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THERE IS NO SYMMETRIC CONFIGURATION 93_{10}

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Abstract. A symmetric configuration v_k is a set of v points and v lines, with k points on every line and k lines through every point, such that two points lie on at most one common line. We prove that no configuration 93_{10} exists. No cyclicity or automorphism is assumed. The parameters have deficiency two, $v = k^2 - k + 3$, and we use the method of our earlier paper on 59_8: the points not collinear with a given point form a 2-regular "leave" graph, its cycle lengths obey exact moment and periodicity conditions, and the incidence matrix gives a factorization $G = BB^\top$ of an explicit integer Gram matrix G built from those cycles. For 93_{10} the conditions leave 37 possible cycle structures. Ten have an even number of cycles and fail a determinant condition, 25 fail a local quadratic-form count modulo 49 or 625, one fails a cross-moment identity, and the last, three 31-cycles, is excluded by a finite analysis of circulant blocks. The entire nonexistence proof, including every finite computation on its formal proof path, is checked by the Lean 4 kernel with the standard axioms only. The text also gives alternative arithmetic cross-checks. We also record smaller results proved in Lean by the same method: every 113_{11} has a Hamiltonian leave graph, and a 75_9 cannot have one. We also formalize, for two 75_9 profiles, the parity theorem of Philbrook and Peet.

1. Introduction

A *symmetric configuration* v_k consists of v points and v lines such that every line contains k points, every point is on k lines, and two distinct points are on at most one common line. Equivalently, it is a $v \times v$ matrix A with entries in $\{0, 1\}$, all row and column sums equal to k, and no all-ones 2×2 submatrix. Counting the points collinear with a fixed point gives $v \geq k^2 - k + 1$, with equality exactly for projective planes of order $k - 1$. The integer $d = v - (k^2 - k + 1)$ is the *deficiency*. For each k the existence of v_k is known for all large v, and the open cases sit just above the projective-plane bound; see Gropp [7] and the tables of Davydov, Faina, Giulietti, Marcugini and Pambianco [2, 3].

Theorem 1. *There is no symmetric configuration* 93_{10}*. Equivalently, there is no* 93×93 *matrix with entries in* $\{0, 1\}$*, exactly ten ones in each row and column, and no all-ones* 2×2 *submatrix.*

Prior work. Davydov et al. distinguish *cyclic* configurations, whose incidence matrix is circulant, from arbitrary ones. Their 2014 tables [3] list 93^c_{10} among the parameters for which no *cyclic* configuration exists; this does not address noncyclic configurations. The parameter-spectrum table of Funk, Labbate and Napolitano [6] marks 93 at degree ten as unresolved for general configurations. In our earlier paper [10] we proved that no 59_8 exists, by the method used here; Kaski and Östergård [8] had earlier excluded 33_6 by exhaustive search, and Philbrook and Peet [9] proved that the leave graph of a deficiency-two configuration has an odd number of cycles (apart from 9_3). The determinant and local quadratic-form techniques used by Francetić, Herke and Horsley [4] for symmetric pair coverings are a methodological precursor of Section 3.4.

In graph terminology the excluded incidence graph is a 10-regular bipartite graph on 186 vertices of girth six: no 4-cycle is possible, and the Moore bound rules out girth eight. Its order is four above the girth-six Moore bound $2(1 + 9 + 9^2) = 182$. Filipovski [5] studied the spectral

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restrictions on such excess-four bipartite graphs via their excess cycles; his cyclic and bicyclic nonexistence theorems do not give the unrestricted girth-six exclusion here.

Companion results. The same front half, with k changed, applies to 113_{11} and 75_9. Two companion statements are complete in Lean. A third, Theorem 4, is a special case of the parity theorem of Philbrook and Peet [9, Theorem 6.1], which we formalize. They are discussed in Section 4.

Theorem 2. *In every configuration* 113_{11} *the leave graph is a single* 113*-cycle.*

Theorem 3. *There is no* $(75, 9)$ *modular Golomb ruler, that is, no* 9*-subset of* $\mathbb{Z}/75\mathbb{Z}$ *whose nonzero differences are distinct; equivalently, there is no cyclic* 75_9*. Moreover no configuration* 75_9 *has a Hamiltonian leave graph.*

Theorem 4 (Philbrook–Peet [9, Theorem 6.1], formalized here)**.** *No configuration* 75_9 *has a leave graph with cycle lengths* $6^{11} + 3^3$ *or* $6^9 + 3^7$*.*

The ruler statement in Theorem 3 is known: Buratti and Stinson [1, Table 1] list no $(v, 9)$ modular Golomb ruler for $74 \le v \le 79$. What is new in Theorem 3 is the machine-checked proof and the reduction from Hamiltonian leaves to rulers. (The equivalence between cyclic configurations and modular Golomb rulers is standard and is not part of the Lean development.) We do not claim a standalone formal nonexistence proof for 113_{11} or any unrestricted result for 75_9; see Remark 20.

Organization. Section 2 recalls the general conditions from [10]. Section 3 proves Theorem 1. Section 4 treats the companion results and the open cases. Section 5 maps each step to its Lean theorem and gives the build instructions and the axiom audit.

2. Necessary conditions at deficiency two

Throughout, A is the incidence matrix of a symmetric configuration v_k with $v = k^2 - k + 3$, rows indexing lines and columns indexing points. Lemmas 5–7 are also in [10, Section 2]; for completeness their proofs are given in Appendix A.

