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ON NORM ATTAINMENT, THE ABSOLUTE VALUE AND POLAR DECOMPOSITION OF BOUNDED FUNCTIONALS ON A C^*-ALGEBRA

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ABSTRACT. The absolute value of a bounded linear functional ϕ on a C^*-algebra $\mathfrak{A}$ is the noncommutative counterpart of the total variation of a complex measure. In the literature, it is usually constructed by considering the normal extension of ϕ to the bidual of $\mathfrak{A}$, and applying Sakai's polar decomposition to this extension. In this article, we provide a direct construction. As our main result, we prove that there is a unique functional ρ on $\mathfrak{A}$ such that

$$\|\phi(\cdot\, B) - \rho\|^2 \leq 2\|\phi\|\,(\|\phi\| - \operatorname{Re}\phi(B)) \quad \text{for every contraction } B \in \mathfrak{A},$$

and moreover, ρ is positive. We refer to ρ as the absolute value of ϕ, and denote it by $|\phi|$. We further show that our definition agrees with the classical one. The right-hand side of the above inequality may be viewed as a measure of the failure of ϕ to attain its norm at B. When ϕ attains its norm at A, we get $|\phi| = \phi(\cdot\, A)$ and further show that $\phi = |\phi|(\cdot\, A^*)$. In general, we obtain an approximate polar decomposition of ϕ with the help of norming sequences. From this we deduce Sakai's theorem, the Banach-space identification of $\mathfrak{A}^{**}$ with the enveloping von Neumann algebra, and the Jordan decomposition of a hermitian functional. The organizing principle is an elementary fact about discs in the unit ball of $\mathfrak{A}$: if $Z + \lambda D$ lies in the unit ball whenever $|\lambda| \leq 1$, then $|\phi(Z)| + |\phi(D)| \leq \|\phi\|$. In particular, a functional which attains its norm at Z vanishes at D. We use four explicit discs: two of them lead to the results above, and the other two give a new proof of Kadison's characterization of the extreme points of the unit ball of $\mathfrak{A}$. To the best of our knowledge, the above characterization of the absolute value without the bidual, and the polar factorization at arbitrary norming contractions, do not appear in this form in the literature. We hope that the elementary approach taken here may be of use in a first course on operator algebras.

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1. Introduction

Let X be a compact Hausdorff space, and let $C(X)$ denote the C^*-algebra of continuous complex-valued functions on X. By the Riesz representation theorem, every bounded linear functional ϕ on $C(X)$ is given by integration against a unique regular complex Borel measure μ on X, that is,

$$\phi(f) = \int_X f \, d\mu \quad \text{for all } f \in C(X).$$

A key tool in studying μ is its *total variation* $|\mu|$, the positive measure defined on Borel sets by

$$|\mu|(E) = \sup \sum_j |\mu(E_j)|,$$

where the supremum is taken over all finite Borel partitions $\{E_j\}$ of E. The total variation gives the norm of the functional represented by μ, namely $\|\phi\| = |\mu|(X)$. Moreover, there is a Borel function h with $|h| = 1$ such that $d\mu = h \, d|\mu|$, which is the polar decomposition of μ. If μ is a real measure, then $\mu^{\pm} := \frac{1}{2}(|\mu| \pm \mu)$ are positive measures, and $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ [12, Theorems 6.12 and 6.14].

Every commutative unital C^*-algebra is of the form $C(X)$, and the positive functionals on $C(X)$ are those represented by positive measures. A bounded linear functional on an arbitrary unital C^*-algebra $\mathfrak{A}$ may therefore be regarded as a noncommutative complex measure, with the positive functionals on $\mathfrak{A}$ in the role of the positive measures. Let $\mathfrak{A}^*$ denote the set of bounded linear functionals on $\mathfrak{A}$. For $\phi \in \mathfrak{A}^*$, the noncommutative counterpart of the total variation $|\mu|$ is the absolute value $|\phi|$, a positive functional on $\mathfrak{A}$ with the same norm as ϕ. It is, however, harder to construct than the total variation of a measure. In the standard treatment [17, Chapter III, § 4], one first identifies the bidual $\mathfrak{A}^{**}$ with a von Neumann algebra, the enveloping von Neumann algebra of $\mathfrak{A}$. One then extends ϕ to a normal functional on $\mathfrak{A}^{**}$, applies Sakai's polar decomposition theorem for normal functionals [13] to this extension, and restricts the resulting positive functional to $\mathfrak{A}$. Thus $|\phi|$ becomes available only once the bidual and Sakai's theorem are in place.

