. The original levels preserve the projection lattices and their probability Haar, so covariance makes the projection commute with their simultaneous action. In particular the displayed tensor is jointly invariant. The direct insertions also commute with this action. In their conjugated bases the original levels preserve the integral box, and the test $\bar{T} e = e$ is equivalent to $b^{-1} \bar{T} a e = e$ for the two residue actions. Hence the tensor with the direct insertions is jointly invariant under the same levels.

Let $u = U_0^{m/2} \varphi$ have unit gauge at $g\delta_0$ for every g, as required by the definition of the preimage lattice. For a fixed terminal point g_m the product integral of the m kernels, summed with $\Psi_\xi(x)$, is

$$\sum_{(g_j)_{j<m}} \sum_x \Psi_\xi(x) K^{\mathrm{d}}_{m-1,x}(g_{m-1}, g_m) \cdots K^{\mathrm{d}}_{0,x}(g_0, g_1) u(g_0).$$

Indeed simultaneous level invariance makes the contracted integrand constant on each product of the original level classes. Their stabilizers are trivial at the old odd principal level and the class

masses in the mean integrals are one. Since N is prime to the twist modulus, $|\jmath(\Psi_\xi(x))|_N \le 1$. The m bounds multiply to N^{mfb}, and the ultrametric inequality gives that same bound for the entire sum in the gauge at $g_m\delta_m$. Multiplication by $c_{\xi,\jmath}$ gives the bound N^{2mfb} before restriction.

For $h \in U(Y_m)(\mathbb{A}_{F_0,f})$ put $\widetilde{h} = \mathrm{diag}(h,1) \in U(X_m)(\mathbb{A}_{F_0,f})$. Let w_τ be the place induced by $\jmath \circ \tau$, and put $\chi_\tau = \mathbf{1}_{\bar{w}_\tau|\pi}$. The weight embedding reads the component at $\bar{w}_\tau$, where $(\widetilde{h}\delta_m)^{-1}$ has blocks h^{-1} and $s_\tau = N^{-\chi_\tau}$. By Lemma 7.9 the last-coordinate exponent of B_Y is $B_\tau = |\mu(X_m)_\tau| - |\mu(Y_m)_\tau| = a_\tau - mh_0$. The split factorization $\pi\bar{\pi} = N$ gives

$$\left| \jmath\left(\prod_\tau \tau(\bar{\pi})^{-B_\tau} \right) \right|_N = \left| \prod_\tau s_\tau^{B_\tau} \right|_N .$$

Indeed both sides are $N^{\sum_\tau \chi_\tau B_\tau}$, for either sign of B_τ. Replacing $\jmath$ by $\jmath \circ \mathrm{conj}$ replaces χ_τ by $1 - \chi_\tau$ on both sides. Thus

$$u_\jmath = \jmath\left(\prod_\tau \tau(\bar{\pi})^{-B_\tau} \right) \prod_\tau s_\tau^{-B_\tau}, \qquad |u_\jmath|_N = 1.$$

In the transported gauges the restriction is $u_\jmath B_Y$ at this reading. The integer matrix B_Y is a contraction, so for the upper lift Φ just bounded,

$$\|(\mathrm{res}\,\Phi)(h)\|_h \le \|\Phi(\widetilde{h})\|_{\widetilde{h}\delta_m} \le N^{2mfb} \qquad (h \in U(Y_m)(\mathbb{A}_{F_0,f})).$$

This is the unit-base gauge bound at every terminal point. Its logarithmic contribution satisfies

$$e_N \log N = 2mfb \log N = O(fm^2bB) = O(faB) = o(faB^3),$$

by $\log N = O(mB)$, $m^2 \le 4x$, and $a = xb$, as in Lemma 7.1.

19.4.3 Integral clearing on the original source

In this subsection, we bound the colength required for the direct lift of Section 19.4.2.

Keep the coefficient embedding $\jmath$ above N. Set $\Delta_N = D_0^{-1}$ on the P half and use its inverse conjugate transpose on the other half. Thus the dual point is $\delta_0^\vee = \delta_0\Delta_N^{-1}$. At a place in $P \cap S$ the original δ_0 has last entry N; on $P \setminus S$ it is the identity. Every embedding in the coefficient gauge reads the point through P. Therefore $(\delta_0^\vee)^{-1}$ acts in the Weyl gauge by integral diagonal matrices. The source representation is polynomial, with determinant exponent zero, so the unit gauge at 1 implies the unit gauge at $\delta_0^\vee$.

For a class representative c of the original source let K_N be the product of the integral groups above N. Choose a rational unitary point γ_c such that

$$\gamma_c c \delta_0^\vee \in K_N \cap \Delta_N K_N \Delta_N^{-1}.$$

Weak approximation here follows from the Cayley transform $X \mapsto (1-X)(1+X)^{-1}$ on skew-Hermitian matrices. The chart is dense in each local unitary group and the coefficients of X can be approximated at the finitely many places.

The required condition is open and nonempty as a condition on γ_c over the local fields. Put $r_c = \gamma_c c$, so $r_c\delta_0^\vee \in K_N$ and $r_c\delta_0 \in \Delta_N K_N$. For k in the original source level, $\delta_0^{-1}k\delta_0$ is integral

above N and fixes the last column on reduction in P. At $P \cap S$ this follows by conjugating the original condition on the last covector, which divides the lower left block by N and multiplies the upper right by N.

At $P \setminus S$ it follows by applying the condition on the last covector to k^{-1} on the other half and taking the inverse conjugate transpose in the unitary coordinates. Both coordinates have unit determinant. Thus the unit gauges at the two points are tested in the same class bases r_c, with the indicated diagonals. Rational equivariance transports these tests to arbitrary representatives.

For a Weyl basis index I let w_I be the total degree in the last variable over all embeddings. Unit gauge at $\delta_0^\vee$ gives integral coordinates, and unit gauge at δ_0 requires them to belong to $N^{w_I} R_\jmath$, where

$$R_\jmath = \{a \in \overline{\mathbf{Q}}_N : v_N(a) \geq 0\} = \jmath(\mathcal{O}_\jmath).$$

Indeed the evaluated class coordinates of the given tuple, and of each of its U_0-iterates, lie in $\jmath(\mathbf{C}) = \overline{\mathbf{Q}}_N$. The integral coordinate tests computed in $\mathbf{C}_N$ therefore give the indicated divisibilities in $R_\jmath$. For a finite tuple of vectors impose these conditions one covector at a time. In a nonzero covector choose a coefficient of least valuation, clear the other coefficients by integral column operations, and multiply the remaining column by N^{w_I}. Every ratio used to clear another coefficient by this pivot has nonnegative valuation and lies in $R_\jmath$. The resulting column operations thus pull back to $\mathcal{O}_\jmath$. A zero covector already satisfies the condition.

The operations preserve all preceding divisibility against covectors. The determinant contribution is at most

$$h_{\mathrm{deg}} = \sum_{c,I} w_I \leq d_X f a / k,$$

where d_X is the dimension of the full original source. Indeed the sum of the degrees in each variable is the same, by permuting the diagonal in the Weyl representation. Summing the degrees over all k variables gives at most fa for each (c, I). The source level is free: a stabilizer at an old odd distinguished prime lies in the torsion-free principal subgroup proved in Section 30.1.

The right translation in U_0 at y is by $u_y = \delta_0^\vee k_y \delta_0^{-1}$. Thus $u_y \delta_0 = \delta_0^\vee k_y$ differs from $\delta_0^\vee$ by an integral invertible right factor above N. For an arbitrary point g the gauge of the corresponding term $u(g u_y)$ at $g\delta_0^\vee$ is precisely its gauge at $g u_y \delta_0$. The coefficient $\psi_N(y_{\mathrm{last}})$ is a unit. It follows that U_0 takes the unit-gauge module at δ_0 into the one at $\delta_0^\vee$. The previous clearing step, applied first to the original tuple and then to the tuple at each U_0-iterate, gives contribution $(m/2+1)h_{\mathrm{deg}}$ to obtain unit gauge at δ_0 after $m/2$ iterations. Each clearing acts on the original tuple by linearity. Since $a = xb$ and $k \geq x$ for sufficiently large x, the contribution is at most $mfbd_X$. Intersecting this preimage with the original ramified source preserves this clearing bound: every operation is an integral linear combination of the original ramified columns.

The preimage is fixed by the level and $\jmath$ before choosing the twist or translated arrow data. The gauge loss bound $2mfb$ has zero saving at N since $N \geq x_1$. The separate branching factor at infinity occurs in the norm calculation, with the same branching exponent $B_\tau = a_\tau - mh_0$.

20 Integral theta sections and local derivatives

In this section, we bound the derivatives of the integral theta sections occurring in Section 19. The exceptional primes require the additional Hecke calculation in Section 21.

20.1 Integral sections and ordinary jets

In this subsection, we construct an integral CM model and control the derivatives of translated theta sections in that model. The derivative estimate applies to the translates in Section 19.3 and retains their actual translation denominators.

Let $K = \mathbb{Q}(\sqrt{-d})$, with d prime and $d \equiv 7 \bmod 8$. Write $d_K = |\operatorname{disc}(K)| = d$ and $\beta = (1+\sqrt{-d})/2$. Let $\Lambda = \Omega_0 \mathcal{O}_K$, with $\Omega_0 = \omega_0$ the period used in the proof of (AJ), for the marked CM lattice, A for its area, and t for its marked half-period. Modulo Λ, the three nonzero points of order two are $t, \beta t, \bar{\beta} t$. The scale is fixed by $\wp''(t) = 2$. Put

$$x(z) = \wp(z+t) - \wp(t), \quad a_1^2 = 12\wp(t), \quad y(z) = \tfrac{1}{2}(\wp'(z+t) - a_1 x(z)).$$

The equation and differential relations are

$$y^2 + a_1 xy = x^3 + x, \qquad x' = 2y + a_1 x, \qquad y' = 1 + 3x^2 - a_1 y.$$

For every ideal class and every normalization at a half-period u_i, the equation in completed-square coordinates is $Y^2 = X(X^2 + 3e_i X + 1)$.

Its invariant is

$$j = 256\,\frac{((3e_i)^2 - 3)^3}{(3e_i)^2 - 4}.$$

Complex multiplication makes j algebraic [49, Theorem 12.8], so each normalized e_i and a_1 is algebraic. The coordinates of the torsion points are algebraic as well, since the multiplication maps are defined over $\overline{\mathbb{Q}}$. The analytic origin $z = 0$ corresponds to the point $T = (0,0)$ on this equation, and $z = t$ to the point at infinity. Thus the invariant differential in this translated coordinate is

$$\frac{dx}{2y + a_1 x} = \frac{dy}{1 + 3x^2 - a_1 y} = dz.$$

The second denominator is one at T, so y is the local parameter there, with $y(0) = 0$ and $y'(0) = 1$. Fix a coefficient place $\kappa = (p, \iota)$ as in Section 2.7, and put $R = \mathcal{O}_{\mathbb{C}_p}$. In the Taylor series below $\iota_p : \mathbb{C} \xrightarrow{\iota} \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}_p$ acts on each coefficient separately.

The discriminant at all primes. Let e_i range over the three half-period values on a CM lattice of any ideal class. If u_i is the corresponding half-period, set

$$s(u_i) = 16\frac{(e_j - e_k)^2}{(e_i - e_j)(e_i - e_k)} \qquad (\{i,j,k\} = \{1,2,3\}).$$

This expression is invariant under homothety. The identity $\wp''(u_i) = 2(e_i - e_j)(e_i - e_k)$ shows that $s(u_i)$ is the discriminant after normalizing $\wp''(u_i) = 2$. Taking a common denominator in the sum and multiplying the three expressions gives

$$\sum_i s(u_i) = -48, \qquad \prod_i s(u_i) = -2^{12}.$$

For the quotient by the point u_i, translate e_i to zero. In the equation $Y^2 = X(X^2 + aX + b)$ one has $a = (e_i - e_j) + (e_i - e_k)$, $b = (e_i - e_j)(e_i - e_k)$. The quotient by $(0,0)$ has equation $Y^2 = X(X^2 - 2aX + a^2 - 4b)$, with dual kernel $(0,0)$. Indeed, the isogeny is given off the kernel by $(X,Y) \mapsto (X + a + b/X, Y(1 - b/X^2))$, as substitution in the equations verifies. The other

two old half-periods map to $(0,0)$. Hence the two values of s at the original and dual kernels are $16(a^2-4b)/b$ and $2^8 b/(a^2-4b)$, respectively.

Index a family of CM lattices by the ideal classes c, including the original lattice in its own class. Since 2 splits in K, the module $(\frac{1}{2}\mathfrak{a})/\mathfrak{a}$ of any invertible ideal $\mathfrak{a}$ is free of rank one over $\mathcal{O}_K/(2) = \mathbb{F}_2 \times \mathbb{F}_2$. Choose for t in each class its unique element nonzero in both summands. The idempotents $\beta, \bar{\beta}$ give the other two nonzero points. Homotheties preserve these three choices. The quotient by βt replaces an ideal $\mathfrak{a}$ by $(1, \beta/2)\mathfrak{a}$ and carries $\bar{\beta}t$ to the nonzero $\bar{\beta}$-half-period of the quotient. In fact β is the projection to the line generated by βt on $\frac{1}{2}\mathfrak{a}/\mathfrak{a}$, so this is the indicated quotient lattice.

Use g for the class of $(1, \beta/2)$. With $s_c, s'_c, s_{0,c}$ the values at $\beta t, \bar{\beta}t, t$, the product calculation gives

$$s_c s'_{gc} = 2^{12}, \qquad s_{gc} s_{0,gc} = -s_c.$$

If $|s_c|_p > 1$, then $|s'_{gc}|_p \le 1$, $|s_{gc} + s_{0,gc}|_p \le 1$, and $|s_{gc} s_{0,gc}|_p = |s_c|_p > 1$. The ultrametric inequality gives $|s_{gc}|_p = |s_{0,gc}|_p = |s_c|_p^{1/2} > 1$. Iterating around the finite cycle contradicts $|s_c|_p > 1$. Interchanging $\beta, \bar{\beta}$ proves $|s'_c|_p \le 1$. The sum shows $|s_{0,c}|_p \le 1$, and multiplying the second identity through all classes yields $\prod_c s_{0,c} = (-1)^{h_K}$. Each $s_{0,c}$ is a unit. In our mark this gives $\Delta = a_1^4 - 64 \in R^\times$ and $a_1 \in R$, including at $p = 2$.

Theta sections and exact multiplication. Let σ_{W} be the Weierstrass sigma function and η_l its quasi-period at $l \in \Lambda$. With A the area of Λ, the Legendre relation gives $\eta_l = \pi \bar{l}/A + \gamma l$ for a constant γ. Use the reduced theta function $\sigma(z) = e^{-\gamma z^2/2}\sigma_{\mathrm{W}}(z)$, so $\sigma'(0) = 1$ and its multiplier has Hermitian form $H(u,z) = \pi \bar{u} z/A$ [8, Example 1.9, eqs. (7)–(8), in the authors' version]. Set

$$(T_v^{(m)} G)(z) = e^{-m(H(v,z) + H(v,v)/2)} G(z+v), \qquad G[u]_m = (T_u^{(m)} G)(0).$$

Put $\theta = T_t^{(1)}\sigma / (T_t^{(1)}\sigma)(0)$. This is even, since σ is odd and the sigma multiplier at $2t$ has sign -1. Its degree-one semicharacter is

$$\chi(l) = \exp(\pi i \operatorname{Im} H(\bar{\beta} l, l)/\pi) = (-1)^{|l|^2 \sqrt{d}/(2A)} \qquad (l \in \Lambda).$$

For the sign calculation, the sigma sign at a period v is $+1$ if $v/2 \in \Lambda$ and -1 otherwise. Twisting sigma by t gives the sign at al, for $a = \beta, \bar{\beta}$, as the negative of the sigma sign at $2t + al$. For $z = t + al/2$ one has $(1-a)z = (1-a)t + |a|^2 l/2 \notin \Lambda$ since $|a|^2 = (d+1)/4$ is even. Thus $\chi(al) = 1$. The semicharacter rule at $l = \beta l + \bar{\beta} l$ gives the displayed formula. The degree-m section law is $T_l^{(m)} G = \chi(l)^m G$.

If multiplication by a is an isogeny into another such marked lattice of area A' and $|a|^2 A = NA'$, then $H'(au, az) = NH(u,z)$ and $\chi'(al) = \chi(l)^N$. Consequently pullback takes degree m sections to degree mN, with

$$(G \circ [a])[u]_{mN} = G[au]_m.$$

Also $(T_v^{(m)} G)[u]_m = e^{\pi i m E(u,v)} G[u+v]_m$ where $E(u,v) = \operatorname{Im} H(u,v)/\pi$. The phase is a root of unity at torsion points since E is integral on Λ. In particular if $\delta u \in \Lambda$, $\delta \in \mathcal{O}_K \setminus \{0\}$, then

$$z \longmapsto (T_u^{(1)}\theta)(\bar{\delta} z) \quad \text{has degree } N = \delta\bar{\delta}.$$

Indeed $T_{\bar{\delta} l}^{(1)}$ commutes with $T_u^{(1)}$ because $E(\bar{\delta} l, u) = E(l, \delta u) \in \mathbb{Z}$. Its remaining multiplier is $\chi(\bar{\delta} l) = \chi(l)^N$.

Interpolation on the marked curve. The classical Weierstrass addition and duplication formulas give

$$\sigma^2 = x\theta^2, \qquad \theta(2z) = \theta(z)^4 - \sigma(z)^4, \qquad \sigma(2z) = x'(z)\theta(z)^4.$$

For the normalization in these formulas, the addition formula at $(z+t, t)$ proves the first identity after comparing the constant at $z = 0$ of $\sigma(z)^2/(x(z)\theta(z)^2)$. Indeed this quotient is elliptic with no zero or pole and tends to one there. Twisting twice by t gives the sigma sign -1 at $2t$. If $c = (T_t^{(1)}\sigma)(0)$, translating $\sigma^2 = x\theta^2$ by t gives $c^4 = x(z)x(z+t) = 1$, the last identity being the addition formula at a half-period with $\wp''(t) = 2$. The sigma duplication formula at $z + t$ now gives the last displayed identity, with the sign at $2t$ cancelling its minus sign. For the middle identity, apply the sigma addition formula first to $(z + t, v)$ and divide out the common exponentials. This gives $\theta(z + v)\theta(z - v) = \theta(z)^2(\theta(v)^2 - x(z)\sigma(v)^2)$. Set $v = z$.

For $w(u) = (\theta[u]_1^2, \sigma[u]_1^2)$, the identities imply $w(2u) = D(w(u))$ with

$$D(U, V) = ((U^2 - V^2)^2, UV(4U^2 + a_1^2UV + 4V^2)).$$

First, the two coordinates of $w(u)$ are algebraic when u is torsion. The identity $\sigma^2 = x\theta^2$ gives its projective coordinate as $[1 : x(u)]$, or $[0 : 1]$ at the unique zero of θ. Write $w(u) = \lambda v$ for a nonzero $v \in \overline{\mathbb{Q}}^2$. Choose $a < b$ with $2^a u \equiv 2^b u \bmod \Lambda$. The section law at torsion gives $D^b(w(u)) = \zeta D^a(w(u))$ for a root of unity ζ. In characteristic zero the two forms of D have no common projective zero, since the curve discriminant is nonzero. For a nonzero coordinate of $D^a(v)$ the corresponding coordinate of $D^b(v)$ is nonzero as well. The homogeneity of D then gives

$$\lambda^{4^b - 4^a} = \zeta \, \frac{D^a(v)_i}{D^b(v)_i} \in \overline{\mathbb{Q}}^\times.$$

This justifies the coefficient-place absolute values on $w(u)$.

The reduction of these forms has no common projective zero. A zero of the first must have $U = \pm V \neq 0$; the second then vanishes only if $a_1^2 = \pm 8$ in the residue field, contrary to $\Delta \in R^\times$. This also applies in characteristic two. Therefore $\|D(w)\|_p = \|w\|_p^4$ in the maximum norm, by reducing after dividing by a coordinate of largest norm. The norm is unchanged modulo Λ because sigma and theta have sign multipliers at periods and the remaining torsion phases have norm one. A torsion orbit under doubling takes only finitely many norms, all positive. The identity forces every one to equal one.

The section $G_3 = y\theta^3$ is entire and its twisted torsion values are in R, by the monic quadratic obtained from the curve equation,

$$G_3[u]_3^2 + a_1\theta[u]_1\sigma[u]_1^2 G_3[u]_3 = \sigma[u]_1^6 + \theta[u]_1^4\sigma[u]_1^2.$$

For $u_i = t/2^i$, $i \geq 1$, the reductions of both coordinates of $w(u_i)$ are nonzero. Indeed both coordinate points map under D to $(1 : 0)$, which is fixed, whereas $D^i(w(u_i))$ reduces to $(0 : 1)$. Thus $x(u_i), y(\pm u_i) \in R$. The differences $x(u_i) - x(u_j)$ are units for $i < j$, by applying D^i to the projective reductions and comparing $(0 : 1)$ with the interior point from u_{j-i}. The duplication identity at the normalized torsion values is

$$\sigma[2u_i]_1 = x'(u_i)\theta[u_i]_1^4.$$

Both theta and sigma coordinates are units at u_i. The sigma coordinate is also a unit at $2u_i$, including $2u_1 = t$. Thus $y(u_i) - y(-u_i) = x'(u_i) \in R^\times$, also at $p = 2$.

Let G be a degree-m section whose twisted torsion values are in R. The elliptic function G/θ^m has its only possible pole at t, of order at most m. Since x, y have pole orders two and three there, successive elimination of leading terms gives

$$G/\theta^m = P_0(x) + yP_1(x), \qquad 2\deg P_0 \le m, \qquad 3 + 2\deg P_1 \le m$$

with missing terms omitted. Elimination uses the fact that a function with a sole pole of order one is impossible on an elliptic curve, by the residue theorem.

Evaluating at $u_i, -u_i$ gives integral values because $\theta[\pm u_i]_1$ is a unit. Their difference is $x'(u_i)P_1(x(u_i))$. Unit Vandermonde interpolation implies $P_1 \in R[X]$, then $P_0 \in R[X]$. The curve equation solves for x in $R[[y]]$ at the origin, since its partial derivative in x at $(0,0)$ is a unit. In particular

$$\frac{(T_u^{(1)}\theta)(\bar{\delta}z)}{\theta(z)^N} \in R[[y(z)]] \qquad (\delta u \in \Lambda).$$

Here the hypothesis on the numerator's torsion values follows from the pullback identity, the torsion translation phase and the norm computation.

For multiplication by any nonzero $a \in \mathcal{O}_K$, apply the same interpolation to the pullbacks of θ and G_3. It gives

$$A_a = \frac{\theta(az)}{\theta(z)^{a\bar{a}}} \in R[[y(z)]], \qquad B_a = \frac{G_3(az)}{\theta(z)^{3a\bar{a}}} \in R[[y(z)]].$$

Since $A_a(0) = 1$, division gives $y(az) = B_a/A_a^3 \in R[[y(z)]]$. Let $Z(Y)$ be the compositional inverse of $y(z) = z + O(z^2)$ and $g = -\log\theta$. For every j, the coefficients of Y^j in

$$y(aZ(Y)), \qquad \theta(aZ(Y))\exp\big(sg(Z(Y))\big)$$

are polynomials in a, s over $\mathbb{C}_p$. Indeed, writing $U(Y) = \theta(Z(Y))^{-1} - 1 \in Y\mathbb{C}_p[[Y]]$, one has

$$\exp(sg(Z(Y))) \equiv \sum_{i=0}^{j} \binom{s}{i} U(Y)^i \pmod{Y^{j+1}}.$$

Each coefficient of $\theta(aZ(Y))$ through degree j is a polynomial in a. The two indicated coefficients belong to R for $(a,s) = (a, a\bar{a})$, $0 \ne a \in \mathcal{O}_K$, and hence at every limit of these pairs because R is closed. At an inert prime choose CM integers $a_n \to \zeta \in \mu_{p^2-1}$ in $\mathcal{O}_{K,p}$. Local conjugation sends ζ to ζ^p, so $(a_n, a_n\bar{a}_n) \to (\zeta, \zeta^{p+1})$. At the ramified prime take rational integers $a_n \to \zeta \in \mu_{p-1} \subset \mathbf{Z}_p^\times$, giving the limit pair (ζ, ζ^2). Thus the same even section with $\theta(0) = 1$ gives both coefficientwise specializations.

For an actual endomorphism $0 \ne a \in \mathcal{O}_K$, the section quotient A_a is even, since θ is even. In the interpolation formula x is even and $y(z) - y(-z) = x'(z) \ne 0$, so $P_1 = 0$. Thus $A_a \in R[x]$ with $2\deg_x A_a \le a\bar{a}$. For odd $a\bar{a}$ the bound is $a\bar{a} - 1$. This applies to $a = p$ at an inert prime and to $a = \sqrt{-d}$ at the ramified prime.

In the same way, the even section $\sigma(az)^2$ has integral quotient $P_a(x) = \sigma(az)^2/\theta(z)^{2a\bar{a}}$ with $\deg P_a \le a\bar{a}$. At least one coefficient of P_a is a unit. At $u = t/a$, its expression before division by $\theta^{2a\bar{a}}$ is a sum of monomial sections $\theta^{2a\bar{a}-2i}\sigma^{2i}$, each with integral torsion evaluation, while the pullback identity gives the unit value $\sigma[t]_1^2$ for their linear combination. The quotient $x(az) = P_a(x)/A_a^2$ is therefore nonzero after reduction and has order at most $2a\bar{a}$ in y. Since $x = y^2 + O(y^3)$ over R, the order of the reduction of $y(az)$ is at most $a\bar{a}$. This proves the needed

degree bound on the original formal parameter from the torsion interpolation itself. For the inert multiplication-by-p calculation at the Teichmüller-averaged parameter, the reduced series has zero linear term and every allowed term has degree $1 \pmod{p^2-1}$; thus its first nonzero term has degree exactly p^2.

A basis with denominator prime to the residue characteristic. Use the cyclotomic field $E = K(\zeta)$, with ζ of order $q = q_1$, and the Weil lattice $\mathfrak{a}_0 = E \cap \alpha_{\mathrm{W}} \widehat{\mathcal{O}}_E$ of Section 7.4.3. In this model $\mathfrak{a}_0$ is the product of the inverse relative different at the primes over β, and the unit ideal elsewhere. It is stable under $\mathrm{Gal}(E/K)$, it contains $\mathcal{O}_E$, and it is self-dual for $\mathrm{Tr}_{E/K}(x\bar{y})$: conjugation interchanges the two sets of primes of the relative different above the split primes dividing q. Write Φ for the embeddings over the fixed embedding of K. The powers of ζ give Gram sum qI under these embeddings by the finite geometric sum. For $p \nmid q$ this already has suitable length $b = q$.

For $p \mid q$, let $j = \lceil \log_2 q \rceil$ and take $e_i = \bar{\beta}^j d_i$, where d_i is trace-dual to ζ^i, for $0 \le i < [E : K]$. Then $e_i \in \mathfrak{a}_0$. Indeed d_i is in the relative codifferent. The codifferent denominators are supported over q and are cleared by q, because $\Phi_q'(\zeta)$ divides q by differentiating the factorization of $X^q - 1$. For every rational $\ell \mid q$, $v_\ell(q) \le j$ and $\beta\bar{\beta}$ is divisible by ℓ.

Since $\beta + \bar{\beta} = 1$, the factor $\bar{\beta}^j$ clears the codifferent at primes over $\bar{\beta}$. Take ζ^{-k} for $0 \le k < q$ and all the vectors $\zeta^k \sigma(e_i)$ with $0 \le k < q$, $\sigma \in \mathrm{Gal}(E/K)$. Their Gram sum is $q(1 + \sum_i \nu_i)I$, where $\nu_i = \mathrm{Tr}_{E/K}(e_i \bar{e}_i)$ is a nonnegative integer by self-duality and reality. Replace each pair ζ^{-i}, e_i with $\zeta^{-i} + e_i$. This adds $(\beta^j + \bar{\beta}^j)I$ by trace duality, since the products of the two square embedding matrices in either order are identity.

The new scalar $b = q(1 + \sum_i \nu_i) + \beta^j + \bar{\beta}^j$ is positive and prime to q. At any prime over $\ell \mid q$, one of $\beta, \bar{\beta}$ is zero and the other is one, so the last two terms sum to one modulo ℓ. In both cases repeat each vector b times to obtain $w_i \in \mathfrak{a}_0$ with Gram sum $b^2 I$.

The finite vacuum expansion. Let $L = \mathfrak{a}_0^n$, embedded in the real Fock coordinates by $r(\gamma)_{\tau,A} = a_d \tau(\gamma_A)$, where $a_d = \sqrt{2\pi/\sqrt{d}}$. This is the self-dual lattice for the Riemann form. Put

$$A_{i,A}(z) = \frac{\Omega_0}{b a_d} \sum_{\tau \in \Phi} \overline{\tau(w_i)} z_{\tau,A}, \qquad S(z) = \prod_{i,A} \theta(A_{i,A} z).$$

The area $A = |\Omega_0|^2 \sqrt{d}/2$ and the basis identity give $\sum A_{i,A}^* H_C = H_{\mathrm{Fock}}$. For $\lambda \in L$ put $s_{i,A} = \mathrm{Tr}_{E/K}(\lambda_A \overline{w_i}) \in \mathcal{O}_K$, so $A_{i,A}(r(b\lambda)) = \Omega_0 s_{i,A}$. The same basis identity gives

$$\sum_{i,A} |s_{i,A}|^2 = b^2 \,\mathrm{Tr}_{E/K}\left(\sum_A \lambda_A \bar{\lambda}_A \right).$$

Since $\chi(\Omega_0 s) = (-1)^{|s|^2}$, the product of the elliptic multipliers is $(-1)^{b^2 \,\mathrm{Tr}_{E/K}(\sum_A \lambda_A \bar{\lambda}_A)}$, the original Fock sign at $r(b\lambda)$, including when $p = 2$ and b is odd. Thus S has the signed period law on bL and $S(0) = 1$.

The same finite Fourier calculation constructs the product vector and resolves the vacuum. Write U_l for signed periods and T_t for ordinary twisted real translations. The groups L/bL and $b^{-1}L/L$ are perfectly paired by the translation commutator and both have order $M_L = b^{2|\Phi|n}$. Let $\mathcal{V}_{\mathrm{int}}$ be the entire theta image of $s_{\mathrm{vac}}^n v_0^{\otimes n}$, where v_0 is the original rank-one unit vacuum of (AV). On the bL-periodic space,

$$P_t S = \sum_{l \in L/bL} U_l T_t S = d_t \mathcal{V}_{\mathrm{int}} \quad (t \in b^{-1}L/L).$$

Indeed the sum is invariant under L, and the line of theta sections on the principally polarized torus is one-dimensional.

The commutator orthogonality gives

$$M_L S = \sum_t T_{-t} P_t S = \sum_t d_t T_{-t} \mathcal{V}_{\mathrm{int}}.$$

This identifies S as the entire theta image of a finite-part vector F. In the finite Weil model a rational real translation by $r(\gamma)$ is induced by the operator at the opposite finite component, by rational invariance of the theta functional. To prove injectivity, let f have zero entire theta image. The Gaussian factor is nonzero, so $\theta(R(z_\infty)(v_\infty \otimes f)) = 0$ for every real displacement z_∞.

For any finite displacement z_f, choose a rational γ with $z_f - \gamma_f$ in an open translation stabilizer of f. Rational Heisenberg invariance identifies the evaluation at $z_\infty + z_f$, up to a unit phase, with that at $z_\infty - \gamma_\infty$. Hence all adelic displacement evaluations vanish. The polarized norm identity of Section 41.5.1 gives $\|v_\infty \otimes f\|^2 = 0$. Since $v_\infty \neq 0$, $f = 0$, and the identities hold in the finite Weil model.

Evaluating at zero, $M_L = \sum_t d_t (T_{-t}\mathcal{V}_{\mathrm{int}})(0)$, and substituting $d_t \mathcal{V}_{\mathrm{int}} = P_t S$ gives

$$\mathcal{V}_{\mathrm{int}} = M_L^{-1} \sum_t (T_{-t}\mathcal{V}_{\mathrm{int}})(0) \sum_{l \in L/bL} U_l T_t S.$$

The coefficients of these translates are integral since $p \nmid b$, the values of the chosen integral vacuum are integral, and the finite translation phases are roots of unity. Every translating point becomes integral after multiplication by b.

The vacuum scalar here is the same nonzero projection scalar in (AV). Its finite values are integral by the torsion computation, and the higher-rank vacuum uses its coordinate product. The basis prime to p changes the expansion and does not change this scalar.

Ordinary integral jets. Let f lie in the integral span of the signed finite translates of this integral vacuum by x with δx integral above p, where $\delta \in \mathcal{O}_K \setminus \{0\}$. Using the preceding expansion, it suffices to treat finitely many translates of F with the same condition. In the standard finite Weil coordinates the image of a rational $\gamma \in E^n$ is $\alpha_{\mathrm{W}}^{-1}\gamma$, where $\mathfrak{a}_0 = E \cap \alpha_{\mathrm{W}} \widehat{\mathcal{O}}_E$. For the fixed shift x, take the intersection of the open translation stabilizer of F, the integral conditions above p, and the kernel of the locally constant finite Heisenberg phase $h \mapsto \beta(x, h)$ of Lemma 17.1. By density choose γ so that $h = \alpha_{\mathrm{W}}^{-1}\gamma - x$ belongs to this open neighborhood. Then $T(x + h)F = T(x)F$ and $\delta \alpha_{\mathrm{W}}^{-1}\gamma$ is integral above p. Only finitely many other primes contribute denominators, so a single integer m prime to p satisfies $m\delta\gamma \in L$ for every term.

For the translate by γ, the factor (i, A) has shift $t_{i,A} = -(\Omega_0/b)\,\mathrm{Tr}_{E/K}(\gamma_A \overline{w_i})$. Choose $k = bm$. Then $k\delta t_{i,A} \in \Lambda$ by trace self-duality. At the argument $\bar{\delta}(a_d/\Omega_0)z$, the translated factor with linear form $(w_i/b)^\dagger z$ is therefore $\theta((w_i/(bk))^\dagger z)^{k^2 N}$ times a series integral in $y((w_i/(bk))^\dagger z)$, with $N = \delta\bar{\delta}$. All terms have exactly the same theta prefactor. Summing preserves the integral factor. These assertions concern the Taylor expansion; the multiplication by k in the argument is the multiplication isogeny from the interpolation argument.

Write $s_{\mathrm{jet}} = p_F a_d/\omega_0$. For each weight embedding $\phi_\tau(a) = \iota_p(\overline{\tau(a)})$, let $\phi_{\tau,f}(\alpha_{\mathrm{W}})$ denote the image of the component of α_{W} at the induced place of E. Then $|\iota_p(p_F)|_p = |\phi_{\tau,f}(\alpha_{\mathrm{W}})|_p^{-1}$. Indeed (AJ) uses $p_F = s_0/u$, with u a unit at every coefficient embedding by the class-cycle argument and $s_0^f = (v(\alpha_{\mathrm{W}})|\alpha_{\mathrm{W}}|^{-1/2})^{-1}$. The algebraic weight-one formula for the finite character gives the product of the absolute values of α_{W} at the f weight embeddings. They are equal since its

ideal is invariant over K. Taking f-th roots gives the stated absolute value of p_F. In particular the fixed jet argument satisfies

$$A_{i,A}(s_{\mathrm{jet}} z) = \frac{p_F}{b} \sum_{\tau \in \Phi} \overline{\tau(w_i)} z_{\tau,A}.$$

Here the real scale a_d and period $\Omega_0 = \omega_0$ cancel before taking the coefficient valuation. Since $w_i/(bk) \in \mathfrak{a}_0 \otimes \mathbb{Z}_p$ and $p \nmid bk$, every coefficient of $\nu = p_F(w_i/(bk))^{\dagger}$ belongs to R.

For completeness, let $g = -\log\theta$, so $g'' = g''(0) + x$, $g'(0) = 0$. For $a = \beta, \bar{\beta}$, evenness and torsion interpolation give $\theta(az)/\theta(z)^{a\bar{a}} = P(x(z))$ with $P \in R[X]$. Since $x(z) = z^2 + O(z^4)$ by $\wp''(t) = 2$, comparison in degree two gives

$$\frac{a(\bar{a} - a)}{2} g''(0) = P'(0) \in R.$$

Subtracting the two values gives $\sqrt{-d}\, g''(0)/2 \in R$. Multiplication by $-2\sqrt{-d}$ gives $dg''(0) \in R$, so $dg'' \in R[[y]]$. For $\mu_i = d\nu_i$, take the ring generated over R by integral series in $y(\mu_i z)$, together with $\theta(\mu_i z)$ and $g'(\mu_i z)$.

Write $Y_i = y(\mu_i z)$ and $X_i = x(\mu_i z) \in R[[Y_i]]$. For $B \in R[[(Y_i)_i]]$, the chain rule gives

$$\partial_j B((Y_i)_i) = \sum_i (\partial_i B)((Y_i)_i)\, \mu_{i,j}(1 + 3X_i^2 - a_1 Y_i).$$

The other derivatives are $\partial_j \theta(\mu_i z) = -\mu_{i,j}\theta(\mu_i z) g'(\mu_i z)$ and $\partial_j g'(\mu_i z) = \nu_{i,j}(dg'')(\mu_i z)$. Hence partial differentiation preserves the ring. Every element of the ring has integral constant term. Applying any polynomial $Q(\partial)$ with coefficients in R and evaluating at zero thus gives an integral value for the actual theta vector at scale $\bar{\delta} d\, s_{\mathrm{jet}}$.

If Q is homogeneous of degree l, rescaling gives the ordinary bound at $c\, s_{\mathrm{jet}}$,

$$\left| Q(\partial)\Theta_f(c\, s_{\mathrm{jet}}\, z)\big|_{z=0} \right|_p \le \left| \frac{c}{\bar{\delta} d} \right|_p^l.$$

In particular, for a multi-index $\mathbf{m}$ with $\mathbf{m}! = \prod_j m_j!$, its Taylor coefficient satisfies

$$\mathbf{m}!\, [z^{\mathbf{m}}]\Theta_f(c\, s_{\mathrm{jet}} z) \;\in\; (c/(\bar{\delta} d))^{|\mathbf{m}|} R$$

at the fixed coefficient embedding. For the original complex Taylor coefficients the same valuation bound holds in $\overline{\mathbb{Q}}_p$, since $R \cap \overline{\mathbb{Q}}_p = \mathcal{O}_{\overline{\mathbb{Q}}_p}$. For the original integral Weyl modules, take a source basis vector v and a target dual basis vector A. The rectangular Schur polynomial $Q_{A,v}(Z) = A\mathcal{S}(Z)v = \sum_{|\alpha|=l} q_\alpha Z^\alpha$ has integral coefficients by the exterior-minor formula. The pairing of Section 18.3 gives

$$AJ_{c\, s_{\mathrm{jet}}}(f)v = \sum_{|\alpha|=l} q_\alpha\, \alpha!\, [z^\alpha]\Theta_f(c\, s_{\mathrm{jet}} z) \;\in\; (c/(\bar{\delta} d))^l R.$$

These are the matrix entries in the original Weyl gauge, also at $p = 2$ and at each $p \mid q_1$.

20.2 Valuations of the derivative coefficients

In this subsection, we use the highest weights to count the columns entering the derivative estimates of Section 20.1. This expresses the local valuation in terms of the prime powers in the arithmetic sum of Section 5.

Fix a coefficient prime $p \neq N$ and an allowed twist ξ with $p \nmid q_\xi$. For each displacement x in the expansion of the evaluated tensor, the translation conditions away from Nq_ξ make $\bar{\pi}_M x$ integral above p; if also $p \nmid M$, then x itself is integral there. At $p = N$ the linked residues have denominators dividing N, and Section 19.4 treats the original gluing.

Use $n_j = k + j(t+1)$, $t = 5r$, $\Delta = 2r^8$, and put $w_j(\tau) = \kappa_\tau + (j + \frac{1}{2})\Delta$. The weight on X_j has determinant exponent zero at $j = 0$ and last entry $\lfloor n_{j-1}/2 \rfloor - w_j$ for $j > 0$. Its columns of height $h < n_j$ have counts

h	number
1	$a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$
k	$w_1(\tau) - \lfloor (k+t+1)/2 \rfloor$
$k + i(t+1),\ i \geq 1$	$\Delta - (t+1)$.

All other counts vanish. The successive blocks of the parameter have size $t+1$, and have centers $-w_i(\tau)$. In each such block the parameter and the highest-weight shift both decrease by one, so the highest weight is constant there. At adjacent block boundaries the parameter centers differ by Δ, while the shifts differ by $t+1$.

For $j \geq 1$, put $l = (t+1)/2$. Subtracting the rank-n_j half-sum of positive roots from the parameter gives the entries on the first two blocks, in zero-based indices,

$$\mu_j(b) = -jl + a_\tau \mathbf{1}_{b=0} \quad (0 \leq b < k), \qquad \mu_j(k+b) = -jl + \lfloor n_1/2 \rfloor - w_1(\tau) \quad (0 \leq b < t+1).$$

The common shift $-jl$ cancels in the difference at height k, giving the table above. It remains in the determinant exponent. For the diagram below n_{j+1}, its first k covectors each include $\kappa_\tau + O(m\Delta)$ boxes from the low columns other than height 1. The remaining covectors have $O(m\Delta)$ such boxes. In particular $2(m+1)\Delta < r^{16}$ gives the additional term on these covectors used in the pointwise size calculation. For part with index zero its additional term $b + a/x_1$ absorbs $\kappa_\tau \leq b$.

For $j < m$ let c_τ, b_τ count the columns and the boxes at heights $1 < h < n_{j+1}$, respectively. The height-k count is between κ_τ and $\kappa_\tau + 2\Delta$, and each subsequent height has at most Δ columns. Thus

$$k\kappa_\tau \leq b_\tau \leq k\kappa_\tau + x^{60}, \qquad c_\tau \leq \kappa_\tau + x^{60}, \tag{CG1}$$

using $2k\Delta + n_m^2\Delta \leq x^{60}$.

Let $p < x_1$ be nonsplit in K, I_p the positive odd exponents if p is inert and all positive exponents if $p = d$. Put $v = 1$ in the inert case and $v = 1/2$ when $p = d$. Write $\tau_e = \langle -p^e \rangle \tau$ and $\gamma_p = v \sum_{e \in I_p} p^{-e}$. For $\mathcal{E} \subset \Phi$ erase the height-one columns at $\mathcal{E}$, setting $u_\tau = a\mathbf{1}_{\Phi_a \setminus \mathcal{E}}(\tau)$. Define $C_p(\mathcal{E})$ as the sum of a/p^e over $\tau, \tau_e \in \Phi_a \setminus \mathcal{E}$, $e \in I_p$. The positive-exponent contribution after erasing is

$$D_{\tau,e} = (u_\tau/p^e - u_{\tau_e} - c_{\tau_e})_+ + (b_\tau/p^e - b_{\tau_e})_+ - c_\tau/p^e.$$

For the full number B_j of boxes, put

$$v \sum_{\tau,\, e \in I_p} D_{\tau,e} + \gamma_p a |\Phi_a \cap \mathcal{E}| - f n_{j+1}^3 (1 + \log_p \max(B_j, 1)).$$

This gives $d_{pj}(\mathcal{E})$ at inert primes. At $p = d$ there is also the exponent-zero subtraction

$$v \sum_{\tau} (\min(u_\tau, u_{\tau_0} + c_{\tau_0}) + c_\tau + \min(b_\tau, b_{\tau_0})), \qquad \tau_0 = \langle -1 \rangle \tau.$$

This specifies the uncharged jet exponent in the ramified case. The inert exponent is derived by the null-direction argument in Section 20.3.

Set $U = a/(pm)$. The saving contribution $(a/p^e - \kappa_{\tau_e})_+$ for $\tau \in \Phi_a$ is at most

$$(u_\tau/p^e - u_{\tau_e} - c_{\tau_e})_+ + x^{60} + ap^{-e}\mathbf{1}_{\mathrm{coll}} + ap^{-e}\mathbf{1}_{\tau \in \mathcal{E}},$$

where coll means that both large parts survive. When $\tau \in \mathcal{E}$ and $e = 1$, the bound still holds with a/x_1 added to the left, since $(a/p - \kappa_{\tau_e})_+ + a/x_1 \le a/p$. When the covector survives and its partner does not, use (CG1) and the Lipschitz bound for the positive part. When both survive the collision term bounds the left side. At $\tau \notin \Phi_a$ there is no long-covector saving. Also (CG1) yields

$$k(\kappa_\tau/p^e - \kappa_{\tau_e})_+ \le (b_\tau/p^e - b_{\tau_e})_+ + x^{60}.$$

For $p^e \ge x_1$ the saving terms vanish because $\kappa_{\tau_e} \ge a/x_1$, so no error is needed in that range.

Summing contributes at most $2x^{60}(1 + \log_2 x_1) \le U$ per place. The negative column term contributes $\gamma_p c_\tau \le 2(b + x^{60})/p \le U$. At the ramified prime τ, τ_0 cannot both be a-places or both be b-places. Thus each minimum in the exponent-zero subtraction is at most $a/(dm)$, using $b_\tau \le ka/x_1 + x^{60} \le a/(dm)$ at a non-b-place. The same bound applies to c_τ, so the subtraction is at most $2U$ per place. Finally $n_{j+1}^3(1 + \log_p \max(B_j, 1)) \le U$. For uniformity in the increasing ranks, use $q_1, f \le x^{12}$, $n_{j+1} \le n_m \le x^6$, $B_j \le f(a + kb + x^{60})$, and $B = \log b$. Then $\log \max(B_j, 1) \ll B + \log x$, while $U = a/(pm) \ge xb/(mx_1)$. Uniformly for $p \le x_1$ and $0 \le j < m$,

$$\frac{x^{60}(1 + \log_2 x_1) + n_{j+1}^3(1 + \log_p \max(B_j, 1))}{U} \ll \frac{mx_1}{xb}\left[x^{60}(1 + \log x_1) + x^{18}(1 + B + \log x)\right] = o(1).$$

Here $x_1 \le b^{1/10^6}$, $m \le 2\sqrt{x}$, and $B = x^\epsilon + O(1)$ for the fixed $\epsilon > 0$. Thus the displayed losses admit one cutoff for all these ranks, primes and indices.

Lemma 20.1 (Local derivative valuation)**.** *Fix $p < x_1$ nonsplit in K, with $p \nmid q_\xi$, and $0 \le j < m$. Let $\mathcal{E} \subseteq \Phi$ index the erased height-one columns of the j-th map. For fixed d, ϵ and sufficiently large x,*

$$d_{pj}(\mathcal{E}) \ge g_p/m - C_p(\mathcal{E}) + v|\Phi_a \cap \mathcal{E}|a/x_1 - faB^2/(mp). \tag{75}$$

Call an inert prime $p < x_1$ exceptional if $p \not\equiv \pm 1 \bmod m$ and $o_0 = \mathrm{ord}_{q_0}(-p)$ is odd with $o_0 \le r_p = \lceil \log_p m \rceil$. For such a prime and a coefficient embedding ι, a full mark is a set Λ containing exactly one prime from each conjugate pair induced by Φ_a via ι and no other primes.

At $p = d$ the multiplier is the class of $-d^e$. Since d has order two modulo q_0 and $d \not\equiv -1$, this class is either -1 or $-d$, and is never 1. Thus it cannot carry an a-place to an a-place. At an inert prime a collision requires an odd e divisible by the order o_0 of $-p \bmod q_0$. For $p \not\equiv \pm 1 \bmod m$ and a nonexceptional prime, such an e is at least $r_p + 1$, where $r_p = \lceil \log_p m \rceil$. For $p \equiv \pm 1 \bmod m$ use $e \ge 1$. The series starting at either lower bound e_0 is at most $2fa/p^{e_0}$. Consequently, at a nonexceptional small nonsplit prime, $mC_p(\varnothing) \le L_p = faB^2 c_p/p$. Indeed the

preceding geometric series gives $mC_p(\varnothing) \le 2fa/p$ when $p \not\equiv \pm 1 \bmod m$ and $mC_p(\varnothing) \le 2mfa/p$ in the two congruence classes. Here $B^2 \ge 2$ for large x.

For an exceptional prime the full mark exists. Let o be the order of $-p \bmod q_1$. It is even because its reduction modulo 2^s is nontrivial and has power-of-two order. Since both $v_2(p \pm 1) < \log_2 m$ and $2^s = m^{22}$, the identity

$$v_2(p^o - 1) = v_2(p - 1) + v_2(p + 1) + v_2(o) - 1$$

gives $m \mid o$. The identity follows by factoring differences of squares after removing the odd part of o. In $\mathrm{Gal}(E/\mathbb{Q}) = \mathrm{Gal}(K/\mathbb{Q}) \times (\mathbb{Z}/q_1)^\times$, write c_K for conjugation on K. The decomposition group is generated by Frobenius (c_K, p), whereas CM conjugation is $(c_K, -1)$. Thus $(1, -p)$ acts on the inducing prime by CM conjugation. If that prime were fixed, $(c_K, -1) = (c_K, p)^j$ for some odd j; then $(-p)^j = 1 \bmod q_1$, contrary to $2 \mid o$. The two induced primes are therefore distinct. For a fixed coefficient embedding let w_τ denote the prime induced by τ. The primes w_σ, w_τ are equal or conjugate precisely when $(1, \sigma\tau^{-1})$ lies in the subgroup generated by (c_K, p) and $(c_K, -1)$. An element of this subgroup fixes K precisely when the two exponents have the same parity. Hence

$$w_\sigma \in \{w_\tau, \bar{w}_\tau\} \quad \Longleftrightarrow \quad \sigma\tau^{-1} \in \langle -p \rangle \subseteq (\mathbb{Z}/q_1)^\times.$$

Even powers of $-p$ fix w_τ, and odd powers give $\bar{w}_\tau$. Multiplication by $(-p)^{o_0}$ preserves Φ_a, so every conjugate pair induced by that set occurs on both sides. Take one member of each pair to get Λ, with $\mathcal{E}$ its inducing set. Then $C_p(\mathcal{E}) = 0$. No b-place induces either half of the mark, because -1 is outside the odd-order subgroup generated by $-p \bmod q_0$. Multiplication by $-p$ interchanges the halves, and $\kappa_\tau = a/x_1$ on both. Thus sums on the two halves agree for any determinant exponent depending only on κ_τ.

For $\tau \in \mathcal{E}$ use $c_\tau \le a/x_1 + x^{60}$. If $p \le m$, then $r_p(a/x_1 + x^{60}) \le a/(pm)$ for large x. If $p > m$, then $r_p = 1 = o_0$. The orbit identity gives $\mathcal{E} \subseteq \Phi_a$, since $-p \equiv 1 \bmod q_0$. Thus $c_\tau \le a/x_1 + a/(pm)$ on $\mathcal{E}$. Thus $r_p \sum_{\tau \in \mathcal{E}} c_\tau$ is at most $|\mathcal{E}|a/(pm) + |\Phi_a \cap \mathcal{E}|a/x_1$. Also $|\Lambda|\nu_p \le L_p - fa/(pm)$ for $\nu_p = fr_p(n_m + r_p)$, with the sharper bound $|\Lambda|\nu_p \le L_p/m - fa/(pm)$ since $a/(pm)$ dominates every fixed power of x. This proves both column comparisons required on the exceptional mark.

At a nonexceptional small nonsplit prime, use the uncharged jet bound $p^{-d_{pj}(\varnothing)}$ from Section 20.3 in the inert case and Section 22.1 in the ramified case. Every translate in the evaluated tensor is standard at that prime, since $p \nmid MNq_\xi$, and the gluing is supported at N. Inequality (75) gives the full-chain gauge $p^{mC_p(\varnothing)+L_p-g_p} \le p^{2L_p-g_p}$. The terminal branching matrix is integral and its scalar has unit norm at $p \ne N$. At a split prime or $p \ge x_1$, except N, the saving is zero. Apply the ordinary jet estimate with $\delta = \bar{\pi}_M$, $c = \pi_M$ and $|c/(\bar{\delta}d)|_p = 1$, since $p \ne d$. In each of these ordinary cases the entire integral ramified source suffices. In particular $-d \equiv 1 \bmod 8$ gives splitting at 2. Although $2 \mid q_1$, the auxiliary Gram scalar in Section 20.1 is odd: it is congruent to $\beta^j + \bar{\beta}^j \equiv 1 \pmod 2$. The finite vacuum average thus has odd denominator. More explicitly, put $n_2 = v_2(q_1)$. A dyadic completion is $U(\zeta_{2^{n_2}})$ with $U/\mathbf{Q}_2$ unramified. The derivative of the Eisenstein polynomial $(1+X)^{2^{n_2-1}} + 1$ at $\zeta_{2^{n_2}} - 1$ generates its different and has valuation $n_2 - 1$ for $v_2(2) = 1$. Thus the two choices of local Weil ideal have valuations $-(n_2 - 1)$ and 0. At either reading the valuation of p_F cancels that of the chosen ideal, as in Section 20.1. For either coefficient reading, $\delta = \bar{\pi}_M$ and $c = \pi_M$ give $c/(\bar{\delta}d) = d^{-1} \in \mathbf{Z}_2^\times$. The clearing matrix is the identity and $e_2 = g_2 = 0$.

20.3 Twisted derivatives at inert primes

In this subsection, we prove the derivative estimate at each inert prime of Section 20.2. At the exceptional primes we then apply the estimate to the Hecke operators of Section 20.3.3.

Fix an inert prime $p < x_1$, a twist with $p \nmid q_\xi$, and a coefficient embedding $\iota : \mathbf{C} \simeq \overline{\mathbf{Q}}_p$. The prime factors of MN exceed x_1 by Section 4, so $p \nmid MNq_\xi$. Let $v(p) = 1$, $\mathcal{O} = \mathcal{O}_{\mathbf{C}_p}$, and let Φ be the chosen CM type. For e odd write $\tau_e = \langle -p^e \rangle \tau$, and put $n_j = k + j(t+1)$. Take any $\mathcal{E} \subseteq \Phi$ and charges satisfying the pairing condition of Section 20.3.2. For $\mathcal{E} = \varnothing$ all charges may be taken to be zero.

The ordinary-coordinate exponential. The following one-variable calculation holds for every inert prime p. Such a prime is odd since 2 splits in K. The restrictions $p < x_1$ and $p \nmid q_\xi$ enter when we apply the result to the translated tensor. By the integral section identities of Section 20.1, the quotient for $\theta(pz)$ is $R(x)$, $R(0) = 1$, with integral coefficients and $2 \deg R < p^2$, and $g = -\log \theta$ satisfies $g'' = g''(0) + x$, $g(0) = g'(0) = 0$. The order bound on the reduced multiplication-by-p series there also applies after the parameter change below.

Set $q = p^2$. Average $y(\zeta z)$ with weight ζ^{-1} and $g(\zeta z)$ with weight ζ^{-p-1} over $\zeta^{q-1} = 1$, obtaining T_0 and $g^\flat$, respectively. The CM pullback formulas extend coefficientwise to the Teichmüller units, for which $\bar{\zeta} = \zeta^p$. Applying the formula at ζ^{-1} and substituting ζz gives

$$\exp(\zeta^{-p-1} g(\zeta z) - g(z)) \in \mathcal{O}[[y]].$$

Multiplying over ζ and taking the unique constant-one $(q-1)$-st root gives $\exp(g^\flat - g) \in \mathcal{O}[[y]]$.

Also $T_0 = y + O(y^2)$, so the two parameters give the same integral series ring. The covariance of T_0 gives

$$T_0(pz) = pT_0(z) + b_1 T_0(z)^q + \sum_{j \geq 2} b_j T_0(z)^{1+j(q-1)}, \qquad b_1 \in \mathcal{O}^\times.$$

The order of the reduced parameter under $[p]$ is at most the degree p^2 of the isogeny. Thus b_1 is a unit. The permitted degrees above the vanishing linear term start at $q = p^2$. Choose $\varpi^{q-1} = -p/b_1$ and set $Y = T_0(\varpi z)/\varpi$. The resulting equation is $Y(pz)/p = Y - Y^q + pE(Y)$, with $E \in U^2\mathcal{O}[[U]]$. In each degree $d \geq 2$, division by $1 - p^{d-1}$ determines an integral coefficient from earlier coefficients. Modulo p, the equation becomes $Y - Y^q = z$, whence

$$Y \equiv \sum_{u \geq 0} z^{q^u} \pmod{p}.$$

For clarity, the logarithm calculation giving the nonintegral exponential is short. Since $T_0 = z + O(z^q)$, integral series in y have integral coefficients in degrees below q. For $3 \leq d < q$, $d(d-1)g_d = [z^{d-2}]x$ gives $pg_d \in \mathcal{O}$. In degree two the CM pullback formula gives $a(\bar{a} - a)g_2 \in \mathcal{O}$. Subtract its instances at the conjugate integers $a = (1 + \sqrt{-d_K})/2$ and $\bar{a}$; since $p \nmid d_K$, this gives $g_2 \in \mathcal{O}$. Thus $qg - g(pz)$ lies in $p\mathcal{O}[[z]]$ through degree $q-1$. Its exponential does also after subtracting the constant, since $v(n!) \leq (n-1)/(p-1)$ for $n \geq 1$. Comparing coefficients with $R(x(z))$, using $x = z^2 + \cdots$ and $2 \deg R < q$, gives $R - 1 \in p\mathcal{O}[X]$. Its formal logarithm shows $qg - g(pz) \in p\mathcal{O}[[T_0]]$. Project this equation to degrees congruent to $p+1$ modulo $q-1$. The same covariance in T_0 gives

$$qg^\flat(z) - g^\flat(pz) = p(a_0 T_0^{p+1} + T_0^{p+q} H(T_0)), \qquad a_0 \in \mathcal{O}, \quad H \in \mathcal{O}[[U]].$$

20.3.1 Integral rescaling of the local exponential

In this subsection, we normalize the exponential in the local coordinates of Section 20.3.

Set $c_\varpi = \varpi^{p+1}/p$ and $\Xi = -g^\flat(\varpi z)/c_\varpi$. After rescaling, the leading scalar $-p\varpi^{p+1}/(qc_\varpi)$ is a unit, and the higher terms have the additional factor $\varpi^{q-1} \in p\mathcal{O}$. The coefficient of Ξ in degree d is multiplied by $1-p^d/q$, a unit congruent to one for $d \geq p+1$, while all lower degrees are zero. Thus Ξ is integral and

$$\theta(\varpi z) = W(z)\exp(c_\varpi \Xi(z)), \qquad W \in \mathcal{O}[[z]],$$
$$\Xi \equiv \lambda Y^{p+1} \equiv \sum_{e \text{ odd}, u \geq 0} \lambda (z^{p^e} z)^{p^u} \pmod{p}.$$

The last equality follows by multiplying the even-index Frobenius series for Y by the odd-index series for Y^p. The degree of a pair with minimum u and positive odd difference e is $l = p^u(p^e+1)$. Conversely $u = v_p(l)$ and $p^e = l/p^u - 1$ recover the two indices. Thus each displayed degree occurs once.

For the lower estimate we also need the order of its leading coefficient. Write $h_j = [z^{p-1}]x(z)^j$. Since $g'' = g''(0) + x$ and the projection $g^\flat$ retains degree $p+1$,

$$[z^{p+1}]\Xi = -p[z^{p+1}]g = -\frac{h_1}{p+1}. \tag{76}$$

We claim $h_1 \in \mathcal{O}^\times$. The Weierstrass equation gives

$$(x^j x')' = (4j+6)x^{j+2} + (j+1)a_1^2 x^{j+1} + (4j+2)x^j.$$

The coefficient of z^{p-1} on the left belongs to $p\mathcal{O}$: it is $p[z^p](x^j x')$, and x, $x' = 2y + a_1 x$ have integral coefficients below degree p^2 by the parameter calculation above. Thus the three-term relation for the h_j vanishes in the residue field. We have $h_0 = 0$ and $h_{(p-1)/2} = 1$ since $x = z^2 + O(z^3)$. For $0 \leq j \leq (p-5)/2$ the factor $4j+6$ is prime to p. If h_1 vanished in the residue field, the relation would force $h_{(p-1)/2}$ to vanish. For $p = 3$, $h_1 = 1$ directly. This proves the claim. Since $Y = z + O(z^2)$, comparison in degree $p+1$ gives $\lambda \in \mathcal{O}^\times$.

For the theta series of standard level-one data on the matrix space, the standard-vacuum resolution writes the series at normalized scale a_d/Ω_0 as a product of $\theta(u_i^* z)^{m_i}$ times an integral series in $y(u_i^* z)$, with u_i integral above p and $\sum_i m_i u_i u_i^* = 1$. Indeed, put $b = q_1$ and repeat each power of ζ_b b times. Fourier orthogonality gives the resulting vectors w_j Gram sum $b^2 I$. For the finite sum of translates take a common k, prime to p, clearing every torsion shift as in Section 20.1. In matrix coordinate A take

$$u_{j,A} = \frac{w_j}{bk} e_A, \qquad m_{j,A} = k^2, \qquad \sum_{j,A} k^2 u_{j,A} u_{j,A}^* = I$$

in the coordinates of the CM type. The finite Fourier denominator is a power of b, with $p \nmid b$. Each translate therefore has the same theta product, and the remaining integral series can be summed before rescaling.

Since $p \nmid dq_1$, every completion of E above p is unramified with uniformizer p. On K the odd Frobenius cancels conjugation, and on the cyclotomic part their composite acts by $-p^e$. Thus for each odd e and each u integral above p, $\overline{\tau(u)}^{p^e} \equiv \tau_e(u) \pmod{p\mathcal{O}}$ at the coefficient embedding. Reduction of the basis identity therefore gives

$$\sum_i m_i (u_i^* z)^{p^e} (u_i^* z) \equiv \Gamma_e(z) := \sum_{\tau,b,a} z_{\tau,b,a}^{p^e} z_{\tau_e,b,a} \pmod{p}.$$

Expanding multiplicities as repetitions gives the same assertion for the p^u-th powers in the exponential. The correction exponential is integral because $v(pc_\varpi) = 1/(p-1)$. In each linear form $A_{i,A}(c\, s_{\mathrm{jet}} z)$ the factors a_d and Ω_0 cancel, as in Section 20.1. All primes dividing q_1 split in K, so here $p \nmid q_1$ and $|\iota(p_F)|_p = 1$. Also $c = \pi_M$ has norm M prime to p, so the remaining scalar $\iota(c\, p_F)$ is a unit.

Writing $s = \iota(c\, p_F)$, the coefficient of $\Gamma_e^{p^u}$ after rescaling can be taken to be $l_{eu} = \lambda s^{(p^e+1)p^u}$. In particular l_{e0} is a unit for every odd $e > 0$ at the unit scale here. Thus the actual Taylor series G, in the ordinary matrix coordinates of the arrow, satisfies

$$G(\varpi z) = W_0(z) \exp\left(c_\varpi \sum_{e \text{ odd} \geq 1,\, u \geq 0} l_{eu} \Gamma_e(z)^{p^u} \right), \qquad W_0 \in \mathcal{O}[[z]], \quad l_{eu} \in \mathcal{O}. \tag{77}$$

For the evaluated tensor, the translations may be represented with support on MNq_ξ. Integral translations at the places above p fix the signed vacuum, since $p \nmid MNq_\xi$. Expanding the gluing adds translations supported at N, with the same coefficients after regrouping. Every arrow in this expansion is therefore standard at p, as used in (77).

20.3.2 Valuations of the jets

In this subsection, we bound the Schur coefficients of the expansion in Section 20.3.1.

Let $n = n_{j+1}$, $\kappa_{\tau,h}$ be the number of columns of height $h < n$, and B_j the total number of their boxes. Erase height-one columns at $\mathcal{E}$, writing B'_j for the remaining number of boxes. Let t_τ, c_τ, b_τ count the remaining height-one columns, the other columns, and the boxes of those other columns, respectively. For each relevant τ, e define

$$E_{\tau,e} = \min(t_\tau/p^e, t_{\tau_e} + c_{\tau_e}) + c_\tau/p^e + \min(b_\tau/p^e, b_{\tau_e}).$$

Consider directions $y_{\tau,a,k} \in \mathcal{O}$ and a fixed target basis I'. Give a covector b of height h at τ a charge $w_{\tau,h,b} \in \mathbf{Z}_{\geq 0}$. Require that one factor has positive charge for an equal-covector pairing with heights $(1, h)$ at (τ, τ_e) if $\tau \in \mathcal{E}$ or if $h = 1$ and $\tau_e \in \mathcal{E}$. Let $C(I')$ sum the charges over its boxes.

Use the heights with $\kappa_{\tau,h} > 0$ in the original diagram, including the erased height-one columns. Variables at the other heights do not occur in the minor product. For each occurring (τ, h) change the first h directions by a matrix $U_{\tau,h}$ over $\mathcal{O}$. For the odd e with $p^e \leq B_j$, the changes make all Frobenius pairings $\sum_a \beta_a^{p^e} \beta'_a$ divisible by p except the charged pairs. The contribution in determinant valuation, with each height weighted by its column count, is at most $\sum_{\tau,e:p^e \leq B_j} E_{\tau,e}$. For fixed (τ, e) use the following constraints.

1. For an unerased height one at the source, either multiply that direction by a p^e-th root of p, or impose at every nonexempt height on the partner side one linear congruence against the original source height-one direction. Choose the cheaper option.

2. For source height above one against partner height one, impose the linear congruence $\sum_a \sqrt[p^e]{y_{\tau_e,a,0}}\, \beta_a = 0 \pmod{p^{1/p^e}}$ on the source. Taking p^e-th powers is additive modulo p. Use the original partner direction even if its place is erased, since this pair has source height above one.

3. For heights above one on both sides, make all directions on the cheaper side divisible by p^{1/p^e} on the source or by p on the partner.

Each linear constraint contributes at most the valuation of its modulus. Choose a coefficient of least valuation and clear the other components by integral subtraction before multiplying the pivot direction by the modulus. Collect the forms in the original directions and impose them successively on the changed directions. All earlier constraints persist under integral operations on covectors. A changed height-one direction is a scalar multiple of its original direction, so the constraints give the required pairings after all changes. The three contributions are the three summands of $E_{\tau,e}$.

In (77) substitute

$$z_{\tau,b,a} = \sum_{\substack{h<n,\,\kappa_{\tau,h}>0\\ k<h}} p^{w_{\tau,h,b}}\beta_{\tau,h,k,a}s_{\tau,h,k,b}.$$

Only equal first indices pair in each coupling. All pairings are zero modulo p by the changes of direction or by a positive charge. Terms with $p^e > B_j$ have degree above B_j. Consequently the substituted series has integral coefficients through degree B_j. There are at most fn^3 variables s. For a multi-index α of total degree B_j, the factorial estimate is

$$v(\alpha!) \geq \frac{B_j}{p-1} - fn^3(1+\log_p \max(B_j,1)).$$

Apply the product of column determinants in the differential variables ∂_s indexed by I'. On a single determinant the chain rule replaces its derivative matrix by the original matrix times $U_{\tau,h}^{\mathsf{T}}$, with covector factors $\varpi p^{w_{\tau,h,b}}$. Thus the derivative is the original minor-product jet J_y times $\varpi^{B_j}p^{C(I')}\prod_{\tau,h}\det(U_{\tau,h})^{\kappa_{\tau,h}}$.

Using the factorial estimate and the determinant contribution gives

$$v(J_y)+C(I') \geq \gamma_p B_j - \sum_{\tau,e:p^e\leq B_j} E_{\tau,e} - fn^3(1+\log_p\max(B_j,1)),$$
$$\gamma_p = \frac{1}{p-1}-\frac{1}{p^2-1}.$$

If $B_j = 0$ use the integral constant term. Otherwise enlarge the nonnegative deduction sum to all positive odd e. Since $\gamma_p = \sum_{e\ \mathrm{odd}\geq 1} p^{-e}$, the resulting bound is

$$v(J_y)+C(I') \geq d_{pj}(\mathcal{E}),$$

Indeed $B'_j = \sum_\tau (t_\tau + b_\tau)$ and $B_j - B'_j = a|\Phi_a \cap \mathcal{E}|$. For the contribution $D_{\tau,e}$ of Section 20.2,

$$\frac{t_\tau+b_\tau}{p^e} - E_{\tau,e} = (t_\tau/p^e - t_{\tau_e} - c_{\tau_e})_+ + (b_\tau/p^e - b_{\tau_e})_+ - c_\tau/p^e = D_{\tau,e}.$$

Summation over odd $e \geq 1$ gives $\gamma_p B_j - \sum_{\tau,e} E_{\tau,e} = \gamma_p a|\Phi_a\cap\mathcal{E}| + \sum_{\tau,e} D_{\tau,e}$. The series converge absolutely.

The same bound applies to each Schur jet entry $J_{I',I}$. Indeed the product of the flag minors of $z_\tau y_\tau$ expands as $\sum_I A_I(y)S_{I',I}(z)$, where S is the matrix of the Weyl map and the A_I are the source flag-minor basis multiplied, at each τ, by $\det(y_{\tau,<n_j})^{\kappa_{\tau,n_j}}$, using the first n_j direction columns of y_τ. For the diagram the only possible heights other than one are $k+h(t+1)$, so there are no heights strictly between n_j and n_{j+1}. The A_I are independent after reduction.

Order the variables by the direction column and then by increasing basis index, putting the earlier variable first in lexicographic order. Every flag minor has leading coefficient one on its

diagonal monomial. In their product the variables of direction k give the multiset of entries in tableau row k; its weak order recovers that row. Distinct tableaux therefore give distinct leading monomials in every residue characteristic. The common determinant factor is nonzero after reduction, so multiplication by it preserves independence.

If all $J_{I',I}$ vanish there is nothing to bound. Otherwise choose I_0 maximizing $|J_{I',I_0}|_p$. The polynomial

$$J_{I',I_0}^{-1} \sum_I A_I(y) J_{I',I}$$

has integral coefficients and nonzero reduction by this independence. The residue field of $\mathbf{C}_p$ is infinite, so some integral y gives it a unit value. Then $|J_y|_p = \max_I |J_{I',I}|_p$. The bound for every J_y gives

$$v(J_{I',I}) + C(I') \geq d_{pj}(\mathcal{E}). \tag{78}$$

20.3.3 The resulting Hecke operators

In this subsection, we insert the valuations of Section 20.3.2 into the integral Hecke action.

Now suppose p is exceptional as in Section 20.2, and take a full mark Λ for ι. Set $\mathcal{E} = \{\tau : \mathrm{ind}_\iota(\tau) \in \Lambda\}$. Odd partners induce conjugate primes, so no two members of $\Phi_a \setminus \mathcal{E}$ are odd partners. At each $\lambda \in \Lambda$ an allowed target neighbor has elementary-divisor matrix $\mathrm{diag}(pI_i, I_{n_{j+1}-i})$ with $0 \leq i \leq r_p$, between two integral invertible matrices. Its lattice index is q_λ^i. Write J_λ for the i positions with divisor p. The integral factors preserve the marked gauge. With the unitary extension at $\bar{\lambda}$, the inverse-gauge exponent of a box with covector in J_λ is $+1$ on the inducing side and -1 on the conjugate-inducing side. For each $\lambda \in \Lambda$, use the charges

$$w_{\tau,h,b} = \begin{cases} \mathbf{1}_{h=1}\mathbf{1}_{b \notin J_\lambda}, & \mathrm{ind}_\iota(\tau) = \lambda, \\ \mathbf{1}_{b \in J_\lambda}, & \mathrm{ind}_\iota(\tau) = \bar{\lambda}. \end{cases}$$

At all remaining embeddings put $w_{\tau,h,b} = 0$. An odd partner sees the same J_λ on the opposite side. In an exempt pair the height on the inducing side is one: $b \notin J_\lambda$ charges that side, and $b \in J_\lambda$ charges the conjugate side. This verifies the charge condition for every neighbor. The determinant weights cancel, since their sums on the two sets of embeddings inducing λ and $\bar{\lambda}$ agree. The sum of the box exponent and the charge is zero on the conjugate side, is one in each height-one box on the inducing side, and contributes at most r_p per other column there by strict increase down each column. If

$$A_\lambda = a\,|\{\tau \in \Phi_a : \mathrm{ind}_\iota\,\tau = \lambda\}|, \qquad L_j(\mathcal{E}) = \sum_{\tau \in \mathcal{E}} \sum_{h \neq 1,\, h < n_{j+1}} \kappa_{\tau,h},$$

then (78) bounds the neighbor arrow by the gauge multiplier

$$p^{\sum_{\lambda \in \Lambda} A_\lambda + r_p L_j(\mathcal{E}) - d_{pj}(\mathcal{E})}.$$

To see this at arbitrary source and target class representatives g, h, write the target neighbor as uDv, with u, v integral and invertible above p and D the preceding unitary diagonal. Let K, K' be the integral compact groups of the source and target at the places above p. By weak approximation in the rational Cayley charts, there are rational unitary elements γ, η of the respective groups with

$$G = \gamma g \in K, \qquad H = \eta h u D \in K' \cap D^{-1} K' D \qquad \text{at all places above } p.$$

Both conditions prescribe nonempty open sets. Rational equivariance then transports the jet to the datum $\omega(G, Hv)f$, which is again a standard datum at p. Indeed the finite vacuum sum in Section 7.4.3 has index prime to p for a change integral and invertible above p. Its multiplier and the splitting characters are units there, and the translated vectors retain their integrality, as in Section 44.4. At the target,

$$\eta h = D^{-1}(DHD^{-1})u^{-1}.$$

The two right factors preserve the integral Weyl lattice, as does G at the source. Thus the target gauge uses the inverse Weyl action at D^{-1}, giving exactly the diagonal box weights above; the source gauge is the gauge at 1. The ultrametric matrix norm is bounded by the largest weighted entry. This proves the same inequality at every pair of representatives. Taking $D = u = v = 1$ gives the corresponding passage for the uncharged jets.

Fix a uniformizer ϖ_λ. Normalize Haar measure on $\mathrm{GL}_d(E_\lambda)$ by $\mathrm{vol}(\mathrm{GL}_d(\mathcal{O}_\lambda)) = 1$. For $d = n_j$, let $T_{j,\lambda,i}$ denote the double-coset operator of $\mathrm{diag}(\varpi_\lambda I_i, I_{d-i})$, $0 \le i \le d$. Write $\chi_{j,\lambda}$ for the local base change of the norm-one character of the lift $X_j \to X_{j+1}$, obtained by composition with $u \mapsto u/\bar{u}$. With $q_\lambda = z_\lambda^2 = p^{2h_\lambda}$ and $2h_\lambda \le f$, put

$$T_{j,\lambda}(U) = \sum_i (-U)^i q_\lambda^{\binom{i}{2}} T_{j,\lambda,i}.$$

The identity is

$$T_{j+1,\lambda}(U)\Theta_{x,j} = R_{j,\lambda}(U)\Theta_{x,j}T_{j,\lambda}(z_\lambda^{t+1}U),$$
$$R_{j,\lambda}(U) = \prod_{k=0}^{t}(1 - q_\lambda^k z_\lambda^{n_j}\chi_{j,\lambda}(\varpi_\lambda)U).$$

In the split type-II correspondence the retained Langlands factors are the contragredients of the smaller factors [50, Corollary 6.3(2)]. For the operator identity on our kernel we compute both double-coset sums. At this split pair the level and twist are unramified, and the allowed translations preserve the integral matrix lattice. Thus the local factor is the indicator of $M_{n',n}(\mathcal{O}_\lambda)$, where $n = n_j$ and $n' = n+t+1$. Put $q = q_\lambda$, $z = z_\lambda$, and $c = \chi_{j,\lambda}(\varpi_\lambda)$. In the split Schrödinger model the determinant change of variables contributes $q^{ni/2}$ on the target and $q^{n'i/2}$ on the inverse source double coset. Write ν for the arrow character, so $\nu^{\mathrm{bc}}(a) = \nu(a/\bar{a})$. In either rank $d = n, n'$ the splitting contributes

$$(\nu\upsilon_1^{d/2})^{\mathrm{bc}}\upsilon^{-d} = \nu^{\mathrm{bc}}.$$

Indeed both ranks are even and $\upsilon_1^{\mathrm{bc}}(a) = \upsilon(a/\bar{a}) = \upsilon(a)^2$ because $\upsilon(a\bar{a}) = 1$ (Section 4.3). Thus the character factor on both cosets is c^i. Explicitly the action is

$$(\omega(g,h)\phi)(Z) = \chi_{j,\lambda}(\det h/\det g)|\det h|_\lambda^{-n/2}|\det g|_\lambda^{n'/2}\,\phi(h^{-1}Zg).$$

The positive target and inverse source cosets therefore have scalars $\alpha^i = (z^n c)^i$ and $\beta^i = (z^{n'}c)^i$.

Write $K_d = \mathrm{GL}_d(\mathcal{O}_\lambda)$ and $D_i^{(d)} = K_d\,\mathrm{diag}(\varpi_\lambda I_i, I_{d-i})K_d$. For each residue i-plane V, choose $\gamma_V \in D_i^{(d)}$ with $\ker\bar{\gamma}_V = V$. The lattice $\gamma_V^{-1}\mathcal{O}_\lambda^d$ determines the coset $K_d\gamma_V$, and reduction of its product by ϖ_λ gives V. Thus

$$D_i^{(d)} = \bigsqcup_V \gamma_V^{\mathsf{t}} K_d, \qquad (D_i^{(d)})^{-1} = \bigsqcup_V \gamma_V^{-1} K_d,$$

with each right coset of measure one. The condition that $Z\gamma_V^{-1}$ is integral is equivalent to the integrality of Z and $V \subseteq \ker \bar{Z}$. Transpose this test for the target cosets. Hence at an integral matrix Z of residue rank s, the target and inverse source counts are the numbers of i-planes in $\ker \bar{Z}^{\mathsf{t}}$ and $\ker \bar{Z}$, respectively. Outside the integral lattice both sums vanish. The finite q-binomial theorem thus gives the polynomials $P_{n'-s}(\alpha U)$ and $P_{n-s}(\beta U)$, where $P_b(V) = \prod_{k=0}^{b-1}(1 - q^k V)$. Since $q^{t+1}\alpha/\beta = z^{t+1}$,

$$P_{n'-s}(\alpha U) = P_{t+1}(\alpha U)\, P_{n-s}(\beta z^{t+1} U).$$

Write $B(\phi, v)$ for the theta lift with datum ϕ and source v, and ρ_d for right translation. Change of variable gives

$$B(\omega(g,1)\phi, v) = B(\phi, \rho_n(g^{-1})v), \qquad B(\omega(1,h)\phi, v) = \rho_{n'}(h)B(\phi, v).$$

Inversion carries the Haar measure on $(D_i^{(n)})^{-1}$ to that on $D_i^{(n)}$ by unimodularity. These identities give the displayed kernel comparison on source forms fixed by $\mathrm{GL}_{n_j}(\mathcal{O}_\lambda)$. On such forms the target is fixed by $\mathrm{GL}_{n_{j+1}}(\mathcal{O}_\lambda)$, since this group fixes the local matrix-lattice indicator and its unramified determinant factor is trivial on units. The initial level is hyperspecial at every $\lambda \in \Lambda$, so the identity applies successively at every stage. To check its coefficient ring, let y be the finite idele equal to ϖ_λ at λ and to one elsewhere. Choose $h > 0$ with $y^h = au$, where $a \in E^\times$ and u is a ray unit annihilated by the base-change character. Automorphy and the infinity weight $-w_{j+1}(\tau)$ express c^h as $\prod_{\tau\in\Phi} \tau(a/\bar{a})^{w_{j+1}(\tau)}$. Since p is unramified and $v_p(p) = 1$, division by h gives

$$v_p(\iota(c)) = \sum_{\mathrm{ind}_\iota(\tau)=\lambda} w_{j+1}(\tau) - \sum_{\mathrm{ind}_\iota(\tau)=\bar{\lambda}} w_{j+1}(\tau) = 0.$$

Indeed $\tau \mapsto \langle -p\rangle\tau$ exchanges the two sets and preserves $\kappa_\tau = a/x_1$ there (Section 20.2). Hence c is a unit. The polynomial is independent of the translate and has constant term one. For the original composite $C = C_{x,j+1} = \Theta_{x,j} \cdots \Theta_{x,0}$, substitution gives

$$\left(\prod_{a=0}^{j} R_{a,\lambda}(z_\lambda^{(j-a)(t+1)} U) \right)^{-1} T_{j+1,\lambda}(U) C = C T_{0,\lambda}(z_\lambda^{(j+1)(t+1)} U).$$

Each factor has constant term one and coefficients in the valuation ring at ι, since q_λ, z_λ are integers and the corresponding local lift character is a unit there. The reciprocal therefore has coefficients in the same ring. Up to sign, the coefficient of U^{r_p} on the right is $p^{u_j} C T_{0,\lambda,r_p}$, with

$$u_j = h_\lambda r_p(r_p - 1 + (j+1)(t+1)) \le f r_p(n_m + r_p) = \nu_p.$$

Here $r_p = \lceil \log_p m \rceil \le m \le k = n_0$, $2h_\lambda \le f$ and $j < m$. Multiplying the coefficient identity by $(-1)^{r_p} p^{\nu_p - u_j}$ gives integral coefficients $c_{j,\lambda,i}$ such that

$$\sum_{i \le r_p} c_{j,\lambda,i} T_{j+1,\lambda,i} C_{x,j+1} = p^{\nu_p} C_{x,j+1} T_{0,\lambda,r_p}, \qquad \nu_p = f r_p(n_m + r_p).$$

The coefficients can depend on the unramified local lift character. The exponent ν_p and the source operators are uniform in the allowed twists and translates.

Put $H_\lambda = p^{A_\lambda} T_{0,\lambda,r_p}$ and $Y_\Lambda = \prod_{\lambda\in\Lambda} H_\lambda$. Move the outgoing Hecke sums into the data by Weil equivariance, using the operators $p^{A_\lambda-\nu_p}\sum_i c_{j,\lambda,i}T_{j+1,\lambda,i}$ on the target side of each datum. The original datum is spherical at every distinguished place, so these operations preserve its original level equivariance. For different λ the operations commute. The resulting arrow applied to $C_{x,j}$ is $C_{x,j+1}Y_\Lambda$, so the modified full chain gives exactly the original chain on Y_Λ^m. The factors $p^{A_\lambda-\nu_p}$ cancel the preceding $\sum A_\lambda$ in the gauge exponent. Taking the maximum over the integral Hecke coefficients gives the per-arrow exponent

$$|\Lambda|\nu_p + r_p L_j(\mathcal{E}) - d_{pj}(\mathcal{E}).$$

The marked column comparison in Section 20.2 gives the following two bounds with $L_p = faB^2/p$ and $e_p = 2L_p$. At these primes $p \not\equiv \pm 1 \bmod m$, so $c_p = 1$ in the definition of L_p.

$$\begin{aligned} d_{pj}(\mathcal{E}) &\geq g_p/m + |\Phi_a \cap \mathcal{E}|a/x_1 - L_p/m, \\ |\Lambda|\nu_p + r_p L_j(\mathcal{E}) &\leq L_p/m + |\Phi_a \cap \mathcal{E}|a/x_1. \end{aligned}$$

The first uses zero collisions on the unerased large parts. Together they give an exponent at most $(2L_p - g_p)/m$. The composite therefore has gauge at most $p^{2L_p-g_p}$ on $Y_\Lambda^m M_\iota$, where M_ι is the original integral source. With no erased columns, the same comparison gives $p^{mC_p+L_p-g_p}$ on M_ι, where $C_p \leq 2fa/p$ by the sum of a/p^e over the colliding large parts. Thus on $p^{j_p}M_\iota$, with $j_p = \lceil 2mfa/p \rceil$, the exponent is at most $L_p - g_p$. Ultrametricity gives $p^{2L_p-g_p}$ on the sum. The terminal restriction uses the original integral Weyl matrix and the original scalar, which is a unit here because $p \neq N$.

Proposition 21.1 gives $H_\lambda M_\iota \subseteq M_\iota$ for the scaled operator $H_\lambda = p^{A_\lambda}T_{0,\lambda,r_p}$ and

$$\operatorname{length}_p\Big(M_\iota/(H_\lambda M_\iota + p^{j_p}M_\iota)\Big) \leq d_X\left(\frac{12A_\lambda}{mp} + \frac{j_p}{m^2 f}\right).$$

Choose a finite extension $K_p/\mathbb{Q}_p$ containing the class-coordinate entries of every H_λ and of the operators defining S. Compute lengths over its valuation ring and divide by the absolute ramification index, so that $v_p(p) = 1$. These normalized lengths are unchanged by extension, by the Smith elementary divisors. Put $R = (\mathcal{O}_{K_p}/p^{j_p})^{d_X}$. For the induced endomorphisms A, B the exact sequence

$$0 \longrightarrow AR/ABR \longrightarrow R/ABR \longrightarrow R/AR \longrightarrow 0$$

has first term a quotient of R/BR, via A. Applying length additivity m times to the product over Λ bounds $\operatorname{length}_p(M_\iota/(Y_\Lambda^m M_\iota + p^{j_p}M_\iota))$ by

$$d_X m \sum_{\lambda\in\Lambda}\left(\frac{12A_\lambda}{mp} + \frac{j_p}{m^2 f}\right) \leq d_X(14fa/p + 1/m) \leq d_X L_p.$$

Here $\sum A_\lambda \leq af$, $|\Lambda| \leq f$, and the last inequality is the parameter inequality $14fa + p/m \leq faB^2$.

Put $J = Y_\Lambda^m M_\iota + p^{j_p}M_\iota$. The equations defining the ramified source S in characteristic zero are

$$\sum_{j=0}^{N-2} R_H(z_w)^j \phi = (N-1)A_w\phi = 0 \qquad (w \mid \pi_N)$$

by Section 6. Over the same K_p, after clearing denominators in the equations, write $S_{K_p} \subseteq K_p^{d_X}$ for their common kernel, $M_{K_p} = \mathcal{O}_{K_p}^{d_X}$, and $J_{K_p} = Y_\Lambda^m M_{K_p} + p^{j_p} M_{K_p}$. The inclusion

$$(M_{K_p} \cap S_{K_p})/(J_{K_p} \cap S_{K_p}) \hookrightarrow M_{K_p}/J_{K_p}$$

bounds the length of its source. Flat extension to R_ι preserves the kernels and intersections. Dividing the lengths over $\mathcal{O}_{K_p}$ by its absolute ramification index gives the normalized colengths over R_ι, including when $p \mid N-1$. The tuple clearing is given explicitly in Section 22.3. The resulting module is fixed after the chosen arithmetic datum and (p, ι) and before any twist or translate. The bounds apply to every twist of level prime to p, and to all evaluated translates. The mark and the module may depend on the coefficient embedding. The loss and clearing bounds depend on p alone.

21 Hecke operators and colength at exceptional primes

In this section, we construct the source Hecke operators that supply the divisibilities in Section 20.3. The local choices of Section 20.2 ensure that the colength bounds hold on the same source lattice for all permitted twists.

Keep the exceptional prime p, its coefficient embedding and the full mark Λ of Section 20.2. The marked pairs are split over the real subfield. Furthermore, $p \nmid dq_1$: the prime d is ramified in K, while each divisor of q_1 splits in K. Hence p is unramified in E. Since $p < x_1 < \min(p_0, p_1, N)$, the source level is hyperspecial at every $\lambda \in \Lambda$. On the rank-k Hermitian space X_0, let X be the full space of algebraic automorphic forms at the original level J_H of Section 6. Put $R_\iota = \{z \in \overline{\mathbb{Q}}_p : v_p(z) \geq 0\}$, with $v_p(p) = 1$. Via the given coefficient embedding ι, let $M = M_\iota$ be the lattice of forms of gauge at most one. We may compute the local matrices over $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$; the last determinant calculation identifies their colength on M. Set $d_X = \dim_{\mathbb{C}} X$, $r = r_p = \lceil \log_p m \rceil$, and $A_\lambda = a\,\#\{\tau \in \Phi_a : \mathrm{ind}(\tau) = \lambda\}$. The operator in question is $Y_\lambda = H_\lambda = p^{A_\lambda} T_{0,\lambda,r}$ of Section 20.3.3, with the unnormalized split double-coset convention.

The Frobenius orbit of embeddings. Choose $\tau \in \Phi_a$ inducing the already marked λ. Put $u = \langle -p \rangle$, $e = \mathrm{ord}_{q_0}(-p)$, and write $N_{\mathrm{orb}} = \mathrm{ord}(u^e)$ for the orbit length in this calculation. The infinite places inducing λ or $\bar{\lambda}$ are the u-orbit of τ, by decomposition at the inert prime of the imaginary quadratic field. The defining q_0-condition of Φ_a is preserved by u^j exactly when $e \mid j$. Thus

$$t(s) = u^{es}\tau, \qquad s \in \mathbb{Z}/N_{\mathrm{orb}}\mathbb{Z}$$

lists precisely the original Φ_a-places above $\lambda, \bar{\lambda}$, each once. The integer e is odd, and the step $u^e = \langle -p^e \rangle$ exchanges the two primes. With $t(0) = \tau$, the even places induce λ. Hence N_{orb} is even and $A_\lambda = aN_{\mathrm{orb}}/2$.

Let W be the valuation ring at λ, and put $P = p^e$. At an even embedding use the embedding of $t(s)$, and at an odd embedding compose that embedding with complex conjugation on the field. These embeddings ι_s all induce λ. On the field the step $\langle -p^e \rangle$ followed by conjugation is the e-th power of arithmetic Frobenius at p. Indeed in $E = K\mathbb{Q}(\zeta_{q_1})$ the step fixes K and takes ζ_{q_1} to $\zeta_{q_1}^{-p^e}$.

Conjugation changes the cyclotomic sign and acts nontrivially on K, where p is inert and e is odd. Since p is unramified, this congruence on the integers extends to W by inverting

elements outside λ. Consequently

$$\iota_{s+1}(x) \equiv \iota_s(x)^P \pmod{p\mathcal{O}}, \qquad q = |W/\lambda W| = p^{eN_{\mathrm{orb}}} = P^{N_{\mathrm{orb}}}.$$

Indeed arithmetic Frobenius is u times conjugation, and conjugation has order two and is independent of $\mathrm{Gal}(E/K)$. The residue degree is $\mathrm{lcm}(\mathrm{ord}(u), 2) = eN_{\mathrm{orb}}$, since N_{orb} is even.

To check the exceptional order condition, write $m = 2^H$. The order eN_{orb} satisfies $u^{eN_{\mathrm{orb}}} \equiv 1 \bmod 2^{2+2H+v_2(q_0)}$. It is even, because an odd exponent would give $v_2(u^{eN_{\mathrm{orb}}} - 1) = v_2(u-1) < H$. For an even exponent the factorization of the difference of powers gives

$$v_2(u^{eN_{\mathrm{orb}}} - 1) = v_2(u-1) + v_2(u+1) + v_2(eN_{\mathrm{orb}}) - 1.$$

One of the first two terms is 1, and both are less than H since $p \not\equiv \pm 1 \bmod m$. It follows that $v_2(eN_{\mathrm{orb}}) \geq H$, so $m \mid eN_{\mathrm{orb}}$. This construction keeps λ and its parameters fixed.

Coordinates for the exact operator. Represent each class x at the original level by a point a_x integral above p. Such representatives exist by weak approximation on the rational Cayley chart of Section 19.4.3. Indeed the local condition for an initial representative c_x is $\gamma_p \in K_{X_0,p}(c_x)_p^{-1}$, where $K_{X_0,p}$ is the product of the integral groups above p. Put $a_x = \gamma c_x$. One may at the same time take γ in the old principal subgroups at the primes dividing M, since $p \nmid M$. Left multiplication by γ preserves the original class and transports its value by rational equivariance. The original level has trivial class stabilizers. Evaluation at these representatives therefore identifies M with the module of all integral coordinate vectors at the classes. Indeed left rational translation in the gauge cancels the rational transformation of the value, and right level translation preserves the integral gauge.

Put $d = \mathrm{diag}(pI_r, I_{k-r})$ at λ. The incoming edges at x are indexed by cosets b of the parahoric

$$\mathcal{P} = \{g \in \mathrm{GL}_k(W) : d^{-1}gd \in \mathrm{GL}_k(W)\}.$$

For the Haar measure put $K_\lambda = \mathrm{GL}_k(\mathcal{O}_{E_\lambda})$ and $\mathcal{P}_\lambda = K_\lambda \cap dK_\lambda d^{-1}$. Reduction modulo p identifies both coset spaces with the $(k-r)$-planes in $\mathbb{F}_q^k$: the upper-right r-by-$(k-r)$ block must vanish modulo p, and invertible residue matrices lift over both the localization W and its completion. With $\mathrm{vol}(K_\lambda) = 1$,

$$K_\lambda dK_\lambda = \bigsqcup_b \beta_b dK_\lambda, \qquad \mathrm{vol}(\beta_b dK_\lambda) = 1, \qquad \mathrm{vol}(\mathcal{P}_\lambda) = s_{\mathcal{P}}^{-1}, \quad s_{\mathcal{P}} = [K_\lambda : \mathcal{P}_\lambda].$$

Thus each edge occurs with coefficient one in the Hecke sum. At λ replace the component of a_x by $\beta_b d$, where β_b represents b, and make the dual change at $\bar{\lambda}$. This gives the same Hecke coset as right multiplication by $a_{x,\lambda}^{-1}\beta_b d$. Write the resulting point as $\gamma_e a_{y(e)} v_e$, with γ_e rational unitary and v_e of original level. At λ this identity gives

$$\gamma_e^{-1} = g_e d^{-1} \beta_b^{-1}, \qquad g_e \in \mathrm{GL}_k(W).$$

Away from the pair $\lambda, \bar{\lambda}$, γ_e and its inverse are integral at every place over p. These are equalities in the fixed rational coordinates of the source, using the localization of the integer ring at λ for W.

For use on the dual edges, the map $e \mapsto (y(e), (g_e^{\mathsf{T}})^{-1}\mathcal{P})$ is a bijection. If two edges have the same image, $g_e^{\mathsf{T}}(g_{e'}^{\mathsf{T}})^{-1} \in \mathcal{P}$. Taking inverse transpose in the conjugation test defining $\mathcal{P}$ shows that $dg_e^{-1}g_{e'}d^{-1}$ is integral with integral inverse. At λ we have

$$\gamma_e\gamma_{e'}^{-1} = \beta_b(dg_e^{-1}g_{e'}d^{-1})\beta_{b'}^{-1} \in \mathrm{GL}_k(W),$$

and the component at $\bar{\lambda}$ is integral by unitarity. Write s_e for the finite right multiplier supported at this pair, so $a_x s_e = \gamma_e a_{y(e)} v_e$. With $\delta = \gamma_{e'}\gamma_e^{-1}$ and $y(e) = y(e')$,

$$a_{x'}^{-1}\delta a_x = s_{e'} v_{e'}^{-1} v_e s_e^{-1}.$$

Away from the pair this belongs to the original level since $s_e = s_{e'} = 1$; at the pair use the integral representatives $a_x, a_{x'}$ and the integral δ. Thus $x' = x$ and freeness gives $\delta = 1$. Now $d^{-1}\beta_b^{-1}\beta_{b'} d = g_e^{-1} g_{e'}$ is integral with integral inverse, so $\beta_b^{-1}\beta_{b'} \in \mathcal{P}$. This proves injectivity, and equal finite cardinalities give bijectivity.

At an embedding τ the one-covector weight has degree $a_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ and determinant exponent zero. The integral tableau coordinates of its evaluation vector are $e(z) = (z^\alpha)_{|\alpha| = a_\tau}$, since every column has height one. For the symmetric-power matrix $C(A)_{\beta,\alpha} = [z^\beta](A^{\mathsf{T}} z)^\alpha$, coefficient comparison gives

$$e(Az) = C(A^{\mathsf{T}})^{\mathsf{T}} e(z).$$

This identity holds over $\mathbb{Z}[A]$, also when $a_\tau = 0$. On the displayed orbit use $\rho_s(g)$ for the symmetric power of $\iota_s(g)$ at even s and of $\iota_s(g^{-\mathsf{T}})$ at odd s, and let $\rho_{\mathrm{let}}(g) = \bigotimes_s \rho_s(g)$. On even embeddings, p times the transpose of the conjugated rational matrix is $p\iota_s(\gamma_e^{-1})$, with middle factor pd^{-1}. On odd embeddings the transpose instead is $\iota_s(\gamma_e^{\mathsf{T}})$, with middle factor d. Outside this orbit, a factor of positive degree lies away from $\lambda, \bar{\lambda}$, where both γ_e and its inverse are integral. A factor of degree zero has trivial action even above $\lambda, \bar{\lambda}$. Therefore the p^{A_λ}-scaled source matrix for this edge is the tensor product of an integral invertible transition on the other Weyl factors and the transpose of

$$\rho_{\mathrm{let}}(g_e)\ \mathrm{diag}_\mu(p^{\mathrm{dep}(\mu)})\ \rho_{\mathrm{let}}(\beta_b^{-1}).$$

Here $\mathrm{dep}(\mu)$ counts exponents in coordinates $r+1, \dots, k$ on even embeddings and $1, \dots, r$ on odd embeddings. This follows by taking symmetric powers of the two diagonal matrices. In particular Y_λ preserves M. The formula is the edge sum on the original class lattice.

Size conditions for the local calculation. Set $\theta = (m^2 f)^{-1}$. The polynomial estimates below use

$$0 < 2r \le k, \quad 0 < \theta \le 1, \quad a^4 \le q, \quad (N_{\mathrm{orb}} k \log q/\theta)^5 P^{3r} \le a.$$

These hold uniformly on the exceptional mark for sufficiently large x. Indeed $m \asymp \sqrt{x}$, $k \asymp x^5$, $f = O(x^{11}\log x)$, $N_{\mathrm{orb}} \le f$, $r = r_p = O(\log x)$, and $e \le r$. In particular $2r \le k$ and $r^2 m \le k$. Since $a = xb$ and $\log b = x^\epsilon + O(1)$,

$$a^4 \le b^5 \le \exp(m) \le p^{eN_{\mathrm{orb}}} = q.$$

Indeed $d \equiv 9239 \bmod 18480$ gives $-d \equiv 1 \bmod 8$ and $-d \equiv 1 \bmod \ell$ for $\ell = 3, 5, 7, 11$. All primes below 13 therefore split in K, whereas p is inert. Also $eN_{\mathrm{orb}} \ge m$. Also $\log q \le eN_{\mathrm{orb}} \log x_1$ and $P^{3r} \le x_1^{3r^2} \le \sqrt{b}$, because $(\log x)^2 \log x_1 = o(\log b)$. Finally $(eN_{\mathrm{orb}}^2 k \log x_1/\theta)^5$ is bounded by a fixed power of x, uniformly in p, λ. Its product with $\sqrt{b}$ is at most a for large x. This gives the second large-degree inequality.

Put

$$C_{\mathrm{div}} = N_{\mathrm{orb}} a \left(P^{-2r} + \frac{r^2}{kP} \right).$$

Let $w^2 = p$ on extending scalars if necessary. Let b_μ be the diagonal symmetric-power pairing, with integral complementary weights b'_μ satisfying $b_\mu b'_\mu = p^{h_0}$. For the exact common exponent

put $h_0 = N_{\mathrm{orb}} k^2 e N_{\mathrm{orb}}$. If $M_s = a!/\prod_i \mu_{s,i}!$, take $b_\mu = p^{h_0}/\prod_s M_s$ and $b'_\mu = \prod_s M_s$. Since $a < q = p^{eN_{\mathrm{orb}}}$, the factorial valuation formula bounds $v_p(M_s)$ by keN_{orb}. This proves integrality. The multinomial theorem proves that substitution by a matrix and by its transpose are adjoint for the weights $1/M_s$, and hence for their scaled tensor product. Finally $k^2 e N_{\mathrm{orb}} P^{2r} \le a$ follows from the displayed large-degree bound and $eN_{\mathrm{orb}} \le \log q$, so $h_0 \le N_{\mathrm{orb}} a/P^{2r} \le C_{\mathrm{div}}$. Write $s_{\mathcal{P}} = [\mathrm{GL}_k(W) : \mathcal{P}]$ and d_{let} for the number of monomials of degree a in each of the N_{orb} embedding factors. Thus

$$d_{\mathrm{let}} = \binom{a+k-1}{k-1}^{N_{\mathrm{orb}}}.$$

Write $W_0 = \mathrm{diag}_\mu(w^{\mathrm{dep}(\mu)})$. A saturated halfway restriction is a vector $W_0 F$ with F over the coefficient fraction field such that $W_0 \rho_{\mathrm{let}}(g) F$ is integral in every parahoric direction. Choose $h_{\mathrm{clr}} \ge 0$ with $F_{\mathrm{int}} = w^{h_{\mathrm{clr}}} F$ integral. The condition is

$$W_0 \rho_{\mathrm{let}}(g) F_{\mathrm{int}} \in w^{h_{\mathrm{clr}}} \mathcal{O}^{d_{\mathrm{let}}} \qquad (g \in \mathrm{GL}_k(W)).$$

21.1 The depth count for the endpoint restrictions

In this subsection, we bound the valuations of the elementary divisors of the two restrictions of Section 21. The proof counts coefficients satisfying the divisibility conditions in coordinates adapted to the parahoric.

Work over the residue coefficient field K_0 of characteristic p for the cycle of embeddings above. Write $k = 2r + k_1$. A variable $X_{b,s}$, $0 \le b < k$, is deep when s is even and $b \ge r$, or s is odd and $b < r$. Let D_j be the space of degree-a forms in each embedding whose translates in all parahoric directions have no coefficient of deep degree less than j. The condition then holds after every $\mathrm{GL}_k(W)$ translate. In fact the parahoric preserves the deep-degree filtration modulo p in both parities. The action on the reduced coefficients factors through $\mathrm{GL}_k(W/pW)$.

Treat the complementary variables as coefficients in the field

$$L_0 = K_0(X_{b,s} : (s \text{ even}, b < r) \text{ or } (s \text{ odd}, b \ge r)).$$

For each deep variable $v = (b, s)$ put

$$\kappa_v = \begin{cases} P, & b < 2r, \\ P^{2r}, & b \ge 2r. \end{cases}$$

Give a monomial in the deep variables weight $\sum_v \mu_v/\kappa_v$. We show that any nonzero form in D_j has a nonzero coefficient of weight at least j.

Let B run through matrices over $\mathbb{F}_q$ supported on entries $(b, b+r)$, $b < r$, and (b, t), $b < r$, $2r \le t < k$. There are $r(k_1+1)$ independent entries. Such matrices satisfy $B^2 = 0$. The translation by $(1+B)^{-1}$ fixes every shallow variable. It takes a deep variable X_v to $X_v - z_v(B)$, where $z_{t,s}$ at even s is $\sum_{b<r} X_{b,s} \iota_s(B_{bt})$, for $t \ge r$, and $z_{b,s}$ at odd s is $-\sum_{t \ge r} X_{t,s} \iota_s(B_{bt})$, for $b < r$. Since the translates of $F \in D_j$ have only terms of deep degree at least j, $F(X - z(B)) \in (X_v : v \text{ deep})^j$. Substitute $-B$ for B to obtain $F \in (X_v - z_v(B) : v \text{ deep})^j$ at every point $z(B)$.

We claim that all the coordinate functions $\iota_i(B_{bt})$, for any fixed embedding i, are L_0-linear combinations of the functions $z_v(B)^{P^u}$ with $s + u = i$ and $P^u < \kappa_v$. For a column $t \ge 2r$,

choose the r distinct integers $0 \le u < 2r$ for which $s = i - u$ is even. The r values $z_{t,i-u}(B)^{P^u}$ have coefficient matrix

$$\left(X_{b,i-u}^{P^u}\right)_{u,b}.$$

This uses $\iota_s(y)^{P^u} = \iota_{s+u}(y)$ on the residue field. The determinant is nonzero. Distinct permutations use different monomials, because within any one column the variables are distinguished either by their embedding or by the distinct exponent P^u.

Thus these values solve for every entry of the column. For the entry $B_{b,b+r}$, use at the embedding i the variable $X_{b+r,i}$ if i is even and $X_{b,i}$ if i is odd. Its coordinate function has a nonzero coefficient at precisely this entry among the columns below $2r$. Subtract the already solved terms in the columns $t \ge 2r$ and divide by the nonzero shallow coefficient. When $k = 2r$, the columns $t \ge 2r$ are absent, and this last calculation determines every $B_{b,b+r}$ directly. When $r = 1$ each interpolation determinant above is a single nonzero shallow coordinate raised to its indicated power. This proves the claim.

It follows first that the points $z(B)$ are distinct. It also follows, by taking the P-power or the P^{2r}-power of the coordinate expression at its original embedding, that for every deep variable v

$$z_v(B)^{\kappa_v} = g_v(z(B)),$$

where g_v is a polynomial of weight strictly less than one. Indeed the power moves the coefficient functions to the corresponding later embedding, where the claim expresses them using the allowed lower powers. The product of the κ_v is

$$P^{N_{\mathrm{orb}} r + N_{\mathrm{orb}} r k_1} = q^{r(k_1+1)},$$

the number of points.

For completeness, the weighted interpolation argument for these points is as follows. Let $\psi_v = X_v^{\kappa_v} - g_v$, of weight one. The ideals of the points are pairwise comaximal, as are their j-th powers. The map to truncated polynomials of degree less than j at all these points is therefore surjective. Division using $X_v^{\kappa_v} = \psi_v + g_v$ expresses each polynomial in the span of $X^\alpha \psi^\gamma$, $0 \le \alpha_v < \kappa_v$. In each replacement the term with g_v strictly lowers weight and the other term removes $X_v^{\kappa_v}$ from the monomial preceding ψ^γ.

This induction terminates, since the weights lie in a discrete nonnegative subgroup of $\mathbb{Q}$. It never increases $\mathrm{wt}(\alpha) + |\gamma|$. Modulo the kernel of the truncated map the terms of $|\gamma| \ge j$ vanish. The number of remaining products equals the dimension of the target, since the number of points is $\prod_v \kappa_v$. Their images are therefore independent. If F had weight less than j, division would use only products with $|\gamma| < j$. Its zero image would force $F = 0$. The change of shallow variables to coefficients is injective, so it follows that a nonzero original coefficient of F has weight at least j.

Projection onto the monomial coordinates of weight at least j is consequently injective on D_j. A monomial of weight u is counted at no more than u positive integer thresholds. In the uniform average over monomials of degree a in each embedding, each variable has mean exponent a/k. There are $N_{\mathrm{orb}} r$ deep variables with $b < 2r$, and at most $N_{\mathrm{orb}} k$ other deep variables. Thus the mean weight is at most

$$\frac{a}{k}\left(\frac{N_{\mathrm{orb}} r}{P} + \frac{N_{\mathrm{orb}} k}{P^{2r}}\right) \le N_{\mathrm{orb}} a \left(P^{-2r} + \frac{r^2}{kP}\right) = C_{\mathrm{div}}.$$

Summing the injective dimension bounds gives $\sum_{j \ge 1} \dim D_j \le C_{\mathrm{div}} d_{\mathrm{let}}$, as required for both endpoint restrictions.

21.2 The rank-two defect estimate

In this subsection, we make the root calculation needed for the source Hecke operator of Section 21. The two-dimensional calculation bounds the coefficients on the saturated restrictions of Section 21.1.

Fix a root (b, b') and an embedding i. In embedding s write X_s, Y_s for the two variables, in this order at even embeddings and in reverse order at odd embeddings. Thus Y_s is the positive variable in the defect. Put $P_s = P^{(s-i) \bmod N_{\mathrm{orb}}}$. For an exponent α define

$$D = q \sum_s P_s \alpha(X_s), \qquad E = q\alpha(Y_i) + \sum_{s \neq i} P_s \alpha(Y_s).$$

The defect of α at i is $(E - D/q)/(q-1)$.

First work over the integral coefficient ring with $w^2 = p$. Let c give weights on the variables with $1/2 \leq c(Y_s) - c(X_s) \leq 1$. For a coefficient a_α, give the term $a_\alpha X^\alpha$ total weight $\mathrm{ord}_w(a_\alpha) + c \cdot \alpha$, where for $a \neq 0$

$$\mathrm{ord}_w(a) = \max\{j \in \mathbf{Z}_{\geq 0} : a \in w^j \mathcal{O}\} = \lfloor 2v(a) \rfloor, \qquad \mathrm{ord}_w(0) = +\infty.$$

We use the corresponding convention over the residue coefficient field, with every nonzero coefficient of order zero. Suppose all terms of all translates of F by the root $\mathrm{SL}_2(W)$ have weight at least $l + c \cdot \alpha$, where $a_\alpha \neq 0$ and $l = \mathrm{ord}_w(a_\alpha) \in \mathbf{Z}_{\geq 0}$. Thus a_α is nonzero modulo w^{l+1}. Over the residue field take $l = 0$ and omit reduction. We prove $Eq \leq D$.

Put $\epsilon_s = -1$ if s has the parity of i, and $\epsilon_s = 1$ otherwise. In a new variable T make the substitution $\mathcal{A}_T$ given by

$$\begin{aligned} X_i &\longmapsto T^q X_i, & Y_i &\longmapsto X_i, \\ X_s &\longmapsto T^{qP_s} X_s - \epsilon_s T^{P_s} Y_s, & Y_s &\longmapsto Y_s - \epsilon_s X_s \quad (s \neq i). \end{aligned}$$

All other variables are fixed. Let $\bar{\alpha}$ be the exponent obtained by merging Y_i into X_i, and let $\Phi(T)$ be the coefficient at $\bar{\alpha}$ of $\mathcal{A}_T F$, reduced modulo w^{l+1}.

For the expansion at infinity put $\rho = (c(Y_i) - c(X_i))/q$. Give T weight $-\rho$, X_s weight $c(X_s) + qP_s\rho$, and every other variable its original weight. The main term of each substituted variable has its original weight. At i this uses $c(X_i) + q\rho = c(Y_i)$. At $s \neq i$ the cross terms have strictly greater weight, since

$$P_s\rho \leq 1/P < 1/2, \qquad qP_s\rho \geq P/2 > 1.$$

These inequalities have a fixed positive additional term. Thus $T^D X^{\bar{\alpha}}$ has weight $c \cdot \alpha$, and its coefficient modulo w^{l+1} comes only from main terms. The merged exponents and D recover the original pair by

$$\alpha(X_i) = D/q - \sum_{s \neq i} P_s \bar{\alpha}(X_s), \qquad \alpha(Y_i) = \bar{\alpha}(X_i) - \alpha(X_i).$$

The other exponents are unchanged. The coefficient is therefore a_α, nonzero modulo w^{l+1}. For T-degree greater than D the weight is strictly less than $c \cdot \alpha$, so the coefficient vanishes modulo w^{l+1}. Hence $\deg \Phi = D$.

For each finite point take $r_0 \in W$, put $t_0 = \iota_i(r_0)$, and shift $T \mapsto T + t_0$. The root matrix is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -r_0 \end{smallmatrix}\right)$ if i is even, with $+r_0$ in the last entry if i is odd. It has determinant one. The

action on the two variables in any embedding sends X_s to a signed Y_s, and Y_s to a signed $X_s + \epsilon_s \iota_s(r_0) Y_s$, with the whole parenthesis sharing its sign. This follows by reading the matrix for even embeddings and its inverse transpose for odd embeddings in the reversed order of the two variables.

Compose that root substitution with $\mathcal{A}_{T+t_0}$. The result on each of X_s, Y_s is the signed image under $\mathcal{A}_T$ of the swapped variable, up to errors on the image of Y_s with coefficients in $p\mathcal{O}[T]$. Indeed for $s \neq i$ the change in $\mathcal{A}_T(X_s)$ modulo p is

$$t_0^{P_s}(X_s - \epsilon_s Y_s) = -\epsilon_s \iota_s(r_0) \mathcal{A}_T(Y_s) \pmod{p}.$$

Here the shifts of T^{qP_s} and T^{P_s} have the same constant modulo p, by the Frobenius cycle. At i the change is $t_0^q X_i = t_0 X_i$ modulo p, and $-\epsilon_i = 1$. The errors consist only of p-multiples of polynomials times X_s, Y_s, or of X_i after merging. This follows by expanding the two shifted p-powers. Over the residue field the errors vanish.

For this finite expansion give T weight $+\rho$, X_s weight $c(X_s)$, Y_s weight $c(Y_s) - P_s\rho$ for $s \neq i$, and Y_i weight $c(Y_i) - q\rho$. The signed image of the swapped variable has weight at least the weight of the source vector. For $s \neq i$ the required two inequalities are $c(Y_s) - P_s\rho \geq c(X_s)$ and $c(X_s) + qP_s\rho \geq c(Y_s)$; at i they hold with the stated ρ. Each error still has weight at least the source weight, because a factor $p = w^2$ contributes two, while the difference lost in either variable is at most $1 + 1/P$. Applying this to the inverse root translate of F shows that all terms of $\mathcal{A}_{T+t_0} F$ have weight at least $l + c \cdot \alpha$ in this finite expansion. Now $T^E X^{\bar{\alpha}}$ has weight $c \cdot \alpha$. Every lower T-degree at that exponent has zero coefficient modulo w^{l+1}. Thus Φ vanishes at t_0 to order at least E.

Choose one lift $r_0 \in W$ of each class in W/pW. Differences of distinct lifts are units in W, so their images under ι_i remain units modulo w^{l+1}. Thus the monic powers $(T - t_0)^E$ are pairwise coprime in this quotient polynomial ring. Their product divides the nonzero polynomial Φ, giving $Eq \leq D$. This proves that the defect of the extreme term is nonpositive. Taking $l = 0$ over the residue field with the height weights proves the least-height fact used in the flag count.

For the edge divisibility number the coordinates $0, \ldots, k-1$. Choose a root across the two blocks, $b < r \leq b'$, and take c to be the ordinary depth. In this case Y_s is deep and X_s is shallow in both parities, so $c(Y_s) - c(X_s) = 1$. Fix i and change these weights by subtracting one third of the defect functional at i. The increments of the changed weights are still in $[1/2, 1]$: the increments of the defect are $1 + 2/(q-1)$ at i and $2P_s/(q-1)$ elsewhere, all in $[0, 3/2]$.

Suppose every root translate initially has order plus ordinary depth at least v_0. If the changed bound failed in even one translate, choose an extreme term of minimal order plus changed weight over all translates. Such a minimum exists, because the exponents lie in a finite set and nonnegative integral orders with weight below v_0 are bounded. The estimate just proved says that the extreme term has nonpositive defect. Its ordinary depth plus order would therefore be below v_0, a contradiction. Thus

$$\operatorname{ord}_w(a_\beta) + \operatorname{dep}(\beta) \;\geq\; v_0 + \operatorname{defect}_i(\beta)/3.$$

In particular let $v = w^{-h_{\mathrm{clr}}} W_0 F$ be a saturated halfway restriction with F integral and $w^{-h_{\mathrm{clr}}} W_0 \rho_{\mathrm{let}}(g) F$ integral in every direction g. Apply the last estimate with $v_0 = h_{\mathrm{clr}}$. The Levi commutes with W_0, since it preserves the block degrees. Consequently in every Levi translate of v, a coefficient of defect at least $3u$ is divisible by w^u. For the pairing with a translated monomial, move its Levi matrix to the transpose on v. By the diagonal pairing formula the resulting scalar is divisible by w^u at the same defect threshold.

21.3 Counting forms with bounded local defect

In this subsection, we use the two-variable estimate of Section 21.2 to bound a dimension in the residue representation. We count the permitted monomials after imposing the same conditions under all elements of the Levi.

Write $b_0 = r - 1$ and take the defect bound $3t$ at the root (b_0, r), where t is an integer with $t \geq 2(k+1)$. Put

$$k_W = W/\mathfrak{m}_W = \mathbb{F}_q, \qquad k_{\mathcal{O}} = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}, \qquad L_r = \mathrm{GL}_r \times \mathrm{GL}_{k-r}.$$

Let S be the $k_{\mathcal{O}}$-space of forms of degree a in each embedding for which every $L_r(k_W)$-translate has all defects along (b_0, r) at most $3t$. The action in embedding s uses the residue of ι_s and the prescribed inverse transpose at odd s. Every other simple root lies within a Levi block.

The rank-two estimate of Section 21.2 gives the following fact at the roots inside one Levi block. Among the nonzero coefficients of all Levi translates of a nonzero polynomial, choose one of least height, with height of $X_{b,s}$ given by b on even embeddings and $-b$ on odd embeddings. The defect at that coefficient is nonpositive at each embedding for every simple root inside the block. The required root calculation applies without a homogeneity requirement on the polynomial, since the root substitutions are linear and preserve the degree at each embedding.

A least-height coefficient occurs at a representative of a Levi flag. Indeed write a Levi element as $g = ux$ with u lower triangular modulo p and x a chosen representative of its left coset. On even embeddings an off-diagonal term of u strictly increases the variable index, and on odd embeddings inverse transpose strictly decreases it. Each increases the height. Thus at the least-height coefficient u contributes only its diagonal action, with a nonzero factor. The coefficient at x is therefore already nonzero.

For a positive integer $M = q^j$, take a family $(F_u)_u$ in S indexed by the vectors u with $0 \leq u_{b,s} < M$. In each embedding put

$$\gamma_{u,b,s} = u_{b,s} + \mathbf{1}_{b=b_0}\, M \left(k - \left\lfloor \frac{\sum_c u_{c,s}}{M} \right\rfloor \right).$$

Then X^{γ_u} keeps the original residue u modulo M, while the degree of γ_u in each embedding belongs to $[Mk, M(k+1))$. Pack the family as

$$F_{\mathrm{pack}}(X) = \sum_u X^{\gamma_u} F_u(X^M).$$

This linear map is injective by distinct exponent residues. The residue entries of every translated matrix lie in the images of $\mathbb{F}_q$, so the substitutions commute with $X \mapsto X^M$. Consequently any nonzero coefficient at α in a Levi translate of the packed polynomial comes from a term with exponent $\alpha = M\beta + \eta$, where β occurs in a translate of some F_u and η occurs in the translate of its monomial X^{γ_u}. The degrees of η/M are in $[k, k+1)$, because linear substitutions preserve them.

The defects along (b_0, r) of β are at most $3t$. In the defect of η/M, the term with factor q is bounded by $(k+1)q/(q-1)$; the singled-out term with factor -1 is nonpositive. Including the other weighted differences gives

$$\frac{k+1}{q-1} \left(q + \sum_{l=1}^{N_{\mathrm{orb}}-1} P^l \right) = (k+1) \left(1 + \frac{1}{P-1} \right) \leq 2(k+1).$$

Thus each of these defects of α/M is at most $3t + 2(k+1) \le 4t$. Its degree at each embedding is in $[a+k, a+k+1]$.

We move the boundary-root bound into the zero-defect cone by a common integral shift. Put $\mu_b = P^{2(b_0-b)}$ for $b \le b_0$ and $\mu_b = 0$ otherwise. Put $\nu_{b,s} = \mu_b$ on even embeddings and $P\mu_b$ on odd embeddings. Put $\varepsilon_s = (-1)^s$. The cyclic steps $\sigma_{b,s} = \varepsilon_s(P\nu_{b,s+1} - \nu_{b,s})$ are $(P^2-1)\mu_b$ on even embeddings and zero on odd embeddings. It is integral and nonnegative, with degree at most $P^{2r} - 1$ at every embedding. Along a simple root, its defects are $P^2\mu_{b+1} - \mu_b$ on even embeddings and $P(\mu_{b+1} - \mu_b)$ on odd embeddings. Thus they are at most -1 at the boundary root and at most zero at all other simple roots.

Write $\mathcal{C}_0$ for the cone in which all simple-root defects are nonpositive, and $\mathcal{D}$ for the nonnegative degree slab with the bounds below. For a nonzero packed family, choose a least-height coefficient α at a flag representative x. The fact about roots inside the Levi and the last estimates show

$$\alpha/M + 4t\sigma \in \mathcal{C}_0 \cap \mathcal{D}, \qquad D_s = a + k + 1 + 4t \sum_b \sigma_{b,s} \le a + 4tP^{2r}.$$

Since $4t\sigma \in \mathbb{Z}^{kN_{\mathrm{orb}}}$, the point belongs to $M^{-1}\mathbb{Z}^{kN_{\mathrm{orb}}}$.

Therefore the linear map that lists the coefficients at $M(y - 4t\sigma)$ in the flag translates, for $y \in \mathcal{C}_0 \cap \mathcal{D} \cap M^{-1}\mathbb{Z}^{kN_{\mathrm{orb}}}$, is injective on the packed families. Points giving negative coordinates can be skipped. It follows that $M^{kN_{\mathrm{orb}}} \dim S$ is bounded by the number of Levi flags times the number of these lattice points. Divide by $M^{kN_{\mathrm{orb}}}$ and let $M = q^j \to \infty$. The bounded polytope has boundary of measure zero, and its normalized lattice counts converge to its volume by the ordinary Riemann sums. This is the required lattice-point bound for the volume calculation.

21.4 The volume factor in the residue-defect estimate

In this subsection, we compute the volume ratio in the dimension estimate of Section 21.3. Ordering the exponents gives interlacing inequalities. Their volume normalization follows by determinant expansion.

Use the same parity sign $\varepsilon_s = (-1)^s$. For permutations $\sigma_s \in S_k$, set

$$(T_\sigma \nu)_{b,s} = \varepsilon_s(P\nu_{b,s+1} - \nu_{\sigma_s(b),s}).$$

Write $q = P^{N_{\mathrm{orb}}}$. The sum of the entries at each embedding is independent of the permutations. Let $\mathcal{D}$ be any slab of nonnegative vectors whose degree in the embedding s is in $[D_s - 1, D_s]$. In coordinates $T_1\nu$, the zero-defect cone $\mathcal{C}_0$ is given by the additional inequalities

$$\begin{cases} P\nu_{b+1,s+1} \le \nu_{b,s}, & \varepsilon_s = 1, \\ \nu_{b+1,s} \le P\nu_{b,s+1}, & \varepsilon_s = -1, \end{cases} \qquad (b < k-1).$$

Indeed put $\beta = T_1\nu$ and $d_s = \nu_{b,s} - \nu_{b+1,s}$. The signed weighted difference of the two relevant beta coordinates is $d_s - Pd_{s+1}$. Summing with weights $P^{(s-i) \bmod N_{\mathrm{orb}}}/(q-1)$ gives $-d_i$, because the cycle closes with $P^{N_{\mathrm{orb}}} = q$. The remaining term of the defect is $\beta_{b+1,i}$ if i is even and $\beta_{b,i}$ otherwise. Thus zero defect is the displayed pair of inequalities. We prove

$$\mathrm{vol}(\mathcal{C}_0 \cap \mathcal{D}) = \frac{\mathrm{vol}(\mathcal{D})}{\prod_{b=1}^{k}(1 + q + \cdots + q^{b-1})}.$$

First take ν with distinct entries in each embedding, avoiding also equality in adjacent comparisons. These restrictions discard only finitely many hyperplanes. For each s let M_s be the k-square matrix of indicators of the inequalities $\varepsilon_s(P\nu_{b,s+1} - \nu_{c,s}) > 0$. The product $\prod_s \det M_s$ is unchanged if the entries in any one embedding are independently permuted, since this permutes the covectors of one determinant and the columns of another with the same sign.

Sort the entries of ν decreasingly in each embedding. For two decreasing sequences x, y, the determinant of $(1_{x_b > y_c})_{b,c}$ is the indicator of interlacing $x_0 > y_0 \geq x_1 > y_1 \geq \cdots$. Indeed each covector consists of zeros followed by ones, and the first positions of the ones are nondecreasing. The determinant vanishes if two covectors coincide or there is a zero covector. In the remaining case the first positions are $0, \ldots, k-1$ and the matrix is triangular with diagonal ones. For the reversed inequality transpose and interchange the two sequences. Thus the product of determinants indicates the cone inequalities and positivity on the sorted arrays.

Conversely the cone inequalities and positivity force the entries of ν to be decreasing in each embedding, by the same interlacing inequalities. Therefore the number of tuples of permutations of the embeddings sending ν to a point satisfying these conditions equals that product of determinants. Expanding the determinants gives the signed sum over the tuples σ of the indicators of $T_\sigma \nu > 0$, with sign $\prod_s \operatorname{sgn} \sigma_s$. Multiply by the degree-slab condition, common to all terms, and integrate.

To compute the resulting volumes, eliminate the embedding vectors cyclically in $P\nu_{s+1} = \Pi_{\sigma_s} \nu_s + \varepsilon_s \beta_s$, where Π_{σ_s} is the corresponding permutation matrix. Eliminating $N_{\mathrm{orb}} - 1$ vectors gives diagonal factors P on each eliminated coordinate. The final block is $P(1 - q^{-1}\Pi_\sigma)$, with Π_σ the product round the cycle. Thus

$$|\det T_\sigma| = P^{kN_{\mathrm{orb}}} |\det(1 - q^{-1}\Pi_\sigma)|.$$

Each determinant is nonzero since permutation eigenvalues have absolute value one and $q > 1$. For a prescribed product π there are $(k!)^{N_{\mathrm{orb}}-1}$ tuples yielding it.

Each reordering of the embedding coordinates has Jacobian of absolute value one. Writing $N = N_{\mathrm{orb}}$ in this calculation, the integrated identity is therefore

$$\frac{(k!)^N}{|\det T_1|} \operatorname{vol}(\mathcal{C}_0 \cap \mathcal{D}) = \operatorname{vol}(\mathcal{D}) \sum_{\sigma \in (S_k)^N} \frac{\prod_s \operatorname{sgn}(\sigma_s)}{|\det T_\sigma|}.$$

Grouping by the product permutation gives

$$\operatorname{vol}(\mathcal{C}_0 \cap \mathcal{D}) = \operatorname{vol}(\mathcal{D}) \, \frac{(1 - q^{-1})^k}{k!} \sum_{\pi \in S_k} \frac{\operatorname{sgn} \pi}{|\det(1 - q^{-1}\Pi_\pi)|}.$$

The indicator determinants and their Jacobians were computed for each sign separately, so this factor also applies to the interlacing cone with one positive embedding, $P = q$, and the unit simplex $\sum_b \beta_b \leq 1$, $\beta_b \geq 0$, of positive volume. In this case the step map is scalar multiplication by $q - 1$, and the cone is the cone of $\beta_b \geq q\beta_{b+1}$, $\beta_{k-1} \geq 0$. Put $d_b = 1 + q + \cdots + q^b$ and $\beta_k = 0$. The triangular map $\beta_b \mapsto d_b(\beta_b - q\beta_{b+1})$ maps this cone to the nonnegative orthant, preserves the sum of coordinates and has determinant $\prod_b d_b$. This proves the claimed factor for every N_{orb}.

Here is the consequence of the corresponding lattice-point bound. Section 21.3 shows that the bounded-defect space S has dimension at most the number F_L of flags of the two-block Levi times $\operatorname{vol}(\mathcal{C}_0 \cap \mathcal{D})$, with $1 \leq D_s \leq a + E$ and $E = 4tP^{2r}$. The parahoric index times F_L is

the number of all flags of $\mathbb{F}_q^k$, namely $\prod_{b=1}^{k}(1+q+\cdots+q^{b-1})$. A complete flag is uniquely its $(k-r)$-plane with the flags on that plane and on the quotient. Thus the volume factor cancels this index and

$$[\mathrm{GL}_k(W):\mathcal{P}]\dim S \le \mathrm{vol}(\mathcal{D}).$$

The nonnegative vectors in one embedding with degree in $[D_s-1, D_s]$ have volume $(D_s^k-(D_s-1)^k)/k! \le D_s^{k-1}/(k-1)!$. If $8N_{\mathrm{orb}}ktP^{2r} \le \theta a$, then $N_{\mathrm{orb}}(k-1)E/a \le \theta/2$. Therefore

$$\mathrm{vol}(\mathcal{D}) \le \exp\!\left(\frac{N_{\mathrm{orb}}(k-1)E}{a}\right)\left(\frac{a^{k-1}}{(k-1)!}\right)^{N_{\mathrm{orb}}} \le (1+\theta)\binom{a+k-1}{k-1}^{N_{\mathrm{orb}}}.$$

The last step uses $0<\theta\le 1$ and $\exp(\theta/2)\le 1+\theta$.

21.5 An integral average in one matrix variable

In this subsection, we construct one integral functional on polynomials in a generic matrix. The invariance and vanishing properties are chosen for the Levi in Section 21, while the congruence to evaluation at the identity will control the pairing in Section 21.6.

Let W be the local ring of the distinguished prime, with residue field of order $q=p^{eN_{\mathrm{orb}}}$, and fix one embedding into the coefficient ring $\mathcal{O}$. Let $L_W \subseteq \mathrm{GL}_k(W)$ be the two-block Levi. For integral weights ω_i, write

$$w_{\mathrm{in}}(\alpha)=\sum_{i,j}\omega_i\alpha_{ij}, \qquad w_{\mathrm{out}}(\alpha)=\sum_{i,j}\omega_j\alpha_{ij}.$$

Call X^α bad if $w_{\mathrm{in}}(\alpha)>t_0$. Suppose $D+k\le q$ and $D<q$. There is an integral functional I on the polynomials of degree at most D, invariant under conjugation by L_W, that kills bad monomials and satisfies

$$I(F)\equiv p^{k^2eN_{\mathrm{orb}}}F(1) \pmod{\mathfrak{b}}$$

for each ideal $\mathfrak{b}\subseteq\mathcal{O}$ and each F of degree at most D whose every Levi conjugate has all bad coefficients in $\mathfrak{b}$.

Here is the construction. Work in degree $d\le D$, on $U_d=(\mathcal{O}^k)^{\otimes d}$. Let $\mathcal{S}_d$ be the Schur algebra with basis the occurrence matrices of pairs of tensor indices with the same sequence of block parameters. For such an occurrence matrix α, the corresponding basis element is the coefficient of X^α in $(X_{\mathrm{block}}^{\mathsf{T}})^{\otimes d}$, where off-block entries of X_{block} are set to zero. The basis elements have disjoint supports in the matrix entries. For independent block matrices X, Y, put $A(X)=(X^{\mathsf{T}})^{\otimes d}$. Comparing coefficients of $X^\alpha Y^\beta$ in $A(X)A(Y)=A(YX)$ expresses each product of two occurrence operators as an integral combination of these operators. For an exponent involving an off-block variable the occurrence operator is zero.

The tensors of matrices in L_W span $\mathcal{S}_d$ over $\mathcal{O}$. To verify this, first work over W. If a homogeneous polynomial of degree d over $\mathbb{F}_q$ in the block entries vanishes on all invertible block matrices, its product with the block determinant vanishes at every point of the affine space. This product is homogeneous of degree $d+k\le q$. A homogeneous polynomial of degree at most q that vanishes on all of this affine space is zero.

Indeed reduction by the relations $X_i^q=X_i$ gives a polynomial of degree less than q in each variable, and such polynomials are tested injectively by evaluation on the grid. In degree q only the pure powers X_i^q have to be changed, and their reductions of degree one cannot cancel the other homogeneous terms. The block determinant is a nonzero polynomial. Thus

the evaluations at invertible block matrices span the dual of the space of degree-d polynomials over $\mathbb{F}_q$. Lift the matrices to L_W, apply Nakayama to the finite free coefficient space over W, and extend along the embedding.

For $|\alpha| = d$, put $u_\alpha = p^{k^2 e N_{\mathrm{orb}}}/\binom{d}{\alpha}$. These weights are integral. By the factorial valuation formula each of the eN_{orb} relevant floors contributes less than k^2 to $v_p\binom{d}{\alpha}$. Use u_α to pair the coefficients of polynomials in the generic matrix variables. Let $F^\sharp \in \mathcal{S}_d$ be the operator whose coefficient at the occurrence matrix α is $u_\alpha[X^\alpha]F$. For $F(X) = \operatorname{tr}(B^{\mathsf{T}}X)^d$, the multinomial expansion gives $F^\sharp = p^{k^2 e N_{\mathrm{orb}}}(B_{\mathrm{block}}^{\mathsf{T}})^{\otimes d}$. In $F(g^{-1}Xg)$ the matrix B^{T} is replaced by $gB^{\mathsf{T}}g^{-1}$. Since g is block diagonal, taking diagonal blocks commutes with this conjugation. Powers of linear forms span the degree-d space over the coefficient field by polarization. Consequently

$$\bigl(F(g^{-1}Xg)\bigr)^\sharp = g^{\otimes d} F^\sharp (g^{-1})^{\otimes d}.$$

For a tensor index $j = (j_1, \dots, j_d)$ let its weight be $\sum_l \omega_{j_l}$. Let U_+ be the submodule generated under $\mathcal{S}_d$ by the basis tensors $e_{j_1} \otimes \cdots \otimes e_{j_d}$ of weight $> t_0$. It is finitely generated and torsion-free, hence free over the valuation ring. It is invariant under both actions in the last conjugation identity.

Define $I_d(F)$ to be the trace of $F^\sharp$ on U_d minus its trace on U_+. This is integral and conjugation invariant. If $F = X^\alpha$ is bad, its operator is zero on tensor indices of weight at most t_0, because the first indices of X^α are the column indices of $F^\sharp$. Its image lies in U_+, so the two traces agree by the cyclic trace identity for a map factoring through U_+. Thus I_d kills bad monomials.

Suppose now all the bad coefficients of every conjugate of F are in $\mathfrak{b}$. For a high-weight basis tensor v, write E_α for the occurrence operator of α. Then

$$F^\sharp v = \sum_{\alpha:\, E_\alpha v \neq 0} [X^\alpha]F \, u_\alpha E_\alpha v \in \mathfrak{b}U_+.$$

Indeed the input indices in each nonzero term have the weight of v, hence α is bad. Also $E_\alpha \in \mathcal{S}_d$, so $E_\alpha v \in U_+$ before reduction. For a translate $g^{\otimes d}v$ the same assertion follows by the conjugation identity. Such translates generate U_+ by the integral Schur spanning just proved. Hence $F^\sharp U_+ \subseteq \mathfrak{b}U_+$, so its trace there belongs to $\mathfrak{b}$. Finally the trace on U_d of $(X^\alpha)^\sharp$ is zero unless α is diagonal. For a diagonal α there are $\binom{d}{\alpha}$ identical diagonal coefficients, so that trace is $p^{k^2 e N_{\mathrm{orb}}}$. Thus the trace on U_d is $p^{k^2 e N_{\mathrm{orb}}}$ times evaluation at 1 in degree d. Summing I_d for $0 \le d \le D$ gives the required average.

Conjugation invariance also gives $I(X^\alpha) = 0$ whenever $w_{\mathrm{in}}(\alpha) \neq w_{\mathrm{out}}(\alpha)$. Indeed conjugation by the diagonal matrix with entries $(1+p)^{\omega_i}$ scales that monomial by a nontrivial integer power of $1+p$ when the weights differ. The coefficient ring has characteristic zero and no zero divisors.

21.6 The simultaneous average on the coefficient space

In this subsection, we tensor the functionals of Section 21.5 across the embeddings in the local representation. The estimates of Sections 21.2 and 21.3 yield both a pairing congruence and a dimension bound for its range after reduction.

Keep $N_{\mathrm{orb}} = 2h$, $P = p^e$, $q = P^{N_{\mathrm{orb}}}$, and r as above. Put $c_0 = k^2 e N_{\mathrm{orb}}$, $h_0 = N_{\mathrm{orb}} c_0$, $c_1 = 5h_0$, and $t = 3kc_1$. Write $\rho_b = 1_{b=r} - 1_{b=r-1}$. For a fixed copy i, give a variable in covector

b and column c of the generic matrix in copy j weight

$$\omega_i(j,b,c) = P^{-h}\begin{cases}\rho_b, & j = i,\\ q^{-1}P^{(j-i) \bmod N_{\mathrm{orb}}}(\rho_b - \rho_c), & j \neq i.\end{cases}$$

Denote the weight of a monomial in all copies by $\omega_i(\nu)$. Write def_i for the defect at an embedding at the boundary root, in the convention of the rank-two calculation.

Take one generic k-square matrix X_j for each copy. At embedding s use the product of the entrywise P^d-th powers of X_{s-d}, in decreasing order of d, with $0 \le d < h$ in small parts and $0 \le d \le h$ in deep covectors. In odd embeddings take the cofactor matrix $\mathrm{adj}(M)^{\mathsf{T}}$ of the corresponding product before taking its covector. Let $\mathcal{B}$ be the tensor product of the symmetric powers of degree a of these covector matrices. Its evaluation at $X_j = 1$ for all j is 1. For a change of basis g, replace the factor in embedding s at position d by

$$g_s\big(g_{s-d}^{-1}X_{s-d}g_{s-d}\big)^{[P^d]}g_s^{-1}, \qquad g_s = \iota_s(g).$$

Brackets denote entrywise power. Call the resulting substitution $\mathcal{B}_g$. Modulo p these conjugations cancel, because the P^d-power map is a ring map there and takes g_{s-d} to g_s. The difference from the original factor has all coefficients divisible by $p = w^2$.

We give the weight estimate on $\mathcal{B}_g$. Fix $|\epsilon| \le 1/2$ and give the generic variables weights $\epsilon\omega_i$, adding the w-order to the weight of each monomial coefficient. Every monomial of any single entry of the factor at d has degree P^d and weight between -1 and 1, since $|\epsilon\omega_i(j,b,c)| \le P^{-h}$ and $d \le h$. Adding w-order two therefore makes every error term weigh at least as much as the original entry monomial. In particular each entry has weight at least $\epsilon P^d\omega_i(s-d,b,c)$.

For the matrix product put

$$A_{s,d}(b) = \frac{\epsilon\rho_b}{P^h}\frac{P^{d+(s-d-i) \bmod N_{\mathrm{orb}}}}{q}, \qquad u = \frac{1}{2P^h}.$$

At $d = 0$, $|A_{s,0}(b)| \le u$. For each $d \le h$ one has

$$\Big|A_{s,d+1}(b) - A_{s,d}(c) - \epsilon P^d\omega_i(s-d,b,c)\Big| \le u.$$

Indeed the expressions agree unless $s - d = i$. At that wraparound the first term supplies exactly the weight of the part, and the remaining column potential has magnitude at most $|\epsilon|P^{d-h}/q \le u$. Along a path in a product of $m \le h+1$ factors, the potentials telescope, so all entries in covector b weigh at least $A_{s,m}(b) - (h+2)u$. In an odd embedding a cofactor omits its own covector and takes one entry in each other covector. The potentials sum to zero over the covectors. Thus in both parities the covector bound is $\varepsilon_s A_{s,m}(b) - k(h+2)u$, with $\varepsilon_s = 1$ for even embeddings and -1 otherwise.

For $m = h+1$ on deep covectors and h on small parts, the term $\varepsilon_s A_{s,m}(b)$ is exactly $\epsilon(1-1/q)$ times the coefficient of the variable (b,s) in the defect at $i+h$. For small parts the power simplifies to $P^{(s-i-h) \bmod N_{\mathrm{orb}}}$ after removing P^h. For deep covectors it is the same power except at $s = i+h$, when it is q, giving the additional deep term of the defect. The signs are those of the two root variables, and the other potentials vanish. Taking the symmetric powers and then the tensor product adds these weights according to the covector exponents β. Hence, for every monomial ν in an entry of covector β of $\mathcal{B}_g$, that coefficient is divisible by w to the nonnegative ceiling of

$$\epsilon\big((1-1/q)\,\mathrm{def}_{i+h}(\beta) - \omega_i(\nu)\big) - N_{\mathrm{orb}}ak(h+2)u.$$

The covector estimate uses exactly a factors in each embedding, one covector factor for each variable counted with its exponent.

The size hypotheses give $N_{\mathrm{orb}}ak(h+2)+a+k<P^h$. Indeed $P^h\ge a^2$, $h+2\le N_{\mathrm{orb}}k$, and $a\ge (N_{\mathrm{orb}}k)^5\ge (N_{\mathrm{orb}}k)^2+2$, with $k<a$. Thus the loss $N_{\mathrm{orb}}ak(h+2)u$ is at most $1/2$. The sum of the cyclic geometric weights of the defect is $1/(P-1)$, and each embedding has degree a, so $|\operatorname{def}_{i+h}(\beta)|/q\le 2a/q\le 1$. Taking $\epsilon=1/2,-1/2$ shows that if the coefficient survives modulo w^{c_1}, then

$$|\operatorname{def}_{i+h}(\beta)-\omega_i(\nu)|\le 2c_1+2\le t.$$

The degree in each generic copy is at most $D=N_{\mathrm{orb}}ak(h+2)P^h$. This follows by adding the degrees of the factors in each product, using at most k factors of product entries for a cofactor and a covector factors in each embedding. The same bound holds in any basis. Our strict size inequality gives $D+k<q$, so the one-copy average applies. At each copy j use its embedding and weights of the parts ρ_b, with bad threshold $2tP^h$. Let I_j be the resulting invariant functional, and let I_j^0 be p^{c_0} times evaluation at the identity. Both are invariant under the Levi.

Apply some such invariant functionals in all copies but j, leaving a polynomial in X_j. If a coefficient of it survives modulo w^{c_1}, some original term survives and its other-copy monomials are not killed by the chosen functionals. Each has equal covector and column weights by invariance under diagonal conjugation. Consequently for the surviving term ω_j is just its weight of the part in copy j, divided by P^h. The last weight bound has two consequences. A defect above $3t$ at $j+h$ forces the weight of the part in this copy above $2tP^h$, and a bad weight of the part forces defect above t.

For g in the Levi, write C_g for the substitution $X_s\mapsto g_s^{-1}X_sg_s$ in every copy, and put $H_s=g_s$ for even s, $H_s=g_s^{-\mathsf{T}}$ for odd s. In each product defining $\mathcal{B}_g$ the adjacent factors $g_s^{-1}g_s$ cancel. At an odd embedding we use the polynomial identity

$$\operatorname{adj}(g_sAg_s^{-1})^{\mathsf{T}}=g_s^{-\mathsf{T}}\operatorname{adj}(A)^{\mathsf{T}}g_s^{\mathsf{T}}.$$

The two products of lengths $h,h+1$ contribute to different depth blocks. Projection onto either block commutes with H_s. Taking the symmetric powers and their tensor product therefore gives the identity over the integral polynomial ring

$$C_g\mathcal{B}=\rho_{\mathrm{let}}(g)^{-1}\mathcal{B}_g\rho_{\mathrm{let}}(g).$$

Fix j and let l_s be any of the invariant functionals above for $s\ne j$. For an integral coefficient vector v, put

$$F_g(X_j)=\left(\bigotimes_{s\ne j}l_s\right)(\mathcal{B}_g)\,\rho_{\mathrm{let}}(g)v.$$

Applying these functionals to the polynomial identity leaves

$$F_1(g_j^{-1}X_jg_j)=\rho_{\mathrm{let}}(g)^{-1}F_g(X_j).$$

Now contract with all the I_j and call the resulting matrix V. Since the I_s are invariant under conjugation in their respective copies, the basis identity gives

$$\rho_{\mathrm{let}}(g)Vv=\left(\bigotimes_{s}I_s\right)(\mathcal{B}_g)\rho_{\mathrm{let}}(g)v.$$

The first weight consequence at $j = i - h$ now shows that every coefficient of defect $> 3t$ at i in every Levi translate of Vv vanishes modulo w^{c_1}. Indeed, once the other copies are contracted the only surviving monomials in copy j are bad, and I_j kills them.

For the pairing, replace I_j^0 by I_j one copy at a time. Fix the other copies and a saturated halfway vector v'. Pair the uncontracted polynomial with v' using the diagonal weights. Its degree is at most D. Write this polynomial as $\Psi(X_j) = \sum_\beta b_\beta v'_\beta F_{1,\beta}(X_j)$. The adjunction of $\rho_{\mathrm{let}}(g^{-1})$ with $\rho_{\mathrm{let}}(g^{-\mathsf{T}})$ for the diagonal pairing gives

$$[X_j^\alpha]\Psi(g_j^{-1} X_j g_j) = \sum_\beta [X_j^\alpha] F_{g,\beta}(X_j)\, b_\beta \big(\rho_{\mathrm{let}}(g^{-\mathsf{T}}) v'\big)_\beta .$$

For a bad α, any first factor surviving modulo w^{c_1} forces the defect of β to exceed $t = 3kc_1$. Since $g^{-\mathsf{T}}$ belongs to the same Levi, the rank-two estimate of Section 21.2 makes the second factor divisible by w^{c_1}. Thus all the bad coefficients of every conjugate of the pairing polynomial are in (w^{c_1}). The congruence property of I_j shows that replacing I_j^0 does not change the pairing modulo w^{c_1}. With all identity evaluations the contraction is $p^{N_{\mathrm{orb}} c_0} 1 = p^{h_0} 1$. This proves the desired pairing congruence for V.

Finally take C to be the integral coefficient vectors whose Levi translates have coefficients of defect $> 3t$ divisible by w^{c_1}. Here the translates in C range over $L_r(W)$. The rank-two estimate puts all saturated halfway restrictions in C, and the full contraction takes values there. Given an integral v with $w^{c_1 - 1} v \in C$, lift any $\bar{g} \in L_r(k_W)$ to $g \in L_r(W)$ by lifting its two invertible blocks. For every exponent α of defect $> 3t$ at some embedding,

$$w^{c_1 - 1} (\rho_{\mathrm{let}}(g) v)_\alpha \in w^{c_1} \mathcal{O} \implies (\rho_{\mathrm{let}}(g) v)_\alpha \in w\mathcal{O} \subset \mathfrak{m}_{\mathcal{O}}.$$

Reduction commutes with each symmetric power and inverse transpose, so the tested coefficient of $\bar{g}\bar{v}$ vanishes. Thus $\bar{v} \in S$. The flag count uses the inequalities: $t \geq 2(k+1)$ and

$$8 N_{\mathrm{orb}} k t P^{2r} = 120 N_{\mathrm{orb}}^2 k^4 e N_{\mathrm{orb}}\, P^{2r} \leq (N_{\mathrm{orb}} k \log q)^5 P^{3r} \leq \theta a.$$

For the first inequality use $e N_{\mathrm{orb}} \leq \log q$, $N_{\mathrm{orb}}, k \geq 2$, and $\log q \geq 40 \log 2 > 27$, which follows from $q \geq a^4$ and $a \geq (N_{\mathrm{orb}} k)^5$. The final bound follows from the large-degree hypothesis and $0 < \theta \leq 1$. The flag estimate therefore gives $s_{\mathcal{P}} \dim S \leq (1 + \theta) d_{\mathrm{let}}$. This supplies both the congruence and the residue dimension used in the edge Smith argument.

21.7 The colength of the source Hecke operator

In this subsection, we combine the two restriction estimates of Section 21.1 with the congruence of Section 21.6. Smith normal form gives an explicit bound on the length of the quotient by the image of the Hecke operator and the prescribed power of p.

The depth count of Section 21.1 bounds the sum of the dimensions of the residue spaces above all positive depths by $C_{\mathrm{div}} d_{\mathrm{let}}$. Use C, V, S from Section 21.6. In particular $s_{\mathcal{P}} \dim S \leq (1+\theta) d_{\mathrm{let}}$, C contains the saturated halfway restrictions, and $w^{c_1 - 1} u \in C$ implies $\bar{u} \in S$. The pairing of V with a saturated halfway restriction agrees with that of $p^{h_0} 1$ modulo w^{c_1}.

Here is the linear algebra that uses those estimates. On an edge the two restrictions are the origin value acted on by $W_0 \rho_{\mathrm{let}}(g_e^{\mathsf{T}})$ and the terminus value acted on by $W_0 \rho_{\mathrm{let}}(\beta_b^{-1})$, respectively. At either endpoint every class is incident to an edge, by the forward coset description and the dual-edge bijection. On a fixed edge the restriction at that class is invertible over the

coefficient field, since W_0 and each change of coordinates are invertible there. Both restrictions therefore have column rank d_X. Factor them as $R_1 = Q_1 D_1$, $R_2 = Q_2 D_2$, with D_i square and the saturated bases Q_i admitting integral left inverses L_i. At a fixed class and an edge direction k_e, the active component of a saturated column of Q_i has the form $v = W_0 \rho_{\mathrm{let}}(k_e) F$ over the fraction field. The column is integral on every incident edge. Thus $W_0 \rho_{\mathrm{let}}(g) F$ is integral on every representative of $\mathcal{P}\backslash \mathrm{GL}_k(W)$ at that class, using the dual-edge bijection at the other endpoint. For $k' \in \mathcal{P}$, the matrix $W_0 \rho_{\mathrm{let}}(k') W_0^{-1}$ is integral: each substitution lowering depth by one has coefficient divisible by $p = w^2$. Thus these tests hold for every $g \in \mathrm{GL}_k(W)$, and in particular for every $g k_e$. Clearing the finitely many denominators of $\rho_{\mathrm{let}}(k_e) F$ by a power of w verifies the all-direction condition in Section 21.2 for v. Consequently every active component of Q_1, Q_2 belongs to C.

Let $A_0 = Q_2^{\mathsf{T}} U_e B$, where B is diagonal multiplication by b_μ and U_e is the direct sum of the integral invertible edge transitions.

Write B_{src} for the same diagonal b_μ on the input class coordinates. On the active coefficient of an edge, write $\rho = \rho_{\mathrm{let}}$. Adjointness gives $B \rho(g_e^{\mathsf{T}}) = \rho(g_e)^{\mathsf{T}} B$, hence

$$(W_0 \rho(\beta_b^{-1}))^{\mathsf{T}} B (W_0 \rho(g_e^{\mathsf{T}})) = (\rho(g_e) W_0^2 \rho(\beta_b^{-1}))^{\mathsf{T}} B.$$

The other tensor factors commute with B. Summing over the edges gives $D_2^{\mathsf{T}} A_0 Q_1 D_1 = Y_\lambda B_{\mathrm{src}}$, whose image lies in $Y_\lambda M$. To bound $v(\det D_i)$, write t_j for the Smith factors of one restriction, and v_j for the corresponding input basis. All parahoric cosets occur at that endpoint by the two edge descriptions above. Thus t_j divides every coefficient of $W_0 \rho_{\mathrm{let}}(g) v_j$ in each active factor at every class, for every $g \in \mathrm{GL}_k(W)$. Indeed passage between representatives acts integrally on the weighted lattice: in either parity a substitution decreasing the deep degree by one has a coefficient divisible by $p = w^2$ for each such crossing. The remaining tensor factors change by integral invertible matrices.

If $2v(t_j) > \delta$ for an integer $\delta \geq 0$, every coefficient of depth at most δ in the unweighted translates of v_j vanishes modulo the maximal ideal. Thus $\bar{v}_j$ belongs to the direct sum of d_X / d_{let} copies of $D_{\delta+1}$. These reduced basis columns are independent. Since $v(p) = 1$ and $w^2 = p$, we obtain

$$\begin{aligned} 2v(\det D_i) &= \sum_j 2v(t_j) \leq \sum_{\delta \geq 0} \#\{j : 2v(t_j) > \delta\} \\ &\leq \frac{d_X}{d_{\mathrm{let}}} \sum_{\delta \geq 0} \dim D_{\delta+1} \leq C_{\mathrm{div}} d_X. \end{aligned}$$

Applying this at both endpoints gives $v(\det D_1) + v(\det D_2) \leq C_{\mathrm{div}} d_X$.

On the edge space put $B' = \mathrm{diag}(b'_\mu)$ and extend V to each active factor. The invertible transitions U_e act on the other tensor factors and commute with B, B', V. Since $BB' = p^{h_0} 1$ and $L_2 Q_2 = 1$, the matrix $B' U_e^{-1} L_2^{\mathsf{T}}$ is a right inverse of A_0 up to p^{h_0}. Apply the average after this matrix, giving $B_0 = V B' U_e^{-1} L_2^{\mathsf{T}}$ with values in C. Then

$$A_0 B_0 - p^{2h_0} 1 = Q_2^{\mathsf{T}} B (V - p^{h_0} 1) B' L_2^{\mathsf{T}} \equiv 0 \pmod{w^{c_1}}.$$

The congruence pairs each saturated column of Q_2 with an integral column of $B' L_2^{\mathsf{T}}$. The values of Q_1 also belong to C.

Take Smith form for the image of B_0 in the complement of Q_1 given by L_1. The projection $1 - Q_1 L_1$ preserves C, since integral linear combinations of the columns of Q_1 belong to C. Keep those nonzero divisors t_j dividing $w^{c_1 - 1}$, and write u_j for their primitive columns. Then

$w^{c_1-1}u_j$ lies in C on each edge, so each u_j reduces into S at every edge. The columns of Q_1 also reduce into S. These two lists together are independent modulo the maximal ideal, since the primitive columns are basis vectors in the complement. If u_{keep} columns are kept, independence and the residue dimension bound give

$$d_X + u_{\mathrm{keep}} \le (d_X/d_{\mathrm{let}})s_{\mathcal{P}} \dim S \le (1+\theta)d_X.$$

Thus $u_{\mathrm{keep}} \le \theta d_X$. The other Smith terms are divisible by w^{c_1-1}.

Let G be the sum of the image of A_0Q_1 and the span of the kept A_0u_j. The near-inverse congruence gives $p^{2h_0}\mathcal{O}^{d_X} \subset G + w^{c_1-1}\mathcal{O}^{d_X}$. Since $h_0 \ge 2$, w^{c_1-1} is a nonunit multiple of p^{2h_0}. Nakayama's lemma gives $p^{2h_0}\mathcal{O}^{d_X} \subset G$. For finitely generated lattices $L' \subseteq L''$ of the same rank over the coefficient valuation ring, successive elimination at an entry of least valuation diagonalizes the inclusion. Thus $\mathrm{length}_p(L''/L') = \sum_i v(t_i)$, where t_i are the elementary divisors and $v(p) = 1$. This normalized length is additive along inclusions by multiplicativity of the determinants of the basis changes. Write $L = \mathcal{O}^{d_X}$, $z = p^{j_p}$, and $\Phi = A_0Q_1$. All the sums of images to which we apply this formula are finitely generated and contain zL.

The containment $p^{2h_0}L \subset G$ bounds the normalized length of $L/(G+zL)$ by $2h_0d_X$. The quotient between $\Phi L + zL$ and $G + zL$ is generated by the kept columns, each killed by z, so its normalized length is at most $j_p\#\{u_j\}$. Replacing ΦL by $\Phi D_1 L$ adds at most $v(\det D_1)$, since the intervening quotient is a quotient of L/D_1L.

Put $T = \Phi D_1 L + zL$. Then $D_2^{\mathsf{T}}T \subseteq D_2^{\mathsf{T}}\Phi D_1 L + zL$ since $D_2^{\mathsf{T}}(zL) \subseteq zL$. Multiplication by D_2^{T} identifies L/T with $D_2^{\mathsf{T}}L/D_2^{\mathsf{T}}T$, whose length is finite since $zD_2^{\mathsf{T}}L \subseteq D_2^{\mathsf{T}}T$. Length additivity thus gives

$$\mathrm{length}_p\big(L/(D_2^{\mathsf{T}}\Phi D_1 L + zL)\big) \le v(\det D_2) + \mathrm{length}_p(L/T).$$

The sum of these endpoint valuations is at most $C_{\mathrm{div}}d_X$ and $h_0 \le C_{\mathrm{div}}$. This proves

$$\mathrm{length}_p\big(M/(Y_\lambda M + p^{j_p}M)\big) \le (3C_{\mathrm{div}} + \theta j_p)d_X.$$

In the class basis of M, let $H = [\,Y_\lambda \mid p^{j_p}I_{d_X}\,]$. The maximal minors of H generate the zeroth Fitting ideal of its cokernel. Unimodular operations preserve this ideal. Diagonalizing Y_λ therefore gives, with $v_p(0) = +\infty$,

$$\min_{\#J=d_X} v_p(\det H_J) = \sum_{i=1}^{d_X} \min\{v_p(s_i(Y_\lambda)), j_p\},$$

where $s_i(Y_\lambda)$ are its Smith factors, including zeros. All entries of H lie in R_ι, since they lie in $\overline{\mathbb{Q}}_p$ and have nonnegative valuation. The same finite list of minors computes the minimum over R_ι and over $\mathcal{O}$. This proves the bound on the original lattice.

Finally $p^r \ge m$, $r \ge 1$, and $P \ge p$ give $P^{2r} \ge mp$. The size bound $r^2m \le k$ gives $r^2/(kP) \le 1/(mp)$. Hence

$$3C_{\mathrm{div}} \le \frac{6N_{\mathrm{orb}}a}{mp} = \frac{12A_\lambda}{mp}.$$

Proposition 21.1 (Colength of the source Hecke operator)**.** *For each $\lambda \in \Lambda$, the operator Y_λ preserves the original source lattice M. With $j_p = \lceil 2mfa/p \rceil$, the normalized colength satisfies*

$$\mathrm{length}_p\big(M/(Y_\lambda M + p^{j_p}M)\big) \le d_X\left(\frac{12A_\lambda}{mp} + \frac{j_p}{m^2 f}\right).$$

22 Norm and index estimates for the translated images

In this section, we combine the derivative estimates of Sections 20 and 21 with the calculations at the level primes. The common scalar of Proposition 7.10 gives the upper estimates on the family of Section 16.1.

22.1 Untwisted derivatives at the ramified prime

In this subsection, we specialize the integral CM calculation of Section 20.1 to $p = d$. The ramified uniformizer changes the exponent in the estimate of Section 20.2, including the contribution at exponent zero.

Put $p = d$, $\pi = \sqrt{-d}$, and normalize $v(p) = 1$. Here $v(\pi) = 1/2$, $p < x_1$, and $p \nmid MNq_1$. For twists with $p \nmid q_Q$, the translations in the evaluated weak-chain tensor may be taken supported on MNq_Q. The original vacuum's signed translation by an integral vector at a place fixes the vacuum there. Thus every arrow is standard of level one at p. Its analytic scale is $\pi_M s_{\mathrm{jet}}$. In the linear forms of Section 20.1, a_d and Ω_0 cancel, leaving the scalar $\iota(\pi_M p_F)$. We have $|\iota(\pi_M)|_p = 1$ since $\pi_M \bar{\pi}_M = M$ is prime to p, and $|\iota(p_F)|_p = 1$ since $p \nmid q_1$.

On the marked CM curve use $g = -\log \theta$. Average $y(az)$ and $g(az)$ over $a^{p-1} = 1$, with weights a^{-1} and a^{-2}, respectively, and denote the results by T and $g^\flat$. At the ramified place these roots lie in $\mathbb{Q}_p$, so their CM conjugates are themselves. The pullback gives $T = y + O(y^2)$ over $\mathcal{O}$ and $Q_a(z) = \exp(a^2 g(z) - g(az)) \in 1 + y\mathcal{O}[[y]]$. Since $y(az) \in y\mathcal{O}[[y]]$, we have

$$\prod_{a^{p-1}=1} Q_{a^{-1}}(az) = \exp((p-1)(g^\flat(z) - g(z))) \in 1 + y\mathcal{O}[[y]].$$

The constant-one $(p-1)$-st root is integral, since $p - 1$ is a unit. Thus $\exp(g^\flat - g) \in \mathcal{O}[[y]] = \mathcal{O}[[T]]$. The integral pullback of the parameter by π has reduced order at most p, by the torsion argument for the original curve. Its root-of-unity covariance then gives

$$T(\pi z) = \pi T(z) + b_1 T(z)^p + \sum_{j \geq 2} b_j T(z)^{1+j(p-1)}, \qquad b_1 \in \mathcal{O}^\times.$$

Choose $\varpi^{p-1} = -\pi/b_1$ and write $Y = T(\varpi z)/\varpi$. The equation $Y(\pi z)/\pi = Y - Y^p + \pi E(Y)$, with $E \in U^2\mathcal{O}[[U]]$, gives

$$Y \in \mathcal{O}[[z]], \qquad Y \equiv \sum_{j \geq 0} z^{p^j} \pmod{\pi}.$$

Indeed the recursion divides in degree $n > 1$ only by the unit $1 - \pi^{n-1}$, and the reduction is the unique zero-constant solution of $Y - Y^p = z$.

The integral even section $\theta(\pi z)$ gives $\exp(pg - g(\pi z)) = R(x)$ with $2 \deg R < p$. For $2 \leq n < p$, $\pi[z^n]g$ is integral. In degree two use the two CM pullbacks at $\beta, \bar{\beta}$ and subtract their quadratic coefficients. For $n > 2$ use $g'' = g''(0) + x$, since the factors $n(n-1)$ are units and $x \in \mathcal{O}[[T]]$ with $T = z + O(z^p)$. It follows that $pg - g(\pi z)$ belongs to $\pi\mathcal{O}[[z]]$ through degree $p - 1$, since $p = -\pi^2$.

Exponentiation preserves the congruence through those degrees because $k/2 - v(k!) \geq 1/2$ for $k \geq 1$. Triangular comparison using $x = z^2 + \cdots$ gives $R - 1 \in \pi\mathcal{O}[X]$. Taking the formal logarithm gives $pg - g(\pi z) \in \pi\mathcal{O}[[T]]$. The component of degree 2 modulo $p - 1$ thus satisfies

$$pg^\flat(z) - g^\flat(\pi z) = \pi(aT^2 + T^{p+1}H), \qquad a \in \mathcal{O}, \quad H \in \mathcal{O}[[T]].$$

With $c_\varpi = \varpi^{p+1}/p$ and $\Xi = -g^\flat(\varpi z)/c_\varpi$, the scalar $-\pi\varpi^2/(pc_\varpi)$ is a unit. The preceding identity gives, coefficientwise, $(1-\pi^n/p)\Xi_n \equiv \lambda[z^n]Y^2 \pmod{\pi}$ for some $\lambda \in \mathcal{O}$.

For $n = 2$ the multiplier is 2; for $n > 2$ it is a unit congruent to one. Hence

$$\Xi \in \mathcal{O}[[z]], \qquad \Xi \equiv \lambda(Y^2 - z^2/2) \pmod{\pi}.$$

In particular the coupling indices now include $e = 0$. The coefficients of $(z^{p^e}z)^{p^j}$ are 2λ for $e > 0$, λ for $e = 0, j > 0$, and $\lambda/2$ for $e = j = 0$.

Use the same resolution of the integral vacuum as for the exceptional jet. The basis entries are integral above p, since $p \nmid q_1$. On K, conjugation is identity modulo π; on the cyclotomic generator, the p^e-power is the corresponding Frobenius. Therefore

$$\overline{\tau(w)}^{p^e} \equiv \tau_e(w) \pmod{\pi}, \qquad \tau_e = \langle -p^e\rangle\tau.$$

This applies first on $\mathcal{O}_E$ and then after clearing denominators prime to p. The basis identity turns the single-curve terms into $\Gamma_e = \sum_{\tau,b,a} z_{\tau,b,a}^{p^e} z_{\tau_e,b,a}$. The congruence error has the form πH with integral coefficients and zero constant term. Since $v(\pi c_\varpi) = 1/(p-1)$ and $v((\pi c_\varpi)^n/n!) = s_p(n)/(p-1) \geq 0$, where $s_p(n)$ is the base-p digit sum, its exponential $\exp(\pi c_\varpi H)$ is integral coefficientwise. Scaling by the actual integral factor $\pi_M p_F$ preserves this form. We obtain the same matrix normal form as in the exceptional case, now for all $e \geq 0$ and with $v(\varpi) = 1/(2(p-1))$.

Use the null-direction proof with no charges or erasures and with each congruence modulus p there replaced by π. The coefficient ring modulo π has characteristic p, so the same linear congruences, using p^e-th roots of π where needed, work also for $e = 0$. Each determinant contribution is multiplied by $v(\pi) = 1/2$. In detail, with $u_\tau = a\mathbf{1}_{\Phi_a}(\tau)$ and c_τ, b_τ as in the column counts, write

$$E_{\tau,e} = \min(u_\tau/p^e, u_{\tau_e} + c_{\tau_e}) + c_\tau/p^e + \min(b_\tau/p^e, b_{\tau_e}).$$

The factorial bound gives the lower bound for every minor jet, and therefore every Schur entry,

$$\frac{B_j}{2(p-1)} - \frac{1}{2}\sum_{\tau,e\geq 0} E_{\tau,e} - fn_{j+1}^3(1 + \log_p \max(B_j, 1)).$$

If $B_j = 0$, the constant term is integral.

The positive contribution is $B_j/(2(p-1)) = \frac{1}{2}\sum_{e\geq 1} B_j/p^e$. Combining it with the deductions for $e \geq 1$ gives exactly the positive-exponent terms $D_{\tau,e}/2$ of the column counts. The $e = 0$ coupling gives solely $-\frac{1}{2}\sum_\tau E_{\tau,0}$, with partner $\langle -1\rangle\tau$. This is the required $d_{pj}(\varnothing)$, including the exponent-zero subtraction. The argument applies to every arrow with the same original tensor. Transport of bases and composition of the finite class sums preserve the gauge estimates. Since $p \neq N$, the final branching matrix is integral and its extra scalar is a unit. Thus the restricted chain has valuation at least $g_p - L_p$ at $p = d$, using the column comparison and the absence of collisions there.

22.2 Integrality at the two original level primes

In this subsection, we treat the two primes dividing M. The tensor in Section 19.3 can have denominators there, so we use the actual translation parameter in Section 20.1 to check the unit bound.

Let $\ell \mid M$, so $\ell = p_0$ or p_1. These primes split in K and are distinct from d, N. For a twist prime to ℓ, the admissible translations of the actual tensor satisfy $\bar{\pi}_M x$ integral above ℓ. This is the datum of standard level $\delta = \bar{\pi}_M$; the argument allows translations with denominators at those places. The genuine arrow scale in the notation of the ordinary jet bound is $c = \pi_M$, so for each homogeneous Schur entry of degree B_j the ordinary multiplier is

$$\left| \frac{c}{\bar{\delta} d} \right|_{\ell}^{B_j} = |d^{-1}|_{\ell}^{B_j} = 1.$$

The rational action permits moving both finite adelic arguments into the integral subgroup at all places above ℓ, by the Cayley chart and weak approximation.

Let A be the resulting finite adelic action on the Hom space, L its vacuum lattice, and $H = L \cap A^{-1}L$. At the places above ℓ, one has $AL_w = L_w$; hence $\ell \nmid [L : H]$. The vacuum expansion of Section 19.3 expresses the translate of $T(x)v_L$ as a sum of translates $T(A(x+y))v_L$, $y \in L/H$. Their coefficients are $\Delta_A/[L : H]$ times the character of the pair and a root of unity. Write $a = \det H_A$ with $H_A L = H$. Put $\vartheta = v_E| \cdot |_{\mathbb{A}_E}^{-1/2}$, where v_E is the full splitting character. Lemma 17.1 gives $\Delta_A = \vartheta_f(a)$. At infinity, $\vartheta_\infty(z) = \prod_v \overline{z_v}^{-1}$. If $a^h = \alpha u$ with $\alpha \in E^\times$ and $u \in \widehat{\mathcal{O}}_E^\times$, then

$$\Delta_A^h = \vartheta_f(a)^h = \prod_v \overline{\Phi_v(\alpha)}.$$

Here h can be taken to be the class number, and α is a unit at every place above ℓ since a is. Thus $v_\kappa(\Delta_A) = 0$ at every $\kappa \mid \ell$. For the determinant t of either finite argument, Section 4.4 similarly gives $t^h = \alpha_1 u_1$ in the norm-one torus for some $h > 0$, with $\alpha_1 \in T_1(F_0)$ and $u_1 \in T_{1,f}^0$. The finite character on u_1 is a root of unity, and its value on α_1 is the inverse infinite weight monomial on a unit at every place above ℓ. Thus the full pair character is a ℓ-unit, even at a place of nontrivial tame character. Finally $\delta A(x + y) = A(\delta x + \delta y)$ is integral there. Thus the span of the allowed translated vacua over the coefficient valuation ring is preserved, including under the actions at the other finite places.

Rational equivariance now gives the entrywise bound in the transported gauges at the original points. The finite class sums have trivial stabilizers and integral tensor coefficients.

The final branching has integral entries and unit scalar at $\ell \neq N$, since $\pi_N \bar{\pi}_N = N$. Consequently the restricted image of the full integral source lies in the terminal unit ball at every place above ℓ. The saving at $\ell \mid M$ is zero because its positive partner set is empty for a split prime. The distinct prime N uses the original preimage clearing and has saving zero from $N \geq x_1$.

22.3 Norm and index bounds for the Poincaré images

In this subsection, we combine the integral estimates with the complex norm bound of Section 7.4. The normalized tensor of Section 19.3 keeps both estimates on the same Poincaré map before applying the scalar of Proposition 7.10.

First consider the class-sum normalization of a finite upper tensor. The tensor is jointly invariant under the upper levels with the last action one. At an intermediate class variable the two adjacent kernels transform by inverse coefficient actions, and the contracted product with the invariant tensor is unchanged by the level. The source factor is already invariant in its original level. Thus the contracted integrand can be summed over the Cartesian product of the original upper classes.

Each quotient measure at a definite stage is multiplied by that stage's level index in the definition of the algebraic lift. Its mass on a class is the reciprocal of the class stabilizer order. All the intermediate levels are free by reduction at an old odd distinguished place, so every class has mass one in this calculation. One can therefore expand the finite tensor after choosing these class representatives. Individual translated vacuum terms need not have the separate level invariance of the combined tensor. Their terms are evaluations at the same representatives in this one finite sum.

For each permitted tuple x, write $V_j(x_j)$ for its translated upper vacuum on arrow j. Let $z(y)$ be the tuple of linked translations in Section 19.3.1, with one residue covector for every pair and every place dividing π_N. The components of $z(y)$ have denominator at most N and vanish at every place away from N. Since x can be taken to vanish above N, the Heisenberg phases between x and $z(y)$ are one. Consequently

$$\sum_{x} \Psi_{\xi}(x) v_{x}^{\mathrm{gl}} = \sum_{x,y} \Psi_{\xi}(x) \bigotimes_{j=0}^{m-1} V_j(x_j + z_j(y)).$$

The coefficients remain integral at every $p \nmid q_{\xi}$, also after equal tuples are collected. The original translation denominators are preserved away from N; above N the new denominators are retained. By linearity, applying the upper lifts and terminal restriction to this tensor identity gives equality on the full source, before intersection with its ramified subspace. At $p = N$ the estimate on that subspace uses the direct lifts of Section 19.4.2. In the charged calculation the local Hecke operators act on the spherical place and commute with the original level actions on these data, so the modified tensor has the same joint invariance and the same class-sum formula. A one-arrow gauge bound in the transported coordinates therefore composes inside each summand in the class sum. The ultrametric inequality then preserves the resulting bound under summing the classes and the integral tensor coefficients.

Proposition 22.1 (Norm and index on the original translates)**.** *Fix d, ϵ. For every sufficiently large integer x there is a spanning set $\mathcal{S}$ of everywhere integral forms in S such that, for every complex automorphism σ and $g \in G$,*

$$\|\sigma \Theta_1 P_g \phi\|^2 \le e^{A - m\mathcal{R} + L} \qquad (\phi \in \mathcal{S}).$$

For each $t \subseteq G$ and each independent tuple y of images $P_g\phi$ with $g \in t$ and $\phi \in S$ everywhere integral,

$$\mathrm{Ind}(y) \le 2|t| \dim(S)\, faB^3.$$

At each coefficient place over p, its clearing matrix works for every allowed twist of level prime to p.

Proof. Fix p and a coefficient place $\kappa = (p, \iota)$. When $p \neq N$ is split or satisfies $p \ge x_1$, use the whole integral ramified module, with gauge bound one. At nonsplit $p < x_1$, use that same module for the ordinary and ramified cases. In the exceptional case use

$$(Y_{\Lambda}^{m} M_{\iota} + p^{j_p} M_{\iota}) \cap S$$

from Sections 20.3 and 21, with M_{ι} the full original integral source. Indeed for $\varphi = Y_{\Lambda}^{m} u + p^{j_p} v$ in this intersection, the tensor identity above applies to $u, v \in M_{\iota}$. Its two terms have gauge at most $p^{2L_p - g_p}$ and $p^{L_p - g_p}$, respectively. The ultrametric bound therefore applies to φ even

when $p \mid N-1$. At N, use the preimage of the unit gauge at δ_0 under $U_0^{m/2}$, intersected with the original integral ramified source. These modules are fixed after the embedding and before choosing any twist or translate. At an exceptional prime put $J_\iota = Y_\Lambda^m M_\iota + p^{j_p} M_\iota$. There is an inclusion

$$(M_\iota \cap S)/(J_\iota \cap S) \hookrightarrow M_\iota/J_\iota.$$

Choose a Smith basis over $\mathcal{O}_{K_p}$ as in Section 20.3.3. After extending to $R_\iota = \iota(\mathcal{O}_\kappa)$, it gives $M_\iota = \bigoplus_i R_\iota e_i$ and $J_\iota = \bigoplus_i d_i R_\iota e_i$. All $d_i \neq 0$ since $p^{j_p} M_\iota \subseteq J_\iota$. On an integral tuple, successively impose divisibility by d_i in coordinate i. In a nonzero coordinate choose a coefficient of least valuation and subtract its multiples from the other columns. The ratios belong to R_ι, so these operations pull back to $\mathcal{O}_\kappa$. Multiply the pivot column by d_i. A zero coordinate requires no operation. All preceding divisibilities and membership in S are preserved by these column operations. The resulting square matrix therefore satisfies

$$v_\iota(\det U) \le \sum_i v_\iota(d_i) = \operatorname{length}_p(M_\iota/J_\iota) \le d_X L_p.$$

At N the successive coordinate restrictions of Section 19.4.3 give $mfbd_X$ directly. In the other cases the whole integral module suffices. The determinant valuation in each case is at most $d_X e_p$. For every allowed twist prime to p the chain gauge after restriction is at most $p^{e_p - g_p}$ on this module.

Let $z = c_{\mathrm{ss}}^{-1} \Pi\varphi$, using the common scalar of Section 19.3. Thus $L_\xi z$ is the normalized upper chain at φ on the original ramified source. Set

$$U_g\varphi = |G|\, \rho(e_0)\rho(g) z.$$

By the lower-chain equivariance of Section 17.3, $\Theta_\xi U_g \varphi$ is the sum of the values of $L_\xi z$ at the finite translates, with coefficients $|G| e_0(s)$. They are algebraic integers: for each irreducible character the coefficient of the central idempotent multiplied by $|G|$ is its degree times the character on the inverse class.

The indicated terminal translations are integral at the places over p. The transported gauge for a point and its right translate by an integral invertible matrix is the same on a fixed value. Thus the same bound $p^{e_p - g_p}$ holds after the projected translate at every terminal point, with no change of the local source module. This uses the original Θ_ξ and its prescribed level, so its unit gauges are exactly the lattices tested in the index bound.

Lemma 7.5, Proposition 7.8 and Lemma 7.3 give

$$\|\sigma \Theta_1 U_g \varphi\|^2 \le \exp\left(A - m\mathcal{R} + 2 \sum_{p \in F} g_p \log p + 2E_0 + x^{99} \right).$$

The same short generators span S. The gauge exponents are $2L_p - g_p = e_p - g_p$ at nonsplit $p < x_1$ other than d, at most $L_p - g_p \le e_p - g_p$ at d, and $2mfb = e_N$ at N, where $g_N = 0$. They are zero elsewhere. Using $\sum_p e_p \log p \le E_0$, the rational rescaling in Proposition 7.10 now gives the short length and index bounds for $P_g = c_B U_g$ on the original family. The span calculation uses these same images, which are unchanged by the nonzero scalar factors. All parameter estimates hold for each fixed d, ϵ at every sufficiently large integer x. □

23 The mixed Gram determinant and the product formula

In this section, we use the product formula to pass from valuations of mixed pairings to a global determinant inequality. We use the upper covolume bounds of Section 22, the critical-value bounds of Section 38, and the lattice completion of Section 44.

Use the normalized absolute value $|\ell|_\kappa = \ell^{-1}$ at a coefficient place $\kappa = (\ell, \iota)$, where $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ is a field isomorphism. Write $\bar{\kappa} = (\ell, \iota \circ \mathrm{conj})$. Say that a family f_κ has height at most j if there is a single $H \neq 0$ and bounds $\alpha \in \mathbb{R},\ e_\ell \geq 0$ such that

$$|\sigma H| \leq e^{\alpha}, \qquad |H f_\kappa|_\kappa \leq \ell^{e_\ell}, \qquad \alpha + \sum_{\ell \in U} e_\ell \log \ell \leq j$$

for every complex automorphism, every coefficient place and every finite set U of primes.

Height upper bounds add under multiplication by multiplying the multipliers and adding the bounds. For a nonzero constant z, if $|\sigma z| \leq e^{u}$ and the constant family z has height at most j, then $0 \leq u + j$. Indeed, use its multiplier H. The bound for the orbit of Hz implies Hz is algebraic. A transcendental element can be sent to one of arbitrarily large absolute value by extending a bijection of transcendence bases to the algebraic closures. The same argument applies to a uniform finite-place bound for a fixed cleared coordinate. A finite list of such algebraic coordinates belongs to one number field, and its embeddings are the restrictions of the existing coefficient or complex maps.

Each embedding of $\mathbb{Q}(Hz)$ in $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$ extends to a map of the required type by matching transcendence bases and then extending to their algebraic closures. Both fields have cardinality continuum. For a transcendence basis B, the algebraic closure of $\mathbb{Q}(B)$ has cardinality $\max(\aleph_0, |B|)$. Hence the two transcendence degrees are equal. Write $K = \mathbb{Q}(Hz)$ and take the finite set

$$U = \{\ell :\ |Hz|_w \neq 1 \text{ for some finite place } w \mid \ell \text{ of } K\}.$$

For every $\ell \in U$, the local degrees above ℓ sum to $[K : \mathbb{Q}]$; the archimedean degrees sum to the same number. Dividing the logarithmic product formula [46, Theorem 7.15] by $[K : \mathbb{Q}]$ therefore gives

$$0 \leq u + \alpha + \sum_{\ell \in U} e_\ell \log \ell \leq u + j.$$

Let $w = (y_1, \dots, y_n, z_1, \dots, z_s)$ be a basis of the target space, and Θ_ξ its injective lift for twist ξ, with trivial twist 1. Set

$$G_\xi = \det \mathrm{Gram}(\Theta_\xi w), \qquad Z_\xi = \det(\Theta_\xi^* \Theta_\xi).$$

Expressing w in an orthonormal basis gives $G_\xi = Z_\xi \det \mathrm{Gram}(w)$.

Proposition 23.1 (Mixed Gram determinant inequality)**.** *For the basis $w = (y, z)$ and the injective comparison lifts above, suppose that the following bounds hold.*

- *For every $\sigma \in \mathrm{Aut}(\mathbb{C})$, the Gram determinant of the coordinatewise conjugates $\sigma\Theta_1 y_i$ is at most e^{c}, and each $\sigma\Theta_1 z_i$ has squared norm at most $e^{b_{\mathrm{short}}}$. The positive definite Hermitian matrix of the trivial lift's coordinate metric is rational.*
- *Fix $a_\ell \geq 0$, independent of the place above ℓ, with $\sum_{\ell \in U} a_\ell \log \ell \leq \delta$ for every finite set of primes U. At each κ above ℓ take a clearing matrix U_κ for the y_i, with κ-integral entries and $|\det U_\kappa|_\kappa \geq \ell^{-a_\ell}$. Its cleared vectors lift integrally at κ for every allowed twist at ℓ.*

- *For each κ choose an allowed twist ξ_κ and write $\gamma = \gamma(\xi_\kappa) \neq 0$. The vectors $\gamma\Theta_{\xi_\kappa} z_i$ are integral at both κ and $\bar{\kappa}$. The pairings of the $\bar{\kappa}$-integral lattice against the κ-integral lattice have absolute value at most 1 at κ.*
- *The families $(\gamma\bar{\gamma})^{-1}$ and Z_1/Z_{ξ_κ} have heights at most j_γ, j_Z.*

Then

$$0 \leq c + 2\delta + s(b_{\mathrm{short}} + j_\gamma) + j_Z.$$

Proof. The terminal mixed Gram form. The pairing in the determinant is at the original lift level, principal at the old marked places and parabolic at the twist places. The dual delta used for the period at the end of the path has the smaller principal level at the twist places. The pointwise Weyl gauge checked by those deltas is the gauge of the same lifted vector. Thus the check applies to its values on the class representatives at the lift level.

Every stabilizer at the lift level is trivial. At any one old odd completely split place a stabilizer embeds in $1 + p_i M_{n_{\mathrm{term}}}(\mathbf{Z}_{p_i})$ after conjugation by the class representative. A finite-order element of this group is one by the binomial-depth calculation. Thus the lift-level Gram pairing is the plain class sum. Write K_{par} for this level and $K_{\mathrm{pr}} = K(\pi_M Q(\xi))$ for the level in (104). Above the class of g, the principal-level classes have representatives gk, $k \in K_{\mathrm{par}}/K_{\mathrm{pr}}$. The two values acquire the same unitary level character at k, so their Hermitian pairing is unchanged. The principal-level class sum would therefore carry the factor $[K_{\mathrm{par}} : K_{\mathrm{pr}}]$, the ratio of the two level indices in Section 44.1. We establish integrality at each representative g of the original level.