Lemma 5 (leave graphs)**.** *Each line (row) is disjoint from exactly two other lines. Let* R *be the adjacency matrix of the graph on rows joining disjoint rows, and* C *the same on columns. Both graphs are* 2*-regular and simple, hence disjoint unions of cycles of length at least* 3*, and*

$$AA^\top = (k-1)I + J - R, \qquad A^\top A = (k-1)I + J - C. \tag{1}$$

Moreover $RA = AC$*, and* R *and* C *have the same multiset of cycle lengths* $L = (\ell_1, \dots, \ell_c)$*, a partition of* v *into parts* ≥ 3*. We call* L *the* profile*.*

Order rows and columns cycle by cycle, in cyclic order around each cycle, so that row cycle i and column cycle i both have length ℓ_i, and write $A = (A_{ij})$ in blocks.

Lemma 6 (block structure)**.** *Each block* A_{ij} *has constant row sum* u_{ij} *and constant column sum* v_{ij}*, with* $\ell_i u_{ij} = \ell_j v_{ij}$ *and* $0 \le u_{ij}, v_{ij} \le k$*. For every row cycle* i*,*

$$\sum_j \ell_j v_{ij} = k\,\ell_i, \qquad \sum_j \ell_j v_{ij}^2 = \ell_i(\ell_i + k - 3), \tag{2}$$

and for distinct row cycles $i \ne h$*,*

$$\sum_j \ell_j v_{ij} v_{hj} = \ell_i \ell_h. \tag{3}$$

Lemma 7 (periodicity)**.** *If $g = \gcd(\ell_i, \ell_j)$ then A_{ij} is invariant under shifting its rows by g. Hence for every row cycle i and $0 < \delta < \ell_i$, $Q_i(\delta) = \sum_{j:\ \gcd(\ell_i,\ell_j)\mid\delta} u_{ij} \le 1$.*

Two consequences are used repeatedly.

Lemma 8 (no 4-cycles)**.** *If $k \ge 4$, the profile has no part equal to 4.*

Proof. For a row cycle with $\ell_i = 4$, (2) gives $\sum_j \ell_j v_{ij} = 4k$ and $\sum_j \ell_j v_{ij}^2 = 4(k+1)$, so $\sum_j \ell_j v_{ij}(v_{ij} - 1) = 4$. Every term with $v_{ij} \ge 2$ is at least $3 \cdot 2 = 6$, so all $v_{ij} \le 1$, and then the two sums are equal, which is false. □

Lemma 9 (profile Gram matrix)**.** *Let Z_L be the adjacency matrix of the disjoint union of cycles of lengths $\ell_1, \dots, \ell_c$, and put $G_L = (k-1)I + J - Z_L$. If a configuration with profile L exists, then $G_L = BB^\top$ for a square $v \times v$ integer matrix B.*

Proof. Take B to be A with rows and columns reordered as above; then $BB^\top = G_L$ by (1). □

Lemma 9 turns any arithmetic obstruction to $G_L = BB^\top$ over $\mathbb{Z}$ into an exclusion of the profile L. For 93_{10} we have $G_L = 9I + J - Z_L$.

3. Proof of Theorem 1

Let $v = 93$ and $k = 10$.

3.1. **Profile coverage.** There are 1,998,184 partitions of 93 into parts ≥ 3. For a partition L and a part $a \in L$, call a vector $(v_j)_j$ indexed by the parts of L *admissible for a* if $v_j \le 10$, $au_j = \ell_j v_j$ for an integer $u_j \le 10$, the two identities (2) hold, and the period bound of Lemma 7 holds.

Proposition 10 (coverage)**.** *If L has no part 4 and every part of L has an admissible vector, then L is one of the 37 profiles in Table 1.*

This is a finite search over partitions. In Lean it is a search tree with 4,176 nodes, each checked by the kernel (Section 5). By Lemmas 6–8, the profile of a 93_{10} is one of these 37.
How to read the coverage search. For one row length a and a column length b, write $g = \gcd(a, b)$. The edge count $au = bv$ permits exactly the integer pairs

$$(u, v) = \big((b/g)t, (a/g)t\big), \qquad 0 \le u, v \le 10.$$

If $g < a$, apply the period bound to the shift g. It forces $u \le 1$. A nonzero choice is therefore possible only on a proper divisor $b = d$ of a, where it is $(u, v) = (1, a/d)$ and requires $a/d \le 10$. No divisor length can be used twice. Two used divisor lengths d, e are compatible exactly when $\operatorname{lcm}(d, e) = a$: a smaller common multiple is a shift that would count both, and these are the only possible violations of the period bound. The remaining nonzero columns have length $b = qa$; each has $(u, v) = (qt, t)$ for $0 \le t \le \lfloor 10/q \rfloor$, with no period restriction.

Thus choose a set S of available, pairwise compatible proper divisors, and one t_j for each available multiple $q_j a$. The two moment identities, after division by a and subtraction, become

$$|S| + \sum_j q_j t_j = 10, \qquad \sum_{d \in S} (a/d - 1) + \sum_j q_j t_j (t_j - 1) = a - 3.$$

In particular, the feasibility test for one row needs only the two budgets $(10, a-3)$. Starting at $(0, 0)$, each multiple offers increments $(qt, qt(t-1))$; discard states exceeding either budget. For each allowed S, test whether the corresponding leftover budget is reached. Repeated multiples can be treated as separate factors, or their multiplicity can be capped at $\lfloor 10/q \rfloor$, since each nonzero choice consumes at least q units of the first budget. Both directions of this reduction hold; it is the same local test, not an additional exclusion. The certified search remains the Lean procedure in Proposition 10.