In this article, we reverse this order. We characterize $|\phi|$ as the unique functional on $\mathfrak{A}$ satisfying an inequality that involves only ϕ and the contractions in $\mathfrak{A}$, that is, the elements of the closed unit ball $(\mathfrak{A})_1$. This characterization serves as our definition of $|\phi|$ and makes no reference to the classical one. Sakai's theorem and the Banach-space identification of the bidual with the enveloping von Neumann algebra are then deduced as consequences, after

which the two definitions are seen to agree (§ 8.2). For $\mathfrak{A} = C(X)$, the functional $|\phi|$ is given by integration against the total variation $|\mu|$ (see Example 5.4).

To obtain this characterization, we study when a functional ϕ on $\mathfrak{A}$ *attains its norm* at a contraction A, that is, when $\phi(A) = \|\phi\|$. In that case, the right translate $\phi(\cdot\, A)$ is positive and does not depend on the choice of A. It is the absolute value $|\phi|$, and $\phi = |\phi|(\cdot\, A^*)$. This is a *polar-type factorization*: the positive part is canonical, while A is an arbitrary contraction at which ϕ attains its norm and need not be a partial isometry. In the normal case, the support projection of the absolute value turns this factorization into Sakai's polar decomposition, in which the contraction is a partial isometry.

When the norm of ϕ is not attained in $(\mathfrak{A})_1$, we take a *norming sequence* for ϕ, that is, a sequence of contractions A_n with $\phi(A_n) \to \|\phi\|$, and we show that the functionals $\phi(\cdot\, A_n)$ converge in norm to a positive functional that does not depend on the norming sequence. Both cases are obtained as a consequence of the following theorem, which is the main result of this article.

Main Theorem (Absolute value of a functional, Theorem 5.1). Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$. Then there is a unique functional $|\phi|$ on $\mathfrak{A}$ such that

$$\|\phi(\cdot\, B) - |\phi|\|^2 \le 2\|\phi\|\big(\|\phi\| - \operatorname{Re}\phi(B)\big) \qquad \text{for every contraction } B \in \mathfrak{A}.$$

The functional $|\phi|$ is positive, has norm $\|\phi\|$, and satisfies

$$\|\phi - |\phi|(\cdot\, B^*)\|^2 \le 2\|\phi\|\big(\|\phi\| - \operatorname{Re}\phi(B)\big) \qquad \text{for every contraction } B \in \mathfrak{A}.$$

In particular, if ϕ attains its norm at A, then $|\phi| = \phi(\cdot\, A)$ and $\phi = |\phi|(\cdot\, A^*)$, with $|\phi|(I - AA^*) = 0$. Moreover, $\phi(\cdot\, A_n) \to |\phi|$ and $|\phi|(\cdot\, A_n^*) \to \phi$ in norm for every sequence $(A_n)_{n\in\mathbb{N}}$ of contractions in $\mathfrak{A}$ with $\phi(A_n) \to \|\phi\|$.

We refer to the convergence $|\phi|(\cdot\, A_n^*) \to \phi$ as an *approximate polar decomposition* of ϕ. When ϕ attains its norm at a contraction A (Theorem 6.1), ϕ vanishes on the left ideal $\mathfrak{A}(I - A^*A)$ and on the right ideal $(I - AA^*)\mathfrak{A}$. Conversely, if ρ is a positive functional and A is a contraction with $\rho(AA^*) = \rho(I)$, then $\rho(\cdot\, A^*)$ attains its norm at A and has absolute value ρ. In summary, for a bounded functional ϕ on $\mathfrak{A}$ and a contraction $A \in \mathfrak{A}$, the following are equivalent:

(i) ϕ attains its norm at A, that is, $\phi(A) = \|\phi\|$;
(ii) $\phi = \rho(\cdot\, A^*)$ for some positive functional ρ on $\mathfrak{A}$ with $\rho(AA^*) = \rho(I)$.

Moreover, when these conditions hold, ρ is uniquely determined and $\rho = |\phi| = \phi(\cdot\, A)$.