Write T for the transport by g^{-1} at the embedding κ, and $T^{\#}$ for the equivalent transport on the conjugate coefficient vector. With a the local matrices of g read at κ, unitarity reads $T = \rho(a^{-1})$, $T^{\#} = \rho(a^{\mathsf{t}})$. The integral rational Weyl Gram matrix J satisfies $\rho(a^{\mathsf{t}})^{\mathsf{t}} J = J\rho(a)$, by the transpose rule on the exterior tensor construction. Hence $(T^{\#})^{\mathsf{t}} J T = J$. Consequently the mixed term at κ is

$$\iota\big(\langle v_-, v_+\rangle_J\big) = \big(T^{\#}\iota(\overline{v_-})\big)^{\mathsf{t}} J\big(T\iota(v_+)\big).$$

Its coordinates are integral when v_- has unit gauge at $\bar{\kappa}$ and v_+ at κ. Summing over the free classes is integral. For the trivial twist the coordinate metric is a direct sum of the same rational Gram matrices, which is the metric used in the archimedean exterior-product inequality.

Use the convention that the first variable of the Hermitian form is conjugate-linear. Fix κ and put $\xi = \xi_\kappa$. Since $\bar{\kappa}$ lies above the same ℓ, its all-twist clearing matrix applies to this ξ as well. Use the block matrices $\mathrm{diag}(U_\kappa, \gamma I_s)$ and $\mathrm{diag}(U_{\bar{\kappa}}, \gamma I_s)$ on the same $\Theta_{\xi_\kappa} w$, with the latter matrix on the conjugated side of the pairing. Taking the determinant of the matrix of integral mixed pairings gives

$$|\det U_{\bar{\kappa}}|_{\bar{\kappa}}\, |(\gamma\bar{\gamma})^s G_{\xi_\kappa}|_\kappa\, |\det U_\kappa|_\kappa \leq 1.$$

Thus $((\gamma\bar{\gamma})^s G_{\xi_\kappa})_\kappa$ has height at most 2δ, with multiplier 1. Now

$$G_1 = \big((\gamma\bar{\gamma})^s G_{\xi_\kappa}\big)\,(\gamma\bar{\gamma})^{-s}\,\frac{Z_1}{Z_{\xi_\kappa}},$$

so the constant G_1 has height at most $2\delta + s j_\gamma + j_Z$.

On the archimedean side, the rational coordinate metric identifies σG_1 with the determinant of the mixed pairing of $(\operatorname{conj}\sigma\operatorname{conj})\Theta_1 w$ and $\sigma\Theta_1 w$. Write $H_\tau(v) = \det \operatorname{Gram}(\tau\Theta_1 v)$. Gram–Schmidt and the norm bounds give

$$H_\tau(w) \le H_\tau(y) \prod_{i=1}^{s} \|\tau\Theta_1 z_i\|^2 \le e^{c+sb_{\mathrm{short}}}.$$

Exterior-power Cauchy–Schwarz, with $\bar\sigma = \operatorname{conj}\circ\sigma\circ\operatorname{conj}$, gives

$$|\sigma G_1|^2 \le H_\sigma(w) H_{\bar\sigma}(w) \le e^{2(c+sb_{\mathrm{short}})}.$$

Since $G_1 \neq 0$, the preceding product-formula argument yields

$$0 \le c + 2\delta + s(b_{\mathrm{short}} + j_\gamma) + j_Z.$$

The same twist is needed on the two sides of each mixed pairing. The individual clearing matrices may depend on the two coefficient places. The volume and denominator heights are separate arithmetic estimates on the allowed twists. □

Proposition 44.3 gives the spanning set and the clearing-scalar height. Proposition 44.2 gives the determinant-ratio height. Section 45 substitutes them into this inequality to prove Proposition 5.2.

Part IV
Selmer bounds and Galois congruences

24 Length bounds after removing auxiliary Selmer conditions

In this section, we estimate the change in the length of the cocycle module when auxiliary local relations are discarded. The local bounds below are used in the polynomial estimate of Section 25.

Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_\ell$. Normalize $v(\ell) = 1$ and put $\mathcal{B} = \mathcal{O}/\ell^M\mathcal{O}$ for $M \ge 1$. In this section lengths are measured by $\lambda(X) = \operatorname{length}_{\mathcal{O}}(X)/v_{\mathrm{ord}}(\ell)$, where v_{ord} gives a uniformizer valuation one.

Lemma 24.1 (Length lost by imposing local relations)**.** *Let ρ be a continuous integral Galois representation of rank n. At each place in a finite list let ϕ_j be an arithmetic Frobenius lift or its inverse, let γ_j belong to inertia, and suppose $\rho \bmod \ell^M$ is unramified there. All these places have residue characteristic different from ℓ and residue cardinality $1 \bmod \ell^N$. Let $s_j, y \in \mathcal{O}$, and let ψ be a continuous scalar character into $\mathcal{B}^\times$. Suppose*

$$\psi(\phi_j) = 1 + y, \quad \psi(\gamma_j) = 1 + s_j y, \qquad s_j \neq 0 \implies \rho(\phi_j) \equiv I \bmod \ell^M.$$

For every element of inertia at the listed places, suppose its scalar α under ψ satisfies

$$\sum_{a=0}^{\ell^N - 1} \alpha^a = 0 \qquad \text{in } \mathcal{B}.$$

Put $R = \psi(\rho \bmod \ell^M)$. Let C be an $\mathcal{O}$-submodule of the continuous R-cocycles with values in $\mathcal{B}^n$. Write C_0 for its submodule defined by $z(\gamma_j) = s_j z(\phi_j)$ for all j. Then

$$\lambda(C/C_0) \le v\left(\prod_j \det((1+y)\rho(\phi_j) - I)\right).$$

Proof. For $z \in C$, write $A_g(u) = z(g) + R(g)u$ for the affine action on $\mathcal{B}^n$. If g belongs to a listed inertia group, $R(g) = \alpha I$, so

$$A_g^{\ell^N}(u) = \alpha^{\ell^N} u + \left(\sum_{a=0}^{\ell^N - 1} \alpha^a\right) z(g) = u.$$

Indeed $\alpha^{\ell^N} - 1 = (\alpha - 1)\sum_{a<\ell^N} \alpha^a$. The finite continuous affine image of wild inertia is trivial: it is a finite group of order a power of the residue characteristic and all its elements have order dividing ℓ^N. In the tame quotient arithmetic Frobenius conjugates inertia by the residue cardinality [16, Lemma 5.1]. Thus A_{ϕ_j} commutes with A_{γ_j}. Commutation persists when the Frobenius lift is inverted.

Put $F_j = (1+y)\rho(\phi_j) - I$. Evaluating the commuting affine maps at zero and applying the condition on $\rho(\phi_j)$ gives, on $\mathcal{B}^n$,

$$F_j z(\gamma_j) = s_j y z(\phi_j), \qquad s_j F_j = s_j y I.$$

These identities give the injection

$$C/C_0 \hookrightarrow \bigoplus_j \ker F_j, \qquad [z] \longmapsto (z(\gamma_j) - s_j z(\phi_j))_j.$$

Here $\ker F_j$ denotes the kernel on $\mathcal{B}^n$. If each integral matrix F_j is nonsingular over $\mathrm{Frac}(\mathcal{O})$ with Smith diagonal $(u_{j,i})_i$, its reduction gives

$$\lambda(C/C_0) \le \sum_j \lambda(\ker F_j) = \sum_{j,i} \min\{M, v(u_{j,i})\} \le v\left(\prod_j \det F_j\right).$$

A singular F_j gives the bound $+\infty$. □

For the auxiliary twist constructed from homomorphisms χ_w into $\mathbb{Z}/\ell^N\mathbb{Z}$, use variables $x_w \in y\mathcal{O}$ and put $\psi(g) = \prod_w (1+x_w)^{\chi_w(g)}$ in $\mathcal{B}$. The scalar condition in Lemma 24.1 follows if $\sum_{a<\ell^N} \alpha^a$ is divisible by ℓ^M for every $\alpha \in 1 + y\mathcal{O}$. In particular each $(1+x_w)^{\ell^N} = 1$ in $\mathcal{B}$. This makes the character well defined. With a first coordinate $x_0 = y$, the other coordinates are $x_w = s_w y$. At each subsequent place, the character values needed for the lemma are 1 in coordinate 0 and 0 elsewhere at ϕ_j, and 1 in coordinate j and 0 elsewhere at γ_j. These statements concern the chosen pairs at both conjugate places, with inverse lifts where used. They give the two evaluations of ψ in the lemma. The polynomial whose valuation bounds the length loss is consequently $\prod_j \det((1+Y)\rho(\phi_j) - I)$.

For a common lower bound $v(y) \ge a_0 > 0$, the choice of N ensuring the scalar condition can be made uniformly. For $z \in y\mathcal{O}$ the polynomial identity

$$\sum_{a=0}^{\ell^N - 1} (1+z)^a = \sum_{j=1}^{\ell^N} \binom{\ell^N}{j} z^{j-1}$$

shows that the right side has valuation at least $N - \sup_{j\ge 1}\{\log_\ell j - (j-1)a_0\}$. Here $j\binom{\ell^N}{j} = \ell^N\binom{\ell^N-1}{j-1}$, and the supremum is finite. At the first place and its conjugate use the untwisted specialization $y = 0$, with slope zero. The scalar condition then reduces to $N \ge M$. In this specialization the kernel matrices only depend on the reductions of the first Frobenius matrices. Thus their length contribution is bounded by $\sum_j v(\det(\rho(g_j) - I))$ for any chosen g_j with $\rho(g_j) \equiv \rho(\phi_j) \bmod \ell^M$, with the sum over the two first-place parameters. This uses the kernel bound before replacing the lifts in the determinant.

24.1 The denominator bound for odd cocycles

In this subsection, we isolate the odd cocycles that contribute to the length bounds of Section 24. We reduce their estimation to the annihilator of Section 24.2 and the detection argument of Section 24.3.

Set $N_c = dq_1$, $t = 5r$, and $y = N_c + t + 3$. The field E of Section 4 embeds in $\mathbb{Q}(\zeta_{N_c})$. For a coefficient DVR $\mathcal{O}$ as above, use ordinary length in this subsection. Write $Z^-_{e,\nu}$ for the module of continuous cocycles $G_E \to (\mathcal{O}/\ell^\nu)(-e)$ that vanish on inertia outside ℓ. For $e > 0$ require

$$z(cgc^{-1}) = -z(g), \qquad z|_{D_v \cap \ker \chi_{\mathrm{cyc}}} = 0$$

for every decomposition group D_v at a prolongation of a place above ℓ, using the full ℓ-adic cyclotomic character. For $e = 0$ require vanishing on inertia above ℓ and impose no sign. Lengths are taken on these modules of cocycles themselves.

For $e = 0$ the action is trivial. Since E has no real places, global reciprocity [48, Ch. V, Theorems 5.3 and 5.5] gives

$$Z^-_{0,\nu} = \mathrm{Hom}(\mathrm{Cl}(E), \mathcal{O}/\ell^\nu).$$

Write $h_E = |\mathrm{Cl}(E)|$. If the class group has cyclic factors of orders d_i, then

$$\mathrm{length}_{\mathcal{O}} Z^-_{0,\nu} = \sum_i \min\{\nu v_{\mathrm{ord}}(\ell), v_{\mathrm{ord}}(d_i)\} \le v_{\mathrm{ord}}(h_E).$$

Detection at small coefficient primes. Suppose that $\ell \le y^2$. There is one list of at most y^{20} elements of G_E whose evaluations detect every cocycle unramified outside ℓ with values in $(\mathcal{O}/\ell^\nu)(-e)$, for all $\nu, e, \mathcal{O}$. Put $m_0 = N_c\ell^2$, $K' = \mathbb{Q}(\zeta_{m_0})$ and $s' = [K' : \mathbb{Q}] \le m_0$. Use representatives of $G_E/G_{K'}$, together with lifts of generators of the maximal abelian quotient of $G_{K'}$ of exponent dividing m_0 unramified outside ℓ. By the idelic class sequence its size is at most

$$h_{K'} \prod_{w|\ell} |{K'_w}^\times/({K'_w}^\times)^{m_0}|.$$

The valuation factor, finite cyclic torsion, and free principal-unit factor of rank $[K'_w : \mathbb{Q}_\ell]$ bound the local quotient by $m_0^{2+[K'_w:\mathbb{Q}_\ell]}$.

Their product is at most $m_0^{3m_0}$. In a number field L' of degree s', each class has an integral ideal of norm at most $\sqrt{|D_{L'}|}$ by Minkowski's bound [46, Theorem 4.3]. There are at most $h^{s'}$ ideals of norm h, since over each rational prime p there are at most s' prime ideals and the number of exponent assignments is at most $(1 + v_p(h))^{s'}$. Hence $h_{L'} \le (2|D_{L'}|)^{s'+1}$. Using $|D_{K'}| \le m_0^{s'}$ bounds the size of the class-field quotient by $2^{m_0^3+5m_0^2+m_0}$. Thus it has a generating set of size at most $m_0^3+5m_0^2+m_0$. With the coset representatives the total is at most $m_0^4 \le y^{20}$, since $m_0 \ge 4$ and $m_0 \le y^5$.

If a cocycle z vanishes at those elements, induct on the power of ℓ annihilating its values. The induction hypothesis applied to $m_0 z$ gives $m_0 z = 0$. On these values $G_{K'}$ acts trivially, since it fixes the m_0-th roots of unity. The restriction of z is now a homomorphism through the indicated class-field quotient. It vanishes on the generators, hence on $G_{K'}$, and the coset representatives give vanishing on all of G_E by the cocycle identity. This includes $\ell = 2$.

Assembly of the bound. For the positive twists, take $\mathcal{O}$ to be the integers of a finite extension of $\mathbb{Q}_\ell$ containing a primitive root of unity of order $\exp((\mathbb{Z}/N_c)^\times)$. For $0 < e < t$, Section 24.2 provides an annihilator $Y_e \le 2^{4y^6}$ independent of $\ell, \nu, \mathcal{O}$ under this condition. The two detection arguments, here for $\ell \le y^2$ and in Section 24.3 for $\ell > y^2$, give at most y^{20} evaluations. Evaluation therefore embeds $Z^-_{e,\nu}$ into at most y^{20} copies of $(\mathcal{O}/\ell^\nu)[Y_e]$, of length at most $v_{\mathrm{ord}}(Y_e)$ each. Together with the zero twist this gives one integer

$$X_{\mathrm{odd}} = h_E \left(\prod_{0<e<t} Y_e \right)^{y^{20}}, \qquad \mathrm{length}_{\mathcal{O}} Z^-_{e,\nu} \le v_{\mathrm{ord}}(X_{\mathrm{odd}}), \qquad \log X_{\mathrm{odd}} \le \log h_E + 4y^{27} \log 2.$$

To remove the assumption on the roots of unity, let $\mathcal{O}'$ be the ring of integers after adjoining them to the fraction field of $\mathcal{O}$. The ring $\mathcal{O}'$ is finite free over $\mathcal{O}$. In any $\mathcal{O}$-basis of $\mathcal{O}'$, the cocycle, inertia, local cyclotomic-kernel and parity conditions hold coefficientwise because the action on the coefficients is scalar. Thus

$$Z^-_{e,\nu}(\mathcal{O}') = Z^-_{e,\nu}(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'.$$

Both the length and $v_{\mathrm{ord}}(X_{\mathrm{odd}})$ multiply by the ramification index of $\mathcal{O}'/\mathcal{O}$. This proves the same bound over the original coefficient ring. Dividing by $v_{\mathrm{ord}}(\ell)$ gives $v_\ell(X_{\mathrm{odd}})$ per exponent. The ideal-class estimate above, applied to E, and $\mathrm{rd}(E) \le N_c$ give

$$\log h_E = O(f^2 \log N_c), \qquad N_c = O_d(x^{11} \log x), \quad t = O(x^5).$$

Consequently, for every finite set of primes U,

$$\sum_{\ell \in U} v_\ell(X_{\mathrm{odd}}) \log \ell \le \log X_{\mathrm{odd}} \le x^{2000}$$

for all sufficiently large x. Summing the odd contributions of at most t ordered exponents therefore contributes at most tx^{2000} to the logarithmic denominator bound.

Application to a paired cocycle test. For $e > 0$ let S be a module of cocycles unramified outside ℓ satisfying the local vanishing above ℓ. Suppose evaluations at g_0 and a finite set of points detect all unramified-outside-ℓ cocycles of this type. Put $d_e = \chi_{\mathrm{cyc}}(g_0)^{-e} - 1$. Assume $d_e z(g_0) = 0$ for $z \in S$, and that at each test point a the paired Frobenius conditions give $2(z + z \circ c)(a) = 0$, where $(z \circ c)(g) = z(cgc^{-1})$. Under these hypotheses $z \mapsto z \circ c - z$ sends S to $Z^-_{e,\nu}$. The local conditions are invariant under conjugation by c.

In the kernel $z = z \circ c$, hence $4z(a) = 0$ at all test points. The cocycle $4d_e z$ therefore vanishes at every test point and at g_0, so it is zero. If $4d_e \mid C$, in particular for the prescribed multiple of $4d_e^2$, the kernel is contained in $S[C]$, and consequently

$$\mathrm{length}_{\mathcal{O}}(S/S[C]) \le \mathrm{length}_{\mathcal{O}} Z^-_{e,\nu}.$$

Supply the paired tests and separation scalar on the same cocycle module over the same coefficient ring, with its roots of unity as above. The length comparison uses integral multiplication throughout, including at $\ell = 2$.

24.2 A uniform annihilator for the positive twists

In this subsection, we construct a common integer that controls the positive exponents appearing in Section 24.1. The distribution identity for the Bernoulli polynomial also supplies the auxiliary units needed in Section 24.3.2.

Fix $e > 0$, abbreviate N_c to N, and put $u = N + e + 3$. We construct an integer $0 < X_e \le 2^{u^6}$. For each prime ℓ it supplies a positive integer $a \equiv 1 \pmod N$, prime to ℓ, with $(a^e - 1)(a^{e+1} - 1) \mid X_e$. For a character ψ of $(\mathbf{Z}/N)^\times$ with $\psi(-1) = -(-1)^e$ and values in $\mathcal{O}$, set

$$S_m(a,\psi) = \sum_{b \in (\mathbf{Z}/Nm)^\times} b^e \psi(b) \left(\left\lfloor a \left\langle \frac{b}{Nm} \right\rangle \right\rfloor - \left\lfloor a \left\langle \frac{-b}{Nm} \right\rangle \right\rfloor \right) \pmod m,$$

where angle brackets denote fractional part. For $m = \ell^\nu$, this scalar in $\mathcal{O}/m\mathcal{O}$ divides $(a^{e+1} - 1)X_e$ for every positive $a \equiv 1 \pmod N$ prime to ℓ.

Construction of the integer. For the Bernoulli polynomials put

$$B(b) = (e+2)! N^{e+1} B_{e+1}(b/N), \qquad P(X) = (e+1)! N^{e+2} (B_{e+2}(X/N) - B_{e+2}(0)).$$

Both have integer coefficients. In fact $(j+1)!B_j$ is integral by the Bernoulli recursion. In the coefficient of X^k in P, $j = e + 2 - k$. For $k \ge 2$, $(j+1)! \mid (e+1)!$, and for $k = 1$ the binomial coefficient contributes $e + 2$. The identities $P' = B$ and $P(b+N) - P(b) = N(e+2)! b^{e+1}$ now give

$$(e+2)! \sum_{i=0}^{m-1} (b + Ni)^{e+1} \equiv mB(b) \pmod{m^2}.$$

Indeed, telescope the difference and use the first-order Taylor expansion of the integer polynomial at increment Nm, then divide by N.

Extend ψ by zero on nonunits. Use the sums

$$T_m = \sum_{0 \le b < Nm} \left\lfloor \frac{ab}{Nm} \right\rfloor b^e \psi(b), \qquad U_\psi = \sum_{0 \le b < N} B(b)\psi(b).$$

Multiplication by a permutes residues modulo Nm and preserves ψ. Modulo $(Nm)^2$, expand

$$\big(ab - Nm\lfloor ab/(Nm) \rfloor\big)^{e+1}.$$

Multiply by $(e+2)!$, sum, group by the residues modulo N, and cancel m in the congruence modulo m^2. This gives

$$(a^{e+1} - 1)U_\psi \equiv (e+2)!(e+1)Na^e T_m \pmod m. \tag{79}$$

On $G = (\mathbf{Z}/N)^\times$ define the integer matrix

$$A_{s,t} = 2B(t/s) + (-1)^e \mathbf{1}_{t=s} + \mathbf{1}_{t=-s},$$

with arguments read in $[0, N)$. Its character vector has eigenvalue $2U_\psi + (-1)^e + \psi(-1)$. Grouping the Dirichlet series by residue and applying the Hurwitz special value gives $U_\psi = -(e+2)!(e+1)NL(-e,\psi)$. If $N = 1$, use $B_{e+1}(0) = B_{e+1}(1)$. The primitive functional equation gives vanishing at $-e$ precisely when $\psi(-1) = (-1)^e$, since the Euler product at $1 + e$

converges absolutely to a nonzero value. Imprimitive Euler factors at $-e$ are nonzero for $e > 0$. The two formulas are [58, (25.11.14) and (25.15.5)]. The eigenvalues are therefore $2(-1)^e$ on the vanishing parity and $2U_\psi \neq 0$ on the other parity, so $\det A \neq 0$.

Take

$$X_e = |\det A| \prod_{a \in \{1+N, 1+2N\}} (a^e - 1)(a^{e+1} - 1).$$

The two choices for a are relatively prime. The Bernoulli Fourier series [58, (24.8.3)] bounds $|B_{e+1}(x)|$ by $17(e+1)!$ on $[0,1]$. With $d' = \varphi(N)$ and $M' = (e+2)!N^{e+1}(e+1)!$, this gives $|\det A| \leq d'!(64M')^{d'}$. Use $d' \leq u$, $M' \leq u^{3u}$ and bound the four factors by $(2u)^u$ each. Then $\log_2 X_e \leq 3u^3 + 5u^2 + 10u \leq u^6$ for $u \geq 5$.

For the odd parity the eigenvalue is $2U_\psi$. The adjugate identity in the identity coordinate of the character vector shows $X_e = 2U_\psi q_\psi$ with $q_\psi \in \mathcal{O}$. Multiply (79) by $2q_\psi$ to obtain

$$(a^{e+1} - 1)X_e \equiv 2wT_m \pmod{m}, \qquad w = q_\psi(e+2)!(e+1)Na^e.$$

The same w works at every power of ℓ. Write $T_{\ell^j} = V_j + \ell^e \psi(\ell) T_{\ell^{j-1}}$, where V_j is the sum on units modulo $N\ell^j$. Subtract the scaled congruence at level $j-1$, using $e \geq 1$, to get

$$(1 - \ell^e \psi(\ell))(a^{e+1} - 1)X_e \equiv 2wV_j \pmod{\ell^j}.$$

For $j = 1$ the level-zero congruence is automatic. Replacing the representative b by that of $-b$ changes $b^e\psi(b)$ to its negative modulo ℓ^j, so $S_{\ell^j}(a, \psi) = 2V_j$ in the quotient. The scalar $1 - \ell^e\psi(\ell)$ is a unit there. This proves the asserted divisibility, also for $\ell = 2$.

24.2.1 The Jacobi quotient

In this subsection, we compute the split-prime Jacobi quotient for the odd cocycles of Section 24.1.

Let $m = \ell^\nu$ with $\nu > 0$, let $p \equiv 1 \pmod{Nm}$ be prime, and fix a place B above p. Write θ_j for the multiplicative characters of $\mathbf{F}_p^\times$, indexed by $j \in \mathbf{Z}/(p-1)$. Fix a nontrivial additive character ξ and put $G_j = \sum_{b \neq 0} \theta_j(b)\xi(b)$. Normalize $v_B(p) = 1$. Inertia fixes the multiplicative characters and acts on ξ by scaling its argument. The resulting scalars for G_j multiply in j. Thus $G_{j+k}/(G_jG_k)$ is inertia-fixed and has integral order, so $j \mapsto v_B(G_j) \bmod \mathbf{Z}$ is additive. For $j \neq 0$, reduction of the additive character to one gives positive order, and $G_j \sum_{b \neq 0} \theta_{-j}(b)\xi(-b) = p$ gives order less than one. The product identity follows by summing first over the ratio of the two nonzero arguments: the inner additive sum is $p-1$ at ratio one and -1 otherwise. For $j = 0$, $G_0 = -1$. Hence $j \mapsto v_B(G_j) \bmod \mathbf{Z}$ identifies the character group with the $(p-1)$-torsion of $\mathbf{Q}/\mathbf{Z}$.

If x corresponds to j, put

$$J(x) = \frac{G_j^a / G_{aj}}{G_{-j}^a / G_{-aj}}.$$

A Galois change of additive character cancels in each ratio G_j^a/G_{aj}. A change of multiplicative character sends j to its cyclotomic multiple. Thus if x has denominator dividing Nm, then $J(x) \in \mathbf{Q}(\zeta_{Nm})^\times$, and its conjugate under $b \in (\mathbf{Z}/Nm)^\times$ is $J(bx)$. The Gauss product shows that these numbers are units off p. By the least nonnegative orders above,

$$v_B(J(x)) = \lfloor a\langle x \rangle \rfloor - \lfloor a\langle -x \rangle \rfloor.$$

24.2.2 Annihilation on character components

In this subsection, we apply reciprocity to the Jacobi quotient of Section 24.2.1.

Write $C = \mathcal{O}/\ell^{\nu}\mathcal{O}$, $K' = \mathbf{Q}(\zeta_{Nm})$ and $\Upsilon = \chi_{\mathrm{cyc}}^{e}\psi$. Let $z_{\psi} : G_{\mathbf{Q}} \to C$ be a cocycle for Υ^{-1}, with $z_{\psi}(c) = 0$ at complex conjugation c. Assume it vanishes on inertia off ℓ lying in $G_{\mathbf{Q}(\zeta_N)}$, and on the elements of decomposition groups over ℓ that fix ζ_N and have $\chi_{\mathrm{cyc}} = 1$. Suppose $\Upsilon(c) = -1$. For this annihilation argument take $a \in \{1+N, 1+2N\}$ prime to ℓ; thus $(a^e-1)(a^{e+1}-1) \mid X_e$ in $\mathbf{Z}$. The restriction of z_{ψ} to $G_{K'}$ is a homomorphism. At any place over ℓ, choose inertia δ with $\chi_{\mathrm{cyc}}(\delta) = a$. This is possible by cyclotomic total ramification over $\mathbf{Q}_{\ell}$. It fixes ζ_N, since inertia fixes the prime-to-ℓ part and $a \equiv 1 \pmod{N}$ handles the ℓ-part. For $g \in G_{K'}$ in that decomposition group, $[\delta^{-1}, g]$ fixes ζ_N and has trivial cyclotomic character. Its cocycle value is $(a^e-1)z_{\psi}(g)$ since g acts trivially on C. It follows that $2(a^e-1)z_{\psi}|_{G_{K'}}$ vanishes on the decomposition groups above ℓ, the inertia groups off ℓ, and any real decomposition groups.

Apply global reciprocity over K' to this homomorphism and the principal idele $J(1/(Nm))$. At the chosen place over p write ϕ for an arithmetic Frobenius and choose lifts s_b of $b \in (\mathbf{Z}/Nm)^{\times}$. The other places above p are $s_b^{-1}B$. Their Frobenius values are

$$z_{\psi}(s_b^{-1}\phi s_b) = b^e\psi(b)z_{\psi}(\phi)$$

because ϕ acts trivially on C. Their orders on the principal idele are $v_B(J(b/(Nm)))$. The sum of local Artin images is zero, giving

$$S_m(a,\psi)\, 2(a^e-1)z_{\psi}(\phi) = 0.$$

Write $(a^{e+1}-1)X_e = S_m k$ in C and $X_e = (a^e-1)(a^{e+1}-1)d_0$ in $\mathbf{Z}$. Multiplication by kd_0 gives $2X_e^2 z_{\psi}(\phi) = 0$.

The cocycle $2X_e^2 z_{\psi}$ is zero on $G_{K'}$ by Chebotarev's density theorem [46, Theorem 8.31]. Indeed take a finite normal quotient carrying its values and the cyclotomic quotient modulo Nm. On the kernel of the latter quotient, conjugacy changes a cocycle value by a unit, and each conjugacy class in the kernel is represented by a Frobenius above a rational prime split in K'. So $2X_e^2 z_{\psi}$ factors through the abelian group $\mathrm{Gal}(K'/\mathbf{Q})$. For a cocycle on that group with value zero at c, commuting an element with c, which acts by -1, shows that twice every value is zero. We obtain $4X_e^2 z_{\psi} = 0$.

24.2.3 Integral recombination of character components

In this subsection, we pass from Section 24.2.2 to the integral lattice.

For $z \in Z^{-}_{e,\nu}$, let s_v be lifts of $v \in (\mathbf{Z}/N)^{\times}$, with $s_1 = 1$. Form the character components

$$z_{\psi}(g) = \sum_{v} \Upsilon(s_v) z(s_v[c,g]s_v^{-1}).$$

Each conjugated commutator lies in $G_{\mathbf{Q}(\zeta_N)} \subseteq G_E$ and acts trivially on C. For $h \in G_E$ acting trivially, the cocycle rule gives $z(khk^{-1}) = \chi_{\mathrm{cyc}}(k)^{-e}z(h)$ for $k \in G_E$. It follows that each summand as a function of s_v is unchanged by left multiplication by $G_{\mathbf{Q}(\zeta_N)}$. Using $[c, gg'] = [c,g]g[c,g']g^{-1}$ and reindexing $s_v g$ proves the cocycle rule for Υ^{-1}. The value at c is zero. At an element g in either prescribed local subgroup, the two conjugates $s_v c g c^{-1} s_v^{-1}$ and $s_v g s_v^{-1}$ are

in the corresponding prescribed local subgroups of G_E. Their values under z vanish, and their difference is the commutator value. Thus the local conditions above hold for every component.

Assume $\mathcal{O}$ contains the roots of unity of order the exponent of $(\mathbf{Z}/N)^\times$. Character orthogonality holds as an identity in $\mathcal{O}$ before reduction. For $g \in G_E$ it gives

$$\sum_{\psi} z_\psi(g) = \varphi(N)(z(cgc^{-1}) - z(g)) = -2\varphi(N)z(g).$$

If the odd components vanish, the even components are invariant under conjugation by c by their cocycle rule. Comparing the sum at g and cgc^{-1} gives $4\varphi(N)z = 0$. Apply this to $4X_e^2 z$, whose odd components vanish by the previous argument. Hence

$$Y_e Z^-_{e,\nu} = 0, \qquad Y_e = 16\varphi(N)X_e^2 \le 2^{4u^6}. \tag{80}$$

The bound is uniform in ℓ, ν, and the coefficient ring. Its integral character argument includes the primes dividing $2\varphi(N)$.

24.3 Detection at large coefficient primes

In this subsection, we use the integer of Section 24.2 to choose a torsion order at a large coefficient prime. Reciprocity and Jacobi derivatives will then detect the nonzero odd component from Section 24.1.

Fix $0 < e < t$, a character ψ with $\psi(-1) = (-1)^{e+1}$ as in Section 24.2, a coefficient ring $O = \mathcal{O}$, and the component depth ν. Let $C_{\psi,\nu}$ consist of the continuous cocycles $z : G_{\mathbf{Q}} \to (O/\ell^\nu O)((\chi_{\mathrm{cyc}}^e\psi)^{-1})$ with $z(c) = 0$ at the fixed complex conjugation. Over $\mathbf{Q}(\mu_{N_c})$ they vanish on inertia away from ℓ and on the kernel of χ_{cyc} in every decomposition group above ℓ, as in Section 24.2.2. Put $h = v_\ell(X_e) + 1$. We prove

$$\exists\, g_1, \dots, g_h \in G_{\mathbf{Q}}, \qquad \{z \in C_{\psi,\nu} :\ z(g_a) = 0 \text{ for } 1 \le a \le h\} = 0.$$

It suffices to detect $C_{\psi,\nu}[\ell]$: every nonzero kernel contains a nonzero element killed by ℓ, obtained by taking the last nonzero ℓ-multiple of any one of its nonzero elements. The inequality $\ell > y^2$ gives $e + 1 < \ell - 1$ and $\ell \nmid N_c$. Here choose $a \equiv 1 \bmod N_c$ with a primitive residue modulo ℓ, by the Chinese remainder theorem. The divisibility $S_{\ell^h}(a, \psi) \mid (a^{e+1} - 1)X_e$ of Section 24.2 holds for this choice as well. Then $2, \varphi(N_c\ell), a^e - 1, a^{e+1} - 1$ are units in O. At a place above ℓ choose γ in inertia, fixing μ_{N_c}, with $\chi_{\mathrm{cyc}}(\gamma) = a$. The existence follows by restricting to the compositum of the unramified prime-to-ℓ cyclotomic extension and the totally ramified ℓ-power cyclotomic extension. The value of the dual character $\Upsilon = \chi_{\mathrm{cyc}}^{e+1}\psi$ at γ is a^{e+1}, with $\Upsilon(\gamma) - 1$ a unit.

Here and below, a cocycle with finite coefficient depth has open zero set, which is a subgroup. If $w(g_0) = 0$ and $(\chi_w(g_0) - 1)w \ne 0$, the zero set cannot contain the subgroup fixing any cyclotomic modulus. Otherwise the cocycle is constant on its right cosets, by the cocycle identity. The cosets of $g_0 g$ and $g g_0$ coincide since the cyclotomic quotient is abelian. Their values are $\chi_w(g_0)w(g)$ and $w(g)$, respectively, giving $(\chi_w(g_0) - 1)w = 0$. For a nonzero odd component take $g_0 = c$, so its character difference is -2. For a nonzero normalized dual cocycle take $g_0 = \gamma$. These differences are units in the large-prime range.

Suppose no h evaluations detect the odd component. For any set T of at most h primes $r \equiv 1 \bmod N_c\ell$, use their Frobenius elements as tests and take the last nonzero ℓ-multiple of a

missed cocycle. This gives a nonzero component z, killed by ℓ, trivial at all test Frobenii. At these split primes the action on the values of z is trivial. It vanishes on inertia, so it vanishes on each tested decomposition group. Conjugating a tested Frobenius multiplies its value by the character at the conjugating element, since the Frobenius action on the coefficients is trivial. The same applies to inertia, so z kills every decomposition group above each $r \in T$.

For sets T of primes split modulo $N_c\ell$, call T tight at depth h when every dual cocycle in O/ℓ^h killed by ℓ and unramified over $\mathbb{Q}(\mu_{N_c})$ outside ℓ, T is unramified there outside ℓ. The empty set is tight. If T is tight and z is nonzero on a decomposition group at a new split r and trivial above T, the modulo-ℓ reciprocity argument in Section 24.3.1 implies that $T \cup \{r\}$ is tight.

Choose a finite set Q' of primes $r \equiv 1 \bmod aN_c\ell^h$ detecting every pair (u, z) in which z is a nonzero ℓ-killed odd component and u is a dual cocycle at depth h, unramified outside ℓ over $\mathbb{Q}(\mu_{N_c})$, and nonzero on the subgroup fixing $\mu_{aN_c\ell^h}$. Both cocycle spaces are finite: restrict to the cyclotomic field killing the coefficient action, take its finite ray-class quotient of bounded exponent unramified outside ℓ, and then add finitely many coset values.

On the subgroup fixing $\mu_{aN_c\ell^h}$ the two zeros are proper subgroups. Their union is proper. If neither contains the other, a product of elements exclusive to the two subgroups lies in neither. Pass to a finite normal quotient through which both cocycles and the cyclotomic quotient factor. Chebotarev gives infinitely many unramified primes in the chosen conjugacy class. They split modulo $aN_c\ell^h$, and at each one a choice of prolongation gives the prescribed Frobenius image with both values nonzero. One choice for each pair gives Q'. The preceding abelian calculation ensures that every nonzero normalized dual cocycle here qualifies for this pair test.

Choose a nonzero odd component killed by ℓ and apply its cyclotomic nonvanishing to the modulus $N_c\ell^h \prod_{r\in Q'} r$. Chebotarev gives a prime p split at this modulus, a distinguished place $B \mid p$, and a nonzero component value at its Frobenius. This uses the same finite-quotient argument as above with one cocycle. In particular $p \notin Q'$, and reciprocity with $T = \varnothing$ makes $\{p\}$ tight. Section 24.3.2 gives κ_S for $S \subseteq Q'$ at depth h, with this p, B, γ. It uses $r \equiv 1 \bmod aN_c\ell^h$ for $r \in Q'$ and $p \equiv 1 \bmod N_c\ell^h \prod_{r\in Q'} r$.

Prove by descending induction for $i = h, \dots, 0$ that $\ell^i\kappa_S$ is unramified outside ℓ over $\mathbb{Q}(\mu_{N_c})$ whenever $|S| \le i$ and $\{p\} \cup S$ is tight. At $i = h$ it is zero. For $i < h$, if $u = \ell^{i+1}\kappa_S \neq 0$, then it is unramified by induction and normalized at γ. Since $|\{p\} \cup S| \le h$, choose $z \neq 0$, killed by ℓ, trivial above $\{p\} \cup S$. Use the pair test to choose $r \in Q'$ with both Frobenius values nonzero. It lies outside S and z makes $\{p\} \cup S \cup \{r\}$ tight. Since $|S \cup \{r\}| \le i+1$, the induction hypothesis at $(i+1, S \cup \{r\})$ gives vanishing on inertia at r. The Frobenius–inertia identity of Section 24.3.3 at precisely that place gives

$$u(\mathrm{Frob}_r) = \ell^{i+1}\kappa_{S\cup\{r\}}(\tau_r) = 0,$$

a contradiction. Thus $u = 0$ as a cocycle and $\ell^i\kappa_S$ is killed by ℓ. By construction it is unramified outside $\ell, \{p\} \cup S$, so tightness removes the additional primes. This proves the induction.

At $i = 0$ the cocycle $\kappa_\varnothing$ kills inertia at B. The base derivative identity gives $(\Upsilon(\gamma) - 1)S_{\ell^h}(a, \psi) = 0$ in O/ℓ^h. Cancel the unit and use the Voronoi divisibility of Section 24.2 to get $\ell^h \mid (a^{e+1} - 1)X_e$ in O. The factor $a^{e+1} - 1$ is a unit. This contradicts $h = v_\ell(X_e) + 1$, in the normalized valuation $v_\ell(\ell) = 1$. This proves existence of the h component tests.

For each ψ with $\psi(-1) = (-1)^{e+1}$, take its h tests g. Evaluate the original cocycle at all $s_u[c, g]s_u^{-1}$ in the character sums of Section 24.2. At most $\varphi(N_c)^2(v_\ell(X_e) + 1) \le y^{20}$ points are used, since $v_\ell(X_e) \le y^6$. If z vanishes there, every component of total character $\chi_{\mathrm{cyc}}^e\psi$ of sign -1 vanishes. Then $4\varphi(N_c)z = 0$ and $4\varphi(N_c) < \ell$, so $z = 0$.

Thus these evaluations detect the original module for $\ell > y^2$.

24.3.1 Local reciprocity at an auxiliary prime

In this subsection, we apply local reciprocity at a split auxiliary prime as part of the detection argument of Section 24.3. The local calculation prescribes how an unramified cocycle is evaluated against the auxiliary units.

Let $K = \mathbf{Q}(\mu_{N_c\ell})$ and suppose a split prime $r \equiv 1 \bmod N_c\ell$ lies outside T. Let x be an ℓ-killed odd component, trivial on the decomposition groups above T and nonzero at one Frobenius over r. On the decomposition groups over r the action on its values is trivial, and x is unramified there. Conjugation therefore shows $2(a^e-1)x(\phi) \neq 0$ for an arithmetic Frobenius ϕ at any $B \mid r$.

Take an inertia element τ at B. Let κ be a dual cocycle for $\Upsilon = \chi_{\mathrm{cyc}}^{e+1}\psi$, killed by ℓ and unramified over $\mathbf{Q}(\mu_{N_c})$ off $\ell, T \cup \{r\}$. Write $z = \kappa(\tau)$. Take an $\mathbf{F}_\ell$-basis b_i of Oz inside the ℓ-torsion of the coefficient module and extend its coordinate maps L_i to that whole $\mathbf{F}_\ell$-space. On G_K the cocycle action on these coefficients is trivial.

With ζ_ℓ fixed, Hilbert 90 gives $r_i \in \overline{\mathbf{Q}}^{\times}$ with

$$g(r_i)/r_i = \zeta_\ell^{L_i(\kappa(g))} \qquad (g \in G_K).$$

Hence $y_i = r_i^\ell \in K^\times$. Apply Artin reciprocity over K to y_i and $2(a^e-1)x|_{G_K}$. This unramified homomorphism is zero on the decomposition groups above ℓ by the cyclotomic commutator calculation in Section 24.2, and on those above T by hypothesis. Elsewhere off r the root r_i is inertia-fixed, so y_i has normalized order divisible by ℓ.

Such orders contribute zero since the homomorphism is killed by ℓ. Thus only the places above r contribute. For a cyclotomic lift s_u, with $u \in (\mathbf{Z}/N_c\ell)^\times$, put $m_i(u) = v_B(s_u y_i)$ with $v_B(r) = 1$. Fix $\pi^\ell = r$ and write $\tau(\pi)/\pi = \zeta_\ell^j$. Since $r_i^\ell = y_i$ and $\pi^\ell = r$, the element $\pi^{-m_i(u)}s_u(r_i)$ has valuation zero at B. Its quotient under τ therefore reduces to 1 there. The quotient lies in μ_ℓ, and reduction on μ_ℓ is injective because $r \neq \ell$. The equality of exponents is therefore

$$jm_i(u) = uL_i(\kappa(s_u^{-1}\tau s_u)) \quad \text{in } \mathbf{F}_\ell.$$

Since τ fixes $\mu_{N_c\ell}$, its dual coefficient action is trivial.

The cocycle identity gives $\kappa(s_u^{-1}\tau s_u) = \Upsilon(s_u)^{-1}z$. Together these identities read in the chosen basis as

$$(u^e\psi(u))\sum_i jm_i(u)b_i = z.$$

The weighted Artin identity gives

$$\left(\sum_u m_i(u)u^e\psi(u)\right)2(a^e-1)x(\phi) = 0.$$

Each scalar in parentheses thus lies in the maximal ideal $\mathfrak{m}_O$. Summing the basis identity over u yields $\varphi(N_c\ell)z \in \mathfrak{m}_O Oz$. Since $\varphi(N_c\ell)$ is a unit in this range, the cyclic module Oz vanishes by Nakayama. This holds at all places over r, so κ is unramified off ℓ, T. If T is tight, it follows that $T \cup \{r\}$ is tight.

24.3.2 The Jacobi derivative cocycles

In this subsection, we differentiate the Jacobi family built from the distribution identity of Section 24.2. The resulting cocycles satisfy the relations needed for the detection calculation of Section 24.3.

We use Kolyvagin's derivatives of Gauss sums in the form discussed by Rubin [62]. Use $N = N_c$, $m = \ell^h$, $L_1 = Nm$, and the primes $r \in Q'$, p and place B from the descent. In the construction of J in Section 24.2 use the chosen multiplier a. All the arguments used below have denominators dividing $p-1$. Since $(a, r) = 1$, the distribution quotient $\prod_{b \bmod r} J(x+b/r)/J(rx)$ is a root of unity. Its valuations disappear by the floor distribution identity at B, its conjugate versions at all primes above p, and the units away from p.

The equality of absolute values $|G_j| = |G_{-j}|$ at each embedding makes every absolute value of the quotient one, so Kronecker's theorem applies. The floor distribution follows by writing each floor as $a\langle x\rangle - \langle ax\rangle$ and using $\sum_{b \bmod r}\langle x + b/r\rangle = (r-1)/2 + \langle rx\rangle$ twice for x and $-x$. If $r \mid p-1$ is prime and $rb = 0$ in $\mathbf{Q}/\mathbf{Z}$, the Gauss sums at the shifted indices for $x + b$ and x are congruent at each place above r.

The extra character takes r-th root values, which reduce to 1, and the Gauss sums are units there. Therefore $J(x + b)/J(x)$ is a principal unit there. For the derivative system, let $m = \ell^h$, choose $y(x)^m = J(x)$, and define $F_g(x)$ by $\zeta_m^{F_g(x)} = \gamma y(x)/(g\gamma g^{-1})y(x)$. On the $M_S = L_1 \prod_{r\in S} r$-torsion this quotient has order dividing m, since the action of the commutator on $J(x)$ is trivial. Use the sum

$$\mathcal{D}_S F = \sum_{b\in(\mathbf{Z}/L_1)^\times} b^e\psi(b)\left[\prod_{r\in S}\sum_{u\in(\mathbf{Z}/r)^\times}\log_r(u)\,T_{u/r}\right]F\left(\frac{b}{L_1}\right), \qquad (T_bF)(x) = F(x+b).$$

For each r, $\log_r$ is a fixed surjective homomorphism to $\mathbf{Z}/m$, available since $m \mid r-1$.

The exponent $F_g(x)$ is independent of the choice of m-th root $y(x)$: the two conjugate Galois operators act alike on μ_m. If b is the action of g on μ_{M_S}, then $g(y(x))$ and $y(bx)$ differ by an m-th root of unity. Cancelling that root gives, for $g' \in G_{\mathbf{Q}}$,

$$F_{gg'}(bx) - F_g(bx) = bF_{g'}(x).$$

The quotient on the left is the g-translate of $\gamma y(x)/(g'\gamma g'^{-1})y(x)$.

For each auxiliary prime q, the quotient $\prod_{u \bmod q} y(x + u/q)/y(qx)$ is a root of unity by the Jacobi distribution identity. The operators γ and $g\gamma g^{-1}$ agree on roots of unity, so $\sum_{u \bmod q} F_g(x + u/q) = F_g(qx)$. For a finite set T of auxiliary primes and $\mathbf{d} \in (\mathbf{Z}/m)^T$, write $\mathcal{D}_T^{\mathbf{d}}$ for the same sum with weights $\log_s(u) + d_s$ at $s \in T$. If $q \notin T$ is an auxiliary prime, put $\boldsymbol{\lambda}_T(q) = (\log_s(q))_{s\in T}$. Since $q \equiv 1 \bmod L_1$, the substitution $u' = qu \bmod s$ in each sum gives

$$\mathcal{D}_T^{\mathbf{d}}(F(q\,\cdot)) = \mathcal{D}_T^{\mathbf{d}-\boldsymbol{\lambda}_T(q)}F.$$

Take $S = T \sqcup \{q\}$ and $F = F_g$. For $h_0 \in \mathbf{Z}/m$, the distribution identity above gives

$$\mathcal{D}_S^{\mathbf{d}+h_0\mathbf{e}_q}F - \mathcal{D}_S^{\mathbf{d}}F = h_0(\mathcal{D}_T^{\mathbf{d}|_T-\boldsymbol{\lambda}_T(q)}F - \mathcal{D}_T^{\mathbf{d}|_T}F),$$

where $\mathbf{e}_q$ is the unit vector at q. Starting at $T = \varnothing$, induction on $|S|$ shows that $\mathcal{D}_S^{\mathbf{d}}F_g = \mathcal{D}_S F_g$ for every $\mathbf{d}$. More generally, for b prime to M_S, the same substitution at the primes of S and the substitution $a' = ba \bmod L_1$ in the base sum give

$$\mathcal{D}_S(F_g(b\,\cdot)) = (b^e\psi(b))^{-1}\mathcal{D}_S F_g.$$

Define $\kappa_S(g) = \mathcal{D}_S F_g$. For the exponent b of g on μ_{M_S}, apply this identity to $F_{gg'}(b\,\cdot) - F_g(b\,\cdot) = bF_{g'}$ to obtain

$$\kappa_S(gg') - \kappa_S(g) = b^{e+1}\psi(b)\kappa_S(g').$$

Thus κ_S is a cocycle with coefficients in O/mO and character $\chi_{\mathrm{cyc}}^{e+1}\psi$. It is continuous because each derivative sum uses finitely many roots. The cocycle at γ is zero termwise. Away from ℓ, p, S, inertia fixing μ_N fixes μ_{M_S}. For g in this inertia, $gy(x)/y(x)$ is a principal unit in μ_m, since $y(x)$ is a unit and its m-th power is fixed. The same holds for $g\gamma y(x)/\gamma y(x)$, since $(\gamma y(x))^m = J(b_S x)$, where $b_S \in (\mathbf{Z}/M_S)^\times$ is the action of γ on μ_{M_S}. We have $b_S \equiv a \pmod{L_1}$ and $b_S \equiv 1 \pmod{r}$ for $r \in S$, since γ lies in inertia above ℓ. Injectivity of reduction on μ_m gives $F_g(x) = 0$.

For the base term $S = \varnothing$, take $x \in L_1^{-1}\mathbf{Z}/\mathbf{Z}$, so $J(x) \in \mathbf{Q}(\mu_{L_1})$ and $b_S x = ax$. At the marked $B \mid p$, take the tame generator τ that multiplies the m-th root ϖ_m of p by ζ_m. It exists by total ramification of the uniformizer extension over the unramified field containing μ_m. It fixes μ_{L_1}. Write

$$s_x = v_B(J(x)) = \lfloor a\langle x\rangle \rfloor - \lfloor a\langle -x\rangle \rfloor.$$

Then $y(x)/\varpi_m^{s_x}$ is a unit whose m-th power lies in $\mathbf{Q}(\mu_{L_1})$. The quotient of this unit under τ is in μ_m and reduces to one. Therefore $\tau y(x)/y(x) = \zeta_m^{s_x}$, and likewise $\tau\gamma y(x)/\gamma y(x) = \zeta_m^{s_{ax}}$, since $\gamma(y(x))^m = J(ax)$. It follows that $F_\tau(x) = as_x - s_{ax}$. Summing over $x = b/L_1$ and reindexing b by ab, one obtains

$$a^e\psi(a)\,\mathcal{D}_\varnothing F_\tau = (a^{e+1}\psi(a) - 1)S_m(a, \psi).$$

This proves the base Stickelberger identity on the same B.

24.3.3 Comparison of Frobenius and inertia

In this subsection, we compute the Frobenius evaluation of the Jacobi cocycles of Section 24.3.2 in terms of inertia. This gives the precise local relation used with the reciprocity calculation of Section 24.3.1.

Retain the Jacobi family and $m = \ell^h$. Fix $r \in Q' \setminus S$. Let A be any place above r, let ϕ be any lift of arithmetic Frobenius at A, and let x be an M_S-torsion point. Choose a unit v mod r with $\log_r(v) = 1$ and choose τ in the inertia group at A acting by v^{-1} on the r-torsion roots of unity. It acts trivially on μ_{M_S} because $r \nmid M_S$. Use additive notation for $\overline{\mathbf{Q}}^\times$ in the following calculation. Write $Y_b = y(x + b/r)$, $Y_* = y(rx)$, and $\lambda(u) \in \mathbf{Z}$ for any integral lift of $\log_r(u)$.

For $\delta \in G_{\mathbf{Q}}$ set

$$W_\delta = \sum_{b \bmod r} Y_b - (\delta\phi\delta^{-1})Y_0 + \sum_{u \in (\mathbf{Z}/r)^\times} \lambda(u)\big((\delta\tau\delta^{-1})Y_{vu} - Y_u\big).$$

At the conjugate place δA, all Y_b are units. Its inertia acts trivially on their reductions and its Frobenius acts on the residue field by the r-th power. With $Z_b = Y_b - Y_0$, reduction of W_δ modulo principal units there gives

$$\sum_{u \in (\mathbf{Z}/r)^\times} \big(1 + \lambda(v^{-1}u) - \lambda(u)\big)Z_u.$$

Each coefficient is divisible by m; the Jacobi congruence says mZ_u is a principal unit at every place above r.

Thus W_δ is a principal unit at δA. This uses the Gauss-sum congruence at the conjugate place as well as at A. Set $R_u = \tau Y_{vu} - Y_u$ and $R_* = \phi Y_0 - Y_*$. The action of ϕ on μ_{M_S} is the r-th power, as reduction is injective there. Thus R_u, R_* are m-torsion by Jacobi equivariance. The distribution quotient gives that $m(\sum_b Y_b - Y_*)$ is torsion, hence $\sum_b Y_b - Y_*$ is torsion.

It follows that W_1 is a root of unity, so γW_1 is a power of W_1 and is still a principal unit at A. Also $\gamma W_{\gamma^{-1}}$ is a principal unit at A. The actions of γ and any of its conjugates agree on all roots of unity. Expand the difference of the two terms, using $\phi Y_0 = Y_* + R_*$ and $\tau Y_{vu} = Y_u + R_u$, to obtain

$$\gamma W_1 - \gamma W_{\gamma^{-1}} = \sum_u \lambda(u)(\gamma Y_u - (\tau\gamma\tau^{-1})Y_u) - (\gamma Y_* - (\phi\gamma\phi^{-1})Y_*).$$

The R_u and R_* terms cancel by the agreement on roots of unity.

The right side is m-torsion and a principal unit. Since $r \neq \ell$, reduction on μ_m is injective. The exponent therefore gives

$$\sum_{u \in (\mathbf{Z}/r)^\times} \log_r(u) F_\tau(x + u/r) = F_\phi(rx).$$

Taking $b = r$ for the chosen Frobenius ϕ gives the local identity

$$\mathcal{D}_{S\cup\{r\}} F_\tau = \mathcal{D}_S(F_\phi(r\,\cdot)) = \mathcal{D}_S F_\phi.$$

Consequently, for this very Frobenius lift ϕ one has $\kappa_{S\cup\{r\}}(\tau) = \kappa_S(\phi)$ with $\tau \in I_A$. This applies at the prolongation used by the simultaneous detection argument in Section 24.3. Thus the Frobenius-to-inertia equality uses the same Jacobi family and marks throughout the descent.

25 Polynomial bounds for the Selmer lengths

In this section, we represent the finite cocycle conditions by a matrix over a polynomial ring in the twist variables. Its determinantal ideals control the lengths from Section 24 after specializing the twist.

Retain the normalized valuation and length from Section 24. Fix a number field E, a finite set Σ of finite places, an integral continuous representation ρ of G_E, and finitely many continuous characters $\chi_w : G_E \to \mathbb{Z}/\ell^N$. For each $g \in G_E$ set

$$R_X(g) = \rho(g) \prod_w (1 + X_w)^{[\chi_w(g)]},$$

where the brackets denote the representative in $[0, \ell^N - 1]$. Fix a collection of linear relations over $\mathcal{O}$, each involving finitely many cocycle evaluations. Over the ring of integers $\mathcal{O}'$ of a finite extension of $\mathrm{Frac}(\mathcal{O})$, denote by $C(x)$ the continuous functions into $(\mathcal{O}'/\ell^M)^n$ satisfying the cocycle equations for R_x, the chosen linear relations, and vanishing on every inertia group at the finite places outside Σ. The equations make sense for every integral x. At an actual character specialization this is the corresponding module of cocycles.

Lemma 25.1 (Polynomial minors for Selmer lengths)**.** *There is a finite set of integral polynomials P_j such that*

$$\lambda(C(x)) = \min_j v(P_j(x))$$

over every such coefficient ring and for each integral x. If j_0 attains the minimum at x_0, then

$$\lambda(C(x_0)) = v(P_{j_0}(x_0)), \qquad \lambda(C(x)) \le v(P_{j_0}(x)) \quad \textit{for every integral } x.$$

Proof. Let H be the open normal subgroup in the kernels of all χ_w and of $\rho \bmod \ell^M$, and let E_H be its fixed field. Every $z \in C(x)$ is additive on H, since $R_x(h) = I \bmod \ell^M$ there. The continuous homomorphisms from H to $\mathbb{Z}/\ell^M$ unramified outside the places above Σ form a finite set. Indeed their kernels cut out cyclic extensions of E_H of degree at most ℓ^M with this ramification bound, and those form a finite set by Hermite's finiteness theorem [46, Theorem 8.42]. Each such field admits finitely many homomorphisms. Let H_1 be the core in G_E of the intersection of their kernels in H. The dual into $\mathbf{Z}/\ell^M\mathbf{Z}$ separates points of the finite abelian group $(\mathcal{O}'/\ell^M)^n$. Applying its characters to each $z|_H$ gives

$$z|_{H_1} = 0, \qquad z(gh) = z(g) + R_x(g)z(h) = z(g) \quad (h \in H_1).$$

Thus z factors through G_E/H_1.

Write the equations on the finitely many coordinates of functions from G_E/H_1. The resulting covectors over $\mathcal{O}[X]$ include cocycle, inertia and linear equations. Their span is finitely generated because $\mathcal{O}[X]$ is Noetherian, so a finite subset of the covectors spans it. Denote the matrix of these covectors by $A(X)$, with q columns. For every specialization the kernel of $A(x)$ acting on $(\mathcal{O}'/\ell^M)^q$ is exactly $C(x)$, since all the equations are equivalent to this finite selection after evaluation. Put

$$A^+(X) = \begin{pmatrix} A(X) \\ \ell^M I_q \end{pmatrix}, \qquad C(x) = \ker\Big(A^+(x) \bmod \ell^M\Big).$$

Over $\mathcal{O}'$, the matrix $A^+(x)$ has full column rank. Its Smith invariants $u_1, \dots, u_q$ divide ℓ^M, since its covectors span a module containing $\ell^M(\mathcal{O}')^q$. For the maximal minors $P_j(X)$ of $A^+(X)$,

$$\lambda(C(x)) = \sum_{i=1}^{q} v(u_i) = \min_j v(P_j(x)).$$ □

25.1 The line through the untwisted point

In this subsection, we restrict the polynomial matrix of Section 25 to a line through the untwisted specialization. The resulting divisibility estimate controls the length after discarding the distinguished local relation.

Write w_0 for the distinguished place and define the line

$$x_{w_0}(Y) = Y, \qquad x_w(Y) = s_w Y \quad (w \neq w_0).$$

The coefficient of Y at w_0 is one even though its untwisted local relation has slope zero. Suppose that the other auxiliary pairs satisfy Lemma 24.1. Write $C_0(x)$ for the module obtained by imposing the auxiliary relations at these pairs, so the first place is excluded from this set of relations. Suppose $\lambda(C_0(0)) \le \beta$. This is the untwisted length bound after accounting for the first place and its conjugate.

Let J_{aux} index the auxiliary pairs where the relations are imposed. For each $w \in J_{\mathrm{aux}}$, put $\phi_{w,1} = \phi_w$, $\phi_{w,2} = (\phi_w^c)^{-1}$. Both evaluations of Lemma 24.1 enter the product

$$D_{\mathrm{aux}}(Y) = \prod_{w \in J_{\mathrm{aux}}} \prod_{i=1}^{2} \det((1+Y)\rho(\phi_{w,i}) - I).$$

Choose by Lemma 25.1 a polynomial G for C_0 that is sharp at 0. For every y satisfying the scalar condition of Lemma 24.1, the exact sequence for $C_0(x) \subset C(x)$ gives

$$\lambda(C(x)) \le v(G(x)) + v(D_{\mathrm{aux}}(y)).$$

The degree of D_{aux} is $D_0 = 2n|J_{\mathrm{aux}}|$, and its leading coefficient is the unit $\prod_{w,i} \det \rho(\phi_{w,i})$. Section 27.3 bounds $|J_{\mathrm{aux}}|$ independently of N. Fix a positive rational radius a_0 and take N large enough for the scalar condition in Section 24 before choosing the auxiliary places and characters at that depth.

For $q(Y) = \sum_k q_k Y^k$ put $v_{a_0}(q) = \min_k(v(q_k) + ka_0)$. Set $g(Y) = G(x(Y))$. Its constant coefficient and the leading coefficient above give

$$v_{a_0}(g) \le v(G(0)) \le \beta, \qquad v_{a_0}(D_{\mathrm{aux}}) \le D_0 a_0.$$

Choose $y_0 \in \overline{\mathbb{Q}}_\ell$ with $v(y_0) = a_0$. For $q = g, D_{\mathrm{aux}}$, divide $q(y_0 U)$ by a coefficient of least valuation. Choose a unit u whose residue avoids the zeros of both resulting nonzero polynomials. A finite extension of the coefficient field contains y_0, u, and at $y = y_0 u$ both evaluations have their respective Gauss valuations. Choose a minor P for C attaining its minimum at $x = x(y)$. It satisfies

$$v(x_w) \ge a_0, \qquad v(P(x)) \le \beta + D_0 a_0, \qquad \lambda(C(x')) \le v(P(x')) \quad \text{for every } x'.$$

25.2 Transfer to divisible coefficients

In this subsection, we apply the finite-length estimate of Section 25 to divisible coefficient modules at torsion specializations. We check that the same inertia and linear conditions identify the finite modules with their torsion subgroups.

At a tuple of ℓ^N-power torsion points $x'_w = \zeta_w - 1$ with $\zeta_w^{\ell^N} = 1$, the specialized twist is a scalar character. Let C_∞ be the module of cocycles for this twist with coefficients in $(\mathrm{Frac}(\mathcal{O}')/\mathcal{O}')^n$, with the same inertia and linear conditions. Division by ℓ^M gives an equivariant isomorphism

$$\mathcal{O}'/\ell^M\mathcal{O}' \longrightarrow (\mathrm{Frac}(\mathcal{O}')/\mathcal{O}')[\ell^M], \qquad \bar{z} \longmapsto \ell^{-M} z \bmod \mathcal{O}'.$$

It commutes with evaluation at inertia and with every prescribed linear relation, so $C(x') \simeq C_\infty[\ell^M]$. Every cocycle in C_∞ is killed by a power of the uniformizer because its continuous image in a discrete module is finite. If $v(P(x')) < M$, there can be no element of C_∞ of order greater than the uniformizer power generating (ℓ^M). Otherwise its cyclic submodule would supply an element of order exactly that power in $C(x')$, forcing $\lambda(C(x')) \ge M$. It follows that $\lambda(C_\infty) = \lambda(C(x')) \le v(P(x'))$.

26 Simultaneous evaluation on torsion

In this section, we choose torsion specializations at which finitely many polynomials satisfy their valuation bounds simultaneously. The choice will apply to the Selmer minors of Section 25 and the group-ring evaluations of Section 27.

Let v be the additive valuation on $\overline{\mathbb{Q}}_\ell$, normalized by $v(\ell) = 1$, and write $\mathcal{O}$ for its valuation ring. The following evaluation lemma gives a torsion exponent before the auxiliary primes or their polynomials are chosen.

Lemma 26.1 (Simultaneous torsion specialization)**.** *Fix a variable bound s_0, an integer $m \geq 1$, a positive rational radius a, a tolerance $\epsilon > 0$, and a finite list of nonnegative bounds B_i. Set*

$$\rho = a/m, \qquad B = m \sum_i B_i, \qquad \delta = \frac{\min(1, \epsilon)}{s_0 + 2}, \qquad \rho_h = \varphi(\ell^h)^{-1} \quad (h \geq 1).$$

Let H_1 be the least positive integer with $\rho_{H_1} \leq \rho$. Any $H \geq H_1$ satisfying

$$B + 1 < \delta \sum_{h=H_1}^{H} \frac{1}{h+1}$$

works. For example one can take

$$H = \lceil (H_1 + 1) \exp((B+1)/\delta) \rceil$$

by integrating $1/t$ on $[H_1 + 1, H + 2]$. For every family of integral polynomials P_i in the same $s \leq s_0$ variables and points x_i such that

$$v(x_{i,w}) \geq a, \qquad v(P_i(x_i)) \leq B_i,$$

there are roots of unity $\zeta_w^{\ell^H} = 1$ with $v(\zeta_w - 1) \leq \rho$ and

$$v\big(P_i((\zeta_w^j - 1)_w)\big) \leq v(P_i(x_i)) + \epsilon \qquad (1 \leq j \leq m, \ \text{all } i).$$

Proof. One coordinate. For a nonzero integral polynomial g, write

$$v_t(g) = \min_b \{v(g_b) + bt\}.$$

This valuation is multiplicative on products in additive notation. To see this for rational t, scale the variable by an element of valuation t, then divide each scaled polynomial by a coefficient of minimum valuation. The reductions are nonzero polynomials over a field and have nonzero product. By factorization, with leading coefficient g_{top},

$$v_t(g) = v(g_{\text{top}}) + \sum_\alpha \min(t, v(\alpha)),$$

where roots are counted with multiplicity. In particular $v_t(g) \leq v(g(y))$ for $v(y) \geq t$, and $v_t(g)$ is increasing in t.

Suppose $v_\rho(g) \leq B + 1$. The polynomial is integral, so $v_0(g) \geq 0$. Subtracting the root formula at radius zero from that at radius ρ gives a sum of the nonnegative terms $\min(\rho, v(\alpha)) - \min(0, v(\alpha))$. Since the radii $\rho_h \leq \rho$ are distinct, the multisets R_h of roots with $v(\alpha) = \rho_h$ obey

$$\sum_{h=H_1}^{H} |R_h| \rho_h \leq v_\rho(g).$$

Thus there is an h with $|R_h|\rho_h(h+1) \leq \delta$, by the harmonic-sum inequality. For each $\alpha \in R_h$, the polynomial

$$F_\alpha(Y) = (1 + \alpha + Y)^{\ell^h} - 1$$

is monic and has linear coefficient of valuation h, because $v(\alpha) > 0$.

Its roots include $Y = \zeta - 1 - \alpha$ for the primitive ℓ^h-th roots ζ. All roots of F_α are of positive valuation. Applying the Gauss valuation at radius 1,

$$\sum_{\zeta \text{ primitive}} \min\{1, v(\zeta - 1 - \alpha)\} \le v_1(F_\alpha) \le h + 1.$$

There are $1/\rho_h$ such roots of unity. Sum over $\alpha \in R_h$ and average to find one ζ with the sum of the truncated valuations at most $|R_h|\rho_h(h+1) \le \delta$.

Since $\delta < 1$, each truncation is inactive. All primitive roots of this order have $v(\zeta - 1) = \rho_h$, by taking the norm of $1 - \zeta$ or using the cyclotomic polynomial at 1. For $y = \zeta - 1$, the roots of g with valuation different from ρ_h have $v(y - \alpha) = \min\{\rho_h, v(\alpha)\}$. The other roots give

$$v(g(y)) - v_{\rho_h}(g) = \sum_{\alpha \in R_h} \big(v(y - \alpha) - \rho_h\big) \le \delta.$$

Now let there be finitely many polynomials g_i, with their old arguments u_i of valuation at least ρ and $\sum_i v(g_i(u_i)) \le B + 1$. Apply this argument to $g = \prod_i g_i$. Since $v_{\rho_h}(g_i) \le v_\rho(g_i) \le v(g_i(u_i))$ and $v(g_i(y)) \ge v_{\rho_h}(g_i)$, we obtain

$$\sum_i \max\{0, v(g_i(y)) - v(g_i(u_i))\} \le \sum_i \big(v(g_i(y)) - v_{\rho_h}(g_i)\big) = v(g(y)) - v_{\rho_h}(g) \le \delta.$$

All coordinates and powers. For each pair (i, j), $1 \le j \le m$, replace $P_i(X)$ by the integral polynomial

$$Q_{i,j}(Y) = P_i(((1 + Y_w)^j - 1)_w).$$

For each old coordinate $x = x_{i,w}$ the monic polynomial $(1 + Y)^j - (1 + x)$ has j integral roots whose product up to sign is x. A root of maximum valuation therefore has valuation at least $a/j \ge \rho$, including the case $x = 0$. Use these roots as the test point of $Q_{i,j}$. Its evaluation is $P_i(x_i)$; the sum of the initial valuations over (i, j) is at most B.

Process the coordinates one at a time and freeze the others at their current values for each test point. On the coordinate currently being replaced the old valuations are at least ρ. The sum of the evaluation valuations after k replacements is at most $B + k\delta \le B + 1$. Hence the one-coordinate argument applies with the same H at every step, including when the previous replacements have valuations below ρ. They remain integral, which is all that is needed for the coefficients of the new one-variable polynomials. The common final points are $(\zeta_w - 1)_w$; the error at each (i, j) is at most $s\delta \le \epsilon$. This proves the assertion. □

27 A common twist for the Selmer and Euler-factor conditions

In this section, we choose one twist for the cocycle conditions of Section 25 and the Euler-factor relations of the norm systems. The auxiliary primes and torsion specialization are arranged so that both estimates apply to this same character.

Fix a coefficient prime ℓ and a finite family of the Selmer data of Section 25. Include finitely many systems of integral group-ring elements Θ_Q, one for each allowed auxiliary set Q, with the following norm relation. For a character ψ factoring through level Q, its evaluation at $\Theta_{Q \cup \{w\}}$ is its evaluation at Θ_Q multiplied by

$$E_w(\vartheta(\phi_w)\psi(\phi_w)), \qquad E_w(A) = \prod_{e \in \mathcal{E}} (1 - Aq_w^{-e})(1 - A^{-1}q_w^{-e}).$$

Here q_w is the residue cardinality and $\mathcal{E}$ is the system's finite list of integer exponents. Each system has a continuous character $\vartheta : G_E \to \overline{\mathbb{Z}}_\ell^\times$, unramified outside the fixed set of excluded places. In particular, $\vartheta(\phi_w)$ is independent of the Frobenius lift at every auxiliary place w.

27.1 A specialization of a norm system

In this subsection, we express each group-ring evaluation from Section 27 as a polynomial in the chosen character coordinates. The norm relation determines the divisibility along the line used for the simultaneous specialization.

Use one list J of auxiliary primes with distinguished prime w_0, equipped with characters χ_w of order dividing ℓ^N as in Section 24. Each χ_w is from a character of the norm-one classes of level w, is zero on the decomposition groups above ℓ, and takes the value 1 at the chosen tame generator at w. The other characters are zero there. At the marked Frobenius and its conjugate inverse for every subsequent prime, the first character has value 1 and the other characters are zero. All q_w are 1 mod ℓ^N. For a norm-one class represented by $u/\bar{u}$, read $\chi_w(\mathrm{Art}\, u)$ using the induced class character. Write $h = \ell^N$ and let $n_w(c) \in \{0, \dots, h-1\}$ represent this value for the class c. If $\Theta_J = \sum_c t_c[c]$, put

$$P_J(X) = \sum_c t_c \prod_{w \in J} (1 + X_w)^{n_w(c)}.$$

Modulo the relations $(1 + X_w)^h = 1$ the substitution $[c] \mapsto \prod_w (1 + X_w)^{n_w(c)}$ is multiplicative, since the exponents add modulo h. At $X_w = \zeta_w - 1$ with $\zeta_w^h = 1$ it evaluates Θ_J at $\prod_w \zeta_w^{\chi_w}$.

Restrict P_J to the line with $X_{w_0} = Y$ and all other variables zero, and call the resulting polynomial $P_J^0(Y)$. Let $R_0(Y)$ denote the polynomial for the same one-variable evaluation of $\Theta_{\{w_0\}}$. Put $j_0 = |\mathcal{E}|(|J| - 1)$. With $A_w = \vartheta(\phi_w)$, the product

$$R_1(Y) = \prod_{w \neq w_0} \prod_{e \in \mathcal{E}} \bigl(1 - A_w q_w^{-e}(1 + Y)\bigr)\bigl(1 + Y - A_w^{-1} q_w^{-e}\bigr)$$

has degree $2j_0$ and unit leading coefficient. The norm relation iterated away from w_0 gives

$$(1 + Y)^{j_0} P_J^0(Y) \equiv R_0(Y) R_1(Y) \pmod{(1 + Y)^{\ell^N} - 1}.$$

Indeed at $Y = \zeta - 1$ the character is $\zeta^{\chi_{w_0}}$. Its value on the marked Frobenius at $w \neq w_0$ is ζ, so each pair of factors of R_1 equals ζ times the Euler-factor pair. This gives equality at every root of the modulus, which is monic and has simple roots. Polynomial division gives the congruence over the valuation ring.

For the first-prime Euler factor, fix g_0 with the following finite bound

$$\beta = v(1(\Theta_\varnothing)) + 2|\mathcal{E}| v(\vartheta(g_0) - 1).$$

The second term is zero for an empty exponent list. Choose $M' > \beta + 1$, and require $N \geq M'$ and $v(\vartheta(\phi_{w_0}) - \vartheta(g_0)) \geq M'$. If $\mathcal{E}$ is nonempty, each of the two factors for each e at the first prime and trivial character has valuation $v(\vartheta(g_0) - 1)$. Indeed $q_{w_0}^{-e}$ is 1 mod $\ell^{M'}$, and $\vartheta(\phi_{w_0})$ and its inverse agree with $\vartheta(g_0)$ and its inverse, respectively, to that precision. The valuation $v(\vartheta(g_0) - 1)$ is strictly less than M' because the augmentation is integral and $\beta < M'$. Thus the norm relation gives $v(R_0(0)) \leq \beta$.

Let $0 < a_0 \le 1$ be rational. The Gauss valuation of $R_0 R_1$ at radius a_0 is at most $\beta + 2j_0 a_0$, by the constant coefficient of R_0 and the leading coefficient of R_1. For $v(y) = a_0$, the binomial expansion shows

$$v((1+y)^{\ell^N} - 1) \ge N a_0.$$

For its j-th term use $v\binom{\ell^N}{j} \ge N - v_\ell(j)$ and $j - 1 \ge v_\ell(j)$, with $v_\ell(j) \le N$. If $\beta + 2j_0 a_0 < N a_0$, take a generic y of valuation a_0 attaining the Gauss valuation of the product, by avoiding the roots of the scaled reduction. The congruence and $v(1+y) = 0$ then give

$$v(P_J^0(y)) \le \beta + 2j_0 a_0.$$

This provides a test point of the required depth for each norm system, on the same auxiliary prime set as the Selmer tests.

27.2 The distinguished auxiliary prime

In this subsection, we use the class-field conditions of Section 27 to choose the first auxiliary prime. Its Frobenius lies in the specified neighborhood, and its local character has the prescribed restrictions at the other places.

Let E be CM and let $g_0 \in G_E$ lie in the normal subgroup generated by inertia at a finite set of places R_{in} away from ℓ. Fix a neighborhood $\mathcal{W}$ of g_0, a finite set of forbidden primes, and a finite set S_{sp} of places at which the new character must be trivial on decomposition groups. Require the conjugate closure of S_{sp} to be disjoint from R_{in}. For each $N \ge 1$ there is a prime of prime residue cardinality 1 mod ℓ^N, outside the forbidden set with its conjugate and distinct from its conjugate, equipped with a character of the kind used in Section 27. Its geometric Frobenius lift can be chosen in $\mathcal{W}$ with character value zero.

To prove this, choose a finite set T, stable under conjugation, containing S_{sp} and the places above ℓ, and avoiding R_{in} and its conjugate. Require the primes in T to generate the ideal class group. Such a set exists by Chebotarev in the Hilbert class field, which gives a prime in every ideal class outside any finite set [48, Ch. V, Theorems 3.23 and 5.5]. Let $\mathbb{A}_{E,f}^{\times}(T)$ denote the finite ideles that are units outside T. Since T generates the class group, the ideal of any finite idele differs from an ideal supported on T by a principal ideal. Thus

$$\mathbb{A}_{E,f}^{\times} = E^{\times} \mathbb{A}_{E,f}^{\times}(T), \qquad E^{\times} \cap \mathbb{A}_{E,f}^{\times}(T) = \mathcal{O}_{E,T}^{\times}.$$

Put $h = \ell^N$ and take the field obtained by adjoining μ_h and the h-th roots of a finite generating set of the T-units. The generating set exists by the unit theorem, since valuation at T has kernel the ordinary unit group and image in $\mathbb{Z}^T$. This Kummer extension is Galois over E and unramified at each place of R_{in}: the T-units are units there and h is prime to the residue characteristic.

Their h-th roots and μ_h can be obtained in an unramified extension by lifting the distinct roots over a residue field extension. Choose a finite Galois extension of E such that the coset of the kernel containing g_0 lies in $\mathcal{W}$, and enlarge it by the Kummer extension. Prescribe the conjugacy class of g_0^{-1} for arithmetic Frobenius in the compositum. Chebotarev gives w_0 and a place above it with this arithmetic element in the quotient [48, Ch. V, Theorem 3.23]. Inverting gives a geometric Frobenius lift in $\mathcal{W}$. We may avoid T, the forbidden primes and their conjugates, and the primes ramified over $\mathbb{Q}$. We may also take w_0 of residue degree one over $\mathbb{Q}$, since prime ideals of higher absolute residue degree have density zero. Then $w_0 \ne \bar{w}_0$,

as an unramified prime fixed by the nontrivial conjugation of E/E^+ has relative residue degree two.

The Frobenius of w_0 fixes the Kummer extension, so $q_{w_0} = 1 + hk$ for an integer k. For each generating T-unit u, its root $u^{1/h}$ reduces to a nonzero element fixed by the residue Frobenius. Hence the reduction of u is an h-th power in the residue field of w_0. The discrete logarithm of residue units modulo h kills these h-th powers and therefore all T-units.

Applying this logarithm to the w_0 component on T-ideles descends to a continuous character θ of the idele classes. By reciprocity [48, Ch. I, Theorem 1.1 and Ch. V, Theorems 5.3 and 5.5], $\theta(u/\bar{u})$ defines an anticyclotomic Galois character, trivial on the decomposition groups at S_{sp} and above ℓ, and unramified away from $w_0, \bar{w}_0$. Write $\log_{w_0} : k_{w_0}^{\times} \to \mathbb{Z}/h$ for the residue logarithm. For a local idele u at a place of S_{sp} or above ℓ, both u and $\bar{u}$ are T-ideles with component one at w_0. For a unit idele supported at w_0, only u contributes there, since $w_0 \neq \bar{w}_0$. Thus

$$\theta(u/\bar{u}) = \begin{cases} 0, & u \in E_v^{\times}, \quad v \in S_{\mathrm{sp}} \text{ or } v \mid \ell, \\ \log_{w_0}(u \bmod w_0), & u \in \mathcal{O}_{E,w_0}^{\times}. \end{cases}$$

Normalize the logarithm to take the value 1 on the fixed tame generator under reciprocity. The local image has order h, also when $\ell = 2$.

Multiplying the Frobenius lift by a power of this tame generator makes the character value of Frobenius zero. This correction does not alter the chosen finite Frobenius, as the compositum used for Chebotarev is unramified at w_0. The corrected lift still lies in $\mathcal{W}$.

27.3 Selmer length descent and the uniform bound

In this subsection, we iterate the local relations of Section 24 until the remaining module can be treated by Section 27.4. The argument bounds the number of added primes independently of the larger torsion order.

Fix the coefficient ring $\mathcal{O}$, its uniformizer ϖ, the finite data indexed by i of ranks n_i, and M. Set $e = \mathrm{ord}_{\mathcal{O}}(\ell^M)$. Assume the residue field has more than $2en_i + 1$ elements for each i. In this subsection lengths are unnormalized. For an auxiliary list with prescribed congruences J and $1 \le a \le e$, let $C_{i,a}(J)$ be the untwisted cocycles modulo ϖ^a with the original linear conditions, allowed ramification at J and its conjugates, and the given congruences at each auxiliary pair. Let $C_{i,a}(\varnothing)$ have just the original ramification and linear conditions. The lengths are finite by the finite-presentation argument.

Apply Section 27.2 at exponent $N > M$ to the inertial reference g_0 in the prescribed common neighborhood $\mathcal{W}$, with $S_{\mathrm{sp}} = \varnothing$. The neighborhood and its precisions are specified below before increasing N. Forbid every place above the rational primes dividing the norms of the original bad places. Start with slope zero at both parameters of the distinguished prime. The terminal comparison for primitive cocycles is proved under the specified space hypotheses in Section 27.4. For datum i at depth a, suppose each eligible new auxiliary prime has the following property.

The two Frobenius values of every member of $C_{i,a}(J)$, at that prime and its conjugate inverse, can be realized as the two tame values of a cocycle with the old congruences and ramification also allowed at the new prime pair. If $C_{i,a}(J)$ has a member with a unit value coordinate, the terminal bound under this property is $G_i + L_i$, where G_i, L_i are the datum's global and original-place terms, in unnormalized length.

An eligible prime P has prime norm, is distinct from its conjugate, and both parameters are outside the forbidden places and the previously chosen prime pairs. The prime P does not lie

over the rational norm of w_0. Its character is trivial on the decomposition groups above ℓ and at both members of each old pair other than $\{w_0, \bar{w}_0\}$, and vanishes on its own Frobenius lift.

On this lift, the old character at w_0 reads 1 and the other old characters read 0. The new prime has the same tame-generator and norm conditions as the original list. For the selected datum i, the representation modulo ℓ^M is trivial on both Frobenius parameters. This makes every nonzero slope admissible in that datum.

Here is the local calculation used to choose a slope. At an eligible new prime pair let C be the enlarged group with the old congruences, and for this datum and depth write $A, B : C \to (\mathcal{O}/\varpi^a)^{2n_i}$ for the two tame and the two Frobenius evaluations respectively. Put

$$C' = \sum_{u \in \mathcal{O}} \ker_C(uA - B).$$

Then $B(C') \subseteq A(C')$. Let L' be the inverse image of $A(C')$ in $\mathcal{O}^{2n_i}$, with reduction $\pi : L' \twoheadrightarrow A(C')$.

It is free of rank $2n_i$ since it contains $\varpi^a \mathcal{O}^{2n_i}$. Lift π through $A : C' \twoheadrightarrow A(C')$ to $\sigma : L' \to C'$, then lift $B\sigma$ through π to an endomorphism U' of L'. This gives $(uA - B)\sigma = \pi(uI - U')$. Exclude the roots of the characteristic polynomial of U' modulo ϖ. There are at most $2n_i$ excluded residues. Outside these residues $uI - U'$ is invertible, so the image of $uA - B$ on C' equals $A(C')$.

Since its kernel on C is contained in C', the length identity for the two surjections onto $A(C')$ gives

$$\operatorname{length} \ker_C(uA - B) = \operatorname{length}(C' \cap \ker A) \leq \operatorname{length}\big(\ker_C A \cap B^{-1}(A(C))\big).$$

The representations are unramified at the new prime pair, and the additive inertia restrictions with these coefficients factor through the pro-ℓ tame quotient. Thus $\ker_C A = C_{i,a}(J)$, since the marked generators detect vanishing on that quotient. For a unit u the new congruence has slope u^{-1}, and the inequality bounds its length by that of the old cocycles satisfying the Frobenius-value property at the new pair.

If the Frobenius-value property fails at any datum and depth, choose an eligible prime where the condition fails. For this datum exclude the union of the bad residues over $1 \leq a \leq e$ and zero. The residue-field bound supplies one unit u for the chosen datum at all depths. Give it slope u^{-1}, and give the other data slope zero. The lengths for the other data also weakly decrease, because the new congruences with slope zero impose the old unramified conditions. The sum of lengths over i, a decreases strictly. For the initial singleton with slope zero this sum is at most

$$B_0 = \sum_i \sum_{a=1}^{e} \operatorname{length} C_{i,a}(\varnothing).$$

The same inertia argument justifies this bound at the first prime. It follows that the process terminates with at most $1 + B_0$ primes, independently of N.

At termination the Frobenius-value property holds at every depth for each eligible prime. To apply the primitive bound to the group at depth e, consider any depth a whose group has no member with a unit value coordinate. Every value then lies in $\varpi\mathcal{O}/\varpi^a$. Multiplication by ϖ on coefficients identifies the group at depth $a - 1$ with the one at depth a: all equations on the values are homogeneous and $\mathcal{O}$-linear. Descend until there is a unit value coordinate or the depth is zero. Thus the terminal bound gives length at most $G_i + L_i$ for $C_{i,e}(J)$. The first-place

bound of Section 24 then bounds releasing the distinguished pair at zero. After dividing by $\operatorname{ord}_{\mathcal{O}}(\ell)$ this gives the bound β_i for the untwisted length used in Section 25. The argument uses the hypotheses on the representation of Section 27.4.

27.4 Duality for the final local representation

In this subsection, we analyze the residual classes at the end of the descent in Section 27.3. The Cartier duality calculation gives the necessary relation between the two representations at the chosen torsion depth.

Consider an untwisted specified space at a torsion depth a with $V = T/\varpi^a T$, $T = \mathcal{O}^n$, and $a \le \operatorname{ord}_{\mathcal{O}}(\ell^M)$. Put $\Gamma = G_E$ and write $g^c = cgc^{-1}$ for the specified complex conjugation. Write $V^* = \operatorname{Hom}(V, \mu_{\ell^N})$ for the Cartier dual. Suppose there is $s \in \Gamma$ satisfying

$$\rho(s) = \rho((s^c)^{-1}) = \xi I, \qquad \chi_{\mathrm{cyc}}(s) = 1, \qquad \xi^2 - 1 \in \mathcal{O}^\times.$$

Suppose s lies in inertia at an original place prime to ℓ and the matrices of ρ span the full matrix algebra over $\mathcal{O}$. The cocycles z satisfy $z(s) = 0$. At termination assume the Frobenius-value property of Section 27.3 at this depth. We prove its length bound for a group containing a cocycle z with a unit coordinate at some group element.

Let S' contain the original ramification set of the representation, including the places above ℓ, and the later auxiliary primes and their conjugates. The first auxiliary pair is outside S'. For $v \in S'$, let A_v be the restrictions to the marked local group of cocycles satisfying the original local relations and the old congruences, with neither a ramification constraint nor the normalization at s.

Assume the original local conditions contain the coboundaries. The congruence conditions also contain them, since inertia acts trivially at the auxiliary primes and Frobenius acts trivially when the slope is nonzero. Let $\mathcal{L}_v$ be the image of A_v in local cohomology. At every finite $v \notin S'$ impose the unramified condition. There V is unramified of order prime to the residue characteristic. The unramified subgroups for V and V^* are exact orthogonal complements under local Tate duality [47, Ch. I, Theorem 2.6]. Thus the dual condition is unramified at the first auxiliary pair and at each new pair used below. The local length estimate needed here is

$$\sum_{v \in S'} \operatorname{length} A_v \le (|S'| + f) \operatorname{length} V + L_i.$$

The old auxiliary places contribute at most $\operatorname{length} V$ each because the Frobenius value determines the tame value by the congruence and determines the rest of the cocycle with that inertia value. Thus the remaining local bounds concern the excess L_i at the original places away from ℓ and the crystalline rank bound at the places above ℓ.

The cocycle module C injects into $H^1_{\mathcal{L}}(E, V)$. At the first pair the zero slope forces unramifiedness, and if a coboundary vanishes at s its defining vector is killed by the unit $\xi - 1$. Write $D^\vee_{\mathcal{L}} = H^1_{\mathcal{L}^\perp}(E, V^*)$ for the ordinary Selmer group using the local Tate orthogonal conditions. Since all infinite places of E are complex, the archimedean factors below apply also at $\ell = 2$. The Greenberg–Wiles formula [25, Theorem 2.19], in unnormalized length, gives

$$\operatorname{length} H^1_{\mathcal{L}}(E, V) = \operatorname{length} D^\vee_{\mathcal{L}} + \sum_{v \in S'} (\operatorname{length} A_v - \operatorname{length} V) - f \operatorname{length} V.$$

Indeed both global invariants vanish by the action of s. Since A_v contains the coboundaries, the local ratio $|\mathcal{L}_v|/|H^0(E_v, V)|$ is $|A_v|/|V|$. Unramified places give ratio one and complex places

give $1/|V|$. Taking logarithms to the residue field cardinality gives the displayed identity. In particular length $C \le \operatorname{length} D_{\mathcal{L}}^{\vee} + L_i$.

Put $U = \ker(\rho \bmod \ell^M) \cap \ker(\chi_{\mathrm{cyc}} \bmod \ell^N)$. For a dual class $[b]$, write $y = b|_U$. This is additive and equivariant under conjugation, and is independent of the representative since U acts trivially. Restriction is injective on dual classes. Indeed its kernel inflates from $H^1(\Gamma/U, V^*)$, where the image of s is central and acts by ξ^{-1}. Comparing a cocycle at sg and gs shows that such a cocycle is a coboundary because $s-1$ acts invertibly on V^*.

Let H be the subgroup generated by commutators with s and s^c. It is normal, stable under c, and lies in U. The identity $z([g,s]) = (1-\xi)z(g)$ and equivariance on U show that the $\mathcal{O}$-span of $z(H)$ is V: its reduction contains a nonzero vector, is stable under all matrices, and Nakayama's lemma applies.

Identify μ_{ℓ^N} additively using one fixed primitive root of unity. For $t \in \mathcal{O}$ and $u, u' \in U \cap U^c$ put

$$F_t(u,u') = y(u')(tz(u)) - y(u^c)(tz(u'^c)).$$

For $h \in H$, apply the prime construction of Section 27.2 to the target $\gamma_{w_0} h$. The commutators defining H lie in the normal subgroup generated by inertia at the original place of s and its conjugate. Together with the first prime these places form R_{in}.

Let J index the old auxiliary pairs, with distinguished representative w_0. Take

$$S_{\mathrm{sp}} = \bigcup_{w \in J \setminus \{w_0\}} \{w, \bar{w}\}.$$

The distinguished pair $\{w_0, \bar{w}_0\}$ belongs to R_{in}, so the splitting conditions apply only to the other pairs. Exclude the old prime pairs, the original forbidden primes and all places above the rational norm of w_0. The target and its conjugate are in U. Every old character kills h, since H is generated by commutators. At γ_{w_0} only the first character is nonzero and reads 1. Prescribe these finitely many residual and character evaluations, as well as the locally constant values of y, z and their conjugates, in the Frobenius neighborhood.

The new character vanishes on its own Frobenius, on every G_{E_v} for $v \mid \ell$, and on G_{E_w} and $G_{E_{\bar{w}}}$ for $w \in J \setminus \{w_0\}$. At an original level place $v \nmid \ell$ it is unramified. Anticyclotomicity preserves the character values on the inverse conjugate parameter. Hence this produces an eligible pair with $\phi, \phi^c \in U$. The cocycles y, z vanish at the first tame pair because both local classes there are unramified and inertia acts trivially. Thus $F_t(h,h) = F_t(\phi,\phi)$.

Choose z' by the terminal property, with $z'(\gamma_P) = z(\phi)$ and $z'((\gamma_P^c)^{-1}) = z((\phi^c)^{-1})$. Apply global reciprocity to $b \cup (tz')$. Its local invariants vanish on S' by orthogonality, and outside S' and the new pair by unramifiedness. At the new pair the coefficient actions are trivial and b is unramified. The local invariant reads the Frobenius value of b on the tame value of tz', with respect to the generator determined by the fixed root of unity.

For this formula, subtract from z' the Kummer cocycle of a uniformizer times the normalized tame value of z'. The difference is unramified and its cup product with b has zero invariant. The remaining cup product is the cyclic pairing of an unramified character with the uniformizer, which gives the stated evaluation by local reciprocity [48, Chapter III, Proposition 3.6]. The inverse conjugate of a normalized tame generator is normalized by the same root of unity at the conjugate place. The two marked generators therefore differ from the normalized generators by the same unit modulo ℓ^N. Multiplying the reciprocity sum by this common unit gives

$$0 = y(\phi)(tz(\phi)) + y(\phi^c)(tz((\phi^c)^{-1})) = F_t(\phi,\phi).$$

Here $z((\phi^c)^{-1}) = -z(\phi^c)$. Hence $F_t(h,h) = 0$ on H.

Additivity makes F_t antisymmetric on $H \times H$. Conjugating the first argument by s replaces t by ξt, and conjugating the second replaces t by $\xi^{-1}t$, by the two scalar actions at s, s^c. Applying antisymmetry with the first argument conjugated, at parameter ξt, gives $F_{(\xi^2-1)t} = 0$. Because $\xi^2 - 1$ is a unit, F_t vanishes for every t. It follows that $z(h) \mapsto y(h^c)$ extends to a unique $\mathcal{O}$-linear map $V \to (V^*)^c$.

In fact any linear relation between the $z(h)$ gives a relation between the $y(h^c)$ after evaluating on the spanning set $z(H)$ and its scalar multiples, using $F_t = 0$. Equivariance follows from the conjugation formulas, and the map depends $\mathcal{O}$-linearly on $[b]$ by uniqueness. If the map vanishes, $y|_H = 0$; for $u \in U$ the commutator $[s,u]$ then gives $y(u) = y(sus^{-1}) = \xi^{-1}y(u)$. Thus $y = 0$. We have proved the injection

$$D_{\mathcal{L}}^{\vee} \hookrightarrow \mathrm{Hom}_{\mathcal{O}[\Gamma]}(V, (V^*)^c).$$

The bound G_i for the length on the right gives the terminal bound $G_i + L_i$ for C. This proof uses the stated local length and hypotheses on the representation. The first-place determinant losses are paid separately at the untwisted specialization.

28 Galois representations, finite types, and local lengths

In this section, we attach Galois representations to the original Hecke eigenspaces and compute the normalizations at the finite places. The resulting representations supply the data for Section 29.

28.1 The original Hecke system and the normalized trace

In this subsection, we isolate the admissible finite type in the original Hecke eigenspace using Section 9.4. We then identify the normalization of the Frobenius trace needed for the attachment in Section 28.2.

Let $z \neq 0$ have Hecke system $b = (b_{w,i})$ in the final good space. On the marked principal level the good idempotent is a sum of the central idempotents attached to admissible tuples of finite parameters. Since the sum fixes z, some term acts nontrivially on z. The corresponding component keeps b: the unramified Hecke operators act at places away from the marked residue group and commute with its action. The joint trace comparison of Section 9.4 gives a nonzero component on the definite group with the same marked finite type and outside-level Hecke system.

The terminal rank-r space W_m is definite with its standard form and finite basis. Its level is the principal subgroup at $(\pi_M) = (\bar{\alpha})$ and the prescribed integral compact elsewhere, as in Section 9.4. The additional covector condition at π_N belongs to the rank-k source X of Section 6. Let σ be the compact coefficient representation of the marked weight, v its unit highest vector and $d_\sigma = \dim \sigma$. On a form reproduced by the real projector set

$$Lz(g_f) = d_\sigma \int_{U(W_m)_\infty} z(k, g_f)\sigma(k)v \, dk, \qquad \int dk = 1.$$

The compact projector kernel in these coordinates is $d_\sigma \langle v, \sigma(k)v \rangle$, with the inner product linear in its second variable. Indeed $\langle v, \sigma(k^{-1})Lz(g_f)\rangle = z(k, g_f)$ by the projector equation and change of variable in Haar. Thus L is injective. For a rational unitary element γ, the change $k \mapsto \gamma_\infty k$

uses left rational invariance of z and gives $Lz(\gamma_f g_f) = \sigma(\gamma_\infty) Lz(g_f)$. This is the coefficient transformation of the algebraic form. The integral commutes with every finite right translation, so it preserves the original level, finite type projector and outside-level Hecke system.

The split polynomial. Let $w, \bar{w}$ be a split pair off the level and twist of a marked descent from a rank r_0 to a rank $n = r_0 + t_0$ covector. The local Schrödinger test there is the matrix-lattice vacuum $\Phi_w = \mathbf{1}_{\mathrm{Mat}_{n\times r_0}(\mathcal{O}_w)}$. The other local factors are unchanged. For an i-space U in a residue vector space of dimension k, choose an integral γ_U with elementary divisors ϖ_w in i positions and 1 elsewhere, whose reduction has kernel U. These kernels parameterize the Hecke representatives. The target representatives are read by transpose and those at the opposite source place by inverse. The split Schrödinger rules give the two terms

$$\nu_w^i \Phi_w(\gamma_U^{-\mathsf{T}} X), \qquad (\nu_w')^i \Phi_w(X \gamma_U^{-1}).$$

The target determinant modulus factor is $q^{r_0 i/2}$, the source one is $q^{ni/2}$, and the splitting characters at the uniformizer give the other factors of $\nu_w^i, (\nu_w')^i$ specified below.

The two terms require X integral. For integral X the tests are $U \subseteq \ker \bar{X}^{\mathsf{T}}$ and $U \subseteq \ker \bar{X}$, respectively, where here $\bar{X}$ denotes reduction. Indeed γ_U^{-1} has only a simple pole and the image of $\varpi_w \gamma_U^{-1}$ modulo ϖ_w is U. Let s be the residue rank of X. For $P_d(Z) = \prod_{j=0}^{d-1}(1 - q^j Z)$, the subspace identity

$$P_d(Z) = \sum_{i=0}^{d} (-Z)^i q^{\binom{i}{2}} \begin{bmatrix} d \\ i \end{bmatrix}_q$$

gives target and source series $P_{n-s}(\nu_w Y)$ and $P_{r_0-s}(\nu_w' Y)$.

28.1.1 Hecke multiplicities and the trace

In this subsection, we compute the multiplicities used to normalize the original trace in Section 28.1.

The identity follows by counting i-spaces contained in and transverse to a fixed hyperplane, which gives the recurrence for multiplying by $1 - q^{d-1}Z$. Because $n = r_0 + t_0$,

$$P_{n-s}(\nu_w Y) = P_{t_0}(\nu_w Y) P_{r_0-s}(q^{t_0} \nu_w Y).$$

Thus on the actual test the target Hecke polynomial action is the source one at $q^{t_0}\nu_w Y/\nu_w'$ times $P_{t_0}(\nu_w Y)$. Moving the source action through the bilinear theta pairing inverts the coset under the inherited Haar, changing its component at $\bar{w}$ to w. Both actions preserve the original data at the other local factors. This proves the split Hecke comparison when the marked lift is nonzero.

We derive the trace identity of the split Hecke polynomial identity on an attached covector. Fix a descent pair with ranks r_0 and $n = r_0 + t_0$, with n even, and let μ be its unitary character before the tame twist ϑ. At an unramified split place w of norm q, use the reciprocal Hecke polynomial

$$P_{b,w}(Y) = \sum_{i=0}^{r_0} (-Y)^i q^{\binom{i}{2}} b_{w,i}.$$

Here $b_{w,0} = 1$. For the lift let $H_w(X)$ be the monic degree-n polynomial in the Galois normalization.

The split Hecke comparison needed here is

$$Y^n H_w(Y^{-1}) = \prod_{a=0}^{t_0-1} (1 - \nu_w q^a Y)\, P_{b,w}(q^{t_0}\nu_w Y/\nu'_w).$$

The nonzero scalars ν_w, ν'_w are the rank-r_0 and rank-n Frobenius values of the splitting characters. For a norm-one character χ and a rank n', this value is

$$q^{n'/2}\chi(\varpi_w/\bar{\varpi}_w)\upsilon(\varpi_w)^{-n'},$$

where υ is the splitting character of the Weil data. In the even rank the character for the lift is $\chi = \mu\vartheta(\upsilon|_{U(1)})^{n/2}$. By Section 4.3, $\upsilon(\varpi_w/\bar{\varpi}_w) = \upsilon(\varpi_w)^2$, so $\nu'_w = q^{n/2}(\mu\vartheta)(\varpi_w/\bar{\varpi}_w)$, where ϖ_w denotes the uniformizer idele.

Comparing linear terms gives the exact formula

$$-\nu_w^{-1}[X^{n-1}]H_w = \frac{q^{t_0}}{\nu'_w} b_{w,1} + \sum_{a=0}^{t_0-1} q^a.$$

In the geometric Frobenius convention, $\chi_{\mathrm{cyc}}(\mathrm{Fr}_w)^{-1} = q$. The scalar q^{t_0}/ν'_w is the product of the characters at Fr_w obtained from ϑ^{-1} and from the algebraic character

$$u \longmapsto \mu^{-1}(u/\bar{u})|u|^{(r_0-t_0)/2}$$

by reciprocity in the same convention. This follows from $|\varpi_w| = q^{-1}$ and $n = r_0 + t_0$. Write θ for the avatar of the latter character and ϑ_ℓ^{-1} for the former avatar.

Let a be the source splitting character of the original pair. The normalizing avatar α_{nor} is that of the algebraic Hecke character

$$x \longmapsto a(x/\bar{x})^{-1}\upsilon(x)^{r_0}|x|^{r_0/2}.$$

On the chosen CM type, the last two factors have infinity type z^{r_0}, by the infinity type of υ. The first factor has integral powers of $z, \bar{z}$ since a is a character of the norm-one torus. Its finite unit action is trivial at the old and twist marks, since the marked source character is a prescribed power of the norm-one restriction of υ. At the selected geometric Frobenius the avatar is ν_w^{-1}. Multiplying the Hecke character by its conjugate gives $|x|^{r_0}$, hence $\alpha_{\mathrm{nor}}\alpha_{\mathrm{nor}}^{c_E} = \chi_{\mathrm{cyc}}^{r_0}$.

If R is the representation attached to b and R_{lift} to H_w, put $\widetilde{R}_{\mathrm{lift}} = \alpha_{\mathrm{nor}} R_{\mathrm{lift}}$. After transporting coefficients by ι, the normalized trace equality is

$$\operatorname{tr}\widetilde{R}_{\mathrm{lift}}(g) = \vartheta_\ell^{-1}(g)\operatorname{tr}((\theta R)(g)) + \sum_{a=0}^{t_0-1} \chi_{\mathrm{cyc}}(g)^{-a}.$$

For this passage it suffices that the coefficient embeddings converge to the corresponding characteristic coefficients along the same split Frobenius filter approaching g.

Take the limit in the linear coefficient identity and use continuity of the avatars. In the usual exact unramified attachment this is also obtained by density of the Frobenius classes of all absolute degree-one places off the finite excluded set, using both split coordinates and Chebotarev [48, Ch. V, Theorem 3.23]. Thus the main summand in this normalization is the single representation θR, independently of the subsequently chosen tame twist.

Removing auxiliary character factors. Suppose two continuous semisimple representations R_i in the same coefficient field have characteristic polynomial at the prescribed Frobenius elements equal, in the limit, to a common monic polynomial times the linear factors of a list of characters U_i. Assume the two character lists are disjoint as Galois characters. On taking first coefficients and limits, the virtual characters $\operatorname{tr} R_i - \operatorname{tr} U_i$ agree on G_E. Characters of pairwise nonisomorphic irreducible representations over an algebraically closed field of characteristic zero are linearly independent.

Indeed the algebra generated by the group on their direct sum is finite-dimensional and its faithful module is semisimple, so its simple factors are the full matrix algebras on the distinct simple modules. Their trace functions are independent on these factors. For any constituent in U_i, comparison with the other list shows that its multiplicity in R_i is at least its multiplicity in U_i. Thus both R_i contain U_i and have the same semisimple remainder R. The summands are continuous as restrictions to invariant subspaces.

This proves the cancellation at the level of representations. The resulting characteristic polynomials satisfy $\operatorname{charpol}(R_i(g)) = \operatorname{charpol}(R(g))\operatorname{charpol}(U_i(g))$. At the specified Frobenius elements, division by the monic character factor recovers the common attachment polynomial in the limit. Indeed coefficients of the monic quotient are obtained by successively subtracting products of higher coefficients, so this division is continuous.

If both larger attachments are unramified, crystalline, or potentially crystalline at a place, these properties pass to the remainder as a direct summand. The jumps at a coefficient embedding are obtained by subtracting the multiset of jumps of the corresponding character list. At an unramified place, if the two larger annihilating polynomials are $P(X)Q_i(X)$ and Q_0, Q_1 are coprime there, P annihilates the remainder by the Bézout identity applied to its Frobenius matrix. These reductions require the two larger attachments and the local polynomial comparisons on the marked form.

28.2 Galois attachment and spherical compatibility

In this subsection, we attach a Galois representation to the spherical eigensystem of Section 28.1. The contragredient compact type fixes the normalization, which is used again for the Frobenius tests of Section 29.2.

Use a marked eigensystem on a definite group of rank n, with the chosen compact coefficient representation σ_{alg}. Its scalar coefficient is $\langle \sigma_{\mathrm{alg}}(k)v, Lz(g_f)\rangle$. At the chosen CM embedding the coefficient action on Weyl coordinates is $k \mapsto S_\mu(\bar{k})$, with the determinant power included in S_μ; here $\bar{k} = (k^{-1})^{\mathsf{T}}$ on the definite unitary group. Right translation acts in the conjugate-linear first variable, so the compact automorphic constituent has type $\sigma_{\mathrm{alg}}^{\vee} \simeq S_\mu$. Its Harish–Chandra entries are $\mu_q + (n-1-2q)/2$, $0 \le q < n$; at the conjugate embedding their multiset is negated. This fixes the sign of the raw jumps in Section 29.1.1. One may choose an irreducible automorphic constituent containing a component of the eigenvector: the space is compact modulo rational points, and the outside-level eigenvalues and the specified finite type are preserved by taking components. By the compact coefficient realization in Section 28.1 this is the same marked finite-level Hecke system.

Apply the base change theorem of Labesse [40, Corollaire 5.3] in the form stated in [31, Theorem 2.2]. The field F_0 has degree at least two, and E/F_0 is unramified at finite places. The definite groups used here have the standard positive Hermitian matrix. At a nonsplit finite place choose a unit u of norm -1, using the unit norm surjectivity of Section 4.3. The vectors

$e_{2j-1}+ue_{2j}$, $1\le j\le\lfloor n/2\rfloor$, span a totally isotropic subspace of maximal dimension. Thus the group is quasi-split there. At a split place it is GL_n. The real groups are compact, so their representations are discrete series and Labesse's condition $(*)$ holds.

The rank-n transfer is an isobaric sum of discrete conjugate-self-dual representations over E. It agrees with the local base change at every split finite place and every inert place where the representation is spherical. Its infinitesimal character is the base change of the compact weight. By Mœglin–Waldspurger [51, theorem of the introduction, pp. 605–606], each block of rank ab is a length-b Speh representation from a unitary cuspidal σ of rank a, with exponents $u_j=(b-1)/2-j$, $0\le j<b$. Conjugate self-duality and uniqueness of the cuspidal support give $\sigma^c\simeq\sigma^\vee$. In both complex coordinates at infinity, the infinitesimal weights of the full base change lie in $(n-1)/2+\mathbb{Z}$ and are pairwise distinct. Subtracting u_j shows that the weights of σ are regular and lie in $(n-b)/2+\mathbb{Z}$.

Use the unitary Hecke character υ of Section 4.3. It is unramified at every finite place, satisfies $\upsilon^c=\upsilon^{-1}$, and has infinity type $z/|z|$ at the chosen CM embeddings. Set

$$\psi=\begin{cases}1, & n-a-b+1 \text{ even},\\ \upsilon, & n-a-b+1 \text{ odd}.\end{cases}$$

Then $\sigma\otimes\psi$ is regular algebraic and conjugate self-dual, since its infinitesimal exponents lie in $(a-1)/2+\mathbb{Z}$.

For its polarizing character on F_0 use 1 if a is even and the quadratic character of E/F_0 if a is odd. Its pullback by the norm is trivial, and its sign at each real place is $(-1)^a$. Hence the hypotheses of Barnet-Lamb–Gee–Geraghty–Taylor [11, Theorem 2.1.1] apply for every coefficient prime ℓ, including 2.

For each j the character

$$\alpha_j=\psi^{-1}|\cdot|^{(a-n)/2+u_j}$$

is algebraic. Write $d_j=a-n+b-1-2j$ and $e_\psi=0$ for $\psi=1$, $e_\psi=1$ for $\psi=\upsilon$. Then $d_j\equiv e_\psi \pmod 2$. On the chosen CM type the infinity character of α_j is

$$z^{(d_j-e_\psi)/2}\bar z^{(d_j+e_\psi)/2};$$

the exponents are exchanged at the conjugate embedding. Let $\rho_{\ell,\iota}$ be the direct sum over all blocks and j of

$$r_{\ell,\iota}(\sigma\otimes\psi)\otimes r_{\ell,\iota}(\alpha_j).$$

Here $r_{\ell,\iota}(\alpha_j)$ is the class field avatar in the same geometric Frobenius convention. The twists total $(1-a)/2+(a-n)/2+u_j=(1-n)/2+u_j$. Thus this direct sum, on the chosen split coordinate, has the local Langlands normalization of the base change twisted by $|\det|^{(1-n)/2}$. This is the direct sum computation in the proof of [31, Theorem 2.3], using the cuspidal attachments for every ℓ. At a nonsplit spherical place $w\nmid\ell$, Labesse likewise identifies the unramified base change. The factors ψ_w and ψ_w^{-1} cancel in each summand. The resulting Weil–Deligne parameter is unramified with zero monodromy, so the actual representation is unramified there.

At an unramified split place w let λ_i be the normalized Satake parameters. With Haar volume one for the maximal compact and right translation by the positive diagonal coset with j uniformizers, the eigenvalue is $t_{w,j}=q^{j(n-j)/2}e_j(\lambda_1,\dots,\lambda_n)$. Indeed, in [44, Ch. V, (2.7) and §3], the Hall–Littlewood polynomial for (1^j) is e_j. The spherical transform of the indicated coset is

$$q^{-\binom{j}{2}}e_j(q^{(n-1)/2}\lambda_1,\dots,q^{(n-1)/2}\lambda_n)=q^{j(n-j)/2}e_j(\lambda_1,\dots,\lambda_n).$$

Proposition 28.1 (Spherical Frobenius polynomial)**.** *For an unramified split place away from ℓ and the finite excluded set of the attached system, the characteristic polynomial of geometric Frobenius, in the chosen split coordinate, is*

$$\begin{aligned}\det(XI_n - \rho_{\ell,\iota}(\mathrm{Frob}_w^{\mathrm{geom}})) &= \sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} t_{w,j} X^{n-j} \\ &= \prod_{i=1}^{n} (X - q^{(n-1)/2}\lambda_i).\end{aligned} \tag{81}$$

All coefficients on the left are computed at w; the inverse Satake tuple belongs to $\bar{w}$. Each parity-corrected cuspidal factor gives a regular compatible system by [11, §5.1, final paragraph]. After a finite extension of its coefficient field, Lemma 5.3.1(3) of that paper gives an integral model at every coefficient place. The compatible systems of the polarized characters α_j are regular because they have rank one. For fixed ℓ take the compositum of the finitely many local coefficient fields. The tensor products and their direct sum then have a stable lattice over its valuation ring. Section 38.2 also proves the local field-of-definition assertion directly for every continuous representation of a profinite group. Thus the coefficients in the ℓ-adic embedding are integral and the polynomial annihilates the actual Frobenius operator. These claims hold at every such split place outside a finite set including the coefficient and level places.

For fixed ℓ, take the union of the finite excluded sets for the original-level systems in Section 29.2 before choosing the depth and auxiliary primes. At the level raised at those primes, Section 30.3 takes the corresponding finite union for its systems before applying Chebotarev there. An existing semisimple attachment specified by convergence to the same Frobenius polynomials has the same trace by continuity on the dense selected Frobenius classes. Independence of irreducible characters as in Section 28.1 identifies the two semisimple representations. In particular (81) applies to the same systems used in the residual comparison. It supplies the spherical part of Section 29.2.

28.3 Inertial parameters of the finite types

In this subsection, we read the inertia eigenvalues of the attachment in Section 28.2 from the original finite types. These local data verify the spectral conditions for the length estimates of Section 28.4.

For a split place of residue characteristic different from ℓ, the attachment of Section 28.2 has the full Frobenius-semisimplified Weil–Deligne comparison in the chosen coordinate. Labesse supplies local base change at split places, and [11, Theorem 2.1.1(2)] gives the comparison for the cuspidal blocks, including the monodromy comparison of Caraiani [19]. For a Speh block the local parameter of its Langlands quotient is the sum of the shifted parameters of the cuspidal datum.

For a cuspidal block of rank $a \geq 2$, the parity-corrected representation $\sigma \otimes \psi$ is cohomological by [21, Lemme 3.14, p. 114], since it is regular algebraic and all the infinite places of the CM field are complex. It is conjugate self-dual, so [19, Theorem 1.2] gives tempered local components. The character ψ is unitary, and for $a = 1$ the datum is itself a unitary character. Thus each σ_w is tempered. Ordering the factors $\sigma_w|\det|^{u_j}$ by decreasing u_j gives the standard module. Its Langlands quotient has parameter $\bigoplus_j \mathrm{rec}(\sigma_w)|\cdot|^{u_j}$, with the monodromy of each summand. Products of unitary representations of general linear groups are irreducible under normalized

parabolic induction by Bernstein's theorem [69, Theorem 2.5(i)], so the local sum across the discrete blocks is the same parameter. The algebraic twists in the direct sum cancel the parity corrections at this very place. This proves the required comparison on the same attachment. Frobenius semisimplification preserves the Weil–Deligne inertia action and the monodromy operator. When the latter vanishes, the inertia action on the original ℓ-adic representation at $w \nmid \ell$ equals its finite Weil–Deligne inertia action.

We describe the level-zero local calculation for the finite type of that attachment. Let $k = \mathbb{F}_q$ be the residue field, $K_w = \mathrm{GL}_n(\mathcal{O}_w)$, and $K_{w,1}$ its principal kernel. For an irreducible factor of the squarefree type polynomial of degree a, the finite type is the genuine cuspidal character $\tau_\theta = (-1)^{a-1} R_T^\theta$ of $\mathrm{GL}_a(k)$, where $T(k) = \mathbb{F}_{q^a}^\times$ and θ is indexed by a root as in Section 28.3.1. Its Frobenius orbit has size a, since the root has degree a and the character parametrization commutes with Frobenius. This is the primitivity hypothesis in [26, §2.4]. Full multiplicative order is required only at the separately chosen primitive type polynomials. The corresponding depth-zero supercuspidal has on inertia the a Frobenius conjugates of this character, with zero monodromy [26, Proposition 2.4.1(i)]. This is a characteristic-zero statement.

For the full parameter, the $K_{w,1}$-invariants of a parabolic induction are the finite Harish–Chandra induction of the inducing $K_{w,1}$-invariants [66, Proposition 6.18]. Indeed the maximal compact and the parabolic give Iwasawa decomposition, and reducing the intersection with the parabolic gives the finite one. Invariants are exact in characteristic zero by compact averaging. For a supercuspidal inducing factor, its nonzero principal-kernel invariants are exactly the finite cuspidal representation of its depth-zero type, by the Cartan calculation in Section 31.1. Finite cuspidal support then determines the multiset of these blocks [66, §6.3]. Applying this to the supercuspidal support shows that the squarefree type polynomial gives one depth-zero factor for each distinct irreducible factor. If the upper type also has a determinant-character block of size t, it gives t rank-one factors on that tame line. There is no nontrivial segment on a line occurring once.

On a repeated line, twist the common tame character out of its block. An essentially square-integrable segment of length $a > 1$ on the resulting rank-one unramified line is an unramified twist of Steinberg. Its invariants under the principal kernel of $\mathrm{GL}_a(\mathcal{O}_w)$ are finite Steinberg. Indeed, for $H_a = \mathrm{GL}_a(k)$ and its upper Borel B_a, compact averaging and the parabolic quotient defining Steinberg give

$$\mathrm{St}_{H_a} = \mathbb{C}[H_a/B_a] \Big/ \sum_{B_a \subsetneq P \subseteq H_a} \mathbb{C}[H_a/P].$$

Here P runs over the standard parabolics, with each summand pulled back along the flag projection. For $a > 1$ the summand $P = H_a$ contains all constant functions, the only H_a-invariants in the numerator. Exactness gives $\mathrm{St}_{H_a}^{H_a} = 0$. By Frobenius reciprocity the finite Harish–Chandra induction from such segments cannot contain the trivial representation of the repeated block.

At a chosen old place $w \in S_M$, let r_0 and t_0 be the ranks of the two blocks. The upper parahoric type on the first block is the prescribed finite type τ_w twisted by $\omega_w^{-j} \circ \det$; it is trivial on the entire complementary group $\mathrm{GL}_{t_0}(k_w)$. The level and its type projector give the same assertion on every upper eigensystem at the original or raised level. Thus finite Frobenius reciprocity gives

$$\mathrm{Hom}_{\mathrm{GL}_{r_0+t_0}(k_w)}\Big(\mathrm{Ind}_{P(k_w)}^{\mathrm{GL}_{r_0+t_0}(k_w)}((\tau_w \otimes (\omega_w^{-j} \circ \det)) \boxtimes \mathbf{1}), \pi_w^{K_{w,1}}\Big) \neq 0,$$

where P stabilizes the first block. If $r_0 = r$, the original type has no root of order dividing

q_2, so none of its rescaled roots equals 1. If $r_0 = 1$, the original polynomial is $X - 1$, and the rescaled line is $\omega_w^{-j} \neq 1$, since $\mathrm{ord}(\omega_w) = q_2 > m \geq j \geq 1$ (Section 4.4). Thus the two blocks have disjoint cuspidal supports in both cases. At an auxiliary place in the ramification support the two determinant characters are also distinct. To pass back to $\mathrm{GL}_n(k)$, group the segments by their distinct supercuspidal inertial lines and use the corresponding parabolic P. The given representation is a subquotient of the induction from the segment representations. Exactness of $K_{w,1}$-invariants embeds each of its finite constituents in the resulting finite induction, by semisimplicity. Apply finite parabolic Mackey theory to the Hom space from the prescribed type, itself induced from the same grouping of the finite cuspidal parameters. In a double coset other than $P(k)$, a block of positive rank moves between two different groups of parameters. The corresponding Jacquet modules would have to have a common irreducible constituent on that block. Each of its cuspidal factors would occur in the supports of both groups, which are disjoint by the cuspidal factorization in [29, §8, Theorem 14]. Only the identity double coset remains. It requires the determinant character on its own block, which the preceding reciprocity calculation excludes if that block has a segment of length greater than one. Thus its monodromy is zero. The same proof applies to a pair of distinct tame determinant characters on two blocks. In particular at a split auxiliary prime with marked ramification the normalized characters are ϑ_ℓ^{-1} on the original rank block and 1 on the added block. For an inertia element g the type is annihilated by $(X - \vartheta_\ell^{-1}(g))(X - 1)$. For $w \nmid \ell$ zero monodromy identifies actual Galois inertia with this finite inertia of the Weil–Deligne parameter. At $w \mid \ell$ the corresponding polynomial acts on the potentially crystalline periods, as proved below.

28.3.1 The inertia characters

In this subsection, we compute the inertia characters of the finite type on the generator used in the trace comparison of Section 28.1.

To specify the action of the generator, fix the faithful multiplicative character $\jmath : \bar{k}^\times \to \mathbb{C}^\times$ from the parameter convention. If $\jmath(\xi) = \exp(2\pi i b/(q^a - 1))$, the Green character of the degree-a field is $\jmath|^b_{\mathbb{F}^\times_{q^a}}$. For each a, let F_a/E_w be unramified of degree a. On inertia set

$$\omega_a(s) = \overline{\mathrm{Art}_{F_a}^{-1}(s^{\mathrm{ab}})} \in \mathbb{F}_{q^a}^\times,$$

using geometric reciprocity and abelianization in W_{F_a}. For $a \mid D$ the norm relation gives $\omega_a = \omega_D^{(q^D-1)/(q^a-1)}$. Take D divisible by all degrees in the finitely many systems at this place. The tame quotient is procyclic, and ω_D maps it onto $\mathbb{F}^\times_{q^D}$. Choose one tame generator s with $\jmath(\omega_D(s)) = \exp(2\pi i/(q^D - 1))$. For every such system the Green character on s then equals $\jmath(\omega_a(s))^b = \jmath(\xi)$.

Its Frobenius conjugates are the q-th powers. On the lower main summand, the character θ of Section 28.1 has smooth inertia character ω_w^{-j} at each chosen old place. On an upper system this factor is already present on the first parahoric block, and the upper normalizing avatar α_{nor} has trivial smooth inertia there. Thus the finite parameter receives the factor only once. The absolute norm factor is unramified. Tensoring by this character multiplies each root of the type polynomial by the corresponding rank-one root, since pullback of a character from $k^\times$ uses the norm with exponent $(q^a - 1)/(q - 1)$. For the rank-one main summand the underlying representation is trivial, so the same normalization gives precisely that rank-one root. The auxiliary twist is unramified at the original primes.

At a distinguished place $w \mid p_\nu$ away from the coefficient prime ℓ, let $q = p_\nu$ be the residue cardinality and let $C_w(X)$ be the normalized parameter of a main summand of rank n. The preceding local comparison gives finite inertia and reads the eigenvalues of the specified tame generator as $\iota\jmath(\xi)$ on the roots of C_w. In rank r, the original parameters are monic squarefree of degree r, with nonzero roots and with no root among the q_2-th roots of unity.

In rank one the original parameter is $X - 1$. Normalization scales each root at lift index j by the corresponding local unit of order dividing q_2. At a marked primitive type polynomial that unit is a power of the inverse parameter of the character of order q_2; in rank one it is its j-th power. Here $1 \le j \le m$ and $q_2^2 \mid q - 1$. In rank r, a normalized root equal to one would make the inverse multiplier a forbidden original root. In rank one the multiplier is nontrivial since $1 \le j \le m < q_2$. Thus every normalized root is nonzero and different from one.

A root of C_w has degree $h \le n$ over the residue field and order dividing $q^h - 1$. It follows that the inertia eigenvalues at the tame generator have orders dividing $q^{n!} - 1$. All the characters of inertia in the preceding decomposition factor through $\omega_{n!}$, by the norm relation for $h \mid n!$. This also holds for the normalizing character, which factors through ω_1. Thus every inertia element has the same image as some power of the chosen generator. The finite inertia representation is semisimple, and every element of its image is killed by the prime-to-q exponent $q^{n!} - 1$.

For any eigenvalue $\alpha \ne 1$ of order dividing $q^h - 1$, the identity $\sum_{a=0}^{q^h-2} \alpha^a = 0$ gives

$$v_\ell(\alpha - 1) \le v_\ell(q^h - 1) :$$

subtract the geometric sum at 1 and divide by $\alpha - 1$, with integral quotient. Since $q^h - 1$ divides the order of $\mathrm{GL}_{rt}(\mathbb{F}_q)$, multiplying the eigenvalue differences proves the determinant bound $nv_\ell(A_{\mathrm{aut}})$. At the conjugate place use the inverse conjugate of the marked element. In the polarization trace relation the cyclotomic factor is trivial on this inertia.

Applying it to every power of the marked element gives equal power traces at the two parameters. Newton's identities give equal characteristic polynomials and hence the same estimate. In this way the determinant bounds concern the specified tame generator at each chosen place and its inverse conjugate. Choosing one distinguished place on a rational prime different from ℓ supplies the common reference element in the bound.

28.3.2 The matrix span at the coefficient place

In this subsection, we use the inertia characters of Section 28.3.1 to obtain the matrix span for the trace comparison.

For the full matrix span, take a primitive type polynomial at a distinguished place above that prime. Its original root generates $\mathbb{F}_{q^r}^\times$. Multiplication by any element of order dividing q_2 preserves this property: the exponent of the multiplier in a generator of this group is a multiple of $(q^r - 1)/q_2$. Every prime dividing $q^r - 1$ divides this quotient, since $q_2^2 \mid q - 1$. Thus the new exponent remains prime to $q^r - 1$. The normalized roots form the q-Frobenius orbit of a generator α_0.

This orbit stays distinct after every ℓ-power. Indeed r is a positive power of 3 with $r \ge 3$. Put $Q' = q^{r/3}$. The factor $(q^r - 1)/(q^{r/3} - 1) = Q'^2 + Q' + 1$ is congruent to 3 mod 9, since $9 \mid q - 1$, and exceeds 3. It has at least two distinct prime divisors. It cannot divide a power of ℓ, so no $\alpha_0^{\ell^a}$ belongs to $\mathbb{F}_{q^{r/3}}$. Every proper subfield over $\mathbb{F}_q$ lies in that field. Under the faithful character map into the coefficient field the orbit consequently has pairwise unit differences: a root of unity reduces to 1 only when its order is an ℓ-power, by the separability of roots of

order prime to ℓ. The same argument applies if the tame generator raises all eigenvalues to an exponent prime to their orders.

At such a place, the orbit is the full spectrum of the inertia generator. To see this from the annihilating polynomial, one eigenvalue occurs, all its q-powers occur by tame Frobenius conjugation, and there are r eigenvalues counted with multiplicity. Raising the generator to the exponent $(q^r-1)/(q-1)$ gives a scalar whose root-of-unity order is $q-1$, since every normalized root is a Frobenius conjugate of a generator.

For the rank-one main pair, its own normalized root already gives a scalar with order $q_2/\gcd(q_2, j)$. Let ξ be the resulting scalar and n_{up} the rank of the upper group. For $a \geq 0$ and $1 \leq h \leq 4n_{\mathrm{up}}+2$, the difference $\xi^{h\ell^a}-1$ is a unit. Otherwise the order of ξ would divide $h\ell^{a+b}$ for some b, and thus $q_2 \mid jh\ell^{a+b}$ in either case. Take $p_* \in \{2,5\}$ different from ℓ. Its s-th power divides $q_2 = (q_0 d)^{s+6}$. The main rank is $n = r$ or 1, while $n_{\mathrm{up}} = r+t$ or $2z \leq 2r$, respectively. Thus for large x,

$$jh \leq (4n_{\mathrm{up}}+2)m \leq (4rt+2)m = O(x^{21/2}) < x^{11} \leq m^{22} = 2^s \leq p_*^s.$$

This contradicts $p_*^s \mid jh$. In particular ξ^2-1 is a unit. The scalar comes from the original tame inertia.

For the lift numbered j and a real place τ, the two conjugate jumps at index u are $(t-1)/2 \pm (\lambda_u(\tau)+w_j(\tau))$ at rank r, and $z-1\pm c_\tau$ at rank one, by (H1) and (H2). Their sums are $t-1$ and $2z-2$, respectively. Thus each pair contains at most one negative jump.

At the coefficient place. Use [10, Theorems 1.1 and 3.1] for each regular algebraic conjugate-self-dual cuspidal block of rank at least two after its parity correction. For every coefficient prime ℓ, including 2, and every $w \mid \ell$, it identifies the semisimplified Weil representations and bounds the Galois monodromy partitions by the automorphic partitions on each inertial orbit. Zero automorphic monodromy means all partitions are minimal. Hence the Galois monodromy is zero and the Frobenius-semisimplified parameters agree.

This applies to the marked split type since the local calculation gives zero monodromy on the sum of the shifted blocks, and hence on each of them. For a rank-one algebraic Hecke character with ℓ-adic avatar δ, put $F = E_w$ and choose a uniformizer ϖ. With geometric reciprocity the definition of the avatar on units is

$$\delta(\mathrm{Art}_F(u)) = \varepsilon_w(u) \prod_{\tau: F \hookrightarrow \overline{\mathbf{Q}}_\ell} \tau(u)^{b_\tau} \qquad (u \in \mathcal{O}_F^\times),$$

where ε_w is the smooth local units character in the coefficient field and the integers b_τ are the exponents of the algebraic factor. Extend ε_w by $\varepsilon_w(\varpi) = 1$ and use reciprocity to view it as a finite Galois character. The Lubin–Tate character χ_ϖ restricts to the identity on units and sends $\mathrm{Art}_F(\varpi)$ to 1. Hence

$$\delta = \delta_{\mathrm{ur}}\, \varepsilon_w \prod_\tau (\tau \circ \chi_\varpi)^{b_\tau}, \qquad \delta_{\mathrm{ur}}|_{I_F} = 1.$$

The unramified factor is crystalline of weight zero. Each $\tau \circ \chi_\varpi$ is crystalline of weight -1 at τ and zero at the other embeddings [38, Example 6.2.6(1),(3), pp. 1095–1096]. Thus the jump at τ is $-b_\tau$. Over the field cut out by ε_w, let e be a basis of the finite-character factor. Its period $1 \otimes e$ has inertia character ε_w. The other factors are crystalline over F, so inertia acts trivially on their period spaces after base change. Hence the linear inertia on the periods of δ is ε_w. For

$F = \mathbf{Q}_\ell$ and $\varpi = \ell$, χ_ϖ is the cyclotomic character. Thus these attachments are potentially crystalline at the marked coefficient places, with the specified tame inertial types.

Suppose $w \mid \ell$ and neither w nor $\bar{w}$ belongs to the original level support. The original level contains the stabilizer of the standard unimodular Hermitian lattice there. This stabilizer also fixes the raised vector since the auxiliary twist and level changes are supported away from ℓ (Sections 27.2 and 30.3). This stabilizer is hyperspecial. Indeed, over the quadratic étale algebra of local integers the integral unitary group becomes GL_n. Projection to one factor determines the other by inverse adjoint. Smoothness and connected reductive geometric fibres descend, also in residue characteristic two. Labesse's local base change Π_w is spherical. Put $K_m = \mathrm{GL}_m(\mathcal{O}_w)$. By the description of the Speh factors in Section 28.2, Π_w is a subquotient of the normalized parabolic induction of all $\sigma_w|\det|^{u_j}$, over all blocks and j. Taking K_n-invariants is exact by compact averaging. The Iwasawa decomposition identifies the invariants of the induction with $\bigotimes_{\text{blocks}} \bigotimes_{j=0}^{b-1} \sigma_w^{K_a}$: restriction to K_n identifies an invariant function with its value at 1, and the modulus and the norm twists are trivial on the integral Levi. Since $\Pi_w^{K_n} \neq 0$, every $\sigma_w^{K_a} \neq 0$. The unramified parity character preserves these fixed vectors.

Theorem 2.1.1(4) of [11] now gives crystalline cuspidal attachments. The smooth local character of α_j is trivial on $\mathcal{O}_w^\times$, so the rank-one calculation above also gives crystallinity of its avatar over E_w. Globally, for $d_j = a - n + 2u_j$ one has $\alpha_j^c \alpha_j = |\cdot|_E^{d_j}$. Hence $\delta_{E/F_0}|\cdot|_{F_0}^{d_j}$ is a polarizing character with sign -1 at every real place. Here δ_{E/F_0} is the quadratic character. The same theorem in rank one gives a crystalline attachment of α_j, which agrees with its class field avatar by equality of their Frobenius values off a finite set. Thus the original direct sum is crystalline over E_w, including for $\ell = 2$. At a coefficient place the geometric Frobenius in the local comparison acts on the Weil–Deligne parameter. Its twist on the j-th summand is $|\cdot|^{(1-n)/2+u_j}$, since the two parity characters cancel. At every $w \mid \ell$, each additive auxiliary character χ_q of Section 27.2 vanishes on G_{E_w}. Thus the selected character $\vartheta = \prod_q \zeta_q^{\chi_q}$ of Section 27 satisfies $\vartheta^h|_{G_{E_w}} = 1$ for every integer h. All its powers preserve these original local representations and their Hodge–Tate numbers.

At a marked coefficient prime $\ell = p_i$ the local field is $\mathbb{Q}_\ell$. Each period eigenvalue for a degree-a parameter has order dividing $\ell^a - 1$, with $a \leq r$. Write $B_w(s)$ for the linear action of inertia on the potentially crystalline periods. Polarization identifies the action at the conjugate place by

$$B_{\bar{w}}(c_E s c_E^{-1}) \sim (B_w(s)^{-1})^{\mathsf{T}}.$$

The Tate power in the polarization has trivial linear period inertia. Consequently $c_E s^{-1} c_E^{-1}$ at $\bar{w}$ gives the same eigenvalues as s at w. Thus the common exponent

$$e_r = \operatorname*{lcm}_{1 \leq a \leq r} (\ell^a - 1)$$

kills inertia on these periods. Put $m_r = \mathrm{lcm}(1, \ldots, r)$. Since $\ell = p_i > r$, one has $\ell \nmid m_r e_r$. The unramified extension of degree m_r contains μ_{e_r}, since $e_r \mid \ell^{m_r} - 1$. Adjoining an e_r-th root of a uniformizer then gives a totally ramified extension of degree e_r. The composite has degree prime to ℓ.

The finite action of wild inertia is already trivial. The tame inertia of the extension maps into the e_r-th powers of the original tame quotient. For a de Rham representation the Weil–Deligne representation is obtained from its potentially semistable periods. Its restriction here has trivial inertia and zero monodromy, so the representation is crystalline over this extension.

The same extension works for the finitely many types under consideration, including the degree-one normalizing characters.

For a fixed inertia element the eigenvalues are common to all systems with this type, and each belongs to an unramified coefficient extension of degree at most r. The product of their monic minimal polynomials gives a common integral annihilator of degree at most $(r+t)r$. The extra tame degree-one characters have the same property. These equations concern the potentially crystalline period actions. They pass to subspaces and sums of the period modules with these types. This supplies the inertial polynomials for the local class of upper lattices in Section 29.1.

The paired jumps. The raw rank-n attachment satisfies $R^{c} \simeq R^{\vee}\chi_{\mathrm{cyc}}^{1-n}$: in each Speh string j pairs with $b-1-j$, and $\alpha_{j}^{c}\alpha_{b-1-j} = |\cdot|^{a-n}$. Combine this with the polarization of the rank-a cuspidal attachment. At the given embedding let v be an infinitesimal exponent of the cuspidal datum σ, and let b_{ψ} be the infinity exponent of the parity character there. It is 0 for $\psi = 1$ and $\pm\frac{1}{2}$ for $\psi = \upsilon$. By [11, Theorem 2.1.1(3)] and the rank-one computation above, the two algebraic factors $r_{\ell,\iota}(\sigma \otimes \psi)$ and $r_{\ell,\iota}(\alpha_{j})$ have respective integral jumps

$$\frac{a-1}{2} - (v + b_{\psi}), \qquad -\left(\frac{a-n}{2} + u_{j} - b_{\psi}\right).$$

Their sum is $(n-1)/2 - (v+u_{j})$. The parentheses run through the infinitesimal exponents of the rank-n base change.

In the two conjugate embeddings the raw jumps match u with $n-1-u$. The normalizing character and its conjugate have product $\chi_{\mathrm{cyc}}^{n-t_{0}}$ in the rank-n attachment of a descent with rank difference t_{0}. Thus the normalized jumps pair as $u, t_{0}-1-u$. The main summand keeps this pairing after subtracting the extra character string, which is invariant under the same involution. Hence at most one jump per pair is negative. The rank-one chart has the same property, with jumps $z - 1 \pm c_{w}$.

28.4 Local length bounds for a main summand

In this subsection, we use the inertial eigenvalues of Section 28.3 and the stable coefficient lattices to bound the lengths needed in Section 27.4. The estimates are normalized by the chosen valuation at ℓ.

Fix the coefficient place $\kappa = (\ell, \iota)$ and the original main summand of rank $n \le r$. We describe the precise data on the summand that suffice for the previous count. Work over a finite ℓ-adic coefficient field, enlarged to contain the marked eigenvalues and the images of the designated embeddings of the local fields. The representation is unramified outside the places above $\ell p_{0} p_{1}$, is de Rham at the places above ℓ, and has trace relation

$$\operatorname{tr}\rho(g^{c}) = \chi_{\mathrm{cyc}}(g)^{-d_{\mathrm{pol}}} \operatorname{tr}\rho(g^{-1}), \qquad 1 \le d_{\mathrm{pol}} + 1 \le t, \quad t = 5r.$$

Assume the marked tame type gives a cyclic Frobenius orbit of n distinct inertia eigenvalues whose pairwise differences are units at a distinguished place prime to ℓ.

There the norm is p_{ν} for $\nu \in \{0,1\}$. Assume also a scalar inertia element s away from ℓ whose scalar ξ has $\xi^{2}-1$ a unit. At the original places above $p_{0}p_{1}$ away from ℓ there is a bound on the exponent of the original inertia image prime to the residue characteristic. At each such place a specified inertial element has determinant of $\rho(g) - I$ of valuation at most $n u_{\ell}$, where

$$A_{\mathrm{aut}} = \prod_{w \text{ marked}} |\mathrm{GL}_{rt}(k_{w})|, \qquad u_{\ell} = v_{\ell}(A_{\mathrm{aut}}).$$

Here w runs through the primes of E dividing π_M: for each $i = 0, 1$ there is one choice above each of the $f = [F_0 : \mathbf{Q}]$ places over p_i. Thus this product has $2f$ factors. In the ramification estimates both w and $\bar{w}$ occur, giving at most $4f$ local terms. At the common reference g_0 and its inverse conjugate require the same determinant bounds. These are local hypotheses on the original summands.

Take a stable lattice, for example the span of the orbit of a coefficient lattice, which is finitely generated because the image is compact and the orbit is bounded. The integral spectral idempotents of the tame generator split it into rank-one summands, by Lagrange interpolation with unit denominators. Frobenius permutes those lines with unit multipliers in integral bases. The idempotents and the powers of Frobenius give all the matrix units over the valuation ring.

For g_1 the n-th power of this arithmetic Frobenius, $\rho(g_1) = \alpha I$ and $\chi_{\mathrm{cyc}}(g_1) = p_\nu^n$. The trace relation and the full matrix span show that

$$\rho(g) = \alpha_g I \quad \Longrightarrow \quad \rho((g^c)^{-1}) = \chi_{\mathrm{cyc}}(g)^{d_{\mathrm{pol}}} \alpha_g I.$$

Indeed for every x, apply $\operatorname{tr} \rho((u^c)^{-1}) = \chi_{\mathrm{cyc}}(u)^{d_{\mathrm{pol}}} \operatorname{tr} \rho(u)$ to $u = g x^{-1}$, then use the formula at x^{-1}. Pairing the difference of the claimed matrices with all $\rho(x^c)$ gives trace zero. The trace pairing on the full matrix algebra is nondegenerate. At s, the cyclotomic value is one by inertiality away from ℓ, giving the two evaluations in Section 27.4.

For $V = T/\varpi^a T$, an equivariant map $V \to (V^*)^c$ is determined by its value on the first coordinate vector, whose orbit spans V. The scalars of g_1 on source and target are α and $\alpha p_\nu^{n(d_{\mathrm{pol}}+1)}$, respectively. Hence the evaluation lies in the torsion killed by $1 - p_\nu^{n(d_{\mathrm{pol}}+1)}$. For $0 \neq u \in \mathcal{O}$, the Cartier dual satisfies

$$V^*[u] = \operatorname{Hom}(V/uV, \mu_{\ell^N}), \qquad \operatorname{length}_{\mathcal{O}} V^*[u] = \operatorname{length}_{\mathcal{O}}(V/uV) = n \min\{a, \operatorname{ord}_{\mathcal{O}}(u)\}.$$

Indeed V/uV has exponent dividing ℓ^N, so the dual has the same cardinality. Division by $\operatorname{ord}_{\mathcal{O}}(\ell)$ gives a bound of $n v_\ell(u)$. For $1 \le n(d_{\mathrm{pol}} + 1) \le rt$, $p_\nu^{n(d_{\mathrm{pol}}+1)} - 1$ divides $|\mathrm{GL}_{rt}(\mathbb{F}_{p_\nu})|$. This bounds the normalized global term by $n u_\ell$.

At an original place away from ℓ, write b_v for the given exponent of the inertia image. On cocycles modulo ϖ^a, the affine action on V shows that every inertia element raised to b_v acts by translation. Thus its order divides $b_v \ell^j$ for some common j, prime to the residue characteristic. The affine action kills wild inertia. Evaluation at a tame generator has image of length at most $\operatorname{length} V$. On its kernel the cocycle vanishes on inertia, and its Frobenius value is inertia-invariant and determines the cocycle. It therefore lies in the kernel of $\rho(g) - I$ on V for the chosen inertia element. By Smith normal form the excess normalized length is at most $v_\ell \det(\rho(g) - I) \le n u_\ell$. Summing over $(S_M \cup \bar{S}_M) \setminus \{v : v \mid \ell\}$ therefore costs at most $4 f n u_\ell$ in excess of the baseline local lengths.

At $v \mid \ell$, impose the full annihilator of the integral cocycles whose rational classes lie in H^1_f. For the de Rham representation W over the coefficient field K, put $d_v = \dim_K(D_{\mathrm{dR}}(W)/\mathrm{Fil}^0)$, summed over all embeddings of E_v into K. Bloch–Kato's formula [15, Corollary 3.8.4] gives

$$\dim_K H^1_f(E_v, W) = d_v + \dim_K H^0(E_v, W), \qquad \dim_K B^1(E_v, W) = n - \dim_K H^0(E_v, W).$$

Thus the preimage of H^1_f in the rational cocycles has dimension $n + d_v$. To see the lattice assertion, finitely many coordinate evaluations embed the rational cocycle space in a finite product of coefficient fields. The integral submodule is then finite free.

Each rational cocycle can be cleared into it by compactness of its image. The quotient of the finite-support evaluation module by the annihilator is a submodule of the dual of this lattice, free of rank at most the same rank. Allowed values modulo ϖ^a define homomorphisms from that quotient to $\mathcal{O}/\varpi^a$, so the local length is at most a times the rank. Thus the sum over the places above ℓ satisfies the required crystalline bound if the total number of negative de Rham jumps is at most fn, as follows from a bound of n at each complex conjugate pair of embeddings.

Adding the global term, the original-place excess and the two first-prime determinant contributions give a normalized bound of at most

$$(4f+3)nu_\ell \le (4f+3)ru_\ell.$$

This bound uses the original lattice and all its original local data. For a finite family of such summands a common torsion cutoff is $M = (4f+3)ru_\ell + 2$. After putting the covectors into one finite coefficient field, an unramified enlargement increases the residue cardinality without increasing the ramification index. It can therefore meet the residue-field requirement in the length descent for all the covectors at once.

The extended local equations still annihilate lattice-valued crystalline cocycles of an allowed twist, at the places above ℓ. There the twist is trivial. To verify the claim after a finite coefficient extension $\mathcal{O} \subseteq \mathcal{O}'$, apply an $\mathcal{O}$-linear functional $\mathcal{O}' \to \mathcal{O}$ to the values of a cocycle. Applying its rational extension to the crystalline period-ring equation shows that the result satisfies the original local condition. Evaluating a finite-support relation commutes with the functional, and these functionals separate the elements of the finite free module $\mathcal{O}'$. This proves the required inclusion of crystalline cocycles in the module tested by the extended equations. Framing at the scalar inertia element is preserved because the twist is unramified at its place.

28.5 The order of the simultaneous choices

In this subsection, we specify which choices in Sections 27.3 and 24 are made before the torsion order grows. This allows the finite Selmer family and the norm systems of Section 27 to use one specialization.

Proposition 28.2 (One auxiliary twist for the Selmer and norm tests)**.** *Fix $m \ge 1$, $0 < \eta < 1$, the original Galois lattices of Section 28.4, and finitely many integral norm systems as in Section 27. Use one inertial reference g_0 for the determinant bounds of the lattices. For each norm system require a nonzero augmentation and, if its exponent list is nonempty, $\vartheta(g_0) \ne 1$. For the signed systems these conditions are verified in Sections 38.1 and 38.3. One can choose a common bound ℓ^H for the torsion orders before the auxiliary level ℓ^N and the data $J, (\chi_w), (P_i)$ at that level. There is then one tuple $(\zeta_w)_{w\in J}$ with*

$$\zeta_w^{\ell^H} = 1, \qquad \psi = \prod_{w\in J} \zeta_w^{\chi_w}, \qquad v(P_i((\zeta_w^j - 1)_w)) \le \beta_i + \eta \quad (\text{all } i,\ 1 \le j \le m).$$

Here $0 < \eta < 1$, and the bounds β_i are independent of N. Each ψ^j is ramified on $J \cup \bar{J}$. For a Selmer test of rank n_i, let O_0 be the coefficient DVR of the original lattices L_i. For every finite extension with ring of integers $O \supseteq O_0$ containing the chosen ζ_w, put $T_{i,j} = (L_i \otimes_{O_0} O) \otimes \psi^j$ and $K = \mathrm{Frac}(O)$. Let $\mathcal{S}_{i,j}$ consist of the continuous cocycles with coefficients in $T_{i,j} \otimes_O (K/O)$ which vanish at the scalar inertia element s_i of the summand and on inertia outside the places above

$\ell p_0 p_1$ and $J \cup \bar{J}$. At each $v \mid \ell$ impose all finite-support relations annihilating the integral cocycles of $T_{i,j}$ whose rational classes lie in $H^1_f(E_v, T_{i,j} \otimes_O K)$. With v_O the uniformizer valuation, $v_\ell(\ell) = 1$, and γ_w a tame generator, one has

$$\frac{\operatorname{length}_O \mathcal{S}_{i,j}}{v_O(\ell)} \le (4f+3) n_i u_\ell + \eta, \qquad 2 \sum_{w \in J} v_\ell(\psi^j(\gamma_w) - 1) \le \eta \qquad (1 \le j \le m).$$

Proof. Use the auxiliary-prime construction of Section 27.3. In choosing the cutoffs below, adjoin zero to each finite list before taking its maximum. After the finite families and the torsion cutoff M for the Selmer groups are fixed, there is a bound on $|J|$ that works for every sufficiently large N. For these J the congruence Selmer lengths at zero satisfy the bounds after the first place has been released as in Section 24. Write $\beta_i^{\mathrm{Sel}} = (4f+3) n_i u_\ell$ for the bound after releasing the distinguished pair, and choose $M > \max_i \beta_i^{\mathrm{Sel}} + 1$ as above. Independently, for each norm system with character ϑ_j choose an integer $M_j' > \beta_j^{\mathrm{norm}} + 1$, using its augmentation and first Euler-factor bound.

In the descent require the first Frobenius in $\mathcal{W}$, the intersection at g_0 of

$$\begin{aligned} \rho_i(g) &\equiv \rho_i(g_0) \pmod{\ell^M}, \\ \rho_i((g^c)^{-1}) &\equiv \rho_i((g_0^c)^{-1}) \pmod{\ell^M}, \\ \vartheta_j(g) &\equiv \vartheta_j(g_0) \pmod{\ell^{M_j'}}, \end{aligned}$$

over all original main data and signed norm systems. This is an open neighborhood by continuity. At $q_{w_0} \equiv 1 \bmod \ell^N$, $N \ge \max_j M_j'$, the last conditions preserve each first norm bound. The M_j' are independent of the Selmer depth M. All these conditions refer to the original main summands. The later trace systems at the raised level retain their marked type and contain the twisted main systems. Write β_i for the combined list of the Selmer and norm bounds.

Take $s_0 \ge 1$ bounding $|J|$, the degrees of the shifted Frobenius products in the Selmer tests, and $2|\mathcal{E}|(|J|-1)$ for each norm system. For a prescribed number m of powers and $0 < \eta < 1$, choose a positive rational a_0 with $s_0 a_0 \le \eta/2$. Apply Lemma 26.1 with s_0, m, a_0, tolerance $\eta/2$, and the list $\beta_i + 1$. This fixes H before the primes or their polynomials. Choose $N \ge H$ with $N > M$, $N \ge \max_j M_j'$ and $N a_0 > \max_j(\beta_j^{\mathrm{norm}} + s_0 a_0)$. Enlarge it for the auxiliary-prime and scalar estimates. Then choose J, the characters, and all the polynomials. Each test has a point of depth at least a_0 where its valuation is at most $\beta_i + s_0 a_0$. One tuple $(\zeta_w)_w$ now suffices for all tests and all powers $1 \le j \le m$.

In particular

$$\psi = \prod_{w \in J} \zeta_w^{\chi_w}, \qquad v(P_i((\zeta_w^j - 1)_w)) \le \beta_i + \eta \quad (1 \le j \le m).$$

Over a coefficient DVR containing the one used to choose the slopes, Lemma 25.1 bounds the specialized normalized length by the valuation of the same minor. Since $\beta_i^{\mathrm{Sel}} + \eta < M$, Section 25 gives the bound with divisible coefficients.

For an arbitrary O in the statement, take the ring of integers R of its compositum with the field used for the slopes. A finite R-linear relation annihilating integral crystalline cocycles also annihilates every scalar extension of an integral crystalline cocycle over O, since its period primitive extends. In an O-basis of R, each coefficient relation over O therefore annihilates these cocycles. Conversely, each O-relation extends to an R-relation by coefficient projection, as in

Section 28.4. The twist values lie in O, so the global cocycle equations, the inertia conditions and the normalization at the scalar element also hold coefficientwise. Consequently

$$\mathcal{S}_{i,j}(R) = \mathcal{S}_{i,j}(O) \otimes_O R.$$

The ramification index multiplies both the length and the valuation of ℓ, giving the asserted normalized bound over O. The norm tests are the values of ψ^j on their systems. Each $v(\zeta_w - 1) \leq a_0/m$. If ζ_w has order ℓ^h and $j = \ell^u j'$ with $\ell \nmid j'$, then

$$\frac{j}{\varphi(\ell^h)} \leq a_0 < 1 \quad \Longrightarrow \quad u < h, \qquad v(\zeta_w^j - 1) = \frac{\ell^u}{\varphi(\ell^h)} \leq a_0.$$

Thus ψ^j is ramified at each $w \in J$. At $\bar{w}$ the values on the marked generators have the same valuation, by anticyclotomicity and invariance under prime-to-ℓ powers. Hence

$$\sum_{w \in J \cup \bar{J}} v(\psi^j(\gamma_w) - 1) \leq 2|J|a_0 \leq \eta. \qquad \square$$

29 Crystalline bounds and auxiliary prime selection

In this section, we bound the relative cocycle lengths for the lattices of Section 28 and select auxiliary primes through paired Kummer evaluations. These choices enter the trace determinant in Section 32.

29.1 Crystalline colength for the relative cocycles

In this subsection, we bound the local indices in the relative cocycle modules from the original potentially crystalline lattices and their Tate duals. Appendices C and E give the local bounds used in Section 28.4.

Let $O = \mathcal{O}$ be the common coefficient DVR. For a main or character lattice L, write $L^\dagger = \mathrm{Hom}_O(L, O)(1)$. Fix a descent pair of ranks $(r', r' + t')$, its finite type, its upper weight, and its lift number j. In the rank-one case this fixes z and the character of weights $-c_\tau$ in (H2). Vary the upper Hecke system x within this family. At each $v \mid \ell$, take $\mathcal{Q}_v$ to be the class of potentially crystalline lattices whose jump support is contained in the upper jump support at each embedding and that satisfy the local conditions below. It contains the restrictions of L and all these upper lattices U_x. Their far Hodge–Tate intervals at an embedding over τ have the common starting magnitude $S_j(\tau)$ of Section 29.1.1. The verification below shows that the class is closed under taking lattices that embed equivariantly into finite direct sums of its members. The class is fixed before the cocycles are chosen. The estimates in Section 29.1.2 are uniform in the descent pair and the upper Hecke system.

Write $Z_f(A)$ for the integral cocycles of a local lattice A that become coboundaries over $B_{\max}$. This agrees with the usual crystalline condition: $\varphi(B_{\max}) \subseteq B_{\mathrm{cris}} \subseteq B_{\max}$ [12, §13, before Theorem 13.1]. Applying $\varphi \otimes 1$ to a $B_{\max}$-primitive gives a B_{cris}-primitive for the same cocycle, and the opposite inclusion uses the ring inclusion.

Define the cup annihilator of $\mathcal{Q}_v$ in $Z_f(L^\dagger)$ using all members $Y \in \mathcal{Q}_v$, all depths ν, and all O-linear forms $Y \otimes L^\dagger \to O(1)$ equivariant modulo ℓ^ν. For each form compose with any $\mathbf{Z}_\ell$-linear functional $O \to \mathbf{Z}_\ell$. The test is that the cup with each $Z_f(Y)$, reduced at $a = \ell^\nu$, vanish in $H^2(E_v, \mu_a)$. By Kummer and Hilbert 90 this is the a-torsion subgroup $\mathrm{Br}(E_v)[a]$.

Let $\mathcal{A}_v \subseteq Z_f(L^\dagger)$ be this annihilator and put $\Delta_v(L) = \mathrm{length}_O(Z_f(L^\dagger)/\mathcal{A}_v)$. We use this O-length in (82); its normalization is $\Delta_v(L)/\operatorname{ord}_O(\ell)$. Write $J_{\mathrm{pair}} = \mathrm{Hom}_O(Y \otimes_O L^\dagger, O(1))$ and $a = \ell^\nu$. Since $Y \otimes_O L^\dagger$ is finite free, every equivariant pairing modulo a lifts as an O-linear form $q \in J_{\mathrm{pair}}$. Then $z_q(g) = (gq - q)/a \in J_{\mathrm{pair}}$ has crystalline primitive q/a. Replacing q by $q + ab$ adds $(g-1)b$ to $z_q(g)$. The map $(j, c) \mapsto (j + cb, c)$ identifies the two extensions and carries the corresponding evaluation $aj + c(q + ab)$ to $a(j + cb) + cq$.

The common integral presentation. A relative cocycle $z : G_E \to L/aL$ obeys all the O-linear relations among $d(g) = (u_x(g) - 1)_x$. Let W be the finite product of the matrix lattices of the u_x, acted on by left multiplication, and let $U \subset W$ be generated by the $d(g)$. The map $d(g) \mapsto z(g)$ defines $U \to L/aL$. It is equivariant since

$$h\, d(g) = d(hg) - d(h).$$

For $\nu = 0$ the quotient is zero. For $\nu > 0$, Smith normal form gives an integer b_1 with $a^{b_1} W \cap U \subseteq aU$. The map extends by zero on $a^{b_1} W$ to $Y = U + a^{b_1} W$. Both summands are stable, so the extended map is equivariant. The matrix columns embed Y equivariantly into copies of the upper lattices. Moreover $a^{b_1} W \subseteq Y \subseteq W$, so $Y[1/\ell] = W[1/\ell]$ as local representations. Thus Y belongs to $\mathcal{Q}_v$, including at the dyadic places. In $Y[1/\ell]$ the identity tuple $I = (1)_x$ satisfies $d(g) = gI - I$. Consequently $d \in Z_f(Y)$. The original relative cocycle at v is the image of a crystalline cocycle in this same class under an equivariant map $t : Y \to L/aL$.

Finite Cartier duality. Write $A_\nu, A_\nu^\dagger$ for the reductions of $Z_f(L), Z_f(L^\dagger)$ at v, and $C_\nu \subseteq A_\nu^\dagger$ for the reduction of the cup annihilator. The trace against a generator of the inverse different of $\mathrm{Frac}(O)/\mathbf{Q}_\ell$ gives a perfect $\mathbf{Z}_\ell$-pairing between $L, L^\dagger$ with values in $\mathbf{Z}_\ell(1)$. Its reduction gives Cartier duality between L/aL and $L^\dagger/aL^\dagger$. Write bars for the images of cocycles in cohomology.

The image of any relative cocycle as above annihilates $\bar{C}_\nu$. Lift t to an O-linear map $q : Y \to L$ and use the form $(y, \lambda) \mapsto \lambda(qy)$. This is one of the allowed finite pairings by equivariance of t. Cup functoriality gives the claimed orthogonality in the Brauer group, hence in $H^2(E_v, \mu_a)$ by the Kummer sequence and Hilbert 90. The same reasoning with the reduction of L itself applies to A_ν.

The two finite conditions $\bar{A}_\nu, \bar{A}_\nu^\dagger$ are exact annihilators. For the cardinality calculation let $V = L[1/\ell]$, $h_0 = \dim_{\mathbf{Q}_\ell} H^0(E_v, V)$, $d_f = \dim_{\mathbf{Q}_\ell} H^1_f(E_v, V)$, and let Q_L be the torsion of $H^1(E_v, L)$. The preimage $H^1_f(E_v, L)$ of the rational finite condition is saturated in $H^1(E_v, L)$ and contains Q_L. Thus $\bar{A}_\nu \simeq H^1_f(E_v, L)/aH^1_f(E_v, L)$. Multiplication by a also gives the exact sequence

$$0 \longrightarrow H^0(E_v, L)/aH^0(E_v, L) \longrightarrow H^0(E_v, L/aL) \longrightarrow Q_L[a] \longrightarrow 0.$$

Since $|Q_L/aQ_L| = |Q_L[a]|$, we obtain

$$|\bar{A}_\nu| = \ell^{\nu d_f} |Q_L[a]|, \qquad |H^0(E_v, L/aL)| = \ell^{\nu h_0} |Q_L[a]|.$$

Use the same identities on the dual. The Bloch–Kato dimension formula and complementary de Rham filtrations give

$$d_f + d_f^\dagger - h_0 - h_0^\dagger = [E_v : \mathbf{Q}_\ell] \operatorname{rank}_{\mathbf{Z}_\ell} L.$$

Rational orthogonality gives integral orthogonality by the injection

$$H^2(E_v, \mathbf{Z}_\ell(1)) \longrightarrow H^2(E_v, \mathbf{Q}_\ell(1)).$$

The cardinality identities and the local Euler characteristic give

$$\bar{A}_\nu \subseteq (\bar{A}_\nu^{\dagger})^{\perp}, \qquad |\bar{A}_\nu|\,|\bar{A}_\nu^{\dagger}| = |H^1(E_v, L/aL)|.$$

The inclusion is an equality by finite local Tate duality [15, Proposition 3.8 and Corollary 3.8.4] [47, Ch. I, Corollary 2.3 and Theorem 2.8].

If S_ν denotes the local image of the relative cocycles, coboundaries modulo a lift to integral coboundaries. The kernel of passage to cohomology on S_ν is therefore contained in A_ν, and

$$\operatorname{length}_O(S_\nu/(S_\nu \cap A_\nu)) \le \operatorname{length}_O(\bar{C}_\nu^{\perp}/\bar{A}_\nu) = \operatorname{length}_O(\bar{A}_\nu^{\dagger}/\bar{C}_\nu) \le \Delta_v(L).$$

For a finite O-module D and $f_O = [O/\mathfrak{m}_O : \mathbf{F}_\ell]$, one has $\log_\ell |D| = f_O \operatorname{length}_O D$. Thus the equality of orders from finite duality gives the equality of O-lengths. The last quotient is an image of the defining defect quotient.

Write $S_{\nu,v}$ for this local image. For a finitely supported family $F_g \in \mathrm{End}_O(L)$, put $R_F(z) = \sum_{g \in G_{E_v}} F_g z(g)$, using reduction modulo a for cocycles valued in L/aL. The integral crystalline relations define

$$B_{\nu,v} = S_{\nu,v} \cap \bigcap_{F:\, R_F(Z_f(L))=0} \ker(R_F \bmod a).$$

Reduction gives $S_{\nu,v} \cap A_\nu \subseteq B_{\nu,v}$, so $\operatorname{length}_O(S_{\nu,v}/B_{\nu,v}) \le \Delta_v(L)$. On the global module S_ν^{glob} of relative cocycles, restriction induces an injection

$$S_\nu^{\mathrm{glob}} \Big/ \bigcap_{v|\ell} \mathrm{res}_v^{-1}(B_{\nu,v}) \;\hookrightarrow\; \bigoplus_{v|\ell} S_{\nu,v}/B_{\nu,v}.$$

Lemma 29.1 (Crystalline colength over the coefficient prime)**.** *If $c_{\mathrm{crys}}(L,\nu)$ denotes the global colength for the crystalline relations at the places above ℓ, then*

$$c_{\mathrm{crys}}(L,\nu) \le \sum_{v|\ell} \Delta_v(L). \tag{82}$$

Summing the marked defects. On the marked descent pair the upper lattice for its own parameter is $T \oplus \bigoplus_{0 \le e < t'} O(-e)$, with $r' = \operatorname{rank}_O T \ge 1$ and $t' \le t$. At a coefficient embedding the jumps of this direct sum are the multiset sum of the original main jumps and $0, \dots, t'-1$. In the Tate dual the jumps u are replaced by $-u-1$ and the inertia operator is the transpose of the inverse period operator. Take the coefficient fraction field to be a finite Galois extension of $\mathbf{Q}_\ell$ containing the matrix entries, the local type eigenvalues, and all conjugates of the marked embedded completions. Such a field exists by taking the normal closure of their finite compositum. For a coefficient embedding extended by $g \in G_{\mathbf{Q}_\ell}$, read the jumps at the embedding $g^{-1}e_v$ of E, where e_v is the chosen local embedding, transported to the complex side by the fixed comparison. This choice depends only on the coefficient embedding, because $g^{-1}e_v(E)$ lies in the coefficient field. The upper class uses the full original upper list at this embedding.

Off M the local class requires crystallinity over E_v. For $v \mid \ell \mid M$ it requires potential crystallinity and, for every $s \in I_v$, the two equations on the linear period action

$$B_Y(s)^{e_r} = 1, \qquad q_{v,s}(B_Y(s)) = 0.$$

Here $e_r = \operatorname{lcm}_{1\le a\le r}(\ell^a-1)$, and $q_{v,s}\in \mathbf{Z}_\ell[X]$ is the common monic polynomial of degree at most $(r+t)r$ for the upper type constructed in Section 28.3.2. An equivariant inclusion into a sum of these lattices at each embedding gives a jump multiset contained, with multiplicity, in that of the ambient direct sum. Its jump support therefore lies in the prescribed union, and the inertia operators satisfy the same polynomials. Indeed one may read the crystalline periods after a common finite crystallizing extension, using the exact crystalline period functor there. In particular the direct summands of the upper lattice for its own parameter belong to that same class.

29.1.1 Hodge–Tate intervals at the coefficient places

In this subsection, we determine the intervals for the marked lattices used in Appendix E.

Write $h=r^8$, $G=xb/x_1$, $Q_*=kt$, $W_*=4t$, and $\eta=h^2+1$. For the fixed lift $1\le j\le m$ put

$$w_j(\tau)=\kappa_\tau+j\Delta+h,\qquad \delta_j=j\Delta+h-2t,\qquad S_j(\tau)=\kappa_\tau+\delta_j.$$

Let ε_σ be 1 if σ restricts to the chosen embedding of K, and -1 otherwise, and let τ be its infinite place. Write J_σ for the multiset of main jumps. For the rank-r marked pair, $t'=t$ and

$$J_\sigma=\left\{\frac{t-1}{2}-\varepsilon_\sigma\bigl(\lambda_q(\tau)+w_j(\tau)\bigr)\,:\,0\le q<r\right\}. \tag{H1}$$

Indeed, if $n=r+t$, the upper highest-weight entry μ_q gives the doubled parameter $P_q=2\mu_q+n-1-2q$.

The raw jump is $(n-1-\varepsilon_\sigma P_q)/2$. The normalizing character subtracts $r-r_-$ on the chosen CM type and r_- on its conjugate, where r_- counts the negative lower signs. Thus the normalized formula replaces $n-1,P_q$ by $t-1,P_q-2r_-+r$. By the parameter correspondence of the marked descent pair, the latter shifted entries consist of $2(\lambda_q+w_j)$ and $t-1-2a$ for $0\le a<t$. In the second block, the ordered parameter entry indexed by a contributes a on the chosen CM type and $t-1-a$ on its conjugate. Reindex the conjugate list by $a\mapsto t-1-a$. The fixed global summand $O(-a)$ has jump a at every embedding, and its Tate dual $O(a+1)$ has jump $-a-1$. Equivalently one gets (H1) directly by lowering the original rank-r jumps by $(r-t)/2+\varepsilon_\sigma w_j$.

For a rank-one marked pair the parameter z and its character of weight $-c_\tau$ satisfy

$$m<z\le r,\qquad w_j+r-t\le c_\tau\le w_j,\qquad t'=2z-1,\qquad J_\sigma=\{z-1-\varepsilon_\sigma c_\tau\}. \tag{H2}$$

The last equality is the jump of the character obtained by normalizing the trivial rank-one system. In both cases $\varepsilon_{\bar\sigma}=-\varepsilon_\sigma$, with τ unchanged. Corresponding main jumps sum to $t'-1$, so their total negative count over the two embeddings is at most r'.

For $q<k-1$, Section 12.10 gives

$$|\varepsilon_\sigma u+w_j|=\left|\varepsilon_\sigma\frac{t-1}{2}-\lambda_q\right|\le\frac{t+k-3}{2}<t.$$

In (H2) the same bound by t follows from $0\le w_j-c_\tau\le t-r$ and $z-1<r$. The remaining m entries of (H1) have $\lambda_{k-2+i}+w_j=(2(j-i)+1)h$, $1\le i\le m$. Thus

$$\frac{h}{2}\le h-\frac{t-1}{2}\le|u|\le(2m-1)h+\frac{t-1}{2}<2mh\le h^2.$$

The jumps with $|\varepsilon_\sigma u + w_j| < t$ satisfy $|u| \geq w_j - t > 2h^2$.

At a coefficient embedding γ, use σ as above and put $\mathcal{U}_\gamma = \mathrm{supp}(J_\sigma) \cup \{0, \ldots, t'-1\}$. For the Tate dual of an individual summand L, let $\mathcal{U}'_{\gamma,L}$ be its actual jump set. These sets lie in opposite intervals:

$$\mathcal{U}_\gamma \subset [-\eta, \eta] \ \cup \ -\varepsilon_\sigma[S_j(\tau), S_j(\tau) + W_*],$$
$$\{-u-1 : u \in \mathcal{U}_\gamma\} \subset [-\eta, \eta] \ \cup \ \varepsilon_\sigma[S_j(\tau), S_j(\tau) + W_*].$$

The 1 in η absorbs the dual shift for low jumps, and the far-interval half-width $2t$ absorbs it for high jumps. For the main summand, every negative member of $\mathcal{U}_\gamma \cup \mathcal{U}'_{\gamma,L}$ is at most $-h/2$. Its Tate dual has at most $\lfloor r'm/k \rfloor$ jumps in $[-\eta, \eta]$, counted with multiplicities: at full rank there are at most m tail indices, and at rank one there are none. For an individual Tate character of jump a, the dual set is $\{-a-1\} \subset [-\eta, 0]$.

Residue-field parameters and sparsity. Write E_v for the embedded local field at $v \mid \ell$, with residue field k_v, degree d_v, ramification index e_v^{ram} and residue degree f_v. Then $e_v^{\mathrm{ram}} f_v = d_v = [G_{\mathbf{Q}_\ell} : G_{E_v}]$ and $d_v \leq [E : \mathbf{Q}] \leq h$, since $[E : \mathbf{Q}] \leq 2q_1 = O(x^{11} \log x)$ and $h = r^8 \geq x^{40}$. Choose a primitive $(\ell^{f_v} - 1)$-st root ζ_v in the maximal unramified subfield of E_v. A coefficient embedding γ has parameter $\gamma(\zeta_v)$. Call a parameter thick if it occurs at an embedding whose read infinite place lies in T, where $\kappa_\tau = b$.

If $\ell \nmid q_1 d$, the completion is unramified and ζ_v generates it. Two embeddings with the same parameter agree on the completion, hence read the same σ. On the indices occurring as $\gamma(\zeta_v)$ this determines both thickness and sign. The full index set is $k_v^\times$, identified with its Teichmüller lifts. The parameters of coefficient embeddings form the orbit $\{\zeta_v^{\ell^a} : 0 \leq a < f_v\}$, since ζ_v generates the unramified extension of degree f_v. Witt Frobenius takes i to i^ℓ, so the level-a test at i reads the index $\varphi^{-a} i$. An index occurring at no embedding is thin and has jump sets $\{0\}$ on the two original sides after adjoining zero. Give it the thin centre $G + \delta_j$, with either sign in the unramified case and the common sign specified below in the ramified case. If $\ell \nmid q_1 d$, then $\ell > 7$, since $2 \cdot 3 \cdot 5 \cdot 7 \mid q_1$. Every prime dividing $q_1 d$ divides $q_0 d$, so is at most Q_* for large x. If $\ell \mid q_1$, the polynomial $X^2 - X + (d+1)/4$ reduces to $X(X-1)$ modulo ℓ, since $(d+1)/4 \equiv 2310 \bmod 4620$.

Thus K splits at ℓ and all the read embeddings have the same sign at fixed v. If $\ell = d$, the local cyclotomic field is unramified and $K_d/\mathbf{Q}_d$ is ramified quadratic. Each embedding of the maximal unramified subfield extends to two embeddings of E_v with opposite CM signs. Two thick embeddings with opposite signs would give a power of d equal to $-1 \bmod q_0$: the action on the cyclotomic coordinates is a power of Frobenius d, and both thick coordinates reduce to $-1 \bmod q_0$ after their places are represented on the chosen CM type. This is impossible since the residue $9239 \bmod 18480$ has order 2 and is neither 1 nor -1. Thus in the ramified cases one can give all parameters the common sign of the thick embeddings, with any choice if there are none. At $\ell = d$, even at a thick residue parameter the thin embedding contributes only to $[-2G, 2G]$, since $G + \delta_j + W_* \leq 2G$.

Suppose $\ell \leq Q_*$ and three thick parameters are $\omega, \omega^{\ell^a}, \omega^{\ell^{a+b}}$, where $1 \leq a, b \leq \lfloor \log_\ell h^3 \rfloor$. Represent them by coefficient embeddings extended to $g_0, g_1, g_2 \in G_{\mathbf{Q}_\ell}$. Put $q' = q_1/\ell^{v_\ell(q_1)}$. The root $e_v(\zeta_{q_1})^{\ell^{v_\ell(q_1)}}$ has order q' and is a power of ζ_v. On these roots, $g_2 g_0^{-1}$ acts by the power ℓ^{a+b} and $g_2 g_1^{-1}$ by ℓ^b. Let τ_i be the place represented by the read embedding $\sigma_i = \iota^{-1} g_i^{-1} e_v$, and write $\varepsilon_i = \varepsilon_{\sigma_i}$. Representing each place on the chosen CM type raises its cyclotomic root to ε_i. Hence, in $(\mathbb{Z}/q')^\times$,

$$\tau_0 \tau_2^{-1} = \varepsilon_0 \varepsilon_2 \ell^{a+b}, \qquad \tau_1 \tau_2^{-1} = \varepsilon_1 \varepsilon_2 \ell^b.$$

Since

$$0 < b < a + b \le 2\lfloor \log_\ell h^3 \rfloor \le 24s < L_0,$$

the sparsity proved in Section 4 excludes this triple. Here $2\lfloor \log_\ell h^3 \rfloor \le 48 \log_2 r \le 24s$ for large x, since $r < 3x^5$ and $s \ge 11 \log_2 x$, while $L_0 > 27s$. At $\ell = 2$ even two thick indices a positive distance $i < L_0$ apart are impossible: their signed ratio would give $4^i \equiv 1 \pmod{3L_0}$, where 4 has order L_0.

For a parameter take the union of the jump sets over its coefficient embeddings and adjoin 0. When $\ell > Q_*$ the resulting intervals are the displayed ones with the sign and center read on the parameter. When $\ell \le Q_*$ the low radius can be enlarged to $2G$, since for a thin embedding $\eta \le G$ and $\delta_j + W_* \le G$ put both of its intervals in $[-2G, 2G]$. The other intervals still have the sign and center read on the thick parameters. For dual sets the signs of the separated intervals are reversed.

The Kummer coordinate and the scale inequalities. Choose a uniformizer π_v of E_v and compatible roots $\underline{\pi_v}$. In Fontaine's ring $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbf{C}_\ell^\flat})$ use $Z = [\underline{\pi_v}]$, and embed $W(k_v)[[Z]]$ by evaluation, which is injective modulo ℓ by the valuation of the first nonzero term. Write θ for Fontaine's map [13, Lemma 3.10 and Remark 3.11]. Lift the unit $-\ell/\pi_v^{e_v^{\mathrm{ram}}}$ to a unit A_v of $W(k_v)[[Z]]$ by the uniformizer expansion. Then $Z^{e_v^{\mathrm{ram}}} A_v + \ell$ generates $\ker\theta$: it lies in that ideal, and its reduction modulo ℓ has the same valuation as the reduction of the generator $[\underline{\ell}] - \ell$.

Their quotient is therefore a unit. If $e_v^{\mathrm{ram}} = 1$, choose $\pi_v = -\ell$ and $A_v = 1$. The group fixing $W(k_v)$ pointwise and fixing π_v is G_{E_v}, since $E_v = W(k_v)[1/\ell](\pi_v)$. If $\underline{\epsilon}$ is a compatible system of primitive ℓ^n-th roots of unity, for every integer $a \ge 0$ some element of G_{E_v} sends Z to $[\underline{\epsilon}]^a Z$. Indeed $X^{\ell^n} - \pi_v$ is Eisenstein over E_v, so the closed sets imposing this action at level n are nonempty and nested. Compactness produces the result. The residue action of the arithmetic Frobenius is $y \mapsto y^{\ell^{f_v}}$.

The size conditions for these intervals are

$$\begin{gathered} Q_* > 0, \quad 99\big(Q_*^2 + (W_* + 1)Q_* + Q_* + W_* + m\big) \le h, \\ 99h^3 \le G, \qquad 99(Gh^3 + G) \le b, \qquad 2\log(4b) < m, \\ 0 \le \delta_j < \eta, \qquad 99((\eta + W_*)Q_* + \eta + h) \le h^3. \end{gathered}$$

Indeed $Q_* \le 5r^2$, $W_* = 20r$, $m < r$, $h = r^8$, and $\delta_j \le (2m+1)h$. For the comparisons with G, b use $b \ge x_1^{10^6}$, $x_1 \ge x^{10^6}$, $G/b = x/x_1$, and $(\log b)^{100} \le x$. At a prescribed level prime $\ell \mid M$, the congruence $\ell \equiv 1 \bmod q_1 d$ makes $E_v = \mathbf{Q}_\ell$, so all coefficient embeddings read the same σ. The tame interval threshold holds because $\ell \ge b^{10}$:

$$S_j(\tau) + 6(W_* + \eta) + r^2((r+t)r + 1)^2 + 2 < \ell.$$

For a local cup exponent q_v, meaning that $\ell^{q_v} Z_f(L^\dagger)$ lies in the cup annihilator, the normalized defect is at most

$$\frac{\Delta_v(L)}{v_{\mathrm{ord}}(\ell)} \le q_v \operatorname{rank}_O Z_f(L^\dagger) \le 2q_v [E_v : \mathbf{Q}_\ell] \operatorname{rank}_O L.$$

Indeed, put $K_O = \mathrm{Frac}(O)$ and $V = L^\dagger \otimes_O K_O$. The rational cocycles $Z_f(V)$ are the inverse image of $H^1_f(E_v, V)$ under passage to cohomology. Their subspace of coboundaries has dimension $\dim_{K_O} V - \dim_{K_O} H^0(E_v, V)$. The Bloch–Kato dimension formula [15, Corollary 3.8.4] gives

$$\dim_{K_O} Z_f(V) = \dim_{K_O} V + \dim_{K_O}\big(D_{\mathrm{dR}}(V)/\operatorname{Fil}^0\big) \le (1 + [E_v : \mathbf{Q}_\ell]) \operatorname{rank}_O L.$$

Finitely many evaluations on G_{E_v} separate the cocycles in this finite-dimensional space. They embed $Z_f(L^\dagger)$ into a finite direct sum of copies of $L^\dagger$, so it is finite free over O. Each rational continuous cocycle has compact image and a power of ℓ makes it integral. Hence $Z_f(L^\dagger)$ has rank $\dim_{K_O} Z_f(V)$.

The estimates needed at v are normalized defects at most $[E_v : \mathbf{Q}_\ell]r'F_\ell$ for T and $[E_v : \mathbf{Q}_\ell]F_\ell/t$ for each character, where

$$F_\ell = 10^6 \frac{m}{k}\left(bv_\ell(M) + v_\ell\left((2b)! \prod_{\substack{b-30h_2^3<j\le b+30h_2^3 \\ j>0}} j^{h_2^3} \right)\right), \qquad h_2 = r^8.$$

29.1.2 Uniform bounds for the local defects

In this subsection, we apply the bounds of Appendices E and D to the intervals from Section 29.1.1.

Put

$$\lambda_\ell = v_\ell\left((2b)! \prod_{\substack{b-30h^3<n\le b+30h^3 \\ n>0}} n^{h^3} \right), \qquad K_\ell = \left\lfloor \frac{2\lambda_\ell}{Q_*} \right\rfloor.$$

Then $\ell(\lambda_\ell + 1) > 2b$, since each multiple of ℓ up to $2b$ contributes at least one to the factorial valuation. If $\lambda_\ell < h^3$, no integer in $(b - 30h^3, b + 30h^3]$ is divisible by ℓ. In this case set $N_\ell = 6(2(\lambda_\ell + 1) + \eta + W_*)$. The scale inequalities give $\delta_j + N_\ell < 30h^3$ and

$$\ell > \frac{2b}{h^3 + 1} > G + \delta_j + N_\ell.$$

For either center $S = \kappa_\tau + \delta_j$, no nonzero multiple of ℓ is within N_ℓ of 0 or of either signed center $\pm S$. For $S = G + \delta_j$ all such integers have absolute value smaller than ℓ. For $S = b + \delta_j$ take the absolute value in the separated intervals to reduce to the preceding exclusion near b.

For $\ell \le Q_*$, the factorial bound also implies

$$Q_*(K_\ell + 3) > 2\lambda_\ell + 2 > 4b/Q_*.$$

Since b/Q_*^2 grows faster than any power of r, this gives $K_\ell \ge 2b/Q_*^2$ for large x. In particular $2(b + \delta_j) < h^2 K_\ell$. For the normalization of the local defect, write ρ for the rank of the summand being tested and $c_* = 1$ on the main summand, $c_* = t$ on each Tate character. With $\mu = \lfloor \rho m/(kc_*) \rfloor$, $Q_* = kt$, and $c_* \le t$, we have

$$0 \le K_\ell \le \frac{2\lambda_\ell}{kt} \le \frac{2\lambda_\ell}{kc_*}.$$

Consequently the crystalline local quantity, after the local degree is removed, is bounded by

$$10^5(\rho K_\ell m + \mu\lambda_\ell) \le 3 \cdot 10^5 \frac{\rho m \lambda_\ell}{kc_*} \le \rho F_\ell / c_*.$$

The off-level estimate in Section E.3 gives the first bound, and the two comparisons use $K_\ell \le 2\lambda_\ell/(kc_*)$ and $\mu \le \rho m/(kc_*)$. At a tame level prime the cup calculation in Appendix C gives

$$q_{\mathrm{tame}} = 3r^2((r + t)r + 1)^2 + 150(h^2 + 4t + 2).$$

For $i = 0, 1$, Proposition 4.1 gives $p_i \geq b^{10} > \max(2b, b + 30h^3)$ at the common cutoff. Thus

$$v_{p_i}(M) = 1, \qquad \lambda_{p_i} = 0, \qquad F_{p_i} = 10^6 \frac{m}{k} b, \qquad k = r + 1 - m.$$

For $v \mid p_i$ we have $E_v = \mathbf{Q}_{p_i}$, so $\operatorname{rank}_O Z_f(L^\dagger) \leq 2\rho$. The inclusion $p_i^{q_{\mathrm{tame}}} Z_f(L^\dagger) \subseteq \mathcal{A}_v$ therefore gives

$$\frac{\Delta_v(L)}{\operatorname{ord}_O(p_i)} \leq 2\rho q_{\mathrm{tame}} \leq \frac{\rho F_{p_i}}{c_*}.$$

The last inequality follows from $c_* \leq t$ and polynomial growth of ktq_{tame} in r, since b dominates every fixed power of r.

For the fixed local class $\mathcal{Q}_v$, use the same cup test to define Δ_v on the sum of the character lattices. Then

$$\Delta_v\left(\bigoplus_{a=0}^{t'-1} O(-a)\right) = \sum_{a=0}^{t'-1} \Delta_v(O(-a)). \tag{83}$$

Indeed, restriction of a test pairing to each dual summand is again a test at the same depth, and the cup of a cocycle tuple is the sum of its component cups. Conversely every test on one summand extends by zero on the others. Thus the cup annihilator is the direct sum of the individual annihilators over the original DVR.

Put $e_O = \operatorname{ord}_O(\ell)$. Summing (82) over the summands of the original descent pair and using $\sum_{v|\ell}[E_v : \mathbf{Q}_\ell] = 2f$ gives, at every depth ν,

$$\frac{c_{\mathrm{crys}}(T, \nu) + \sum_{0 \leq j < t'} c_{\mathrm{crys}}(O(-j), \nu)}{r' e_O} \leq 2f\Big(1 + \frac{t'}{r't}\Big) F_\ell \leq 4f F_\ell.$$

This is the crystalline contribution per unit rank in (RB1).

For every finite set I of coefficient primes, put $B = \log b$ and $P_h = \prod_{b - 30h_2^3 < j \leq b + 30h_2^3,\ j > 0} j$. Factorization of the integers in F_ℓ gives

$$\sum_{\ell \in I} 4f F_\ell \log \ell \leq 4 \cdot 10^6 f \frac{m}{k} \big(b \log M + \log((2b)!) + h_2^3 \log P_h\big).$$

The interval is open on the left and closed on the right, so it contains at most $60h_2^3$ positive integers. As $h_2 = r^8$ and $r < 3x^5$, for fixed d, ϵ and sufficiently large x,

$$h_2^3 \log P_h \leq 60 h_2^6 \log(b + 30h_2^3) \leq 120 h_2^6 B \leq bB.$$

Both primes dividing M exceed $2b$, which bounds every prime factor of $(2b)! P_h$. At the other primes the branches $\ell \leq Q_*$, $\ell > Q_*$, $\lambda_\ell \geq h_2^3$ and $\ell > Q_*$, $\lambda_\ell < h_2^3$ are disjoint. Each λ_ℓ enters the sum once. Also $M \leq b^{22}$ and $\log((2b)!) \leq 4bB$. Dividing the right side by $fxbB^3/(mk)$ gives at most $108 \cdot 10^6 m^2/(xB^2)$, which tends to zero because $m^2 \leq 4x$ and $B \to \infty$.