profile	c	excluded by	profile	c	excluded by
3^{31}	31	contrast mod 625	$6^{14}+3^3$	17	contrast mod 625
5^9+3^{16}	25	contrast mod 49	$12^5+6^4+3^3$	12	c even
6^8+3^{15}	23	contrast mod 625	$12^6+6^2+3^3$	11	contrast mod 625
6^9+3^{13}	22	c even	12^7+3^3	10	c even
$5^{12}+3^{11}$	23	contrast mod 625	24^3+12+3^3	7	contrast mod 625
$6^{10}+3^{11}$	21	contrast mod 625	21^4+3^3	7	contrast mod 625
7^9+3^{10}	19	Prop. 12	15^5+5^3+3	9	contrast mod 625
9^7+3^{10}	17	contrast mod 625	$30^2+15+10+5+3$	6	c even
$6^{11}+3^9$	20	c even	$6^{15}+3$	16	c even
$6^{12}+3^7$	19	contrast mod 625	12^5+6^5+3	11	contrast mod 625
$12^5+6^2+3^7$	14	c even	12^6+6^3+3	10	c even
9^8+3^7	15	contrast mod 625	$9^{10}+3$	11	contrast mod 625
12^6+3^7	13	contrast mod 625	18^4+9^2+3	7	contrast mod 625
15^5+3^6	11	contrast mod 49	27^3+9+3	5	contrast mod 625
$6^{13}+3^5$	18	c even	15^6+3	7	contrast mod 625
$12^5+6^3+3^5$	13	contrast mod 625	30^3+3	4	c even
9^9+3^4	13	contrast mod 625	31^3	3	Sec. 3.5
18^4+9+3^4	9	contrast mod 625	93	1	contrast mod 625
27^3+3^4	7	contrast mod 625			

TABLE 1. The 37 profiles left by Proposition 10, with the number c of cycles and the argument that excludes each. Exponents denote repeated parts.

3.2. **Profiles with an even number of cycles.**

Proposition 11. *If $G_L = 9I + J - Z_L$ has an integral factorization $G_L = BB^\top$ with B a square integer matrix, then c is odd.*

Proof. The eigenvalues of G_L are $9+93-2=100$ on the all-ones vector; $9-2=7$ on the $(c-1)$-dimensional space of vectors that are constant on each cycle and orthogonal to the all-ones vector; and $9-2\cos(2\pi t/\ell)$ for $1 \le t \le \ell-1$ on each cycle of length ℓ. Write $9=\alpha+\alpha^{-1}$ and $V_\ell = \alpha^\ell + \alpha^{-\ell}$, so $V_0=2$, $V_1=9$, $V_{\ell+1}=9V_\ell - V_{\ell-1}$. Since $\prod_{t=0}^{\ell-1}(x-2\cos(2\pi t/\ell)) = V_\ell - 2$ for $x=\alpha+\alpha^{-1}$,

$$\det G_L = 100 \cdot 7^{c-1} \prod_{i=1}^{c} \frac{V_{\ell_i}-2}{7}.$$

Moreover $V_\ell - 2 = (\alpha^{\ell/2} - \alpha^{-\ell/2})^2$ equals $7W_\ell^2$ for odd ℓ and $77U_\ell^2$ for even ℓ, with integers U_ℓ, W_ℓ. Hence $\det G_L = 100 \cdot 7^{c-1} \cdot 11^e \cdot N^2$ for an integer N, where e is the number of even parts. The 7-adic valuation of $\det G_L = (\det B)^2$ is even, so $c-1$ is even. □

This is the $k=10$ case of the parity theorem of Philbrook and Peet [9, Theorem 6.1]. It excludes the ten profiles of Table 1 with c even. In Lean the valuation is computed profile by profile rather than by the closed formula.

3.3. **The profile 7^9+3^{10}.**

Proposition 12. *The profile 7^9+3^{10} is impossible.*

Proof. Take a row cycle h of length 3. By (2), $\sum_j \ell_j v_{hj} = 30 = \sum_j \ell_j v_{hj}^2$, so every $v_{hj} \in \{0,1\}$. For a column cycle j of length 7, $3u_{hj} = 7v_{hj}$ then forces $v_{hj}=0$. Take a row cycle i of length 7. For a column cycle j of length 3, $7u_{ij} = 3v_{ij}$ makes v_{ij} a multiple of 7, and $3v_{ij}^2 \le \sum_j \ell_j v_{ij}^2 = 98$ forces $v_{ij}=0$. So $v_{ij}v_{hj}=0$ for every j, and the left side of (3) is 0, not $7\cdot 3 = 21$. □

3.4. **Twenty-five odd profiles: local counts.** The remaining arithmetic step uses a counting invariant of quadratic forms over a finite ring. Fix an odd prime p and $m \geq 1$, put $R = \mathbb{Z}/p^{2m}\mathbb{Z}$, $s = p^m$ and $t = p^{2m-1}$. For a symmetric integer matrix M let $q_M(x) = x^\top M x$ on R^n, and define the *contrast*

$$\kappa(M) = \#\{x \in R^n : q_M(x) = 0\} - \#\{x \in R^n : q_M(x) = t\}.$$

We use $(p, m) = (7, 1)$, so $R = \mathbb{Z}/49\mathbb{Z}$ and $t = 7$, and $(p, m) = (5, 2)$, so $R = \mathbb{Z}/625\mathbb{Z}$ and $t = 125$. Write $|y|^2 = y^\top y$.