The constraints imposed on a functional by norm attainment all stem from one elementary inequality, which we describe below. Let $\mathbb{D}$ denote the closed unit disc $\{\lambda \in \mathbb{C} : |\lambda| \le 1\}$. For $Z, D \in \mathfrak{A}$ such that $Z + \lambda D$ is a contraction for every $\lambda \in \mathbb{D}$, we refer to the set $Z + \mathbb{D}D$ as a *disc in the unit ball* of $\mathfrak{A}$ with centre Z and direction D. In Lemma 3.2, we note that for such a disc and every functional ϕ on $\mathfrak{A}$, we have the *disc inequality*

$$|\phi(Z)| + |\phi(D)| \le \|\phi\|.$$

In particular, a functional which attains its norm at Z must vanish at D. We use the following four discs (Proposition 3.4).

(D1) $H + \mathbb{D}(I - H)$, where $0 \le H \le I$;
(D2) $A + \mathbb{D}\big(\frac{1}{3}(I - AA^*)X(I - A^*A)\big)$, where $A, X \in (\mathfrak{A})_1$;
(D3) $W + \mathbb{D}\big((I - Q)X(I - P)\big)$, where $W \in \mathfrak{A}$ is a partial isometry with initial projection P and final projection Q, and $X \in (\mathfrak{A})_1$;

(D4) $A + \mathbb{D}\big(A(I - |A|)\big)$, where $A \in (\mathfrak{A})_1$.

The article is organized around these four discs. The first two lead to the Main Theorem, and the last two to the extreme points of the unit ball. The proof of the Main Theorem runs as follows. By (D1) and the disc inequality, a functional which attains its norm at a positive contraction attains it at I and is positive (Proposition 4.2). In particular, we recover the classical fact that a functional attains its norm at I if and only if it is positive (Corollary 4.4). Note that if ϕ attains its norm at an arbitrary contraction A, then $\phi(\,\cdot\, A)$ and $\phi(A\,\cdot\,)$ attain their norms at I, and are therefore positive. Furthermore, using (D2), we observe that ϕ vanishes on $(I - AA^*)\mathfrak{A}(I - A^*A)$. In Lemma 4.7, we combine these facts to show that $\phi = \phi(\,\cdot\, A^*A)$. This helps us to derive the estimates of the Main Theorem when ϕ attains its norm. The general case is reduced to this one by extending ϕ to a norm-attaining functional on the C^*-algebra $\ell^\infty(\mathfrak{A})$ of bounded sequences in $\mathfrak{A}$ (Lemma 4.9). In Theorem 6.1, we note that the statements about norm-attaining functionals correspond to the special case in which the right-hand sides of the estimates vanish.

We now turn to the consequences. By compactness, a normal functional ω on a von Neumann algebra attains its norm, so that the polar-type factorization applies to it and yields Sakai's polar decomposition $\omega = |\omega|(\,\cdot\, V)$ [13, Theorem 2] (Theorem 7.2). Using any approximate polar decomposition, we show that every functional ϕ on $\mathfrak{A}$ is a vector functional in the GNS representation π of its absolute value, namely $\phi = \langle \pi(\,\cdot\,)\eta, \xi\rangle$ with $\|\eta\| = \|\xi\| = \|\phi\|^{1/2}$ (Proposition 8.1). Since $\|\phi\| \leq \|\eta\|\,\|\xi\|$ holds for any such representation, these constants are best possible. The resulting equality identifies $\mathfrak{A}^*$ isometrically with the predual of the enveloping von Neumann algebra of $\mathfrak{A}$, and hence $\mathfrak{A}^{**}$ with that algebra as a Banach space (Theorem 8.2), without recourse to the Kaplansky density theorem. In Proposition 8.4, we then observe that our definition of the absolute value agrees with the classical one. Furthermore, in Proposition 8.5, we obtain the Jordan decomposition of a hermitian functional. Finally, in Theorem 9.7, we use (D3) and (D4) to prove Kadison's characterization of the extreme points of $(\mathfrak{A})_1$ (see [7, Theorem 1]), and the theorem of Akemann and Russo which asserts that these coincide with the complex extreme points (see [2, Theorem 3.2]).