29.2 Cyclotomic-kernel evaluations for the auxiliary primes

In this subsection, we refine the finite detection argument of Section 24.1 to choose elements in the cyclotomic kernel. The normalization of Section 28.1 lets these elements test the representations in the prime selection of Section 29.4.

The finite set of evaluation elements in Section 24.1 constructed by adjoining $\mu_{N_c \ell^2}$ works for every ℓ. The bound y^{20} on its size was used only for small ℓ. To put its points in the

cyclotomic kernel, let P be that finite set and suppose g_0 has cyclotomic value generating the image of G_E topologically. For $p \in P$ choose q_p in the closure of $\langle g_0 \rangle$ with $\chi_{\mathrm{cyc}}(q_p) = \chi_{\mathrm{cyc}}(p)$. Such a choice exists by continuity and compactness. Set $s_p = pq_p^{-1}$, so $s_p \in \ker \chi_{\mathrm{cyc}}$. The zero set of a continuous cocycle is a closed subgroup. If it contains g_0 and every s_p, it contains all $p = s_p q_p$, so the cocycle is zero. This procedure detects all unramified-outside-ℓ cocycles in the stated cyclotomic characters, without a parity condition. The chosen points are independent of $e, \nu, \mathcal{O}$.

Put $Y_{\mathrm{tr}} = r^8 = \Delta/2$, so $N_c + t + 3 \le Y_{\mathrm{tr}}$ for large x. The image of χ_{cyc} on G_E is procyclic, since it is a closed subgroup of $\mathbb{Z}_\ell^\times$ for odd ℓ, and of $1 + 4\mathbb{Z}_2$ at $\ell = 2$ since $i \in E$. Choose $\gamma_\ell \in G_E$ whose cyclotomic value ξ_ℓ generates that image topologically. If w is any value in the image, $\xi_\ell^a - 1$ divides $w^a - 1$, by taking limits of powers of ξ_ℓ in the closed ideal generated by $\xi_\ell^a - 1$. For $\ell \le Y_{\mathrm{tr}}^2$ take $w = p_i^{\mathrm{lev}}$ from arithmetic Frobenius at a distinguished place not above ℓ. For $0 < a < t$, the integer $w^a - 1$ divides A_{aut} since it is a factor of the order of the marked general linear group of rank rt. For $\ell > Y_{\mathrm{tr}}^2$, inertia at ℓ fixes $E \subseteq \mathbb{Q}(\zeta_{N_c})$ and its cyclotomic reduction fills $\mathbb{F}_\ell^\times$. Choose w reducing to a generator. Then $w^a - 1$ is a unit for $0 < a < t$. Thus, with

$$h_\ell = \begin{cases} v_\ell(A_{\mathrm{aut}}), & \ell \le Y_{\mathrm{tr}}^2, \\ 0, & \ell > Y_{\mathrm{tr}}^2, \end{cases}$$

every $\xi_\ell^a - 1$, $0 < a < t$, divides ℓ^{h_ℓ}.

For the triangle use the reference $g_{\mathrm{tr}} = \gamma_\ell^{-1}$ and characters $\chi_e = \chi_{\mathrm{cyc}}^{-e}$. Then $\chi_e(g_{\mathrm{tr}}) = \xi_\ell^e$. Define

$$\Delta_e = \prod_{\substack{0 \le j < t \\ j \ne e}} (\xi_\ell^e - \xi_\ell^j), \qquad C = \ell^{Y_{\mathrm{tr}} h_\ell}.$$

For $0 < e < t$ the two scalars $4\Delta_e$ and $4(\xi_\ell^e - 1)^2$ divide C in $\mathbb{Z}_\ell$. Indeed each difference has valuation at most h_ℓ. At $\ell = 2$, $4 \mid \xi_\ell - 1$ implies $v_2(4) \le h_\ell$; at odd primes 4 is a unit. The inequalities $t - 1 + 1 \le Y_{\mathrm{tr}}$ and $2 + 1 \le Y_{\mathrm{tr}}$ prove the claims.

Let R be the rank-n source attachment of Section 28.1 before its character normalization. The twisted main summand is $T = \vartheta_\ell^{-1}(\theta R)$. Sections 28.3.1 and 28.3.2 supply an off-ℓ element γ such that $T(\gamma) = \xi I$, $\chi_{\mathrm{cyc}}(\gamma) = 1$, and $\xi^{a\ell^\nu} - 1$ is a unit for $1 \le a \le n$. Here the auxiliary character is unramified at the original inertia used for γ. Fix any $g \in G_E$ and depth ν. The eigenvalues of $T(g)$ are integral, since the image is compact and preserves a lattice over its coefficient field. For λ in this multiset, at most one $i \in \{0, \dots, n\}$ satisfies $\overline{\xi^{i\ell^\nu}\lambda} = 1$, where bars denote residue classes. Indeed the $n + 1$ powers of $\overline{\xi^{\ell^\nu}}$ are distinct. Choose i outside all the forbidden indices, and set $\delta = \gamma^i$. Then

$$\chi_{\mathrm{cyc}}(\delta) = 1, \qquad \det(1 - T(\delta^{\ell^\nu} g)) = \prod_\lambda (1 - \xi^{i\ell^\nu}\lambda) \in \mathcal{O}^\times.$$

The unit condition persists on an open neighborhood of $\delta^{\ell^\nu} g$ by continuity. This is the determinant condition when selecting new Frobenius points for the triangle.

For an original-level system x with normalized representation N_x, call x a weak lift if $\det(1 - N_x(g)) = 0$ for every $g \in G_E$. At an off-level split spherical place with residue norm q, let $N_H = r_0 + t$ be the rank of a weak lift and $t > 0$ the chain length. Write H for its monic Hecke polynomial. Suppose the normalized weak-lift representation has eigenvalue 1 at geometric Frobenius ϕ, and the unnormalized representation at ϕ is killed by H in the coefficient

embedding. If $R^{\mathrm{nor}} = \chi R$, the unit eigenvalue of $R^{\mathrm{nor}}(\phi)$ gives the root

$$X_0 = \chi(\phi)^{-1}, \qquad |X_0| = q^{r_0/2}$$

in the complex realization, by the normalization of Section 28.1.

At this split place the local representation is identified with the spherical component of the original unitary automorphic representation. It is therefore unitarizable before the arithmetic shift. Tadić's spherical unitary classification [70, Theorem 3.2] expresses the Satake roots in strings. In arithmetic normalization a string of length h has terms

$$aq^{(N_H-1)/2+\theta-(h-1)/2+j}, \qquad 0 \le j < h, \quad |a| = 1, \quad -\tfrac{1}{2} < \theta < \tfrac{1}{2}.$$

This follows by taking the Satake parameters of the unitary characters of determinant and of the complementary pairs in the classification, and applying the common arithmetic shift $(N_H - 1)/2$. In the string containing X_0, subtract that common shift from the exponents. The exponent of X_0 is $(1 - t)/2$. If β is the bottom of its string, then $\beta \le (1 - t)/2$ and the top is $2\theta - \beta$. The number of steps from X_0 to the top therefore exceeds $t - 2$. It is an integer and hence at least $t - 1$. Thus $X_0 q^j$ is a root of H for each $0 \le j < t$; these roots are distinct.

Put $B_t(X) = \prod_{0 \le j < t}(X - q^j)$. In the coefficient embedding the exact chain gives

$$H(\chi^{-1}X) = B_t(X)U(X), \qquad \chi \in \mathcal{O}^{\times}, \qquad \deg H = N_H,$$

where χ is the normalizing character at the chosen ϕ. Let $C(X)$ be its normalized Galois characteristic polynomial. Assume the coefficientwise comparison $C \equiv \chi^{N_H} H(\chi^{-1}X)$ in the residue field, with the rescaled polynomial integral, and a residual factorization $C \equiv SB_t$ with $S(1)$ a unit. Division by the monic integral B_t gives

$$\chi^{N_H} U \in \mathcal{O}[X], \qquad \chi^{N_H} U \equiv S.$$

Thus $U(1)$ is a unit and $U/U(1)$ is an integral polynomial of value one at 1. For finitely many residual-near weak lifts at the same prime, multiply these polynomials to obtain a single such polynomial divisible by their normalized quotients.

For an original-level system x with normalized representation N_x, put $C_x(g; X) = \det(X - N_x(g)) \in \mathcal{O}[X]$. The finite set used in the product consists precisely of the systems satisfying, for every $g \in G_E$,

$$C_x(g; 1) = 0, \qquad C_x(g; X) \equiv C_{x_0}(g; X) \pmod{\mathfrak{m}_{\mathcal{O}}},$$

where x_0 is the distinguished lift and the congruence is coefficientwise. Section 28.2 gives spherical annihilation and coefficientwise compatibility for these systems outside a common finite set of places. Before choosing the auxiliary primes, take a basis of each original isotypic space and exclude the finitely many places where any basis vector fails to have the specified spherical eigenvalues. At every remaining split place, each spherical operator thus acts by the specified scalar on the entire original isotypic space, as required in Section 32.3. Apply the trace identity of Section 28.1 to the positive powers of g. Newton's identities over $\mathrm{Frac}(\mathcal{O})$ give

$$C_{x_0}(g; X) = \det(X - T(g)) \prod_{0 \le j < t} (X - \chi_{\mathrm{cyc}}(g)^{-j}).$$

Both sides are integral. Reducing this identity at any coefficient prime gives the residual factorization for every system in the set. On the chosen neighborhood $\det(1 - T(g))$ is a unit, so the local argument applies to all of them.

Eligible Frobenius elements of prime norm are dense in this neighborhood, also after any further finite exclusion. In an open subset take a coset from a finite Galois quotient of G_E, and pass to a normal closure of that extension over $\mathbb{Q}$. A representative in the normal closure fixes E, which is Galois over $\mathbb{Q}$. Apply Chebotarev over $\mathbb{Q}$ to the inverse conjugacy class for arithmetic Frobenius [46, Theorem 8.31]. Choosing the prolongation realizes the prescribed geometric Frobenius in the original coset. The resulting rational primes split completely in E, so yield prime norm and allow distinct primes of F_0 as required. This proves abundance at $\delta^{\ell^\nu} g$ for Section 29.4, on the stated scalar inertia and distinguished lift data. The finite family and its off-level comparisons are fixed before choosing the depth ν.

29.3 The fresh Kummer maps

In this subsection, we construct the Kummer maps for adding a prime to the places allowed to ramify. Their local restrictions are the ones prescribed in the inductive selection of Section 29.4.

Fix $\nu > 0$ and $e > 0$. By a ν-split place we mean a degree-one prime $w \nmid \ell$ of E, with a prolongation and a geometric Frobenius at that prolongation fixing μ_{ℓ^ν}. Take k_0 earlier such places of type e whose restrictions to $F_0 = E^+$ are distinct. Let S_0 be the set of their primes in F_0 together with the primes over ℓ. Put $F_\nu = E(\mu_{\ell^\nu})$, $U = G_{F_\nu}$, and $\kappa = \chi_{\mathrm{cyc}}\chi_{E/F_0}$ on G_{F_0}, where the second factor has values in $\{\pm 1\}$. Then $\kappa|_{G_E} = \chi_{\mathrm{cyc}}$ and $\kappa(c) = 1$ at every real conjugation. It is unramified off ℓ since E/F_0 is finite unramified. Let

$$M_\nu = (\mathbb{Z}/\ell^\nu\mathbb{Z})(\kappa^{-e}), \qquad A = M_\nu[\ell], \qquad M_\nu^\vee = \mathrm{Hom}(M_\nu, \mu_{\ell^\nu}).$$

The earlier primes of F_0 split in F_ν, so both coefficient modules are trivial there.

Use primal local conditions of zero at real places and at the earlier primes, vanishing on $D_v \cap \ker\kappa$ at $v \mid \ell$, and unramified elsewhere. These give conditions on cohomology classes as coboundaries obey the required vanishings. Let P be a finite set of auxiliary primes split in F_ν, outside S_0, where the primal condition is relaxed to the full local group. Write $Z_\nu(P)$ for the resulting group of actual cocycles. The same conditions for A give $Z_A(P) = Z_\nu(P)[\ell]$.

To choose P, take all residual dual classes with coefficients in $A^\vee = \mathrm{Hom}(A, \mu_\ell)$ unramified outside S_0 and infinity. These classes form a finite group. Their restrictions to U are homomorphisms. For each nonzero restriction, take an element of U detecting it. In a common finite quotient that records these classes and F_ν/F_0, Chebotarev gives distinct primes of P with Frobenius equal to those elements after choosing their prolongations.

At each finite $v \notin S_0 \cup P$, the coefficients are unramified of order prime to the residue characteristic. The unramified subgroups for the module and its Cartier dual are exact annihilators [47, Ch. I, Theorem 2.6], both for A and for M_ν. At P the full primal condition has zero annihilator. Thus each residual dual Selmer class is unramified outside S_0 and infinity and vanishes on the chosen Frobenius elements at P. By the choice of P, its restriction to U is zero. Inflation–restriction puts it in $H^1(\mathrm{Gal}(F_\nu/F_0), A^\vee)$. The Galois group is generated by at most two elements. For odd ℓ it is a subgroup of the product of the order-two group of E/F_0 and the cyclic units modulo ℓ^ν. For $\ell = 2$, use $E = F_0(i)$: at $\nu = 1$ the group has order two and for $\nu \geq 2$ it is a subgroup of the units modulo 2^ν. Cocycles are determined by their values on generators, so the residual dual group has dimension at most two.

Apply Greenberg–Wiles to A with the relaxed conditions [25, Theorem 2.19], which applies to finite modules of arbitrary order. At P the local factor is ℓ by the unramified and tame directions in local H^1. Each earlier prime contributes ℓ^{-1}. At a real place v, the local condition is zero and $\kappa(c) = 1$, so $|\mathcal{L}_v|/|H^0(F_{0,v}, A)| = \ell^{-1}$, also for $\ell = 2$. At $v \mid \ell$, an allowed cocycle factors through the procyclic $\kappa(D_v)$ and has at most ℓ possible values on a lift of a generator. After division by $\ell/|H^0(F_{0,v}, A)|$ coboundaries the local factor is at most one. Elsewhere the unramified factor is one. At $\ell = 2$, the character of $E = F_0(i)$ is the sign modulo 4 of χ_{cyc}, so κ takes values in $1 + 4\mathbb{Z}_2$ and its local images are procyclic. The global coboundary factor $\ell/|H^0(F_0, A)|$ cancels the numerator of the global invariant ratio. With the residual dual bound this gives

$$|Z_\nu(P)[\ell]| \le \ell^{|P|-k_0-f+3} \le \ell^{|P|-k_0-1},$$

since $f \ge 4$.

At full depth let $B \subset M_\nu^P$ be the image of tame evaluation at P on $Z_\nu(P)$. Let $B^\vee \subset (M_\nu^\vee)^P$ be the Frobenius evaluations of the global dual classes with the orthogonal conditions for the unrelaxed primal group. The local pairing at P pairs an unramified dual value on Frobenius with the primal tame value [47, Ch. I, Corollary 2.3 and Remark 2.5(b)]. Use a common geometric Frobenius convention and a tame generator acting on an ℓ^ν-th root of a uniformizer by the chosen primitive root of unity. The ordering of the cup product fixes one common sign. Global cup reciprocity gives $B^\vee \subset B^\perp$. The kernels of tame and Frobenius evaluation are the unrelaxed primal and the strict dual groups, respectively. The ratio of the two Greenberg–Wiles identities for M_ν with and without relaxation at P yields

$$|B|\,|B^\vee| = \ell^{\nu|P|}.$$

Thus $B^\vee = B^\perp$.

Every character of $A^P/(B \cap A^P)$ extends to M_ν^P/B by divisibility of $\mathbb{Q}/\mathbb{Z}$, and therefore arises from one of the dual classes. The coefficient map $M_\nu^\vee \to A^\vee$ restricts a character to $A = \ell^{\nu-1}M_\nu$. In root-of-unity coordinates, $b(\ell^{\nu-1}x) = b(x)^{\ell^{\nu-1}}$, so restriction reduces the exponent modulo ℓ. Coboundaries vanish on U, and the Frobenius elements of P lie in U. Hence the reduced restriction of the dual class to U determines its values on A^P. Moreover $|B[\ell]| \le |Z_\nu(P)[\ell]|$, by surjectivity on the quotients by multiplication by ℓ and the equality of the kernel and cokernel sizes of multiplication on a finite abelian group. Hence there are at least ℓ^{k_0+1} different reductions of the restricted dual classes. Some restriction is outside the span of the k_0 earlier maps modulo ℓ.

The restricted cocycle on U is a homomorphism ψ to $\mathbb{Z}/\ell^\nu\mathbb{Z}$ in the chosen root-of-unity coordinates. Conjugation acts through $\chi_{\mathrm{cyc}}\kappa^e$. This gives oddness under c and exponent $e + 1$ under conjugation by γ_ℓ. The global dual class is unramified outside S_0. The exploratory set P can be discarded.

For the reciprocity law, let z be an even cocycle over E allowed to ramify at the later split pairs and above ℓ, vanishing on $D_v \cap \ker \chi_{\mathrm{cyc}}$ for every $v \mid \ell$, with earlier tame and type-e Frobenius tests zero. Extend it to G_{F_0} by $\widetilde{z}(gc) = z(g)$ for $g \in G_E$. This is a cocycle by evenness and $\kappa(c) = 1$. Use $2\widetilde{z}$ as the primal cocycle. At every real conjugation $2\widetilde{z}$ vanishes by squaring.

For $g \in D_v \cap \ker \kappa$ at $v \mid \ell$, $2\widetilde{z}(g) = \widetilde{z}(g^2) = 0$, since $g^2 \in D_v \cap G_E$ has cyclotomic value one. At an earlier prime of type e, unramifiedness and the zero Frobenius value give zero on its decomposition group. Thus the terms at S_0 and infinity vanish by the orthogonal local conditions. At every earlier prime of another type both classes are unramified and their local

pairing is zero, as it is at the other finite primes outside S_0 and the new pairs. At each new prime of F_0 split in E, choose τ_i° in inertia for which $\tau_i^\circ(\varpi_i^{1/\ell^\nu})/\varpi_i^{1/\ell^\nu} = \zeta_{\ell^\nu}$, using a uniformizer of F_{0,v_i} and the same primitive root at every place. In the chosen root-of-unity coordinates, represent the local class of z by a Kummer element $b_i \in F_{0,v_i}^\times$. An ℓ^ν-th root of its unit part lies in the maximal unramified extension, so $z(\tau_i^\circ) = v_i(b_i) \bmod \ell^\nu$. Local reciprocity pairs this class with the unramified dual value as $\pm\psi(\phi_i)v_i(b_i)$, with one sign for the choice of geometric Frobenius and cup-product order. Thus the term for $2\widetilde{z}$ is $\pm 2\psi(\phi_i)z(\tau_i^\circ)$. This proves the reciprocity relation. For O/ℓ^ν-valued cocycles apply the relation after all linear functionals $O/\ell^\nu \to \mathbb{Z}/\ell^\nu\mathbb{Z}$, which separate values since O/ℓ^ν is free over $\mathbb{Z}/\ell^\nu\mathbb{Z}$. This proves the fresh-map assertion at every positive depth.

29.4 The ordered prime selection from the Kummer relations

In this subsection, we combine the cyclotomic-kernel evaluations of Section 29.2 with the Kummer maps of Section 29.3. The order of the choices ensures that each newly selected prime satisfies the conditions imposed by the earlier ones.

Keep g_{tr}, $C = \ell^{Y_{\mathrm{tr}}h_\ell}$, and a finite set of evaluation elements in the cyclotomic kernel P_0 as in Section 29.2. For $\ell \le Y_{\mathrm{tr}}^2$, the finite detection bound of Section 24.1 gives

$$m_0 = N_c\ell^2 \le Y_{\mathrm{tr}}^5, \qquad |P_0| \le m_0^4 \le Y_{\mathrm{tr}}^{20}.$$

For the actual rank increment $t' \le t$, order the pairs (e, s), $0 < e < t'$, $s \in P_0$, lexicographically with the exponent first. The uniform bound uses t. This list is fixed before the depth is chosen. For each $\nu > 0$, put $A_\nu = \mathbb{Z}/\ell^\nu\mathbb{Z}$ and

$$G_\nu = G_{E(\mu_{\ell^\nu})} = \{g \in G_E : \chi_{\mathrm{cyc}}(g) \equiv 1 \pmod{\ell^\nu}\}.$$

At each step Section 29.3 gives a Kummer map ψ_j on G_ν with values in A_ν for the earlier split pairs of type e, and a finite excluded set of primes of F_0. The map is a continuous homomorphism, is odd under complex conjugation, and conjugation by γ_ℓ multiplies its values by ξ_ℓ^{e+1}. Its reduction modulo ℓ is outside the span of the previous maps of type e.

The required reciprocity law is as follows. Let an even cocycle z of type e vanish on the full cyclotomic kernel in the decomposition groups above ℓ. Suppose it is unramified at the previous pairs and outside ℓ and a finite family $\{w_i, \bar{w}_i\}$ of further ν-split pairs whose restrictions to F_0 are pairwise distinct and lie outside the excluded set. If it vanishes on the earlier Frobenius elements of type e, then

$$\sum_i 2\psi_j(\phi_i)z(\tau_i^\circ) = 0.$$

Here ϕ_i is geometric Frobenius and τ_i° is normalized as in Section 29.3.

The primes over ℓ and the previously specified local tests belong to the excluded set of the new map. The set of usable Frobenius points must be abundant near a translate of every prescribed point. At each depth ν and $g \in G_E$ there must be $\delta \in \ker \chi_{\mathrm{cyc}}$ such that every open neighborhood of $\delta^{\ell^\nu} g$ contains usable Frobenius elements of new places of prime norm outside any given finite excluded set. At those places use the determinant-unit and marked Hecke conditions of Section 29.2.

Under this Frobenius condition, consider the index $j = (e, s)$. The earlier maps of type e and the new ψ_j are independent modulo ℓ. Their joint evaluation on G_ν has image all of

the appropriate power of A_ν: modulo ℓ this follows by the absence of an annihilating linear combination, and Nakayama's lemma gives the assertion over A_ν. Choose $a \in G_\nu$ on which the new map reads one and the earlier maps of type e read zero. For each $k \in \{0, \dots, t-1\} \setminus \{e\}$, replace a successively by

$$(\gamma_\ell a \gamma_\ell^{-1})^{\xi_\ell^{-1}} a^{-\xi_\ell^k}.$$

The exponents are integer representatives modulo ℓ^ν. Every map of type d is multiplied by $\xi_\ell^d - \xi_\ell^k$. Thus the resulting $a_* \in G_\nu$ has every earlier value zero and $\psi_j(a_*) = \Delta_e \bmod \ell^\nu$.

Apply the auxiliary-prime theorem at $g = scsc^{-1}a_*(ca_*c^{-1})^{-1}$. All factors and δ^{ℓ^ν} lie in G_ν. An odd Kummer map reads $2\psi(a_*)$ at $\delta^{\ell^\nu} g$, since the first two factors cancel and ℓ^ν kills its value on δ. An even cocycle of type e, restricted to G_ν, is additive and reads $2z(s)$ there. The finitely many Kummer evaluations persist on an open neighborhood.

For the even evaluations take the finitely many A_ν-valued cocycles unramified outside ℓ using the finite evaluation set. Their values persist on one open neighborhood. Applying the additive A_ν-linear duals of O/ℓ^ν, which separate its elements, gives the same equalities for the O/ℓ^ν-valued cocycles, on that neighborhood for every coefficient ring. Intersect also with G_ν. This intersection contains $h = \delta^{\ell^\nu} g$. Choose a finite Galois extension L/E such that the coset of h modulo G_L lies in the intersection. The restriction of h to the normal closure of $L/\mathbb{Q}$ thus prescribes all the evaluations together. Apply Section 29.2 to choose a geometric Frobenius in that coset, avoiding the primes of F_0 under all earlier pairs and the finite exclusions up to this step. Its prime norm implies it is split over F_0, and $\ell^\nu \mid \mathrm{N}w_j - 1$. It has

$$\psi_i(\phi_j) = 0 \ (i < j), \qquad \psi_j(\phi_j) = 2\Delta_e, \qquad z(\phi_j) = 2z(s)$$

for the stated unramified even cocycles.

For an even cocycle ramified only on the new pairs and above ℓ, first impose vanishing at all earlier tame generators. Evenness gives vanishing at their conjugates, so the cocycle is unramified at those earlier pairs. Impose the earlier Frobenius tests of type e. The reciprocity law for ψ_j now applies to the current and later pairs. The later coefficients vanish by construction. The remaining term is $2\psi_j(\phi_j)z(\tau_j^\circ) = 4\Delta_e z(\tau_j^\circ)$. Thus $Cz(\tau_j^\circ) = 0$, since $4\Delta_e \mid C$. At $w_j \nmid \ell$ the coefficient character is unramified. The restriction of z to inertia factors through its cyclic tame quotient of order ℓ^ν, generated by the image of τ_j°. For the original tame generator τ_j we therefore have $z(\tau_j) = a_j z(\tau_j^\circ)$ with $a_j \in A_\nu$, and hence $Cz(\tau_j) = 0$. This gives the ordered tame test and the reads of the original evaluation map at the new Frobenius elements. The total loss exponent, counting the indices and evaluations points, and the reference point, is bounded by

$$((t-1)|P_0| + |P_0| + 1)Y_{\mathrm{tr}} h_\ell \le Y_{\mathrm{tr}}^{22} h_\ell.$$

When $\ell > Y_{\mathrm{tr}}^2$ this holds because $h_\ell = 0$. There is no bound on the norms of the new primes in this argument.

29.5 The paired tame and Frobenius tests

In this subsection, the successive primes of Section 29.4 give two local evaluations for each relative cocycle. We combine the tame and Frobenius evaluations with the original local conditions to bound the length of their common kernel.

At depth ν, let Z be a submodule of the relative character cocycles for $(O/\ell^\nu)(-e)$ of Section 29.1, with the same upper lattices. For the original scalar inertia element γ_π, the

reference element g_{tr} of Section 29.2, and $\chi_e = \chi_{\mathrm{cyc}}^{-e}$, assume

$$z \in Z, \quad z(\gamma_\pi) = 0 \quad \Longrightarrow \quad (\chi_e(g) - 1)z(g_{\mathrm{tr}}) = 0 \qquad (g \in G_E).$$

The calculation with the normalizer below gives this implication for the congruence modules used in the determinant. Suppose the old marked tame tests and their conjugate inverses vanish and cover all original and auxiliary ramification away from ℓ except the new pairs $w_j, \bar{w}_j$. Outside these places the matrices $u_x(g) - 1$ vanish for inertia g, so evaluation at g is a relative linear relation. At an old place away from ℓ, the character cocycle on inertia is a homomorphism to an ℓ-primary finite group. It factors through pro-ℓ tame inertia. The marked topological tame generator and its conjugate inverse therefore give vanishing throughout those groups. Thus Z is unramified outside ℓ and the new pairs.

Let τ_j generate pro-ℓ tame inertia at w_j, and let the chosen ϕ_j fix μ_{ℓ^ν}. Set $\tau_j' = (c\tau_j c^{-1})^{-1}$ and $\phi_j' = (c\phi_j c^{-1})^{-1}$. At the conjugate prolongation $c\phi_j c^{-1}$ is geometric Frobenius and ϕ_j' is arithmetic Frobenius. Consequently, for $q_j = \mathrm{N}w_j$,

$$\chi_{\mathrm{cyc}}^{-1}(\phi_j) = q_j, \qquad \chi_{\mathrm{cyc}}^{-1}(\phi_j') = q_j^{-1}, \qquad q_j \equiv q_j^{-1} \equiv 1 \pmod{\ell^\nu}.$$

Let $Z_0 \subseteq Z$ require vanishing on the local sets above ℓ used for $Z_{e,\nu}^-$ in Section 24.1. This concerns inertia for $e = 0$, and the full cyclotomic kernel in each decomposition group for $e > 0$. The crystalline linear relations imply this vanishing. For positive e, the cyclotomic character has infinite image on G_{E_v}, and $t_{\mathrm{cyc}}^e \otimes v_{-e}$ generates the de Rham module in Fil^0, where v_{-e} is a basis of the Tate character. Both the invariant space and the de Rham quotient are zero, so Bloch–Kato's formula gives $H_f^1(E_v, \mathrm{Frac}(O)(-e)) = 0$. An integral crystalline cocycle is then a rational coboundary and vanishes on the cyclotomic kernel.

At $e = 0$, the formula gives dimension one over $\mathrm{Frac}(O)$ [15, Corollary 3.8.4]. For $f_v = [k_v : \mathbb{F}_\ell]$, solve $\varphi^{f_v} u - u = 1$ in $W(\overline{\mathbb{F}}_\ell)$ successively modulo powers of ℓ, using surjectivity of $x \mapsto x^{\ell^{f_v}} - x$ on $\overline{\mathbb{F}}_\ell$. Under its inclusion in B_{cris}, arithmetic Frobenius at E_v acts on this Witt ring by φ^{f_v}. Thus the unramified cocycle sending Frobenius to 1 has primitive u, and the finite condition is the unramified line. Each asserted evaluation is therefore an integral crystalline relation. It remains a relation modulo ℓ^ν, including for cocycles that only obey the relations. Conjugation transfers vanishing from the designated decomposition group to its conjugates since the character at an element of the local set is one. Consequently

$$\mathrm{length}_O(Z/Z_0) \le c_{\mathrm{crys}}(O(-e), \nu).$$

For $z \in Z_0$, paired agreement at either (τ_j, τ_j') or (ϕ_j, ϕ_j') is equivalent to vanishing of $z + z \circ c$ at the first member. In fact the character acts trivially on both pairs and hence inversion negates the value. The sum $z + z \circ c$ is an even cocycle with the same ramification and local vanishing. Suppose the new places have the ordered positive types and satisfy the conditions of Section 29.4, with the separation scalar $C = \ell^{Y_{\mathrm{tr}} h_\ell}$. Its even-cocycle test at an index of type e says that vanishing on earlier tame generators and earlier Frobenius elements of type e forces the value at the current tame generator to be killed by C. For the paired tests this gives agreement up to C at τ_j, τ_j', using all earlier tame pairs and only earlier Frobenius pairs of type e.

Now let $S \subseteq Z_0$ require the two tame values at every new pair to vanish individually and the Frobenius values to agree for type e. Killing the tame generators removes all new ramification as above. For the reference test use the normalizing element for the covector congruences. Write

$d(g)$ for simultaneous augmentation coordinates in a finite generating list g_i for the tuples $(u_x(g)-1)_x$. Thus

$$u_x(g)-1=\sum_i d(g)_i(u_x(g_i)-1) \qquad \text{for every } x.$$

For a group-algebra covector v, its value is $\sum_i v_i(g_i-1)$. The congruence ideal contains the common action kernel of the u_x and the left multiples of the normalizer N_0 by the augmentation ideal.

Here the normalizer is

$$E_\pi=c_\pi(\gamma_\pi-1), \qquad N_0=E_\pi+(1-E_\pi)(g_{\mathrm{tr}}-1),$$

where $c_\pi=(y_\pi-1)^{-1}$ for the scalar y_π of γ_π. Its inverse is integral since the reduction of the scalar differs from one. The covector $v_0=c_\pi d(\gamma_\pi)+(1-E_\pi)d(g_{\mathrm{tr}})$ has value N_0 modulo the common action kernel. Hence $(g-1)v_0$ satisfies the congruences for every g. Put $\chi_e=\chi_{\mathrm{cyc}}^{-e}$. Evaluation at χ_e kills E_π, since γ_π is in inertia off ℓ. The character cocycle given by a functional annihilating the vectors satisfying the congruences therefore satisfies

$$(\chi_e(g)-1)\big(c_\pi z(\gamma_\pi)+z(g_{\mathrm{tr}})\big)=0.$$

This relation holds on the character cocycles from the vectors satisfying the congruences and their further quotients.

For $z\in S$ the value on γ_π is zero by unramifiedness, giving $(\chi_e(g_{\mathrm{tr}})-1)z(g_{\mathrm{tr}})=0$. For $e>0$, use the triangle's detecting points besides g_{tr}, its prescription that twice the even value at each point is an appropriate Frobenius value of type e, and its divisibility $4(\chi_{\mathrm{cyc}}(g_{\mathrm{tr}})^{-e}-1)^2 \mid C$. The paired-cocycle argument in Section 24.1 gives $\mathrm{length}_O(S/S[C])\le \mathrm{length}_O Z^-_{e,\nu}$. For $e=0$ there is no index of type zero, and S is already everywhere unramified, hence a submodule of $Z^-_{0,\nu}$. Thus in both cases

$$\mathrm{length}_O(Z/Z_0)+\mathrm{length}_O(S/S[C])\le c_{\mathrm{crys}}(O(-e),\nu)+v_{\mathrm{ord}}(X_{\mathrm{odd}}). \tag{84}$$

Together with the tests for unramified cocycles, evaluations at both tame generators of each new pair detect all of Z: their vanishing removes the new ramification, and the remaining tests give zero. The reference test uses the original congruence ideal. The prime triangle uses the scalar inertia, distinguished lift and residual comparison of the same representation.

30 Measured norm tests and the integral trace algebra

In this section, we normalize the two theta norm tests for the critical values and construct the integral trace algebra on the raised levels. The norm tests are used in Section 38, and the algebra is used in Section 31.

30.1 Norm tests from a definite line

In this subsection, for $z\ge 2$, the theta lift from the definite line gives a measured norm involving the critical value at z. We compute the finite factors and the archimedean normalization here, and establish integrality for the same test in Section 44.4.

Let $p = (\nu, z)$ be a critical pair on the prescribed old level with $z \geq 2$, allowing trivial residue characters at old places. The infinity weights of ν are $-c_v$ with $c_v \geq z$. Use the standard definite line and the standard definite group of rank $a_0 = 2z$ over the same E/F_0. In this section S_M denotes the chosen halves of the old split level places. Let Q be a set of the auxiliary split places allowed in the marked tests, disjoint from the old places and their conjugates. Write

$$q_Q = \prod_{w \in Q} q_w, \qquad S_Q = (S_M \cup Q) \cup (\bar{S}_M \cup \bar{Q}), \qquad C_G = \prod_{w \in S_M} |\mathrm{GL}_{rt}(k_w)|.$$

For a norm-one character χ write $\chi^{\mathrm{bc}}(a) = \chi(a/\bar{a})$. We use the finite L-function of this base change and omit the places in S_Q.

Use the marked vacuum multiplied by $s_{\mathrm{vac}}^{a_0}$. At each old place restrict the column to the reduction of the specified basis vector. At a place of Q prescribe the distinguished coordinate to be one. Use the same arrow scale $\pi_M = \bar{\alpha}$ and $s_{\mathrm{jet}} = \gamma = p_F a_d/\omega_0$ from (AJ). Use the jet of weight $\bigotimes_v \mathrm{Sym}^{c_v - z}(\mathbf{C}^{a_0})$ at scale $\pi_M s_{\mathrm{jet}}$, with the fixed character ν.

Denote the resulting kernel by $K_Q(g, h)$. Here g is on the definite line and h on the rank-a_0 space, both with the original measures. For line classes g_i take the old principal-level classes times integral lifts of inverse unit residues at Q. Set $u_i = \det(1, g_i)$. The values of

$$\Phi_\tau(h) = \sum_i \tau(u_i)^{-1} K_Q(g_i, h)$$

are the class lift of the line character, since summing over the inverse unit residues gives the test on which the distinguished coordinate is nonzero and weighted by the tame character. The sum on the line has no stabilizer denominators at the old principal level. Its quotient stabilizer is trivial by reduction at an old odd split prime in rank one.

On the target the level J_Q is the old vector fixer and the parabolic preserving the complementary hyperplane at each new place. For $k \in J_Q$, let $t_w(k) \in k_w^\times$ be its action on the one-dimensional quotient by that hyperplane, and let $\psi_{w,\tau}$ be the residue character of $\nu\tau$ at $w \in Q$. Since $(k^{-1}X)_1 = t_w(k)^{-1} X_1$ in the residue field, the character-weighted column transforms by

$$\Phi_\tau(hk) = \prod_{w \in Q} \psi_{w,\tau}(t_w(k))^{-1} \Phi_\tau(h).$$

The old vector fixers preserve the prescribed column. The finite character has modulus one, so the squared Weyl norm descends to the target classes. The class pairing sums the transported Weyl pairings with weights given by the inverse finite stabilizer orders. Every such order divides C_G. Indeed the stabilizer is finite, since the real group is definite and its rational integral points are discrete. After conjugation into an old marked compact group at an odd split place with $E_w = \mathbf{Q}_{p_i}$, the kernel of reduction is torsion-free. In that kernel a nonidentity $1 + p_i^s A$, for $s \geq 1$ and A nonzero modulo p_i, has the same depth s after raising to a power prime to p_i and depth $s+1$ after raising to p_i, by the binomial expansion. Reduction therefore injects the finite stabilizer into the finite linear group. The rank is at most rt.

Proposition 30.1 (Measured norm of the line lift)**.** *For the marked line lift Φ_τ with the original class pairing at J_Q,*

$$\|\Phi_\tau\|_{J_Q}^2 = c_p \, q_Q^{a_0 - 1} L^{S_Q}(z, ((\nu\tau)^{\mathrm{bc}})^{-1}), \qquad c_p > 0. \tag{85}$$

The scalar c_p is independent of Q and τ.

Proof. Put $u = v|_{U(1)}$, $b = (\nu\tau)u^z$. Use the column test f after the sum over inverse unit residues at Q, and put $f_g = \omega_b(g,1)f$, where the source action includes $b(\det g)^{-1}$. Let $\eta = (1,\eta_f)$, $\eta_f\bar{\eta}_f = -1$, be the opposite-form coordinate of Section 40.2.1. In the two original blocks the doubled evaluation on theta translates is

$$f_g[X]\,\overline{f_{g'}[\eta^{-1}Y]},$$

using the fixed opposite-space transport of Section 40.2.1.

If the auxiliary character on the common target in the first factor is a, its opposed character in the rank-one negative factor is $a^{-1}u$, by the intrinsic determinant multiplier of that factor. Their product is u. Thus this is the target action of Section 17.2 with Siegel rank one, with both source translations already in the test. The partial Fourier transform into the split polarization uses the original dual additive measures. The doubled line has a one-dimensional Lagrangian. In [77, Theorem 2] the Siegel index is therefore $n = 1$, the Hermitian dimension is $a_0 = 2z$, and the Witt index is zero. Thus $a_0 > n$, $s_0 = (2z-1)/2$, and evaluation at the identity of the double has coefficient one for target quotient probability Haar.

Every rational isotropic line of the doubled source is a graph of an isometry δ, since both projections are injective by definiteness. Evaluation at zero in the transverse polarization evaluates the two blocks at $(M\widehat{\delta}, \eta M)$ in the original point coordinates of Section 40.2.1. Thus the second argument returns to M. Rational invariance and unitarity give

$$\langle f_{g'}, \omega_b(\delta,1)f_g\rangle = \langle f, \omega_b(g'^{-1}\delta g,1)f\rangle.$$

The source character here is $b(\det(g'^{-1}\delta g))^{-1}$, since $b(\det\delta) = 1$. Use the original target volume to undo the probability normalization. The quotient formula unfolds the rational graph sum to the ambient line group, leaving one original source quotient volume. The absolute local sum at an unmarked split place is bounded by $\sum_{j\in\mathbb{Z}} q_w^{-z|j|}$. The product converges for $z > 1$, and the remaining local factors are finite. Together with the compact real groups this justifies the absolute unfolding and factorization.

Here are the local quantities, with each standard finite compact of mass one. At a split place outside S_Q, put $q = q_w = |k_w|$. For the line element of valuation pair $(j,-j)$, the square root of the Jacobian times the overlap volume of the two integral column boxes is

$$q^{-a_0 j/2} q^{-a_0 \max(0,-j)} = q^{-a_0|j|/2}.$$

The split Weil character factor, after multiplication by the source character, is A_w^{-j} for $A_w = (\nu\tau)^{\mathrm{bc}}(\varpi_w)$: indeed the determinant factor from the splitting is $v(\varpi_w)^{2zj}$ and the source factor contains its inverse.

The sum of the overlaps is

$$\sum_{j\in\mathbb{Z}} A_w^{-j} q_w^{-z|j|} = \frac{1 - q_w^{-2z}}{(1 - A_w q_w^{-z})(1 - A_w^{-1} q_w^{-z})}.$$

At a nonsplit finite place v of F_0, let w be the place of E above v and put $q = |k_v|$. The norm-one line group is compact and fixes the test, so its integral is one. Since $E_w/F_{0,v}$ is unramified quadratic, $|k_w| = q^2$ and a uniformizer of $F_{0,v}$ is also one of E_w. It has base-change character value one. The Euler factors are consequently $L_w(z) = (1-q^{-2z})^{-1} = \zeta_{F_0,v}(2z)$. Let S_Q^+ be

the set of finite places of F_0 below S_Q, and write I_v for the local integral. Since $a_0 = 2z$ and $S_M \cup Q$ contains one place above each $v \in S_Q^+$,

$$\prod_{v \notin S_Q^+,\ v \nmid \infty} I_v = \frac{L^{S_Q}(z, ((\nu\tau)^{\mathrm{bc}})^{-1})}{\zeta_{F_0}^{S_Q^+}(2z)} = \frac{L^{S_Q}(z, ((\nu\tau)^{\mathrm{bc}})^{-1})}{\zeta_{F_0}(2z)} \prod_{w \in S_M \cup Q} (1 - q_w^{-a_0})^{-1}.$$

At a distinguished place a nonunit moves the primitive column support off itself. The old column has squared mass $q_w^{-a_0}$, and among source units its support is fixed exactly by reduction one. Every allowed old unit character factors through $k_w^\times$, so it is identically one on this stabilizer. The old matrix coefficient integral is $q_w^{-a_0}/(q_w - 1)$, also when the character is trivial on all units. At a new place the sum over the inverse unit residues replaces the point condition by the character-weighted condition $X_1 \neq 0$ on the distinguished coordinate X_1.

Every source unit preserves this support. The residue multiplier cancels the source character, so the coefficient integral is the mass $(q_w - 1)/q_w$. The same statement holds with a trivial new-place character: the residue sum still gives the same support with coefficient one. Thus in that case the new quotient level and volume factor are the same as for an exact-support place. The passage to the target class pairing multiplies by $q_w^{a_0} - 1$ at an old place and $(q_w^{a_0} - 1)/(q_w - 1)$ at a new place. These are the coset counts for the two target stabilizers. Turning the line Haar lift into the old principal-level class sum multiplies its norm square by the square of the old source index, whose old local factor is $q_w - 1$.

At infinity the line character cancels the rotation of the selected highest monomial $X_1^{c_v - z}$ and its squared norm is $(c_v - z)!$ in the Fock pairing. For a highest covector of norm one, Schur orthogonality on the compact group gives the squared scalar coefficient integral as the squared Weyl norm divided by $\dim(\bigotimes_{v|\infty} \mathrm{Sym}^{c_v - z}(\mathbb{C}^{a_0}))$. Thus passing to the Weyl norm multiplies by this dimension. At an old place the two class-sum source indices, the source coefficient integral, the target index and the removed Euler numerator give

$$(q_w - 1)^2 \frac{q_w^{-a_0}}{q_w - 1} \frac{q_w^{a_0} - 1}{1 - q_w^{-a_0}} = q_w - 1.$$

At a new place the corresponding factor is

$$\frac{q_w - 1}{q_w} \frac{q_w^{a_0} - 1}{q_w - 1} (1 - q_w^{-a_0})^{-1} = q_w^{a_0 - 1}.$$

In particular, with V_j the volume of the standard definite rank-j quotient for these Haar measures, put

$$t_v = c_v - z, \qquad d_\infty = \prod_{v|\infty} \binom{t_v + a_0 - 1}{a_0 - 1}.$$

For the vacuum and jet scales specified above the constant in (85) is

$$c_p = \frac{V_1 V_{a_0}}{\zeta_{F_0}(a_0)}\, d_\infty \prod_{v|\infty} t_v! \prod_{w \in S_M} (q_w - 1) \left| s_{\mathrm{vac}}^{a_0} (\pi_M s_{\mathrm{jet}})^{\sum_{v|\infty} t_v} \right|^2.$$

All factors are positive and independent of Q, τ. □

For the integral coefficient test put

$$d_p = M^{2fa_0}(Md_K)^{\sum_v (c_v - z)}, \qquad F_{p,Q}(\tau) = C_G d_p^2 q_Q^{1-a_0} \|\Phi_\tau\|_{J_Q}^2.$$

Section 44.4 proves the unit bound on $d_p K_Q$ in the transported coordinates at every embedding away from q_Q, including the old level primes and 2. To apply it in the pairing, let A be the tuple of original coefficient embeddings of h, G_{W} the integral Weyl Gram matrix, and S the weight representation. The conjugate value of h^{-1} is A^t, since the standard definite form is unitary. The pairing identity

$$S(A^t)^t G_{\mathrm{W}} S(A^{-1}) = G_{\mathrm{W}}$$

follows on tensor powers of the coordinate pairing and hence on the Weyl space.

For each target class at level J_Q, fix a representative h once for every character of modulus Q. Evaluating the class norm of the full lift at these representatives and expanding the Gram form, linear in its second entry, gives

$$F_{p,Q}(\tau) = \sum_{i,j} C_{ij}(Q)\tau(u_j u_i^{-1}),$$
$$C_{ij}(Q) = \frac{C_G d_p^2}{q_Q^{a_0-1}} \sum_{[h]} \frac{\mathrm{Gram}(K_Q(g_j,h), K_Q(g_i,h))}{|\operatorname{Aut}(h, J_Q)|}.$$

Indeed $\overline{\tau(u_j)^{-1}}\tau(u_i)^{-1} = \tau(u_j u_i^{-1})$. At each fixed h, read the second cleared vector at κ and the first at $\bar{\kappa}$. The integral Gram form, the stabilizer bound and $\ell \nmid q_Q$ give $v_\kappa(C_{ij}(Q)) \geq 0$. The unitary level character cancels in the squared norm of the complete lift, so the expansion may use the same class representatives for every τ. Equation (85) proves

$$\frac{F_{p,Q}(\tau)}{F_{p,\varnothing}(1)} = \frac{L^{S_Q}(z, ((\nu\tau)^{\mathrm{bc}})^{-1})}{L^{S_\varnothing}(z, (\nu^{\mathrm{bc}})^{-1})}.$$

The denominator is positive by absolute convergence and pairing the two conjugate Euler factors at each split place. Using τ and τ^{-1} gives the two marked sign tests with the same clearing constant.

30.2 The rank-two determinant norm

In this subsection, when $z = 1$, we instead evaluate a determinant period on the rank-two group. Its ratio to the untwisted period will enter the same integral comparison as the norm of Section 30.1.

Use the same marked finite bases, places S_M, Q, S_Q, and q_Q, C_G as in the preceding subsection. Let $p = (\nu, 1)$ be a ramified critical pair, with infinity weights $-c_v$, where $c_v \geq 1$. Put $k_v = c_v - 1$ and

$$\alpha = \nu\tau\, \upsilon|_{U(1)}, \qquad P(X) = \prod_{v|\infty} \det(X_v)^{k_v}.$$

Use the original oscillator on the pair of standard definite rank-two spaces. Write $H = U(I_2)$, and K_2 for the stabilizer of its standard finite lattice. Give each finite compact factor of K_2 and the real compact group mass one. Write v_2 for the volume of $[H] = H(F_0)\backslash H(\mathbb{A}_{F_0})$ for this Haar normalization.

The finite test is $\chi_{\mathfrak{m}}(\det X)$ in the original split Schrödinger coordinates at the modulus $\mathfrak{m} = (\pi_M)^2 \prod_{w \in Q} \mathfrak{p}_w$. The residue character $\chi_{\mathfrak{m}}$ is the finite unit character of α in these coordinates, extended by zero on nonunits. At every other place the finite test is the original vacuum. Denote by f_Q this finite test with the real Fock vector P, and by $\theta_{f_Q}(g,h)$ its kernel with the two characters α, α. The determinant period is

$$\Lambda_Q(\tau) = \sum_{h \in H(F_0) \backslash H(\mathbb{A}_f)/K_2} \frac{\theta_{f_Q}(1,h)}{|\operatorname{Aut}(h,K_2)|}. \tag{86}$$

The local actions and the quotient integral. At a split place let X be the 2-by-2 Schrödinger variable. Identify both groups there with $\mathrm{GL}_2(F_{0,v})$ by the same chosen member of the split pair. The action is

$$(\omega(g,h)f)(X) = B(\det h)B(\det g)^{-1} |\det g|_v |\det h|_v^{-1} f(h^{-1}Xg), \qquad B = \alpha^{\mathrm{bc}} v^{-2} = (\nu\tau)^{\mathrm{bc}}.$$

Here B and v are read on the chosen split factor. The powers of the absolute values are the square roots of the two matrix Jacobians. The original split self-dual coordinate measure gives the integral matrix box mass one. The splitting factor on each side is v^{-2}, and $v(t/\bar{t}) = v(t)^2$ on this split coordinate.

On integral units B is precisely the finite character. In particular the test is fixed by both unit compacts: multiplying the argument of $\chi_{\mathfrak{m}} \circ \det$ on the left by h^{-1} and on the right by g cancels the two unit character factors. This holds also on the noninvertible residue matrices, on which both sides vanish. Away from the marked places, the original unit compacts fix the vacuum and the characters are unramified. At each real place the compact substitution on the Fock polynomial $\det(X_v)^{k_v}$ contributes $(\det h_v)^{k_v}(\det g_v)^{-k_v}$. Since $\alpha_v(z) = z^{-k_v}$ on $U(1)$, the factor $\alpha_v(\det h_v)\alpha_v(\det g_v)^{-1}$ cancels this multiplier. The oscillator vector and the two theta variables are therefore invariant under their respective real compact groups and under K_2 at finite places.

The kernel is left rationally invariant in each adelic variable. In the finite evaluation in (86), this includes the real factor of a rational move: substitution in P multiplies it by the inverse power of the rational determinant, which cancels the real multiplier of that move. The definite quotients have compact sets of representatives. The finite translates of the fixed test over each such set are a finite family, by its open stabilizer. On the real compact groups its Schwartz seminorms are uniformly bounded. Accordingly the rational theta sum converges absolutely and uniformly on these sets of representatives, and the kernel is continuous and bounded. Haar disintegration of a class of representative h gives the mass $|\operatorname{Aut}(h,K_2)|^{-1}$, since the real compact and K_2 have mass one. Thus $\Lambda_Q(\tau) = F_Q(1)$, where

$$F_Q(g) = \int_{[H]} \theta_{f_Q}(g,h)\, dh.$$

Constancy on the original group. Fix a chosen split place and regard the integral over h as a linear functional in the local test, with all other factors and the source point fixed. On $C_c^\infty(\mathrm{GL}_2(F_{0,v}))$ this functional satisfies

$$T(f(l^{-1}X)) = \lambda(\det l)\, T(f), \qquad \lambda(t) = B(t)^{-1}|t|_v.$$

Indeed right translation by l of the target point preserves the quotient Haar, and its oscillator action has scalar $B(\det l)|\det l|_v^{-1}$. For a small compact open L on which $\lambda \circ \det$ is trivial, replace each test by its finite expansion in indicators of left cosets lL.

This shows that on every such test the functional is a constant times $\int f(X)\lambda(\det X)\,d^{\times}X$. More explicitly, for any finite set of tests one may choose one L so that all their compact supports are disjoint finite unions of its cosets. The constant for these tests is $T(\mathbf{1}_L)/\operatorname{vol}(L)$, by the displayed covariance. Both the marked test and its conjugate by any fixed matrix satisfy this condition since their supports are compact subsets of the open invertible locus. Simultaneous translation of source and target by the same matrix sends the test to its conjugate and has scalar one.

Conjugation preserves the weighted integral. Absorb the target translation into its Haar integral to obtain right invariance of F_Q by the full source split group at this place. The argument applies at every place of $S_M \cup Q$, including a place of a larger modulus at which τ is unramified.

The group $\mathrm{SU}(I_2)$ is simply connected and absolutely almost simple. At an old split place it is SL_2, which is noncompact. Strong approximation [60, Theorem 7.12] away from this place implies $F_Q(sg) = F_Q(g)$ for every $s \in \mathrm{SU}(I_2)(\mathbb{A}_{F_0})$. Indeed, approximate s by rational s_0 away from the place so that $g^{-1}s_0^{-1}sg$ is integral at all other finite places and tends to the identity at the real places. Apply rational left invariance, right invariance by the full group at the omitted place, and continuity. The determinant map is onto the adelic norm-one group, by $u \mapsto \operatorname{diag}(u,1)$. The lift therefore factors through the determinant.

At an unmarked split place both tests are the same vacuum. The minuscule source and target actions on it agree when the target acts at w and the source acts at $\bar{w}$. Namely choose integral elementary matrices γ_U with determinant valuation one and reduction kernel U, for the residue lines $U \subset k_w^2$. Use γ_U^{T} on the target coordinate at w, and γ_U^{-1} on the source coordinate at w, the latter being the positive elementary class at $\bar{w}$.

Take the corresponding right coset representatives. The target and source translates are respectively

$$qA\,\mathbf{1}_{\mathrm{int}}(\gamma_U^{-\mathsf{T}}X), \qquad qA\,\mathbf{1}_{\mathrm{int}}(X\gamma_U^{-1}), \qquad A = B(\varpi_w).$$

They require integral X, and then require U to lie respectively in the kernel of $\bar{X}^{\mathsf{T}}$ and $\bar{X}$. Indeed the image of $\varpi_w\gamma_U^{-1}$ on reduction is U. The two sums are therefore equal because these kernels have the same dimension.

The group on the integrated variable can be moved through the integral, including the inverse transpose if that coordinate is used. Thus the corresponding source Hecke operator acts on F_Q by $q+1$. On a determinant function it acts by $q+1$ times translation by the determinant of the diagonal with one uniformizer (or its inverse). This proves invariance by that translation and hence by the whole split group. At a nonsplit finite place every norm-one element is integral and $\operatorname{diag}(u,1)$ belongs to the standard compact. At infinity the real compact invariance was verified above. This proves that F_Q is constant, so

$$\|F_Q\|^2_{[H]} = v_2|\Lambda_Q(\tau)|^2.$$

30.2.1 Doubling and the local integrals

In this subsection, we apply the doubled action of Section 17.2 to the determinant norm.

For this equal-rank calculation put $u = v|_{U(1)}$ and write θ_c for the kernel at the same vector f_Q with pair characters (u,u). With $\chi = (\nu\tau)\circ\det$ one has

$$\theta_{f_Q}(a,h) = \chi(a)^{-1}\chi(h)\theta_c(a,h).$$

On negating a rank-two factor, conjugation inverts its character b, and the intrinsic Weil multiplier on the conjugate Hom action contributes u^2 on each determinant in the respective numerator or denominator. Hence the opposed character is $b^{-1}u^2$. Both centered characters u are unchanged. On doubling the integrated group, the pair characters are u for the doubled source and u^2 for the common rank-two target, in the convention that the action has the target determinant in the numerator.

For every idele y, the identity $\upsilon(\bar{y}) = \upsilon(y)^{-1}$ gives $u^{\mathrm{bc}}(y) = \upsilon(y/\bar{y}) = \upsilon(y)^2$. The corresponding splitting characters are trivial, since $u^{\mathrm{bc}}\upsilon^{-2} = 1$ and $(u^2)^{\mathrm{bc}}\upsilon^{-4} = 1$. In the doubled model the source-character factor contributes $(u^{\mathrm{bc}})^{-1} = \upsilon^{-2}$ to the untwisted Siegel Levi factor of Section 17.2, so the centered section has the trivial splitting character. The doubled theta kernel on the pullback is the product of the two centered theta kernels with the first factor conjugated. Both inputs use f_Q before conjugation. In the corresponding real Fock models the two factors are $\overline{P}$ and P. In the common finite graph coordinates the tensor is $\overline{f_{Q,f}} \otimes f_{Q,f}$, with $f_{Q,f}$ the determinant test at $\mathfrak{m} = (\pi_M)^2 \prod_{w \in Q} \mathfrak{p}_w$. On the diagonal this gives $|f_{Q,f}(X)|^2$, for any order of the determinant character. In taking the partial Fourier transform, the passage from midpoint and difference to the two arguments is

$$(Z, U) \longmapsto (Z + U/2, Z - U/2).$$

Its determinant in each scalar coordinate is -1, also at the places over two. The transformed test uses the image of its lattice and the original dual measures.

At each real place, the positive and negative definite rank-two summands of the doubled source define the maximal compact $U(2) \times U(2) \subset U(2,2)$. Use the corresponding oscillator complex structure on Hom with the definite target. This compact group preserves the line of the tensor vacuum and the space of creation operators. The vector $\overline{P} \otimes P$ is a finite sum of products of those operators applied to the vacuum. The partial Fourier change to the mixed polarization carries the vacuum and its creation operators together, by differentiating covariance of the Weyl displacements. Thus the transformed test belongs to the polynomial Fock space for this maximal compact. To compare with the compact K_∞ of [77, pp. 236–237], write the compact for the doubled definite splitting as $K'_\infty = p_\infty K_\infty p_\infty^{-1}$, with p_∞ in the real Siegel parabolic. Indeed the real group acts transitively on the positive and negative orthogonal splittings and has Iwasawa decomposition $P_\infty K_\infty$. Write Φ for the transformed tensor. Weil covariance transports its vacuum line and creation operators, so $\Phi_0 = \omega(p_\infty^{-1})\Phi$ is polynomial in the standard Fock model. With the same compacts at the finite places the Iwasawa heights satisfy

$$|a_K(gp_\infty)| = |a_{K'}(g)|\,|a_K(p_\infty)|, \qquad f^{(s)}_{\Phi,K'}(g) = |a_K(p_\infty)|^{-s} f^{(s)}_{\Phi_0,K}(gp_\infty).$$

Here $s_0 = 0$, and $\Theta(g, h; \Phi) = \Theta(gp_\infty, h; \Phi_0)$. The scalar is one at zero. The graph arguments are evaluations of the section attached to the original transformed tensor.

Write E_c for the Eisenstein series of this section and $\iota(h, h')$ for the graph embedding. The norm integrand carries $\chi(h)^{-1}\chi(h')$ outside the Eisenstein series, because $\overline{\theta_{f_Q}(a,h)}\theta_{f_Q}(a,h')$ has that factor. The Hermitian and Siegel ranks are both two and the Hermitian space is definite. Thus [77, Theorem 2] at $s = 0$, with the probability target measure of [76, §1.3 and Proposition 5.8(i)], gives

$$\|F_Q\|^2_{[H]} = \frac{v_2}{2} \int_{[H]\times[H]} \chi(h)^{-1}\chi(h') E_c(\iota(h,h'),0)\,dh\,dh'.$$

The factor v_2 changes the probability measure to the original quotient measure.

All rational maximal isotropic subspaces are graphs of rational isometries by definiteness of the two factors. The graph zero term is the centered coefficient $\langle f_Q, \omega_c(1,g) f_Q\rangle$ at $s = 0$: the self-dual partial Fourier transform at zero gives the inner product on the common polarization. For $\Re s > 1$, extend the coefficient by the parabolic height to the power s, using the compact stabilizing the original doubled self-dual lattice.

Its Iwasawa decomposition is the integral Witt completion of an isotropic direct summand. The change to the classical skew coordinates in Section 17.2 can be nonintegral, so the compact is transported with the lattice. The choice of extension leaves the value at zero unchanged: two holomorphic sections agreeing at zero differ by s times a holomorphic section, and in rank two with trivial inducing character [77, Theorem 1(2)] gives no pole at zero for the series of the latter section. The diagonal copy of H stabilizes the identity graph. Its action on this Lagrangian has determinant of absolute value one over E, since $\det h \, \overline{\det h} = 1$. The centered inducing character is trivial, so left diagonal translation preserves the flat section for every s. Thus on $\Re s > 1$ the rational graph sum unfolds with relative variable $g = h^{-1}\gamma h'$, $\gamma \in H(F_0)$.

The external factor equals $\chi(g)$ since $\chi(\gamma) = 1$. The relative variable has ambient Haar, and the remaining quotient variable has volume v_2. Writing $Z(s)$ for the unfolded integral including $\chi(g)$, continuation and the constancy already proved give

$$v_2 |\Lambda_Q(\tau)|^2 = \|F_Q\|^2_{[H]} = \frac{v_2^2}{2} Z(0).$$

For evaluation under the integrals, choose finite open compacts in the two factors fixing the pullback of the flat section and the external characters. The definite finite class sets at these levels are finite. At a fixed pair of representatives the real restriction is in a fixed finite sum of compact matrix coefficient spaces. Indeed the embedded real groups lie in the doubled maximal compact and the flat height is right invariant under that compact.

The doubled section at infinity is compact-finite: in the Fock model the original polynomial tensor belongs to finitely many homogeneous degrees, each preserved by the real maximal compact. The flat extension has the same restriction to that compact. Finitely many point evaluations recover the coefficients in this fixed space, and every such evaluation is holomorphic at zero by [77, Theorem 1(2)]. Integration against the external characters on the compact quotients is a fixed linear functional on the finite sum of these spaces. It therefore commutes with evaluation at zero.

Here are the local integrals of this section. Invert the ambient variable, preserving its unimodular Haar. The height is unchanged by inversion since the unitary matrix and its inverse have equal absolute lattice intersection index. At the split places the centered operator is $\omega_c(1, g^{-1})$ and acts by $X \mapsto gX$ with its intrinsic half-density. The external character becomes $\chi(g)^{-1}$. In particular on elementary divisors j, k it is A^{-j-k}, where $A = (\nu\tau)^{\mathrm{bc}}(\varpi_w)$. In fact $\chi(g)^{-1}\omega_c(1, g^{-1}) = \omega_{\alpha,\alpha}(1, g^{-1})$. Thus on the marked compact supports the coefficient is the squared norm of the original test, which that compact fixes. At a finite place the doubled compact preserves the doubled self-dual lattice. A graph intersects it in

$$\{(gx, x) : x \in \Lambda \cap g^{-1}\Lambda\}, \qquad \Lambda = \mathcal{O}_{E,v}^2.$$

The inverse index of $\Lambda \cap g^{-1}\Lambda$ in Λ is the absolute determinant of the graph coordinate in the Levi, hence the parabolic height.

This follows by taking an integral basis of the graph intersection and completing it to a hyperbolic basis of the doubled lattice. A primitive isotropic summand in the split or unramified

self-dual lattice admits such a completion: choose vectors paired dually with its basis and subtract multiples of the isotropic basis to cancel their Gram matrix. In the unramified case the diagonal equations use surjectivity of the integral trace, also at residue characteristic two.

At an unmarked split pair with norm q and $A = B(\varpi_w)$, the Cartan representative with elementary divisors $j \geq k$ has height $q^{-|j|-|k|}$, using both members of the pair. The overlap of the two integral matrix boxes, with the square-root Jacobian in the local action, gives the coefficient

$$A^{-j-k} q^{-|j|-|k|}.$$

For example in a coordinate with valuation j the matrix Jacobian factor is q^{-j}, and the overlap condition in the two covector entries has probability $q^{-2\max(0,-j)}$. The double coset has Haar mass 1 if $j = k$, and mass $q^{j-k}(1+q^{-1})$ otherwise, by counting free lines over $\mathcal{O}_v/\varpi_v^{j-k}$. With $x = q^{-1-s}$, $t = A^{-1}x$, $u = Ax$, summation over the Cartan double cosets gives

$$\begin{aligned} &\frac{1}{(1-t)(1-qt)} + \frac{1}{(1-u)(1-qu)} - 1 + \frac{(1+q^{-1})q^2 tu}{(1-qt)(1-qu)} \\ &\qquad = \frac{(1-x^2)(1-qx^2)}{(1-Ax)(1-A^{-1}x)(1-qAx)(1-qA^{-1}x)}. \end{aligned}$$

The first two terms come from the two closed equal-sign quadrants, with the origin subtracted once. The last term is the strict opposite-sign quadrant.

At an unramified nonsplit place let q be the norm in F_0, so the norm in E is q^2. An integral hyperbolic basis of the standard local plane exists because the unit norm is onto. It may be constructed by first taking a primitive isotropic vector and then making its integral dual vector isotropic using the trace. The special unitary group in the hyperbolic basis is $\mathrm{SL}_2(F_{0,v})$ under conjugation by a suitable diagonal matrix. Indeed determinant one forces the diagonal entries to be fixed by conjugation and the off-diagonal ones to be skew. Choose a skew integral unit δ, for example $z - \bar{z}$ with the residue of z outside the fixed residue field. Conjugating by $\mathrm{diag}(\delta, 1)$ gives the asserted isomorphism and identifies the integral compact with $\mathrm{SL}_2(\mathcal{O}_v)$, including above two.

Let K_v be the unitary integral compact, let $K_v^1 = K_v \cap \mathrm{SU}(2)$, and put $a_j = \mathrm{diag}(\varpi^j, \varpi^{-j})$, $j \geq 0$. The determinant of $K_v \cap a_j K_v a_j^{-1}$ is all norm-one units. In fact the integral isometries $\mathrm{diag}(z, \bar{z}^{-1})$ with $z \in \mathcal{O}_{E_w}^\times$ commute with a_j and have determinant $z/\bar{z}$. A Hilbert 90 preimage of a norm-one unit can be made a unit by dividing by a power of the fixed-field uniformizer. In particular every unitary double coset has special unitary Cartan representative a_j and

$$[K_v : K_v \cap a_j K_v a_j^{-1}] = [K_v^1 : K_v^1 \cap a_j K_v^1 a_j^{-1}] = q^{2j}(1+q^{-1}) \quad (j > 0).$$

The last equality counts free lines modulo ϖ^{2j}. The identity coset has mass one.

The graph height at a_j is q^{-2j} by the intersection lattice formula. In the split-plane polarization the centered matrix coefficient is the overlap of the integral box in E_w^2 with its dilation by the rank-one Levi. In the convention $X \mapsto \varpi^j X$, the old box is the smaller box. The intrinsic half-density is $|\varpi^j|_{E_w}^{2/2} = q^{-2j}$ and the centered splitting character is trivial. The external character is one on a_j. Thus the vacuum coefficient is q^{-2j} in this same model. The local sum is therefore

$$1 + (1+q^{-1}) \sum_{j \geq 1} q^{-2sj} = \frac{1 + q^{-1-2s}}{1 - q^{-2s}}.$$

At a chosen split place, overlap of the two determinant tests requires $X, gX \in \mathrm{GL}_2(\mathcal{O}_v)$ and thus $g \in \mathrm{GL}_2(\mathcal{O}_v)$. Conversely for every such g the finite determinant phase cancels by the local action checked above. The height is one, the ambient compact has mass one, and the integral of the squared test is exactly

$$\frac{|\mathrm{GL}_2(k_w)|}{q_w^4} = (1 - q_w^{-1})(1 - q_w^{-2}).$$

The same value holds at the old squared modulus, since each invertible matrix modulo w has equally many lifts modulo w^2 and each supported character value has modulus one. At infinity the parabolic height is one on the two compact groups, and the local integral is $\| \det(X_v)^{k_v} \|^2$. In the Bargmann model [9], polar coordinates give

$$\frac{1}{\pi} \int_{\mathbb{C}} z^a \bar{z}^b e^{-|z|^2} \, dA(z) = \delta_{ab} \, a! \qquad (a, b \in \mathbb{Z}_{\geq 0}).$$

Applying this identity in the four matrix entries gives

$$\| \det(X)^k \|^2 = \sum_{j=0}^{k} \binom{k}{j}^2 (j!)^2 ((k-j)!)^2 = k!(k+1)!.$$

Let $\beta = (\nu\tau)^{\mathrm{bc}}$ and let ξ be the quadratic character of E/F_0. The local formulas give the absolute Euler product for $\Re s > 1$. For example the nonidentity contributions at places away from S_Q are $O(q^{-\Re s})$ uniformly on closed half-planes $\Re s \geq 1 + \delta$. The remaining finite local factors are constant in s. The product away from S_Q is

$$\frac{L^{S_Q}(s, \beta^{-1}) L^{S_Q}(s+1, \beta^{-1})}{L_{F_0}^{S_Q}(2s+1, \xi) \, \zeta_{F_0}^{S_Q}(2s+2)}.$$

In a superscript on an F_0-function, omit the places below S_Q. At a nonsplit place β is trivial on the fixed-field uniformizer, which explains the inert factor. At each such place the local integral has a pole at zero. We first multiply the local identities in $\Re s > 1$, then continue the global identity meromorphically by the Eisenstein series and the Hecke L-functions.

The continued Eisenstein pullback may be evaluated under the compact quotient integrals at the holomorphic Siegel–Weil point. At zero the local masses at the omitted places insert exactly the omitted reciprocal denominator factors. The factorization $\zeta_E(s) = \zeta_{F_0}(s) L_{F_0}(s, \xi)$ and the positive residues of the two Dedekind zeta functions give $L_{F_0}(1, \xi) > 0$. Hence the denominator $L_{F_0}(1, \xi) \zeta_{F_0}(2)$ is nonzero. We obtain

$$|\Lambda_Q(\tau)|^2 = D_p L^{S_Q}(0, \beta^{-1}) L^{S_Q}(1, \beta^{-1}), \qquad D_p > 0, \tag{87}$$

where D_p depends only on the original field, Haar and real test at p. With the stated probability real measures and unscaled Bargmann test it is $v_2 \prod_{v|\infty} k_v!(k_v+1)!/(2 L_{F_0}(1, \xi) \zeta_{F_0}(2))$. Using the fixed normalization of the theta vector throughout multiplies this constant by a fixed positive factor. The same normalization is used with and without Q.

For a twist of exact support Q, the conductor of β^{-1} contains both members of every marked pair with exponent one. The old factors are fixed, since p is ramified at each old place and the auxiliary support is disjoint. The archimedean gamma factors are $\Gamma_{\mathbb{C}}(s + c_v)$ and are finite and nonzero at zero and one. Each new split pair contributes q_w^2 to the conductor norm, so its

square root contributes q_w. The functional equation for the completed Hecke L-function has a root number of modulus one. Since β is unitary, the absolute values at one for β and β^{-1} agree. Thus [71, §§2.5, 4.4–4.5] gives

$$|L(0,\beta^{-1})| = D'_p q_Q |L(1,\beta^{-1})|, \qquad D'_p > 0.$$

In this calculation the omitted factors of the full Hecke function at the exact ramified support are all one.

Nonvanishing at one for the nontrivial unitary Hecke character follows from the usual Hecke nonvanishing theorem. It also follows by applying $3 + 4\cos\theta + \cos(2\theta) \geq 0$ to the logarithmic Euler products for $\zeta_E(s), L(s,\beta^{-1})$, and $L(s,\beta^{-2})$ as $s \downarrow 1$, since β^{-2} has nonzero infinity weights. These facts give $\Lambda_\varnothing(1) \neq 0$ and

$$\frac{|\Lambda_Q(\tau)|^2}{|\Lambda_\varnothing(1)|^2} = q_Q \left| \frac{L^{S_Q}(1, ((\nu\tau)^{\mathrm{bc}})^{-1})}{L^{S_\varnothing}(1, (\nu^{\mathrm{bc}})^{-1})} \right|^2 . \tag{88}$$

For an unramified enlargement at a new split place w, put $q = q_w$, $A = \beta(\varpi_w)$, and $t = (qA)^{-1}$. To compare the periods themselves, take each subspace $U \subset k_w^2$ of dimension i and choose h_U with

$$h_U \mathcal{O}_w^2 = \{y \in \mathcal{O}_w^2 : \bar{y} \in U^\perp\}.$$

The local target action on the vacuum is $(qA)^i$ times the indicator of integral matrices X such that $U \subset \ker \bar{X}^{\mathsf{T}}$. Indeed the lattice has index q^i, and its columns satisfy precisely the displayed reduction condition. Multiply this translate by $(-t)^i q^{\binom{i}{2}}$ and sum over U. If $c = \dim \ker \bar{X}^{\mathsf{T}}$ equals $0, 1, 2$, the resulting values are respectively $1,\ 1-1,\ 1-(q+1)+q$. Thus the sum is the new test $\mathbf{1}_{\mathrm{GL}_2(\mathcal{O}_w)}$. In the original Haar integral defining $\Lambda_Q(\tau)$, each target translate has the same integral by right invariance of the quotient measure. The dimensions $0, 1, 2$ occur with multiplicities $1, q+1, 1$, so

$$\Lambda_{Q\cup\{w\}}(\tau) = (1 - (q+1)t + qt^2)\Lambda_Q(\tau) = (1 - q^{-1}A^{-1})(1 - A^{-1})\Lambda_Q(\tau).$$

Since $|A| = 1$, the corresponding multiplier of the squared period is

$$(1 - A)(1 - A^{-1})(1 - q^{-1}A)(1 - q^{-1}A^{-1}). \tag{89}$$

This also follows by deleting the split Euler factors at $s = 0, 1$ in (87). Both identities hold when the multiplier vanishes.

30.2.2 Integral coefficients of the rank-two norm

In this subsection, we express the norm of Section 30.2.1 in the original finite coordinates.

Let h_c be representatives in (86). For $y \in \mathrm{GL}_2(\mathcal{O}_E/\mathfrak{m})$, let $\widetilde{y}$ be the split integral lift in the original coordinates at the marked places, with identity components elsewhere, and set $h_{c,y} = h_c\widetilde{y}$, $u_{c,y} = \det h_{c,y}$. Let $K^{\mathrm{pt}}_{Q,1}(h)$ be the kernel at the same modulus with the point indicator of the identity and with $\tau = 1$. Expand the determinant character as the sum of the point indicators at invertible matrices y, weighted by $\chi_{\mathfrak{m}}(\det y)$. Translation of the identity indicator by $\widetilde{y}$ on the target is exactly this weighted indicator at y, by the displayed split action.

Twisting the two group characters from $\nu v|_{U(1)}$ to α then multiplies the point kernel at $(1, h_{c,y})$ by $\tau(u_{c,y})$. Thus

$$\Lambda_Q(\tau) = \sum_{c,y} \frac{K_{Q,1}^{\mathrm{pt}}(h_{c,y})\tau(u_{c,y})}{|\operatorname{Aut}(h_c, K_2)|}.$$

The stabilizer order here divides C_G, by reduction at an old odd completely split place as in the preceding subsection.

Use the fixed nonzero integral-jet scale

$$s_p = (M^2)^{8f} s_{\mathrm{vac}}^4 (M^2 d_K s_{\mathrm{jet}})^{2\sum_v k_v}$$

for the original rank-two point test at modulus $\mathfrak{m}$. Section 44.4 gives

$$\iota(s_p K_{Q,1}^{\mathrm{pt}}(h_{c,y})) \in \mathcal{O}_{\overline{\mathbb{Q}}_\ell} \qquad (\ell \nmid q_Q)$$

at every coefficient embedding, with the same s_p for all Q, c, y.

Set

$$a_{c,y}(Q) = \frac{C_G s_p K_{Q,1}^{\mathrm{pt}}(h_{c,y})}{|\operatorname{Aut}(h_c, K_2)|}.$$

Its coefficients are integral away from q_Q by this jet bound. With i, j ranging over the pairs (c, y), the squared norm system is

$$F_{p,Q}(\tau) = |C_G s_p \Lambda_Q(\tau)|^2 = \sum_{i,j} a_i(Q)\overline{a_j(Q)}\,\tau(u_i u_j^{-1}).$$

The conjugate coefficients are integral by applying the bound at the conjugate embedding. Formula (88) gives its exact-support ratio, with the same scale s_p on numerator and denominator.

30.3 The integral trace algebra on the auxiliary levels

In this subsection, the attachment of Section 28.2 identifies traces with Hecke data at a cofinite set of primes. We use the integral Hecke action there to place traces at all Galois elements in one coefficient algebra.

Fix a coefficient embedding above ℓ and any of the finitely raised levels under consideration. The prescribed type and the characters at the old and twist places stay fixed. Raising changes the invariance conditions only at auxiliary split places away from ℓ. Write $V_{\mathrm{orig}} \subset V_{\mathrm{raised}}$ for the corresponding spaces and $L_{\mathrm{orig}}, L_{\mathrm{raised}}$ for their Weyl lattices in the common transported gauge. Then

$$L_{\mathrm{orig}} = V_{\mathrm{orig}} \cap L_{\mathrm{raised}}, \qquad V_{\mathrm{orig}} \cap aL_{\mathrm{raised}} = aL_{\mathrm{orig}} \quad (a \neq 0).$$

The second equality follows by dividing by the scalar a in V_{orig}. On the ambient integral space all local translations supported away from ℓ preserve this gauge. In fact the gauge of a value at a point is read in the transported coordinates of that point at the places above ℓ. Right translating by a point with identity components above ℓ changes neither those coordinates nor the list of points where the gauge is bounded. The finite local sums that preserve the raised invariants therefore act on the raised integral lattice.

Index the eigenspaces by the systems of spherical Hecke eigenvalues at almost all split places, identifying systems that agree away from a finite set. For such a system x, let W_x be the space

of forms with these eigenvalues at almost all split places. For either level V its contribution is $V \cap W_x$. Thus the inclusion of levels induces

$$V_{\mathrm{orig}} \cap W_x \ \subseteq \ V_{\mathrm{raised}} \cap W_x.$$

Let $Y = \{x : V_{\mathrm{raised}} \cap W_x \neq 0\}$; it is finite by finite dimensionality and orthogonality of distinct systems. Local translations at the auxiliary places preserve each W_x since they commute with the spherical operators at all other places. Define $\mathbb{T}$ to be the ring of functions $Y \to \overline{\mathbb{Q}}_\ell$ whose diagonal operators preserve the raised integral lattice. It contains the scalar valuation ring. It is closed in the product of the coefficient topologies: for each vector in the integral lattice and each gauge-coordinate test, the result of the diagonal operator is a continuous linear function of the finitely many scalars. The condition that this result be in the valuation ring is closed, and the intersection of all these conditions is closed. This argument uses the original gauge at every class point, so it is valid with stabilizers as well.

Choose a basis of each $V_{\mathrm{raised}} \cap W_x$. Away from the finite union of the exceptional places of these basis vectors and the levels, the spherical operators act on each summand by its prescribed scalars. At every such split $w \nmid \ell$, the minuscule operators are finite sums of the integral translations above, so their scalar functions on Y belong to $\mathbb{T}$. Here the auxiliary primes have already been fixed; this cofinite set is used for the subsequent Frobenius comparison. The coefficients of the normalized Hecke polynomial are integral multiples of these operators, with powers of q_w as in Section 28.2. Fix $g \in G_E$. The cofinite Frobenius comparison for each member of Y, and Chebotarev, approximate the coefficients of $\det(X - R_x(g))$ simultaneously, for all $x \in Y$, by these Hecke coefficients.

Indeed for any required precision, continuity gives an open normal subgroup on whose coset at g the coefficients agree to that precision. The functions are conjugacy invariant. Chebotarev's theorem [46, Theorem 8.31] gives positive density for the conjugacy class in the finite quotient. The primes of E of residue degree greater than one over their rational primes have zero density, so one can take prime norm and hence split places over F_0, excluding the finite raised and bad levels. Closedness now gives each coefficient function in $\mathbb{T}$.

For an integral character a, putting $N_x = aR_x$ multiplies the coefficient of X^{n-i} by $a(g)^i$; hence these twisted characteristic coefficients also belong to $\mathbb{T}$. In particular $x \mapsto \mathrm{tr}(N_x(g))$ belongs to $\mathbb{T}$ for every g, including products involving the selected tame generators and Frobenius lifts. The trace functions used below are integral linear combinations of these. For instance expansion of p in monomials writes

$$\mathrm{tr}\big(p(N_x(\phi))(N_x(\tau) - 1)N_x(g)\big) = \sum_i p_i\big(\mathrm{tr}(N_x(\phi^i \tau g)) - \mathrm{tr}(N_x(\phi^i g))\big).$$

All of these diagonal operators commute with the local translations preserving the decomposition into outside-level eigensystems. Thus the raised local comparison uses the same integral endomorphism ring and original invariant lattice throughout.

31 Hecke operators and the common trace coefficient

In this section, we identify the tame and Frobenius operators in the algebra of Section 30.3. Spherical compatibility gives a common Hecke coefficient to which the Fitting and trace arguments of Section 32 apply.

31.1 Tame and Frobenius operators at a raised prime

In this subsection, we represent tame inertia and Frobenius by operators on the K_1-invariants at the same split place. The resulting trace identities refine the integral trace comparison of Section 30.3 by retaining the local finite type.

Fix a coefficient prime ℓ, a chosen split place $w \nmid \ell$ off the original level, and a congruence depth ν. Write $k = k_w$, $q = |k|$, $H = \mathrm{GL}_n(k)$, $K = \mathrm{GL}_n(\mathcal{O}_w)$, and $K_1 = \ker(K \to H)$. At w the original level is K and the raised level is K_1. The other local factors stay fixed. We assume $\ell^\nu \mid q - 1$. Choose the uniformizer ϖ with local Artin class the geometric Frobenius lift ϕ. Let u be a unit with local Artin class that of the chosen tame generator τ, and let $\lambda : k^\times \to A = \mathbb{Z}/\ell^\nu\mathbb{Z}$ be the chosen homomorphism with $\lambda(\bar{u}) = 1$. Put $d = \mathrm{diag}(\bar{u}, 1, \dots, 1)$.

Write ρ for the H-action on the raised space and $\mathcal{A}$ for the integral endomorphisms of its lattice. The raised lattice is the relevant invariant space in the ambient integral lattice on which all translates at w act. Section 30.3 gives the integral diagonal trace action on this same lattice. The outside-level eigensystems decompose the raised space over characteristic zero, and the local operators at w preserve each summand.

On every raised system x use its semisimple attachment R_x and the full split-place comparison from Section 28.3. At a split level place this comparison follows from the same split base change on each constituent of the eigenvector. The cofinite comparison of Section 28.2 identifies the semisimple attachments of all constituents with system x. Choose the normalization $N_x = aR_x$, where $a(\tau) = 1$. For each Galois element g, write

$$F_x = N_x(\phi), \qquad T_x = N_x(\tau), \qquad M_x = N_x(g),$$

and let $C_p, D_p \in \mathcal{A}$ be the diagonal operators whose entries for an integral polynomial p are

$$C_p(x) = \mathrm{tr}(p(F_x)M_x), \qquad D_p(x) = \mathrm{tr}(p(F_x)(T_x - 1)M_x).$$

All scalars are evaluated at the same embedding.

Let $\bar{P} \subset H$ be the stabilizer of ke_1, $J \subset K$ its inverse image, and $I_1 \subset K$ the inverse image of the upper unitriangular group. For a character ψ of A, put $\chi = \psi \circ \lambda$. On the H-type $\chi \circ \det$, the inverse determinant twist has a K-fixed vector. Its parameter is spherical. Twisting back gives zero monodromy, and τ acts on the entire Galois representation by $\psi(1)$, the eigenvalue of $\rho(d)$ on the same type.

Consider now the J-type $j \mapsto \chi(\bar{j}_{11})$ with $\chi \neq 1$. In an irreducible local constituent π, finite Frobenius reciprocity gives a common constituent of π^{K_1} and $\mathrm{Ind}_{\bar{P}}^{H}(\chi \boxtimes 1)$. The latter embeds into the Borel induction from $(\chi, 1, \dots, 1)$ by transitivity of induction. Write π as a subquotient of an induction from its supercuspidal support. Iwasawa decomposition and exactness of compact invariants give the K_1-invariants of that induction as finite induction from the corresponding principal-kernel invariants of its factors.

A nonzero principal-kernel invariant space of a supercuspidal factor is the finite cuspidal representation in its depth-zero compact-induction description [55, Propositions 6.6 and 6.8]. Indeed, in a block of rank b, write $K_b = \mathrm{GL}_b(\mathcal{O}_w)$ and take Cartan representatives $s = \mathrm{diag}(\varpi^{v_1}, \dots, \varpi^{v_b})$, with $v_1 \le \cdots \le v_b$, modulo $E_w^\times K_b$ on the left and K_b on the right. For any strict cut $v_i < v_{i+1}$, the reduction of $K_b \cap sK_{b,1}s^{-1}$ contains the upper rectangular unipotent radical at that cut, since conjugation by s^{-1} multiplies the entry in covector $a \le i$, column $b' > i$, by $\varpi^{v_{b'} - v_a}$. The intersection is the same for sk_0, $k_0 \in K_b$, because $K_{b,1}$ is normal in K_b. Cuspidality forces the value of every $K_{b,1}$-invariant vector at those cosets to vanish. The central Cartan class gives the finite cuspidal factor.

31.1.1 The restriction to the tame subgroup

In this subsection, we calculate the tame subspace using the local Hecke data of Section 28.2.

Now use permutation representatives in finite Mackey theory between the Borel and the block parabolic. For each representative, the intersection in each conjugated Levi block contains the unipotent radical of a Borel of that block. A finite cuspidal of rank greater than one has no such invariants, because that radical contains the radical of a proper parabolic. Thus every block has rank one. Comparing the characters on the diagonal torus gives the tuple $(\chi, 1, \dots, 1)$, up to permutation. The ramified factor occurs once. Unramified twists preserve its restriction to $\mathcal{O}_w^\times$, so its segment has length one and lies on a separate cuspidal line. The classification by segments and the product over distinct lines give

$$\pi = \operatorname{Ind}_P^{\mathrm{GL}_n(E_w)}(\chi_0 \boxtimes \pi_1) \quad \text{with normalized induction,}$$

where P is the stabilizer of the first coordinate line, $\chi_0|_{\mathcal{O}_w^\times} = \chi$ via reduction, and π_1 has all-unramified rank-one support [78, Theorem 6.1 and Proposition 8.6, pp. 188, 197].

Apply finite Mackey theory to $\bar{P}\backslash H/\bar{P}$. For $n \geq 2$ the two representatives are 1 and $s = (12)$. Choose $b \in k^\times$ with $\chi(b) \neq 1$. The element $h = \operatorname{diag}(1, b, 1, \dots, 1)$ belongs to $\bar{P} \cap s\bar{P}s^{-1}$ and acts trivially by the prescribed character $j \mapsto \chi(j_{11})$. On the conjugated inducing representation it acts by $\chi(b)$, since $s^{-1}hs = \operatorname{diag}(b, 1, \dots, 1)$ acts on the first block alone. The intertwining space for this coset is therefore zero. The identity coset gives $\pi_1^{\mathrm{GL}_{n-1}(\mathcal{O}_w)}$. In particular π_1 is spherical and evaluation at 1 is injective on the required J-type. For $n = 1$ there is only χ_0 and the evaluation is immediate.

Set $t = \operatorname{diag}(\varpi, 1, \dots, 1)$ and let U be the unnormalized operator of $I_1 t I_1$. For $b \in k^{n-1}$, put $u_b = \left(\begin{smallmatrix} 1 & \widetilde{b} \\ 0 & I_{n-1} \end{smallmatrix}\right)$. Both $I_1 \cap tI_1t^{-1}$ and $J \cap tJt^{-1}$ impose precisely the extra divisibility by ϖ on the first covector off-diagonal entries. Thus the cosets are $u_b t I_1$ and $u_b t J$, respectively. The J-character agrees with its t-conjugate on the intersection because the $(1,1)$ entry is unchanged. Left multiplication by $j \in J$ permutes the cosets and contributes its character to each term, so U preserves the type. In the induced model every term has value

$$f(u_b t) = q^{-(n-1)/2}\chi_0(\varpi) f(1).$$

There are q^{n-1} terms. Evaluation at 1 therefore shows that the U-scalar is $q^{(n-1)/2}\chi_0(\varpi)$.

Both blocks have zero monodromy, and the complementary block has trivial inertia. On the unique $\psi(1)$-inertia line the Frobenius scalar after the twist $|\det|^{(1-n)/2}$ is $q^{(n-1)/2}\chi_0(\varpi)$. Frobenius semisimplification preserves this scalar on the one-dimensional line. Hence $T_x - 1$ acts only on this line, where F_x has the scalar of $\widetilde{U} = a(\phi)U$. For arbitrary g, on the given J-type the operator D_p is $p(\widetilde{U})D_1$. This and the determinant-type calculation apply on the corresponding types of every raised eigensystem, by taking irreducible components.

For an integer-valued weight f on A, define

$$L_J(f) = \sum_{k \in \bar{P}} f(\lambda(k_{11}))\rho(k), \quad L_H(f) = \sum_{h \in H} f(\lambda(\det h))\rho(h),$$
$$B(f) = p(\widetilde{U})L_J(f).$$

Left multiplication by I_1 preserves the line weight, so the image of $L_J(f)$ is I_1-fixed. The translates in the coset sum U act on the ambient integral lattice, and their sum on I_1-invariants preserves those invariants. As $K_1 \subset I_1$, $B(f)$ is an integral operator on the raised lattice

made from translates at w. Let $\mathcal{E}(f) = B(f)D_1 - L_J(f)D_p$. Fourier expansion in A and the evaluations on the two types give

$$L_H(f)D_p = (\rho(d) - 1)L_H(f)C_p, \qquad \mathcal{E}(f(\cdot - n!) - f) = 0.$$

The weight ψ^{-1} has image in the ψ-type, and its coefficient in $f(\cdot - n!) - f$ is multiplied by $\psi(n!) - 1$. The equality is tested in characteristic zero between the integral operators.

Let $\omega : A \to \mathbb{Z}$ give the signed representatives. Taking $f(s) = \omega(s)(\omega(s) + n!)$ gives $f(s - n!) - f(s) \equiv -2n!\,\omega(s) \pmod{\ell^\nu}$, so $2n!\mathcal{E}(\omega) \in \ell^\nu \mathcal{A}$. Let σ be a reduction homomorphism on $\mathcal{A}$ which kills every X with $2n!X \in \ell^\nu \mathcal{A}$. For example, use reduction modulo $\ell^\nu/(2n!)$ when that quotient belongs to the coefficient valuation ring. Write $\mathcal{N}_w = \sum_{h \in H} \rho(h)$. The determinant identity with the constant weight gives $\mathcal{N}_w D_1 = 0$ exactly.

Choose representatives r_s with $r_s \bar{P}$ running over the lines, and set

$$C = \sum_s \rho(r_s), \quad C^\vee = \sum_s \rho(r_s^{-1}), \quad \partial = -CB(\omega)C^\vee.$$

Write $hr_s = r_{hs}k_{h,s}$ with $k_{h,s} \in \bar{P}$. In the group algebra the coefficient of h in $CL_J(\omega)C^\vee$ is $\sum_s \omega(\lambda((k_{h,s})_{11}))$. The product $\prod_s (k_{h,s})_{11}$ is a homomorphism in h by the coset cocycle identity and is independent of the representatives because their changes telescope. On $\mathrm{diag}(b, 1, \dots, 1)$ the product is b.

For each line other than ke_1 choose a representative vector with its first nonzero coordinate among coordinates 2 through n equal to 1. The diagonal element preserves this normalization, so the only contribution is that of ke_1. A transvection has residue-characteristic order, prime to $|k^\times|$, and therefore contributes 1. These elements generate H, so the product is the determinant. It follows that

$$CL_J(\omega)C^\vee \equiv L_H(\omega), \qquad (\rho(d) - 1)L_H(\omega) \equiv -\mathcal{N}_w \pmod{\ell^\nu \mathcal{A}}.$$

The second congruence uses $\omega(s - 1) - \omega(s) \equiv -1 \pmod{\ell^\nu}$.

As the diagonal operators commute with the local ones, we get

$$\sigma(\partial D_1) = \sigma(\mathcal{N}_w C_p).$$

If $p(X) = (X - q^b)H_0(X)$, with $b \geq 0$ and H_0 integral, then $q^b \equiv 1 \pmod{\ell^\nu}$ and $\sigma(\partial D_1) = \sigma(\mathcal{N}_w C_{(X-1)H_0})$. The operator ∂ is independent of g and uses the original unnormalized norm on the full raised lattice. At every selected place the argument uses the same depth and reduction, with the hypotheses stated at the start of the subsection.

31.1.2 Integral clearing from the tame action

In this subsection, we use the tame restriction of Section 31.1.1 to pass to integral vectors.

For use with clearing, suppose an integral endomorphism Ψ and scalar c satisfy

$$\sum_{k \in \bar{P}} \rho(k^{-1})\Psi\rho(k) = c\,1.$$

Let $\mathcal{N}_{\mathrm{all}}$ be the product of the selected full norms, including $\mathcal{N}_w$, and suppose the spherical annihilation on the supported original systems gives $\delta\mathcal{N}_{\mathrm{all}}B(1) = 0$, with 1 the constant weight. For $h \in H$ set $A_h = \sum_s \omega(\lambda((k_{h,s})_{11}))\rho(r_s^{-1})$. The equations

$$B(f)\rho(k) = B(f(\cdot - \lambda(k_{11}))), \qquad r_{hs}^{-1}h = k_{h,s}r_s^{-1}$$

and the signed weight congruence give

$$\partial(\rho(h)-1) \equiv CB(1)A_h \pmod{\ell^{\nu}\mathcal{A}}.$$

Here the difference of the weights at $k_{h,s}$ is congruent to the negative of its logarithm.

The double-coset calculation for the trivial J-character gives $\rho(k)B(1) = B(1)$. Applying the identity for Ψ to $B(1)$ yields $cCB(1) = \mathcal{N}_w \Psi B(1)$. Set $Z = \Psi B(1)A_h$. Then

$$\sigma\big(c\partial(\rho(h)-1)\big) = \sigma(\mathcal{N}_w Z).$$

Before applying C to that identity for Ψ, multiply on the left by $\delta\mathcal{N}_{\mathrm{all}}$. Norm invariance gives

$$c\delta\mathcal{N}_{\mathrm{all}}B(1) = |\bar{P}|\delta\mathcal{N}_{\mathrm{all}}\Psi B(1).$$

The left side vanishes. Integral endomorphisms have no integer torsion, so $\delta\mathcal{N}_{\mathrm{all}}Z = 0$.

31.2 Spherical root chains on the original level

In this subsection, we factor the normalized spherical polynomial into successive roots in the pro-unipotent Iwahori space. The construction preserves the original spherical vector and keeps the local normalization used in Section 31.1.

In this subsection r is the local rank and t is the length of the specified chain of roots. Let $K_0 = \mathrm{GL}_r(\mathcal{O}_w)$ with residue field of size q, and use the fixed ordered coordinates and uniformizer ϖ at the auxiliary place. Let $I_1 \subset K_0$ be the pro-unipotent Iwahori. Work in the given smooth unitary representation with positive invariant pairing B, conjugate linear in the first argument. Let $x \neq 0$ be the original K_0-fixed eigenvector, with eigenvalues t_j for the unnormalized spherical coset sums of $\mathrm{diag}(\varpi I_j, I_{r-j})$.

The polynomial is

$$P(X) = \sum_{j=0}^{r} a_j X^{r-j}, \qquad a_j = (-1)^j q^{j(j-1)/2} t_j, \quad t_0 = 1.$$

Put $d = \mathrm{diag}(\varpi, 1, \dots, 1)$. The operator U in the derivative congruence is the unnormalized I_1-double-coset sum of d. On I_1-fixed vectors, it is $q^{r-1}M_{I_1}\rho(d)M_{I_1}$, where M_{I_1} is probability Haar averaging. Indeed $I_1 \cap dI_1 d^{-1}$ imposes the vanishing modulo ϖ of the off-diagonal entries in the first covector, giving index q^{r-1}. The adjoint is $U^{\dagger} = q^{r-1}M_{I_1}\rho(d^{-1})M_{I_1}$.

For $m, j \geq 0$, let $T_{m,j}$ be the sum of $\rho(g)x$ over the lattices $L = g\mathcal{O}_w^r$ between $\varpi\mathcal{O}_w^r$ and $\mathcal{O}_w^r$, of colength j, containing the first m coordinate vectors. A representative g gives a well-defined vector because x is K_0-fixed. In particular $T_{0,j} = t_j x$. Put $T_j = T_{1,j}$. These vectors are fixed by I_1, which permutes the relevant lattice sets.

The interval of lattice sums gives a forward recursion. For each residue covector ξ with first entry 1, take $n_\xi \in I_1$ the identity with first covector lifted from ξ. For a lattice in $T_{0,j+1}$, its occurrence in $n_\xi^{-1}dT_j$ means that its reduction lies in the hyperplane $\xi = 0$. To check the condition, $n_\xi L$ must have first residue coordinate zero; then $d^{-1}n_\xi L$ is integral, contains $\varpi\mathcal{O}_w^r$ and contains e_1 since $n_\xi L \supset \varpi\mathcal{O}_w^r$. The colength has dropped by one. There are no such covectors if L contains e_1. Otherwise the requirement that ξ vanish on a subspace of dimension $r-j-1$ and take value 1 on e_1 leaves j free coordinates, hence q^j covectors. Applying M_{I_1}, which kills the action of each n_ξ^{-1} on the left, gives

$$UT_j = q^j(t_{j+1}x - T_{j+1}). \tag{SC1}$$

In this notation $T_0 = x$ and $T_r = 0$.

For the inverse count use the representation $\rho'(g) = \rho((g^{\mathsf{T}})^{-1})$. Under duality the lattice L becomes $L^\vee = (g^{\mathsf{T}})^{-1}\mathcal{O}_w^r$ for the standard integral bilinear pairing. For L containing the first m vectors, the first m dual coordinates are integral. With $C_m = \mathrm{diag}(I_m, \varpi^{-1}I_{r-m})$, the lattice $C_m^{-1}L^\vee$ is between $\varpi\mathcal{O}_w^r$ and $\mathcal{O}_w^r$, contains those same coordinate vectors, and its colength is $r-m-j$. This is a bijection, by applying duality and the inverse diagonal scaling. It proves

$$T'_{m,j} = \rho(C_m)T_{m,r-m-j},$$

where primes refer to ρ'. In the forward count for ρ' test by the linear map $\Psi = M_{I_1}\rho(C_1^{-1})$.

For every first-covector unipotent n_ξ of that count, $C_1^{-1}n_\xi^{\mathsf{T}}C_1$ is in I_1: its off-diagonal first-column entries are multiplied by ϖ. Hence $\Psi\rho'(n_\xi^{-1}) = \Psi$, as required in the same hyperplane count. Use $d = C_0^{-1}C_1$. For $1 \le i < r$, take the dual colength $j = r-1-i$. Replacing each of the three dual sums by its original counterpart gives

$$U^\dagger T_i = q^{r-1-i}(q^{-(r-1)}t_iU^\dagger x - T_{i-1}). \tag{SC2}$$

At the end, T_{r-1} is the single lattice of C_1^{-1}, and $t_rx = \rho(\varpi I_r)x$. Thus $t_r \neq 0$,

$$U^\dagger x = q^{r-1}t_r^{-1}T_{r-1}, \qquad UT_{r-1} = q^{r-1}t_rx, \qquad UU^\dagger x = q^{2r-2}x.$$

Write $P_i(X) = \sum_{j=0}^i a_jX^{i-j}$. The recursion $P_{i+1} = XP_i + a_{i+1}$ and (SC1) give the following identities.

Lemma 31.1 (The spherical root relation)**.** *For the marked spherical vector x, the Hecke operator U, and its polynomials P_i, one has*

$$P_i(U)x = (-1)^i q^{\binom{i}{2}}T_i \quad (0 \le i \le r), \qquad P(U)x = 0.$$

For $1 \le i \le r$, (SC2) and the endpoint identity give

$$U^\dagger P_i(U)x = a_iq^{-i}U^\dagger x + q^{r-2}P_{i-1}(U)x. \tag{SC3}$$

For $i = r$ both terms on the right cancel, using the formula for $U^\dagger x$. Put $P_{r+1}(X) = XP_r(X)$. For any scalar c, set $D_i(X) = (P_i(X) - P_i(c))/(X-c)$ for $0 \le i \le r+1$. Then $D_{i+1} = P_i + cD_i$ for $0 \le i \le r$. By (SC3), induction gives

$$U^\dagger D_{i+1}(U)x = q^{-i}P_i(qc)U^\dagger x + q^{r-2}D_i(U)x, \qquad 0 \le i \le r.$$

The case $i = 0$ is $D_1 = 1$, $D_0 = 0$. In the step use $q^{-i}P_i(qc) = a_iq^{-i} + cq^{-(i-1)}P_{i-1}(qc)$.

If $P(c) = 0$, put $z = D_r(U)x$. Since the constant term of P is nonzero, $c \neq 0$. The divided difference identity and $P(U)x = 0$ give $Uz = cz$ and

$$cU^\dagger z = q^{-r}P(qc)U^\dagger x + q^{r-2}z.$$

Pair this last identity first with $U^\dagger x$ in the first variable. From $UU^\dagger x = q^{2r-2}x$ one obtains

$$c(q^r - 1)B(x,z) = P(qc)B(x,x).$$

Pairing with z in the second variable instead and substituting this equality gives

$$(|c|^2 - q^{r-2})B(z,z) = \frac{|P(qc)|^2}{q^r(q^r-1)}B(x,x).$$

Since $x \neq 0$ and B is positive definite, $B(x,x) > 0$. If $P(qc) \neq 0$, the first identity gives $z \neq 0$. The second, with $B(z,z) > 0$, then gives $|c|^2 > q^{r-2}$.

If $P(qc) = 0$, the first identity gives $B(x,z) = 0$, and the operator identity gives $U^\dagger z = q^{r-2}c^{-1}z$. Therefore $B(U^j x, z) = 0$ for every j, by moving the adjoint. Since z is a polynomial in U applied to x, it is orthogonal to itself, and $z = 0$. Thus when c, qc are both roots, $(P/(X-c))(U)x = 0$. Suppose now $P(X/a)$ is $\prod_{i=0}^{t-1}(X - q^i)\,R(X)$, with $a \neq 0$. Let H be any polynomial divisible by R. At the auxiliary prime indexed by j we use the normalized annihilating polynomial H_j constructed from the complementary factors in Section 29.2. The minimal polynomial of U on the cyclic space $\mathcal{C} = \mathbb{C}[U]x$ divides $P/(X - q^i/a)$ for every $i < t-1$, by the calculation for adjacent roots.

The roots in this set are distinct. Taking the greatest common divisor cancels one copy of each such factor from P. Thus $(aU - q^{t-1})R(aU)$ kills $\mathcal{C}$, and so does $p(aU)$ for $p(X) = (X - q^{t-1})H(X)$. The space $\mathcal{C}$ is spanned by $T_0, \dots, T_{r-1}$ and is stable under $U^\dagger$ by (SC2). The adjoint of $p(aU)$ applied to x belongs to $\mathcal{C}$ and is orthogonal to it, hence is zero. For every I_1-fixed y, therefore, $B(x, p(aU)y) = 0$. This is the orthogonality required for the operator with prescribed congruences on the original spherical block, with $a = a(\phi_w)$.

The same marked calculation also supplies the full root chain from its initial modulus. The adjoint of the j-th spherical double-coset sum is the $(r-j)$-th such sum times $\rho(\varpi^{-1}I_r)$, by inverting the diagonal representative and permuting coordinates. Thus $\overline{t_j} = t_r^{-1}t_{r-j}$, since the original vector is a joint eigenvector and t_r is the unitary central eigenvalue. Substitution into P gives the reciprocal identity, for $z \neq 0$,

$$q^{r(r-1)/2}\overline{P(z)} = (-\bar{z})^r \bar{t}_r\, P(q^{r-1}/\bar{z}).$$

For example, reindex the right-side term of order $r-j$ by j and use $\binom{r}{2} + \binom{j}{2} = \binom{r-j}{2} + (r-1)j$. The roots are therefore stable under $z \mapsto q^{r-1}/\bar{z}$. If $P(c) = 0$ and $|c|^2 q^k = q^r$, this reflection interchanges cq^i and cq^{k-1-i}. For $0 < i \leq (k-1)/2$, the previous root cq^{i-1} has squared modulus $< q^{r-2}$. The endpoint bound forces cq^i to be a root. Induction covers the lower half and reflection covers the upper half.

31.3 The common coefficient of the Hecke polynomial

In this subsection, we construct one polynomial over the integral Hecke algebra of Section 30.3 on the entire finite set of systems. The coefficient of interest is isolated by specializing the common relations on the selected lifts.

Fix the descent pair, the full set Y of outside-level eigensystems on its raised space as in Section 30.3, its lift subset $\mathcal{X} \subseteq Y$, and the evaluation system $\pi \in \mathcal{X}$. Write r' for the main rank and t' for the rank difference. Use the ℓ-adic embedding in the congruence construction and put

$$G = C_G, \qquad v = v_\ell(G), \qquad H = Y_{\mathrm{tr}}, \qquad h = h_\ell, \qquad C = \ell^{Hh}, \qquad 0 \leq h \leq v.$$

Let J be the ordered set of new primes and $\mathrm{ty}\colon J \to \{1, \dots, t'-1\}$ the monotone type.

The coefficient of $X_J = \prod_{j\in J} X_j$ in the common Ribet polynomial at π is d. The bound for the actual passage to the Hecke algebra is

$$v_\ell(G^2 d) \le \mathrm{sel} + 2t'\mathrm{ram} + r'\mathrm{loc}_\ell + \mathrm{err}_\ell, \qquad \mathrm{loc}_\ell = 4fF_\ell, \tag{RB1}$$

where f is the number of infinite places of E, and

$$\mathrm{err}_\ell = t\, v_\ell(X_{\mathrm{odd}}) + 8ft\, v + 2v + \big((4t+1)H^{22} + H\big)v.$$

All data in this bound refer to the actual marked family.

For the relative traces one may take any common multiple of the ambient stabilizer orders. In the application of Section 32.3 take $c = G = C_G$. Both the stabilizer orders and the order of the finite residue group defining the type projector divide C_G. Thus the same c clears that projector, and (RB1) bounds $v_\ell(c^2 d)$. The normalization of the valuation is $v_\ell(\ell) = 1$.

For this raised family take the finite compositum of the fields of definition of the representations N_y, $y \in Y$, the prescribed lattices, characters, congruence polynomials and block identifications. Include the chosen torsion values and all conjugates of the marked completions required in Section 29.1. Let O be the valuation ring of its normal closure over $\mathbf{Q}_\ell$, used for every member of the family. On the distinguished lift π take $L_\pi = L_0 \otimes_{O_0} O$, where L_0 is the main-summand lattice over O_0 carrying the selected Selmer bound. Let $\mathcal{S}_0$ be the divisible-coefficient Selmer cocycle module for L_0, normalized by $z(\gamma_\pi) = 0$. In (RB1) we use

$$\mathrm{sel} = \frac{\mathrm{length}_{O_0} \mathcal{S}_0}{v_{O_0}(\ell)}, \qquad \mathrm{ram} = \sum_{w\in Q} v_\ell(\psi(\sigma_w) - 1),$$

where Q contains one place of each twist pair and ψ is the power of the auxiliary character used in T_π. Replacing the tame generator raises its ℓ-power root of unity to an ℓ-adic unit power and leaves the valuation unchanged. The scalar-extension inequality of Section 38 bounds the normalized length for L_π by sel.

For each lift $x \in \mathcal{X}$, denote by R_x^{main} its rank-r' main summand before the auxiliary twist. For the source attachment R of Section 28.1.1 this summand is θR. Write $R_\pi = R_\pi^{\mathrm{main}}$. On each other lift take its specified twisted lattice. In the semisimple block decomposition add the standard Tate lattices, in the prescribed order. Thus for $x \in \mathcal{X}$ the integral model is

$$u_x = T_x \oplus \bigoplus_{e=0}^{t'-1} \xi_e, \qquad T_x = \psi R_x^{\mathrm{main}}, \quad \xi_e = \chi_{\mathrm{cyc}}^{-e}.$$

For each $y \in Y \setminus \mathcal{X}$ choose any G_E-stable O-lattice in N_y and let u_y be its representation. Such a lattice is the O-span of the translates of an initial lattice. Compactness of the continuous image bounds all translates in one scalar multiple of the initial lattice, so this span is a full finite free O-module and is stable. All these models have the specified rank $n = r' + t'$.

The marked γ_x acts on T_x by y_x with $y_x - 1 \in O^\times$, and trivially on all ξ_e. The $T_\pi(g)$ span $\mathrm{Mat}_{r'}(O)$. The original local conditions in Section 29.1 hold on every u_y by inclusion in N_y. The auxiliary primes are away from ℓ. At places over ℓ the original type projector and the original determinant conditions are kept on the raised space, so all u_y belong to the same class $\mathcal{Q}_v$, by the inertial and de Rham comparisons there. In the chosen basis the integral main actions satisfy $T_\pi(g) = R_\pi(g)$ for every $g \in G_{E_v}$: the chosen auxiliary character is trivial on this whole group. Globally $T_\pi = \psi R_\pi$ carries the selected Selmer bound.

Use the doubled pairs (τ_j, ϕ_j) and $(\overline{\tau_j}^{-1}, \overline{\phi_j}^{-1})$, with the same variable X_j for the two pairs. The raw rank-n attachments have polarization exponent $1-n$. The normalizing avatar of Section 28.1 has product with its conjugate $\chi_{\mathrm{cyc}}^{r'}$. Consequently

$$N_y^{c_E} \simeq N_y^{\vee} \chi_{\mathrm{cyc}}^{1-t'} \qquad (y \in Y).$$

Put $\chi = \chi_{\mathrm{cyc}}^{-1}$ and $g^\sharp = (c_E g c_E^{-1})^{-1}$. On each $y \in Y$, polarization and cyclicity of trace give

$$\operatorname{tr} N_y((b^\sharp)^i g) = \chi(g)^{t'-1} \chi(b^\sharp)^{i(t'-1)} \operatorname{tr} N_y(b^i g^\sharp),$$

since the conjugate inverse of $(b^\sharp)^i g$ is $g^\sharp b^i$. At j, let H_j be the product of the normalized quotients $U/U(1)$ constructed in Section 29.2. Thus $H_j \in O[X]$ and $H_j(1) = 1$, also for the empty product. Write $P = (X-1)H_j = \sum_i P_i X^i$. With $u = (u_y)_{y \in Y}$, use $R_j(g) = \operatorname{tr}\big(u((\tau_j - 1 - X_j P(\phi_j))g)\big)$ for the tuple of trace relations on Y, and put

$$R_j^\sharp(g) = \operatorname{tr}\big(u((\tau_j^\sharp - 1 - X_j P(\phi_j^\sharp))g)\big).$$

Tame inertia gives $\chi(\tau_j^\sharp) = 1$. The auxiliary Frobenius condition gives $\chi(\phi_j^\sharp) \equiv 1 \bmod \ell^\nu$. Therefore $b_i = (\chi(\phi_j^\sharp)^{i(t'-1)} - 1)/\ell^\nu$ belongs to O, and

$$R_j^\sharp(g)_x = \chi(g)^{t'-1} \left(R_j(g^\sharp)_x - X_j \sum_i P_i b_i \ell^\nu \operatorname{tr} N_x(\phi_j^i g^\sharp) \right).$$

The terms in parentheses belong to the original trace ideal by its ordinary congruences and its ℓ^ν-multiples of traces. The coefficients are common to Y. The same trace identities hold after changing to the integral realizations u_y.

Since the auxiliary Frobenius fixes the ℓ^ν-th roots of unity, its value under each cyclotomic power ξ_e is 1 mod $\ell^\nu O$, also at the conjugate inverse. By integrality and $H_j(1) = 1$, this gives $H_j(\xi_e(\phi)) - 1 \in \ell^\nu O$ on both sides. Take $a = g_{\mathrm{tr}} = \gamma_\ell^{-1}$ from Section 29.2. Its values are $\xi_e(a) = \xi_\ell^e$. The cyclotomic image has finite index in $\mathbb{Z}_\ell^\times$ since $E/\mathbb{Q}$ is finite, so its topological generator has infinite order. The values are pairwise distinct. By Section 29.2, each $\xi_\ell^b - 1$ for $0 < b < t$ divides ℓ^h. Each difference below is a unit multiple of such a factor. Since $t' - 1 \le H$, its product satisfies

$$\prod_{e' \neq e} (\xi_e(a) - \xi_{e'}(a)) \mid C \qquad (0 \le e < t').$$

Index the tame elements δ_w by the places above M away from ℓ and by the twist support Q, and include their conjugate inverses as in Section 31.3.2. At each of these places χ_{cyc} is unramified. Thus all $\xi_e(\delta_w) = 1$, and the stated local types give $(u_y(\delta_w) - 1) o_w(u_y(\delta_w)) = 0$ for every $y \in Y$. At an old place over ℓ, Section 29.1 uses the inertia polynomials on the potentially crystalline periods. Let s' consist of the test set P_0 of Section 29.4, the reference a, and the two tame generators at each new prime pair. The triangle gives, with n_{test} counting the tests and a,

$$(|J| + n_{\mathrm{test}}) H h \le H^{22} h, \qquad n_{\mathrm{det}} := |s'| \le 2(|J| + n_{\mathrm{test}}).$$

31.3.1 The main-block covectors

In this subsection, we use the trace relations above and the congruence of Section 29.2 to select the main-block covectors for Section 32.

Put $\Lambda = O[X_j : j \in J]$ and $A = \Lambda[\mathrm{Gal}(\overline{E}/E)]$. Use one finite list g_i, $i \in I$, generating the augmentation module

$$\mathcal{M} = \mathrm{span}_O\{(u_y(g) - 1)_{y \in Y} : g \in G_E\} \subseteq \prod_{y \in Y} \mathrm{Mat}_n(O).$$

This module is finitely generated over the DVR O. Include the old tame elements and s' in the list. Let $z(g) \in O^I$ express $(u_y(g) - 1)_{y \in Y}$ in these generators. The common action kernel is the kernel of the tuple action on Y. For a covector $r \in A^I$, its value is $\sum_i r_i(g_i - 1)$. The vector satisfying the congruences module $\mathcal{N}$ requires this value to belong to the left ideal generated by the common kernel, ℓ^ν times the augmentation ideal, the paired congruences, and right multiplication of the augmentation ideal by

$$\rho_0 = \varepsilon_\pi + (1 - \varepsilon_\pi)(a - 1), \qquad \varepsilon_\pi = (y_\pi - 1)^{-1}(\gamma_\pi - 1).$$

Let ψ_0 be evaluation at $X_j = 0$ followed by the T_π action.

The spanning hypothesis makes ψ_0 onto $\mathrm{Mat}_{r'}(O)$. The zero-variable image of the covector $z(\tau) - X_j H_j(\phi) z(\phi)$ is $z(\tau)$ acting by scalar entries. The covector

$$(\gamma_\pi - 1)\big((y_\pi - 1)^{-1} z(\gamma_\pi) + (1 - \varepsilon_\pi) z(a)\big)$$

satisfies the congruences and has image $z(\gamma_\pi)$ with scalar entries, since $\psi_0(\varepsilon_\pi) = 1$. These statements about values are read modulo the common kernel. The same kernel contains the values of the cocycle vectors $z(gg') - z(g) - gz(g')$, the coordinate covectors $z(g_i) - \mathbf{e}_i$, and the covectors of all linear relations among $(u_y(g) - 1)_{y \in Y}$.

Let $L \subset O^{I \times r'}$ be the span of the individual scalar covectors of $\psi_0(\mathcal{N})$. For $w \in (K/O)^{I \times r'}$ annihilating L, $K = \mathrm{Frac}(O)$, put $c_w(g) = \sum_i z_i(g) w_i$. The displayed relations give the cocycle rule for T_π, the simultaneous linear relations, and $c_w(\tau) = c_w(\gamma_\pi) = 0$. Also $c_w(g_i) = w_i$, so this map is injective. The augmentation covectors multiplied by ℓ^ν belong to $\mathcal{N}$, so $\ell^\nu c_w = 0$. Under the identification $(K/O)[\ell^\nu] \simeq O/\ell^\nu O$, $x + O \mapsto \ell^\nu x + \ell^\nu O$, write $\widetilde{c}_w$ for the resulting cocycle in $L_\pi/\ell^\nu L_\pi$. The simultaneous relations make its evaluation factor through $\mathcal{M}/\ell^\nu \mathcal{M}$. Smith form in the finite free ambient matrix lattice W gives n such that $\mathcal{M} \cap \mathfrak{m}_O^n W \subseteq \ell^\nu \mathcal{M}$. Continuity of the original representations now gives continuity of $\widetilde{c}_w$ and c_w.

The primes J of the raised Hecke tests are chosen off the level of the selected twist, as in Sections 29.4 and 31.1. In particular $J \cup \bar{J}$ is disjoint from $Q \cup \bar{Q}$ and $T_\pi = \psi R_\pi$ is unramified there. The inertia restriction of the cocycle at these test primes is a continuous homomorphism to an ℓ-primary group. Its wild and prime-to-ℓ tame restrictions are zero. The two tame generators test the pro-ℓ tame quotients at the two places of each pair. Off the prescribed support all the tuple differences vanish on inertia. Let D_ν be the image of $w \mapsto \widetilde{c}_w$ in the relative cocycles for this tuple, and let $D_{\nu,f}$ consist of its members satisfying the crystalline relations. Inclusion in the full relative module and (82) give

$$\mathrm{length}_O(D_\nu / D_{\nu,f}) \le c_{\mathrm{crys}}(T_\pi, \nu) \le \sum_{v \mid \ell} \Delta_v(T_\pi).$$

The inverse identification $z + \ell^\nu L_\pi \mapsto \ell^{-\nu} z + L_\pi$ embeds $D_{\nu,f}$ in the normalized Selmer module with divisible coefficients, since all its defining relations are O-linear.

At this depth, over the coefficient ring O of this raised family, the annihilator has ordinary O-length at most

$$b_T = \mathrm{length}_O \mathcal{S}_O + \sum_{v \mid \ell} \Delta_v(T_\pi),$$

where $\mathcal{S}_O$ is the divisible-coefficient Selmer module with $z(\gamma_\pi) = 0$. For every choice of depth and raised primes, the scalar-extension inequality of Section 38 and the local bounds of Section 29.1.2 give

$$\frac{b_T}{v_O(\ell)} \le \mathrm{sel} + 2fr'F_\ell.$$

The finite-depth injection above is applied separately to each raised family. The common twist and its Selmer bound were fixed at the earlier cutoff M. Smith form identifies this annihilator with $\mathrm{Hom}_O(O^{I\times r'}/L, K/O)$. A free summand in the quotient would give a K/O summand of infinite length in the annihilator. For a summand $O/(\varpi^a)$ the two lengths are both a. It follows that L has full rank and colength at most b_T.

Lift an O-basis of L to scalar covectors of block covectors as follows. Each basis covector is a finite sum of individual covectors of $\psi_0(\mathcal{N})$. For the c-th position of a block covector, multiply each contributing vector satisfying the congruences on the left by a lift of the matrix unit that puts its chosen scalar covector in position c, then sum. Such lifts exist by surjectivity of ψ_0, and the left multiplication preserves the congruences on the covector. Grouping all basis covectors into $|I|$ blocks gives a matrix satisfying the congruences B_0 with

$$v_O(\det \psi_0(B_0)) = \mathrm{length}_O(O^{I\times r'}/L) \le b_T. \tag{RB2}$$

Thus the whole matrix, rather than a multiple of its determinant by a rank-dependent denominator, realizes the cocycle length bound.

The character matrices and the top coefficient. Order the generator coordinates as the disjoint blocks $I_{\mathrm{old}} \sqcup I_{\mathrm{aux}} \sqcup s'$. The first block consists of the old tame tests and the last of the test set. Adjoin a finite list I_{aux} of generators as needed to span the simultaneous augmentation tuples. The coordinate relation $z(g_i) - \mathbf{e}_i$ remains valid if a generator is repeated. Let $\eta_e : A \to \Lambda$ evaluate each group element by ξ_e, with the polynomial variables fixed. We need covectors satisfying the congruences $U_i \in \mathcal{N}$, $i \in I_{\mathrm{aux}}$, whose I_{aux}-parts are the respective unit vectors under every η_e.

First fix e. Let P_e be the image of $\mathcal{N}$ under η_e restricted to the I_{aux} coordinates. It contains $\ell^\nu \Lambda^{I_{\mathrm{aux}}}$. Put $B_e = O^{I_{\mathrm{aux}}}/(P_e \cap O^{I_{\mathrm{aux}}})$. For a functional $b : B_e \to O/\ell^\nu O$, extend it by zero on the other two constant coordinate blocks. The cocycle, simultaneous linear, coordinate and normalizer rows belong to $\mathcal{N}$ and have η_e-values in O^I. Their auxiliary parts therefore lie in $P_e \cap O^{I_{\mathrm{aux}}}$, so the extended functional kills these rows. Thus $c(g) = b(z(g))$ satisfies the cocycle and simultaneous linear relations. The Smith form argument for the simultaneous tuple module gives continuity. The coordinate relations give zero at all old tame and test points, including both tame generators of each new pair. The vectors $(g-1)((y_\pi - 1)^{-1} z(\gamma_\pi) + (1-\varepsilon_\pi) z(a))$ give

$$(\xi_e(g) - 1)((y_\pi - 1)^{-1} c(\gamma_\pi) + c(a)) = 0.$$

These are precisely the conditions for the injectivity criterion in Section 29.5, so $c = 0$. Evaluating at the coordinate generators gives $b = 0$. Since B_e is a finite-length O-module killed by ℓ^ν, finite DVR duality gives $B_e = 0$. Thus each unit vector of I_{aux} has a lift $U_{e,i} \in \mathcal{N}$ under this evaluation.

For simultaneous lifting, start with $U_i = 0$. At the next character e replace U_i by

$$U_i + \sum_{j \in I_{\mathrm{aux}}} (\mathbf{1}_{j=i} - (U_i)_j)\, U_{e,j}.$$

The coefficients multiply on the left in A, so the result still satisfies the congruences. Its evaluation on I_{aux} is the unit vector at e. At every previously processed character all coefficients of the correction evaluate to zero. Induction gives U_i as required. This construction is integral.

For each e eliminate the auxiliary block of a covector w satisfying the congruences by subtracting $\sum_{i\in I_{\mathrm{aux}}} w_i U_i$ before evaluating. This operation makes the auxiliary part zero at e. It defines a test-coordinate image in $\Lambda^{s'}$, the same image used in the next subsection. For $J_e = \{j : \mathrm{ty}(j) = e\}$, its Fitting calculation gives determinant sums $d_e(X)$ and polynomials Γ_e, ζ_e such that

$$d_e + \zeta_e = \Gamma_e \prod_{j\in J_e} X_j, \qquad \zeta_e \prod_{\mathrm{ty}(j)<e} X_j \in (X_j^2 : j \in J), \qquad v_O(\Gamma_e(0)) \le b_e + (|J_e| + n_{\mathrm{det}}) v_O(C). \tag{RB3}$$

Write each d_e as a weighted sum of test-block determinants. Each is formed from covectors that lift to the congruence module after elimination at e. Choose one such matrix B_e^{test} for each e and denote the eliminated lifts by $w_{e,l}$, $l \in s'$. In the common test block use $\sum_e q_e w_{e,l}$, with the separation element

$$q_e = \frac{C}{\prod_{e'\neq e}(\xi_e(a) - \xi_{e'}(a))} \prod_{e'\neq e} (a - \xi_{e'}(a)).$$

The quotient is integral and the evaluations are $\xi_{e'}(q_e) = C\mathbf{1}_{e=e'}$. Use the common U_i in the auxiliary block. For $w \in I_{\mathrm{old}}$, use the covector supported in that coordinate with entry $o_w(\delta_w)$. The congruences follow since $(\delta_w - 1)o_w(\delta_w)$ annihilates every system. In the order $I_{\mathrm{old}}, I_{\mathrm{aux}}, s'$, evaluation at η_e gives

$$\begin{pmatrix} \mathrm{diag}_{w\in I_{\mathrm{old}}}(o_w(1)) & 0 & 0 \\ * & I_{I_{\mathrm{aux}}} & * \\ * & 0 & CB_e^{\mathrm{test}} \end{pmatrix}.$$

The lower middle block vanishes by the elimination with the common U_i. Expanding first along the old coordinates gives determinant $c_0 \det B_e^{\mathrm{test}}$, where $c_0 = (\prod_w o_w(1)) C^{n_{\mathrm{det}}}$. In particular the separation factor is used only on the test block. This establishes the simultaneous specialization for each tuple of determinant summands.

Let B_{char} be the common matrix just constructed for one tuple of test-block determinants. In the integral group algebra set

$$p_{\mathrm{char}} = \prod_{x\in\mathcal{X}}^{\mathrm{ord}} \frac{y_x - \gamma_x}{y_x - 1}, \qquad B = (1 - p_{\mathrm{char}})B_0 + p_{\mathrm{char}} B_{\mathrm{char}},$$

using left multiplication on each entry. The denominators are units. On T_x the factor indexed by x is zero, and on each ξ_e every factor is one. Thus B satisfies the same left-module congruences, acts as B_0 on every T_x, and has the prescribed character matrices on all lifts. The trace determinant is taken on the full set Y with this same B. Sum over tuples with the product of the weights of the chosen determinants. The common trace determinant construction applied to these congruence matrices gives a single multiplier f for all $\det(1 - u_x(g))$, with specialization on $\mathcal{X}$

$$f_x(X) = C\, c_0^{t'} D_x(X) \prod_e d_e(X),$$

where D_x is the block determinant. The first factor C is the value of q_0 on the trivial character. In the common determinant construction one takes the trace of the corresponding projector

times the Ribet polynomial evaluated at ρ_0. Since $\xi_0(\rho_0) = 0$, only its constant term contributes on $\mathcal{X}$. This gives the displayed block product.

Replace the factors d_e in increasing order by $d_e + \zeta_e$. At each replacement the difference is a multiple of $\zeta_e \prod_{\mathrm{ty}(j)<e} X_j$. It is zero modulo $(X_j^2 : j \in J)$. Hence on $\mathcal{X}$ all the coefficients of proper divisors of X_J vanish and

$$d = C c_0^{t'} D_\pi(0) \prod_e \Gamma_e(0).$$

Together with (RB2)–(RB3) this gives the exact DVR bound

$$v_O(d) \le b_T + \sum_e b_e + t' \left(\sum_w v_O(o_w(1)) + 2 n_{\det} v_O(C) \right) + (|J| + 1) v_O(C). \tag{RB4}$$

31.3.2 The old tame relations in the coefficient comparison

In this subsection, we use the tame operators of Section 31.1 to bound the remaining coefficient contributions.

Fix the coefficient prime ℓ, the chosen embedding, and the prescribed twist at this embedding. Recall $C_G = A_{\mathrm{aut}}$ and $v = v_\ell(C_G)$. Use all the places above M prime to ℓ, together with the twist support Q. At an old place above p_i, the normalized upper representations have finite inertia and every inertia eigenvalue has order dividing $p_i^a - 1$ for some $1 \le a \le r$. The conjugate-inverse element has the same bound, by the marked local types on the conjugate place. The integer $p_i^a - 1$ divides the order of the corresponding general linear group in C_G. Finite order in characteristic zero implies semisimplicity. Thus, on every upper representation and for either choice of inertia element, $X^{C_G} - 1$ annihilates its action. Take $o_w(X) = \sum_{j=0}^{C_G - 1} X^j$, so $o_w(1) = C_G$.

The old local projector is imposed on the whole raised space. At a twist place the prescribed two determinant types give the quadratic annihilator of Section 28.3 on every raised system. Choose the tame generator σ_w used in the ramification bound. For $g = \sigma_w$ or its conjugate inverse the quadratic relation permits $o_g(X) = \psi(g) - X$. The norm-one character satisfies $\psi((cgc^{-1})^{-1}) = \psi(g)$. Hence the two copies give the same valuation $v_\ell(\psi(\sigma_w) - 1)$. These annihilations pass to all the fixed integral models by their injections into the representations over the coefficient field. There are at most $4f$ old places above the two rational level primes and at most twice that many old inertia elements in the list. Thus on this same family

$$\sum_g v_\ell(o_g(1)) \le 2\mathrm{ram} + 8fv.$$

Places above ℓ omitted here are handled by the crystalline colength bound. The auxiliary triangle uses separate places outside both the old and the twist level.

Divide (RB4) by $v_O(\ell)$. In normalized units,

$$\frac{b_T}{v_O(\ell)} = \mathrm{loss}_T + \mathrm{sel}_T, \qquad \frac{b_e}{v_O(\ell)} = \mathrm{loss}_e + \mathrm{odd}_e.$$

The local defect estimate bounds the normalized sum of all losses by

$$\sum_{v|\ell} [E_v : \mathbf{Q}_\ell](r' + t'/t) F_\ell = 2f(r' + t'/t) F_\ell \le 4 f r' F_\ell = r' \mathrm{loc}_\ell.$$

Here $r' \geq 1$ and $t' \leq t$. The normalized sel_T is at most sel; the normalized odd_e at most $v_\ell(X_{\mathrm{odd}})$; and the normalized old tame sum at most $2\mathrm{ram} + 8fv$. Finally the two triangle inequalities give

$$(2t' n_{\mathrm{det}} + |J| + 1)Hh \leq (4t+1)(|J| + n_{\mathrm{test}})Hh + Hh \leq ((4t+1)H^{22} + H)v.$$

Adding $2v$ to (RB4) proves (RB1). The next two subsections establish the character Fitting estimate and construct f on Y. The annihilation of f by the detecting determinants modulo the trace ideal is proved on Y, and the coefficient bound is computed by the block evaluation at π. The quantitative bound uses the local defect bounds of Section 29.1 and the Selmer bound on the prescribed representation.

32 Fitting ideals and trace determinant specialization

In this section, we specialize the common coefficient of Section 31 on the chosen characters. The resulting Fitting inclusion and trace determinant give the finite coefficient used with the norm tests in Section 38.

32.1 The Fitting element for the ordered characters

In this subsection, the specialization on a character summand of Section 31.3 reduces to an inclusion between Fitting ideals. We derive it by eliminating the unit relations and applying the bounds on the paired evaluations of Section 29.5.

Fix the character ξ_e in the common Ribet construction and write $I = J_e$, $s = s'$, $R = O[X_j : j \in J]$, $p = \ell^\nu$ and $F = R^s$. Read the vectors satisfying the congruences on ξ_e, drop the old tame coordinates, and eliminate the auxiliary spanning coordinates by the unit covectors. Write $N \subset F$ for this image plus

$$\left((X_j^2 : j \in J) : \prod_{\mathrm{ty}(j) < e} X_j \right) F.$$

Write $\Phi : O^{I_{\mathrm{coord}}} \to F$ for the inclusion of constant coordinate covectors followed by this evaluated elimination and projection to the test block. Here I_{coord} is the full list of the common augmentation generators.

Write $y(g) = \Phi(z(g))$. The cocycle vectors show that the forms on $O^{I_{\mathrm{coord}}}$ modulo p which annihilate $\Phi^{-1}N$ give the character cocycles by z. The coordinate relations $z(g_i) - \mathbf{e}_i$ show that z recovers each such form from its cocycle. On the test positions $g \in s$ one has $y(g) \equiv \mathbf{e}_g \pmod N$. The two congruences are $y(\tau_j^{\pm}) \equiv X_j y(\phi_j^{\pm}) \pmod N$, because $H_j(\xi_e(\phi_j^{\pm})) \equiv 1 \pmod p$ and $pF \subset N$. For an earlier $i < j$ with $j \in I$, either $i \in I$ or $\mathrm{ty}(i) < e$; in the latter case $X_i F \subset N$ by the colon ideal.

For each R-submodule L containing N, put

$$A_L = O^{I_{\mathrm{coord}}}/\Phi^{-1}L, \qquad Z_L = \mathrm{Hom}_O(A_L, O/(p)).$$

The finite module A_L is killed by p. Via cocycle evaluation the dual Z_L is exactly the corresponding module of character cocycles; this uses the coordinate relations just noted. We use the following determinant operation throughout. For $w \in F$, if $aw \in L$ for $a \in R$, then

$a\,\mathrm{Fitt}_0(F/(L+Rw)) \subseteq \mathrm{Fitt}_0(F/L)$. Expand a determinant of covectors in $L+Rw$. Terms with two or more covectors proportional to w are zero, and in each remaining term the factor a can be placed in the w-covector if it is present. The Fitting ideal here is generated by all determinants of $|s|$ covectors, so this proves the inclusion.

For $b \in Z_L$ put $c_b(g) = b([z(g)])$ in the notation of A_L. An old tame generator $\delta = g_i$ satisfies $\Phi(z(\delta) - \mathbf{e}_i) \in N$ and $\Phi(\mathbf{e}_i) = 0$, so $c_b(\delta) = 0$. Write $a = g_{\mathrm{tr}}$. The covector $(g-1)v_0$ of Section 29.5 satisfies the congruences. Its evaluation at ξ_e is the constant covector

$$m_g = (\xi_e(g) - 1)(c_\pi z(\gamma_\pi) + z(a)), \qquad \xi_e(\gamma_\pi) = 1.$$

Deletion of the old coordinates and elimination of the auxiliary ones gives $\Phi(m_g) \in N \subseteq L$. Applying b gives $(\xi_e(g) - 1)(c_\pi c_b(\gamma_\pi) + c_b(a)) = 0$. Thus Z_N and every $Z_L \subseteq Z_N$ satisfy the normalization and the old tame conditions of Section 29.5. Take $Z = Z_N$ there. Let $A \subset G$ be the union of the inertia groups at $v \mid \ell$ if $e = 0$, and of $D_v \cap \ker \chi_{\mathrm{cyc}}$ at $v \mid \ell$ if $e > 0$. Then $Z^0 = Z_N \cap Z_0$ is the submodule that vanishes on A. The O-submodule of A_N spanned by the $z(a)$, $a \in A$, has length $b_A = \mathrm{length}(Z_N/Z^0)$, by finite DVR duality. In fact $O/(p)$ identifies with the p-torsion of K/O, and duality with K/O is exact on finite O-modules. Choose cyclic direct summands of that submodule, and lift their generators to constant vectors w_i in the preimage of the submodule in $O^{I_{\mathrm{coord}}}$. If the summand has length a_i, $\varpi^{a_i}\Phi(w_i) \in N$, where ϖ is a uniformizer. Set $K_1 = N + \sum_i R\Phi(w_i)$. Applying the determinant operation to the successive augmentations gives

$$\varpi^{b_A}\,\mathrm{Fitt}_0(F/K_1) \subseteq \mathrm{Fitt}_0(F/N).$$

The generators $y(a)$, $a \in A$, belong to K_1. Consequently $Z_L \subseteq Z^0$ whenever $L \supset K_1$.

For $j \in I$ set $v_j = y(\phi_j^+) - y(\phi_j^-)$. Starting at K_1, add Rv_j in increasing order of j. Immediately before adding v_j, call the current module L. Every $c \in Z_L$ vanishes on the differences of the earlier tame generators: for $i \in I$ this follows from $v_i \in L$ and the two congruences, and for $i \notin I$ from $X_i F \subset N$. It also vanishes on the earlier Frobenius differences of type e, by $v_i \in L$. The triangle condition therefore gives $C(c(\tau_j^+) - c(\tau_j^-)) = 0$. The finite dual Z_L separates classes of A_L. Applied to the constant vector $C(z(\tau_j^+) - z(\tau_j^-))$, this gives

$$C(y(\tau_j^+) - y(\tau_j^-)) \in L, \qquad CX_j v_j \in L.$$

After all augmentations, let T be the resulting module. The determinant operation gives

$$C^{|I|}\left(\prod_{j\in I} X_j\right)\mathrm{Fitt}_0(F/T) \subseteq \mathrm{Fitt}_0(F/K_1).$$

Let $L_0 \subset O^s$ be the image of T at $X = 0$. It contains pO^s. The image of Φ at zero surjects onto O^s/L_0, since its values on $z(g)$, $g \in s$, are the test basis modulo N. Thus the annihilator $Z_1 = Z_{T+(X_j : j\in J)F}$ is the dual of O^s/L_0. Every member of Z_1 is in Z^0, vanishes at both tame generators for each j by the two congruences specialized at zero, and vanishes on the Frobenius differences for $j \in I$ by $v_j \in T$.

Thus $Z_1 \subseteq S$ for the module S of Section 29.5, and $Z_1/Z_1[C]$ injects into $S/S[C]$. Also $b_A \le \mathrm{length}_O(Z/Z_0)$, by the inclusion $Z_N \subseteq Z$. Equation (84) therefore gives

$$b_A + \mathrm{length}_O(Z_1/Z_1[C]) \le c_{\mathrm{crys}}(O(-e), \nu) + v_{\mathrm{ord}}(X_{\mathrm{odd}}) = b_e.$$

Since $Z_1 \subset (O/(p))^s$ via the test basis, $\operatorname{length}(Z_1[C]) \le |s| v_O(C)$. Choose a basis of L_0 and lift its elements to T. Their determinant $d_T \in \operatorname{Fitt}_0(F/T)$ satisfies

$$v_O(d_T(0)) = \operatorname{length}(O^s/L_0) = \operatorname{length}(Z_1) \le b_e - b_A + |s| v_O(C).$$

Take $\Gamma_e = \varpi^{b_A} C^{|I|} d_T$. The two determinant inclusions prove

$$\Gamma_e \prod_{j \in I} X_j \in \operatorname{Fitt}_0(F/N), \qquad v_O(\Gamma_e(0)) \le b_e + (|I| + |s|) v_O(C).$$

Adding the colon-ideal covectors to the original character image enlarges its Fitting ideal by at most that colon ideal, by expansion of the determinant of the covectors. This recovers the determinant sum d_e, the correction ζ_e and the claimed ordered character multiplier, with the local and paired tests on this same cocycle module.

32.2 The trace determinant under the Frobenius relations

In this subsection, we use the trace pairing on the algebra of Section 31.3 to obtain a determinant over its coefficient ring. Its annihilation property holds on the full family of representations and permits specialization at the chosen system.

Use the coefficient DVR O fixed for this raised family in Section 31.3. With $\mathbb{T}$ acting on the full set Y as in Section 30.3, put

$$\mathbb{T}_O = \mathbb{T} \cap O^Y, \qquad S = \mathbb{T}_O[X_j : j \in J] \subseteq L = \prod_{y \in Y} O[X_j : j \in J].$$

The characteristic coefficients of each tuple $u(g)$ belong to $\mathbb{T}$ by the Hecke comparison and to O^Y by the chosen integral models. All local relation polynomials have coefficients in O. The resulting identities map into $\mathbb{T}[X_j : j \in J]$ by inclusion. In this trace calculation write $G = G_E$, and extend the tuple of representations to $u : S[G] \to \operatorname{Mat}_n(L)$. Define the scalar ideal and its trace annihilator by

$$\mathcal{J} = \left(\ell^\nu \operatorname{tr} u(g) : g \in G\right)_S + \left(R_j(g) : j \in J,\ g \in G\right)_S,$$

$$\mathfrak{a} = \{w \in S[G] : \operatorname{tr}(u(wv)) \in \mathcal{J} \text{ for every } v \in S[G]\}.$$

This is a two-sided ideal by cyclicity of the trace, and it contains $\ker u$. Write $y_i = g_i - 1$, and let $\mathcal{I} = \sum_i S[G] y_i$ be the left ideal they generate.

If B is a matrix of the previously constructed vectors satisfying the congruences, the trace relations imply

$$By \equiv B_1 y\, \rho_0 \pmod{\mathfrak{a}}$$

for a matrix B_1 over $S[G]$. Indeed the common kernel, the $p = \ell^\nu$ augmentation multiples, and the generators of the paired congruences all lie in $\mathfrak{a}$. Any multiple on the left of the remaining augmentation relation can be written modulo $\ker u$ as a combination of the y_i followed by ρ_0, by the coordinates of the full tuple in $\mathcal{M}$.

The characteristic coefficients of $u(g)$ lie in S. The same holds for $u(w)$, $w \in S[G]$, and for all flattened block matrices over this image, also with polynomial coefficients. For square matrices $A_1, \dots, A_h$ of the same size over a commutative ring, use Amitsur's identity [4], in the primitive-word form of [61, p. 301, (6)]. As formal power series in t,

$$\det(1 - t \textstyle\sum_i A_i) = \prod_{[v] \text{ primitive}} \det(1 - t^{|v|} A_v),$$

where the product is over primitive cyclic words and A_v is the ordered product of the matrices of a representative word.

Only words of length at most k contribute to the coefficient of t^k. It is an integral polynomial in the characteristic coefficients of the word matrices. Applying it to scalar multiples of $u(g)$ proves the assertion for $u(w)$. For block matrices, split into individual blocks $E_{ij} \otimes u(w_{ij})$. Every product is zero or $E_{ab} \otimes u(w)$. If $a \neq b$, the matrix $1 - tE_{ab} \otimes u(w)$ is block unitriangular after ordering the blocks with a before b. Its determinant is one over the coefficient ring. If $a = b$ its only contributing diagonal block is $u(w)$. This proves coefficient integrality in S and in $S[T]$.

Put

$$B_T = B - TB_1, \qquad P(T) = \det(u(B_T)) \in S[T].$$

Here and below $u(B_T)$ is flattened into one ordinary matrix. Cayley–Hamilton writes its adjugate as a polynomial in $u(B_T)$ with coefficients in $S[T]$, hence gives a matrix B'_T over $S[G][T]$ such that

$$B'_T B_T \equiv P(T)\mathbf{1} \pmod{\ker u}$$

coefficientwise. View $S[G]/\mathfrak{a}$ as an $S[G][T]$-module by left multiplication from $S[G]$ and right multiplication by ρ_0 for T. The two actions commute by associativity. The covector relation says that B_T annihilates y in this module, so the adjugate relation gives

$$y_i P(\rho_0) \in \mathfrak{a}, \qquad \mathcal{I}P(\rho_0) \subset \mathfrak{a}.$$

Taking $T = 0$ in the definition yields $P(0) = \det(u(B))$.

Let p_{char} be the ordered product of the factors $(y_x - \gamma_x)/(y_x - 1)$ over the lift systems x, as in Section 31.3. On each lift it acts by zero on the main summand and by one on the characters. The separation element q_0 acts on ξ_e by $C\mathbf{1}_{e=0}$. Thus $\rho_1 = q_0 p_{\mathrm{char}}$ acts on ξ_0 by $\mu_0 = C$ and vanishes on all other summands of each lift. Now put $f_B = \mathrm{tr}(u(\rho_1 P(\rho_0)))$. The trace belongs to S, by the characteristic coefficient result applied to the group-algebra element inside the trace. For $w \in \mathcal{I}$ put $b = \det(u(w)) \in S$. The adjugate of $u(w)$ is the image of an element $w' \in S[G]$, again by Cayley–Hamilton with coefficients in S.

Thus $u(w'w) = b\mathbf{1}$, and

$$bf_B = \mathrm{tr}(u(\rho_1\, w' w P(\rho_0))) \in \mathcal{J},$$

since $w'wP(\rho_0) \in \mathfrak{a}$. At a lift system x, one has $\xi_0(\rho_0) = 0$, and $u_x(\rho_1)$ is zero off that character. Hence

$$f_B(x) = \mu_0 P(0)(x) = \mu_0 \det(T_x(B)) \prod_{e=0}^{t'-1} \det(\xi_e(B)).$$

Every determinant $\det(1 - u(g))$ is covered by the same f_B, because $1 - g$ belongs to the left ideal $\mathcal{I}$ modulo $\ker u$. Combining the matrices satisfying the congruences and summing their integral weights gives the common f used above. The argument does not introduce factorial denominators or require changing the coefficient embedding or the marked system set.

32.3 The highest squarefree coefficient and the original test space

In this subsection, the polynomial of Section 31.3 must be evaluated on the space at the original level. We use the spherical root construction of Section 31.2 to track the top squarefree coefficient under this restriction.

The type projection and the order of the congruence. Write r' for the source rank of this descent pair and $G_c = \prod_{w\in S_M} \mathrm{GL}_{r'}(k_w)$. For the irreducible finite types λ with the prescribed parameters the coefficient of $g \in G_c$ in the projector is

$$|G_c|^{-1} \sum_{\lambda} \dim(\lambda)\ \mathrm{tr}\,\lambda(g^{-1}).$$

Each trace is a sum of roots of unity. The residue action of g is right translation by an integral lift of the first block of g, multiplied by $\prod_{w\in S_M} \psi_w(\det g_w)$. The scalar is a root of unity, and right translation by the lift permutes the points integral over ℓ.

The lift is supported at the specified old places and their conjugates. The block embeddings in the groups defining C_G show $|G_c| \mid C_G$. Hence $C_G Q y$ is integral for integral y, also when ℓ is an old prime. This conclusion holds in the original level with the type condition imposed: one takes the given level and character space at the old and twist places, its Q-fixed subspace, and the spherical invariants outside those places. Raising auxiliary level changes only the last invariance condition.

Fix the chosen original system x_0. There are finitely many systems at the original level in the fixed type space. For each original system x_k outside the weak lifts, Section 29.2 gives, by definition, an element $g_k \in G_E$ with $\lambda_k = \det(1 - N_{x_k}(g_k)) \neq 0$. These values are integral: the powers of the continuous Galois matrix are bounded by compactness, so its eigenvalues have absolute value at most one. Put $n_0 = \prod_k \lambda_k$, with the empty product equal to one.

The triangle construction first gives its index set I, with a choice of primes and root chains at every positive depth for the weak original systems congruent to x_0. Put

$$n_1 = C_G^2 n_0, \qquad n_2 = 2n! n_1^{|I|+1}.$$

Choose a positive integer ν with $v_\ell(n_2) \le \nu$. Only now select the actual depth-ν primes and the common raised trace algebra. At each raised system use $\beta_k(x) = \det(1 - N_x(g_k))$ with the same g_k. The scalar $\eta = \ell^\nu/(2n!)$ satisfies

$$v_\ell(\eta) = \nu - v_\ell(2n!) \ge (|I| + 1)v_\ell(n_1).$$

On passing to the coefficient DVR O of any resulting raised family, the valuation of $\eta/n_1^{|I|+1}$ is

$$v_O(\ell)(\nu - v_\ell(2n!) - (|I| + 1)v_\ell(n_1)) \ge 0.$$

Thus $\eta \in n_1^{|I|+1} O$ at the depth fixed from the original systems and their index set.

Use the full raised trace algebra $\mathbb{T}$ acting on its integral lattice L. The raised places are indexed by I. Write $\mathcal{N}_j, \partial_j$ for the local norm and derivative from the local comparison at j, and $\mathcal{N} = \prod_{j\in I} \mathcal{N}_j$. The groups at different places commute, as do their local operator algebras. The original-level space is the fixed space for the product $H = \prod_j G_j$ of the finite residue groups. Use $c = C_G$ for the common multiple of the ambient compact stabilizer orders.

Let H_j be the polynomial of Section 31.3 and put $P_j(X) = (X - 1)H_j(X)$. For every raised system $x \in Y$ and $g \in G_E$, put

$$t_{j,g}(x) = \mathrm{tr}\big((N_x(\tau_j) - 1)N_x(g)\big), \qquad s_{j,g}(x) = \mathrm{tr}\big(P_j(N_x(\phi_j))N_x(g)\big).$$

Under the identification $J = I,\ X_j = \alpha_j$, extend the scalar ideal from S to $\mathcal{J}_{\mathbb{T}} = \mathcal{J}\,\mathbb{T}[\alpha_j : j \in I]$ by the coefficient inclusion $\mathbb{T}_O \hookrightarrow \mathbb{T}$. The generators are $\ell^\nu \,\mathrm{tr}\, N(g)$ and $t_{j,g} - \alpha_j s_{j,g}$. In this

calculation reductions are modulo $\eta = \ell^{\nu}/(2n!)$ in the coefficient valuation ring. Section 31.1.1 gives on the full raised lattice

$$\mathcal{N}_j t_{j,g} = 0, \qquad \partial_j t_{j,g} \equiv \mathcal{N}_j s_{j,g} \pmod{\eta}.$$

For the second identity the local operator uses $(X - q_j^{t'-1})H_j(X)$, where q_j is the residue norm. The replacement by P_j in the trace follows from $q_j \equiv 1 \bmod \ell^{\nu}$. Let $\delta' \in \mathbb{T}$ specify the original systems on which the spherical root chain is required.

Section 31.1.2 reduces the construction of $Z_{j,h}$ to the relative trace and spherical annihilation verified below. Its conclusions, for $h \in G_j$, are

$$c\partial_j(\rho(h) - 1) \equiv \mathcal{N}_j Z_{j,h} \pmod{\eta}, \qquad \delta' \mathcal{N} Z_{j,h} = 0.$$

For the local and full relative traces (cf. [34]), take K to be the ambient standard maximal finite compact. Reduction at the original auxiliary primes gives $\pi : K \to H$, where $H = \prod_{j \in I} G_j$ uses the full rank n at each place. Put $K^1 = \ker \pi$. On $\Gamma \backslash G_f / K^1$, with Γ the rational left group, let D be H or a line stabilizer in one factor. The stabilizer at $x = \Gamma g K^1$ is

$$D_x = D \cap \pi(K \cap g^{-1}\Gamma g).$$

Indeed a lift in K of a residue element fixing x differs from a rational stabilizer by an element of K^1. Its order therefore divides the order of the ambient compact rational stabilizer. At an old split place over the odd prime p_i its integral action injects under reduction into $\mathrm{GL}_n(\mathbb{F}_{p_i})$: the principal congruence kernel in $\mathrm{GL}_n(\mathbb{Z}_{p_i})$ is torsion-free. Block inclusion for $n = r' + t' \le rt$ and the definition of C_G in Section 30.1 give the asserted divisibility.

On each D-orbit define an integer-valued weight m_D equal to $c/|D_x|$ at one point and zero elsewhere. Each orbit point occurs $|D_x|$ times under translation by D, so $\sum_{d \in D} m_D(xd^{-1}) = c$. Multiplication by the pullback of this weight commutes with every right K^1-translation. The old and twist compact factors, including conjugate values, lie in K^1 since their places are off the auxiliary support. The multiplier therefore respects the original projection Q, both character equations and the auxiliary principal levels. It preserves L because it is an integer scalar at each gauge point. For this endomorphism Ψ_D the orbit identity gives

$$\sum_{d \in D} \rho(d^{-1}) \Psi_D \rho(d) = c\, 1_L.$$

In particular $\mathcal{N}\Psi_H v = cv$ on the original-level invariants. Use D equal to the line stabilizer in the local derivative calculation.

For the spherical annihilation at j, write $B_j(1) = p(aU)L_J(1)$ for the local operator, so $L_J(1)y$ is I_1-fixed. For an original spherical vector x on a system supported by δ', the full norm $\mathcal{N}$ is self-adjoint and acts on x by $|H|$. Therefore

$$\langle x, \mathcal{N} B_j(1) y \rangle = |H| \langle x, B_j(1) y \rangle = 0$$

by the adjoint orthogonality in Section 31.2. The image of $S_j = \mathcal{N} B_j(1)$ lies in the full original-level invariants. For every original-level w, the vector $\delta' w$ is a sum of supported vectors, and hence $\langle \delta' w, S_j y \rangle = 0$. The local operators defining S_j commute with the diagonal trace operator δ'. Taking $w = S_j y$ and replacing the second variable y by $\delta' y$ gives $\|\delta' S_j y\|^2 = 0$. This proves $\delta' \mathcal{N} B_j(1) = 0$ on the characteristic-zero space and on the integral lattice. Together with the local Higman operator it gives the two stated properties of $Z_{j,h}$.

Put $\Delta = c^2\delta'$ and

$$\widetilde{\mathcal{N}} = \prod_{j\in I}(\mathcal{N}_j + \alpha_j\partial_j), \qquad \widetilde{\mathcal{N}}_S = [\alpha_S]\widetilde{\mathcal{N}}, \qquad \alpha_S = \prod_{j\in S}\alpha_j.$$

In the polynomial operator ring modulo η and $\alpha_j^2 = 0$, the two local trace equations give $\widetilde{\mathcal{N}}(t_{j,g} - \alpha_j s_{j,g}) = 0$. Coefficients in $\mathbb{T}$ commute with the local operators. Hence $[\alpha_I](\widetilde{\mathcal{N}}B) \equiv 0 \pmod{\eta}$ for every $B \in \mathcal{J}_{\mathbb{T}}$. The quotient by α_j^2 does not change any squarefree coefficient.

For $i \geq |S|$ there is an integral $Y_{S,i}$ such that

$$\Delta^i\widetilde{\mathcal{N}}_S \equiv Y_{S,i}\mathcal{N} \pmod{\eta}, \qquad Y_{\varnothing,i} = \Delta^i.$$

Indeed the empty case is $\widetilde{\mathcal{N}}_\varnothing = \mathcal{N}$. For $S \neq \varnothing$ put $X = c\delta'\Delta^{i-1}\widetilde{\mathcal{N}}_S$. For a place $j \notin S$, $X(\rho(h) - 1) = 0$ for $h \in G_j$, since $\mathcal{N}_j$ is a factor. For $j \in S$, apply the local derivative congruence at j and commute δ' to the right of $\widetilde{\mathcal{N}}_{S\setminus\{j\}}$. By induction, $X(\rho(h) - 1)$ reduces to $Y_{S\setminus\{j\},i-1}\mathcal{N}\delta' Z_{j,h} = 0$. Thus $X\rho(h) \equiv X$ for all $h \in H$. Multiplying the full Higman identity on the left by X gives $cX \equiv X\Psi_H\mathcal{N}$. Since $cX = \Delta^i\widetilde{\mathcal{N}}_S$, take $Y_{S,i} = X\Psi_H$.

Let $D(\alpha) = f((\alpha_j)_{j\in I})$, with f the common trace multiplier of Section 31.3 for the same raised space and distinguished lift $\pi = x_0$. The trace determinant gives $\beta_k D \in \mathcal{J}_{\mathbb{T}}$ for every k. On each lift system x, the ordered factorization modulo $(\alpha_j^2 : j \in I)$ gives

$$D(x) \equiv [\alpha_I]D(x)\,\alpha_I, \qquad [\alpha_I]D(x_0) = d.$$

Write $D_S = [\alpha_S]D$, and put $i = |I|$, $b = i + 1$. Let $A \in \mathbb{T}$ be the indicator of the residual congruence class of the chosen lift. It belongs to the integral diagonal algebra. To construct it, for each system outside this class choose an integral characteristic-polynomial coefficient separating it modulo the maximal ideal from the chosen lift. Subtract the value at the outside system and divide by the unit value at the chosen lift.

The product T of these factors is zero outside the class and is a unit on the class. The polynomial

$$1 - \prod_{x \text{ in the class}} (1 - X/T(x))$$

evaluated at T gives the indicator. All scalars here are in the coefficient valuation ring. Put

$$\delta_0 = \prod_k(\lambda_k - \beta_k), \qquad n_0 = \prod_k \lambda_k, \qquad n_1 = c^2 n_0, \qquad \delta' = A\delta_0.$$

Here the β_k vanish on the lift systems, the $\lambda_k \neq 0$ are the detecting values at the non-weak original systems, and δ_0 vanishes on those systems. Consequently an original system with $\delta'(x) \neq 0$ is weak and residually congruent to x_0. For each j, its complementary spherical factor divides H_j by the product construction in Section 29.2. This verifies the root-chain hypothesis used in $\delta'\mathcal{N}B_j(1) = 0$. The trace relation $\beta_k D \in \mathcal{J}_{\mathbb{T}}$ gives $(\delta_0 - n_0)D \in \mathcal{J}_{\mathbb{T}}$ by expanding the product. The first-order lower coefficient vanishing means $D_S = 0$ on the lift systems when $S \neq I$. The original-level choice above gives $2n!\, n_1^b \mid \ell^\nu$ in the coefficient valuation ring before the raised prime selection.

Let l, v be original-level vectors with $d_0 l \in L$ for an integral scalar d_0. Assume $\delta_0 v = n_0 v$, $D_S v = 0$ for $S \neq I$, and $A\delta_0 l = n_0 A v$. On the raised lattice of the prescribed old type take the coefficient

$$G = [\alpha_I](\widetilde{\mathcal{N}}D) = \sum_{S\subseteq I} \widetilde{\mathcal{N}}_{I\setminus S} D_S \in \mathrm{End}_O(L).$$

Each $D_S \in \mathbb{T}$ commutes with the local operators. The equality also holds after reduction modulo η. Since $\Delta^b - n_1^b A^b$ is divisible by $\delta_0 - n_0$, the vanishing modulo the congruence ideal and the factorization of $\widetilde{\mathcal{N}_S}$ yield the following congruence. In each summand write $\Delta^b \widetilde{\mathcal{N}}_{I\setminus S} D_S = (\Delta^i \widetilde{\mathcal{N}}_{I\setminus S})\Delta D_S$ before applying the factorization on the left. Thus

$$n_1^b A^b G \equiv \Delta^b G \equiv \sum_{S\subseteq I} Y_{I\setminus S,i} \mathcal{N} \Delta D_S \pmod{\eta}.$$

Apply this congruence to $u = \Psi_H(d_0 l) \in L$, so $\mathcal{N}u = cd_0 l$. The term for S on the right is

$$n_1 cd_0\, Y_{I\setminus S,i} A D_S v.$$

Only $S = I$ survives. Since $Y_{\varnothing,i} = \Delta^i$, the sum is $n_1^b cd_0 A^b D_I v$. Write $L = L_{\text{raised}}$. Since $A^b G u \in L$ and

$$n_1^b (cd_0 A^b D_I v - A^b G u) \in \eta L \subseteq n_1^b L,$$

cancellation in the ambient vector space gives $cd_0 A^b D_I v \in L$. The functions A, D_I act by scalars on each $V_{\text{orig}} \cap W_x$. Hence this vector belongs to $V_{\text{orig}} \cap L = L_{\text{orig}}$ by Section 30.3. Thus the exact clearing function before the type denominator is

$$s = A^b D_I = A D_I.$$

At the chosen lift system x_0 it has value $D_I(x_0)$.

For the fixed descent pair at the original level, let Z be the span of a collection of good lift eigenvectors as above, each of a fixed old type and cofinite Hecke system. Choose a good ambient type idempotent e_0 whose image contains their types. The distinguished system x_0 above is the system of one of these generators. Taking all the generators gives the full good lift space. At a split place w outside the descent level and its conjugate, write $q = \mathrm{N}w$ and let ν_w be the source splitting-character value of Section 28.1.1. With $T_{w,a}$ the spherical double-coset operators on the original space and $T_{w,0} = 1$, set

$$K_w(Y) = \sum_{a=0}^{n} (-Y)^a q^{\binom{a}{2}} T_{w,a}, \qquad X_w = (\nu_w q^{t'-1})^{-1}.$$

On a simultaneous eigenvector at w with monic polynomial H, this acts as $Y^n H(Y^{-1})$. The false-lift subspaces are

$$F_w = \operatorname{ran} e_0 \cap Z^{\perp} \cap \ker K_w(X_w).$$

Let Q be the self-adjoint projection to the fixed lift type used for x_0, with $\operatorname{ran} Q \subseteq \operatorname{ran} e_0$.

Each lift generator of Z has a fixed old type and is an outside-level Hecke eigenvector by Section 28.1. Orthogonality of the type idempotents makes Q zero or the identity on each generator. Thus QZ is spanned by lift vectors of the selected type, and all its systems belong to $\mathcal{X}$ on inclusion in the raised space. Let P_x be the orthogonal projection to the original summand of a cofinite system x. The spherical operators away from the level form a commuting algebra closed under adjoints. Simultaneous diagonalization and grouping the eigenspaces by their cofinite systems give these projections. Each P_x commutes with the old type projections and preserves Z and $Z^{\perp}$, since the generators of Z are outside-level eigenvectors. We use P_x only over characteristic zero. For an original integral test $y \perp F_w$ for all w, put $l = Qy$, $v = Q\pi_Z y$, with π_Z the orthogonal projection. On $\mathcal{X}$ all β_k vanish and $D_S = 0$ for $S \neq I$. Hence

$$\delta_0 v = n_0 v, \qquad D_S v = 0 \quad (S \neq I).$$

Choose a split $w \nmid \ell$ outside the levels, the exceptional places of a basis of each original isotypic summand (Section 30.3), and those of the Galois comparison (Section 28.2). At this w the monic polynomial $H_{x,w}$ is common to the entire summand of x. For every weak x, $\det(1-N_x(\mathrm{Fr}_w)) = 0$. The normalizing avatar takes the value ν_w^{-1} at geometric Frobenius. Section 29.2 therefore gives

$$H_{x,w}(\nu_w q^j) = 0 \quad (0 \le j < t'), \qquad K_w(X_w)|_x = X_w^n H_{x,w}(\nu_w q^{t'-1}) = 0.$$

For each weak x, the vector $v_x = P_x Q(y - \pi_Z y)$ lies in $\operatorname{ran} Q \subseteq \operatorname{ran} e_0$, in $Z^\perp$ and in $\ker K_w(X_w)$. Thus $v_x \in F_w$. Since $QP_x v_x = v_x$,

$$\|v_x\|^2 = \langle y - \pi_Z y, QP_x v_x \rangle = \langle y - \pi_Z y, v_x \rangle = 0.$$

Here $y \perp F_w$ and $\pi_Z y \in Z$. On the other systems δ_0 vanishes by the detecting factors. Thus $\delta_0(l - v) = 0$ and $A\delta_0 l = n_0 A v$.

The integral realization of Q at its base level uses $cQy \in L$ for integral y. Taking $d_0 = c$ gives the following bound.

Proposition 32.1 (Clearing on the original test space)**.** *The operator $c^2 sQ\pi_Z$ is integral on the original integral tests perpendicular to F_w for all w. Its coefficient at the selected system satisfies*

$$v_\ell(c^2 s(x_0)) \le \mathrm{sel} + 2t'\mathrm{ram} + r'\mathrm{loc}_\ell + \mathrm{err}_\ell.$$

Here r' is the rank of the source, $t' = n - r'$, and the inequality is the quantitative first-order bound of Section 31.3 on $c^2 D_I(x_0)$. The coefficient extraction introduces no further valuation loss. The simultaneous clearing argument in Section 42.1.3 combines the restrictions to the original space for all basis generators and gives this allowance at every basis vector.

Part V
Period comparison and the lower determinant bound

33 Period vanishing and the integral comparison lattice

In this section, we use the finite support of Section 18.1 to determine the terms that survive in the comparison periods. The integral first factors and their common scalar give the lattice used in Section 34.

33.1 Vanishing of the marked comparison periods

In this subsection, the finite support of the data in Section 18.1 forces certain kernels to vanish after integration. We establish the vanishing with the original local characters and measures before using it in Section 34.1.

Use the marked pair $W, V = I_n$, the original inclusion $\iota_0 : W \hookrightarrow V$ on first indices, and the split sets S_M, Q of chosen and twist places. The places in S_M are the divisors of π_M, and every residue character on the exact support Q is nontrivial. These sets and their conjugates are disjoint in the construction. Let $S = S_M \cup Q$. At $w \in S$ write k_w for its residue field and χ_w

for the exact residue character in the descent pair. Fix the false lift Ξ: it belongs to the good idempotent, is perpendicular to the good lift image at this level, and is killed by the marked Hecke polynomial at a split place off $S \cup \bar{S}$.

At a chosen place and a nonzero residue covector λ of length $r = \dim W$, put

$$H_\lambda = \{1 + u\lambda : \det(1 + u\lambda) \neq 0\}, \qquad \mathcal{A}_\lambda = \sum_{s \in H_\lambda} \chi_w(\det s)^{-1} s.$$

In a basis with $\lambda = (1, 0, \ldots, 0)$, H_λ is the product of the lower unipotent radical for the $(1, r-1)$ parabolic and the first $\mathrm{GL}_1(k_w)$. The operator $|H_\lambda|^{-1}\mathcal{A}_\lambda$ first projects to unipotent invariants and then to type χ_w on the first GL_1.

For a squarefree Green parameter c_w, the finite Harish–Chandra support is the collection of cuspidal types of the irreducible factors of c_w. Any GL_1 of type χ_w in those unipotent invariants would give, by finite parabolic induction and Frobenius reciprocity, the factor of degree one named by χ_w. This contradicts the good-parameter condition: $\chi_w^{q_2} = 1$ and no q_2-torsion root of $k_w^\times$ occurs in the polynomial. The q_2-power condition is for the original χ_w. The auxiliary twist is trivial on the unit group at each old chosen place, since its conductor is supported on Q. Thus the weighted sum kills every good type at w and hence kills Ξ. This is the parabolic product in Green's classification [29, §2, pp. 410–411, and §8, Theorem 14].

Let D_η be the normalized descent for the fixed twist. For an integral finite source element k, let s_w be its reduction at $w \in S_M$. On the target take an integral lift of $\mathrm{diag}(s_w, I_{n-r})$ at these places and the identity outside the old split pairs. The simultaneous action fixes the original inclusion matrix modulo each old place. At $w \in Q$, right multiplication of the selected square submatrix by k_w multiplies its residue character by $\chi_w(\det k_w)$, which cancels the inverse factor in the Weil action. Thus the block datum is multiplied by $\prod_{w \in S_M} \chi_w(\det s_w)^{-1}$. Moving the target translation through the class pairing gives

$$(D_\eta \Xi)(gk) = D_\eta(\rho_\eta(s)\Xi)(g),$$

where $\rho_\eta(s)$ is translation by this lift multiplied by $\prod_{w \in S_M} \chi_w(\det s_w)$. Taking $s = 1$ gives the original principal source level. The twist has weight zero. The real coefficient projection and the square-integrability argument of Sections 33.1.1 and 40.3 therefore apply termwise to the finite target class sum at this twist. Unfolding that sum pairs $D_\eta \Xi$ against each source form y by pairing Θy with Ξ, up to the fixed nonzero normalizing scalar.

Insert the self-adjoint idempotent e_0 on Ξ and move it to Θy. The finite intertwining relation gives $e_0 \Theta y = \Theta(e_0 y)$, a good lift, so the pairing is zero. The normalized descent lies in that source space and therefore vanishes. At the original block finite datum the kernel at the harmonic highest vector is exactly this normalized descent up to conjugation and a nonzero scalar.

33.1.1 The real type in the bottom descent

In this subsection, we determine the type for the descent used in Section 34.1.

At a place τ, write the decreasing source stem as $\varsigma_0, \ldots, \varsigma_{r-1}$.

Write $a = k + j - 1$, $b_- = m - j$, $n = r + t$, and $w = w_j(\tau)$. Then $a + b_- = r$. The matrices for the positive source sector use the first a stem coordinates in decreasing order. Those for the negative sector use the last b_- in increasing order. The two families of column counts are

$$k^+(h) = \big(\varsigma_{h-1} - \max(\varsigma_h, (t-1)/2 - w) - 1\big)_+,$$
$$k^-(h) = \big(\min(\varsigma_{r-h-1}, -w - (t-1)/2) - \varsigma_{r-h} - 1\big)_+.$$

Here one uses the first formula only through height a and the second only through height b_-. At a missing endpoint the respective middle block endpoint is used. The inequality $\varsigma_i + w \geq \Delta/2$ for $i < a$, and $\varsigma_i + w \leq -\Delta/2$ for $i \geq a$, makes all entries beyond these heights zero.

In the compact target the column multiplicity at height h is $k^+(h) + k^-(n-h)$, and the determinant exponent is $-\sum_h k^-(h)$. The differences of consecutive twice Harish–Chandra parameters are twice these column multiplicities plus two. At position a, the determinant exponent cancels the whole negative contribution. It follows that after subtracting $2u - 1$, where $u = b_- + w - \lfloor r/2 \rfloor$, the compact twice parameter is the source twice stem with the inserted block

$$-2w + t - 1,\ -2w + t - 3,\ \ldots,\ -2w - t + 1$$

between the two sectors. This is the regular discrete-series theta correspondence with one inserted compact block, after the known determinant shifts of the Weil action. After removing the determinant twists, the retained Harish–Chandra entries are $\eta_i = \varsigma_i + w$. The inserted entries are $\gamma_b = (t-1)/2 - b$, $0 \leq b < t = n - r$. In these units,

$$\min_i |\eta_i| \geq \Delta/2 \geq (t+1)/2, \qquad \min_{i,b} |\eta_i - \gamma_b| \geq (\Delta - t + 1)/2 > 500(r+n)^3$$

for sufficiently large x. The positive source sector has a positive entries and the negative sector has b_- negative entries. Thus the retained entries all have positive signature on the target, and the inserted string adds t positive directions, giving $U(a + b_- + t, 0) = U(n, 0)$. Li's regular discrete-series correspondence therefore gives nonvanishing on this target as well as the indicated compact type [42], in the parameter convention of [59, Theorem 2.7(b)]. See also [37, Lemma 8.2].

The polynomial is the product of powers of leading minors in the two sectors. For the positive sector use the first target spaces in decreasing order and the positive source order just given. For the negative sector use the last target spaces in reverse order and the negative source order just given. These two sets of target spaces are disjoint because $a + b_- = r < n$.

The determinant polynomials are nonzero. A raising operation either kills a given minor or duplicates one of its covectors or columns. A mixed Laplacian differentiates one positive variable and one negative variable with the same target space. This gives zero by the disjoint covector sets. Thus the polynomial is jointly harmonic and highest in the two compact orders. Its target weight of the part is obtained by adding the positive part sizes and subtracting the negative reversed lengths, plus the vacuum determinant shift, which is canceled by the splitting character.

It is exactly the compact representation with column multiplicities and determinant exponent described above. The polynomial is the image of its marked highest vector by the Cartan product, since evaluating both highest vectors on the ordered coordinate arrays gives these same leading minors. Thus projection to that compact target highest line is nonzero. Howe's compact-member duality makes the projected oscillator one irreducible representation of the source group. This proves the occurrence used in the compact bump argument.

The local oscillator correspondence expresses the lift with any polynomial Fock vector at that same finite datum as a real convolution of the harmonic lift. The normalized descent therefore vanishes there for every polynomial Fock vector. For this assertion let $K = U(n)$ at one real place, with probability Haar measure, and let v be the unit highest vector in the compact type τ of the harmonic. Set $c(k) = \langle v, \tau(k)v \rangle$, $d_\tau = \dim \tau$. Schur orthogonality gives the matrix unit

$$p = \int_K d_\tau \overline{c(k)}\, \omega(1,k)\, dk$$

on the oscillator. The conjugate scalar coefficient y of Ξ satisfies $R(d_\tau c)y = y$. In the bilinear descent, changing h to hk^{-1} replaces $c(k)$ by $c(k^{-1}) = \overline{c(k)}$; hence the descent of Q equals the descent of pQ. Take the product over the real places.

The projected joint harmonic is the original highest vector. Its orbit under the smaller real group has dense linear span in the matched irreducible theta representation. A polynomial vector projected to it is compact-type finite. For its finitely many compact types the image of compactly supported smooth bump averages of the projected harmonic is dense and has finite dimension after the compact-type projection. It is therefore the entire finite-type subspace.

The compact-type projection can be absorbed in the smooth bump by averaging over the real maximal compact. This expresses the projected polynomial exactly as one compact bump average. Moving that average to the source variable gives the claimed convolution. The finite vector stays fixed and the compact support justifies the interchange on the definite target quotient. Trivializing the second character of the kernel multiplies the descent by a character of $\det g$, so leaves its vanishing intact.

Here is the extension from the block datum to every finite test off the original vacuum places. Fix a real polynomial and a source point g. Call the corresponding descent pairing $B(\Psi, \Xi)$, linear in Ψ and conjugate-linear in Ξ. At $w \in S$, write a matrix as $z = (z', z'')^{\mathsf{t}}$ in the marked blocks, with z' an r-by-r block. Say that the block ratio is allowed at w if z' is invertible over the local field and $z''z'^{-1}$ is integral, with zero reduction at $w \in S_M$. The block function Ψ_{bl} on integral matrices is the indicator of the marked inclusion modulo $w \in S_M$, multiplied at $w \in Q$ by $\chi_w(\det z')$, extended by zero at singular reductions. It has the original vacuum at the other places. The conjugate place is prescribed by the split unitary identification.

Suppose first that z has allowed block ratio at every place of S. Take the finite source element A whose component at each $w \in S$ is z'^{-1} and which is the identity off $S \cup \bar{S}$. Then $\Psi_0(x) = \Psi_{\mathrm{bl}}(xA)$ takes the value one at z. A source translation shows $B(\Psi_0, \Xi)$ is the original block descent at gA times an invertible Weil scalar. Thus it is zero.

The nonzero support of Ψ_0 is one orbit of the marked target level, with its character action, in the S-coordinates. Indeed after right multiplication by A, a matrix in the support extends to an integral invertible block matrix with that matrix as its first columns. At S_M these columns are the marked coordinate columns modulo the residue place, and at Q the completed matrix is block lower triangular with the indicated first block. These are exactly the two level conditions. On this orbit all equivariant functions are determined by their value at the original inclusion multiplied by A^{-1}.

Let $C = \prod_{w \in S}(z_w + \varpi_w^{a_w} M_{n,r}(\mathcal{O}_w))$. First suppose that the ratio is allowed at every point of C. Then every possible first block $z' + \varpi_w^{a_w} B$ is invertible. In particular every entry of $\varpi_w^{a_w} z'^{-1}$ lies in the maximal ideal. Otherwise, for an entry u of modulus at least one, take B to have the single transposed entry $-u^{-1}$. The rank-one determinant formula then gives determinant zero. It follows that $x'z'^{-1} \equiv I \bmod \varpi_w$ and $x''z'^{-1} \equiv z''z'^{-1} \bmod \varpi_w$ for every $x \in C$. Thus C is contained in the orbit supporting Ψ_0. Average $\mathbf{1}_C$ over this same compact target level, including the conjugate of its level character. The translates have finite span and factor through a finite quotient. Equivariance of the Weil action and covariance of Ξ give the same value of $B(\cdot, \Xi)$ before and after averaging. The averaged function is supported on this orbit. The average is therefore a scalar multiple of Ψ_0. It follows that $B(\mathbf{1}_C, \Xi) = 0$.

Otherwise choose a point $z \in C$ where the ratio fails at a place w, and use that point as the center of the same ball. There exists an integral covector $l = (\lambda, \lambda')$ with $lz = 0$, $\bar{\lambda} \neq 0$, and $\bar{\lambda}' = 0$ when $w \in Q$. If z' is singular take an integral multiple of a nonzero covector of its left

kernel, normalized to have a unit entry. Otherwise use a covector of $(z''z'^{-1}, -I)$ whose first part has a coefficient of modulus at least one at S_M, or strictly greater than one at Q.

Divide by an entry of largest modulus in that first part. This gives exactly the asserted integrality and residue conditions. For each $s = 1 + \bar{u}\bar{\lambda}$ in $H_{\bar{\lambda}}$, lift $\bar{u}$ to an integral column u and take the target shear

$$h_s = 1 + \iota_0(u)l.$$

It is integral invertible since its determinant reduces to $\det s$, and fixes z. Extend it by the identity at the other split places with the corresponding conjugate components.

Since h_s fixes z and preserves $M_{n,r}(\mathcal{O}_w)$, it fixes each such additive ball. Its level decomposition is upper block triangular at a chosen place, and at a twist place the reduction of the off-diagonal top-right block is zero. By that decomposition, its action on Ξ is $\chi_w(\det s)^{-1}$ times the action of s at a chosen place, and just $\chi_w(\det s)^{-1}$ at a twist place.

The finite Weil translation identity implies $B(\mathbf{1}_C, h_s\Xi) = B(\mathbf{1}_C, \Xi)$. At a chosen place the sum of these actions kills Ξ by $\mathcal{A}_{\bar{\lambda}}$. At a twist place it also kills Ξ, since $\det : H_{\bar{\lambda}} \to k_w^{\times}$ is onto with equal fibers, and $\chi_w \neq 1$ on its exact support. Adding the pairings therefore gives $|H_{\bar{\lambda}}|B(\mathbf{1}_C, \Xi) = 0$.

Every compactly supported locally constant test on the finite split coordinates is a finite linear combination of these cosets, each of which is covered by one of the two cases. This proves the bottom vanishing for all finite Schwartz–Bruhat factors with the prescribed vacuum off $S \cup \bar{S}$. Polynomial Fock vectors and such factors have dense span in that vacuum subspace. For each fixed source point the theta pairing with Ξ is continuous on the datum space, by the moderate kernel estimate and the compact definite target quotient. Passing to the closure gives the all-datum bottom vanishing used in the tower.

33.1.2 Extension through the hyperbolic planes

In this subsection, we extend the type of Section 33.1.1 through the original tower.

Keep the original pair, level, support S and false lift Ξ of the bottom-descent argument above. Write $V = I_n = I_{d_1} \oplus I_{d_2}$ for the prescribed gluing blocks, with $2r + 2d_1 + 3 < n$, and let $\mathbb{H}^l$ be the standard marked hyperbolic summand of rank $2l$. In the rank-r chain $n = r + t$, $t = 5r$, and $d_1 = r + 9m + 1$. The displayed inequality is then $4r + 18m + 5 < 6r$, which follows from the original bound $100m + 9 < k$ and $r + 1 = k + m$.

Let Y denote the conjugate of the matrix coefficient of Ξ at its original highest vector. The chosen split place of the false-lift test is w with norm q, off $S \cup \bar{S}$. Conjugation of this scalar coefficient commutes with right translation and conjugates the spectral argument of the Hecke polynomial. Thus Y is spherical at w and $\mathcal{P}_{n,w}(X)Y = 0$, where

$$|X| = q^{1-n+r/2}, \qquad \mathcal{P}_{a,w}(X) = \sum_i (-X)^i q^{i(i-1)/2} T_{w,i}. \tag{F1}$$

The $T_{w,i}$ are the original right-coset sums, using compact mass one.

The parameter X in (F1) is $\overline{(q^{n-r-1}a_{r,w})^{-1}}$, where $a_{r,w}$ denotes the rank-r Frobenius value of the original source character at w. Its modulus comes from the rank-r character value at w, of modulus $q^{r/2}$, and the exponent $n - r - 1$ in the last root of the complementary Hecke string.

First suppose that the descents to $W \oplus \mathbb{H}^{l-1}$ vanish for all data vacuum off $S \cup \bar{S}$, for the source character $a\upsilon^{l-1}$ where $a = \chi_W$ and υ denotes its original norm-one splitting character here. Set $W_l = W \oplus \mathbb{H}^l$, $r_l = r + 2l$. Split off the last plane, of Gram matrix $J_{\mathbb{H}} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For its

finite adelic coordinates take the fixed integral trace-one element $e = c_{\mathrm{tr}}$ and a unit c of norm -1 by Section 4.3. The matrix

$$B_f = \begin{pmatrix} 1 & -c \\ e & c\bar{e} \end{pmatrix}, \qquad B_f^* J_{\mathbb{H}} B_f = I_2, \qquad \det B_f = c$$

identifies the hyperbolic lattice with the original finite unitary lattice, including at dyadic places. At infinity use $2^{-1/2}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, which is orthogonal and carries $J_{\mathbb{H}}$ to $\mathrm{diag}(1,-1)$. Removing successive planes only reorders these blocks and leaves the summand W fixed.

Write $g = n_0(u, g_0)k$ in Iwasawa form. At a split place the integral compact moves an isotropic line to the specified line by extending its primitive vector inside the kernel of its primitive dual covector to a basis. At a nonsplit finite place the extension is unramified. For a primitive isotropic vector take an integral dual vector with pairing one, then subtract a multiple of the isotropic vector to make the dual vector isotropic, using the surjective integral trace. These two vectors split off a unimodular plane. Its orthogonal complement admits an integral orthonormal basis: lift a vector of nonzero norm over the quadratic residue field, normalize by surjectivity of the unit norm, and repeat. A nondegenerate residue Hermitian form has such a vector even in characteristic two, by nondegeneracy of the residue trace pairing. These local choices are integral at almost all places and give the adelic decomposition. Let θ_0 denote the isotropic zero slice of the theta functional. The compact factor carries the vacuum subspace at every place off $S \cup \bar{S}$ to itself, and the radical n_0 acts as the identity on the zero slice.

The slice of the Levi action is the Weil action of the smaller pair at g_0, times

$$\big((1)_n(\det u)\big)^{-1} |\det u|_{\mathbb{A}_E}^{n/2}. \tag{F2}$$

Here $(1)_n$ is the rank-n space character associated to the trivial target automorphic character. The restriction of the enlarged source character is av^{l-1} in this slice. This is the original mixed-model Levi formula, including its change-of-variables determinant. The zero-slice datum is still vacuum off the same set, so its pairing with Y is zero by induction.

The remainder gives a bounded descent on $U(W_l)$. Here are the analytic details. Write $\Lambda = \theta - \theta_0$ and denote by k_0 the rational isotropic column of the plane. The zero slice is the probability mean of the theta functional along the adelic isotropic translations

$$Z = \{z k_0^* J : z \in E^{n\times 1}\} \subset \mathrm{Hom}_E(W_l, V).$$

Indeed averaging the mixed-model translation characters removes all nonzero rational basis indices, leaving that slice. In particular Λ is invariant under the rational translations $Z(E)$, and its probability mean along $Z(\mathbb{A})/Z(E)$ vanishes on every datum.

33.1.3 The mixed-model height bound

In this subsection, we bound the mixed theta sum used in the vanishing argument after Section 33.1.2.