Lemma 13 (cosets of sR^n)**.** *For every $y_0 \in R^n$, $\#\{z : |y_0 + sz|^2 = 0\} \geq \#\{z : |y_0 + sz|^2 = t\}$, where z runs over $(\mathbb{Z}/p^m\mathbb{Z})^n$, so that $y_0 + sz$ runs once over the coset $y_0 + sR^n$.*

Proof. If every coordinate of y_0 is divisible by s, every element of the coset is sy' and $|sy'|^2 = p^{2m}|y'|^2 = 0$, so the right side is 0. Otherwise let p^j $(0 \leq j < m)$ be the largest power of p dividing every coordinate of y_0 modulo s, and write $y_0 = p^j u + sy_1$ with u having a coordinate prime to p. Replacing z by $z + y_1$ we may take $y_1 = 0$. Then

$$|p^j u + sz|^2 = p^{2j}|u|^2 + 2p^{m+j}\langle u, z\rangle + p^{2m}|z|^2 \equiv p^{2j}|u|^2 + 2p^{m+j}\langle u, z\rangle \pmod{p^{2m}}.$$

As z varies, $\langle u, z\rangle$ is uniformly distributed modulo p^m, because u has a unit coordinate, so (as 2 is a unit) the value is uniformly distributed over the residue class $p^{2j}|u|^2 + p^{m+j}\mathbb{Z}$ modulo p^{2m}. Both 0 and $t = p^{2m-1}$ are divisible by p^{m+j}, since $m + j \leq 2m - 1$. Hence either both lie in that class and are hit equally often, or neither is hit. □

Lemma 14 (transfer)**.** *Let G be a symmetric integer matrix, and suppose there are an integer matrix N and integers $d \neq 0$, w with $NG = dI$ and $wd \equiv s \pmod{p^{2m}}$. If $G = BB^\top$ with B a square integer matrix, then $\kappa(G) \geq 0$.*

Proof. From $(NB)B^\top = dI$, the square matrix $B^\top$ is invertible over $\mathbb{Q}$ with inverse NB/d, so also $B^\top(NB) = dI$. Reducing modulo p^{2m}, the map $f : x \mapsto B^\top x$ on R^n satisfies $f(wNBy) = wd\,y = sy$, so its image L contains sR^n and is a union of cosets of sR^n. Since $q_G(x) = |B^\top x|^2$, each value $y \in L$ is taken by exactly $\#\ker f$ vectors x, and

$$\kappa(G) = \#\ker f \cdot \sum_{\text{cosets } y_0 + sR^n \subseteq L} \Big(\#\{y \in y_0 + sR^n : |y|^2 = 0\} - \#\{y \in y_0 + sR^n : |y|^2 = t\}\Big),$$

which is ≥ 0 by Lemma 13. □

Proposition 15. *For each of the 25 profiles marked "contrast" in Table 1, with the modulus shown there, there are N, d, w as in Lemma 14 for $G = G_L$, and $\kappa(G_L) < 0$. For the two profiles $5^9 + 3^{16}$ and $15^5 + 3^6$ (modulus 49) the contrast is -7^{105} and -7^{98} respectively. Hence, by Lemmas 9 and 14, none of these 25 profiles occurs.*

Proposition 15 is a computation. The certificates N are scaled inverses of G_L, stored as integer matrices and checked exactly, and the contrasts are evaluated exactly from the cycle structure of L; both steps are checked in Lean. The list includes the Hamiltonian profile 93. Lemma 13 and Lemma 14 are the paper versions of the Lean statements `gram_contrast_nonneg_of_scaled_image`, `integer_gram_has_reduced_scaled_image` and `no_integral_gram49_of_negative_contrast` / `no_integral_gram625_of_negative_contrast`.

For an independent way to recompute the signs, let $R = p^{2m}$ and diagonalize the explicit G_L by symmetric elimination modulo p^{2m+8}. Choose a pivot of least p-adic valuation v; if it occurs only off the diagonal at (i, j), add basis vector j to i to obtain a diagonal pivot of valuation v (the prime is odd). For each remaining column subtract a multiple of the pivot column to zero its pivot entry and make the same row operation. The multiple is obtained by dividing

both entries by p^v and inverting the unit part of the pivot. Write a_i for the resulting diagonal residues modulo R. Start $H_0(0) = 1$, $H_0(t) = 0$ for $t \neq 0$, and iterate

$$H_i(t) = \sum_{x \in \mathbb{Z}/R\mathbb{Z}} H_{i-1}(t - a_i x^2).$$

Then the desired contrast is $H_{93}(0) - H_{93}(p^{2m-1})$. The pivots in all these cases have valuations at most m. This also supplies the hypothesis about N, d, w, not just the signs. Indeed G_L is nonsingular (its smallest eigenvalue is at least 7). The reduction modulo p^{2m} has elementary-divisor valuations at most m, by the invertible row and column operations just described. Thus all the p-adic valuations of the integer Smith invariants of G_L are at most m: an invariant divisible by p^{m+1} would still be detected modulo p^{2m}. If d is the largest integer Smith invariant, then $N = dG_L^{-1}$ is integral and $v_p(d) \leq m$. This is exactly the solvability condition for $wd \equiv p^m \pmod{p^{2m}}$. Neither the sign check nor this image check requires a supplied witness matrix.

3.5. **The profile 31^3.** The remaining profile has three cycles of length 31. Here $\gcd(\ell_i, \ell_j) = 31$, so Lemma 7 gives nothing, and this profile is not disposed of by either of the preceding arithmetic rosters. We argue directly on the blocks. Index each cycle by $\mathbb{Z}/31\mathbb{Z}$ in cyclic order, so that two rows r, r' of the same row cycle are disjoint exactly when $r' - r = \pm 1$, and similarly for columns.