The principal new contributions of this article are the characterization of $|\phi|$ via the inequalities in the Main Theorem, which yields a construction of the absolute value without the bidual, and the factorization $\phi = |\phi|(\,\cdot\, A^*)$ at an arbitrary contraction A at which ϕ attains its norm. Furthermore, the description of the norm-attaining functionals and our derivation of Theorem 8.2 appear to be new in this form. Although the remaining results are classical, it is worthwhile to note that all of them are obtained here in a single logical sequence, with the disc inequality as the main tool. We hope that even experts, to whom these results are familiar, will find this unified treatment to be of some interest.

The relationship between Sakai's theorem and the geometry of the unit ball has been studied in the literature by Akemann and Russo in [2], by Harris in [6] in the context of J^*-algebras, and by Brown and Pedersen in [3] in the context of extremally rich C^*-algebras. In [5, Theorem 2.12] and [9, Theorem 7.3.2], Sakai's theorem itself has been deduced from geometric considerations by Friedman and Russo, and by Kadison and Ringrose, respectively. Our proof of Sakai's theorem is closest to Takesaki's [17, Theorem III.4.2], the difference being that the factorization of Theorem 6.1 holds at an arbitrary norming contraction, so that extreme points are not needed.

We freely use the basic theory of unital C^*-algebras and von Neumann algebras. Our general references are [8, 9, 11], together with [12] for the Riesz representation theorem, and precise references are given where less elementary facts are used. § 2 fixes the notation. Sections 3–8 are best read in order, whereas § 9 depends only on § 3.

2. Notation and preliminaries

In this section, we fix the notation and record some conventions and basic facts that are used throughout the article.

2.1. **General conventions.** Throughout this article, $\mathfrak{A}$ denotes a unital C^*-algebra with identity I, and $\mathscr{R}$ denotes a von Neumann algebra. (The non-unital case of the Main Theorem is discussed in Remark 5.6.) The closed unit ball of $\mathfrak{A}$ is denoted by $(\mathfrak{A})_1$, and its elements are called *contractions*. We use the standard notation $\mathfrak{A}^*$ for the dual of $\mathfrak{A}$. By a "functional", we always mean a bounded linear functional. We typically use the symbols ϕ, ρ, and ω for general, positive, and normal functionals, respectively. We write $\mathbb{N} := \{1, 2, \dots\}$ for the set of positive integers, $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ for the unit disc, and $\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ for the unit circle. For a Hilbert space $\mathscr{H}$, the set of bounded operators on $\mathscr{H}$ is denoted by $\mathcal{B}(\mathscr{H})$. Our inner products are linear in the first variable. By a *representation* of $\mathfrak{A}$ on $\mathscr{H}$, we mean a unital $*$-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathscr{H})$. A *state* of $\mathfrak{A}$ is a positive functional of norm 1.

For $T \in \mathfrak{A}$, we define the left and right multiplication operators $L_T, R_T : \mathfrak{A} \to \mathfrak{A}$ by

$$L_T(X) = TX, \qquad R_T(X) = XT \qquad (X \in \mathfrak{A}).$$

We use $\phi \circ L_T$ and $\phi \circ R_T$ interchangeably with $\phi(T\,\cdot\,)$ and $\phi(\,\cdot\,T)$, respectively. Note that $L_S \circ L_T = L_{ST}$ and $R_S \circ R_T = R_{TS}$. We also write $\phi^*(X) := \overline{\phi(X^*)}$, and note that $\|\phi^*\| = \|\phi\|$.

Definition 2.1. We say that a functional ϕ on a C^*-algebra $\mathfrak{A}$ *attains its norm at a contraction* $A \in \mathfrak{A}$ if $\phi(A) = \|\phi\|$. Such an A is called a *norming contraction* for ϕ. This condition is stronger than $|\phi(A)| = \|\phi\|$, which is often used to define norm attainment. For a general contraction, the latter condition reduces to the former after replacing A by γA for a suitable $\gamma \in \mathbb{T}$. When A is required to be positive, however, this replacement is not available (see Remark 4.3).

To measure how far a functional is from attaining its norm at a given contraction, we use the following quantity. For a functional ϕ on $\mathfrak{A}$ and a contraction $B \in \mathfrak{A}$, the *defect* of ϕ at B is defined as

$$\delta_\phi(B) := \Big(2\|\phi\|\big(\|\phi\| - \operatorname{Re}\phi(B)\big)\Big)^{1/2}. \tag{2.1}$$

Since $|\phi(B)| \le \|\phi\|$, the quantity under the square root is non-negative. For $\phi \neq 0$, it vanishes precisely when ϕ attains its norm at B, and for $\phi = 0$ it vanishes identically.