Let δ be the first Euclidean minimum of the moved integral source lattice, with its original positive Euclidean archimedean metric.

Put $H = \max(1, \delta^{-1})$. If $\delta < 1/4$, the E-span of the vectors of length $< 4\delta$ is totally isotropic. Indeed, the pairings of such vectors are integral at every finite place and have absolute value < 1 at every embedding, so vanish by the product formula. At each infinite place τ, with sign

matrix J_τ, the completion I_τ of this isotropic subspace and $J_\tau I_\tau$ are Euclidean orthogonal and dual for the signed pairing. At every τ simultaneously, in the Euclidean splitting into $I_\tau, J_\tau I_\tau$, and their common orthogonal complement, multiply the first factor by two, the second by $1/2$, and the third by one.

This is a signed unitary map of norm at most two. It doubles the lengths of the short vectors and decreases the others by at most a factor of two. The new first minimum is therefore at least 2δ. Iteration of the inverse moves on the adelic point puts its minimum above $1/4$, using a purely real product of norm $O(H)$. For the resulting point, the lattices in the Euclidean coordinates have fixed covolume, since g is unitary, and first minimum at least $1/4$. Minkowski's second theorem bounds their last minimum. Choose a full set of independent lattice vectors of bounded length. They span a sublattice of bounded index by Hadamard's inequality. Its possible enlargements, in the coordinates of these vectors, lie between $\mathbb{Z}^N$ and $\frac{1}{D}\mathbb{Z}^N$, where N is the real dimension and D is a common multiple of the possible indices. There are finitely many such enlargements. Choosing a basis in each gives a uniformly bounded $\mathbb{Z}$-basis of the moved lattice, with uniformly bounded inverse basis matrix since its covolume is fixed.

In the corresponding rational basis, the matrices for multiplication by a fixed integral basis of $\mathcal{O}_E$ have bounded integer entries. The signed Hermitian pairings of the basis vectors lie in the fixed fractional ideal of pairing values and are bounded at every embedding, so they also have only finitely many possible values. Two bases with the same multiplication matrices and pairings are carried into one another by an E-linear unitary map defined over F_0. Fix one rational basis and one finite lattice representative for each realized set of these values. After the corresponding rational left translation, the finite point lies in that representative times the compact stabilizer of the original finite lattice. The real point and its inverse are bounded because they carry the fixed real basis to the bounded moved basis. This gives a compact set of representatives for all points with first minimum at least $1/4$. The definite target has a positive uniform lower bound on its first minimum, by integrality of the signed pairing and the product formula. The same argument gives a compact set of representatives there.

The theta distribution on a rational Schrödinger polarization is a rational lattice sum. On a fixed finite Schwartz space it is bounded by a finite-order Schwartz seminorm at infinity, by summing a decaying power on the rational lattice. Compact sets of finite translates of such a space are contained in another fixed finite Schwartz space. At infinity compact sets of Weil translates act boundedly on each fixed seminorm, allowing a higher fixed seminorm on the source vector.

For a general real move the conjugation formula on the position and differentiation operators is its linear symplectic action. Together with unitarity on L^2, this bounds a fixed Hermite Sobolev norm of the image by a polynomial in its matrix norm times a fixed Hermite Sobolev norm of the source vector. The real symplectic action of the pair is the linear Hom action, whose norm is polynomial in the two matrix norms. Rational invariance of theta and the preceding compact reduction consequently give

$$|\theta(\omega(g,h)\phi)| \le C H^A p(\phi)$$

with one fixed Schwartz seminorm p on the finite Schwartz spaces in use, uniformly in the definite target h.

In the short-translation argument each difference is conjugated past $\omega(g,h)$, after choosing the rational representative with the fixed short isotropic vector. The transformed finite shifts fix the test because its finite stabilizer contains the fixed ideal used for the coefficient

representatives. The real shifts have size $O(H^{-1})$ and are in one isotropic family, so commute. The fundamental theorem of calculus along a real Heisenberg one-parameter group bounds a difference by its length times the seminorm of the corresponding position or derivative operator. Repeating for A or more differences gives a factor H^{-A} in a higher fixed seminorm of the original polynomial Fock test. The following averaging identity applies this estimate directly to the full theta functional.

If $H > 2$, choose a shortest rational integral source vector of length at most $2/H$. Its norm pairing is integral at finite places and of absolute value less than one at the archimedean places, so it is isotropic by the product formula. Witt's theorem moves it to k_0 by a rational translate γ of the source. Rational invariance of the full kernel allows the descent at γg in place of g; the zero-slice integral at that point is again zero. At the definite target, a rational basis in its moved lattice has uniformly bounded lengths: Euclidean covolume is fixed and the first minimum is bounded below, so Minkowski's second theorem bounds the last minimum. With this basis d_j, the maps $d_j k_0^* J$ span Z, are integral in the moved Hom lattice, and have real lengths $O(H^{-1})$.

Choose a fixed compact open additive ideal in the finite coefficient coordinates which makes the transformed finite shifts act trivially on the fixed Schwartz vector. Such an ideal works uniformly because the moved finite rank-one maps are integral. Additive approximation gives representatives for the adelic quotient with finite coefficients in this ideal, and the archimedean coefficients can then be taken in a fixed compact fundamental domain for the corresponding rational coefficient lattice.

Give $[Z] = Z(E)\backslash Z(\mathbb{A})$ probability Haar measure. For $j \geq 1$ and the moved datum f, rational periodicity and isotropy give

$$(\theta - \theta_0)(f) = \int_{[Z]^j} \theta\left(\prod_{i=1}^{j}(1 - \varrho(z_i))f\right) \prod_{i=1}^{j} dz_i.$$

Indeed each nonempty term of the expanded product averages to $\theta_0(f)$, since translation preserves Haar measure on $[Z]$; their signed coefficients sum to -1. The rational basis $d_j k_0^* J$ identifies this probability measure with that on the coefficient quotient, independently of the chosen basis. After conjugation each finite shift fixes the test and each real shift has length $O(H^{-1})$. Integrating its Heisenberg generator gives one factor H^{-1} per difference in the Schwartz seminorm bound.

For the polynomial Fock datum all the resulting Schwartz seminorms are finite. Taking $j \geq A$ bounds the remainder uniformly, by $C' H^A H^{-j}$. For $H \leq 2$ use the initial operator norm bound directly. The short translations act in the Schwartz model of the same pair, and their seminorm bounds apply to every fixed finite Schwartz factor at S. Integrating against bounded Y on the compact definite target proves boundedness of the descent.

The local Hecke transfer now kills this bounded function. For a polynomial Fock datum with the required off-level vacua write F_l for its descent to W_l, $k = n - r_l$, $\alpha = \mathrm{frob}_{r_l}(a\upsilon^l, \bar{w})$, and $\beta = \mathrm{frob}_n(1, \bar{w})$. The vacuum at $w, \bar{w}$ makes F_l spherical and gives the exact adjoint transfer

$$0 = \prod_{i=0}^{k-1}(1 - q^i \alpha X)\mathcal{P}_{r_l,w}(X')F_l, \qquad X' = q^k \alpha \beta^{-1} X. \tag{F3}$$

In the transpose descent the original target Hecke operator moves by right-invariant Haar integration to its inverse double cosets at $\bar{w}$.

The split inverse-transpose relation reads these at w in the source group. The determinant factors on the vacuum have moduli $|\alpha| = q^{r_l/2}$, $|\beta| = q^{n/2}$. There is no conjugation of X at this step, since the kernel integration with Y is linear. To check the local rectangular vacuum identity in (F3), use the split coordinate at $v = \bar{w}$, whose residue field has order q. Put $\Lambda = \mathcal{O}_v^a$. For each $U \subseteq k_v^a$ choose γ_U with elementary divisors $1, \varpi_v$ and $\ker \bar{\gamma}_U = U$. Then

$$\gamma_U^{\mathsf{t}} \Lambda / \varpi_v \Lambda = U^{\perp}, \qquad \varpi_v \gamma_U^{-1} \Lambda / \varpi_v \Lambda = U,$$

where the orthogonal complement is for the standard dot product. These equalities classify the right cosets by their image lattices. Thus the representatives in the coordinate at v are γ_U^{t} for the positive elementary double coset and γ_U^{-1} for its inverse at the conjugate place. On an n-by-r_l matrix Z, the large group's inverse left substitution is $\gamma_U^{-\mathsf{t}} Z$ and the small group's right substitution is $Z \gamma_U^{-1}$. Hence in the vacuum test the matrix is $Y' \gamma_U^{-1}$, with $Y' = Z^{\mathsf{t}}$ and $a = n$ on the large side, and $Y' = Z$ and $a = r_l$ on the small side. In a basis adapted to U,

$$Y' \gamma_U^{-1} \text{ integral} \quad \Longleftrightarrow \quad Y' \text{ integral and } U \subseteq \ker \overline{Y'}.$$

Indeed the columns in the directions of U must be divisible by ϖ_w; the remaining columns must be integral. Thus both operators vanish off the integral matrices. On an integral matrix of residue rank a_0, the contributing subspaces lie in kernels of dimensions $n - a_0$ and $r_l - a_0$, respectively. Both base change characters at $\bar{w}$ are unramified, since the original source character and υ are unramified there, and the target character is trivial. A subspace of dimension i therefore contributes determinant factor α^i or β^i. For a residue space of dimension d, the finite binomial identity is

$$\sum_{U \subset k_w^d} (-s)^{\dim U} q^{\binom{\dim U}{2}} = \prod_{j=0}^{d-1} (1 - q^j s).$$

With $\beta X' = q^k \alpha X$, the two sums are consequently related by

$$\prod_{j=0}^{n-a_0-1} (1 - q^j \alpha X) = \prod_{j=0}^{k-1} (1 - q^j \alpha X) \prod_{j=0}^{r_l - a_0 - 1} (1 - q^j \beta X').$$

This gives (F3) for all X in the original Haar normalization.

The factors in (F3) are nonzero: by (F1),

$$q^k |\alpha X| = q^{1-l} \le 1, \qquad q^{r_l+1} |X'| = q^{(r_l + r + 4 - n)/2} \le 1.$$

The second inequality follows from $2r + 2d_1 + 3 < n$ and $l \le d_1$. For a bounded spherical function F on the source, the absolute sum of the coefficients of the Hecke right-coset operator $\mathcal{P}_{r_l, w}(X')$ is

$$\sum_{U \subset k_w^{r_l}} q^{\binom{\dim U}{2}} |X'|^{\dim U} = \prod_{i < r_l} (1 + q^i |X'|) \le \exp(q^{r_l} |X'|) \le e^{1/2} < 2.$$

The term from $U = 0$ has weight one and acts as the identity.

Isolating it proves $\|F_l\|_\infty \le (e^{1/2} - 1) \|F_l\|_\infty$, so $F_l = 0$. This uses no spectral decomposition of the noncompact source. Density of polynomial and finite Schwartz factors extends this to every datum fixed by the off-level vacuum projection. Induction starting with the bottom-descent argument above gives vanishing at $l = d_1$. Every conjugate matrix coefficient of Ξ has

the same vanishing: translates of the highest vector span the irreducible Weyl space, and a real translation of the coefficient corresponds to the inverse real translation of the datum, leaving its finite projection fixed.

Next we pass from these descents to the prescribed block period. Write T for the complement of $S \cup \bar{S}$. For a continuous function Y_0 on $[U(V)]$ fixed by the integral group at T, suppose first that its descents to the split space $\mathbb{H}^{d_1}$ with character υ^{d_1} vanish for all data vacuum at T. Fix $h_1 \in U(I_{d_1})(\mathbb{A})$ and set

$$g_0 = \operatorname{diag}(h_1, 1), \qquad I(h) = \int_{[U(I_{d_2})]} Y_0(\operatorname{diag}(h_1, h_2)h)\, dh_2.$$

Let o_1 be the original orthonormal first-block basis and define

$$x(g) = g^{-1} o_1 h_1, \qquad W(x) = |\mathrm{vac}(\alpha_0 x)|^2, \qquad \psi_u(x) = \psi\left(\delta \sum_{i,j} u^{\circ}_{ij} x_{ij}\right).$$

Use the rational skew element $j_{\mathrm{W}} = -1/\sqrt{-d}$ and the finite idele α_{W} of Section 8.1. The idele α_0 has finite component $\alpha_f = (\alpha_0)_f = \alpha_{\mathrm{W}}$ and component $(2\pi \operatorname{Im} \tau(j_{\mathrm{W}}))^{-1/2}$ at τ. The vacuum support is $\alpha_f \widehat{\mathcal{O}}_E$ in each finite coordinate, so $W(x) > 0$ precisely when x_f is integral. The idele $\delta = j_{\mathrm{W}} \alpha_0 \bar{\alpha}_0$ generates the inverse different at finite places, and u° removes the finite components of u at T and their conjugates.

Parametrize the Siegel center by the skew block $C - C^*$, for $C \in \mathrm{Mat}_{d_1}(\mathbb{A}_E)$. Write $G_z = z^* J z$ for a rational matrix z in the mixed theta sum and ψ_E for the additive character through the absolute trace. The mixed term has multiplier $\psi_E(-j_{\mathrm{W}} \operatorname{tr}(G_z C))$: in its half-trace formula $\operatorname{tr}(G_z C^*) = \overline{\operatorname{tr}(G_z C)}$ and $\bar{j}_{\mathrm{W}} = -j_{\mathrm{W}}$. Multiply the zero descent by $\psi_E(j_{\mathrm{W}} \operatorname{tr}(I_{d_1} C))$ and average over adelic matrices modulo rational matrices with probability Haar. Additive orthogonality gives

$$\int \psi_E(j_{\mathrm{W}} \operatorname{tr}((I_{d_1} - G_z)C))\, dC = \mathbf{1}_{G_z = I_{d_1}}.$$

For $d_1 > 0$ this kills the zero matrix and all matrices of deficient rank. For $d_1 = 0$ the empty matrix is the unique term. The surviving matrices are isometric rational embeddings. They are one rational orbit by Witt's theorem, with stabilizer the original rational $U(I_{d_2})$. Fix the Siegel Levi point and take compact sets of representatives for the definite quotient and for the additive unipotent quotient. The finite supports of the translated mixed slices impose $z \in \mathfrak{a}^{n d_1}$ for a fixed fractional ideal $\mathfrak{a}$ of E. Their real Schwartz seminorms are uniformly bounded on these compact sets, and the unipotent character has modulus one. Thus the sum of absolute values is at most

$$C_N \sum_{z \in \mathfrak{a}^{n d_1}} (1 + \|z\|)^{-N} < \infty, \qquad N > n d_1 [E : \mathbb{Q}].$$

The last inequality follows by counting lattice points in dyadic annuli. The bounded factor Y_0 preserves this majorant. We may therefore take the coefficient term by term and unfold the surviving orbit to the quotient by its rational stabilizer with the original ambient Haar measure.

Read the coefficient at the Siegel Levi element with matrix $\alpha_0 h_1$, and use the Heisenberg translate of the original vacuum by the isotropic vector whose adapted coordinates are $\alpha_0 \overline{u^{\circ}}$. It is vacuum at T. Its mixed slice at $\alpha_0 x$ is a common nonzero scalar times $W(x)\psi_u(x)$: the slice of the vacuum is the squared Gaussian with integral finite support, and the translation supplies the displayed character, since the half pairing on the isotropic slice doubles there to δ.

The Levi scalar is the nonzero factor (F2) for $\alpha_0 h_1$. Neither scalar depends on the basis being integrated. After canceling them and integrating first over the unchanged stabilizer quotient, the vanishing coefficient gives

$$\int_{U(I_{d_2})(\mathbb{A})\backslash U(V)(\mathbb{A})} W(x(g))\psi_u(x(g))I(g_0^{-1}g)\,d\dot{g} = 0 \qquad \text{for every } u. \tag{F4}$$

The quotient measure is the quotient of the original group Haar measures, and the inner integral defining I uses the original adelic quotient Haar measure.

The positive weight in (F4) detects the period at the chosen basis. On its support the finite basis is integral and orthonormal. Integral Witt extension at each finite place extends this basis to an integral unitary matrix. At a split place this is completion of a direct summand; at an inert place the basis splits off a unimodular orthogonal summand, whose complement is isometric to the standard complement (unramified Hermitian lattices are classified by rank when unimodular).

This applies also at residue characteristic two, where the unramified trace map is still onto. Here the nonsplit finite extensions are unramified: primes dividing q_1 split in K, and at d the total conjugation is nontrivial on the unramified cyclotomic component. At other primes the compositum is unramified. Taking the compact real extension as well, every nonzero-weight basis is read on the product of the real compact and the finite integral group.

More precisely, put $G = U(V)$, $H_2 = U(I_{d_2})$, and $K = G_\infty \prod_{v<\infty} G(\mathcal{O}_v)$ for the standard lattice. When $W(x(g)) > 0$, choose $k \in K$ with $x(g) = k^{-1}o_1$. Then $gk^{-1}o_1 = o_1 h_1$, whence

$$g \in H_2(\mathbb{A})g_0K.$$

The support of the integral in (F4) is therefore contained in the compact image of g_0K in $H_2(\mathbb{A})\backslash G(\mathbb{A})$. The inner period is unchanged by left multiplication of g by $H_2(\mathbb{A})$, by right invariance of its quotient measure. Two integral representatives with the same characters ψ_u have identical basis coordinates off T, by nondegeneracy of the additive pairing. Multiplying on the right by the T-component of their relative integral matrix then identifies their full bases and leaves Y_0 fixed. Thus $\overline{I}$ is a continuous function of the array of these characters on the compact character image.

The character span on this compact image contains $1 = \psi_0$, is closed under products and conjugation by $\psi_u\psi_v = \psi_{u+v}$ and $\overline{\psi_u} = \psi_{-u}$, and separates points. By Stone–Weierstrass choose finite character sums p_j converging uniformly to $\overline{I}$. Equation (F4) gives $\int W(x)p_j I = 0$, and

$$\left|\int W(x)(p_j - \overline{I})I\right| \le \|p_j - \overline{I}\|_\infty \int W(x)|I| \longrightarrow 0$$

by the unfolded absolute convergence. Thus $\int W(x)|I|^2 = 0$. Continuity and full support of quotient Haar measure give $W(x)|I|^2 = 0$ everywhere. At $g = g_0$ the basis is o_1, which has strictly positive vacuum weight. Hence $I(1) = 0$, the desired block period. This argument uses the chosen basis and complement throughout.

33.1.4 Vanishing for the gluing adjoint

In this subsection, we apply the vanishing calculation of Section 33.1.3 to the original false-lift coefficient.

Allow a finite right translate γ of Ξ supported on S, as in the prescribed divided lattice. The proved vanishing for $W \oplus \mathbb{H}^{d_1}$ remains valid after this translate because the transported datum is unchanged at T. Let ϕ_2 be the actual W, I_{d_2} datum of the slice, with target character b, and fix its source point g.

Choose a rational Schrödinger polarization for (W, I_{d_1}), with coordinate space L over F_0. Take the original finite vacuum f_f. Its translate f'_f by $(g_f, h_{1,f})$ is a nonzero Schwartz–Bruhat function. By additive approximation some $x_0 \in L(F_0)$ satisfies $f'_f(x_0) \neq 0$. The rational points in its compact finite support lie in a fixed fractional lattice. Choose $f'_\infty \in C_c^\infty(L(F_{0,\infty}))$ equal to one at x_0 and supported in a ball containing no other point of that lattice. In the real factor of ϕ_1 take the inverse oscillator translate of f'_∞ by $(g_\infty, h_{1,\infty})$. The rational theta sum at (g, h_1) has the single nonzero term $f'_f(x_0)$; the normalization and splitting character multiply it by a nonzero scalar. This supplies the required first kernel for the target character b^{-1}, with the original vacuum at T.

For the conjugate coefficient Y and $\Phi = \phi_1 \boxtimes \phi_2$, put

$$Y_0(h) = Y(h\gamma)\, \theta^{a,1}_{W,V}(\Phi; g, h), \qquad 1 = b^{-1}b.$$

Both factors are continuous and left invariant under $U(V)(F_0)$. If k is integral and supported on T, then $k\gamma = \gamma k$ and $Y(h\gamma k) = Y(h\gamma)$. The second factor is fixed because the vacuum projection at T fixes Φ and the character a is unramified there. This applies at every source point g, since the two group actions on the Weil representation commute. The split-space descents of Y_0 vanish, since the product with the $\mathbb{H}^{d_1}, V$ kernel is the $W \oplus \mathbb{H}^{d_1}, V$ kernel with the box datum and source character av^{d_1}. On the block $I_{d_1} \oplus I_{d_2}$, the W, V kernel factors as the product of the two chosen kernels. Its first factor is the fixed nonzero value at (g, h_1). Cancel it from the block period and conjugate to obtain zero for the pairing of the original translated matrix coefficient with the conjugate W, I_{d_2} kernel. Let b_t range over the negative tableau minor products and write $\theta_t(g; u, h)$ for their kernels at second argument (u, h), with u real and h finite. Put $v(g, h) = (\overline{\theta_t(g; 1, h)})_t$. For the conjugate Weyl action ρ_- on the second block,

$$[\rho_-(u^{-1})v(g, h)]_t = \overline{\theta_t(g; u, h)}.$$

Indeed the source character has weight $m - j$ for the original pair of index j, the number of negative source directions. It cancels the compact vacuum character. The remaining action on a negative tableau polynomial substitutes u^{-1} in its variables; expansion in the basis of minors gives the displayed formula.

For fixed first-block vector y_1 write $C(v) = \mathrm{Cartan}(y_1, \bar{v})$. If ρ_V is the original conjugate Weyl action, then

$$\rho_V(\mathrm{diag}(1, u))C(v) = C(\rho_-(u)v).$$

Indeed a complemented minor of height $n - h$ contributes one determinant and a minor of height h of the inverse transpose. At each infinite place the full exponent $-\sum_h \mu_\tau^-(h)$ cancels these determinants. The first block of $\mathrm{diag}(1, \bar{u})$ is the identity, and the inverse transpose on the second block is u. Apply this identity with u^{-1} to both vectors in the invariant Hermitian pairing. Conjugate-linearity of C and of the first entry of that pairing gives a sum of the coordinate periods above, each at the prescribed finite translates.

Write $\varpi = \pi_M$. The finite function is fixed by $J(\varpi^2)$. On the kernel vector, a right integral action precomposes the residue function by the inverse of the matrix reduction at that level. This reduction is the identity, and the determinant factor in the kernel transformation is one for

these integral translates, since the source splitting character is unramified there. Conjugation by a divided lattice point with $\varpi\mathcal{O}^n \subset L_i\mathcal{O}^n \subset \mathcal{O}^n$ loses at most one factor of ϖ in the conjugated difference from the identity.

Hence it is the identity modulo ϖ at the chosen places. At a twist place the divided lattice point is the identity and the second block fixes the original first-block covectors, with level character one. This verifies the exact right invariance in the original finite level. Normalize the maximal finite compact and the compact real group to have mass one, or use the same common positive constant on both sides.

Each class at $J(\varpi^2)$ then has measure $[K_{\mathrm{std}} : J(\varpi^2)]^{-1}/|\Gamma_c|$. The level $J(\varpi^2)$ is principal at the old split place, so Lemma 6.1 gives $\Gamma_c = 1$. Thus the vanishing measured integral is the plain sum over the second-block classes of the given gluing map times one positive constant. For a second-block class c, let h_c be its finite representative, and put $\kappa(g, h_c)$ equal to the vector of *conjugates* of the second-block theta kernels in the negative tableau basis. Here the two characters are $\upsilon^{m-j}, \upsilon^{m/2-j}$ and the residue test is the inclusion modulo ϖ^2. Write $\eta_{i,g}$ for the tuple supported on choice i with value $\overline{\kappa(g, h_c)}$ at (i, c). We use the inner product linear in the second variable.

Lemma 33.1 (Vanishing against the gluing adjoint)**.** *For the gluing map G_ϑ of Section 34.1, each false lift Ξ at this twist satisfies, for every i, g,*

$$\overline{K^\circ_{(G^*_\vartheta \Xi)_i}(g)} = \langle \eta_{i,g}, G^*_\vartheta \Xi \rangle = 0.$$

Here K° is the component pairing of Section 39.4.3.

Indeed, put $l_c = \overline{\kappa(g, h_c)}$ and let G_- be the matrix of the Weyl form in negative tableau coordinates. At each infinite place, let $a_1, \dots, a_{m-j}$ index the negative source coordinates in their prescribed reverse order. For a tableau column of height h with entries $1 \le i_1 < \cdots < i_h \le d_2$, the corresponding minor is

$$\det\bigl(Z_{i_\alpha, a_\nu}\bigr)_{1 \le \alpha, \nu \le h}.$$

By Section 33.1.1, $h \le m - j$. The substitution uses distinct source coordinates and retains every target coordinate $1, \dots, d_2$. It therefore carries the standard minor products b_t injectively to a basis of the indicated polynomial tests. For $j = m$ the diagram is empty and its basis is 1. Since G_- is real positive definite, the polynomials $\sum_t (G_- z)_t b_t$, as z ranges over the complex Weyl module, span precisely these tests. Hence, for any tuple ζ in the i-th component,

$$K^\circ_\zeta(g) = \sum_c l_c^{\mathsf{T}} G_- \overline{\zeta_c}, \qquad \overline{K^\circ_\zeta(g)} = \sum_c \overline{l_c}^{\mathsf{T}} G_- \zeta_c.$$

Write $y_i(h) = \overline{c'' \Upsilon_{\vartheta,i}(h)}$ for the first factor constructed below. By adjunction the inner product is

$$M^{l_1} \sum_{[h] \bmod J_\Lambda} \sum_c \Bigl\langle \mathrm{Cartan}\bigl(y_i(h), \overline{\kappa(g, h_c)}\bigr), \Xi\bigl(\mathrm{diag}(h, h_c) L_i\bigr) \Bigr\rangle_{\mathrm{W}}.$$

The stabilizer weights are one at these free levels. For each fixed h, the inner sum is the vanishing block period just proved.

33.2 The lattice for the comparison operator

In this subsection, we decompose the target lattice into the two Hermitian summands used in Section 34.1. The calculation identifies the subgroup and quotient of the original level needed for the tensor contraction.

On the j-th pair W_j, V_j the target splits in its fixed order of the covectors as $V'' \oplus V^\circ$, of ranks

$$d_+ = r + 9m + 1, \qquad d_- = t - 9m - 1.$$

The first r coordinates in the definite space V'' carry the index set of W_j. In the separate signed space

$$W' = W_j \perp I_j^- \perp I_{3m-j} \perp I_{6m+1},$$

the complement of W_j has j negative and $3m-j$ positive directions from the first two summands and $6m+1$ definite directions from the short-covector class. We use the same coordinate index set for W' and V''. The first comparison factor takes values on the definite space V''.

The second factor uses V° with its original level $J(\pi_M^2)$. Let I_e, I_z denote the inclusions of r columns in the fixed coordinates of V'', V°, respectively. Let $a_0 = 3m$ be the rank of the carrier. The residue choices are matrices $\Lambda \in M_{a_0,d_+}(\mathcal{O}_E/(\pi_M^2))$ and the marked $6m+1$ covector class with its short covectors. For each choice use the level J_Λ on V'' that is 1 modulo π_M, fixes Λ modulo π_M^2, and has the identity columns off W_j modulo $\pi_M^2 Q$. Thus J_Λ contains the indicated principal level and is contained in the original free chosen level.

Here is the matrix of the lattice point in this description. It suffices to construct it at each place dividing π_M, in its split matrix realization, and extend by inverse conjugate transpose. Put $\pi = \pi_M$ locally and lift Λ to an integral matrix. Let D be a basis matrix for the lattice of covectors generated by the covectors of $\binom{\Lambda}{\pi I_{d_+}}$. That lattice of covectors contains $\pi\mathcal{O}^{d_+}$ and lies in $\mathcal{O}^{d_+}$, so both D and $B_+ = \pi D^{-1}$ are integral. The equality of the covector lattices gives, for all matrices X, Y of compatible size, the equivalence modulo π^2

$$DX = DY \quad \Longleftrightarrow \quad \begin{pmatrix} \Lambda \\ \pi I_{d_+} \end{pmatrix} X = \begin{pmatrix} \Lambda \\ \pi I_{d_+} \end{pmatrix} Y.$$

Write $J_z = I_z I_z^{\mathrm{t}}$.

Set

$$A = \begin{pmatrix} D & -DI_e I_z^{\mathrm{t}} \\ 0 & \pi J_z + 1 - J_z \end{pmatrix}, \qquad C = \begin{pmatrix} B_+ & I_e I_z^{\mathrm{t}} \\ 0 & J_z + \pi(1 - J_z) \end{pmatrix}.$$

The product is $CA = \pi I$. Permute the target covectors of A and the source columns of C inversely, so that the bottom z-covectors of A are the prescribed inclusion covectors of the original pair. Take $L_i = C$ at this place, and use the identity away from the chosen pairs. In particular L_i is chosen using the old modulus only and is independent of the twist.

To verify the exact block congruence, write x_+, x_- for the blocks of x and put $x_z = I_z^{\mathrm{t}} x_-$. The condition $(\pi L_i^{-1})x = \pi I_{\mathrm{incl}}$ modulo π^2 says

$$x_z \equiv I \pmod{\pi}, \qquad (1 - J_z)x_- = 0, \qquad D(x_+ - I_e x_z) = 0 \pmod{\pi^2}.$$

The first condition determines the unique unit $Y = x_z^{-1}$ modulo π^2 with $Y \equiv I$ modulo π for which $x_- Y = I_z$. Then the last condition is $D(x_+ Y - I_e) = 0$. By the covector-lattice equivalence it is exactly

$$x_+ Y \equiv I_e \pmod{\pi}, \qquad \Lambda x_+ Y = \Lambda I_e \pmod{\pi^2}.$$

The implications reverse by multiplication by Y^{-1}.

Thus the original inclusion indicator equals the sum over $Y \equiv I \bmod \pi$ of the two block indicators. At every residue point this sum has zero or one term. The character of a member of this congruence group is trivial, since the original determinant twist factors through reduction modulo π. When the class periods are assembled, the right changes by these matrices give equal integrals on the original good source, which is fixed by this old congruence group. Consequently the unnormalized sum in that period contributes its actual order, once.

For the denominator estimate, a change of columns in V'' puts the kernel of $\Lambda \bmod \pi$ on all but at most a_0 coordinates. In these coordinates the corresponding columns of D are zero modulo π, by the same equality of lattices of covectors. Thus $L_i^{-1} = \pi^{-1} A$ is integral on every first-block basis vector except at most a_0; on the second block all d_- vectors may be divided once.

Cofactor matrices at the conjugate places are given by the unitary inverse transpose convention, and the integral inverse there follows from the displayed integral C. For a height-s column of the first or second Weyl diagram, the exterior minor contains at most a_0 or d_- divided variables, respectively. This gives the common clearing exponent

$$l_1 = \sum_{\tau} \left(a_0 \sum_{s<d_+} \mu_\tau^+(s) + d_- \sum_{s<d_-} \mu_\tau^-(s) \right).$$

Since $\pi_M \bar{\pi}_M = M$, the rational integer M^{l_1} clears these denominators at every coefficient embedding. To check the Cartan map on the dual tableau lattices, put $n = d_+ + d_-$ and $\check{\mu}_\tau^-(h) = \mu_\tau^-(n-h)$ for $1 \le h < n$. Let ∇_κ denote the $\mathbf{Z}$-span of the standard products of minors of shape κ. Multiplication has an integral right inverse:

$$\nabla_{\mu^+} \otimes \nabla_{\check{\mu}^-} \longrightarrow \nabla_{\mu^+ + \check{\mu}^-}.$$

For each standard tableau T of the combined shape and each height, give the first $\mu_\tau^+(h)$ columns to T_+ and the others to T_-, in their original order. Each subtableau is semistandard, and $b_{T_+} b_{T_-} = b_T$. The transpose on the dual lattices is therefore a split injection. Jacobi's complementary-minor identity changes the second factor from $\check{\mu}^-$ to μ^- with inverse-transpose action and the determinant twist. On exterior coordinates this change and its inverse take complementary index sets with signs, so both are integral. Denote the resulting embedding by C and an integral left inverse by B, so $BC = 1$.

The tensor of the two highest vectors on the disjoint coordinate blocks is the image under C of the combined highest vector: evaluation of the minor products on the ordered bases gives this identity, with the opposite order in the second block. Independent block changes of basis and linearity give the same inclusion in $\operatorname{im} C$ for the tensor of any two block vectors, since the highest vectors generate their representations in characteristic zero. The Cartan product on these tensors is thus given by B. After a unitary move the two Schur actions read the first columns of its matrix and the second columns of its inverse transpose; the latter are read at the conjugate embedding. The preceding exterior-minor estimates clear these columns by the stated M^{l_1} on the tensor. The integral matrix B preserves the resulting lattice, including at $\ell \mid M$.

33.3 A common integral scale for the first factors

In this subsection, from the lattice decomposition of Section 33.2, we bound denominators and archimedean norms of the first factors on one integral scale. These bounds are uniform over all

the finite classes in the subsequent comparison.

Write d_K for the positive quadratic parameter d of the datum. In this subsection the letter d denotes a polynomial degree. Fix the original j and the gluing choices of Section 33.2. The space V'' has rank $d_+ = r + 9m + 1$. Use the same modulus $\mathfrak{m}_\vartheta = \varpi^2 Q$ and the original split residue coordinates. The first factor is a vector $\overline{c''\Upsilon_{\vartheta,i}(h)}$ of weight μ^+ at $h \in U(V'')(\mathbb{A}_f)$, where $\Upsilon_{\vartheta,i}$ is the left class series on the original levels. The scalar c'' is independent of ϑ, i, h.

We use the following normalized definite kernel. Let $K(P, \Psi; g, h)$ be the kernel on the standard definite pair (I_l, I_n), with its two automorphic characters of the prescribed integral weights and tame finite parts, at the finite points (g, h). Let P be homogeneous of degree d. Put

$$s_{l,n,d} = (M^2)^{2fln}\, s_{\mathrm{vac}}^{ln} (M^2 d_K s_{\mathrm{jet}})^d. \tag{I1}$$

Here s_{vac} is the rank-one vacuum scale, s_{jet} the marked jet scale, and d_K the positive integer of the original quadratic field. All factors are nonzero. Let $q_Q = \mathrm{Nm}\, Q$.

If $q_Q M^2$ belongs to the modulus ideal, the residue values of Ψ are integral away from q_Q, and $\ell \nmid q_Q$, then at integral points above ℓ one has

$$|s_{l,n,d} K(P, \Psi; g, h)|_\iota \le 1 \tag{I2}$$

whenever P has integral coefficients at ι. Indeed the residue datum applied to the integral vacuum is a finite Fourier expansion in translates of that vacuum, with translation coordinates integral after multiplication by $q_Q M^2$.

Put $B_{l,n} = \mathrm{Mat}_{n\times l}(\mathcal{O}_E/\mathfrak{m})$. Fourier inversion on this additive group has denominator $|B_{l,n}|$. Since $\ell \nmid q_Q$, localization of $(q_Q M^2) \subseteq \mathfrak{m}$ at ℓ gives

$$v_\ell(|B_{l,n}|) \le 2fln\, v_\ell(M^2).$$

The character values are roots of unity, and multiplication by M^2 makes the dual translates integral above ℓ. For the vacuum scaled by s_{vac}^{ln}, multiplication by $(M^2)^{2fln}$ therefore puts the finite datum in the ℓ-integral span of its translates used in Section 20.1 with $\delta = M^2$. The two integral group points preserve it by the finite Weil action and the prescribed norm-one character weights. The jet at scale $M^2 d_K s_{\mathrm{jet}}$ is integral on that module. Applying the homogeneous polynomial gives precisely the scale (I1), by the homogeneous jet formula in Section 20.1.

We also use the conjugate size part of the same jet estimate. Suppose $l, n \le \Delta$, $d \le M$, and the split modulus $\mathfrak{m}$ satisfies $M^2 \in \mathfrak{m}$. A bound A_0 for all conjugates of the residue values and a bound B_0 for the conjugate polynomial on the polydisc of radius M^4 give

$$|\sigma(s_{l,n,d} K(P, \Psi; g, h))| \le e^{x_1} A_0 B_0. \tag{I3}$$

The total conjugate absolute value of the Fourier coefficients is at most

$$|\mathcal{O}_E/\mathfrak{m}|^{ln} A_0 \le (M^2)^{2fln} A_0,$$

since $(M^2) \subseteq \mathfrak{m}$ and $[E : \mathbb{Q}] = 2f$. This includes the unit modulus of K_0 in (I5), whose constant datum gives just the vacuum.

33.3.1 Bounds for the normalized derivatives

In this subsection, we apply the derivative estimates of Part III to the first factors of the original lift.

The normalized derivative estimate applies to each translated vacuum and to every finite point of the definite groups. The usual homogeneous rescaling uses

$$\frac{(M^2 d_K)^2}{\Omega}(2d+2) \le (M^4)^2$$

in its coefficient ball, where Ω is the original positive period parameter. At radius R in the squared norm of that ball, homogeneity bounds the polynomial by $B_0 e^{dR}$, because $\sqrt{R} \le e^R$. The jet exponential factor is

$$\exp\bigl(C_0(q_1 l n+1)^2 + f l n \log(2d+2)\bigr)$$

in the original numerical notation. Multiplying by the Fourier cardinality and the denominator-clearing factor gives at most e^{x_1} by the cutoff inequality.

Here is the polynomial to which to apply (I2)–(I3). Write $w_{\mathrm{off}} = a/x_1$ for the weight at the non-b-places. The parameter bounds give $w_{\mathrm{off}} \le b$. The total determinant count in the specified space is bounded by q_*, where

$$q_* = f(b + w_{\mathrm{off}} + \Delta + m\Delta).$$

Let Y be the square matrix of short covectors of the $W^\flat = I_{6m+1}$ class in the choice i. The product of stacked determinants consists of constant covectors, some given by Y, over variable covectors.

Separate the W_j positive variables from the ones on

$$W^{\mathrm{add}} = I_{3m-j} \oplus W^\flat \oplus I_{3m}.$$

Let Z denote the positive variables of W_j, let $b_s(Z)$ be the standard tableau products of minors in those variables, and let G_{W} be the Gram matrix of the corresponding integral Weyl basis. For the stacked determinant product P, the Weyl coordinates $\zeta(Y)$ are characterized over the coefficient field by

$$P = \sum_s \bigl(G_{\mathrm{W}} \zeta(Y)\bigr)_s b_s(Z).$$

Both the coefficients of this expansion and the entries of $\zeta(Y)$ are polynomials in the added variables. The number of variable columns remaining in a determinant attached to height h is $d_+ - h$. For $1 \le h < k-1$ the consecutive stem entries $\varsigma_{h-1}, \varsigma_h$ of Section 33.1 differ by one. Thus

$$\mu_\tau^+(h) = \bigl(\varsigma_{h-1} - \max(\varsigma_h, (t-1)/2 - w_j) - 1\bigr)_+ = 0.$$

The height-zero count also vanishes. Every contributing height therefore satisfies $d_+ - h \le d_+ - (k-1) = 10m+1$. Consequently their common homogeneous degree is

$$d = \sum_{\tau,h} (d_+ - h)\mu_\tau^+(h) \le (10m+1) q_* \le M.$$

The constant covectors contain only $0, 1$ and entries $\Phi_\tau(Y_{ab})$ at the complex embeddings of the CM type.

Let $d_s(Y) = (G_{\mathrm{W}}\zeta(Y))_s$. Integral straightening for products of minors [2, Theorem II.2.16] puts the coefficients of each $d_s(Y)$ in the ring generated by the constant covectors. In the convention of that theorem the lengths of the exterior factors are the column heights used here; its Schur map alternates each column and multiplies the entries in each horizontal line, giving exactly the products of minors.

33.3.2 Inverse Gram matrices in the chosen coordinates

In this subsection, we pass from the derivative bounds of Section 33.3.1 to the integral Gram scale.

To pass to the dual Weyl coordinates ζ_s, let $\Delta_u(T_\tau)$ be the leading principal minor of order u and put

$$F_\vee(T) = \prod_\tau \prod_{1\le u<d_+} \Delta_u(T_\tau)^{\mu_\tau^+(d_+-u)} = \sum_\alpha a_\alpha T^\alpha, \qquad H = F_\vee(\partial)F_\vee(0) = \sum_\alpha \alpha! a_\alpha^2.$$

Here $F_\vee(I) = 1$, so H is a positive integer. Complementation in exterior powers, with reversed order of the covectors and application of the determinant tensor, is an integral isometry between the Gram spaces for the complementary shapes. On the exterior tensor basis it permutes the complementary column fillings with signs.

It takes the highest generated rational subspace to the corresponding complementary subspace, by the exterior inverse-transpose law. For either shape let $F_{T,s}$ be the coefficient of b_s in the straightening of the exterior filling T, as in Section 34.1.1. For a standard filling S, $F_{S,s} = \delta_{S,s}$. Projection π to the coordinates of these fillings satisfies $\pi F = I$. Hence, if x is rational and Fx is integral, then $x = \pi F x$ is integral. This identifies the Weyl lattice of each shape with the intersection of its rational subspace and its exterior tensor lattice. Thus the signed isometry and its inverse are integral on the Weyl lattices. By the Fock Gram identity in Section 34.1.1, applied to this complementary shape, its Fock matrix inverts its Gram matrix up to the factor H.

Conjugating by that integral isometry shows that H times the inverse Gram matrix of μ^+ is integral. Thus every $H\zeta_s(Y)$ has integral coefficients whenever those entries of Y are integral, at the same embedding. The complementary shape has d boxes. Expanding each determinant by permutations gives $\sum_\alpha |a_\alpha| \le d!$, since the product of the factorials of the column heights divides $d!$. As $\alpha! \le d!$, we obtain $H \le (d!)^3 \le d^{3d}$, with $H = 1$ when $d = 0$. All choices of covectors have the same d, H.

For the norm estimate, specialize the added variables to a point of the M^4-polydisc at the chosen conjugate embedding. The short-covector entries of Y also have absolute values at most M^4 there. For one determinant of height h, a block unipotent covector operation clears all but the first h variable columns from the positive block. Indeed the remaining positive columns meet fixed unit covectors, which subtract their coordinate entries from each variable covector.

Let C be the matrix of the remaining specialized columns. Laplace expansion in the first h columns gives the exterior coordinates $\pm\det C_{I^c}$, where I ranges over the h-subsets of the variable covectors. Cauchy–Binet and Hadamard give

$$\sum_I |\det C_{I^c}|^2 \le \det(C^*C) \le \prod_s \|C_s\|_2^2.$$

The number of columns of C is $\dim W^{\mathrm{add}} + \mathrm{posRank}(W_j) - h \le 12m + 1$. Each squared column

length is at most $(r+11m+1)M^{8}\le M^{9}$. The Cartan product has norm at most the product of norms, since on the exterior tensor model it is orthogonal projection of the tensor product.

For a polynomial V, write V^{σ} for the result of applying σ to its coefficients before evaluating the added variables. Thus for $V=H\zeta(Y)$ and $\|z\|_{\infty}\le M^{4}$,

$$\|V^{\sigma}(z)\|_{\mathrm{Gram}}\le B:=HM^{9(12m+1)q_{*}/2}. \tag{I4}$$

For a linear functional $L(z)=\sum_{s}a_{s}z_{s}$ of Gram dual norm one, put $Q=\sum_{s}\sigma^{-1}(a_{s})V_{s}$. Then

$$Q^{\sigma}(z)=L(V^{\sigma}(z)),\qquad \sigma(K(Q))=L\big((\sigma(K(V_{s})))_{s}\big),$$

where the remaining arguments of the kernel K are fixed. The first identity and (I4) give the bound B on the polydisc. Apply (I3) to Q. The common scalar $s_{l,n,d}$ commutes with L after applying σ, so taking the supremum in the second identity gives the same bound for the vector of normalized kernel values.

To expand the first factor, let

$$c_{0}=s_{m,3m,0},\qquad c_{1}=s_{j,d_{+},0},\qquad c_{2}=s_{\dim W^{\mathrm{add}},d_{+},d}.$$

For the scalar kernel K_{0} on (I_{m},I_{3m}), the finite modulus is the unit ideal, the polynomial and the finite datum are both 1, and the two characters are $\chi_{M}^{j}\vartheta$ and 1. The second kernel K_{1} is on (I_{j},V'') with polynomial one and the prescribed inclusion indicator acted on by the norm-minus-one residue unit. Its second character is the inverse of the first-factor character times $\upsilon^{d_{+}}$. For the corresponding kernel K_{-} with source I_{j}^{-}, the negative-source identity in the Weil model reads

$$\epsilon K_{-}=\overline{K_{1}},\qquad |\epsilon|=1.$$

The scalar ϵ depends only on the fixed pair and is independent of ϑ,i,h. The third kernel K_{2} is the linear map on polynomials for (W^{add},V'') with the first-factor character on both sides. Write its residue matrix in the three added blocks as $(z_{0},z_{\flat},z_{3})$. On the locus where $(z_{0},z_{\flat})$ equals the prescribed inclusion modulo $\mathfrak{m}_{\vartheta}$, its datum is

$$\sum_{A\in\mathrm{GL}_{3m}(\mathcal{O}_{E}/(\varpi^{2}))}\psi_{\Lambda}\big((z_{3}\bmod\varpi^{2})A\big),$$

where ψ_{Λ} is the trace character of the chosen residue matrix Λ. The datum is zero elsewhere. Its values are integral and every conjugate absolute value is at most

$$\mathcal{N}=|\mathrm{GL}_{3m}(\mathcal{O}_{E}/(\varpi^{2}))|.$$

The double class sum is over I_{3m} and I_{m} at their integral levels. Both class weights are the reciprocals of the actual stabilizer orders. Put $n_{A}=C_{G}$: reduction at an old odd split prime embeds each stabilizer into a finite general linear group of rank at most rt, so its order divides n_{A}. Expanding that sum and the coordinate minor product gives the identity

$$\begin{aligned} c''&=n_{A}^{2}c_{2}Hc_{0}\overline{c_{1}}\epsilon\neq 0,\\ \overline{c''\Upsilon_{\vartheta,i}(h)}&=n_{A}^{2}\sum_{w,v}^{\mathrm{cl}}\overline{(c_{0}K_{0}(v,w))\,\overline{c_{1}K_{1}(h)}\,c_{2}K_{2}(H\zeta(Y);w,h)}.\end{aligned} \tag{I5}$$

In (I5) the arguments of K_{2} on the first group are $\mathrm{diag}(1,h^{\flat},w)$.

For (I2), integral representatives can be chosen by rational translation at all places over ℓ. When moving the representative of the short covectors by γ_1, transport the matrix Y to $Y\gamma_1^{\mathsf{t}}$. This matrix is integral at the conjugate embedding because

$$\gamma_1 Y^{\mathsf{t}} = (\gamma_1 h^\flat)((h^\flat)^{-1}Y^{\mathsf{t}})$$

is a product of two integral matrices there. The last factor is integral by the original choice of short covectors from that moved lattice. Rational covariance of the kernel brings the transformed value back to the gauge at the original h. For source and target rational elements Γ, γ' and character exponents e_τ, e'_τ, the exact complex identity is

$$\begin{aligned} K_P(\Gamma g, \gamma' h) &= \left(\prod_\tau \tau(\det \gamma')^{-e_\tau}\tau(\det \Gamma)^{e'_\tau}\right) K_{P^{A,B_1}}(g,h), \\ P^{A,B_1}(X) &= P\big((A_\tau^{\mathsf{t}} X_\tau B_{1,\tau}^{\mathsf{t}})_\tau\big), \\ A_\tau &= \tau(\gamma'^{-1}), \qquad B_{1,\tau} = \tau(\Gamma). \end{aligned}$$

Rational invariance of theta and the compact Fock action at the inverse real pair give the identity before any coefficient embedding. In the stacked case use $B' = \operatorname{diag}(1, B_1)$ on the full column set. Each stacked determinant changes by $\det(A)\det(B')$. Write $e_j^{(1)} = j + \lfloor (t+k-1)/2 \rfloor$ for the first-factor character exponent. There are exactly $\kappa_j(\tau) - e_j^{(1)}$ such determinants at τ. The exponent of the first-factor character is the negative of this number, so these determinants cancel the two character factors of the kernel. The remaining Schur action on conjugate coordinates is exactly the gauge transformation at $\gamma' h$. Thus no polynomial coefficient integrality is asserted before moving the rational representatives.

In the conjugated product in (I5), use the embedding $\iota \circ \mathrm{conj}$ for K_0, K_2 and ι for K_1. The scalar kernels have the indicated integral values after moving their representatives, with constant polynomials. Both exponents of K_0 are zero. In K_1 the source stays at one and the character on the target determinant is trivial. Thus the rational moves preserve each scalar value. The vector kernel is integral in the original gauge by the covariance just proved. The factors in the two class denominators divide n_A^2. Therefore (I5) has gauge at most one for every ι above every $\ell \nmid q_Q$.

Suppose the auxiliary conductor is empty, so that $M^2 \in \mathfrak{m}_\vartheta$; this includes $\vartheta = 1$. For $\phi \in \mathrm{Aut}(\mathbf{C})$ put $\sigma = \phi \circ \mathrm{conj}$. For every prescribed pair $(h^\flat, Y)$, trace matrix Λ, and finite points h, w, v, the K_0, K_1, K_2 factors of the ϕ-image of the summand in (I5) are evaluated at σ, ϕ, σ, respectively. Their bounds, in that order, are

$$e^{x_1}, \qquad e^{x_1}, \qquad e^{x_1}\mathcal{N}B$$

by (I3)–(I4), using the short covectors chosen for $h^\flat$. The triangle inequality and $|\Gamma_c|^{-1} \le 1$ bound the squared Weyl norm by

$$\left(n_A^2 h_{3m} h_m e^{3x_1}\mathcal{N}B\right)^2,$$

where h_a denotes the number of classes of I_a at its integral level. The lattice count in the proof of Lemma 7.2 applies at ranks $m, 3m$ with this level. Thus $\log h_a = O(x^{52})$. Also $\mathcal{N} \le (M^2)^{2f(3m)^2}$ and $\log B \le 3d \log M + \frac{9}{2}(12m+1)q_* \log M$. The logarithm of the displayed square is thus at most

$$4\log n_A + O(x^{52}) + 6x_1 + 8f(3m)^2 \log M + \big(6d + 9(12m+1)q_*\big)\log M \le fmb(\log b)^2.$$

Indeed $q_* = O(fb)$, $d = O(fmb)$, and $\log M \le 22 \log b$. The definition of n_A gives $\log n_A \le f(rt)^2 \log M$. The ranks and f grow at most polynomially in x, while $x_1 \le b^{1/10^6}$ by Section 4.1. The displayed left side is therefore $o(fmb(\log b)^2)$ for fixed d_K, ϵ as x tends to infinity. Thus the scale in (I5) gives both first-factor bounds for the marked gluing. Its nonvanishing precedes both the three-exchange identity and the carrier-peak injectivity used to prove $c'_\vartheta \neq 0$.

34 The comparison operator and the first theta lifts

In this section, we define the contraction on the lattice of Section 33 and compute the Gram forms and paired periods that enter the exchange of Section 36.

34.1 The comparison operator on the original lattice

In this subsection, we contract the target tensor against the first factors of Section 33.3. The operator takes values in the measured direct sum over the original classes, and the vanishing of Section 33.1 identifies the period terms it retains.

Keep the actual twist ϑ, original descent level and quotient measures fixed. The tuple space $\mathcal{H}$ is the direct sum over the gluing choices and the $J(\pi_M^2)$-classes in V° of the second-factor Weyl spaces with their Gram forms. This direct sum has its plain class metric, since the second-factor level is free. Put $n_A = C_G$. For the gluing classes the bottom covectors $\pi_M I$ of the stabilizer matrix force the reduction of every element of J_Λ to be the identity at each old place. A definite class stabilizer is finite.

At one old odd split place it injects into a conjugate of the torsion-free principal congruence subgroup, so it is trivial. The same argument applies to the second-factor $J(\pi_M^2)$-classes. Consequently the reciprocal stabilizer weights in the original gluing operator G_ϑ are all one, and the operator is

$$G_\vartheta f = M^{l_1} \sum_{i,c} \sum_{[h] \bmod J_\Lambda} \delta_{(h,c_f)L_i}(\phi''_{\vartheta,i}(h) \cdot f(i,c)).$$

The dot is the integral Cartan product in the original block order.

The notation $\delta_z(v)$ means the class form obtained by K-equivariant averaging of the specified value v at z; its values sum over the class stabilizer without division. In the target pairing against an equivariant value this multiplicity cancels the reciprocal target stabilizer order, so $\langle \delta_z(v), a \rangle = \langle v, a(z) \rangle$. The first-factor coefficient is $\overline{c'' \Upsilon_{\vartheta,i}(h)}$ from Section 33.3. Its gauge is at most one. For the second factor we use the integral vectors constructed below. Section 33.2 shows that the power M^{l_1} clears the divided lattice in the Cartan product; the point-form evaluation sums unitary quotient transports with unit level characters, so preserves gauge. This verifies the integral gluing for these definite coordinates.

To construct this finite family, work first with the fixed column shape μ^-. Let $b_i(X)$ denote the standard minor products and F the integral matrix of the embedding of their dual coordinates into the tensor product of the exterior powers for the columns. Explicitly, the covector of F at a tableau T is the list of coefficients of its product of minors in the b_i.

The matrices here use the original space enumerations. Writing G for the Gram matrix in the coordinate space, Cauchy–Binet in every column gives

$$K(X,Y) = \sum_T b_T(X) b_T(Y) = \sum_{i,k} G_{ik} b_i(X) b_k(Y), \qquad G = F^{\mathsf{t}} F.$$

Here $K(X,Y)$ is the product of the prescribed leading minors of $Y^{\mathsf{t}}X$. In the sum over T, take an increasing list of indices independently in each exterior column, including each choice for repeated columns. The indices i,k enumerate the semistandard tableaux. Integral straightening expresses the coefficients of each exterior column filling in that basis and gives the integral matrix F. Let

$$H_{ik} = b_i(\partial)b_k(0), \qquad h_- = b_{\mathrm{high}}(\partial)b_{\mathrm{high}}(0).$$

The last integer is positive because the highest minor product takes the value one on the standard array, and the Fock pairing of a polynomial with real coefficients with itself is a positive weighted sum of its squared coefficients.

34.1.1 The comparison in the Fock model

In this subsection, we evaluate the comparison operator on the marked monomials in the metric of Section 33.3.2.

For $P = \sum_\alpha p_\alpha X^\alpha$ and $Q = \sum_\alpha q_\alpha X^\alpha$, put

$$B(P,Q) = P(\partial)Q(0) = \sum_\alpha \alpha! p_\alpha q_\alpha, \qquad (S_a P)(X) = P(aX).$$

The chain rule gives $B(S_a P, Q) = B(P, S_{a^{\mathsf{t}}} Q)$. The Hermitian Fock form $\langle P,Q\rangle_F = B(\overline{P},Q)$ therefore has $S_a^* = S_{\overline{a}^{\mathsf{t}}}$. Both forms give H on the integral basis (b_i). The algebraic kernel obeys the transpose rule for B. Therefore HG commutes with the dual Schur action on the coordinate space, and is scalar by irreducibility. At the standard array Y_0, write $v = (b_i(Y_0))_i$. Then $v^{\mathsf{t}}Gv = K(Y_0,Y_0) = 1$ and $K(\cdot,Y_0) = b_{\mathrm{high}}$ in the prescribed enumeration. Consequently

$$v^{\mathsf{t}}GHGv = h_-, \qquad HG = GH = h_- I.$$

Thus for a vector e_T of the orthonormal basis of the full exterior tensor product, the vector $\eta_T = HF^{\mathsf{t}}e_T$ has integral coordinates and

$$F\eta_T = h_- \operatorname{pr}_{\operatorname{im} F}(e_T).$$

It follows that

$$\sum_T \langle \eta_T, \xi\rangle_G \eta_T = h_-^2 \xi, \qquad \|\eta_T\|_G^2 \le h_-^2. \tag{A}$$

For a given prime $\ell \nmid \operatorname{Nm} Q$, move each original second-factor representative h_c by a rational unitary element δ_c, so that $\delta_c h_c$ is integral above ℓ.

Here weak approximation is over $F_0 = E^+$. For an adelic point g, let K_v be the integral lattice stabilizer. The conditions $\delta_v \in K_v g_v^{-1}$ at $v \mid \ell$ are open. At any other finite set of places one may also require $\delta_v \in g_v J_v g_v^{-1}$ for prescribed open level subgroups J_v, so that $\delta_v g_v \in g_v J_v$. Choose a point u_v in each such open set with $\det(1+u_v) \neq 0$. Indeed a norm-one scalar arbitrarily close to one avoids the finitely many forbidden eigenvalues. The Cayley parameters $A_v = (1-u_v)(1+u_v)^{-1}$ are skew-Hermitian. Approximate them by a single rational point A of this F_0-vector space. Then $\delta = (1-A)(1+A)^{-1}$ satisfies all the open conditions. The inverses here are in the local fields, also at $v \mid 2$.

Only finitely many places above ℓ are involved. Transport (A) at coordinate (i,c) by

$$\eta_{c,T} = \overline{\rho(\delta_c^{-1})\overline{\eta_T}}, \qquad \rho(\delta_c)\overline{\eta_{c,T}} = \eta_T.$$

Here ρ is the Weyl action of weight μ^- and determinant exponent zero. The transport is Gram-unitary, since the rational point is unitary at every embedding and the Gram matrix is real. After any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the entries of the transport are read at the embedding $\sigma\tau$ of the CM field; it is again Gram-unitary.

Therefore each tuple f_a, supported at the single coordinate (i, c) with value $\eta_{c,T}$, has squared norm at most h_-^2 at every σ, and these tuples have sum of rank-one operators $h_-^2 I$ in $\mathcal{H}$. Rational covariance of the gauge and the displayed integral coordinate identity show that the value of f_a at h_c has gauge at most one at every embedding above ℓ.

In this calculation the embedding applied to the unbarred coordinate is the conjugate of the embedding applied to $\rho(\delta_c)\overline{\eta_{c,T}}$. The same spanning family therefore works at both coefficient embeddings. Write $v_a = G_\vartheta f_a$. All v_a are integral at every embedding above ℓ when ℓ is prime to the twist modulus. Define the comparison operator by

$$U_\vartheta = n_A h_-^2 G_\vartheta G_\vartheta^* = n_A \sum_a v_a \langle v_a, \cdot \rangle.$$

The equality follows by summing the rank-one operators in (A) over the class coordinates.

At the chosen embedding use the conjugate coefficient embedding in the first Gram factor. The two transported coordinates cancel by the transpose identity for the Weyl Gram matrix. The rational class weights have denominators dividing n_A. For integral z, the coefficients $n_A\langle v_a, z\rangle$ are therefore integral. This proves preservation of the original integral target lattice by U_ϑ.

For the fixed twist the comparison identity is $c'_\vartheta G_\vartheta^* u_{j,\vartheta} = P$, proved below, with $c'_\vartheta \neq 0$ on a nonzero good source. The individual kernel pairing of $G_\vartheta^*\xi$, for $\xi \in F_w$, reduces to the original block period in Section 33.1. Write $\eta_{i,g}$ for the tuple supported on the fixed gluing choice i, with entries the second-block kernel vectors at the source point g. Expanding the original gluing adjoint gives

$$\langle \eta_{i,g}, G_\vartheta^*\xi\rangle = \sum_a c_{i,a} \sum_b \langle V_{i,a,b}(g), \xi(x_{i,a,b})\rangle = 0.$$

Here a, b index the first- and second-block classes, $x_{i,a,b}$ is their original glued point, and $V_{i,a,b}(g)$ is the Cartan product of the fixed first-block vector and the second-block kernel vector. The weights $c_{i,a}$ are those of the gluing adjoint. For each fixed i, a, g the inner sum is the class sum of the vanishing block period at $J(\pi_M^2)$. The second-block stabilizers are trivial, so this is the required weighted class sum and is zero.

For $z = u_{j,\vartheta} y$, the comparison identity gives $U_\vartheta z = n_A h_-^2 (c'_\vartheta)^{-1} G_\vartheta P y$. For $\xi \in F_w$, move G_ϑ to its adjoint. The period for $\langle G_\vartheta^*\xi, Py\rangle$ is zero by the individual kernel pairing vanishing. Hence

$$\langle \xi, U_\vartheta z\rangle = 0 \qquad (\xi \in F_w,\ z = u_{j,\vartheta} y,\ \text{every } w). \tag{90}$$

Thus U sends integral vectors in Z to integral tests for the same false-lift subspaces. In particular a clearing operator Λ with the bound above makes $\Lambda \pi_Z U|_Z$ preserve the original integral lift lattice. This proves the required compatibility of the clearing and comparison operators on that lattice.

Nonvanishing of the real first-arrow constant. At one real place let $P \neq 0$ be the prescribed joint harmonic polynomial. In the source action the variables are grouped by target index. In each such group, the source signs give the two Fock signs up to a common reversal. The polynomial is termwise harmonic: $\partial_\alpha \partial_\beta P = 0$ when the two variables have opposite signs in a

single such group. Use the source discrete-series parameter λ so that its lowest compact highest weight is the weight of P for the oscillator action, including the determinant twist. In the real factor of the doubled integral $\overline{e_j(g)}/e_j(1)$ is then the conjugate of the normalized coefficient $c_\lambda(g)$ of the unit highest vector of this discrete series. This convention is the contragredient on the scalar form side.

If z is a positive definite element of the source unitary group, its action on the matrix space is positive definite, since on each group of variables with a common target index it is a copy of z, its transpose, or its inverse transpose in the indicated sign conventions. Its blocks on the positive and negative Fock variables are positive definite. Its determinant twist is one, since $\det z = 1$.

In the Gaussian Bargmann kernel for this action the two pure quadratic terms involve only opposite-sign pairs with the same target index. The mixed bilinear term has positive definite coefficient given, up to transposition with the variables, by the inverse of the complex-linear part of the action. Indeed that linear part is the direct sum of the positive sign blocks, with complex conjugation on the negative variables. The scalar of the kernel is the positive vacuum coefficient. It is the reciprocal determinant of the negative principal block in the unitary normalization.

34.1.2 The Gaussian pairing

In this subsection, we evaluate the corresponding Gaussian integral in the Fock model of Section 34.1.1.

Pair the Gaussian kernel against P in both Bargmann variables by differentiation at zero. Every contribution from a pure quadratic term vanishes by termwise harmonicity. The remaining mixed kernel pairs P with its linear substitution by the positive definite coefficient matrix T. For the positive definite square root $D = T^{1/2} = D^*$,

$$\langle P, S_T P\rangle_F = \langle S_D P, S_D P\rangle_F > 0,$$

since $P \neq 0$ and D is invertible. Consequently

$$\langle P, \omega(z)P\rangle > 0.$$

The determinant twist and the scalar multiplying the Bargmann kernel are fixed by continuity at the identity on the positive cone.

To deduce the assertion about the projector, let K_∞ be the maximal compact of this source group, with probability Haar, and d_K the dimension of its lowest compact type. Insert Sun's positive-factor formula for this coefficient, as in the real coefficient decay calculation, and average the integrand over $K_\infty \times K_\infty$. At a positive factor z the result is

$$\begin{aligned}\int_{K_\infty^2} &\overline{c_\lambda(k_1^{-1} z k_2)}\langle P, \omega(k_1^{-1} z k_2)P\rangle \, dk_1 \, dk_2 \\ &= \frac{1}{d_K}\int_{K_\infty} m_{-\lambda-\rho}(u^{-1}zu)^{-1}\langle P, \omega(u^{-1}zu)P\rangle \, du.\end{aligned}$$

In fact Sun's formula is an integral of rank-one compact projectors times the displayed weight of the positive factor. For a fixed u, substitute $k_1 \mapsto uk_1$, $k_2 \mapsto uk_2$ and apply compact Schur orthogonality on each side of the oscillator matrix coefficient.

The two factors d_K^{-1} cancel one of the factors from the compact integral formula, leaving the displayed d_K^{-1}. A preceding compact factor in the polar decomposition is absorbed by

Haar in k_1. For $Z = u^{-1}zu$ we have $\det Z = 1$ and $Z^{-1} = JZJ$. Jacobi's complementary-minor identity gives $D_I(Z) = D_{I^c}(Z)$. Since $\{i : -a_i \geq t\} = \{i : a_i \geq 1 - t\}^c$ for integral a, the definition in Section 12.7 gives $m_{-a}(Z) = m_a(Z) > 0$. Thus either sign of $a = \lambda + \rho$ gives the displayed weight, with ρ indexed in decreasing λ order. The right side is strictly positive. Absolute integrability follows from integrability of the original sufficiently separated discrete-series coefficient and the unitary norm bound for ω. Integrating over the real group proves nonvanishing. The tensor product over the real places gives the nonzero constant in the original doubled first-arrow formula.

34.2 Gram forms for the first lifts

In this subsection, we compute the Gram form of the initial theta lift with the finite and archimedean data fixed. Along with the paired period of Section 34.3, this identity determines the normalization for composing the lifts.

We use the doubling seesaw of [28, §11.3, pp. 820–821]. At the ranks below the strict-range formula of Section 17.2 applies to every translated Schwartz–Bruhat test. Unfolding its graph terms gives the Gram identity with the local masses of the specified vectors and compact subgroups.

Put $j = i+1$, $\mathfrak{m} = \pi_M Q(\xi)$ and let S be the union of its support and the conjugate support. Write r for the original source rank and $d = r + t$ for the rank at the first arrow, so $d > 2r$. Let K_0 be the original standard compact in the finite source group. The finite bases identify this group and K_0 for every space in the calculation. Give every local factor of K_0 mass one. Denote the source principal level at the old places by J_M and at the whole modulus by $J_{\mathfrak{m}}$.

The character on the target side of the first lift in the sequence indexed by i is the vertical character of index i evaluated at the rank d. It is

$$b_j = \mu_j \xi^j(\upsilon|_{U(1)})^{\lfloor d/2 \rfloor}.$$

For the single lift on W_j at ξ^j, the rank twist of its defining character is the same expression. This is the character in the source factor of each Weil matrix coefficient. The first-column identification sends the Hermitian form J to $-J$ and keeps the ordered Harish–Chandra parameter λ. The equations $g^*Jg = J$ and $g^*(-J)g = -J$ define the same group in these coordinates. The compact roots and the inducing character for λ are unchanged. Thus the normalized coefficient and the Haar measure agree, and pullback is a complex-linear isometry commuting with finite right translations. The contragredient parameter is used separately for a reversed theta arrow in Section 34.3.2. The finite target character is unramified off S on both source spaces.

34.2.1 The masses of the first lifts

In this subsection, we compute the local measures required for Proposition 34.1.

At a chosen member of a chosen split pair use the original Schrödinger matrix $X \in M_{d,r}(E_w)$. For g in the source integral compact, the action on the residue test is

$$F(X) \longmapsto b_{j,w}(\det g)^{-1} F(Xg).$$

Here the finite character is read on the chosen split coordinate. The absolute-value factor on such g is one and the unramified Weil splitting has trivial unit factor. The opposite coordinate is already accounted for by the dual basis in this model. On the triangle the test at every

distinguished place is the point $\binom{I_r}{0}$. On the single lift it is this point at an old place, and at a new place it is

$$F_w\left(\binom{U}{V}\right) = \chi_w(\det U), \qquad U \in M_r(k_w), \quad V \in M_{t,r}(k_w).$$

The value is zero for noninvertible U. Here χ_w is the unit character of $b_{j,w}$. At the other places both data are the original vacuum.

Every matrix in either prescribed support has an integral left inverse. If X and Xg belong to their respective supported integral boxes, applying these left inverses shows g, g^{-1} are integral. For the point support the overlap then occurs exactly when $g \equiv I_r$. On the new block support the transformation $\chi_w(\det(Ug)) = \chi_w(\det U)\chi_w(\det g)$ cancels the phase. The stabilizer of the coefficient of the block test is therefore the entire integral source group at that place.

At an old place the stabilizer is the principal subgroup. Normalizing the residue measure by the original integral matrix box of mass one, put

$$B_\xi = |M_{d,r}(\mathcal{O}_E/\mathfrak{m})|, \qquad w_0 = [K_0 : J_\mathfrak{m}] = |\mathrm{GL}_r(\mathcal{O}_E/\mathfrak{m})|.$$

The matrix-coefficient integrals on the two marked products are exactly

$$c_\triangle = \frac{B_\xi^{-1}}{w_0}, \qquad c_{\mathrm{single}} = B_\xi^{-1}\frac{\prod_{w|Q(\xi)} |\mathrm{GL}_r(k_w)| q_w^{rt}}{\prod_{w|\pi_M} |\mathrm{GL}_r(k_w)|}.$$

Indeed the coefficient is the squared norm of the local test on its stabilizer and zero elsewhere. The squared norm of a block test is its support proportion, and that support has rt unconstrained entries and an invertible r-block. There is no extra Haar factor at a conjugate distinguished place.

For the single lift multiply c_{single} by the target level index and by the Weyl dimension, to pass from the scalar Petersson pairing to the algebraic class pairing. Compact Schur orthogonality gives the Weyl dimension for the original normalized real coefficient. The Weyl dimension is independent of the twist. Write

$$F_w = \prod_{h=t+1}^{t+r} (1 - q_w^{-h}).$$

At an old place the index of the target vector-basis stabilizer is the number of full-rank d-by-r matrices, namely $q_w^{dr} F_w$.

At a new place the single-lift level is the opposite parabolic $P^- = \{\left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)\}$. Its action on the first block of $\binom{U}{V}$ factors through A, so the test transforms by its determinant character. Conversely, let $(A\ B)$ be the upper blocks of a residue matrix preserving the support. Testing $(U, V) = (I_r, 0)$ gives $A \in \mathrm{GL}_r(k_w)$. If $Bb \neq 0$, take $a = -A^{-1}Bb \neq 0$ and choose V with $Va = b$. Then $(A + BV)a = 0$, contradicting support preservation at (I_r, V). Thus $B = 0$, which proves the asserted stabilizer. Inverse transpose identifies P^- with the parabolic P^+ preserving the first r-dimensional subspace. The action of P^+ on its ordered bases is transitive, with stabilizer the pointwise fixer above. Thus the index of P^- is the old pointwise-fixer index divided by $|\mathrm{GL}_r(k_w)|$. The target index times c_{single} consequently contributes $F_w/|\mathrm{GL}_r(k_w)|$ at each old place and $q_w^{rt} F_w$ at each new one. The first group of factors is fixed and the second is precisely

$$c_Q(\xi) = \prod_{w|Q(\xi)} q_w^{rt} F_w.$$

For the triangle Petersson pairing there is no such target conversion at the first arrow.

34.2.2 Doubling and the common series

In this subsection, we use the masses of Section 34.2.1 to compare the two Gram forms.

Let A be the complement of S and put $G_A = U_r(\mathbb{A}_{F_0,A})$, with the finite restricted product interpreted in the original finite bases. Let K_A be its standard integral compact. Take representatives γ for G_A/K_A, with 1 representing K_A. Extend them by the identity at S and write

$$u_j(\gamma) = \big(b_j(\det\gamma)\Delta(\gamma)^d\big)^{-1}, \qquad R_j = \sum_{\gamma \in G_A/K_A} u_j(\gamma) R(\gamma).$$

The coefficient is bi-K_A-invariant. Both source levels contain K_A, so $R(\gamma k)y = R(\gamma)y$ for $k \in K_A$ and for each source vector y to which the series is applied. On each of the d target covectors the source acts through $\gamma^{-\mathsf{t}}$. Transposition preserves the intersection elementary divisors, and $\Delta(\gamma^{-1}) = \overline{\Delta(\gamma)}$. The vacuum coefficient $1/\overline{\Delta(\gamma^{-\mathsf{t}})}$ therefore contributes $\Delta(\gamma)^{-1}$ per covector. In particular its absolute value is

$$|u_j(\gamma)| = \left(\prod_{w \in A} [\gamma\Lambda_w : \gamma\Lambda_w \cap \Lambda_w] \right)^{-d/2},$$

where the product includes both members of split pairs. This follows by multiplying the integral lattice overlap by its unitary square-root factor. The splitting character in the overlap is the same one in both cases, since b_j, d, and the matrix in the chosen coordinates are identical.

Absolute convergence and injectivity can both be seen by a lattice count in the present ranks $t = 5r$, $d = 6r$. At a split pair a lattice at the chosen place determines its dual at the other. If it is not contained in Λ_w, choose x in it with $\varpi_w x$ primitive in Λ_w. Enlarge Λ_w by x and take its dual at the opposite place. The intersection at the chosen place is multiplied by a factor q_w, and the old intersection at the opposite place is still contained in the new dual, by the integral pairing with x. If containment is strict in the other direction, start on the opposite member. Each enlargement is specified by the line of $\varpi_w x$ in the residue space of dimension r. Thus there are at most $2(1 + q_w + \cdots + q_w^{r-1})$ moves at one split pair.

At a nonsplit unramified place the two lattices are self-dual. Choose $\varpi_w = \varpi_v$ in the fixed field. If the lattices differ, take x as above in Λ'_w and put $y = \varpi_w x$. The lattice $\Lambda_0 = \{z \in \Lambda_w : h(y,z) \in \varpi_w \mathcal{O}_w\}$ has index q_w. The integral lattice $\Lambda_0 + \mathcal{O}_w x$ is self-dual, since it also contains Λ_0 with index q_w. The original intersection lies in Λ_0, and the new intersection is multiplied by a factor at least q_w by adjoining x. For a fixed residue line of y, two choices of x differ by $z \in \Lambda_w$ after fixing that line's representative. Modulo ϖ_v one has

$$\varpi_v\big(h(x+z, x+z) - h(x,x)\big) \equiv \operatorname{Tr}_{k_w/k_v} h(\bar{y}, \bar{z}).$$

The residue pairing with $\bar{y} \neq 0$ identifies Λ_w/Λ_0 additively with k_w. Since k_w/k_v is finite separable of degree two, its trace is surjective, also in characteristic two. Thus at most $|k_v| = \sqrt{q_w}$ classes give an integral square for this line. Each new self-dual lattice is isometric to the original one. Indeed its reduction is a nondegenerate Hermitian space over k_w/k_v. If every vector had zero norm, polarization with $x + ay$ for every $a \in k_w$ would give $\operatorname{Tr}_{k_w/k_v}(a h(x,y)) = 0$ for all a, x, y, contradicting nondegeneracy of the residue trace pairing. Lift a vector of nonzero norm and scale its norm to one by surjectivity of the norm on units (Section 4.3). Its orthogonal complement in the lattice is self-dual. Induction gives an integral orthonormal basis, also at places above two.

For each such fixed move choose an isometry taking the new standard lattice back to the original one. Repeat on the moved Λ'. The product of indices in $|u_j|$ decreases by a factor at least q_w at each step. Reversing the resulting word recovers the original lattice coset. Its coefficient is bounded by the product of the step weights $q_w^{-d/2}$. The sum of those step weights over all choices of a single step is

$$\eta \ll [E:\mathbb{Q}] \sum_{n\geq 2} n^{-2r} \ll [E:\mathbb{Q}]\, 2^{-2r}.$$

Indeed either count above is $O(q_w^r)$, and there are at most $[E:\mathbb{Q}]$ places of a given prime power norm.

Here $[E:\mathbb{Q}] \ll x^{12}$ and $r \geq x^5$, so $\eta < 1/3$ for all sufficiently large x. Choose one word for each nonidentity coset. The geometric series gives absolute convergence in operator norm and

$$\|R_j - I\| \leq \sum_{\gamma\neq 1} |u_j(\gamma)| \leq \frac{\eta}{1-\eta} < \frac{1}{2}.$$

If $R_{j,<L}$ contains the representatives with chosen word length less than L, the same count gives

$$\|R_j - R_{j,<L}\| \leq \sum_{h\geq L} \eta^h = \frac{\eta^L}{1-\eta} \qquad (L \geq 1).$$

These estimates hold on the full space, uniformly in the prescribed support, which only removes places from the count. In particular R_j is injective.

The doubled action is identified in Section 17.2. The doubled source has dimension $2r$ and Siegel rank r. For the target Witt index $i(V)$, we have $d - i(V) \geq d/2 > r$. The all-Schwartz formula of [75, §52, Theorem 5] therefore applies to the separately translated factors, as verified in Section 17.2. The Siegel–Weil factor in this range is one. The doubled datum is the original tensor times the conjugate tensor on the opposite source. In a rational Schrödinger polarization the partial Fourier transform in the difference variable gives the product of the theta kernels on the diagonal group, and its graph zero term gives the original unitary matrix coefficient. The finite opposite-basis scale cancels against its inverse in the conjugate tensor. The graph change of variables preserves the probability measure on the additive quotient, by the calculation in the principal-level doubling calculation.

Let $f_g = \omega_{b_j}(g,1)f$ for the original source vector. If C is the antiunitary map to the opposite space, the doubled test on a pair of translates is $f_g \otimes \eta_0 C f_{g'}$, where $\theta_-(\eta_0 C w) = \overline{\theta_+(w)}$. The opposite source character is $b_j^{-1}\upsilon_1^d$: conjugation inverts b_j, and changing the Hom sign contributes υ_1^d on the source determinant. The graph parameter $Z \mapsto X_T^{-1} J_T^{-1} Z J_S X_S$ transports probability Haar on the rational additive quotients. Thus the term for $\delta \in U_r(F_0)$ is exactly $\langle f, \omega_{b_j}(g'^{-1}\delta g, 1)f\rangle$, by Section 40.2.1.

For clarity, the other rational Lagrangians give no term on either of the present tests. If a Lagrangian of the doubled source is not a graph, its intersection with the first source has a nonzero vector. Indeed if its second projection is injective it is an isomorphism, and the isotropy equation then makes the first projection an isometry. A vector in that intersection yields a Heisenberg translation in the first tensor factor alone under which the doubled zero-term functional is invariant.

In the split Schrödinger coordinates at an old place this translation is multiplication by $\psi_w(\mathrm{tr}(ZXc))$, for arbitrary Z, where $c \neq 0$ is the transported first-source vector at w. Invariance

forces the functional to be supported on $Xc = 0$. To see this directly, partition a compact open support in X into finitely many compact open pieces on each of which some one multiplier is a constant different from one. Test invariance on each piece. The support has injective X, so all pieces are of this kind. This is the usual open-graph unfolding with a full-rank Schrödinger test.

In the convergent range the absolute sum of the Siegel–Weil sections is bounded by a Schwartz seminorm of the transformed datum times the convergent spherical Eisenstein series at the identity. This follows by writing each section in Iwasawa coordinates and bounding its restriction to the maximal compact by that seminorm. At reduced representatives of the two source quotients the seminorm of the tensor of the Weil translates grows polynomially in the reduction height. For the original two theta kernels the required absolute bound is the integral over the target of their absolute product. If $\Phi_{g,g'}$ is the mixed realization of the translated doubled test, this integral is at most

$$J(\Phi_{g,g'}) = \int_{[U_d]} \sum_{X \in \mathrm{Hom}_E(E^r, V)} |\Phi_{g,g'}(h^{-1}X)| \, dh.$$

Section 40.2.1 bounds J by a Schwartz seminorm for each fixed finite factor. This applies to both original residue tests; in reduced source representatives their finite translates range over a finite set.

The source forms in the prescribed real coefficient types are rapidly decreasing by the coefficient-projector argument in Section 18.1. That argument applies to every bounded finite-level form of these types. Hence the graph expansion can be integrated absolutely on those quotients. The same argument applies to the other test form even if it is not fixed by $J_{\mathfrak{m}}$. Unfolding the rational graph sum by the quotient Haar formula leaves the integral over the original adelic source of the Weil coefficient times $\langle x, R(g)y\rangle$, multiplied by the original target quotient volume.

At infinity a rank-one discrete-series coefficient projector gives

$$\langle x, R(g_\infty)z\rangle = \frac{\overline{e_j(g_\infty)}}{e_j(1)} \langle x, z\rangle.$$

This is the compression of the real right regular action to the unit highest coefficient line, by Schur orthogonality. The source form in the second variable is fixed by $J_{\mathfrak{m}}$ for the triangle and by J_M for the single lift. Fubini therefore gives the local masses computed above and the same series over the remaining places. The real factor is

$$\mathrm{vol}([U_d]) \int_{U_r(F_0 \otimes \mathbb{R})} \langle v_j, \omega(g_\infty, 1)v_j\rangle \frac{\overline{e_j(g_\infty)}}{e_j(1)} \, dg_\infty.$$

For the two pairs use their respective joint harmonic and quotient volume. These constants are independent of ξ, since the real parameters and vectors do not include the tame twist. At a real place let d_λ be the formal degree for the given Haar and $p_\lambda = \omega(d_\lambda \overline{c_\lambda})$. Schur orthogonality makes p_λ an orthogonal projection. For the prescribed harmonic P, the local integral is

$$\int \langle P, \omega(h)P\rangle \overline{c_\lambda(h)} \, dh = d_\lambda^{-1} \|p_\lambda P\|^2 > 0,$$

where strict positivity follows from the compact average of Section 34.1.2. The same calculation applies to the first true return: the rank $r+1$ source and rank $r+t$ target have the common

entries and inserted block computed in the signed parameter comparison. Their joint harmonic has only one Fock sign in each group of variables with the same target index, so both pure quadratic terms vanish in the Bargmann pairing as above.

For x, y in the original full triangle base, y is fixed by $J_{\mathfrak{m}}$, as required in the unfolding. This gives the following two identities in the original Petersson metrics.

Proposition 34.1 (Gram forms of the first lifts)**.** *On the full triangle base, with $\mathcal{V}_i$ the convergent spherical series,*

$$w_0\langle a_{i,0}j_i x, a_{i,0}j_i y\rangle = C_i B_\xi^{-1}\langle x, \mathcal{V}_i y\rangle.$$

For $j = i+1$, the restriction of $\mathcal{V}_i$ to the good base is R_j. On the good single-lift base,

$$\langle u_{j,\xi^j} x, u_{j,\xi^j} y\rangle_{\text{class}} = A_j c_Q(\xi)\langle x, R_j^W y\rangle,$$

where R_j^W uses the same coefficients and cosets as R_j, acting on W_j. On each right-hand side use the Petersson pairing on the corresponding source. The two restrictions satisfy $\det R_j = \det R_j^W$.

For either lift U, write K for its spherical operator and c for the scalar multiplying its quadratic form. The Haar, Gaussian and finite factors give $c > 0$. The preceding series estimate gives

$$\|Uy\|^2 = c\,\mathrm{Re}\langle y, Ky\rangle \geq c(1 - \|K - I\|)\|y\|^2 \geq \tfrac{c}{2}\|y\|^2$$

in the respective source and target metrics.

34.2.3 The two spherical actions

In this subsection, we compare the actions of the coefficient series of Proposition 34.1 on the good spaces.

For a double coset $K_A\gamma K_A$, its right cosets are represented by $k\gamma$ with k ranging over $K_A/(K_A \cap \gamma K_A \gamma^{-1})$. Because u_j is bi-invariant, the portion of the series on this double coset is exactly $u_j(\gamma)$ times its unnormalized Hecke operator. These operators commute with the finite residue action on the good subspace.

Each finite sum over double cosets acts through the common spherical Hecke algebra on both source good spaces. Absolute convergence of the vacuum weights implies operator-norm convergence by unitarity. The joint image of the Hecke algebra in the two finite-dimensional endomorphism spaces is closed, so the limit is one common image element. Apply (4) to the powers of this common element. Newton's identities give $\det R_j = \det R_j^W$ on the two original good spaces. The trace identity uses the finite Hecke action; each source keeps its own Petersson pairing in the two Gram formulas.

Finally the first-column adjoint return preserves the original base level. Write $\mathrm{red}_{\mathfrak{m}}$ for reduction at the chosen split places of the full triangle modulus. For each integral source element u, lift $\mathrm{diag}(\mathrm{red}_{\mathfrak{m}} u, I_t)$ to an integral target element h, using inverse transpose at the conjugate places. With $X_0 = \binom{I_r}{0}$ in the residue module,

$$(\mathrm{red}_{\mathfrak{m}} h)^{-1} X_0 \,\mathrm{red}_{\mathfrak{m}} u = X_0.$$

Off the modulus the integral actions preserve the vacuum line. Thus the simultaneous Weil action on the finite test is a scalar χ of absolute value one. In the quotient integral the first arrow a satisfies $R(h)a = \chi\, aR(u)$. In particular

$$\langle aR(u)x, aR(u)y\rangle = \langle R(h)ax, R(h)ay\rangle = \langle ax, ay\rangle,$$

which gives $a^{*}aR(u) = R(u)a^{*}a$. Hence $a^{*}a$ preserves the old principal level. On this entire base, the preceding lower bound gives

$$\langle y, a^{*}ay\rangle = \|ay\|^{2} \geq \tfrac{c}{2}\|y\|^{2}.$$

Thus the restriction of $a^{*}a$ to the original base level is injective. The first adjoint descent gives this same return on the next isometric base copy.

34.3 The period pairing for the four unitary groups

In this subsection, we write the composition of the two theta lifts as a paired period on the direct sum of the source and target spaces. This places both choices of intermediate space in the normalization fixed in Section 34.2.

Let U_1 and U_2 be the source and target Hermitian spaces and A and A' the two middle spaces. All quotient integrals below use the measures of the theta operators. Write $[B] = U(B)(E^{+})\backslash U(B)(\mathbb{A}_{E^{+}})$. Set $W = U_1 \perp U_2$, in the given order of the covectors. Write $X = (X_1, X_2)$ and $Y = (Y_1^{\mathrm{t}}, Y_2^{\mathrm{t}})^{\mathrm{t}}$ for the two marked point arrays. Write

$$T_X = D_{2,X_2}L_{1,X_1}, \qquad T'_Y = L'_{2,Y_2}D'_{1,Y_1}.$$

Each arrow uses the indicator of the indicated block at the modulus and the original vacuum away from it. The two lower real polynomials are P'_1, P'_2, and the two upper real polynomials are P_1, P_2.

We use the following measured period comparison on W. Put $\gamma = \nu^{-1}\upsilon^{\dim(W)/2}$, where υ is the splitting character. For the fixed lower real tensor $v_{-} = P'_1P'_2$ (with its standard Fock Gaussian), the comparison supplies a real Fock vector v_{+} of finite moments, before choosing the admissible modulus or character. For matched finite arrays (Φ, Ψ) it gives

$$\int_{[A]} K_{W,A}^{\gamma,b_1}(\Phi, v_{+}; g, h)\, dh = c_{\xi} \int_{[A']} K_{A',W}^{b'_1,\gamma}(\Psi, v_{-}; h', g)\, dh' \qquad (g \in [W]). \tag{91}$$

Here $c_{\xi} \neq 0$. If $\dim A - \dim A' = 2t$, the comparison asserts

$$|c_{\xi}|^{2} = C \prod_{w|\mathfrak{m}} q_w^{t^2} \prod_{k=1}^{t} \left|L^{S_{\mathfrak{m}}}(k, \nu)\right|^{2}, \qquad C > 0.$$

The constant C and v_{+} may depend on the ordered real spaces, the fixed lower datum and archimedean weight. The finite characters and arrays are varied after this choice. Equation (91) is a period comparison for two kernels, with the individual original quotient measures of A and A'.

Write (a_i, b_i) for the upper splitting pairs and (b'_i, a'_i) for the lower splitting pairs. The rank corrections give

$$a_1a_2 = a'_1a'_2 = b_1b'_1 = b_2b'_2 = \gamma, \qquad \nu_0 = b_1/b_2 = b'_2/b'_1.$$

In each product the two ranks sum to $\dim W$, and the product of the corresponding rank-specific space characters is $(\nu^{\mathrm{bc}})^{-1}$. The norm-one character calculation preceding Proposition 36.1 gives the common value $\nu^{-1}\upsilon^{\dim W/2}$. The calculation for the shallow squares is in Section 36.2, and that for the deep squares is in Section 43.2. In particular $b_1b'_1 = \gamma$ is the condition for

applying Proposition 36.1 in (91). Our convention is $K^{a,b}(g,h) = a(\det h)b(\det g)^{-1}K^{1,1}(g,h)$. Multiplying the upper kernel with a pure real tensor by $\nu_0(\det g_2)$ changes the factor $b_1(\det g_2)^{-1}$ to $b_2(\det g_2)^{-1}$. Multiplying the lower kernel by the same factor changes $b_1'(\det g_2)$ to $b_2'(\det g_2)$. The untwisted product formula in both cases follows by splitting the rational theta sum into the two matrix blocks. The Gaussian, the real pure tensor and the finite point indicator are products in those blocks. Thus the two corrections have the same sign.

Take the source vector f and the target vector z in their original real types, and set $f_1 = f$, $f_2 = \overline{z}$. Pairings of automorphic forms are linear in the second argument. For a real Fock vector v set

$$H_{\xi,X,f,z}(v) = \int_{[U_1]\times[U_2]\times[A]} f_1(g_1)f_2(g_2)\nu_0(\det g_2)K_{W,A}^{\gamma,b_1}(\mathbf{1}_X, v; g_1 \oplus g_2, h)\, dg_1\, dg_2\, dh.$$

Only primitive X are needed. Indeed, the support of a matched Φ is formed by the last covectors of k^{-1}, and the support of its partner Ψ by the first columns of k, for invertible k. Thus X has a right inverse and Y has a left inverse at every factor of the nonempty modulus.

We prove a bound uniform in the integrated middle variable. Write $\rho = \mathrm{ht}_W(g)$ and $H = \mathrm{ht}_A(h)$ for the reciprocal minima as in Section 18.1. In the real Schrödinger model write u for the image of v. With the finite test fixed, reduction in both variables and Weil covariance of position and differentiation give the bound

$$|L_{g,h}(u)| \le C(\rho H)^A s_j(u)$$

for the evaluation $L_{g,h}$ of the kernel and a fixed Schwartz seminorm s_j. Indeed the reduced finite translates range over a finite set and the real changes of variable have matrix and inverse norms bounded by powers of ρH. Take A to be a positive integer.

34.3.1 Uniform bounds for the two kernels

In this subsection, we bound the kernels in (91) by averaging over a rational isotropic subspace.

When $H > 2$, take a shortest vector d of the moved middle lattice. Its integral Hermitian norm has absolute value below one at every infinite place, so it is isotropic. In the kept lattice take an E-basis of vectors c_j with lengths $O(\rho^a)$, for a fixed $a \ge 1$. Indeed the real lattice has fixed covolume and first minimum at least ρ^{-1}, so Minkowski's second theorem bounds its last minimum by $O(\rho^{D-1})$ in real dimension D. Among D independent $\mathbb{Z}$-lattice vectors one can take an E-basis.

For the upper kernel average the theta functional over the adelic quotient of the rational isotropic space spanned by $dc_j^* J_W$, with probability Haar. In the moved finite coordinates these displacements are integral. As in Section 36.1.1, the modulations at a split place read $d_h^* \widetilde{X} C_g$, where C_g is the moved basis matrix and $\widetilde{X}$ is a local matrix in the support with residue X. This covector is nonzero over the local field: right invertibility of the residue implies right invertibility of $\widetilde{X}$.

For the lower kernel take the short isotropic vector c in the middle source and a basis d_j in the kept target. The modulations read $D_g^* \widetilde{Y} c_{h'}$, nonzero by left invertibility of $\widetilde{Y}$ over the local field. In either case, partition the compact finite support into pieces on each of which one modulation is a constant different from one. Invariance of the averaged functional under this modulation makes it vanish on the piece for every real vector u.

Rational Heisenberg invariance and additive strong approximation allow computing the average with bounded real coefficients of the rational generators and with finite coefficients in a fixed ideal of $\widehat{\mathcal{O}}_E$. The ideal can be chosen so that every such finite displacement fixes the finite test, by integrality of the generators in the moved coordinates. This yields an average A_h of real translations and modulations of size $O(\eta)$, where $\eta = \rho^a/H$. The vanishing gives $L_{g,h}(A_h u) = 0$ for every u. For $\eta \le 1$, differentiating the real Heisenberg law along the segment from zero to each displacement gives $s_j((1 - A_h)u) \le C_j \eta s_{j'}(u)$ with a higher fixed seminorm $s_{j'}$. Now

$$|L_{g,h}(u)| = |L_{g,h}((1 - A_h)^A u)| \le C(\rho H)^A \eta^A s_{j''}(u) = C\rho^{A(a+1)} s_{j''}(u).$$

For $\eta > 1$ use $H < \rho^a$ in the initial bound. For $H \le 2$ the initial bound suffices.

In the Fock coordinates the coherent-state formula is

$$v(w) = \pi^{-d} \int_{\mathbb{C}^d} v(y) e^{-(|w|^2+|y|^2)/2+\bar{y}\cdot w}\, dy.$$

The inverse Bargmann transform of the exponential is a translated and modulated normalized Gaussian, up to a unit scalar, by the Gaussian integral in the Bargmann transform. By Leibniz's rule and translation of the polynomial weights, its $s_{j''}$-seminorm is $O((1 + |y|)^N)$ for a fixed N.

Integration in that seminorm gives $s_{j''}(u) \le C p_N(v)$. Thus

$$\left| K^{\gamma,b_1}_{W,A}(\mathbf{1}_X, v; g, h) \right| \le C_X p_N(v)\, \mathrm{ht}_W(g)^B \qquad \text{uniformly in } h.$$

For every integer $D \ge 0$, the same argument applies to the lower kernel, with an integral left inverse at a chosen split place, and gives rapid decay in the moving source. Indeed apply $A + D$ differences instead of A. When $H > 2$ and $\eta = \rho^a/H \le 1$, this replaces the bound by

$$C_D(\rho H)^A \eta^{A+D} s_{j_D}(u) = C_D \rho^{A+a(A+D)} H^{-D} s_{j_D}(u).$$

If $H \le 2$ or $\eta > 1$, then $H \le 2\rho^a$ and the initial bound gives the same estimate up to a constant. The bound for the upper kernel, with an integral right inverse, follows identically. This argument also applies to a finite sum of residue tests supported on matrices of the indicated rank. It applies to $K^{b_1',\gamma}_{A',W}(\mathbf{1}_Y, v_-; h', g)$. Here p_N is the weighted absolute Fock L^1 moment in the prescribed normalization. The direct sum height of $g_1 \oplus g_2$ is bounded by a power of the product of the two heights. The source tests are rapidly decreasing, and the quotient measures are finite. These inequalities give absolute integrability and, for each fixed $j = (\xi, X, f, z)$, the continuity bound

$$|H_j(v) - H_j(v')| \le C_j p_{N_j}(v - v').$$

The constants C_j, N_j can vary with j. In particular these bounds do not change the normalization of the quotient measures.

With lower real datum v_-, the block product formula and Fubini give the integral of the two transposed lower kernels against f_1, f_2, paired bilinearly on $[A']$. By transposition of the second lower arrow this is exactly $\langle z, T'_Y f\rangle_{[U_2]}$. The identical upper calculation with datum P_1P_2 gives $\langle z, T_X f\rangle_{[U_2]}$. These transpositions use the product quotient measure and introduce no indices.

For an arbitrary upper pure polynomial tensor Q_1Q_2, the real edge convolution identity proved below expresses the two lifts on $[A]$ as $C_{\alpha_1}F_1$, $C_{\alpha_2}F_2$, where F_j are the lifts with P_j, and

$$C_\alpha F(h) = \int_{U(A)_\infty} \alpha(k) F(hk)\, dk.$$

The smoothing functions depend on the real edge and Q_1, Q_2, and precede the finite arrays and forms. The real types of F_1 and F_2 are conjugate. Primitivity concerns the combined matrix $X = (X_1, X_2)$ or $Y = (Y_1^{\mathsf{t}}, Y_2^{\mathsf{t}})^{\mathsf{t}}$. For the individual blocks we use the Gaussian estimate below to justify each coefficient projection and prove square integrability.

Consider one edge of source and target ranks a, b', with the joint harmonic P and a rapidly decreasing source test. At each infinite place use the lowest compact type coefficient projectors e_S, e_T for the source and target lines, with the inverse in the source convolution as prescribed by the invariant pairing. Both kernels are integrable with weight $\|k\|^{ab'}$. Indeed the opposite-sign gaps in the marked types exceed $500d_*^3$, where d_* bounds the individual ranks, by the half-gap comparison of the base and inserted blocks. Let $t_0 = 5r$ be the block parameter. At every node of either grid,

$$N = r + z + i(t_0 + z) + j(t_0 - z) \le d_* := r + 1 + 2mt_0 \qquad (0 \le i, j \le m,\ z = 0, 1).$$

Since $m < r$ and $\Delta = 2r^8$, the lower bound $\Delta/2 - (5r + 2 + 2m)$ exceeds $500d_*^3$ for large x. Reversing or conjugating the parameters preserves the gaps. Section 12.7 gives at least the exponent $ab' + 8d_*^2 + 1$.

The Cartan density of $U(p, q)$, $p+q \le d_*$, in coordinates $t_1 \ge \cdots \ge t_{\min(p,q)} \ge 0$ is a product of powers of $\sinh \alpha(t)$. The restricted roots have at most $2d_*^2$ total multiplicity and $|\alpha(t)| \le 2t_1$, so the density is $O(e^{4d_*^2 t_1})$. Since $\|k\| = e^{t_1}$ on the Cartan part, the stated moment is integrable. The compact cases are immediate.

For a fixed finite test the rational theta functional is bounded by the real Fock L^1 norm. In the coherent integral above each normalized coherent vector is a translated and modulated Gaussian. Its absolute rational sum is bounded uniformly in the modulation and the real translate, by summing a Gaussian on each fixed fractional lattice occurring in the finite test. Reduce the source point using Section 18.1. At one real place put $d_0 = ab'$, let v_0 be the normalized vacuum and write $H(u)$ for Heisenberg translation by u. For the signed Hom action σ the coherent integral in these coordinates is

$$\omega(\sigma)v = \Delta(\sigma) \int_{\mathbb{C}^{d_0}} v(y) H(\sigma y) v_0 \, \frac{dy}{\pi^{d_0}}.$$

For the bound on the scalar, denote the integral operator by B_σ. The coherent resolution of the identity and $|\det \sigma| = 1$ make B_σ bounded by Cauchy–Schwarz. The Heisenberg multiplication law gives $B_\sigma H(u) = H(\sigma u) B_\sigma$, by changing y to $y + u$. Irreducibility gives $B_\sigma = c_\sigma \omega(\sigma)$. The Gaussian integral of its vacuum coefficient gives

$$0 < \langle v_0, B_\sigma v_0 \rangle = \det_{\mathbb{R}} \big((I + \sigma^{\mathsf{t}} \sigma)/2 \big)^{-1/2} \le |c_\sigma|.$$

Here σ^{t} is the transpose for the real Euclidean structure of the signed coordinates. Since σ is symplectic, its $2d_0$ singular values pair as s, s^{-1}, with $s \ge 1$. Each pair contributes $(s + s^{-1})/2 \le \|\sigma\|$ to the square root of the determinant. Thus $|\Delta(\sigma)| = |c_\sigma|^{-1} \le \|\sigma\|^{d_0}$. The Gaussian sum just estimated is uniform in the displacement σy.

For a fixed target h the resulting kernel at (g, hk) is bounded by

$$C_h \, \mathrm{ht}_S(g)^B \prod_{v \mid \infty} \|k_v\|^{ab'}.$$

This follows from $\|\sigma_v\| \le \|(hk)_v\| \|g_v^{-1}\|$ and the finitely many finite translates in the reduction. Integrating the rapid source test gives the same bound in k for the lift. The analogous bound

in a source real translation justifies moving the source projector through the source integral. Indeed, write $L_f(\phi)$ for the lift of f with the original finite datum and real vector ϕ. Right invariance of quotient Haar gives

$$L_f(\omega_S(u)\phi) = L_{R(u^{-1})f}(\phi), \qquad L_f(\omega_T(h)\phi) = R(h)L_f(\phi).$$

Thus $L_f(p_S\phi) = L_f(\phi)$ for the prescribed source line: its convolution kernel on scalar forms is $e_S(u^{-1})$. On the target $L_f(p_T\phi) = R(e_T)L_f(\phi)$.

34.3.2 The real line projections

In this subsection, we derive the real convolution identity, using the uniform bounds of Section 34.3.1 to justify integration.

On the real oscillator, let p_S, p_T be the line projections given by these coefficients, with the dual source representation used in the pairing. We first verify the parameter rule and $p_S P \neq 0$ for the following block insertions. Write $z = 0, 1$ for the two grids. For i up blocks and j down blocks the node has rank $N = r + z + i(t_0 + z) + j(t_0 - z)$. Its unlowered doubled entries are the stem $2\lambda_a + z$ and the block entries $2c + d - 1 - 2b$, $0 \leq b < d$, with $d = t_0 + z$ for an up block and $d = t_0 - z$ for a down block. The lowering ϵ_N is 1 at odd $i + j$ in grid 0, and 0 otherwise. For either kind of edge the opposite rank N' has parity ϵ_N.

For a down insertion at c, the opposite determinant exponent is $c + \lfloor N'/2 \rfloor$. Its shifted entry is therefore $(u - \epsilon_N) - 2(c + \lfloor N'/2 \rfloor) + N' = u - 2c$ for an unlowered entry u. For an up insertion at c, the conjugate parameter and opposite exponent $-c + \lceil N'/2 \rceil$ give $(-u + \epsilon_N) - 2(-c + \lceil N'/2 \rceil) + N' = -u + 2c$. Thus the matched shifted entries coincide. The inserted entries form the length d compact block with a single signature sign. The signature of a retained entry contains one factor -1 per absent down centre above it and per absent up centre below it. Inserting c changes this factor precisely for a negative shifted entry. Here are the insertions in both grids, before reversing any arrow. A node (i, j) has its first i up blocks. With offset o its down blocks have indices $o + 1, \ldots, o + j$.

step	centre c	length d
$(i, j) \longrightarrow (i + 1, j),\ o = 0$	w_{i+1}	$t_0 + z$
$(i, j) \longrightarrow (i, j + 1)$	$-w_{o+j+1}$	$t_0 - z$
$(o + 1, 0, j) \longrightarrow (o, 0, j + 1)$	$-w_{o+1}$	$t_0 - z$

The triples on the last line list offset, number of up blocks, and number of down blocks. In the rectangular grids $o = 0$, and in the triangles of Section 34.3.3 there are no up blocks. At stem index p, put $a = (p + 1 - k)_+$. The stem in Section 42.2 lies below the first a down centres. Of these, $\min((a - o)_+, j)$ are present. Its signature is therefore

$$(-1)^{o+a+\min((a-o)_+, j)}.$$

A present up block has $o = 0$ and no missing centre contributes to its sign. A present down block has exactly o missing down centres above it, which cancel the offset sign. Thus each inserted block is positive in the arithmetic signatures. If two indices at the same node have different signs, their missing-centre counts differ modulo two. Some missing centre therefore lies strictly between their doubled entries. Each entry lies at distance at least $\Delta - t_0 - 2$ from that doubled centre, by the half-gap for the stem and the block widths. Their separation in the

undoubled parameter is at least $\Delta - t_0 - 2$. This is the opposite-sign gap used for the coefficient moment in Section 34.3.1.

On the last line of the table, give the smaller space the opposite form to put both signature rules over $(-1)^o$, then insert the missing first down centre.

For the prescribed opposite characters and parameters use the substitution

$$(l, l', s, s'; \gamma, \gamma') \longmapsto (-l, -l', -s, s'; N - \gamma, N' - \gamma'),$$

where s, s' are the two sign vectors for this insertion. The shifted entries change sign and the smaller signature returns to its original one. The retained rule is again equal signs at positive entries and opposite signs at negative entries. The two commuting actions on the same oscillator give the reverse arrow with the same harmonic P. Its source and target coefficient kernels are $(\overline{e_T}, \overline{e_S})$, since interchanging the source and target in the bilinear theta pairing takes both automorphic representations to their contragredients.

To apply Li's discrete-series correspondence [42] in the form of [59, Theorem 2.7(b)], remove the determinant twists and use the tensor realization $\mathrm{Hom}(S, T) = S^\vee \otimes T$: on the source dual the compact weights change sign and their order in each compact factor reverses. More explicitly, if s_a, s'_b are the two signatures, the vacuum weights after removing the scalar twists are $-s_a \sum_b s'_b/2$ and $s'_b \sum_a s_a/2$ in the Hom model. Dualizing the first factor gives exactly the two vacuum weights of [59, Lemma 2.5]. The resulting genuine Harish–Chandra entries are the shifted entries divided by two. They are integral for odd rank difference and half-integral for even rank difference. Let $n_{\epsilon,\eta}$ count the retained entries with signature ϵ and shifted sign η on the smaller side. They contribute

$$(n_{+,+} + n_{-,-},\ n_{-,+} + n_{+,-})$$

to the larger signature. The remaining entries are $(d-1)/2, (d-3)/2, \dots, (1-d)/2$, all in one compact factor. The distance of a retained entry from the centre is $|\xi_a|/2 \geq (d+1)/2$. In undoubled units the separation bound

$$(\Delta - t_0 - 2)/2 > 500(2d_*)^3 \geq 500(N + N')^3$$

holds for large x, since $d_* = r + 1 + 2mt_0 = O(r^2)$ and $\Delta = 2r^8$. The retained-entry bound $|\xi_a| \geq d + 1$ and the placement of the compact string verify the two alternatives in Paul's Theorem 2.7(b). The stronger cubic gap supplies the coefficient moment in Section 34.3.1. Simultaneous reversal of the two Hermitian forms preserves the correspondence. Reversing just one form conjugates the oscillator and takes the contragredient on both representation factors [59, Proposition 2.9 and Corollary 2.10].

At retained indices write ξ_a for the doubled shifted entry, and use ξ_b also for the inserted string in the larger parameter. In the convention of (TP1), let C_a and R_b be the relative lowest compact weights of the dual smaller and the larger factors. The formula $\lambda + \rho - 2\rho_K$ and the vacuum weights give

$$2C_a = -\xi_a + s_a \left(\sum_{a'} s_{a'} \operatorname{sgn}(\xi_a - \xi_{a'}) + \sum_{b'} s'_{b'} \right),$$
$$2R_b = \xi_b - s'_b \left(\sum_{b'} s'_{b'} \operatorname{sgn}(\xi_b - \xi_{b'}) + \sum_{a'} s_{a'} \right).$$

Set $\eta_a = \operatorname{sgn} \xi_a$, so that $s'_a = s_a \eta_a$ at a retained index. In $2(R_a + C_a)$ the contribution of another retained index a', divided by $s_a s_{a'}$, is

$$(1 - \eta_a \eta_{a'}) \operatorname{sgn}(\xi_a - \xi_{a'}) + \eta_{a'} - \eta_a = 0.$$

The contribution of an inserted index b, divided by $s_a s'_b$, is $1 - \eta_a \operatorname{sgn}(\xi_a - \xi_b) = 0$. At an inserted index b the retained terms in $2R_b$ cancel since $\operatorname{sgn}(\xi_b - \xi_a) = -\eta_a$. The signed sum over the other inserted positions equals ξ_b after multiplication by s'_b, because the string has a common sign and spacing two. Therefore

$$R_a = -C_a, \qquad R_b = 0 \text{ on the inserted string}, \qquad \operatorname{sgn} R_a = \eta_a.$$

For a retained stem or block the distance from the new doubled centre satisfies

$$|\xi_a| \geq \min\{2w_v - k - 1,\ \Delta - 1,\ 2\Delta - t_0\} > 2d_* \geq N + N'$$

for sufficiently large x, with $w_v = \kappa_\tau + \Delta/2$ as in Section 42.2. Indeed the head has modulus at most $k + 1$. At a tail entry the distance from a down centre is $|(2h + 1)\Delta + z|$ for an integer h, and the up centres lie on the opposite side. A previously inserted block has radius at most t_0 and a different doubled centre at distance at least 2Δ. Since $|2R_a - \xi_a| \leq N + N' - 1$, this gives the asserted sign.

In each of the four sign quadrants take the product of minors on the matched positions with $|R_a| \geq h$, for positive integers h. These are initial segments in the decreasing compact order for positive weights and in the dual order for negative weights. They are nested on both sides because the relative weights have equal magnitudes at the matched positions and are dominant in their compact orders. In the signed Fock coordinates each compact raising operator either kills a minor or replaces an index by one already present in it. Evaluation at the matrix with ones on the matched positions and zeros elsewhere gives a product of determinants ± 1. Each individual source or target index occurs with just one oscillator sign. The mixed derivatives with opposite oscillator signs and a common index thus kill P for both compact groups. The construction therefore gives the nonzero harmonic for the two lowest compact types. The pair occurs with multiplicity one in the joint harmonics, so the prescribed highest polynomial spans the same line [59, Theorem 2.2(1) and Lemma 2.5].

The positive Bargmann pairing and compact Schur averages of Section 34.1.2 now apply to either group as source. In the Hom model a positive source factor g acts on the matrix variables by $X \mapsto X g^{-1}$ and a positive target factor h acts by $X \mapsto hX$. The induced matrices $1 \otimes g^{-\mathsf{T}}$ and $h \otimes 1$ are Hermitian positive. At either positive factor the determinant twist is one, the pure quadratic terms vanish by the mixed derivative calculation, and the mixed term gives the squared Fock norm after an invertible substitution, times the positive vacuum coefficient. The two compact Schur averages against the source matrix coefficient consequently have strictly positive integral at each positive factor. Integrate in the polar decomposition and multiply by the positive formal degree. The coefficient moment verified above justifies this step and gives $\langle P, p_S P \rangle > 0$.

Write $c_S(g) = \langle v_S, \pi_S(g) v_S \rangle$ for the coefficient in the source oscillator convention. The convolution calculation of Section 12.3 gives $p_S \omega(g) p_S = c_S(g) p_S$, and similarly for the target. For any unit $u \in \operatorname{im} p_S$, its source translates therefore have the same Gram matrix as the translates of v_S. Their closed span is a copy of V_S under $\pi_S(g) v_S \mapsto \omega(g) u$. In particular $p_S P$ belongs to the full source discrete-series isotypic component.

Let Q_S project onto this component of the oscillator Hilbert space $\mathcal{H}_\omega$. This is a central projection in the source von Neumann algebra. On $Q_S\mathcal{H}_\omega = V_S\widehat{\otimes}M$, the source algebra is $\mathcal{B}(V_S)\overline{\otimes}1$. Howe's commutant theorem [36, Theorem 6.1], restricted by this central projection, identifies the target algebra with $1\overline{\otimes}\mathcal{B}(M)$. Thus M is irreducible and admissible. Scalar twists leave these algebras unchanged.

The map Q_S intertwines both derived actions on Fock polynomials and has dense polynomial image in $V_S\widehat{\otimes}M$. Projection onto any fixed $K_S \times K_T$-type preserves the polynomials, since the compact action preserves degree. The image fills this finite-dimensional isotypic space by density and admissibility. Consequently $(V_S)_{K_S}\boxtimes M_{K_T}$ is a quotient of the polynomial oscillator module. Its target factor is the one identified by the preceding discrete-series correspondence, using the smaller group first and reciprocity for the reversed orientation.

On this component $p_S = P_{v_S} \otimes 1$, where v_S spans the source compact highest line. Since p_S commutes with the target action, $0 \neq p_S P$ has the target compact highest weight computed above. This lowest compact type has multiplicity one by [5, Theorem 8.5] on the derived group: adjoining the compact central circle with its fixed character preserves the multiplicity. For a definite group this is ordinary highest-weight theory. Its highest line is consequently one-dimensional, so $p_T p_S P = p_S P$.

The nonzero vector $p_S P$ is cyclic in the target representation. Smearing its translates by smooth compactly supported functions has dense linear span. Project this span onto the finite sum of target compact types of $p_S Q$, for a polynomial Q. This sum is finite dimensional by admissibility. The projected span is therefore the whole sum, and its projection is still obtained by smooth compactly supported convolution, by averaging the convolution functions over the target compact group with the compact-type character projectors. Thus one such function sends $p_S P$ to $p_S Q$.

Use the product over infinite places and linearity for polynomials in the product Fock space. The integrability bounds above permit evaluation after the source projection and this compactly supported smoothing, on the same finite datum. This proves the asserted replacement of each real polynomial lift. The same bounds let us evaluate the target coefficient projector on the lift of P. To justify evaluation of the oscillator identities, let $p_0(v) = \int |v(z)|\, dz/\pi^d$ in the Gaussian-included Fock realization of complex dimension d. Use the complete norm $\|v\|_2 + p_0(v)$ on the intersection of the Hilbert Fock space with L^1. In the coherent integral for $\omega(k)v$, the normalized vacuum at displacement kz has constant p_0-norm by translation invariance. The determinant multiplier bound above therefore gives

$$p_0(\omega(k)v) \leq C\|k\|^{ab'} p_0(v)$$

at one real place, and the product bound at all places. Translations of a modulated Gaussian depend continuously on the displacement in L^1; dominated convergence in the coherent integral gives strong L^1 continuity. Hilbert continuity follows from unitarity and the continuous oscillator action. The weighted coefficient moments and compact support of the smoothing functions thus give convergence of all these integrals in both norms. Their Hilbert equalities identify their L^1 representatives almost everywhere. For our fixed finite test the theta functional is bounded by Cp_0, by the shifted Gaussian sum in Section 34.3.1. At a fixed target point the extra source height factor is polynomial and integrable against the rapid source form. This proves the evaluated identities by Fubini for the same finite test. Consequently the lift F satisfies $R(e_T)F = F$ as an absolutely convergent pointwise integral, with the same finite datum and quotient Haar.

For the fixed finite datum choose a compact open subgroup J of the target that fixes its finite Weil vector, including the splitting-character action. Then F and its truncations

$F_j^{\mathrm{tr}} = F\mathbf{1}_{\{|F|\le j\}}$ are right J-invariant. The quotient has finite volume, so $F_j^{\mathrm{tr}} \in L^2$. Put $u_j = R(e_T)F_j^{\mathrm{tr}}$. Real convolution preserves level J. Schur orthogonality gives $e_T * e_T = e_T$, so

$$R(e_T)u_j = u_j, \qquad \|u_j\|_\infty \le j\|e_T\|_1.$$

Section 18.1.3 puts each u_j in the cuspidal space with the prescribed compact type, infinitesimal character and central character. At level J there are finitely many arithmetic quotient components. On each derived real group, Harish-Chandra's finiteness theorem for the specified compact type and infinitesimal character gives $\dim \operatorname{span}\{u_j : j \ge 1\} < \infty$ [32, Theorem 1].

The continuous representatives of the projected truncations are given pointwise by convolution, since $e_T \in L^1$ and F_j^{tr} is bounded. Dominated convergence with the weighted estimate above gives $R(e_T)F_j^{\mathrm{tr}} \to F$ pointwise. Take finitely many evaluations that are injective on the finite-dimensional space. Convergence at these points gives convergence of the coefficients in a basis, so F lies in that same space. In particular the two individual upper lifts and the two transposed lower lifts are square-integrable. They are rapidly decreasing by Section 18.1.3.

For fixed real elements s, t of the source and target, respectively, the translated harmonic satisfies

$$p_S\omega_S(s)P = b_s p_S P, \qquad L_f(\omega_T(t)\omega_S(s)P) = b_s R(t)F.$$

Indeed $p_S\omega_S(s)P$ lies in the source line tensored with the same target compact highest line as $p_S P$, since both operators commute with the target compact action. This line is one-dimensional and $p_S P \neq 0$. For $e_T^t(k) = e_T(t^{-1}kt)$, unimodularity gives $R(e_T^t)R(t) = R(t)R(e_T)$. The real translation preserves the finite level and commutes with the left unipotent integral defining each constant term. The two reciprocal lattice minima $H(ht)$ and $H(h)$ differ by factors bounded in terms of t, t^{-1}, so the translated lift has the same rapid decay for each fixed t.

Write e_A for the normalized matrix coefficient of right translation on the real line of F_1. With the projector kernel e of this type, its argument convention is $e_A(g) = e(g^{-1})/\overline{e(1)}$. Then

$$\int_{[A]} F_2(u)F_1(ug)\,du = e_A(g)\int_{[A]} F_2(u)F_1(u)\,du \qquad (g \in U(A)_\infty).$$

To see the independence of this identity from multiplicity, pair the real isotypic summands: the real right action acts on the representation factor and trivially on the multiplicity factor. On the two distinguished lines the bilinear pairing is therefore the normalized representation coefficient times the pairing on the multiplicity factors.

The change $u = hk_2$ preserves the Haar measure of $[A]$. Consequently, for real edge data Q_1, Q_2,

$$H_j(Q_1Q_2) = r(Q_1, Q_2)\langle z, T_X f\rangle_{[U_2]},$$
$$r(Q_1, Q_2) = \int_{U(A)^2_\infty} \alpha_1(k_1)\alpha_2(k_2)e_A(k_2^{-1}k_1)\,dk_1\,dk_2.$$

Fubini here also follows directly from Cauchy–Schwarz, unitarity of the real translations on $[A]$, and compact support of the smoothing functions. The integrals are bilinear, so neither smoothing function is conjugated. Every polynomial in both blocks is a sum of pure tensors, and the same conclusion holds with a scalar $r(Q)$.

To approximate v_+, use the marked signed Fock coordinates $\zeta(w)$ and Gaussian $G(w) = e^{-|w|^2/2}$. For each m, the degree-m Taylor kernel

$$G(w)G(y)\sum_{k=0}^{m}\frac{(\overline{\zeta(y)}\cdot\zeta(w))^k}{k!}$$

integrated against $v_+(y)\,dy/\pi^d$ gives a polynomial Fock vector $v_m(w)$, where d is the complex dimension. The full exponential is the Fock reproducing kernel, by the Gaussian integral for orthogonal monomials. The absolute value of each partial kernel and of its limit is at most $e^{-(|w|-|y|)^2/2}$.

For every N there is a constant $C_{d,N}$ such that

$$\int_{\mathbb{C}^d} (1+|w|)^N e^{-(|w|-|y|)^2/2}\,dw \le C_{d,N}(1+|y|)^{N+2d-1}.$$

Indeed, use polar coordinates and bound the polynomial in the radial variable r by a constant times $(1+|y|)^{N+2d-1}(1+|r-|y||)^{N+2d-1}$. For $d=0$ the Fock space consists of constants. Thus dominated convergence in (y,w) proves $p_N(v_m - v_+) \to 0$ for every N, using the finite moments of v_+. Set $B_j = \langle z, T_X f\rangle_{[U_2]}$.

If some $B_{j_0} \neq 0$, the identities for polynomials give

$$H_j(Q)B_{j_0} = H_{j_0}(Q)B_j.$$

Continuity for each j gives the same formula at v_+; put $\kappa = H_{j_0}(v_+)/B_{j_0} \in \mathbf{C}$. If all the B_j vanish, all the $H_j(v_+)$ vanish, and we may take $\kappa = 0$. In both cases

$$H_j(v_+) = \kappa\langle z, T_X f\rangle_{[U_2]} \qquad \text{for every } j.$$

This gives one κ for the entire family.

Integrate (91) at $g = g_1 \oplus g_2$ against $f_1(g_1)f_2(g_2)\nu_0(\det g_2)$. Expanding the two finite point arrays and applying the preceding identities yields

$$\kappa \sum_X \Phi(X)\langle z, T_X f\rangle_{[U_2]} = c_\xi \sum_Y \Psi(Y)\langle z, T'_Y f\rangle_{[U_2]}.$$

The target pairing is nondegenerate. Thus the two composites on the original four Hermitian spaces satisfy the matched operator identity

$$\sum_Y \Psi(Y)T'_Y = \frac{\kappa}{c_\xi}\sum_X \Phi(X)T_X.$$

This argument permits $\kappa = 0$. Its needed nonvanishing for the forward triangle follows from injective first returns and the weighted chain argument.

34.3.3 Application to the marked unitary groups

In this subsection, we apply the two pairings of Section 34.3.2 with the original levels and measures.

For the forward square the nodes U_1, U_2, A, A' are

$$(i, n+1), \qquad (i+1, n+1), \qquad (i, n+2), \qquad (i+1, n),$$

respectively. For grid parameter $z_0 \in \{0,1\}$, the rank in column j is $d_j = r + z_0 + j(t_0 - z_0)$. Hence $\dim W = 2d_{n+1} = d_n + d_{n+2}$ and the exchange defect is $t = t_0 - z_0$. The signs of the added blocks are positive. On a fixed base coordinate with excess a, the sign in column j at offset i is

$$(-1)^{i+a+\min((a-i)_+, j)}.$$

Moving $(i, n+1)$ to $(i, n+2)$ and $(i+1, n+1)$ to $(i+1, n)$ flips both signs exactly when $a - i \geq n+2$. In that case the two old signs are opposite. Thus the number of negative signs agrees between the two diagonals, as the period comparison requires. That number is at most $2(r + z_0) < d_{n+1}$, using $t_0 \geq r + 6$. The inverse quotient character in the comparison has weight $(n+1)$ times the original gap at each real place. The inequality $d_{n+1} \leq (n+1)\Delta$ gives $\dim W \leq 2(n+1)\Delta + 2q$, with q the number of negative signs in W.

For the reversed square in the $z_0 = 0$ triangle, the source and target are $(i+1, n)$ and $(i, n+2)$. The upper and lower middle nodes in the ordered comparison are then $(i+1, n+1)$ and $(i, n+1)$. Their ranks are equal, so $t = 0$ in the measured period comparison. Interchanging the two diagonals keeps their common negative index q and their common total rank $2d_{n+1}$. The inverse quotient character still has weight $(n+1)\Delta$; hence

$$2q + 2 \leq 2d_{n+1} \leq 2(n+1)\Delta + 2q.$$

Apply the same paired calculation with the two reversed Fourier edge data, in their chosen order of the covectors. The modulus of the period scalar is constant on this family.

To identify the adjoint consequence, let A_0, A_1 be the two reversed lifts with the Fourier data of the original test and its extended test, B_0, B_1 the corresponding transposed descents, and L_+, L_- the upper and lower ordinary lifts. Let S_n, S_{n+1} be the two original central reversal translations. Write the two reflection identities, the reversed square and the central intertwining as

$$\begin{aligned} \langle B_0 x', y\rangle &= d_0 \langle x', A_0 S_n^{-1} y\rangle, \\ L_+ A_0 &= g_0 A_1 L_-, \\ L_- S_n^{-1} &= \eta S_{n+1}^{-1} L_-, \qquad |\eta| = 1, \\ \langle x, A_1 S_{n+1}^{-1} u\rangle &= d_1 \langle B_1 x, u\rangle. \end{aligned}$$

For $x' = L_+^* x$, the second-linear convention gives $\langle x', v\rangle = \langle x, L_+ v\rangle$. It follows directly that

$$\langle B_0 x', y\rangle = d_0 g_0 \eta d_1 \langle B_1 x, L_- y\rangle.$$

In particular none of the four scalars is conjugated in this passage. The finite products of field sizes in the moduli of d_0, d_1 belong to the two reflection identities. The reversed square itself contributes the modulus of its real constant divided by the modulus of its equal-rank period scalar.

35 Real restrictions and self-dual Fourier tests

In this section, we compute the local restrictions of the paired periods of Section 34.3. The real slice and the finite self-dual transform give the normalization for the exchange in Section 36.

35.1 The real restriction of the larger period

In this subsection, the real integrals in the paired periods of Section 34.3 involve the original Fock vectors. We evaluate the larger and smaller real slices with their quotient measures to determine the factor in the exchange.

Let $W_0 = H \perp W$ have rank $2j$, let $B_0 = U^-$ have rank b, and let $X = H^b \perp W_0$. Use the original two isotropic bases of X and the original mixed Schrödinger–Fock models for its pairs with $B_0 \perp W_0$ and B_0. Fix the lower real Schwartz vectors v for (W_0, B_0) and v_1 for (W_0, W_0). Fix the common infinity weight $\mu = (\mu_\tau)_\tau$. It is positive on the admissible family, so the exponents $2\mu_\tau$ below are nonnegative integers. All matrices in the real formula are families at the infinite places. Let $J = \operatorname{diag}(J_1, J_2)$ be the matrix of $B_0 \perp W_0$, and set $x_0 = (1; 0)$.

The two real vectors. Near x_0, choose a smooth unitary $k(y)$ carrying $\operatorname{im} x_0$ onto $\operatorname{im} y$. For the graph of $T : \operatorname{im} x_0 \to W_0$, the derivative at zero of the graph chart on the action of

$$\exp \begin{pmatrix} 0 & -J_1^{-1} T^* J_2 \\ T & 0 \end{pmatrix}$$

is $T \mapsto T$. The real inverse function theorem produces the section on a neighborhood. Multiplying its Lie-algebra coordinates by a cutoff and exponentiating extends k smoothly while keeping it unitary. Take a smooth compactly supported nonnegative bump σ in a smaller neighborhood of x_0, with $\sigma(x_0) = 1$.

For $x^\dagger = x^* J$ choose a smooth matrix function ι equal to inversion on a neighborhood of $x_0^\dagger x_0$ and of all the Gram matrices from the support of σ. This is obtained by multiplying inversion by a smooth cutoff supported where the determinant is nonzero. Set

$$P(x) = x\, \iota(x^\dagger x) x^\dagger x_0, \qquad x^\flat = k(P(x))^{-1} x.$$

The large real vector Φ and small real vector Λ have the following slices in the initial polarization:

$$\Phi[x] = \kappa\, \sigma(P(x)) \sigma(x^\flat) \prod_\tau \overline{\det(x_\tau^\flat)_1}^{\,2\mu_\tau}\ \omega_{j-\mu}(1, k(P(x)))(v \boxtimes v_1), \tag{92}$$

$$\Lambda[B] = \kappa\, \sigma(x_0 B) \prod_\tau |\det B_\tau|^{2(j+\mu_\tau)}\, v. \tag{93}$$

In the first formula $(x_\tau^\flat)_1$ is the first b-block. The second-group determinant exponent in $\omega_{j-\mu}$ is $j - \mu_\tau$ at each place.

These sections are Schwartz in the mixed model. In the first formula $x = k(P(x)) x^\flat$, so the support is contained in the compact image of the two bump supports. All scalar factors are smooth, including at rank-deficient matrices, since no inverse is taken outside the invertible Gram neighborhood. On that compact support the Weil operators form a smooth compact family on Schwartz functions. For every derivative and Schwartz seminorm this follows locally from the linear-change, quadratic-multiplier and Fourier formulas for the metaplectic action.

The small section is supported on a compact set because $B \mapsto x_0 B$ is a closed linear embedding. The mixed-model isomorphism therefore gives Schwartz vectors with these slices. Equivalently, in signed Fock coordinates the condition on the real source can be stated as finiteness of all weighted absolute moments. To see that this condition gives the Schwartz estimates, use the coherent-state reproducing integral. The modulus of its kernel in real coordinates x, y is $e^{-|x-y|^2/2}$, and differentiation contributes a polynomial in x, y. For any prescribed weighted derivative, bound the resulting Gaussian times the weight by a fixed polynomial in $|y|$, uniformly in x, and use finiteness of the corresponding moment.

In the opposite source polarization the zero slice is the integral of the slices in the initial polarization, with the Fourier measure of the chosen bases. To compute its normalization for Λ, write $t_\Lambda(B) = \sigma(x_0 B) \prod_\tau |\det B_\tau|^{2(j+\mu_\tau)}$. Choose κ so that the real integral of κt_Λ is the

real integral of the scalar vacuum slice. The integral of t_Λ is positive since it is continuous, nonnegative, and equal to one at the identity.

The Gaussian vacuum integral is nonzero. With any finite complementary test, the adelic opposite zero slice factors into this real integral and the same finite slice integral as for the vacuum. The marked linear change of slice variables carries the self-dual Haar to the same positive multiple of the product real and finite Haar in these two evaluations. Equality of the real integrals therefore gives equality for every finite test. The vacuum has the normalized opposite zero slice equal to the smaller vacuum. Consequently the opposite zero slice of Λ is v. This normalization uses the actual mixed model and places no condition on the finite data later used in the period.

A single translate for every test. Fix a real point $h \in U(B_0 \perp W_0)_\infty$ and put $x_h = h^{-1}x_0$, $y = P(x_h)$, and $\rho(h) = \sigma(y)$. Here P uses the true inverse, since the Gram matrix of x_h equals that of x_0. The function ρ is nonnegative and equals one at the identity. It depends only on the plane $\mathrm{im}(h^{-1}x_0)$.

If $\rho(h) \neq 0$, y is a basis spanning the plane of x_h. Since $k(y)x_0$ and x_h have the same Gram matrix and span, there is $u_1 \in U(B_0)_\infty$ such that $k(y)x_0u_1 = x_h$. The isometry $hk(y)$ preserves the first summand and its orthogonal complement. Write

$$hk(y) = \mathrm{diag}(u_1^{-1}, u_2), \qquad p = \mathrm{diag}(u_1, u_2^{-1}).$$

Regard p as an adelic point with finite component 1. For every invertible B in the chosen adelic basis, the relevant real basis is

$$x = (ph)^{-1}x_0B = x_hu_1^{-1}B, \qquad k(y)^{-1}x = x_0B, \qquad phk(y) = 1.$$

If $\sigma(x_0B) \neq 0$, the Gram matrix of x is that of x_0B and the inversion in $P(x)$ is genuine.

The orthogonal projection is independent of the choice of basis of the same plane, so $P(x) = y$. If $\sigma(x_0B) = 0$, a nonzero value in (92) would force $k(P(x))^{-1}x$ to lie in the bump. Its Gram matrix is the Gram matrix of x, so again the projection is genuine and $P(x) = y$. This makes the second bump $\sigma(x_0B) = 0$, a contradiction. If $\rho(h) = 0$, choose $p = 1$. The same argument shows that the large slice at $(ph)^{-1}x_0B$ vanishes, since otherwise the first bump would equal $\rho(h)$.

Lemma 35.1 (Real slice at a common translate)**.** *For each $h \in U(B_0 \perp W_0)_\infty$, there is $p = p(h) \in (U(B_0) \times U(W_0))_\infty$, with finite component 1, such that for every invertible B in the chosen adelic basis,*

$$\omega_{j-\mu}(1, ph)\Phi[(ph)^{-1}x_0B] = \rho(h)\kappa\,\sigma(x_0B)\prod_\tau \overline{\det B_\tau}^{2\mu_\tau}\ (v \boxtimes v_1). \tag{94}$$

The marked product and denominator. Let ν be an admissible split tame character of weight μ, with finite residue character χ. Let S be its nonempty tame support on the chosen split side, so that $\chi_w \neq 1$ for every $w \in S$, and put $\gamma_0 = \nu^{-1}\upsilon^j$. For (W_0, W_0) at the finite places use the original determinant datum $\phi_\chi = \mathbf{1}_{\mathrm{GL}_{2j}(O_S)}\chi^{-1}(\det)$ at S, with the original vacuum off the support. Let F be any finite Schwartz vector for (W_0, B_0). The real vectors are evaluated against the theta distribution on the finite vector $F \boxtimes \phi_\chi$ and its small member F, respectively. In the large functional the Heisenberg translate is $(c, 0)$, the group point is $(g, y_0 \oplus l)$, and the splitting characters are $(\gamma_0, \gamma_0\beta_0)$.

On the small functional these data read $c, (g, y_0)$ and (γ_0, β_0), for any remaining splitting character β_0. The block product of the rational lattice sums, together with the direct-sum Heisenberg action, factors the theta functional of $v \boxtimes v_1$ into that small functional times the equal-rank kernel of v_1, ϕ_χ at (g, l) with characters (γ_0, γ_0). This formula is for the original coordinates. The second Heisenberg move is zero and the determinant twist is the product of the twists of the two blocks.

The equal-rank kernel has absolutely convergent period on $[W_0]$ by the primitive support of ϕ_χ and the marked target-bounded theta estimate. Its period is independent of g, as follows. For a source element s supported at the chosen split places, simultaneously translate the target by s. The split Weil action on the square finite matrix reads $\phi(X) \mapsto \phi(s^{-1}Xs)$ with scalar one, since the two ranks and characters are equal.

Set $\lambda(a) = \nu^{\mathrm{bc}}(a)|a|^j$ on the split local factors. If $I_g(\phi)$ is the integrated kernel, the split target action and right invariance give

$$I_g(\phi(l^{-1}\,\cdot)) = \lambda(\det l) I_g(\phi).$$

Thus I_g on compactly supported locally constant functions on the invertible matrices depends through $\int \lambda(\det X)\phi(X)\, d^\times X$. Indeed, choose a common compact open subgroup K on whose right cosets the tests are constant and on which $\lambda \circ \det = 1$.

The equality on each coset lK follows from the target-action formula, giving the assertion by finite linearity. This weighted integral is invariant under conjugation by s. The determinant period is therefore right invariant at the chosen split places, and off them it is invariant under the original standard compact level away from both selected split halves.

Strong approximation [60, Theorem 7.12], applied to $\mathrm{SU}(W_0)$ now gives invariance under left translation by $\mathrm{SU}(W_0)(\mathbb{A}_{E^+})$. Since W_0 contains the hyperbolic plane H, we have $2j \geq 2$. Its special unitary group is simply connected and absolutely almost simple. At each of the selected split places it is SL_{2j}, which is noncompact. Explicitly, for $s' \in \mathrm{SU}(W_0)(\mathbb{A}_{E^+})$, approximate s' outside the selected places by a rational point s_0 so that $g^{-1}s_0^{-1}s'g$ lies in the original level at the other finite places and tends to the identity at infinity. At the selected places the full right factor is allowed. Automorphy and continuity give the claimed left invariance.

The period therefore equals its normalized radical mean on the one-plane parabolic of W_0. For this equal-rank datum the marked one-plane unfolding has lower source and target ranks equal, with empty deleted block. At its source Levi point (a_0, g_0), the unfolded integrand uses the idele variable a solely in the combination aa_0: the source and target Levi powers combine to $|aa_0|^{h+1}$, multiplied by the mixed slice factor $|aa_0|^{-h/2}$ in lower rank h.

The character and the slice variable are likewise read at aa_0. Right translation in the absolutely convergent idele-class integral proves that the mean is independent of a_0. Hence the period at $\mathrm{levi}(a_0, 1)$ equals its value I_0 at 1. The determinant of the Levi point is $a_0/\bar{a}_0$. Local Hilbert 90, with unit representatives away from finitely many places, gives every norm-one idele this way. As two points of equal determinant differ by the special unitary group, the equal-rank period is the constant I_0. This uses the equal-rank one-plane unfolding on the same finite determinant datum.

Finally let z_B be the idele with component $\det B_\tau$ at infinity and 1 at every finite place. With the normalized complex absolute value,

$$|z_B|^j = \prod_\tau |\det B_\tau|^{2j}, \qquad \nu^{\mathrm{bc}}(z_B) = \prod_\tau (\det B_\tau/|\det B_\tau|)^{2\mu_\tau}.$$

Consequently

$$\prod_{\tau} \overline{\det B_\tau}^{2\mu_\tau} = \frac{\prod_\tau |\det B_\tau|^{2(j+\mu_\tau)}}{\nu^{\mathrm{bc}}(z_B)|z_B|^j}.$$

Use (93) and (94), and integrate the equal-rank factor over its original $[W_0]$-measure. The result is the small functional of $\Lambda[B]$ multiplied by

$$\frac{\rho(h) I_0}{\nu^{\mathrm{bc}}(z_B)|z_B|^j}.$$

This holds simultaneously for every B, β_0, F, c, g, y_0, with the fixed choice of $p(h)$. The real vectors, bump, normalizing constant κ, and correction $p(h)$ are chosen before the character and modulus.

35.2 Extension of the self-dual Fourier tests

In this subsection, to pass from the real computation of Section 35.1 to an adelic identity, the finite data must match under self-dual Fourier transform. We establish the required Fourier identity on the full-rank locus and then extend the period formula.

Let D be the b-plane basis of the large source X, and A the additional marked plane inside its complement. The D-basis embeds from the complement of A. All conjugate-transpose and Fourier coordinates are those of these bases. Let S be the nonempty set of selected split places of E, and let P be the finite places outside $S \cup \bar{S}$. Use the marked compact level at P and normalized probability measure for all unipotent rational quotient averages.

In this subsection write $N = \dim X = 2b+2j$ and $a = \dim(B_0 \perp W_0) = b+2j$, so $N = a+b$. In the ordered hyperbolic basis (K_D, I_D, L_D), the coordinate covector $s = L_D^\vee$ is the Hermitian pairing with K_D. In particular $sK_D = 0$ and $sL_D = I_b$. At S put

$$\Psi_0(Y) = \chi^{-1}(\det(sY)) \quad (Y \text{ integral, } \det(sY) \text{ a unit}),$$

with zero value elsewhere.

The common-section space and the equivariant difference. Let $F_S = \prod_{w\in S} E_w$, let O_S be its integral subring and put $\lambda(u) = \nu^{\mathrm{bc}}(u)|u|^{N/2}$ on $F_S^\times$. For a full-column-rank matrix Y, define a section as follows. In each rank write $G = \mathrm{GL}(F_S)$ and $K = \mathrm{GL}(O_S)$. Choose a fixed set of representatives simultaneously for G/K and $K\backslash G$. Indeed each compact double coset has equally many left and right cosets by unimodularity, with each coset of measure one when K has measure one.

Each left and right coset in KgK meet, since $k_1 g k_2 \in k_1 gK \cap Kgk_2$. Pair the two finite sets and select an element in each intersection. Define

$$S_Y(\Psi) = \sum_r \lambda(\det r)\Psi(Yr).$$

On X^{t} use the same construction with GL_a. This defines

$$S_X^{\mathrm{cov}}(\Phi) = \sum_r \lambda(\det r)\Phi(r^{\mathsf{t}}X).$$

Use Schwartz–Bruhat functions compactly supported in the full-rank locus. For fixed full-rank Y, the map $r \mapsto Yr$ identifies $\mathrm{GL}_b(F_S)$ with a closed subset of that locus, cut out by

the condition that every column belongs to $\operatorname{im} Y$. The preimage of the support is therefore compact. The preimage meets only finitely many right K-cosets, so the section sum is finite. The covector section has the same property by transposition. In the column space $W \subset E_w^N$ of a full-rank matrix, the intersection $W \cap O_w^N$ is a saturated lattice. Its basis gives the integral reference for comparing the fractional lattice generated by the matrix columns.

Indeed, set $L(Y) = \{v \in F_S^b : Yv \in O_S^N\}$. If $AY = 1$, then for Y' sufficiently close to Y the matrix $u = 1 + (Y' - Y)A$ belongs to $\mathrm{GL}_N(O_S)$ and $uY = Y'$. Thus $L(Y') = L(Y)$. Writing $Y = Y_0 t$ with Y_0 primitive gives $L(Y) = t^{-1} O_S^b$, which determines the coset Kt. Only finitely many such cosets occur on a compact support. Transposition gives the covector statement. The integral determinant covariance is $\Psi(Yu) = \lambda(\det u)^{-1}\Psi(Y)$ for $u \in \mathrm{GL}_b(O_S)$, and $\Phi(vX) = \lambda(\det v)^{-1}\Phi(X)$ for $v \in \mathrm{GL}_a(O_S)$. Integral matched functions have these covariances by changing the complementary block of an integral completion. Both are preserved by all split source translations. With them each weighted summand is independent of the choice within its right coset rK.

For $k \in \mathrm{GL}_N(F_S)$, take Y to be its first b columns and X the last a covectors of k^{-1}. Take pairs of Schwartz–Bruhat functions on the full-rank loci with those integral covariances and satisfying

$$S_X^{\mathrm{cov}}(\Phi) = \lambda(\det k) S_Y(\Psi) \quad \text{for every } k.$$

For an integral matched pair, choose bases $a_0 X, Y b_0$ of the saturated covector and column modules. The corresponding covector and column sections each have only one potentially nonzero term, up to the integral change of bases. Its weight cancels the determinant covariance on the integral group. Put $Y_0 = Y b_0$ and $X_0 = a_0 X$. Since Y_0 spans the saturated lattice in $\ker X$, its integral image equals $\ker X_0$. Choose an integral matrix B_0 with $X_0 B_0 = I_a$. Then $k_0 = [Y_0 \ B_0]$ is unimodular and the last a covectors of its inverse are X_0. The block upper triangular matrix $k^{-1}k_0$ has diagonal blocks b_0, a_0^{-1}, so $\det k_0 = \det k \det b_0 / \det a_0$. The integral matching law at k_0 proves the section identity. All bases here can be chosen separately at the places of S. The section space is stable under

$$(\Phi, \Psi) \longmapsto (\lambda(\det g)\Phi(\,\cdot\, g), \Psi(g^{-1}\,\cdot\,)).$$

In fact its identity at k is the old one at $g^{-1}k$, multiplied by $\lambda(\det g)$.

35.2.1 Factorization through the Fourier test

In this subsection, we use the real restriction of Lemma 35.1 to compare the large period and its Fourier test.

The large period $\mathcal{L}(\Phi)$ factors linearly through Ψ on this space. To prove this, split the compact support of Φ into the finitely many clopen coset pieces Φ_r for which $X^{\mathsf{t}} r^{-1}$ is primitive integral. If $X^{\mathsf{t}} = Y_0 t$ for a primitive basis Y_0, this is exactly $tr^{-1} \in K$. Thus the partition uses the left-coset property of the common representatives. At a primitive covector X, the term $\Phi_r(r^{\mathsf{t}} X)$ equals $\Phi(r^{\mathsf{t}} X)$, and the same representatives enumerate the right cosets in the section sum.

Translate each piece to primitive integral covectors, with datum $\Phi_r(r^{\mathsf{t}} X)$. By changing the target variable at the selected split places and using the determinant multiplier of the Weil action,

$$\mathcal{L}(\Phi_r) = \lambda(\det r)\mathcal{L}\big(\Phi_r(r^{\mathsf{t}}\,\cdot\,)\big).$$

Consequently $\mathcal{L}(\Phi)$ is the large period of the function $S_X^{\mathrm{cov}}(\Phi)$ on primitive integral covectors, extended by zero. At each such X, choose an integral completion k_X with that last block.

The section identity reads this function directly from Ψ as $\lambda(\det k_X) S_{Y(k_X)}(\Psi)$. Thus it depends linearly on Ψ. All changes of target variable are justified by the full-rank target convergence, and they use its original Haar measure. Fix ν and use the scalar $c_{\mathrm{top},\nu}$ of the self-dual coefficient calculation (1). If $\mathcal{L}_{\mathrm{sm}}(\Psi)$ is the small period on the original column datum, set

$$\mathcal{D}(\Psi) = \mathcal{L}(\Phi) - c_{\mathrm{top},\nu}\mathcal{L}_{\mathrm{sm}}(\Psi).$$

The factorization above makes this independent of the covector partner Φ. It is linear in Ψ on the image of the section space, including when $c_{\mathrm{top},\nu} = 0$.

Let d be the finite idele equal to $\det g_S$ at S and one elsewhere. On the large side translate the covectors by $X \mapsto X g_S^{-1}$, with section multiplier $\lambda(d)^{-1}$. Since $a + b = N$, the resulting scalar is

$$\big((\nu^{\mathrm{bc}})^{-1} \upsilon_E^b\big)(d)^{-1} |d|^{a/2} \lambda(d)^{-1} = \big(\upsilon_E(d)|d|^{1/2}\big)^{-b}.$$

On the small side the transposed columns at the conjugate places read the determinant $\bar{d}^{-1}$. Their rank-b character is υ_E^b; hence their scalar is

$$\upsilon_E(\bar{d}^{-1})^{-b} |\bar{d}^{-1}|^{b/2} = \big(\upsilon_E(d)|d|^{1/2}\big)^{-b},$$

since $\upsilon_E(d\bar{d}) = 1$. The identity holds with the original quotient measures in $\mathcal{L}$ and $\mathcal{L}_{\mathrm{sm}}$.

The self-dual tests first force only those central coefficients of $f = \mathcal{D}(\Psi_0)$ whose index is primitive at the selected split places to vanish. The finite Fourier combination B will then satisfy $\mathcal{D}B = 0$, and the translates of its dilate span the locally constant matched column data. For rank $b = 0$ there is one column datum up to scalar and one central coefficient with empty determinant equal to one. The self-dual test already gives its vanishing on the full parabolic, which in this case is the whole group. The line shifts and the Gauss sum are unnecessary in that case.

In these coordinates the covariance is

$$\mathcal{D}(\Psi \circ g_S)(x g_S) = \big(\upsilon_E(\det g_S)|\det g_S|^{1/2}\big)^{-b} \mathcal{D}(\Psi)(x).$$

Its values are continuous and automorphic with the marked compact level at P, by the absolute summability in the calculation of the self-dual coefficients.

Interchange of the two line averages. For $T \in M_b(E)$, write $n_D(M) = 1 + K_D(M - M^*)s$. In the original bases the central coefficient is

$$F_T(g) = \int_{M_b(E)\backslash M_b(\mathbb{A}_E)} \psi_E(\mathrm{tr}(TM))\, f(n_D(M) g)\, dM, \qquad \int dM = 1.$$

Translation of M gives $F_T(n_D(Z)g) = \psi_E(-\mathrm{tr}(TZ)) F_T(g)$. Since every Hermitian Z acts trivially, a nonzero coefficient requires $T^* = -T$, by nondegeneracy of the trace pairing. Let $n_+(a)$ and $n_-(b')$ be the shifts using the two opposite isotropic lines of A, where a, b' are covectors of length b. Define

$$\phi_g^+(a) = \int_{E^b\backslash \mathbb{A}_E^b} F_T(n_-(b') n_+(a) g)\, db',$$

and use the opposite order for ϕ_g^-. Multiplication of the two unipotent block matrices gives

$$n_+(a)n_-(b') = n_D(-a^*b')\, n_-(b')n_+(a).$$

To move $n_+(a)$ to the left in the opposite mean, use this identity in reverse order. The intervening $n_D(a^*b')$ contributes $\psi_E(-\operatorname{tr}(Ta^*b'))$. Hence

$$\phi^+_{n_+(a)g}(u) = \phi^+_g(u+a), \qquad \phi^-_{n_+(a)g}(b') = \psi(-\operatorname{tr}(Ta^*b'))\phi^-_g(b').$$

For a rational covector a_0, move $n_+(a_0)$ across the opposite shift and use rational automorphy. The value $\phi^+_g(a+a_0)$ is then the Fourier coefficient at $-Ta_0^*$ of $b' \mapsto F_T(n_-(b')n_+(a)g)$, with the displayed Fourier sign. If $\det T \neq 0$, these are all the characters of the covector quotient. Therefore ϕ^+_g determines these translated coefficients by Fourier uniqueness on the compact covector quotient. The same proof with the two lines exchanged shows that ϕ^+_g and ϕ^-_g determine one another, compatibly with scalar multiplication.

Shears on the actual standard datum. At a point fixing the D-basis away from P, the partial mean at a shifted point, as a linear evaluation on the equivariant data, is invariant under

$$Y \longmapsto Y + uwsY \qquad (w \in M_{1,b}(E_S)),$$

where u is the moved extra-line vector at S. Indeed these are the split components of the opposite root shifts used in the mean. They have determinant one in the split covariance, and right translation of the covector quotient preserves its Haar measure. The covector here is still s: it is the adjoint covector of the fixed basis, and the added extra-line vector is orthogonal to it. Consequently $su = 0$. For a general allowed rank-one split shear the change in the local datum is $Y \mapsto Y + urY$, with $ru = 0$. The other split half has the conjugate unitary coordinates.

Such a shear also fixes the value of this linear evaluation on Ψ_0. Average the two functions over the restricted maps $Y \mapsto Y + urtsY$ with t in the compact additive group $M_{\dim X,b}(O_S)$. If for some covector ρ the point $Y + u\rho$ lies in $\operatorname{supp}\Psi_0$, then sY is integral invertible and $t_0 = (Y + u\rho)(sY)^{-1}$ is integral. Since $ru = su = 0$,

$$Y + urY + urtsY = Y + ur(t + t_0)sY.$$

Translation by t_0 preserves Haar on the compact group, so the two averages agree. If no such ρ exists the averages vanish. The averages are finite linear combinations of translates because the datum is smooth for the compact group. The restricted invariance therefore proves the needed invariance under the full split shear with the exact datum.

For either extra line this gives partial-mean invariance under the adelic line radical, between the points under consideration. First, its centre fixes the partial mean for rational parameters by automorphy and for parameters supported on the selected split half by the shear argument. The centre is an additive vector group over $F_0 = E^+$. At a selected split place the coordinate on the chosen half determines the full local central parameter. Additive strong approximation away from these places shows that the sum of the rational parameters and the parameters supported there is dense in the full adelic centre. Continuity gives invariance under that centre. Averaging over this centre now writes the partial mean as the coefficient of the combined isotropic basis with index $\operatorname{diag}(0, T)$.

The mixed blocks of its centre give the opposite line shift already integrated in the mean. A rational parameter in the line radical acts on the combined basis by an upper block unipotent

and preserves this bordered index by congruence. The parameters supported on the selected split half fix the same coefficient by the preceding calculation. Additive strong approximation in the remaining radical coordinates, with commutator errors in its centre, gives the stated invariance by continuity.

35.2.2 Integral extension of the Fourier test

In this subsection, we extend the test of Section 35.2.1 in the local self-dual coordinates.

Suppose $F_T(g) \neq 0$ at a point fixing the D-basis away from P, with $\det T \neq 0$. The line interchange gives a nonzero shifted mean. At the places in P, act on the right by the marked integral group so that the action on the combined isotropic basis is upper block triangular for the flag given by the extra line and the D-plane. This is the integral flag decomposition: intersect the two isotropic spaces of the transported flag with the standard lattice, take primitive bases in succession, and complete to an integral Witt basis.

The orthogonal complement at each step is unimodular. By Section 4.3, every nonsplit finite place of E/E^{+} is unramified. For a primitive isotropic column e choose an integral dual column v with $h(e,v)=1$. Take z integral with $z+\bar{z}=h(v,v)$; then $v-ze$ is isotropic and still dual to e. Such z exists also above 2, since the residue trace is surjective and a lift of a residue element of trace one has unit trace. At a split place one extends a basis and its dual. The compact action is trivial at the other places. The complementary factor of the upper-triangular action is in the extra-line radical, which may be discarded by the invariance just proved. This leaves a point which at P agrees with a block action on the chosen bases with scalars t, t' on the two lines and matrix γ on the D-basis, where $(t')^{*}t=1$. Call it a straight point. It still fixes the D-basis off P.

Put $\delta = j_{\mathrm{W}}\alpha_0\bar{\alpha}_0$, the additive scale, and set $C=\delta^{-1}\gamma^{*}T\gamma$ at P. At every straight point where the shifted mean is nonzero, C is integral at P. Indeed some shifted value of the coefficient is nonzero at a point with the same D-basis action. The compact level allows all integral central translations at P there. At $w\in P$, take the integral matrix parameter $M=aE_{ji}$, $a\in\mathcal{O}_w$, supported at that place. The central translation has integral skew block $M-M^{*}$. Put $T'=\gamma^{*}T\gamma=\delta C$. Since $(T')^{*}=-T'$,

$$\tfrac{1}{2}\operatorname{Tr}_{E/F_0}\operatorname{tr}\bigl(T'(M-M^{*})\bigr)=\operatorname{Tr}_{E/F_0}\operatorname{tr}(T'M).$$

If $gK_D=K_D\gamma$ on P, then $gn_D(M)g^{-1}=n_D(\gamma M\gamma^{*})$ for M supported there. The level fixes the right translate, whereas central translation on the left gives

$$F_T(gn_D(M))=\psi_{E,w}(-\delta_w a C_{ij,w})F_T(g).$$

Here $\operatorname{tr}(MC)=aC_{ij,w}$ at w and $\psi_{\mathrm{sd},w}(u)=\psi_{E,w}(\delta_w u)$. The element used for this test is integral also above 2. The multiplier fixes the nonzero coefficient. Since the annihilator of $\mathcal{O}_w$ is $\mathcal{O}_w$, every entry of C is integral, also at places above two. Choose a straight point with nonzero shifted mean minimizing the positive integer

$$I(C)=\prod_{w\in P} q_w^{v_w(\det C)}.$$

This is the lattice index in the dual, with the conventional choice of counting both conjugate components if present. The same minimum applies with the two lines exchanged by the line interchange above.

For a covector a with $\phi_g^+(a) \neq 0$, one has

$$\gamma^{-1} a^* t \in O_P^b. \tag{95}$$

Otherwise the shifted point has basis matrix with lower left block $-a^*t$ and lower right block γ. Straightening again expresses this basis matrix as a block upper-triangular matrix of lower right block γ' times an integral matrix M, the basis action of the inverse right compact move. Thus $-a^*t = \gamma' M_{21}$ and $\gamma = \gamma' M_{22}$. Failure of (95) implies that M_{22} is a nonunit at some place in P. The new Gram is integral since the new mean is nonzero, and

$$C = M_{22}^* \, (\delta^{-1} (\gamma')^* T \gamma') \, M_{22}.$$

Writing $C' = \delta^{-1}(\gamma')^* T \gamma'$, and using $\bar{P} = P$, we obtain

$$I(C) = I(C') \prod_{w \in P} q_w^{2 v_w (\det M_{22})} > I(C').$$

This contradicts minimality. The opposite argument gives $\gamma^{-1}(b')^* t' \in O_P^b$ whenever $\phi_g^-(b') \neq 0$.

Take any column x with $Cx \in O_P^b$, and set $a = (\gamma x (t')^*)^*$ on P and zero at the other places. On the support of ϕ_g^- the multiplier of the shift $n_+(a)$ is one. Indeed the scaled argument of its additive character is $-(\gamma^{-1}(b')^* t')^* C x$, which is integral. Off P the argument is zero. Thus a is a period of ϕ_g^+ by the interchange. Apply (95) to a nonzero point of ϕ_g^+ and its translate by a, and subtract. Since $(t')^* t = 1$, this gives $x \in O_P^b$. Use the columns of C^{-1} as x. The minimizing Gram is therefore unimodular at every place in P.

A nonzero shifted mean gives a nonzero coefficient at a point with this same D-basis action on P and identity on that basis elsewhere, because both line shifts fix D. Suppose now that $\delta^{-1} T$ is primitive at S. Both T and δ are skew under the adjoint, so $D_0 = \delta^{-1} T$ and D_0^{-1} are Hermitian. Integrality of each at $w \in S$ gives integrality at $\bar{w}$ by conjugate transposition. Write $T = j_{\mathrm{W}} \beta$ and let $q = \alpha_0 q_0$ be the Levi matrix of the coefficient point. It acts by γ at P and by the identity at the selected split pairs. Consequently

$$Q = (q_0^* \beta q_0)_f = (\delta^{-1} q^* T q)_f, \qquad Q|_P = C, \qquad Q|_{S \cup \bar{S}} = D_0|_{S \cup \bar{S}}.$$

Thus $Q \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ is Hermitian. The split and unramified integral basis constructions in Section 36.1.1 give $k \in \mathrm{GL}_b(\widehat{\mathcal{O}}_E)$ with $k^* Q k = I_b$. If β is outside the real congruence class of the marked $B_0 = U^-$, use the off-class vanishing in the self-dual test. For the chosen real slice this follows from its support and the empty small orbit in Section 36.1.1. For the arbitrary real vectors used in Section 36.1.6, that section proves the requisite vanishing by compact averaging.

Otherwise Landherr's theorem [41, pp. 246–247] gives $c \in \mathrm{GL}_b(E)$ with $\beta = c^* J_{B_0} c$: the finite forms have orthonormal bases and their real signatures agree. Let $m_D(c)$ be the rational Levi element acting by c on K_D. Substituting $M = c M' c^*$ in the central integral gives

$$F_{j_{\mathrm{W}} J_{B_0}}(m_D(c) g) = F_T(g).$$

Indeed $n_D(c M' c^*) m_D(c) = m_D(c) n_D(M')$, and rational automorphy removes $m_D(c)$ from the integrand. The substitution preserves rational matrices and the normalized Haar on their additive quotient. Its local Jacobians also have product one by the product formula for $(\det c \, \overline{\det c})^b$. The new Levi matrix is $\alpha_0 c q_0 = \alpha_0 X_{B_0} Y$ with $Y = X_{B_0}^{-1} c q_0$. The original finite basis satisfies $(X_{B_0}^* J_{B_0} X_{B_0})_f = I_b$, hence

$$(Y_f k)^* (Y_f k) = k^* Q k = I_b.$$

This is the self-dual coefficient at the prescribed index and Levi matrix, so it vanishes. The contradiction proves that $F_T(1, g_f) = 0$ when $\delta^{-1}T$ is primitive at S and g_f is trivial there and at the conjugate split places.

Finite detection of the other coefficients. Use the opposite ordering $(K, L) = (L_D, K_D)$ in this calculation. Then $s = K^\vee = L_D^\vee$, $K^\vee L = L^\vee K = 0$, and $K^\vee K = L^\vee L = I_b$. Thus $LzK^\vee$ is the original central block $K_D z L_D^\vee$ at the selected split places. For $z \in M_b(E_S)$ put

$$\Psi_z(Y) = \Psi_0(Y + LzK^\vee Y).$$

At the indicated points $(1, g_f)$ the determinant-one central shear acts on the datum by composition. Split covariance therefore acts on the coefficient point by the inverse shear. The central translation formula gives

$$\Lambda_{T,g}(\Psi_z) = \psi_S(\mathrm{tr}((\delta^{-1}T)_S z))\Lambda_{T,g}(\Psi_0),$$

where $\Lambda_{T,g}$ is the central coefficient of $\mathcal{D}$ at g as a linear form of the datum. Here $\psi_S(u) = \prod_{w \in S} \psi_{E,w}(\delta_w u_w)$. Each local character has conductor $\mathcal{O}_w$, so $r \mapsto \psi_S(\varpi^{-1}\widetilde{r})$ on scalar residues is well defined and nontrivial on each field factor. Let k_S be the product of residue fields, and let $\widetilde{r}$ be integral lifts of their matrices with $\widetilde{1} = 1$. Put

$$B(Y) = \sum_{r \in M_b(k_S)} \chi^{-1}(\det r)\Psi_{\varpi^{-1}\widetilde{r}}(Y).$$

The character is extended by zero on nonunits.

If $\Lambda_{T,g}(\Psi_0) \neq 0$, invariance under integral shears forces $(\delta^{-1}T)_S$ integral. Its reduction is singular somewhere by the primitive test already proved. At a singular factor choose an invertible u fixing the matrix on the right and scaling one line in its kernel by a unit of nontrivial χ-value. Reindex the Gauss sum by $r \mapsto ur$. The additive term is unchanged and the determinant character changes by a nontrivial scalar. Thus the sum is zero. This also handles a singular rational T, and all other terms with $\Lambda_{T,g}(\Psi_0) = 0$ are already zero.

The skew coefficients exhaust the characters of the central quotient in the original convention. Fourier uniqueness gives $\mathcal{D}B(g) = 0$ at these points. Their rational translates are dense by weak approximation for the unitary group at the specified real and split finite places, with all other finite components unrestricted. Weak approximation here follows from its rational Cayley chart and weak approximation for the vector space of skew matrices. Continuity and automorphy give $\mathcal{D}B = 0$.

Translate by $Y \mapsto -\varpi^{-1}Y$ on the split source. If a summand at that point is nonzero, there is an integral argument X of Ψ_0 with $Y = L\widetilde{r}(K^\vee X) - \varpi X$. It follows that Y is integral, $L^\vee Y \equiv r(K^\vee X) \pmod{\varpi}$, and the value of the summand is $\chi^{-1}(\det(L^\vee Y))$. Thus $B(-\varpi^{-1}Y)$ is the opposite standard datum times a nonnegative integer multiplicity. That multiplicity is positive at $Y = L - \varpi K$, where the term $r = 1$ has argument K.

Let Ψ be the locally constant column function of any integral matched pair. Put $P(Y) = B(-\varpi^{-1}Y)$, $p = L^\vee$, and $x_0 = L - \varpi K$. The kernel of $\mathcal{D}$ is stable under the split translations, so $P \in \ker \mathcal{D}$. We have $px_0 = I_b$ and $P(x_0) > 0$. Choose $c \geq 1$ so that translation by ϖ^c times any integral matrix preserves Ψ. Such a depth exists by local constancy and compact support. Let

$$d_c = x_0 p + \varpi^{-c}(I - x_0 p).$$

For $k \in \mathrm{GL}_N(O_S)$, if $Z = d_c k Y$ lies in the support of P, then $g = pZ \in \mathrm{GL}_b(O_S)$ and

$$Y = k^{-1} x_0 g + \varpi^c k^{-1}(Z - x_0 g).$$

Thus $\Psi(Y) = \chi(\det g)^{-1}\Psi(k^{-1}x_0)$. Since $P(Z)$ is a nonnegative integer times $\chi(\det g)^{-1}$, it follows that

$$\Psi(k^{-1}x_0)P(d_c k Y) = |P(d_c k Y)|\Psi(Y)$$

for every k, Y, including the zero terms. The average over k on the left is a finite linear combination of translates of P; the translates of this Schwartz–Bruhat function are constant on cosets of an open subgroup of the compact group. The integral of $|P(d_c k Y)|$ is constant on primitive integral columns by transitivity under $\mathrm{GL}_N(O_S)$. Its value there is positive since $d_c x_0 = x_0$ and $|P(x_0)| > 0$. The average is therefore a nonzero scalar multiple of Ψ in $\ker \mathcal{D}$. This proves the period comparison for all the integral matched data used below.

36 The normalized period exchange

In this section, we compare the periods using the local factors of Section 35. The real identity and its extension to the finite tests supply the comparison used in Section 39.

36.1 The normalized comparison of the two periods

In this subsection, we compare the paired periods of Section 34.3 by removing hyperbolic planes. The real factor from Section 35.1 and the finite Fourier transform of Section 35.2 fix the final scalar.

Let W, V, U be the marked Hermitian spaces of ranks N, a, b, with their original standard unimodular models at the finite places. Each space has a fixed signature at all infinite places. Write $q(W), q(V), q(U)$ for their respective common negative indices. Assume that $N = a + b$ and $a - b = 2t$ for some $t \in \mathbf{Z}$, and that

$$q(W) = q(V) + q(U), \qquad 2q(W) + 2 \le N \le 2\mu_\tau + 2q(W) \quad (\tau \mid \infty).$$

Fix the ordered splitting into b, a coordinates and a real Fock vector y for (U, W) of finite moments. For the admissible split tame character ν, put

$$\gamma = \nu^{-1}\upsilon^{N/2}, \qquad M_\nu(j) = \prod_{w \in S} q_w^{j^2} \prod_{k=1}^{j} |L^S(k, \nu)|^2 \quad (j \ge 0), \qquad M_\nu(-j) = M_\nu(j)^{-1}.$$

Here υ_E is the fixed splitting Hecke character and υ is its restriction to the norm-one ideles. The modulus has nonempty split support S, disjoint from $\bar{S}$. Its tame character satisfies $\chi_w \ne 1$ for each $w \in S$. The base-change character is ramified exactly at $S \cup \bar{S}$, whose Euler factors are omitted in L^S. The characters have weight μ at infinity, which is fixed. The Haar measures and vacua throughout are the original ones for the marked models.

We give the assembly from the Fourier and constant-term identities, specifying the versions needed to normalize the comparison. For the top calculation use

$$W_0 = H \perp W, \qquad X = H^b \perp W_0, \qquad V_{\mathrm{top}} = U^- \perp W_0.$$

The small side is (X, U^-), with U^- the opposite space. Write $j_0 = \dim W_0/2$. We use the real slice of Section 35.1 with $j = j_0$. Given the two fixed real vectors, one for the small side and one z_0 for (W_0, W_0), the marked slice identity on the original self-dual finite lattices has coefficient

$$\frac{\rho(h) I_{W_0,\nu}(z_0)}{\nu^{\mathrm{bc}}(z_B)|z_B|^{j_0}}.$$

Here B is the invertible small slice matrix, z_B is $\det B$ at infinity and 1 at finite places, and $I_{W_0,\nu}(z_0)$ is the equal-rank $[W_0]$-period for the finite determinant datum $\chi^{-1}(\det)$.

The real factor and the real coordinate corrections are chosen before ν, B and the finite data. The self-dual Fourier unfolding and the primitive-coefficient vanishing and spanning on its marked test space give the identity on standard partners with scalar

$$c_{\mathrm{top},\nu} = R_0 \kappa_0 \varepsilon\, \nu^{\mathrm{bc}}(d_0) I_{W_0,\nu}(z_0), \qquad \kappa_0 = (v_{\mathrm{big}}/v_{\mathrm{sm}})|d_0|^{j_0}. \tag{1}$$

The nonzero vacuum constants $v_{\mathrm{big}}, v_{\mathrm{sm}}$ and the coordinate-change idele $d_0 = \det(\alpha_0 X_{U^-})$ belong to the original splittings, with X_{U^-} the small adelic basis.

At S let B_D be the integral basis matrix with columns (L_D, K_D, I_D) in the prescribed order, and put $s = L_D^\vee$. For a matrix Z of covectors, adjoining s to Z in that order gives a square matrix $C_s(Z)$. In block coordinates

$$C_s(Z) B_D = \begin{pmatrix} I_b & 0 \\ Z L_D & Z(K_D, I_D) \end{pmatrix}, \qquad \det C_s(Z) \det B_D = \det\big(Z(K_D, I_D)\big).$$

On the integral unit support this gives

$$\chi^{-1}(\det C_s(Z)) = \varepsilon \chi^{-1}\big(\det\big(Z(K_D, I_D)\big)\big), \qquad \varepsilon = \chi(\det B_D).$$

Hence $|\varepsilon| = 1$. The column partner is $\chi^{-1}(\det(sY))$ on its original unit support. The positive integral R_0 is the integral of $\rho(h)\mathbf{1}_{H_f K_f^{\mathrm{std}}}(u)$ on the quotient by the block stabilizer. In particular the compact set in this integral is the full fixed standard level. The exponent here is j_0 because the complement in the mixed model is W_0.

36.1.1 Absolute summability of the mixed terms

In this subsection, we justify the unfolding needed for Proposition 36.1.

Write $A = (K, I)$ for the rational basis of $K^\perp$ in the fixed source splitting D. At a source point g in its parabolic let A_g be its coordinates in the moved source lattice. Write Y for a matrix in the finite Schrödinger model. It is sufficient that $Y A_g$ have full rank on the dual side on the datum's support at one selected split place. For the large standard-covector datum in the self-dual test, the support is given by a unit minor of $Y(K, I)$ in the finite basis. For the small standard-column datum, read its transpose at the conjugate split places. Its support requires the square block on K to be invertible. The parabolic preserves $K^\perp$, and the transported basis is obtained by an invertible change of columns on (K, I). Thus both support conditions give the stated rank condition. The rank-zero target has one term and needs no estimate.

For a fixed source point let q_x denote the theta functional of the mixed slice at the rational matrix x. Let h be the target point and $H = H(h)$ its reciprocal lattice minimum cut off below

at one. There are A_0, M and C_g such that, for an arbitrary real Schwartz vector u and the fixed finite datum f_f,

$$\sum_x |q_x(\omega(g,h)(u \otimes f_f))| \le C_g H^{A_0} p_M(u).$$

Here p_M is a fixed finite-order Schwartz seminorm.

Indeed the absolute sum is invariant under rational target translation, which permutes the rational matrices x. Reduce h to a point in a fixed adelic compact set followed by a real move of norm $O(H)$, by the minimum-doubling reduction. In any rational polarization of the smaller mixed block, the absolute sum of the slice theta evaluations is bounded by the sum of the absolute Schwartz values over the rational coordinates of both blocks.

The finite coordinates in play are in fixed compact subsets, so these rational coordinates lie in a fixed fractional lattice. Summing $(1 + \|x\| + \|y\|)^{-M}$ for sufficiently large M gives the claimed bound before the real move. The position and differentiation operators transform by the real symplectic matrix under Weil conjugation. Their finite products, Sobolev's inequality and unitarity give the stated polynomial factor for the real move, increasing the fixed order M if needed. The finite translates in the compact reduction are taken from a finite set by smoothness.

For $H > 2$ choose a nonzero rational isotropic vector d in the moved target lattice with length at most $2/H$. It is obtained by taking a shortest vector, since its integral Hermitian norm has absolute value below one at every infinity place. Multiply the columns c_j of the fixed rational basis of $K^{\perp}$ by a single positive integer to put them in the moved source lattice.

Their lengths are bounded in terms of g. Let Z be the rational isotropic span of the maps

$$d\, c_j^* J_X.$$

These maps have integral finite coordinates and real lengths $O_g(H^{-1})$ in the moved Hom space. Each kills K, so translation by its rational points fixes every q_x: in the mixed model the slice stays at x, its residual translation is rational, and its additive phase is the value at a rational number. Therefore any finite phase-weighted sum L of the q_x's, with phases of modulus one, has a probability mean L_0 over $Z(E)\backslash Z(\mathbb{A}_E)$. Use this mean on the translated datum, with Weil covariance for the source and target action.

The mean annihilates the translated datum. In the moved finite coordinates, the covector $d_h^* Y A_g$ is nonzero at a selected split place whenever the datum at Y is nonzero. Indeed d_h^* is a nonzero covector there and $Y A_g$ has a right inverse. In the opposite split component the corresponding Heisenberg shift multiplies the Schrödinger datum by the additive character paired with this covector. For every support point one can choose a parameter for which this character is different from one. It is constant on a small open neighborhood. A finite partition of the compact support into such neighborhoods, and invariance of L_0 under the corresponding shifts, show that L_0 vanishes on each piece, with any real part.

For clarity the estimate for the difference from this mean is as follows. Choose a positive integer m_0 such that all finite Heisenberg shifts in $m_0 \widehat{\mathcal{O}}_E$ in the chosen integral coordinates fix the fixed finite datum. For each coefficient of the displayed generators, reduce modulo E to an adele with finite part in $m_0 \widehat{\mathcal{O}}_E$ and bounded real part. This uses additive strong approximation followed by a fundamental parallelepiped of $m_0 \mathcal{O}_E$.

It gives measurable representatives of the mean for which the translated finite part is fixed and the real shift s has length $O_g(H^{-1})$. Denote by A_h the average of these real Heisenberg operators. On the fixed finite tensor factor, $L - L_0$, after the Weil translate, annihilates $A_h u$

for every real u, by translation invariance of the mean. For every integer $n \geq 0$, its value on u therefore equals its value on $(1 - A_h)^n u$.

The real translation and modulation formulas, by integrating their first derivatives along ts, give

$$p_j((1 - A_h)u) \leq C_{g,j} H^{-1} p_{j'}(u), \qquad p_M((1 - A_h)^n u) \leq C_{g,M,n} H^{-n} p_{M_n}(u).$$

All translated real vectors here stay in a bounded set of translation parameters, so each derivative requires only a higher fixed Schwartz seminorm. The initial bound applies to L and to the average L_0, with constants independent of the finite set of x's and their phases.

Taking phases so that the finite sum is the sum of the absolute values, then increasing the finite set, gives

$$\sum_x |q_x(\omega(g,h)(u \otimes f_f))| \leq C_{g,n} H^{A_0 - n} p_{M_n}(u).$$

At $H \leq 2$ use the first bound. In particular this is bounded on the target quotient, which has finite volume. The marked radical quotient is compact. Over a compact set of representatives its source points stay in the parabolic, so the same rank proof applies.

The constants can be taken uniformly there. The bounded representative set has compact finite projection, so a single integer clears the coordinates of the fixed rational basis throughout, and its real basis coordinates are uniformly bounded. The initial Schwartz bound is uniform on that compact set as well. The central parameter itself only multiplies each mixed term by a unit phase. Thus the absolute sum is integrable also in the radical variables.

Tonelli and Fubini now justify both the extraction of the central coefficient term by term and the subsequent unfolding of the surviving rational orbit. Replacing (K, I) by a full rational source basis and replacing q_x by the whole theta functional gives the same argument for the ordinary theta integral of any compactly supported finite datum of full rank on the dual side. In that version the initial bound and the source lattice basis may be taken uniformly on compact sets of source points. This gives continuity of the ordinary period by dominated convergence. All constants in this paragraph are for the chosen finite datum. No bound as its level changes is required.

Here is the scalar computation on the original self-dual tests. Put $B_0 = U^-$, let $\beta = J_{B_0}$ be its rational Hermitian matrix, and take the skew Fourier index $T = j_{\mathrm{W}} \beta$. Let the parabolic Levi matrix be $q = \alpha_0 X_{B_0} Y_0$, where $Y_{0,f} k$ is unitary for an integral finite invertible matrix k. For a rational matrix x of Gram matrix G_x, use the central coordinates $C - C^*$ of Section 33.1.3. Orthogonality on $\mathrm{Mat}_b(\mathbb{A}_E)/\mathrm{Mat}_b(E)$ with probability Haar gives

$$\int \psi_E(\mathrm{tr}((T - j_{\mathrm{W}} G_x)C))\, dC = \mathbf{1}_{G_x = \beta}.$$

Over $F_0 = E^+$, Witt extension gives one orbit of the ordered isometric embeddings on each side, with representatives $(1; 0)$ and 1, and stabilizers $U(W_0)(F_0)$ and 1. For the radical parameter $c \in \mathrm{Mat}_{2j_0 \times b}(\mathbb{A}_E)$, set $d_c = X_{B_0}^{-1} c^* J_{\mathrm{std}}$, where J_{std} is the Gram matrix of W_0 in its adelic basis. The two Heisenberg translations are $(d_c; 0)$ and d_c. For the unreduced radical parameter z, the multiplier of a representative of Gram matrix G_x is $\psi_E(\frac{1}{2}\mathrm{tr}(zG_x))$. The skew block $C - C^*$ is represented by the change $z \mapsto z - 2j_{\mathrm{W}} C$, since

$$-\frac{1}{2j_{\mathrm{W}}}((-2j_{\mathrm{W}} C) + (-2j_{\mathrm{W}} C)^*) = C - C^*.$$

This gives $\psi_E(-j_{\mathrm{W}} \mathrm{tr}(G_x C))$. Both representatives have $G_x = \beta$, so their multipliers agree before integration. This gives the large unfolded quotient by $U(W_0)(F_0)$ and the small unfolded

integral over $U(B_0)(\mathbb{A}_{F_0})$, with the original Haar normalizations. The bump in (92) may be supported where the Gram matrix has the signature of β at every real place. A nonzero large slice then has $x^\flat \in \operatorname{supp}\sigma$, and $(x^\flat)^* J x^\flat = x^* J x$ because $k(P(x))$ is unitary. The source basis change acts by invertible congruence, preserving inertia and rank. Thus an index of different signature or of deficient rank contributes zero. For a nondegenerate index of different signature the small orbit is empty. For a singular index an associated small square matrix is singular, so $\sigma(x_0 B) = 0$.

A nonzero finite slice forces the large group point to lie in $H_f K_f^{\mathrm{std}}$, where H_f is the block group for $B_0 \perp W_0$. Indeed multiply the integral basis by k to obtain an integral orthonormal basis. It extends to a unitary integral basis by splitting off its unimodular orthogonal complement. At split places extend the chosen basis and its dual by a complementary basis. At an unramified nonsplit place, reduction of the complement is a nondegenerate Hermitian space over the quadratic residue extension. A vector of nonzero norm exists by the residue trace pairing. Lift it and scale its unit norm to one using surjectivity of the local unit norm. Induction on dimension diagonalizes the complement. This proves the claim in all the finite models, which are split or unramified.

At $w \in S$ write $\Phi_w(Z) = \chi_w^{-1}(\det C_s(Z))$ on the integral unit support. For an integral target element k_w, changing Z to $k_w^{-1} Z$ multiplies Φ_w by $\chi_w(\det k_w)$. The target Weil multiplier on units is $\chi_w^{-1}(\det k_w)$, since the character is $\nu^{-1} \upsilon^{\dim X/2}$. At finite places away from the selected split pairs the integral group fixes the vacuum. Thus the full finite target compact fixes the large test. Reduce to the block point and put $B = u^{-1} Y_0$, $X_f = B_f$, where $u \in U(B_0)(\mathbb{A}_{F_0})$ is its small group variable. Both data vanish unless X_f is integral. On this support $X_f k$ is integral unitary, so its adjoint is its integral inverse. As k is integral invertible, $x = \det X_f$ is a unit at every finite place. In the complementary variables Z_1, Z_2 at $w \in S$, the large determinant is

$$\det \begin{pmatrix} X_{f,w} & Z_1 \\ 0 & Z_2 \end{pmatrix} = \det X_{f,w} \, \det Z_2.$$

Its slice is the integral vacuum on Z_1 times the original determinant datum on Z_2, with scalar $\varepsilon_w \chi_w^{-1}(\det X_{f,w})$, where $\varepsilon_w = \chi_w(\det B_{D,w})$. On the small side the residual variable carries the same integral vacuum, with scalar $\chi_w^{-1}(\overline{\det X_{f,\bar{w}}})$. Conjugation here identifies $E_{\bar{w}}$ with E_w. The two character factors have quotient

$$\frac{\chi_w^{-1}(x_w)}{\chi_w^{-1}(\overline{x_{\bar{w}}})} = \chi_w(x_w / \overline{x_{\bar{w}}})^{-1}.$$

The product over S is $\nu^{\mathrm{bc}}(x)^{-1}$, since away from these split pairs the base-change character is trivial on units. The large side also has the phase ε. The small rank twist changes by the factor $\nu(\det u)^2$ between the two formulas. The ratio of the Levi factors contributes $\nu^{\mathrm{bc}}(\det q)|\det q|^{j_0}$. Since $z_B = \det(Y_0)(\det u)^{-1}(\det X_f)^{-1}$ as ideles, and $\nu^{\mathrm{bc}}(\det u) = \nu(\det u)^2$, the resulting phase and positive factor are exactly

$$\frac{\nu^{\mathrm{bc}}(\det q)}{\nu^{\mathrm{bc}}(z_B)\nu^{\mathrm{bc}}(\det X_f)\nu(\det u)^2} = \nu^{\mathrm{bc}}(d_0), \qquad \frac{|\det q|^{j_0}}{|z_B|^{j_0}} = |d_0|^{j_0}.$$

The remaining block stabilizer quotient integral of the bump gives the factor R_0 of (1).

36.1.2 The measure on the open orbit and the signed comparison

In this subsection, the first stage of the comparison in Section 36.1 unfolds over the full-rank orbit. We keep track of the quotient measure and the source involution to identify its signed scalar.

Let $G = U(B_0 \perp W_0)(\mathbb{A}_{E^+})$, $L = U(B_0)(\mathbb{A}_{E^+})$, $H_1 = U(W_0)(\mathbb{A}_{E^+})$, and $H = L \times H_1$ inside G. Let $\Gamma = U(W_0)(E^+)$ in its block embedding. The groups G and H are unimodular, and H is closed. Give the homogeneous space $H\backslash G$ the invariant quotient measure determined by dG and $dL\,dH_1$ via [27, Theorem 2.51 and (2.52)]. The same Haar disintegration, first by Γ and then by H, gives for an integrable unfolded term F

$$\int_{\Gamma\backslash G} F = \int_{H\backslash G} \int_L \int_{\Gamma\backslash H_1} F((l,h_1)g)\,dh_1\,dl\,d\bar{g}.$$

This follows from the quotient integral formula first for compactly supported lifts to G, then for integrable functions by applying the positive formula to their absolute values. Right translation by each component of $p(h)$ in the two inner integrals preserves the original Haar, including the measured $[W_0]$. Fix the source point and the matched self-dual test. Denote by A_{sm} its small Fourier coefficient, unfolded as the integral over L of the term at the identity basis. Applying (94) inside the two inner integrals gives

$$\int_L \int_{\Gamma\backslash H_1} F((l,h_1)g)\,dh_1\,dl = \kappa_0 \varepsilon \nu^{\mathrm{bc}}(d_0) I_{W_0,\nu}(z_0) A_{\mathrm{sm}}\,\rho(g_\infty) \mathbf{1}_{H_f K_f^{\mathrm{std}}}(g_f).$$

The H_1-integral contributes the equal-rank period, and the L-integral is precisely A_{sm} with its original Haar measure. The only dependence on the outer coset is in the last two factors.

This weight descends to $H\backslash G$. The finite indicator is left H_f-invariant. At infinity $\rho(h)$ depends only on the span of $h^{-1}x_0$. On its support $hk(P(h^{-1}x_0)) \in H_\infty$, so the corresponding cosets lie in the compact image of $k(\operatorname{supp}\sigma)^{-1}$. At finite places the supported cosets are the image of K_f^{std}, also compact. The weight is positive on a neighborhood of the identity coset. Its integral is therefore the positive finite R_0 in (1), before the modulus and character. This construction even gives the same R_0 when $I_0 = 0$.

For completeness, at each plane removal the old target need not initially exhibit the next marked hyperbolic summand. If its rank and negative count are those of the new target plus one plane, form $H \perp V$ with its standard chosen basis and permute the basis so that its signs agree entry by entry with the old target. At infinity the forms are thus identified.