Lemma 16 (blocks are circulant or Hankel)**.** *Let X be a 0/1 matrix indexed by $\mathbb{Z}/31\mathbb{Z} \times \mathbb{Z}/31\mathbb{Z}$ with $Z_{31}X = XZ_{31}$. Then either $X[r, s]$ depends only on $s - r$ (*circulant*) or $X[r, s]$ depends only on $r + s$ (*Hankel*). If X has a row sum other than 0 and 31, exactly one of these holds.*

Proof. The hypothesis says $X[r+1, s] + X[r-1, s] = X[r, s+1] + X[r, s-1]$. Put $D(r, s) = X[r+1, s] - X[r, s+1]$; the hypothesis is $D(r, s) = D(r-1, s-1)$, so D depends only on $k = r - s$, say $D = \delta(k)$. Let $g_k(n) = X[n+k, n]$ be the k-th diagonal. Then $g_{k+1}(n) = g_{k-1}(n+1) + \delta(k)$ for all n. Since entries are 0 or 1, if $\delta(k) = \pm 1$ then g_{k+1} and g_{k-1} are constant, and if $\delta(k) = 0$ then g_{k+1} is a shift of g_{k-1}. Because 31 is odd, the steps $k \mapsto k + 2$ pass through every diagonal. So if some $\delta(k) \neq 0$, some diagonal is constant, and then every diagonal is constant, that is, X is circulant. If every $\delta(k) = 0$, then $g_{k+2j}(n) = g_k(n + j)$, so $X[r, s] = X[\frac{r+s+k}{2}, \frac{r+s-k}{2}]$ (halving in $\mathbb{Z}/31\mathbb{Z}$), which depends only on $r + s$. A matrix that is both circulant and Hankel is invariant under $(r, s) \mapsto (r, s + 2)$, hence has constant rows, so its row sums are 0 or 31. □

Lemma 17 (normal form)**.** *After reversing the orientation of some row and column cycles, every block A_{ij} is circulant, $A_{ij}[r, s] = \mathbf{1}_{F_{ij}}(s - r)$ for subsets $F_{ij} \subseteq \mathbb{Z}/31\mathbb{Z}$. The sizes $|F_{ij}|$ form a Latin square on $\{2, 3, 5\}$, and for $\lambda_F(\delta) = \#\{(x, y) \in F^2 : x - y = \delta\}$ and $\mu_{F,F'}(\delta) = \#\{(x, y) \in F \times F' : x - y = \delta\}$,*

$$\sum_j \lambda_{F_{ij}}(\delta) = \begin{cases} 0, & \delta = \pm 1, \\ 1, & \delta \neq 0, \pm 1, \end{cases} \tag{4}$$

$$\sum_j \mu_{F_{ij}, F_{hj}}(\delta) = 1 \qquad \text{for all } \delta \text{ and } i \neq h. \tag{5}$$

Proof. With all lengths 31, Lemma 6 gives $u_{ij} = v_{ij}$, $\sum_j v_{ij} = 10$ and $\sum_j v_{ij}^2 = 38$, whose only solution is $\{5, 3, 2\}$; and (3) gives $\sum_j v_{ij} v_{hj} = 31$ for $i \neq h$, which among permutations of $(5, 3, 2)$ holds exactly when the two rows differ in every position. So the degrees form a Latin square, and no block has row sum 0 or 31. By Lemma 16 each block has a well-defined type, circulant or Hankel.

Rows r of cycle i and r' of cycle $h \neq i$ meet in exactly one column. If A_{ij} and A_{hj} have the same type, their contribution $(A_{ij} A_{hj}^\top)[r, r']$ depends only on $r - r'$; if the types differ, it

depends only on $r+r'$. The total is $\phi(r-r')+\psi(r+r')=1$, and since $(r-r', r+r')$ takes every value in $(\mathbb{Z}/31\mathbb{Z})^2$, ψ is constant. Its value is $\sum v_{ij}v_{hj}/31$ over the blocks j of different type, and no nonempty proper subset of the products $\{15, 6, 10\}$ has a sum divisible by 31. So the types of rows i and h either agree in every column or differ in every column. Reversing a row cycle ($r \mapsto -r$) swaps the type of every block in its row; reversing a column cycle swaps every block in its column. Reverse the row cycles whose types differ from row 0, then the column cycles where row 0 is Hankel. Reversal keeps adjacency $r \pm 1$, so it commutes with Z_{31}, and all blocks are now circulant.

For circulant blocks, rows r and $r+\delta$ of cycle i share $\sum_j \lambda_{F_{ij}}(\delta)$ columns, which is 0 for $\delta = \pm 1$ (disjoint rows) and 1 otherwise; this is (4). Rows r of cycle i and $r+\delta$ of cycle h share $\sum_j \mu_{F_{ij},F_{hj}}(\delta)$ columns, which is 1; this is (5). □

Proposition 18. *There are no subsets F_{ij} of $\mathbb{Z}/31\mathbb{Z}$ ($i \in \{0,1\}$, $j \in \{0,1,2\}$) with sizes given by two rows of a Latin square on $\{2,3,5\}$ that satisfy (4) for $i = 0, 1$ and (5) for $(i,h) = (0,1)$. Hence the profile 31^3 is impossible.*

This is a finite check, which we describe in a form a reader can rerun, and which is also certified in Lean.

Row families. By (4), each row (F_{i0}, F_{i1}, F_{i2}), with sizes $5, 3, 2$ in some order, is a difference family in $\mathbb{Z}/31\mathbb{Z}$ covering every nonzero difference except ± 1 exactly once: $20 + 6 + 2 = 28$ differences. Up to translation there are 182 admissible 5-sets and 104 admissible 3-sets; the 2-set is then forced, since the two missing differences form a pair $\pm x$. This gives 136 row families, reflections counted separately.