2.2. **Von Neumann algebras.** For $A \in \mathfrak{A}$, we write $|A| := (A^*A)^{1/2}$. If $\mathfrak{A}$ acts on a Hilbert space $\mathscr{H}$, we denote the projection onto the closure of the range of A by $R(A)$. This use of the letter R is local to operator-theoretic statements and should not be confused with the right-multiplication operators R_T.

Remark 2.2. Recall that the *ultraweak topology* on $\mathcal{B}(\mathscr{H})$ is the locally convex topology determined by the seminorms $T \mapsto \big|\sum_n \langle T\xi_n, \eta_n\rangle\big|$, where (ξ_n) and (η_n) are sequences of vectors in $\mathscr{H}$ satisfying $\sum_n \|\xi_n\|^2 < \infty$ and $\sum_n \|\eta_n\|^2 < \infty$. By a von Neumann algebra

acting on $\mathscr{H}$, we mean a unital $*$-subalgebra $\mathscr{R}$ of $\mathcal{B}(\mathscr{H})$ which is closed in the ultraweak topology. Since the ultraweak topology and the weak-operator topology coincide on bounded subsets of $\mathcal{B}(\mathscr{H})$, the limit of a bounded net in $\mathscr{R}$ which converges in the strong-operator topology (or in the weak-operator topology) again lies in $\mathscr{R}$. This is the only closure property of $\mathscr{R}$ used in this article. For the same reason, the unit ball $(\mathscr{R})_1$ is ultraweakly compact, the unit ball of $\mathcal{B}(\mathscr{H})$ being compact in the weak-operator topology.

A functional ω on $\mathscr{R}$ is said to be *normal* if it is ultraweakly continuous, and the set of normal functionals on $\mathscr{R}$ is denoted by $\mathscr{R}_*$. A bounded monotone net (H_α) of self-adjoint operators in $\mathscr{R}$ converges in the strong-operator topology to an operator H (see [8, Lemma 5.1.4]), which lies in $\mathscr{R}$ by Remark 2.2. The convergence is also ultraweak, so that $\omega(H_\alpha) \longrightarrow \omega(H)$ for every $\omega \in \mathscr{R}_*$.

3. Discs in the unit ball of a C^*-algebra

This section contains the two tools used throughout the article. The first is the disc lemma (Lemma 3.2): a functional which attains its norm at the centre of a disc in the unit ball vanishes at the direction of the disc. The second is a list of four explicit discs in the unit ball of a C^*-algebra (Proposition 3.4). Most of the results in this article are obtained by combining the two.

Definition 3.1. Let $\mathfrak{A}$ be a C^*-algebra. For $Z \in (\mathfrak{A})_1$ and $D \in \mathfrak{A}$, we write $Z + \mathbb{D}\, D := \{Z + \lambda D : \lambda \in \mathbb{D}\}$. When this set lies in $(\mathfrak{A})_1$, we call it a *disc in* $(\mathfrak{A})_1$ with *centre* Z and *direction* D.

Lemma 3.2. *Let $\mathfrak{A}$ be a C^*-algebra. Let $Z + \mathbb{D}\, D$ be a disc in $(\mathfrak{A})_1$, and let ϕ be a bounded linear functional on $\mathfrak{A}$. Then we have the following:*

(a) $|\phi(Z)| + |\phi(D)| \leq \|\phi\|$;
(b) *If ϕ attains its norm at Z, then $\phi(D) = 0$, and hence ϕ has the constant value $\|\phi\|$ on the disc $Z + \mathbb{D}\, D$;*
(c) *If $D \neq 0$, then Z is not an extreme point of $(\mathfrak{A})_1$.*

Proof. (a) We choose $\alpha, \beta \in \mathbb{T}$ with $\alpha\phi(Z) = |\phi(Z)|$ and $\beta\phi(D) = |\phi(D)|$. Since $\alpha Z + \beta D = \alpha\big(Z + \bar{\alpha}\beta D\big)$ is a contraction, we have

$$|\phi(Z)| + |\phi(D)| = \phi(\alpha Z + \beta D) \leq \|\phi\|.$$

(b) Since $\phi(Z) = \|\phi\|$, part (a) implies that $\phi(D) = 0$. Thus $\phi(Z + \lambda D) = \phi(Z) + \lambda\phi(D) = \|\phi\|$ for every $\lambda \in \mathbb{D}$.
(c) Note that $Z + D$ and $Z - D$ are distinct elements of $(\mathfrak{A})_1$, and Z is their midpoint. □