At finite places the marked forms have the same standard unimodular model, and Landherr's theorem [41, pp. 246–247] gives a rational isometry C. If X_{old} and X_{ad} are their adelic bases, write $X_{\mathrm{old}} = C^{-1}X_{\mathrm{ad}}u$ with u adelic unitary. Their rational groups in chosen coordinates are conjugate by u. Their theta distributions have the same rational vectors after the corresponding unitary change of point.

The real representation and the finite vacuum and test, in chosen coordinates, are unchanged because they depend on the common signs and on the given covector ordering. Transport the theta functional by this isometry. Then the two kernels are related by left translation $h \mapsto uh$ and the factor $\alpha(\det u)^{-1}$, with α the character of the kept source. If the two normalized functionals have already been chosen, uniqueness of rational summation and its unit-vacuum normalization in Section 17.1 contribute a single constant of modulus one to this comparison. Conjugation preserves the counting measure on rational points and left translation preserves ambient Haar, so this map preserves the induced quotient measure.

Thus the same finite covector datum occurs, and only a unit phase changes in the scalar. In the block-reindexed adapted model the plane basis and the new target inclusion have their prescribed integral and orthonormal forms. We compute the two radical constant terms in these coordinates. Through the entire source sequence use the fixed plane blocks and fixed complement in the original adapted basis. Successive zero slices are evaluations in the product mixed model, so they give its combined zero slice. The same choice of coordinates thus supplies the adapted vectors at every removal.

For a single plane removal write e for the new defect $h - N_1/2$, using the new target rank h and source rank N_1. The old defect is $e + 1$. Under the quotient radical mean the standard partner covector p is read on the old spaces as $p\,\mathrm{pr}_{W_1}$, where pr_{W_1} is the projection in the marked dual basis. The two constant-term formulas on this datum have the form

$$\mathrm{CT}\,\Theta^{+}_{\mathrm{old}} = \gamma_{\mathrm{pl},\nu}\Theta^{+}_{\mathrm{new}}, \qquad \mathrm{CT}\,\Theta^{-}_{\mathrm{old}} = \kappa_A\Theta^{-}_{\mathrm{new}}. \tag{2}$$

The lower small vector is the zero slice of the old one. The large vector is the cone slice at $s(e) = e + 1$ for $e \geq 0$, or its dual cone slice at $s(e) = -e$ for $e < 0$. The small vacuum slice scalar $\kappa_A \neq 0$ depends on the splitting and the fixed measures. These operations choose the new real vectors before the finite data and twist.

36.1.3 The two radical terms after removing one hyperbolic plane

In this subsection, one hyperbolic-plane removal involves two unipotent integrals. We unfold each separately and apply Section 35.2 to see how the lower-rank datum and normalization arise.

Use the adapted decompositions $\widetilde{W} = H \perp W_1$ and $\widetilde{V} = H \perp V$ on the original tower. Put $N_1 = \dim W_1$, $h = \dim V$, and $e = h - N_1/2$. Retain the given middle and isotropic bases. Let p be the integral covector parameter in the lower basis. The two upper data use $p\,\mathrm{pr}_{W_1}$, where the projection is the middle dual covector of the source decomposition. The upper determinant test on covectors and its column partner are the functions $\chi^{-1}(\det[p\,\mathrm{pr}_{W_1}; Y])$ and $\chi^{-1}(\det(p\,\mathrm{pr}_{W_1}\,Z))$, respectively, restricted to integral matrices with unit determinant in the indicated formula.

Finite restriction of the upper space. Write (K, I, L) and $(K^\vee; I^\vee; L^\vee)$ for the column and dual covector bases of the source, and use the same notation with subscript V for the target. At the selected split places these are integral inverse bases. Extend the target basis to the stacked test by the identity on the fixed covectors of p. The resulting square change of basis G takes the first and last line coordinates to K_V, L_V on the moving block and acts on the middle by the identity on the fixed covectors and by I_V on the moving covectors.

Let $R = (K^\vee; I^\vee; L^\vee)$; both G and R have integral inverses. In the two partial Fourier models, the matrix inside the determinant is $GM_{\pm}R$, with

$$M_{+} = \begin{pmatrix} a & z_1 & 0 \\ 0 & [p; Y] & z_2 \\ 0 & 0 & u \end{pmatrix}, \qquad M_{-} = \begin{pmatrix} u & z_1 & 0 \\ 0 & [p; Y] & z_2 \\ 0 & 0 & a \end{pmatrix}.$$

The middle column z_2 is zero on the fixed covectors. These block expressions are in the chosen order of the covectors: multiplication by R reads the three dual source coordinates, and multiplication by G restores the target coordinates on the three moving blocks. On the fixed covectors the diagonal line blocks vanish and the middle p is read by precisely $\mathrm{pr}_{W_1} = I^\vee$.

Thus restriction to the marked line leaves the old fixed covectors on the same middle coordinates. Integrality of the determinant test is equivalent to integrality of $M_{\pm}$, and $\det(GM_{\pm}R) = \det(RG)\, a\, u\, \det[p;Y]$. The basis phase is $\varepsilon = \chi^{-1}(\det(RG))$, of absolute value one. The two diagonal line entries merely switch places.

The slice at $\alpha_0 a$ in the self-dual measure Fourier transforms z_1, z_2, u, with zero frequency except at u, where the frequency is $\bar{a}$ in the scaled coordinates. The integrals along z_1, z_2 are one since their Haar gives their integral lattice measure one. Let $\varpi = (\varpi_w)_{w\in S}$ on the chosen split side. The u-integral at $w \in S$ is

$$\int_{O_w^\times} \chi_w(u)^{-1}\psi_w(\bar{a}_w u)\, du = \begin{cases} g_w\, \chi_w(\varpi_w \bar{a}_w), & \varpi_w \bar{a}_w \in O_w^\times, \\ 0, & \text{otherwise,} \end{cases} \qquad |g_w| = q_w^{-1/2}.$$

The additive character has conductor O_w in this coordinate. The formula follows by integration on residue classes of $O_w^\times$ and finite Gauss orthogonality. Together with the diagonal a in $M_{\pm}$ it gives the standard lower covector datum on the same $[p;Y]$, with scalar $l_{\pm}\mathfrak{g}\, F_{\varpi}(a_f)$. Here $l_{\pm}$ is the nonzero vacuum slice constant in the indicated Fourier model,

$$\mathfrak{g} = \varepsilon \prod_{w\in S} g_w, \qquad F_{\varpi}(a_f) = \chi^{-1}(a_S)\chi((\bar{a}\,\varpi)_S)$$

on $a_S, (\bar{a}\,\varpi)_S \in O_S^\times$ and a integral outside $S \cup \bar{S}$, and zero otherwise. Away from the two split halves the integral is the original vacuum slice. The sign choice affects the fixed $l_{\pm}$, while the same F_{ϖ} occurs on both sides. In particular both line data have zero value at the zero line coordinate.

The large radical mean. At a source Levi point (u, g), take $u = 1$, or allow arbitrary u if $h = N_1$. In the mixed theta expansion the central average of the source line retains exactly the rational vectors of zero Hermitian norm in $\widetilde{V}$. Its other radical coordinates produce the line zero-term functional for each retained nonzero vector. More explicitly, for an isotropic vector ξk_V with $\xi \in E^\times$, the change of the additive radical coordinate by ξ and conjugation is an automorphism of its compact rational quotient. The probability mean is therefore the linear functional of the indicated partial Fourier line term, with no rational scale factor. The zero vector does not contribute: its finite determinant of the covectors has the zero line block, as in $M_{\pm}$.

36.1.4 Unfolding the isotropic line orbit

In this subsection, we group the nonzero radical terms before the equal-rank vanishing in Section 36.1.6.

Witt extension gives one orbit of rational isotropic lines. Group the nonzero vectors by this orbit, then by their coordinate $\xi \in E^\times$ on the line. Use the inverse line coordinate a in the target Levi. The quotient density $|a|^{h+1}$ is computed below in this basis.

Let $\ell_{\pm}(z; h_0, k)$ be the lower kernel at the line slice at the actual parameter z, target point h_0 and old target compact point k; the retained source point is g. The two partial Fourier models use the arguments au and $\overline{au}^{-1}$. The Weil half-densities are $|au|^{-N_1/2}$ and $|au|^{-(N_1+2)/2}$, respectively. When u is present, the source Levi contributes $|u|^{h+1}$ precisely in the equal-rank case permitted here.

These half-densities against the orbit density give the powers $e+1$ and e. The space character of the rank-twisted source is $(\nu^{\mathrm{bc}})^{-1}$, so its inverse contributes $\nu^{\mathrm{bc}}(au)$ in both formulas. Thus

the unfolded normalized mean is c_{unf} times the value at $s = e + 1$ of

$$Z(s) = \int \nu^{\mathrm{bc}}(au)|au|^s \sum_{\xi \in E^\times} \ell_+(\xi au; h_0, k)\, d^\times a\, dh_0\, dk,$$

where $a \in E^\times \backslash \mathbb{A}_E^\times$. The scalar $c_{\mathrm{unf}} \neq 0$ is fixed by the original orbit and quotient measures.

All the quotients and the compact factor retain their prescribed Haar measures. The Fourier formula in the two opposite line coordinates gives

$$|a| \sum_{\xi \neq 0} \ell_+(\xi a; h_0, k) = \sum_{\xi \neq 0} \ell_-(\xi \bar{a}^{-1}; h_0, k).$$

Indeed let F be the common zero slice in the complementary coordinates, with the lower and compact points fixed. In the residual hyperbolic plane, let $S_+(\xi)$ and $S_-(\xi)$ denote the two rational line slices, and let θ_0 be the lower theta functional. Their normalizations give

$$\sum_{\xi \in E} \theta_0(S_+(\xi)F) = \theta_{H \perp \mathrm{Hom}(W_1, V)}(F) = \sum_{\xi \in E} \theta_0(S_-(\xi)F).$$

In either polarization the individual terms are the Fourier coefficients of the same rational theta evaluation translated along the corresponding isotropic line. The zero coefficient is its mean on $E\backslash\mathbb{A}_E$ with probability Haar, and Fourier inversion at the identity gives the middle value in both expansions. In the trace pairing with conjugation the two line coordinates are Fourier dual. For fixed lower and compact variables the resulting line function is Schwartz–Bruhat: summing over the lower rational lattice preserves each Schwartz bound in the line variable by taking sufficiently many additional powers of decay in that lattice. With the self-dual Haar measure, dilation $f(x) \mapsto f(ax)$ has transform $y \mapsto |a|^{-1}\widehat{f}(\bar{a}^{-1}y)$. Adelic Poisson summation on E [71, Lemma 4.2.4] now gives the displayed equality in the rational coordinate, before passing to the finite test at z/α_0. Both evaluations at zero vanish by the support calculation above. Put $a' = \overline{au}^{-1}$. Then $\nu^{\mathrm{bc}}(a') = \nu^{\mathrm{bc}}(au)$ and $|a'| = |au|^{-1}$. Multiplicative Haar on the idele class group is invariant under translation, conjugation and inversion. Substituting the Poisson formula shows that $Z(s)$ uses the negative line sum with exponent $1 - s$. In the equal-rank case, $a \mapsto au$ shows directly that the constant term is independent of u.

For $s > 1$ unfold the sum over $E^\times$ in the positive-sign formula for $Z(s)$ or in the negative-sign formula for $Z(1 - s)$. From the finite factor just obtained, the idele translation is $K = \alpha_0 c_0^{-1}$, where c_0 is $\bar{\varpi}_w$ at $\bar{w}$ and one elsewhere. Thus $|K| = |\alpha_0| Q_S$, with $Q_S = \prod_{w \in S} q_w$. At the original parameter z both line slices use z/α_0. Substitute $z = Ka$. Since $(c_0)_\infty = 1$,

$$(z/\alpha_0)_\infty = a_\infty, \qquad F_\varpi((z/\alpha_0)_f) = F_1(a_f),$$

where F_1 denotes F_ϖ with $\varpi = 1$. The same substitution in the multiplicative character and norm contributes $\nu^{\mathrm{bc}}(K)|K|^s$; multiplicative Haar measure is invariant.

Normalize the local multiplicative measures by $\mathrm{vol}_{d^\times a_w}(\mathcal{O}_{E_w}^\times) = 1$ and take their restricted product on the finite ideles. The original finite measure is $v_\pm$ times this product, where $v_\pm > 0$ is its mass on $\widehat{\mathcal{O}}_E^\times$, fixed before S. At $w \in S \cup \bar{S}$ the translated function is supported on units and $\nu^{\mathrm{bc}} F_1 = 1$ there, so the normalized local integral is one. Elsewhere the integral is $\sum_{n \geq 0} \nu_w^{\mathrm{bc}}(\varpi_w)^n q_w^{-ns}$, since each multiplicative shell has mass one. The full finite factor is therefore $v_\pm L^S(s, \nu^{\mathrm{bc}})$, abbreviated to $v_\pm L^S(s, \nu)$ below. In absolute value the sum is bounded by the partial Dedekind zeta series at $s > 1$. The real integral is the cone vector $C_\pm(s)$ formed

by integrating the real line slice against $\nu^{\mathrm{bc}}(a_\infty)|a_\infty|^s$ and the original target compact Haar. For a finite point k of the original target compact, let k_S be its components at the chosen split places. Write $\phi_p(Y) = \chi^{-1}(\det[p;Y])$ on the integral unit support and zero elsewhere. Since $[p;k_S^{-1}Y] = \mathrm{diag}(I,k_S^{-1})[p;Y]$, the support is preserved and

$$\phi_p(k_S^{-1}Y) = \chi(\det k_S)\phi_p(Y), \qquad (\nu^{-1}\upsilon^{\dim(\widetilde{W})/2})(\det k) = \chi(\det k_S)^{-1}.$$

The second identity uses the unramified splitting character and the prescribed tame character on units. Thus the full finite compact fixes the vector in the twisted Weil representation.

For $s > 0$ the real vector integral is convergent in each Schwartz seminorm. In a compact set of such s, a power $|a_\tau|^{2\min s}$ suffices at zero, and the Schwartz slice gives arbitrary decay at infinity. The bounds hold uniformly on the compact factor. To give the bound before integrating in the lower target variable, write $U_\pm(z_\infty,k_\infty)$ for the two real line slices and L_{g,h_0} for the lower theta functional with the retained finite covector test. In the support of that test the stacked matrix $[p;Y]$ is invertible, so its last h rows admit a right inverse. The whole-kernel estimate of Section 34.3.1, using all source basis vectors, therefore gives $\sup_{h_0}|L_{g,h_0}(v)| \le C_g p_M(v)$ for some fixed Schwartz seminorm. For every $B > 0$, restriction of the Schwartz vector to the line in either Fourier model gives

$$\sup_{k_\infty} p_M(U_\pm(z_\infty,k_\infty)) \le C_B \prod_\tau (1+|z_\tau|)^{-B}.$$

Indeed the compact translates have bounded Schwartz seminorms and the partial Fourier transform preserves the Schwartz space. At $z = \alpha_0 a$ this bounds $|\ell_\pm(z;h_0,k)|$ by a fixed multiple of $|F_\varpi(a_f)|\prod_\tau(1+|a_\tau|)^{-B}$, uniformly in h_0,k. The lower target quotient has finite volume. For positive real part the real majorant is integrable against $|a_\infty|^s d^\times a_\infty$ with B sufficiently large; for $\Re s > 1$ the finite integral is bounded by a constant times $\zeta_E(\Re s)$.

This justifies termwise unfolding and Fubini for each sign when its own exponent has real part greater than 1. For the Mellin integral on the original quotient, write

$$D(a,h_0,k) = \nu^{\mathrm{bc}}(au)\sum_{\xi\in E^\times}\ell_+(\xi au;h_0,k), \qquad r = |au|.$$

The positive unfolding bounds the integral of $|D(a,h_0,k)|r^{\sigma_+}$ for $\sigma_+ > 1$. For $\sigma_- < 0$, apply Poisson summation to the sum before taking its absolute value. The zero terms vanish as above. With $a' = \overline{au}^{-1}$, Poisson summation gives

$$|D(a,h_0,k)|\,r^{\sigma_-} = |a'|^{1-\sigma_-}\left|\sum_{\xi\in E^\times}\ell_-(\xi a';h_0,k)\right| \le |a'|^{1-\sigma_-}\sum_{\xi\in E^\times}|\ell_-(\xi a';h_0,k)|.$$

The integral on the right is finite by the negative unfolding, since $1-\sigma_- > 1$ and $d^\times a' = d^\times a$.

Consequently $r^\sigma \le r^{\sigma_-} + r^{\sigma_+}$ for $\sigma_- \le \sigma \le \sigma_+$ gives an integrable majorant for the grouped integral. Widening the interval absorbs every power of $\log r$, which gives holomorphy of this Mellin integral. This justifies the unfolded constant term at $e+1$, including 0 and 1. To identify this integral with the upper constant term, first integrate the whole upper theta kernel over the compact radical and the target quotient. The bottom rows of the integral invertible upper stack have a right inverse. Apply the whole-kernel bound of Section 34.3.1 using all source basis vectors. The source heights are bounded on a compact set of radical

representatives, so this bound justifies Fubini there. At each fixed target point its Schwartz expansion is absolutely summable, uniformly over those representatives. Take the radical mean there and group its nonzero terms by their rational isotropic line. For the chosen line $L = Ev_L$, let $\Phi_L = \sum_{\xi \in E^\times} \mathrm{CT}_{\xi v_L}$, where CT_x denotes the radical mean of the term with vector x. Rational covariance carries this sum to that for each other line and makes it independent of the generator. Put $P_L = \mathrm{Stab}(L)$, with unipotent radical N_L. For $n \in N_L(\mathbb{A}_{E^+})$, the first slice stays at ξv_L, since $nv_L = v_L$. On the target zero slice, n acts trivially on $L^\perp/L$, so it commutes with the remaining Heisenberg action and acts by a scalar. Conjugation by a line dilation of factor 2 fixes this scalar character and acts by 2 on the vector quotient and 4 on the center of N_L. Its commutators generate first the center and then the vector quotient. Thus the scalar is one and $\Phi_L(nh) = \Phi_L(h)$.

With the inverse line coordinate a, Levi conjugation on the radical coordinates is $(c, b) \mapsto (h_0 c/\bar{a}, b/(a\bar{a}))$. Its Jacobian $|a|^{-h-1}$ gives the quotient density $|a|^{h+1}$. The rational unipotent quotient is compact: its vector quotient and its center are additive adelic quotients by rational points. Integrating first on this radical quotient gives its fixed positive volume. The two remaining rational Levi quotients are $E^\times \backslash \mathbb{A}_E^\times$ and $[U(V)]$. Thus the integral of $|\Phi_L|$ on $P_L(E^+)\backslash U(\tilde{V})(\mathbb{A}_{E^+})$ is the integral of $r^{e+1}|D(a, h_0, k)|$ times the fixed positive measure factor. This integral is finite by the two bounds above. Tonelli applied to the absolute values of the line sums now permits unfolding the orbit of L at $s = e + 1$.

The value required in both cases is $Z(e+1)$. For $e \geq 0$, the positive cone has Mellin exponent $s(e) = e+1$; for $e < 0$, the negative cone has exponent $s(e) = 1-(e+1) = -e$. At $e = 0, -1$ the two boundary values are $Z(1) = \lim_{t \downarrow 1} Z(t)$ and $Z(0) = \lim_{t \downarrow 1} Z(1-t)$, respectively. The real cone factors are continuous at one by their Schwartz bounds. The nontrivial Hecke L-function is regular there. The exact scalar multiplying the lower measured period in the respective model is

$$\gamma_{\pm,\nu} = c_{\mathrm{unf}}\, l_\pm\, v_\pm\, (\mathfrak{g}\sqrt{Q_S})\, \nu^{\mathrm{bc}}(K)\, |\alpha_0|^{s(e)} Q_S^{s(e)-1/2} L^S(s(e), \nu).$$

In particular $|\mathfrak{g}\sqrt{Q_S}| = 1$. This gives the first identity of (2) and its stated constant with all measure-dependent quantities $c_{\mathrm{unf}}, l_\pm, v_\pm$ fixed before the twist.

36.1.5 The small radical mean and its zero term

In this subsection, we compute the second radical integral of Section 36.1.3 and show that only its zero term remains.

For the small side read the transposed column datum at $\bar{S}$. In the original paired bases the middle projection read at S is the transpose of the middle inclusion read at $\bar{S}$. Consequently the test in the old basis only reads the lower complementary block of the transpose inside $\det(pZ)$. All other factors are the integrality conditions of the original lattice. Its source-line slice at $\alpha_0 x$ is zero unless x_f is integral, and otherwise equals the standard lower column datum, times the real slice at x_∞ and the fixed vacuum slice constant $\kappa_A \neq 0$. Indeed the extra sliced coordinates have exactly the constant integral-indicator test of the vacuum in this projection basis. In particular the slice at zero uses exactly the same lower finite datum at $\bar{S}$.

Take the source radical mean of this small kernel at $\mathrm{levi}(1, g)$. The central average again retains the isotropic rational vectors of the fixed small space U. At a nonzero isotropic vector, its line term is obtained by first taking the source slice, then its target zero term on the isotropic line. The latter, composed with the theta distribution, is invariant under all Heisenberg translations along the transported target line.

The transported line vector at any selected place is nonzero. In the finite Schrödinger polarization at $\bar{S}$ this invariance is invariance under all multiplications by the additive characters paired against $x^{*}Z^{\mathrm{t}}$, where x is that line coordinate and Z the column variable. On the support, pZ is invertible, so Z is left-invertible; hence $x^{*}Z^{\mathrm{t}} \neq 0$. Nondegeneracy of the additive pairing makes one of these multipliers different from one at each support point.

It is constant on a sufficiently small compact open neighborhood. Compactness gives a finite disjoint refinement of this cover, and on each part the invariant functional must be zero. This proves vanishing of the whole nonzero isotropic term, with no size estimate and no change of finite datum. Anisotropic terms have already vanished in the central mean.

Only the zero source slice remains. At that slice the retained group action is the old one restricted to W_1 and U. The source splitting character drops by the prescribed one power of υ when the rank drops by two, which is the lower rank twist. The finite part is exactly the same standard column datum and the small real vector is the zero slice. Thus the small radical mean is κ_A times the lower measured period, as asserted in (2). For $U = 0$ this argument simply retains the unique zero vector on the zero-group quotient.

In particular,

$$|\gamma_{\mathrm{pl},\nu}| = C_{\mathrm{pl}} L_{\nu}^{\sharp}(s(e)), \qquad L_{\nu}^{\sharp}(s) = |L^{S}(s,\nu)| \prod_{w \in S} q_{w}^{s-1/2}, \tag{3}$$

where $C_{\mathrm{pl}} > 0$ includes the original line-orbit, slice and unit-volume factors, all fixed before the twist.

For the real section needed below, let x_B be the target marked isotropic vector. Give the lower vector v a target-line mixed-model extension v_B by tensoring with a Schwartz function of value one at zero in the removed coordinates. Write $x_B = (x_{B,v})_{v|\infty}$. At each real place $x_{B,v} \neq 0$, and Witt extension makes $U(\widetilde{V}_v)$ transitive on its nonzero isotropic vectors. The orbit of x_B is consequently

$$\prod_{v|\infty} \{x_v \in \widetilde{V}_v \setminus \{0\} : h_v(x_v, x_v) = 0\}.$$

The local orbit maps are submersions. Their smooth sections near $x_{B,v}$ give $u_x x_B = x$ on this product near x_B, with $u_{x_B} = 1$. Extend the sections in local Lie-algebra coordinates to an ambient neighborhood. Choose a smooth compactly supported nonnegative function β there, with $\beta(x_B) = 1$ and $\|x_{B,v}\|/2 < \|x_v\| < 2\|x_{B,v}\|$ on its support at every real place. The product below then extends by zero to a Schwartz slice. Prescribe on the source-line mixed model the slice

$$\beta(x)\omega_{N_1/2-\mu}(1, u_x)v_B.$$

On the support of the cone integral at (κ, a_∞) we have $x = \kappa_\infty^{-1} x_B a_\infty$. Then $\kappa_\infty u_x$ sends x_B to $x_B a_\infty$.

On the orthogonal quotient $(E_\infty x_B)^{\perp}/(E_\infty x_B)$ this element induces $h \in U(V)_\infty$. In an adapted Witt basis its diagonal blocks are $a_\infty, h, \bar{a}_\infty^{-1}$. On the zero slice the unipotent radical fixes the remaining Heisenberg coordinates, so its action there is scalar. Conjugation by the Levi dilation of factor 2 preserves this scalar and acts by 2 on the vector quotient and 4 on the centre. The corresponding differences are surjective on both additive spaces, so the scalar is one. At τ put $u_\tau = a_\tau/|a_\tau|$, with $|a_\tau|_{\mathbb{C}} = |a_\tau|^2$. On $\mathrm{Hom}(W_{1,\tau}, \mathbb{C}x_{B,\tau})$ the isotropic block is $a_\tau I_{N_1}$. By (58) its intrinsic multiplier is $u_\tau^{-N_1}|a_\tau|_{\mathbb{C}}^{N_1/2}$. The determinant ratio to h_τ is $a_\tau/\bar{a}_\tau = u_\tau^2$. The

source slice lowers the determinant exponent from $N_1/2+1-\mu_\tau$ to $N_1/2-\mu_\tau$, so the relative scalar at τ is

$$u_\tau^{-N_1}|a_\tau|_{\mathbb{C}}^{N_1/2}(u_\tau^2)^{N_1/2-\mu_\tau}=|a_\tau|_{\mathbb{C}}^{N_1/2}u_\tau^{-2\mu_\tau}.$$

The product of these phases cancels $\nu^{\mathrm{bc}}(a_\infty)$. The translation by h disappears from the lower period by right invariance on its original quotient.

The cone period of the constructed real vector is therefore the prescribed lower period times

$$B(s)=\int_{\kappa,a_\infty}\beta(\kappa_\infty^{-1}x_Ba_\infty)|a_\infty|^{s+N_1/2}\,d\kappa\,d^\times a_\infty.$$

On the support the a_τ's are in a compact annulus by invariance of the Euclidean norm under κ_∞. Thus $B(s)$ is entire, positive at every real s, and independent of the modulus and finite character. At a fixed positive exponent we may divide the real vector by $B(s)$ to have exactly the desired period. For the continued plus identity at a negative exponent the same section works without altering its finite test.

The Euler product proves nonvanishing for $s>1$. At 1, the positive logarithmic Euler-product inequality from $3+4\cos\theta+\cos 2\theta\ge 0$, applied to $\zeta_E^3|L(s,\nu^{\mathrm{bc}})|^4|L(s,(\nu^{\mathrm{bc}})^2)|$, rules out a zero. The square character is nontrivial because the weights are positive, so its L-function is regular at 1 by the Hecke continuation theorem.

Since $M_\nu(j+1)/M_\nu(j)=L_\nu^\sharp(j+1)^2$ for $j\ge 0$, and $M_\nu(-j)=M_\nu(j)^{-1}$, (3) gives

$$|\gamma_{\mathrm{pl},\nu}|^2M_\nu(e)=C_{\mathrm{pl}}^2M_\nu(e+1).$$

In (2) the new identity therefore has scalar $c_{\mathrm{new},\nu}=c_{\mathrm{old},\nu}\kappa_A/\gamma_{\mathrm{pl},\nu}$. The positive normalizing factor changes from D_{old} to $D_{\mathrm{old}}C_{\mathrm{pl}}^2/|\kappa_A|^2$, so

$$\frac{D_{\mathrm{new}}|c_{\mathrm{new},\nu}|^2}{M_\nu(e)}=\frac{D_{\mathrm{old}}|c_{\mathrm{old},\nu}|^2}{M_\nu(e+1)}. \tag{4}$$

Under the rank and signature conditions this identity holds for either sign of the integral defect e. In (3) the Mellin exponent is $e+1$ for $e\ge 0$ and $-e$ for $e<0$. At $(J,0,J)$ the last removal has $e=-k$. In (3), $s(e)\ge 1$ and ν is nontrivial, so the plane coefficient $\gamma_{\mathrm{pl},\nu}$ is nonzero. The scalar $c_{\mathrm{old},\nu}$, and hence $c_{\mathrm{new},\nu}$, may vanish until the equal-rank vector is chosen below.

At the top the factor is $D_{\mathrm{top}}=(R_0|\kappa_0|)^{-2}$ and the value of this invariant is

$$R_{W_0,\nu}(z_0)=|I_{W_0,\nu}(z_0)|^2/M_\nu(j_0). \tag{5}$$

The top pair has $b+1$ extra planes on both sides. The sum of the negative signs at the top is correct because $q(U^-)=b-q(U)$, hence $q(V_{\mathrm{top}})=q(V)+b+1$. Successive source zero slices compose, since they are evaluations of separate blocks in the product mixed model. Choose the enlarged small vector in the extra coordinates with the prescribed terminal zero slice, using a fixed Schwartz tensor of value one. After j removals the target rank is $b+N+2-2j$, so its defect relative to the small rank b is

$$e_j=(N+2)/2-j,\qquad e_{b+1}=(a-b)/2=t.$$

At each step (3) uses e_j+1 if $e_j\ge 0$, and $-e_j$ otherwise. Thus (4) reaches W,V,U^- with the prescribed small real vector for either sign of t.

From standard data to the actual finite data. In the product $F_S = \prod_{w\in S} E_w$, put $O_S = \prod_{w\in S} \mathcal{O}_w$. Let p be an integral block of size b and use the two standard data $\chi^{-1}(\det[p; X])$ and $\chi^{-1}(\det(pY))$, each supported on integral arguments with unit determinant in the indicated formula. For X the last covectors of k^{-1} and Y the first columns of k, one has

$$\det[p; X] \det k = \det(pY).$$

Thus these are matched, with factor $\chi(\det k)$.

For use with the equivariant difference of the two theta periods, we prove finite generation. A locally constant matched column function Ψ is supported on primitive integral columns. It satisfies $\Psi(Yu) = \chi(\det u)^{-1}\Psi(Y)$ for $u \in \mathrm{GL}_b(O_S)$: replace a completion k by $k\,\mathrm{diag}(u, 1)$. Choose x_0 primitive and an integral p with $px_0 = 1$. Let $F(Y) = \chi^{-1}(\det(pY))$ on its standard support. Take a common depth $c \geq 1$ of local constancy of Ψ, and the product uniformizer $\varpi = (\varpi_w)_{w\in S}$.

Set

$$d = x_0 p + \varpi^{-c}(1 - x_0 p), \qquad K = \mathrm{GL}_N(O_S).$$

If $F(dkY) \neq 0$, put $Z = dkY$ and $u = pZ$. Then u is invertible and

$$Y = k^{-1}x_0 u + \varpi^c k^{-1}(Z - x_0 u), \qquad \Psi(Y) = \Psi(k^{-1}x_0)F(dkY).$$

Conversely any nonzero value of $F(dkY)$ forces Y primitive integral. Thus

$$\int_K \Psi(k^{-1}x_0)F(dkY)\, dk = c_F \Psi(Y)$$

everywhere, where $c_F = \int_K |F(dkx_0)|\, dk > 0$. To see constancy of c_F on primitive Y, use the transitivity of K on primitive columns. Positivity follows on a neighborhood of $k = 1$.

All functions of k in the average are locally constant. The translated F's themselves range over finitely many functions on K, by smoothness of the translation action on locally constant compactly supported functions. Consequently this is a finite linear combination of translates of F. The split covariance of the difference $\mathcal{D}$ takes the form

$$\mathcal{D}(\Psi \circ g_\Sigma)(x g_\Sigma) = \lambda_b(\det g_\Sigma)^{-1}\mathcal{D}(\Psi)(x), \qquad \lambda_b(u) = (\upsilon_E(u)|u|^{1/2})^b.$$

Its kernel is therefore stable under the translates. The common-section calculation in Section 35.2 identifies the domain with the original integral matching and its split translates. This proves the comparison for every locally constant matched pair.

For a modulus at S, the reduction $O_S \to \mathcal{O}_E/\mathfrak{m}$ is surjective and detects units by the Chinese remainder theorem. Each invertible residue matrix lifts to an invertible integral matrix by lifting its entries and using the determinant test. A primitive integral covector or column block extends to such a matrix by completion over each local ring. The residue matching law and its support thus lift exactly, by pullback and extension by zero, to the same law and support on O_S. The unit character is the same product of nontrivial residue characters on the selected support S. Inflation to arbitrary positive powers of those primes changes neither this support nor the conductor. Outside $S \cup \bar{S}$ the prescribed vacua are identical. In particular the change from the finite ring to the local integral model introduces no level indices or measure ratios.

36.1.6 The equal-rank normalizer

In this subsection, we use the invariant period ratio (4) to normalize the comparison in Section 39.4.

Write $N = 2k + 2q(W)$, so that $\mu_\tau \geq k \geq 1$. First take the definite space J of rank $2k$ and $Y = H \perp J$. For each ν the equal-rank period $I_{Y,\nu}(z)$ with determinant datum is constant in its unitary variable by the special-unitary invariance and the one-plane Levi calculation in Section 35.1.

Fix a split tame character of the indicated weight. Suppose that every equal-rank period of (Y, Y) with the finite determinant datum is zero. In the large space $X = H^{2k} \perp Y$, with target $J^- \perp Y$, consider an arbitrary real Schwartz vector. In the self-dual test with rational Hermitian index J_{J^-} and the prescribed adelic basis, the representative $(1; 0)$ has orthogonal complement Y. Its mixed orbit integral in the Y-block therefore has the original determinant datum and zero Heisenberg displacement. Rationally congruent indices give the same calculation after the corresponding change of basis in Section 35.2.2.

It is zero even for nondecomposable real vectors. To see this, denote the moved mixed slice by w. Use the coherent-state resolution of the identity on the real Fock product to write

$$w = \int w(u_1, u_2)\, v_{u_1} \boxtimes v_{u_2}\, du_1\, du_2,$$

where $w(u_1, u_2) = \langle v_{u_1} \boxtimes v_{u_2}, w\rangle$ for the unit coherent vectors v_{u_i}. In each block of complex dimension d_i the notation du_i denotes π^{-d_i} times Lebesgue measure on $\mathbf{C}^{d_i}$, as in Section 34.3.1. With the first finite vector and first test point held fixed, the small theta evaluation $T(v_{u_1})$ has at most polynomial growth in u_1, by the finite order Schwartz bound for the theta evaluation.

Moreover $p_M(v_{u_2}) \ll_M (1 + \|u_2\|)^M$, since its absolute value is a displaced Gaussian. It follows by the finite moments of w that the contraction

$$w_2 = \int w(u_1, u_2) T(v_{u_1}) v_{u_2}\, du_1\, du_2$$

is a real Fock vector with all moments finite. At the fixed source point, the determinant datum is supported on invertible matrices at a selected split place. For its theta evaluation L_h, Section 34.3.1 therefore gives $\sup_{h\in[Y]} |L_h(v)| \leq C p_N(v)$. Writing $u = (u_1, u_2)$, with the product measure du, gives

$$\int_{[Y]} \int |w(u) T(v_{u_1}) L_h(v_{u_2})|\, du\, dh \leq C' \operatorname{vol}([Y])\, p_{N'}(w) < \infty$$

for a sufficiently large weighted absolute moment $p_{N'}$. The tensor kernel product therefore identifies the inner integral with the assumed zero period of w_2 at each outer point. By the absolute unfolding of Section 36.1.1, integrating these zero inner periods gives a zero coefficient for every upper real vector.

Off the negative real class, that vanishing requires a separate argument for arbitrary real vectors. If the orbit is nonempty, choose a real place where the nondegenerate span of the basis vectors contains a positive vector. By a unitary change of coordinates take this vector to the first positive coordinate of the target. In the smaller mixed Fock space the source Y has positive and negative dimensions $p_Y = 2k + 1$ and $q_Y = 1$, respectively. The target has $b + q_Y$ negative coordinates, where $b = 2k$. Its real determinant exponent after the source slice is $m_\tau = k + 1 - \mu_\tau \leq 1$.

Fix a point of the signed polynomial Fock coordinates. Choose a unit covector r in the negative target coordinates which kills the negative projections of the b basis columns and the conjugates of the q_Y evaluation columns with negative source sign. There are at most $b-1+q_Y$ independent conditions, since the span of the basis vectors contains the first positive coordinate. Hence such r exists.

The negative line corresponding to $\bar{r}$ is orthogonal to the basis, so the circle rotating it by $\bar{\eta}$ lies in the basis stabilizer and the adapted real maximal compact. The negative-block vacuum multiplier and the twist give together $\eta^{p_Y-m_\tau}$. In the signed polynomial coordinates the components along this line scale by $\bar{\eta}$ on the negative source columns and by η on the positive source columns.

The former components vanish at the chosen evaluation point. If a polynomial vector is invariant under the stabilizer's compact subgroup, its evaluation at this point is therefore the mean on this circle of a polynomial in η all of whose exponents are at least $p_Y - m_\tau > 0$. This mean is zero. As the point was arbitrary, there is no invariant polynomial vector. The compact group preserves each homogeneous polynomial subspace. Finite Hermite sums converge in every Schwartz seminorm, and the compact average acts continuously in these seminorms. Its vanishing on polynomials therefore gives its vanishing on every Schwartz vector.

Unfold the Fourier term over the rational basis stabilizer as in the self-dual calculation. Disintegrate through the adelic stabilizer and average over the compact subgroup just used, conjugated back to the given basis. Right multiplication by the compact subgroup preserves the stabilizer Haar, since the modular character is trivial on compact groups. The mixed-model slice commutes with this stabilizer action at the chosen vector, with the lower source rank twist. Thus its compact average vanishes by the Fock calculation. The coefficient therefore vanishes off the negative class for an arbitrary upper real vector. If no rational basis realizes the index, it is an empty sum.

Apply the primitive coefficient argument of Section 35.2.2 to the large period alone, taking the small functional in the equivariant difference to be zero. The preceding two cases give the required vanishing at every self-dual test. Hence the large period vanishes at every point for every upper real vector and standard covector datum. To descend this vanishing, take any proposed lower period I and the common real bump section on the cone, whose scalar is $B(s) > 0$ for every real s. The positive-sign Mellin integral for $s > 1$ is

$$d_\nu |\alpha_0|^s Q_S^{s-1/2} L^S(s,\nu) B(s) I, \qquad d_\nu \neq 0.$$

Its left side is holomorphic in s by the two Poisson summation formulas and absolute integrability established in the plane-removal proof.

The right side is holomorphic on $\Re s > -k$ by Tate's continuation theorem [71, §§4.4–4.5]; the compact annulus defining $B(s)$ gives an entire factor. The identity theorem therefore extends the equality to this half-plane. At each needed integer $s = e + 1 > -k$ this Hecke factor is nonzero. For $s > 1$ the Euler product converges absolutely; at $s = 1$ use the nonvanishing argument following the definition of $B(s)$ above. For $s \leq 0$ the infinity factors are $\Gamma_{\mathbb{C}}(s + \mu_\tau)$ and $\Gamma_{\mathbb{C}}(1 - s + \mu_\tau)$ at the two reflected points. Their arguments are positive, and the Hecke functional equation reduces the assertion to nonvanishing at $1 - s \geq 1$ for the dual character.

The Euler factors removed at $S \cup \bar{S}$ are already the ramified factors in that equation. By the upper vanishing the left side is zero at $s = e + 1$, so $I = 0$. Apply this at the successive $2k + 1$ planes from $(H^{2k} \perp Y, J^- \perp Y)$ to $(J, 0)$. The successive lower defects are $k, k-1, \dots, -k$, so the smallest argument used in the continued positive formula is $s = 1 - k > -k$. Whenever

$s \le 0$ in this list, $s + \mu_\tau \ge 1$ and $1 - s + \mu_\tau \ge 1 + \mu_\tau$ at every infinite place. At the bottom choose the source point 1, the real vacuum, and the identity covector array $p = I_{2k}$ in the given order. The zero Hom space has just one rational point and its finite factor is $\chi^{-1}(\det I_{2k}) = 1$. Its period is therefore the positive volume of the rank-zero quotient. This contradiction proves the required nonvanishing of some $I_{Y,\nu}(z)$.

Fix a vector z on (Y, Y) for which the ratio is positive at one admissible ν. The standard exchange constructed by (1)–(4), allowing its scalar to vanish, applies at $(J, J, 0)$ and $(J, 0, J)$, with defects k and $-k$. At the latter triple the upper target $J^- \perp H \perp J$ is split: the diagonal and antidiagonal in $J^- \perp J$ are complementary rational isotropic subspaces. In the common finite unimodular coordinates of the two copies of J, choose an integral c with $c + \bar{c} = 1$. The integral trace is surjective at split and unramified quadratic places, including those above two. The two isotropic spaces with columns (u, u) and $(-cu, (1-c)u)$ have cross pairing $\langle u, v \rangle_J$; their block change of basis has determinant one. This gives an integral hyperbolic decomposition of the original finite lattice. Its $2k + 1$ plane removals end at the zero target with $e = -k$, using $L^\sharp_\nu(k)$ in (3). Use the real image under the first map as the argument of the second, before varying ν. In both applications $N = 2k$. For $b = 0$ the empty Gram matrix is nondegenerate, with determinant one. The two shift spaces and the central Fourier quotient for this block are singletons, so the unique coefficient is the function itself. The real slice uses $B \in \mathrm{GL}_0 = \{1\}$, and the finite matched pair is

$$\Phi(X) = \mathbf{1}_{\mathrm{GL}_N(O_S)}(X)\chi^{-1}(\det X), \qquad \Psi(0) = 1.$$

The open orbit quotient in (1) is then a point; its quotient measure and the inner $U(0)$ measure together give the original ambient Haar. For $a = 0$ the top Fourier index has size $b = N$; the zero target occurs only after the last removal. Taking $p = I_N$ gives

$$\Phi(0) = 1, \qquad \Psi(Y) = \mathbf{1}_{\mathrm{GL}_N(O_S)}(Y)\chi^{-1}(\det Y).$$

Each displayed pair spans the integral matched pairs at its endpoint, by determinant covariance on $\mathrm{GL}_N(O_S)$. The zero Hom space has one rational term, and integration over $[U(0)]$ contributes the original mass $v_0 = \mathrm{vol}([U(0)])$.

The adjoint conversion to the original small quotient below applies to the preliminary identity for either sign of the defect, including when its scalar vanishes. Use the original (J, J) real vector and the same square determinant datum in both applications, with $(\alpha, \alpha') = (1, \gamma)$. Write $K_\nu = K^{\gamma,\gamma}_{J,J}$ for this kernel. In the $(J, J, 0)$ identity,

$$K^{\gamma,1}_{J,J}(g, h) = \gamma(\det g)K_\nu(g, h).$$

The zero-source kernel contributes the same $\gamma(\det g)$ times the quotient mass v_0. Cancel this character. In the $(J, 0, J)$ identity the small kernel is $K_\nu(h, g)$ and the rank-zero target kernel with characters $(\gamma, 1)$ is the fixed scalar coordinate ζ_0. With fixed real constants B, C, fixed zero-space scalar ζ_0 and positive zero-group quotient volume v_0, they read

$$\int_{[J]} K_\nu(g, h)\, dh = c_+ v_0, \qquad B|c_+|^2 = M_\nu(k)R_{Y,\nu}(z),$$
$$c_- \int_{[J]} K_\nu(h, g)\, dh = v_0\zeta_0, \qquad C|c_-|^2 = M_\nu(-k)R_{Y,\nu}(z).$$

On the compact product $[J]^2$, the theta kernel is continuous and bounded: over compact sets of representatives its finite translates form a finite set and its real Schwartz seminorms are uniformly bounded. Write $V_J = \mathrm{vol}([J])$ in the original quotient measure. Integrating in the two orders gives

$$c_- c_+ v_0 V_J = c_- \int_{[J]^2} K_\nu(g,h)\, dg\, dh = v_0 \zeta_0 V_J.$$

Since $v_0 V_J > 0$, this gives $c_+ c_- = \zeta_0$. Multiplication, using $M_\nu(k) M_\nu(-k) = 1$, gives

$$R_{Y,\nu}(z)^2 = BC|\zeta_0|^2.$$

Its initially positive value is therefore independent of ν.

An equal-rank step from rank $2j$ to rank $2j+2$ uses (2) on the standard equal-rank determinant datum. The larger period is constant, hence equals its normalized constant term. For every given lower vector a real isotropic-line bump supplies the same upper cone section before ν, normalized by its positive real integral at $s = j+1$. Equations (3)–(4) say that the ratio at the upper vector is a fixed positive multiple of the lower ratio.

Iterate through $q(W)$ planes. Identify the resulting marked space with W_0 by the rational Hermitian isometry and changes of adelic basis in the marked model. The rational group, adelic compact normalization and quotient Haar are transported, and the constant period changes at most by a phase. The rational isometry follows from Landherr's theorem [41, pp. 246–247], since the signatures and the finite local marked forms agree. This gives a single z_0 for which (5) is a fixed positive R_0'. If the admissible character family is empty the assertion has no instances and a vacuum can be chosen.

Return to the original small quotient. The standard small comparison uses (W, U^-). Let c_f be the fixed integral unit of norm -1 in the finite Weil model. In the scaled finite coordinates the adjoint is $B \mapsto c_f B^*$; at infinity it is the signed adjoint. It gives the rational involution $B \mapsto J_W^{-1} B^* J_U$ on the corresponding rational maps. The opposite Hermitian sign cancels the sign from adjunction in the Heisenberg form. The adjoint preserves the finite vacuum lattice and the real Gaussian, so the two normalized theta distributions differ by a phase. Indeed, in a rational Schrödinger polarization, rational modulation and translation characterize the distribution as a multiple of the sum over the rational points. The normalized mean square on vacuum translates fixes the absolute value of that multiple.

For characters (a_1, a_2) on (U, W), the flipped characters on (W, U^-) are

$$(a_2^{-1} \upsilon^N,\ a_1^{-1} \upsilon^b).$$

The real complementary-minor identity and the finite elementary divisors give the multiplier ratio

$$\frac{\upsilon(\det h)^N}{\upsilon(\det g)^b}, \qquad (h,g) \in U(U) \times U(W).$$

The absolute determinant powers cancel because the unitary determinants have norm one. For a matched column datum Ψ, replacing the finite $c_f B^*$ by B^* changes the value by the unit constant $\chi((c_f)_S)^b$, by the right covariance proved above.

The rational groups of U and U^- have the same matrix realizations, since their marked points differ by a scalar. Their compact subgroups and Haar normalizations likewise coincide.

Thus the original small integral is related by a unit scalar and no volume ratio. More precisely, starting with (γ,γ) on (U,W), the standard small period obeys

$$\gamma(\det g)\Theta_{\mathrm{small}}(g)=\varepsilon_{\mathrm{ad}}\int_{[U]}K_{U,W}^{\gamma,\gamma}(\Psi,y;h,g)\,dh,\qquad |\varepsilon_{\mathrm{ad}}|=1.$$

On the standard large side, multiplication by $\gamma(\det g)$ changes the characters from (γ,γ) to $(\gamma,1)$.

Absorb $\varepsilon_{\mathrm{ad}}$ into the scalar in (4). Since (5) is $R_0'>0$, the resulting scalar c_ν is nonzero, with

$$|c_\nu|^2=(R_0'/D_{\mathrm{final}})M_\nu(t).$$

Finally, the product of the two rank-specific space characters equals $(\nu^{\mathrm{bc}})^{-1}$ precisely when $\alpha\alpha'=\gamma$. This follows by multiplying the two powers of the splitting character and using the surjectivity of $x\mapsto x/\bar{x}$ on the norm-one ideles, by local Hilbert 90, choosing units at the unramified finite places.

For $\mathfrak{m}=\prod_{w\in S}\mathfrak{p}_w^{e_w}$, reduce entries by

$$\rho:\prod_{w\in S}\mathcal{O}_w\longrightarrow R=\prod_{w\in S}\mathcal{O}_w/\mathfrak{p}_w^{e_w}.$$

An element and its reduction are units simultaneously. Thus every $k\in \mathrm{GL}_N(R)$ lifts to an integral invertible matrix, by lifting its entries at each place. Pull back each residue array by ρ on integral matrices and extend by zero elsewhere. On the support, the columns have an integral left inverse and the covectors an integral right inverse, since they belong locally to invertible completions modulo $\mathfrak{p}_w$. In an integral completion the matching identity follows by reduction, as the character on units factors through $\prod_{w\in S}k_w^\times$. These locally constant functions are exactly the factors of the two prescribed finite Weil data at S, with their original vacua off the split pairs. This proves the identity at the full exponents e_w.

Replacing $(1,\gamma)$ by (α,α') on the two middle groups multiplies both kernels by $\alpha(\det g)^{-1}$.

Proposition 36.1 (Normalized period exchange)**.** *Suppose $a=b+2t$ with $t\ge 0$. For these spaces, Haar measures, infinity weight μ, and real Schwartz vector y for (U,W), one can choose a real Schwartz vector x for (W,V) and $R_0'/D_{\mathrm{final}}>0$ before the split tame modulus and character ν. For every such modulus $\mathfrak{m}$ with nonempty chosen split support S, and every admissible ν whose tame character on $k_w^\times$ is nontrivial for every $w\in S$, there is a scalar $c_\nu\neq 0$ such that, for all α,α' with $\alpha\alpha'=\gamma$, all matched arrays (Φ,Ψ) over $\mathcal{O}_E/\mathfrak{m}$ for its induced residue character, and every adelic g,*

$$\int_{[V]}K_{W,V}^{\gamma,\alpha}(\Phi,x;g,h)\,dh=c_\nu\int_{[U]}K_{U,W}^{\alpha',\gamma}(\Psi,y;h,g)\,dh,$$
$$|c_\nu|^2=(R_0'/D_{\mathrm{final}})M_\nu(t).$$

36.2 The finite extension of the period identity

In this subsection, after the analytic exchange of Proposition 36.1, finite Fourier inversion gives an identity on the entire matrix space. We determine the resulting next datum, including the inverse determinant character and its scalar.

Let $R=\mathcal{O}_E/(\pi_M Q(\xi))$, a product of finite fields at the chosen split members, and let χ be the character of $R^\times$ of the quotient character. Extend multiplicative characters by zero

on nonunits. A paired residue datum on a-by-$(b+a)$ matrices X and $(b+a)$-by-b matrices Y is supported, respectively, on the last a covectors of k^{-1} and the first b columns of k for $k \in \mathrm{GL}_{a+b}(R)$, and satisfies

$$\Phi(X) = \chi(\det k)\Psi(Y)$$

on such pairs. The order of the covectors in this definition is b, a. The matched point data from the measured square is the following. The scalar tested composites $P(X), P'(Y)$ on the prescribed four Hermitian spaces satisfy

$$P(gX) = \chi(\det g)P(X), \qquad P'(Yh) = \chi(\det h)P'(Y),$$
$$\sum_{Y} \Psi(Y)P'(Y) = c\sum_{X} \Phi(X)P(X) \qquad \text{for paired data } (\Phi, \Psi).$$

Here the test functional in the top covector is arbitrary; the scalar c is independent of it and of the residue arrays. The first line is the finite determinant covariance. The second line is the measured identity for the paired data.

For the shallow triangle, $i, n \geq 0$ and $i+n+2 \leq m$. In column n the rank is $r+nt$. The two kept spaces at $(i, n+1)$ and $(i+1, n+1)$ have rank $r+(n+1)t$; the two integrated spaces at $(i+1, n)$ and $(i, n+2)$ differ by rank $2t$. The negative counts on the two diagonal pairs agree. For a stem entry with excess parameter a_*, its sign in covector o and column j is

$$(-1)^{o+a_*+\min(j,(a_*-o)_+)}.$$

If $a_* \leq i+n+1$, the two column changes leave both signs unchanged.

If $a_* \geq i+n+2$, both signs change and they were opposite. All added entries are positive. The common negative count is at most $2r < r+(n+1)t$. Moreover the positive weight of the inverse quotient is $(n+1)\Delta$, greater than $r+(n+1)t$. These are the rank, sign and weight inequalities for the exchange. In the reversed square interchange the two roles of the diagonal pairs. The two integrated ranks now agree, so the rank defect is zero.

Write $u = v|_{U(1)}$ and $\mu_j = \mu_0(u^{-\Delta}\chi_M)^j$. The horizontal and left-pair determinant twists at ranks p, q in the square respectively have factors

$$\mu_{i+n+2}\xi^{i+n+2}u^{\lfloor p/2 \rfloor}, \qquad \mu_{i+1}^{-1}\xi^{-(i+1)}u^{\lceil q/2 \rceil}.$$

The second is the inverse of the first-column character at rank q, multiplied by u^q. Since $p+q = 2(r+(n+1)t)$, its product with the first is the rank twist of $(\mu_{i+n+2}/\mu_{i+1})\xi^{n+1}$. In the split Weil characters the two rank corrections therefore cancel, leaving the base change of that quotient. The paired finite determinant character is its inverse. The individual lift indices satisfy $1 \leq i+1 < i+n+2 \leq m$, and the quotient has exponent $1 \leq n+1 \leq m-1$. Thus it is nontrivial at every old and new chosen place by the order constraint for powers up to m. Thus the finite character used above is exactly the one on $R^\times$ of the marked square.

36.2.1 Fourier transform in the complementary blocks

In this subsection, we calculate the finite transform for the exchange identity in Proposition 36.1.

Write $X = (X_1, X_2)$ and $Y = \binom{Y_1}{Y_2}$ with square blocks X_2, Y_1. For a fixed $w \in M_{a,b}(R)$ use

$$\Phi(X) = \chi^{-1}(\det X_2)\,\mathbf{1}_{\{X_2^{-1}X_1 = w\}}, \qquad \Psi(Y) = \chi^{-1}(\det Y_1)\,\mathbf{1}_{\{-Y_2Y_1^{-1} = w\}},$$

defined to be zero if their square block is not invertible. These functions are paired. For supported pairs $XY = 0$. For a given k, Y_1 is invertible if and only if X_2 is invertible, by complementary inverse blocks. When they are invertible, block elimination gives

$$X_2^{-1}X_1 = -Y_2Y_1^{-1}, \qquad \det k = \det Y_1 / \det X_2.$$

Conversely, a supported X is completed to the last covectors of an inverse by adjoining a complementary covector basis, since X_2 is invertible. A supported Y is completed to a column basis in the same way. The verification works over R factor by factor.

Using the covariance on each supported basis, the two weighted sums reduce to

$$|\mathrm{GL}_b(R)|P'\begin{pmatrix} I_b \\ -w \end{pmatrix} = c\,|\mathrm{GL}_a(R)|P(w, I_a). \tag{96}$$

For example $Y = \left(\begin{smallmatrix} I_b \\ -w \end{smallmatrix}\right)Y_1$; its character cancels $\chi^{-1}(\det Y_1)$ and every Y_1 contributes the same scalar. The left block is similar.

Now take the successive ranks $N, N+t, N+2t$ at the square. In the transposed-lift square Y has N columns, X has $N+2t$ covectors, and the fixed outer blocks adjoined to each datum are the original inclusions. The abstract sizes in the next calculation are $n = N+t$, $a = N$, and $b = t$. Write p for a datum of size n-by-a and define

$$p^+\begin{pmatrix} \eta & s \\ 0 & I_b \end{pmatrix} = \frac{p(-\eta)}{|R|^{nb}}, \qquad s \in M_{n,b}(R), \tag{97}$$

with value zero on all other matrices. In the order n, a, b, the two full arguments are

$$Y = \begin{pmatrix} -\eta \\ I_a \\ 0 \end{pmatrix}, \qquad X = \begin{pmatrix} I_n & \eta & s \\ 0 & 0 & I_b \end{pmatrix}.$$

To obtain them in (96), use the decomposed order $a, (n, b)$ and take $w = \left(\begin{smallmatrix} \eta \\ 0 \end{smallmatrix}\right)$.

Multiplying the covector argument by $\left(\begin{smallmatrix} I_n & s \\ 0 & I_b \end{smallmatrix}\right)$ introduces the displayed s. This shear has determinant one, so the covariance preserves the tested value. Average over s and then sum with weight $p(-\eta)$ over η. It follows that the weighted sum of the lower composite at p times $|\mathrm{GL}_a(R)|$ equals c times the weighted sum of the other composite at p^+ times $|\mathrm{GL}_{n+b}(R)|$. Thus the weights on the actual square are the orders at the two outer ranks, and the datum of the next descent is exactly (97). In column notation the weights are w_n, w_{n+2}.

For the forward square let $k_{i,n}$ be the archimedean scalar κ of Section 34.3.2 on this square. It is independent of ξ, and the paired-data scalar is $c = k_{i,n}/c_\xi$. The two intermediate ranks differ by $2t$. In Proposition 36.1 the real constant is fixed, the old primes occur in both moduli, and each new place contributes $q_w^{t^2}$ to $M_\nu(t)$. Thus

$$\frac{|c_\xi|^2}{|c_1|^2} = q_\xi^{t^2} Q_{i+1,i+n+2}(\xi)^2,$$

where $Q_{i+1,i+n+2}$ is the product over the critical points $1, \dots, t$ of the absolute ratios for the quotient character at ξ^{n+1}. For a nonzero original source space, (GC2) gives $k_{i,n} \neq 0$, and $\beta'_{i,n} = \overline{c_\xi / k_{i,n}}$ has the same squared-modulus ratio.

Fourier extension for the reversed square. Write $\mathcal{F}p(Z) = \sum_X p(X)\psi(\mathrm{tr}(ZX))$ for the unnormalized matrix Fourier transform with the original primitive additive character of R. For block $Z = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ compatible with (97), character orthogonality gives

$$\mathcal{F}(p^+)(Z) = \mathbf{1}_{\{C=0\}}\mathcal{F}p(-A)\,\psi(\mathrm{tr}\,D).$$

Indeed the trace in the Fourier phase is $\mathrm{tr}(A\eta) + \mathrm{tr}(Cs) + \mathrm{tr}(D)$. The mean over s is zero unless $C = 0$, by primitivity and the matrix trace pairing. The change of variable $\eta \mapsto -\eta$ accounts for the sign.

36.2.2 The finite determinant Gauss sum

In this subsection, we evaluate the determinant factor in Section 36.2.1.

We use the following exact determinant Gauss sum over a residue field k of size q. If a function f on $M_b(k)$ transforms by the nontrivial character $\chi(\det g)$ under left multiplication, it is zero on singular matrices. A singular matrix has a left stabilizer with arbitrary determinant, obtained by scaling a line complementary to its image. Thus $f(D) = \chi(\det D)f(I_b)$ on all matrices with the zero extension convention.

The additive Fourier transform of $\chi \circ \det$ transforms by the inverse determinant character on the opposite side. It is also zero on singular matrices, using their right stabilizer. Consequently it equals $\mathcal{G}\,\chi^{-1}\circ\det$, where $\mathcal{G}$ is its value at I_b. In the normalization $\mathcal{F}f(Y) = \sum_D f(D)\psi(\mathrm{tr}(YD))$, Average the defining sum for $\mathcal{G}$ over left multiplication by upper unitriangular u. This preserves $\chi(\det D)$, while $\mathrm{tr}(uD) = \mathrm{tr}\,D + \sum_{i<j} u_{ij}D_{ji}$. The average kills every matrix with a nonzero entry below the diagonal. The upper triangular invertible matrices give

$$\mathcal{G} = q^{\binom{b}{2}}\left(\sum_{t\in k^\times}\chi(t)\psi(t)\right)^b,$$

the full-rank case of [43, Theorem 2.1]. Parseval for the unnormalized Fourier transform on the b^2-dimensional matrix space gives

$$|\mathcal{G}|^2 = q^{b^2}.$$

The original modulus is squarefree, and the quotient character is nontrivial in each field factor, because its old and new tame powers have full support. The additive character is primitive in each factor. Taking products proves the same formula with $|R|$ in place of q.

In the reversed square the abstract block sizes are $n = N$, $a = N + t$, $b = t$. Thus the two linear-group orders in (96) are equal after the shear extension, since $a = n + b$. Put $p_* = \mathcal{F}(\bar{p})$. In the sum weighted by $\mathcal{F}(\overline{p^+})$, the last formula leaves only matrices with lower-left block zero. At fixed η, s, the tested value as a function of the lower-right block D transforms by $\chi(\det g)$ under $D \mapsto gD$, by applying $\mathrm{diag}(I_n, g)$ to the whole covector argument.

The sum over D, with its Fourier phase, is therefore $\mathcal{G}$ times the value at $D = I_b$. The extra factor $|R|^{nb}$ converts the unrestricted sum over s into the mean in p_*^+. Hence

$$\sum_X \mathcal{F}(\overline{p^+})(X)P(\iota, X) = \mathcal{G}|R|^{nb}\sum_X p_*^+(X)P(\iota, X),$$

where ι is the fixed inclusion adjoined to the covector argument. The squared modulus of the coefficient here is $|R|^{b(b+2n)}$. Combining it with the paired-data law for p_* gives the reversed

commutation coefficient with the additional inverse factor $(\mathcal{G}|R|^{nb})^{-1}$. The remaining paired-data scalar for that square has twist-independent modulus because its two middle ranks agree, and the critical product is empty. Thus the reversed commutation scalar has modulus a fixed constant times a fixed half-integral power of q_ξ.

Matrix reversal in the original two bases. Let S, T have ranks u, v, and let $R = \mathcal{O}_E/\mathfrak{m}$ have the original selected split support. Denote by $K^{a,b}_{S,T}(D, P; g, h)$ the theta kernel with residue datum D and real Fock polynomial P. Write $\psi_0(\bar{x}) = \psi_f(j_{\mathrm{W}}\alpha_f\bar{\alpha}_f x/t_{\mathfrak{m}})$, where $t_{\mathfrak{m}}$ is the chosen generator on this support. Let $s_{\mathfrak{m}} = \overline{t_{\mathfrak{m}}\alpha_f}/(t_{\mathfrak{m}}\alpha_f)$ at the finite places. With the unnormalized matrix Fourier transform

$$\widehat{\bar{D}}(Z) = \sum_{X \in \mathrm{Mat}_{v\times u}(R)} \overline{D(X)}\,\psi_0(\mathrm{tr}(ZX)),$$

the exact kernel identity is

$$\overline{K^{a,b}_{S,T}(D, P; g, h)} = \tau_{\mathfrak{m}} K^{b,a}_{T,S}(\widehat{\bar{D}}, \mathrm{flip}(\bar{P}); h, gs_{\mathfrak{m}}), \qquad |\tau_{\mathfrak{m}}|^2 = |R|^{-uv}. \tag{MR}$$

The signed real transpose is the line calculation on each coordinate, $X_{i,j} \mapsto -s_j t_i X_{j,i}$ for the respective source and target signs.

First conjugate with the source replaced by S^-. Negating the Hermitian form negates the symplectic form on $\mathrm{Hom}(S, T)$, so conjugation intertwines the Heisenberg actions with the two opposite central characters. In the chosen coordinates the real sign of every variable changes, so conjugation replaces its holomorphic polynomial coefficients by $\bar{P}$ on the opposite signs. At the finite places the coordinate change is multiplication by u_0^{-1}, for the fixed unit $\bar{u}_0 u_0 = -1$.

Its preservation of the integral boxes changes $D(X)$ to $\overline{D(u_0^{-1}X)}$. The two determinant weights become $a^{-1}\upsilon^u, b^{-1}\upsilon^v$. These powers are the change from each split Weil normalization to its complex conjugate: the underlying splitting character of $E^\times\backslash\mathbb{A}_E^\times$ is conjugated to its inverse, while the fixed standard splitting on a rank h space contributes υ^h. The rational matrices coincide under passage to S^- when the coordinate change is undone. Equivariance for rational Heisenberg translations identifies the transported functional with a scalar multiple of the normalized theta functional. To determine the modulus, write J for the unitary or antiunitary transport. It carries the rational translation lattice onto the new one and preserves probability Haar on the quotient. In the self-dual Schrödinger measures,

$$\int |\theta'(\rho'(x)Jv_0)|^2\,dx = \|Jv_0\|^2 = 1.$$

Indeed, integration in the modulation variable gives the sum of squared Schrödinger values by Parseval and integration in the displacement variable unfolds this to their squared norm. The same identity holds for the original unit vacuum. Consequently the scalar has modulus one. The adjoint transport below has the same property before the finite dilation.

Next apply the Hermitian adjoint on the opposite-source pair, which returns to the original S as the second space. In rational coordinates it is $X \mapsto J_{S^-}^{-1}X^*J_T$, a bijection of the rational matrix sets. The signed real transpose and the original finite basis scalar are the entrywise transports in the line calculation. On the original source group the basis scalar moves the finite argument by $\bar{\alpha}_f/\alpha_f$. Swapping again inverts the determinant weights and inserts the same respective rank powers, so the two weights are now b, a.

To verify the finite factor, use a point indicator at X_0 after the opposite-source transport. Its pre-dilation vacuum displacement has box $X - X_0/t_{\mathfrak{m}}$ integral at the selected places. The

split source dilation by $t_{\mathfrak{m}}^{-1} I_u$ gives the box $(X - X_0)/t_{\mathfrak{m}}$ integral. In rank u the determinant norm of the dilation is $|R|^u$. The split Weil factor is its $v/2$-power, giving squared modulus $|R|^{uv}$ for the multiplier from the displaced vacuum to the point datum.

The Hermitian transpose sends the displacement to the conjugate places, where the Heisenberg relation turns it into multiplication on the original integral box by $\psi_0(\mathrm{tr}(Z(-\bar{u}_0)X_0))$. This is the entrywise trace pairing, so the computation for one column extends to the full matrix without a second measure factor on the conjugate split side. The normalized theta transport under adjoint is a unit phase. The source dilation moves to the group argument and combines with the basis scalar, giving precisely $s_{\mathfrak{m}}$. Solving for the original kernel inverts the displacement multiplier, giving the stated value of $|\tau_{\mathfrak{m}}|^2$.

Finally sum the point identities with coefficients $\overline{D(u_0^{-1} X_0)}$. The substitution $X_0 = u_0 X$ cancels the shift in the additive character because $-\bar{u}_0 u_0 = 1$. This gives exactly $\widehat{\bar{D}}$ under the unchanged primitive character ψ_0, proving (MR). For the two tested adjoint identities use (MR) on the two left-pair kernels, then trade the reversed polynomial for the distinguished reversed joint harmonic by the fixed real edge convolution. In the first comparison the tested factor is $\tau_{\mathfrak{m}} \bar{k}_0$, and in the second it is $\tau_{\mathfrak{m}}^{-1} k_1$, with the relevant ranks in $\tau_{\mathfrak{m}}$ on each edge. Both k_j are independent of the finite modulus and data. In (GC3) these two factors are read at $R = R_0$. Their product with the reversed-square factor has squared modulus $C|R_0|^{t^2}$, with $C \geq 0$ independent of the twist.

37 The leading coefficient and the four-block test

In this section, we estimate the leading Taylor coefficient on the prescribed Weyl lattice and construct the four-block test. The reference norm determines the normalization of the critical ratios in Section 38.

37.1 A lower bound for a coefficient in twelve coordinates

In this subsection, the four-block test of Section 37.2 requires the leading Taylor coefficient from its first twelve coordinates. We compare this coefficient with its Weyl norm in the prescribed lattice and normalization.

Write $V_0 = s_{\mathrm{vac}}$, $\Lambda_{12} = \mathfrak{a}_0^{12}$, and $\delta = \bar{\pi}_M$. Use the original 12-coordinate Weil vacuum v. The Taylor map $\mathcal{T}$ includes the scale $\pi_M s_{\mathrm{jet}}$. For nonnegative integers λ_τ satisfying $\kappa_\tau \leq \lambda_\tau \leq \kappa_\tau + x^{42}$, put

$$n_{\#} = \lceil 6b/k \rceil, \quad B_\tau = \prod_{j=1}^{n_{\#}} ((\lambda_\tau - n_{\#})_+ + j), \qquad P_p = \sum_\tau S_p(\kappa_\tau, \tau).$$

We construct one nonzero t_0, depending on the datum and the λ_τ, with

$$\log |\sigma t_0| \leq f m b/k \qquad \text{for every } \sigma.$$

At each embedding above a prime p, we will find an exponent α satisfying

$$\sum_c \alpha_{\tau,c} = \lambda_\tau$$

and a residue cell x_0 mod π_M with indicator I_{x_0} such that

$$v_p\left(\frac{V_0^{12}}{t_0}\prod_{\tau}\lambda_\tau!\,[Z^\alpha]\mathcal{T}I_{x_0}\right)\le P_p. \tag{98}$$

Lemma 37.1 (Field of definition of the coefficient test)**.** *For every residue cell x_0 and multi-index α, the coefficient $c_{\alpha,x_0}=V_0^{12}[Z^\alpha]\mathcal{T}I_{x_0}$ is algebraic. For any embedding $\iota:\overline{\mathbb{Q}}\hookrightarrow\mathbf{C}_p$, its image is the coefficient obtained from the corresponding CM sections using the base-changed invariant differential, torsion translations and finite character sums.*

Proof. Work over a number field defining the CM elliptic curve, its invariant differential, its CM action and the distinguished point $t=-t\neq 0$. The bundle $\mathcal{O}([t])$ is symmetric, its period sign has order at most two, and its Hermitian form satisfies $H(az,w)=H(z,\bar{a}w)$ for $a\in\mathcal{O}_K$. The section has value one at the origin. Consequently its reduced theta series and its normalized torsion translates have algebraic coefficients in the logarithm coordinate z of this differential [8, Prop. 2.1 and Thm. 2.9 in the authors' version]. More precisely, let F be the field of definition above, enlarged to contain K. Since $0\notin[t]$, the local equation at zero can be taken to be 1; the normalized value there is 1. For $N_u u\in\Lambda_C$, set $n=2N_u$, write π_C for the uniformization and retain the division point $Q_u=\pi_C(u/(2n))\in C[4N_u^2]$. The symmetric line bundle and the pair $(Q_u,2Q_u)$ determine the rational function $R_u\in F(C[4N_u^2])(C)$ given by

$$R_u(\pi_C(v))=\frac{(U_u^M\vartheta_C)(nv)}{\vartheta_C(nv)},\qquad (U_u^M\vartheta_C)(z)=e^{-\pi(z\bar{u}+|u|^2/2)/A_C}\vartheta_C(z+u)$$

[8, Cor. 2.4 and Lem. 2.7 in the authors' version]. Indeed, the auxiliary trivialization at Q_u and its inverse enter the symmetric translation together, so their scalar factors cancel. Under ι, the curve, divisor and division point therefore give the rational function ιR_u on C^ι. For the semicharacter χ_C and $\lambda\in\Lambda_C$, the period law gives

$$U_{u+\lambda}^M\vartheta_C=\chi_C(\lambda)\exp\left(\frac{\pi(u\bar{\lambda}-\lambda\bar{u})}{2A_C}\right)U_u^M\vartheta_C.$$

The phase is a root of unity, since $\mathrm{Im}(u\bar{\lambda})/A_C\in N_u^{-1}\mathbb{Z}$. Let y_{W} denote the affine coordinate in the Weierstrass model with origin at infinity. The parameter at the analytic origin is $y(P)=y_{\mathrm{W}}(P+t)$, as in Section 20.1. It satisfies $y(0)=0$ and $y'(0)=1$ in the invariant logarithm coordinate. Take $g_u=y$ if $\pi_C(u)=t$, when the translated divisor has a simple zero at the origin, and $g_u=1$ otherwise. Theorem 2.9 of the same reference gives $g_u^{-1}U_u^M\vartheta_C\in F(C[4N_u^2])[[z]]^\times$. Multiplication by g_u gives the translated series, including its zero constant term when $\pi_C(u)=t$, for every torsion order.

The forms in Lemma 7.6 satisfy

$$A_{k,i}(\pi_M s_{\mathrm{jet}}Z)=\frac{\pi_M p_F}{q_1}\sum_{\tau}\overline{\tau(\zeta_{q_1}^k)}Z_{\tau,i},\qquad p_F=s_0/u\in\overline{\mathbb{Q}}^\times.$$

Here s_0 has a power in K, and u is obtained by taking a root of the product of algebraic isogeny multipliers and discriminants in the same lemma. The finite translates of its seed are products of the preceding torsion series in these forms, up to roots of unity. This holds at every finite translate by rational approximation modulo the open stabilizer. Tensoring (AV) twelve times therefore gives the asserted algebraicity for every translate of $V_0^{12}v$.

On the residue group D at the prescribed places the identity

$$\mathbf{1}_{\bar{x}=x_0} = |D|^{-1} \sum_{\chi \in \widehat{D}} \chi(-x_0)\chi(\bar{x})$$

expresses the cell indicator times the vacuum as a finite combination of its translates by the dual lattice. The perfect pairing is given in Section 37.1.2. All coefficients and substitutions in this expression lie in a number field. In the model of Section 20.1, let $x(Y) \in YF[[Y]]$ be the unique solution of $x^3 + x - a_1xY - Y^2 = 0$. The invariant differential gives

$$z = \log_C(y) = \int_0^y \frac{dY}{1 + 3x(Y)^2 - a_1Y} = y + O(y^2).$$

In each degree, inversion of this series and substitution of $y(z/n)$ in R_u commute with every characteristic-zero embedding. Thus the same translated section gives $\iota(c_{\alpha,x_0})$ after coefficient extraction. □

The one-variable tests. Use the marked integral parameter $y = z + O(z^2)$ and $R = \mathcal{O}_{\mathbf{C}_p}$. The formal group is integral: the translated identity is the section $(0, 0)$ of the good Weierstrass model, and the group law transported by this section gives a series in $R[[y_1, y_2]]$. Differentiation preserves $R[[y]]$ since $y' = 1 + 3x^2 - a_1y$. Let (π, q) be (p, p) if p splits, (p, p^2) if it is inert, and $(\sqrt{-d}, p)$ if $p = d$. Section 20.1 gives $y(az) \in R[[y]]$ for $a \in \mathcal{O}_K$, and the reduced order $i(a)$ is at most $a\bar{a}$ when $a \neq 0$, by the unit torsion value of $\sigma(az)^2$ at t/a. For split p, take a principal power $\mathfrak{p}^h = (a)$ of the chosen split prime. Then $a\bar{a} = p^h$, $\bar{a}$ is a unit at the chosen embedding, and $i(\bar{a}) = 1$. Composition of reduced parameter series gives $i(p)^h = i(a)i(\bar{a}) \leq p^h$. Thus $i(\pi) \leq q$ in every case.

Average $a^{-1}y(az)$ over $a^{q-1} = 1$. These Teichmüller values are limits of CM integers at the chosen embedding, so the result $T = y + O(y^2)$ is integral. Root-of-unity covariance gives

$$T(\pi z) = \pi T(z) + b_1T(z)^q + \sum_{j \geq 2} b_jT(z)^{1+j(q-1)}, \qquad b_1 \in R^\times,$$

where the reduced order bound gives the unit assertion. This also works at $p = 2$, when $q - 1 = 1$. For $\varpi^{q-1} = -\pi/b_1$ put $\rho = v_p(\varpi) = v_p(\pi)/(q-1)$. The coefficients of $Y = T(\varpi z)/\varpi$ are integral. Its equation is $Y(\pi z)/\pi = Y - Y^q + \pi E(Y)$ with $E \in X^2R[[X]]$, and the recursion divides in degree $n > 1$ by the unit $1 - \pi^{n-1}$. Thus $Y \equiv \sum_{j \geq 0} z^{q^j} \pmod{\pi}$.

Put $c_n = \max_{H \in R[[y]]} |n![z^n]H|_p$. Since $y = z + O(z^2)$, the ultrametric inequality gives

$$c_n = \max_{0 \leq i \leq n} |n![z^n]y^i|_p, \qquad c_0 = 1.$$

Thus one of the rational functions $1, y, \ldots, y^n$ attains the maximum. Integral differentiation gives $c_n \leq 1$, and the integral group law gives $c_{m+n} \leq c_mc_n$ by taking the z^mw^n coefficient of $H(z+w)$ and multiplying by $m!n!$. Since $T(z) = \varpi Y(z/\varpi)$ belongs to $R[[y]]$, $[z^n]T = \varpi^{1-n}[z^n]Y$ and $[z^{q^j}]Y$ is a unit. Taking $H = T$ and using Legendre's formula gives

$$c_{q^j} \geq p^{-\gamma_p(q^j-1)}, \qquad \gamma_p = \frac{1}{p-1} - \rho.$$

For $n \geq 1$, write $q^j = an + b$ with $0 \leq b < n$. Submultiplicativity gives $c_{q^j} \leq c_n^a$. Let j tend to infinity to get $c_n \geq p^{-\gamma_p n}$. Consequently in each degree n some $H \in R[[y]]$ has $v_p(n![z^n]H) \leq \gamma_p n$.

Put $g = -\log\theta$. The normalized pullback $\exp(a\bar{a}\,g(z) - g(az)) \in R[[y]]$ extends to limits of $(a, a\bar{a})$. For $\delta' \in \mathbf{Z}_p$ approximate δ' by ordinary integers, obtaining the same assertion with $\bar{a}a = (\delta')^2$. There are four elements of $\mathbf{Z}_p$ whose squares sum to -1, by four squares modulo p^u and compactness. Adjoin 1 and 0 to this list to give $\delta_0, \dots, \delta_5$, with $\delta_0 = 1$, and $\sum_c \delta_c^2 = 0$. It follows that $\prod_{c=0}^{5} \theta(\delta_c z) \in R[[y]]^\times$ by multiplying the six pullbacks.

The weighted Gram identity. Use the finite family $(w_i)_i$ in the trace-dual lattice $\mathfrak{a}_0 = E \cap \alpha_{\mathrm{W}} \widehat{O}_E$, whose positive integral multiplicities c_i satisfy

$$\sum_i c_i \tau(w_i)\overline{\sigma(w_i)} = M\delta_{\tau,\sigma}, \qquad c_i \geq 6fb.$$

Combine the family of powers of ζ_{q_1} and the family constructed from the trace-dual basis in Section 20.1, before squaring either scalar. Their scalar constants are q_1 and h respectively, with $(h, q_1) = 1$. They can both be taken at most b. In the mixed basis the dual vectors are $\bar{\beta}^{\lceil \log_2 q_1 \rceil} b_i / \Phi'_{q_1}(\zeta)$, with b_i the coefficients of $\Phi_{q_1}(X)/(X - \zeta)$. The bound for h follows from $\Phi_{q_1}(X) = \Phi_{q_0}(X^{q_1/q_0})$, which bounds $|b_i|$ by $q_1 2^{q_0}$ at every embedding and gives a lower bound $c(q_0)q_1/q_0$ for the derivative there. Thus h is polynomial in $q_1, d^{\lceil \log_2 q_1 \rceil}$, whereas b exceeds every fixed power of x. Put $m_0 = \lceil 6fb \rceil$. The scale $M \geq b^{20}$ ensures $M - m_0(q_1 + h) \geq q_1 h$. Write the left side as $uq_1 + vh$ with $u, v \geq 0$: choose $0 \leq v < q_1$ in the required residue class first. Give each vector of the first family multiplicity $m_0 + u$ and each of the second multiplicity $m_0 + v$. The two scalar Gram matrices then add to MI.

Write $\mu_i(Z) = p_F \sum_\tau \overline{\tau(w_i)} Z_\tau$. At every coefficient embedding these forms have integral coefficients by the period-factor valuation. At least one μ_i has only unit coefficients. At weight embeddings outside β-primes use the vector 1. At the primes over β use the vector $\zeta\bar{\beta}^{\lceil \log_2 q_1 \rceil} / \Phi'_{q_1}(\zeta)$. This is ζe_{f-1} in the notation of Section 20.1, since the top coefficient of $\Phi_{q_1}(X)/(X - \zeta)$ is 1. Its entry in the family of translates survives the replacements at $k = 0$. The denominator cancels the period factor in valuation. Either case applies to all weight embeddings together, since their restrictions to K are equal. At an inert prime the μ_i span the coordinate forms integrally up to a unit, by the powers basis and its Vandermonde determinant, since $p \nmid q_1$.

Put $\Theta(Z) = \prod_i \theta(\mu_i Z)^{c_i}$ and let $\mathcal{A}$ be the integral power series in $y(\mu_i Z)$. For a degree vector ν write $q_N(\nu) \leq u$ if some coefficient of combined degree ν in a product of N independent copies $H\Theta$, $H \in \mathcal{A}$, has valuation at most u. Products on distinct copies add bounds. An integral linear substitution of copies within each coordinate preserves the combined degree, and each new coefficient is a finite integral combination of the old ones.

For the six-copy general bound, $U(Z) = \prod_{c=0}^{5} \Theta(\delta_c Z)$ is a unit of $\mathcal{A}$ by the one-variable identity. In the first copy take $H = H_1(\mu_{i_0} Z)/U(Z)$ with μ_{i_0} the unit form, and in the other five take $H = 1$. Substitute $Z_c = \delta_c Z$. The product becomes $H_1(\mu_{i_0} Z)$, whose coefficient is obtained from the one-variable test and multinomial formula. Thus

$$q_6(\nu) + v_p(\nu!) \leq \gamma_p |\nu|, \qquad \nu! = \prod_\tau \nu_\tau!. \tag{99}$$

At inert p, the spanning property also gives $q_6(\nu) \leq 0$: multiply the integral series with linear terms Z_τ, to powers ν_τ, by Θ. The lowest term in total degree is the desired monomial with unit coefficient.

37.1.1 The inert expansion and valuation

In this subsection, we apply the original expansion of Section 20.3 to the coefficient in twelve coordinates.

The inert marked normal form has $\Theta(\varpi Z) = W(Z)\exp(K\Xi(Z))$, $K = \varpi^{p+1}/p$, with W an integral unit and Ξ reducing modulo p to

$$\sum_{e\ \mathrm{odd},\, j\ge 0} \lambda_{e,j} \left(\sum_{\tau} Z_{\tau}^{p^e} Z_{-p^e\tau} \right)^{p^j}, \qquad \lambda_{e,0} \in R^{\times}.$$

Equation (76) gives the unit coefficient on the original elliptic curve for every inert p. For odd $e > 0$, put $u_e(\tau) = -p^e\tau$. Since $p \nmid dq_1 M$, the completions of E above p are unramified and the w_i are integral there. Frobenius on the cyclotomic factor raises ζ_{q_1} to $\zeta_{q_1}^{p}$; on K it induces conjugation. Hence $\overline{\tau(w_i)}^{p^e} \equiv u_e(\tau)(w_i) \pmod{pR}$ at the chosen embedding. The weighted Gram identity gives

$$\sum_i c_i \mu_i(Z)^{p^e} \mu_i(Z) \equiv M p_F^{p^e+1} \sum_{\tau} Z_{\tau}^{p^e} Z_{u_e(\tau)} \pmod{pR[[Z]]}.$$

At $j = 0$ the coefficient is therefore multiplied by the unit $M p_F^{p^e+1}$.

Let selected starts τ have odd exponents e_τ and ends $u_\tau = -p^{e_\tau}\tau$. For each selected τ, assume that every selected σ and every positive odd $e \le e_\tau$ satisfy

$$-p^e\sigma = u_\tau \quad \Longrightarrow \quad (\sigma, e) = (\tau, e_\tau).$$

Use this coupling j_τ times, with degrees $\nu' = \sum_\tau p^{e_\tau} j_\tau[\tau]$ and $\nu'' = \sum_\tau j_\tau[u_\tau]$; put $J_c = \sum j_\tau$. Apply the substitution using

$$G(Z', Z'') = \Theta(\varpi(Z' + Z'')) \prod_{c=1}^{5} \Theta(\varpi \delta_c Z').$$

The logarithmic derivative in Z''_u, divided by K, is

$$(\partial_u \Xi)(Z' + Z'') + K^{-1}(W^{-1}\partial_u W)(Z' + Z'').$$

Here 2 splits in K, so an inert p is odd and $v_p(K^{-1}) = 1 - 1/(p-1) > 0$. Iterating the derivative gives $K^{-J_c}\partial_{Z''}^{\nu''} G = GH$ with H integral, whose reduction modulo the maximal ideal is the product of the reduced first derivatives with the specified multiplicities.

Set $Z'' = 0$ and all Z'_τ outside the starts to zero. The unique lowest term of each relevant derivative is $\bar{\lambda}_{e_\tau,0}(Z'_\tau)^{p^{e_\tau}}$: terms with $j > 0$ vanish on differentiation, and the pairing hypothesis excludes smaller exponent terms. Also $G(Z', 0) = U(\varpi Z')$ is integral with constant term 1, by $T(\varpi z) = \varpi Y(z)$. Consequently the coefficient of $Z'^{\nu'} Z''^{\nu''}$ in G has valuation exactly $J_c(1/(p-1) - 1) - v_p(\nu''!)$. Undo the scale ϖ and use the integral substitution back to the six independent copies. With $\nu = \nu' + \nu''$ this gives

$$q_6(\nu) \le -J_c, \qquad q_6(\nu) + v_p(\nu'!) + v_p(\nu''!) \le \gamma_p|\nu| - J_c.$$

The first inequality uses $|\nu| \ge (p+1)J_c$, and the second uses $v_p(\nu'!) \le |\nu'|/(p-1)$. The exact coefficient here is that of the substituted series. On the original independent variables this proves existence at the indicated valuation bound.

We now show

$$q_{12}(\lambda) + \sum_{\tau} v_p(\lambda_\tau!) \le P_p + 2\sum_{\tau} v_p(B_\tau). \tag{100}$$

At a split prime $\gamma_p = P_p = 0$, so use (99) and six constant coefficients. For inert $p > k$ use every start with $e_\tau = 1$, $i_\tau = \lfloor (\lambda_\tau - n_\#)_+/p \rfloor$, and $j_\tau = \min(i_\tau, \lambda_{-p\tau})$. Here 1 is the least positive odd exponent, and multiplication by $-p$ permutes the places, so each end has a unique start. Since $i_\tau \le n_\#$, the outgoing plus incoming degree at each place is at most $(\lambda_\tau - n_\#)_+ + \min(n_\#, \lambda_\tau) = \lambda_\tau$. The remaining six copies give valuation at most zero on unused degree. Legendre's formula and $n_\#! \mid B_\tau$ give $v_p(\lambda_\tau!) \le i_\tau + 2v_p(B_\tau)$: the factorial below the last block has valuation $i_\tau + v_p(i_\tau!)$, and the last block divides B_τ. Also $i_\tau - j_\tau \le (\kappa_\tau/p - \kappa_{-p\tau})_+$ because $n_\# \ge x^{42}$. Summing proves (100).

For nonsplit $p \le k$, select the pairs with $2e < L_0$ whose two places have weight b, with e among the allowed positive exponents. For $p = d$ all positive exponents occur in S_p, with coefficient $1/2$. Nevertheless the selected set is empty: the weight-b places all equal $-1 \bmod q_0$, whereas $d \equiv 9239 \bmod 18480$ and $d^2 \equiv 1 \bmod 18480$ give $-d^e \not\equiv 1 \bmod q_0$ for every $e \ge 1$. Use six constant terms for that empty set. At inert p the partners are $(-p)^e\tau$. Sparsity along this orbit forbids two pairs with the same start, two with the same end, or a start which is an end.

These would give triples with exponents $e, e', e'-e, e'$, or $e, e+e', e'$, respectively, all below L_0. A smaller positive odd exponent e' from a selected start to a selected end would give a second selected pair. This also excludes a shorter return from the same start: in the two congruences of Section 4, take $\tau_1 = \tau$, $\tau_2 = \tau_3 = u_\tau$, and $(i, j) = (e', e_\tau)$. Take $j_\tau = \lfloor \lambda_\tau/p^e \rfloor$ on these pairs; their ends have degree at least $b \ge j_\tau$. Use the second coupling bound and (99) on the remaining degrees. At each place write $\lambda_\tau = u + v + w$ for the three degree contributions. Since $\lambda_\tau \le 2b$, Legendre's formula gives

$$v_p\left(\frac{(u+v+w)!}{u!v!w!}\right) \le 2\lfloor \log_p(2b) \rfloor \le 3(\log b + 2).$$

Indeed subtracting the three floors at a power of p introduces at most two. Summing over the places gives the stated factorial bound.

For each τ,

$$R_\tau = \sum_{e \text{ allowed}} \min(\kappa_\tau/p^e, \kappa_{-p^e\tau})$$

has its selected term at most $j_\tau + 1$; all other terms with $2e < L_0$ are at most a/x_1, and the tail is at most $2b/(kp)$. For the tail, $p^{\lceil L_0/2 \rceil - 1} \ge 2^s \ge k$. The identity $\gamma_p \sum_\tau \kappa_\tau - P_p = v_p(\pi) \sum_\tau R_\tau$ accounts for the missing couplings. The excess weights and the factorial errors are paid by $v_p(B_\tau)$, since

$$3(\log b + 2) + x^{42} + 1 + L_0 a/x_1 + 2b/(kp) \le 3b/(kp) \le v_p(B_\tau).$$

Here the marked scales give $3(\log b + 2) + x^{42} + 1 \le a/x_1$, $(L_0 + 1)a/x_1 \le b/k^2$, and $b/(kp) \ge 1$. The last inequality follows by counting at least $\lfloor n_\#/p \rfloor$ multiples of p in B_τ. This proves (100) also here.

The exact geometric section and finite cells. At a coefficient furnished by (100), expand each H in its monomials in the $y(\mu_i Z_c)$. Only parameter degrees at most $|\lambda|$ can contribute. Choose contributing monomials with no higher valuation. Writing their exponents $u_{i,c}$, the corresponding curve sections are

$$(y\theta^3)^{u_{i,c}} \theta^{c_i - 3u_{i,c}}.$$

The powers are nonnegative since $3|\lambda| \le 6fb \le c_i$. The sections have degree c_i and integral twisted torsion values. Evaluate them in the Fock coordinates at the forms $\Omega_0 w_i^{\dagger}/(\bar{\delta}\, a_d)$ on the corresponding coordinates. Their weighted pullback Hermitian forms sum to the original Fock form by the basis identity and $|\bar{\delta}|^2 = M$.

In the real coordinates $r(\lambda)_{\tau,c} = a_d \tau(\lambda_c)$ the forms satisfy

$$A_{i,c}(r(\bar{\delta}\lambda)) = \Omega_0 \operatorname{Tr}_{E/K}(\lambda_c \bar{w}_i) \in \Omega_0 \mathcal{O}_K \qquad (\lambda \in \Lambda_{12}).$$

The inclusion follows from trace self-duality. Write $H_C(l,z) = \pi \bar{l} z / A_C$ and let H_{Fock} denote the sum of the pullbacks of $c_i H_C$. The Gram factor is $\pi|\Omega_0|^2 M/(A_C|\delta|^2 a_d^2) = 1$. The sign $\chi_C(l) = \exp(i \operatorname{Im} H_C(\bar{\beta} l, l))$ computed for the marked two-torsion point gives

$$\prod_{i,c} \chi_C(A_{i,c} L)^{c_i} = \exp\!\big(i \operatorname{Im} H_{\mathrm{Fock}}(\bar{\beta} L, L)\big).$$

This is exactly the prescribed signed period law on $\bar{\delta}\Lambda_{12} = \pi_M \Lambda_{12}$. The principal-vacuum Fourier inversion on this lattice produces a finite-part vector F_* with this entire theta section. At $\pi_M s_{\mathrm{jet}}$ its forms are exactly μ_i because $\bar{\delta} = \pi_M$. A finite translate of F_* can be read after rational approximation modulo its open stabilizer. Each factor then evaluates at the twisted value at $-A_{i,c} r(\gamma)$, up to a root of unity, for the rational displacement γ. Choose $b_0 \ge 1$ with $b_0 \gamma \in \Lambda_{12}$. Then

$$M b_0 A_{i,c} r(\gamma) = \Omega_0 \delta \operatorname{Tr}_{E/K}(b_0 \gamma_c \bar{w}_i) \in \Omega_0 \mathcal{O}_K.$$

The normalized torsion values of θ and $y\theta^3$ are integral by Section 20.1. Thus F_* has integral values on the original finite slice.

37.1.2 The finite coefficient pairing

In this subsection, we pair the resulting coefficient of Section 37.1.1 against the finite test.

Put $L = \Lambda_{12}$, $A = L/\bar{\delta}L$ and $B = \delta^{-1}L/L$. The Weil commutator pairs A and B perfectly, and $|A| = |B| = M^{12f}$. For rational representatives a, u, use the finite operators $S(a) = T_\sharp(\alpha_{\mathrm{W}}^{-1} a)$ and put $T(u) = T_{\mathrm{f}}(\alpha_{\mathrm{W}}^{-1} u)$, where T_{f} denotes ordinary displacement in normalized finite coordinates. Rational Heisenberg invariance identifies the signed period law above with the corresponding finite translation law, as in Section 20.1. The sign in $T_\sharp$ also applies at the places above 2. Since $\bar{\delta}L$ fixes F_* under the signed translations and pairs trivially with B, the sum over A is well defined on $T(-u)F_*$. For each u, the sum

$$d_u v = \sum_{a \in A} S(a) T(-u) F_*$$

satisfies the signed period law on L, whose theta space is the vacuum line. Conjugation by $T(u)$ multiplies $S(a)$ by the commutator character at u. Its sum over B is zero for $a \neq 0$ and $|B|$ for $a = 0$. Thus

$$M^{12f} F_* = \sum_{u \in B} d_u\, T(u) v.$$

If the vacuum vanished on the whole finite slice, rational translation would make it vanish at every rational real argument and every finite argument. The rational real arguments are dense, so continuity would give $v = 0$. Thus the finite slice has a nonzero value.

Fix such a point z_f^* and put $C_v = V_0^{12} v(0, \alpha_{\mathrm{W}} z_f^*) \neq 0$. At this point $d_u v(0, \alpha_{\mathrm{W}} z_f^*)$ is a sum of finite evaluations of F_* multiplied by roots of unity. It is therefore integral also at the primes dividing M. In the normalized finite coordinates $x = \alpha_{\mathrm{W}}^{-1} u$ we have δx integral. The nonintegral components of x are at the conjugates of the places of π_M. Subtracting the other components acts on the vacuum by a root-of-unity phase. On the chosen Schrödinger lattice the remaining translation multiplies the vacuum by an additive character with coefficient $\bar{x}$.

This character factors through reduction modulo π_M because $\pi_M \bar{x} = \overline{\delta x}$ is integral. Write $T(u)v = \sum_x \omega_u(x) I_x$ on the original residue cells, where each nonzero $\omega_u(x)$ is a root of unity. For $v_* = v(0, \alpha_{\mathrm{W}} z_f^*)$ the Fourier identity gives

$$M^{12f} C_v [Z^\alpha] \mathcal{T} F_* = \sum_x \left(\sum_{u \in B} d_u v_* \omega_u(x) \right) V_0^{12} [Z^\alpha] \mathcal{T} I_x.$$

Every parenthesized coefficient is integral. At the chosen nonzero coefficient on the left, the ultrametric inequality therefore supplies an original cell with

$$v_p(V_0^{12} [Z^\alpha] \mathcal{T} I_{x_0}) \leq v_p(M^{12f} C_v) + v_p([Z^\alpha] \mathcal{T} F_*).$$

Set $t_0 = M^{12f} C_v \prod_\tau B_\tau^2$.

It is fixed before the coefficient prime and embedding, and nonzero. Put $C_{\mathrm{int}} = 2 + \log M_K$ and fix $C_P \geq \max\{C_K, C_{\mathrm{int}} + 2d + 6\}$, with the constants from the proof of Lemma 7.7. They depend only on K. Applying its finite-value bound in the twelve coordinates gives

$$\log |\sigma t_0| \leq 12f \log M + 12 C_P q_1^2 + 2f n_\# \log(2b + n_\#) \leq fmb/k.$$

Indeed q_1, k grow at most polynomially in x, while $b = \exp(x^\epsilon + O(1))$, $\log M \leq 22 \log b$ and $\log b = O(x^\epsilon) = o(m)$. Using $n_\# \leq 6b/k + 1$, division of the left-hand side by fmb/k gives a quantity tending to zero for fixed d, ϵ. Now (100) proves (98). The polynomial degree, section, and finite Fourier steps all use the same $\Lambda_{12}, \alpha_{\mathrm{W}}, \pi_M$ as the numerator test.

37.1.3 CM transport of the untwisted numerator

In this subsection, we apply Lemma 7.6 to the coefficient of the original numerator test.

Fix $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$, and define p_σ by $\sigma \Phi_\tau = \Phi_{p_\sigma(\tau)}$. For the untwisted datum X, put $J_X = \sum_\tau \lambda_{X,\tau}$. Its conjugate datum Y has the same level, with $\lambda_{Y,\tau} = \lambda_{X, p_\sigma^{-1}(\tau)}$ and $\nu_{Y,f} = \sigma \circ \nu_{X,f}$. These finite and infinite characters agree on rational norm-one elements by the automorphy of ν_X and the permutation of its integral infinity weights. The ray class calculation in Section 37.2.5 gives $|\sigma(\nu_{X,f}(g))| = 1$ on every finite norm-one idele. At a tame place the conjugate character is $\sigma \circ \psi_w$, with the same kernel and conductor.

Instantiate the exact translated jet identity (AJ) and the vacuum identity (AV) on the standard Hom space of rank $2z$. Their proof depends on the rank only via the coordinate product, so the same construction uses the normalized scale t_σ with $|t_\sigma|^2 = \Omega^{-1}$. Keep the exact identities (JV1) and the finite multiplier law, rather than their subsequent bounds on coefficient sizes. Write the resulting scalar translate as $b_\sigma = u_\sigma a_\sigma$ where $u_\sigma \bar{u}_\sigma = 1$ and a_σ is an integral unit equal to 1 on the old chosen places. This is possible because $b_\sigma \bar{b}_\sigma$ is a unit. At every split pair choose one member w, taking the prescribed member at each old pair. Set $a_{\sigma,w} = 1$ and $a_{\sigma,\bar{w}} = \overline{(b_\sigma \bar{b}_\sigma)_w}$. At a conjugation-fixed place b_σ itself is a unit, so use it as a_σ.

In the untwisted column datum the finite coefficient function is the original chosen-column indicator and has values $0, 1$. After removing the scalar action of u_σ, the exact semilinear operator transports each signed translate by x to the one by $a_\sigma x$, and multiplies the vacuum by one fixed nonzero scalar c_σ. Let D be the residue group of the column at the chosen places. Use the expansion

$$\mathbf{1}_{y=y_0} = |D|^{-1} \sum_{\chi \in \widehat{D}} \chi(-y_0)\chi(y).$$

The modulation for χ is a finite translation in the opposite split component. There its parameter is multiplied by a_σ. Since $a_\sigma = 1$ on the chosen component and $a_\sigma \bar{a}_\sigma = b_\sigma \bar{b}_\sigma$, the new modulation is $\sigma\chi$ by (JV1). Semilinearity changes $\chi(-y_0)$ to $(\sigma\chi)(-y_0)$. Reindexing the finite sum gives the same indicator times c_σ. Away from the level the integral unit preserves the vacuum line.

Write $b_X = \nu_X \upsilon_1^z$, $b_Y = \nu_Y \upsilon_1^z$, where $\upsilon_1 = \upsilon|_{U(1)}$, and let g' denote the inverse source action on the $2z$ Hom coordinates. For a finite norm-one element g, let $D_1 = \Delta_1(g^{-1})$ be the multiplier on one coordinate. Its squared modulus is a rational lattice index and $D_1^2 = |D_1|^2 \upsilon_1(g)$. The normalized vacuum is a tensor product on the $2z$ coordinates, so $\Delta(g') = D_1^{2z}$. Hence

$$d_\sigma(g') = \frac{\sigma(\Delta(g'))}{\Delta(g')} = \left(\frac{\sigma(D_1^2)}{D_1^2}\right)^z = \left(\frac{\sigma(\upsilon_1(g))}{\upsilon_1(g)}\right)^z.$$

After writing $\nu_Y(g) = \sigma(\nu_X(g))$ on the finite ideles, the source factor in a corresponding kernel term is

$$\sigma\big(b_X(g)^{-1}\big) d_\sigma(g') = b_Y(g)^{-1} = b_Y(u_\sigma^{-1})\, b_Y(g u_\sigma^{-1})^{-1}.$$

The last inverse character belongs to the transported kernel at $(g u_\sigma^{-1}, h)$. Multiplication by u_σ^{-1} permutes the classes of the norm-one torus at the same old level. Summing over this permutation leaves $b_Y(u_\sigma^{-1})$ outside the class sum.

Write $(R_\sigma Z)_{\tau,j} = Z_{p_\sigma^{-1}(\tau),j}$ and $i_\tau^\sigma = i_{p_\sigma^{-1}(\tau)}$. The Schur polynomials have integer coefficients and are products over the places. Thus $P_{Y,i^\sigma}(R_\sigma Z) = P_{X,i}(Z)$, and the jet of total degree J_X contributes $t_\sigma^{J_X}$. With $A_X = s_{\mathrm{jet}}^{J_X} X^{\mathrm{raw}}$, the exact transport at the unchanged target point h is

$$\sigma(A_X(h)_i) = t_\sigma^{J_X} c_\sigma b_Y(u_\sigma^{-1}) \frac{\sigma(\Delta(h'))}{\Delta(h')} Y^{\mathrm{raw}}(h)_{i^\sigma}.$$

At fixed $2z$ and old level, c_σ and the Taylor scale were fixed before the character and its local degrees. This proves the transport identity needed for the mixed pairing in the untwisted reference norm. It has no analog requiring a simultaneous CM transport of the twisted numerator at $\kappa, \bar{\kappa}$: there one constructs the two point tests separately by the same coefficient theorem, keeping the same η, multiplier t_0, and main line.

37.2 The four-block test and the untwisted reference norm

In this subsection, we insert the coefficient of Section 37.1 in the original four-block theta test. The resulting numerator is measured against the untwisted norm on the same space.

Fix a main pair $p = (\nu, z)$, its marked twist η of exact support Q, and the original lift X from $U(1)$ to $U(2z)$. Let C be the old $U(1)$ class group, $h_1 = |C|$, and $V_0 = s_{\mathrm{vac}}$. Use its original ordered blocks of sizes $12, 4, 4, 2z - 20$. Write $\lambda_\tau = c_\tau - z$ for the part sizes and $J = (\pi_M s_{\mathrm{jet}})^{\sum_\tau \lambda_\tau}$. The normalized main lift is

$$\phi_\eta = V_0^{2z} J \sum_{g \in C} K_X(g),$$

where $K_X(g)$ is the raw kernel form. The class stabilizers in this source sum are trivial.

37.2.1 The marked derivative estimate

In this subsection, we use the coefficient bound of Section 37.1 for the point test.

The required scalar $t_0 \neq 0$ depends on the part sizes and the fixed datum, with $\log|\sigma t_0| \leq fmb/k$ for every complex automorphism. At each coefficient embedding $\kappa = (\ell, \iota)$, the sharp derivative bound gives an exponent α on the first twelve coordinates with $\sum_i \alpha_{\tau,i} = \lambda_\tau$, and a cell x_0 in the original residue space modulo π_M, such that

$$\left| \frac{V_0^{12}}{t_0} \left(\prod_\tau \lambda_\tau! \right) [Z^\alpha] \mathcal{T} I_{x_0} \right|_\kappa \geq \ell^{-P_\ell}, \qquad P_\ell = \sum_\tau S_\ell(\kappa_\tau, \tau). \tag{NB1}$$

Here I_{x_0} is the point indicator at the chosen places and the original vacuum elsewhere, and $\mathcal{T}$ reads the Taylor series at scale $\pi_M s_{\mathrm{jet}}$. Lemma 37.1 identifies the coefficient here with its value at κ. Its degree-zero calculation on a translated vacuum likewise gives $C_v \in \overline{\mathbb{Q}}^\times$, hence $t_0 \in \overline{\mathbb{Q}}^\times$. For a main pair the defining inequalities of Section 38.2 give $w_j(\tau) + r - t \leq c_\tau \leq w_j(\tau)$ and $m < z \leq r$. Since $w_j(\tau) = \kappa_\tau + (2j+1)r^8$ and $t = 5r$, we have

$$\kappa_\tau \leq w_j(\tau) - t \leq c_\tau - z \leq w_j(\tau) \leq \kappa_\tau + x^{42}.$$

The last inequality follows for all $1 \leq j \leq m$ from $(2m+1)r^8 = O(x^{81/2})$ by Section 4.1. Thus Section 37.1 applies at the actual degree $\lambda_\tau = c_\tau - z$; it fixes t_0 before the coefficient embedding and gives (NB1) on the original residue space.

37.2.2 The point test on the original target

In this subsection, we insert the derivative of Section 37.2.1 in the test on four blocks.

Use the empty rank-four pair $p_{\mathrm{aux}} = (\chi_M \upsilon_1^{-2}, 2)$ in the second block at twist η, and two untwisted empty covectors in the third and fourth blocks. For an even rank $2z'$ and an old weight-zero character ψ, the empty partition is the pair $(\psi \upsilon_1^{-z'}, z')$, with rank-twisted character ψ. These even ranks lie between 4 and t, since $m < z \leq r$. The character on the first block is the quotient of the target character of X by the other three characters.

Set $T_\alpha = \prod_\tau \lambda_\tau!/\alpha!$, with $\alpha! = \prod_{\tau,i} \alpha_{\tau,i}!$. Let P_i be the dual Schur basis of products of minors for this weight. Its column minors have size one. In the original Weyl coordinates take $v_i = [Z^\alpha] P_i$, so v is integral. The Fock polynomial $\mathcal{P}(v)$ in these coordinates satisfies

$$\langle \mathcal{P}(v), P_i \rangle_{\mathrm{Fock}} = \left(\prod_\tau \lambda_\tau! \right) v_i, \qquad \langle Z^\alpha, P_i \rangle_{\mathrm{Fock}} = \alpha! [Z^\alpha] P_i.$$

The first identity is the Weyl to dual Schur pairing with the one-covector hook. The second follows by monomial orthogonality. The pairing with all P_i therefore gives $\mathcal{P}(v) = T_\alpha Z^\alpha$. The contragredient substitution fixes the coefficient functional when the first twelve columns are identity columns. Equivalently the transpose substitution fixes its monomial in the Fock pairing. This gives invariance of v under the last three embedded groups. There is no determinant factor in this Weyl weight.

At the old chosen places write $Y = (x_0, e_2, e_3, e_1)^{\mathsf{t}}$, with e_i the included unit vector in the indicated empty block. Write e_{old} for the original included vector of X and e_{second} for the included vector of the second block. Choose the reduction of the integral point lift A by

$$Ae_{\mathrm{old}} = Y \quad (w \in S_M), \qquad e_{\mathrm{old}}^{\mathsf{t}} A^{-1} = e_{\mathrm{second}}^{\mathsf{t}} \quad (w \in Q).$$

At an old place Y is nonzero by its last component and extends to a basis. At a new place the prescribed nonzero covector extends to an invertible matrix of covectors. Split reduction gives the local integral lifts. They define A by taking the identity at all remaining finite places of F_0.

Put $x = \mathrm{diag}(1, h_2, h_3, h')A$ and let J_Q, χ be the target level and its character. Write the covariance of a form as $f(\gamma y k) = \chi(k)\rho(\gamma)f(y)$, where Γ is the rational group and ρ its coefficient action. Define $F(h', h_2, h_3) = F_x(v)$ by

$$F_x(v)(y) = \sum_{\substack{\gamma \in \Gamma \\ x^{-1}\gamma y \in J_Q}} \chi(x^{-1}\gamma y)\rho(\gamma^{-1})v.$$

The sum is finite by definiteness and discreteness. It has the stated covariance by changing variables and is supported on the class of x. For $k = x^{-1}\gamma y$, transporting the summand to the gauge at y acts on v by $k^{-1}x^{-1}$ and a root of unity. The inverse block diagonal fixes v. Both A^{-1} and k^{-1} belong to the integral compact and preserve the Weyl lattice at every coefficient place above ℓ. Thus each summand has gauge at most one there, as does their sum.

At x the sum is over $S_x = \Gamma \cap xJ_Q x^{-1}$. For $\gamma \in S_x$, put $T_\gamma = \chi(x^{-1}\gamma x)\rho(\gamma^{-1})$. It is unitary in the original definite pairing and fixes $f(x)$ by covariance. Since the class weight at x is $|S_x|^{-1}$, the pairing gives

$$\langle F_x(v), f\rangle_{J_Q} = \frac{1}{|S_x|} \sum_{\gamma \in S_x} \langle T_\gamma v, f(x)\rangle = \langle v, f(x)\rangle.$$

Move A from the kernel argument into the finite datum by the split Weil action. In that action the source character for X is trivial, so this move has scalar one. The datum now acts on A^{-1} times the residue column. On $\pi_M^2 Q$ the two displayed residue conditions give the prescribed column indicator at S_M and the tame character of the second-block covector at Q.

This is exactly the product of the four block functions, pulled back to that modulus. The archimedean polynomial is $T_\alpha Z^\alpha$ on the first block and 1 on the other three. The rational theta sum and the Weil character both factor across these orthogonal blocks. If K_i are the raw empty kernels and K_0 is the selected first kernel, this proves

$$\langle F(h', h_2, h_3), K_X(g)\rangle = T_\alpha K_0(g) K_2(g, h_2) K_3(g, h_3) K_1(g, h'). \tag{NB2}$$

In this formula the kernel is in the linear leg of the pairing. The same g serves every block.

For an empty block p_i of rank a_i, let $\phi_i = V_0^{a_i}\phi_i^{\mathrm{raw}}$. The denominator $d_i = d_{p_i} = M^{2fa_i}$ is that of Section 30.1. Its local jet degree is zero at each infinite place, since its polynomial is the constant 1. Put

$$A_i = C_G d_i^2 \|\phi_i\|^2 / Q_i^{a_i - 1}, \quad Q_2 = \mathrm{N}Q \text{ for the twisted block}, \quad Q_1 = Q_3 = 1.$$

Here Q_i in this display denotes only the norm of the auxiliary modulus on that block. The class averaging operator on F weights its value at h by

$$\frac{C_G d_i^2}{|\operatorname{Aut}(h)|\, Q_i^{a_i - 1}} \langle 1, \phi_i(h)\rangle_{\mathrm{Weyl}}.$$

These weights are integral at all the embeddings above ℓ, since $\ell \nmid \mathrm{N}Q$, d_i clears the normalized empty lift and C_G clears the class weights.

In the pairing against $K_X(g)$ the weights are conjugated. They replace $K_i(g,h)$ in (NB2) by $A_i/(h_1 V_0^{a_i})$. To see this, a normalized central translation by u^{-1} sends $K_i(g)$ to $K_i(gu)$: the simultaneous scalar cancels on the Hom variable, leaving the inverse source twist. This translation is unitary and fixes $\phi_i^{\mathrm{raw}} = \sum_g K_i(g)$, as right translation permutes the source classes. It follows that $\langle \phi_i^{\mathrm{raw}}, K_i(g) \rangle$ is independent of g; summing over g gives its value $\|\phi_i^{\mathrm{raw}}\|^2/h_1$. Combining this with the conjugated scalar $\overline{V_0^{a_i}}$ in the weights gives precisely $A_i/(h_1 V_0^{a_i})$.

37.2.3 The last-block class sum

In this subsection, we evaluate the last block of the test in Section 37.2.2.

Apply the measured period argument in Section 33.1 to the original rank-one descent datum O of the four-block lift X. The source has rank 1, the target has rank $n = 2z$, and the two blocks for this application are the retained first $12 + 4 + 4 = 20$ coordinates and the last $n - 20$ coordinates. Thus the dimension condition is $2 + 2 \cdot 20 + 3 = 45 < n$. At infinity the line parameter is zero at every place, the determinant exponent is zero, and the target space of length λ_τ is matched by the homogeneous Fock polynomials of that degree. Indeed the inverse source scalar acts on that homogeneous degree by its $-\lambda_\tau$-th power, and the inverse character of source weight $-\lambda_\tau$ cancels it. The positive Fock variables give the single covector $(\lambda_\tau, 0, \dots, 0)$, with no negative-sector columns or mixed Laplacians. In each degree this is the irreducible symmetric power on the compact target, including the trivial representation in degree zero. Distinct degrees have distinct source circle characters. This proves the required rank-one correspondence.

Fix a split w with $w, \bar{w}$ off the original chosen and twist places. For this test a false lift y lies in

$$\operatorname{im}(e_{X-1}) \cap \ker \mathcal{P}_{n,w}(X_*) \cap \ker D_O, \qquad X_* = \left(q_w^{n-2} \operatorname{frob}_1(1,w)\right)^{-1},$$

where D_O is the normalized descent for O, and $\operatorname{frob}_1(1,w)$ is the rank-one unramified value of the trivial χ_W with the original splitting character included. If Υ is the original Hecke splitting character of E, then $\operatorname{frob}_1(1,w) = q_w^{1/2}\Upsilon(\varpi_w)^{-1}$. In particular $|X_*| = q_w^{3/2-n}$. The conjugate matrix coefficient of y is killed at $\overline{X_*}$, as in (F1). The coefficient operator here is the one of this rank-one datum, with $\chi_W = 1$, even when η is nontrivial in the other character.

The idempotent e_{X-1} is the product of the means over the old groups $\mathrm{GL}_1(k_v) = k_v^\times$. Thus its image consists of their invariant vectors. If λ is a nonzero residue covector at a chosen place v, the rank-one set $1 + u\lambda$ with nonzero determinant is all of $k_v^\times$. The action of its hyperplane sum on y is therefore $\left(\sum_{s \in k_v^\times} \chi_v(s)^{-1}\right) y = 0$. Here χ_v is the exact old main-pair tame character, which is nontrivial, while η has trivial old tame components.

The same weighted sum in the product of old local groups is zero if any one of its covectors is nonzero. At the new places the exact-support hypothesis gives nontrivial residue characters. The bottom descent vanishing for the original block is the condition $D_O y = 0$. These are the three conditions for the bottom extension to arbitrary Schwartz data vacuum off the original support. They replace the rank-r argument with good Green parameters and the orthogonal complement of the good image.

For the following tower argument, the source at step l, $1 \le l \le 20$, has rank $1 + 2l$. With $k_l = n - (1 + 2l)$, the estimates of (F3) read

$$q_w^{k_l} |\alpha X| = q_w^{1-l} \le 1, \qquad q_w^{1+2l+1} |X'| = q_w^{(6+2l-n)/2} \le 1,$$

where $X = \overline{X_*}$. The latter follows from $n \geq 46$. Hence the same hyperbolic-plane induction gives vanishing of all descents through twenty planes, and the argument from (F4) gives the measured second-block periods with retained dimension 20. The bottom and tower arguments use the fixed O, fixed target point, original finite vacua and original right-coset operators throughout.

For the application to the test, take the fixed first-block point $\mathrm{diag}(1, h_2, h_3)$ of the retained twenty coordinates. Take the additional right translate to be 1. The integral point lift A in $F_x(v)$ then gives the period point $\mathrm{diag}(1, h_2, h_3, h')A$, with finite translation supported at the old and auxiliary split places. The last-block kernel $K_1(g, h')$ is the degree-zero kernel, whose polynomial is 1, whose target dimension is $n - 20$, and whose rank-twisted character is the untwisted old factor ψ_1. In the argument of Section 33.1.4 this character is b. The retained factor uses b^{-1}, so their product has target character 1 for every choice of ψ_1. The vanishing hypothesis is $D_O y = 0$ for the fixed main pair with twist η.

In $K_1(g, h')$ the finite Schwartz datum has the original vacuum at the new places and elsewhere off the old support. Here g is the free source argument of Section 33.1.4. At this fixed g, that section constructs a nonzero auxiliary first-block factor to cancel in the period. The resulting sum over h' vanishes on the false lifts and allows arbitrary values of the original last-block kernel. The datum can be read on the nominal modulus π_M^2, with the prescribed column indicator depending only on reduction modulo π_M. In the measured period use the conjugate kernel vector $\overline{K_1(g, h')}$ and the conjugate-linear coefficient map $c \mapsto \bar{c}\, v$. This map is equivariant for the real last-block group: the empty weight is trivial, and the retained coefficient vector v is fixed by that group. Since the class pairing is conjugate-linear in its first argument, the resulting scalar integrand is

$$\overline{K_1(g, h')}\, \langle F(h', h_2, h_3), y \rangle.$$

The point-form reproducing identity evaluates the second form at the finite point

$$\mathrm{diag}(1, h_2, h_3, h')A.$$

Use the actual last-block vector-fixer level J_1 of the untwisted degree-zero lift. It is integral and fixes e_1 at the old chosen places, so fixes the last-block datum. At an old place the embedded element fixes the prescribed column $(x_0, e_2, e_3, e_1)^{\mathsf{t}}$. Consequently $A^{-1}\,\mathrm{diag}(1, k)A$ fixes the originally included column. At a new place this block embedding fixes the prescribed twist covector because that covector lies in the second block of the four-block splitting. Conjugating by A therefore fixes the originally included covector there and has level character 1. Off those places the element is integral. This is the right invariance needed for the period-to-class-sum passage. Rational left invariance follows from the same kernel and form covariance.

The measured period is zero. Compact integration at J_1 multiplies its class sum by the common nonzero factor $[K_{\mathrm{std}} : J_1]^{-1}$ with maximal compact mass one. Retaining the actual stabilizer weights gives

$$\sum_{[h']_{J_1}} \frac{\overline{K_1(g, h')}}{|\operatorname{Aut}_{J_1}(h')|} \langle F(h', h_2, h_3), y \rangle = 0.$$

This is for every original g, h_2, h_3, before the outer three sums. The last-block sum has its vector-fixer level, so there is no need to assert freeness of this class space. Expanding the degree-zero lift in its source class sum now gives the required orthogonality of the averaged

test to y. The two remaining scalar averages preserve this orthogonality, and all three averages preserve integrality.

Denote the test after the three averages by Ξ. Substituting the replacement identities into (NB2) gives

$$h_1^3 V_0^{2z-12}\langle \Xi, K_X(g)\rangle = T_\alpha K_0(g) A_1 A_2 A_3. \tag{NB3}$$

37.2.4 Selection of the class character

In this subsection, we use the class sum of Section 37.2.3 to choose characters for the numerator estimate.

Before selecting the two untwisted empty-block characters, let χ_0 be the quotient of the target character of X by the twisted rank-four character. Let $s(g)$ be the unscaled first kernel for χ_0, x_0, α. Choose the actual source representatives g_a of C, and let $z_0 = g_{1_C}^{-1}$. The representatives $g_a z_0$ have the identity point in the identity class. For $\psi \in \widehat{C}$ put

$$B_\psi = h_1^{-1} \sum_{a \in C} \psi(\det(g_a z_0)) s(g_a z_0).$$

Fourier orthogonality gives $\sum_\psi B_\psi = s(1)$. For each a, the sum of $\psi(\det(g_a z_0))$ over all characters is $h_1 \mathbf{1}_{a=1_C}$, so the identity holds on the chosen representatives. Even when $\ell \mid h_1$, the ultrametric inequality gives a ψ with $|B_\psi|_\kappa \geq |s(1)|_\kappa$. The three class weights defining the test remain integral.

Let $\mathcal{C}_0$ be the finite group of allowed old tame weight-zero characters. Each $\psi \in \widehat{C}$ belongs to this group by inflation. Indeed, on a norm-one idele take ψ of the class of its finite component. This character is trivial on rational points, on the archimedean torus and on the old principal level. Its finite unit restrictions thus factor through the residue groups at S_M and are trivial elsewhere. Fix $L_{\mathrm{geom}} \neq 0$ by choosing a root

$$L_{\mathrm{geom}}^{|\mathcal{C}_0|} = \prod_{\chi \in \mathcal{C}_0} A_1(\chi) A_3(\chi).$$

These are the untwisted degree-zero values on the last and third blocks, on the same old level. Proposition 30.1 applies to every $\chi \in \mathcal{C}_0$, including $\chi = 1$, since $z' \geq 2$. Each value is nonzero, and the Fourier denominator estimate of Section 44.4 applies with $q_Q = 1$. The permutation $\chi \mapsto \psi\chi^{-1}$ shows that one factorization $\psi = \psi_1\psi_3$ has $|A_1(\psi_1)A_3(\psi_3)|_\kappa \geq |L_{\mathrm{geom}}|_\kappa$. Use these two characters on the corresponding blocks. The target-leg character of the first block becomes χ_0/ψ. Its factor on the source determinant is

$$(\chi_0/\psi)(\det g)^{-1} = \psi(\det g)\chi_0(\det g)^{-1}.$$

Thus $K_0(g) = \psi(\det g)s(g)$. The source splitting character remains trivial.

Write $L_{\mathrm{twist}} = A_2(1)$, and let $\rho_\eta = A_2(\eta)/A_2(1) > 0$. For the empty rank-four pair $p_{\mathrm{aux}} = (\chi_M \upsilon_1^{-2}, 2)$, the exponents $c_v - z$ vanish, so the denominator $d_2 = M^{8f}$ is independent of η. The norm identity (85) gives

$$\frac{\|\phi_2(\eta)\|^2}{\|\phi_2(1)\|^2} = q_Q^3 R_{p_{\mathrm{aux}},+}(\eta), \qquad \rho_\eta = R_{p_{\mathrm{aux}},+}(\eta).$$

On the source, rational equivariance and the trivial source weight give left invariance of the full kernel K_X. The old principal level fixes its old column. At a new place, multiplication of the residue covector value by the unit character cancels the inverse character in the source Weil action. At all remaining places the units fix the vacuum. Thus K_X descends to the old class group C, and ϕ_η can be evaluated at $g_a z_0$. Sum (NB3) there and take its linear-leg normalization. The result is

$$\frac{h_1^2}{\rho_\eta}\langle \Xi, \phi_\eta\rangle = V_0^{12} J T_\alpha L_{\mathrm{twist}} A_1 A_3 B_\psi. \tag{NB4}$$

Here the third power of h_1 in (NB3) loses one power in passing to the Fourier average.

No complex conjugate of V_0 or J occurs in (NB4). The coefficient identity at $g = 1$ is $(J/\alpha!)s(1) = [Z^\alpha]\mathcal{T} I_{x_0}$, since differentiation multiplies the Taylor coefficient by $\alpha!$. Apply (NB1) and the two selection estimates to (NB4). In valuation form this gives

$$v_\kappa\left(\frac{\langle \Xi, \phi_\eta\rangle}{c_0 \rho_\eta}\right) \le P_\ell - 2v_\ell(h_1), \qquad c_0 = t_0 L_{\mathrm{twist}} L_{\mathrm{geom}}.$$

For $\ell = 2, p_0, p_1$ one has $P_\ell = 0$, since all three primes split in K and $1 - \chi_K(\ell^e) = 0$ for every $e \ge 1$.

Proposition 37.2 (The four-block numerator test)**.** *For each coefficient embedding κ above ℓ and each allowed twist η of exact support Q with $\ell \nmid q_Q$, there is a test $\Xi_{\kappa,\eta}$ integral at every coefficient place above ℓ and orthogonal to the false lifts. It satisfies*

$$|\langle \Xi_{\kappa,\eta}, \phi_\eta\rangle|_\kappa \ge |c_0 \rho_\eta|_\kappa \ell^{-P_\ell}, \qquad c_0 = t_0 L_{\mathrm{twist}} L_{\mathrm{geom}} \neq 0.$$

The constant c_0 is fixed for the original main pair and $\log|\sigma c_0| \le J_*$ at every complex automorphism σ.

Write h_{a_i} for the number of classes at the untwisted vector-fixer level of the rank-a_i block. This level and the old principal level of the source contain principal level (M). Their ranks are at most $2z \le 2r \le n_m$, so the lattice count in the proof of Lemma 7.2 gives $\log h_1 + \log h_{a_i} = O(x^{52})$. The translated-vacuum bound in the proof of Lemma 7.7 applies in a_i coordinates with constant polynomial 1. At either complex embedding the transport of the $V_0^{a_i}$-scaled vacuum gives a scalar of absolute value at most $\exp(C_{\mathrm{int}} q_1^2 a_i)$ multiplying the ordinary unit vacuum. The Gaussian bound for the latter is $\exp((2d+6)(q_1 a_i + 1)^2)$. Including the degree-zero integral and at most $M^{f a_i}$ residue characters in the untwisted column expansion gives at both σ and $c\sigma c$ the kernel bound

$$U_i = M^{f a_i} \exp(C_P (q_1 a_i + 1)^2 + f a_i \log 4).$$

Each source sum has at most h_1 terms, and each target class has weight at most one. The coefficient representation in degree zero is one-dimensional, so the conjugated class pairing gives

$$|\sigma A_i(\chi)| \le C_G d_i^2 \, h_{a_i} (h_1 U_i)^2, \qquad \log(C_G d_i^2) \le f\big((rt)^2 + 4a_i\big) \log M.$$

Here $f, q_1 = O(x^{11} \log x)$, $r, t = O(x^5)$, and $\log M = O(x^\epsilon)$. Consequently $\log|\sigma A_i(\chi)| \le x^{54}$ at the fixed old level, uniformly in the tame character. The defining product for L_{geom} and the bound for t_0 now give

$$\log|\sigma L_{\mathrm{geom}}| \le 2x^{54}, \qquad \log|\sigma c_0| \le fmb/k + 3x^{54} \le J_*.$$

Indeed $(fmb/k)/J_* = m^2/(x(\log b)^3) \to 0$ and $x^{54}/J_* \to 0$.

For a fixed twist η of exact support Q prime to ℓ, the construction can be applied twice to the same η at κ and $\bar{\kappa}$; only the local exponent, cell, Fourier character and test may change. The fixed multiplier and the main line stay the same.

37.2.5 Untwisted norms at both conjugate embeddings

In this subsection, we control the untwisted norms in the finite comparison of Proposition 37.2.

Keep the half-rank $m < z \le r$ of the fixed main pair, put $a = 2z$, and take its original old level with $Q = \varnothing$. In (NB5)–(NB6) take $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$; then both σ and $\sigma^{\#} = c\sigma c$ preserve the chosen CM type. For the untwisted datum X write $J_X = \sum_\tau \lambda_{X,\tau}$ and $A_X = s_{\mathrm{jet}}^{J_X} X^{\mathrm{raw}}$, with X^{raw} the source class sum. The CM transport of Section 37.1.3, evaluated at the same target point h, gives

$$\sigma(A_X(h)_i) = t_\sigma^{J_X} c_\sigma b_Y(u_\sigma^{-1}) d_\sigma(h')\, Y^{\mathrm{raw}}(h)_{i^\sigma}. \tag{NB5}$$

Here b_Y is the rank-twisted character of the conjugate datum, i^σ permutes the real-place indices, and $d_\sigma(h') = \sigma(\Delta(h'))/\Delta(h')$ for the target Hom action h'. The scale satisfies $|t_\sigma|^2 = \Omega^{-1}$. The vacuum scalar $c_\sigma \neq 0$ depends only on σ, the rank and the original level.

$\sigma^{\#} = c\sigma c$ gives the same datum Y. Its permutation agrees with that of σ since $E/\mathbb{Q}$ is abelian. To check the finite character, write any finite norm-one idele as $y/\bar{y}$ by local Hilbert 90, taking unit preimages at almost all places. Choose a finite ray modulus so that its ray units map into the kernel of the finite character. A power y^h is a principal idele a times a ray unit. Automorphy expresses the corresponding h-th power of the finite character value as a product of integer powers of the embeddings of $a/\bar{a}$. All conjugates of these numbers have absolute value one. Thus for each finite character value v, $\sigma^{\#}(v) = \overline{\sigma(v^{-1})} = \sigma(v)$. Permuting its infinity type and conjugating its finite part therefore gives one automorphic character, on the same level, for both complex automorphisms.

In each Gram term, $D_{h'} = \Delta(h')\overline{\Delta(h')}$ is a positive integral lattice index. Hence

$$d_\sigma(h')\overline{d_{\sigma^{\#}}(h')} = \frac{\sigma(D_{h'})}{D_{h'}} = 1.$$

The class weights and Gram matrix are rational and the place permutation reindexes the basis. Using (NB5) in the two legs gives the equality

$$|\sigma\langle A_X, A_X\rangle| = |c_\sigma c_{\sigma^{\#}}|\, \Omega^{-J_X} \|Y^{\mathrm{raw}}\|^2. \tag{NB6}$$

The two character values in (NB5) have modulus one.

Apply the proof of Proposition 30.1 at the untwisted level to the raw kernel. For the transported datum Y, let $\lambda_{Y,\tau} = c_{Y,\tau} - z$ denote its local degree and put $W_Y = \bigotimes_\tau \mathrm{Sym}^{\lambda_{Y,\tau}}(\mathbb{C}^a)$. Use a highest covector of norm one for the compact invariant Hermitian form. The local monomial has Bargmann square norm

$$\pi^{-a} \int_{\mathbb{C}^a} |Z_1|^{2\lambda} e^{-\|Z\|^2}\, dZ = \lambda!.$$

For a unit highest covector, compact Schur orthogonality gives the scalar coefficient's square integral as the squared Weyl norm divided by $\dim W_Y$. Write $v_j = \mathrm{vol}([U(j)])$ in the original

measures of Section 30.1 and $S^0 = S_M \cup \bar{S}_M$. Then

$$C_{a,S_M} = \frac{v_a v_1}{\zeta_{F_0}(2z)} \prod_{w \in S_M} (q_w - 1),$$

$$\|Y^{\mathrm{raw}}\|^2 = C_{a,S_M} \dim W_Y \left(\prod_{\tau} \lambda_{Y,\tau}! \right) L^{S^0}(z, (\nu_Y^{\mathrm{bc}})^{-1}).$$

Indeed, for $q = \mathrm{N}v$ at a split pair $w, \bar{w}$ off S^0, put $A_w = \nu_Y^{\mathrm{bc}}(\varpi_w)$. The two Euler factors of the base change multiply to $|1 - A_w q^{-z}|^{-2}$, with $|A_w| = 1$. At an inert place the base change is trivial on the uniformizer of $F_{0,v}$, and its factor $(1-q^{-2z})^{-1}$ cancels the local zeta denominator. All finite nonsplit places are inert by Section 4.2. At an old split pair, even for a ramified tame character, the character is trivial on the principal-unit stabilizer. The local mass, the target index, the square of the source index and the removed zeta factor contribute together

$$\frac{q^{-a}}{q-1}(q^a - 1)(q-1)^2(1-q^{-a})^{-1} = q - 1.$$

Thus the partial Euler product is positive, and its absolute logarithm is bounded by $[E : \mathbb{Q}] \log \zeta(2) < 4f$. The vacuum and jet factors satisfy

$$\|\phi_X\|^2 = |V_0^a|^2 M^{J_X} \langle A_X, A_X \rangle, \qquad |\pi_M|^2 = M \in \mathbb{Q}_{>0}.$$

In particular the common term at σ is

$$C_{\sigma,a,S_M} = \log\Big(\Big| \sigma\Big(|V_0^a|^2 \Big) \Big| \; |c_\sigma c_{\sigma\#}| \, C_{a,S_M} \Big).$$

Equation (NB6) and the permutation of the local degrees give

$$\log |\sigma \|\phi_X\|^2| = C_{\sigma,a,S_M} + r_X + \sum_{\tau} (\log(\lambda_{X,\tau}!) + \lambda_{X,\tau} \log(M/\Omega)),$$

where $r_X = \log \dim W_Y + \log L^{S^0}(z, (\nu_Y^{\mathrm{bc}})^{-1})$. Thus $r_X \geq -4f$, and $r_X \leq 4f$ for the empty partition, when $\dim W_Y = 1$.

Take the fixed empty partition $q_{\mathrm{ref}} = (v_1^{-z}, z)$ of the same rank, with trivial old unit characters and infinity weight $-z$ at each place. Its rank-twisted character is $\nu_{\mathrm{ref}} v_1^z = 1$. The source still has its old principal level and the target still has the vector fixer $J_\varnothing$. Since $J_{\mathrm{ref}} = 0$ and $\dim W_{\mathrm{ref}} = 1$, the norm on these classes is

$$\|\phi_{\mathrm{ref}}\|^2 = |V_0^a|^2 C_{a,S_M} L^{S^0}(z, (\nu_{\mathrm{ref}}^{\mathrm{bc}})^{-1}) > 0.$$

Here $\nu_{\mathrm{ref}} = v_1^{-z}$; the same positive Euler product formula with the main character gives $\|\phi_X\|^2 > 0$. By Section 44.4 at $Q = \varnothing$, the values of $d_{\mathrm{ref}} \phi_{\mathrm{ref}}$, with $d_{\mathrm{ref}} = M^{4fz}$, are integral at every coefficient embedding above each ℓ. In the class pairing, use the two embeddings κ and $\bar{\kappa}$ for the conjugate factors. The Weyl Gram matrix is integral and every stabilizer order divides C_G, by Section 30.1. Hence

$$I_{\mathrm{ref}} = C_G d_{\mathrm{ref}}^2 \|\phi_{\mathrm{ref}}\|^2 \neq 0, \qquad v_\kappa(I_{\mathrm{ref}}) \geq 0$$

for every κ. The fixed multiplier $H = I_{\mathrm{ref}} / \|\phi_X\|^2$ therefore clears the untwisted norm at every prime. Subtract the two conjugate logarithms. The common constant cancels and the main part sizes are at least κ_τ. Monotonicity of $J(n) = \log(n!) + n \log(M/\Omega)$ gives

$$\log |\sigma H| \leq \log(C_G d_{\mathrm{ref}}^2) + 8f - \sum_{\tau} J(\kappa_\tau) \leq J_* - \sum_{\tau} J(\kappa_\tau).$$

For the last inequality, $\log(C_G d_{\mathrm{ref}}^2) \le f((rt)^2 + 8z)\log M$. Here $z \le r$, $\log M \le 22\log b$, and $J_* = fxb(\log b)^3/(mk)$ dominates this expression and $8f$ for fixed d, ϵ and all sufficiently large x. If σ acts nontrivially on K, then $c\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and $|\sigma H| = |(c\sigma)H|$. Applying the preceding bound with $c\sigma$ gives it for every complex automorphism on this same old level. This is the fixed multiplier estimate for the untwisted term of the reciprocal height.

38 The common twist and bounds on critical values

In this section, we apply the Selmer and trace estimates of Part IV to select one twist per coefficient place. The two measured norm tests of Section 30 then bound the critical values for all the prescribed pairs and both signs.

Fix the admissible datum. For a coefficient place $\kappa = (\ell, \iota)$ put $\bar\kappa = (\ell, \iota\circ\mathrm{conj})$ and normalize $v_\kappa(\ell) = 1$. Write S_M for the old chosen places and $A_{\mathrm{aut}} = \prod_{w\in S_M} |\mathrm{GL}_{rt}(k_w)|$. For a ramified critical pair $p = (\nu, z)$, sign $\epsilon = \pm 1$, and twist η of exact support Q, put $q_Q = \mathrm{N}Q$ and

$$R_{p,\epsilon}(\eta) = \frac{|L^{S_M\cup Q}(z, ((\nu\eta^\epsilon)^{\mathrm{bc}})^{-1})|}{|L^{S_M}(z, (\nu^{\mathrm{bc}})^{-1})|}.$$

The superscript omits the conjugates of the indicated places as well. Let β be either inverse base-change character in the ratio, and omit a conjugation-stable set S' containing its ramified places. For real $s > 1$, the factors at a split pair and an inert place above a place of norm q are respectively

$$|1 - \beta(\varpi_w)q^{-s}|^{-2}, \qquad (1 - q^{-2s})^{-1}.$$

At the inert place a uniformizer from F_0 has trivial base-change character. Thus the absolutely convergent product is positive. At the endpoint $z = 1$ we have $c_v \ge 1$ at every infinite place by Section 2.9. In the complex coordinate of the CM type, the finite-order twist leaves the restriction of β to the unit circle equal to $u \mapsto u^{2c_v}$. Both β and β^2 are therefore nontrivial on the ideles of module one, so their Hecke functions are holomorphic at one [71, §§4.4–4.5]. For real $s > 1$,

$$\zeta_E^{S'}(s)^3 |L^{S'}(s, \beta)|^4 |L^{S'}(s, \beta^2)| \ge 1$$

by the logarithmic Euler-product argument of Section 30.2.1. A zero at one would make the left side tend to zero, since ζ_E has a simple pole. Removing finitely many Euler factors preserves nonvanishing at every $z \ge 1$. The common twist chosen below controls the valuations of these ratios.

38.1 Signed systems for the critical pairs

In this subsection, we form the systems for the common torsion choice of Proposition 28.2.

Write each of the two marked coefficient formulas as $F_{p,Q}(\eta) = \sum_i c_i(Q)\eta(u_i)$. At $\ell \nmid q_Q$ the coefficients are integral at every coefficient embedding. Use the signed array

$$\Theta_{p,\epsilon,Q} = \sum_i \iota(c_i(Q))[u_i^{-\epsilon}]_Q.$$

For the inverse avatar $\psi = \eta_\ell^{-1}$, evaluate a norm-one class $[u]_Q$ on an Artin preimage of $u = y/\bar y$. The norm-one quotient law makes the value independent of its preimage and gives $\iota(\eta(u)^{-1})$.

Therefore $\psi(\Theta_{p,\epsilon,Q}) = \iota(F_{p,Q}(\eta^{\epsilon}))$. The sign is in the array, so each sign is evaluated with the same ψ.

The unit-valued character in the Euler norm relation is the avatar ϑ_{ϵ} of $(\nu^{-\epsilon})^{\mathrm{bc}}$. Put $A_w = \iota((\nu\eta^{\epsilon})^{\mathrm{bc}}(\varpi_w))$. In the geometric Artin convention we have

$$(\vartheta_{\epsilon}\psi)(\mathrm{Fr}_{w,\mathrm{geom}}) = A_w^{-\epsilon}, \qquad (\vartheta_{\epsilon}\psi)(\mathrm{Fr}_{w,\mathrm{ar}}) = A_w^{\epsilon}.$$

Indeed $\nu^{-\epsilon}\eta^{-1} = (\nu\eta^{\epsilon})^{-\epsilon}$, and the two Frobenius elements are inverses. Each Euler-factor pair is invariant under $A_w \mapsto A_w^{-1}$. On an unramified enlargement of the nominal modulus the rank-one system deletes the pair at exponent z, and the rank-two system deletes the pairs at exponents $0, 1$. The rank-one normalizing power of q_Q cancels the local mass at each new place even when the residue character is trivial. In rank two the local multiplier at $0, 1$ is (89), with the same fixed jet scalar for the two levels. These are multiplication identities even if a deleted factor vanishes.

For clarity, let $\mathcal{C}_Q$ be the finite norm-one class group with maximal integral units off Q and principal units at Q. The old level is part of the fixed norm test and of ν. Write $P : \mathcal{C}_{Q\cup\{w\}} \to \mathcal{C}_Q$ for the quotient and let b_w be the class of $y_w/\bar{y}_w$ where y_w is a uniformizer at the new split place $w \nmid \ell M$ and is one elsewhere. Put $a_w = \iota(\nu^{\mathrm{bc}}(y_w))$. The ℓ-adic components of y_w are one, so its geometric Artin image has this value under the avatar of ν^{bc}. The continuous avatar has compact image in the multiplicative group of a finite coefficient field, so its valuation is zero. Away from ℓ and the old conductor its restriction to local units is trivial, since the algebraic correction defining the avatar uses only the components at ℓ. Thus it is unramified at all auxiliary places, as required in Section 27. The value at $\bar{w}$ is a_w^{-1}, since base change is evaluated on $y/\bar{y}$. Put $\mathcal{E}_p = \{z\}$ for $z \geq 2$, $\mathcal{E}_p = \{0, 1\}$ for $z = 1$. Then

$$P_*\Theta_{p,\epsilon,Q\cup\{w\}} = \Theta_{p,\epsilon,Q} \prod_{e\in\mathcal{E}_p} (1 - q_w^{-e}a_w^{-\epsilon}[b_w])(1 - q_w^{-e}a_w^{\epsilon}[b_w]^{-1}).$$

Evaluating at η^{-1} gives the multiplication identity above with $A_w = a_w\iota(\eta^{\epsilon}(b_w))$. The kernel of P is the image of the local units at the new split place. If a power of a character of $\mathcal{C}_{Q\cup\{w\}}$ is trivial there, apply this identity to the power descended to $\mathcal{C}_Q$. Every character of $\mathcal{C}_Q$ has trivial residue part at w, and these characters separate its group algebra in characteristic zero. Indeed, take a finite extension K_ℓ containing the displayed coefficients and all character values of $\mathcal{C}_Q$. If $D = \sum_c d_c[c]$ is the difference of the two sides, Fourier inversion in K_ℓ gives

$$|\mathcal{C}_Q|d_c = \sum_{\chi\in\widehat{\mathcal{C}_Q}} \chi(c)^{-1}\chi(D) = 0.$$

Thus each $d_c = 0$ in K_ℓ. The identity holds in its valuation ring coefficient by coefficient, including when $\ell \mid |\mathcal{C}_Q|$, since q_w, a_w are units and the arrays are integral. It supplies the norm relation of Section 27.

For $Q = J$ and the auxiliary characters χ_w of that section, choose finite idele representatives y_i with $u_i = y_i/\bar{y}_i$ in $\mathcal{C}_Q$. With N the auxiliary exponent, put $n_{iw} = [-\epsilon\chi_w(\mathrm{Art}\, y_i)]_{\ell^N}$, the representative between 0 and $\ell^N - 1$. The integral polynomial

$$P_{p,\epsilon,Q}(X) = \sum_i \iota(c_i(Q)) \prod_{w\in Q} (1 + X_w)^{n_{iw}}$$

has value $\iota(F_{p,Q}(\eta^{j\epsilon}))$ at $X_w = \zeta_w^j - 1$ whenever $\zeta_w^{\ell^N} = 1$ for each w and $\prod_{w\in Q}\zeta_w^{\chi_w}$ is the inverse avatar of η. The signs therefore use one torsion tuple.

For exact support the two marked norm formulas give

$$\frac{F_{p,Q}(\eta^{\epsilon})}{F_{p,\varnothing}(1)} = \begin{cases} R_{p,\epsilon}(\eta), & z \geq 2, \\ q_Q R_{p,\epsilon}(\eta)^2, & z = 1. \end{cases}$$

At $Q = \varnothing$ the two values are

$$F_{p,\varnothing}(1) = \begin{cases} C_G d_p^2 \|\Phi_1\|_{J_\varnothing}^2, & z \geq 2, \\ |C_G s_p \Lambda_\varnothing(1)|^2, & z = 1. \end{cases}$$

The first uses the old vector fixer $J_\varnothing$; the second uses the determinant test modulo $(\pi_M)^2$ and the full standard level K_2 of (86). Positivity follows from (85) in the first case and from the functional equation and nonvanishing at one in Section 30.2.1 in the second. Write $a_p = F_{p,\varnothing}(1) > 0$ and let $\overline{\mathbb{Z}}$ be the ring of algebraic integers. The two integral expansions give

$$c_i(Q) \in \overline{\mathbb{Z}}[1/q_Q], \qquad a_p \in \overline{\mathbb{Z}} \setminus \{0\}.$$

Indeed, fix $\ell \nmid q_Q$. If $c_i(Q)$ were transcendental, an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$ could send it to u/ℓ, for a transcendental unit $u \in \mathbb{Z}_\ell^\times$, by extending transcendence bases. This contradicts integrality at every embedding above ℓ. For an algebraic coefficient, integrality at every prime away from q_Q gives the displayed localization. At $Q = \varnothing$ one has $q_Q = 1$. In particular a_p is fixed before the auxiliary places and may have positive valuation at the coefficient prime.

38.2 Original Hecke systems and their integral models

In this subsection, we define the lift systems corresponding to the arrays of Section 38.1.

Fix the coefficient embedding above ℓ. Let $\mathcal{A}$ be the set of pairs (j, b), where $1 \leq j \leq m$ and b is an outside-level Hecke system in the good space on W_m. Let $\mathcal{B}$ be the set of (j, p), where $p = (\nu, z)$ is main for j, so that

$$m < z \leq r, \qquad \nu_{\mathrm{tame}} = \chi_M^j, \qquad \mathrm{wt}_v(\nu) = -c_v, \qquad w_j(v) + r - t \leq c_v \leq w_j(v).$$

The spaces of principal-level forms are finite dimensional. For a fixed infinity type and fixed finite level the characters in $\mathcal{B}$ differ by characters of a finite norm-one class quotient. The bounds on c_v, z and j therefore make $\mathcal{B}$ finite.

For $(j, b) \in \mathcal{A}$, take a nonzero form of system b in the image of a good central type idempotent on W_m. This is possible because the sum of the good idempotents fixes the good space, and each idempotent commutes with the off-level Hecke operators. The residue action on its algebraic form is the one obtained by reducing at the prescribed level, since the principal kernel fixes the form and the algebraic conversion commutes with right translation.

The attached lower representation R_b has the local type parameters of the chosen constituent on W_m. Set

$$T_{j,b} = \theta_j R_b, \qquad \theta_j = \mathrm{av}_\ell(u \longmapsto \mu_j^{-1}(u/\bar{u})|u|^{(r-t)/2}),$$

where μ_j is the original descent character of lift j. The local–global comparison gives the scaled lower parameters of $T_{j,b}$. On each nonzero lift, Section 28.1 identifies the normalized upper trace, with the string χ_{cyc}^{-a}, $0 \leq a < t$. The normalization changes the raw polarization exponent $1 - r$ to $1 - t$, since $\theta_j \theta_j^c = \chi_{\mathrm{cyc}}^{r-t}$. Its jumps are (H1).

For $(j,p)\in\mathcal{B}$, use the constant system on the original norm-one line. The raw attachment is the trivial character, with jump 0 and parameter $X-1$ at every distinguished place. Put

$$t_p=2z-1,\qquad T_{j,p}=\mathrm{av}_\ell(u\longmapsto\nu^{-1}(u/\bar{u})|u|^{\,1-z}).$$

This is the same normalization with lower rank 1 and upper rank $2z$. Its polarization exponent is $1-t_p$ and its single jump at σ is $z-1-\varepsilon_\sigma c_v$. In particular

$$|\varepsilon_\sigma(z-1-\varepsilon_\sigma c_v)+w_j(v)|\le t.$$

Indeed $0\le w_j(v)-c_v\le t-r$ and $z-1<r$. As $w_j(v)>2(r^8)^2+t$, there are no small jumps, and the two conjugate jumps add to $2z-2$. For every allowed twist the lift of the constant line form is nonzero: its squared norm in Proposition 30.1 is a positive scalar times an absolutely convergent, nonvanishing unitary Euler product at $z>m\ge1$. Section 28.1 therefore gives the string of length t_p for its upper system.

The unscaled parameter $X-1$ is the trivial character of each $\mathrm{GL}_1(k_w)$. It also fixes the upper lift in the normalized residue action. For a tuple g of residue scalars choose the indicated lower integral lift and the upper integral lift of its first block. Their simultaneous Weil action on the marked column multiplies it by $\prod_{w\in S_M}\nu_w(\det g_w)^{-1}$. In the marked split matrix coordinates, if i_0 is the original block inclusion, the two lifts act on the support by $X\mapsto k'Xk^{-1}$, up to the fixed transpose convention.

Its reduction fixes i_0, since $\mathrm{diag}(g_w,1)i_0=i_0g_w$. The integral matrix lattice is preserved and both determinants are units. The two splitting-character factors have opposite rank powers of the splitting character and leave $\nu_w(\det g_w)^{-1}$ at w. The same identity holds after the auxiliary twist, whose tame part is trivial on S_M. Changing the source variable in the theta integral shows that the upper translation multiplied by $\prod_{w\in S_M}\nu_w(\det g_w)$ acts on the lift as the lower translation acts on the source. The constant vector is fixed. Hence the mean over the $\mathrm{GL}_1(k_w)$ fixes the line, which is the required $X-1$ idempotent. Its normalized inertial root instead names $(\chi^{j}_{M,w})^{-1}$ in the original parameter convention. It is nontrivial since $1\le j\le m<q_2$.

At rank r, the same marked residue equivariance on the vector column gives the corresponding assertion for every good type component of a lower eigenform on W_j. Decompose the good space on W_j into simultaneous eigenspaces for the split spherical Hecke operators at every place off the level and its conjugate, and for the finite central type projectors. These operators commute and are normal. For any nonzero piece, its original system occurs on W_m: the common Hecke test of Section 9.4, projected to this tuple and this system, has the same trace on W_j and W_m.

The hypotheses of that trace argument hold for both source spaces, since the same two primitive finite marks and the prescribed common regular real parameter occur there. For every nonzero vector y in such a piece, Proposition 34.1 on the original single-lift space gives, at the twist $\eta=\xi^j$ used below,

$$\|u_{j,\eta}y\|^2_{\mathrm{class}}\ge\tfrac12A_jc_Q(\xi)\|y\|^2>0.$$

The residue equivariance preserves its specified upper type. Fix the original choice of R_b on the definite space W_m. For each resulting lifted component of system b and type c, the split polynomial calculation of Section 28.1.1 gives, for its normalized upper attachment $\widetilde{R}_{\mathrm{lift}}$,

$$\operatorname{tr}\widetilde{R}_{\mathrm{lift}}(g)=\eta_\ell^{-1}(g)\operatorname{tr}T_{j,b}(g)+\sum_{a=0}^{t-1}\chi_{\mathrm{cyc}}(g)^{-a}\qquad(g\in G_E).$$

Here η_ℓ^{-1} is the avatar of the inverse auxiliary character. The formula follows at the cofinite split Frobenius classes from the eigenvalues of b and extends by density and continuity. The component retains its own type projector, by the residue equivariance above. Thus the same $T_{j,b}$ supplies the trace comparison for every such component and every allowed twist. The lifts of the pieces span the original good lift image.

They lie in the image of their prescribed upper type projectors, and their split Hecke polynomials are the lift polynomials of the corresponding systems in $\mathcal{A}$. Therefore the finite list supplies a main summand for every generator used in clearing on the original lift image.

Integral models and the full matrix span. For the original systems take $K_\ell/\mathbf{Q}_\ell$ to be the normal closure of a finite field containing all their matrix entries, the prescribed lattices, scalar values and marked eigenvalues. For completeness a continuous representation of a profinite group in $\mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ has its image defined over a finite extension. Exhaust $\overline{\mathbf{Q}}_\ell$ by an increasing countable sequence of finite extensions, using finiteness of the extensions of each fixed degree of $\mathbf{Q}_\ell$. The inverse images of the corresponding matrix groups are closed subgroups. By Baire one has interior and hence finite index. Adjoin the matrix entries of finitely many coset representatives to its coefficient field. If ρ is the representation and L_0 is an initial O_{K_ℓ}-lattice, compactness gives an integer $a \geq 0$ such that

$$L_0 \subseteq \sum_g \rho(g) L_0 \subseteq \ell^{-a} L_0.$$

The middle module is invariant and finite free over the DVR. It gives a model over every larger coefficient ring by scalar extension.

Let L be any such invariant lattice over a ring O containing the relevant inertia eigenvalues. At rank r choose a primitive place above an old prime distinct from ℓ. The preceding primitive-parameter calculation gives r mutually incongruent eigenvalues for the specified tame generator s. Their Lagrange idempotents belong to the O-span of the group image and split L into free lines of rank one.

An arithmetic Frobenius ϕ at this place permutes the lines in one cycle and has unit entries between the lines in integral bases, since its inverse preserves the lattice. Thus the products of the idempotents and powers of ϕ give all matrix units up to units. The span is an algebra and is $\mathrm{End}_O(L)$. The operator ϕ^r acts on every line by the same product of the r unit multipliers, with cyclotomic value p_i^r.

At rank one the full span follows from the identity and the same Frobenius is itself scalar, with cyclotomic value p_i. After reduction over any extension of the residue field, the matrix units still span. Consequently every lattice in either rank has absolutely irreducible reduction. A scalar unit-valued twist preserves the span, since each matrix is multiplied by a unit.

The old type calculation bounds the inertial determinants at every original place prime to ℓ, with the same chosen tame generator at a distinguished place and its conjugate inverse. In each summand one may choose the scalar inertia element supplied by that calculation, with scalar ζ satisfying $\zeta^{h\ell^a} - 1 \in O^\times$ for $a \geq 0$, $1 \leq h \leq 4\,\mathrm{rank}(L) + 2$. This element is chosen for the summand and need not be the common reference tame generator g_0. The twists are unramified at the scalar inertia place, so on the twisted lattice the same element acts by ζ. The unit $\zeta - 1$ gives the normalization $z(\gamma) = 0$. Adding the coboundary of $-(\zeta-1)^{-1} z(\gamma)$ imposes this condition uniquely.

Use one common g_0 and torsion cutoff for $\mathcal{A} \amalg \mathcal{B}$. The joint choice applies every power of the selected character to *every* original summand in this union. The ramified critical pairs

and their signs give the separate norm tests. The character is trivial on the full decomposition groups above ℓ, so the selected local lattice is the scalar extension of the original one there.

The finite-depth bound passes to divisible coefficients. Let L_0 be the selected twisted lattice over O_0, and let $O \supset O_0$ be a finite coefficient DVR. Write K_0, K_O for the fraction fields. After the torsion tuple and the raised systems have been chosen, take for the joint trace and crystalline calculations the common normal closure specified in Section 31.3. For every finite coefficient extension, the scalar-extended lattice $L_O = L_0 \otimes_{O_0} O$ satisfies

$$(K_O/O) \otimes_O L_O = \big((K_0/O_0) \otimes_{O_0} L_0\big) \otimes_{O_0} O.$$

The coordinate projections for an O_0-basis of O commute with Galois and preserve the scalar normalization and the inertia restrictions. At $v \mid \ell$ they take a crystalline cocycle and its period-ring primitive to ones over the original coefficient field. Thus the old linear relations extend to relations on the enlarged crystalline module, by the argument in Section 28.4. Each coordinate of a cocycle satisfying the enlarged Selmer conditions lies in the original module $\mathcal{S}_0$. For the enlarged module $\mathcal{S}_O$, this gives

$$\mathcal{S}_O \subseteq \mathcal{S}_0 \otimes_{O_0} O, \qquad \frac{\operatorname{length}_O \mathcal{S}_O}{v_O(\ell)} \le \frac{\operatorname{length}_{O_0} \mathcal{S}_0}{v_{O_0}(\ell)}.$$

The last inequality follows because the ramification index multiplies both lengths after scalar extension and the uniformizer valuation of ℓ.

38.3 The common specialization and its valuations

In this subsection, we use Proposition 28.2 for the signed systems of Section 38.1.

The norm tests include each ramified critical pair with both signs. Let $\mathcal{P}_{\mathrm{ram}}$ denote these pairs. Their critical integers and infinity weights lie in finite sets, and characters of any one weight differ through the fixed finite norm-one class quotient at the old level. For $\mathcal{A}, \mathcal{B}$ as in Section 38.2, the number of polynomial evaluations, counted with multiplicity over the powers $1 \le j \le m$, is

$$m\big(|\mathcal{A}| + |\mathcal{B}| + 2|\mathcal{P}_{\mathrm{ram}}|\big).$$

Choose an old split place $w_* \in S_M$ away from ℓ; one exists since the two rational level primes are distinct. Use the common tame generator g_0 from Section 28.3.1 at w_*. With $q = \mathrm{N}w_*$, choose D divisible by the degrees of all irreducible factors of the original type polynomials there, so that $\omega_D(g_0)$ generates $\mathbb{F}_{q^D}^{\times}$. The determinant bounds of Section 28.4 then hold at both g_0 and $(g_0^c)^{-1}$ for every original summand. The norm of $\omega_D(g_0)$ generates $\mathbb{F}_q^{\times}$ and equals $\omega_1(g_0)$. For each ramified pair the residue character there is nontrivial, so either signed avatar takes g_0 to a root $\zeta \ne 1$ of order $h \mid q_{w_*} - 1$. The cyclotomic character is one on this inertia group. Thus the untwisted value of each of the two Euler factors at exponent e has valuation $v_\kappa(1 - \zeta^{\pm 1}) = v_\kappa(1 - \zeta)$. The identity

$$h = (1 - \zeta) \sum_{i=0}^{h-1} (h - 1 - i)\zeta^i$$

gives $v_\kappa(\zeta - 1) \le v_\ell(q_{w_*} - 1)$. Thus the first-prime bound is the augmentation valuation plus at most $4v_\ell(q_{w_*} - 1)$, as there are at most two Euler exponents.

Fix the norm systems, the coefficient ring for the original Selmer lattices and avatar congruences, the Selmer truncation, and the neighborhood of g_0 before choosing the auxiliary exponent. The uniform bound on the number of auxiliary places permits this exponent to be chosen before the actual auxiliary primes. After selecting that finite prime set, the polynomials for specialization have integral coefficients in $\overline{\mathbb{Q}}_\ell$. They and the finitely many test points lie in a finite coefficient extension. In Proposition 28.2 the order bound ℓ^H precedes the auxiliary character level $\ell^{N_{\mathrm{tor}}}$, and the set $Q = J$ of auxiliary places precedes the polynomials formed from $F_{p,Q}$. Lemma 26.1, applied in that proposition, selects a single torsion tuple at tolerance $2^{-\ell}$ for both signs and all powers $1 \le j \le m$. For the triangle and turn factors of Section 38.6, the auxiliary powers are respectively $j - i$ and $j + 1$. Their old tame powers are $j - i$ and $i + j + 2$, between 1 and $2m < q_2$, and their weight bounds include the required critical points. The same list contains $p_{\mathrm{aux}} = (\chi_M \upsilon_1^{-2}, 2)$ from Section 37.2.2. Its old unit character at each $w \in S_M$ is the character of exact order q_2 from Section 4.3. Its infinity weights are -2, and its new twisting power is the same j as for the main pair.

In choosing the auxiliary primes, exclude all places above $\ell p_0 p_1 N$, with p_0, p_1, N the three rational level primes fixed in Section 4. The resulting degree-one support Q is disjoint from its conjugate and avoids this set. The norm-one quotient law identifies the selected product of characters with the inverse avatar of one torus character ξ_κ. Its powers $1, \dots, m$ have support exactly Q, since the grid excludes every trivial tame component at all these powers.

Write $u_\ell = v_\ell(A_{\mathrm{aut}})$. Proposition 28.2 gives the following bounds in the trace comparison for each selected power $\eta = \xi_\kappa^j$, with ψ the inverse avatar of ξ_κ:

$$\begin{aligned} \mathrm{sel} &\le (4f + 3) r u_\ell + 2^{-\ell} = e_\ell, \\ \mathrm{ram} &= \sum_{w \in Q} v_\ell(\psi^j(\gamma_w) - 1) \le |Q| a_0 \le 2^{-\ell - 1}. \end{aligned}$$

The first line bounds the normalized Selmer lengths with divisible coefficients, since each main summand has rank at most r. The second sum uses one member of each auxiliary pair. Since ψ factors through $y \mapsto y/\bar{y}$, we have $\psi(cgc^{-1}) = \psi(g)^{-1}$ for complex conjugation c. Thus at the conjugate tame generator the valuation is the same, by $v_\ell(u^{-1} - 1) = v_\ell(u - 1)$ for a unit u. At every $v \mid \ell$ the twisting character restricts trivially to G_{E_v}, so Section 29.1 applies to the original local lattice after coefficient extension. At $w \in Q \cup \bar{Q}$ the residue characteristic differs from ℓ, so the contribution above comes from the tame character. The norm tests independently give

$$v_\kappa F_{p,Q}((\xi_\kappa^j)^\epsilon) \le v_\kappa F_{p,\varnothing}(1) + 4 v_\ell(q_{w_*} - 1) + 2^{-\ell}.$$

The right side is finite because $a_p \neq 0$, so $F_{p,Q}((\xi_\kappa^j)^\epsilon) \neq 0$. For $\eta = \xi_\kappa^j$ the exact-support identity of Section 38.1 gives

$$R_{p,\epsilon}(\eta)^{1 + \mathbf{1}_{z=1}} = q_Q^{-\mathbf{1}_{z=1}} \frac{F_{p,Q}(\eta^\epsilon)}{F_{p,\varnothing}(1)} \in \overline{\mathbb{Q}}^\times.$$

Thus the positive real value $R_{p,\epsilon}(\eta)$ is algebraic. Since $v_\kappa(q_Q) = 0$, the exact-support formula gives

$$\begin{aligned} (1 + \mathbf{1}_{z=1}) v_\kappa R_{p,\epsilon}(\xi_\kappa^j) &= v_\kappa F_{p,Q}((\xi_\kappa^j)^\epsilon) - v_\kappa F_{p,\varnothing}(1) \\ &\le 4 v_\ell(q_{w_*} - 1) + 2^{-\ell}. \end{aligned}$$

Consequently

$$v_\kappa R_{p,\epsilon}(\xi_\kappa^j) \le e_\ell. \tag{RT1}$$

More precisely, put $d_z = 1+\mathbf{1}_{z=1}$ and $b_\ell = 4v_\ell(q_{w_*}-1)+2^{-\ell}$. At the selected power, integrality of $F_{p,Q}(\eta^\epsilon)$ gives

$$-\frac{v_\kappa(a_p)}{d_z} \le v_\kappa R_{p,\epsilon}(\eta) \le \frac{b_\ell}{d_z}.$$

The same bounds hold at $\bar\kappa$ for this η, since $R_{p,\epsilon}(\eta)$ and a_p are real. In particular, at $z=1$,

$$\max_{\lambda\in\{\kappa,\bar\kappa\}} |R_{p,\epsilon}(\eta)^{-1}|_\lambda \le \ell^{b_\ell/2}, \qquad |x|_\lambda = \ell^{-v_\lambda(x)}.$$

Here $q_{w_*}-1 \mid A_{\mathrm{aut}}$, so $4v_\ell(q_{w_*}-1) \le (4f+3)r\, v_\ell(A_{\mathrm{aut}})$. In particular, at the same $\eta=\xi_\kappa^j$ the second block of Section 37.2.4 satisfies

$$\frac{A_2(\eta)}{A_2(1)} = \frac{F_{p_{\mathrm{aux}},Q}(\eta)}{F_{p_{\mathrm{aux}},\varnothing}(1)} = R_{p_{\mathrm{aux}},+}(\eta).$$

38.4 Integral clearing and projected pairings

In this subsection, we apply Proposition 32.1 to the models of Section 38.2.

The first-order comparison on the selected twisted summands gives the good-image clearing operator and the rank-one projected pairing estimate below. In the notation of its local defect F_ℓ, the odd-class integer X_{odd}, and the original rank-one level class number h_1, and $Y_{\mathrm{tr}} = r^8$ from Section 29.2, put

$$\begin{aligned} C_\ell &= 4fF_\ell + (1+2t)e_\ell + tv_\ell(X_{\mathrm{odd}}) \\ &\quad + (8ft+3+(4t+1)Y_{\mathrm{tr}}^{22}+Y_{\mathrm{tr}})v_\ell(A_{\mathrm{aut}}), \\ B_\ell &= C_\ell + v_\ell(h_1), \qquad J_\ell = B_\ell \log \ell. \end{aligned}$$

The good-image operator clears the image of the orthogonal projection on the integral tests orthogonal to the false lifts, with determinant valuation at most $r\dim(V)C_\ell$. In rank one let Π project to the normalized line $\mathbb{C}u$ for $\eta=\xi_\kappa^j$. The two tests a, a' must be integral at every coefficient embedding over ℓ and perpendicular to the false lifts defined by the kernel of the full descent.

The following pairing and central-average calculation gives the line estimate

$$v_\kappa\langle \Pi a, \Pi a'\rangle \ge -B_\ell. \tag{RT2}$$

Integral pairings after the first-order comparison. Let D_Z be the good-image clearing operator obtained from the first-order coefficients. The original class pairing is the sum of the transported Weyl pairings with denominators the stabilizer orders. The transport and the integral Weyl evaluations use the same original bases. The left test is integral also at the conjugate embedding used for complex conjugation. Each stabilizer order divides C_G by the reduction argument in Section 30.1. Thus

$$v_\kappa\langle a, C_G D_Z \pi_Z a'\rangle \ge 0$$

for the original tests, with the right test perpendicular to the false lifts. The integer multiplication does not affect the integral image property. The valuation per eigenvalue is at most

$$\begin{aligned} &\mathrm{sel} + 2t'\mathrm{ram} + 4fr'F_\ell + \mathrm{err}_\ell + u_\ell \\ &\quad \le r'(4fF_\ell + (1+2t)e_\ell + \mathrm{err}_\ell + u_\ell). \end{aligned}$$

Here $\mathrm{sel},\mathrm{ram} \le e_\ell$ and $1 \le r' \le r$, $t' \le t$. This is the allowance $r'C_\ell$ in the good-image determinant. The C_G for this pairing is in addition to the C_G^2 in the first-order coefficient.

Rank-one averaging on the original line. Let $T = U(1)$, let K_1 be the original chosen line level, and let s_c represent the h_1 classes of $T(F_0)\backslash T(\mathbf{A}_{F_0,f})/K_1$. Take the representatives integral at ℓ. To do this, write the inverse local component of each proposed representative as $a_v/\bar{a}_v$ by Hilbert 90, using a split pair of factors at a split place. Approximate the a_v at the places above ℓ by a single $a \in E^\times$. Replacing the representative by its product with $a/\bar{a}$ puts the local components in the open integral compact.

For the main pair (ν, z) and its twist η, let $\beta = (\nu\eta)\upsilon_1^z$ be the full marked target-leg character. For a norm-one scalar x let $T(x)$ act by right translation by xI_{2z}, multiplied by $\beta(x)$. This operator is unitary, commutes with the type projector and spherical operators, and has adjoint $T(x^{-1})$. For the original line kernel $K(g)$ one has

$$T(x^{-1})K(g) = K(gx).$$

Indeed the simultaneous scalars $(x^{-1}, x^{-1}I_{2z})$ fix the Hom matrix variable. In the Weil action their multiplier is $\beta(x^{-1})^{-1}$, which cancels that of $T(x^{-1})$. The line stabilizers are trivial. Put

$$u_{\mathrm{raw}} = \sum_c K(s_c), \qquad s_\eta = V_0^{2z}(\pi_M s_{\mathrm{jet}})^{\sum_\tau (c_\tau - z)}, \qquad u = \phi_\eta = s_\eta u_{\mathrm{raw}}.$$

The scalar s_η is nonzero. The same identity and translation of classes give $T(x)u_{\mathrm{raw}} = u_{\mathrm{raw}}$ and $T(x)u = u$.

Let $\mathcal{D}$ denote normalized descent to the full line form space. Its value at g is the pairing with $K(g)$. Set $S = \sum_c T(s_c)$. Commutativity of the torus gives

$$\mathcal{D}(Sy)(g) = \left\langle \sum_c T(s_c^{-1})K(g), y \right\rangle = \langle u_{\mathrm{raw}}, y\rangle = \overline{s_\eta}^{-1}\langle u, y\rangle.$$

The equality of the kernel sum uses that gs_c runs over precisely the same line classes.

Apply Proposition 32.1 with $Z = \mathbf{C}u$ and $e_0 = Q = e_{X-1}$. Indeed, u is a nonzero multiple of the lift of the original constant line form, and Section 38.2 gives its Hecke system x_0 and its $X-1$ type. Thus $\pi_Z = \Pi$. At every split off-level place w, the preceding descent identity and the commutation of S with Q and $K_w(X_w)$ give

$$S\big(\operatorname{ran} Q \cap u^\perp \cap \ker K_w(X_w)\big) \subseteq \operatorname{ran} Q \cap \ker \mathcal{D} \cap \ker K_w(X_w).$$

The original test a' is perpendicular to the right-hand side. Hence $S^*a' \perp F_w$ for this line at every such w. Also S^*a' is integral at every embedding above ℓ: each scalar translation by s_c^{-1} preserves the integral local Weyl gauge. For a finite norm-one idele x integral above ℓ, its class in the original quotient is killed by h_1. Write $x^{h_1} = \gamma k$ with $\gamma \in T(F_0)$ and $k \in K_1$. Then γ is integral above ℓ, and

$$\beta_f(x)^{h_1} = \beta_\infty(\gamma)^{-1}\beta_f(k).$$

The first factor is a monomial in conjugates of an ℓ-unit. The second is a root of unity, since β_f is smooth on the compact K_1. Thus $v_\kappa(\beta_f(x)) = 0$ at every embedding above ℓ. Since $T(s_c)u = u$, we have $\Pi S^*a' = h_1\Pi a'$. On this line the original type projector satisfies $Qu = u$. The operator $C_G c^2 sQ$ therefore acts by the nonzero scalar $d_u = C_G c^2 s(x_0)$, with $v_\kappa(d_u) \le C_\ell$ by Proposition 32.1 and $v_\ell(C_G) = u_\ell$. Applying the original class pairing to a and S^*a' gives

$$v_\kappa\big(d_u h_1 \langle \Pi a, \Pi a'\rangle\big) = v_\kappa\langle a, C_G c^2 sQ\Pi S^* a'\rangle \ge 0.$$

This proves (RT2).

For the class-number factor, put $T(\widehat{\mathcal{O}}_E) = \prod_{v<\infty} T(\mathcal{O}_{E_v})$, with the analogous interpretation at split places. The norm-one ideal map from classes at this maximal compact into $\mathrm{Cl}(E)$ has kernel of order at most 2^f. For a principal ideal (a) in the kernel, $a\bar{a}$ is a unit of F_0. Changing a by a unit changes this unit by a norm, and if the resulting unit class is trivial there is a norm-one generator.

Thus the kernel injects into $\mathcal{O}_{F_0}^{\times}/N_{E/F_0}(\mathcal{O}_E^{\times})$. The quotient has size at most 2^f because the norm subgroup contains the squares. The index from this maximal compact to K_1 is at most $(MN)^{2f}$, since K_1 contains the principal level MN, and reduction modulo $MN\mathcal{O}_E$ has at most that many values. Hence

$$h_1 \le 2^f h_E (MN)^{2f}.$$

By the Minkowski bound [46, Thm. 4.3], each class has an integral representative of norm at most $D = (4/\pi)^f (2f)!\sqrt{|\Delta_E|}/(2f)^{2f}$. Thus

$$h_E \le \sum_{\mathrm{N}\mathfrak{a}\le D} 1 \le D^2 \zeta_E(2) \le 2^{2f} |\Delta_E| \zeta(2)^{2f}.$$

All logarithms on the right-hand side of the bound for h_1 are polynomially bounded in x by the parameter bounds in Section 4.

For every finite set U of rational coefficient primes, factorization gives $\sum_{\ell\in U} v_\ell(H)\log\ell \le \log H$ for each positive integer H. Since $C_G = A_{\mathrm{aut}}$ and $\sum_{\ell\in U} 2^{-\ell}\log\ell \le \sum_{n\ge1} n2^{-n} = 2$, the bound in Section 29.1.2 gives

$$\begin{aligned}\sum_{\ell\in U} J_\ell \le{}& 108\cdot 10^6 \frac{fm}{k}\, b\log b + (1+2t)\big((4f+3)r\log A_{\mathrm{aut}} + 2\big)\\ &+ t\log X_{\mathrm{odd}} + \log h_1\\ &+ \big(8ft + 3 + (4t+1)Y_{\mathrm{tr}}^{22} + Y_{\mathrm{tr}}\big)\log A_{\mathrm{aut}}.\end{aligned}$$

Dividing the first term by $J_* = fxb(\log b)^3/(mk)$ gives $108\cdot 10^6 m^2/(x(\log b)^2) \to 0$, since $m^2 \le 4x$. All other terms grow at most polynomially in x, whereas J_* and x_1 exceed every fixed power of x for fixed d, ϵ. Each summand J_ℓ bounds the local contribution at every embedding over the same rational prime.

For every finite set U of coefficient primes,

$$\sum_{\ell\in U} e_\ell \log\ell \le (4f+3)r\log A_{\mathrm{aut}} + 2 \le x_1, \qquad \sum_{\ell\in U} J_\ell \le J_* = \frac{fa(\log b)^3}{mk}. \tag{RT3}$$

Here $\log A_{\mathrm{aut}} \le (rt)^2 f \log M$, since the original set contains one split half above each place over each of the two level primes. These allowances are independent of the coefficient embedding, so one may make the choice of ξ_κ independently at each κ. The first-order comparison is applied at κ on that chosen twist and does not use the twist selected at $\bar{\kappa}$.

38.5 Reciprocal heights of the main critical values

In this subsection, we use the local estimates of Section 38.4 and the numerator test of Proposition 37.2.

Fix j, p with $p = (\nu, z)$ main. At both conjugate embeddings take $\eta = \xi_\kappa^j$ and the normalized lift $u = \phi_\eta$ of Section 37.2. Summing the inverse unit residues in the class formula of Section 30.1 gives this same vector Φ_η, with vacuum and jet scalar $V_0^{2z}(\pi_M s_{\mathrm{jet}})^{\sum_\tau (c_\tau - z)}$. Use $u_1 = \phi_1$ at $Q = \varnothing$ and $\rho = R_{p_{\mathrm{aux}},+}(\eta) > 0$. The marked numerator estimate gives a fixed $c_0 \neq 0$ with $\log|\sigma c_0| \leq J_*$ for every complex automorphism and, for any coprime modulus and any coefficient embedding, an integral test satisfying

$$v_\kappa\langle a, u\rangle \leq v_\kappa(c_0\rho) + P_\ell, \qquad P_\ell = \sum_\tau S_\ell(\kappa_\tau, \tau). \tag{RT4}$$

Here S_ℓ is the local prime sum of Section 7. The test is integral at every coefficient place above this ℓ and has the false-lift orthogonality of (RT2). This is the four-block numerator test on the original rank-one lift.

Apply (RT4) at $\bar\kappa$ to the same η, u and call the resulting test a'. Using that ρ is real and conjugating this inequality gives, for $I = \langle a, u\rangle\langle u, a'\rangle \neq 0$,

$$v_\kappa(I) \leq v_\kappa(c_0\bar c_0) + 2v_\kappa(\rho) + 2P_\ell.$$

Apply (RT2) to these two tests at κ. Proposition 30.1 uses J_Q for u and the old vector fixer $J_\varnothing$ for u_1. At the trivial twist $Q = \varnothing$ and $q_\varnothing = 1$. The vacuum and jet scales agree in both norms. Thus

$$\frac{\|u\|^2}{\|u_1\|^2} = q_Q^{2z-1} R_{p,+}(\eta), \qquad \langle \Pi a, \Pi a'\rangle = \frac{I}{\|u\|^2}.$$

Both norms are nonzero by the positive Euler products. It follows that

$$R_{p,+}(\eta)^{-1} = q_Q^{2z-1}\|u_1\|^2 \frac{\langle \Pi a, \Pi a'\rangle}{I}. \tag{RT5}$$

Here $v_\kappa(\overline{a}) = v_{\bar\kappa}(a)$ for every algebraic scalar a. The numerator test at $\bar\kappa$ applies to this same η, since $\ell \nmid q_Q$. The projection estimate (RT2) is applied at κ with that same η.

For the untwisted term use the integral algebraic norm $I_{\mathrm{ref}} \neq 0$ and the fixed empty reference pair of rank $2z$ from Section 37.2.5. Its multiplier $H = I_{\mathrm{ref}}/\|u_1\|^2$ satisfies

$$H\|u_1\|^2 \text{ integral at every coefficient embedding}, \qquad \log|\sigma H| \leq J_* - \sum_\tau J(\kappa_\tau). \tag{RT6}$$

Use the single fixed multiplier $Hc_0\bar c_0$ in (RT5). Its archimedean allowance is

$$3J_* - \sum_\tau J(\kappa_\tau).$$

Since $v_\kappa(q_Q) = 0$, (RT5) gives

$$\begin{aligned} v_\kappa\Big(Hc_0\bar c_0 R_{p,+}(\eta)^{-1}\Big) &\geq v_\kappa(I_{\mathrm{ref}}) - B_\ell - 2v_\kappa(\rho) - 2P_\ell \\ &\geq -B_\ell - 2e_\ell - 2P_\ell. \end{aligned}$$

The last inequality uses (RT1) at p_{aux} and $v_\kappa(I_{\mathrm{ref}}) \geq 0$. Also

$$2\sum_{\ell\in U} P_\ell \log\ell \leq \sum_\tau S(\kappa_\tau, \tau).$$

Indeed $2S_\ell(\kappa_\tau,\tau)\log\ell$ is the sum of the terms of $S(\kappa_\tau,\tau)$ at $n=\ell^e$. Terms with $\ell^e>a$ vanish, since $\kappa_\tau/\ell^e<1$ whereas every weight is at least one. All terms are nonnegative. Thus the archimedean allowance plus the finite-prime sum is at most

$$3J_*-\sum_\tau J(\kappa_\tau)+J_*+2x_1+\sum_\tau S(\kappa_\tau,\tau)\le 6J_*-\sum_\tau\bigl(J(\kappa_\tau)-S(\kappa_\tau,\tau)\bigr),$$

by (RT3) and $x_1\le J_*$.

Proposition 38.1 (Reciprocal height of the main critical ratio)**.** *Let $(\xi_\kappa)_\kappa$ be the chosen family. For every main pair p and its lift index j,*

$$\mathrm{Height}\bigl((R_{p,+}(\xi_\kappa^j)^{-1})_\kappa\bigr)\le 6J_*-d_{\mathrm{dir}},\qquad d_{\mathrm{dir}}=\sum_\tau\bigl(J(\kappa_\tau)-S(\kappa_\tau,\tau)\bigr).$$

38.6 Height of the exchange product

In this subsection, we use the same auxiliary twists as in Proposition 38.1 for the exchange factors.

Use the untwisted lift character $\mu_j^0=\chi_0(\upsilon_1^{-\Delta}\chi_M)^j$. Its weight at τ is $-w_j(\tau)$, where $w_j(\tau)=\kappa_{j,\tau}=\kappa_j(\tau)=\kappa_\tau+(j+\frac12)\Delta$. The following table lists the characters and critical endpoints in reverse order of the measured exchanges of Section 39.4.

character	upper point	sign
$\mu_j^0\upsilon_1^{j+(t-11m-1)/2}$	$7(m/2)+k/2$	$+$
$\upsilon_1^{-m}\chi_M^{\,j}$	m	$+$
$\upsilon_1^{-m/2}\chi_M^{-j}$	$m/2$	$-$

Here $\upsilon_1=\upsilon_E|_{T_1(\mathbb{A}_{F_0})}$. Lowering the point keeps the character and its tame part. For the interval with upper endpoint $(7m+k)/2$, put $s_j=j+(t-11m-1)/2$. For sufficiently large x,

$$0\le s_j\le t-r,\qquad m<7m/2<u\le(7m+k)/2=(r+6m+1)/2\le r.$$

At each such integer u the negative weight is $c_\tau=w_j(\tau)-s_j\in[w_j(\tau)+r-t,w_j(\tau)]$, and the tame part is ω^j. These last $k/2$ pairs are therefore main for j by Section 38.2, and Proposition 38.1 applies. For the remaining $7m/2+m+m/2=5m$ pairs, (RT1) suffices. Their tame parts are the original nontrivial powers $\pm j$ of the old marked character. Write (χ_i,z_i,ϵ_i) for the three rows of the table, with $\epsilon_i\in\{1,-1\}$. The three exchange identities of Section 39.4 give

$$\frac{H_j(\eta)}{H_j(1)}=q_Q^{\sum_i z_i^2}\left(\prod_{i=1}^{3}\prod_{1\le u\le z_i}R_{(\chi_i,u),\epsilon_i}(\eta)\right)^2.$$

To see this, write $M_\lambda^S(z)$ for the factor $M_\lambda(z)$ of Section 36 with split support S. Base change commutes with inversion. Thus the quotient for exchange i is

$$\frac{M^{S_M\cup Q}_{(\chi_i\eta^{\epsilon_i})^{-1}}(z_i)}{M^{S_M}_{\chi_i^{-1}}(z_i)}=q_Q^{z_i^2}\prod_{u=1}^{z_i}R_{(\chi_i,u),\epsilon_i}(\eta)^2.$$

The numerator omits both places above each new split prime. Its conductor factor counts the chosen place once. The denominator retains both Euler factors there. For the selected twist put

$$U_{i,u} = \frac{F_{(\chi_i,u),Q}(\eta^{\epsilon_i})}{F_{(\chi_i,u),\varnothing}(1)}.$$

Section 38.3 gives $U_{i,u} \in \overline{\mathbb{Q}}^{\times}$, with $R_{(\chi_i,1),\epsilon_i}(\eta)^2 = q_Q^{-1} U_{i,1}$ and $R_{(\chi_i,u),\epsilon_i}(\eta) = U_{i,u}$ for $u \geq 2$. Consequently

$$\frac{H_j(\eta)}{H_j(1)} = q_Q^{\sum_i z_i^2 - 3} \prod_{i=1}^{3} \left(U_{i,1} \prod_{u=2}^{z_i} U_{i,u}^2 \right) \in \overline{\mathbb{Q}}^{\times}.$$

Since $\ell \nmid q_Q$, the conductor power is a unit at κ. For each main pair p with $7m/2 < u \leq (7m+k)/2$ in the third measured exchange, put $h_p = H_p c_{0,p} \overline{c_{0,p}}$, using (RT4)–(RT6). An explicit multiplier for the inverse exchange ratio is $h_j = \prod_p h_p^2$, with multiplicity over those points. All other inverse ratios have multiplier one by (RT1)–(RT3). Thus

$$\mathrm{Height}\left(\left(\frac{H_j(\xi_\kappa^j)}{H_j(1)} \right)^{-1} \right) \leq 10mx_1 + k(6J_* - d_{\mathrm{dir}}).$$

For the triangle and the two orientations in Section 43.4, the characters on $U(1)$ and the powers at which their ratios are evaluated are

	untwisted character	old tame part	auxiliary twist
$1 \leq i < j \leq m$	μ_j^0/μ_i^0	ω^{j-i}	ξ_κ^{j-i}
$0 \leq i, j < m$	$\mu_{i+1}^0 \mu_{j+1}^0$	ω^{i+j+2}	ξ_κ^{j+1}

Indeed a turn at (a,b) uses the power $b+1$ in the first orientation and $a+1$ in the reflected orientation. The same cell (i,j) is read at $(a,b) = (i,j)$ and (j,i), respectively. At each old place ω_w has order $q_2 > 2m$, so the two displayed tame parts are nontrivial there and trivial off the old support. By Section 4.4, the negative weights are $(j-i)\Delta$ and $w_{i+1}(\tau) + w_{j+1}(\tau)$, respectively. For large x, $(2m+1)r^8 \leq b$, $\kappa_\tau \leq b$ and $\Delta = 2r^8 \geq t$. Both weights thus lie in $[t, 4b]$. Every point $1, \ldots, t$ occurs in the signed list, and the prescribed powers of the selected twist all have the same exact support Q. Half-integral powers of q_Q in these formulas also have valuation zero, since their squares are integral powers of q_Q.