Pairs of rows. Permute the block columns so that row 0 has sizes $(5, 3, 2)$. Row 1's sizes are then a derangement, $(3, 2, 5)$ or $(2, 5, 3)$. Translating the whole of row 1 fixes the shift in its first block; the other two shifts s_1, s_2 range over $\mathbb{Z}/31\mathbb{Z}$. Equation (5) says that the three cross-difference multisets $F_{0j} - F_{1j} - s_j$ partition $\mathbb{Z}/31\mathbb{Z}$.

Count. Of the $136 \times 136 \times 2 = 36{,}992$ cases, only 1,120 have all three cross-difference multisets free of repetitions. Among these, 120 partial shift choices, spread over 116 of the 1,120 cases, make the first two multisets disjoint. No completed choice covers $\mathbb{Z}/31\mathbb{Z}$; among all completions of those partial choices, the best covers 27 of the 31 residues.

An independent short test. For a repetition-free candidate let D_j be its unshifted cross-differences and let $b_j = |D_j|$ (a permutation of $15, 6, 10$). A partition of $\mathbb{F}_{31}$ has vanishing power sums in degrees one through four. Fix $s_0 = 0$. Given s_1, the first moment forces

$$s_2 = b_2^{-1} \left(\sum_{j=0}^{2} \sum_{x \in D_j} x - b_1 s_1 \right) \pmod{31}.$$

For each of the $1{,}120 \cdot 31 = 34{,}720$ choices form all $x - s_j$ and check the next three power sums successively modulo 31. Exactly 1,152 choices survive the second moment, 44 survive the third and none survives the fourth. Here translation orbits can be enumerated simply by choosing the lexicographically least of $F - a$ ($a \in F$), after sorting residues in $0, \ldots, 30$. No orientation quotient or division by a stabilizer is used. This test uses necessary conditions only; its zero count suffices.

The certified version is the Lean route. It encodes the same conditions on the first rows of the blocks (the structure `TwoBank31`), derives them without first aligning orientations, and closes every case by a recorded local witness checked by the kernel.

Proof of Theorem 1. By Lemma 5 a 93_{10} has a profile L, and by Lemmas 6–8 and Proposition 10, L is in Table 1. Propositions 11, 12, 15 and 18, with Lemma 9, exclude every entry. □

Remark 19. An earlier written version of this proof used a different arithmetic step: Hasse–Minkowski invariants at 2, 3, 5 and 7 for the 25 profiles of Proposition 15. That version was not formalized and is not relied on here.

4. Companion results and open cases

4.1. 113_{11}. For $v = 113$, $k = 11$ the same front half gives 19 profiles after coverage (17,635,015 partitions, a 7,635-node search). Two or three rows of the moment and period conditions exclude 17 of them. The profile $3 + 5 + 15^7$ needs more: the 15-blocks are circulant, a degree census over row and column types is checked in shards, and a class-sum identity modulo 15 gives a contradiction. Lean proves the two halves (only 113 or $3{+}5{+}15^7$; not $3{+}5{+}15^7$), and Theorem 2 is their immediate combination. *Written corollary using published computation.* With a single 113-cycle the incidence matrix comprises one block. The wave argument of Lemma 16 uses only odd cycle length, so the block is circulant after a possible reversal. Its first row therefore gives a $(113, 11)$ modular Golomb ruler: repeated nonzero differences would give two rows meeting in two columns. Buratti and Stinson [1, Theorem 2.3(9) and Table 1] exclude such rulers for $111 \le v \le 119$. Combined with Theorem 2 this excludes 113_{11} as a written corollary, not as a second standalone kernel-checked nonexistence theorem. The Lean endpoint that draws this conclusion takes `NoModularGolombRuler113` as a named hypothesis; it does not recheck the published ruler computation.

4.2. 75_9. For $v = 75$, $k = 9$ the front half and a 33-profile coverage over 235,858 partitions, followed by 18 exclusions, leave 15 profiles; one is the Hamiltonian profile 75, which reduces to a $(75, 9)$ modular Golomb ruler. An exhaustive search, split into 49 kernel-checked pieces, shows that none exists. This is Theorem 3, and leaves 14 profiles. Four of them, $15^4 + 5^3$, $12^6 + 3$, $12^5 + 6^2 + 3$ and $10^6 + 5^3$, are excluded in Lean by a clique argument on admissible row vectors.

Two more are excluded by the parity theorem of Philbrook and Peet [9, Theorem 6.1], which applies to profiles with an even number of cycles. We formalize it for these two profiles, using the determinant argument of Proposition 11 at $k = 9$.[1]

Proof of Theorem 4. Let L be either profile, with c cycles ($c = 14$ or 16), and let Z_L be the adjacency matrix of its disjoint union of cycles. By Lemma 9 with $k = 9$, $G_L = 8I + J - Z_L = BB^\top$ for a square integer matrix B. Put $H = 8I - Z_L$. Every row of H sums to 6, so $G_L = H(I + \frac{1}{6}\mathbf{1}\mathbf{1}^\top)$ and $\det(I + \frac{1}{6}\mathbf{1}\mathbf{1}^\top) = 1 + \frac{75}{6}$. Hence $6 \det G_L = 81 \det H$. The matrix H is block diagonal, one block $8I - Z_\ell$ per cycle, and

$$\det(8I - Z_3) = 486 = 2 \cdot 3^5, \qquad \det(8I - Z_6) = 238140 = 2^2 \cdot 3^5 \cdot 5 \cdot 7^2.$$

So $v_3(\det H) = 5c$ and $v_3(\det G_L) = 4 + 5c - 1 = 3 + 5c$, which is odd for $c = 14$ and $c = 16$. But $\det(BB^\top) = (\det B)^2$ has even valuation at every prime. □

The 3-adic computation applies exactly to the profiles made of 3- and 6-cycles with c even. Among the 14 remaining profiles these are the two above. With them, 8 of the 14 profiles remain.