39 Determinant factors and the gluing operator

In this section, we track the determinant and measure factors in Section 36 through three period identities. The resulting adjoint is the common period used in Section 41.

39.1 Products of determinants in the period comparison

In this subsection, we check how the product of archimedean determinants transforms in the exchange of Section 36.1. The resulting coefficient is used on the three pairs in Section 39.4.

Let the kept real space have positive and negative dimensions m, ℓ, the old small space dimensions p, ℓ, and the definite target dimension a. The ordered positive basis satisfies $m =$

$p + a$. At each infinite place τ let c_τ be the number of stacked determinants. In the large and small Fock spaces use the polynomials

$$P(\eta) = \prod_{\tau,i} \det[\eta_{\tau i}; Z_\tau], \qquad Q(\eta) = \prod_{\tau} \det(Z_\tau^-)^{c_\tau + 2s_0} \prod_{\tau,i} \det(\eta_{\tau i} Y_\tau), \qquad 2s_0 = \dim W - 2\ell.$$

Here Z_τ is the a-by-m block of positive columns, Y_τ is the m-by-p positive block of the small vector, and Z_τ^- is its ℓ-by-ℓ negative block. The basis used for the first and second covectors of each determinant is the marked one. Put $\eta_0 = [I_p\ 0]$ in that basis, at every place and for every index.

Apply the measured partner comparison with the fixed small vector $Q(\eta_0)$. Let v be its large real vector. We show that v can be replaced by $P(\eta)$ for every η, with one scalar multiplying the large side. At this point the scalar may vanish. Write α, β for the two determinant characters of the large kernel. For the kernels of (W, I_a) and (V', W) the pairs of characters are (α, β) and $(\alpha/\beta, \alpha)$, respectively. At (g, h) and (x, g), both sides are multiplied by $\beta(\det g)^{-1}$ relative to the choice $\beta = 1$. We cancel this common factor throughout the compact group calculation. For fixed matched finite data and g, the large period is a continuous functional L_g in a real Fock moment seminorm.

The matched support in the first coordinate is primitive, so the kernel estimate at exponent zero in the integrated variable bounds its absolute integral by $C_g p_N(v)$. The functionals L_g are invariant under the definite target real group by right translation in the original quotient. The standard covariance under right translation in the kept group gives $L_g(\omega(k, u)v) = L_{gk}(v)$. Write $S_{gk}(Q(\eta_0))$ for the small period. Proposition 36.1 gives $L_{gk}(v) = c_\nu S_{gk}(Q(\eta_0))$, with $c_\nu \neq 0$. The scalar κ obtained by projecting v below is chosen before ν and may vanish. For the projection, remove the common factor $\beta(\det g)^{-1}$ on both sides. Indeed the large pair of characters is (α, β) and the small pair is $(\alpha/\beta, \alpha)$. Thus in the circle calculation we take $\beta = 1$, and restore the common factor after the averaging.

Let $t(z)$ on the positive kept basis be diagonal with z^{-2} on the p coordinates and z^{-1} on the a coordinates, with $|z| = 1$. Use its identity extension on the negative block, and put $u(z) = z^{-1} I_a$ in the target. In the real representation the target determinant exponent is $\ell - c_\tau$. The vacuum contributes $\det(u_\tau)^{-\ell}$, hence their product contributes $\det(u_\tau)^{-c_\tau}$. For the operator

$$v \longmapsto z^{-a \sum_\tau c_\tau} \omega(t(z), u(z)) v$$

these scalar factors cancel. On polynomial vectors the substitution scales all variables outside the marked a columns by z and fixes the variables in those a columns. Indeed on the positive columns the substitutions use $u(z)$ and $t(z)^{-1}$; on the negative columns the signed Fock coordinate uses the conjugate factor of $u(z)$. Thus averaging over z sets all unretained variables to zero. The remaining polynomial is in the square a-by-a block at each place.

Under the definite target average this square polynomial is projected to its left determinant weight c_τ. This weight space consists of the multiples of $\prod_\tau \det(Z_\tau)^{c_\tau}$. Indeed, a polynomial with that left covariance under unitary matrices has it under the complexification $\prod_\tau \mathrm{GL}_a(\mathbb{C})$, by differentiation and complex linearity. On invertible square matrices its value is determined by its value at the identity, and polynomial continuation gives the statement.

The unit circle and the compact target average are bounded operators on the Fock Hilbert space. Polynomials are dense, and the resulting line is closed. Hence for the fixed v these two averages give $\kappa P(\eta_0)$ for one scalar κ, independent of all the finite data, the modulus, the character and g. The averages can also be evaluated inside the period.

To obtain polynomial approximation in the moments used by L_g, write $p_k(f;A)$ for the integral of $|f(z)|(1+\|z\|)^k e^{-c\|z\|^2}$ over $A \subset \mathbb{C}^{d_{\mathrm{F}}}$ in the fixed Fock coordinates, and $p_k(f) = p_k(f;\mathbb{C}^{d_{\mathrm{F}}})$. For $0 < t_0 \le t < 1$, the dilation $D_t f(z) = f(tz)$ satisfies

$$p_k(D_t f; \|z\| > R) \le t_0^{-2d_{\mathrm{F}}-k} p_k(f; \|z\| > t_0 R).$$

This is the substitution $w = tz$, with the Gaussian decreasing under the substitution. The tail bound and local uniform convergence give $p_k(D_t f - f) \to 0$ as $t \uparrow 1$. Circle extraction gives $p_k(f_j) \le p_k(f)$ for each homogeneous term f_j. Thus, for fixed $0 < t < 1$,

$$p_k\left(D_t f - \sum_{j=0}^{J} t^j f_j\right) \le \sum_{j>J} t^j p_k(f_j) \le \frac{t^{J+1}}{1-t}\, p_k(f).$$

Orthogonality of the degrees gives the two limits in Fock norm as well. The compact averages are bounded in the radial moments because their substitutions are unitary and their character factors have modulus one. Applying the bound for L_g passes the polynomial identities to the period of v on the original finite level.

For $k = (k_\tau) \in \prod_\tau U(m)$ define $\eta_k = \eta_0 k^{\mathrm{t}}$. Direct substitution in the two determinant formulas gives their real covariance. In the large polynomial apply k^{-1} in the kept positive space to $P(\eta_0)$ to obtain $P(\eta_k)$ times $\prod_\tau \det(k_\tau)^{c_\tau}$. This follows by factoring k_τ^{t} out of the full stacked matrix, so its determinant is counted c_τ times. In the small formula the positive block changes by left multiplication by k_τ^{t}, the negative block stays fixed, and its vacuum and splitting factor is $\prod_\tau \det(k_\tau)^{-c_\tau}$.

Apply the measured comparison at the real point $kt(z)$ and use these covariances. The small basis there is $z^{-2}\eta_k$, so homogeneity contributes $z^{-2p\sum c_\tau}$. The factor $\det(t(z))^{-c_\tau}$ contributes $z^{(2p+a)\sum c_\tau}$, which cancels the preceding power and the additional $z^{-a\sum c_\tau}$. The determinant powers of k_τ cancel between the covariances. Hence the circle-averaged comparison is the asserted identity at η_k, with large polynomial $P(\eta_k)$, namely $\kappa L_g(P(\eta_k)) = c_\nu S_g(Q(\eta_k))$. This identity also holds if $\kappa = 0$.

It remains to justify the passage from η_k to arbitrary families of coefficient matrices. At a fixed place consider the polynomial space in m-by-p matrices Y with $F(Yu) = \det(u)^c F(Y)$. For $p = 0$ this is the constant line. For $p > 0$, it is a finite-dimensional representation of GL_m under left substitutions and of $U(m)$ by restriction. It is irreducible. Indeed, in every irreducible summand of this polynomial representation a highest weight line is fixed by the lower unitriangular subgroup, using the corresponding Borel. A polynomial fixed by that subgroup is determined by one scalar.

On the dense set where the first p covectors form an invertible block A, a lower block shear sends Y to $(A;0)$. The right covariance then gives $F(Y) = \det(A)^c F((I_p;0))$. Thus the entire unitriangular invariant space has dimension one, and complete reducibility under $U(m)$ gives irreducibility. The finite space of homogeneous degree pc is preserved by the action, and complexified Lie algebra invariance identifies the summands under $U(m)$ and GL_m. Consequently $\prod_i \det(\eta_i Y)$ is a finite linear combination of the corresponding polynomial at the matrices $\eta_0 k^{\mathrm{t}}$. The same holds over all places by tensor products.

The same coefficients work for the two-sided stacked determinant. At a place with $c_\tau = 0$ the empty products equal one. For $c_\tau > 0$, if the lower matrix Z has rank smaller than a, every stacked determinant vanishes. Otherwise extend its covectors to an invertible square matrix. Multiplying on the right by its inverse turns Z into $[0\ I_a]$. It follows that $\det[\eta_i; Z] = d \det(\eta_i Y_Z)$

for a fixed Y_Z, consisting of the first p columns of that inverse, and a scalar d depending on Z alone. Apply the polynomial linear combination at Y_Z, and use the same coefficients in each large evaluation and in the small polynomial. For every family η of covectors and every matched pair of residue functions we therefore have

$$\kappa L_g(P(\eta)) = c_\nu S_g(Q(\eta)).$$

Writing $A = R_0'/D_{\mathrm{final}} > 0$ as in Proposition 36.1, the coefficient multiplying the large class sum is

$$\frac{\kappa}{c_\nu}, \qquad \left|\frac{\kappa}{c_\nu}\right|^2 = \frac{|\kappa|^2/A}{M_\nu(t)}, \qquad 2t = \dim V - \dim U.$$

Both A and κ are fixed before the finite character and the covectors. This formula also applies when $\kappa = 0$.

39.2 Constants in the three finite comparisons

In this subsection, we compute the finite orbit and stabilizer measures for the three applications of Section 36.1. This identifies the determinant character in each Hecke product and fixes the finite scalar in Section 39.4.

We use the covector, column and theta character conventions of the marked partner comparison. In the three class-sum applications the matching character is χ^{-1} if χ is the determinant character appearing in the upper datum. Let R be the original residue ring, with its full exponent at each selected place, and write $X = (A, B)$ and $Y = (C; D)$ for the matched blocks of k^{-1} and k, in that order.

Here B, C are square. In the order of Section 36.2, X consists of the last covectors of k^{-1} and Y of the first columns of k. They are respectively a split surjection and injection. The identities

$$AC + BD = 0, \qquad \det C = \det k \, \det B$$

hold over R, by block multiplication and complementary minors. For any Ψ' invariant under the right action of the integral column basis group, put

$$\Phi(A, B) = \chi(\det B)\Psi'(B^{-1}A), \qquad \Psi(C, D) = \chi(\det C)\Psi'(D),$$

with value zero when the determinant of the corresponding square block is a nonunit of R.

When either block has unit determinant, both do, and $D = -B^{-1}AC$. As $-C$ is invertible, $\Psi'(D) = \Psi'(B^{-1}A)$. Thus $\Phi(X) = \chi^{-1}(\det k)\Psi(Y)$. Conversely B invertible makes the upper array split surjective, and C invertible makes the lower array split injective. This proves support and matching over the actual residue ring, including the chosen prime-square coordinates.

For a definite standard target, normalize the real compact group and the full integral finite group K_{std} to have mass one. For a function F invariant under the real compact group and a finite-index subgroup $J \subseteq K_{\mathrm{std}}$, quotient disintegration gives

$$\int_{[U]} F(h)\, dh = \frac{1}{[K_{\mathrm{std}} : J]} \sum_{[h]_J} \frac{F(h)}{|U(F_0) \cap hJh^{-1}|}.$$

At full level the prefactor equals one. For the stacked determinant, the real target action contributes $\det(u_\tau)^{\ell - c_\tau} \det(u_\tau)^{-\ell} \det(u_\tau)^{c_\tau} = 1$. For a finite target translation $u \in K_{\mathrm{std}}$, the

displayed array satisfies $\Phi(u_S^{-1}A, u_S^{-1}B) = \chi(\det u_S)^{-1}\Phi(A,B)$. The finite determinant character of the target action on the kernel contributes $\chi(\det u_S)$. Thus the full integral compact fixes the twisted test. Hence these class weights apply to the actual kernels.

For rank zero the quotient is a single point with one automorphism, so its mass is one. The kernel on the zero Hom space with characters $(\alpha/\beta, \alpha)$ is $(\alpha/\beta)(\det g)$ on the retained point. In the first exchange $\beta = \alpha$, so it equals one. Write χ_α for the residue character of the source character α. The matched data are $\chi_\alpha \circ \det$ and 1, since $X = k^{-1}$ and Y has zero columns:

$$\chi_\alpha(\det X) = \chi_\alpha(\det k)^{-1}.$$

In Section 39.3 the source character is $\alpha = c^{-1}$, so $\chi_\alpha = \chi^{-1}$ when χ denotes the residue character of c. The empty small polynomial is one. The normalization of this application is therefore a unit-mass constant, with no unrecorded Haar factor.

For the second and third comparisons, the graph matching above splits the small kernel into its two blocks. The standard determinant factor is the kernel in the proper definite class variable, and the remaining factor is the polynomial kernel in the retained block. On the support of the product datum the combined column $Y = \binom{C}{D}$ has left inverse $(C^{-1}, 0)$ in the original residue ring. The product formula identifies the two factors with one theta kernel on the orthogonal sum. Section 34.3.1 therefore gives rapid decrease of that kernel in the integration variable for each fixed pair of retained points. Its integral converges absolutely. One may therefore insert the preceding measured identity under the quotient integral and interchange the finite class sum, with its same weights, and the integral. This accounts for multiplication of the exchange constants.

In the third comparison Ψ' is the sum of translates of the trace character by the group over the untwisted base ring $R_0 = \mathcal{O}_E/(\pi_M^2)$, inflated to R. It is invariant under the right basis group over R because its reduction permutes those terms. Each element of $\mathrm{GL}_{3m}(R_0)$ lifts to the original standard level at the chosen split places by lifting entries and testing determinants.

Extend the lift by its conjugate unitary coordinate on the other split half and by identity elsewhere. The splitting character of the retained small kernel has trivial finite tame part, and thus has value one on the determinant of this integral move. The full-level carrier is unchanged by it. The carrier is rapidly decreasing by Section 41.3, whose proof uses the first two exchanges. Each fixed translated trace datum has moderate growth by Section 34.3.1. Their product is absolutely integrable on the carrier quotient. Right translation in that quotient consequently makes the integral of each translated trace character equal to the unshifted integral. The finite sum multiplies by $\mathcal{N} = |\mathrm{GL}_{3m}(R_0)|$. This is the untwisted group order, even if the modulus R has acquired new twist places.

With $e = m/2$, $z_* = 7e + k/2$ and $n_j = j + (t - 11m - 1)/2$, the three defects and period normalization arguments are

kept, large, small	defect	character in the Hecke product
$I_m,\ I_m,\ 0$	e	$c\upsilon^e$
$I_m \perp S,\ I_{3m},\ I_m^-$	m	$c^{-1}\upsilon^m$
$I_{3m} \perp W',\ V'',\ S$	z_*	$(ca_j)^{-1}\upsilon^{3m+z_*}$.

Here m, k are even, W_j has rank $r = k + m - 1$, and $W' = W_j \perp I_j^- \perp I_{3m-j} \perp I_{6m+1}$ has rank $k + 10m = 3m + 2z_*$. The negative rank of each kept space equals that of its small member, respectively $0, m, m$. For the third character, $ca_j = \mu_j^0 \vartheta\, \upsilon^{m+j+\lfloor (t+k-1)/2 \rfloor}$; the exponent left

after inversion is

$$3m + z_* - m - j - \lfloor (t + k - 1)/2 \rfloor = -j - (t - 11m - 1)/2 = -n_j.$$

The first conversion to the negative copy of I_m takes the conjugate of its scalar and multiplies by the unit phase $\lambda_0 \chi(\eta)^{-m}$, with η the marked norm-minus-one residue unit. Thus the squared moduli multiply and the factor from trace averaging is $\mathcal{N}^{-2}$. In particular the coefficient of the third left class sum taking it to the original carrier period is $\gamma_3 \gamma_{12}/\mathcal{N}$, with square modulus equal to a fixed constant divided by the product of the three marked Hecke factors in the table.

39.3 The rank-zero comparison and the opposite Hermitian form

In this subsection, the first application of Section 36.1 has a rank-zero smaller space. We compute the source-negation factor with the original residue character before applying the comparison to the other two pairs.

Use the modulus $\mathfrak{m} = \pi_M^2 \mathfrak{q}_\vartheta$, the character $c = \chi_M^j \vartheta$ of weight zero, and its residue character χ. Let $R = O_E/\mathfrak{m}$. Write $m = 2e > 0$ and $P = 1$. Apply the measured exchange in Section 36.1 with kept space I_m, positive space I_m, and smaller space of rank zero. Take the characters on the positive kernel to be $\alpha = \beta = c^{-1}$, and its finite function to be $\Psi_+(X) = \chi^{-1}(\det X)$. The two finite functions are matched.

Indeed the block of an inverse matrix $k^{-1} \in \mathrm{GL}_m(R)$ is the entire matrix, and $\Psi_+(k^{-1}) = \chi(\det k)$, the matching factor for the residue character χ^{-1} of α. The other block is the unique empty column, with value one. On singular matrices the extended residue character is zero. Both real polynomials are one. The rank-zero kernel has vacuum one and only its zero lattice vector.

The surviving character on the kept group is $(\alpha/\beta)(\det g) = 1$ since $\alpha = \beta$. The rank-zero group's quotient is a point of mass one. Thus

$$\gamma_1 \sum_{[h] \in [U(I_m)]/K_{\mathrm{std}}} \frac{\Theta^{I_m, I_m}_{c^{-1}, c^{-1}}[\Psi_+, 1](g, h)}{|\operatorname{Aut}(h)|} = 1, \qquad |\gamma_1|^2 = \frac{C_1}{\mathcal{L}(c\upsilon^e, e)}.$$

The first equality is the measured exchange, with its class weights unchanged. The second is its scalar formula, since $\alpha^{-1} \upsilon^e = c \upsilon^e$. In particular the scalar used in the first equality is nonzero. The point of evaluation g is arbitrary on the marked I_m.

For the source-negation step, take the fixed finite unit u with $u\bar{u} = -1$ and put $\eta = u \bmod \mathfrak{m}$. At split pairs one can choose u with coordinates $1, -1$; at an unramified quadratic place, unit-norm surjectivity gives a norm-minus-one unit. Use the original such unit and the original point of the negative space. In chosen coordinates its point matrix is $X_{I_m^-} = (1, u) X_{I_m}$; its rational form is the negative of the original form. On the Hom space, with the target unchanged, the new point matrix is therefore $(1, u^{-1}) X_{\mathrm{Hom}}$. The coordinate map $\phi(x) = (1, u^{-1})x$ carries the new rational lattice onto the old one.

On the coefficient realization of the oscillator data, put $Jf(x) = \overline{f(\phi x)}$. This transports the Heisenberg action to the opposite source, because $\beta_{\mathrm{old}}(\phi x, \phi y) = \overline{\beta_{\mathrm{new}}(x, y)}$. At infinity the sign matrix in the bicharacter changes sign and ϕ is the identity. At finite places the norm of u^{-1} is -1. The Gaussian and standard integral box are preserved, so this is a vacuum-preserving transport with the same Haar normalization. The standard holomorphic coordinate of each entry has been conjugated, so P becomes its coefficientwise conjugate. For a residue datum

Ψ, reduction of ϕ multiplies every entry by η^{-1} at the selected places. The integral support is preserved since u is a unit, and the transported residue function is

$$\Psi^{J}(X) = \overline{\Psi(\eta^{-1}X)}.$$

The theta functionals on the two Hom spaces are proportional under this transport. One way to see uniqueness in a rational Schrödinger polarization is to use rational modulations and translations. Invariance under the modulations implies that the functional is supported on the rational diagonal, which is its own annihilator for the adelic character. There are no derivatives at a rational point: locally the multipliers $\psi(\langle x, \lambda_i\rangle) - 1$, for a rational dual basis (λ_i), generate the ideal of smooth test functions vanishing at that point.

Indeed the finite parts are constant near the point and the real parts have independent linear terms. Rational translations then force the same coefficient at all rational points. This gives the usual rational summation functional on compactly supported tests, and on the Schwartz space by continuity and approximation. Here J matches the rational Heisenberg operators and their vacuum translates. The two normalized theta functionals have the same squared mean, one, on the vacuum translates over their additive adelic quotients. The induced quotient isomorphism preserves probability Haar measure. Consequently for a fixed $|\lambda_0| = 1$,

$$\overline{\theta_{\mathrm{old}}(f)} = \lambda_0 \theta_{\mathrm{new}}(Jf).$$

The scalar depends on the two marked spaces and their theta normalizations alone.

To track the splitting characters, write $A(x) = hxg^{-1}$. Its matrix stays the same on passing to the opposite source. At a real place the multiplier is the conjugate of the negative principal minor of A. Jacobi's complementary-minor identity and $A^{-1} = J_{\mathrm{sign}} A^{*} J_{\mathrm{sign}}$ give $\overline{\Delta_{\infty,\mathrm{old}}(A)} = \det A\, \Delta_{\infty,\mathrm{new}}(A)$. At the finite places let L be the original standard lattice and choose b with $bL = L \cap A^{-1}L$.

The finite multiplier is $\chi_f(\det b)$, where $\chi_f = (\upsilon_E)_f |\cdot|_f^{-1/2}$ on $\mathbb{A}_{E,f}^{\times}$, with $(\upsilon_E)_f$ the finite component of the splitting Hecke character. The matrix Ab represents $L \cap AL$, so

$$\Delta_f(A^{-1}) = \chi_f(\det A)\Delta_f(A).$$

Since $A^{-1} = A^{*}$, the determinant ideal defining $\Delta_f(A^{-1})$ is the conjugate of the one defining $\Delta_f(A)$, by local Smith form. The character χ_f is unramified on units and takes complex-conjugate values on conjugate ideles. Thus $\Delta_f(A^{-1}) = \overline{\Delta_f(A)}$. Since $\det A$ has idele norm one, the total multiplier under the conjugating transport acquires $\upsilon(\det A)$.

In the oscillator integral this is the only scalar change: the coordinate substitution intertwines the matrices, conjugates the Heisenberg bicharacter, and preserves the Gaussian and Haar measure. As $\det A = (\det h)^m (\det g)^{-m}$, the original pair character $a(\det h)b(\det g)^{-1}$ becomes the pair character with entries $a^{-1}\upsilon^m$ and $b^{-1}\upsilon^m$.

For $a = b = c^{-1}$ and $\Psi_{+} = \chi^{-1} \circ \det$ this gives exactly

$$\Psi_{+}^{J}(X) = \chi(\eta)^{-m}\chi(\det X).$$

The same formula holds at a singular matrix since both sides vanish. The level and class stabilizers do not change, and the class weights are rational. Conjugating the first measured identity gives

$$\overline{\gamma_1}\lambda_0\chi(\eta)^{-m} \sum_{[h]\in[U(I_m)]/K_{\mathrm{std}}} \frac{\Theta^{I_m^{-},I_m}_{c\upsilon^m,c\upsilon^m}[\chi\circ\det,1](g^{-},h)}{|\operatorname{Aut}(h)|} = 1.$$

In particular the same $\mathcal{L}(cv^e, e)$ controls the square modulus. The factor λ_0 is one fixed complex phase and $\chi(\eta)^{-m}$ is the residue phase at the actual modulus. This proves the normalization used when inserting the first class sum under the second exchange.

39.4 Three period identities for the gluing operator

In this subsection, we apply the normalized partner comparison to the three pairs defining the operator of Section 34.1. The real factors of Section 39.1 and the finite factors of Section 39.2 identify the resulting common period.

These identities use the marked partner comparison on the three stated pairs. In particular the scalar formula is used with its original vacuum normalization at the top and after all plane removals. Use the same j, original W_j, V'', V°, and choice set $\mathcal{I}$ as in Section 34.1. Put $I_a = (E^a, 1)$, $S = I_{2m} \oplus I_m^-$ and

$$W' = W_j \oplus I_j^- \oplus I_{3m-j} \oplus I_{6m+1}.$$

The marked space underlying V'' is the definite space with these indices. Set $e = m/2$ and $z_* = 7e + k/2$, an integer since r is odd and m is even. Then $\dim V'' = 3m + 2z_*$. In these three comparisons write $\upsilon = \upsilon_E|_{T_1(\mathbb{A}_{F_0})}$, where υ_E is the splitting Hecke character of Section 4.3. Let $c = \chi_M^j \vartheta$ be the weight-zero character and a_j the twist-free character on the larger pair. Write $\mathfrak{m} = \pi_M^2 \mathfrak{q}_\vartheta$ for the modulus in the twisted intermediate kernels. The finite character of c is the one used in their determinant data and is nontrivial at every place of the prescribed support of $\mathfrak{m}$.

Section 39.3 gives the measured first exchange on the two copies of I_m. Its weighted class sum is normalized to one by γ_1. Its squared modulus is a constant divided by $\mathcal{L}(cv^e, e)$, where $\mathcal{L}(\nu, z) = M_\nu(z)$ is the normalizing factor of Proposition 36.1 on the split support of $\mathfrak{m}$. Conjugating the first group to I_m^- gives the scalar $\overline{\gamma_1}\lambda_0\chi(\eta)^{-m}$ of that section, with the prescribed reduction and $|\lambda_0| = 1$.

The next exchange uses

$$(I_m \oplus S, I_{3m}; I_m^-)$$

and the source character cv^m. Denote its normalized coefficient by γ_2, so that

$$|\gamma_2|^2 = C_2/\mathcal{L}(c^{-1}\upsilon^m, m).$$

Insert the first constant class sum under this second exchange and multiply the kernels for the direct sum $I_m \oplus S$. The remaining kernel on (I_m^-, S), integrated against the constant function on $[U(I_m^-)]$, is the common carrier F°.

Its datum is the original vacuum and the polynomial on the negative block is the $2m$-th power of the determinant minor. For a finite representative h of I_{3m}, use the kernel K_0 of (I5) to put

$$A_c(h) = \sum_{[v]} \frac{K_0(v, h)}{|\operatorname{Aut}(v)|} = c(\det h) A_1(h),$$

where $[v]$ ranges over the integral-level classes of I_m. With the same finite and real tests, the determinant characters give

$$K^{a,b}(g, h) = a(\det h) b(\det g)^{-1} K^{1,1}(g, h),$$
$$\Theta^{S, I_{3m}}_{\upsilon^m, 1}[\chi \circ \det, 1](x, h) = c(\det x)\Theta^{S, I_{3m}}_{\upsilon^m, c}[\chi \circ \det, 1](x, h).$$

The small kernel at (y,x) acquires the same factor from its first character cv^m. After canceling it, the coefficient is

$$\gamma_{12}=\gamma_2\overline{\gamma_1}\lambda_0\chi(\eta)^{-m}.$$

Here η is the residue of the fixed finite unit of norm -1. Since $|\lambda_0|=|\chi(\eta)|=1$, its squared modulus is

$$|\gamma_{12}|^2=\frac{C_{12}}{\mathcal{L}(c^{-1}v^m,m)\mathcal{L}(cv^e,e)}.$$

Thus for every $x\in U(S)(\mathbb{A}_{F_0})$ the carrier is

$$F^\circ(x)=\gamma_{12}\sum_{[h]}\frac{A_c(h)}{|\operatorname{Aut}(h)|}\,\Theta^{S,I_{3m}}_{v^m,c}[\chi\circ\det,1](x,h). \tag{101}$$

Here $[h]$ ranges over the integral-level classes of I_{3m}. At that level the factor $c(\det k)$ in $A_c(hk)$ cancels the factor $\chi(\det\bar{k})^{-1}$ from the residue determinant.

For $i=(\Lambda,h^\flat,\sigma)$, where σ is the chosen tuple of short covectors, use the third exchange on

$$(I_{3m}\oplus W',V'';S).$$

Write $e_j^{(1)}=j+\lfloor(t+k-1)/2\rfloor$ and μ_j^0 for the untwisted lift character. On the first block the character is

$$b''_{j,\vartheta}=\mu_j^0\vartheta\,v^{e_j^{(1)}},\qquad ca_j=b''_{j,\vartheta}v^m.$$

Thus the third kernel has characters $(ca_j,b''_{j,\vartheta})$. The finite matching pairs the trace character of Λ with the unnormalized invariant sum of the trace character on the ratio of the two square blocks. The left datum is $\chi(\det x'')\psi_\Lambda^{\mathrm{inv}}((x'')^{-1}x')$.

39.4.1 Matching the real factors

In this subsection, we form the common real polynomial for the period comparisons of Section 39.4.

The real matched polynomial on W' is the product P_jQ_σ of Section 41.1.4. For $r_+=k+j-1$, at each positive height $h<r_+$ take the $2m$ rows indexed by $0\le a<2m$: row $h+a$ of W_j if $h+a<r_+$, and row a of I_{3m-j} otherwise. For each of the Δ' further full-height factors take the prescribed short rows of $W^\flat$ at $h^\flat$. Extend these rows by zero on the initial I_{3m} block. The ordered negative covectors are row a of I_j^- for $0\le a<j$ and row $k+a-1$ of W_j for $j\le a<m$. Writing $\mu_\tau=\sum_{h<r_+}n_\tau^+(h)+\Delta'$, the negative determinant exponent in the small polynomial is

$$\mu_\tau+\dim(I_{3m}\perp W')-2m=\mu_\tau+d_1+m\qquad(d_1=\dim W').$$

Apply Section 39.1 in these orders. The third exchange scalar γ_3 has squared modulus a constant divided by $\mathcal{L}((ca_j)^{-1}v^{3m+z_*},z_*)$. Summing the first block using the carrier identity and then reducing the trace-character data modulo the untwisted π_M^2 gives

$$\gamma_3\gamma_{12}\,\Upsilon_i(g)=\mathcal{N}F_i(g),\qquad\mathcal{N}=|\mathrm{GL}_{3m}(\mathcal{O}_E/(\pi_M^2))|.$$

For a fixed retained point, the kernel of each residue translate has moderate growth in the carrier variable by the fixed Schwartz seminorm estimates of Section 34.3.1. The full-level carrier is rapidly decreasing by Section 41.3, using the invertible determinant datum in its

expansion from the first two exchanges. Thus each product is absolutely integrable on $[U(S)]$, and right translation in this quotient gives the same integral for each residue translate. Their unnormalized sum has $\mathcal{N}$ terms, with the original quotient measure. Equivalently

$$c'_{\vartheta} = \gamma_3\gamma_{12}/\mathcal{N}, \qquad |c'_{\vartheta}|^2 = \frac{C_3C_{12}/\mathcal{N}^2}{H_j(\vartheta)}.$$

Here the product in the denominator is

$$H_j(\vartheta) = \mathcal{L}(c\upsilon^e, e)\,\mathcal{L}(c^{-1}\upsilon^m, m)\,\mathcal{L}((ca_j)^{-1}\upsilon^{3m+z_*}, z_*).$$

To match these factors with the signed critical values of Section 38, write $n_j = j+(t-11m-1)/2$, μ_j^0 for the untwisted lift character, and

$$\nu_+ = \upsilon^{-m}\chi_M^{j}, \qquad \nu_- = \upsilon^{-e}\chi_M^{-j}, \qquad \nu_* = \mu_j^0\upsilon^{n_j}.$$

Then $c^{-1}\upsilon^m = (\nu_+\vartheta)^{-1}$, $c\upsilon^e = (\nu_-\vartheta^{-1})^{-1}$, and $(ca_j)^{-1}\upsilon^{3m+z_*} = (\nu_*\vartheta)^{-1}$.

The marked first-factor pairing. Write $R = \mathcal{O}_E/\mathfrak{m}$, $R_0 = \mathcal{O}_E/(\pi_M^2)$, and let $\phi : R \longrightarrow R_0$ be reduction. Let T be the coordinate set of V'', $I \subset T$ the coordinates of W_j, and $J_I : R^I \to R^T$ their coordinate inclusion. Write w for the covectors Λ stacked over $\pi_M 1_T$ in the marked residue coordinates of $\Xi_{\vartheta,i}$. The top covector projection u satisfies $uw = \Lambda$. Define

$$H = \{A \in \mathrm{GL}_T(R) : Ae_b = e_b \ (b \in T \setminus I),\ w\phi(A) = w\}.$$

Here ϕ is applied entry by entry. The inverse image of H under reduction from the full integral subgroup K on V'' is the prescribed level J_Λ. Reduction onto $\mathrm{GL}_T(R)$ is surjective, since each place in the modulus is split in the chosen coordinates and reduction of the integral general linear group onto each local residue ring is surjective.

Let Y, Z be the two residue blocks of the third kernel, with Z square on T and the columns of Y indexed by I_{3m}. The invariant trace function is the unnormalized sum

$$B(Y) = \sum_{D \in \mathrm{GL}_{3m}(R_0)} \psi\big(\operatorname{tr}(\Lambda\phi(Y)D)\big).$$

For $A \in H$, $\Lambda\phi(A) = uw\phi(A) = \Lambda$, so $B(AY) = B(Y)$. The ratio datum $\Phi(Y, Z)$ is $\chi(\det Z)B(Z^{-1}Y)$ for invertible Z, and vanishes otherwise. Put $\Phi_H = 1_{Z\in H}\Phi$. For $\sigma = ZJ_I$ set

$$C_w(\sigma) = \begin{cases} \chi(\det(J_I^{\mathrm{t}}\sigma)), & w\phi(\sigma) = wJ_I, \\ 0, & \text{otherwise.} \end{cases}$$

The character of the residue ring is extended by zero on nonunits. When the columns of Z off I are those of the identity, Z is block triangular in $I \sqcup I^c$ and $\det Z = \det(J_I^{\mathrm{t}}ZJ_I)$. Also $w\phi(Z) = w$ is equivalent to $w\phi(ZJ_I) = wJ_I$. On H, $B(Z^{-1}Y) = B(Y)$. Thus

$$\Phi_H(Y, Z) = C_w(ZJ_I)B(Y)1_{Ze_b = e_b \ (b\in I^c)}. \tag{*}$$

For $A \in \mathrm{GL}_T(R)$, define

$$(T_A\Psi)(Y, Z) = \chi(\det A)\Psi(A^{-1}Y, A^{-1}Z).$$

By the finite Weil action, right translation of the kernel by $k \in K$ with residue A replaces Ψ by $T_A\Psi$. The determinant factors give $T_A\Phi = \Phi$. Hence $T_A\Phi_H = 1_{Z\in AH}\Phi$ and

$$\Phi = \sum_{AH\in \mathrm{GL}_T(R)/H} T_A\Phi_H.$$

For a full-level class representative h, put $\Gamma_h = U(V'')(F_0) \cap hKh^{-1}$. The J_Λ-classes above h are the orbits of Γ_h on K/J_Λ under the action through $h^{-1}\Gamma_h h$.

The restricted kernel at hk is constant on each such orbit by rational invariance and right J_Λ-invariance. Dividing the sum over K/J_Λ by $|\Gamma_h|$ therefore gives the sum over the orbits with weights equal to the reciprocal orders of their stabilizers. The real compact component introduces no further weight. At a real place $u \in U(V'')$ left multiplies the bottom covectors of each stacked determinant. The polynomial contributes $\det(u)^{c_\tau}$; the vacuum and splitting give $\det(u)^{-c_\tau}$. Their product is one in the full kernel. This proves equality of the full-level class sum of Φ and the J_Λ-class sum of Φ_H, with the original class measures.

Use the ordered block decomposition of the kept space

$$I_{3m} \oplus W' = W_j \oplus I_j^- \oplus W^{\mathrm{add}},$$

where W^{add} includes I_{3m}. The equality is by permutation of the marked coordinate parameters. In (*) the columns on W_j carry C_w, those on I_j^- carry the inclusion indicator there, and the remaining columns carry the added-block datum of Section 33.3. Indeed the complementary identity columns split into the negative pad and the fixed positive columns on W^{add}, while B depends only on the I_{3m} columns. At each infinite place the positive column count vanishes for $0 \le h < k-1$ by Section 33.1.1: on $1 \le h < k-1$ the successive stem difference is one. At every contributing height $h < r_+$ we therefore have $1 \le r_+ - h \le j \le m$. In the stacked determinant the selected rows remove exactly the W_j columns numbered $h, \dots, r_+ - 1$ and otherwise remove added positive columns. Expanding in the remaining first h columns gives height-h minors of W_j. In a full-height factor the rows on $W^\flat$ leave all r_+ positive columns of W_j. Accordingly the real polynomial is

$$P(x_{W_j}, x_{\mathrm{add}}) = \sum_\lambda L_\lambda(x_{W_j})\zeta_\lambda(x_{\mathrm{add}}).$$

Here L_λ are the marked positive Weyl coordinate polynomials, and ζ_λ are the coordinates used in (I5), obtained by expanding the stacked determinants in the W_j columns with coefficients in the added variables. The polynomial is constant on the negative pad.

The tensor-product normalization of Section 17.1 applies after the indicated permutation of coordinates. Applying it twice gives the product of the three kernels. In the Schrödinger model the integral vacuum and the residue functions factor on the three Hom spaces. In the real Fock model the vacuum and the displayed polynomial factor in the same way. The rational summation set is a Cartesian product, and the finite class sums and Schwartz theta series converge absolutely.

In the order $W_j, I_j^-, W^{\mathrm{add}}$, the three pairs of characters are

$$(\upsilon^{m-j}, b''_{j,\vartheta}), \qquad (\upsilon^j, b''_{j,\vartheta}), \qquad (b''_{j,\vartheta}, b''_{j,\vartheta}).$$

Their first characters multiply to ca_j. The three first points are g, the identity of I_j^-, and $\mathrm{diag}(1, h^\flat, h')$ in the added block, with the same second point h in each. After multiplication by $A_c(h')$ and weighted summation over h' as in (101), the coefficient of L_λ equals the negative-pad

kernel at h times the carrier sum of the added kernel with ζ_λ. By (I5) these are the coordinates of $\Upsilon_{\vartheta,i}(h)$ before the scale and conjugation. Thus $\Upsilon_i(g)$ is the J_Λ-class sum of the W_j, V'' kernel with residue function C_w and polynomial vector $\Upsilon_{\vartheta,i}(h)$.

At a new split place w, an element of J_Λ has residue $A = \left(\begin{smallmatrix} D_w & 0 \\ E_w & 1 \end{smallmatrix}\right)$ in the order I, I^c. Since $J_I^{\mathsf{t}} A^{-1} = D_w^{-1} J_I^{\mathsf{t}}$, we have

$$C_w(A^{-1}\sigma) = \chi_w(\det D_w)^{-1} C_w(\sigma).$$

The first splitting character of the W_j, V'' kernel is unramified, so this is its multiplier under target translation by the element. The negative-pad and added-block inclusion conditions are preserved, and the trace datum is read on R_0. The two other kernels together contribute $\chi_w(\det D_w)$ through their target characters. Thus the product in the class sum is invariant. The conjugate of $c''\Upsilon_{\vartheta,i}(h)$ has character $\chi_w(\det D_w)^{-1}$, the level character of the target lift under the block embedding. At the old places all these residue characters are trivial on J_Λ.

The conjugate of the first factor is $c''\Upsilon_{\vartheta,i}(h)$ by (I5). Linearity in the polynomial, followed by the third exchange, gives

$$\Xi_{\vartheta,i}(g) = c''\Upsilon_i(g), \qquad c'_\vartheta \Xi_{\vartheta,i}(g) = c'' F_i(g).$$

39.4.2 Ranks and levels in the three comparisons

In this subsection, the applications of Section 36.1 require simultaneous inequalities at every real place and the stated finite levels on each pair. We verify these conditions for the characters just used.

Let $c = \chi_M^j \vartheta$ and let α denote the actual source character in one class-sum comparison. If the kept rank is N, its number of negative signs is ℓ, and the number of real stacked determinants is c_τ, then

$$\mathrm{wt}_\tau(\alpha) = \ell - c_\tau, \qquad \nu = \alpha^{-1} \upsilon^{N/2}, \qquad \mu_\tau = N/2 - \ell + c_\tau.$$

This is the same μ_τ used in the real slice proof and in the normalization. It gives $N \le 2\mu_\tau + 2\ell$. The comparison needs $N \ge 2\ell + 2$, and the small member must have exactly ℓ negative signs since the large target is definite. In the order of the three comparisons the small members are 0, I_m^-, and S of signature $(2m, m)$. Thus the third small period is taken on the indefinite group $U(2m, m)$, with the measures of Section 36.1.

For the first comparison the ranks (N, a, b) are $(m, m, 0)$, $\ell = 0$, $\alpha = c^{-1}$ and $c_\tau = 0$. The inequality is $m \ge 2$. For the second they are $(4m, 3m, m)$, $\ell = m$, $\alpha = c\upsilon^m$, and again $c_\tau = 0$. For the third they are $(6m + 2z_*, 3m + 2z_*, 3m)$, $\ell = m$, and $\alpha = ca_j$. The $m - j$ negative coordinates of W_j and the j negative pad coordinates give the kept negative rank. The real number of stacked determinants is explicitly

$$c_\tau = w_j(\tau) - j - \lfloor (t + k - 1)/2 \rfloor,$$

the cardinality of the marked set of determinant indices.

Since ca_j has weight $m + j + \lfloor (t + k - 1)/2 \rfloor - w_j(\tau)$ this is exactly the preceding weight condition. The parameters give m, k even, so $z_* = 7m/2 + k/2$ and all three ranks and defects are integers as written. At each old distinguished place the residue character of χ_M is ω_w, of order q_2. Thus $\omega_w^{\pm j}$ has order $q_2 / \gcd(q_2, j) > 1$ for $1 \le j \le m < q_2$. At an additional twist place the character is nontrivial by exact twist support.

The fixed norm-one splitting and fixed a_j have trivial finite tame parts. Thus the same semilocal matching proof applies to the full modulus $\pi_M^2\mathfrak{q}_\vartheta$ in all three cases. At an old place the determinant character factors through $k_w^\times$, while the test may distinguish classes modulo ϖ_w^2. Proposition 36.1 applies to these locally constant matched tests. The reversed matrix Gauss sum in Section 36.2 is taken on its squarefree residue ring. The polynomial transfer uses a fixed real vector before the modulus, and it applies to the same determinant indices for every finite test.

39.4.3 The common period as the full gluing adjoint

In this subsection, with the three comparisons established, we pair their common period against arbitrary tuples in the target of Section 34.1. The identity identifies the adjoint used in the injectivity argument of Section 41.

Let y be in the original good part of W_j and let $\zeta = (\zeta_{i,c})$ be a tuple in the original second-factor Weyl spaces. Write $K^\circ_{\zeta_i}(g)$ for the sum over the original V°-classes c of the second-block kernel at $(g,(1,c))$, with the polynomial in the negative coordinates associated to $\overline{\zeta_{i,c}}$. This kernel uses the untwisted W_j-character, the indicated power $\upsilon_{U(1)}^{m/2-j}$, modulus π_M^2, and the original inclusion indicator. The class weights and real test of that block are the ones in the gluing map.

Use the inner products linear in the second argument. Moving the lift integral through the gluing adjoint pairs $y(g)$ against the conjugate normalized descent of $G_\vartheta\zeta$ at g. The product formula for the original glued lattice expands this integrand into the finite sum over i and the congruence group $\bar{J}^1 = \{u \bmod \pi_M^2 : u \equiv 1 \bmod \pi_M\}$ of translates by the fixed lifts $k(u)$ of

$$M^{l_1} c_{L_i}\, y(g)\Xi_{\vartheta,i}(g)K^\circ_{\zeta_i}(g).$$

Let δ_i be the determinant of L_i on the old split members and let $\chi_{W,r}$ be the fixed split Weil character of W_j at rank r. The finite action on the matrix variable of the glued kernel is

$$f(X) \longmapsto c_{L_i} f(L_i^{-1}X), \qquad c_{L_i} = \chi_{W,r}(\delta_i)|\delta_i|^{-r/2}.$$

At an old chosen place w, let $s_{i,w}$ be the rank of Λ modulo w. The row lattice of $\binom{\Lambda}{\pi_M I_{d_+}}$ has $s_{i,w}$ elementary divisors of valuation zero and $d_+ - s_{i,w}$ of valuation one. Thus the first diagonal block $B_+ = \pi_M D^{-1}$ of L_i has determinant valuation $s_{i,w}$. The other diagonal block $J_z + \pi_M(1-J_z)$ has determinant valuation $d_- - r$. With $v_w(\varpi_w) = 1$, we obtain

$$v_w(\delta_i) = s_{i,w} + d_- - r, \qquad c_{L_i,w} = \big(\chi_{W,r,w}(\varpi_w)q_w^{r/2}\big)^{s_{i,w}+d_--r}.$$

In the convention of (F2), the target character υ^{m-j} of Section 33.1.4 contributes $\upsilon_E^{2(m-j)}$ by base change as in Section 20.3.3. The rank-r matrix variable contributes the split Weil factor υ_E^{-r} from Lemma 17.1. Hence $\chi_{W,r} = \upsilon_E^{2(m-j)-r}$ is unramified at the old split places. Both the character and L_i are fixed. At the twist places L_i and the lifts $k(u)$ are identity.

At an old split place, put $A = \pi_M L_i^{-1}$; both L_i and A are integral. For $k_+ \in J_\Lambda$, the old congruences imply $D(k_+ - 1) \equiv 0 \bmod \pi_M^2$ for the row basis D above. In the block order of Section 33.2 this gives

$$L_i^{-1}\operatorname{diag}(k_+,1)L_i - 1 = \pi_M^{-1}A\operatorname{diag}(k_+-1,0)L_i \equiv 0 \pmod{\pi_M}.$$

Thus the block embedding preserves the principal level at the glued point. Moreover $L_i^{-1}X \equiv I_{\mathrm{incl}} \pmod{\pi_M}$ implies X is integral. For integral X this congruence is equivalent to $AX \equiv \pi_M I_{\mathrm{incl}} \pmod{\pi_M^2}$. Section 33.2 decomposes the latter condition into the two residue conditions. For the first-block column σ on the support of C_w, the $\pi_M I$ covectors give $\sigma \equiv J_I \pmod{\pi_M}$, so the old tame character of $\det(J_I^{\mathrm{t}}\sigma)$ equals one. At a twist place the same determinant carries the prescribed residue character, extended by zero on nonunits. The second block retains its inclusion indicator at π_M^2. Its splitting character is $b_j^{\circ} = \upsilon^{m/2-j}$, and the product on the original target is

$$b''_{j,\vartheta} b_j^{\circ} = \mu_j^0 \vartheta\, \upsilon^{(r+t)/2}, \qquad e_j^{(1)} + m/2 - j = (r+t)/2.$$

The good form y is invariant by $k(u)$. These translates therefore have the same integral by right invariance of the original quotient measure. The integrals converge absolutely: each class sum is finite and its theta kernels have moderate growth for the fixed Schwartz vectors, while y decreases rapidly on Siegel sets. For the moderate growth use continuity of the lattice sum in the Schwartz seminorms and the polynomial bounds on these seminorms under the real oscillator action. The powers can be absorbed by arbitrarily high rapid-decay powers of y. It follows that

$$\langle \zeta, G_{\vartheta}^{*} u_{j,\vartheta} y \rangle = \sum_i M^{l_1} c_{L_i} |\bar{J}^1| \int_{[U(W_j)]} y(g) \Xi_{\vartheta,i}(g) K_{\zeta_i}^{\circ}(g)\, d\mu(g).$$

The translates on W_j are indexed by $1 + \pi_M X \bmod \pi_M^2$ with $X \in M_r(\mathcal{O}_E/(\pi_M))$, so $|\bar{J}^1| = |\mathcal{O}_E/(\pi_M)|^{r^2}$. No change of target measure or twist-level index is used in this calculation.

Substitute $c'_{\vartheta}\Xi_{\vartheta,i} = c'' F_i$. Define P by the displayed pairing with $c'' F_i$ in place of $\Xi_{\vartheta,i}$. This is the twist-independent period on the original good source; the kernel formulas determine it for every ζ in the full tuple space.

Proposition 39.1 (Common period and the gluing adjoint)**.** *On the original good source, at each allowed twist,*

$$c'_{\vartheta} G_{\vartheta}^{*} u_{j,\vartheta} = P.$$

The period P is independent of ϑ.

For the parameters of Proposition 16.2, its proof gives $D > 0$, and the identity test in (4) gives $\dim V_j = D$ on the good source at every j. Take $0 \neq y \in V_j$. The injectivity established in Section 41 gives $Py \neq 0$. The displayed identity therefore gives $c'_{\vartheta} \neq 0$ for every allowed twist. It also gives $\ker u_{j,\vartheta} \subseteq \ker P = 0$. This joins the same three measured exchanges to the comparison operator before applying the false-lift and determinant tests.

40 Coefficient reproduction on the descent space

In this section, we establish real coefficient separation and the inner-product identities for the original periods of Section 39.4. These give the coefficient tests used in the proof of injectivity in Section 41.

All measures, classes, Fock polynomials and finite lattices in the following argument belong to the original specified space. The sharp inner-product calculation below includes an arbitrary rapidly decreasing left test and the compact-source carrier. The adelic overlap calculation gives the polarized Fock overlap on the same residue ring, including the unit coefficient at zero displacement.

The original good form is read in its holomorphic coefficient by the untwisted sharp injection and its marked adjoint.

40.1 Separation by the circle mean

In this subsection, we prove a reproducing identity by integrating over a circle, then use it to estimate the reference coefficient at a peak. Section 41.1.6 applies the estimate to the common period.

Fix the original signed real product $H = \prod_{\tau\in\Phi} U(J_\tau)$ of rank n, with the marked product Haar dh, and let $f = |\Phi|$. Use the same injective order o on the basis, with the positive block first. The weights u, u' decrease weakly within each sign block. At every place the difference of any negative entry and any positive entry of each weight is at least $300n_*$, where

$$1 \le n \le t, \qquad n_* = 2ft^3, \qquad |u_{\tau,i}| \le 5b.$$

For either weight v define m_v as the product over the real places of the marked minor quotients.

In the order o, its local factor is

$$m_{v_\tau}(h) = \det(h)^{v_{\tau,n}} \prod_{a=1}^{n-1} \Delta_a(h)^{v_{\tau,a}-v_{\tau,a+1}}, \qquad \Delta_a(h) = \det(h_{[1,a],[1,a]}). \tag{CM1}$$

The only possible negative exponent on a proper minor is at the end of the positive block. That minor never vanishes on $U(J_\tau)$. Formula (CM1) is also the definition by quotient of two dominant minor products: add $2\sum_i |v_{\tau,i}|$ uniformly to the positive block in the numerator, and divide by the minor product for that shift. In a compact sign group use the same formula with no negative proper-minor exponent.

An elementary majorant. For a mixed signature write $h = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the positive-first order. The equations $h^*Jh = J = hJh^*$ imply that the singular values of A, D are at least one, $\|B\| = \|C\|$, and $\|A\|^2 = \|D\|^2 = 1 + \|C\|^2$. Consequently $\|h\| \le \|A\| + \|C\| \le 2|\det A|$. Every positive-initial segment minor has absolute value at most $|\det A|$, by the singular-value bound for an exterior power of the compression of A. For a longer initial segment, the complementary minor identity and $h^{-1} = Jh^*J$, $|\det h| = 1$, express its absolute value as that of a trailing minor of D. The same singular-value bound applies, and $|\det D| = |\det A|$ by the complementary identity at the end of the positive block. The nonnegative exponents and the one cross-block exponent in (CM1) therefore give

$$|m_{v_\tau}(h)| \le |\det A|^{v_{\tau,1}-v_{\tau,n}} \le \min\{1, (2/\|h\|)^{300n_*}\}. \tag{CM2}$$

In compact signature $\|h\| = 1$ and every proper minor has absolute value at most one, giving the same statement. This argument also proves analyticity locally around each group point by (CM1).

Let $I_\tau = \int |m_{u_\tau}|^2\, dh_\tau$. Take $L = 2\lfloor 5b\rfloor + n$, $\beta = n + L + 1$, and $\delta_1 = (200\beta^5)^{-1}$. For $\|h - 1\| \le \delta_1$, the logarithm series in each principal compression gives $|\log \Delta_a(h)| \le 2n\delta_1$. The sum of absolute exponents in (CM1) is at most $2n \max_i |u_{\tau,i}|$. Hence $|m_{u_\tau}(h)| \ge 1/2$ on that ball. A maximal δ_1/Y-separated set in the group ball $\|h\| \le Y$, $Y \ge 1$, has at most $(2000\beta^6Y^2)^{2n^2}$ points, by packing disjoint ambient matrix balls in dimension $2n^2$. The corresponding right translates of the identity ball cover the group ball, since $\|h^{-1}\| = \|h\|$.

In particular ball volumes have polynomial growth of order $4n^2$. With (CM2) this first proves $0 < I_\tau < \infty$. Then the small-ball estimate gives the sharper bound

$$dh_\tau\{\|h\| \le Y\} \le 4I_\tau(2000\beta^6Y^2)^{2n^2}.$$

Put $c = 10n^3$. Integrating the last bound against $cY^{-c-1}\,dY$ gives

$$\int (2/\|h\|)^c \, dh_\tau \le 8 I_\tau (2000\beta^6)^{2n^2} 2^c \le I_\tau 2^{13t^3} b^{14t^2}.$$

The first inequality uses $c/(c-4n^2) \le 2$. The second uses the original size condition $2000(n+L+1)^6 \le b^7$ and $n \le t$. No formula for the numerical formal degree is needed for this estimate.

Write $X(h) = \max_\tau \|h_\tau\|$ and $I = \int_H |m_u|^2 \, dh = \prod_\tau I_\tau > 0$. For each h, use $6n_*$ extra powers in (CM2) at any place attaining $X(h)$, and c powers at the others. Since $c + 6n_* \le 300 n_*$,

$$X(h)^{6n_*} |m_{u'}(h)| \le 2^{6n_*} \prod_\tau (2/\|h_\tau\|)^c.$$

This bound is independent of the choice of maximizing place. Integrating gives

$$\int_H |m_{u'}(h)| X(h)^{6n_*} \, dh \le 2^{6n_*} (2^{13t^3} b^{14t^2})^f I \le b^{n_*} I. \tag{CM3}$$

In the last step use $b \ge 2^{25}$, $t \ge 14$, and $n_* = 2ft^3$.

The circle mean and reproduction. Let a function A on H be holomorphic of weight v in the sense of the marked model. Its analytic extension to neighborhoods of hR, with R invertible upper triangular in order o, transforms on hR by the product of diagonal characters $\chi_v(R) = \prod_{\tau,i} R_{\tau,ii}^{v_{\tau,i}}$. Use the circle with probability Haar $d\theta$, the simultaneous rotation $r_s = \mathrm{diag}(s^{o(i)})$ at every real place, and let $\chi_v(s) = \chi_v(r_s)$. Then

$$\int_{\mathbb{T}} A(r_s h) \chi_v(s)^{-1} \, d\theta(s) = A(1) m_v(h). \tag{CM4}$$

Here is a direct verification. For h close to the identity write each $h_\tau = L_\tau R_\tau$ with L_τ lower unitriangular and R_τ upper triangular.

For $|s| = 1$,

$$r_s h_\tau = (r_s L_\tau r_s^{-1})(r_s R_\tau).$$

All strictly lower entries of the first factor have positive powers $s^{o(i)-o(j)}$. The factor extends polynomially to the closed disc, is identity at its center, and stays in the analytic neighborhood of the identity for h sufficiently close to it. Write $L_s = r_s L r_s^{-1}$ and let $\widetilde{A}$ be the holomorphic extension near the identity. For $|s| = 1$, the equality $L_s = (r_s h)(r_s R)^{-1}$ has its first factor in H and its second upper triangular. Hence

$$\chi_v(s)^{-1} A(r_s h) = \chi_v(R) \widetilde{A}(L_s).$$

The mean value property gives $\chi_v(R) A(1)$, which equals $m_v(h) A(1)$ by the leading-minor formulas for triangular elimination. For arbitrary h join identity to h by a real analytic path in the connected product H.

Such a path is obtained from the compact and positive factors of the polar decomposition. The two sides of (CM4) are real analytic along it. For the left side the local analytic extension is uniform in the circle variable by its compactness. Analytic continuation along the path proves (CM4).

If $A\overline{m_v}$ is integrable, apply (CM4) after the left translations $h \mapsto r_s h$, which preserve the original Haar. The principal minors in (CM1) transform by their diagonal weights, so

$m_v(r_s h) = \chi_v(s) m_v(h)$. Joint integrability follows from the same left translations and the probability circle. Hence

$$\int_H A(h)\overline{m_v(h)}\,dh = A(1)\int_H |m_v(h)|^2\,dh. \tag{CM5}$$

The peak estimate. Let F be continuous and bounded on the marked adelic group, with $h \mapsto F(x(h,1))$ holomorphic of weight u for every x. Let z be a peak, so $F(z) \neq 0$ and $\sup|F| \le 2|F(z)|$. Suppose K_{ref} is bounded on H, and at every peak x some K_x of holomorphic weight $u - u'$ satisfies

$$|K_x(h) - K_{\mathrm{ref}}(h)| \le \delta X(h)^{6n_*}, \qquad \int_H \overline{F(x(h,1))}K_x(h)m_{u'}(h)\,dh = 0. \tag{CM6}$$

Lemma 40.1 (The circle-mean peak estimate)**.** *Under the holomorphic weight, peak, and approximation conditions above, the reference function satisfies*

$$|K_{\mathrm{ref}}(1)| \le (4b^{n_*} + 1)\delta.$$

Proof. Every $z_s = z(r_s, 1)$ is again a peak: its value differs from $F(z)$ by the unit character $\chi_u(s)$. Put $K_* = K_z$ and

$$W_s(h) = \overline{F(z(r_s h, 1))\chi_u(s)^{-1}}.$$

Equation (CM4) at z gives $\int W_s(h)d\theta(s) = \overline{F(z)m_u(h)}$.

Also $|W_s(h)| \le 2|F(z)|$, and (CM6) eliminates the integral with K_{z_s} in place of K_*. Thus (CM3) bounds each fixed slice by

$$\left|\int_H W_s(h)K_*(h)m_{u'}(h)\,dh\right| \le 4\delta|F(z)|b^{n_*}I.$$

Only K_* occurs in the joint integral. It is jointly measurable and integrable by (CM3), since K_* is bounded by a constant times X^{6n_*}. No regularity in the choice of K_{z_s} as a function of s is needed. Average the slice bound and use Fubini and (CM5) on the product $A = K_* m_{u'}$, which is holomorphic of weight u. Since $m_{u'}(1) = 1$, this gives

$$I|F(z)|\,|K_*(1)| \le 4\delta|F(z)|b^{n_*}I.$$

Consequently,

$$|K_{\mathrm{ref}}(1)| \le (4b^{n_*} + 1)\delta. \tag{CM7}$$

Indeed, cancellation gives $|K_*(1)| \le 4\delta b^{n_*}$, and (CM6) at $h = 1$ bounds $|K_{\mathrm{ref}}(1) - K_*(1)|$ by δ, since $X(1) = 1$. □

40.2 The inner product formula at the principal levels

In this subsection, we unfold the inner product with an arbitrary rapidly decreasing left test on the original levels. The calculation extends the use of the Gram identity in Section 34.2 to the left tests needed for injectivity.

Let S, T be the original Hermitian spaces of ranks n, s, with $s > 2n$, and retain their finite unitary bases. Write $G = U(S)$, $H = U(T)$, and use the original quotient measures. On the finite group normalize the integral compact at each place of F_0 to have mass one. On additive

compact quotients in the Weil calculation use probability Haar. These conventions leave the factor $V_T = \mathrm{vol}([H])$ when passing from the probability average to the original target pairing. The argument applies to the pairs

$$(n, s) = (r, t - 9m - 1), \quad (3m, 6m + 1), \quad (m, 3m).$$

In the first pair $s - 2n = 3r - 9m - 1 > 0$ for sufficiently large x, since $t = 5r$, $r \geq x^5$ and $m < 2\sqrt{x}$. In the second and third pairs $s - 2n = 1$ and m, respectively. The first two pairs use the original point datum at $\mathfrak{m} = (\pi_M^2)$. The third uses the unit-modulus vacuum and has compact real source.

The sharp datum and its mass. At the chosen member w of each chosen split pair the Schrödinger variable X is an s-by-n matrix. The point datum is the indicator of $X \equiv \iota \pmod{\mathfrak{m}}$ for the marked inclusion ι, with vacuum at the other places. Write $R = \mathcal{O}_E/\mathfrak{m}$. The finite inner product is the average on $M_{s,n}(R)$ in the unit vacuum scale. Thus the squared norm of the point datum is $|R|^{-sn}$. Let J be the original principal level in the standard compact K^0 at this modulus. The integral action on the slice is

$$\Psi(X) \longmapsto b(\det g)^{-1}\Psi(Xg).$$

Here b is the source-actuating character of the lift. Its tame part is trivial away from the prescribed support, and on J it evaluates to one. In the two point tests its powers are, respectively, $b = \upsilon^{m/2-j}$ and $b = 1$. In particular there is no extra conductor from these characters.

Let g be integral off the marked pairs and suppose it gives nonzero overlap. On the support, X and Xg both have integral left inverses. Applying them to Xg and X, respectively, makes g and g^{-1} integral at the chosen member of every marked pair. At the opposite member this follows by inverse transpose. The congruences $X \equiv Xg \equiv \iota$, followed by a left inverse of ι in the residue ring, give $g \equiv 1$.

Conversely J fixes the test. The character on the marked groups therefore gives no factor on its coefficient support. The coefficient on the points integral off the marked pairs is exactly $|R|^{-sn}\mathbf{1}_J$. Its integral is

$$c_{\mathfrak{m}} = \frac{|R|^{-sn}}{|\operatorname{GL}_n(R)|}.$$

At a selected place w of exact modulus exponent two the local mass is

$$\frac{q_w^{-2sn}}{q_w^{n^2}|\operatorname{GL}_n(k_w)|}.$$

Indeed reduction from exponent two to exponent one is surjective with kernel $1 + \varpi_w M_n(k_w)$. The split group over F_0 is one copy of $\operatorname{GL}_n(E_w)$. The opposite matrix is determined by the inverse transpose and does not give a second volume. At unit modulus $J = K^0$, the finite vacuum is a unit vector, and $c_{(1)} = 1$.

40.2.1 The adelic doubling identity

In this subsection, we identify the doubled kernel in the original quotient measures after the mass calculation of Section 40.2.

Let v be the original real Fock vector and $f = v \otimes F$ the datum with the finite part just specified. Denote the rational Gram matrices by J_S, J_T and the adelic point matrices by X_S, X_T.

Put $u = (1, u_f)$ with $u_f\overline{u_f} = -1$, so the opposite space has $J_{S^-} = -J_S$ and $X_{S^-} = uX_S$. Let C be the antiunitary oscillator transport to the opposite space. At infinity the coordinate transport is the identity, and complex conjugation of the oscillator values reverses the phase. At finite places $Y \mapsto u_f^{-1}Y$ has norm factor -1 and reverses the alternating form there. Write θ_+ and θ_- for the fixed theta functionals from Section 17.1. Rational Heisenberg invariance characterizes each functional up to a scalar. Their squared means on unit vacuum translates are both one. Consequently a fixed scalar η_0, $|\eta_0| = 1$, satisfies

$$\theta_-(\eta_0 Cw) = \overline{\theta_+(w)} \qquad \text{for every } w.$$

Use $f \otimes \eta_0 Cf'$ in the two oscillator blocks. If $f[X] = \theta_+(T(X)f)$ denotes evaluation by displacement, then $T(Y)C = CT(u^{-1}Y)$ and the product rule for the theta functional give the doubled evaluation

$$f[X]\,\overline{f'[u^{-1}Y]}.$$

For the pair characters (a, b) the opposite pair has characters $(a^{-1}\upsilon_1^n, b^{-1}\upsilon_1^s)$, where $\upsilon_1 = \upsilon_E|_{T_1(\mathbb{A}_{F_0})}$. Indeed the multiplier calculation of Section 39.3 applies to $X \mapsto hXg^{-1}$, of determinant $(\det h)^n(\det g)^{-s}$. Thus the two kernels on the diagonal target give the original kernel times the conjugate of the other original kernel. Their target characters have product $\upsilon_1(\det h)^n$, which cancels the mixed-model Levi scalar $\upsilon_1(\det h)^{-n}$ in Section 17.2.

In the split polarization the rational Lagrangian

$$L_\delta = \{(\delta^{-1}y, -y) : y \in S\}, \qquad \delta \in G(F_0),$$

is parameterized by $y \mapsto (\delta^{-1}y, -y)$. For a matrix Z in the adelic zero term set

$$M = X_T^{-1}J_T^{-1}ZJ_SX_S, \qquad \widehat{\delta} = X_S^{-1}\delta X_S.$$

The adjoint of this parametrization gives the blocks $(M\widehat{\delta}, uM)$, since $(\delta^{-1})^*J_S = J_S\delta$ and $(-1)^*(-J_S) = J_S$. The second tensor factor therefore has argument M in its original coordinates. The change $Z \mapsto M$ maps the standard rational matrices bijectively onto

$$\{X_T^{-1}WX_S : W \in \mathrm{Mat}_{s\times n}(E)\},$$

because J_T^{-1} and J_S are rational and invertible. The induced isomorphism of additive quotients preserves probability Haar. The zero term is consequently the mean of $f[M\widehat{\delta}]\overline{f'[M]}$. Rational Weil transport gives the action of δ in the first factor, and the quotient inner-product identity of Section 41.5.1 gives the oscillator pairing with f'. Translating the two data by g, g' and using unitarity gives the coefficient

$$\langle f', \omega_b(g'^{-1}\delta g)f\rangle.$$

Only these graphs contribute. If L is a Lagrangian and its second projection is injective, it is an isomorphism since $\dim L = n$; the isotropy equation then makes the first projection an isometry. Otherwise L contains $(c, 0)$ with $c \neq 0$. In the compact-source case this is impossible by definiteness. For a point test, the zero-term functional is invariant under the Heisenberg translations dual to this vector, acting on the first tensor factor alone.

Indeed in a split basis for L the zero term is the mean over the complementary directions, and these translations act by a change of the averaged variable. In the marked split Schrödinger

slice, after moving the source point through the Heisenberg action, the same translations act by the multipliers

$$X \longmapsto \psi_w(\mathrm{tr}(ZXc_w)),$$

where Z varies over the dual local vector space and $c_w \neq 0$ is the coordinate in the chosen basis of c.

An invariant functional kills a compactly supported, locally constant test supported where $Xc_w \neq 0$: partition its support into finitely many compact open pieces on each of which one such multiplier is a constant different from one, and apply invariance on each piece. The point datum is supported on matrices with integral left inverse, so $Xc_w \neq 0$ throughout. The graph terms therefore exhaust the doubled expansion.

The doubled action is identified in Section 17.2. Apply the Siegel–Weil formula [77, Theorem 2] to the split polarization of rank n of $S \oplus S^-$, with T as the Hermitian space of dimension s. The doubled splitting uses the Hecke character υ_E^s. Its restriction to $\mathbb{A}_{F_0}^\times$ is ε_{E/F_0}^s. Yamana's division algebra is E, and his Witt index of T is at most $s/2$. Hence the convergent clause $s - \mathrm{Witt}(T) > n$ applies with coefficient one and $s_0 = (s-n)/2 > n/2$.

The real Weil translates and the change of polarization give Schwartz–Bruhat sections. In this strict range $s > 2n$, use the formula for all such sections in [75, §52, Theorem 5], with its quotient probability measure. Returning to the original target Haar measure gives

$$\int_{[H]} \overline{\theta_{f'}(g',h)}\, \theta_f(g,h)\, dh = V_T \sum_{\delta \in G(F_0)} \langle f', \omega_b(g'^{-1}\delta g) f \rangle.$$

For the two source integrations we need absolute bounds in the original quotient measures. In the split Schrödinger model and for a doubled datum Φ, set

$$J(\Phi) = \int_{[H]} \sum_{X \in T^n(F_0)} |\Phi(h^{-1}X)|\, dh.$$

Here the rational points of T^n are taken over F_0 as for the restriction of scalars. For a fixed finite factor this is finite on the real Schwartz space.

Indeed the absolute value of the real factor is bounded by a nonnegative Schwartz function, by taking smooth annular cutoffs with rapidly decreasing coefficients. Use that majorant and the absolute value of the finite factor in the convergent theta integral [75, §51, Proposition 8]. The target acts by substitution in the split model of Section 17.2. The involution of E is of the second kind, so Weil's type (I) parameter is $1/2$ by [75, §§26–27]. His convergence condition is therefore $s - \mathrm{Witt}(T) > n$, as verified above.

The absolute integral J is bounded by a continuous real Schwartz seminorm with the finite factor fixed. To see this, choose a measurable quotient section and take finite partial sums of the rational evaluations, with measurable multipliers of modulus at most one. After integration over the section these are continuous linear functionals, each bounded by the real supremum norm times a finite constant. Their common pointwise bound is J. Uniform boundedness on the Schwartz Fréchet space gives a common continuous seminorm. Taking the supremum over partial sums and multipliers proves the claim. In particular J bounds the absolute product of the original theta kernels integrated over $[H]$.

The absolute graph sum is bounded by the convergent spherical Eisenstein series at $s_0 > n/2$ times $\sup_k |(\omega(k)\Phi)(0)|$ for the maximal compact. That supremum is a continuous real Schwartz seminorm up to a fixed constant. The real compact acts continuously on Schwartz data and the

finite compact has finitely many translates of the fixed finite factor. The absolute graph sum is rationally invariant in the two source variables by reindexing. The corresponding absolute product on the target side is invariant by rational theta invariance. Thus we may reduce both source representatives as in Section 18.1. Their finite translates range over finite sets.

Multiplication and differentiation operators are conjugated by the real Weil action into linear combinations with coefficients from the symplectic matrix and its inverse. By unitarity their products give polynomial bounds for the translated weighted Sobolev norms, which generate the Schwartz topology. The two seminorm bounds for the doubled translate are therefore at most $C\mathcal{H}(g)^{\nu}\mathcal{H}(g')^{\nu}$ for the original reduction heights. Multiplying by the rapid source functions gives integrable majorants on the two finite-volume quotients. This argument also applies to the strict-range doubled first arrow.

For rapid y', y, multiply by $\overline{y'(g')}y(g)$ and unfold the rational graph sum. For each fixed g', the sum of the translated source quotient integrals is the $G(\mathbb{A})$-integral in $a = g'^{-1}\delta g$. Left automorphy gives $y(g) = y(g'a)$ on that graph term. Thus

$$\langle L_{f'}y', L_f y\rangle_T = V_T \int_{G(\mathbb{A})} \langle f', \omega_b(a)f\rangle \langle y', R(a)y\rangle_S \, da.$$

In particular the first test need not be fixed by J.

40.2.2 Vacuum coefficients away from the marked places

In this subsection, we evaluate the local factors in Section 40.2.1.

Let A be the union of conjugation orbits outside the support of $\mathfrak{m}\bar{\mathfrak{m}}$. At A put $\Lambda = \widehat{\mathcal{O}}_E^n$ in the original finite basis. For γ supported on A put $\Delta(\gamma) = (\upsilon_E)_f(\det B)|\det B|^{-1/2}$, where $B\Lambda = \Lambda \cap \gamma^{-1}\Lambda$. This expression is independent of the integral basis of the intersection since $(\upsilon_E)_f$ is unramified there. The index formula is

$$I(\gamma) = |\Delta(\gamma)|^2 = [\Lambda : \Lambda \cap \gamma^{-1}\Lambda] = [\gamma\Lambda : \Lambda \cap \gamma\Lambda].$$

In the normalized finite lattice model the uncharactered Weil operator for an isometry a of the displacement space is

$$\omega(a)u = \Delta_{\mathrm{disp}}(a) \int u(a^{-1}z)\mathcal{R}(z)v_{\mathrm{vac}}\, dz,$$

where z is a finite Heisenberg displacement, $\mathcal{R}(z)$ its operator, and the vacuum lattice has additive measure one.

In the vacuum matrix coefficient the integrand is supported on the intersection of the vacuum lattice and its image under a. The vacuum multiplier on one factor is the complex conjugate of that on the other, since a preserves the Hermitian norm. It follows that

$$\langle v_{\mathrm{vac}}, \omega(a)v_{\mathrm{vac}}\rangle = \Delta_{\mathrm{disp}}(a)|\Delta_{\mathrm{disp}}(a)|^{-2} = \overline{\Delta_{\mathrm{disp}}(a)}^{-1}.$$

On $\mathrm{Hom}(S,T)$ the action of γ is $X \mapsto X\gamma^{-1}$. The s covectors give $\Delta_{\mathrm{disp}}(a) = \Delta(\gamma^{-1})^s = \overline{\Delta(\gamma)}^{s}$. For the last equality, the inverse of a unitary matrix is its conjugate transpose. Transposition preserves the elementary divisors, and conjugation takes the idele splitting multiplier to its complex conjugate in the fixed unramified normalization. The character $b(\det\gamma)^{-1}$ therefore gives the exact coefficient

$$a(\gamma) = \big(b(\det\gamma)\Delta(\gamma)^s\big)^{-1}, \qquad |a(\gamma)| = I(\gamma)^{-s/2}.$$

It multiplies over disjoint conjugation orbits and is bi-invariant by the standard integral group on A.

40.2.3 Summability and injectivity

In this subsection, we use Kneser's method of neighboring lattices to bound the coefficients of Section 40.2.2. For the Hermitian neighbor construction, see Greenberg and Voight [30, Propositions 5.3–5.4].

At a split pair a lattice at w determines the opposite lattice by duality. If the moved lattice is not contained in the standard one at w, take a vector x in it with $\varpi_w x$ primitive in the standard lattice. Enlarge the standard lattice by x and use its dual at $\bar{w}$. The intersection is multiplied by a factor at least q_w at w and does not decrease at $\bar{w}$, by pairing with x. If containment is strict in the other direction, choose the vector at $\bar{w}$. Each enlargement at a chosen member is determined by the line of $\varpi_w x$ in k_w^n, giving at most $(q_w^n - 1)/(q_w - 1)$ steps for that member.

At a nonsplit finite place the extension $E_w/F_{0,v}$ is unramified. Take $\varpi_w = \varpi_v$ in the fixed field. Both lattices are self-dual. Choose x in the moved lattice with $y = \varpi_w x$ primitive in the standard lattice Λ_w, and put

$$\Lambda_0 = \{z \in \Lambda_w : \langle y, z\rangle \in \varpi_w \mathcal{O}_{E_w}\}.$$

The lattice $\Lambda_0 + \mathcal{O}_{E_w} x$ is integral and self-dual, since its two successive indices relative to Λ_0 and Λ_w are equal to q_w. The old intersection lies in Λ_0, and adjoining x increases its index by at least q_w.

For a fixed residue line choose a representative of $\bar{y}$; choices of x then differ by $z \in \Lambda_w$. The integrality of the two Hermitian squares requires $\mathrm{Tr}_{k_w/k_v}\langle \bar{y}, \bar{z}\rangle = 0$. This leaves at most $\sqrt{q_w}$ classes modulo Λ_0 for each residue line. These neighboring self-dual lattices admit integral orthonormal bases. One way to see this last assertion is to find a vector of unit norm by nondegeneracy of the residual Hermitian form, normalize it by surjectivity of the unramified norm on units, and continue on its unimodular orthogonal complement. The residual norm polynomial is not identically zero on a nondegenerate Hermitian space: if every self-pairing vanished, polarizing after multiplication by all elements of k_w and using the nondegenerate trace would make every pairing zero. This argument also applies in residue characteristic two.

Write $L' = \gamma\Lambda$, and let L_1 be the neighbor just constructed. At a nonsplit place the map

$$\frac{L'_w \cap L_{1,w}}{L'_w \cap \Lambda_w} \longrightarrow L_{1,w}/\Lambda_0$$

is injective since $L'_w \cap \Lambda_w \subset \Lambda_0$, and surjective since the class of x generates the target. At a split pair the same argument uses $L_{1,w}/\Lambda_w$ on the enlarged side. On the opposite side the two intersections agree, by the integral pairing with x. Fix an isometry ρ taking each neighbor back to the standard lattice. All intersection indices except the one at w are unchanged, so

$$I(\rho\gamma) = q_w^{-1} I(\gamma), \qquad |a(\gamma)| = q_w^{-s/2} |a(\rho\gamma)|.$$

The positive integral index reaches one after finitely many steps, and the inverse word recovers the original coset. The total weight of one-step choices at w is bounded, since $s \geq 2n + 1$, by

$$a_w \leq \begin{cases} \dfrac{q_w^n - 1}{q_w - 1} q_w^{-s/2} \leq \dfrac{1}{(q_w - 1)\sqrt{q_w}}, & w \text{ split}, \\ \sqrt{q_w}\,\dfrac{q_w^n - 1}{q_w - 1} q_w^{-s/2} \leq \dfrac{1}{q_w - 1}, & w \text{ nonsplit}. \end{cases}$$

Every $q_w \geq 4$, since E contains both a primitive third and a primitive fourth root of unity. A split orbit has two members of equal norm, so its total is at most $1/3$, as is the weight at a nonsplit orbit. The weights are summable. At split places they are $O(q_w^{-3/2})$, and at nonsplit places $q_w \geq p^2$ for the rational prime p underneath. There are at most $[E : \mathbb{Q}]$ places over p.

For any union B of conjugation orbits of weight $\sum_{w \in B} a_w \leq 1/3$, sum the weights over the descending words. The identity contributes one and the entire remaining mass is at most

$$\sum_{h \geq 1} \Big(\sum_{w \in B} a_w \Big)^h \leq \frac{1}{2}.$$

This proves absolute convergence and injectivity of $\mathcal{V}_B = \sum_{\gamma \in G_B / K_B} a(\gamma) R(\gamma)$ on bounded functions right invariant under K_B. Indeed $\|\mathcal{V}_B u - u\|_\infty \leq \|u\|_\infty / 2$. Choose finitely many conjugation orbits so that the total weight of all remaining places is at most $1/3$. Intersecting these orbits and the tail with A gives the same bound uniformly over the omitted supports. Absolute convergence and multiplicativity identify $\mathcal{V}_A$ with the product of these finitely many factors and the tail factor. Each factor preserves the other factors' integral levels by disjoint supports, and its own by bi-invariance. On the supremum-norm space of bounded functions with those levels, $a(1) = 1$, and for each piece B

$$\|1 - \mathcal{V}_B\| \leq \tfrac{1}{2}, \qquad \mathcal{V}_B^{-1} = \sum_{d \geq 0} (1 - \mathcal{V}_B)^d.$$

The partition into individual orbits and the tail was fixed before intersecting with A. Its finitely many inverse factors therefore give injectivity for every omitted support at these ranks.

40.2.4 Real convolution and the original measures

In this subsection, we combine the absolutely convergent integrals of Section 40.2.3 with the real projectors.

Take $f' = f$ and y of level J, and put $k_\infty(h) = \langle v, \omega_b(h) v \rangle$. For fixed real h, $\langle y', R(h\gamma) y \rangle$ is right J-invariant in the finite variable, even for a test y' of a different level. On each coset γK_A the compact at A has mass one and the coefficient is $a(\gamma)$ times the sharp coefficient at the marked pairs. Fubini and the exact sharp mass calculated above give

$$\langle Ly', Ly \rangle_T = V_T c_{\mathfrak{m}} \left\langle y', \mathcal{V}_A \left(\int_{G(\mathbb{R})} k_\infty(h) R(h) y \, dh \right) \right\rangle_S .$$

Here $L = L_f$. In each holomorphic test let $M_u = m_u$ be the minor quotient of (CM1), with u the source weight of v for ω_b. The coefficient $h \mapsto \langle v, \omega_b(h) v \rangle$ is holomorphic of weight u: in the highest minor, the strictly upper triangular operators within a sign block annihilate its flag minors, and the mixed second derivatives vanish because only one sign of source variable occurs in each of the two definite-target tests. The diagonal gives the weight u. The extension to the complex upper triangular group is the one proved in Section 40.3. Unitarity and the circle action on the first vector give

$$k_\infty(r_s h) = \chi_u(s) \, k_\infty(h).$$

With the pairing linear in its second variable, the circle average in (CM4) therefore yields $k_\infty(h) = k_\infty(1) m_u(h) = \|v\|^2 M_u(h)$. The opposite-sign gap gives $M_u \in L^1$. On the compact

real carrier source the coefficient is also integrable. Absolute summability of the finite series and integrability on the real group justify interchanging them.

If $Ly = 0$, the formula shows that $z = \mathcal{V}_A \int k_\infty(h) R(h) y \, dh$ is orthogonal to every rapid test. Each right translate of y is such a test. The function z is continuous and bounded by absolute convergence and has finite L^2 norm on the finite-volume quotient. Expressing the second occurrence of z in $\langle z, z \rangle_S$ by this absolutely convergent series and integral of translates gives zero. Thus $z = 0$. The real convolution is right integral-level invariant on A, so the injectivity above gives

$$\int_{G(\mathbb{R})} k_\infty(h) y(xh) \, dh = 0 \qquad \text{for every } x.$$

This is the real convolution implication for both marked tableau reproductions and the carrier.

40.3 The adjoint of a single lift and its holomorphic type

In this subsection, we apply the Gram identity of Section 34.2 to the original lift and its adjoint. Tracking the complex conjugation of the real coefficient gives the weight of the projected source used in Section 41.

Fix j and the untwisted marked descent pair $W = W_j$, $V_j = I_{r+t}$. Write $d_{\mathrm{tgt}} = r + t$, Y^{pr} for the full old principal-level type space on W, and $Y = e_0 Y^{\mathrm{pr}}$ for its good part. Write $L : Y^{\mathrm{pr}} \to \mathcal{A}(V_j)$ for the single lift into the original algebraic form space and $L_Y = L|_Y = u_{j,1}$. The metric on Y^{pr} is its source Petersson metric, and that on $\mathcal{A}(V_j)$ is the marked target class metric.

The first-arrow formula of Section 34.2 on this pair gives, for $x, y \in Y$,

$$\langle L_Y x, L_Y y \rangle_{\mathrm{class}} = A_j \langle x, R_j^W y \rangle_W.$$

Here $A_j \neq 0$ is the nonzero real edge constant with its positive class and sharp-level factors. Its real datum is the joint harmonic of the untwisted pair. The first-arrow calculation establishes the nonvanishing before the comparison period is constructed. The vacuum series R_j^W preserves Y: the unmarked spherical finite operators commute with the old residue action, and real and finite right translations commute. It is injective by the finite vacuum estimate. Thus $L_Y y = 0$ implies $R_j^W y = 0$ by applying the displayed identity with $x = R_j^W y$, and then $y = 0$. This deduction uses only the two spaces of the single lift.

Let $\mathcal{K}_g \in \mathcal{A}(V_j)$ denote the vector theta kernel at $g \in U(W)(\mathbb{A})$. For a Weyl coordinate vector ζ, its value at the finite target point c is characterized by

$$\langle \zeta, \mathcal{K}_g(c) \rangle_{\mathrm{W}} = \theta_{P_{\bar{\zeta}}, F_{\mathrm{bl}}}(g, (1, c)).$$

Here P_ξ is the marked mixed Cartan polynomial, F_{bl} is the original untwisted block datum, and the characters are v^{m-j} on the target determinant and $\mu_j^0 v^{d_{\mathrm{tgt}}/2}$ on the source determinant in the inverse-character convention of the Weil action.

In particular $P_{\bar{\zeta}}$ is linear in $\bar{\zeta}$. The right reduction actions on the block test and the Weyl action on the polynomial give the stated level and rational covariance of $\mathcal{K}_g$. Explicitly the simultaneous integral source and target lifts preserve the inclusion reduction, and the target rational change is the polynomial's Weyl substitution. Moving the source principal level alone fixes the residue block, and the tame character is trivial on its determinant.

Let C run over the finite target classes at the original level and c_C be their representatives. The integral and the adjoint are

$$\begin{aligned} Ly &= \int_{[U(W)]} y(g)\mathcal{K}_g \, dg, \\ (L^*\Xi)(g) &= \sum_C \frac{\langle \mathcal{K}_g(c_C), \Xi(c_C)\rangle_{\mathrm{W}}}{|\operatorname{Aut}(c_C)|} \\ &= \sum_C \frac{\overline{\theta_{P_{\overline{\Xi(c_C)}},F_{\mathrm{bl}}}(g,(1,c_C))}}{|\operatorname{Aut}(c_C)|}. \end{aligned}$$

Indeed the scalar expression in the last two lines is the normalized descent. For the compact target, its highest matrix unit cuts out the smaller group's irreducible partner (Section 33.1.1). The harmonic belongs to the highest line of the lowest compact type of this partner, which occurs once. Thus its smaller-group line projector fixes it. The Cartan map is equivariant, so every $P_{\bar{\zeta}}$ is a linear combination of compact target translates of that harmonic. These translations commute with the smaller-group projector. Conjugating the scalar kernels in the displayed adjoint therefore gives the original source coefficient projector. The finite covariance above gives principal level. The weighted convolution and truncation argument of Section 34.3.2 applies to the finite class sum and places it in Y^{pr}. Rapid decay of y and moderate growth of each of the finitely many kernels give absolute integrability. Pairing the first integral with Ξ and moving the finite class sum inside it proves the adjoint formula. This pairing has no target level index, because the original metric is the class sum with reciprocal stabilizer weights.

It suffices to use the good restriction. The orthogonal projection from Y^{pr} to Y is the original unitary finite projector e_0, so $L_Y^* = e_0 L^*$. The injective map L_Y has invertible $L_Y^* L_Y$. Thus for every $y \in Y$ there is $w \in Y$ with

$$y = e_0 L^* L w.$$

Conjugating L^*Lw yields a finite combination of the scalar kernels with their chosen Fock polynomials. The operator obtained by conjugating e_0 is a finite linear combination of finite right translations. It preserves the real coefficient law at every adelic left translate. The real weight can therefore be checked on the scalar kernels of this descent.

At τ let $n(c)$ be the numbering in the descent datum, counting the positive coordinates in their original order and the negative coordinates in reverse. Use the order

$$o(c) = \begin{cases} -1 - n(c), & c \text{ positive}, \\ n(c), & c \text{ negative}. \end{cases}$$

At τ, write $\mu_\tau^{\pm}(h) = k^{\pm}(h)$ for the two column multiplicities of Section 33.1, and put $\gamma_\tau = \kappa_{j,\tau} - d_{\mathrm{tgt}}/2$. In the highest polynomial each height-h minor uses the source columns numbered $0, \dots, h-1$ in its sign block. The source diagonal action contributes minus the number of such columns containing c for the positive variables, and plus the corresponding number for the negative variables. The real vacuum adds d_{tgt} on negative source columns, one for each definite target space, and zero on positive ones. The inverse determinant character contributes the common shift $-\mathrm{wt}(\mu_j^0 \upsilon^{d_{\mathrm{tgt}}/2})_\tau = \gamma_\tau$.

The resulting weight is

$$u_\tau(c) = \begin{cases} \gamma_\tau - \displaystyle\sum_{n(c)<h<d_{\mathrm{tgt}}} \mu_\tau^+(h), & c \text{ positive}, \\ \gamma_\tau + d_{\mathrm{tgt}} + \displaystyle\sum_{n(c)<h<d_{\mathrm{tgt}}} \mu_\tau^-(h), & c \text{ negative}. \end{cases}$$

The compact highest weight calculation on the same minors identifies it with the lowest compact weight of the negated doubled source parameter. Equivalently writing $\lambda_{W,\tau}$ for the doubled source parameter, the formula $2u_\tau = -\lambda_{W,\tau} + 2\rho_{\mathrm{n}} - 2\rho_{\mathrm{c}}$ gives the same entries: successive differences within each sign block are the two displayed staircase differences and the vacuum and character shifts give their initial constants.

The strictly upper triangular root operators for o in one sign block kill the joint highest polynomial by the minor rule. In different sign blocks an upper triangular operator is in the positive-to-negative position of this order. Its oscillator action is a sum of mixed second derivatives in a fixed target space. The positive and negative minor diagrams use disjoint target space sets, so every summand is zero. Thus every strictly upper triangular operator kills the highest polynomial and the diagonal acts by u_τ.

For completeness, this infinitesimal statement gives the holomorphic law of the actual matrix coefficients. Let $G_\tau = U(J_\tau)$ and let B_τ be the complex invertible upper triangular group for o. Their product $G_\tau B_\tau$ is open, since $\mathfrak{g}_\tau + \mathfrak{b}_\tau = \mathfrak{gl}_r(\mathbb{C})$ as real spaces. The intersection $G_\tau \cap B_\tau$ is the diagonal unitary torus, by elimination in the ordered form with all diagonal entries nonzero.

For the highest Fock vector v, extend the real orbit map on this product by $\rho(gb)v = \chi_{u_\tau}(b)\rho(g)v$. The torus weight makes the definition independent of the decomposition. Its real differential at 1, on a tangent $Z = Y + X$ with $Y \in \mathfrak{g}_\tau$, $X \in \mathfrak{b}_\tau$, is

$$d\rho(Y)v + d\chi_{u_\tau}(X)v = d\rho_{\mathbb{C}}(Z)v,$$

which is complex linear. The equality uses upper triangular annihilation and the diagonal weight. Left translation by G_τ and right translation by B_τ transport this calculation throughout the open product.

The same calculation applies to the theta evaluation at each fixed adelic point x and finite target class C. Indeed this evaluation is a continuous linear functional $T_{x,C}$ on the real Schwartz space: in a rational polarization its absolute lattice sum is bounded by a Schwartz seminorm. In the real Schrödinger model, linear changes of variables, multiplication by quadratic phases and their Fourier conjugates give local coordinates for the oscillator action. Differentiating these operators on a polynomial times the Gaussian in any Schwartz seminorm gives the smooth real orbit. Thus on the product of the open sets $G_\tau B_\tau$ the function

$$(gb) \longmapsto \chi_u(b)\, T_{x,C}(\rho(g)v)$$

is smooth, by local smooth sections of multiplication. Its differential is complex linear by the calculation above, so it is holomorphic. The compact target action commutes with ρ and its translates of the highest vector span the prescribed Weyl space. The same formula therefore applies to every Cartan polynomial in the descent. Taking the finite class sum and the finite translations of the conjugate projector proves this law for $h \mapsto \overline{y(x(h,1))}$ at every adelic x.

Finally, put $p_- = m - j$, $d_- = t - 9m - 1$. The complementary kernel weight is

$$u_\tau^\circ(c) = \begin{cases} j - m/2, & c \text{ positive}, \\ j - m/2 + d_- + \displaystyle\sum_{n(c)<h<d_-} \mu_\tau^-(h), & c \text{ negative}. \end{cases}$$

For $p_- > 0$, reading the marked stem at either side of its j-th gap gives the last negative column count

$$\mu_\tau^-(p_-) = \Delta/2 - \lfloor t/2 \rfloor - 1 \geq 300 n_*.$$

In fact the two stem entries are $\Delta/2 - \kappa_{j,\tau}$ and $-\Delta/2 - \kappa_{j,\tau}$, while the truncating endpoint for the negative count is $-\lfloor t/2 \rfloor - \kappa_{j,\tau}$. For the numerical bound, $t = 5r$, $n_* = 2ft^3$, $f \leq x^{12}$ and $r \geq x^5$ give

$$\frac{300 n_* + t}{\Delta/2} = \frac{75000 f}{r^5} + \frac{5}{r^7} \leq 75000 x^{-13} + 5x^{-35} < 1$$

for sufficiently large x. Every negative numbering is smaller than $p_- < d_-$, so the count is included at every negative index. The two weights are dominant on each sign block because the counts are nonnegative and the positive numbering is reversed in o. They both have the required gap $300 n_*$ between a negative and a positive entry. When $p_- = 0$ this gap condition is empty. The doubled source parameter has entries of modulus at most $2\kappa_{m+1,\tau}$; the signed sum defining $\rho_{\mathrm{n}} - \rho_{\mathrm{c}}$ contributes at most r in each index. Thus

$$|u_\tau(c)| \leq 2\kappa_{m+1,\tau} + r \leq 5b.$$

These are the weights and estimates used in the two peak and circle calculations.

41 Injectivity and estimates at a maximum

In this section, we use the reproducing coefficients of Section 40 to show that the common period of Section 39.4 detects every nonzero vector in the distinguished source subspace. At a maximum, the leading term dominates the other terms in the period sum.

41.1 Reproduction and separation at a maximum

In this subsection, we apply the period identity of Proposition 39.1 at a point of maximum norm.

Fix the marked j-th descent pair, put $W = W_j$, and let V° be the positive definite complementary space of rank $t - 9m - 1$. All integrals on the adelic groups below use the original quotient measures. Put $\mathfrak{m} = (\pi_M^2)$. For a class c in $U(V^\circ)$ at this level take its prescribed representative c_f. For a tuple $\zeta = (\zeta_c)_c$ of Weyl vectors of the negative diagram, write $K_\zeta^\circ(g)$ for the sum over c of the kernel of (W, V°) at (g, c_f). Its finite test is the indicator of the marked inclusion of W modulo $\mathfrak{m}$. Its polynomial is the evaluation in the negative coordinates of $\overline{\zeta_c}$ for the original numbering of W. Its splitting characters are υ^{m-j} and $\upsilon^{m/2-j}$. This pairing is conjugate-linear in ζ.

Write F_i for the twist-free first-factor functions of Section 39.4, with the choices specified below, and P for the common period operator. The marked seesaw identity gives

$$\langle \eta, Py \rangle = \sum_i C_i \int_{[U(W)]} y(g) F_i(g) K_{\eta_i}^\circ(g)\, d\bar{g}, \qquad C_i = M^{l_1} c_{L_i} |\bar{J}^1| c_{\mathrm{first}}.$$

Each $C_i \neq 0$. Here c_{first} is the nonzero multiplier c'' chosen for the integral first-factor form, and the divided-lattice Weil scalar c_{L_i} of Section 39.4.3 is nonzero. The separate c'_ϑ is the scalar

whose nonvanishing follows from the injectivity of P below. Taking η supported in one choice yields

$$Py = 0 \quad \Longrightarrow \quad \int_{[U(W)]} y(g) F_i(g) K_\zeta^\circ(g)\, d\bar{g} = 0 \quad \text{for every } i, \zeta. \tag{CP1}$$

This uses the actual complementary tuples at level $\mathfrak{m}$.

41.1.1 Real reproduction on the complementary block

In this subsection, we use the sharp-level inner product formula of Section 40.2.4 to express real convolution against the complementary kernel as a sum over the original classes. The peak argument in Section 41.1.6 applies the circle mean to this formula.

Let M_{u° be the holomorphic quotient of principal minors for the complementary kernel weight u° on W. The parameter inequalities give $2r < \dim V^\circ$ and the required holomorphic gap for $M_{u^\circ} \in L^1$. The complementary block reproducing formula says that for each adelic x, there are $a_{c,b}(x)$ for which

$$\int_{U(W)(\mathbb{R})} A(xh) M_{u^\circ}(h)\, dh = \sum_{c,b} a_{c,b}(x) \int_{[U(W)]} A(g) K^\circ_{\zeta(c,b)}(g)\, d\bar{g}. \tag{CP2}$$

Here A can be any rapidly decreasing automorphic function of level $\mathfrak{m}$. Let G_- be the Weyl Gram matrix of Lemma 33.1 and e_b the coordinate vector of the b-th negative tableau. Put

$$\zeta(c,b)_{c'} = \mathbf{1}_{c'=c}\, \overline{G_-^{-1} e_b}.$$

The class stabilizers at $J(\pi_M^2)$ are trivial by Lemma 6.1, so this is a tuple on the original level. Its polynomial has coefficient vector $G_- \overline{\zeta(c,b)_c} = e_b$. When $j = m$, the diagram is empty and the sole polynomial is 1. The definite Fock translates of the highest-weight vector in the negative Weyl space lie in this tableau span at each class. For completeness, the reproducing formula follows by viewing the left side and the finitely many right-side integrals as linear functionals on the rapid space. Vanishing of the latter implies vanishing of the highest-vector theta lift, by the tableau span and the class description at $\mathfrak{m}$. The sharp-level Siegel–Weil real convolution formula then gives vanishing of the left side. Factor the left-side functional through the finite evaluation map of the right-side functionals. This gives (CP2) with the same inclusion, two characters, and measures. It uses the preceding sharp-level convolution before the finite-dimensional factorization.

For a good form y with $Py = 0$, each yF_i is rapidly decreasing of level $\mathfrak{m}$. Equations (CP1) and (CP2) then give, for any constants a_i,

$$\int_{U(W)(\mathbb{R})} y(xh) \Big(\sum_i a_i F_i(xh) \Big) M_{u^\circ}(h)\, dh = 0. \tag{CP3}$$

To verify the level used in (CP2), specialize the class-sum identity of Section 39.4.3 at the trivial twist. It writes $F_i = (c_1'/c'')\Xi_{1,i}$ with $c'' \neq 0$. Each summand is a theta kernel in the variable g with residue datum modulo $\mathfrak{m}$. At a place dividing $\mathfrak{m}$, a source translation $k \equiv I \bmod \mathfrak{m}$ fixes that datum and its tame determinant character is one. Away from the modulus, the integral group fixes the vacuum and the character is unramified. Thus each summand has level $K(\mathfrak{m})$. Its moderate growth and the rapid decrease of y, whose old level contains $K(\mathfrak{m})$, give the assertion for yF_i and justify the finite sum.

41.1.2 The finite first-factor choices

In this subsection, we choose the finite test for the real reproduction of Section 41.1.1.

Let S have signature $(2m, m)$, $\widetilde{W} = W \oplus E_-^j \oplus E_+^{3m-j}$, and $W^\flat = E_+^{6m+1}$ with their marked lattices. The carrier ϕ is the theta lift of 1 from E_-^m to S, with trivial finite modulus, finite vacuum, splitting characters $(\upsilon^m, 1)$, and $\prod_\tau \det(y_\tau^-)^{2m}$ as Fock polynomial. It is rapidly decreasing of full finite level and holomorphic of weight $(0; 3m)$ on S. It is nonzero. Indeed the real coefficient of its Fock vector on the source compact group is a nonzero constant, since the determinant factor and its conjugate cancel on that group. The sharp vacuum convolution formula at unit modulus would give zero for its integral against 1 if the lift vanished. This contradicts the positive Fock norm times the compact source volume.

A choice $i = (\Lambda, \sigma)$ consists of the frequency matrix Λ modulo $\mathfrak{m}$ and a class of $W^\flat$ at $\mathfrak{m}$ with its specified choice of short covectors σ. Define P_j on $(S, \widetilde{W})$ as the product of the full negative minor and the positive minors, with exponents read from the positive diagram of the untwisted pair. Let Q_σ be the paired-determinant polynomial of the prescribed short covectors in the positive letters of $(S, W^\flat)$. Then $F_i(g)$ is the integral on $[U(S)]$ of ϕ times the theta kernel of $(S, \widetilde{W} \oplus W^\flat)$, at $(g, 1, c_{\sigma,f})$ in the second variable, with polynomial $P_j Q_\sigma$, test $\psi_\Lambda(B) = \psi_{\mathfrak{m}}(\mathrm{tr}(B\Lambda))$, and characters $(\upsilon^m, \chi_{\widetilde{W},j})$. Here

$$\chi_{\widetilde{W},j} = \chi_0 \upsilon^{\,m+e_j^{(1)}-j\Delta}, \qquad e_j^{(1)} = j + \lfloor (t+k-1)/2 \rfloor.$$

Thus the second character has trivial tame part and weight $m + e_j^{(1)} - \kappa_{j,\tau}$ at τ. On $R = \mathcal{O}_E/\mathfrak{m}$ we use the additive character

$$\psi_{\mathfrak{m}}(x \bmod \mathfrak{m}) = \psi_f(j_{\mathrm{W}} \bar{\alpha}_{\mathrm{W}} \alpha_{\mathrm{W}}\, e_{\mathfrak{m}} s_0^{-1} x), \qquad x \in \widehat{\mathcal{O}}_E,$$

where s_0 generates $\mathfrak{m}$ in the finite adeles and $e_{\mathfrak{m}}$ is the idempotent of its chosen split places as in Section 41.5.2.

Let θ_{Ψ,P_j} be the $(S, \widetilde{W})$ kernel with any function Ψ modulo $\mathfrak{m}$ and the same two characters. The analogous reproducing formula on $(S, W^\flat)$ uses the weight $u_1 = (-\Delta'; 6m+1)$, where $\Delta' = (\Delta - t - 1)/2$, and $2 \dim S < \dim W^\flat$. In each class the paired-determinant tests in the short basis span the relevant highest-vector translates. Apply this reproduction to the rapidly decreasing function

$$x \longmapsto \phi(x) \theta_{\Psi,P_j}(x, (g, 1)).$$

Indeed ϕ has full finite level. On S, a translation in $K(\mathfrak{m})$ acts trivially on the residue argument of Ψ at the marked places and preserves the vacuum elsewhere. The finite determinant character is one on this subgroup. For each fixed g the theta kernel has moderate growth in the S-variable, so its product with ϕ belongs to the same rapid space at $K(\mathfrak{m})$. The resulting kernels on $(S, W^\flat)$ use the indicator δ_{incl} of the marked inclusion modulo $\mathfrak{m}$ and polynomials Q_σ.

The product of the two kernels is the kernel of the orthogonal sum, with test $\Psi(B_1)\delta_{\mathrm{incl}}(B_2)$ and polynomial $P_j Q_\sigma$. Finite Fourier inversion writes that product test as a linear combination of the ψ_Λ on the whole residue matrix space, with its original primitive additive character. Therefore, for each $x_\phi \in U(S)(\mathbb{A})$ and each Ψ, the function of g

$$\int_{U(S)(\mathbb{R})} \phi(x_\phi h')\, \theta_{\Psi,P_j}(x_\phi h', (g, 1)) M_{u_1}(h')\, dh' \tag{CP4}$$

belongs to the span of the actual F_i. The reproduction coefficients depend on x_ϕ and the finite Fourier coefficients on Ψ. All quotient measures and class representatives are as above.

The class test used in both reproductions can be checked before taking any finite Fourier expansion. Write $K_T = K_T(\pi_M^2)$ and let c_f be the chosen representative of a target class. Every adelic target point a can be written

$$a = \gamma(1, c_f)(1, k)(h, 1), \qquad k \in K_T, \quad h = (\gamma^{-1}a)_\infty,$$

using the finite class equality $a_f = \gamma_f c_f k$. For a Fock vector v fix the marked inclusion indicator and denote the value of its lift at $(1, c_f)$ by $L_c(v)$.

The lift at a is

$$L_c(\omega_\infty(1, h)v).$$

Indeed rational left invariance removes γ. Covariance puts the two right translations into the Weil datum. The finite action of k fixes the datum because k reduces to the identity modulo π_M^2 and the source character has trivial tame part. The real translation therefore acts on the Fock vector alone. In the complementary block, writing the translated polynomial in the negative coordinates in the tableau basis uses the same c_f, and the tuple supported at c must take the conjugated Weyl vector as in (CP2).

In the carrier reproduction the fixed matrix of short covectors A_c is the transpose of the chosen rational short basis of the lattice at this same c_f. It is invertible at every infinite embedding. Thus each covector functional appearing in a translated highest minor is a linear combination of the covectors of A_c. Multilinearity of the determinant expands it into the minors indexed by the recorded covector lists. Multiplying the expansions in all factors and at all infinite places yields precisely the span of the polynomials Q_σ at that class. Hence vanishing of all finite class tests kills the lift of the highest vector at every a in either case.

41.1.3 Approximation at a point of maximum norm

In this subsection, we bound the error after applying the test of Section 41.1.2.

Put $n_* = 2ft^3$ and b equal to the marked large weight scale. Choose a peak x_ϕ of the nonzero carrier ϕ. Thus $\sup|\phi| \le 2|\phi(x_\phi)|$, and its first minimum is at least $(5b\sqrt{b})^{-1/2}$ by the bounds for the first minima. If $\lambda_1(W, x) \ge (5b^{24})^{-1/2}$, extending x by the identity on the adjoined definite summands leaves the truncated first minimum unchanged or increases it. The product of the two minima is at least b^{-13}, since $5b^{51/4} \le b^{13}$ at the marked large scale.

Let $c_P(h', h)$ be the real Fock coefficient of P_j on $(S, \widetilde{W})$, with h embedded by the identity on the adjoined summands. On this pair and these points the sparse kernel estimate gives a test Ψ_x modulo $\mathfrak{m}$ with

$$\left|\theta_{\Psi_x, P_j}(x_\phi h', (xh, 1)) - c_P(h', h)\right| \le b^{-20n_*}|h'|^{6n_*}|h|^{6n_*}|c_P(1, 1)|, \tag{CP5}$$

where the absolute values of real points denote the maximum of the operator norms. The estimate can be reduced to one explicit finite average.

Use as Ψ_x the average of the additive characters modulo $\mathfrak{m}$, weighted by the conjugates of their original kernel values at $(x_\phi, (x, 1))$. The product formula with the Heisenberg translates expands its kernel by rational differences. The zero difference gives $c_P(h', h)$; finite character orthogonality restricts all other differences to the kernel of reduction on the marked lattice. Multiplication by q_1/π_M^2 clears this difference lattice, because the finite Weil idele satisfies $q_1\alpha_{\mathrm{W}} \in \widehat{\mathcal{O}}_E$ by Section 8.1.

Section 41.4 gives separation b^6 in the real norm of the nonzero differences. The real Fock coefficient bound for a homogeneous polynomial of degree $d_P \le b\sqrt{b}$, proved below, with $q = 2n_*$, contributes

$$(4(d_P + 2q))^q \, |h'|^{3q} |h|^{3q} |c_P(1,1)| \, \|v\|^{-2q}$$

at a nonzero difference of real coordinate v. The sum of $\|v\|^{-2q}$ is at most $(5/b^6)^{2q}$. Indeed, pack balls of radius $b^6/2$ around the points in the annuli with radii $2^k b^6$ and $2^{k+1} b^6$; if the real dimension is $2d < 2q$, the number in the k-th annulus is at most $(5 \cdot 2^k)^{2d}$, and the geometric sum is at most 5^{2q}. The inequality between the parameters $100(d_P + 2q) b^{-12} \le b^{-10}$ now gives (CP5). This derivation uses the finite Heisenberg overlap identity, which makes the zero-difference term the Fock coefficient with unit coefficient, and the real Fock decay bound.

Define one reference function

$$F_*(h) = \int_{U(S)(\mathbb{R})} \phi(x_\phi h') c_P(h', h) M_{u_1}(h') \, dh'.$$

It is bounded by the unitary Fock coefficient bound and the integrability of M_{u_1}.

41.1.4 The nonzero reference coefficient

In this subsection, we use the approximation of Section 41.1.3 to construct the comparison function.

In the numbering of W_j, put $r_+ = k + j - 1$. The negative covectors used in $\widetilde{W}$ are indexed by $0 \le a < m$: for $a < j$ take the covector a of the adjoined negative space, and for $a \ge j$ take covector $k + a - 1$ of W_j. The latter covectors lie in the negative block, since $k + a - 1 \ge r_+$, and their indices run through the original negative covectors. The two disjoint pieces give m distinct covectors. For each height $0 \le h < r_+$, take the $2m$ positive covectors indexed by $0 \le a < 2m$: use covector $h + a$ of W_j if $h + a < r_+$, and covector a of the adjoined positive space otherwise. The positive space has dimension $3m - j \ge 2m$. These covectors are distinct, and both choices have positive sign.

Let Y_τ^- be the matrix on the selected negative covectors and all m negative columns of S, and let $Y_{\tau,h}$ be the matrix on the height-h positive covectors and all $2m$ positive columns. Write $n_\tau^\pm(h)$ for the number of columns of height h in the positive and negative diagrams, respectively. Put

$$\begin{aligned} \nu_\tau &= \sum_{h < r_+} n_\tau^+(h), \\ \mu_\tau &= \nu_\tau + \Delta', \\ d_1 &= r + 9m + 1, \\ c_\tau &= \mu_\tau + d_1 + m. \end{aligned}$$

This is precisely the polynomial

$$P_j = \prod_\tau \det(Y_\tau^-)^{c_\tau} \prod_{h < r_+} \det(Y_{\tau,h})^{n_\tau^+(h)}.$$

Each determinant has independent variables in all its positions and is nonzero. The polynomial ring is a domain, so their product is nonzero even when different minors share some variables. Every variable in the product pairs identical signs on the target and source coordinates. It thus

has positive oscillator sign, and every contraction involving two opposite oscillator signs kills P_j.

The degree at τ is $mc_\tau + 2m\nu_\tau$. The determinant count on the original positive diagram satisfies

$$\mu_\tau + e_j^{(1)} = \kappa_{j,\tau}.$$

Indeed the number of columns at height h is a successive difference of the shifted entries $\lambda_{h-1} + \max(h, 1)$, with λ_i numbered from zero at τ as in Section 12.10. The initial and final shifted parameters are $k/2$ and $k + j - 1 + \Delta/2 - \kappa_{j,\tau}$, respectively. The sum telescopes, with the extra Δ' counted at full positive height. The admissible bound $d_1 + m \le e_j^{(1)}$ now gives $c_\tau \le \kappa_{j,\tau} \le 2b$. In particular $2c_\tau \le 5b$. Since $\nu_\tau \le c_\tau$, the total degree is at most $10fmb \le b\sqrt{b}$ on the chosen scale. The second character on the padded pair has weight $e_\tau = m + e_j^{(1)} - \kappa_{j,\tau} = m - \mu_\tau$, and the first has weight m.

Here is the matrix coefficient calculation at one place. For $u \in U(2m, m)$, write $A = \det u_{++}$, $D_- = \det u_{--}$, and $\delta = \det u$. Both block minors are nonzero. The complex-linear substitution for the inverse source action on the full sign columns of each minor is multiplication on those columns by the corresponding block of u in transposed variable notation. Thus the product of the substitution scalars at u is $A^{\nu_\tau} D_-^{c_\tau}$. The scalar entering the coefficient is the conjugate inverse of this expression at u^{-1}, namely $A^{-\nu_\tau} D_-^{-c_\tau}$, because $u^{-1} = Ju^*J$.

The vacuum factor is the inverse of the minor of the Hom action on the negative oscillator variables. Each positive target space uses the negative source columns of u^{-1}, and each negative target space uses the positive source columns. By complementary minors on u^{-1}, that minor is

$$(A/\delta)^{r+2m} (D_-/\delta)^m.$$

These are the two diagonal factors of the normalized Fock coefficient. To see that no contraction term remains, use the creation, complex-linear, and annihilation factorization of the Fock Gaussian kernel. The annihilation factor contracts opposite oscillator signs, and hence acts as the identity on the indicated polynomial. Its adjoint creation factor disappears in the pairing with that polynomial for the same reason. The normalized complex-linear factor is the substitution above, with vacuum value the indicated inverse minor. Finally the splitting character on the source multiplies by δ^{-e_τ}.

It follows that the local multiplier of $c_P(1, 1)$ is

$$\delta^{r+3m-e_\tau} A^{-r-2m-\nu_\tau} D_-^{-m-c_\tau}.$$

For a weight constant on the two blocks one has $m_{(a;b)}(u) = \delta^b A^{a-b}$. The complementary identity $A = \delta\overline{D_-}$ and $|\delta| = 1$ therefore give, on multiplying over all places,

$$c_P(u, 1) = c_P(1, 1) M_{u_2}(u) \overline{M_{u_3}(u)},$$
$$u_2 = (\Delta'; \mu_\tau + r + 2m), \quad u_3 = (0; \mu_\tau + r + 11m + 1).$$

At the identity $c_P(1, 1) = \|v_{P_j}\|^2 > 0$ by injectivity of polynomial realization and positivity of the Fock inner product. Unitarity gives $|c_P(u, h)| \le c_P(1, 1)$ on both real groups.

The carrier weight and the two factors satisfy $(0; 3m) + u_1 + u_2 = u_3$, with $u_1 = (-\Delta'; 6m+1)$. The weights u_1, u_2, u_3 are dominant within blocks. Since $\Delta/2 \ge 300n_* + t$ and $\Delta' = \Delta/2 - (t + 1)/2$, the cross-block gaps of u_1, u_3 are at least $300n_*$. Also $|u_{3,i}| \le c_\tau + m \le 2c_\tau \le 5b$. The minor-quotient moment bound thus applies with $(u, u') = (u_3, u_1)$ on S in the original product

Haar. In particular $J_3 = \int |M_{u_3}|^2$ is finite and positive. Apply the holomorphic reproducing formula to the product $h \mapsto \phi(x_\phi h) M_{u_1}(h) M_{u_2}(h)$, of weight u_3. Its pairing with $\overline{M_{u_3}}$ is integrable by the unitary bound on c_P, the carrier peak bound, and the moment bound. Since $M_{u_i}(1) = 1$, it gives

$$F_*(1) = J_3\, \phi(x_\phi)\, c_P(1,1) \neq 0.$$

Here x_ϕ is the one carrier peak chosen before the later source space point and its finite test. This proves nonvanishing for the same reference function that appears in the uniform approximation.

The weight comparison gives $\int |M_{u_1}(h')||h'|^{6n_*}\, dh' \leq b^{n_*} J_3$. Apply (CP4) with Ψ_x, (CP5), and the carrier peak bound.

Lemma 41.1 (Uniform approximation at a carrier peak)**.** *For each adelic x with $\lambda_1(W,x) \geq (5b^{24})^{-1/2}$, there are coefficients $a_i(x)$, independent of h, such that*

$$\left| \sum_i a_i(x) F_i(xh) - F_*(h) \right| \leq 2b^{-19n_*} |F_*(1)|\, |h|^{6n_*} \quad (h \in U(W)_\infty). \tag{CP6}$$

The reference function F_* is fixed and $F_*(1) \neq 0$. The test and the coefficients may depend on x.

41.1.5 Coefficients and weights of the approximants

In this subsection, we check the weight and source of the approximants in Lemma 41.1.

In the carrier reproduction the kernel on $(S, W^\flat)$ has characters $(\upsilon^m, 1)$, while that on $(S, \widetilde{W})$ has $(\upsilon^m, \chi_{\widetilde{W},j})$. The product is the kernel on the target sum with $(\upsilon^m, \chi_{\widetilde{W},j})$. In fact the first character acts separately on each target determinant, and the two factors on the common source determinant multiply. Let $a_r(x_\phi)$ be the coefficients of flat reproduction at the fixed carrier point, with r running over the original class and short-covector choices. These coefficients work for all rapidly decreasing functions of the given level. With R the marked residue ring, let

$$f_\Psi(\Lambda) = \frac{1}{\#\operatorname{Mat}_{d_1 \times 3m}(R)} \sum_{B \in \operatorname{Mat}_{d_1 \times 3m}(R)} \Psi(B_1)\, \delta_{\mathrm{incl}}(B_2) \overline{\psi_\Lambda(B)}.$$

Fourier inversion and the product identity give coefficients $a_r(x_\phi) f_\Psi(\Lambda)$ of $F_{(\Lambda,r)}$ in (CP4). These coefficients serve every evaluation g, hence the entire real translate gh once $\Psi = \Psi_x$ has been chosen for the approximation.

The untwisted class-sum identity $F_i = (c_1'/c'')\Xi_{1,i}$, with the fixed $c'' \neq 0$ of Section 33.3, reads each F_i as a finite combination of (W_j, V'') kernels in the original order, with polynomial in the positive coordinates of the diagram with column counts $n_\tau^+(h)$. The target is positive definite of rank d_1 and its splitting weight on the source is $e_j^{(1)} - \kappa_{j,\tau} = -\mu_\tau$. The highest-weight polynomial in the positive coordinates gives the holomorphic weight with entries

$$\mu_\tau - \sum_{h > n(a)} n_\tau^+(h) \quad \text{on positive } a, \qquad \mu_\tau + d_1 \quad \text{on negative } a.$$

Here the positive entries come from the staircase minors and the negative entries from the vacuum and splitting weights. The definite-group translates of this highest polynomial span

the Weyl space in the positive coordinates, so the same weight applies to every coefficient in that class sum. Put $d_- = t - 9m - 1$. The complementary weight at τ has entries

$$\begin{cases} j - m/2, & \text{on positive } a, \\ j - m/2 + d_- + \displaystyle\sum_{n(a)<h<d_-} n_\tau^-(h), & \text{on negative } a. \end{cases}$$

The full staircase shift is $\kappa_{j,\tau} - (r+t)/2$. Indeed, if e_j is the first-factor character exponent, then

$$e_j = (r+t)/2 + j - m/2, \qquad \mu_\tau = \kappa_{j,\tau} - e_j.$$

The rank terms add because $d_1 + d_- = r + t$, and the positive columns of height at least d_1 and the negative columns of height at least d_- have count zero. Thus the displayed weights add to u on the actual covector. Thus each function $h \mapsto F_i(xh)$, and hence the sum $K_x(h) = \sum_i a_i(x) F_i(xh)$ in (CP6), is holomorphic of weight $u - u^\circ$ in the order of Lemma 40.1.

41.1.6 Injectivity of the common period

In this subsection, we combine Lemmas 40.1 and 41.1 to prove injectivity.

Suppose $0 \neq y \in \ker P$. Since $\bar{y}$ is bounded, there is a point z with $0 < \|y\|_\infty/2 \le |\bar{y}(z)|$. Every such peak satisfies the minimum bound in (CP6). This is the holomorphic peak bound at level $M\sqrt{-d}$, with the marked conjugate weight and positive gap. Both that weight and the complementary kernel weight have the dominance and gap required in the circle-of-peaks estimate. Their difference is the weight of each $F_i(x\,\cdot)$. Thus (CP3) and (CP6) apply at every peak on the original space. Lemma 40.1 gives

$$|F_*(1)| \le (4b^{n_*} + 1)\, 2b^{-19n_*} |F_*(1)| < |F_*(1)|,$$

since $b^{n_*} > 10$. This contradiction proves P injective. The separation uses the actual first-factor products on the fixed descent pair. The sharp-level convolution and the finite Heisenberg overlap are the earlier analytic hypotheses for the two reproducing formulas and the sparse kernel step, respectively. In particular the reverse-grid comparison concerns a different spanning argument and is unnecessary here.

41.2 Bounds for the first minima at the maxima

In this subsection, we bound the minimum of each moved lattice at points satisfying the half-supremum bound in Section 41.1.6. This separates the nonzero Hom-lattice terms of Section 41.1.3 using the original level and Hermitian form.

Let V have the chosen adelic basis, with Hermitian matrix $J = \mathrm{diag}(\epsilon_i)$ at infinity, and order the positive coordinates before the negative ones. Use the Euclidean norm $\|v\|^2 = \sum_\tau \sum_i |v_{\tau,i}|^2$ in the archimedean coordinates of a rational lattice vector. The first minimum $\lambda_1(V, g)$ is cut off at 1. Suppose that F is a bounded automorphic function, adelically holomorphic of weight u_τ in the following sense: at each adelic translate it extends along hB, with $h \in U(J)$, B upper triangular, and with value $\prod_i B_{ii}^{u_{\tau,i}}$ times the value at h. Assume

$$|u_{\tau,i}| \le A, \quad A \ge 2, \qquad u_{\tau,i} + 1 \le u_{\tau,k} \quad (\epsilon_i = 1,\ \epsilon_k = -1).$$

Let $p = -\bar{p}$ be an algebraic integer with $0 < \operatorname{Im} \tau(p) \le A'$ at each chosen infinite place, $A' \ge 1$, and assume F is invariant under the principal level $K(p)$. If $C = \sup |F| > 0$ and $|F(g)| \ge C/2$, then

$$\lambda_1(V, g) \ge (AA')^{-1/2}. \tag{PM1}$$

Suppose instead that an original nonzero rational lattice vector v in the basis moved by g has length $< (AA')^{-1/2}$. Its Hermitian square is integral, because the adelic coordinates at every finite place are integral and the finite Hermitian matrices are integral. Its absolute value at every archimedean embedding is at most $\|v\|^2 < 1$. The integral norm therefore forces the square to vanish.

In the moved coordinates v_τ at any real place are nonzero and isotropic. The transvection

$$t = I - pvv^* J$$

in the moved adelic basis is unitary. Its conjugate back to the original basis is the rational transvection defined by the same rational vector and the rational Hermitian form of V. Its finite component is in $K(p)$, since the moved coordinates of v are integral at every finite place. Writing $t = (t_\infty, t_f)$, rational left invariance and right $K(p)$-invariance give

$$F(g(t_\infty h, 1)) = F\big((gtg^{-1})\, g(h,1)(1, t_f^{-1})\big) = F(g(h,1)). \tag{PM2}$$

This last equality relates the transvection to the original automorphic function at g.

Here is the triangular factorization and bound for the complexified transvection. At a distinguished place τ write $\tau(p) = ia_\tau$, $a_\tau > 0$, and put $\eta_\tau = a_\tau \|v_\tau\|^2 > 0$. Then $A \sum_\tau \eta_\tau \le 1$, so $\eta_\tau \le 1/2$. For z in the upper half-plane set

$$X_\tau(z) = I - d_\tau(z) v_\tau v_\tau^* J, \qquad d_\tau(z) = ia_\tau(z - i), \qquad h_z = \operatorname{Im} z.$$

The isotropy gives $X_\tau^* J X_\tau = J - 2\operatorname{Re}(d_\tau) J v_\tau v_\tau^* J$. For an initial segment of the ordered coordinates of length j, the determinant of its Gram block, divided by the determinant of the corresponding block of J, is

$$p_j = 1 - 2\operatorname{Re}(d_\tau) \sum_{i \le j} \epsilon_i |v_{\tau,i}|^2.$$

The partial sums in this expression increase to $\|v_\tau\|^2/2$ on the positive block and then return to 0.

The p_j are positive. They start and end at 1, with value $p_* = 1 + \eta_\tau(h_z - 1)$ at the end of the positive block. Thus Hermitian elimination without pivoting gives $X_\tau = h_\tau B_\tau$ with $h_\tau^* J h_\tau = J$ and B_τ upper triangular. Indeed the successive elimination pivots have values $\epsilon_i p_i / p_{i-1}$, and normalizing their absolute values gives $|B_{\tau,ii}| = (p_i/p_{i-1})^{1/2}$.

Write $L_i = \log(p_i/p_{i-1})$ and $q_\tau = 1 + \eta_\tau h_z$. If $h_z \ge 1$, the L_i are nonnegative on the positive block and nonpositive on the negative block. Their total is zero and their positive-block sum is $\log p_*$. The weight gap gives $\sum_i u_{\tau,i} L_i \le -\log p_*$. Moreover $p_* = q_\tau - \eta_\tau \ge (1 - \eta_\tau) q_\tau$ and $-\log(1 - \eta_\tau) \le 2\eta_\tau$. If $0 < h_z < 1$, the signs of the L_i reverse, so

$$\tfrac{1}{2} \sum_i u_{\tau,i} L_i \le -A \log p_* \le 2A\eta_\tau(1 - h_z) \le 2A\eta_\tau - \tfrac{1}{2} \log q_\tau.$$

The last inequality follows from $\log q_\tau \le \eta_\tau h_z$. Both cases imply

$$\left| \prod_i B_{\tau,ii}^{u_{\tau,i}} \right| \le \frac{\exp(2A\eta_\tau)}{\sqrt{1 + \eta_\tau \operatorname{Im} z}}. \tag{PM3}$$

These factorizations give one holomorphic function along the whole upper half-plane. Indeed two factorizations $hB = h'B'$ differ by an element of $U(J)$ in the triangular subgroup. Such an element is diagonal unitary, since it preserves the successive nondegenerate coordinate spaces and their orthogonal complements. The weight rule identifies the two values $F(g(h,1))\chi_u(B) = F(g(h',1))\chi_u(B')$. Locally at hB, right translation by B^{-1} transports the holomorphic germ at h, multiplied by $\chi_u(B)$. These germs thus give the holomorphic composition $f(z)$ along $z \mapsto (X_\tau(z))_\tau$. Equation (PM3) and $A\sum_\tau \eta_\tau \le 1$ give $\sup|f| \le Ce^2$, and the same bound with the denominators in (PM3) gives $f(z) \to 0$ uniformly as $\operatorname{Im} z \to \infty$. The identity $X(z+1) = t_\infty X(z)$ follows because $(v_\tau v_\tau^* J)^2 = 0$. The triangular character and (PM2) give $f(z+1) = f(z)$. In the variable $q = e^{2\pi i z}$, the function is holomorphic on $0 < |q| < 1$, extends by zero at the origin, and is bounded by Ce^2. Schwarz's lemma at $z = i$ gives $|F(g)| = |f(i)| \le Ce^{2-2\pi} < C/2$, contradicting the peak condition. This proves (PM1) in the moved lattice and the original level.

41.3 The auxiliary automorphic form at a maximum

In this subsection, we apply the first two exchanges of Section 39.4 at the point where the leading carrier term is attained. The formula identifies that term with the original real coefficient and expresses the other terms as a lattice sum.

Put $S = I_{2m} \perp I_m^-$, and write $K_0 = U(S)(\widehat{\mathcal{O}}_F)$ for the stabilizer of the stated standard finite lattice. For the carrier identity, take the same tame character c of weight zero as in the first two exchanges, with nonempty support $\mathfrak{m}$ and residue character χ. The class sum (101) expresses $\phi = F^\circ$ as a finite linear combination, with coefficients independent of x, of

$$\Theta_{v^m,c}^{S,I_{3m}}[\chi \circ \det, 1](x, h_a)$$

at the original class representatives h_a.

The inverse stabilizer orders are part of these coefficients. For $k \in K_0$, the integral Weil action on the finite datum is

$$c(\det k)^{-1}\chi(\det \bar{k}) = 1,$$

where $\bar{k}$ denotes residue reduction. Thus every summand and ϕ itself are right K_0-invariant. In particular they have every principal level contained in K_0. The determinant character extended by zero on nonunits is supported on invertible residue matrices.

At a place in $\mathfrak{m}$ the lifted finite datum therefore has support on matrices with an integral left inverse. Applying Section 34.3.1 with $H = \mathrm{ht}_S(x)$ and $\rho = \mathrm{ht}_{I_{3m}}(h_a)$ gives, for each $D \ge 0$, a bound on H^D times this summand by a fixed Fock moment times a fixed power of ρ. These constants are finite for the polynomial 1 and the finitely many h_a. Hence ϕ is rapidly decreasing.

Continuity and rational left invariance follow from the theta sum. At infinity the Fock polynomial 1 is killed by the holomorphic lowering operators. Its diagonal weight on S, with weight-zero second splitting character and positive definite rank $3m$ on the other side, is 0 on the positive block and $3m$ on the negative block. For the order numbering the positive block by $-1, -2, \dots$ and the negative block by $0, 1, \dots$, the Fock holomorphic continuation in the triangular variable gives precisely the continuation of weight $(0; 3m)$. The theta sum has this continuation for every adelic left translate, by local uniform convergence of the Schwartz theta series. Every summand uses the same order and weight, so ϕ has these as well.

To prove nonvanishing use the original presentation of ϕ as the lift of 1 from I_m^- to S, with characters $(v^m, 1)$. The finite datum is the integral vacuum at every place. For the sharp convolution formula take the unit modulus, empty prescribed support, and full integral source

level. The finite sharp condition at the empty support is vacuous, the vacuum fixes the integral level, its mass there is positive, and the second character is trivial. In the sharp inner-product formula the finite factor is now the vacuum series on all finite places. Its neighbour count applies at each conjugation orbit, so the same absolute convergence and injectivity argument applies. The rank inequality is $3m > 2m$, and the constant source is rapidly decreasing since I_m^- is definite at infinity and its adelic quotient is compact.

Let v be the original Fock vector of $P = \prod_\tau \det(Y_\tau^-)^{2m}$. This polynomial is nonzero. For $u = (u_\tau)$ on the compact negative-definite source, put $\delta_\tau = \det u_\tau$. The $2m$ positive target coordinates give the vacuum factor δ_τ^{2m} from the negative Hom block. On the negative target coordinates the Fock substitution is right multiplication by $(u_\tau^{-1})^{\mathsf{T}}$, so that $\det(Y_\tau^-)^{2m}$ acquires δ_τ^{-2m}. The source splitting character is trivial and the target is evaluated at the identity. Consequently

$$\omega(u,1)v = v, \qquad k(u) = \langle v, \omega(u,1)v\rangle = \|v\|^2.$$

If the carrier vanished, the sharp formula and the injectivity of its finite vacuum series would give

$$0 = (k * 1)(1) = \|v\|^2 \operatorname{vol}(U(I_m^-)(F_\infty)) > 0.$$

The Haar measure is that of the original real convolution. This proves nonvanishing on the same adelic realization as in the carrier identity. In particular ϕ is the single nonzero full-level holomorphic form to which the peak minimum is applied, common to the later source space and class tests.

Apply this first to the carrier ϕ of signature $(2m, m)$. Its nonzero rapid full-level form is holomorphic of weight $(0; 3m)$. Full level contains $K(\sqrt{-d})$; the given CM embeddings read $\sqrt{-d}$ with positive imaginary part $\sqrt{d}$. The admissible scale gives $3m \le 5b$ and $d \le b$. Therefore $A = 5b$, $A' = \sqrt{b}$ give, at every carrier peak x_ϕ,

$$\lambda_1(S, x_\phi) \ge (5b\sqrt{b})^{-1/2}. \tag{PM4}$$

For a nonzero good form y on W_j, use $F = \bar{y}$ on the original W_j. Its good type is cuspidal and the form is rapidly decreasing. The good form is fixed by $K(\pi_M)$. Both $K(M\sqrt{-d})$ and $K(\pi_M^2)$ lie in this group, since $M\sqrt{-d} = \pi_M(\bar{\pi}_M\sqrt{-d})$. The first inclusion gives the transvection level for $\bar{y}$, and the second gives the level used for the product yF_i in (CP3). Conjugating values preserves level invariance. Both uses keep the automorphic form and the lattice on W_j at their original level.

The adjoint calculation gives this holomorphic weight on the original good space. Its proof uses only the good single-lift Gram operator and the marked adjoint. The opposite-sign gap is at least $300n_*$ by the last negative column count, with the definite case having no opposite signs.

Now $p = M\sqrt{-d}$ is purely imaginary in the required sense and

$$0 < \operatorname{Im}\tau(p) = M\sqrt{d} \le b^{22}\sqrt{b} \le b^{23}$$

by the original level-prime interval and $d \le b$. Thus (PM1) with $A = 5b$, $A' = b^{23}$ gives

$$\lambda_1(W_j, g) \ge (5b^{24})^{-1/2} \quad \text{at every peak of } \bar{y}. \tag{PM5}$$

The product of (PM4) with (PM5), after adjoining the fixed definite summands to W_j, is at least $(5b^{51/4})^{-1} \ge b^{-13}$ at the chosen scale. Each definite summand at the identity has first

minimum at least 1, so the orthogonal sum has the same truncated first minimum as W_j. For $z_s = z(r_s, 1)$ the finite lattice is the one at z, and each archimedean coordinate v_τ becomes $r_s^{-1} v_\tau$ of the same Euclidean norm. Hence $\lambda_1(W_j, z_s) = \lambda_1(W_j, z)$ throughout the circle of Lemma 40.1.

41.4 Finite overlaps and Gaussian decay

In this subsection, the error in Section 41 is summed over nonzero maps between two moved lattices. A first-minimum estimate for this Hom lattice gives a summable Gaussian bound in the original normalizations.

The comparison of the first minimum on the Hom space with the two first minima uses the original self-dual Hermitian lattices. Let L be the source lattice in its Euclidean real coordinates, of dimension $D_{\mathbb{R}} = 2f \operatorname{rk} S$. Take $N_c = dq_1$, so that $E \subseteq \mathbb{Q}(\zeta_{N_c})$ and $N_c = O_d(x^{11} \log x)$ by Section 24.1. Then N_c belongs to the different of E. Indeed it belongs to the different of the cyclotomic field by differentiating $X^{N_c} - 1$ at a primitive root and using the cyclotomic factor.

Transitivity of the different and contraction give the claim for E. The Euclidean dual of L is $2J_\infty \mathfrak{D}_E^{-1} L$. The trace of the Hermitian pairing is twice the real Euclidean pairing with the sign matrix J_∞, and the finite Hermitian lattice is self-dual. Consequently the Euclidean dual of $2L/N_c$ is contained in $J_\infty L$, so its nonzero vectors have length at least the cut-off first minimum λ_S.

Banaszczyk's transference theorem [7, Theorem 2.2, p. 632] gives covering radius at most $D_{\mathbb{R}}/(2\lambda_S)$, hence at most $D_{\mathbb{R}}^2/\lambda_S$, for $2L/N_c$. The vectors of a lattice with covering radius at most $R > 0$ and length at most $3R$ span its real space. Otherwise a vector of length $2R$ in their orthogonal complement is farther than R from all those vectors, and a longer lattice vector is also farther than R.

Thus the source lattice contains an E-basis with each norm at most $6N_c(f \operatorname{rk} S)^2/\lambda_S$. A nonzero integral Hom vector sends at least one of these basis vectors to a nonzero vector of the target lattice. If λ_T is the target minimum, the operator norm bound by the Euclidean coordinate norm, even with the weaker factor t^2, gives

$$\lambda_{\mathrm{Hom}(S,T)} \geq \frac{\lambda_S \lambda_T}{6N_c (ft)^2 t^2}.$$

Let X_S, X_T be the adelic basis matrices of Section 40.2.1. At the fixed points g, h, a rational map $Z \colon S \to T$ has adelic coordinate

$$z(Z) = (X_T h)^{-1} Z (X_S g).$$

Write $z = z(Z)$. For a surviving difference, $\alpha_{\mathrm{W}}^{-1} z_f = \pi_M^2 v_f$, with v_f integral. The rational map $w = (q_1/\pi_M^2) Z$ belongs to the moved Hom lattice since $z(w)_f = (q_1 \alpha_{\mathrm{W}}) v_f$ is integral by Section 8.1. Write $\|w\|^2 = \sum_{\tau \in \Phi} \sum_{i,k} |z(w)_{\tau,ik}|^2$. Since $\pi_M \bar{\pi}_M = M$, one has $|\tau(\pi_M^2)| = M$ for every τ. In the chosen CM type, $j_{\mathrm{W}} = -1/\sqrt{-d}$ from Section 8.1 gives the real Fock scale $(2\pi/\sqrt{d})^{1/2}$. Therefore

$$\|z_{\mathbb{R}}\| = \left(\frac{2\pi}{\sqrt{d}}\right)^{1/2} \frac{M}{q_1} \|w\| \geq \left(\frac{2\pi}{\sqrt{d}}\right)^{1/2} \frac{M \lambda_S \lambda_T}{6 q_1 N_c (ft)^2 t^2} \geq b^6.$$

The last inequality uses $M \geq b^{20}$, $\lambda_S \lambda_T \geq b^{-13}$, fixed d, and polynomial growth of q_1, N_c, f, t in x.

41.5 The normalized overlap on the adelic Hom space

In this subsection, we calculate the overlap of two adelic tests as a product of a residue-ring factor and a polarized real Fock factor. This supplies the coefficient estimates used to compare the leading term and the error in Section 41.

Use the original pair S, T, with $T = \widetilde{W}$ for the sparse test, the modulus $\mathfrak{m} = (\pi_M^2)$, and the Weil parameter $(\psi_f, j_{\mathrm{W}}, \alpha)$, where $\alpha = \alpha_{\mathrm{W}}$ is the finite idele of Section 8.1. Put $R = \mathcal{O}_E/\mathfrak{m}$. Let $D = \mathrm{rk}_E \operatorname{Hom}_E(S, T)$. An adelic displacement z is read at the finite places as $a(z) = \alpha^{-1} z_f$ and at infinity as $z_{\mathbb{R}} = (\sqrt{2\pi \operatorname{Im} \tau(j_{\mathrm{W}})}\, z_\tau)_\tau$. Let $\mathcal{R}(z)$ be the Heisenberg operator and v_{vac} the finite unit vacuum.

41.5.1 The probability mean and Fourier inversion

In this subsection, we express the overlap by Fourier inversion in the normalization used to bound the error in Section 41.5.3.

In the rational symplectic displacement space choose a rational Schrödinger polarization $X \oplus X^\vee$ over F_0. Give $X(\mathbb{A})$ the Haar measure for which $X(F_0)$ has covolume one, and give $X^\vee(\mathbb{A})$ its Fourier-dual measure for the fixed additive character. The annihilator of $X(F_0)$ is $X^\vee(F_0)$, so Poisson summation gives covolume one on the dual quotient as well. The theta functional in this realization is rational summation, up to a unit scalar.

Indeed rational modulation invariance of a functional forces its support to be on $X(F_0)$, by adelic self-duality. Locally at a support point, multiplication by the modulation factors removes derivatives of point masses in the real variables: the differentials of their characters span the real cotangent space. Rational translation invariance then gives the same multiplier at every rational point. Thus it is proportional to rational summation. The unit normalization follows from the mean calculation below, since the original theta functional is normalized to have mean square one on the unit vacuum.

For a Schwartz vector f, the value at (u, v) of its theta translate in these coordinates is, up to that scalar,

$$\theta(\mathcal{R}(u,v)f) = \sum_{\xi \in X(F_0)} f(\xi + u)\psi\big(v(\xi + u/2)\big).$$

Multiply by the conjugate of the same expression for f'. The two half-shift factors cancel. Integration over $v \in X^\vee(\mathbb{A})/X^\vee(F_0)$ kills each term whose two rational indices differ. Integration over $u \in X(\mathbb{A})/X(F_0)$ then unfolds the common rational index. Consequently, with dz the probability Haar on the displacement quotient,

$$\int \theta(\mathcal{R}(z)f)\overline{\theta(\mathcal{R}(z)f')}\, dz = \int_{X(\mathbb{A})} f(u)\overline{f'(u)}\, du = \langle f', f\rangle.$$

The Schwartz series and its derivatives converge uniformly on compact sets in this calculation. In particular the norm-one vacuum gives one for the mean square and fixes the modulus of the theta scalar.

Let $A_{f,f'}(z)$ denote the integrand. Rational Heisenberg invariance and cancellation of the common unit phase make it periodic. For a rational displacement γ, use the convention $\mathcal{R}(z)\mathcal{R}(\gamma) = \beta(z,\gamma)^2\mathcal{R}(\gamma)\mathcal{R}(z)$. Applying the last mean identity to $(\mathcal{R}(\gamma)f, f')$, rational invariance gives

$$\int \beta(z,\gamma)^2 A_{f,f'}(z)\, dz = \langle f', \mathcal{R}(\gamma)f\rangle.$$

The inverse characters $\beta(z,\gamma)^{-2}$ are the dual of the quotient.

The series of these Fourier coefficients is absolutely convergent. To see this choose an open finite additive compact group fixing both f, f' under its Heisenberg displacements. The product is invariant under that group because the remaining commutator factors have modulus one and cancel. The quotient by this group is a real torus, by additive approximation. The product descends to a smooth function there, and integration by parts bounds its Fourier coefficients by every inverse power of the frequency norm. Fourier inversion at zero therefore yields

$$\theta(f)\overline{\theta(f')} = \sum_{\gamma \text{ rational}} \langle f', \mathcal{R}(\gamma) f \rangle.$$

In particular the zero-frequency coefficient is $\langle f', f \rangle$ with its original scale.

41.5.2 The finite character and its translate

In this subsection, we compute the finite component of the transform in Section 41.5.1.

Write s_0 for the finite generator of $\mathfrak{m}$ in this paragraph and $e_{\mathfrak{m}}$ for the idempotent of its chosen split places. Define $\psi_0(x) = \psi_f(j_{\mathrm{W}} \bar{\alpha} \alpha x)$. The marked primitive character on R is the reduction of $\psi_0(e_{\mathfrak{m}} s_0^{-1} x)$. For a residue frequency matrix t, let $\widetilde{t}$ be an integral lift of its transpose, indexed by the Hom coordinates. Its normalized coordinate is

$$y_t = \overline{e_{\mathfrak{m}} s_0^{-1} \widetilde{t}},$$

and its actual adelic displacement is $Y_t = (0, \alpha y_t)$. The finite part of the datum of $\psi_t(X) = \psi_{\mathfrak{m}}(\mathrm{tr}(Xt))$ is $\mathcal{R}(Y_t) v_{\mathrm{vac}}$. In fact the projection onto the integral Fourier box fixes v_{vac}. The translation above has support on the conjugate member of each chosen split pair, so in the selected Schrödinger box it is multiplication by the indicated character. The dot product with $\widetilde{t}$ is the trace of Xt. This also shows independence from the integral lift of t.

For an arbitrary adelic displacement z, move $\mathcal{R}(Y_t)$ across the inner product. Its commutator with $\mathcal{R}(z_f)$ contributes $\beta(z_f, Y_t)^2$, with

$$\beta(z_f, Y_t)^2 = \psi_f\left(j_{\mathrm{W}} \sum_{\nu} \bar{\alpha}\, e_{\mathfrak{m}} s_0^{-1} \widetilde{t}_{\nu} z_{f,\nu} \right).$$

When $a(z)$ is integral this is $\psi_t(a(z) \bmod \mathfrak{m})$, by $\bar{\alpha} z_f = \bar{\alpha} \alpha a(z)$. The other finite factor is the marked vacuum coefficient

$$\mathbf{1}_{a(z) \in \widehat{\mathcal{O}}_E^D}\, \psi_f\left(\frac{c_{\mathrm{tr}} j_{\mathrm{W}}}{2} \sum_{\nu} \overline{z_{f,\nu}} z_{f,\nu} \right), \qquad c_{\mathrm{tr}} = (1 - j_{\mathrm{W}}^{-1})/2.$$

For real Fock vectors v, w the real factor is $\langle v, \mathcal{R}_{\infty}(z_{\mathbb{R}}) w \rangle$. Finite character orthogonality is for the perfect trace pairing of the two rectangular matrix spaces over R. The probability average over all t therefore restricts the product of these factors to $a(z) \in \widehat{\mathcal{O}}_E^D$ and $a(z) \equiv 0 \bmod \mathfrak{m}$. At $z = 0$, the finite factor, all commutators, and the normalized average of the constant function are one.

41.5.3 The sparse test and its error

In this subsection, we apply Section 41.5.2 to the marked test used in the comparison.

Fix the original two adelic points g, h and let p be the induced isometry of $\mathrm{Hom}_E(S,T)$ over the adeles, given by $X \mapsto hXg^{-1}$. Let v be the Fock vector for the polynomial P. Write $K_t(g,h)$ for its kernel with finite test ψ_t. Use the single finite test

$$\Psi(X) = \frac{1}{|M_{\mathrm{rk}\,S,\mathrm{rk}\,T}(R)|} \sum_t \overline{K_t(g,h)}\psi_t(X).$$

At $g(u,1), h(u',1)$ with real u, u', put $w = \omega_\infty^{a,b}(u,u')v$, including the determinant weights of the original characters a, b. Linearity in the finite test writes its kernel as the same average of the product of the theta value at w with the conjugate theta value at v, both moved by p.

The starting pair contributes the same unit scalar to both theta values and cancels in the product. Move $\omega(p)$ through the Heisenberg coefficient by unitarity and covariance. For a rational map Z, the displacement is therefore $p^{-1}(X_T^{-1}ZX_S) = z(Z)$ as in Section 41.4. Both its finite reduction and its scaled real coordinate are evaluated at the fixed pair (g,h). The real translations u, u' enter through w. The Fourier identity just proved applies for each t. The resulting average is thus a sum over precisely the rational differences that satisfy the displayed finite support and congruence, with unit-modulus finite phases. Its zero difference equals $\langle v, \omega_\infty^{a,b}(u,u')v\rangle$. This is the original Fock matrix coefficient with mass one.

For the nonzero differences let N be the number of complex Fock variables. The polynomial used here is $P = P_j$, a product of minors homogeneous of degree $d \le b\sqrt{b}$, as computed in Section 41.1.4. The coherent expression of the normalized real Weil operator is

$$\omega_\infty(\sigma)v = \Delta_\infty(\sigma) \int v(x)\mathcal{R}_\infty(\sigma x)v_{\mathrm{vac},\infty}\, dx.$$

The measure is the original coherent model measure and the multiplier is the product of the conjugate determinants of the negative-sign blocks of σ, so $|\Delta_\infty(\sigma)| \le \|\sigma\|^N$.

In the vacuum pairing, Heisenberg covariance bounds the displaced coefficient by

$$\|\sigma\|^N \int |v(x)|\,|v(\sigma x + z_{\mathbb{R}})|\, dx.$$

The determinant-character factors are units. If $q > N$, put $A_0 = d + N$ and $c^2 = 1 + q/A_0$. For $G(x) = |c^d v(x/c)|^2$, homogeneity gives

$$\int G \le e^q \|v\|^2, \qquad |v(x)| = G(x)^{1/2} e^{-q\|x\|^2/(2(A_0+q))}.$$

The first inequality uses the dilation Jacobian c^{2N}. The signed unitary matrix preserves real volume and $\|\sigma\| \ge 1$. The inequality

$$\|x\|^2 + \|\sigma x + z_{\mathbb{R}}\|^2 \ge \|z_{\mathbb{R}}\|^2/(2\|\sigma\|^2)$$

follows from the triangle inequality and the two-square bound.

Apply $2ab \le a^2 + b^2$ to the two square roots of G. For $z_{\mathbb{R}} \ne 0$, the remaining Gaussian is at most e^{-qt_z}, where

$$t_z = \frac{\|z_{\mathbb{R}}\|^2}{4(A_0+q)\|\sigma\|^2}.$$

Since $e^{-qt_z} \le e^{-q} t_z^{-q}$ and σ preserves volume, integration cancels the factor e^q. This gives

$$\|\sigma\|^{N+2q}(4(d+N+q))^q \|v\|^2 \|z_{\mathbb{R}}\|^{-2q}.$$

On Hom, $\sigma(X) = u' X u^{-1}$ and $\|\sigma\| \le |u'|\,|u|$, since the operator norm of a signed unitary matrix equals that of its inverse.

On the marked reduction kernel the original multiplier q_1/π_M^2 sends every surviving difference into the Hom lattice. When the two given first minima have product at least b^{-13}, the Hom-lattice calculation gives separation $\mathcal{D} = b^6$ for their Fock coordinates, including distance at least $\mathcal{D}$ from zero. Packing balls of radius $\mathcal{D}/2$ bounds the count in the annulus of radii $2^i\mathcal{D}$ and $2^{i+1}\mathcal{D}$ by $(5 \cdot 2^i)^{2N}$. Summing the geometric series and using $q \ge N+1$ gives

$$\sum_{z_{\mathbb{R}} \neq 0} \|z_{\mathbb{R}}\|^{-2q} \le (5/\mathcal{D})^{2q}.$$

Here S and $\widetilde{W}$ have ranks $3m$ and $r+3m$. Thus $N = f(3m)(r+3m) \le ft^2 < q = 2n_*$, since both ranks are at most $t = 5r$. The error in the sparse test is at most

$$(|u|\,|u'|)^{6n_*}\|v\|^2 \left(\frac{100(d+2q)}{b^{12}}\right)^q \le b^{-20n_*}(|u|\,|u'|)^{6n_*}\|v\|^2.$$

Indeed $100(d+2q) \le b^2$ for all sufficiently large x at fixed discriminant and ϵ: one has $100b^{3/2} \le b^2/2$ once $b \ge 200^2$, and $400n_* \le b^2/2$ eventually by polynomial growth in x and $\log b = x^{\epsilon} + O(1)$. The same finite test works for every u, u'. In particular, at $u = u' = 1$, writing $\mathcal{T} = \mathrm{Mat}_{\mathrm{rk}\,S \times \mathrm{rk}\,T}(R)$, its defining average satisfies

$$\frac{1}{|\mathcal{T}|}\sum_{t \in \mathcal{T}} |K_t(g,h)|^2 \ge (1 - b^{-20n_*})\|v\|^2.$$

This is positive for the prescribed nonzero Fock vector. The error estimate uses $\|v\|^2 = c_P(1,1)$ at every real translate.

42 Compression, separation, and reduction at cusps

In this section, we compress the comparison operator of Section 34 and establish separation on the initial unitary group. The cusp estimates justify the decay needed for the deeper periods of Section 43.

42.1 Compression of the comparison operator

In this subsection, we restrict the comparison of Section 34.1 to the image of the injective lift and project with respect to the Petersson form. We bound the determinant of this operator using the original norm on each factor.

Fix one original descent pair and write $u_\eta = u_{j,\eta}$ for the injective marked lift on the good source W of dimension D. Write $Z_\eta = \operatorname{im} u_\eta$ and let Π_η be the Petersson projection onto Z_η. Write $W_{j,b,c}$ for the simultaneous subspace of the good source of Hecke system b and old finite type c. For the rank-r arrow, the residue equivariance and decomposition of Section 38.2 give

$$Z_\eta = \sum_{b,c} u_\eta(W_{j,b,c}).$$

Each summand is in the corresponding upper type at the original level. These are the generators of the space Z in Proposition 32.1. Use the original gluing operator G_η, its original tuple space, and the operator $U_\eta = n_A h_-^2 G_\eta G_\eta^*$ of Section 34.1. Put

$$\Delta_\eta = \det(\Pi_\eta U_\eta|_{Z_\eta}), \qquad Z_j^{\mathrm{single}}(\eta) = \det(u_\eta^* u_\eta).$$

42.1.1 The period and the Hecke factor

In this subsection, we use Proposition 39.1 in the compressed pairing.

Lemma 42.1 (Determinant of the compressed comparison)**.** *Set $\rho_\eta = H_j(\eta)/H_j(1)$ for the allowed twists. If $D > 0$, then $\rho_\eta > 0$ and there is a fixed positive definite operator U° on the source such that*

$$u_\eta^* U_\eta u_\eta = \rho_\eta U^\circ, \qquad Z_j^{\mathrm{single}}(\eta)\Delta_\eta = \det(U^\circ)\rho_\eta^D, \qquad \frac{Z_j^{\mathrm{single}}(1)}{Z_j^{\mathrm{single}}(\eta)} = \frac{\Delta_\eta}{\Delta_1}\rho_\eta^{-D}.$$

If $D = 0$, all the determinants and ratios of determinants are 1.

Proof. For $D > 0$, let P be the injective common-period map and $c'_\eta \neq 0$ the scalar in $c'_\eta G_\eta^* u_\eta = P$. The three exchanges of Section 39.4 give

$$|c'_\eta|^2 = C_{\mathrm{ex}}/H_j(\eta),$$

where $C_{\mathrm{ex}} \neq 0$ is independent of η, and H_j is the marked product of Hecke values including the local conductor factors. In particular

$$\rho_\eta = \frac{H_j(\eta)}{H_j(1)} = \frac{|c'_1|^2}{|c'_\eta|^2} > 0, \qquad u_\eta^* U_\eta u_\eta = \rho_\eta U^\circ, \qquad U^\circ = \frac{n_A h_-^2}{|c'_1|^2} P^* P > 0.$$

Indeed the pairing of the middle operator against $y', y \in W$ is $n_A h_-^2 \langle G_\eta^* u_\eta y', G_\eta^* u_\eta y\rangle$; using the two scalar factors gives the formula. Since $u_\eta^*(1 - \Pi_\eta) = 0$,

$$u_\eta^* U_\eta u_\eta = (u_\eta^* u_\eta)(u_\eta^{-1}\Pi_\eta U_\eta u_\eta).$$

Taking determinants yields

$$Z_j^{\mathrm{single}}(\eta)\Delta_\eta = \det(U^\circ)\rho_\eta^D, \qquad \frac{Z_j^{\mathrm{single}}(1)}{Z_j^{\mathrm{single}}(\eta)} = \frac{\Delta_\eta}{\Delta_1}\rho_\eta^{-D}.$$

For the covectors f_a of (A), transported to the tuple space as in Section 34.1.1, and $z = u_\eta y$,

$$\sum_a |\langle \Pi_\eta G_\eta f_a, z\rangle|^2 = h_-^2 \|G_\eta^* z\|^2 = h_-^2 |c'_\eta|^{-2}\|Py\|^2.$$

Thus the projected vectors $\Pi_\eta G_\eta f_a$ span Z_η, and positivity gives all the nonzero determinants used above. In dimension zero the determinant is the empty product. □

42.1.2 The projection at complex embeddings

In this subsection, we bound the conjugate projection for the operator of Section 42.1.1.

Let B be the class sum with the original rational weights and Weyl Gram matrices on the ambient class coordinates, with conjugation c in the first variable. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ put

$$B_\sigma(v,w) = \sigma^{-1}B(\sigma v, \sigma w), \qquad \|v\|_\sigma^2 = \sigma B_\sigma(v,v) = B(\sigma v, \sigma v) \geq 0,$$

where σ acts on the form values coordinatewise. The first variable of B_σ uses $\sigma^{-1}c\sigma$. For $\gamma \in U(I_n)(E^+)$, each $\sigma\tau(\gamma)$ is unitary: complex conjugation on the CM field commutes with every embedding into $\mathbb{C}$. The rational Weyl matrix formula therefore makes the transport by γ an isometry for $\|\cdot\|_\sigma$. Let J be the original target level and $K \subseteq J$ a common compact open subgroup fixing the two test forms. On the finite adelic quotient $U(I_n)(E^+)\backslash U(I_n)(\mathbf{A}_{E^+,f})$, let $b(x)$ be their pointwise Weyl Gram pairing, with first-slot involution $\sigma^{-1}c\sigma$. It is locally constant and left invariant by the rational isometry just proved. With the original level normalization,

$$\frac{1}{\mathrm{vol}(J)} \int b(x)\, dx = \frac{\mathrm{vol}(K)}{\mathrm{vol}(J)} \sum_{[x]_K} \frac{b(x)}{|\operatorname{Aut}_K(x)|}.$$

For forms of the original level character χ, the pairing is J-invariant since $\overline{\sigma\chi(j)}\,\sigma\chi(j) = 1$. For a J-class represented by x, put $\Gamma_x = x^{-1}U(I_n)(E^+)x \cap J$. Its refinements are the Γ_x-orbits on J/K, with stabilizers $(\Gamma_x)_y$. Thus

$$\frac{1}{[J:K]} \sum_{y \in \Gamma_x\backslash J/K} \frac{1}{|(\Gamma_x)_y|} = \frac{1}{|\Gamma_x|}.$$

This recovers the original class weight after refinement. Right invariance of the quotient measure gives inverse translation as the adjoint in this pairing, also after σ.

The original finite group acts by right translations and finite-order character factors, so its action is unitary at each embedding. The central idempotent e_0 is self-adjoint there: conjugation of coefficients gives a central idempotent in the same finite group algebra and its operator acts as 0 or 1 on each irreducible unitary summand. Choose a rational prime $p \equiv 1 \bmod dq_1$ away from the finite support, using [52, Corollary 4.12(c)]. Since $E \subseteq \mathbb{Q}(\zeta_{dq_1})$, every place $w \mid p$ of E lies over a split place of E/E^+, and both $w, \bar{w}$ are off the level. At one such w use the reciprocal operator of Section 32.3,

$$A_w = K_w(X_w), \qquad K_w(Y) = \sum_{i=0}^{n} (-Y)^i q_w^{\binom{i}{2}} T_{w,i}, \qquad X_w = (\nu_w q_w^{t'-1})^{-1}, \quad t' = n - r'.$$

Here r' is the source rank. On a spherical system x, $K_w(Y)|_x = Y^n H_{x,w}(Y^{-1})$. On each good lift the factor $\prod_{a=0}^{t'-1}(1 - \nu_w q_w^a Y)$ of Section 28.1.1 vanishes at X_w.

With compact mass one the adjoint of each spherical double-coset sum is the sum for the inverse double coset. In the split coordinate at $\bar{w}$, inverse transpose takes this coset to that for the same elementary Hecke operator. Thus the B_σ-adjoint of A_w is $K_{\bar{w}}((\sigma^{-1}c\sigma)(X_w))$. Both operators lie in the spherical algebra on the original invariants. It is commutative since transpose reverses convolution and fixes each $\mathrm{GL}_n(\mathcal{O}_w)$-double coset by elementary divisors. Normality gives $\ker A_w^* = \ker A_w$, so $\operatorname{im} A_w \perp_\sigma \ker A_w$.

In particular $Z_1 \subseteq \operatorname{im} e_0 \cap \ker A_w$. For

$$F_w = Z_1^{\perp} \cap \operatorname{im} e_0 \cap \ker A_w$$

the orthogonals in the original metric satisfy $F_w^{\perp} = Z_1 + \ker e_0 + \operatorname{im} A_w$. The last two summands are orthogonal to Z_1 for every conjugate pairing. Therefore, for $v \in F_w^{\perp}$, its Z_1-component is the original $\Pi_1 v$, and

$$\|\Pi_1 v\|_\sigma \le \|v\|_\sigma.$$

Equation (90) with $\vartheta = 1$ gives $U_1 z \in F_w^{\perp}$ for $z \in Z_1$.

42.1.3 The norm at the trivial twist

In this subsection, we combine the projection bound of Section 42.1.2 with the original gluing sum.

Write $A_{\mathrm{glue}} = fmbB^2$, N_{glue} for the number of marked gluing choices times the number of second-factor classes, and l_1 for the denominator exponent of the splitting. The original gluing sum has the factor M^{l_1}. For each of the N_{glue} coordinates, it has at most e^b first-factor classes. Each point-form evaluation is a sum of at most n_A terms with unitary rational transports and root-of-unity level factors. Let T be the Cartan map from the tensor product of the two block Weyl spaces in Section 33.2. The operator T^*T commutes with $\prod_{\tau\mid\infty}(\mathrm{U}(d_+) \times \mathrm{U}(d_-))$ by block equivariance at each place. The inverse transpose on the second block and the determinant character preserve the Hermitian forms. The Weyl tensor is irreducible for this product group. In the prescribed orders, reversed on the second block, the tensor of the normalized highest vectors maps to the combined highest vector, also of Weyl norm one. Schur's lemma gives $T^*T = 1$. The coordinate matrices of T are integral, so at every embedding

$$\|\sigma T(y \otimes v)\|^2 = \|\sigma y\|^2 \, \|\sigma v\|^2.$$

At the trivial twist the first-factor level J_Λ, both principal levels in the choice of a tuple coordinate, and the target level contain $K(\pi_M^2)$ in their respective ranks (Section 33.2). Since $\pi_M \bar{\pi}_M = M$, they contain $K(M^2)$. For any of these levels J of rank $n \le r + t$, the lattice count in the proof of Lemma 7.2 gives

$$h(J) \le h(K_{\mathrm{int}})[K_{\mathrm{int}} : K(M^2)] \le h(K_{\mathrm{int}})(M^2)^{[E:\mathbb{Q}]n^2}, \qquad \log h(J) = O(x^{52}) < b.$$

Using $\|\phi''_{1,i}(h)\|_\sigma^2 \le e^{A_{\mathrm{glue}}}$ from Section 33.3, we obtain

$$\|G_1 \zeta\|_\sigma^2 \le M^{2l_1} e^b (N_{\mathrm{glue}} e^b n_A)^2 e^{A_{\mathrm{glue}}} \|\zeta\|_\sigma^2.$$

The coefficient bound here uses the unweighted direct sum over the second-factor classes, which are free at their original principal level. There are at most e^b choices in each of the two principal-level class sets defining N_{glue}. The residue matrix has $3md_+$ entries, since $\dim_E S = 3m$ and $\dim_E V'' = d_+$. Its residue ring has order at most M^{4f}. At each of the f embeddings the index specifies $\Delta' = (\Delta - t - 1)/2$ ordered tuples of $2m$ covectors from the chosen basis of $W^\flat$, whose rank is $6m + 1$. Consequently

$$N_{\mathrm{glue}} \le e^{2b} M^{4fd_+(3m)} (6m+1)^{2mf\Delta'}.$$

The column estimate for the denominator exponent is

$$l_1 \le 3fm(b + a/x_1 + \Delta + m\Delta) + ft^2\Delta.$$

Here $B = \log b$, $\log M \le 22B$, $a/x_1 \le b$, and $\log n_A \le f(rt)^2 \log M$. Since f, r, t, m, Δ are polynomially bounded in x, for a fixed exponent C we have

$$2l_1 \log M + 3b + 2\log(N_{\mathrm{glue}} n_A) \le 264 fmbB + 7b + O(x^C B) = o(A_{\mathrm{glue}}).$$

The same bounds give $n_A h_-^2 \le e^{A_{\mathrm{glue}}}$, since the minus diagram has at most $ft^2\Delta$ boxes and its hook factor is bounded by $L_-^{3L_-}$ for $L_- \ge 2$ bounding that count.

For precision about adjunction, use the full spaces of class values with their rational block Gram matrices. The tuple space of G_1 is the full tuple-coordinate space, since its stabilizers are trivial. The image of G_1 lies in the original target, and its ambient adjoint restricts there to the marked adjoint. With C_S, C_T the rational Gram matrices, the adjoint matrix is $C_S^{-1} c(G_1)^{\mathsf{t}} C_T$. Therefore $\sigma(G_1^*)$ is the ordinary Gram adjoint of $(c\sigma c)(G_1)$ in the ambient coordinates. On conjugated class values, the adjoint norm bound follows from the uniform bound for G_1 just proved on its full tuple domain. Thus

$$\|U_1 z\|_\sigma \le n_A h_-^2 e^{2A_{\mathrm{glue}}} \|z\|_\sigma \le e^{3A_{\mathrm{glue}}} \|z\|_\sigma.$$

An eigenvector of $\Pi_1 U_1|_{Z_1}$ with eigenvalue λ now gives $|\sigma\lambda| \le e^{3A_{\mathrm{glue}}}$. Multiplying over the eigenvalues with algebraic multiplicity gives $\log|\sigma\Delta_1| \le 3DA_{\mathrm{glue}}$.

One operator on the full good lift space. Choose a basis $v_1, \dots, v_s$ from the lift generators of Z. For each i, construct the first-order operator D_i by using the type and original system of v_i. Every generator v_j has its own fixed type parameter. Its projection to the type of v_i is either zero, if the types differ, or v_j itself. In the latter case the spherical system remains the same after inclusion in the raised level used for D_i. Thus $D_i v_j = d_{ij} v_j$, with the bound (RB1) on $v_\ell(d_{jj})$. By Proposition 32.1, $D_i \pi_Z y$ is integral at the chosen embedding for every original integral test y satisfying $y \perp F_w$ at every split place w off the level and its conjugate.

Write ι for that embedding and, for each j, select $a_j \ne 0$ from the numbers $\iota(d_{ij})$ of maximum absolute value. Reduce the integral ratios $\iota(d_{ij})/a_j$ in the residue field of $\overline{\mathbf{Q}}_\ell$. Each resulting linear form $\sum_i X_i \overline{\iota(d_{ij})/a_j}$ is nonzero. All these ratios lie in one finite extension of $\mathbb{Q}_\ell$. For $s > 0$, enlarge it unramified so its residue field has order $q > s$. In s variables the s kernels contain at most $sq^{s-1} < q^s$ points. For a tuple in their complement let $b_i = 0$ at a zero coordinate and take the Teichmüller lift at each nonzero coordinate. Since $b_i^{q-1} = 1$ in the latter case, all b_i belong to $\iota(\overline{\mathbb{Q}})$ and are integral at ℓ. The operator $D_Z = \sum_i \iota^{-1}(b_i) D_i$ maps the projections of the same integral tests to integral forms. On v_j its eigenvalue has absolute value $|a_j|_\ell$, at least $|\iota(d_{jj})|_\ell$. In the same basis every original off-level Hecke polynomial is diagonal, so D_Z preserves all its eigenspaces. The determinant bound follows by multiplying the eigenvalue bounds. If $Z = 0$, use the identity on the zero space, with determinant one.

42.1.4 Compression on the original local lattice

In this subsection, we apply the integral clearing of Proposition 32.1 to the local determinant.

Fix $\kappa \mid \ell$, normalize $v_\kappa(\ell) = 1$, and take the selected twist η with auxiliary modulus of norm prime to ℓ. Use the clearing operator $\Lambda = D_Z$ just constructed on $Z = Z_\eta$. The selected-twist bounds in Section 38 give $v_\kappa(\det \Lambda) \le DrC_\ell$. It takes projections of the integral tests in $\bigcap_w F_w^\perp$

to integral vectors. The marked U_η preserves integrality at κ. For this same twist let Θ denote the full single lift and e_W, e_V the good idempotents on its source and target. The identity $e_V\Theta = \Theta e_W$ of Section 33.1 gives

$$\operatorname{im} e_V \cap Z_\eta^{\perp} = \operatorname{im} e_V \cap \ker \Theta^*.$$

Indeed, for $h = e_V h$, pairing h with Θy amounts to pairing it with $\Theta(e_W y)$. Intersecting with $\ker K_w(X_w)$ gives exactly F_w used by the clearing operator. Equation (90) gives $U_\eta Z_\eta \subseteq \bigcap_w F_w^{\perp}$. The first Gram argument of U_η uses the integral gluing vectors at $\bar\kappa$ on the fixed twist η; Proposition 32.1 is applied at κ. Consequently $\Lambda\Pi_\eta U_\eta|_{Z_\eta}$ preserves integral vectors in Z_η.

Every nonzero eigenvector of this composite can be scaled to an integral vector with a unit coordinate. Indeed choose one representative of each original finite class integral above ℓ, by weak approximation in the rational unitary group. In these bases the gauge is the maximum of the norms of the finitely many Weyl coordinates. Rational covariance, the integral level action and its unit level character give the same gauge test on every representative.

Divide by a coordinate of largest norm. The eigenvalue then has nonnegative valuation, because the composite preserves integrality and the normalized vector has a unit coordinate. Taking the product of the eigenvalues gives

$$v_\kappa(\Delta_\eta) \geq -v_\kappa(\det \Lambda) \geq -DrC_\ell, \qquad \log|\Delta_\eta|_\kappa \leq DrC_\ell \log \ell.$$

In particular the finite logarithmic allowance is $\beta_{j,\ell} = C_\ell \log \ell$, or the larger $B_\ell \log \ell$ used for the paired rank-one test. Since $B_\ell = C_\ell + v_\ell(h_1)$ and h_1 is an integer, in either case $\beta_{j,\ell} \leq B_\ell \log \ell = J_\ell$. Thus (RT3) applies also to $\sum_{\ell \in U} \beta_{j,\ell}$ for every finite U. These bounds only use the clearing operator on the same selected twist. The fixed multiplier for the ratio Δ_η/Δ_1 is Δ_1, which does not depend on η and serves as one height multiplier at every coefficient embedding.

42.2 Separation on the initial unitary group

In this subsection, we show that the inclusion at the initial deep level separates the chosen automorphic coefficients. The reproducing coefficient of Section 12.3 gives the local test, with the scale selected in Section 42.3.

Write $V = Y_0$, $Z = V \perp \langle 1 \rangle$, and $\iota(h) = h \oplus 1$ in the original rational basis. In particular, $\iota^{-1}(K_Z(a)) = K_V(a)$ at each principal modulus a. The marked grid numbers the first r entries of Z exactly as for V. Its last entry has index r and sign $(-1)^{r+1-k} = (-1)^m = +1$. Thus the real group inclusion is the block inclusion of this rational orthogonal sum. Set $a_0 = \pi_M^{b_0}$, $b_0 = \lfloor b \rfloor$, and denote the original principal source level by $J_M = K_V(\pi_M)$. Let W be its source forms of the prescribed type, and W_Z the forms of the initial true grid type at $K_Z(a_0)$. We will construct one $\omega \in U(Z)(F_\infty)$ such that

$$y \in W, \quad \int_{[U(V)]} \overline{y(h)}\, \psi(\iota(h)(\omega, 1))\, dh = 0 \quad (\psi \in W_Z) \quad \Longrightarrow \quad y = 0. \tag{DS0}$$

Here F is the maximal real field of E. The Haar measures are those in the marked spaces.

Weights and the archimedean scale. At the infinite place v indexed by τ, put

$$w_v = \kappa_\tau + \Delta/2, \qquad \Delta = 2r^8,$$

with the integer weight κ_τ of Section 2.3. Let l, l_Z denote the numbered doubled grid parameters of V, Z. The actual form types of W, W_Z use $l, -l_Z$, respectively. In ranks $r + z$, $z = 0, 1$, the

numbered doubled entries of the grid are $2\lambda_i + z$. On $0 \le i < k-1$, $\lambda_i = k/2 - 1 - i$; on the tail $i = k - 1 + j$, $\lambda_i = -w_v - (j + 1/2)\Delta$. Both initial blocks have sign $+$, and the tails alternate starting with $+$. The correction from doubled parameter to doubled lowest-weight entry is the signed count of later entries minus earlier entries, signs taken relative to the entry in question.

On the initial block this equals $k - 2 - 2i + z$, since the tail has signed excess z. Thus the lowest-weight entry there is zero. The correction everywhere is bounded in absolute value by $r + 1$. Consequently both lowest-weight tails are strictly decreasing in $[-3w_v, -w_v]$. Indeed $w_v \ge \Delta^2$, $m\Delta = o(w_v)$, and $\Delta > r+1$. The doubled entries have modulus at most $L_{\mathrm{sep}} = \lceil 20b \rceil$, since $w_v \le 2b$ and $(m+1)\Delta = o(b)$. The opposite-sign gaps for the undoubled parameters are at least Δ before subtracting $500(r+1)^3$. The same gap bounds hold after negating a parameter.

Put $A_{\mathrm{sep}} = r^4$, $D_{\mathrm{sep}} = (2r+1)^2$, $H = 20D_{\mathrm{sep}}$, $\beta = (\sqrt{M})^{\lceil x \rceil}$, $B_d = \beta^{20A_{\mathrm{sep}}(f+1)}$, and let $\kappa = 8fr\,\mathrm{rd}(E) + 48r^6 f^2 N_D + 1$, where $f = [F : \mathbb{Q}]$ and $N_D = q_1 d$ lies in the different. The parameter bounds $r \asymp x^5$, $m \le 2\sqrt{x}$, $b^{20} \le M \le b^{22}$, and $\log b = x^{\epsilon} + O(1)$ give

$$\begin{gathered} A_{\mathrm{sep}} > 8(r+1)^2, \quad A_{\mathrm{sep}} \ge 2f(r+1) + 2, \quad H < A_{\mathrm{sep}}, \quad \beta \ge \max\{10^{12}, f, 2r + 1 + L_{\mathrm{sep}}\}, \\ \beta \ge \max\{2000(r + L_{\mathrm{sep}} + 1)^6, M\kappa^2\}, \\ (r+1)^4 (6w_v)^{2m+2} \le \beta, \qquad 10^{30}(D_{\mathrm{sep}} \log \beta)^3 = O(x^{33+3\epsilon}) \le r^7, \\ |a_0|_v = (\sqrt{M})^{b_0} \ge B_d^{3r+2}. \end{gathered} \tag{DS1}$$

For the first line use the polynomial bounds on $f, q_1 d, \mathrm{rd}(E)$ of the marked field and $\beta \ge b^{10\lceil x \rceil}$; in particular the admissible bound on f gives $2f(r+1) + 2 \le r^4$. For the basis bound, $r + 1, 6w_v \le \sqrt{M}$ and $2m + 6 \le \lceil x \rceil$. For the logarithmic comparison we use $\log M = O(x^{\epsilon})$. Finally $b_0 \ge 20A_{\mathrm{sep}}(f+1)(3r+2)\lceil x \rceil$ since its right side is polynomial in x. The excess gap is at least r^7, and in particular at least $4A_{\mathrm{sep}}$ in the smaller group and $2A_{\mathrm{sep}}$ in the larger group.

The local stationary chosen element. Let μ_i, $0 \le i \le m$, be the decreasing tail of the lowest weight for l_Z, and ν_j, $0 \le j < m$, the increasing tail for $-l$. Then $\mu_i \in [-3w_v, -w_v]$ and $\nu_j \in [w_v, 3w_v]$. Write

$$p(t) = \prod_i (t - \mu_i), \qquad a(t) = \prod_j (t - \nu_j), \qquad [R]_i = \frac{R(\mu_i)}{\sqrt{|p'(\mu_i) a(\mu_i)|}}.$$

In the tail sign models define

$$v_* = [a], \qquad u_j = \sqrt{|p(\nu_j)/a'(\nu_j)|}\,[a(t)/(t - \nu_j)].$$

Since m is even, $p'(\mu_i)a(\mu_i)$ has sign $(-1)^i$. The identity $\sum_i R(\mu_i)/p'(\mu_i) = [t^m]R(t)$ for $\deg R \le m$ gives $(v_*, v_*) = 1$, $(v_*, u_j) = 0$, and $(u_j, u_k) = 0$ if $j \ne k$. The signed square norm of $[a(t)/(t - \nu_j)]$ is $-a'(\nu_j)/p(\nu_j)$: sum the residues of $a(t)/((t - \nu_j)^2 p(t))$.

Its sign is $(-1)^j$. Multiplication by t on the polynomial vectors gives $\mu u_j = \nu_j u_j + \sqrt{|p(\nu_j)/a'(\nu_j)|}\, v_*$. Hence the isometry with columns u_j compresses μ to ν. Extend this isometry by the identity on the common zero-weight positive block X, and append v_* last to obtain $w \in U(Z_v)$. All entries of w have modulus at most $(6w_v)^{m+1}$, using the integral node separation and the bound $6w_v$ on all node distances. For $\omega_v = w^{-1}$, the equality $\|w^{-1}\| = \|w\|$ and (DS1) give $\|\omega_v\|^2 \le \beta$.

Put $g = w(h \oplus 1)w^{-1}$, $h \in U(V_v)$. Then $\mathrm{tr}(g - 1) = \mathrm{tr}(h - 1)$, and the compression gives $\mathrm{tr}(\mu(g - 1)) = \mathrm{tr}(\nu(h - 1))$, where both weight matrices are extended by zero on X. The two

embeddings contain the diagonal $U(X)$. Let $\theta = c_{l_Z}(g)\overline{c_{-l}(h)}$. Since $c_{-l} = \overline{c_l}$ and $c_{-l_Z} = \overline{c_{l_Z}}$, this is exactly $c_l(h)\overline{c_{-l_Z}(\omega_v^{-1}\iota(h)\omega_v)}$.

We apply the one-factor coefficient computations of Sections 12.3 and 12.9. Here are the two-factor localization details. Let E_g, E_h be the opposite-sign blocks, $v_0^2 = \|E_g\|_{\mathrm{HS}}^2 + \|E_h\|_{\mathrm{HS}}^2$, $T = D_{\mathrm{sep}} \log \beta$, and $\eta = 10^{-13}/T$. The signed covector identity, weighted by $-\mu_i$ and ν_i, bounds the squared norm of the covectors of $g-1$ off X by $6v_0^2$. Explicitly,

$$2\Re(g-1)_{ii} = 2\sum_j |(E_g)_{ij}|^2 - \sum_j |(g-1)_{ij}|^2$$

and likewise for h; add the weighted identities, use the equality of weighted traces, discard the nonnegative terms from the second squared norm, and divide by w_v.

Apply the same argument to the inverse pair for the columns. If $v_0 \le \eta$, the polar decomposition $g_{XX} = up$ gives $\|p-1\|_{\mathrm{HS}} \le 6\eta^2 \le \eta$, by the signed Gram identity on the X-columns. Left multiply by the diagonal u^*. Both highest vectors are fixed by $U(X)$: the weight-minor thresholds on this zero-weight block cancel its determinant contribution. Thus θ is unchanged. The moved first squared distance to 1 is at most $13\eta^2$.

The unweighted covector identity and trace equality give the second squared distance at most $15\eta^2$. Both distances are at most 4η. At the moved pair the equality of weighted traces holds on all positive powers. Finite differences give it for each power of $g-1, h-1$, hence for the convergent logarithm series. The quadratic approximation in Section 12.9 has error at most $a/10 + e^{-100T}$ from a number $a \in [0,1]$ after removing its imaginary log phase in each factor.

Since the two phases cancel, this gives $|\theta| - 2\Re\theta \le 12e^{-100T}$ by multiplying the two error bounds. If $v_0 \ge \eta$, the two one-factor decay estimates give

$$|\theta|^4 \le [(1+\|E_g\|_{\mathrm{HS}}^2)(1+\|E_h\|_{\mathrm{HS}}^2)]^{-r^7} \le (1+\eta^2)^{-r^7} \le e^{-400T}.$$

For this region $|\theta| - 2\Re\theta \le 3|\theta|$ gives the same cone bound. With $J(h) = g \oplus h$ and $Q = (2/\|J(h)\|)^{9D_{\mathrm{sep}}}$, one-factor norm decay at the larger of the two norms gives $|\theta| \le Q^2$. Taking the geometric mean of the two bounds on the positive part, and applying the one-factor small-ball estimate at both matrices, gives

$$|\theta| \le 2\Re\theta + 6\beta^{-50D_{\mathrm{sep}}}Q, \qquad |\theta| \ge \tfrac{1}{4} \quad (\|J(h)-1\| \le (200\beta^5)^{-1}).$$

The volume of $\|J(h)\| \le R$ is at most $(3R^2/\delta)^{2D_{\mathrm{sep}}}$ times that of $\|J(h)-1\| \le \delta$ for $R \ge 1$, $0 < \delta \le 1$.

Indeed pack δ/R-separated matrices, and use $\|J(h)^{-1}\| = \|J(h)\|$ to convert each covering ball to a translate of the δ-ball. Let V_b, V_s be the volumes for $R = \beta + 1$, $\delta = (200\beta^5)^{-1}$. Then $V_b \le \beta^{16D_{\mathrm{sep}}} V_s$. The same packing with $\delta = 1$, integrated in R against $9D_{\mathrm{sep}}\, 2^{9D_{\mathrm{sep}}} R^{-9D_{\mathrm{sep}}-1}\, dR$, gives $\int Q \le 16^{9D_{\mathrm{sep}}} V_b$. Thus

$$V_s/4 \le 2\left|\int \theta\right| + 6\beta^{-50D_{\mathrm{sep}}} 16^{9D_{\mathrm{sep}}} V_b, \qquad V_b \le 16\beta^{16D_{\mathrm{sep}}}\left|\int \theta\right|.$$

For the second inequality, multiply the first by $4\beta^{16D_{\mathrm{sep}}}$ and absorb the resulting error into $V_b/2$.

The h-ball of radius 1 about 1 lies in the large ball since $\|w\|^2 \le \beta$. Under its original Haar, write its volume as $v_{1,v}$. We obtain the required chosen element in the original first-path embedding

$$\left|\int_{U(V_v)} c_l(h)\overline{c_{-l_Z}(\omega_v^{-1}\iota(h)\omega_v)}\, dh\right| \ge v_{1,v}\beta^{-H}. \tag{DS2}$$

Take the product of these choices over the real places. The resulting real point depends only on the chosen arithmetic data and works for every source form of the specified type.

42.3 Quantitative reduction near the cusp

In this subsection, we quantify the size of a cusp neighborhood in terms of the first minimum of the moved integral lattice. This permits the localization argument of Section 42.2 with a uniform cutoff.

Let W be V moved to the point under consideration. Write $n = \dim_E W$, $f = [E^+ : \mathbb{Q}]$, D_E for the absolute discriminant, and $\mathrm{rd}_E = |D_E|^{1/(2f)}$. Use the rational lattice Λ of vectors with integral finite coordinates in the moved adelic basis, and Euclidean lengths in this basis at infinity. Let $N_D > 0$ be the chosen integer in the different, and set

$$\kappa = 8fn\,\mathrm{rd}_E + 48n^6 f^2 N_D + 1.$$

For $0 < \delta \le 1$ and a positive integer M, suppose the first minimum cut off at one satisfies $\lambda(W) < [\delta/(M\kappa^2)]^n$. We construct $0 \neq A \subset W$ rational isotropic with

$$N_A(\mathbb{A}_{E^+}) \subset N_A(E^+) B_\infty(\delta) J_W(M) \tag{CF1}$$

in this basis. Here $B_\infty(\delta)$ is the product of the operator-norm balls about the identity, and $J_W(M)$ is the original finite standard principal level M in these coordinates.

First let $y_1, \dots, y_n$ be successively shortest lattice vectors independent over E, and let their lengths be $\alpha_1 \le \cdots \le \alpha_n$. These minima exist since the ordinary real lattice is discrete and spans its ambient space. For the ordinary real successive minima μ_j, one has $\alpha_i \le \mu_{2f(i-1)+1}$, by the real dimension $2f(i-1)$ of the archimedean extension of the span of the first $i-1$ vectors. Hence

$$\prod_{i} \alpha_i^{2f} \le \prod_{j=1}^{2fn} \mu_j.$$

The Euclidean covolume is $2^{-fn}|D_E|^{n/2}$.

In fact the covolume in the archimedean basis multiplies the standard covolume by the inverse of the adelic absolute value of the basis determinant, and the latter is one by the Hermitian basis identity and the product formula for $\det J$. Compute the covolume from an integral basis of O_E, the local indices relative to O_E^n, and the real Jacobian of the inverse infinity basis. Minkowski's second theorem for the Euclidean ball and the contained cube of side $2/\sqrt{2fn}$ give

$$\prod_{i} \alpha_i^{2f} \le (2fn\,\mathrm{rd}_E)^{fn}.$$

In particular $\prod_i \alpha_i \le \kappa^n$.

Put $\theta = \delta/(M\kappa^2)$, and choose $0 \neq z \in \Lambda$ of length less than θ^n. The Hermitian pairing of two vectors of Λ is integral in E by the finite basis identity. If their two lengths have product less than one, the pairing has absolute value less than one at every embedding, so is zero. Nondegeneracy of the pairing against the basis y_i therefore gives $1 \le \|z\| \alpha_n$.

In particular $n \ge 2$ under the assumed shortness, since $\theta\kappa \le 1$. It follows that

$$\prod_{i<n} \alpha_i \le \|z\| \prod_{i} \alpha_i < (\theta\kappa)^n.$$

Pair the $n-1$ lengths by $i \leftrightarrow n-i$. At least one pair satisfies $\alpha_i \alpha_{n-i} < \theta\kappa$, since otherwise the square of the last product would be at least $(\theta\kappa)^{n-1}$. Choose its index $i \le n-i$. Then the first i vectors span an isotropic subspace A, orthogonal to the first $n-i$ vectors. The latter span $A^\perp$ by dimension. Write $a = \alpha_i$, $b' = \alpha_{n-i}$, so $a \le b'$ and $ab' \le \delta/(M\kappa)$. The bases consist of rational vectors in the lattice at the point.

For the radical reduction let X, Y be the column matrices of these rational bases of $A, A^\perp$, and let x, y be their adelic coordinates. Write J for the standard adelic Hermitian form matrix in the transported basis. For any matrix z_0 with columns in $A^\perp$, in this basis define

$$\mathcal{R}(z_0) = 1 + 2\big(z_0 x^* - x z_0^* - x z_0^* J z_0 x^*\big) J.$$

It is in the radical and satisfies

$$\mathcal{R}(w_0)\mathcal{R}(z_0) = \mathcal{R}(w_0 + z_0 - x w_0^* J z_0). \tag{CF2}$$

The product follows on expanding and using $x^* J x = x^* J z_0 = x^* J w_0 = 0$.

For surjectivity, use a rational Witt decomposition $A \oplus W_1 \oplus A'$ with X the first basis and the dual basis in A'. Adjoint in the original form becomes adjoint in the Witt Gram matrix in this calculation. The radical matrices in Witt coordinates have middle-right block C, top-middle block $-C^* J_1$, top-right block D with $D + D^* + C^* J_1 C = 0$, and identity diagonal. For $z_0 = (Z_1, Z_2, 0)$ the corresponding blocks of $\mathcal{R}(z_0)$ have $C = 2Z_2$ and $D = 2(Z_1 - Z_1^*) - \frac{1}{2} C^* J_1 C$.

Take $Z_2 = C/2$ and $Z_1 = (D + \frac{1}{2} C^* J_1 C)/4$. Thus every $u \in N_A$ is $\mathcal{R}(z_0)$, and since the columns of Y span $A^\perp$ one may write $z_0 = Myc$ for an adelic matrix c. These identities are in the original basis after transporting back from the Witt coordinates.

There is a rounding function r on adelic scalars such that $c - r(c)$ is finite integral and has archimedean absolute values at most $4f^2 N_D$. To see this, approximate the finite adele by $t \in E$ modulo $\widehat{O}_E$ by the Chinese remainder theorem. At infinity use the lattice $\Lambda_0 = (2/N_D) O_E$ with the real Euclidean inner product. Its dual is $N_D \mathfrak{D}_E^{-1} \subset O_E$: the Euclidean bilinear form is half the trace pairing with conjugation.

Its nonzero vectors have length at least one by the product formula. For any Euclidean lattice in dimension d, dual minimum at least one implies covering radius at most d^2. Indeed Minkowski's convex body theorem applied to the lattice and its dual, using reciprocal covolumes and an inscribed cube, gives a nonzero lattice vector of length at most $2d$. Use a shortest vector a, which is primitive.

Orthogonal projection to $a^\perp$ is a lattice since a extends to a lattice basis, and its dual is the intersection of the original dual with $a^\perp$. By induction the projected remainder has length at most $(d-1)^2$. Lift the approximant and round its coefficient along a, adding at most $\|a\|/2 \le d$ to the error. This proves the bound. Apply it to the archimedean residue divided by N_D and multiply back by N_D. The approximant then lies in $2O_E$, so it preserves finite integrality. The archimedean remainder in each coordinate has modulus at most $4f^2 N_D$. Apply r entrywise, so the matrix remainder has operator norm at most $4n^2 f^2 N_D$ at each real factor.

Choose $d = -M(yr(c))^* J y c$ in adelic coordinates. The entries of $r(c)$ and $r(d)$ in the expression $w_0 = M(xr(d) - yr(c))$ are embedded rational coefficients; thus $\mathcal{R}(w_0)$ corresponds to $\gamma \in N_A(E^+)$. Since $x^* J y = 0$, one has $w_0^* J z_0 = Md$. Equation (CF2) yields

$$\gamma u = 1 + 2(MT - (MT)^* - (MT)^* J (MT)) J, \qquad T = (y(c - r(c)) - x(d - r(d)))x^*.$$

The finite entries of T are integral. The operator norms of x, y at infinity are at most $n^2 a, n^2 b'$, so

$$\|T_v\| \le 4n^6 f^2 N_D (b' + a) a \le 8 n^6 f^2 N_D a b' \le \delta/(6M).$$

Since every J_v is a sign involution, $\|(\gamma u)_v - I\| \le 2\delta/3 + \delta^2/18 \le \delta$. The finite formula gives $\gamma u \equiv I \bmod M$. This proves (CF1) in the original basis at level M.

42.4 Vanishing of the constant term under the reproducing kernel

In this subsection, the reproducing kernel of Section 12.3 annihilates the relevant constant term. We prove this pointwise and apply it on the neighborhood just constructed in Section 42.3.

Now let e be the small-type reproducing convolution kernel on the real group G_∞ of V. It is integrable, idempotent, and has an integrable first norm moment, by the prescribed large gap. Write $Tf(g) = \int e(u) f(gu)\, du$. We prove the local vanishing for bounded continuous functions invariant under the real radical of the nonzero rational isotropic A above. The proof uses only the idempotence and the indicated moment.

For the product of real groups, let $\mathcal{D}$ consist of pairs (a, c) with $c > 0$ such that each a_v multiplies A_v by c and acts as the identity on $A_v^\perp / A_v$. Its kernel over $c = 1$ is $N_{A,\infty}$. Each real factor of A is nonzero. Fix a real place v_0 and choose $0 \ne x_{v_0} \in A_{v_0}$. Put $H(g) = \log \|g_{v_0}^{-1} x_{v_0}\|$. Then $H(ag) = H(g) - \log c$ for $(a, c) \in \mathcal{D}$ and

$$|H(gu) - H(g)| \le \|u_{v_0}\|.$$

The latter follows by submultiplicativity in both directions and $\|u_{v_0}^{-1}\| = \|u_{v_0}\|$.

For each g and each $\epsilon > 0$, take at each place the Euclidean orthogonal projection P onto $g_v^{-1} A_v$, and put $Q = J_v P J_v$. The two projections have orthogonal images by isotropy. The operator $b_v = I + \epsilon P - \epsilon Q/(1+\epsilon)$ is unitary for J_v, acts by $c = 1+\epsilon$ on $g_v^{-1} A_v$, and acts trivially on its orthogonal quotient for J_v. Its distance from I is at most ϵ; hence $(gbg^{-1}, c) \in \mathcal{D}$. Translation is continuous on $L^1(G_\infty)$. Choose ϵ so small that $\|e(b^{-1}\cdot) - e\|_1 < 1/2$ throughout this neighborhood of I.

Let $\mathcal{B}$ consist of the bounded continuous functions fixed by T and left invariant under $N_{A,\infty}$, with the supremum norm. For a fixed $(a, c) \in \mathcal{D}$ with $c = 1 + \epsilon$, left translation $Uf(g) = f(ag)$ preserves $\mathcal{B}$. Given g, the element b just constructed satisfies $gbg^{-1}a^{-1} \in N_{A,\infty}$, hence $f(ag) = f(gb)$. Reproduction and the L^1 translation bound give $\|I - U\| \le 1/2$ on $\mathcal{B}$. Put $\tau = \log c$, take a large positive integer k, and set $\chi(g) = \exp(-i\pi H(g)/(k\tau))$ and $Sf = T(\chi f)$.

The multiplier is invariant under $N_{A,\infty}$, T commutes with left translations and $T^2 = T$, so S preserves $\mathcal{B}$. The shift in H gives $US = e^{i\pi/k} SU$. It follows that $US^k = -S^k U$; the operator identity $2S^k = (I - U)S^k + S^k(I - U)$ and $\|I - U\| \le 1/2$ then give $S^k = 0$. On the other hand $Sf = 0$ implies

$$\|f\|_\infty \le \frac{\pi}{k\tau} \left(\int |e(u)|\, \|u_{v_0}\|\, du \right) \|f\|_\infty,$$

by subtracting $\chi(g) Tf(g)$ in the integral. The moment is finite and the coefficient is less than one for large k. Thus S is injective, and $S^k = 0$ proves $\mathcal{B} = \{0\}$. In particular T kills every bounded continuous left-$N_{A,\infty}$-invariant function.

Write $[N_A] = N_A(E^+) \backslash N_A(\mathbb{A}_{E^+})$ for the radical quotient in the original rational coordinates, and put $N_{A,g} = g^{-1} N_A(\mathbb{A}_{E^+}) g$. Conjugation identifies it with $(g^{-1} N_A(E^+) g) \backslash N_{A,g}$. By (CF1) this latter quotient has representatives in the compact set $N_{A,g} \cap B_\infty(\delta) J_W(M)$. Use Haar probability on $[N_A]$ and its image under conjugation. For $h \in G_\infty$ the constant term is

$$C_g(h) = \int_{[N_A]} F(ng(h, 1))\, dn = \int_{(g^{-1} N_A(E^+) g) \backslash N_{A,g}} F(gu(h, 1))\, du.$$

Right invariance of the quotient measure makes C_g left invariant under $g_\infty^{-1} N_A(E_\infty^+) g_\infty$, the radical in the transported real basis used in the preceding argument. The mean is bounded and continuous. By Fubini and reproduction of F, it is fixed by T, so the preceding real vanishing argument gives $C_g = 0$. Since F is automorphic and right invariant under $J_W(M)$, (CF1) makes every radical value differ from $F(g)$ by no more than the oscillation on the real δ-ball. Averaging against the probability normalization of the same quotient Haar measure gives

$$|F(g)| \le \sup_{\|h-1\| \le \delta} |F(g(h,1)) - F(g)|.$$

This is the required bound at the marked cusp.

42.5 Removing the smoothing at the marked period

In this subsection, we combine the constant term vanishing of Section 42.4 with the quantitative reduction of Section 42.3.

Throughout this subsection write $A = A_{\mathrm{sep}} = r^4$ and $L = L_{\mathrm{sep}} = \lceil 20b \rceil$ as in (DS1). Let G_V, G_Z and Γ_V, Γ_Z be the adelic and rational unitary groups in the given bases. Write $\jmath = \iota$, $c_v = c_l$ and $c'_v = c_{-l_Z}$ at place v. Put $e(u) = \prod_v d_v \overline{c_v(u_v)}$, with d_v the formal degree. Assume the orthogonality in (DS0). For an adelic point a of G_V and an infinite point α, put

$$\Phi_{\alpha,\omega}(g) = \prod_v \overline{c'_v(\alpha_v^{-1} g_v \omega_v)},$$

$$\psi_\alpha = \Phi_{\alpha,\omega} \circ \jmath, \qquad (R_\alpha y)(a) = \int \psi_\alpha(k) y(a(k,1))\, dk.$$

For clarity, fix the center $q = \iota(a)(\alpha, 1)$ in G_Z. The periodization of the conjugate coefficient kernel times $\mathbf{1}_{K_Z(a_0)}$, at arguments $q^{-1}\gamma g$, is the conjugate of a bounded form of the required type and level, divided by the fixed product of formal degrees for normalized coefficients.

To verify this, restrict the supported terms $a_\gamma = q^{-1}\gamma g$ to $\|a_{\gamma,v}\| \le R_v$ at every real place, $R_v \ge 1$. For a fixed one of these terms a_{γ_*}, the matrices $a_\gamma a_{\gamma_*}^{-1} = q^{-1}\gamma\gamma_*^{-1} q$ have integral finite coordinates in the center basis. They are distinct points of the fixed moved rational matrix lattice and have real norms at most R_v^2. Packing its disjoint balls of fixed radius less than half its first minimum gives at most $C_q \prod_v R_v^{4(r+1)^2}$ such points. Indeed the balls fit in the product of local balls whose radii are R_v^2 plus the fixed packing radius.

Dyadic summation converges uniformly in g, since the coefficient decay exponent $2A$ exceeds $4(r+1)^2$. It converges locally uniformly term by term by increasing the real radii by a fixed factor on a compact neighborhood and restricting the finite neighborhood to $gK_Z(a_0)$.

Reindexing gives automorphy and right level invariance. For the type, conjugate and replace γ by γ^{-1}. The conjugate uses arguments $g^{-1}\gamma q$. The idempotent kernel $e_Z = \prod_v d'_v \overline{c'_v}$ gives

$$\int e_Z(u) e_Z(u^{-1} g^{-1} \gamma q)\, du = e_Z(g^{-1}\gamma q)$$

at the real places. Boundedness of the absolute sum and integrability of e_Z justify convolution term by term. Thus the conjugate expression is fixed by right reproduction. Inserting $g = \iota(h)(\omega, 1)$ in this test in W_Z and conjugating the vanishing hypothesis of (DS0) gives zero for the period against $y(h)$. Its kernel is exactly $\Phi_{\alpha,\omega}((\iota(a)^{-1}\gamma\iota(h))_\infty)$ times the finite level indicator.

The source vector $y = R(e)y$ has a bounded continuous representative. Indeed, set $y_j = y\mathbf{1}_{\{|y|\le j\}}$. Then

$$\|R(e)y_j\|_\infty \le j\|e\|_1, \qquad \|R(e)y_j - y\|_2 \le \|e\|_1\|y_j - y\|_2 \longrightarrow 0.$$

The projected truncations retain level J_M and are continuous by translation continuity of e in L^1. They are cuspidal by Section 18.1.3. At this level, compact type and infinitesimal character, the cuspidal space is finite dimensional by the finiteness argument of Section 34.3.2. Their span is therefore closed in L^2 and contains y.

Take any global bound $|y| \le S$, and set $\lambda(a)$ equal to the first minimum, cut off at 1, in the lattice of V moved to a. If $B_d^2 \le B_d^{3r+2}\lambda(a)$, unfold the period by the cosets Γ_Z/Γ_V of the embedded rational subgroup (equivalently by the orbits of right multiplication). Only cosets which meet the principal level support contribute. Such a coset, with representative ζ, is determined by the last column ζe_{last} of the rational matrix in the fixed rational decomposition.

Equal columns mean $\zeta^{-1}\zeta'$ fixes the last unit vector and belongs to Γ_V. Write $c(\zeta)$ for this column in the Euclidean real coordinates moved to $\jmath(a)$. The finite coordinates of every supported column are e_{last} modulo a_0. Hence the difference of two supported columns, divided by a_0, is in the moved rational lattice of Z. The added definite line has first minimum at least 1 and the sum is orthogonal in the original metrics.

Consequently

$$\|c(\zeta) - c(\zeta')\| \ge B_d^{3r+2}\lambda(a) \ge B_d^2 \quad (\zeta\Gamma_V \ne \zeta'\Gamma_V), \qquad \|c(1)\| = \sqrt{f} \le B_d. \tag{D3}$$

The last column does not vary under multiplication on the right by $\jmath(h)$. At any archimedean point of a contributing term, the product of the operator norms of the local group elements is at least $\|c(\zeta)\|/f$, since each norm is at least 1. By (DS1) it is also at least $\|c(\zeta)\|/B_d$.

For clarity, the L^1 slice estimate follows from decay and a volume calculation. For $R \ge 1$ and a fixed ambient group element q, take a maximal $1/R$-separated set in the subset of matrices $\jmath(h)q$ of operator norm at most R. Its size is at most $(1 + 2R^2)^{2(r+1)^2}$ by packing in the real vector space of complex matrices, using the operator norm. Two such matrices a, b at distance at most $1/R$ satisfy $\|ab^{-1} - 1\| \le 1$ because $\|b^{-1}\| = \|b\| \le R$.

Thus the measure of the original subset of h's is at most this same number times $v_{1,v}$, by right translation in the subgroup. Sum over $2^j \le \|\jmath(h)q\| < 2^{j+1}$, $j \ge 0$. With $d_{\mathrm{pack}} = 2(r+1)^2$ the sum for the weight $(2/\|\jmath(h)q\|)^A$ is at most

$$v_{1,v}\, 2^A 9^{d_{\mathrm{pack}}} \sum_{j\ge 0} 2^{-j(A-2d_{\mathrm{pack}})} \le v_{1,v}\, 16^A,$$

using $A > 4d_{\mathrm{pack}}$. The corresponding bound on the left-translated slice follows by inversion of the unitary group and invariance of its Haar measure.

Write

$$C_{\alpha,\omega} = \prod_v [2(\|\alpha_v\|\|\omega_v\|)^2]^A 16^A v_{1,v}.$$

Here and in the following estimates the unit-ball volume in each real-place factor is its own $v_{1,v}$. The local large-type bound $|c_v'(g)| \le (2/\|g\|)^{2A}$ and submultiplicativity give one power of degree A in the slice for integration as above, and one power for $\|c(\zeta)\|/B_d$. For the latter, use the product of the operator norms before applying the translations α^{-1} and ω.

For a supported coset, the set of finite h in its term is a single left coset of $K_V(a_0)$: choose one h_0 in the support, and use $\jmath^{-1}(K_Z(a_0)) = K_V(a_0)$. Thus its total absolute integral is at most

$$\operatorname{vol}(K_V(a_0))\, C_{\alpha,\omega} S(B_d/\|c(\zeta)\|)^A \qquad (\zeta\Gamma_V \neq \Gamma_V).$$

This retains the volume in the unfolded main term.

From (D3) all nonbase norms and pairwise distances are at least $R_0 = B_d^2/2$. In dimension $2f(r+1)$ the count in $2^j R_0 \le \|c(\zeta)\| < 2^{j+1}R_0$ is at most $(5\cdot 2^j)^{2f(r+1)}$, by packing balls of radius $R_0/2$. Since $A - 2f(r+1) \ge 2$, summing gives

$$\sum_{\zeta\Gamma_V \neq \Gamma_V} (B_d/\|c(\zeta)\|)^A \le (10/B_d)^A \le 10/B_d.$$

These estimates justify the period unfolding absolutely. For the base coset use $h = ak$; the finite indicator is then that of $K_V(a_0)$ and y is invariant under this group since π_M divides a_0. The base coset contributes precisely $\operatorname{vol}(K_V(a_0))\, R_\alpha y(a)$. The full period is zero. Cancelling the positive volume gives

$$|R_\alpha y(a)| \le C_{\alpha,\omega} S(10/B_d) \qquad \text{if } B_d^2 \le B_d^{3r+2}\lambda(a). \tag{D4}$$

Removing the smoothing.

Put $\mathfrak{M}(u) = \prod_v d_v (2/\|u_v\|)^{2A}$. The integral of each real factor is at most $8[2000(r+L+1)^6]^{2r^2}2^{2A}$. The Casimir calculation in Section 12.3 gives a translation Lipschitz bound $100(r+L+1)^5$ for each small-type coefficient. In detail, for a unit lowest-type vector and any real Lie algebra element Z it gives $\|d\pi(Z)v\| \le \sqrt{3r}(r+L+1)\|Z\|_{\mathrm{HS}}$. For $\|u-1\| \le 1/2$ apply this to $Z = \log u$ and integrate along $\exp(tZ)$, using $\|Z\|_{\mathrm{HS}} \le 2\sqrt{r}\|u-1\|$.

For larger distances the coefficient difference is at most 2. The bound is uniform in the other argument by unitarity. Now if $K_0 = 2000(r+L+1)^6$, the small-type coefficient has absolute value at least $1/2$ in the ball $\|u-I\| \le 4/K_0$ by its Lipschitz bound $100(r+L+1)^5$. The volume of that ball is at most $4/d_v$ by the formal degree identity. Packing the ambient matrices as above with gap $(4/K_0)/R$ gives $\mu\{\|u\| \le R\} \le (K_0R^2)^{2r^2}(4/d_v)$ for $R \ge 1$, since $1 + K_0R^2/2 \le K_0R^2$. Layer integration gives the claimed bound: $2A/(2A-4r^2) \le 2$. In particular (DS1) gives

$$\int \mathfrak{M}(u)\, du \le \frac{B_d}{16\beta\,[\beta^H(2\beta)^{3A}]^f}. \tag{D5}$$

For this use $1 + 2r^2 + 2A + H + 6A \le 20A$, $16 \le \beta$, and the product measure of the real factors.

Put $\lambda_0 = (B_d^{-2}/(M\kappa^2))^r$. Since $M\kappa^2 \le \beta \le B_d$, one has $\lambda_0 \ge B_d^{-3r}$. Thus $\lambda(a) \ge \lambda_0$ implies $B_d^{3r+2}\lambda(a) \ge B_d^2$, including equality at the cutoff. For a real u, change variables on the subgroup to obtain

$$(R_\omega y)(a(u,1)) = (R_{\jmath(u)\omega} y)(a).$$

Average this equality against $e(u)\,du$. By Schur orthogonality, convolution by e followed by translation by k and convolution by e again is multiplication by $\overline{e(k)}/e(1)$ on its image. Since right convolution reproduces the form, Fubini gives

$$\int e(u)(R_\omega y)(a(u,1))\, du = y(a) \int \psi_\omega(k)\overline{e(k)}/e(1)\, dk = \nu\, y(a),$$
$$|\nu| \ge \beta^{-fH} \prod_v v_{1,v}.$$

Boundedness of y and integrability of both kernels justify Fubini. Since $\overline{e(k)}/e(1) = \prod_v c_v(k_v)$, the scalar is

$$\nu = \prod_{v|\infty} \int_{U(V_v)} c_v(k_v)\, \overline{c'_v(\omega_v^{-1}\jmath(k_v)\omega_v)}\, dk_v.$$

Its lower bound is exactly (DS2) in the same local measures. At $\alpha = \jmath(u)\omega$,

$$|e(u)|\, C_{\jmath(u)\omega,\omega} \le \mathfrak{M}(u)(2\beta)^{3Af} \prod_v v_{1,v}$$

Indeed $\|\jmath(u_v)\omega_v\|\,\|\omega_v\| \le \beta\|u_v\|$. Multiplying the local factor $16^A[2(\beta\|u_v\|)^2]^A$ by $(2/\|u_v\|)^{4A}$ gives $(2/\|u_v\|)^{2A}(128\beta^2)^A$, and $128\beta^2 \le (2\beta)^3$. By (D4) and (D5), dividing by $|\nu|$ cancels $\prod_v v_{1,v}$ and gives $|y(a)| \le 10S/(16\beta) \le S/2$.

If instead $\lambda(a) < (B_d^{-2}/(M\kappa^2))^r$, use the quantitative inclusion of Section 42.3 at radius B_d^{-2} and level $K_V(M)$ on V. It gives a rational isotropic subspace whose unipotent radical, read in the basis at a, is covered by its rational translates of the product of the real ball of radius B_d^{-2} and $K_V(M)$. Because the reproduced form has zero constant term by the argument of Section 42.4 and $K_V(M) \subset J_M$,

$$|y(a)| \le \sup_{\|u-1\| \le B_d^{-2}} |y(a(u,1)) - y(a)|.$$

The product of small-type coefficients has Lipschitz constant at most β^2 under this translation, with each factor of absolute value at most one.

Combine the Lipschitz bound with the $4A$-decay using $\min(U,V) \le \sqrt{UV}$. Including the formal degrees gives

$$|e(u^{-1}z) - e(z)| \le (\beta/B_d)[\mathfrak{M}(u^{-1}z) + \mathfrak{M}(z)].$$

Applying reproduction of y and left invariance of Haar measure, then (D5), bounds the oscillation by $2S(\beta/B_d)\int \mathfrak{M} \le S/8$. This proves $|y(a)| \le S/2$ for all adelic points a. Repeated halving of any initial bound for y gives $y = 0$.

Compose the deep restriction at this ω with the weighted transpose of the true deep lifts from the definite end of the grid. Equation (TP4) in Section 43.2 proves $\Theta^\flat(\mathcal{H}_m) = \mathcal{H}_0 = W_Z$. Thus for every nonzero $y \in W$ the separating test in (DS0) has the form $\Theta^\flat\eta$ for some $\eta \in \mathcal{H}_m$. The terminal test may depend on y. The level $K_Z(a_0)$ and the tuple $\omega = (\omega_v)_{v|\infty}$ are fixed throughout.

43 Theta compositions at deeper principal levels

In this section, we transfer the separation of Section 42.2 along two longer theta compositions. The paired period of Section 34.3 compares them after identifying the opposite Hermitian forms and gives the terminal pairings used in Section 44.

Put $\mathfrak{b} = (\pi_M^{b_0})$ and $R_b = \mathcal{O}_E/\mathfrak{b}$. Use the trivial twist in the true triangle, with this same ring at every node. Its rank in column n is $d_n = r + 1 + n(t-1)$. The identifications j_i of its base copies come from passage to the opposite Hermitian form, with the original finite bases, quotient Haar measures, and principal levels. Use forward triangle forms of the conjugate type to the true-grid forms $\mathcal{H}_j$ of level $K_{Z_j}(\mathfrak{b})$. In particular $\mathcal{H}_0$ consists of the tested forms W_Z of Section 42.2. Write $a_{i,n}$ for the forward point lift and $w_n = |\mathrm{GL}_{d_n}(R_b)|$.

43.1 The first-column return

In this subsection, we compute the first return on the level used in Section 42.2.

At the first arrow put $u = d_0 = r + 1$, $v = d_1 = r + t$ and $X_0 = \binom{I_u}{0}$ over R_b. The characters at the two ranks are $\mu_{i+1}^{0}\upsilon^{\lceil u/2\rceil}$ and $\mu_{i+1}^{0}\upsilon^{\lceil v/2\rceil}$. They are unramified off the marked split pairs. The finite datum is the indicator of X_0 at the marked places, with the integral vacuum elsewhere. At a split place dividing $\mathfrak{b}$, suppose X and Xg both belong to the support of the residue-point datum. Each has an integral left inverse, obtained from its first u covectors, which are congruent to I_u.

It follows that g and g^{-1} are integral. The overlap is then equivalent to $g \equiv I_u \bmod \mathfrak{b}$, and the character of g on the overlap is one. Give the integral matrix box and the integral source compact K_Z° mass one. The squared norm of the indicator is $|R_b|^{-uv}$. Its source stabilizer has mass

$$\operatorname{vol} K_Z(\mathfrak{b}) = w_0^{-1}, \qquad w_0 = [K_Z^{\circ} : K_Z(\mathfrak{b})] = \prod_{w \mid \pi_M} |\mathrm{GL}_u(\mathcal{O}_{E,w}/\mathfrak{b}\mathcal{O}_{E,w})|.$$

Indeed reduction is onto at each chosen split factor by lifting representatives whose determinant is nonzero modulo the maximal ideal. The conjugate factor is determined by unitarity. The space W of Section 42.2 has level $K_V(\pi_M)$; the return here acts on the conjugate of W_Z at level $K_Z(\mathfrak{b})$.

Write $A_i = a_{i,0} j_i$. The open-graph doubling calculation of Section 34.2 applies at these ranks. The doubled source has Siegel rank $u = r+1$. Since $v = 6r$, we have $v/2-u = 2r-1 > 0$ and $v - \iota(T) \geq v/2 > u$. Thus the convergent formula applies to the separately translated Schwartz factors [75, §52, Theorem 5]. Its parameter is $(v-u)/2$, with coefficient one [77, Theorem 2]. At each prescribed split place every supported matrix has an integral left inverse as above, hence full column rank over the local field. A nongraph orbit would force a rank drop there, so it contributes zero to the global doubling integral. This gives

$$w_0 \langle A_i x, A_i y\rangle = C_i |\operatorname{Mat}_{v\times u}(R_b)|^{-1} \langle x, \mathcal{V}_i y\rangle, \qquad C_i \neq 0. \tag{TP1}$$

Use x, y in the true forward base $\overline{\mathcal{H}_0}$ at level $K_{Z_0}(\mathfrak{b})$. The projected truncation argument of Section 42.5 supplies bounded cuspidal representatives throughout this space. They are rapidly decreasing by Section 18.1.3. At the fixed finite datum, Section 40.2.1 bounds the absolute theta-product integral and the absolute graph sum separately by $C\mathcal{H}(g)^B\mathcal{H}(h)^B$ in the two source heights. Multiplication by $|x(g)y(h)|$ therefore justifies both source integrations and the graph unfolding on this base.

The scalar C_i is the original target quotient volume times the integral of the real joint-harmonic coefficient against $\overline{e_i(h)}/e_i(1)$, with e_i the source projector. We verify the real harmonic at this edge. Fix an infinite place, let L be the doubled forward stem $2\lambda_j + 1$ of Section 42.2. The doubled target parameter L' adds to this stem the block $-2w_{i+1} + t - 2 - 2b$, $0 \leq b < t-1$, with positive signs. Here $w_j = \kappa_\tau + (j+1/2)\Delta$ in the indexing of the grid and $w = w_{i+1}$. The source and target splitting weights are $u/2 - w, v/2 - w$. Both cross shifts are $2w$. At tail stem index $k - 1 + a$ the common shifted entry is

$$\xi_a = \Delta - 2w_{a+1} + 1 + 2w = \Delta + 1 - 2(a-i)\Delta.$$

The ratio of the two signature signs there is $(-1)^{\min(1,(a-i)_+)} = \operatorname{sgn}\xi_a$. The head has positive shifted entries and unchanged signs. The $v - u = t - 1$ inserted entries have positive signature

and shifted values $t-2, t-4, \dots, 2-t$. The common shifted entries have modulus at least $\Delta-1$. The gap between opposite signature signs is therefore at least $(\Delta-t-2)/2 > 500(u+v)^3+uv$ in the undoubled parameters.

Use s_a, s'_b for the source and target signs. In the oscillator the doubled source parameter is $-L$ and the target one is L'. After subtracting the vacuum exponents, the lowest compact relative weights are

$$2R_b^{\mathrm{wt}} = \xi_b - s'_b\left(\sum_{b'} s'_{b'}\operatorname{sgn}(\xi_b-\xi_{b'}) + \sum_{a'} s_{a'}\right),$$
$$2C_a^{\mathrm{wt}} = -\xi_a + s_a\left(\sum_{a'} s_{a'}\operatorname{sgn}(\xi_a-\xi_{a'}) + \sum_{b'} s'_{b'}\right).$$

Indeed apply $\lambda+\rho-2\rho_K$ to the two parameters and subtract the respective vacuum weights $-w+s'_b\sum_a s_a/2$, $w-s_a\sum_b s'_b/2$. At matched indices, the contributions of an index with the same sign of ξ cancel using $s'_a = s_a\operatorname{sgn}\xi_a$. For the opposite sign they cancel using $\operatorname{sgn}(\xi_a-\xi_b)=\operatorname{sgn}\xi_a$. The inserted-block contributions also cancel at matched indices. At an inserted index the matched terms cancel, and the signed sum of the other inserted positions equals its shifted value. It follows that $R_a^{\mathrm{wt}} = -C_a^{\mathrm{wt}}$ at matched indices, $R_b^{\mathrm{wt}} = 0$ at inserted indices, and $\operatorname{sgn} R_a^{\mathrm{wt}} = \operatorname{sgn}\xi_a$.

In each sign quadrant take the product, over positive integers h, of the minors on the matched indices with $|R_a^{\mathrm{wt}}| \ge h$, omitting empty factors. Compact dominance puts these indices in initial segments of the decreasing compact orders for positive weights and of the dual compact orders for negative weights. A compact raising operator either kills a minor or duplicates an index in it. The diagonal weights are the $R_b^{\mathrm{wt}}, C_a^{\mathrm{wt}}$ just computed. For each fixed first or second matrix index the polynomial uses coordinates of a single oscillator sign. Thus mixed derivatives with opposite signs and a common index vanish. The matched diagonal monomial has nonzero coefficient.

The specified pair of compact types occurs once in the space of joint harmonics by Howe's theorem; its highest line is therefore spanned by this polynomial [59, Theorem 2.2(1) and Lemma 2.5]. For the discrete-series correspondence, remove the scalar determinant twists and dualize the source as in Section 34.3.2. The source parameter in [59, Theorem 2.7(b)] is then $(\xi_a/2)_a$ in its two decreasing compact orders. Its entries are half-integral since $r = 3^{\lceil \log_3(x^5)\rceil}$ is odd and $v-u = t-1$ is even, and $|\xi_a|/2 \ge (\Delta-1)/2 \ge t/2 = (v-u+1)/2$. The signature is preserved at the positive entries and reversed at the negative entries. All $v-u$ new entries are positive-sign coordinates and form the inserted string from $(t-2)/2$ to $(2-t)/2$ with step -1. These are the hypotheses of the cited correspondence for the given real groups. At each real place τ, let $P_{i,\tau}$ denote this polynomial, $e_{i,\tau}$ the source projector just used in (TP1), and ω_τ the Weil action. Set

$$Z_{i,\tau} = \int_{U(Z_0)(F_{0,\tau})} \langle P_{i,\tau}, \omega_\tau(h,1)P_{i,\tau}\rangle \frac{\overline{e_{i,\tau}(h)}}{e_{i,\tau}(1)}\, dh.$$

The source coefficient has the oscillator-side parameter $-L/2$ with the scalar-form convention above. The Gaussian pairing at a positive Cartan factor and the two compact averages of Section 34.1.2 give $Z_{i,\tau} > 0$. Hence

$$C_i = \operatorname{vol}([U(T)])\prod_{\tau|\infty} Z_{i,\tau} > 0.$$

The operator $\mathcal{V}_i$ has the original vacuum weights away from the prescribed support. The local lattice count of Section 34.2, with source rank u, gives $O(q_w^u)$ neighbor choices of weight $q_w^{-v/2}$ at each place.

Here $v/2-u\ge r$ for large x. Thus the sum of step weights is $O([E:\mathbb{Q}]\sum_{q\ge 2} q^{-r})=o(1)$. The sum of nonidentity word weights is $o(1)$ by the geometric series. This proves absolute operator-norm convergence and injectivity of $\mathcal{V}_i$. The series preserves the base level and the real projector because it is spherical away from the prescribed support and acts by finite translations. By (TP1), on this fixed-type subspace at $K_Z(\mathfrak{b})$ the adjoint satisfies

$$A_i^* A_i = \frac{C_i}{w_0\,|R_b|^{uv}}\,\mathcal{V}_i.$$

The scalar is positive and the neighbor sum gives $\|\mathcal{V}_i - I\|_{2\to 2}\le 1/2$ for sufficiently large x, uniformly in the allowed levels. Hence

$$\|A_i y\|^2 \ge \frac{C_i}{2w_0|R_b|^{uv}}\,\|y\|^2,$$

so A_i is injective.

The first descent $d_{i,0}$ is the adjoint return to the next opposite base copy. To realize it with the original moduli, let P_i be the upward joint harmonic and $P_i^\flat$ the opposite real edge harmonic. Conjugation of the Weil theta kernel carries its residue datum Ψ to

$$\Psi_0(X)=\overline{\Psi(u_R^{-1}X)},\qquad u_R=u_0 \bmod \mathfrak{b}.$$

Here $u_0\in\widehat{\mathcal{O}}_E^\times$ is the fixed unit satisfying $u_0\bar{u}_0=-1$ in the matrix reversal (MR). The scalar c_i^{op} in this conjugation is fixed by the normalized theta functionals of the two opposite bases.

The quotient Haar is carried to the same Haar. Replace the conjugate polynomial by $P_i^\flat$ in the tested period using the fixed real edge convolution. If its constant is k_i', the identity with the resulting descent B_0 is $\langle x, A_i y\rangle = \overline{c_i^{\mathrm{op}}}\,k_i'\langle B_0 x, j_{i+1} y\rangle$. We use inner products linear in their second argument. Thus the datum $c_i^{\mathrm{op}}\overline{k_i'}\Psi_0$ gives the exact adjoint. This identity uses the opposite-basis conjugation and the real edge convolution of Section 34.3. The conjugation and real replacement take place before any residue expansion, so the initial adjoint datum is a function on R_b.

Let $y^\flat$ be the return on the base. Then $\langle x, y^\flat\rangle = \langle A_i x, A_i y\rangle$. For an integral base element k, let k_b be its reduction in $\mathrm{GL}_u(R_b)$. Choose an integral target element with reduction $g_b=\mathrm{diag}(k_b, I_{v-u})$. In the marked Schrödinger coordinates $g_b^{-1}X_0 k_b = X_0$, and the determinant character has modulus one. Consequently the source and target translations intertwine A_i up to that unit scalar. Pairing with all x shows that $A_i^*A_i$ commutes with the integral base action. In particular $y^\flat$ has the deep level when y does. If $y^\flat=0$, take $x=y$ and use (TP1) to conclude $y=0$.

43.2 The measured square and transposition

In this subsection, we use the local return of Section 43.1 in the measured square.

At the square (i,n), the kept ranks are d_{n+1}, d_{n+1} and the other ranks are d_n, d_{n+2}. The exchange defect is $t-1$. The stem sign with excess parameter a_* at (o,j) is $(-1)^{o+a_*+\min(j,(a_*-o)_+)}$.

Passing from the kept pair to the other pair leaves both signs unchanged if $a_* \le i+n+1$. Otherwise the two changed signs were opposite, so the negative count is preserved. The common negative count q_- satisfies $q_- \le 2(r+1) < d_{n+1}$. The quotient character is

$$\nu = \mu^0_{i+n+2}/\mu^0_{i+1} = (\upsilon^{-\Delta}\chi_M)^{n+1}.$$

Its inverse has positive weight $(n+1)\Delta$, and $d_{n+1} \le (n+1)\Delta$.

At the marked split places its inverse tame part is the inflation of $\chi_M^{-(n+1)}$, nontrivial by $1 \le n+1 \le m$, and elsewhere it is unramified. The forward and opposite left-pair factors at ranks p, q, $p+q = 2d_{n+1}$, have product

$$\mu^0_{i+n+2}\upsilon^{\lceil p/2\rceil}(\mu^0_{i+1})^{-1}\upsilon^{\lfloor q/2\rfloor} = \nu\upsilon^{d_{n+1}}.$$

The first adds the center $-w_{i+n+2}$. For the left pair, pass to the opposite form and add the center $-w_{i+1}$. Conjugating both parameters replaces the old determinant exponent at rank h by h minus that exponent, giving the floor on the left pair.

The character of positive infinity weight to which Section 36.1 applies is ν^{-1}. Its splitting character on the kept space is therefore $(\nu^{-1})^{-1}\upsilon^{d_{n+1}} = \nu\upsilon^{d_{n+1}}$. Together with the rank, signature and real edge calculations, this gives the measured period on this square. Its ratio is $k_{i,n}/c_{i,n}$, where $c_{i,n} \neq 0$ is the measured partner scalar and $k_{i,n}$ comes from the real pairing of the four harmonic vectors. The real replacement is the one in Section 34.3.

We prove that the point extends over R_b. In the decomposition b_1, a of the full matrix, invertible blocks Y_1, X_2 in the first b_1 columns of a matrix and the last a covectors of its inverse give

$$X_2^{-1}X_1 = -Y_2Y_1^{-1}, \qquad \det k = \det Y_1/\det X_2.$$

These follow by block elimination over each finite local ring. Jacobi's complementary minor identity makes invertibility of either square block equivalent to invertibility of the other. For fixed s, use the determinant-character weighted indicators of $X_2^{-1}X_1 = s = -Y_2Y_1^{-1}$ in the partner period. Covariance cancels the character on the two invertible blocks. Thus the respective point values are multiplied by $|\mathrm{GL}_a(R_b)|$ and $|\mathrm{GL}_{b_1}(R_b)|$. Here only invertible blocks contribute. The argument uses finite groups over R_b itself and requires no Gauss sum with a primitive multiplicative character at the deep power.

For the actual square let $a_* = d_n$, $d_+ = d_{n+1}$, and $b_* = t-1$. In the order d_+, a_*, b_* the two arguments are

$$Y = \begin{pmatrix} -\eta \\ I_{a_*} \\ 0 \end{pmatrix}, \qquad X = \begin{pmatrix} I_{d_+} & \eta & s \\ 0 & 0 & I_{b_*} \end{pmatrix}.$$

They are obtained by choosing the common graph parameter $\binom{\eta}{0}$ in the order $a_*, (d_+, b_*)$ and applying the determinant-one shear with off-diagonal block s. The point extension therefore uses the new descent datum

$$p^+\begin{pmatrix} \eta & s \\ 0 & I_{b_*} \end{pmatrix} = p(-\eta)|R_b|^{-d_+b_*}, \qquad p^+ = 0 \text{ on the other matrices.}$$

Indeed the shear fixes the tested covector value by covariance. Average over s and sum over η with weight $p(-\eta)$. Starting with the adjoint datum above and applying this recursion gives descents $d_{i,n}$ with the measured identity

$$w_n a_{i+1,n} d_{i,n} = (k_{i,n}/c_{i,n}) w_{n+2} d_{i,n+1} a_{i,n+1}. \tag{TP2}$$

Injectivity and transposition. Write $X_{i,h}$ for h weighted forward arrows in sequence i and $X_{i,0} = I$. Iterating (TP2) gives

$$X_{i+1,h} d_{i,0} a_{i,0} = \alpha_{i,h} d_{i,h} X_{i,h+1},$$
$$\alpha_{i,h} = \frac{w_{h+1}}{w_0 w_1} \prod_{0 \le n < h} \frac{k_{i,n}}{c_{i,n}}. \tag{TP3}$$

Indeed $\alpha_{i,0} = w_0^{-1}$. Multiplying by $w_h a_{i+1,h}$ on the left and using $X_{i,h+2} = w_{h+1} a_{i,h+1} X_{i,h+1}$ gives $\alpha_{i,h+1} = \alpha_{i,h}(k_{i,h}/c_{i,h}) w_{h+2}/w_{h+1}$. If $X_{i,h+1} j_i y = 0$, (TP3) makes $X_{i+1,h} j_{i+1} y^\flat = 0$. Induction on the length of the chain, starting with $X_{i,0} = I$ and using the injective return, proves $y = 0$. In particular the full true forward chain $F = X_{0,m}$ is injective on the original deep base. This implication only uses (TP2) in its displayed direction. One may now also deduce nonvanishing of the real factors used there. The good subspace of W has positive dimension by Section 16.2, so (DS0) gives $W_Z \neq 0$. For a nonzero forward base vector y, the return $y^\flat = A_i^* A_i y$ is nonzero. Thus $X_{i+1,h} j_{i+1} y^\flat \neq 0$ for each admissible i, h, and (TP3) forces $\alpha_{i,h} \neq 0$. Its product formula gives $k_{i,n} \neq 0$ for all the factors occurring there.

On every forward edge of $F = X_{0,m}$, the residue point is the specified inclusion matrix. Its left and right translations by the two principal subgroups at $\mathfrak{b}$ fix this matrix. Their determinant character is one, since it factors through the marked tame reduction. Thus each forward and transposed arrow has the asserted principal level on the target. Edgewise transposition of the same kernel, against the original quotient measures, with the source weight w_j on both directions, gives

$$\int_{[Z_m]} (Fy)\phi = \int_{[Z_0]} y(\Theta^\flat \phi). \tag{TP4}$$

The transpose here is bilinear and has the conjugate endpoint types. If $z \in \mathcal{H}_0$ is perpendicular to its image, take $y = \bar{z}$ and $\phi = \overline{F\bar{z}} \in \mathcal{H}_m$. Then (TP4) gives $\|F\bar{z}\|^2 = 0$. Injectivity gives $z = 0$. The fixed-type fixed-level spaces are finite dimensional, so $\Theta^\flat : \mathcal{H}_m \to \mathcal{H}_0$ is onto.

43.3 The first-path array

In this subsection, we write the finite array for the transposed lift of Section 43.2.

Use the point ω from Section 42.2. Let $R_0(\xi) = \mathcal{O}_E/(\pi_M Q(\xi))$ and $R_\xi = \mathcal{O}_E/(\mathfrak{b} Q(\xi))$. The false first column uses its original twist and $R_0(\xi)$. The true first column uses the trivial twist and R_b. Both can be read on R_ξ, since the reduction maps $R_\xi \to R_0(\xi), R_b$ are onto. Indeed, with $R_Q = \mathcal{O}_E/Q(\xi)$, the Chinese remainder theorem gives $R_\xi = R_b \times R_Q$. On the true half the map is projection onto R_b. On the false half it is reduction of each old prime power to its residue field and the identity at the new places, so its kernel is nilpotent. Pullback of an edge datum from either quotient is the same adelic datum: its value on an integral matrix depends on exactly the indicated class, and it has zero value off the integral box at the marked places. At the extra places on the true side the pulled-back value on the integral box is constant, hence is the vacuum.

For a tuple S of matrices on the two first columns put

$$P_0(S) = \prod_{\substack{z \in \{0,1\} \\ 0 \le i < m}} \mathbf{1}_{\{\bar{S}_{z,i} = \iota_{z,i}\}},$$

where $\iota_{z,i}$ is the original inclusion matrix, and the bar denotes reduction to the ring of that column. Put $D_{z,i}$ for the group order at the source of that column edge over $R_0(\xi)$ or R_b, respectively. Let $T_{z,i}(S_{z,i})$ be the unweighted transposed point lift at $S_{z,i}$ over R_ξ.

Pullback and expansion in the indicators of individual matrices give the weighted arrow exactly as

$$D_{z,i}\sum_{S_{z,i}}\mathbf{1}_{\{\bar{S}_{z,i}=\iota_{z,i}\}}T_{z,i}(S_{z,i}).$$

The matrices in a tuple are independent choices on disjoint edges. Multiplying the two chains and using bilinearity of the diagonal period at ω gives

$$\left(\prod_{z,i<m}D_{z,i}\right)\sum_{S}P_0(S)\mathcal{P}_\omega(S;y',\eta). \tag{TP5}$$

Here $\mathcal{P}_\omega(S;y',\eta)$ is the original diagonal pairing of the two unweighted transposed point chains, at the same real ω, of the false-terminal dual form y' and true-terminal form $\eta\in\mathcal{H}_m$.

Write $G_{z,i}$ for the invertible group in the node's rank over R_ξ. Set $L_{0,i}=\{1\}$ and $L_{1,i}=\ker(G_{1,i}\to \mathrm{GL}_{d_i}(R_b))$. Use the finite set

$$\mathcal{C}=\prod_{z,i<m}G_{z,i}/L_{z,i}.$$

Choose representatives, with identity at the terminal nodes. In the acting family $g(c)$ replace the representative at the true bottom node by the block sum of the false bottom representative and 1. The unused coset coordinate is still counted in $\mathcal{C}$.

For an edge $e:p\to q$, put $(S^g)_e=g_q^{-1}S_eg_p$. Let $\chi(g)$ be the product of the edge characters at $\det(g_q)/\det(g_p)$, using the characters of their respective columns. The finite Weil action on the original boxes gives

$$\mathcal{P}_\omega(S^g;y',\eta)=\chi(g)\mathcal{P}_\omega(S;y',\eta)$$

for all admissible families g. Indeed choose integral lifts of these invertible matrices at each node. Moving a lift through the transposed edge contributes its determinant character and replaces the edge matrix by $g_q^{-1}S_eg_p$.

Every interior group translation cancels between the adjacent edges. The bottom translations match by the block relation, commute with ω, and disappear by right invariance of the diagonal quotient measure. At the two ends the translations fix the endpoint forms by their stated levels. All these equalities hold at the given marked finite points.

Set

$$P_1(S)=\sum_{c\in\mathcal{C}}\chi(g(c))P_0(S^{g(c)}).$$

This array has the required covariance $\chi(g)P_1(S^g)=P_1(S)$. To see this, use $(S^g)^h=S^{gh}$ and send c to gc. The representative of gc differs from $g\,g(c)$ by the node-level subgroups on the right. These latter groups fix P_0 and have trivial character, since they reduce to identity in the ring of their column. At the true bottom node the discrepancy is identity by the block relation and $L_{0,0}=\{1\}$.

This also verifies independence of representative choices. By reindexing S in each summand and using the preceding period covariance,

$$\sum_S P_1(S)\mathcal{P}_\omega(S;y',\eta)=|\mathcal{C}|\sum_S P_0(S)\mathcal{P}_\omega(S;y',\eta).$$

Each entry contains at most $|\mathcal{C}| \le E_\xi^{2m}$ roots of unity, where E_ξ bounds the number of residue matrices at the allowed grid ranks. If an entry is nonzero, some translate has nonzero indicator.

Fix such a translate g. On a false edge $e : p \to q$, let π project onto the first $\mathrm{rk}(p)$ coordinates and put $C = \pi g_q^{-1} S_e g_p$. Then $C \equiv I$ over $R_0(\xi)$. The reduction $R_\xi \to R_0(\xi)$ has nilpotent kernel, so C is invertible over R_ξ. In the original coordinates the left inverse is

$$(g_p C^{-1} \pi g_q^{-1}) S_e = I.$$

Let $n_i = r + it$ be the rank at the false first column's node i. Its factor in $|\mathcal{C}|$ is $|\mathrm{GL}_{n_i}(R_\xi)|$, while the true factor is the order over R_b, by onto reduction. The order over $R_0(\xi)$ divided by the order over R_ξ is $(|R_0(\xi)|/|R_\xi|)^{n_i^2}$. Indeed a matrix is invertible if and only if its image is, and each matrix has a fiber of size $(|R_\xi|/|R_0(\xi)|)^{n_i^2}$. Thus the ratio of the product in (TP5) to $|\mathcal{C}|$ is the fixed positive number

$$A_{\mathrm{col}} = \prod_{i<m} \left(\frac{\mathrm{Nm}(\pi_M)}{\mathrm{Nm}(\mathfrak{b})} \right)^{n_i^2}. \tag{TP6}$$

For $\eta \in \mathcal{H}_m$, take $r_0\eta$ to be the Riesz representative on the marked base W of the restriction $(\Theta^\flat \eta)((h \oplus 1)(\omega, 1))$. Use Hermitian pairings for the Riesz representative. Take a test v of the real type of W and level $K_V(\pi_M Q(\xi))$, and pair with its conjugate in the bilinear integral. Its average at $K_V(\pi_M)$ belongs to W, since the finite average preserves the real type. For $k \in K_V(\pi_M)$ write $k = k_b k'$, where k_b is supported at the original places and their conjugate places, and k' is the identity there. Because the twist support is disjoint from these places, k_b fixes every test of level $K_V(\pi_M Q(\xi))$.

The component k' lies in $K_V(\mathfrak{b})$, so its block embedding fixes the true function of deep level, and it commutes with ω. The restriction paired with v therefore equals the restriction paired with its average. Apply the Riesz identity in W to that average, and remove the average from the pairing with $r_0\eta$, which has the base level. The image of r_0 is all W by (TP4) and separation. The bilinear transpose of the false chain against y' is the conjugate of a test of the required real type. It has the required source level, since the principal groups fix the point data on both ends. Applying r_0's identity and (TP5)–(TP6) yields

$$\int_{[Y_m]} y'(h)(T_\xi r_0 \eta)(h)\, dh = A_{\mathrm{col}} \sum_S P_1(S) \mathcal{P}_\omega(S; y', \eta). \tag{TP7}$$

43.4 Transport across the consecutive squares

In this subsection, we transport the arrays of Section 43.3 through the marked squares.

Parameterize each half path from its definite endpoint by right and down steps toward a common diagonal node (j, j), recording the true half with its coordinates transposed. At a square fix all residue matrices on unchanged edges.

In the ordered direct sum of the two kept node spaces the old two edge blocks form a column Y and the new blocks a covector X. Their sizes are the ranks of the removed and added node, respectively. At an off-diagonal square there is no line variable. At a diagonal square write v for the removed column of the line and v' for the new covector. In the prescribed covector and column coordinates define

$$P^{\mathrm{F}}(Y, v') = \sum_v \psi(\mathrm{tr}(v'v)) P_{\mathrm{old}}(Y, v). \tag{TP8}$$

The sum for an off-diagonal turn has the one zero line coordinate and is the old value. The character ψ is the one in the line reversal below.

Here is the matched finite array for the marked partner comparison of Section 36.1. For a completion k in the general linear group of the kept direct sum, write Y for its first columns and X for the complementary covectors of k^{-1}. Define

$$P_{\mathrm{new}}(X, v') = \chi(\det k) P^{\mathrm{F}}(Y, v') \tag{TP9}$$

and set it equal to zero for covectors admitting no completion. This is independent of the completion: another completion with the same inverse covectors has the first columns Yg, where g is invertible on $\ker X$, and its determinant is multiplied by $\det g$.

Indeed the change of basis of the two completions has block form $\left(\begin{smallmatrix} g & * \\ 0 & I \end{smallmatrix}\right)$. The old path covariance at the removed node gives $P^{\mathrm{F}}(Yg, v') = \chi(\det g)^{-1} P^{\mathrm{F}}(Y, v')$; that node acts on Y on the right and fixes the line coordinate and the retained edges. Call an edge vertical when it changes the second index on the false half or the first index on the true half, in the coordinates of $n_z(a, b)$ below. The left-inverse condition on vertical edges also holds at $\xi = 1$. At every turn at least one old block is vertical. If (TP8) is nonzero, some old summand is nonzero, hence that block has a left inverse. If the old edge has an extra line column, restrict its left inverse to the retained columns and project to their coordinates. Over each local factor of R_ξ, complete the residue columns of Y to a basis and lift the extra columns. The resulting determinant is a unit in that factor. The Chinese remainder theorem assembles these completions into one k, allowing a different pivot minor at each place.

The node covariance of (TP9) follows by the same change of completions. Explicitly, the retained-node block element G and the new-node element h change X to $h^{-1} X G$. One may replace k by $G^{-1} k \operatorname{diag}(1, h)$, so that Y changes to $G^{-1} Y$ and the determinant acquires $\det h / \det G$. At a diagonal turn the action on the old column and new covector of the line is inverse at their common subdiagonal node.

Changing v in (TP8) therefore preserves $\operatorname{tr}(v' v)$. In detail, let χ_1, χ_2 be the characters of the two old edges, with target determinants d_1, d_2. The removed node determinant is the same on its line extension. Thus $\chi = (\chi_1 \chi_2)^{-1}$. If C_w is the product from the retained edges, the Fourier value after the move with the removed element equal to the identity is multiplied by $[C_w \chi_1(d_1) \chi_2(d_2)]^{-1}$.

Write $d_h = \det h$. The completion contributes the factor $\chi(d_h / (d_1 d_2))$. Opposite edges have equal characters, so the new edge-character product is $C_w \chi_2(d_h / d_1) \chi_1(d_h / d_2)$, which cancels the product of these two factors. Thus the new array has the required covariance. Each new vertical block is opposite a split injective old block. Over each residue field, if the projection of Y into one retained summand is injective, then its image intersects the other retained summand trivially.

Since $\ker X = \operatorname{im} Y$, X on this other summand is injective. Lifting the resulting minor over the finite local factors makes this a split injection over R_ξ. If the new edge also contains the covector v', extend the left inverse by zero on that coordinate. The retained edges keep their support conditions. Each new entry has at most as many root-of-unity terms as the old bound times the number of choices of v, which is at most V_ξ.

For half $z = 0, 1$, the rank at (a, b) is $n_z(a, b) = r + z + (t + z)a + (t - z)b$. The opposite sums give the same even rank N, and the integrated ranks differ by $2t$. With $s = m - b + z$, both negative counts are $\lfloor s/2 \rfloor + \lfloor (s - 1)/2 \rfloor = m - b - 1 + z$, so $2q + 2 \le N$. The product character before inversion is $\mu^0_{a+1} \mu^0_{b+1} \xi^h$, where $h = b + 1$ on the false half and $h = a + 1$ on

the true half. Its inverse has weight at least $\Delta \ge N/2$. The old tame character is χ_M^{a+b+2} with $2 \le a+b+2 \le 2m$, and the new one is ξ^h with $1 \le h \le m$. Each is nontrivial at its marked places, including after inflation to the deep old powers. For the edge of index a at rank p and the edge of index b at rank q', the ordered exponents of the splitting character are

$$\left(\left\lfloor \frac{p+1-z}{2} \right\rfloor, \left\lfloor \frac{q'+z}{2} \right\rfloor \right), \qquad p+q' = N.$$

Their sum is $N/2$, since N is even. The reflected square uses $z=1$ and $(a,b)=(j,i)$, so both halves use $h=j+1$. The same square calculation applies at (j,j) before the line flip.

Fix a square $s=(i,j)$ and the unchanged edge matrices. In the recorded coordinates of either half the retained nodes are $W_1=(a,b+1)$ and $W_2=(a+1,b)$. The removed and added nodes are $V'=(a,b)$ and $V=(a+1,b+1)$. Write L_B for a point lift and ${}^{\mathsf{t}}L_B$ for its transpose under the bilinear period pairing. In the notation $Y=\binom{Y_1}{Y_2}$ and $X=(X_1\ X_2)$, the matched operator identity of Section 34.3.2, obtained from (91), gives

$$\begin{aligned} &\sum_Y P^{\mathrm{F}}(Y,v')\,\lambda(L_{Y_2}{}^{\mathsf{t}}L_{Y_1} f) \\ &\quad = \frac{k_s}{c_{\mathrm{sq},s}(\xi)} \sum_X P_{\mathrm{new}}(X,v')\,\lambda({}^{\mathsf{t}}L_{X_2} L_{X_1} f). \end{aligned}$$

Here f is the incoming form on W_1. For an off-diagonal square, λ is the remaining path from W_2 followed by the bilinear diagonal pairing with the unchanged half. That half occupies the second argument when $i<j$ and the first when $i>j$. The false form is already of the conjugate type. The original quotient measures enter in all four lifts. Thus $c_s = c_{\mathrm{sq},s}$ off the diagonal and k_s depends only on the real edge data. On the diagonal the same identity applies to the two blocks with v' fixed.

The square rank defect is t; the square law is evaluated at the same determinant shifts and common Haar as in the marked comparison. Write

$$S_\xi = S_M \cup \operatorname{supp} Q(\xi), \qquad \nu_{ij,\xi} = (\mu^0_{i+1}\mu^0_{j+1}\xi^{j+1})^{-1}, \qquad T_{ij}(\xi) = \prod_{k=1}^{t} \frac{|L^{S_\xi}(k,\nu_{ij,\xi})|}{|L^{S_M}(k,\nu_{ij,1})|}.$$

Here each incomplete base-change series omits the indicated places and their conjugates, as in Section 36.1. The ratio of the two partner factors $M_{\nu_{ij,\xi}}(t)/M_{\nu_{ij,1}}(t)$ gives

$$\frac{|c_{\mathrm{sq},s}(\xi)|^2}{|c_{\mathrm{sq},s}(1)|^2} = \mathrm{Nm}(Q(\xi))^{t^2}\, T_{ij}(\xi)^2. \tag{TP10}$$

The fixed product of the old residue cardinalities cancels in this ratio. The appearance of $Q(\xi)$ only to t^2 uses that the conductor factor is a product over the support, and the new support is squarefree. Thus (TP10) also holds with the deep old powers in R_ξ.

At the diagonal j, the two intermediate spaces are identified in their chosen coordinates. Give this space rank $n_j = r+(2j+1)t$. The old and new lines have signs $-\epsilon$ and ϵ, $\epsilon=(-1)^{j+1}$. Choose the finite idele t_ξ equal to one off the marked support with $t_\xi \widehat{\mathcal{O}}_E = \mathfrak{b} Q(\xi)\widehat{\mathcal{O}}_E$. If α_f is the fixed finite Weil scale, the old and new finite line points are u_ξ^j, u_ξ^{j+1} for $u_\xi = \overline{t_\xi \alpha_f}/(t_\xi \alpha_f)$. In the original sign coordinates the polynomial reversal is $X_{v,i,1} \mapsto -\epsilon s_i X_{v,1,i}$, where s_i are the intermediate signs.

In fact the Hermitian adjoint transposes and conjugates, while reversing the line changes its sign and thus the conjugation in the holomorphic Fock coordinate. The Gaussian is unchanged. Let u_0 be the fixed finite norm-minus-one unit and $c_0 = u_0^{\epsilon}$. On the self-dual coordinates use

$$\psi(\mathrm{red}\, z) = \psi_f(j_{\mathrm{W}}\, \alpha_f \overline{\alpha_f}(-\bar{c}_0) z/t_\xi)\,.$$

The integral box shifts by v/t_ξ and the line dilation changes it to $(X - v)/t_\xi$ integral.

Its squared Weil Jacobian is $|R_\xi|^{n_j}$. The conjugate-place Heisenberg displacement gives the multiplier $\psi(\mathrm{tr}(v'v))$ on the reversed integral box. Thus the line kernels are related by an unnormalized finite Fourier sum with scalar $c_{\mathrm{line}}(\xi)$ of squared modulus $|R_\xi|^{-n_j}$. More precisely, the rational Hermitian transpose identifies the rational summation spaces. In the finite bases it has scalar $-c_0\alpha_f/\overline{\alpha_f}$. Conjugating the line dilation adds the factor $\overline{\alpha_f}/\alpha_f$ on the group argument, giving u_ξ between the two line points. The transport of the normalized theta functional has unit modulus. For the determinant phases put $u = v|_{U(1)}$ and set

$$H_j(a) = \mu_{j+1}^0 u^{\lceil a/2 \rceil}, \qquad V_j(a) = \mu_{j+1}^0 \xi^{j+1} u^{\lfloor a/2 \rfloor}.$$

The ranks $h_j = r + 2jt$ and h_{j+1} are odd, so $H_j(h_j + 1) = H_j(h_j)$ and $V_j(h_{j+1} + 1) = uV_j(h_{j+1})$. With $k = u_\xi^j$ the determinant factors on the two products are

$$\frac{H_j(h_j)(\det g_{\mathrm{mid}})}{H_j(n_j)(k \det g_{\mathrm{old}})}, \qquad \frac{V_j(n_j)(k u_\xi \det g_{\mathrm{new}})}{(uV_j(h_{j+1}))(\det g_{\mathrm{mid}})}.$$

They are exactly the true-edge factors by the parity identities. Reversing the column replaces its character $H_j(n_j)$ by $H_j(n_j)^{-1}u^{n_j}$. The other character is already u. Thus the ratio to the required line kernel on the new side is

$$\frac{H_j(n_j)^{-1}u^{n_j}}{V_j(n_j)}(ku_\xi) = ((\mu_{j+1}^0)^{-2}\xi^{-(j+1)})(ku_\xi).$$

This factor has complex absolute value one at the fixed line point. It is part of $c_{\mathrm{line}}(\xi)$. The coefficients of the finite sum in (TP8) are the values of ψ. Permuting the block and line coordinates preserves the quotient Haar. The normalized restriction of additive Haar to the integral box induces uniform probability measure on the finite quotient modulo t_ξ. The sum in (TP8) uses counting measure. In the finite pairing the annihilator of the integral box is $t_\xi \widehat{\mathcal{O}}_E$ by self-duality and the unit $-\bar{c}_0$. In particular the finite pairing is perfect: for columns $v, w \in R_\xi^{n_j}$,

$$\sum_{v' \in (R_\xi^{n_j})^\vee} \psi\big(v'(v - w)\big) = |R_\xi|^{n_j} \mathbf{1}_{v=w}.$$

Indeed a nonzero coordinate of $v - w$ generates a nonzero ideal on which the primitive character is nontrivial, so the corresponding sum vanishes by translation. For fixed ξ take a cyclotomic field containing the values of the additive character and of the residue determinant characters at all the turns. The matrix $F_{v',v} = \psi(v'v)$ satisfies $\overline{F}^{\mathsf{t}} F = |R_\xi|^{n_j} I$ in this field. At a coefficient embedding ι the first matrix is read by $\iota \circ c$, the second by ι. The unitary reversal supplies the reciprocal scalar in $|c_{\mathrm{line}}(\xi)|^2$. Since $R_\xi = R_b \times \prod_{w \in Q(\xi)} k_w$,

$$\frac{|c_{\mathrm{line}}(1)|^2}{|c_{\mathrm{line}}(\xi)|^2} = \left(\frac{|R_\xi|}{|R_b|}\right)^{n_j} = \left(\prod_{w \in Q(\xi)} q_w\right)^{n_j}.$$

The common old factor cancels even when $\ell \mid |R_b|$; the displayed rational integer is a unit whenever $\ell \nmid q_\xi$.

For the measured diagonal exchange apply the two seesaws on the intermediate groups. On the old side the true-edge datum is the product of its original false block indicator and its line-column indicator. On the new side the factorization is into the false block and the line-covector indicator. The factorizations have the actual line points just computed.

Both seesaws use the original quotient measures. Write $x = (\xi, y', \eta)$ for the entire triple, z for the fixed real point of the intermediate group, and p for a complex polynomial in the archimedean line variables. Denote by $\mathcal{C}_{z,p}(x)$ and $\mathcal{B}_{z,p}(x)$ the old column and new covector periods with this line weight. Write S and $\widetilde{S}$ for the false and true intermediate spaces of rank n_j, and $s_j : U(\widetilde{S}) \to U(S)$ for their fixed reindexing. In both finite sums the pairing with the two incoming block forms f_+, f_- is

$$\int_{[U(\widetilde{S})]} K(g)\, f_+(g)\, f_-\big(s_j(g)(z,1)\big)\, dg.$$

Here $K(g)$ is the column or covector line kernel evaluated at $s_j(g)$, with the finite line point u_ξ^j or u_ξ^{j+1}, respectively. The same quotient Haar and the same real translate z occur term by term. The line point is fixed throughout the integration. Summing with the old or new array gives $\mathcal{C}_{z,p}$ or $\mathcal{B}_{z,p}$, respectively.

Suppose that the new path period at every real point is a scalar multiple of a function $\Xi(x)$, with the same scalar for all x at that point. The new seesaw implies, for each z, p, that $\mathcal{B}_{z,p}(x) = \beta(z,p)\Xi(x)$ for all x. Here are the details of the passage to real tests. Fix the line polynomial p first. For a polynomial A on the false block, the product Ap, in the reindexed orthogonal blocks, gives the whole-edge kernel whose product factorization is the false block kernel of A times the line kernel of p. The residue point datum on the whole edge is the product of the two point indicators on those blocks. Substituting this identity and applying Fubini turns the seesaw double integral into an average of new path periods. The averaging function is the compactly supported real smoothing for Ap on that edge, as constructed in Section 34.3.

It depends on the fixed real data. The finite sums stay inside the average. If $\Xi(x_0) \neq 0$, the proportionality coefficient at each averaging point is the period there at x_0 divided by $\Xi(x_0)$. It is integrable against the smoothing function by the same fixed-triple bound. If $\Xi = 0$, the average vanishes. In either case the integrated expression belongs to $\mathbb{C}\Xi$ for every A.

To pass to the actual real translate of the joint harmonic, regard each summed double integral as a continuous linear functional of its real Fock vector. For fixed x this follows from the moment estimates of Section 34.3 and rapid decrease against the fixed moderate line kernel. Only finitely many residue terms occur for fixed x. If $\Xi(x_0) \neq 0$, a family of such integrals $F_v(x)$ is in $\mathbb{C}\Xi$ precisely when

$$\Xi(x_0)F_v(x) - \Xi(x)F_v(x_0) = 0 \qquad \text{for all } x.$$

Each of these equations is continuous in the real vector v. If $\Xi = 0$, use the equations $F_v(x) = 0$. Polynomial density in the moment topology then proves the assertion for the real translate. In particular $\beta(z,p)$ is independent of every component of x.

The line Fourier law gives (TP8). For each fixed new finite covector and each tuple on the unchanged edges, apply (TP9) and the measured partner identity to the false block. The archimedean coefficient k_s is the fixed one of that square and the line reversal $p \mapsto \tau p$ depends on its signed real coordinates only. The measured square denominator $c_{\mathrm{sq}}(\xi) = c_{\mathrm{sq},(j,j)}(\xi)$ and the

Fourier scalar $c_{\mathrm{line}}(\xi)$ are independent of the finite covector and the unchanged edges. Summing the resulting identities gives

$$\mathcal{C}_{z,p}(x) = \frac{k_s c_{\mathrm{line}}(\xi)}{c_{\mathrm{sq}}(\xi)} \mathcal{B}_{z,\tau p}(x) = k_s \beta(z,\tau p) \frac{\Xi(x)}{c_s(\xi)}, \qquad c_s(\xi) = \frac{c_{\mathrm{sq}}(\xi)}{c_{\mathrm{line}}(\xi)}.$$

Thus (TP10) and the line norm give

$$\frac{|c_{(j,j)}(\xi)|^2}{|c_{(j,j)}(1)|^2} = \mathrm{Nm}(Q(\xi))^{t^2+n_j}\, T_{jj}(\xi)^2.$$

For the old seesaw, products Ap in the false block and old line variables span all the whole-edge polynomials. Each product has the factorization into the two block point indicators and the two real polynomials on those blocks. Use Fubini and the compactly supported real smoothing for its false polynomial A to express the tested old integral as an average of the summed $\mathcal{C}_{z,p}(x)$. The averaging preserves the common proportionality to $\Xi(x)/c_s(\xi)$. The same continuous equations, now on the old side and with this function of x, extend the assertion to the actual translated whole-edge harmonic. This proves that at every fixed real point the old path period is proportional to $\Xi(x)/c_s(\xi)$ by one scalar for all triples. The off-diagonal turn has this property directly by its tested square identity.

43.5 Composition and terminal measure

In this subsection, we compose the transports of Section 43.4 and retain the original terminal measure.

For a pair of paths ending on the diagonal at j, take the cells in the j-by-j diagonal square, together with the cells to the left of each down step in both halves, reflecting the true half. A down step at horizontal coordinate a across sequence b contributes the cells (h,b), $0 \le h < a$, in its half's recorded coordinates. At an off-diagonal turn the down step moves one column to the right, so the union contains one additional cell.

At a diagonal turn the two strips of length j next to the j-by-j square are already included, and the enlarged diagonal square adds just (j,j). Initially the union is empty. A half with a down step before a right step contains an adjacent down–right pair and therefore admits the turn. If neither half has one, all right steps precede the down steps. If $j < m$, both last steps are down steps and give a diagonal turn. If $j = m$, the last path has been reached. Thus an arbitrary successive choice of available corners crosses each square exactly once, ending on the last covector of each half.

Choose the arrays by (TP8)–(TP9) in this order, starting with P_1. Their choices depend on the twist and marked spaces before the terminal forms are chosen. An off-diagonal turn only multiplies a selected entry by a character value. The diagonal turn at j sums over $|R_\xi|^{n_j}$ columns. Thus the number of roots of unity in each terminal entry is at most

$$E_\xi^{2m} \prod_{j=0}^{m-1} |R_\xi|^{n_j} \le E_\xi^{2m} V_\xi^{m^2}.$$

The backward implications compose by multiplying the $c_s(\xi)$. If $c_\xi = \prod_s c_s(\xi)$, then $c_\xi \neq 0$ and

$$\left|\frac{c_\xi}{c_1}\right|^2 = \mathrm{Nm}(Q(\xi))^{m^2t^2+m^2t+mr} \prod_{i,j<m} T_{ij}(\xi)^2. \tag{TP11}$$

Here the sum of the line dimensions is $\sum_{j=0}^{m-1}(r+(2j+1)t) = mr+m^2t$. At the trivial twist all critical ratios are equal to one. This follows directly from the nonzero Hecke denominators and the definition of the normalized ratios.

For the last path let $I_{z,j}$ be the original finite level index at node (z,j,m). Every such index other than $I_{0,0}$ is the general linear group order at $\mathfrak{b}$, because the level there is the kernel of $\mathrm{GL}(R_b)\times\mathrm{GL}(R_Q)\to\mathrm{GL}(R_b)$ in the corresponding rank. At $(0,0,m)$ the level is the identity over R_ξ and the rank is $r+mt$, so $I_{0,0} = |\mathrm{GL}_{r+mt}(R_\xi)|$. The last nodes are the original definite models.

On each edge with fixed polynomial degree D_e, the scalar analytic image of a source algebraic form is the analytic realization of its jet lift times

$$\frac{\beta_e}{s_e I_{z,j}}. \tag{TP12}$$

In detail, the original jet matrix applied to a source vector is paired with the highest target covector. The homogeneous coefficient of degree D_e gives $a_e s_{\mathrm{jet}}^{-D_e}$ when converting from the last-path jet scale of Section 44.3.1 to the chosen joint harmonic, where a_e depends only on the real datum.

The original compact real integral, using the unit highest source vector, gives $1/\dim V_{\mathrm{src}}$ by Schur orthogonality. The finite class integral of the algebraic lift gives the source level mass one, whereas in the analytic measure its mass is $I_{z,j}^{-1}$. Finally the algebraic datum multiplies the same matrix indicator by $s_e = s_{\mathrm{vac}}^{\dim(\mathrm{Hom})}$. These are precisely the original source, target and polynomial conventions in the definite jet lift.

Thus $\beta_e = a_e s_{\mathrm{jet}}^{-D_e}/\dim V_{\mathrm{src}}$ and s_e are fixed independently of the residue point and the twist. Iterating (TP12) gives, after j edges, the algebraic chain's analytic realization times $b_{z,j}I_{z,j}/I_{z,0}$, with

$$b_{z,0}=1, \qquad b_{z,j+1} = \frac{\beta_e b_{z,j}}{s_e I_{z,j+1}}.$$

The $b_{z,j}$ are fixed, because the last index in this recursion has $j+1>0$. The recursion reads one original tuple at a time.

At the final diagonal the true form's finite point is the block point $\mathrm{diag}(h,u_\xi^m)$ in its given reindexing. The extra line has sign $+1$ since m is even. The two nodes here have ranks $n_{\mathrm{diag}} = r+2mt$ and $n_{\mathrm{diag}}+1$. We construct the bilinear branching pairing for these two ranks. Fix one infinite place, suppress it from the notation and write $w_i = \kappa_\tau + i\Delta + \Delta/2$ and $s_i = 2\lambda_{k-1+i} = \Delta - 2w_{i+1}$ for $0\le i\le m$, extending the same arithmetic progression for its final term. The doubled parameters to interlace are the false parameter before duality and the negative of the true grid parameter.

For an integer threshold b', write $\phi(y) = \mathbf{1}_{b'<y}$ and $g_n(y) = \max(0,\min(n,\lfloor (y-b')/2\rfloor))$. A descending block $d, d-2,\dots,d-2(n-1)$ contributes $g_n(d+1)$ entries above the threshold. With i pairs of blocks included, the counts on the false and true sides are respectively

$$C_i^- = \sum_{a<k-1+i}\phi(2\lambda_a) + \sum_{j=1}^{i}\{g_t(2w_j+t)+g_t(-2w_j+t)\},$$

$$C_i^+ = \sum_{a<k+i}\phi(-2\lambda_a-1) + \sum_{j=1}^{i}\{g_{t+1}(-2w_j+t+1)+g_{t-1}(2w_j+t-1)\}.$$

These are the base entries and the two progressions inserted at each step of the diagonal grid, with lengths t, t on the false side and $t+1, t-1$ on the true side. We claim that

$$\phi(s_i) \le C_i^+ - C_i^- \le \phi(-s_i - 1).$$

For $i = 0$, the central false progression is $k-2, k-4, \dots, 2-k$. The central true progression consists of the same entries diminished by one, together with $-s_0 - 1$ above it. Here $s_0 \le -\Delta < 1-k$ and $-s_0 - 1 \ge \Delta - 1 > k-2$, using $w_1 \ge \Delta$ and $k+t+1 \le \Delta/2$. The shifted and unshifted central progressions have counts differing by 0 or -1, before adding the extra true entry $-s_0 - 1$. This gives the two endpoints of the claim. For the induction use the following block count. The difference between the counts of the centered progressions of lengths $n+1, n$ at center c is $g_{n+1}(c+n+1) - g_n(c+n)$. Their entries alternate, so this is zero or one, equals one below $c-n$, and vanishes at or above $c+n$. At $c = -2w_{i+1}, n = t$, this difference lies between $\phi(s_{i+1})$ and $\phi(s_i)$. At $c = 2w_{i+1}, n = t-1$ it lies between $\phi(-s_i - 1)$ and $\phi(-s_{i+1} - 1)$. Indeed $w_{i+2} - w_{i+1} = \Delta$ and $t+1 \le \Delta/2$. In passing from i to $i+1$, add the first difference, subtract the second, and add $\phi(-s_{i+1} - 1) - \phi(s_i)$ from the two new base entries. This proves the claim. At $i = m$, since $r = k - 1 + m$, the counts of the actual terminal parameters differ by zero or one at every threshold.

Let the descending highest weights of the larger module and the smaller primal module be $\Lambda_1, \dots, \Lambda_{n_{\mathrm{diag}}+1}$ and $\nu_1, \dots, \nu_{n_{\mathrm{diag}}}$, respectively. The descending doubled parameters are

$$\ell_i^+ = 2\Lambda_i + n_{\mathrm{diag}} + 2 - 2i, \qquad \ell_i^- = 2\nu_i + n_{\mathrm{diag}} + 1 - 2i.$$

Their opposite parity and the count inequality give $\ell_i^+ > \ell_i^- > \ell_{i+1}^+$. Thus $\Lambda_i \ge \nu_i \ge \Lambda_{i+1}$. Put $e = \Lambda_{n_{\mathrm{diag}}+1}$ and $e_0 = \nu_{n_{\mathrm{diag}}}$. Of the $\Lambda_i - \Lambda_{i+1}$ upper columns at height i, retain $\nu_i - \Lambda_{i+1}$ and contract $\Lambda_i - \nu_i$ against the last unit vector. For $i < n_{\mathrm{diag}}$, the retained columns at i and contracted columns at $i+1$ give exactly $\nu_i - \nu_{i+1}$ lower columns. At $i = n_{\mathrm{diag}}$, the retained full columns contribute $e_0 - e$ to the lower determinant power. The extra line acts with exponent

$$L_\tau = e + \sum_{i=1}^{n_{\mathrm{diag}}} (\Lambda_i - \nu_i).$$

Let E_+, E_- be the integral inclusions of the upper and lower Weyl lattices in their exterior tensor models. Let C be the tensor of the retained-column projections and the last-covector contractions with these counts, absorbing the retained full lower columns into the lower determinant line. With $G_- = E_-^{\mathsf{t}} E_-$ and c_- the Fock squared norm of the full lower highest polynomial, set

$$b_{\mathrm{grid}} = c_- G_-^{-1} E_-^{\mathsf{t}} C E_+.$$

By the Fock calculation of Section 7.5, applied to the full lower diagram, $c_- G_-^{-1}$ is the integral Fock matrix on the dual tableau polynomials. The other matrices are integral, and the column operations and Weyl projection commute with the smaller compact group. Laplace expansion gives one last-line factor per contraction. Pairing the contracted evaluation tensor of a generic upper flag with the lower highest tensor gives a product of nonzero minors: a retained column uses its leading rows, while a contracted column uses its shortened leading rows and the last row. In particular $b_{\mathrm{grid}} \neq 0$. The number of lower boxes after removing determinant powers is at most the upper count H, since $\nu_i - e_0 \le \Lambda_i - e$. The minor and factorial bounds give $c_- \le H^{3H}$, with value one for $H = 0$. Before multiplication by c_-, the map is a composition

of exterior operations and an orthogonal Weyl projection, each of norm at most one. Thus $\|b_{\mathrm{grid}}\| \le H^{3H}$ and $|L_\tau| \le H + |e|$.

Compose b_{grid} with the integral invariant duality of the false node to obtain the integer bilinear matrix $B_{\mathrm{grid}} = J_{\mathrm{diag}} b_{\mathrm{grid}}$. Here complementary coordinate wedges, with the volume sign, identify the primal representation isometrically with the conjugate of its dual; the dual tensor Gram matrix composed with this signed identification gives J_{diag}, as in Section 44.3.4 at this rank. Cauchy–Schwarz bounds the resulting bilinear pairing by H^{3H} times the product of the two Weyl norms. At a fixed real point g the real coefficient integral averages the rank-one map $h_0 \ell_g$ against the target and source compact actions.

Here h_0 is the unit highest vector of the primal false type and $\ell_g(v)$ is the true highest matrix entry of the translate by g^{-1}. The average is an intertwiner for the false compact group. After removing a common determinant twist, specialize the Schur branching identity to the smaller alphabet and the last variable [44, Ch. I, (5.8)–(5.9)]. Each interlacing diagram occurs once because its skew Schur polynomial in one variable is a single monomial. Hence the average is $a(g) b_{\mathrm{grid}}$. It is independent of the twist and the finite point and can be zero.

At each finite point this leaves the bilinear pairing $F_-(h)^{\mathsf{t}} B_{\mathrm{grid}} F_+(\mathrm{diag}(h, u_\xi^m))$. The compact finite measure converts to the class integral of level mass one with coefficient $I_{0,m}^{-1}$. For an individual locally constant summand, take an open subgroup $J' \subset J$ of the displayed level that fixes it. A J'-coset has mass $[J : J']^{-1}$ in the Haar normalization giving J mass one, and mass $[K_{\mathrm{std}} : J']^{-1}$ in the original finite measure. Dividing the latter mass by the former gives $[K_{\mathrm{std}} : J]^{-1} = I_{0,m}^{-1}$, also when the individual summand has smaller stabilizer than J. After multiplying by the array, invariance under right translations at the internal nodes follows by reindexing their residue matrices.

The edge phases cancel by the covariance of (TP9). Let J be the product of the integrated levels, counting the common terminal variable once. At that variable, $h \mapsto hk$ acts on the true point by

$$\mathrm{diag}(hk, u_\xi^m) = \mathrm{diag}(h, u_\xi^m)\,\mathrm{diag}(k, 1).$$

The extra-line value is fixed. Let F be the combined integrand, which is right J-invariant by the array covariance, and take $J' \subseteq J$ fixing all individual summands. The original old principal level gives trivial rational stabilizers, so each fiber of the class map $\mathcal{C}(J') \to \mathcal{C}(J)$ has cardinality $[J : J']$. With J of mass one,

$$\frac{1}{[J : J']} \sum_{h \in \mathcal{C}(J')} F(h) = \sum_{h \in \mathcal{C}(J)} F(h).$$

In particular the finite sum carries no inverse refinement index. The two outer levels are respected: the false outer deep principal level is contained in the original $K(\pi_M Q(\xi))$, and the true outer level is $K(\mathfrak{b})$.

The factor $I_{0,m}^{-1}$ cancels the false terminal numerator of the chain conversion. The true ratio $I_{1,m}/I_{1,0}$ is fixed. By the same calculation as (TP6), $I_{0,0}$ differs from $I_\xi = |\mathrm{GL}_{r+mt}(R_0(\xi))|$ by the fixed ratio $I_\xi / I_{0,0} = (\mathrm{Nm}(\pi_M)/\mathrm{Nm}(\mathfrak{b}))^{(r+mt)^2}$. Consequently the last-period function at any real g is a scalar depending only on g and the datum times $\Xi_\xi(y', \eta)/I_\xi$, where Ξ_ξ is the final array paired against those algebraic class sums. Apply the composed backward implication to that function and evaluate at ω in (TP7). With one scalar A_* independent of the twist and both forms, we obtain the following identity.

Proposition 43.1 (The measured terminal period)**.** *Let ξ be a full-chain twist as in Section 2.6, let y' be a dual test at the original full principal terminal level, and let $\eta \in \mathcal{H}_m$. With the original paths, real points and quotient measures, one scalar A_*, independent of ξ, y', η, gives*

$$I_\xi \int_{[Y_m]} y'(h)(T_\xi r_0 \eta)(h)\, dh = A_* \, \Xi_\xi(y', \eta)/c_\xi. \tag{TP13}$$

Here $c_\xi \neq 0$ and I_ξ is the index of the original false terminal test level, computed by surjectivity of split reduction.

44 Chain determinants and completion of the lattice

In this section, we compute the determinant by composing the individual lifts, using the terminal periods of Section 43. The same calculation gives the auxiliary vectors and the clearing scalar needed in Section 23.

44.1 Determinants of the composition

In this subsection, we compute the determinant of the weighted composition by reducing it to the determinants of the individual lifts. This gives the ratio whose height is estimated in Section 44.2.

Fix the admissible datum and the coefficient embeddings $\kappa = (\ell, \iota)$. Let W be the original full source of level $K(\pi_M)$, and $V \subset W$ its good space of dimension D. Write $T_\xi = L_\xi$ for the full weighted chain of Section 17.3 on W, and $\Theta_\xi = T_\xi|_V$ for its good restriction at the allowed m-fold twist ξ. The source weights below are $w_n = A_{\xi,n}$. The terminal algebraic conversion is that of the same chain. The target inner product is the terminal class pairing. Write

$$Z_\xi = \det(\Theta_\xi^* \Theta_\xi), \qquad Z_j(\xi) = Z_j^{\mathrm{single}}(\xi^j) = \det(u_{j,\xi^j}^* u_{j,\xi^j}),$$

where u_{j,ξ^j} is the j-th single lift on the good space of the specified space W_j, also with its marked target class pairing. The same source metrics are used at all twists. Set

$$B = \log b, \qquad J_* = \frac{faB^3}{mk}, \qquad L = faB^4, \qquad A = mk\, d_{\mathrm{dir}}.$$

Here Section 38 gives

$$d_{\mathrm{dir}} = \sum_\tau \big(\log(\kappa_\tau!) + \kappa_\tau \log(M/\Omega) - S(\kappa_\tau, \tau)\big).$$

Thus $A = mk\, d_{\mathrm{dir}}$ is the same quantity as in Section 5. Each height bound uses one multiplier for all coefficient embeddings.

Finite factors in the chain. For a twist ξ put $q_\xi = \mathrm{N}Q(\xi)$ and let w_n be the order of the general linear group of a triangle node in column n modulo $\pi_M Q(\xi)$. Let $a_{i,n}$ be the horizontal arrow and $d_{i,n}$ the downward arrow from column $n+1$ of sequence i to column n of sequence $i+1$. Denote the composite of the first n arrows of sequence i, weighted by $w_0, \dots, w_{n-1}$, by $X_{i,n}$. Let j_i be the isometric base identification of sequence i.

Injectivity before dividing by the archimedean scalar. The first-source calculation in Section 34.2 applies to every $y \in W$. In fact y is invariant under the point stabilizer at $\pi_M Q(\xi)$,

which is contained in $K(\pi_M)$, and is invariant under the integral finite group off the support. Writing $A_i = a_{i,0} j_i$, we get

$$w_0 \langle A_i x, A_i y \rangle = C_i B_\xi^{-1} \langle x, \mathcal{V}_i y \rangle \qquad (y \in W),$$

where x may be any form of the original source real type. Put $R_0 = R_0(\xi) = \mathcal{O}_E/(\pi_M Q(\xi))$ and $B_\xi = |R_0|^{r(r+t)}$. The constant C_i is nonzero. For these ranks $t = 5r$, the count in Section 34.2 gives $\|\mathcal{V}_i - I\| \le \eta/(1-\eta) < 1/2$ on the full source space, uniformly in the set of omitted places. On the good subspace it is R_{i+1}. In particular the first-source injectivity applies to all W: if $A_i y = 0$, take $x = \mathcal{V}_i y$. This vector has the source type and level, and $\mathcal{V}_i$ is injective.

For the shallow descents pick the first datum by adjointness on the two isometric copies, and propagate it by the finite shear formula (97) on R_0. The marked square gives the equality

$$w_n a_{i+1,n} d_{i,n} = u_{i,n} w_{n+2} d_{i,n+1} a_{i,n+1}, \qquad u_{i,n} = k_{i,n}/c_{i,n}, \quad c_{i,n} \neq 0. \tag{GC1}$$

Here the measured partner scalar $c_{i,n}$ is nonzero from the start. The archimedean pairing factor $k_{i,n}$ is allowed to be zero in (GC1). As in the deep case, the target block lift of any integral source element gives a simultaneous action fixing the point indicator up to its unit determinant phase. Thus $A_i^* A_i$ preserves the original fixed-level subspace W. In fact it commutes with each integral finite translation $R(u)$ by the first-column calculation following Proposition 34.1. It therefore commutes with their linear combination $p = \rho(e_0)$. The exact first adjoint identity says

$$B_i = j_{i+1}^{-1} d_{i,0} a_{i,0} j_i = A_i^* A_i$$

on W. Thus B_i is injective on W and

$$B_i = \frac{C_i}{w_0 B_\xi} \mathcal{V}_i \quad \text{on } W, \qquad B_i|_V = \frac{C_i}{w_0 B_\xi} R_{i+1}, \qquad V = pW.$$

These equalities follow by testing the first-lift identity against all $x \in W$. The adjoint here is represented by the original transposed kernel on the indicated types, with the original Haar.

If $X_{i,n}$ is the first n weighted arrows, iterating (GC1) from $n = 0$ gives

$$X_{i+1,n} d_{i,0} a_{i,0} = \frac{w_{n+1}}{w_0 w_1} \left(\prod_{s<n} u_{i,s} \right) d_{i,n} X_{i,n+1}. \tag{GC2}$$

In particular the implication $X_{i,n+1} j_i y = 0 \Rightarrow X_{i+1,n} j_{i+1} B_i y = 0$ involves no inversion of $k_{i,n}$. Induction on n from the isometry j_i proves each chain injective on its base copy. If $W \neq 0$, the left side of (GC2) is nonzero on $j_i y$ for every $y \neq 0$. Taking $n = s + 1$ shows $u_{i,s} \neq 0$ for every square. We may therefore use $\beta'_{i,n} = \overline{c_{i,n}/k_{i,n}}$ in the Gram computation. When $W = 0$, every chain and single-lift determinant on V is the empty determinant.

The terminal level of the chain image. Reduce an element of the terminal level modulo $\pi_M Q(\xi)$. Its matrix g over R_0 is upper block triangular of sizes $r, t, \dots, t$. Write g_j for its leading corner of rank $r + jt$, $g^{(j+1)}$ for the next diagonal block, and I_j for the coordinate inclusion. Over R_0,

$$g_{j+1} I_j = I_j g_j, \qquad \det(g_{j+1}) / \det(g_j) = \det(g^{(j+1)}).$$

Choose integral lifts of the corners by split reduction. The first identity is then a congruence modulo $\pi_M Q(\xi)$, which fixes the residue point, and the second determines the character of

the determinant on local units. The simultaneous Schrödinger action on the original point indicator therefore moves the terminal translation through arrow j, multiplying by the $j+1$st lift character at $\det(g^{(j+1)})$.

This is the determinant ratio of the target and source actions, with the same tame character at the two ranks. Its other local factors are trivial on integral units. Iterate this equality. At old places the terminal element is the identity after reduction, so g_0 fixes the base form. At each new place g_0 is integral, so it again fixes the base form. The resulting product of the diagonal-block characters is the original terminal level character. The passage from scalar forms to the algebraic compact model intertwines these translations. Thus the terminal algebraic map with this character is the very same injective weighted chain.

Write $I_{\mathrm{par}}(\xi)$ for the index of this terminal level. Its factor at a twist place is the index of the block upper triangular group. The compact real Schur average and the finite quotient volume convert the Petersson norm of the forward chain to its algebraic class norm by $d_Y I_{\mathrm{par}}(\xi)$. This differs from the dual-delta test conversion $d_Y I_\xi$, where $I_\xi = |\mathrm{GL}_{r+mt}(R_0)|$. For that period the dual test has the full principal level at the twist places. Both conversions apply to the same forward function.

The adjoint square and the Gram scalar. The tested adjoint of the upper point arrow is the arrow with conjugate kernel and opposite determinant characters, by Fubini on the two original quotients. Use the original reversed local kernels with Fourier datum $\mathcal{F}(\bar{p})$, and let $\mathcal{R}_n$ be right translation by the inverse of the finite central reversal scalar on rank $r+nt$. In the notation where D_0, D_1 are the two descents and B_0', B_1' their reversed lifts, the opposite-kernel identities in Section 36.2 read

$$\langle D_0 x', y\rangle = s_0 \langle x', B_0' \mathcal{R}_n y\rangle,$$
$$\langle x, B_1' \mathcal{R}_{n+1} z\rangle = s_1 \langle D_1 x, z\rangle.$$

They are tested identities and the real polynomial replacement constants in s_0, s_1 may vanish.

The reversed finite square on R_0 gives $a_{\mathrm{up}} B_0' = \gamma B_1' a_{\mathrm{low}}$. The two integrated groups both have rank $N+t$, where $N = r+nt$. Thus the rank defect in Proposition 36.1 is zero: $M_\nu(0) = 1$, and the normalized period scalar has squared modulus R_0'/D_{final}, fixed before the tame character. The determinant Gauss sum and the Fourier shear supply the indicated power of $|R_0|$. Put $s = s_{\mathfrak{m}}$ as in (MR). For the characters (a, b) of the lower arrow from rank N to rank $N+t$, simultaneous central translation fixes the Hom variable and the original residue indicator. Its kernel satisfies

$$K(gs, hs) = \eta K(g, h), \qquad \eta = a_{\mathrm{f}}(s^{N+t}) b_{\mathrm{f}}(s^N)^{-1}, \qquad |\eta| = 1.$$

Centrality preserves both right levels. Changing the source variable by s in its quotient measure gives $a_{\mathrm{low}} \mathcal{R}_n = \eta \mathcal{R}_{n+1} a_{\mathrm{low}}$, where both translations are by s^{-1}. For $x' = a_{\mathrm{up}}^* x$ this gives

$$\langle D_0 a_{\mathrm{up}}^* x, y\rangle = s_0 \gamma \eta s_1 \langle D_1 x, a_{\mathrm{low}} y\rangle. \tag{GC3}$$

After removing fixed real factors, the squared moduli of s_0, γ, s_1 have powers of $|R_0|$

$$-N(N+t), \qquad -t(t+2N), \qquad (N+t)(N+2t).$$

The first and last powers come from (MR). The middle one comes from the determinant Gauss sum on $M_t(R_0)$ and the shear in Section 36.2.2; the two middle unitary groups have equal rank.

Their sum is t^2. The adjoint square has no critical ratio. Its fixed real factor may vanish during the iteration. No adjoint between ambient unbounded conjectural operators is needed: the conjugate kernel represents its tested functional on the original cusp types.

The modulus of $\beta'_{i,n}$ is a fixed positive constant times $q_\xi^{t^2/2} Q_{i+1,i+n+2}(\xi)$. Here $Q_{ij}(\xi)$ is the product of t critical ratios for the quotient character of lifts i, j, at ξ^{j-i}. In the notation of Section 38, with $\nu_{ij} = \mu_j^0/\mu_i^0$ the quotient in the square law, this is

$$Q_{ij}(\xi) = \prod_{z=1}^{t} R_{(\nu_{ij},z),+}(\xi^{j-i}).$$

The same inverse quotient is used in both orientations of the balanced exchange. Put $\beta_{i,n} = \prod_{s<n} \beta'_{i,s}$ and $\epsilon'_{i,n} = s_0\gamma\eta s_1$ as in (GC3).

The corresponding adjoint law, starting with the first-column adjointness, is

$$\langle v, a_{i,n} X_{i,n} j_i y\rangle = \epsilon_{i,n}\langle d_{i,n} v, X_{i+1,n} j_{i+1} y\rangle, \qquad \epsilon_{i,n} = \prod_{s<n} \epsilon'_{i,s}.$$

It follows inductively by moving the last upper arrow to its adjoint and using the displayed adjoint square. If $j_{i+1}x' = d_{i,0}a_{i,0}j_i x$, the Gram form of $X_{i,n+1}j_i$ becomes the Gram form of $X_{i+1,n}j_{i+1}$ at (x', y), times

$$\epsilon_{i,n}\beta_{i,n}\frac{w_0 w_1 w_n}{w_{n+1}}.$$

The conjugation of the first argument changes $\overline{\beta_{i,n}}$ to $\beta_{i,n}$.

With v in the good subspace, substitute the first-lift formula for $w_0\langle x', v\rangle$. The second argument acquires the operator $C_i B_\xi^{-1} R_{i+1}$. Consequently the terminal Petersson Gram form is

$$g_\xi \frac{w_0 w_1^m}{w_m B_\xi^m}\langle x, R_1 \cdots R_m y\rangle.$$

The weights telescope. At step i the squares are indexed by $0 \le s \le m-i-2$. The correspondence $(i,s) \mapsto (a,b) = (i+1, i+s+2)$ lists each pair $1 \le a < b \le m$ once, with twist power $b-a = s+1$. Relative to the trivial twist, the forward square contributes $q_\xi^{t^2/2} Q_{ab}(\xi)$ and the adjoint square contributes $q_\xi^{t^2/2}$ in absolute value. Indeed $|R_0(\xi)| = |R_0(1)| q_\xi$ by the Chinese remainder theorem. Thus, with the Rallis operators in the indicated order,

$$|g_\xi| = |g_1|\, q_\xi^{\binom{m}{2}t^2} \prod_{1\le a<b\le m} Q_{ab}(\xi).$$

The squared norm at a nonzero $y \in V$ is positive by injectivity. Thus the final Gram factor and its untwisted value are nonzero on $V \neq 0$.

At a place of $Q(\xi)$ write G_n for the order of $\mathrm{GL}_{r+nt}(k_w)$. The parabolic for the terminal level has blocks $r, t, \ldots, t$ and order

$$|P_n| = G_0 |\mathrm{GL}_t(k_w)|^n q_w^{nrt+\binom{n}{2}t^2}.$$

The local single-arrow factor before division by the matrix count is

$$\delta_w = [G_1 : P_1] G_0 q_w^{rt} = G_1/|\mathrm{GL}_t(k_w)|.$$

It follows that $[G_m : P_m] = q_w^{-mrt-\binom{m}{2}t^2}\delta_w^m G_m/(G_0 G_1^m)$. At the fixed chosen places the terminal index is the full general linear group order.

At a new place put $q = q_w$, $B_w = q^{r(r+t)}$ and $F_w = \prod_{t<h\le t+r}(1-q^{-h})$. The factors from the node weights, terminal index, successive squares and matrix counts give, in this order,

$$\frac{G_0 G_1^m}{G_m}\,[G_m : P_m]\,q^{\binom{m}{2}t^2}B_w^{-m} = q^{-mrt}(\delta_w/B_w)^m = q^{-mrt}(q^{rt}F_w)^m.$$

The last equality follows from $G_1/|\mathrm{GL}_t(k_w)| = q^{(r+t)^2-t^2}F_w$. Each single lift contributes $\delta_w/B_w = q^{rt}F_w$ at this same place, by Section 34.2.1. Hence all the factors F_w cancel exactly in the chain-to-single ratio, including at coefficient primes dividing $q^h - 1$. The Weyl dimension d_Y in the class norm is independent of the twist. At an old place the pointwise fixer of the inclusion has index equal to the number of full-rank matrices, by transitivity.

The sharp mass has the additional inverse source index G_0^{-1}, so the fixed old factor for each single lift is $G_0^{-1}\prod_{t<h\le t+r}(1-q_w^{-h})$. Let $c_Q(\xi) = \prod_{w|Q(\xi)} q_w^{rt}F_w$. For $V \neq 0$, positivity of the injective chain Gram form above forces the product of the archimedean constants to be nonzero. The Gram operator of the chain is

$$C_\xi c_Q(\xi)^m R_1 \cdots R_m, \qquad |C_\xi| = C q_\xi^{-mrt}\prod_{1\le i<j\le m} Q_{ij}(\xi), \qquad C > 0.$$

The single-lift identities for this comparison are the marked Gram operators $C_j c_Q(\xi) R_j^W$, where C_j is independent of the twist. The sharp datum at the chosen places fixes the inclusion matrix, and the datum at the twist places has invertible first block and arbitrary lower block. The Haar index and the squared residue mass therefore give exactly the same $c_Q(\xi)$ above.

It is the same at every ξ^j since these powers have the full modulus. For the Rallis series put A equal to the complement of the old and twist supports, including their conjugates. For a finite subset $\mathcal{D} \subset K_A\backslash G_A/K_A$, the partial series on both source spaces is the action of the same finite Hecke test

$$h_{\mathcal{D}} = \sum_{K_A\gamma K_A\in\mathcal{D}} u_j(\gamma)\frac{\mathbf{1}_{K_A\gamma K_A}}{\mathrm{vol}(K_A)},$$

extended by normalized integral identity factors at the twist places.

This expression uses every finite place of A, with the same original bases and vacuum coefficient. The vacuum coefficient is constant on each double coset: integral left and right multiplication preserve the lattice index, and b_j is unramified on A. Each $K_A\gamma K_A$ consists of $[K_A : K_A \cap \gamma K_A\gamma^{-1}]$ cosets gK_A. Hence on either good space the operator norm of the tail is at most

$$\sum_{K_A\gamma K_A\notin\mathcal{D}} [K_A : K_A \cap \gamma K_A\gamma^{-1}]\,|u_j(\gamma)| \longrightarrow 0$$

by the absolute sum over G_A/K_A in Section 34.2.2. For every positive integer s, apply (4) to the convolution power $h_{\mathcal{D}}^{*s}$, and take the limit. Continuity of powers and traces in finite dimensions gives

$$\mathrm{tr}(R_j^s) = \mathrm{tr}((R_j^W)^s).$$

The dimensions are equal. Newton's identities give the same characteristic polynomial and hence $\det R_j = \det R_j^W$.

Set $W_\xi = q_\xi^{-mrt} \prod_{i<j} Q_{ij}(\xi)$. The two factorizations just proved give

$$Z_\xi = C^D W_\xi^D c_Q(\xi)^{mD} \prod_j |\det R_j(\xi)|,$$

$$\prod_j Z_j(\xi) = \left(\prod_j |C_j|\right)^D c_Q(\xi)^{mD} \prod_j |\det R_j^W(\xi)|.$$

For $D > 0$ each $Z_j(\xi) > 0$ by injectivity of the single lift, and $Z_\xi > 0$ by (GC2). The same holds at $\xi = 1$. Thus the two factorizations and $\det R_j = \det R_j^W$ give

$$\frac{Z_\xi}{\prod_j Z_j(\xi)} = \left(\frac{C}{\prod_j |C_j|}\right)^D W_\xi^D.$$

Compare this quotient at ξ and at 1, where $W_1 = 1$.

Proposition 44.1 (The chain determinant ratio)**.** *For a full-chain twist ξ as in Section 2.6, the weighted chain and its marked single lifts satisfy*

$$\frac{Z_1}{Z_\xi} = \left(q_\xi^{-mrt} \prod_{1\le i<j\le m} Q_{ij}(\xi)\right)^{-D} \prod_{j=1}^m \frac{Z_j(1)}{Z_j(\xi)}. \tag{102}$$

The formula includes the empty determinant.

44.2 Height of the chain determinant ratio

In this subsection, we apply the critical-value comparisons of Section 38 to Proposition 44.1.

Use the family ξ_κ of Section 38 for the finite list of critical inverse ratios and all single lifts. Its modulus is prime to ℓ and every elementary critical inverse ratio in the square and turn lists has height at most x_1. On this same family, Section 42.1 gives operators $U_j(\xi)$ with

$$u_{j,\xi^j}^* U_j(\xi) u_{j,\xi^j} = \rho_j(\xi) U_j^\circ$$

for a fixed positive definite U_j°. The ratio in Lemma 42.1 is

$$\rho_j(\xi) = \rho_{\xi^j} = \frac{H_j(\xi^j)}{H_j(1)}.$$

The exchange-product comparison in Section 38.6 bounds its reciprocal on this family. Write $u = u_{j,\xi^j}$ and let Π be orthogonal projection onto its image.

Set $\Delta_j(\xi) = \det(\Pi U_j(\xi)|_{\operatorname{im} u})$. On the source, conjugate the compression by the isomorphism u onto its image. Multiplying the conjugated operator by u^*u gives $u^*\Pi U_j(\xi)u = u^* U_j(\xi) u$. Taking determinants gives

$$Z_j(\xi)\Delta_j(\xi) = C_j' \rho_j(\xi)^D, \qquad C_j' = \det U_j^\circ \neq 0.$$

The required bounds for this comparison are

$$\log|\sigma\Delta_j(1)| \le 3Dfmb B^2 \quad \text{for all } \sigma,$$
$$\log|\Delta_j(\xi_\kappa)|_\kappa \le Dr\beta_{j,\ell}, \qquad \sum_{\ell\in U} \beta_{j,\ell} \le J_* \quad (U \text{ finite}),$$
$$\operatorname{Height}(\rho_j(\xi_\kappa)^{-1}) \le 10mx_1 + k(6J_* - d_{\mathrm{dir}}).$$

The second line is a logarithmic allowance, independent of the embedding above ℓ. Taking positive parts of these allowances preserves the finite-sum bound: apply that bound to the subset of primes with positive allowance. These are respectively the untwisted compression estimate, the tested clearing estimate, and the fixed numerator and inverse ratio bound on the same regular family.

Dividing the comparison identity by its untwisted value gives

$$\frac{Z_j(1)}{Z_j(\xi_\kappa)} = \frac{\Delta_j(\xi_\kappa)}{\Delta_j(1)}\rho_j(\xi_\kappa)^{-D}.$$

For the first ratio use $\Delta_j(1)$ as its fixed height multiplier. Multiplication gives the allowance

$$D\{3fmbB^2 + rJ_* + 10mx_1 + k(6J_* - d_{\mathrm{dir}})\}.$$

On dividing the positive terms by L/m, using $a = xb$, their sum becomes

$$\frac{3m^2}{xB^2} + \frac{r}{kB} + \frac{10m^2x_1}{fxbB^4} + \frac{6}{B}.$$

It tends to zero under the admissible choices $m^2 \le 4x$, $r/k \le 2$, $\log b \sim x^\epsilon$, and $\log x_1 \le x^{\epsilon/6}$. This gives height at most $D(L/m - kd_{\mathrm{dir}})$. For each j, the $k/2$ pairs at $7m/2 < u \le (7m+k)/2$ in the third measured exchange of Section 39.4 use Proposition 38.1. Each occurs twice in the squared product of Section 38.6. Their combined negative degree term is $\sum_{j=1}^{m} 2\sum_{i=1}^{k/2}(-d_{\mathrm{dir}}) = -mkd_{\mathrm{dir}} = -A$.

The triangular product in (102) contains one factor for each $1 \le i < j \le m$ and $1 \le z \le t$, at twist power $j - i$. Since $\ell \nmid q_{\xi_\kappa}$, (RT1) gives

$$v_\kappa(W_{\xi_\kappa}^{-1}) = -\sum_{1\le i<j\le m} v_\kappa Q_{ij}(\xi_\kappa) \ge -\binom{m}{2} te_\ell.$$

For $D > 0$ let $h_j \ne 0$ be the fixed height multiplier for $\rho_j(\xi_\kappa)^{-1}$ in Section 38.6. Since $Z_j(1) > 0$, $\rho_j(1) = 1$ and $\det U_j^\circ \ne 0$, one has $\Delta_j(1) \ne 0$. A single multiplier for the chain is

$$H_{\mathrm{chain}} = \prod_{j=1}^{m} \Delta_j(1) h_j^D.$$

In fact, at $\xi = \xi_\kappa$ the preceding identities give

$$H_{\mathrm{chain}} \frac{Z_1}{Z_\xi} = W_\xi^{-D} \prod_{j=1}^{m} \Delta_j(\xi) \left(\frac{h_j}{\rho_j(\xi)}\right)^D.$$

The conjugated coefficients in these comparisons are evaluated at $\bar\kappa$ on the same ξ_κ^j, as in (RT4)–(RT5). Equations (RT1) and (RT3) give multiplier 1 for each elementary triangle inverse, and the conductor power is a unit at the chosen embedding. For $D = 0$ take $H_{\mathrm{chain}} = 1$. Before taking the D-th power, the height is at most $\binom{m}{2}tx_1$ by (RT3). Thus (102) gives

$$D\binom{m}{2}tx_1 + \sum_{j=1}^{m} D(L/m - kd_{\mathrm{dir}}) \le D(b + L - A), \qquad A = mk\, d_{\mathrm{dir}}.$$

Proposition 44.2 (Uniform height of the chain ratio)**.** *Fix d, ϵ. For sufficiently large x, choose the common twists $(\xi_\kappa)_\kappa$ as in Section 38. Then*

$$\mathrm{Height}\big((Z_1/Z_{\xi_\kappa})_\kappa\big) \le D(b + L - A). \tag{103}$$

All choices in this bound are on one fixed admissible datum. The numerical inequalities used above hold before the coefficient embedding and the testing twist are chosen.

44.3 Spanning vectors from the terminal period

In this subsection, we use Proposition 43.1 to bound the norms and denominators of the vectors used in Section 23.

Let W be the full principal-level source, with good subspace V, and T_ξ its marked chain. Let $\mathcal{H}$ be the algebraic space at the true outer node, of level $K(\pi_M^{b_0})$. Use the map $r_0 : \mathcal{H} \to W$, array Ψ_ξ, and scalar c_ξ of Section 43. The deep transpose and separation prove surjectivity of r_0, with the same period at all the false levels $K(\pi_M Q(\xi))$. The terminal dual delta identity below gives, for a fixed β_0,

$$v^{\mathsf{t}} J_{\mathrm{W}}\big((c_\xi/c_1) T_\xi r_0 \eta\big)(x) = \beta_0\, \Xi_\xi(\delta_x(v), \eta).$$

The terminal delta and η have the original false and true outer levels. By (TP11),

$$|c_\xi/c_1| = q_\xi^{(m^2t^2+m^2t+mr)/2} \prod_{0\le i,j<m} T_{ij}(\xi).$$

The conductor exponent here is integral, since m is even. Each T_{ij} consists of the t critical ratios at power $j+1$ on that square. For $i \neq j$ the reflected square (j, i) uses the same untwisted character at power $i+1$.

44.3.1 Bounds for the two terminal jets

In this subsection, we apply the jet estimates of Section 7.4 to both factors in the measured period.

Put $D_0 = M^{b_0}$, $\mathfrak{b} = (\pi_M)^{b_0}$, and $\mathfrak{m} = \mathfrak{b}Q(\xi)$. Use the original edge in either half, with source and target ranks n, n', respectively, and fix its residue matrix $S \in \mathrm{Mat}_{n'\times n}(\mathcal{O}_E/\mathfrak{m})$. The indicated point test is the characteristic function $\mathbf{1}_{\{S\}}$ on this matrix ring. On its original standard definite Hom space, the edge datum Φ_S is this test on the vacuum already scaled by $s_{\mathrm{vac}}^{nn'}$. Its jet scale is s_{jet}. The π_M-factor in the earlier forward-chain jet scale does not enter the last-path kernel.

On each half take the compatible Weyl diagrams at the path ranks. Above the initial rank the only permitted heights are the successive node ranks. The number at each positive new height is the gap parameter minus t, and minus one further on the true half. This is the column form of appending the lower parameter block at each edge: for a fixed height list the determinant exponent at the source is the weight of the upper character plus the multiplicity at the source rank.

The lower character gives the next determinant exponent. Thus at the edge there are no columns of height strictly between n and n', and the Schur jet entries are homogeneous of the same degree H, the number of boxes below height n' in that edge's diagram. The determinant

weights are included in the two endpoint gauges. Since the doubled parameters have absolute values at most $\lceil 20b \rceil$, the part sizes below full height are at most $\lceil 20b \rceil$.

Here $n_m = k + m(t+1) = r + mt + 1$ is the upper-chain rank of Section 7.4. The part size is the difference from the last lowest-weight entry, and subtracting the usual descending rank shifts can only decrease the parameter difference. Consequently

$$n, n' \le 2n_m, \qquad H \le f(2n_m)\lceil 20b \rceil \le b_0 r^4.$$

The last inequality follows from $f \le x^{12}$, $n_m \le x^6$, $r \ge x^5$ and $b_0 \ge b/2$ eventually.

Fix a prime $\ell \nmid q_\xi$ and an arbitrary coefficient embedding. The integer $q_\xi D_0$ belongs to $\mathfrak{m}$. The finite Fourier calculation of Section 44.4 therefore expresses $D_0^{2fnn'}\Phi_S$ as an integral combination at the chosen coefficient embedding of translates of the normalized vacuum. Each Fourier phase is a root of unity. Each translating vector a satisfies $D_0 a_w \in \mathcal{O}_{E,w}^{nn'}$ for every $w \mid \ell$, and the ℓ-part of the Fourier denominator divides $D_0^{2fnn'}$. The residue test is still taken modulo $\mathfrak{b}Q(\xi)$. For the additive group $B_\mathfrak{m} = \mathrm{Mat}_{n'\times n}(\mathcal{O}_E/\mathfrak{m})$, Fourier inversion gives

$$D_0^{2fnn'}\mathbf{1}_{\{S\}}(X) = \frac{D_0^{2fnn'}}{|B_\mathfrak{m}|}\sum_{\chi\in\widehat{B}_\mathfrak{m}} \chi(-S)\chi(X).$$

The prefactor is integral above ℓ. Each $\chi(X)$ on the vacuum is the modulation from the dual half of the Heisenberg polarization, up to a root of unity in the signed convention. This gives the claimed integral span also at old primes and at 2.

Suppose that both finite points are integral at all places above ℓ. There the pair action carries the Hom lattice onto itself and carries each Fourier translation a to a translation a' with $D_0 a'_w$ integral for every $w \mid \ell$. The vacuum expansion of Section 7.4.3 at the other places has lattice index prime to ℓ. The endpoint characters and the Weil multiplier are ℓ-adic units by Section 44.4. Thus the Fourier-cleared datum belongs to the same ℓ-integral span of translated vacua. At the scale $D_0 d\, s_{\mathrm{jet}}$, the ordinary integral jet calculation applies to each Schur matrix entry, since the Schur polynomials have integral coefficients and degree H.

At the actual scale s_{jet} homogeneity introduces $|D_0 d|_\ell^{-H}$. Multiplying the edge matrix by

$$D_0^{2fnn'}(D_0 d)^H$$

therefore sends the source unit lattice into the target unit lattice. For arbitrary two finite points move the source and target separately by rational unitary matrices into the open sets of points integral above ℓ. The Cayley chart gives both weak approximations. Rational equivariance of the jet changes its argument and its value by the same weight representations at the two endpoints.

These changes cancel exactly in the respective transported gauges. This proves the bound at every pair of points. In particular the common integer

$$D_{\mathrm{jet}} = D_0^{2f(2n_m)^2}(D_0 d)^{b_0 r^4}$$

works for every edge, matrix and allowed twist at that embedding. Iterating gives the factor D_{jet}^m on each half, on the original outer levels and in their transported Weyl gauges.

For the complex bounds take the trivial twist. Fourier inversion of the single indicator on the finite matrix ring of cardinality J has J coefficients of absolute value $1/J$ at every

complex embedding. Thus the total coefficient norm of the translated integral vacua is one. The pointwise conjugate vacuum bound for Hom, of rank nn', is

$$\exp(C_K(q_1nn'+1)^2).$$

Apply Lemma 7.6 at the Hom rank nn' and the Gaussian Schur calculation of Section 7.4.4 with $c=L=1$.

In particular the squared scale in the integral is Ω^{-1}. Write $\lambda_{\tau,i}\geq 0$ for the part sizes in the edge Schur polynomial. They sum to H. The conjugated Weyl operator norm is at most the exponential of

$$C_K(q_1nn'+1)^2+fnn'\log 4+\frac{1}{2}\sum_{\tau,i}(\lambda_{\tau,i}+\log(\lambda_{\tau,i}!)+\lambda_{\tau,i}\log(\Omega^{-1})).$$

In fact a matrix coefficient against a covector of dual Weyl norm one is bounded by the product of the singular values with the indicated covector exponents.

The factorial estimate with $L=1$ in the Gaussian integral leaves the factor $4^{fnn'}$. This proof therefore bounds the operator directly between the conjugated Weyl norms. Using $\mathrm{rd}(E)\leq q_1d$ and the explicit formula for Ω, we have $|\log\Omega|\ll_d 1+\log q_1$. The sum of the factorial logarithms is at most $H\log\max(1,H)$. With $q_1,f\leq x^{12}$, $n_m\leq x^6$, $r<3x^5$ and $\log b\leq x$, the exponent is $O_d(x^{48}+bx^{21})$, hence at most $x^{99}b^2$. This bound is uniform over the original edge points and holds at every complex embedding.

At the trivial twist each last-path level contains the principal congruence subgroup of rational level D_0. Apply the counting argument of Section 7.2 at each rank $n\leq 2n_m$, with D_0 as level. There $u=2fn\ll x^{18}$ and the maximal-level count has logarithm $O(u^3\log x)=O(x^{54}\log x)$. Refining to the path level contributes at most $2fn^2\log D_0=O(x^{25}b)$ in the logarithm. Thus each class count and each unscaled edge norm have the common bound $G=\exp(x^{99}b^2)$ used below.

44.3.2 Local transport and terminal branching

In this subsection, we use the bounds of Section 44.3.1 in the terminal comparison.

Let t_ξ generate the finite ideal of $\mathfrak{m}$ and write $u_\xi=\overline{t_\xi\alpha_{\mathrm{W}}}/(t_\xi\alpha_{\mathrm{W}})$. In the original terminal charts the false point is g and the true one is $\mathrm{diag}(g,u_\xi^m)$. Write $u'=u_\xi^{-m}$. If v',v at these points have unit gauge, their Weyl-coordinate transports w',w by g^{-1} and $\mathrm{diag}(g^{-1},u')$ have integral entries. Use the integral bilinear matrix B_{grid} constructed for the two terminal grid nodes in Section 43.5. Its equivariance for the common diagonal group and the extra line gives

$$(w')^{\mathsf{t}}B_{\mathrm{grid}}w=\left(\prod_\tau \iota_\tau(u')^{L_\tau}\right)\iota((v')^{\mathsf{t}}B_{\mathrm{grid}}v),$$

with the weight embeddings read on the corresponding finite components. The original pairing is bilinear in these two already-dual Weyl models, as in the displayed coordinate product.

The scalar u_ξ and its inverse become integral after multiplication by $q_\xi D_0q_1$. For u_ξ, write this product as

$$(q_\xi D_0/t_\xi)\,\overline{t_\xi(q_1\alpha_{\mathrm{W}})}\,\alpha_{\mathrm{W}}^{-1}.$$

Each factor is integral on the finite adeles. For the inverse conjugate this expression, since $\bar{u}_\xi = u_\xi^{-1}$. Above $\ell \nmid q_\xi$, either sign of L_τ is therefore cleared by $(D_0 q_1)^{m|L_\tau|}$. Because the left-hand side of the equivariance identity is integral, the original pairing is cleared by

$$M_0 = \prod_\tau (D_0 q_1)^{m|L_\tau|}.$$

Combining with the two half paths gives $N_{\mathrm{path}} = M_0 D_{\mathrm{jet}}^{2m}$ on every summand of the original finite expansion.

The true terminal diagram has at most $b_0 r^4$ boxes by Section 44.3.1. Its determinant exponent is the last highest-weight entry; the bound $\lceil 20b \rceil$ for the doubled parameters and $n_{\mathrm{diag}} + 1 \le 2n_m \le \lceil 20b \rceil$ give $|e| \le \lceil 20b \rceil$ after removing the rank shift. Hence the exponents of this pairing satisfy $|L_\tau| \le b_0 r^4 + \lceil 20b \rceil$. The rank and box bounds and $\log M = O(\log b)$ give $\log N_{\mathrm{path}} \le x^{99} b^2$. By (TP13), the original internal integrals are free class sums. Reindexing the combined array at the internal nodes cancels the adjacent kernel characters against the array character. At the linked terminal node the basis changes are u and $\mathrm{diag}(u, 1)$. Each array entry is a sum of roots of unity. Thus the edge and branching bounds give integrality of $N_{\mathrm{path}} \Xi_\xi(\delta_x(v), \eta)$ on integral outer data at all embeddings above $\ell \nmid q_\xi$.

At the trivial twist, put $G = \exp(x^{99} b^2)$ and $h_0 = b_0 r^4$. Use the bounds G for each class count and each point-indicator edge operator in the conjugate Weyl metrics, and $h_0^{3h_0}$ for the terminal branching pairing. In the residue ring of the grid modulus, put

$$N_b = |\mathcal{O}_E/(\pi_M)^{b_0}|, \qquad E_0 = N_b^{(2n_m)^2}, \qquad V_0 = N_b^{2n_m}.$$

There are at most E_0^{2m} edge arrays. The first path supplies at most E_0^{2m} root-of-unity terms per entry and each of the m^2 turns multiplies that bound by at most V_0.

Hence at most $V_0^{m^2} E_0^{4m} G^{2m+1}$ terms remain after summing over the $2m+1$ original class variables. Each term has absolute value at most

$$h_0^{3h_0} G^{2m} \|\sigma v\|_{\mathrm{W}} \sqrt{B_\eta},$$

by the two compositions of m edge operators, where B_η bounds the squared norm of every σ-conjugate value of η. The numerical bound

$$V_0^{m^2} E_0^{4m} h_0^{3h_0} G^{4m+1} \le \exp(x^{200} b^2)$$

now gives the corresponding bound for $|\sigma \Xi_1(\delta_x(v), \eta)|$. Indeed $f, q_1 \le x^{12}$, $n_m \le x^6$, $r \le 3x^5$, $m \le 2\sqrt{x}$ and $\log M \ll \log b \le x$ for large x. Thus $\log N_b \le x^{13} b$. With $h_0 = b_0 r^4$, each term in the logarithm of the count bound is $o(x^{200} b^2)$. The same estimates give $\log(M_0 D_{\mathrm{jet}}^{2m}) = o(x^{99} b^2)$ using the bound on $|L_v|$. This estimate concerns the same path period as the finite calculation.

44.3.3 The dual vector on the terminal lattice

In this subsection, we construct the dual vector for the terminal pairing of Section 44.3.2.

Write $K_\xi = K(\pi_M Q(\xi))$ on the original terminal definite group Y_m, and let K_{std} denote its maximal integral group. Put $I_\xi = [K_{\mathrm{std}} : K_\xi]$, and let d_Y be the dimension of the terminal Weyl module. Reduction at the marked split places is onto, with kernel K_ξ. Hence I_ξ is the order of the general linear group of the false terminal node over the residue ring of $\pi_M Q(\xi)$. In particular this is the terminal divisor appearing in the last-path period. The measures in this

calculation are the original quotient measures, with the compact real group and the integral finite group at the terminal space each of mass one.

Use the dual module of type obtained by reversing and negating the terminal parameter. Write $B(v,z)=v^{\mathsf{t}}J_{\mathrm{W}}z$ for its normalized invariant bilinear pairing with the terminal module. Let $\delta_x(v)$ be the dual form of level K_ξ whose value at x is v, extended by equivariance on the class of x and zero elsewhere. This level is principal at the old split place, so Lemma 6.1 gives trivial stabilizers and the equivariant extension has precisely one value at the representative. Use its conjugate-type real realization in the period.

For an irreducible compact representation of dimension d_Y, let w be the marked unit highest vector. The average of the rank-one projector on w under the unitary compact action is $d_Y^{-1}I$, by invariance and trace. Consequently the product of the two opposed highest coefficients at a finite point h integrates over the real compact group to

$$d_Y^{-1}B(\delta_x(v)(h),(T_\xi y)(h)).$$

The rational left actions cancel in this expression by invariance of B. Both forms are fixed by K_ξ, and the finite mass of the class at x is I_ξ^{-1} because its rational stabilizer is trivial. Thus

$$I_\xi\int_{[U(Y_m)]}\delta_x(v)(h)(T_\xi y)(h)\,dh=d_Y^{-1}B(v,(T_\xi y)(x)). \tag{104}$$

This formula pairs the algebraic form with its dual bilinearly. The dual real type has already absorbed the complex conjugation.

In Proposition 43.1, use $y'=\delta_x(v)$. Its right stabilizer contains the false outer deep subgroup $K(\pi_M^{b_0}Q(\xi))$. The true form η has level $K(\pi_M^{b_0})$. Combining (TP13) with (104) gives

$$B(v,((c_\xi/c_1)T_\xi r_0\eta)(x))=\beta_0\,\Xi_\xi(\delta_x(v),\eta),\qquad \beta_0=\frac{d_Y A_*}{c_1}. \tag{105}$$

The finite index has canceled and the compact dimension is retained in β_0. In particular $\beta_0\neq 0$ if $D>0$ and T_1 is injective on V. In that case take $y\in V\setminus\{0\}$, lift it by r_0, and choose a dual vector separating a nonzero value of T_1y.

44.3.4 Integral dual coordinates and pointwise bounds

In this subsection, we estimate the coordinates of the dual vector of Section 44.3.3.

In the two exterior tensor models, replace each coordinate wedge by the complementary coordinate wedge, with the sign from the unit volume wedge. This is an integral isometry. For an invertible basis matrix g, the complement identity on each exterior power moves g to $(g^{-1})^{\mathsf{t}}$ and supplies one factor of $\det g$ per column. The prescribed dual determinant exponents cancel those factors. In the ordering prescribed by the parameters, the signed complement takes the primitive highest tensor w to one of the two signs of the reversed highest tensor $w^\vee$. Fix its overall sign by requiring $w\mapsto w^\vee$. Both tensors are real in the specified integral bases and have norm one.

Applying this identity to invertible bases proves that the tensor isometry restricts to the Weyl spans, since the evaluation vectors of those bases span. The Weyl coordinates are integral on these tensors: the standard-tableau coefficient matrix has an integral left inverse, obtained from its unitriangular minor submatrix. The same holds for the inverse complement map.

Denote this unimodular isometry by J and the dual Weyl Gram matrix by $G_\vee$. The pairing has matrix $J_{\mathrm{W}} = G_\vee J$, so in the convention linear in the second argument

$$B(v,z) = \langle \overline{Jz}, v\rangle_{\mathrm{W}}, \qquad \|\overline{Jz}\|_{\mathrm{W}} = \|z\|_{\mathrm{W}}.$$

In particular $B(w^\vee, w) = 1$, as required in (104).

The Fock pairings of the integral dual minor basis form an integral matrix $F_\vee$ with $F_\vee^{\mathsf{t}} G_\vee = H_\vee I$, where $H_\vee > 0$ is the Fock square of the dual highest minor polynomial. To see the scalar, compare the Fock metric with the tensor-induced invariant metric on the irreducible polynomial module. The highest evaluation vector has tensor norm one. Pairing with each basis minor therefore gives $H_\vee$ times the corresponding Weyl coordinate. Thus

$$C_{\mathrm{W}} J_{\mathrm{W}} = H_\vee I, \qquad C_{\mathrm{W}} = J^{-1} F_\vee^{\mathsf{t}} \in M(\mathbb{Z}).$$

There are at most $h_0 = b_0 r^4$ boxes in the dual diagram. The determinant expansion bounds the coefficient ℓ^1-norm of its highest polynomial by $h_0^{h_0}$, and each monomial factorial by $h_0^{h_0}$. Consequently $H_\vee \le h_0^{3h_0}$.

If x is integral above ℓ and v is an integer dual coordinate vector, $\delta_x(v)$ has gauge at most one. Indeed x^{-1} is integral at those places, as it is the Hermitian adjoint of x, and the polynomial representation and determinant units preserve the unit lattice. Equivariance gives this gauge at every point of the class. At any coefficient automorphism σ, its pointwise Weyl norm is at most $\|\sigma v\|_{\mathrm{W}}$. A rational unitary matrix acts unitarily after taking every embedding into account, and the Weyl Gram matrices are rational.

Use the fixed integer $N_{\mathrm{path}} = M_0 D_{\mathrm{jet}}^{2m}$ of Section 44.3.2. For $\beta_0 \neq 0$, put $s_0 = H_\vee N_{\mathrm{path}}/\beta_0$, and write v_i^{t} for the i-th row of C_{W}. Since $v_i^{\mathsf{t}} J_{\mathrm{W}} = H_\vee e_i^{\mathsf{t}}$, (105) gives the coordinate identity

$$\big((c_\xi/c_1) T_\xi(s_0 r_0 \eta)(x)\big)_i = N_{\mathrm{path}}\, \Xi_\xi(\delta_x(v_i), \eta).$$

For integral outer data and x integral above ℓ, the right side is integral at every coefficient embedding above $\ell \nmid q_\xi$, including when $\ell \mid H_\vee$. If $\beta_0 = 0$, put $s_0 = 1$; nondegeneracy of J_{W} gives zero in every coordinate. In either case $s_0 \neq 0$ is independent of ξ and ℓ.

For arbitrary x, use the Cayley-chart weak approximation of Section 34.1.1 to choose $\gamma \in U(Y_m)(F_0)$ with $\gamma_\ell x_\ell \in K_{\mathrm{std},\ell}$, where the subscript ℓ denotes all local components above ℓ. Apply the integral estimate at γx. Writing $F = (c_\xi/c_1) T_\xi(s_0 r_0 \eta)$ and ρ_{W} for the terminal Weyl action in each coefficient reading above ℓ, rational equivariance gives

$$\rho_{\mathrm{W}}\big((\gamma_\ell x_\ell)^{-1}\big) F(\gamma x) = \rho_{\mathrm{W}}(x_\ell^{-1}) F(x).$$

Thus the transported coordinates are integral at the original point.

At the empty twist, if the path bound is

$$|\sigma \Xi_1(\delta_x(v), \eta)| \le e^{x^{200} b^2} \|\sigma v\|_{\mathrm{W}} \sqrt{B_\eta},$$

then the dual isometry and (105) give the squared norm bound

$$\big(H_\vee N_{\mathrm{path}} e^{x^{200} b^2}\big)^2 B_\eta \le e^{2x^{201} b^2} B_\eta.$$

The last bound uses $\log N_{\mathrm{path}} \le x^{99} b^2$ and the hook estimate just proved. Indeed, at the given point put $z = (T_1(s_0 r_0 \eta))(x)$. For fixed σ, choose $v = \sigma^{-1}(\overline{J(\sigma z)})$ coordinatewise. Since B has integral matrix, $\sigma B(v,z) = B(\sigma v, \sigma z) = \|\sigma z\|_{\mathrm{W}}^2$ and $\|\sigma v\|_{\mathrm{W}} = \|\sigma z\|_{\mathrm{W}}$. Cancel this norm if it is nonzero. If $\beta_0 = 0$, all of these terminal values vanish by the same nondegenerate pairing.

44.3.5 Integral generators at the outer node

In this subsection, we use the dual coordinates of Section 44.3.4 to bound the original generators.

Let $n_H \le 2n_m$ be the original outer rank. For each free true outer class g write $L(g) = E^{n_H} \cap g\widehat{\mathcal{O}}_E^{n_H}$. The short-basis calculation at the marked source supplies $Y(g) \in \mathrm{GL}_{n_H}(E)$ whose columns lie in $L(g)$ and whose entries have absolute value at most $R = 6(q_1 d)(f n_H)^2$ at every complex embedding. Here the columns form an E-basis, and may span a proper $\mathcal{O}_E$-sublattice of $L(g)$. Only invertibility over E and $g^{-1}Y(g) \in M_{n_H}(\widehat{\mathcal{O}}_E)$ are used. For each actual determinant exponent e_τ apply the same construction at g^{e_τ}.

At the representative g, multiply the polynomial Weyl action of $(\iota_\tau Y(g))_\tau$ on the evaluation vector by

$$\prod_\tau \det \iota_\tau Y(g^{e_\tau}).$$

After moving into the local gauge at g, the determinant at any coefficient embedding is evaluated on

$$\det(g)^{-e_\tau} \det Y(g^{e_\tau}) = \det\left((g^{e_\tau})^{-1} Y(g^{e_\tau})\right).$$

The identity also holds for negative e_τ. Its right-hand side and the polynomial Weyl action of $g^{-1}Y(g)$ are integral. For root-of-unity evaluation arrays all initial coordinates are integral at every coefficient embedding.

The rational left action and right level action show that this is the gauge at every class point. Writing such a point as $\gamma g k$, with k in the deep compact group and γ rational, equivariance cancels the weight factor of γ against transport by $(\gamma g k)^{-1}$. The remaining k^{-1} acts integrally, with inverse also integral. The free class is supported at a single class and thus incurs no stabilizer sum. A corresponding conjugate value has the same norm as at g, because the rational definite unitary action remains unitary in the simultaneous conjugated Weyl coordinates, and the class level has trivial character.

Use arrays with entries in a single group μ_L for $L > b_0 r^4$. The finite-grid argument of Lemma 7.5 applies since $b_0 r^4$ bounds the total polynomial degree across all places. The matrix of the polynomial action is invertible and the determinant factors are nonzero. Taking the finitely many classes gives a finite generating set $\mathcal{S}_H$ defined in algebraic coordinates over a number field before any twist or coefficient place is chosen. Use the exterior-power estimate in the proof of Lemma 7.5 at rank n_H. Multiplying $Y(g)$ by the square block of the evaluation array gives entries of modulus at most $n_H R$, hence squared Frobenius norm at most $n_H^4 R^2 \le R^6$. The polynomial action on the unit highest vector contributes at most $R^{6b_0 r^4}$ to the squared norm, since its degree is at most $b_0 r^4$. By Hadamard the f determinant factors together contribute at most $(\sqrt{n_H} R)^{2 f n_H} \le R^{6 f n_H}$. Thus the squared norm is at most $R^{6(f n_H + b_0 r^4)}$. This is at most $\exp(x^{99} b^2)$ by the original scale bounds. Transport to the analytic form space is by the same coefficient-type isomorphism at the actual deep level.

As r_0 is onto and $s_0 \neq 0$, the vectors $s_0 r_0 \eta$ so obtained span the original full source.

44.3.6 Projection to the good space

In this subsection, we project the spanning set of Section 44.3.5 to the good space, retaining the bounds needed in Section 45.

Use the integral spanning set of $\mathcal{H}$ just constructed. Its gauges are integral and its squared values are at most $e^{x^{99} b^2}$ at every complex embedding. The vectors $y = s_0 r_0 \eta$, $\eta \in \mathcal{S}_H$, span W.

Section 44.3.4 gives unit gauge for $\gamma_\xi T_\xi y$, where $\gamma_\xi = c_\xi/c_1$, above every $\ell \nmid q_\xi$ and pointwise squared norm at most $e^{x^{202}b^2}$ for every conjugate of $T_1 y$. Set

$$\mathcal{Z} = \{\, |\mathcal{G}| p\, s_0 r_0 \eta : \eta \in \mathcal{S}_H \,\} \subset V,$$

where $\mathcal{G} = G_0 \times G_1$ is the original finite group and $p = \rho(e_0)$ projects its principal-level space W onto V, as in Section 2.4.

This map is onto V and is a sum of integral right translations with coefficients algebraic integral at every coefficient embedding. Indeed its coefficient at a group element is the sum of dimension times the character value of the retained simple types. Its conjugate absolute value is at most $|\mathcal{G}|$, by the bound of character values by dimension and the sum-of-squares identity. By Lemma 7.3 with the identity translate, the lifted projection is the same sum of terminal right translates by integral first-block lifts. Each preserves the unit gauge at every place above ℓ, so the same γ_ξ clears every projected form at these places. For $\xi = \xi_\kappa$, apply this result at ι and at $\iota \circ \mathrm{conj}$ to the same form $\gamma_\xi \Theta_\xi z$, for each $z \in \mathcal{Z}$. The second application supplies the conjugated coordinates in the Gram pairing at κ. The class count at the original terminal level from Section 7.2 and $|\mathcal{G}|^4$ multiply the squared pointwise bound by at most $e^{x^{99}}$. The total conjugate squared norm of each projected form is therefore at most $e^{2x_1^2 b^2}$, by $x^{99} + x^{202}b^2 \le 2x_1^2 b^2$.

For $\xi = \xi_\kappa$ and $p_{ij,k} = (\mu^0_{i+1}\mu^0_{j+1}, k)$, put

$$U_{ij,k} = \frac{F_{p_{ij,k},Q(\xi)}(\xi^{j+1})}{F_{p_{ij,k},\varnothing}(1)}.$$

Section 38.3 identifies the square of the first factor of $T_{ij}(\xi)$ with $q_\xi^{-1} U_{ij,1}$. For $2 \le k \le t$ the factor itself is $U_{ij,k}$. Consequently (TP11) gives

$$\gamma_\xi \bar\gamma_\xi = q_\xi^{m^2t^2 + m^2 t + mr - m^2} \prod_{0 \le i,j < m} \left(U_{ij,1} \prod_{k=2}^{t} U_{ij,k}^2 \right) \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}.$$

Its valuations at $\bar\kappa$ and κ agree on this same twist. Since $\ell \nmid q_\xi$, (RT1) gives

$$v_\kappa\big((\gamma_\xi \bar\gamma_\xi)^{-1}\big) = -2 \sum_{0 \le i,j < m} v_\kappa T_{ij}(\xi) \ge -2m^2 t e_\ell.$$

By (RT3), with height multiplier 1,

$$\mathrm{Height}\big((\gamma_{\xi_\kappa} \bar\gamma_{\xi_\kappa})^{-1}\big) \le 2m^2 t x_1 < b.$$

Indeed $m^2 \le 4x$, $t = 5r < 15x^5$, and for fixed $\epsilon > 0$ and sufficiently large x,

$$\log(2m^2 t x_1) \le \log 120 + 6 \log x + x^{\epsilon/6} < x^\epsilon < \log b.$$

Proposition 44.3 (Spanning vectors for the lattice completion)**.** *Choose $(\xi_\kappa)_\kappa$ as in Section 38. There is a set $\mathcal{Z} \subset V$ with*

$$\operatorname{span} \mathcal{Z} = V, \qquad \|\sigma \Theta_1 z\|^2 \le e^{2x_1^2 b^2} \quad (z \in \mathcal{Z},\ \sigma \in \mathrm{Aut}(\mathbb{C})).$$

For each κ, the scalar $\gamma_{\xi_\kappa} = c_{\xi_\kappa}/c_1$ is nonzero and $\gamma_{\xi_\kappa} \Theta_{\xi_\kappa} z$ is integral at both κ and $\bar\kappa$ for all $z \in \mathcal{Z}$. Moreover,

$$\mathrm{Height}\big(((\gamma_{\xi_\kappa} \bar\gamma_{\xi_\kappa})^{-1})_\kappa\big) \le b.$$

44.4 Integral derivatives in the two norm formulas

In this subsection, we prove the integrality used by the two norm constructions in Sections 30.1 and 30.2. Fourier inversion first reduces the point indicators to translated vacuum vectors. At coefficient primes outside the auxiliary support, the same denominator clears each expansion.

Fix a coefficient prime $\ell \nmid q_Q$ and any embedding above it. For a marked finite modulus $\mathfrak{m}$ with $q_Q D \in \mathfrak{m}$, Fourier inversion on the finite matrix coordinates expresses its integral indicator datum as a combination of translates, with roots of unity in the numerator. The Fourier denominator divides

$$D^{2fn'n}$$

in the chosen valuation, since $\mathcal{O}_E/\mathfrak{m}$ at ℓ is a quotient of $\mathcal{O}_E/(D)$ at ℓ and $[E : \mathbb{Q}] = 2f$. The Fourier dual translations are integral after multiplication by D above ℓ, by the self-dual translation pairing. Thus multiplying by $D^{2fn'n}$ gives the standard datum of level $\delta = D$, once the vacuum has been normalized by $s_{\mathrm{vac}}^{n'n}$. This argument uses coprimality with q_Q alone. For a finite unitary change integral above ℓ, both it and its inverse preserve the local lattice there. In the vacuum sum of Section 7.4.3 the index is therefore prime to ℓ. For the norm-one determinant u, choose $h > 0$ with $u^h = ak$ for rational a and an integral norm-one idele k, as in that section. Every conjugate of a is a unit above ℓ. For each endpoint character α of integral infinity weight,

$$\alpha_{\mathrm{f}}(u)^h = \alpha_{\mathrm{f}}(k)\alpha_\infty(a)^{-1}.$$

Here $\alpha_{\mathrm{f}}(k)$ is a root of unity, and $\alpha_\infty(a)$ is a monomial in conjugates of a. Both are ℓ-adic units. Applying the same argument to the splitting character and using the square identity for the Weil multiplier in Section 7.4.3 proves that the latter is a unit as well. The Heisenberg phases are roots of unity, also above two. Thus this change preserves the integral span of translates with Dx integral above ℓ.

For the rank-one source and its rank-a_0 column target take $D = M$ and $(n', n) = (a_0, 1)$. The column polynomial has integral coefficients and degree $u = \sum_v (c_v - z)$. Using the ordinary jet bound at the integral scale $Md_K s_{\mathrm{jet}}$ and homogeneity at the actual scale $\pi_M s_{\mathrm{jet}}$ shows that the Fourier-cleared column kernel has gauge at most $|\pi_M/(Md_K)|_\ell^u$ at its actual target point. Here the column kernel already includes the vacuum scalar $s_{\mathrm{vac}}^{a_0}$. Thus

$$M^{2fa_0}(Md_K)^u K_Q(g, h)$$

has gauge at most one in the Weyl coordinates transported by h. The source vector is the constant 1 in the trivial source weight. Here $|\pi_M|_\ell \le 1$ at every weight embedding since $\pi_M \in \mathcal{O}_K$. For arbitrary g, h, apply weak approximation to put each point into the integral subgroup above ℓ by rational left multiplication. Rational equivariance cancels both changes in the transported source and target gauges, as in Section 44.3.1. In particular the bound applies to both coefficient embeddings in the Gram pairing and to the old level primes and 2.

For the rank-two determinant point test take $D = M^2$, $(n', n) = (2, 2)$, and $Q(Z) = \prod_v \det Z_v^{k_v}$, where $k_v = c_v - 1$. This polynomial has integer coefficients and degree $l = 2\sum_v k_v$; the datum is the indicator of the identity residue matrix. The point kernel uses the unscaled vacuum and the unscaled homogeneous jet. The normalized kernel is exactly the jet of the scalar-cleared datum at the scale $c = M^2 d_K s_{\mathrm{jet}}$. By homogeneity,

$$s_p K_{Q,1}^{\mathrm{pt}}(h) = Q(\partial)\Theta_{F_h}(cz)\big|_{z=0}, \qquad s_p = (M^2)^{8f} s_{\mathrm{vac}}^4 c^l.$$

Here F_h is the paired Weil translate at $(1,h)$ of the identity-residue datum multiplied by $(M^2)^{8f}s_{\mathrm{vac}}^4$. By weak approximation choose $\gamma \in U(I_2)(F_0)$ such that γh is integral at all places above ℓ. Under this rational move the determinant character contributes $A_\gamma = \prod_v \Phi_v(\det \gamma)^{k_v}$; substitution of $\Phi_v(\gamma^{-1})^{\mathrm{t}} Z_v$ in Q contributes A_γ^{-1}. Hence $K_{Q,1}^{\mathrm{pt}}(h) = K_{Q,1}^{\mathrm{pt}}(\gamma h)$. The datum $F_{\gamma h}$ is standard of level $\delta = M^2$. Applying (I2) with both ranks two gives $s_p K_{Q,1}^{\mathrm{pt}}(h) \in \mathcal{O}$. This scalar is the same for all point representatives and all moduli Q. The statement applies before the squared-modulus coefficient system is formed and supplies integrality at both embeddings.

45 Proof of the determinant estimate

In this section, we combine the determinant-ratio height of Proposition 44.2 with the spanning vectors of Proposition 44.3. At each coefficient place, the same auxiliary twist works in both factors of the terminal Hermitian pairing. This permits the use of the product formula in Section 23 and proves Proposition 5.2.

Proof of Proposition 5.2. The proof applies to the original datum of Section 16.1 at every sufficiently large integer x for fixed d, ϵ. Use the weighted chain $T_\xi = L_\xi$ of Section 17.3 and restrict it to V. The injectivity argument in Section 44.1 uses the first-arrow return and (GC1) before division by the real square factor. The single lifts are injective by Section 41.1.6 and Proposition 39.1.

On this datum $c_{\mathrm{meas}} \neq 0$. Indeed, at $C(S)$ the first moment of Proposition 9.2, together with the codimension bound of Section 6, gives

$$\|p(v_G\nu)^{-1/2}\Pi C^{-1} P_{C(S)}\|_{\mathrm{HS}}^2 \geq (1 - 5b^{-4} - b^{-8}) d_X > 0.$$

Choose $\phi \in S$ with $p\Pi\phi \neq 0$. If $c_{\mathrm{meas}} = 0$, (72) would give $L_1\Pi\phi = 0$, then $L_1 p\Pi\phi = 0$ by Lemma 17.2. This contradicts injectivity on V.

For each κ use the twist ξ_κ of Section 38. Its modulus is prime to the rational prime under κ. It supplies the same signed critical ratios and main Selmer tests for all the original lift indices. The compression in Section 42.1 therefore uses the regular twist and its powers on the original good subspaces. The finite spherical trace identity (4) identifies the corresponding Rallis determinants. Equation (103) gives $j_Z \leq D(b + L - A)$ on this family.

Proposition 44.3 supplies a spanning set with

$$b_{\mathrm{short}} = 2x_1^2 b^2, \qquad j_\gamma \leq b.$$

At each κ, the one scalar γ_{ξ_κ} clears these forms at both κ and $\bar{\kappa}$. The height multipliers are independent of the coefficient place.

Fix an independent tuple $y = (y_1, \ldots, y_n)$ as in the proposition. The images of $\mathcal{Z}$ span $V/\langle y_1, \ldots, y_n\rangle$. Choose $z_1, \ldots, z_s \in \mathcal{Z}$ whose images form a basis of that quotient. Then (y, z) is one basis of V for all coefficient places, and $s = D - n \leq 4D/b^2$. The matrices $U_\kappa, U_{\bar{\kappa}}$ supplied by $\mathrm{Ind}(y)$ both clear the lifted columns at the same ξ_κ, since its modulus is prime to ℓ. In the application to translated images, Proposition 22.1 supplies these matrices for the fixed tuple, simultaneously for every such twist. The critical-value and Selmer estimates used in the determinant ratio are those selected at κ. The free terminal classes give integral pairings at κ,

as proved in Section 23. Proposition 23.1 gives

$$\begin{aligned} 0 &\le c + 2\delta + s(2x_1^2 b^2 + b) + D(b + L - A) \\ &\le c + 2\delta + D(2b + L - A) \\ &\le c + 2\delta + D(3L - A), \end{aligned}$$

Here $L = faB^4 \ge b$, and

$$s(2x_1^2 b^2 + b) \le D(8x_1^2 + 4/b) \le Db$$

by $s \le 4D/b^2$ and $x_1^{10^6} \le b$ for sufficiently large x. Rearranging gives $D(A - 3L) \le c + 2\delta$, with the cutoff depending only on the fixed d, ϵ. □

Part VI
The analytic conclusion

46 The analytic criterion

In this section, we give an analytic criterion in terms of a one-sided prime sum. We choose the auxiliary quadratic twist so that the resulting Dirichlet L-functions have no zeros in a prescribed rectangle. Section 47 derives the estimate used in the Mellin transform argument from Proposition 5.3.

46.1 The one-sided Mellin criterion

In this subsection, we use the prime sums of Section 2.1 to rule out zeros by positivity.

Fix $8 \mid q_0$, a reduced class $d_0 \equiv 7 \bmod 8$, and $1/2 < \beta < 1$. The required bound is $R_{d,q_0}(x) \le x^{\beta-1}$ for all sufficiently large real x, for each prime $d \equiv d_0 \bmod q_0$. The threshold may depend on d. Here, with $\varphi = \varphi(q_0)$,

$$\lambda(n) = \mathbf{1}_{n \equiv 1 \bmod q_0} \Lambda(n)(1 - (n/d)), \qquad R_{d,q_0}(x) = \sum_{n \le x} \lambda(n)(n^{-1} - x^{-1}) - \frac{\log x}{\varphi} - c.$$

For a character χ, let F_χ be $L(s, \chi)$ with its pole removed by multiplication by $s - 1$ when χ is principal.

Dirichlet nonvanishing gives $F_\chi(1) \ne 0$ for nonprincipal χ [52, Theorem 4.9]. In the principal case, $F_\chi(1) = \prod_{p \mid \mathrm{mod}(\chi)} (1 - p^{-1})$. Set

$$D(s) = \sum_{\chi \bmod q_0} \left(\frac{F'_{\chi(\cdot/d)}(s)}{F_{\chi(\cdot/d)}(s)} - \frac{F'_\chi(s)}{F_\chi(s)} \right), \qquad c = \frac{D(1) - 1}{\varphi}.$$

The same definitions with modulus $q = 1$ give $R_{d,1}$ and $c_{d,1}$. The character set is closed under conjugation, so $D(1)$ is real. No twisted character in this sum is principal.

Lemma 46.1 (One-sided remainder criterion)**.** *Fix $1/2 < \beta < 1$, $H_0 > 0$, and a prime d in the prescribed class. Suppose the twisted factors have no zeros in $\Re s \ge \beta$, $|\Im s| \le H_0$ and $R_{d,q_0}(x) \le x^{\beta-1}$ for all sufficiently large real x. Then the untwisted factors F_χ have no zeros at*

$$\Re s > \beta, \qquad |\Im s| \le H_0.$$

Proof. For $N = \lfloor x \rfloor \geq 2$, the identity $\sum_{n \leq N} \Lambda(n) \lfloor N/n \rfloor = \log(N!)$ and $\lfloor N/n \rfloor \geq N/(2n)$ give

$$0 \leq \sum_{n \leq x} \frac{\lambda(n)}{n} \leq 2 \sum_{n \leq N} \frac{\Lambda(n)}{n} \leq \frac{4 \log(N!)}{N}.$$

Thus $R_{d,q_0}(x) = O_{d,q_0}(1 + \log x)$. For $\Re s > 1$, the Mellin integral is absolutely convergent. Orthogonality, the Euler products and termwise integration give

$$\sum_n \lambda(n) n^{-s} = \frac{D(s) + (s-1)^{-1}}{\varphi}, \qquad \int_n^\infty (n^{-1} - x^{-1}) x^{-s}\, dx = \frac{n^{-s}}{s(s-1)}.$$

The transforms of $\log x$ and 1 are $(s-1)^{-2}$ and $(s-1)^{-1}$, respectively. Therefore

$$\mathcal{M}_R(s) = \int_1^\infty R_{d,q_0}(x) x^{-s}\, dx = \frac{D(s) - 1 - s(D(1) - 1)}{\varphi s(s-1)} \qquad (\Re s > 1). \tag{106}$$

The singularity at $s = 1$ is removable. If the twists have no zeros in $\Re s \geq \beta$, $|\Im s| \leq H_0$, then for $s \neq 1$ in this region the residue is

$$\operatorname{Res}_{w=s} \mathcal{M}_R(w) = \frac{\sum_\chi \operatorname{ord}_s F_\chi}{\varphi s(1-s)}.$$

The rectangle includes the real interval $(\beta, 1)$, so each twisted logarithmic derivative is holomorphic there, including those of the real characters. At a real zero s of an untwisted factor in this interval the residue is strictly positive, since $s(1-s) > 0$ and $\sum_\chi \operatorname{ord}_s F_\chi > 0$. For $s \geq 1$ it vanishes by holomorphy at 1 and nonvanishing of the untwisted factors on $\Re s > 1$.

For $x \geq X_0$ sufficiently large put $h(x) = x^{\beta-1} - R_{d,q_0}(x) \geq 0$. For $\Re s > 1$, its Mellin transform is

$$\int_{X_0}^\infty h(x) x^{-s}\, dx = \frac{1}{s - \beta} - \mathcal{M}_R(s) - \int_1^{X_0} (x^{\beta-1} - R_{d,q_0}(x)) x^{-s}\, dx. \tag{107}$$

The last integral is entire. If the real abscissa of convergence were $a > \beta$, the transform would have a singularity at a. Indeed, if it were holomorphic there, a Taylor expansion about $a + \delta$ would reach $a - \delta/2$ for sufficiently small $\delta > 0$. The Taylor coefficients with the argument written as $a + \delta - s$ are the nonnegative moments $\int_{X_0}^\infty h(x) x^{-a-\delta} (\log x)^n\, dx/n!$. Tonelli would then give convergence at $a - \delta/2$. The only allowed singularity at $a > \beta$ is a pole of negative residue, which would make the nonnegative Mellin integral tend to $-\infty$ from the right. Hence the transform converges on $\Re s > \beta$. If an untwisted factor has a zero ρ with $\Re \rho > \beta$ and $|\Im \rho| \leq H_0$, the residue formula contradicts holomorphy there. □

46.2 Choosing a twist with no zeros in the rectangle

In this subsection, we choose the prime needed for the Mellin criterion of Section 46.1.

Proposition 46.2 (A quadratic twist avoiding a fixed box)**.** *Fix $1/2 < \beta < 1$ and $H_0 > 0$. There is a prime $d \equiv d_0 \bmod q_0$ such that*

$$L(s, \chi(\cdot/d)) \neq 0 \quad (\chi \bmod q_0,\ \Re s \geq \beta,\ |\Im s| \leq H_0).$$

Proof. For simultaneous twist selection set

$$\mathcal{P} = \{d \text{ prime} : d \equiv d_0 \pmod{q_0}\}.$$

Fix a character $\xi \bmod q_0$, and let ξ^* be its primitive inducing character. Since d_0 is a reduced class, $d \nmid q_0$ for $d \in \mathcal{P}$. The two primitive characters in $\chi = \xi^*(\cdot/d)$ have coprime conductors, so their product has conductor $Q = \mathrm{cond}(\xi^*)d > 1$. The Euler factors removed from $L(s,\chi)$ in passing to the twist modulo $q_0 d$ are nonzero on $\Re s > 0$. For $d \le X$ put $T_{\mathrm{an}} = (q_0 X)^{1/2} \ge \sqrt{Q}$, and let $j \in \{0,1\}$ be the parity of χ. For $a = \chi, \mu\chi$ set

$$\theta_a(y) = y^{1/2} \sum_{n \ge 1} a(n)(ny)^j e^{-\pi(ny)^2}, \qquad J_d = \int_{1/T_{\mathrm{an}}}^{\infty} \big(|\theta_\chi(y)| + |\theta_{\mu\chi}(y)|\big) \frac{dy}{y}.$$

Here $J_d = J_{d,\xi}(T_{\mathrm{an}})$ depends on both the character and the cutoff. Put $\Gamma_j(s) = \pi^{-(s+j)/2}\Gamma((s+j)/2)/2$. For $\Re s > 1$ the Gaussian Mellin integral gives

$$\Gamma_j(s) L(s,\chi) = \int_0^{\infty} \theta_\chi(y) y^{s-1/2} \frac{dy}{y}.$$

By the theta functional equation [52, Theorem 10.6, with $z = Qy^2$], $\theta_\chi(y) = \varepsilon_\chi \theta_{\bar\chi}(1/(Qy))$, where $|\varepsilon_\chi| = 1$. This continues the integral to the whole strip. At $\Re s = 1/2$, substitute $y \mapsto 1/(Qy)$ in the integral below $Q^{-1/2}$. Since $|\theta_{\bar\chi}(y)| = |\theta_\chi(y)|$ and $1/T_{\mathrm{an}} \le Q^{-1/2}$, we obtain $|\Gamma_j(s)L(s,\chi)| \le 2J_d$. Let $M(s)$ be the Mellin integral of $\theta_{\mu\chi}(y) y^{s-1/2}$ from $1/T_{\mathrm{an}}$ to ∞ with measure dy/y. Then $K(s) = \Gamma_j(s)L(s,\chi)M(s) - \Gamma_j(s)^2$ satisfies $|K| \ll 1 + J_d^2$ on $\Re s = 1/2$. On $\Re s = 2$ the full reciprocal integral equals $\Gamma_j(s)/L(s,\chi)$. Uniformly on this line the removed tail is bounded by

$$\int_0^{1/T_{\mathrm{an}}} y^{3/2} O(y^{-1/2}) \frac{dy}{y} = O(T_{\mathrm{an}}^{-1}).$$

The Euler integral for Γ_j and absolute convergence of the Dirichlet series bound $\Gamma_j(s)L(s,\chi)$ uniformly.

Thus $|K| \ll T_{\mathrm{an}}^{-1}$ there. Theta reflection gives exponential decay at 0 as well as at ∞. Thus the absolute Mellin integral bounds $\Gamma_j L$ on the closed strip for fixed χ. The truncated integral and the gamma factor are bounded there by their absolute integrals. The two boundary estimates above are uniform in χ, T_{an}. Suppose $J_d \le T_{\mathrm{an}}^{\kappa}$ with $\kappa = (\beta - 1/2)/8$.

For $\lambda = (\Re s - 1/2)/(3/2)$, the three-lines theorem gives

$$|K(s)| \ll T_{\mathrm{an}}^{2\kappa(1-\lambda)-\lambda} \le T_{\mathrm{an}}^{-5(\beta-1/2)/12} \quad (\beta \le \Re s \le 1).$$

In particular $K = o(1)$ for the fixed range $\beta \le \Re s \le 1$, $|\Im s| \le H_0$. On a zero in this box $K = -\Gamma_j(s)^2$, whose absolute value has a positive lower bound depending only on β, H_0 for the two parities. Thus, for sufficiently large T_{an}, a zero forces $J_{d,\xi}(T_{\mathrm{an}}) > T_{\mathrm{an}}^{\kappa}$, uniformly in $Q \le T_{\mathrm{an}}^2$. This includes a possible real zero. The Euler product handles $\Re s > 1$.

For $d \equiv 7 \bmod 8$, quadratic reciprocity gives $(n/d) = (-d/n_{\mathrm{odd}})$, where n_{odd} is the odd part of n. The parity j depends only on ξ in the prescribed class. For each of $b(n) = \xi^*(n)$, $\mu(n)\xi^*(n)$ put

$$\Theta_b(h,y) = y^{1/2} \sum_{n \ge 1} b(n) \Big(\frac{-h}{n_{\mathrm{odd}}}\Big) (ny)^j e^{-\pi(ny)^2}.$$

At $h = d \in \mathcal{P}$ the two sums are respectively $\theta_\chi(y)$ and $\theta_{\mu\chi}(y)$. For either sum,

$$\sum_{\substack{d \le X \\ d \in \mathcal{P}}} |\Theta_b(d, y)|^2 \le \sum_{1 \le h \le X} |\Theta_b(h, y)|^2.$$

For $n \le A,\ m \le B$, there are $O(\sqrt{AB}\log(2\min(A, B)))$ pairs for which $(nm)_{\mathrm{odd}}$ is a square.

Indeed write $n = 2^a k h_1^2,\ m = 2^b k h_2^2$ with k odd squarefree and sum the upper bound $\sqrt{AB}/(k2^{(a+b)/2})$. These pairs contribute at most X each before applying the weights. For every other pair $u = (nm)_{\mathrm{odd}}$ has a prime factor of odd multiplicity, so $h \mapsto (h/u)$ is a nonprincipal character modulo u. The constant sign $(-1/u)$ has modulus one. Pólya–Vinogradov [52, Theorem 9.18] therefore bounds the sum over $h \le X$ by $O(\sqrt{u}\log(2u))$, also for an imprimitive character. Summing over dyadic Gaussian ranges in ny, my gives

$$\sum_{1 \le h \le X} |\Theta_b(h, y)|^2 \ll_\eta X\log(2/y) + y^{-2-2\eta} \qquad (1/T_{\mathrm{an}} \le y \le 1).$$

In this sum the factor y from the squared normalization reduces the diagonal count at $A, B \asymp y^{-1}$ to $O(\log(2/y))$.

The off-diagonal bound before that factor is $O(y^{-3-2\eta})$. The polynomial parity factor is absorbed by reducing the Gaussian exponent. For $y > 1$ the same expansion decays exponentially. Cauchy–Schwarz gives

$$\sum_{\substack{d \le X \\ d \in \mathcal{P}}} J_d \ll_\eta X\left(1 + \int_{1/T_{\mathrm{an}}}^1 \sqrt{\log(2/y)}\,\frac{dy}{y}\right) + \sqrt{X}\int_{1/T_{\mathrm{an}}}^1 y^{-2-\eta}\,dy \ll_{q_0,\eta} T_{\mathrm{an}}^{2+\eta}. \tag{108}$$

Let $\mathcal{B} \subseteq \mathcal{P}$ consist of the primes for which at least one prescribed twist has a zero in the fixed box. For each sufficiently large X, use the same $T_{\mathrm{an}} = \sqrt{q_0 X}$ in (108) for every ξ. With $\eta = \kappa/2$,

$$\#\{d \le X : d \in \mathcal{B}\} \le T_{\mathrm{an}}^{-\kappa} \sum_{\xi \bmod q_0} \sum_{\substack{d \le X \\ d \in \mathcal{P}}} J_{d,\xi}(T_{\mathrm{an}}) \ll_{q_0,\beta,H_0} X^{1-\kappa/4}.$$

Here the outer sum runs over characters. Partial summation gives $\sum_{d \in \mathcal{B}} d^{-1} < \infty$. The reciprocal sum of primes in the prescribed reduced class diverges [52, Corollary 4.12(c)], so some prime from the class avoids all the bad zeros. □

With $q_0 = 18480$ and $d_0 = 9239$, the principal character modulo q_0 suffices for classical RH. Its Euler factors relative to $\zeta(s)$ are nonzero for $\Re s > 0$. The functional equation gives the other half of the critical strip. The bound on R_{d,q_0} is the needed conclusion of the common-datum comparison in Section 5.

47 The bound for the prime remainder

In this section, we apply the cancellation estimate of Proposition 5.3 to the prime sum of Section 46. The error from the auxiliary scale x_1 is estimated with the fixed-modulus prime number theorem. We then pass from integer to real arguments before applying the criterion.

Proposition 47.1 (The one-sided prime-remainder bound)**.** *Set $q_0 = 18480$ and $d_0 = 9239$. For each prime $d \equiv d_0 \pmod{q_0}$ and $0 < \epsilon < 1/100$, there is a constant $C_{q_0,d,\epsilon}$ with*

$$R_{d,q_0}(y) \le C_{q_0,d,\epsilon} y^{-1/2+6\epsilon}$$

for all sufficiently large real y.

Proof. Write $B = \log b$, $u = \#\Phi_a$, $v = |T| > 0$, and $\mathcal{R}$ for the total remainder defined in Section 5. The exact expansion is

$$\frac{\mathcal{R}}{av} = R_{d,q_0}(x) - R_{d,q_0}(x_1) + \frac{u}{v} R_{d,1}(x_1).$$

The datum has $u/v \le B$ and $f = \varphi(q_0)u$. Thus $m\mathcal{R} \le faB^5$ gives

$$R_{d,q_0}(x) \le \frac{\varphi(q_0)B^6}{m} + |R_{d,q_0}(x_1)| + B|R_{d,1}(x_1)|.$$

For each $q \in \{q_0, 1\}$, apply the prime number theorem in arithmetic progressions [52, Corollary 11.19] modulo qd. The reduced classes above $1 \bmod q$ run through $(\mathbb{Z}/d)^\times$, on which $\sum_a \chi_d(a) = 0$. Powers of d contribute $O_d(\log t)$. Hence for every $A > 0$,

$$S_q(t) := \sum_{n \le t} \lambda_{d,q}(n) = \frac{t}{\varphi(q)} + O_{q,d,A}\big(t(\log t)^{-A-1}\big).$$

Termwise integration gives $\sum_{n \le y} \lambda_{d,q}(n)(1/n - 1/y) = \int_1^y S_q(t) t^{-2}\, dt$. Consequently, for some constant C_q,

$$R_{d,q}(y) = C_q - c_{d,q} + O_{q,d,A}\big((\log y)^{-A}\big).$$

By boundedness and dominated convergence,

$$\lim_{s \downarrow 1} (s-1) \int_1^\infty R_{d,q}(y) y^{-s}\, dy = \lim_{s \downarrow 1} \int_0^\infty R_{d,q}\big(e^{u/(s-1)}\big)\, e^{-u}\, du = C_q - c_{d,q}.$$

Formula (106), with D_q in place of D, is regular at 1, so $C_q = c_{d,q}$ for both moduli.

Choose $0 < \epsilon < 1/100$ and use a datum at each large integer x, with $B \le 2x^\epsilon$, $m \ge \sqrt{x}$, and $\log x_1 \ge x^{\epsilon/12}$. Take $A > 24/\epsilon$ in the fixed-modulus bound. Both remainders at x_1 are at most x^{-2} in absolute value for sufficiently large x. Consequently

$$R_{d,q_0}(x) \le (64\varphi(q_0) + 3) x^{-1/2+6\epsilon}.$$

For real $y \ge 2$, put $n = \lfloor y \rfloor$. The defining finite sum has no new term between n and y, so

$$R_{d,q_0}(y) - R_{d,q_0}(n) = S_{q_0}(n)\frac{y-n}{ny} - \frac{\log(y/n)}{\varphi(q_0)} \le \frac{S_{q_0}(n)}{ny} \ll_{q_0,d} y^{-1}.$$

For $t = -1/2 + 6\epsilon \in (-1, 0)$ we have $n^t \le 2^{-t} y^t$ and $y^{-1} \le y^t$. The integer bound therefore gives $R_{d,q_0}(y) \le C_{q_0,d,\epsilon} y^t$. □

Proof of Theorem 1.1. Suppose $\zeta(\rho) = 0$ with ρ nontrivial and $\Re\rho > 1/2$. Choose

$$\tfrac{1}{2} < \beta < \min(1, \Re\rho), \qquad H_0 > |\Im\rho|.$$

Use Proposition 46.2 to fix a prime d whose twists have no zeros in the rectangle. Choose $0 < \epsilon < 1/100$ with $6\epsilon < \beta - 1/2$. Proposition 47.1 gives, for all sufficiently large real x,

$$R_{d,q_0}(x) \le C_{q_0,d,\epsilon} x^{-1/2+6\epsilon} \le x^{\beta-1}.$$

Lemma 46.1 excludes ρ. The functional equation gives the same exclusion for $\Re\rho < 1/2$. □

A Period rings and integral comparison matrices

In this appendix, we construct the coordinate maps and comparison matrices for the local lattices of Section 29.1. The approximation and divisibility results give the integral period bases used in Appendix B.

Use the local representations, jump convention and normalized defect of Section 29.1. Here p is the residue characteristic at the coefficient place.

A.1 Integral coordinates for the period basis

In this subsection, we construct coordinate maps from A_{inf} to the power series ring S. The construction uses the graded modules for the ideal (p, z), and it extends to the bounded period ring. Section A.5 uses these maps to preserve divisibility at the de Rham period.

Write $A = A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_p^\flat})$, $S = W(k)[[z]] \subset A$, and $I = (p, z)A$. Use the chosen Teichmüller coordinate z and generator $E = p + z^e u$ of $\ker \theta$ from Section 29.1.1, or $E = p - z^e$ in the tame chart constructed below. Here k is the chart residue field, e the absolute ramification index of the chart field, and $u \in S^\times$.

Lemma A.1 (Graded coordinates of the period ring)**.** *The ring A is complete and separated for $I = (p, z)A$. For each $n \geq 0$, there is an isomorphism*

$$\bigoplus_{i=0}^{n} A/I \xrightarrow{\sim} I^n/I^{n+1}, \qquad (\bar{a}_i)_i \longmapsto \left[\sum_{i=0}^{n} a_i p^i z^{n-i}\right].$$

Proof. The reduction of z in the tilt is nonzero and is a nonunit. The ring A is p-torsion free and A/p is a complete rank-one valuation ring. Multiplication by z is injective on A/p^m for each m, by induction using A/p. Also A/p^m is complete and separated for its z-adic topology. To see the induction, use $p^{m-1}A/p^m \simeq A/p$, with its induced topology: if $z^h a$ is in $p^{m-1}A$, then $a \in p^{m-1}A$ by z-injectivity modulo p^{m-1}. Thus the induced topology on the kernel of $A/p^m \to A/p^{m-1}$ is its own z-adic topology. Lift the limit in the quotient and then take the limit in this kernel, using successive corrections modulo z^h. These completions and the p-adic completeness of A give I-adic completeness because the ideals (p^m, z^h) are cofinal with the I^n.

For each n, the monomials $p^i z^{n-i}$, $0 \leq i \leq n$, identify I^n/I^{n+1} with $(A/I)^{n+1}$. Surjectivity follows by expansion. For injectivity reduce a relation modulo p and cancel z^n to see that its coefficient of z^n lies in I. Subtract that term. The intersection $pA \cap I^{n+1} = pI^n$ follows by reducing a generating expression for I^{n+1} modulo p: its pure z-term has coefficient divisible by p. Divide the remaining relation by p and induct on n. This proves the asserted graded formula. □

Choose a k-basis of A/I containing 1, and lift its vectors to A, using 1 for the distinguished lift. The graded formula shows that the resulting map $\bigoplus_i S/(p, z)^n \to A/I^n$ is an isomorphism for each n. Passing to compatible quotients gives the S-linear coordinates $F_i : A \to S$. They separate elements, preserve both p and z divisibilities, and the coordinate of the lift 1 is the retraction $F_0|_S = \mathrm{id}_S$. The same construction with the finite quotient filtrations gives coordinates continuous in I.

For the bounded period ring, realize $B^+ = A_{\mathrm{max}}[1/p]$ with A_{max} the p-adic completion of $A[E/p]$. For any S-linear $F : A \to A$, first extend by inverting p. Since $E \in S$, the extended map sends $(E/p)^j a$ to $(E/p)^j F(a)$. For a polynomial relation multiply by a power

of p clearing the denominators, use S-linearity in A, and cancel that power. The extension preserves $A[E/p]$, hence extends continuously to $A_{\max}$ and B^+. It commutes with multiplication by $S[E/p][1/p] = S[1/p]$ and, by continuity, with multiplication by its closure S_B in B^+. For the coordinate maps $F_i : A \to S$ the extension takes values in S_B. This also identifies S_B with the scalars whose multiplication commutes with every S-linear endomorphism of B^+ carrying $A_{\max}$ into itself. Indeed such endomorphisms are continuous and commute with $S[1/p]$, hence with its closure S_B. Conversely, for any scalar b in this commutant, the distinguished coordinate satisfies $b = b\,\widehat{F}_0(1) = \widehat{F}_0(b) \in S_B$. Frobenius preserves S_B: it preserves S, and $\varphi(E)/p = 1+(z^e)^p\varphi(u)/p$ if $E = p + z^e u$. The relation $z^e/p = (E/p - 1)/u$ puts this expression in $S[E/p]$. For any finite tuple y over B^+ and any precision $p^j A_{\max}$, approximate its entries by polynomials in E/p after clearing p-denominators.

Expand the finitely many coefficients of these polynomials in the chart basis modulo a high power of (p, z). Since $(p, z)^e A \subset pA_{\max}$, truncating the coordinate expansion of y to a common finite set of indices approximates y to the specified precision. In particular, if $Uy_j = E^h x_j$ with $y_j \in S_B^d$, an A-linear relation among the x_j is the same as the relation among the y_j. Applying each extended coordinate to the latter produces its S-coordinate relation among the x_j. Thus an S-independent family in the solution module is A-independent, and its cardinality is at most d.

For a matrix U invertible over the fraction field of B^+, let $\mathcal{P}$ be the submodule of A^d of vectors x for which $Uy = E^h x$ has a solution $y \in S_B^d$. It is (p, z)-adically separated inside A^d, and p-saturated since $1/p \in S_B$. Multiplication by p is injective, and z is injective modulo $p\mathcal{P}$, since $\mathcal{P}$ is p-saturated. Hence its associated graded module has the same monomial formula with $\mathcal{P}/(p, z)\mathcal{P}$ as coefficient space. Any independent family of residue vectors lifts to an independent family over S, by taking the least (p, z)-order in a presumed relation. So the residue space has dimension at most d. Lift a basis, and express any element of $\mathcal{P}$ successively modulo $(p, z)^n$ in those lifts. The finitely many coefficients converge in S. Separation of $\mathcal{P}$ now gives an exact expression. In particular the solution module is finite free over S.

Finally a principal ideal (δ) with $p \nmid \delta$ is closed in A for the topology of (p, z). There is $k > 0$ such that $\bar{\delta} \mid \bar{z}^k$ in the rank-one valuation ring A/p. Write $\delta u = z^k + pv$. Modulo p^m, the geometric sum gives

$$\delta u \sum_{i=0}^{m-1} z^{k(m-1-i)}(-pv)^i = z^{km}.$$

If δa_j is a Cauchy sequence in A for (p, z), then modulo p^m it is Cauchy for z. Multiplying any difference in $z^{L+km}(A/p^m)$ by the displayed multiplier and cancelling the nonzero divisor z^{km} puts the difference of the a_j's in $z^L(A/p^m)$. This works for every L, m. The a_j therefore converge in A, and their product with δ has limit in (δ). The topology of (p, ξ) is the same here, since ξ is associated to $E = z^e a + p$ with a a unit of the chart. This supplies the closedness used to pass from the finite coordinate partial sums to divisibility in the basis proof.

A.2 Integral approximation in the fixed period chart

In this subsection, we use the coordinate maps of Section A.1 and the Ax–Sen–Tate theorem to approximate fixed tilt vectors. The approximation gives equality in the prescribed tilt component modulo p. It is used to prove that the determinant in Section B.1 is nonzero modulo p.

For the chart $S \subset A_{\mathrm{inf}}$ of the preceding construction and each integer $j \geq 0$, put $\theta_j = \theta \circ \varphi^{-j}$ and $E_j = \varphi^j E \in S$. The map $\theta_j : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ is surjective, with kernel (E_j), since E is associated to a generator of the kernel of θ. For any S-linear map $F : A_{\mathrm{inf}} \to S$, $\alpha \in S$, $r \in A_{\mathrm{inf}}$, the implication

$$\theta_j(\alpha) \mid \theta_j(r) \quad \Longrightarrow \quad F(r) - \alpha x \in E_j S \quad \text{for some } x \in S$$

follows by lifting an integral quotient to $a \in A_{\mathrm{inf}}$. Then $r - \alpha a = E_j b$ and one can take $x = F(a)$. This gives the preservation of divisibility by p in the residue at the j-th component for every chart coordinate map used in the period basis.

Now let $y \in \mathcal{O}_{\mathbb{C}_p^\flat}$ be fixed by the pointwise fixer of S. For each $m \geq 0$ put $t = \theta_m([y])$, with $[y]$ the Teichmüller lift. It is integral and fixed by the same group. Put $K_0 = W(k)[1/p]$, $\pi_j = \theta_j(z)$, and $K_j = K_0(\pi_j)$. Here π_0 is a uniformizer of the chart field, and $\pi_{j+1}^p = \pi_j$. If $P \in W(k)[X]$ is the Eisenstein polynomial of π_0 over K_0, then $P(X^{p^j})$ is Eisenstein for every j. Thus $\mathcal{O}_{K_j} = W(k)[\pi_j]$. The map θ_j acts on $W(k)$ through inverse Witt Frobenius to the power j, and a series in z evaluates to a convergent series in π_j. Consequently

$$R_j = \theta_j(S) = \mathcal{O}_{K_j}, \qquad K_\infty = \bigcup_{j \geq 0} K_j \subset \overline{\mathbb{Q}}_p.$$

Its fixer equals the pointwise fixer of S. For the converse direction, if a Galois element fixes every $\theta_j(s)$, then all θ_j kill $gs - s$. Those values modulo p are all the components of the first Witt coordinate of $gs - s$ in the tilt. That coordinate is zero. Divide $gs - s$ by p, and repeat using p-torsion freeness of $\mathcal{O}_{\mathbb{C}_p}$ and p-adic separatedness of A_{inf}. It follows that $gs = s$.

By the Ax–Sen–Tate fixed field theorem [6, theorem, p. 417], t is in the closure of K_∞. Choose $x \in K_\infty$ with $|x - t|_p < |p|_p$, so x is integral. Choose j_0 with $x \in K_{j_0}$. For every $j \geq j_0$, integrality gives $x \in R_j$, so some $s_j \in S$ satisfies $\theta_j(s_j) = x \equiv t \bmod p$. The j-th tilt component of $s_j \bmod p$ therefore equals the m-th component of y. For a fixed finite tuple of such y, taking the largest of the finitely many thresholds gives these equalities simultaneously at every sufficiently late component.

A.3 Uniform denominator clearing in period coordinates

In this subsection, we combine division in $A_{\max}$ with a Frobenius relation to obtain a bound for integral denominators in the solution module of Section A.1. The bound applies uniformly to all its coordinate maps and is used in Section A.6.

In $A = A_{\mathrm{inf}}$ choose the cyclotomic generator of $\ker \theta$ as in [13, §3.3, before Definition 3.22]. Explicitly,

$$\tau = [\epsilon^{1/p}] - 1, \qquad \xi = \sum_{i=0}^{p-1} (1+\tau)^i, \qquad \varphi(\tau) = [\epsilon] - 1 = \mu = \xi\tau.$$

Thus $E/\xi \in A^\times$. Put $R = A[X]/(pX - \xi)$. If $pF = (pX - \xi)G$ in $A[X]$, then $\bar{\xi}\,\bar{G} = 0$ in $(A/p)[X]$. Since $\bar{\xi} \neq 0$ in the tilt domain, $G = pG_1$. Cancelling p shows that R is p-torsion free. Its p-adic completion $A_{\max}$ is also p-torsion free: an element killed by p has component modulo p^{m+1} in $p^m R/p^{m+1}R$ for every m, hence is zero. Dividing compatible components by p at one higher precision also gives $A_{\max}/pA_{\max} = R/pR = (A/(p,\xi))[X]$. A polynomial with monic reduction acts injectively on this quotient. Induction proves that its multiplication is p-saturated at every finite depth.

An element $a \in A$ that lies in $p^m A_{\max}$ belongs to $(p,\xi)^m A$. Indeed for $m > 0$, $\theta(a) \in p^m \mathcal{O}_{\mathbb{C}_p}$; lift the quotient under θ to $c \in A$. Then $a = \xi w + p^m c$ with $w \in A$. Using $\xi = pX$ and cancelling p shows $Xw \in p^{m-1} A_{\max}$. The preceding regularity and induction put w in $(p,\xi)^{m-1}$. The reverse inclusion uses $\xi = pX$ directly.

Since $\xi - p \in \tau A$, we have $p(X-1) = \tau u$ for an integral u. Every $y \in A_{\max}$ is $a + (X-1)b$ with $a \in A$, $b \in A_{\max}$. This follows by polynomial division modulo p and successive lifts modulo all powers of p. Inducting on j and using $p(X-1) = \tau u$ gives $p^j y \in A + \tau^j A_{\max}$.

Let $V \subset A_{\max}$ and suppose for each $v \in V$ there is $w \in V$ such that $\xi^h v = u_0 \varphi(w)$ with $u_0 \in A$ fixed. Then for every $m \geq 0$ there exists $a_m \in A$ with

$$p^{h+1} v - a_m \in \tau^{h+1} p^m A_{\max}.$$

The case $m = 0$ follows from the preceding polynomial division. For $m+1$, apply φ to the congruence for w and multiply by u_0. The error acquires $(\xi\tau)^{h+1}$. After factoring out ξ^h, the remaining factor $\xi = pX$ contributes one extra p. Here $u_0 \varphi(a_m(w)) \in A \cap \xi^h B^+ = \xi^h A$. Indeed, if $a = \xi^h y \in A$ with $y \in B^+$ and $h > 0$, evaluate at $X = 0$ to obtain $\theta(a) = 0$. Thus $a = \xi a_1$ with $a_1 \in A$. Cancelling ξ in B^+ gives $a_1 = \xi^{h-1} y$, and induction proves the intersection identity. This gives the next integral approximation. The difference of two approximations at precision m lies in $p^m A_{\max} \cap A = (p,\xi)^m A$. They converge in A, by Section A.1. Their limit in $A_{\max}$ is $p^{h+1} v$. Thus $p^{h+1} V$ has integral A-coordinates.