Remark 20 (Scope of companion results). The unconditional headline is the impossibility of 93_{10}. No unrestricted 75_9 or 113_{11} nonexistence result is claimed as one of the kernel-checked conclusions of this paper. In particular, the certified result for 113_{11} restricts the leave profile; using a separate modular-ruler nonexistence result is an additional inference. For 75_9 our profile exclusions alone do not exhaust the surviving list.

[1] The profile $12^5 + 6^2 + 3$ also has an even number of cycles ($c = 8$), so [9, Theorem 6.1] excludes it as well; its Lean exclusion above uses a different argument.

5. Formal verification and reproducibility

The headline theorem and the listed formal companion statements are formalized in Lean 4 with Mathlib in one standalone Lake project, `symconf-deficiency-two-lean`. The main statement is about the literal incidence matrix:

```
theoremSymconf93.no_unrestricted_incidence93: ¬ ∃ A:Int93,Incidence93A
```

where `Int93` is the type of 93×93 integer matrices and `Incidence93A` says that every entry is 0 or 1, every row and column has sum 10, and two distinct rows share at most one column. Nothing about cycles, blocks or profiles appears in the hypotheses; the leave graphs and their profile are derived. No `sorry`, `axiom`, `native_decide`, `implemented_by` or `extern` occurs in the sources. Finite searches are checked by the kernel with `decide+kernel`.

step	Lean file	theorem
Lemmas 5–7, profile facts	`Symconf93/ActualSieveInput` and front-half modules	`physical_profile_mem_parity_free`
Lemma 8	`Symconf93/ProfileSieve93`	`true_profile_passes_sieve`
Lemma 9	`Symconf93/Terminal93`	`physical_profile_gram_factor`
Prop. 10 (4,176 nodes)	`Symconf93/ExactCoverageParallel93`	`exact_original_coverage93`
Prop. 11 (10 profiles)	`Symconf93/EvenCatalogueProof93`	`no_original_even_profile_gram`
Prop. 12	`Symconf93/ActualSieveInput`	`no_three_seven_original_profile`
Lemma 14	`Symconf93/GramCongruence93`	`no_integral_gram49_of_negative_contrast`, `no_integral_gram625_of_negative_contrast`
Prop. 15, modulus 49 (2)	`Symconf93/PivotFreeQ49Terminal`	`AffineFiber49.no_integral_gram49_without_pivots`
Prop. 15, modulus 625 (23)	`Symconf93/PivotFreeQ625Terminal`	`AffineFiber625.no_integral_gram625_without_pivots`
Prop. 18, reduction	`Symconf93/Actual31Bridge`	`actual_31cubed_to_two_bank`
Prop. 18, finite analysis	`Symconf93/DegreeFiveEndgame31` (29 `DoubleCell` files)	`no_two_bank31`
assembly	`Symconf93/Terminal93`, `OriginalModularJoin93`	`no_unrestricted_incidence93_of_certified_gates`
Theorem 1	`Symconf93/OriginalIncidence93Complete`	`no_unrestricted_incidence93`
Theorem 2	`Symconf113/OneProfile113`, `ResidualFinal113`	`incidence113_two_profiles`, `no_residual_incidence113`
Theorem 3	`Symconf75/NoRuler75`, `Hamiltonian75`	`noModularGolombRuler75`, `no_hamiltonian_incidence75`
75_9: 14 profiles	`Symconf75/Reduction75`	`incidence75_fourteen_profiles`
75_9: four exclusions	`Symconf75/VecCliqueChecks75`	`VecClique.no_profile15`, `no_profile12a`, `no_profile12b`, `no_profile10`
Theorem 4	`Symconf75/EvenParity75`	`no_ordered_profileSix11`, `no_ordered_profileSix9`, `incidence75_twelve_profiles`

Table 2. Where each step is proved. Theorem names are in the namespace of their file's directory (`Symconf93`, `Symconf113`, `Symconf75`).

Versions. Lean `v4.31.0-rc1` (commit `fd009949`); Mathlib at commit

`0ed37c9258f123a2c76a2175596f75c2bd50f941`;

every transitive dependency is pinned in `lake-manifest.json`.

Build. Download the original Lean proof archive. It contains the complete Lake project, pinned dependencies, axiom audit, and build receipt. Unpack the archive before running these commands. The manuscript gives the data and algorithms for its independent finite checks.

```
lakebuild  (default target SymconfDeficiencyTwo: 59_8 and 93_10)
lakebuildSymconf113Symconf75
lakeenvleanAuditAxioms.lean
```

Recorded times on a 4-core, 15 GB Linux machine (24 November 2026): from an empty build directory with Mathlib compiled from source (the Mathlib cache was unreachable), the default target built in about 57 minutes (3,330 jobs). With compiled Mathlib reused, a fresh clone rebuilt every `Symconf` module of all three targets in 10 minutes 24 seconds (3,348 jobs); the longest single module is `ProfileExclusions113` (about 6.5 minutes). Some 75 and 113 modules peak near 11 GB of memory. The module `EvenParity75`, added afterwards, builds in about 10 seconds; `lakebuildSymconf75` then completes with 3,122 jobs.