For the period matrix U, apply this to the coordinates of the solutions w of

$$Uy = E^h x, \quad x \in A^d, \qquad U \widehat{F}(y) = E^h w$$

as y and the S-linear maps $F : A \to A$ vary. There is one power p^m putting all such w in $A_{\max}$: the inverse of U up to $(E\tau)^h$ first bounds a denominator of y, and integrality of the extended coordinate maps gives the same bound for $F(y)$.

Applying U/E^h requires one further fixed power. For the denominator claim, note that $\xi \equiv \tau^{p-1} \bmod p$, so $\tau^{p-1} = p(X - b_0)$ with $b_0 \in A$. If $\mu z \in A_{\max}$ then multiplication by τ^{p-2} gives $p^2 X(X - b_0) z \in A_{\max}$. Multiplication by the monic polynomial $X(X - b_0)$ is p-saturated, by its reduction modulo p. Clearing an initial power of p in B^+ and using this saturation yields $p^2 z \in A_{\max}$. Iteration proves the claimed denominator bound for any fixed power of μ, and hence for any fixed power of a divisor of μ.

Section A.6 verifies Frobenius descent for the retracted solutions and applies the contraction with a uniform exponent.

A.4 The two period matrices over the bounded ring

In this subsection, we construct the crystalline comparison matrix and its partial inverse over B^+. The key scalar assertion is divisibility by μ in the common kernel of the Fontaine evaluations at nonnegative Frobenius powers. This gives the matrix equations used in Section B.1.

For the same chart write $\mu = \xi\tau$. In $A = A_{\mathrm{inf}}$, the common kernel of $\theta\varphi^j$, $j \geq 0$, is μA. The individual kernels are generated by the distinct primes $\varphi^{-j}\xi$. Distinctness follows by evaluating the cyclotomic geometric sum for ξ at the different roots of unity. The value is zero at the primitive p-th root and nonzero at the other indicated powers. The intersection of any finite initial set of these principal prime ideals is their product. Its generator is

$$\prod_{j=0}^{m-1} \varphi^{-j}(\xi) = \mu / \varphi^{-m}(\mu).$$

The intersection of these products for $m \geq 1$ is μA by [13, Lemma 3.23]. Applying this result to $\varphi(a)$ gives the kernel τA for $\theta\varphi^j$, $j \geq 1$.

For $y \in B^+$, choose $n \geq 0$ such that $w = p^n y \in A_{\max}$, and put $X = \xi/p$. Division by $X - 1$ and by X modulo p, followed by successive lifting as in Section A.3, gives $w = a+(X-1)(c+Xd)$ with $a, c \in A$, $d \in A_{\max}$. Since $\tau \mid \xi - p$ and $\mu = \xi\tau$,

$$p^{n+2}y = p^2 a + p(\xi - p)c + \xi(\xi - p)d \in A + \mu A_{\max}.$$

Since μ belongs to all the displayed kernels, the divisibility statements pass to B^+. In particular if $\varphi(y)$ is divisible by $\varphi(\tau)$ in B^+, then y is divisible by τ. This can be repeated for every power of τ. The same approximation proves $\ker(\theta : B^+ \to \mathbb{C}_p) = \xi B^+$ using the single kernel on A.

For comparison with the usual period ring, take the tilt element $p^\flat$ from compatible p-power roots of p, with $v_\flat(p^\flat) = 1$. Since $[p^\flat] - p$ also generates $\ker\theta$, $A[\xi/p] = A[[p^\flat]/p]$. In the Witt coordinates of $A[1/p]$, the unit ball for the norm of [12, §13] is this algebra: a term $p^{-j}[x]$ of negative index satisfies the unit-ball condition exactly when $v_\flat(x) \geq j$, in which case it belongs to $([p^\flat]/p)^j A$. The finitely many negative indices and the integral tail give the assertion. Thus Berger's bounded ring is $B^+ = A_{\max}[1/p]$. Also $t = \log(1 + \mu)$ and t/μ is a unit of B^+. In fact $\mu = pX\tau$, so the logarithm quotient converges and its nonconstant terms lie in $pA_{\max}$, using $j - 1 - v_p(j) \geq 1$ for $j \geq 2$ except $p = j = 2$. In that case $\xi \equiv \tau \bmod 2$ and $\xi = 2X$ give $\mu/2 = X\tau \in 2A_{\max}$. Hence $B^+[1/\mu] = B^+[1/t] = B_{\max}$.

Let $a \geq 0$ be an integer. If $y \in B_{\max}$ satisfies $\varphi^j y \in t^{-a}B^+_{\mathrm{dR}}$ for every $j \geq 0$, then $\mu^a y \in B^+$. Write $y = b/\mu^k$ with $b \in B^+$. Every $\varphi^j\mu$ has de Rham order one, by its binomial expansion at $\mu = 0$. If $k > a$, every $\theta\varphi^j(b) = 0$, so that $\mu \mid b$ in B^+. Cancel and repeat to reach $k = a$.

For a crystalline lattice on this chart choose an invariant matrix $M \in \mathrm{GL}_d(B_{\max})$ whose columns are the coordinates of a K_0-basis of $D_{\mathrm{cris}}(T[1/p])$ in a $\mathbf{Z}_p$-basis of T, where $K_0 = W(k)[1/p]$. Crystalline comparison makes M invertible and gives $\rho(g)g(M) = M$ for $g \in G_K$ [12, §§4, 13–14]. Frobenius commutes with G_K and fixes $\rho(g)$, so each $\varphi^j M$ also has invariant columns. It is invertible, and the crystalline fixed-field theorem [12, §13, corollary to Theorem 13.1] gives $C_j = M^{-1}\varphi^j(M) \in \mathrm{GL}_d(K_0)$. With the filtration-adapted basis matrix Y of Section A.5, one has

$$\varphi^j(M)C_j^{-1}Y = MY = W\,\mathrm{diag}(t^{k_i}), \qquad W \in \mathrm{GL}_d(B^+_{\mathrm{dR}}).$$

Thus for jumps in $[-a, b]$ each $\varphi^j M$ has entries in $t^{-a}B^+_{\mathrm{dR}}$ and its inverse has entries in $t^{-b}B^+_{\mathrm{dR}}$. The bounded-ring criterion and $E\tau \sim \mu$ give the matrices

$$U = (E\tau)^a M, \qquad W' = (E\tau)^b M^{-1} \quad \text{in } \mathrm{Mat}_d(B^+).$$

They satisfy $UW' = (E\tau)^{a+b}I$.

The transition C_1 and its inverse have entries in K_0, which is contained in the chart bicommutant. Also $\varphi(E)$ is a unit in that bicommutant: $E = z^e u + p$ with u a chart unit, and $E = pX'$ in $A_{\max}$. Thus $z^e \in pA_{\max}$ and $\varphi(E)/p \equiv 1 \bmod p$. Its inverse is given by the convergent geometric series in the closure defining S_B. Writing $v_0 = \mu/(E\tau) \in A^\times$, we get

$$\varphi(U) = v_0^a U(\varphi(E)^a C_1)$$

with an invertible last factor over the bicommutant.

For the Galois equation, E/gE is a unit since both elements generate $\ker\theta$. Also $\tau/g\tau$ is a unit by the cyclotomic binomial formula. The integral representation matrix, scaled by the

unit $(E\tau/g(E\tau))^a$, takes $g(U)$ back to U. Finally the divisibility condition under coordinate retraction is proved in Section A.5. This gives the period-matrix hypotheses of the integral basis for each lattice in the modular cup product, on the crystallizing extension in the common local class.

A.5 Divisibility under the coordinate retraction

In this subsection, we prove that the retraction of Section A.1 respects the Hodge filtration relevant to the equation $Uy = E^h x$. Approximation of the change-of-basis coefficients in the bounded ring reduces the assertion to divisibility by a power of E.

Let K be the local chart field and keep the fixed crystalline comparison matrix M of the original lattice. The de Rham comparison gives

$$M = W\ \mathrm{diag}(t^{k_j})\, Y^{-1} \quad \text{in } B_{\mathrm{dR}}, \qquad W \in \mathrm{GL}_d(B_{\mathrm{dR}}^+), \quad Y \in \mathrm{GL}_d(K).$$

Here the change of basis Y takes the invariant period basis to a filtration-adapted basis d_j with jumps k_j. The columns $t^{-k_j} d_j$ are integral for the de Rham lattice. The dual basis λ_j has filtration steps $-k_j$ by filtered duality, so $t^{k_j}\lambda_j$ are integral covectors. They give the inverse matrix over B_{dR}^+. The corresponding period matrix of height h over B^+ is $U = (E\tau)^a M$.

For any $u \in K$, its image in B_{dR}^+ is approximable at every ξ-adic precision by elements of the chart bicommutant S_B. Indeed $\theta(S_B) = K$. The chart constants and the image of its uniformizer generate K, and p is invertible in S_B. Conversely, the uncompleted ring $S[E/p][1/p]$ has image in K, since $\theta(E) = 0$ and series in the uniformizer converge in K. Continuity of θ in the bounded-ring p-adic topology and completeness of K give the assertion after completion. Start with $b \in S_B$ having residue u. For a polynomial $P \in \mathbb{Q}_p[X]$ having a simple root u, lift $P'(u)^{-1}$ to $c \in S_B$ by this same residue map. If $b \equiv u \bmod \xi^j$, Taylor expansion gives $b - cP(b) \equiv u \bmod \xi^{j+1}$. This yields the required approximations.

For $y_i \in B^+$ and $\beta_i \in K$, let $\widehat{F}$ be the extension of any S-linear $F : A \to A$ constructed in Section A.1. If $\sum_i \beta_i y_i$ is divisible by ξ^j in B_{dR}^+, approximate each β_i modulo ξ^j by $b_i \in S_B$. Then $\sum_i b_i y_i$ is divisible by E^j in B^+, since divisibility by ξ in B_{dR}^+ descends to B^+ by the kernel of θ, and E is associated to ξ. Apply $\widehat{F}$, which commutes with S_B and multiplication by E^j, and replace b_i by β_i again modulo ξ^j. It follows that

$$\mathrm{ord}_\xi\left(\sum_i \beta_i \widehat{F}(y_i)\right) \geq \mathrm{ord}_\xi\left(\sum_i \beta_i y_i\right).$$

If the latter sum is zero, this says that the former is also zero, by separatedness.

Apply this to each covector of Y^{-1}. There are $s_j \in B_{\mathrm{dR}}^+$ such that $(Y^{-1}\widehat{F}(y))_j = s_j (Y^{-1}y)_j$. Since the two diagonal factors commute, we obtain

$$U\widehat{F}(y) = W\ \mathrm{diag}(s_j)\, W^{-1} U y.$$

Consequently divisibility of Uy by E^h implies the same for $U\widehat{F}(y)$, first in B_{dR}^+ and then in B^+ as above. This proves the retraction property used in Section A.3.

A.6 Divisibility of the period matrix against integral covectors

In this subsection, we use the denominator bound of Section A.3 and the retraction of Section A.5 to extend divisibility from the basis solutions to all bounded-ring solutions. Section B.1 uses this assertion to prove full rank and then to obtain the integral span.

Work on the same chart and take the symmetric height bound b. Put $h = 2b$, $A = A_{\mathrm{inf}}$, and write $U = (E\tau)^b P$ for the matrix of Section A.4. There are matrices W, F, F^{-1} over B^+ such that

$$UW = WU = (E\tau)^h I, \qquad \varphi(U) = vUF,$$

where $v \in A^\times$ and F has entries in S_B. Write $\widehat{f}$ for the extension of an S-linear $f : A \to A$ to B^+. For any solution $Uy = E^h x$ with $y \in (B^+)^d$ and $x \in A^d$, the evaluation divisibility in Section A.5 gives

$$U\widehat{f}(y) = E^h w.$$

A single power p^N, depending on the chart and matrix, puts every such w in A^d.

Choose $c_U, c_W \geq 0$ with $p^{c_U}U, p^{c_W}W$ having entries in $A_{\max}$. Since $\tau^h y = Wx$, the division bound of Section A.3 gives $p^{c_W+2h}y \in A^d_{\max}$. Every $\widehat{f}$ preserves $A_{\max}$. Applying the same bound to $E^h w = U\widehat{f}(y)$ gives

$$p^{N_2} w \in A^d_{\max}, \qquad N_2 = c_U + c_W + 4h.$$

The two applications are valid because E and τ each divide μ in A.

For Frobenius contraction set $x' = \varphi^{-1}(vE^h x) \in A^d$. Apply φ to $UWx' = (E\tau)^h x'$ and cancel vU to get

$$F\varphi(Wx') = \varphi(E\tau)^h y.$$

The inverse of F is defined over B^+, so $\varphi(Wx')$ is divisible there by $\varphi(\tau)^h$. The cyclotomic kernel argument of Section A.4 gives $y' = \tau^{-h}Wx' \in (B^+)^d$, with $Uy' = E^h x'$. Take the S-linear $f' = \varphi^{-1} f \varphi$ and write $U\widehat{f'}(y') = E^h w'$. The extensions satisfy $\varphi\widehat{f'} = \widehat{f}\varphi$. The equality on A extends to the divided generators using the extension of the chart maps and then to the completion by continuity. The S_B-linearity permits $\widehat{f}$ to pass through F and $\varphi(E)^h$. Since $F\varphi(y') = \varphi(E)^h y$, we get

$$\varphi(E)^h \varphi(w') = vUF\widehat{f}(\varphi(y')) = v\varphi(E)^h U\widehat{f}(y), \qquad \varphi(w') = vE^h w.$$

The entries of $p^{N_2}w$ thus belong to a single subset of $A_{\max}$ closed under the descent relation $\xi^h a = u\varphi(a')$, with the fixed unit $u = (\xi/E)^h v^{-1}$. The contraction of Section A.3 yields $N = N_2 + h + 1$.

Let $d_0 = \operatorname{rank}_S \mathcal{P} \leq d$, and let $Q \in M_{d \times d_0}(A)$ have as its columns an S-basis of the solution module $\mathcal{P}$ of Section A.1. For every integral covector l and every $\delta \in A$ with $p \nmid \delta$ we claim that

$$\delta \mid lQ \quad \Longrightarrow \quad \delta \mid lx \qquad (Uy = E^h x,\ y \in (B^+)^d,\ x \in A^d). \tag{1}$$

Choose the topological chart basis $(b_i)_i$. For $f = \mathrm{coord}_i$, the entries of $\widehat{f}(y)$ are in S_B.

The uniform exponent gives $x_i = p^N w_i \in A^d$ and

$$U\big(p^N \widehat{\mathrm{coord}_i}(y)\big) = E^h x_i.$$

Thus $x_i = Qs_i$ with $s_i \in S^{d_0}$. Take $k = c_U + 2h$, using $p^{c_U}U \in M_d(A_{\max})$, $E \mid \mu$ and the estimate $\mu^h y \in A_{\max} \Rightarrow p^{2h}y \in A_{\max}$ for $y \in B^+$ from Section A.3. For any m choose one finite subset I_m of the basis with

$$y - \sum_{i \in I_m} b_i \widehat{\mathrm{coord}_i}(y) \in p^{m+k} A^d_{\max}.$$

This is the simultaneous finite approximation of the finitely many entries.

Set

$$t = p^N x - \sum_{i \in I_m} b_i Q s_i \in A^d.$$

Apply $p^N U$ to the last approximation and use the definition of k to obtain $t \in p^m A_{\max}^d$. The identity $p^m A_{\max} \cap A = (p, \xi)^m$ puts lt in $(p, \xi)^m$. If $lQ = \delta g$, then the multiples $\delta \sum_{i \in I_m} b_i g s_i$ approximate $p^N l x$ to every (p, ξ)-adic order. By closedness of δA from Section A.1, $p^N l x \in \delta A$. Since p is prime in A and $p \nmid \delta$, repeatedly cancel p in this divisibility. This proves (1) for all solutions on the same chart, including those over B^+.

B Integral crystalline bases and filtration orders

In this appendix, we use the comparison matrices of Appendix A to construct an integral period basis and describe its filtration under the divided Galois action. The resulting orders are used in the cup calculation of Appendix C and the lifts of Appendix D.

B.1 The integral period basis

In this subsection, we construct an integral basis from the solution module of Section A.1. The fixed-vector approximation of Section A.2 rules out a kernel modulo p. We then obtain the two partial inverses and the equations for Frobenius and Galois.

For a lattice T free of rank r_T over the coefficient DVR O, write $L = \mathrm{Frac}(O)$. All determinants below are over $A = A_{\mathrm{inf}}$ or S on the underlying $\mathbf{Z}_p$-lattice of rank $d = r_T[L : \mathbf{Q}_p]$. Coefficient multiplication preserves the two modules of Section B.2 by the pullback of their defining forms, as in Section B.2.2.

Use the S-basis of the solution module from Section A.1, with its columns in Q and solutions in Z. Initially the number of columns is at most $d = \mathrm{rank}_{\mathbf{Z}_p} T$. Thus $UZ = E^h Q$ with Z over S_B. The divisibility against covectors of Section A.6 holds before asserting that Q is square. If the rank of Q were less than d, there would be a nonzero covector $l \in A^d$ with $lQ = 0$. Use this covector and $\delta = z^k$ for every $k \geq 1$ in the divisibility against covectors, with the solutions $UW = (E\tau)^h I$. Then $l\tau^h I \subseteq \bigcap_k z^k A^d = 0$, a contradiction. The covector argument proves the following rank statement.

Lemma B.1 (Full rank of the integral solution basis)**.** *The solution matrices Q, Z are square of size $d = \mathrm{rank}_{\mathbf{Z}_p} T$, and $\det Q \neq 0$.*

If $Qs = px$ for $s \in S^d$ and $x \in A^d$, then $U(Zs/p) = E^h x$ and $Zs/p \in S_B^d$. It follows that $x = Qs'$ for $s' \in S^d$, and $s = ps'$ by injectivity of Q. Thus the chart-coordinate covectors of Q separate vectors over S/p: if all coordinate covectors kill a vector modulo p, the z-adic coordinate expansion implies $Qs = 0 \bmod p$, hence $s = 0 \bmod p$. Over the fraction field of $S/p = k[[z]]$, these covectors therefore span the full dual space. Choose d of them with nonzero determinant modulo p, and let N be their matrix.

The equation $\varphi U = vUF$ and the invertibility of $\varphi(E)$ in S_B give

$$U\left((E/\varphi E)^h F \varphi Z\right) = E^h v^{-1} \varphi Q.$$

By the solution basis, $QB = v^{-1}\varphi Q$ for some B over S. The reduction of $\det B$ is nonzero. Otherwise the determinant equation and invertibility of φ on A would put $\det Q$ in every $p^n A$.

Suppose $\det Q = 0 \bmod p$. Over the valuation ring A/p choose a nonzero kernel vector a of minimal support and divide by a coordinate of smallest valuation to make that coordinate 1. If g fixes S pointwise, it fixes Z: it is continuous for the bounded ring topology and fixes $S[E/p]$ pointwise. Applying g to $UZ = E^h Q$ and using the equivariance of U shows $Q = R_g g Q$ for a matrix R_g over A. Thus $g(a)$ is again in the kernel. Subtracting a gives a kernel vector of strictly smaller support unless $g(a) = a$. The normalized vector a is consequently fixed by the pointwise fixer of S.

Put $C_n = B\varphi(B)\cdots\varphi^{n-1}(B)$ and $D_0 = \overline{\det B}\,\overline{\det N}$. The Frobenius equation and the determinant product imply

$$\bar{Q}\bar{C}_n a^{p^n} = 0, \qquad D_0\,\overline{\det C_n} \mid D_0^{p^n}.$$

For the divisibility use $1 + p + \cdots + p^{n-1} \le p^n - 1$. Fix a component m of the tilt. For some common component $m+n$, integral approximation of the fixed a gives $s \in S^d$ whose component $m+n$ equals the component m of a in every coordinate. The component $m+n$ of $QC_n s$ is therefore zero modulo p. Chart-coordinate maps preserve this vanishing, so the component $m+n$ also kills $NC_n s$. The selected coordinate of s equals 1 there. Multiplication by the adjugate gives $\det(NC_n) = 0$ in that component, even though the component ring need not be a domain. It follows that $D_0^{p^n}$ has zero component $m+n$, so D_0 has zero component m. This is true for every m, contrary to $D_0 \neq 0$. Hence $\det Q \bmod p \neq 0$ in $\mathcal{O}_{\mathbf{C}_p^\flat}$.

This conclusion is independent of the two initial bases. Changing the integral basis by $H \in \mathrm{GL}_d(\mathbf{Z}_p)$ and the crystalline basis by $C \in \mathrm{GL}_d(K_0)$ gives

$$P_1 = H^{-1} P C, \qquad U_1 = H^{-1} U C, \qquad \mathcal{P}_1 = H^{-1}\mathcal{P}.$$

Indeed $U_1 y = E^h x$ is equivalent to $U(Cy) = E^h H x$, and C, C^{-1} have entries in $K_0 \subset S_B$. Consequently $Q_1 = H^{-1} Q V$ for some $V \in \mathrm{GL}_d(S)$; the change is integral even when O is ramified.

Apply the divisibility against covectors with $\delta = \det Q$ and the covectors of $\mathrm{adj}(Q)$. Every solution $Uy = E^h x$ with $x \in A^d, y \in (B^+)^d$ then has $x \in QA^d$. Taking y from the columns of W gives $QQ' = \tau^h I$ with Q' integral. Conversely, $UZ = E^h Q$ gives a bounded-ring solution for every $x \in QA^d$. Thus the full solution module is

$$QA^d = A \otimes_S QS^d = \{x \in A^d : \mathrm{adj}(Q)x \in (\det Q)A^d\},$$

where the tensor product is identified with its image using the independence of the columns. The last description and the closedness of $(\det Q)A$ give closedness in the (p,z)-adic topology. It also gives p-saturation, since p is prime and $p \nmid \det Q$.

Put $v_0 = \mu/(E\tau) \in A^\times$. To construct the second partial inverse, take $y \in B^+$ and $s, k \ge 0$. For some $c \ge 0$, $a \in A$, and $b \in A_{\max}$,

$$p^c y = a + p^{c+k}\mu^s b.$$

Indeed choose c_0 with $p^{c_0} y \in A_{\max}$. Approximate by a polynomial in ξ/p modulo $p^{c_0+k+2s}A_{\max}$, then multiply by p^{c-c_0} so that the resulting polynomial in ξ/p belongs to A. The resulting error is $p^{c+k+2s}b_0$. Successive use of the polynomial-division argument with $X-1$ and X gives

$$p^s b_0 = a_1 + \tau^s b_1, \qquad p^s b_1 = a_2 + \xi^s b, \qquad a_1, a_2 \in A.$$

The remaining error after absorbing integral terms is $p^{c+k}\tau^s\xi^s b = p^{c+k}\mu^s b$.

Apply this entrywise to U with $k = 0, s = h$. Take one common c and obtain $p^cU = A_0 + p^c\mu^h V_0$ with integral A_0 and V_0 over $A_{\max}$. Then

$$A_0 = p^cU(I - v_0^h W V_0), \qquad p^c\mu^h V_0 = p^c(Ev_0)^h QQ'V_0.$$

The first identity has A_0 on the right side of an equation $Uy = E^hA_0$, after multiplying by E^h. The integral span just proved puts A_0 in Q times an integral matrix. The second identity has the same factor Q, so $U = QY$ with Y over B^+. Cancel Q in $QYZ = E^hQ$ to get $YZ = E^hI$.

Finally let g fix the chart field, so it fixes the Witt constants and $\theta(z)$. If $U = A_ggU$ is the period equation, then $UgZ = (gE)^hA_ggQ$, so $UgZ = E^hR_ggQ$ with $R_g = (gE/E)^hA_g$ integral. The integral span gives $R_ggQ = QG_g$ for G_g integral. Cancel U in the resulting equality to obtain $gZ = ZG_g$. Altogether the basis satisfies

$$QQ' = \tau^hI, \qquad UZ = E^hQ, \qquad YZ = E^hI, \qquad \varphi Q = vQB, \qquad gZ = ZG_g.$$

B.2 The period basis, divided action, and tensor orders

In this subsection, we use the de Rham evaluations of the integral basis in Section B.1 to control division of Galois differences by $z\tau$. The retraction preserves each de Rham filtration, and the tensor estimates give the orders in Section C.2.

Keep the chart of Sections A.1 and A.3, and the original crystalline $\mathbf{Z}_p$-lattice T. Write $A = A_{\inf}$, $\mathrm{Fil}^j = t^jB_{\mathrm{dR}}^+$, $E_a = \varphi^aE$, and let K' be the local fixed field of the chart subgroup. We use the symmetric height bound b for the Hodge jumps. For an invariant crystalline period matrix P, use the matrices of Section B.1 with $h = 2b$, so

$$U = (E\tau)^bP, \qquad UZ = E^{2b}Q, \qquad QQ' = \tau^{2b}I, \qquad YZ = E^{2b}I.$$

The columns n_i of $\tau^{-b}Q$, in the original lattice basis, satisfy

$$E_a^b\lambda\Big(\sum_i c_i\varphi^an_i\Big) = (\lambda_{\mathrm{lat}}\varphi^aP)\varphi^a(Z)c \qquad (a \geq 0). \tag{1}$$

For the integral tests the coefficients of c lie in $A_{(p)} = (A \setminus pA)^{-1}A$, viewed in B_{dR}. The identity also holds for $c \in B_{\mathrm{dR}}^d$ after extending the chart-equivariant covector λ by linearity.

Its covector λ_{lat} consists of its evaluations on the original lattice basis. Equivariance puts the entries of $\lambda_{\mathrm{lat}}\varphi^aP$ in $B_{\mathrm{dR}}^{G_{K'}} = K'$. For a crystalline functional, $\lambda(T) \subset B_{\max}$, so the same products lie in $B_{\max}^{G_{K'}} = K_0 \subset S_B$ by Section A.4. For each standard covector e_j^{T}, define $(\lambda_{a,j})_{\mathrm{lat}} = e_j^{\mathsf{T}}(\varphi^aP)^{-1}$. Its entries are crystalline periods, and its product with φ^aP is e_j^{T}. Thus $\lambda_{a,j}$ is a chart-equivariant crystalline functional. For $x_a = \sum_i c_i\varphi^an_i$, put $L_a(x_a) = (\lambda_{a,j}(x_a))_j$. Substitution in (1) and $YZ = E^{2b}I$ give

$$\varphi^a(Z)c = E_a^bL_a(x_a), \qquad E_a^{2b}c = \varphi^a(Y)\,E_a^bL_a(x_a). \tag{2}$$

In particular, for any B^+-submodule $J \subset B_{\mathrm{dR}}$, $E_a^bL_a(x_a) \in J^d$ implies $E_a^{2b}c \in J^d$, since $\varphi^a(Y)$ has entries in B^+.

The A-span M of the n_i is exactly the part of $A[1/\tau] \otimes T$ on which every chart-equivariant form takes integral values after each strictly positive power of Frobenius. In one direction (1)

proves the integrality, since E_a has de Rham order zero for $a > 0$. In the other direction $QQ' = \tau^{2b} I$ first gives a presentation $x = \tau^{-m} \sum_i c_i n_i$, $c_i \in A$. Take m least. If $m > 0$, for every $a > 0$ the values of $\varphi^a(\tau^m x)$ have order at least one, since $\theta\varphi^a(\tau) = 0$.

Apply (2) at that level to $\varphi^a(c_i)$ and use $\theta(E_a) \neq 0$. Then $\theta\varphi^a(c_i) = 0$ for all $a > 0$. The cyclotomic intersection of these kernels is τA. This contradicts the least choice of m. The basis is free by $QQ' = \tau^{2b} I$. In particular

$$\tau^b(A \otimes T) \subset M \subset \tau^{-b}(A \otimes T). \tag{3}$$

The normalizer of the chart subgroup preserves these tests, since it permutes the equivariant forms and commutes with Frobenius. This includes the tame generator, since the tame chart field is Galois over $\mathbf{Q}_p$.

Put $N = \sum_i S n_i$ and invert all E_a to obtain N_{loc} and M_{loc} from N and M. The integral change of basis is

$$E^b \varphi(n) = nB, \qquad B \in M_d(S). \tag{4}$$

Indeed the scalar in $\varphi Q = vQB$ is $v = (\varphi(\tau)/(E\tau))^b$ by Section A.4, so $E^b \varphi(\tau)^{-b} \varphi Q = \tau^{-b} QB$. In $W(\mathbf{C}_p^\flat)$ the element τ is a unit, since its reduction modulo p is nonzero. Thus $QQ' = \tau^{2b} I$ makes n and $\varphi(n)$ invertible over this Witt ring, and $\det B \neq 0$. In particular $\varphi^a(n) = nC_a$ for $C_a \in \mathrm{GL}_d(\mathrm{Frac}\, S)$. Define the value field

$$\mathcal{K}_a = K'(S_B, \varphi^{-a} S) \subset B_{\mathrm{dR}}.$$

By (1), forms take values in $\mathcal{K}_0$ on n_i; by the identity $\varphi^{-a}(n) = n\varphi^{-a}(C_a^{-1})$, they take values in $\mathcal{K}_a$ on $\varphi^{-a}(N_{\mathrm{loc}})$. The inverse Frobenius here acts on Witt coefficients, before taking their de Rham images.

B.2.1 Integral divided operators

In this subsection, we use the integral basis of Lemma B.1 to control the divided operators.

For the scalar integral cancellation, let $y = z$ or $y = \tau$. In each case $p \nmid y$, $\theta(y) \neq 0$, and $\xi - pu \in yA$ for some $u \in A$. For z use $E = p + z^e u'$ and $E \sim \xi$. For τ use $\xi \equiv p \bmod \tau$. The map $A[X]/(pX - \xi) \to A/(y, p^j)$ defined by $X \mapsto u$ extends to the p-adic completion. If $a \in A \cap yB^+$, clear p^m from its quotient by y in B^+ and apply these maps.

It follows that $p^m a \in yA + p^j A$ for every j. The ideal yA is closed for the p-adic topology: since $\bar{y} \neq 0$ in the tilt domain, divisibility of yc by p^j implies divisibility of c by p^j. The approximating quotients therefore converge. The same regularity cancels p^m and gives $a \in yA$. First contract $E^r a \in yB^+$ to $E^r a = yc$ with $c \in A$. Since E is prime and $\theta(y) \neq 0$, E^r divides c. This proves

$$E^r a \in yB^+ \implies a \in yA. \tag{5}$$

Applying (5) successively to the integral quotients gives the same statement for $y = \tau^m z^n$.

Write $d = z\tau$ and let H be the subgroup fixing the chart field K', so $g \in H$ fixes the Witt constants and $\theta(z)$. For $g \in H$ the chart action satisfies

$$gs - s \in dEA \quad (s \in S), \qquad gy - y \in dB^+ \quad (y \in S_B). \tag{6}$$

For the first inclusion gz/z is the Teichmüller lift of a tilt element of untilt one. Its difference from one is divisible by $\mu = \varphi(\tau)$, by the cyclotomic kernel criterion. Thus $gz - z \in z\mu A = dEA$, and this extends from z to power series since the ideals are closed and the constants are fixed.

For the second inclusion, first divide $gy - y$ by z in B^+. To see that the quotient exists, approximate y by elements of $S[E/p][1/p]$. On S the difference has a factor z, since g fixes the constant term. For the divided generator, $(gE - E)/(pz)$ is bounded. Indeed $E = p + z^e u$, and $gz/z - 1 \in \mu A$ and $gu - u \in z\mu A$ show $gE - E \in z^e \mu A$. Now $\mu/p \in A_{\max}$. For convergence of the divided differences, multiplication by z in B^+ has a continuous inverse on its image: if $zv \in A_{\max}$, then $z^e v \in A_{\max}$. Put $q = z^e/p = (E/p - 1)/u$. With $X = \xi/p$, the image of q in $A_{\max}/p = (A/(p, \xi))[X]$ has unit leading coefficient $\overline{E/\xi}/\bar{u}$, hence is a nonzero divisor. Thus $qw \in p^n A_{\max}$ implies $w \in p^n A_{\max}$ for $w \in A_{\max}$, by induction. Clearing a power of p in $q(pv) = z^e v \in A_{\max}$ now gives $pv \in A_{\max}$.

Thus the quotient passes to the limit. Next $\theta(\varphi^j y) \in K'$ for every $j \geq 1$: Frobenius preserves S_B, and its θ-image lies in K', by density of $S[E/p][1/p]$ and completeness of K'. It follows that $\theta\varphi^j(gy - y) = 0$. Since $\theta\varphi^j(z) \neq 0$, the bounded-ring cyclotomic kernel criterion puts $(gy - y)/z$ in τB^+. This proves the second inclusion. For $x \in N$, (1) gives $h = E^b \lambda(x) \in S_B$ for each crystalline λ.

Equivariance and (6) give

$$E^{2b}\lambda(gx - x) = \frac{E^b}{(gE)^b}(E^b(gh - h) + (E^b - (gE)^b)h) \in dB^+,$$

since gE/E is an integral unit. Write $gx - x = \sum_i c_i n_i$, with $c_i \in A$ by stability of M. At $a = 0$, (2) yields

$$E^{3b}c = YE^{2b}L_0(gx - x) \in (dB^+)^d.$$

Equation (5) gives $c_i \in dA$ for each i, hence $(gx - x)/d \in M$. For $x = y/D \in N_{\mathrm{loc}}$, write

$$\frac{gx - x}{d} = D^{-1}\left(\frac{gy - y}{d} - \frac{gD - D}{d}\,gx\right).$$

Each factor gE_a is associated to E_a and (6) applies to D. Thus the quotient belongs to M_{loc}. Further $\theta\varphi^{-a}(d) \neq 0$ for every $a \geq 0$, as both z and τ have nonzero values there. Therefore division by d preserves every recorded de Rham order and the coefficient index.

B.2.2 Retraction at each order

In this subsection, we construct the retraction compatible with the operators of Section B.2.1.

Let $F : A \to S$ be any S-linear coordinate map, $F_a = \varphi^{-a} F \varphi^a : A \to A$. On A it commutes with multiplication by $\varphi^{-a}(s)$ for every $s \in S$, since F is S-linear. In particular F_a is S-linear by $\varphi^a S \subset S$. Extend it to B^+ by applying it to coefficients of divided polynomials in E/p and then taking limits. Each scalar $\varphi^{-a}(s)$ belongs to A, so its commutation survives this extension. The extended map is also S_B-linear. Since $\theta(S_B) \supset K'$, successive Newton approximation lifts each constant of K' to S_B modulo any specified positive de Rham order. Precisely, for $c \in K'$ take its minimal polynomial $P \in \mathbb{Q}_p[X]$. The root c is simple. Choose lifts in S_B of c and of $P'(c)^{-1}$. Subtracting the latter lift times the polynomial evaluated at the current approximation raises the precision by one.

Consequently, for $u_i \in B^+$, $c_i \in K'$, replacing u_i by $\widehat{F}_a(u_i)$ can only increase the order of $\sum_i c_i u_i$. For positive precision, approximate the c_i by elements of S_B, use $B^+ \cap \mathrm{Fil}^j = E^j B^+$, and apply S_B-linearity. Order zero is automatic. For the values $v_i \in \mathcal{K}_a$, clear a common nonzero denominator and expand each numerator as a finite sum of $c\,b\,\varphi^{-a}(s)$ with $c \in K', b \in S_B, s \in S$.

Applying the preceding argument to the resulting single finite sum and using both scalar actions gives

$$\operatorname{ord}_t\Big(\sum_i \varphi^{-a}(F(r_i))v_i\Big) \;\geq\; \operatorname{ord}_t\Big(\sum_i \varphi^{-a}(r_i)v_i\Big), \qquad r_i \in A. \tag{7}$$

The clearing factor shifts both orders equally. Separatedness includes the case of an exact relation.

Take $F = \operatorname{coord}_0$, which fixes 1. At each cut the exact integral intersection inside B_{dR} is

$$A \cap \mathcal{K}_a = \varphi^{-a} S.$$

Indeed, for b in the intersection apply (7) to the zero sum $b - \varphi^{-a}(\varphi^a b)$, with values $b, -1$ and coefficients $1, \varphi^a b$. This gives $b = \varphi^{-a} F(\varphi^a b)$. The reverse inclusion follows from the definition of $\mathcal{K}_a$.

If $x \in M$ has all its form values in $\mathcal{K}_0$, apply (7) to the relation $-x + \sum_i r_i n_i = 0$. Crystalline forms separate vectors by the inverse period matrix. Thus $x = \sum_i F(r_i) n_i \in N$, proving the intrinsic description of N. For the positive Frobenius basis at level l, apply the same argument to $x = \sum_i \varphi^l(r_i)\varphi^l(n_i)$. By (1) the form values on $\varphi^l(n_i)$ belong to $\mathcal{K}_0$, so the coefficients of a vector satisfying the value-field test can be replaced by $F(\varphi^l(r_i)) \in S$. Thus the tested vectors in $\varphi^l M_T$ form the S-span of $\varphi^l(N_T)$.

Let D be a finite product of the E_j, allowing repetitions and the empty product. On the localized modules define

$$R_T\left(D^{-1}\sum_i r_i n_i\right) = D^{-1}\sum_i F(r_i) n_i, \qquad r_i \in A.$$

Passing to a common denominator gives the same expression, by the independence of the n_i and S-linearity of F. In particular $R_T(ay) = F(a)y$ for $a \in A$ and $y \in N_{\mathrm{loc}}$, and R_T fixes N_{loc}. For a chart-equivariant form λ, put $v_a(y) = \lambda(\varphi^{-a} y)$. For $y \in N_{\mathrm{loc}}$ we have $v_a(y) \in \mathcal{K}_a$ by (1) and $\varphi^{-a}(n) = n\varphi^{-a}(C_a^{-1})$. Write $x = \sum_i r_i (n_i/D)$. Applying (7) to the values $v_a(n_i/D)$ and the additional value $-v$ with coefficient $1 = F(1)$ gives, for every $a \geq 0$ and $v \in \mathcal{K}_a$,

$$\operatorname{ord}_t(v_a(R_T x) - v) \geq \operatorname{ord}_t(v_a(x) - v).$$

An equivariant lattice map $f : T \to T'$ carries M_T to $M_{T'}$ and N_T to $N_{T'}$ by pulling back the forms in their defining tests. Thus it sends n_i/D into $N_{T',\mathrm{loc}}$; applying the formula above term by term gives $R_{T'} f = f R_T$. In particular the retraction commutes with lattice projections and with the spectral idempotents, which are polynomials in the coefficient action over $W(k)$. For a coefficient automorphism γ, put $T^\gamma = O_\gamma \otimes_O T$, and write $f_\gamma(x) = 1 \otimes x$, where the right scalar action on O_γ is $b \cdot a = b\gamma(a)$. This map is equivariant over $\mathbf{Z}_p$, so

$$R_{T^\gamma} f_\gamma = f_\gamma R_T, \qquad f_\gamma e_i(\zeta) = e_i(\gamma\zeta) f_\gamma,$$

where $e_i(\zeta)$ projects onto the $[i]$-eigenspace of multiplication by the coefficient root ζ. The second identity follows by applying f_γ to its projector polynomial. For a semilinear action on the chart, the congruence below gives the required control after taking residues.

For $v \in \mathcal{K}_a \cap \mathrm{Fil}^0$, choose $r \in A$ and $m \geq 0$ with $p^m v \equiv r \bmod \mathrm{Fil}^1$, using the surjectivity of $\theta : A \to \mathcal{O}_{\mathbb{C}_p}$. In (7) use the two values $p^m v, -1$ and the two coefficients $1, \varphi^a(r)$. This replaces r by $\varphi^{-a} F \varphi^a(r)$ modulo Fil^1. With $\pi_a = \theta(\varphi^{-a} z)$ this gives

$$\theta(v) = p^{-m}\theta\varphi^{-a} F(\varphi^a r) \in K'(\pi_a).$$

Indeed $\theta\varphi^{-a}S = W(k)[\pi_a]$, by Weierstrass division with the Eisenstein polynomial of π_a, obtained by substituting X^{p^a} in the polynomial of $\theta(z)$. Conversely $\varphi^{-a}S$ is in $\mathcal{K}_a$, so the two residue images are

$$\theta\varphi^{-a}(S) = \mathcal{O}_{K'(\pi_a)}, \qquad \theta(\mathcal{K}_a \cap B_{\mathrm{dR}}^{+}) = K'(\pi_a).$$

For chart actions $g, t_1, \dots, t_m$, suppose $(g-1)\sum_i t_i(s) \in E_a A$ for every $s \in S$. Given $v \in \mathcal{K}_a \cap B_{\mathrm{dR}}^{+}$, choose $n \geq 0$ and $s \in S$ with $p^n v \equiv \varphi^{-a} s \bmod \mathrm{Fil}^1$. The actions commute with Frobenius and preserve filtration. Hence $(g-1)\sum_i t_i(v) \in \mathrm{Fil}^1$, since p^n is a de Rham unit. In particular $\pi_0 \in K'$, so an integral value of $x \in N_{\mathrm{loc}}$ has difference of order at least one under H. This permits the additional division by E there, which has order zero at every other level. This proves the two divided differences on the standard-order modules used for the level-zero and level-one lift ideals. The tame generator acting nontrivially on $\theta(z)$ uses the separate polynomial calculation of Section C.2.

B.2.3 Tensor pairings and orders

In this subsection, we apply the order estimates of Section B.2.2 to a marked pairing.

Let $b_0 : T_1 \times T_2 \to T_3$ be an equivariant pairing, bilinear over the coefficient ring. Its scalar extension to $W(\mathbf{C}_p^{\flat})$ is equivariant for Frobenius and the chart Galois action. The localized vectors and their Frobenius translates have coefficients in $A_{(p)} = (A \setminus pA)^{-1}A$: each inverted period has nonzero reduction in $\mathcal{O}_{\mathbf{C}_p^{\flat}}$. The inclusions

$$W(\mathbf{C}_p^{\flat}) \longleftarrow A_{(p)} \longrightarrow B_{\mathrm{dR}}$$

give, on these coefficients, both the Witt reduction modulo p^m and the de Rham evaluation. The coefficient idempotents for an index i commute with this base change, and bilinearity implies $b_0(e_i y, x) = e_i b_0(y, x)$. At level a the relevant index is $j = i^{p^{-a}}$. A form of type u on T_s takes the lattice projected to j into Fil^{-u}.

For each projected factor, choose a K'-basis of the invariant de Rham dual adapted to its filtration. If λ_s has jump u_s, the functionals $t^{u_s}\lambda_s$ form an integral de Rham dual basis, as proved in Section B.3. On the two parts indexed by j, expand the pullback of a form λ of type at most v as

$$\lambda \circ b_0 = \sum_{s,h} \alpha_{sh} \lambda_s \otimes \lambda'_h, \qquad \alpha_{sh} \in K'.$$

Indeed the tensor products give a de Rham basis, and its coefficients are fixed by the chart subgroup. Bilinearity on the original lattices makes $t^v(\lambda \circ b_0)$ integral; in the rescaled dual bases this means $t^{v-u_s-u'_h}\alpha_{sh} \in B_{\mathrm{dR}}^{+}$. A nonzero scalar in K' has de Rham order zero, so $\alpha_{sh} = 0$ if $u_s + u'_h > v$. For two vectors whose orders are bounded below by c_1, c_2, it follows that

$$\operatorname{ord}_t \lambda(\varphi^{-a} b_0(x_1, x_2)) \;\geq\; \min_{u+u' \leq v} \big(c_1(a,u) + c_2(a,u')\big),$$

where u, u' run through the jumps at j, and the minimum of an empty set is $+\infty$.

For $x_s \in M_{T_s}$, bilinearity on the original lattices first gives $b_0(x_1, x_2) \in A[1/\tau] \otimes T_3$. For every $n > 0$, expand the pullback of any invariant form on T_3 in products of invariant dual bases of the two factors. All coefficients lie in K'; both evaluations on $\varphi^n x_s$ lie in B_{dR}^{+} by the defining test for M_{T_s}. This proves $b_0(x_1, x_2) \in M_{T_3}$. For localized vectors with denominators D_1, D_2, the product $D_1 D_2$ clears the paired vector, giving the same conclusion in $M_{T_3,\mathrm{loc}}$.

B.3 Standard orders and projections on the chart

In this subsection, we identify the lattices defined by the filtration in Section B.2 with $\varphi^l M_T$. Frobenius transports the projection cokernel along the residue-field embeddings, as used in Section D.2.

For a lattice T crystalline on the chart subgroup use the two bases $M_T = \sum An_i$ and $N_T = \sum Sn_i$ just constructed. Write $E_a = \varphi^a E$ for $a \geq 0$, and localize both bases at the E_a. A residue index is a nonzero residue-field element i. The projector to its part is defined by the eigenvalue $[i]$ for the chosen coefficient root of unity. Its polynomial formula has coefficients in $W(k)$, since differences between distinct Teichmüller lifts are units. It therefore commutes with the coordinate retraction. At $a \geq 0$ a vector in the part i has φ^{-a}-translate in the part $i^{p^{-a}}$.

Let λ be a chart-equivariant de Rham form of type u on that latter part, meaning that its restriction to the original lattice, after projection, takes values in $t^{-u} B^+_{\mathrm{d}R}$. The Hodge filtration is split after choosing an adapted basis of invariant forms. In particular an element of the de Rham vector space belongs to the original $B^+_{\mathrm{d}R}$-lattice precisely when its values on the basis forms of types u are in $t^{-u} B^+_{\mathrm{d}R}$.

Indeed the reductions of the rescaled forms $t^u \lambda$ are independent over $\mathbf{C}_p$. In a minimal dependence, normalize one coefficient to 1. The remaining coefficients transform under H by the differences of the powers of the cyclotomic character, by uniqueness of these coefficients. Tate's identities $\mathbf{C}_p^H = K'$ and $\mathbf{C}_p(n)^H = 0$ for $n \neq 0$ apply to the cyclotomic powers here. The cyclotomic tower over K' has bounded residue degree: the tower over $\mathbb{Q}_p$ is totally ramified and adjoining K' increases the degree by at most $[K' : \mathbb{Q}_p]$. Choose a finite cyclotomic layer L/K' above which the residue degree is constant and the cyclotomic character χ has torsion-free image. For $n \neq 0$, the restrictions of χ^n and χ to G_L have the same kernel. The field fixed by this kernel is totally ramified over L with Galois group $\mathbf{Z}_p$, also for $p = 2$. Tate's Theorem 2 therefore gives $\mathbf{C}_p(n)^{G_{K'}} \subseteq \mathbf{C}_p(n)^{G_L} = 0$. The identities force all the types in the dependence to coincide and all its coefficients to belong to K'. Lifting this relation with these constant coefficients gives a relation in the corresponding graded space, contrary to the adapted basis [72, §3, Theorems 1 and 2].

The rescaled forms therefore give a basis of the $B^+_{\mathrm{d}R}$-dual: their number is the dimension of the projected space by de Rham comparison. For each residue index j, fix the finite jump set $\mathcal{J}_j$ of the filtered invariant dual there. A vector $x \in e_i M_{T,\mathrm{loc}}$ has order bounds c when for all $a \geq 0$, $j = i^{p^{-a}}$, $u \in \mathcal{J}_j$, and all chart-equivariant forms of type at most u on the part at j,

$$\operatorname{ord}_t \lambda(\varphi^{-a} x) \geq c(a, u).$$

The submodule with these orders consists of the intersection of the respective linear conditions. Its intersection with $N_{T,\mathrm{loc}}$ is the module of chart vectors with those orders. The coordinate retraction maps the first module onto the latter, preserves congruences of the values to constants, and commutes with an equivariant projection of lattices.

Define $\mathrm{std}_l(a, u) = 0$ for $a < l$, and $\mathrm{std}_l(a, u) = -u$ for $a \geq l$. The part at i having these orders is precisely the part at i of $\varphi^l M_T$. To prove the direct inclusion, $\varphi^l M_T$ is contained in $M_{T,\mathrm{loc}}$ by $E^b \varphi(N_T) \subset N_T$ and induction. Write a vector there as $\varphi^l y$. At $a < l$, positive Frobenius integrality of y gives order at least zero. At $a \geq l$, a power of τ makes y an original integral vector.

That power has order zero under every $\theta \varphi^{-(a-l)}$, so the lattice condition gives order at least $-u$. Conversely, write x_α for the coordinates of x in an original $\mathbf{Z}_p$-basis of T. Choose

$D = \tau^s \prod_a E_a^{m_a}$ with finite support and $Dx_\alpha \in A$ for all α. For $a \geq l$ the adapted-basis test gives $\varphi^{-a}(x_\alpha) \in B_{\mathrm{dR}}^+$. Whenever $E_a \mid D$, it follows that

$$\theta\varphi^{-a}(Dx_\alpha) = 0, \qquad Dx_\alpha \in E_a A.$$

Replacing D by D/E_a preserves integrality for every coordinate. Repeating removes all factors with $a \geq l$.

For $a < l$ the remaining E_a divide $\varphi^l\tau$, since $\varphi^l\tau = \tau \prod_{j=0}^{l-1} \varphi^j\xi$. The denominator of $\varphi^{-l}x$ is therefore cleared by a power of τ. For $1 \leq n \leq l$, the test on $\varphi^n(\varphi^{-l}x)$ has order at least zero at $a = l - n$. For $n > l$, choose a product D' of the E_a with $D'x \in M_T$. The defining positive-Frobenius test applies to $\varphi^{n-l}(D'x)$, while $\varphi^{n-l}(D')$ is a unit in B_{dR}^+. Indeed, write $E = p + z^e U$ with $U \in \mathfrak{S}^\times$. For any integer $h \neq 0$, the two terms of $\theta\varphi^h(E)$ have distinct valuations 1 and p^h, so

$$v_p(\theta\varphi^h(E)) = \min\{1, p^h\}.$$

In particular its residue in $\mathbf{C}_p$ is nonzero. This also gives $\mathrm{ord}_a(E_b) = 0$ for $a \neq b$, whereas $\mathrm{ord}_a(E_a) = 1$ since E_a generates $\ker(\theta\varphi^{-a})$. Thus $\varphi^{-l}x \in M_T$, proving the equality.

For use on extensions, let $X \to C$ be the projection to the trivial parameter lattice in any of the three sides. The localized equivariant projection and the retraction commute. Define $I_l(i)$ as the ideal of multipliers of the parameter lattice of component i into the image of the standard-order module. With a modulus $q = p^b$, $f \in I_l(i) + (q)$ means that a common scalar $w \in A$ exists such that $(f - qw)c_i$ lifts for every parameter constant c_i in that component. If $f \in S$, applying the coordinate retraction puts w in S and the lifts in $N_{X,\mathrm{loc}}$, with their orders. This uses the formula $R(ax) = F_0(a)x$ for $x \in N_{C,\mathrm{loc}}$. Indeed C has trivial action and jump zero, so (3) with $b = 0$ gives $M_C = A \otimes_{\mathbf{Z}_p} C$. Equivariance puts the values of every form on $1 \otimes c$ in $B_{\mathrm{dR}}^H = K'$. Thus $1 \otimes c \in N_C$. Every element of N_C is fixed by R_C, so applying F_0 to its coefficients in a $\mathbf{Z}_p$-basis of C gives $N_C = S \otimes_{\mathbf{Z}_p} C$. Its component $e_i N_C$ is generated over S by $c_i = e_i(1 \otimes c)$.

Frobenius on the Witt coefficients carries $e_i\varphi^l M_X$ bijectively onto $e_{i^p}\varphi^{l+1}M_X$ and sends c_i to c_{i^p}. For $u, w \in A$, the lift equation becomes

$$\pi(x) = (u - qw)c_i \quad \Longleftrightarrow \quad \pi(\varphi x) = (\varphi(u) - q\varphi(w))c_{i^p}.$$

The same $\varphi(w)$ works for every c. Since Witt Frobenius is bijective on $A = W(\mathcal{O}^\flat_{\mathbf{C}_p})$ and fixes q, this gives $\varphi(I_l(i) + (q)) = I_{l+1}(i^p) + (q)$. Multiplication by E_a raises precisely the level-a orders by 1. Indeed the distinct primes E_j have $\theta\varphi^{-j}(E_a) \neq 0$ for $a \neq j$, and E_a has order one at its own level. These facts justify the level-one lift after multiplying by E^ω.

For g in the chart subgroup and $x \in N_{T,\mathrm{loc}}$ of standard order l, Section B.2.1 gives $(gx - x)/(z\tau) \in M_{T,\mathrm{loc}}$ with the same lower bounds on its orders. When $l = 1$, we may also divide by E while preserving these bounds. To see this, apply at level zero the coordinate congruence of the preceding section. A chart-form value of order zero has residue in the fixed field of the chart, so its g-difference has order at least one. The other levels are unaffected by division by E. On the trivial parameter, the same quotient of a chart scalar $w \in S$ is integral by $gw - w \in z\tau EA$. Thus division and subsequent retraction give the integral lift-ideal operations with divisor $z\tau$ at level zero and $z\tau E$ at level one, for the original parameter, index, and modulus.

C The tame cup-product estimate

In this appendix, we apply the filtration orders of Appendix B on the tame chart. The level-zero estimates control the four terms in the cup calculation and give the bound at the tame level primes in Section 29.1.

C.1 Level-zero steps for tame periods

In this subsection, we use the divided difference in Section B.3 to raise the order of a Galois difference at $E = 0$. The valuation of the scalar $(\omega^j - 1)/(z\tau)$ determines the power of p required at each step. The operator preserves every filtration bound already attained.

Let K' be the tame chart field and $H = G_{K'}$. At the level primes of Section C.2.2, $K' = K_0(\pi)$, where $K_0/\mathbf{Q}_p$ is unramified of degree $r!$, $e = \operatorname{lcm}_{1\le b\le r}(p^b - 1)$, and $\pi^e = p$. Write $E = p - z^e$, $\Delta_z = z\tau$, and $\mu = E\tau v_0$ with $v_0 \in A^\times$. Choose $g \in H$ such that $gz = [\epsilon]z$. Indeed, $X^{p^a} - \theta(z)$ is Eisenstein over K', so at each finite level such a conjugate of the chosen root exists. The conditions are nested and H is compact. Write $gE = E\omega$, where $\omega \in A^\times$ since g preserves $\ker\theta$. Then

$$\frac{\omega - 1}{\Delta_z} = -v_0 z^{e-1} \sum_{i=0}^{e-1} [\epsilon]^i.$$

Its value under θ has normalized p-adic valuation $(e-1)/e$, since e is prime to p. Let $\delta = 1$ for $e > 1$ and $\delta = 0$ for $e = 1$. For every nonzero $j \in \mathbf{Z}$, put $X_j = (\omega^j - 1)/\Delta_z$. Since $v_p(\theta(z)) = 1/e$ and $v_p(\theta(\tau)) = 1/(p-1)$, we have

$$v_p(\theta(\omega) - 1) = \frac{p}{p-1}, \qquad v_p(\theta(X_j)) = \frac{e-1}{e} + v_p(j).$$

Indeed, for $v_p(a) > 1/(p-1)$ the unique lowest term of $(1+a)^p - 1$ is pa. A power prime to p preserves $v_p(a)$ by the geometric sum, and inversion also preserves it. In the following steps $p \nmid j$. Thus there is $v_j \in A$ with $v_j X_j \equiv p^\delta \bmod E$, by surjectivity of θ. These are precisely the nonzero steps used at a tame level prime, since the arithmetic radius excludes all nonzero multiples of p.

If y is fixed by H modulo $\operatorname{Fil}^j B_{\mathrm{dR}}$, it is congruent to an element of K' modulo that filtration. Indeed, a nonzero leading class below degree j is a fixed vector in $\operatorname{gr}^n B_{\mathrm{dR}} \simeq \mathbf{C}_p(n)$. By Tate's invariant theorem it must have $n = 0$ and coefficient in the chart field. Subtract that constant. Any remaining nonzero leading class below j would have positive degree and is excluded by the same theorem. See [72, Theorems 1–2].

Suppose x is a localized chart point over a constant parameter. The depth bound at j means that its selected values differ from constants by Fil^j, by the preceding criterion. Let $y = \lambda(x) - c$ be one such difference with $c \in K'$. By (1), for a sufficiently large integer $b_0 \ge \max(0, -j)$ one has

$$E^{b_0} y = \sum_i \beta_i s_i, \qquad \beta_i \in K', \quad s_i \in S_B.$$

In fact x is in the chart span of the period basis localized at the E_a, and $E_a \in S_B^\times$ for $a > 0$, by the Frobenius computation in Section A.4. Approximate the β_i by elements of S_B to order $b_0 + j + 1$, using Section A.5. The resulting sum lies in S_B and is divisible by E^{b_0+j} in B^+, by the residue-kernel property. Its quotient still commutes with each extended chart endomorphism,

by multiplying the commutation identity by E^{b_0+j} and canceling. The quotient therefore lies in S_B and has residue in K'. This residue equals that of $E^{-j}y$. Since g fixes K' and $gE = \omega E$, we have

$$gy - \omega^j y \in \mathrm{Fil}^{j+1} B_{\mathrm{d}R}.$$

Consequently

$$x \longmapsto R\Big(p^{\delta}x - \upsilon_j(gx - x)/\Delta_z\Big)$$

moves the values to the next filtration step modulo the scaled constants. The divided difference and its retraction preserve the old orders and lie over zero. This follows from Section B.3, since g fixes the parameter constant and Δ_z has order zero at all places of the chart. At $j = 0$ a chart value that is integral at E has residue fixed by the chart group, so $p^{\delta}x$ already advances this step. Thus the contribution of each step is at most δ.

Perform the steps in increasing order of j. For a line whose target depth is $j+1$, all earlier needed steps have then been performed, while the operator preserves previously obtained depths at every line. Indeed a value congruent to a constant modulo Fil^h has Galois difference of order at least h. Division by Δ_z has order zero, and the retraction preserves the congruence. This advances the orders over the finite union of step intervals. If a form value originally had positive order, the constant in its higher-depth congruence must be zero. Otherwise the new guaranteed order is the minimum of 0 and the new depth. In both cases the operator also retains the initially imposed orders. In the tame application the starting order is $-u$, and the final depth at zero is $\eta - u$. The final order is at least $-u$. On the negative-far side $u \le \eta$, so its final order at zero is at least $\max(0, -u - \eta)$ and in particular is nonnegative.

C.2 The four terms of the crystalline cup calculation

In this subsection, we prove the tame cup estimate by comparing the two pairs of crystalline lifts before and after Frobenius. The order estimates of Section C.1 place four bilinear terms in the image of the Witt vectors of the maximal ideal of the tilt. The Artin–Schreier–Witt equation then produces a continuous cochain whose differential is the required cup product.

On both sides choose the resulting points at each index with the same constant multiplier, multiplying by extra powers of p to reach the common exponent. Their orders are at least $-u$ at every level and their depths are $\eta - u$ at zero and $-u$ above. A difference of two points over the same parameter belongs to the kernel lattice with those orders.

For strictness of the filtration, take an invariant form λ of type u on the kernel and a crystalline primitive $b(c)$ for the cocycle of a basis element c of the trivial parameter lattice over $\mathbf{Z}_p$. Since

$$(g-1)\lambda(b(c)) = \lambda(z(c)(g)) \in \mathrm{Fil}^{-u},$$

the invariant calculation in Section C.1 gives $k(c) \in K'$ with $\lambda(b(c)) - k(c) \in \mathrm{Fil}^{-u}$. Extend λ on (v, c) by adding this last value to $\lambda(v)$, and extend linearly over that parameter basis. The resulting form is equivariant and has type at most u. The integral localized basis of the kernel is the intersection with its ambient subspace, by clearing the E_a and applying the positive Frobenius tests. Hence the order bounds on a difference apply to its original kernel components.

At index i, write x, x' for the two current points and y, y' for the Frobenius translates from the preceding index. These old points have order 0 at level zero and at least $-u$ at every positive level. Their differences under Galois have order at least 1 at level zero: for a chart vector in N_{loc}, the values of its Frobenius translate under θ belong to the chart field, by the change of

basis, the invariance of the period covector, and the residue map on S_B. At higher levels the old differences inherit the depth at the preceding index.

Suppose the first side is the negative-far side. Take the four-move path with intermediate points $w = x, w' = x'$. The first move $y - x$ has order at least zero at level zero because x and y both do. Use the Frobenius translate of the preceding third period, of order zero at level zero. The second-side resting difference at y' has order at least 1 there, giving total at least 1.

For the trilinear form with target the Tate twist, the nonzero coefficients in adapted bases have jumps satisfying

$$u + u' + u_3 \leq -1$$

by the tensor estimate of Section B.2.3. On each such term at a positive level, the first two orders sum to at least $-u - u'$, while the translated third period has order at least $-u_3 \geq u + u' + 1$. In the second move $y' - x'$, the moving side has $u' \geq -\eta$, so its order is at least $-u' - \eta$ at zero. The resting difference at x has depth $\eta - u$, and the current third period contributes $u + u' + 1$. Thus total order is at least 1, also at higher levels using the standard orders.

The last two moves are zero. If the second side is negative-far, use $w = y, w' = x'$. The first and fourth moves are zero, the second uses the preceding third period, and the third uses the current third period. The same order bounds apply with the sides interchanged. The pair of cocycle differences at two current points also passes the order test with the current third period, since both depths dominate their respective standard orders.

In each active test the factor Δ_z comes from the Galois difference of the resting point, or one of the two cocycle differences. Dividing by Δ_z preserves its orders and the integral localized span by the basis calculation. The third extension form is equivariant, so orders add by the filtered tensor rule. The resulting Tate value has its standard orders 1 at every level. It is therefore in the Tate basis at level zero by Section B.3. The Tate basis bound says that multiplication by τ puts it in $A \otimes \mathbf{Z}_p(1)$. The remaining factor z places the reduced pairing in the small Witt ideal. This verifies all the order tests for the shared Frobenius cochain calculation. Summing over indices preserves the tests because the equivariant bilinear pairing is O-balanced, so distinct spectral idempotents have zero cross pairing.

C.2.1 The Frobenius cup calculation

In this subsection, we use the four terms and depth bounds in Section C.2 to construct the cochain.

Fix two extensions $Y \oplus O, D \oplus O$ for z_1, z_2 and the third period of the pairing. Work over the chart subgroup H. Write x, w for first-side points over p^{e_1}, and x', w' for second-side points over p^{e_2}. The two endpoints and intermediate points give the path

$$(\varphi x, \varphi x') \longrightarrow (w, \varphi x') \longrightarrow (w, w') \longrightarrow (x, w') \longrightarrow (x, x').$$

In the following computation the order tests for the four moves mean that their four cross pairings, with the resting difference as the other argument, lie in the subgroup

$$I = W_m(\mathfrak{m}_{\mathbf{C}_p^\flat}) \otimes \mathbf{Z}_p(1) \ \subset \ W_m(\mathbf{C}_p^\flat) \otimes \mathbf{Z}_p(1).$$

Here $W_m(\mathfrak{m}_{\mathbf{C}_p^\flat})$ is the kernel of reduction on the integral Witt ring $W_m(\mathcal{O}_{\mathbf{C}_p^\flat})$. Congruences in the ambient module below use its additive subgroup I.

Fix the projection onto the first summand of each presentation $X_1 = Y \oplus O$, $X_2 = D \oplus O$. For the chosen $\mathbf{Z}_p$-linear functional $\lambda : O \to \mathbf{Z}_p$, put

$$\psi_m = ((\lambda \otimes 1) \circ \psi) \bmod p^m : (Y/p^m Y) \times (D/p^m D) \longrightarrow (\mathbf{Z}/p^m)(1).$$

Let B be the Witt-bilinear extension of $p^{e_3}\psi_m$. On arguments from X_1, X_2, apply their respective projections before evaluating B. For differences over zero these projections identify the actual kernels. The third period over p^{e_3} gives this multiplier by its extension form. These reduced pairings take values in $W_m(\mathbf{C}_p^\flat)(1)$: the inverted elements τ, E_a have nonzero reductions in the field $\mathbf{C}_p^\flat$, so they are units of $W(\mathbf{C}_p^\flat)$. The smallness order test is applied to points over zero. The sum of the three orders must be at least 1, the Tate standard order. A difference of Galois translates has the additional factor $z\tau$, which puts the integral Tate value in I. Indeed τ clears the Tate basis. Write $z = [\pi^\flat]$ with $\pi^\flat \in \mathfrak{m}_{\mathbf{C}_p^\flat}$. On integral Witt coordinates $(b_j)_{0 \le j < m}$, multiplication by z sends b_j to $(\pi^\flat)^{p^j} b_j$, which belongs to that maximal ideal for every j.

Solve

$$(\varphi - 1)y = B(w, \varphi x' - w') + B(x, w' - x').$$

Frobenius acts on the Witt scalar and commutes with Galois. Its difference is onto by Witt coordinate induction and the Artin–Schreier equation over the algebraically closed field $\mathbf{C}_p^\flat$ [13, p. 223]. Write $a = \mathrm{pr}_Y x$ and $b = \mathrm{pr}_D x'$. For $g \in H$ set $A_g = gx - x$, $D_g = gx' - x'$ and $c_{i,g} = p^{e_i} z_i(g)$, so that

$$A_g = ga - a + c_{1,g}, \qquad D_g = gb - b + c_{2,g}.$$

Define

$$F(g) = B(\mathrm{pr}_Y(gx), gb) - g(B(a,b) - y) - B(a, D_g) - y = B(c_{1,g}, gb) - B(a, D_g) + (g-1)y.$$

Here gb is the linear action on the kernel of the second extension. Since φ fixes $c_{i,g}$ and commutes with g, expanding $(\varphi - 1)F(g)$ and substituting the equation for y gives the four terms

$$\begin{aligned}(\varphi - 1)F(g) = {} & B(gw - w, g(\varphi x' - w')) \\ & - B(\varphi x - w, g\varphi x' - \varphi x') \\ & - B(w - x, gw' - w') + B(gx - x, g(w' - x')).\end{aligned}$$

Each term lies in I by the indicated path.

Whenever $(\varphi - 1)d \in I$, there is a unique $\gamma \in (\mathbf{Z}/p^m)(1)$ with $d - \gamma \in I$. Indeed Frobenius acts only on the Witt factor. On the successive Verschiebung quotients of $W_m(\mathfrak{m}_{\mathbf{C}_p^\flat})$, $\varphi - 1$ induces $v \mapsto v^p - v$. For each u in the maximal ideal, $v^p - v = u$ has a unique root there by Hensel's lemma at zero. Induction up this finite filtration makes $\varphi - 1$ bijective on I. Subtract from d the unique solution in I with the same Frobenius difference. Its remainder lies in $W_m(\mathbf{F}_p)(1) = (\mathbf{Z}/p^m)(1)$, the Frobenius-fixed submodule. Apply this to the summed value $d = F(g)$ to define $\gamma(g)$.

The differential of F satisfies

$$\begin{aligned}(dF)(g,h) &= B(c_{1,g}, gD_h) - B(c_{1,g}, g(hb - b)) - B(A_g, gD_h) \\ &= B(c_{1,g}, gc_{2,h}) - B(A_g, gD_h).\end{aligned}$$

The first term on the last line is $p^{e_1+e_2+e_3}(z_1 \smile_{\psi_m} z_2)(g,h)$ by the definition of B. The same order test applied to the two cocycle differences puts the final term in I. As I is stable

under the Galois action, the identity modulo I is exactly $p^{e_1+e_2+e_3}(z_1 \smile_{\psi_m} z_2) = d\gamma$ on the chart subgroup. This is an identity of continuous cochains. The Galois action on the tilt is continuous. For an integral tilt element, its class to any fixed valuation precision is determined by a sufficiently high component in $\mathcal{O}_{\mathbf{C}_p}/p$. The action on that component is continuous. Quotients give continuity for arbitrary tilt elements. The input cocycles are continuous and the finite Witt operations are polynomial, so $F(g)$ is continuous at 1. Since $F(1) = 0$ and the tilt maximal ideal is open, $F(g) \in I$ and $\gamma(g) = 0$ on a neighborhood of 1.

Take h in this neighborhood with $c_{2,h} = 0$ modulo p^m, a further open condition. The differential identity gives $\gamma(gh) = \gamma(g)$, so γ is locally constant. For the O/p^mO-valued pairing before taking scalar readings, choose a $\mathbf{Z}_p$-basis (b_i) of O with coordinate forms λ_i. Then

$$\Gamma_O = \sum_i b_i \gamma_{\lambda_i}, \qquad d\Gamma_O = p^{e_1+e_2+e_3}(z_1 \smile_{\psi \bmod p^m} z_2) \quad \text{in } (O/p^mO)(1)$$

on the chart subgroup. Thus the finite-level cochain is continuous and integral over the original coefficient ring. Let Γ be the Galois group of the original completion and $\alpha \in H^2(\Gamma, \mu_{p^m})$ the scaled cup class just computed. Then

$$[\Gamma : H]\alpha = \operatorname{cor}_H^\Gamma \operatorname{res}_H^\Gamma \alpha = 0.$$

At $p \mid M$ the index is $e\,r!$, which is prime to p by Section C.2.2. In Section E.3 one takes $H = \Gamma = G_{E_v}$. In both cases $p^m\alpha = 0$ gives $\alpha = 0$.

C.2.2 The three extensions at a prescribed level prime

In this subsection, we apply Section C.2.1 on the original local group and bound the common exponent.

For the chosen infinite place τ_∞, put $p = \ell \mid M$, $\beta = S_j(\tau_\infty)$, $W_* = 4t$, $N = (r+t)r$, $k_0 = r^2(N+1)^2$, and $n_0 = 6(\eta + W_*)$. Each prime $v \mid p_0p_1$ has $E_v = \mathbf{Q}_p$ by Section 6. Proposition 4.1 gives $p \geq b^{10}$. Since $\beta = \kappa_{\tau_\infty} + \delta_j$ with $\kappa_{\tau_\infty} \leq b$, while δ_j, n_0, k_0 are polynomially bounded in r, we have $p > \beta + n_0 + k_0 + 2$ at the common cutoff. Set $D = L^\dagger$ for the original summand in the colength estimate. Fix $Y \in \mathcal{Q}_v$, crystalline cocycles z_1, z_2 of Y, D, and an integral bilinear form ψ giving one of the pairings modulo p^m. The third lattice is $J = (Y \otimes_O D)^\vee(1)$. Its cocycle is

$$z_3(g) = (g\psi - \psi)/p^m.$$

It is integral by finite-level equivariance, and crystalline with period primitive ψ/p^m. On these three lattices $V_1 = Y, V_2 = D, V_3 = J$, use the split-as-modules extensions

$$X_i = V_i \oplus O, \qquad g(v, c) = (gv + c\, z_i(g), c).$$

The constructions are O-linear in c. For X_3 keep this parameter even if $z_3 = 0$. Its extension form is

$$E_3((\lambda, c), y \otimes d) = p^m\lambda(y \otimes d) + c\,\psi(y \otimes d).$$

The map $e_m : X_3 \to J$, $e_m(\lambda, c) = p^m\lambda + c\psi$, is equivariant since $p^m z_3(g) = g\psi - \psi$. If f is a chart-equivariant form of type u on J, then $p^{-m} f \circ e_m$ extends f to X_3 with the same type: $\psi \in J$ and p has de Rham order zero. The positive Frobenius tests and the orders at every cut

therefore restrict to the original kernel J. The integral map e_m commutes with the coordinate retractions by Section B.2.2. Its reduction modulo p^m is $c\psi$. Take $1 \in O$ as the third parameter. A standard lift over $p^{k_0} \cdot 1$ gives $p^{k_0}\psi$ modulo p^m, also when $z_3 = 0$.

Let $e = \operatorname{lcm}_{1 \le a \le r}(p^a - 1)$. Put $q = p^{r!}$ and $K_0 = W(\mathbf{F}_q)[1/p]$. The unramified field of degree $\operatorname{lcm}(1, \dots, r)$ used in Section 28.3.2 lies in K_0. Since $a \mid r!$ for $1 \le a \le r$, we have $e \mid q - 1$ and $\mu_e \subset K_0$. Take $K' = K_0(\pi)$ with $\pi^e = p$, and put $H = G_{K'}$. The Eisenstein polynomial $X^e - p$ has all its roots in $\mu_e \pi \subset K'$, and $K_0/\mathbf{Q}_p$ is Galois. Thus $K'/\mathbf{Q}_p$ is Galois. On the original completion, H is exactly the pointwise stabilizer of $W(\mathbf{F}_q)$ and π, independently of the choice of this root. Its index is

$$[G_{E_v} : H] = [K' : K_0][K_0 : \mathbf{Q}_p] = e\, r!, \qquad (p, e\, r!) = 1$$

since $p > r$. The extension K'/K_0 is totally ramified, so K_0 is the maximal unramified subfield of K'. On inertia I_p the character $g \mapsto g(\pi)/\pi$ has image μ_e and kernel $I_p \cap H$: I_p fixes K_0 and $X^e - p$ remains Eisenstein over every finite unramified extension. The type in Section 28.3.2 and the defining polynomials of $\mathcal{Q}_v$ give $X^e - 1 = 0$ on the linear inertia operators of Y. This identity also holds on $D = L^\dagger$, since inversion preserves μ_e and $\mathbf{Q}_p(1)$ is crystalline with trivial linear inertia on its period line. Wild inertia has finite p-power image on the periods of Y, D; the prime-to-p exponent e makes that image trivial. In the procyclic tame quotient the kernel of the displayed character consists of the e-th powers, which act trivially on these periods.

For $V = Y[1/p]$ or $D[1/p]$, choose a finite extension $L/\mathbf{Q}_p$, Galois and containing K', over which V is crystalline, by taking a Galois closure. On $D_{\mathrm{cris,L}}(V)$, the inertia subgroup of $\mathrm{Gal}(L/K')$ acts trivially by the preceding polynomial identity. The remaining action is semilinear over L_0/K_0, where L_0 is the maximal unramified subfield of L. Galois descent gives

$$L_0 \otimes_{K_0} D_{\mathrm{cris,K'}}(V) \simeq D_{\mathrm{cris,L}}(V),$$

so Y, D, and their tensor dual J are crystalline over H. Each $z_i|_H$ is a coboundary over the crystalline period ring, hence the three extensions X_i are crystalline there as well. Use the same Kummer coordinate z with $\theta(z) = \pi$, $E = p - z^e$, and an element s of inertia inducing a generator of the tame quotient of order e. It fixes the unramified chart coefficients and satisfies $s(z) = \zeta z$, where ζ has order e. Here $\zeta \bmod p$ has that same order.

To choose s, fix compatible roots π_a of π of degrees p^a, and let $\zeta_a^{p^a} = \zeta$ in μ_e. The polynomial $X^{ep^a} - p$ is Eisenstein over every finite unramified field, so inertia takes π_a to $\zeta_a \pi_a$. The closed conditions at successive a are nested; compactness gives one s for all a. Their Teichmüller ratios give $s(z) = \zeta z$. The different allowance of this chart is at most 1.

Take the first two annihilating polynomials q_i at s over $\mathbf{Z}_p$. Each inertia eigenvalue of these types has order dividing $p^a - 1$ for some $1 \le a \le r$. The value and its inverse are Teichmüller units in an unramified extension of degree a, hence have monic minimal polynomials over $\mathbf{Z}_p$ of degree at most a. Multiplying these polynomials over the original upper type gives degree at most $N = (r + t)r$. Finite inertia is semisimple in characteristic zero. This gives the indicated linear-period annihilators, with the inverse type for D. They hold on the periods of the underlying $\mathbf{Z}_p$-lattice as well. Indeed, after scalar extension the factors of $K_0 \otimes_{\mathbf{Q}_p} \mathrm{Frac}(O)$ are read at pairs of embeddings. Conjugating the coefficients preserves $q_i \in \mathbf{Z}_p[X]$, and crystalline Frobenius cycles the embeddings of K_0 with the coefficient embedding fixed. It commutes with the linear action of s. Thus q_i vanishes on every factor, and faithful scalar extension gives the claimed identity on the original rational period space. For the third side take annihilators of Y, D at s^{-1}. The characteristic polynomial of multiplication by $X \otimes X$ on the tensor product

of their two polynomial quotient algebras is monic of degree at most N^2. By Cayley–Hamilton it kills the tensor period operator at s^{-1}, hence kills the dual operator at s.

The Tate factor has trivial linear inertia on its crystalline period. This is the polynomial q_3. On X_i the polynomial $Q_i = (X-1)q_i$ kills the period action, by the exact sequence with trivial quotient. The possible jump sets for the extensions are the two original intervals with 0 adjoined and $\{0\} \cup \{-u-u'-1\}$ for the third side. Here u, u' range over the two jump sets at the corresponding indices, already with 0 adjoined. The opposite far signs give

$$-(u+u') \le \beta + W_* + 2\eta.$$

Every jump of every extension is thus at most $\omega = \beta + n_0$. All coefficient embeddings restrict to the same chosen embedding of the local field $E_v = \mathbf{Q}_p$. Thus the union of their jump sets still has the specified far sign and center on each side, even if the coefficient ring is ramified. Take the coefficient root of unity in O to be 1. For $i \in \mathbf{F}_q^{\times}$, the projector of Section B.3 therefore acts on each of the three extended lattices by the scalar

$$e_i = \frac{1}{q-1} \sum_{n=0}^{q-2} [i]^{-n} = \begin{cases} 1, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

All coefficient embeddings occur in the component $i = 1$, which Frobenius fixes. The primitive e-th root $\zeta \in W(\mathbf{F}_q)^{\times}$ gives the separate inertia action $s(z) = \zeta z$ on the chart.

The exact cokernel calculation on the basis. Use the integral crystalline basis on each X_i, and the coordinate retraction on its localization by all the $\varphi^a E$. At level l, the standard orders are 0 at $a < l$, and $-u$ at $a \ge l$, where a is Frobenius level and u the jump. At an index let $I_l \subset A_{\mathrm{inf}}$ be the annihilator of the quotient of the trivial parameter module by the image of the standard-order module. Put $J_l = I_l + (p^{k_0+1})$. The basis includes $\tau^b(A_{\mathrm{inf}} \otimes X_i)$ in its standard module for some symmetric jump bound b, where $\tau = [\underline{\epsilon}^{1/p}] - 1$.

Thus $\tau^b \in J_0$ at every index. Modulo p the tilt is a valuation ring with rank-one valuation, and z and τ both have strictly positive value. Consequently z is in the radical of (p, τ). There is a least integer $\nu \ge 0$ for which $z^{\nu} \in J_0$ at every index. Multiplying by E^{ω} passes to level one. Since $z^e \in (E, p)$, binomial expansion puts $z^{\nu+e(\omega+k_0)}$ into J_1. Indeed a term of the added power has either E-degree at least ω or p-degree at least $k_0 + 1$.

For level-one lifts the same divided Kummer difference uses $z\tau E$ as divisor. Since $[\underline{\epsilon}]^j - 1$ is a unit times $\tau \prod_{a=0}^{v_p(j)} \varphi^a E$, division by $z\tau E$ leaves $v_p(j)$ factors $\varphi^a E \equiv p \bmod z$, $1 \le a \le v_p(j)$. Thus (LI1) lowers z^j to z^{j-1} after multiplication by $p^{v_p(j)}$. Each step from an exponent n to $p\lfloor n/p \rfloor$ is therefore free. Inverse Frobenius passes from the level-one lift ideal to the level-zero lift ideal at the preceding index and divides this last exponent by p. Minimality now gives $(p-1)\nu \le e(\omega + k_0)$, so $\nu < e$ by the bound on p.

Suppose $1 \le j \le \nu$ and $p^c z^j \in J_0$. By the definition of J_0, the lift over every constant parameter d may be taken with a common scalar multiplier $p^c z^j - p^{k_0+1} u$. After coordinate retraction $u \in S$. Write $u = w + zv(z)$, where $w \in W(\mathbf{F}_q)$ and $v(z) \in S$. The tame element s fixes both w and the projected constant d. Apply $Q_i(s)$ to the lift. For a crystalline form λ, put $\lambda_a = s^{-a} \lambda s^a$. In an H-invariant period basis $(P_b)_b$, write $x = \sum_b h_b P_b$. The values $\lambda(s^a P_b)$ lie in the unramified constant field, which s fixes. Thus $\lambda_a(x) = \sum_b h_b \lambda(s^a P_b)$ evaluates the a-th power of the linear period operator. The coefficients b_a of $Q_i = \sum_a b_a X^a$ are in $\mathbf{Z}_p$ and commute with the covector. The annihilator identity gives $\sum_a b_a \lambda_a(x) = 0$ on the lifted point x. Thus the value on $Q_i(s)x$ is $\sum_a b_a (s^a - 1)\lambda_a(x)$.

The retracted lift lies in $M \cap N_{\mathrm{loc}} = N$. For a symmetric jump bound b, put $h_a = E^b \lambda_a(x) \in S_B$ by (1). Every s^a fixes K_0 and E/p, and sends z to $\zeta^a z$. Hence $(s^a - 1)S[E/p][1/p] \subset zB^+$. The continuity of division by z on its image, proved in Section B.2.1, extends the inclusion to S_B. Consequently

$$E^b \lambda(Q_i(s)x) = \sum_a b_a (s^a - 1) h_a \in zB^+.$$

Normality of H gives $Q_i(s)x \in M$. Applying (2) at level zero and (5) to its basis coefficients gives $Q_i(s)x \in zM$. Division by z preserves each order since $\theta\varphi^{-a}(z) \neq 0$. Since $Q_i(1) = 0$ and $s^a(z) = \zeta^a z$, the projection of the divided point is

$$\left(p^c z^{j-1} Q_i(\zeta^j) - p^{k_0+1} \sum_{a=0}^{\deg Q_i} b_a \zeta^a s^a(v(z)) \right) d.$$

The sum lies in S and is independent of d. Thus $p^c z^{j-1} Q_i(\zeta^j) \in J_0$. If $Q_i(\bar{\zeta}^j) \neq 0$, then $Q_i(\zeta^j)$ is a unit in $W(\mathbf{F}_q)$ and $p^c z^{j-1} \in J_0$. In all cases the Kummer difference $g(z) = [\underline{\epsilon}]z$ also gives

$$p^c z^j \in J_0 \quad \Longrightarrow \quad p^{c+v_p(j)+1} z^{j-1} \in J_0.$$

Indeed modulo z one has $E \equiv p$, and $\tau(p^{v_p(j)+1} + zB) = A([\underline{\epsilon}]^j - 1)$ with $A, B \in A_{\mathrm{inf}}$. Multiply this identity by $p^c z^j$ and divide the lifted Kummer difference by $z\tau$; the correction proportional to the old multiplier lies in the lift ideal.

The monic reduction of Q_i vanishes on at most $\deg q_i + 1$ of the distinct $\bar{\zeta}^j$. Since

$$j < e \leq \prod_{a=1}^{r} (p^a - 1) < p^{r(r+1)/2} \leq p^{r^2}, \qquad v_p(j) + 1 \leq r^2,$$

the total exponent on each extension is bounded by

$$r^2 \deg Q_1 \leq r^2(N+1), \quad r^2 \deg Q_2 \leq r^2(N+1), \quad r^2 \deg Q_3 \leq r^2(N^2+1) \leq k_0.$$

Therefore $p^{k_0} \in J_0$ for each of the three sides. Nakayama applied to the cyclic image of (p^{k_0}) modulo I_0 removes the modulus p^{k_0+1}. Explicitly, for some $w \in A$ we have $p^{k_0}(1 - pw) \in I_0$, and the geometric series in pw inverts $1 - pw$ in A. This proves

$$p^{k_0} \text{ kills each of the three standard-order projection cokernels at each index.}$$

Application of the chart depth and cup calculation. Use the three cokernel annihilators with E-exponent 0. The negative-jump exclusion $[-0, 0)$ is empty. There is no nonzero multiple of p within n_0 of 0 or $\pm\beta$, since $p > \beta + n_0$. The level-zero depth increase from $-u$ to $\eta - u$ uses steps in the union of these intervals. The contribution at a step $j \neq 0$ is $v_p(j) + \delta$, where $\delta \leq 1$ is the tame different allowance, and step 0 contributes δ. Summing over at most four intervals, each of length $2n_0 + 1$, contributes at most $(8n_0 + 4)\delta$ per side.

The negative-far side can rest with orders $\max(0, -u - \eta)$ at level zero and $-u$ thereafter. The other side, whose jumps are $\geq -\eta$, can move from its preceding index: at level zero its Frobenius difference has order at least $-u - \eta$. The improved resting depth $\eta - u'$ makes the orders add to at least $-u - u'$ there, and the unmodified higher orders give this same bound there. The third lift supplies the complementary dual order $u + u' + 1$. The first and last moves use the same order test on differences.

Write $s_0 = (8n_0 + 4)\delta$. The two moving points lie over $p^{k_0+s_0}$ times their respective parameters, and the third lies over p^{k_0}. The coboundary calculation of Section C.2.1 therefore uses exponent

$$2(k_0 + s_0) + k_0 \le 3k_0 + 150(\eta + W_* + 1).$$

Restriction to H is injective on these p-power torsion cohomology classes by corestriction and $(p, [G_{\mathbf{Q}_p} : H]) = 1$. Thus the bound descends to the original local group.

Proposition C.1 (The tame cup-product exponent)**.** *Under the tame chart and annihilator hypotheses above, let z_1, z_2 be the crystalline cocycles of an allowed cup test defining $\Delta_v(L)$, with pairing ψ and depth m. For each $\mathbf{Z}_p$-linear functional $\lambda : O \to \mathbf{Z}_p$, let ψ_m be the reduction of $(\lambda \otimes 1) \circ \psi$ modulo p^m. Then*

$$p^{\,3k_0+150(\eta+W_*+1)}[(\psi_m)_*(\bar{z}_1 \cup \bar{z}_2)] = 0 \quad \text{in } H^2(E_v, \mu_{p^m}).$$

D Depth operators and crystalline lifts

In this appendix, we extend the divided operators from Appendix C to the higher Frobenius levels. We use the projection ideals and the period basis to construct lifts with prescribed filtration orders. The construction gives uniform cup exponents for the colength estimates in Appendix E.

D.1 Depth operators on the integral chart

In this subsection, we extend the divided operators of Section C.1 to $E_a = \varphi^a(E)$, including at $p = 2$. Each operator raises the order of the Galois difference at E_a by one and preserves the earlier orders. Its factor of p enters the annihilator bounds of Section D.2.

Keep the integral localized basis of Section B.3. For the side of any pairing write x for a chart point over a constant. Say that it has depth $d(a, u)$ if its Galois differences have those orders. By the invariant calculation in Section C.1, this is equivalent to each corresponding form value being congruent to a constant modulo the indicated filtration. A previous positive order forces that constant to vanish whenever the new depth is positive. Otherwise the new order bound from depth d is $\min(0, d)$.

Let $D_E = 0$ at ramification index $e = 1$, and $D_E = 1 + v_p(e)$ otherwise. A depth step from n to $n + 1$ at level a requires

$$C(a, n) = v_p(n) + D_E + 3a, \qquad v_p(0) = 0.$$

Here is a construction including the case $p = 2$. For $n \neq 0$ choose $gz = [\epsilon]^{p^a} z$ with g in the local chart group. Write $gE = E\omega$; then $gE_a = E_a\varphi^a(\omega)$. Put $\Delta_z = z\tau$. The chart action gives $\Delta_z \mid \omega - 1$, and $\Delta_z \mid \varphi^a(\Delta_z)$. Thus $X = (\varphi^a(\omega)^n - 1)/\Delta_z$ is integral, also for negative n.

Let F be the Eisenstein minimal polynomial of $\theta(z)$ over the unramified coefficient field. Weierstrass division in S identifies the kernel of specialization at $\theta(z)$ with ES. Thus $F(z) = EG$ for $G \in S$. The reductions of $F(z)$ and E both have order e in z, so $G \in S^\times$. The subgroup H fixes $\theta(G)$. Taylor expansion of $F(gz)$ at z, followed by division by E and Fontaine evaluation, therefore gives

$$\theta(\omega - 1)\theta(G) = F'(\theta z)\,\theta((gz - z)/E).$$

The derivative has valuation at most D_E. If $e > 1$, the nonzero derivative terms have distinct valuations modulo $\mathbf{Z}$, and the term of degree $e-1$ has valuation $v_p(e)+(e-1)/e$; for $e = 1$ the derivative is a unit.

The geometric sum for $[\underline{\epsilon}]^{p^a} - 1$ shows that $\theta((gz - z)/(E\Delta_z))$ has valuation a. Hence

$$v_p(\theta(\omega - 1)) \le D_E + a + v_p(\theta\Delta_z).$$

Moreover $v_p(\theta(\omega-1)) \ge v_p(\theta\Delta_z) > 1/(p-1)$, since $\theta(\tau) = \zeta_p - 1$ and $\theta(z)$ has positive valuation. If $v_p(y) > 1/(p-1)$, the linear term py has uniquely smallest valuation in $(1+y)^p - 1$. Iterating and using the geometric sum for a prime-to-p exponent gives $v_p((1+y)^n - 1) = v_p(y) + v_p(n)$ for positive n. For $n = -m < 0$, use $(1+y)^{-m} - 1 = -((1+y)^m - 1)/(1+y)^m$; the denominator is a unit. At $p = 2$ this uses $v_2(\theta\Delta_z) = 1 + 1/e > 1$. Set $d_\Delta = 1/e + 1/(p-1) \le 2$. The compatible roots of π_v and of unity give $v_p(\theta\varphi^{-a}(\Delta_z)) = p^{-a}d_\Delta$. Since $\varphi^{-a}(X)\varphi^{-a}(\Delta_z) = \omega^n - 1$,

$$v_p(\theta\varphi^{-a}(X)) \le v_p(n) + D_E + a + (1 - p^{-a})d_\Delta \le v_p(n) + D_E + 3a = C(a, n).$$

Thus $\theta\varphi^{-a}(X)$ divides $p^{C(a,n)}$. Lift the quotient to $\upsilon \in A$ by the surjectivity of $\theta\varphi^{-a}$, so that $\upsilon X \equiv p^{C(a,n)} \pmod{E_a}$.

At level a the selected g fixes $K'(\pi_a)$, since $\theta\varphi^{-a}([\underline{\epsilon}]^{p^a}) = 1$. Write $v_{a,\lambda}(x) = c + y$, with $c \in K'$ and $y \in \mathrm{Fil}^n \cap \mathcal{K}_a$. The element $\varphi^{-a}(E_a) = E$ has filtration order one. The residue of $y/E^n \in \mathcal{K}_a \cap \mathrm{Fil}^0$ lies in $K'(\pi_a)$ by Section B.2.2. Hence $(g-1)(y/E^n) \in \mathrm{Fil}^1$. Multiplying by $g(E^n)$ gives

$$(g-1)y \equiv (\varphi^{-a}(g(E_a)/E_a)^n - 1)y \pmod{\mathrm{Fil}^{n+1}}.$$

Division by $\varphi^{-a}(\Delta_z)$ has order zero. Thus the divided difference has value $\varphi^{-a}(X)y$ modulo the next grade, for either sign of n. Therefore the operator

$$p^{C(a,n)}x - R(\upsilon(gx - x)/\Delta_z)$$

advances the depth at that step. Indeed, let y_1 be the unretracted expression $p^{C(a,n)}x - \upsilon(gx - x)/\Delta_z$. Since R fixes x in the localized chart module, the displayed operator is $R(y_1)$. For $n \neq 0$ set $c_1 = p^{C(a,n)}c \in K'$. Section B.2.2 gives

$$\mathrm{ord}_t(v_{a,\lambda}(Ry_1) - c_1) \ge \mathrm{ord}_t(v_{a,\lambda}(y_1) - c_1) \ge n + 1.$$

As c_1 is fixed, all Galois differences have order at least $n+1$ on this line. On every other line, the correction $R(\upsilon(gx - x)/\Delta_z)$ retains the prescribed orders of the differences. Its projection to the parameter module is zero by naturality of R.

For $n = 0$, choose $g_j z = [\underline{\epsilon}]^j z$ for $0 \le j < p^{D_E+3a}$. In the reduction at E_a, the action on z is multiplication by the j-th power of a primitive p^a-th root. On the K'-basis $(\pi_a^i)_{0 \le i < p^a}$, the sum multiplies π_a^i by

$$\sum_{j=0}^{p^{D_E+3a}-1} \zeta_{p^a}^{ij} = \begin{cases} p^{D_E+3a}, & i = 0, \\ 0, & 0 < i < p^a. \end{cases}$$

Thus every integral value at this level maps to one with residue in K'. Thus use the operator

$$p^{D_E+3a}x + \sum_j R(g_j x - x).$$

This is $R(\sum_j g_j x)$. For each form, let $c_1 \in K'$ be the residue of its value on $\sum_j g_j x$, obtained by the displayed sum. The same retraction inequality gives $v_{a,\lambda}(R(\sum_j g_j x)) - c_1 \in \mathrm{Fil}^1$; the

preceding divided-action argument preserves the earlier depths. At every step the actual quotient parameter satisfies $\pi(R(\sum_j g_j x)) = p^{D_E+3a}\pi(x)$, by the equivariance of π, the invariance of $\pi(x)$, and the naturality of the retraction.

For each (a, n), fix the Kummer elements and, for $n \neq 0$, the scalar lift υ, independently of the vector and the equivariant form. Both operators are $\mathbf{Z}_p$-linear.

For finitely many required depth increases take the union of their steps (a, n) over the jump lines and perform them in increasing order of n. Each operation handles every line at the given level having the indicated depth. It preserves the previous depth bounds on all lines, because its correction lies in the kernel with those order bounds. Thus the total contribution is the sum over distinct steps, independent of the rank.

D.2 Ideals controlling the crystalline lifts

In this subsection, we bound the annihilator of the cokernel of projection to the trivial parameter module in Section B.3. The divided operators of Section D.1 give a power of p in the sparse case and an E-power under the unramified Hodge bounds.

We give the calculation of the two types of annihilator used at a cut. Write $p = \ell$, $\mathfrak{S} = W(\mathbf{F}_{p^f})[[z]]$, $E_a = \varphi^a E$, $\tau = [\underline{\epsilon}^{1/p}] - 1$, and $D_E = 0$ if $e = 1$ and $\mathrm{ord}_p(e) + 1$ otherwise. The coefficient ring of the chart is a subring of A_{inf}. Keep the extension $0 \to Y \to X \xrightarrow{\pi} C \to 0$ of one fixed side, in particular any of the two universal sides or the Tate-dual side. Use the crystalline $\mathfrak{S}$-module obtained from its periods, localized by all E_a. The A_{inf}-module of order r at j is $e_j \varphi^r M_X$. Its chart vectors are its intersection with $N_{X,\mathrm{loc}}$. On a jump line u at $j^{p^{-a}}$, the two modules have the same order tests at E_a, namely ≥ 0 for $a < r$ and $\geq -u$ otherwise.

For a nonnegative integer R let $J_r(j; R) \subset A_{\mathrm{inf}}$ be the ideal of scalars lifting constants at order r, plus (R). Concretely, a member s means that for some $t \in A_{\mathrm{inf}}$ and every $c \in C$, the constant $(s - Rt)c_j$ is $\pi(x)$ for some order-r vector x in component j of the A_{inf}-module. The following properties of the period basis are used here. The basis spans the order-zero module over A_{inf}, and each way between that module and $A_{\mathrm{inf}} \otimes X$ is bounded by a power of τ.

Section B.2.2 gives $R_C \pi = \pi R_X$. Applying this to $\pi(x) = (s - Rt)c_j$ with $s \in \mathfrak{S}$ gives

$$\pi(R_X x) = (s - RF(t))c_j, \qquad F(t) \in \mathfrak{S}.$$

The same $F(t)$ works for every c, and each $R_X x$ retains its prescribed orders. Frobenius transports the standard-order module separately, as in Section B.3.

For such a localized chart vector y, the divided difference $(gy - y)/(z\tau)$ lies in the localized A_{inf}-module by Section B.2.1. The divisor has order zero at every E_a; equivariance therefore preserves the prescribed lower bounds after division. At order 1 we may also divide by E. At $a = 0$ the residues of the integral form values lie in K' by Section B.2.2. The chart subgroup fixes K', so each value of $gy - y$ at this level is in Fil^1. At $a > 0$ the factor E has order zero. These properties apply to each of the three extensions. They are the properties of the basis used here. Suppose $F \in \mathfrak{S} \cap J_r(j; R)$, and

$$\alpha(gF - F) = D(s + \beta F)$$

for $D = z\tau$ at order 0, or $D = z\tau E$ at order 1.

Then $s \in J_r(j; R)$. Indeed lift $(F - Rt)c_j$ over $\mathfrak{S}$ and apply $\alpha(g - 1)/D$. As g fixes c_j, the new lift has constant $(s + \beta F - R\alpha(g - 1)t/D)c_j$. The difference of $\mathfrak{S}$-elements is divisible by

D for the corresponding chart elements g; subtract the known contribution of βF. This yields the assertion.

Choose a chart element g with $gz = [\epsilon]z$. The cyclotomic factorization for $n > 0$ gives

$$\alpha([\epsilon]^n - 1) = \tau E(p^{\mathrm{ord}_p(n)} + \beta z). \tag{LI1}$$

Indeed $[\epsilon]^n - 1$ is a unit times $\tau \prod_{a=0}^{\mathrm{ord}_p(n)} E_a$. For the prime-to-$p$ part of n the geometric sum is a unit. For its p-part the consecutive cyclotomic factors give the displayed generators. Each $E_a \equiv p \bmod z$, which proves (LI1). Applying the difference operator just established to $F = p^c z^n$ gives the two implications

$$\begin{aligned} p^c z^n \in J_1(j;R) &\implies p^{c+\mathrm{ord}_p(n)} z^{n-1} \in J_1(j;R), \\ p^c z^n \in J_0(j;R) &\implies p^{c+\mathrm{ord}_p(n)+1} z^{n-1} \in J_0(j;R). \end{aligned}$$

The additional factor p in the second line is $E \bmod z$. Put $d_{n,m} = \mathrm{ord}_p(n!) - \mathrm{ord}_p(m!)$ for $0 \le m \le n$. Iterating from n down to m increases the exponent of p by at most $d_{n,m}$ at level one and $n - m + d_{n,m}$ at level zero.

There is a second order-one operation. For $n \neq 0$ use $gE = \omega E$, $z\tau X = \omega^n - 1$, $\upsilon X \equiv p^{\mathrm{ord}_p(n)+D_E} \pmod E$. The chart supplies this by choosing $gz = [\epsilon]z$. To see the valuation, write $F_0(z) = EG_0$ with F_0 the minimal polynomial of the chosen uniformizer over the unramified field. As in Section D.1, $G_0 \in S^\times$, so g fixes $\theta(G_0)$. Write $gz - z = Ew$. Taylor expansion and reduction by Fontaine's map θ give

$$\theta(\omega - 1)\theta(G_0) = F_0'(\theta z)\theta(w), \qquad \theta(w) = \theta(z\tau) \cdot \text{unit}.$$

The different bound is $v_p(F_0'(\theta z)) \le D_E$.

For $e > 1$ the nonzero derivative terms have distinct valuations modulo $\mathbf{Z}$; the leading term has valuation $v_p(e) + (e-1)/e$. For $e = 1$ the derivative is a unit. The derivative calculation of Section D.1 applies also at $p = 2$: $v_p(\theta(z\tau)) > 1/(p-1)$ and $\omega - 1$ is divisible by $z\tau$. Hence $v_p(\theta X) \le D_E + \mathrm{ord}_p(n)$. Lift a quotient $p^{D_E + \mathrm{ord}_p(n)}/\theta X$ by the surjectivity of θ. Inserting υ in the difference operator at order 1 trades E^n for zE^{n-1} at contribution $D_E + \mathrm{ord}_p(n)$. In detail, if $F = p^c z^a E^n$, then $gz^a/z^a = 1 + \tau E b$ for some $b \in A_{\mathrm{inf}}$. It follows that

$$\frac{z\upsilon(gF - F)}{z\tau E} = p^{c + D_E + \mathrm{ord}_p(n)} z^{a+1} E^{n-1} + \beta F.$$

This is the asserted trade on $J_1(j;R)$.

Frobenius carries the order-zero ideal at j to the order-one ideal at φj. Thus $p^c z^{pa} \in J_1(\varphi j;R)$ is equivalent to $p^c z^a \in J_0(j;R)$. No integer between $p\lfloor n/p \rfloor + 1$ and n is divisible by p. Thus lowering to that multiple of p is free, and subsequent inverse Frobenius gives $p^c z^{\lfloor n/p \rfloor}$ in the preceding ideal.

Put $I = \bigcap_j J_0(j; p^{\kappa+1})$, and let b_X bound the absolute jumps of the extension X. For each $c \in C$ choose $x_c \in X$ with $\pi(x_c) = c$. By (3), the projection to component j of $\tau^{b_X}(1 \otimes x_c)$ has standard orders and lies over $\tau^{b_X} c_j$. Thus $(\tau^{b_X}, p^{\kappa+1}) \subseteq I$. Write bars for reduction in the valuation ring A_{inf}/p. The reduction $\bar{z}$ has positive tilt valuation since z comes from compatible roots of a uniformizer. Choose $m_0 \ge 1$ with $m_0 v_\flat(\bar{z}) \ge b_X v_\flat(\bar{\tau})$. Then $z^{m_0} = \tau^{b_X} u + pv$ with $u, v \in A_{\mathrm{inf}}$, so

$$z^{m_0(\kappa+1)} \in (\tau^{b_X}, p^{\kappa+1}) \subseteq I.$$

Let a be the least nonnegative integer with $z^a \in I$. If ω_0 bounds all jumps above, multiplying the lifts by E^{ω_0} puts them at order 1. The relation $z^e = uE + vp$ with $u, v \in \mathfrak{S}$ then puts $z^{a+e(\omega_0+\kappa)}$ in every order-one ideal modulo $p^{\kappa+1}$. Indeed, every term of its binomial expansion contains E^{ω_0} or $p^{\kappa+1}$. Coming down and using minimality yields

$$(p-1)a \le e(\omega_0 + \kappa). \tag{LI2}$$

For an unramified chart suppose $p \ge 5$, $E = z + p$, and $\omega_0 \le (p-3)(s+t)$. Set $\kappa = s + t$. Then

$$(p-1)a \le (p-2)\kappa, \qquad a + \operatorname{ord}_p(a!) \le \kappa,$$

by $(p-1)\operatorname{ord}_p(a!) \le a$. The second monomial operation puts $p^{\kappa-m}z^m$ in $J_0(j; p^{\kappa+1})$ for $0 \le m \le \kappa$: when $m < a$ use the accumulated contribution formula, and when $m \ge a$ use ideal closure. In particular the ideal contains p^κ. Modulo $J_0(j;0)$ this means $p^\kappa = p^{\kappa+1}h$. Since $1 - ph$ is a unit, $p^\kappa \in J_0(j;0)$, so $J_0(j;p^{\kappa+1}) = J_0(j;0)$. Expand $p^s E^t = p^s(z+p)^t$. Each term is one of those monomials times its integer binomial coefficient. Thus $p^s E^t$ kills the constant cokernel.

For a cut followed by J indices of jump bound ω_1, assume $8e \le p^{J+1}$. Use (LI2) with the global jump bound ω_0. At each of the J next indices multiply the lifts by E^{ω_1}, trade its factors for z successively, and come down by Frobenius. The exponent of z changes from n to $\lfloor (n+\omega_1)/p \rfloor$, at contribution $D_E\omega_1 + \operatorname{ord}_p(\omega_1!)$. After J steps the monomial $p^c z^n$ in the ideal at the cut satisfies

$$c \le J\big(D_E\omega_1 + \operatorname{ord}_p(\omega_1!)\big), \qquad n \le a/p^J + \omega_1/(p-1).$$

Since $p \ge 2$, (LI2) gives $a/p^J \le 2e\omega_0/p^{J+1} + \kappa/4$.

Removing the remaining z^n requires $n + \operatorname{ord}_p(n!) \le 2n$. Therefore p^κ lies in the exact ideal as soon as

$$2JD_E\omega_1 + (2J+4)\frac{\omega_1}{p-1} + \frac{8e\omega_0}{p^{J+1}} \le \kappa. \tag{LI3}$$

For the stated cut parameters the global bound is $2b_0$ and the bound for a thin next component is G_1 in the cut-size conditions. They impose $D_E G_1 \le X_0$, $G_1/(p-1) \le X_0$, and $2eb_0/p^{J+1} \le D_0$. The left side of (LI3) is at most $(4J+4)X_0 + 8D_0$, below $\kappa = 12((J+1)X_0 + D_0) + 1$.

At a shifted cut the unramified argument applies with $s = \kappa$, $t = s_j$, since the global jump bound is at most $(p-3)(\kappa + s_j)$. This proves the two annihilator assertions on all three sides. It also gives the annihilator with no p-factor E^A in the separated-interval regime, using the jump bound $(p-3)A$.

The basis used for these divisibilities is built in Sections B.1 and B.3. For the Breuil–Kisin module of a crystalline representation see Kisin [39]. Restrict to the chosen finite crystallizing field K'. Its residue field is perfect. In every residue characteristic, including two, the underlying $\mathbf{Z}_p$-lattice is therefore the étale realization of a prismatic F-crystal on $\operatorname{Spf}\mathcal{O}_{K'}$. Evaluation at the Breuil–Kisin prism gives a finite free $\mathfrak{S}$-module $\mathfrak{M}$ with $(\varphi^*\mathfrak{M})[1/E] \simeq \mathfrak{M}[1/E]$ [14, Example 4.3, Theorem 5.6 and §7]. The A_{inf}-module formalism is developed in [13, §4]. The order bounds used here are the coordinate divisibilities proved above.

D.3 The convergence radius of the period basis

In this subsection, we prove a generation bound for positive Frobenius translates of the basis in Section B.1. The stronger de Rham orders at $E = 0$ supply the intermediate lifts in Section D.4.

Use the chart and period comparison of Sections A.1 and A.3. Write $\Lambda = W(\mathbf{F}_{p^f})$, $S = \Lambda[[z]]$, $A = A_{\max}$, $B^+ = A[1/p]$, and $S_B \subset B^+$ for the p-adic closure of $S[E/p]$ with p inverted. The extended first coordinate $\mathcal{R} : B^+ \to S_B$ is the identity on S_B and is S_B-linear. It restricts to $\mathrm{coord}_0 : A_{\inf} \to S$ on $A_{\inf}$. It is compatible with approximation by finite chart coordinates.

Here are two scalar consequences. If $\nu \in S$ and $\xi - pb_0 \in \nu A_{\inf}$ for some $b_0 \in A_{\inf}$, then for $r \in A$ and $M \geq 0$ there are $s \in S$, $y \in S_B$ with

$$p^M \mathcal{R}(r) = s + \nu^M y. \tag{109}$$

To see this, express r as a p-convergent sum of polynomials in ξ/p with integral coefficients and expand in $\xi/p - b_0$. After multiplication by p^M the summands of degree $< M$ have coefficients in $A_{\inf}$, and the remaining terms are divisible by $(\xi - pb_0)^M$ in A. Both sums converge using the original p-divisibility of the approximating polynomials. Apply $\mathcal{R}$. This applies to $\nu = E$ with $b_0 = 0$. It applies to $\nu = z$ because $E = p + z^e u$ and E is associated to ξ. It follows, using $\nu = z$, that every element of S_B is constant in $\Lambda[1/p]$ modulo zS_B.

For g fixing Λ and $\theta(z)$, one has

$$z\tau \mid gy - y \quad \text{in } B^+, \qquad y \in S_B. \tag{110}$$

This is the divided action (6). In that proof $\varphi^m(S_B) \subset S_B$ and $\theta(S_B) = K'$, the chart field fixed by g, give $\theta\varphi^m(gy - y) = 0$ for $m \geq 1$. The cyclotomic kernel criterion applied after division by z then gives divisibility by τ. The cancellation (5) gives the contraction used below: if $z^r\tau \mid E^m x$ in B^+ for $m, r \geq 0$ and $x \in A_{\inf}$, then $z^r\tau \mid x$ in $A_{\inf}$.

Suppose Z has entries in S_B, $Z' \in M_d(B^+)$ satisfies $Z'Z = E^h I$, and $tZ = ZG$ for $G \in M_d(A_{\inf})$, where t fixes $K' = \theta(S_B)$ and $tz = [\underline{\epsilon}]z$. Apply $\mathcal{R}$ entrywise to Z', which preserves $Z'Z = E^h I$. The constant term of $p^{-h}\mathcal{R}(Z')$ modulo z gives a matrix B_0 over $\Lambda[1/p]$ with

$$Y = B_0 Z = I + zW, \qquad W \in M_d(S_B).$$

Indeed $p^{-h}E^h \equiv 1 \bmod z$, and multiplication by Z preserves the discarded ideal. Moreover $E^h(G - I) = Z'(tZ - Z)$, hence $G - I$ is divisible by $z\tau$ over $A_{\inf}$, by (110) and contraction. The relation $tY = YG$ holds since $tB_0 = B_0$.

For clarity the successive approximation proving the radius starts with the period coefficients Y_i. Suppose

$$A_n = q_n Y_i - z^{n+1} W_n \in A_{\inf}^d, \qquad W_n \in S_B^d, \quad q_n \in \mathbf{N}.$$

The initial values satisfy $q_0 = 1$, $W_0 = W_i$, $A_0 = I_i$. The period term cancels from $A_n G - tA_n$, leaving

$$z^{n+1}(([\underline{\epsilon}]^{n+1} - 1)W_n + [\underline{\epsilon}]^{n+1}(tW_n - W_n) - W_n(G - I)).$$

The last two terms in parentheses are divisible by $z\tau$. For $v_n = v_p(n+1)$, the cyclotomic factorization (LI1) and $E = p \bmod z$ give a multiplier $\alpha \in A_{\inf}$ with

$$\alpha([\underline{\epsilon}]^{n+1} - 1) = \tau(p^{v_n+1} + zu), \qquad u \in A_{\inf}.$$

Contraction gives

$$\frac{\alpha(A_n G - tA_n)}{z^{n+1}\tau} \in A_{\inf}^d.$$

In B^+ it equals $p^{v_n+1}W_n + zV_n$. Applying $\mathcal{R}$ produces $s_n \in S^d$ and $W_{n+1} \in S_B^d$ such that

$$p^{v_n+1} W_n = s_n + zW_{n+1}.$$

Consequently $A_{n+1} = p^{v_n+1}A_n + z^{n+1}s_n$ and $q_{n+1} = p^{v_n+1}q_n$ obey the same recurrence.

Take $N = p^l \geq 3e$ and write $E = p + z^e u$ with $u \in S^\times$. In $S[E/p] \subset A_{\max}$ we have

$$q = \frac{z^e}{p} = u^{-1}(E/p - 1), \qquad \frac{z^N}{p^2} = pz^{N-3e}q^3 \in pS[E/p].$$

Consequently evaluation at $X = z^N/p^2$ defines $\mathrm{ev} : S[[X]] \to \widehat{S[E/p]} \subset A_{\max}$ by p-adic convergence. Put $a_n = n - v_p(n!) + 2$. Applying φ^l to the recurrence for the period coefficients gives

$$p^{a_n}\varphi^l W_n = p^{a_{n+1}-2}\varphi^l s_n + X\, p^{a_{n+1}}\varphi^l W_{n+1}.$$

Here $a_{n+1} - 2 \geq 0$ and $a_{n+1} + v_n + 1 = a_n + 2$, by the factorial identity. Iterating, the formal series with coefficients $p^{a_{n+1}-2}\varphi^l s_n$ evaluates to $p^2\varphi^l W_0$: the difference is divisible by every power of X in B^+, with $z \mid X$.

To see that B^+ is z-adically separated, write $\xi = u_0 E$ with $u_0 \in A_{\mathrm{inf}}^\times$ and set $X_0 = \xi/p$. In B^+ the element z^e is a unit multiple of $X_0 - u_0$, since $E = p + z^e u$. If a nonzero element were divisible by every power of z, rescale by a power of p to obtain an element of $A_{\max} \setminus pA_{\max}$ with the same divisibility. Each power of $X_0 - u_0$ has monic reduction in $(A_{\mathrm{inf}}/(p,\xi))[X_0]$. By the p-saturation proved in Section A.3, division by that power takes place in $A_{\max}$. Reduction would make a nonzero polynomial divisible by monic polynomials of arbitrarily high degree. This proves separatedness.

Collecting all covectors gives a matrix F over $S[[X]]$ and

$$\varphi^l(B_0)\varphi^l(Z) = \mathrm{ev}(I + XF).$$

Also $\varphi^l(E) = p\,\mathrm{ev}(\varepsilon)$ with $\varepsilon = 1 + pX\varphi^l((E-p)/z)$. The series $I + XF$ and ε are invertible over $S[[X]]$, with constant terms I and 1. Thus for any $b \geq 0$,

$$\varphi^l(Z) = \varphi^l(E)^b\, C\, \mathrm{ev}(V), \qquad C = (p^b\varphi^l(B_0))^{-1}, \qquad V = \varepsilon^{-b}(I + XF).$$

The matrix C is fixed by every g fixing Λ. Its inverse exists because $\varphi^l(B_0)\varphi^l(Z)$ is invertible over B^+.

If $eA_0 \leq N\delta$, approximate each formal series $H \in S[[X]]$ by its truncation of degree $< \delta$. After multiplying by $p^{2\delta}$, the truncated part evaluates into S. Write the tail as

$$z^{N\delta}\mathrm{ev}(H_1) = p^{A_0}r, \qquad r = z^{N\delta - eA_0}q^{A_0}\mathrm{ev}(H_1) \in \widehat{S[E/p]} \subset A_{\max}.$$

Since $r \in S_B$, the retraction fixes r. Applying (109) with $\nu = E$ to r gives

$$p^{2\delta}\mathrm{ev}(H) \equiv s \pmod{E^{A_0}B^+}, \qquad s \in S.$$

Apply this entrywise to V^{-1} and V, obtaining matrices $M, M' \in M_d(S)$ congruent modulo $E^{A_0}B^+$ to $p^{2\delta}\,\mathrm{ev}(V)^{-1}$ and $p^{2\delta}\,\mathrm{ev}(V)$, respectively. In that quotient,

$$\mathrm{ev}(V)M = p^{2\delta}I, \qquad \mathrm{ev}(V)(p^{4\delta}I - MM') = 0.$$

The columns of M and $p^{4\delta}I - MM'$ have images $p^{2\delta}e_i$ and 0, respectively, under $\mathrm{ev}(V)$ in this quotient, where e_i is the corresponding coordinate vector. For every $y \in S^d$,

$$p^{4\delta}y = (p^{4\delta}I - MM')y + M(M'y),$$

so these columns span a module containing $p^{4\delta}S^{d}$.

For the period basis of a lattice, take b to be the nonnegative uniform jump bound used in choosing the invariant period matrix P. The basis vectors $n_\nu = \tau^{-b}Q_\nu$ satisfy the formula for every equivariant de Rham form λ and column y of basis coordinates:

$$\lambda\left(\sum_{\nu} y_\nu \varphi^{l} n_\nu\right) = (\lambda_{\mathrm{lat}}\varphi^{l}P)\,\frac{\varphi^{l}Z}{\varphi^{l}(E)^{b}}\,y.$$

This follows from $UZ = E^{2b}Q$ and $U = (E\tau)^{b}P$, in the given integral basis. The covector $\lambda_{\mathrm{lat}}\varphi^{l}P$ is fixed by the chart subgroup and has entries in $\mathrm{Fil}^{0}B_{\mathrm{dR}} = B_{\mathrm{dR}}^{+}$. Using the last factorization, if $\mathrm{ev}(V)y$ is an integer vector modulo E^{A_0}, the value of the form is congruent to an element of K' modulo $\mathrm{Fil}^{A_0} = t^{A_0}B_{\mathrm{dR}}^{+}$. The span statement therefore gives the precise $p^{4\delta}$ radius property at level l used in the long-gap coefficient extraction. Every operation is on the basis of the given extension and is compatible with projection to the given index.

D.4 Crystalline lifts and termination of the construction

In this subsection, we apply the annihilator estimates of Section D.2 to construct lifts of constants at selected coefficient summands. Between successive summands we transport the lifts by Frobenius. The radius estimate from Section D.3 supplies the additional depth needed for the longer intervals.

Use the marked period basis and its depth operators on each of the three sides of the pairing. At an index j let β_j be the far start, let $\eta \ge 1$ be the low radius, let V be the width of the separated intervals, and fix $W_0 \ge V + 4\eta$, $J \ge 1$, n_0, b_0, G_1, X_0 and D_0. The numerical requirements, verified for the two schemes below, are

$$\begin{gathered}\beta_j + s_j + W_0 \le 2b_0, \quad s_j + W_0 \le pn_0, \quad G_1 \le (p-1)n_0, \quad 2eb_0 \le p^{J+1}D_0,\\ n_0 + \lfloor \log_p(2b_0)\rfloor + (J+1+D_E)W_0 + D_E G_1 \le X_0,\\ \kappa = 12((J+1)X_0 + D_0) + 1.\end{gathered}$$

At a cut we use one of the two estimates of Section D.2. Either its shift $s_j = 0$ and the next J indices have global upper jump bound G_1, with $8e \le p^{J+1}$, or the chart is unramified, $p \ge 5$ and $s_j < \beta_j$ with all jumps bounded above by $(p-3)(\kappa + s_j)$. In either case $p^{\kappa}E^{s_j}$ kills all three standard projection cokernels. The third-side jump set has upper bound $\beta_j + W_0$, because the two separated intervals are opposite and the lower-radius contributions are bounded by 2η.

At a cut j let $l \le 2J+1$ be the distance to the preceding cut and s_j its shift. The annihilator $p^{\kappa}E^{s_j}$ gives a point over $p^{\kappa}c_j$ by lifting the multiple of the constant and dividing by E^{s_j}. Its orders and depth are $-u - s_j\mathbf{1}_{a=0}$. Increase the depth to

$$D(a,u) = -u + s_j\mathbf{1}_{a=0} + \eta\mathbf{1}_{a\le l}.$$

Call the improved orders $C_j(a,u)$, obtained from $-u - s_j\mathbf{1}_{a=0}$ and $D(a,u)$ by the constant-term rule above.

At the cut j and level $0 \le a \le l$, put $i = F^{-a}j$ and $r_0 = s_j\mathbf{1}_{a=0}$, where $F(i) = i^{p}$. Let ϵ_i be the sign of the far interval for the moving side being considered. A filtration step $n \mapsto n+1$ on a line of jump u satisfies

$$-u - r_0 \le n < -u + r_0 + \eta.$$

For $|u| \le \eta$ this gives $|n| \le r_0 + 2\eta$; for $u \in \epsilon_i[\beta_i, \beta_i + V]$ it gives $|n + \epsilon_i\beta_i| \le r_0 + \eta + V$. Thus the two integer intervals have centers $0, -\epsilon_i\beta_i$ and common radius $R = r_0 + W_0 \le pn_0$. There are no steps for $a > l$, and $l \le 2J + 1$. If $r_0 > 0$, then $a = 0$ and $D_E = 0$. The sum of $v_p(n)$ for nonzero n in either interval of radius R is at most $2R/(p-1) + \lfloor \log_p(2b_0) \rfloor$, where $2b_0$ bounds all absolute endpoints.

Indeed, for either such interval I this sum equals

$$\sum_{q=1}^{\lfloor \log_p(2b_0) \rfloor} \#\{n \in I \cap \mathbb{Z} : n \neq 0,\ p^q \mid n\}.$$

Each inner count is at most $2R/p^q + 1$. Thus the two intervals contribute at most $4R/(p-1) + 2\lfloor \log_p(2b_0) \rfloor$ from the valuations. Since $R \le pn_0$ and $p \ge 2$, the first term is at most $8n_0$. If $r_0 > 0$, the constant part $D_E + 3a$ of the cost is zero. Thus both intervals on one side at one level have total exponent at most

$$8n_0 + 2\lfloor \log_p(2b_0) \rfloor + 30(J + 1 + D_E)W_0 \ \le\ 30X_0.$$

For the last term use $4W_0 + 2 \le 5W_0$ and $D_E + 3a \le 6(J + 1 + D_E)$ when the shift is zero. The size conditions

$$n_0 + \lfloor \log_p(2b_0) \rfloor + (J + 1 + D_E)W_0 + D_E G_1 \le X_0, \qquad 12((J + 1)X_0 + D_0) < \kappa$$

give $\kappa + 4D_0 + (2J + 2)30X_0 < 25\kappa$. Hence on each moving side the standard points may be taken over $p^{25\kappa}c_j$, using any unused powers of p as scalars. The cuts and their distances depend only on the Hodge intervals on the finite Frobenius cycle. For fixed parameters on the two sides, choose these standard points at every cut now. Each segment will use the points already chosen at its two ends. The Tate-dual side uses its own projection annihilator as in the order criterion below. At $a = 0$, the absence of a jump in $[-s_j, -\eta)$ shows that $C_j(0, u) \ge -u - \eta$: for $u < -s_j$, the initial order is positive and improves to $D(0, u)$; otherwise $u \ge -\eta$, the required order is nonpositive and is bounded by $\min(D(0, u), 0)$. At all other levels $C_j(a, u) \ge -u$.

D.4.1 Termination at a plain cut

In this subsection, we use the ideal bounds of Section D.2 for a plain cut.

For the stop at a plain cut, choose a sign such that for every index at distance $a < l$ the far interval has that sign or lies inside the low interval. The side with negative separated interval for the chosen sign has all its jumps $v \le \eta$ at those levels; the moving side has all $u \ge -\eta$. Use the former side's standard point as the stop. It has the orders

$$O_l(a, v) = \begin{cases} \max(0, -v - \eta), & a < l, \\ -v, & a \ge l. \end{cases}$$

At level 0 this uses the stronger improved orders just proved: for $v \ge -\eta$ its improved order is nonnegative, and for $v < -\eta$ its initial order is positive.

At the other levels it follows from the depth increase. For a point x_0 at the preceding cut and $g \in G_{K'}$, set

$$w_g = (gx_0 - x_0)/(z\tau) \in M_{T,\mathrm{loc}} \cap \ker \pi$$

by the divided action of Section B.2.1. Since Frobenius commutes with g,

$$(g-1)\varphi^{l}x_0=\varphi^{l}(z\tau)\,\varphi^{l}w_g.$$

For $n=l-a>0$, choose $P_E=\prod_{b\geq 0}E_b^{m_b}$ of finite support with $P_Ew_g\in M_T$. For each chart-equivariant form λ, (1) gives

$$\lambda(\varphi^{n}w_g)=\frac{\lambda(\varphi^{n}(P_Ew_g))}{\prod_b E_{n+b}^{m_b}}\in B_{\mathrm{dR}}^{+}.$$

Indeed the numerator is integral by the positive Frobenius test for M_T, and $\theta(E_{n+b})\neq 0$ for every b. Moreover $\theta\varphi^{l-a}(\tau)=0$, so the displayed difference has order at least one there. At $a\geq l$, $\theta\varphi^{-(a-l)}(z\tau)\neq 0$, and the preceding depth bound is transported without change. The order of the difference of two points over the same constant is bounded below by the minimum of their two orders. For the move from the preceding cut to the present point on the moving side, that minimum is at least $\min(0,-u-\eta)$ at level 0, $\min(0,-u)$ at $0<a<l$, $-u-\eta$ at $a=l$, and $-u$ above. Adding the depth of the stop gives at least s_j-u-v at $a=0$ and $-u-v$ above. Thus the annihilator of the Tate-dual side at the current cut satisfies the order criterion below.

For a long gap, the preceding cut has shift 0. A pivot j_* occurs $L=J+1$ steps after it, with the rest of the indices at distance $<l$ of opposite sign or bounded by η. Let g_0 bound the positive jumps of the resting side at the L indices up to the pivot and let E^{s_0} kill the pivot's cokernel. The gap imposes $N=s_0+g_0\sum_{0\leq a<L}p^{a}<\beta_{j_*}$.

On the unramified chart take $A=2b_0$, with $N<A\leq p^{L}D_0$. Start with the point y at j_* over c_{j_*} obtained from E^{s_0}. Multiply it by

$$f(E)=E^{s_0}\prod_{0\leq a<L}E_a^{g_0},\qquad E_a=(E-p)^{p^{a}}+p.$$

This is monic of degree N in E. The polynomial product has the standard orders of level L and, at level 0, has order at least g_0-u. Write T for the lattice of the extension on the resting side. Since $y\in N_{T,\mathrm{loc}}$ and $f(E)\in\mathfrak{S}$, every equivariant form has value in $\mathcal{K}_0$ on $f(E)y$. Its standard orders place it in $\varphi^{L}M_T$ by Section B.3. The value-field test of Section B.2.2 therefore puts $f(E)y$ in $\sum_i\mathfrak{S}\varphi^{L}(n_i)$. Since $A\leq p^{L}D_0$ and $3\leq p^{L}$ on this unramified chart, Section D.3 expresses $p^{4D_0}f(E)y$ as an $\mathfrak{S}$-linear combination of vectors of this module whose form values are fixed modulo Fil^{A}. Project this equality to the summand at j_*. The projector fixes its left side, preserves the module and preserves the filtration condition by pullback of forms. Since $E=z+p$, expand the $\mathfrak{S}$-coefficients in powers of E over $W(\mathbf{F}_{p^{f}})$ and get

$$p^{4D_0}f(E)y=\sum_{n=0}^{N}E^{n}\xi_n+E^{N+1}y_1$$

Here each ξ_n and the remainder y_1 have the standard orders at level L. At level zero the value of every chart-equivariant form on ξ_n is congruent to a constant in K' modulo Fil^{A}.

If $u<-N$, comparing the values on the line of jump u in this expansion forces each ξ_n, $0\leq n\leq N$, to have order at least A there. Indeed the total has order $>N$; reduction modulo E forces the constant of ξ_0 to vanish, giving order at least A. An invariant residue modulo Fil^{A} is represented by a constant in the chart field, by Section C.1. If its order is positive, that

constant is zero. Subtract the coefficient, divide by E, and repeat. For the projection π to the trivial quotient, put $f_n = [X^n]f(X)$ and $w_n = \pi(\xi_n) - p^{4D_0} f_n c_{j_*}$. The projected expansion is the exact identity

$$\sum_{n=0}^{N} E^n w_n = -E^{N+1}\pi(y_1).$$

The w_n satisfy the same congruences to constants. Since $N < A$, successive reduction and division by E give order at least A at level zero for every w_n. Monicity gives $f_N = 1$, hence

$$\pi(\xi_N) = p^{4D_0} c_{j_*} + w, \qquad \operatorname{ord}_{a,u}(w) \geq \operatorname{std}_L(a,u) + A\mathbf{1}_{a=0}.$$

Both terms defining w lie in the localized chart module, where E is invertible. As $\operatorname{ord}_a(E) = \mathbf{1}_{a=0}$, w/E^A belongs to the integral standard module at level L in the trivial quotient of the pivot component.

Over A_{inf}, φ^L carries the preceding summand and its level-zero standard module to the pivot summand and its level-L standard module. Projection to the trivial quotient commutes with φ^L, which fixes p^κ. The only jump of the trivial quotient is zero. Its standard-order vectors at level L therefore lie in $e_{j_*}(A_{\mathrm{inf}} \otimes_{\mathbf{Z}_p} C)$, which is generated over A_{inf} by the projected parameter constants. The transported multiplier thus lifts $p^\kappa w/E^A$. Apply the coordinate retraction of Section D.2 to a lift. It preserves the standard orders and fixes w/E^A in the localized $\mathfrak{S}$-module of the trivial quotient. This gives a point x_1 of the same orders with $\pi(x_1) = p^\kappa w/E^A$. Then $p^\kappa \xi_N - E^A x_1$ lies over the exact constant $p^{\kappa+4D_0} c_{j_*}$, has depth A at the pivot, and has order at least A there for $u < -N$. Write q_* for this point and $t = l - L$. Its transport $\varphi^t q_*$ is at the current cut. For $a < t$, put $h = t - a > 0$. For every g in the chart subgroup write $gq_* - q_* = (z\tau)y_g$, where $y_g \in M_{T,\mathrm{loc}}$ by Section B.2.1. For any chart-equivariant form λ,

$$\lambda\Big(\varphi^{-a}(g-1)\varphi^t q_*\Big) = \varphi^h(z\tau)\,\lambda(\varphi^h y_g) \in \mathrm{Fil}^1.$$

Indeed $\varphi^h(z)$ has order zero and $\varphi^h(\tau) = (1+\mu)^{p^{h-1}} - 1$ has order one. The second factor is integral by the positive-Frobenius test for M_T, since all inverted $\varphi^h(E_j)$ are de Rham units for $h > 0$.

D.4.2 Raising depth before termination

In this subsection, we apply the depth step to the remaining indices before checking the two schemes in Section D.4.3.

At the levels $t < a < l$ raise depth from 0 to 1 by the depth step $n = 0$. These steps are among the same interval steps as above and bring the contribution to at most 25κ. The resting side has $v \geq -\eta$ off the pivot below l; at the pivot either $v \geq -\eta$ or $v < -N$. Thus the point has orders $O_l(a,v)$ and depth 1 below l off the pivot, A at the pivot, and $-v$ at $a \geq l$.

At a long gap use the preceding cut's Tate-dual annihilator. Off the pivot the moving side has $u \leq \eta$ below l, so $C_j(a,u) \geq 0$ there. Its old point carried from the preceding cut has order ≥ 0 below l. At $a = t$ the order of their difference is at least $\min(0, \eta - u)$, and $u < \eta + A$. At $a \geq l$ the order is at least $-u$; the shift of the old cut is zero. Adding the depth of the stop yields at least 1 below l and $-u - v$ thereafter, as required for the move.

Here is the order criterion used in both cases. The dual jumps $-u-v-1$ can lie in both far intervals. At the cut l_1 steps before the current index use the Tate-dual projection annihilator

$p^{\kappa}E^{s}$. Lift the multiple $p^{\kappa}E^{s}$ of the constant, divide by E^{s}, and carry the resulting point over p^{κ} by Frobenius. At the dual jump $-u-v-1$ its order is bounded below by $T_{l_1,s}(a;u,v)$. The complementary order required of the two moving sides is

$$1-T_{l_1,s}(a;u,v)=\begin{cases}1, & a<l_1,\\ s-u-v, & a=l_1,\\ -u-v, & a>l_1.\end{cases}$$

If the mover's order and the resting point's depth sum to this bound or more, their contraction with the Tate-dual point has order at least 1, the order of $\mathbf{Z}_p(1)$.

The tensor calculation follows by taking filtered bases of the de Rham spaces: the weight on a pair is the sum of its weights, and the displayed dual orders decrease with the dual weight. This also handles an invariant form in a higher filtration step by taking a sum in the filtered basis. In the Galois difference of the resting point divide by $z\tau$ before taking this contraction.

This keeps the orders by the chosen basis. The contraction has τ times its value in the integral Tate lattice. Returning the removed $z\tau$ places the value in $W(\mathfrak{m}^{\flat})$, since z lies in that ideal. Reduction modulo the pairing exponent kills the component of the Tate-dual extension over zero, so the remaining value is precisely p^{κ} times the desired reduced pairing. This proves safety for either moving side.

The moves to and from a stop also satisfy this criterion. The move from the old cut to the stop uses $l_1=l$ and that cut's shift s_{old}. Below l its order is ≥ 0 and the other old point's depth is ≥ 1. At l the lower bounds are $-u-\eta$ and $-v+s_{\mathrm{old}}+\eta$, and above l their sum is at least $-u-v$. The move from the stop to the new point uses $l_1=0$ and $s=s_j$.

At 0 its order is at least $-u-\eta$ and the depth is at least $-v+s_j+\eta$; above 0 their sum is at least $-u-v$. Finally the pair of cocycle differences of two standard points transported from a cut satisfies the criterion with that cut's annihilator. Their depths are at least 1 below the transport length, at the transport length the order of each increases by the shift of the cut, and above it each is at least its standard order. The cochain calculation now applies to either cut scheme.

D.4.3 Termination in the two filtration schemes

We check the hypotheses of Section D.4 for the two distributions of Hodge jumps in Section 29.1. In the first case the large jumps occur sparsely in the Frobenius cycle. In the second case we use the two possible magnitudes of the centers of their intervals.

Work in the Frobenius cycle of one original chart. Let B_0, G_0 be the larger and smaller starting points of the separated intervals, of width V. Index a segment by $0,1,\dots,l$ in the Frobenius direction, with the preceding cut at 0 and the current cut at l. An index at either end is counted in this direction even if the segment wraps around the cycle.

Sparse cuts. Here the low radius is η_0, and $G_0+V\leq\eta_0$. On the periodic extension of the cycle to $\mathbf{Z}$, let T be the positions assigned the larger start B_0 in Section 29.1.1. Call positions in T thick and all other positions thin. The sparsity hypothesis is

$$i,\ i+a,\ i+a+b \text{ cannot all belong to } T \quad (i\in\mathbf{Z},\quad 1\leq a,b\leq J).$$

This includes coincident residue indices after reduction modulo the cycle length. A cut is an index whose next J indices are thin. If a thick index w fails the cut condition, some thick index

w' occurs a steps after it with $1 \le a \le J$. The J indices immediately before w must all be thin, since any thick one would give a forbidden triple ending at w'. Therefore $w-(J+1)$ is a cut.

For any index i, either $i-(J+1)$ is a cut or there is a thick w in the interval from $i-J$ to $i-1$. In the second case one of w and $w-(J+1)$ is a cut. Thus the nearest strictly preceding cut has $l \le 2J+1$. Within its segment, every thick intermediate index occurs exactly $J+1$ steps after that cut. It occurs at least $J+1$ steps later by the definition of cut. If it occurred more than $J+1$ steps later, the preceding argument would put another cut strictly inside the segment.

Suppose all thick indices in $1,\dots,l$ have the same sign. The plain stop applies since every thin separated interval is contained in the low interval. This includes the constant-sign alternative. Otherwise there are exactly two thick indices in $1,\dots,l$. Take the earlier one, at step $L=J+1$, as the pivot. The only other thick index is the current cut, whose sign is opposite to the pivot sign.

The chart is unramified, $p \ge 7$, and $10p^J\eta_0 \le B_0$. Both cut shifts are zero. The low radius satisfies $\eta_0 \ge 1$. For the resting side take $g_0=\eta_0$, including at the pivot, where its separated interval is negative. The integer

$$s_0 = \left\lfloor \frac{B_0+10\eta_0}{p-3} \right\rfloor + 1$$

is an E-annihilator exponent for the resting side at the pivot. Indeed, at every index its jumps are at most $\max(\eta_0, B_0+V) \le B_0+\eta_0 \le (p-3)s_0$. This is the global upper bound required by the unramified lift-ideal estimate. The long-stop degree satisfies

$$N = s_0 + \eta_0 \sum_{a=0}^{J} p^a \le (B_0+10\eta_0)/4 + 1 + 2p^J\eta_0 < B_0.$$

For the last inequality, $10p^J\eta_0 \le B_0$, $p^J \ge 7$, and $B_0 \ge 70$. Set $b_0 = B_0+5\eta_0$. The radius parameter in the long stop is $A = 2b_0 \le p^L D_0$, by the sparse depth hypothesis with $e=1$. In particular $N < A$. This proves the geometric and numerical hypotheses of the long stop at every sparse cut.

For the sparse lift ideals and the step bound take the auxiliary width $W_0 = 5\eta_0$, and take $G_1 = n_0 = 10\eta_0$. The J successors of a cut satisfy $\beta + W_0 \le G_1$. Also $V + 4\eta_0 \le W_0$, $W_0 \le pn_0$, $G_1 \le (p-1)n_0$, and $\beta + W_0 \le 2b_0$. The depth hypothesis gives $2eb_0 \le p^{J+1}D_0$. The ramification inequality $8e \le p^{J+1}$ is assumed in the constant-sign case and follows from $e=1$, $p \ge 7$, $J \ge 1$ in the other. Finally

$$n_0 + \lfloor \log_p(2b_0) \rfloor + (J+1+D_E)W_0 + D_E G_1 \le 20(J+1+D_E)\eta_0 + \lfloor \log_p(2b_0) \rfloor \le X_0.$$

These bounds give the stated estimate wherever the following jump set is thin.

Two far starts. Here put $\widetilde{W} = V + 4\eta \le K$, $q = \lfloor B_0/p \rfloor$, and let the shift at an index be $\max(0, q-7K)$ if its successor is thick, and zero otherwise. An index is a cut when its shift is smaller than its far start. Every thick index is a cut, since its shift is at most $q < B_0$. Every index with thin next component is a cut, since its shift is zero. Therefore if the immediate predecessor is no cut, it is thin and the second predecessor is a cut. Thus $l \in \{1,2\}$.

For $l=1$, or $l=2$ with equal signs on the indices at steps $1,2$, use the plain stop. When $l=1$, the preceding cut may be thick with the opposite sign. Only the current index enters the sign condition at $a<l$. At $a=l$ the two jump indices are both $F^{-l}j$, as in Section D.4.1.

Otherwise $l = 2$, the index at step 1 is thin and the current index is thick. The preceding cut has shift zero. The no-cut condition at the thin index gives $G_0 + 7K \le q$, since $G_0 > 0$. The pivot is the current index. Take $L = 2$ (the long-stop parameter is $J = 1$), $g_0 = G_0 + \widetilde{W}$, and $s_0 = q + 12K$. At every index the jumps of the resting side are at most $\max(\eta, B_0 + V)$, and

$$(p-3)s_0 - (B_0 + V) \ge -3q + 12K(p-3) - (p-1) - V > 0.$$

Here $B_0 < Kp^2$ gives $q < Kp$, and $V \le K$, $K \ge 1$, $p \ge 7$ imply the last strict inequality. Also $\eta \le K \le (p-3)s_0$. Thus the global jump bound gives the annihilator E^{s_0} at the pivot. The bound on g_0 uses the thin start at step 1 and the negative separated interval of the resting side at the pivot. Since $G_0 + \widetilde{W} \le q - 6K$,

$$\begin{aligned} N &= s_0 + g_0(1+p) \\ &\le (p+2)q - (6p-6)K \le B_0 + 2q - (6p-6)K < B_0. \end{aligned}$$

With $b_0 = B_0 + \widetilde{W}$, one has $N < 2b_0 \le 4Kp^2$, so the radius parameter $A = 2b_0$ meets the long-stop bound with $D_0 = 4K$. The same integer $s_0 = q+12K$ bounds the upper jumps $\beta+\widetilde{W}$ in the sense $\beta + \widetilde{W} \le (p-3)s_0$. Indeed the calculation above applies with $\widetilde{W} \le K$ in place of V.

For the step bound use $n_0 = K$, $G_1 = G_0+\widetilde{W}$, $J = 1$, $D_0 = 4K$, and $X_0 = 3K+\lfloor \log_p(2b_0) \rfloor$. We have $G_1 \le (p-1)K$ by $2G_0 < Kp$. Also $s + \widetilde{W} \le pK$ by $q < pK$ and the two cases of $\max(0, q - 7K)$, and $\beta + s + \widetilde{W} \le 2b_0$. Because $D_E = 0$, the step-bound follows from $K + 2\widetilde{W} \le 3K$. For a cut with a thin next component use the lift-ideal estimate for a thin next component with shift zero. For a cut with thick successor (even if its shift happens to be zero), $s_0 - s \le 19K < \kappa$, so the unramified lift-ideal estimate gives $p^\kappa E^s$. All cuts and stops use the original signs and coefficient intervals of the original components.

The total exponent. For each index j, let $l(j)$ be the distance to the nearest preceding cut strictly before it, and let $d(j) = 0$ at a cut and $d(j) = l(j)$ elsewhere. Transport the two selected points at the last cut by $\varphi^{d(j)}$. Write $Fj = j^p$. Since $l(j) = 1 + d(F^{-1}j)$, for either family (x_i) of points at the cuts,

$$\varphi\left(\sum_j \varphi^{d(j)} x_{F^{-d(j)}j}\right) = \sum_j \varphi^{l(j)} x_{F^{-l(j)}j}.$$

Reindex the finite sum by the permutation F. For a parameter c on either moving side and $a = d(j)$ or $l(j)$, the transported point projects to

$$\pi(\varphi^a x_{F^{-a}j}) = p^{25\kappa}\varphi^a e_{F^{-a}j}(1 \otimes c) = p^{25\kappa} e_j(1 \otimes c).$$

Summing gives $p^{25\kappa}(1 \otimes c)$ since $\sum_j e_j = 1$. Off a cut the transported points agree because $d(j) = l(j)$. At a cut the two points have equal projections, including when the preceding cut represents the same residue index. Write (y, y') for the transported pair and (x, x') for the chosen pair there. In every summand use the same order of changes,

$$(y, y') \longrightarrow (w, y') \longrightarrow (w, w') \longrightarrow (x, w') \longrightarrow (x, x').$$

If the first side has the stopping point q, take $(w, w') = (q, x')$; the fourth change is zero. If the second side has the stopping point q', take $(w, w') = (y, q')$; the first change is zero. In each

case the other three changes satisfy the order inequalities just proved. Thus each position in this sequence can be summed across coefficient summands.

For the cocycle-difference test, take the third period from the cut supplying the endpoints and transport it by $\varphi^{d(j)}$ from $j^{p^{-d(j)}}$. At a cut this is the current third period. Off a cut it is the transported preceding period. At the fixed pairing depth m, the third period is over $p^{\kappa}e_j$, where e_j is the projector to index j. The reduction of its extension form is $p^{\kappa}\psi(e_j y, y')$, followed by the same $\mathbf{Z}_p$-linear functional on O for every test. On index j this is $p^{\kappa}\psi(y, y')$.

The cross pairings of distinct indices vanish: balance for the coefficient root of unity multiplies such a pairing by the difference of two Teichmüller lifts, which is a Witt unit. Thus the sum over all indices satisfies the four move tests and the two-cocycle difference test of Section C.2.1. Including the two moving sides and the Tate-dual period gives the factor

$$(p^{25\kappa})^2 p^{\kappa} = p^{51\kappa}.$$

In the sparse regime $\kappa = 12((J+1)X_0 + D_0) + 1$, so $51\kappa \le 10^3((J+1)X_0 + D_0)$. In the two-start regime $\kappa = 12(2X_0 + 4K) + 1$, where $X_0 = 3K + \lfloor \log_p(2b_0) \rfloor$. Consequently

$$51\kappa \le 10^4(K + \lfloor \log_p(2b_0) \rfloor).$$

Here $K \ge 1$.

Proposition D.1 (Uniform cup annihilation in the two filtration schemes)**.** *Let z, z' be crystalline cocycles of an allowed partner and the fixed Tate-dual lattice, respectively, with the pairing ψ and depth m of Section D.4. For any $\mathbf{Z}_p$-linear functional $\lambda : O \to \mathbf{Z}_p$, put $\psi_m = ((\lambda \otimes 1) \circ \psi) \bmod p^m$. Use one of the two filtration schemes under the interval and cut hypotheses of Section D.4.3. For the original lattices the Hodge intervals and the sparsity at each coefficient place are derived in Section 29.1.1. The two applications are given in Section E.3. Then*

$$p^{51\kappa}[(\psi_m)_*(\bar{z} \cup \bar{z}')] = 0 \quad \text{in } H^2(E_v, \mu_{p^m}), \qquad 51\kappa \le \begin{cases} 10^3((J+1)X_0 + D_0), & \text{sparse regime,} \\ 10^4(K + \lfloor \log_p(2b_0) \rfloor), & \text{two-start regime.} \end{cases}$$

E Local colength away from the original level

In this appendix, we apply the lifts of Appendix D to the local coefficient lattices. We handle separated Hodge–Tate weights and the cases of exact integrality, then combine the bounds at the primes away from the original level in Section E.3.

E.1 Colength for separated Hodge–Tate weights

In this subsection, we treat the unramified case in which the nonzero multiples of p avoid the intervals of filtration orders used by the divided operators. Using the E-power annihilator of Section D.2, we construct one submodule of crystalline cocycles on which all the requisite cup products vanish. Its colength gives the local estimate.

Assume the unramified chart with $p \ge 5$, and opposite separated intervals of low radius $\eta \ge 1$ as in Section 29.1.1. Let A be a nonnegative integer with

$$\beta_j + W + \eta \le (p-3)A, \qquad 6(A + \eta + W) < \beta_j$$

at every index j. Require that no nonzero multiple of p lie within $6(A+\eta+W)$ of $0, \pm\beta_j$. Here W is the width of the separated intervals. For jumps u, v of the two sides at the same index, one of them is at least $-\eta$, since their separated intervals have opposite signs. Both are at least $-(\beta_j + W)$, as $\beta_j > \eta$. It follows that

$$-u - v - 1 \le \beta_j + W + \eta - 1 \le (p-3)A.$$

The trivial quotient contributes only the jump 0. Since this upper bound holds at every index, the unramified annihilator estimate applies to all three extensions. Write $R(t)$ for the orders $R(t)(a,u) = -u + t\mathbf{1}_{a=0}$.

Let L be the coefficient lattice of the fixed side, so $C = Z^1_f(L)$ is its module of crystalline cocycles. Use the single extension on $L \oplus C$, with trivial action on C and with

$$g(y,c) = (gy + c(g), c).$$

The cocycle equation gives the action law. The module C is finite free: integral H^1 is finitely generated, the coboundaries have finite rank, and C is torsion free over O. A crystalline primitive for each element of an O-basis of C splits this extension over B_{cris}. In the de Rham grading it adds copies of jump zero. The nonzero-jump estimates in the extension thus use the multiplicities of L; the depth estimates over zero use the kernel L itself.

For the projector e_j of Section B.3, put $q_j(z) = e_j(1 \otimes z)$ for $z \in C$. Let $\mathcal{D}_j(t,c) \subseteq q_j(C)$ consist of the values $q_j(z)$ admitting a point of the extension over $q_j(z)$ with orders $R(t)$ and depth c. The module C and these maps are fixed before choosing the partner or the pairing. For a partner or a Tate-dual period in an individual cup test one may use the extension by O defined by that single cocycle. Let n_j, r_j bound f times the low-jump and all-jump multiplicities of L at index j.

We first prove the integral linear algebra for one order step. Suppose a system of orders c is being raised at (a,u), and $c(a,v) > c(a,u)$ for $v < u$ at the same index and level. Take $fm_u + 1$ vectors of orders c in the localized chart module N_{loc}, on which the coordinate retraction is the identity. Here m_u is the multiplicity of the jump. Consider the leading coefficient covectors on a basis of the m_u-dimensional graded space of de Rham forms at u. Lower lines make no contribution by the strict inequality on orders. Over C_p choose a maximal-rank minor with largest absolute determinant among minors of that rank. Cramer's rule then writes every covector in terms of its $d \le m_u$ selected covectors with coefficients in $\mathcal{O}_{C_p}$. Lift the coefficients by Fontaine's map and apply φ^a, so that the lifted identity is valid after evaluating at E_a. The difference is a vector whose order is raised at (a,u).

Write $M(c)$ for the localized module for the system c. Let x_j denote the original vectors and $x_{s(k)}$ the d selected ones. Retraction preserves the raised order and takes the product of a lifted scalar and $x_{s(k)}$ to the product by its coordinate in $\mathfrak{S}$. It gives

$$x_j - \sum_{k=1}^{d} b_{jk} x_{s(k)} \in M(c + \mathbf{1}_{(a,u)}), \qquad b_{jk} \in \mathfrak{S}.$$

Modulo E, the d-tuples of coefficients are in $W(\mathbf{F}_{p^f})^d$, of $\mathbf{Z}_p$-rank $fd \le fm_u$. Divide a nonzero integral relation by a coefficient of smallest valuation. The resulting coefficients a_j include 1 and satisfy

$$\sum_j a_j b_{jk} \in E\mathfrak{S}, \qquad \sum_j a_j x_j \in M(c + \mathbf{1}_{a=0}) + M(c + \mathbf{1}_{(a,u)}).$$

Here E raises exactly the level-zero orders. This argument is also valid for the actual extension lattice. Its multiplicity at any nonzero u is that of the sublattice L, since the parameter quotient is trivial.

E.1.1 Uniform comparison of the depth steps

In this subsection, we compare successive depths using the lift construction of Section D.4.

The depth version compares $\mathcal{D}_j(0,c)$ with the constants for the stronger depth c^+ at (a,u). Assume all current depth orders are $\geq -u$, so the order module of vectors over zero with those orders is contained in the no-pole order module. Denote this kernel order module by M. Write $X(c)$ for the points with no pole, depth c, and projection in $q_j(C)$. Its projection is $\mathcal{D}_j(0,c)$. Every vector of M has parameter zero and depth c by equivariance. Thus the quotient of constants is a quotient of

$$X(c)/(X(c^+)+M).$$

The depth step of contribution $C(a,c(a,u))$ on each point proves that this quotient is killed by $p^{C(a,c(a,u))}$.

To bound its p-torsion, take fm_u+1 points x_j and decompositions $px_j = x'_j + m_j$, where x'_j has the raised depth and $m_j \in M$. By the strictness of the kernel inclusion in Section C.2, each m_j comes from the localized kernel lattice with the same orders. Apply the preceding relation there. It gives a primitive combination x of the x_j for which $px-w$ has the raised depth and w is over zero with all level-zero orders raised by 1.

Suppose further that no multiple of p lies strictly between $c(0,v)$ and $c(0,v)+p^a$ at any line v. All the depth operations that raise w from the first level-zero order increment to p^a increments are free. Use their common composite B on $x, w, px-w$. The factors are $\mathbf{Z}_p$-linear with scalar multiplier 1. On a point of depth c their corrections have parameter zero and orders at least c, so

$$B(px-w) = pBx - Bw, \qquad Bx - x \in M.$$

Since w lies over zero and has the increased orders, it has at least those depths as well.

The free steps and improvement of orders give $Bw = E^{p^a} w_1$ with $w_1 \in M$. In checking this improvement, either $c(0,v)+1>0$ or $c(0,v)+p^a \leq 0$, because the interval contains no multiple of p, in particular no 0. Use $E^{p^a} = E_a + pF$ with $F \in \mathfrak{S}$. Then

$$p(Bx - Fw_1) = B(px-w) + E_a w_1.$$

The first term has the raised depth because the free operator preserves depths already acquired. The second term has that depth because w_1 has orders c and the multiplication raises every order at a by 1.

At each period prime E_b, the scalar p has order zero, so $Bx - Fw_1$ has the raised depth as well. It differs from x by an element of M. Thus our p-torsion classes satisfy a primitive relation and their dimension is at most fm_u. The filtration by annihilators p^i now bounds the quotient length by $C(a,c(a,u))fm_u$: each successive layer injects into the p-torsion by multiplication by p^{i-1}.

One may iterate this estimate toward a target depth that is antitone in the jumps at the level being raised. At each step take the least jump u whose current depth is short of the target. Every smaller jump has already reached its target and therefore has strictly greater current depth than the one being raised. For level 0 the open intervals of length $p^0 = 1$ are always

empty of integers. For level 1 we retain the same depth at level 0 on every line, a multiple of p, so the intervals of length p contain no multiple in their interiors.

For $0 \le s \le A$, take $t = A - s$ and $\kappa = s + t = A$ in Section D.2. The scalar $p^s E^{A-s}$ kills each of the three standard-order projection cokernels at every index. On either of the two original sides, fix a parameter component d and choose a standard-order lift of $p^s E^{A-s} d$. The coordinate retraction preserves its orders and fixes its projection. Division by E^{A-s} then gives a point over $p^s d$ with orders $R(-(A-s))$.

Raise its depth to $R(2A+\eta)$. Every level-zero step is within $6(A+\eta+W)$ of 0 or $\pm\beta_j$, so none changes the parameter multiplier p^s. At $s = 0$ the gap $[-A, -\eta)$ improves the orders to at least $R(-\eta)$ over d. At $s = A$ the orders are at least $R(0)$ over $p^A d$. For the tensor-dual extension the depth $R(A)$ used below is verified for its own jumps in Section E.1.2.

E.1.2 Pole removal and the final integral lift

In this subsection, we apply the comparison of Section E.1.1 at each pole order.

Remove the pole one step at a time. For $t = -\eta, \dots, -1$, compare $\mathcal{D}_j(t, R(2A+\eta))$ with $\mathcal{D}_j(t+1, R(2A+\eta))$. Their quotient is killed by p^A. If t is absent from the jump set, every point already has the stronger orders. Indeed at $v < t$ its old order $t - v > 0$ forces its constant term to vanish, and at $v > t$ the retained depth gives order at least $\min(0, 2A+\eta-v) \ge t+1-v$. This step has zero colength. If t occurs, take $fm_t + 1$ representatives in the extension and apply the single-order relation at t. It applies because $R(t)(0,v) = t - v$ is strictly decreasing. Raising all level-zero orders by 1 already implies the order increase at $(0,t)$; hence the relation has this increase. For $v < t$ the old order is positive, so the depth forces the full new required order $t+1-v$, and for $v > t$ that required order is nonpositive and follows from the depth too. The combination therefore lies in the smaller module. The primitive relations bound the number of generators by fm_t, and the length by Afm_t. All these jumps are low and the trivial quotient adds no jump at $t < 0$. Summing over $-\eta \le t < 0$ bounds the total $\mathbf{Z}_p$-colength of pole removal by $Af \sum_{-\eta \le t < 0} m_t \le An_j$.

At an index where the separated interval of the fixed side is positive, retain the depth $R(2A+\eta)$. At a negative-interval index seek the depths

$$\begin{aligned} d_j(0,u) &= \beta_j + W + A + \eta, \\ d_j(1,u) &= W + 1 - u + 2A\mathbf{1}_{|u| \le \eta}, \\ d_j(a,u) &= -u \qquad (a \ge 2). \end{aligned}$$

At level zero first increase the depth to the least multiple T of p at least $\beta_j + W + A + \eta$, allowing a fall where the old depth was bigger. The steps for a jump u are $2A+\eta-u \le y < T$. For $|u| \le \eta$ they lie in $[0, T)$, with contribution zero at $y = 0$. Since $T < \beta_j + W + A + \eta + p \le (p-2)A + p$, Legendre's formula gives

$$\sum_{2A+\eta-u \le y < T} C(0,y) \le v_p((T-1)!) \le (T-1)/(p-1) \le A+1.$$

For $-u \in [\beta_j, \beta_j + W]$, a multiple of p in this interval would satisfy $\beta_j + 2A + \eta \le y \le T - p < \beta_j + W + A + \eta$. The exclusion near β_j rules this out. By the depth version of the order estimate the contribution in length is at most $(A+1)n_j$.

Set $i = \varphi^{-1} j$. At level one the jumps and their multiplicities are those at i, whose separated interval can have either sign. Keep T at level zero; the open interval $(T, T + p)$ contains no multiple of p. For a jump u at i, the first-level steps are the integers

$$-u \le y < -u + W + 1 + 2A\mathbf{1}_{|u| \le \eta}.$$

For $|u| \le \eta$, we have $|y| \le \eta + W + 2A$. Otherwise $-u$ is within W of a signed center $\pm\beta_i$, so y is within $2W$ of that center. The exclusion at i implies that each such y is zero or prime to p, hence $C(1, y) = 3$ in the convention $C(1, 0) = 3$. The target $d_j(1, u)$ decreases on each interval. Across the two intervals it is antitone because $\beta_i - \eta > 2A$. Thus the first-level increment in length is at most $3(W + 1)r_i + 6An_i$. Finally the depth at level zero may be lowered from T to $d_j(0, u)$ without any loss. The contribution at j is therefore at most

$$(2A + 1)n_j + 3(W + 1)r_{\varphi^{-1} j} + 6An_{\varphi^{-1} j}.$$

Write $D_j \subseteq q_j(C)$ for the resulting module of constants at index j. The maps q_j induce the injection

$$V = \bigcap_j q_j^{-1}(D_j) \subseteq C, \qquad C/V \hookrightarrow \bigoplus_j q_j(C)/D_j.$$

Since $j \mapsto \varphi^{-1} j$ permutes the indices,

$$\operatorname{length}_{\mathbf{Z}_p}(C/V) \le (8A + 1) \sum_j n_j + 3(W + 1) \sum_j r_j \le 11 \sum_j \big(n_j(A + 1) + r_j(W + 1)\big).$$

Thus V has full $\mathbf{Z}_p$-rank in C. The quotient bound uses the jump multiplicities of L alone, including when the partner stays fixed during the middle change at a cut.

For any allowed partner and pairing, the constructed points give safe moves, with a cut at every index. On the partner take points over every constant of depth $R(2A + \eta)$, starting from E^A and using the free steps. Their orders are the improvement of $R(-A)$ by that depth. In particular their orders at level zero are at least $-u - \eta$. Suppose the fixed side has a positive separated interval at j. Write u, v for the partner and fixed jumps at the index read at the tested level. With x_k, y_k the partner and fixed-side points at k and $i = \varphi^{-1} j$, the active moves are

$$(\varphi x_i, \varphi y_i) \longrightarrow (x_j, \varphi y_i) \longrightarrow (x_j, y_j).$$

They occupy the first two slots of the four-move path. First move the partner with the old fixed point at rest. At level zero the difference of the moving points has order at least 0, since the current partner has $u \le \eta$ and its improved order is nonnegative. The resting depth is at least 1. At level one, read both jumps at $i = \varphi^{-1} j$. The moving order is at least $-u - \eta$, and the old resting depth is at least $A + \eta - v$ for either sign of the separated interval at i. At levels $a \ge 2$ the sum is at least $-u - v$. Thus the Tate-dual lift with shift A at i meets the order criterion of Section D.4.2.

Next move the fixed side with the current partner at rest. At level zero the moving order is at least $\min(0, -v)$. Since $v \ge -\eta$, the sum with the resting depth is at least

$$\min(0, -v) + 2A + \eta - u \ge A - u - v.$$

At level one the old fixed point has order at least $-v$ by the pole removal at i. At all positive levels the sum is therefore at least $-u - v$. This time use the Tate-dual lift with shift A at j.

If the fixed side has a negative separated interval at j, write x_k, y_k for the partner and fixed-side points at k, and put $i = \varphi^{-1} j$. The active moves are

$$(\varphi x_i, \varphi y_i) \longrightarrow (\varphi x_i, y_j) \longrightarrow (x_j, y_j).$$

They occupy the second and third slots of the four-move path. In the first active move use the Tate-dual point from the E^A-annihilator at i, transported by Frobenius. At level zero the moving difference is integral since both φy_i and y_j have nonnegative order. For y_j this follows from $d_j(0, v) > 0$. The resting depth of φx_i is at least one by the transported $z\tau$ divided difference. At level one the moving difference has order at least $-v$ by pole removal at i, and the resting depth there is $2A + \eta - u$. Above level one the orders sum to at least $-u - v$. Thus the sums dominate $1, A - u - v, -u - v$ at the three ranges of the transported dual test.

In the second active move the fixed point y_j rests. At level zero $-v \le \beta_j + W$, so $d_j(0, v) \ge A + \eta - v$, and at level one $d_j(1, v) \ge -v$. Its no-pole orders dominate $R(-\eta)$. The partner's orders dominate $R(-\eta)$ by the gap above and its depth is $R(2A + \eta)$. For this move take a Tate-dual point at $i = \varphi^{-1} j$ with depth $R(A)$ and with orders given by the improvement of $R(-A)$ by $R(A)$. Its jumps are 0 and $-u - v - 1$ for partner and fixed-side jumps u, v at i. Two low jumps or two opposite far jumps give a dual jump within $2\eta + W + 1$ of zero. If only one jump is far, the dual jump is within $\eta + W + 1$ of $\pm\beta_i$. The exclusion at i thus gives the required free steps. After Frobenius transport from i to j, the level-zero order is zero, while $\beta_j + W + A + \eta$ exceeds the possible pole of the moving point. At levels > 1 the three orders sum to the standard Tate order 1.

At level 1 of j, both u and v are again read at $i = \varphi^{-1} j$. Their far intervals have opposite signs and the same starting magnitude β_i. Write

$$E_q(w) = \begin{cases} q - w, & w < -A, \\ \min(q - w, 0), & w \ge -A. \end{cases}$$

The sum of the three orders there is at least

$$E_A(-u - v - 1) + \min(-u, E_{2A+\eta}(u)) + W + 1 - v + 2A\mathbf{1}_{|v| \le \eta}.$$

This is positive. For $|v| \le \eta$, use $E_A(-u - v - 1) \ge u + v + 1 - A$ and the moving order $\ge -u - A$, giving $W + 2$. If v is far and u is low, the gap gives $E_A(-u - v - 1) \ge A + u + v + 1$, again giving $W + 2$. If both are far, then $|u + v| \le W$, the moving order equals $-u$, and $E_A(-u - v - 1) \ge u + v - W$. The sum is at least 1. The safe-move criterion, with the dual point in place of its standard lift, now applies. Fix a partner cocycle c_1 and $c \in V$. The points x_j, y_j lie over $e_j(1 \otimes 1)$ and $e_j(1 \otimes c)$, respectively. With $x = \sum_j x_j = (a, 1)$, $y = \sum_j y_j = (b, c)$ and $i(j) = \varphi^{-1} j$, one has

$$\sum_j \varphi x_{i(j)} = \varphi x, \qquad \sum_j \varphi y_{i(j)} = \varphi y, \qquad \varphi(e_{i(j)}) = e_j.$$

For the intermediate first point take $w_j = x_j$ when the fixed side has positive far sign and $w_j = \varphi x_{i(j)}$ when it has negative far sign. Take $w'_j = y_j$ in both cases. These points have the same respective projections as the endpoints. The differences therefore lie in the localized kernel with the asserted orders by Section C.2. The chart group fixes the idempotents and preserves the orders, including after the Galois translation in a second-side move.

For the endpoint cup test, at j pair $gx_j - x_j$ with $g(hy_j - y_j)$. The depth sum at level zero is at least

$$(2A + \eta - u) + (A + \eta - v) \geq A - u - v,$$

for both fixed-side signs. At positive levels it is at least $-u - v$. Thus the current E^A Tate-dual lift gives total order at least one. Dividing the first difference by $z\tau$ and restoring the factor after contraction gives a value in the small Witt ideal by Section D.4.2.

For the original pairing ψ modulo p^m, the third cocycle is $(g\psi - \psi)/p^m$ in $(Y \otimes_O L)^\vee(1)$. Each third point, including a transported one, projects to $e_j(1 \otimes 1)$ and gives $\psi \bmod p^m$ on the j-components. Distinct indices have zero cross pairing by O-balance. Summation therefore preserves the four move tests and the endpoint test, also after any $\mathbf{Z}_p$-linear reading of the coefficient ring. The kernel differences are $ga - a + c_1(g)$ and $gb - b + c(g)$. Section C.2.1 with $e_1 = e_2 = e_3 = 0$ proves vanishing on the chart for every $c \in V$, partner, depth and pairing. The annihilator is O-linear: multiplying a cocycle by a coefficient scalar composes each integral coefficient functional in the test with multiplication by that scalar. This allows the constructed $\mathbf{Z}_p$ sublattice to bound the colength of the O-annihilator.

Write $K_O = \mathrm{Frac}(O)$ and let d_j count its coefficient embeddings at index j. These are counted in the de Rham grading over $\mathbb{C}_p$, so each embedding occurs at one index. Restriction of scalars in the order steps is over $\mathbf{Z}_p$: the multiplicity at a jump is the sum of the graded multiplicities of those embeddings, and $\mathfrak{S}/(E)$ has rank f over $\mathbf{Z}_p$. Thus $n_j = fd_j\mu$, $r_j = fd_j\rho$ suffice for the respective bounds per embedding μ, ρ, with $\sum_j d_j = [K_O : \mathbf{Q}_p]$. For the coefficient field chosen in Section 29.1, the primitive Teichmüller root ζ_v generates the unramified subfield of degree $f = f_v$. Each of its f embeddings extends to exactly $[K_O : \mathbf{Q}_p]/f$ embeddings of K_O. Thus $d_j = [K_O : \mathbf{Q}_p]/f$ on this Frobenius orbit and $d_j = 0$ elsewhere. The multiplicities are computed over $\mathbb{C}_p$. The length comparison below takes place on the original integral O-lattice. Let $V \subseteq C$ be the common $\mathbf{Z}_p$-sublattice constructed above, and let $\mathcal{A}_v$ be the O-linear cup annihilator. Then $V \subseteq \mathcal{A}_v$. In the application to an original summand L_{orig}, the fixed lattice in C is $L = L_{\mathrm{orig}}^\dagger$. Write e_O, f_O for the ramification index and residue degree of $K_O/\mathbf{Q}_p$. Since $\mathrm{length}_{\mathbf{Z}_p}(O/\mathfrak{m}_O) = f_O$ and $\mathrm{ord}_O(p) = e_O$,

$$\frac{\Delta_v(L_{\mathrm{orig}})}{e_O} = \frac{\mathrm{length}_{\mathbf{Z}_p}(C/\mathcal{A}_v)}{[K_O : \mathbf{Q}_p]} \leq \frac{\mathrm{length}_{\mathbf{Z}_p}(C/V)}{[K_O : \mathbf{Q}_p]} \leq 11f\big(\mu(A+1) + \rho(W+1)\big).$$

This is a common colength estimate with cup exponent zero inside the sublattice.

E.2 Exact integrality in two special cases

In this subsection, in two special cases the filtration bounds in Section E.1 let us use the full crystalline cocycle module. We verify the order conditions for the coboundary calculation of Section C.2.1 with lifts over the unscaled constants.

Keep the unramified chart and separated-interval hypotheses of Section E.1. At each index i the two jump sets fit opposite signed intervals of far start β_i. Put $R = 6(A + \eta + W)$. The required arithmetic hypothesis is that no nonzero multiple of p lies within R of 0 or $\pm\beta_i$. The three upper jump bounds $(p-3)A$ give annihilators E^A for both side cokernels and the Tate-dual cokernel. Use the integral lift-ideal estimate of Section D.2. Write P_s for the order function

$-u - s\mathbf{1}_{a=0}$, where a is Frobenius level, and put

$$D_{s,h,h_1}(a,u) = -u + (s+h)\mathbf{1}_{a=0} + h_1\mathbf{1}_{a=1},$$
$$O_h(a,u) = \begin{cases} \max(0, -u-h), & a = 0, \\ -u, & a > 0. \end{cases}$$

A point of order P_s has a pole of order at most s at level zero. All points below lie over the given constant in the given index, with no p-multiplier.

If neither side has a jump in $[-A, 0)$, lift using E^A and divide by E^A. The initial orders and depths are P_A. Raise the depth to $D_{A,\eta,0}$. The steps lie at level zero in $[-u-A, A+\eta-u)$. A low jump gives a step within $A+2\eta$ of zero and a far jump gives one within $A+\eta+W$ of its opposite far start. Both are within R. The contribution is zero by the unramified depth calculation. If $u < -A$, the initial order $-u-A > 0$ eliminates its constant term and the new order is at least $A+\eta-u$. Otherwise $u \geq 0$, and the improved order is at least $\min(A+\eta-u, 0) \geq -u$. Thus the new points have order P_0.

At each index let the side with the negative separated interval rest. Its jumps $v \leq \eta$ imply that its just-constructed point has order O_η: for $v < -\eta$ use P_0, and otherwise use the nonnegative depth $A+\eta-v$ to rule out a pole. The moving side has $u \geq -\eta$ at level zero and both its current and preceding points have order P_0. The difference $\varphi x_{ip^{-1}} - x_i$ has orders at least $\min(0, -u)$ at level zero and $-u$ at levels ≥ 1. Adding the resting depth gives at least $A-u-v$ at zero and $-u-v$ at the other levels. These are exactly the order bounds required for the Tate-dual annihilator E^A at the current index.

Now let the second side have jump set in $[-\eta, 0]$, and put $h = A+\eta$. Its lift ideal contains (1), by the small-jump estimate with maximum jump 0. Give the points on the first side orders P_h and depth $D_{A,h,0}$ by starting at P_A and raising the depth for free. Give the points on the second side orders P_0 and the same depth by starting at P_0. The steps in either case are within $A+h+\eta+W \leq R$ of the three centers. For the first side the improved orders dominate P_h because $h \geq A$: if $-u-A > 0$ the order becomes $A+h-u$, and otherwise the improved order $\min(0, A+h-u)$ is at least $-u-h$. For the second side $u \leq 0$, so the same constant-term rule retains P_0.

Where the first side has negative separated interval, its point has stop orders O_h by the constant-term rule and $u \leq \eta \leq h$. The moving second side has pole-free preceding and current points, with $v \geq -h$. The sum of the moved order and resting depth is at least $A-u-v$ at level zero and $-u-v$ above.

Where the first side has positive separated interval, transport the preceding point of the second side by φ. Its depth is initially 1 at level zero, $A+h-v$ at level one, and $-v$ above. Its orders are 0 at zero and $-v$ above. Raise the depth at zero to $A+h-v$ by free steps; these lie between 1 and $A+h+\eta$. The result has order at least O_h, since $v \geq -\eta \geq -h$. Its depth dominates $D_{A,h,h}$, including the extra h at level one. The moving first side has $u \geq -h$ at level zero. The difference between its preceding translated point and its present point has orders bounded below by

$$-u-h \quad (a = 0, 1), \qquad -u \quad (a > 1).$$

The resting depth therefore gives $A-u-v$ at zero and $-u-v$ at all other levels. This explains the additional first-level depth required when the moving point has a pole at the preceding index.

Let $i = \varphi^{-1}j$ and write x_k, y_k for the two points at k. If the first side has negative far sign, the active path passes through $(x_j, \varphi y_i)$. In the first case above, when the second side has

negative far sign, the active path passes through $(\varphi x_i, y_j)$. In the second case above, when the first side has positive far sign, write $\widehat{y}_j$ for the level-zero improvement of φy_i. The path is

$$(\varphi x_i, \varphi y_i) \longrightarrow (\varphi x_i, \widehat{y}_j) \longrightarrow (x_j, \widehat{y}_j) \longrightarrow (x_j, y_j).$$

The improved point keeps its transported level-one depth.

Write $h_0 = \eta$ in the first case and $h_0 = h$ in the second. For the move to the stop, the moving difference has order at least zero at level zero, where the transported resting point has depth at least one. At level one, for moving jump s and resting jump t, the lower bounds are $-s - h_0$ and $A + h_0 - t$. Above level one their sum is at least $-s - t$. Use the transported Tate-dual point from i. For the final move from the stop use the current Tate-dual point. The order and depth bounds are $-s - h_0$ and $A + h_0 - t$ at zero and sum to at least $-s - t$ at positive levels. For the endpoint test the two current depths at zero dominate $A + h_0$ minus the respective jump, and the positive-level depths dominate minus that jump. Hence the endpoint differences pass the test with the current dual point, also with the second difference Galois translated. All points used in the four-move cochain have unscaled parameters. The four-move calculation gives a cup coboundary on the chart. In the off-level application of Section E.3 the chart group is G_{E_v} itself, so the original cup vanishes and the normalized defect is zero.

E.3 The bound for the original coefficient lattices

In this subsection, we apply the preceding estimates to the coefficient primes away from the original level. The bounds from Sections D.4, E.1 and E.2 yield the required normalized colength on the local class of Section 29.1.

Use the original local class in Section 29.1, with $p = \ell \nmid M$, fixed local dual $L^\dagger$, and $d_v = [E_v : \mathbf{Q}_\ell] = e_v^{\mathrm{ram}} f_v$. Write $\rho = \operatorname{rank}_O L$; write $c_* = 1$ for its main summand and $c_* = t$ for each additional character. Only the actual jumps of $L^\dagger$ are used on the fixed side. The test-side jump set is the full original upper set. The Kummer chart is that of E_v, with working group G_{E_v} itself. Denote the fixed lift number by j_0 and the variable index in the residue Frobenius cycle by j. Thus $\delta_{j_0} = j_0\Delta + h - 2t$ is constant along the cycle, whereas β_j is its far starting magnitude at index j. For a residue parameter i, let U_i and V_i be the unions, over coefficient embeddings reading i, of the allowed partner jumps and the actual jumps of $L^\dagger$, respectively, with 0 adjoined. Put $T_1 = Y$, $T_2 = L^\dagger$ and $T_3 = (Y \otimes_O L^\dagger)^\dagger$. Off M, T_1, T_2 are crystalline over E_v by Section 29.1; tensor products and Tate duals give the same property for T_3.

For a cup test at depth ν, take the two crystalline cocycles c_1, c_2 and put $c_3(g) = (g\psi - \psi)/p^\nu$. Equivariance modulo p^ν makes $c_3(g)$ integral. The three extensions $T_i \oplus O$ have action $g(v, s) = (gv + sc_i(g), s)$. Write $c_i(g) = gb_i - b_i$ over B_{cris}: for $i = 1, 2$ apply Frobenius to a $B_{\max}$-primitive, and take $b_3 = \psi/p^\nu$. Then $(-b_i, 1)$ is an invariant lift of 1, proving crystallinity of each original lattice. For the extension by the whole cocycle module in Section E.1, choose these primitives on an O-basis and extend linearly. The trivial parameter contributes only the jump 0. The third extension has jumps contained in

$$\{0\} \cup \{-u - v - 1 : u \in U_i,\ v \in V_i\}.$$

Indeed an O-bilinear pairing pairs only factors at the same coefficient embedding, and the union displayed above contains every such pair.

Small coefficient primes. Suppose $\ell \le Q_*$. Use the sparse scheme of Section D.4.3 with

$$\eta_0 = 2G, \quad B_0 = b + \delta_{j_0}, \quad G_0 = G + \delta_{j_0}, \quad J = \lfloor \log_\ell h^3 \rfloor, \quad D_0 = 2K_\ell, \quad X_0 = K_\ell.$$

The actual far width is W_*. The thin intervals fit inside $[-\eta_0,\eta_0]$, since $G_0+W_*\le 2G$. The thick-index sparsity for successive gaps at most J is the one proved for these coefficient parameters. For the unramified alternative, $20\ell^J G\le 20h^3G<B_0$. In the constant-sign alternative, $8e_v^{\mathrm{ram}}<h^3<\ell^{J+1}$. The sign assigned to each residue parameter agrees with the CM sign at every thick embedding. At $\ell=d$ the argument of Section 29.1.1 gives one such sign for all thick embeddings; all thin intervals already lie in $[-2G,2G]$. At $\ell\mid q_1$, in particular at $\ell=2,3,5$, the quadratic field K splits at ℓ. The CM sign is constant at the fixed completion, so every segment uses the plain stop of Section D.4.1. More explicitly, put $u_3=\lfloor\log_3 s\rfloor$. Since $q_0=2^4\cdot 3\cdot 5\cdot 7\cdot 11$ and $q_1=3^{u_3+5}2^s q_0$, the ramification indices and the depth constants of Section D.1 are

ℓ	2	3	5
e_v^{ram}	2^{s+3}	$2\cdot 3^{u_3+5}$	4
D_E	$s+4$	u_3+6	1.

Indeed the roots of order prime to ℓ give an unramified extension, whereas $\Phi_{\ell^n}(1+X)$ is Eisenstein for $n=v_\ell(q_1)$. Thus $e_v^{\mathrm{ram}}=\varphi(\ell^n)$ and $D_E=1+v_\ell(e_v^{\mathrm{ram}})$. In particular the dyadic logarithmic valuation identity for the depth step applies because $v_2(\theta\Delta_z)=1+1/e_v^{\mathrm{ram}}>1$. For $\ell=2$, let j be any thick position on the periodic Frobenius cycle. The order of 4 mod $3L_0$ and $J<L_0$ exclude another thick position in the J places on either side. Consequently j and $j-J-1$ are cuts, and each intervening position has j among its next J positions. The preceding cut is exactly $J+1$ steps back, with both shifts zero. If j is a thin cut, the positions $j,\dots,j+J-1$ are thin, so $j-1$ is a cut and the preceding distance is 1. This includes every position on an orbit without thick indices. Both cases use the divisor $z\tau$ on the integral localized kernel at this completion.

The sparse depth inequality follows from

$$B_0+5\eta_0<2B_0,\quad 2B_0<h^2K_\ell,\quad e_v^{\mathrm{ram}}\le h,\quad h^3<\ell^{J+1}.$$

These give $2e_v^{\mathrm{ram}}(B_0+5\eta_0)<\ell^{J+1}(2K_\ell)$. For the other sparse condition, $J+1+D_E\le h$ and $\lfloor\log_\ell(2(B_0+5\eta_0))\rfloor<m\le h$. This uses $e_v^{\mathrm{ram}}\le h$, $h\ge 99m$, and $2\log(4b)<m$. Thus the contribution sum is at most $41hG<K_\ell$, using $b\ge 99G(h^3+1)$ and $2B_0<h^2K_\ell$. The uniform exponent is at most $10^3K_\ell(J+3)\le 10^4K_\ell m$. In particular at $\ell=2$ the cut parameters give $\kappa=12(J+3)K_2+1$. For every permitted partner lattice, pairing and finite depth, Proposition D.1 therefore kills the cup by $2^{51\kappa}$. The points may depend on that test. The exponent depends only on the original parameters, so $2^{51\kappa}Z_f(L^\dagger)\subseteq\mathcal{A}_v$.

Large coefficient primes with large loss. The rational primes ramifying in $E=K(\zeta_{q_1})$ divide q_1d. Since $q_1/q_0=3L_0\cdot 2^s$ and $6\mid q_0$, they lie in the prime support of the fixed integer q_0d. For large x, $Q_*=kt$ exceeds every prime in that support and exceeds 7. Both large-prime branches thus use unramified charts with $\ell\ge 7$.

Suppose $\ell>Q_*$ and $\lambda_\ell\ge h^3$. Use the two-start scheme, with $B_0=b+\delta_{j_0}$, $G_0=G+\delta_{j_0}$. Since $K_\ell\ge\lambda_\ell/Q_*$ and $b<\ell\lambda_\ell$, the required sizes follow from

$$\begin{aligned}K_\ell\ell^2-b&>(\ell\lambda_\ell/Q_*)(\ell-Q_*)\ge h^3>\delta_{j_0},\\ K_\ell\ell&>b/Q_*>2(G+\delta_{j_0}),\\ K_\ell&\ge h^3/Q_*>W_*+4\eta.\end{aligned}$$

The argument of the logarithm in the two-start estimate is at most $4b$. Since $2\log(4b)<m$ and $\log\ell>1/2$, the integer $\lfloor\log_\ell(2(B_0+W_*+4\eta))\rfloor$ is at most $m-1$. With $K_\ell\ge 1$, the resulting uniform exponent is at most $10^4K_\ell m$.

Large coefficient primes with small loss. For $\ell > Q_*$ and $\lambda_\ell < h^3$, put $A = 2(\lambda_\ell + 1)$. The missing-multiple radius $6(A + \eta + W_*)$ is exactly the one proved above. It is smaller than G, so in particular smaller than both far starts. Also

$$\beta_j + W_* + \eta < 2b < \ell A/2 \le (\ell - 3)A.$$

The last inequality uses $\ell \ge 7$. These are the separated-interval hypotheses. For each original character $O(-a)$, $0 \le a < t'$, the fixed dual $O(a + 1)$ has jump $-a - 1 \in [-\eta, 0]$. Section E.2 therefore applies to its full cocycle module. By (83) the defect of the original sum of characters also vanishes. For the main summand, if $\lambda_\ell < \lfloor h/5 \rfloor$, then $A < h/2$ for the marked scales. Every negative jump on either side is at most $-h/2$, so the same section gives zero defect by the gap below zero.

Otherwise use the common submodule from Section E.1, with at most $\mu = \lfloor \rho m/(kc_*) \rfloor$ low jumps per embedding. Its normalized colength is at most

$$11 f_v\big(\mu(A + 1) + \rho(W_* + 1)\big) \;\le\; 11 d_v\big(6\mu\lambda_\ell + \rho K_\ell\big).$$

If $K_\ell = 0$, then $2\lambda_\ell < Q_*$, and the scale bounds imply $\lambda_\ell < \lfloor h/5 \rfloor$. That case was covered by the zero-defect argument. Here $A + 1 \le 6\lambda_\ell$ and $W_* + 1 \le K_\ell$. For the second inequality use $\lambda_\ell \ge \lfloor h/5 \rfloor$ and $99(W_*+1)Q_* \le h$ to see $2\lambda_\ell \ge Q_*(W_*+1)$. The bound comes from the colength of this single submodule, whose cup vanishes unscaled for every partner and every pairing.

On the two branches with a uniform exponent the rank bound for the crystalline cocycle lattice gives normalized defect at most $2d_v\rho \cdot 10^4 K_\ell m$. Combining with the separated-interval calculation gives the bound.

Proposition E.1 (Off-level crystalline colength)**.** *For the original coefficient lattice and the local class above at $\ell \nmid M$, the normalized defect satisfies*

$$\frac{\Delta_v(L)}{v_{\mathrm{ord}}(\ell)} \le 10^5 d_v(\rho K_\ell m + \mu\lambda_\ell).$$

If $\ell > Q_*$, then $\Delta_v(L) = 0$ for a character when $\lambda_\ell < h^3$, and for the main summand when $\lambda_\ell < \lfloor h/5 \rfloor$.

The comparison with $d_v\rho F_\ell/c_*$ in Section 29.1 now applies.

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