Axiom audit. `lakeenvleanAuditAxioms.lean` reports, for each endpoint,

```
Symconf93.no_unrestricted_incidence93: [propext,Classical.choice,Quot.sound]
Symconf113.incidence113_two_profiles: [propext,Classical.choice,Quot.sound]
Symconf113.no_residual_incidence113: [propext,Classical.choice,Quot.sound]
Symconf75.noModularGolombRuler75: [propext,Classical.choice,Quot.sound]
Symconf75.no_hamiltonian_incidence75: [propext,Classical.choice,Quot.sound]
Symconf75.incidence75_fourteen_profiles: [propext,Classical.choice,Quot.sound]
Symconf75.VecClique.no_profile15: [propext,Classical.choice,Quot.sound]
Symconf75.VecClique.no_profile12a: [propext,Classical.choice,Quot.sound]
Symconf75.VecClique.no_profile12b: [propext,Classical.choice,Quot.sound]
Symconf75.VecClique.no_profile10: [propext,Classical.choice,Quot.sound]
Symconf75.no_ordered_profileSix11: [propext,Classical.choice,Quot.sound]
Symconf75.no_ordered_profileSix9: [propext,Classical.choice,Quot.sound]
Symconf75.incidence75_twelve_profiles: [propext,Classical.choice,Quot.sound]
```

that is, only `propext`, `Classical.choice` and `Quot.sound`.

Provenance. John Tripodo directed the project. OpenAI and Anthropic models contributed to the mathematics, manuscript, search programs, formalization and internal checks. The headline nonexistence theorem was checked by the Lean kernel; the formal statements and the scope of the companion results are specified above. No independent human referee report is claimed.

Appendix A. Proofs of Lemmas 5–7

Proof of Lemma 5. A line has k points, each on $k-1$ further lines, and two lines share at most one point, so a line meets exactly $k(k-1)$ other lines and is disjoint from $v-1-k(k-1)=2$. Hence $(AA^\top)$ has diagonal k, entry 1 for meeting lines and 0 for disjoint ones, which is (1) with R symmetric, 0/1, zero on the diagonal and with row sums 2; the same holds for points, by duality. A simple 2-regular graph is a disjoint union of cycles of length at least 3. Since A is square with all line sums k, $JA = AJ = kJ$, and expanding $(AA^\top)A = A(A^\top A)$ with (1) gives $RA = AC$.

The matrices $AA^\top$ and $A^\top A$ have the same eigenvalues. Their common constant direction has eigenvalue k^2, and on its orthogonal complement $J = 0$. Using (1) on this complement (and the common eigenvalue 2 on constants), we get the same spectrum for R and C. The spectrum of a disjoint union of cycles determines its cycle lengths: for $n \geq 3$, the eigenvalue $2\cos(2\pi/n)$

has multiplicity 2 in C_ℓ if $n \mid \ell$ and 0 otherwise, so the spectrum gives $F(n) = \#\{i : n \mid \ell_i\}$ for every $n \geq 3$, and $\#\{i : \ell_i = n\} = F(n) - \sum_{q\geq 2} \#\{i : \ell_i = qn\}$ is recovered by downward induction on n. □

Proof of Lemma 6. In block form $RA = AC$ reads $Z_i A_{ij} = A_{ij} Z_j$, where Z_ℓ is the adjacency matrix of the cycle C_ℓ. Applying both sides to $\mathbf{1}$ gives $Z_i(A_{ij}\mathbf{1}) = 2A_{ij}\mathbf{1}$; the eigenvalue 2 of a cycle has the constants as its eigenspace, so A_{ij} has constant row sum u_{ij}. Applying $\mathbf{1}^\top$ on the left gives constant column sums v_{ij} in the same way, and counting the ones of A_{ij} gives $\ell_i u_{ij} = \ell_j v_{ij}$. Summing the row sums over the ℓ_i lines of cycle i gives $\sum_j \ell_i u_{ij} = k\ell_i$, the first part of (2). For the second, sum $(AA^\top)_{rr'}$ over all pairs r, r' in cycle i. Column c in cycle j contributes v_{ij}^2, so the total is $\sum_j \ell_j v_{ij}^2$; by (1) it is $(k-1)\ell_i + \ell_i^2 - 2\ell_i = \ell_i(\ell_i + k - 3)$. Summing instead over r in cycle i and r' in cycle $h \neq i$ gives $\sum_j \ell_j v_{ij} v_{hj}$ on one side and $\ell_i \ell_h$ on the other, since R has no entries between different cycles. This is (3). □

Proof of Lemma 7. Work over $\mathbb{C}$. If $Z_j y = \mu y$ then $Z_i(A_{ij}y) = \mu A_{ij} y$, so each column of A_{ij} lies in the sum of the eigenspaces of Z_i whose eigenvalues are also eigenvalues of Z_j. The eigenspace of Z_i for $2\cos(2\pi a/\ell_i)$ is spanned by $r \mapsto e^{\pm 2\pi \mathrm{i} ar/\ell_i}$. If $2\cos(2\pi a/\ell_i) = 2\cos(2\pi b/\ell_j)$ then $a\ell_j \equiv \pm b\ell_i \pmod{\ell_i\ell_j}$, so $\ell_i \mid a\ell_j$ and $(\ell_i/g) \mid a$. Then $e^{2\pi \mathrm{i} ar/\ell_i}$ has period g in r. So every column of A_{ij} is g-periodic, which is the first claim.

Now take lines r and $r+\delta$ of cycle i, $0 < \delta < \ell_i$. They are distinct, so they share at most one point. For each j with $\gcd(\ell_i, \ell_j) \mid \delta$, the two lines have the same row in A_{ij}, so they share all u_{ij} of its ones. Summing over such j gives $Q_i(\delta) \leq 1$. □

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