Remark 3.3. The statement and the proof of Lemma 3.2 involve only the norm of $\mathfrak{A}$, and are valid verbatim in any complex normed space. Part (b) is the maximum modulus principle: the function $\lambda \mapsto \phi(Z + \lambda D)$ is analytic on the open unit disc, continuous on $\mathbb{D}$, bounded in modulus by $\|\phi\|$, and takes the value $\|\phi\|$ at the interior point 0. Hence it is constant. The corresponding maximum modulus theorem for analytic functions with values in a Banach space is due to Thorp and Whitley [18], and it led them to the notion of a complex extreme point. A point $Z \in (\mathfrak{A})_1$ is said to be a *complex extreme point* of $(\mathfrak{A})_1$ if there is no disc in $(\mathfrak{A})_1$ centred at Z with non-zero direction (cf. Theorem 9.7).

Proposition 3.4. *Let A and X be contractions in a unital C^*-algebra $\mathfrak{A}$, let H be a positive contraction in $\mathfrak{A}$, and let W be a partial isometry in $\mathfrak{A}$ with initial projection $P = W^*W$ and final projection $Q = WW^*$. Then each of the following sets is a disc in $(\mathfrak{A})_1$:*

(D1) $H + \mathbb{D}(I - H)$;
(D2) $A + \mathbb{D}\big(\frac{1}{3}(I - AA^*)X(I - A^*A)\big)$;
(D3) $W + \mathbb{D}\big((I - Q)X(I - P)\big)$;
(D4) $A + \mathbb{D}\big(A(I - |A|)\big)$.

Proof. Let $\lambda \in \mathbb{D}$. The discs (D1) and (D4) involve only functions of a single positive contraction, and for these we use the continuous functional calculus.

(D1) Let $f_\lambda : [0,1] \to \mathbb{C}$ be the continuous function given by $f_\lambda(t) := t + \lambda(1-t)$. For $t \in [0,1]$, note that $f_\lambda(t)$ is a convex combination of 1 and λ, whence $|f_\lambda(t)| \le 1$. Since the spectrum of H is contained in $[0,1]$, we observe that $H + \lambda(I - H) = f_\lambda(H)$ is a normal element of $\mathfrak{A}$ whose norm is the supremum of $|f_\lambda|$ over the spectrum of H. Thus $\|H + \lambda(I - H)\| \le 1$.

(D2) Let $S := I - A^*A$ and $D := \frac{1}{3}(I - AA^*)XS$. Since $A^*(I - AA^*) = SA^*$, we have $A^*D = \frac{1}{3}SA^*XS$. A straightforward computation tells us that

$$(A + \lambda D)^*(A + \lambda D) = A^*A + SMS,$$

where $M := \frac{1}{3}\big(\lambda A^*X + \bar{\lambda}X^*A\big) + \frac{1}{9}|\lambda|^2X^*(I - AA^*)^2X$. Note that M is self-adjoint and $\|M\| \le \frac{2}{3} + \frac{1}{9} \le 1$. Since $0 \le S \le I$, we have $SMS \le S^2 \le S$, and it follows that $(A + \lambda D)^*(A + \lambda D) \le A^*A + S = I$.

(D3) Let $D := (I - Q)X(I - P)$. Since $QW = W = WP$, we have $W^*D = 0 = D^*W$. Moreover, since $X^*(I - Q)X \le X^*X \le I$, we have

$$D^*D = (I - P)X^*(I - Q)X(I - P) \le I - P.$$

As $W^*W = P$, we conclude that

$$(W + \lambda D)^*(W + \lambda D) = W^*W + |\lambda|^2D^*D \le P + (I - P) = I,$$

that is, $W + \lambda D \in (\mathfrak{A})_1$.

(D4) Note that $A + \lambda A(I - |A|) = A\,g_\lambda(|A|)$, where $g_\lambda(t) := 1 + \lambda(1 - t)$. Since $g_\lambda(|A|)$ commutes with $|A|$ and $A^*A = |A|^2$, we get

$$\big(A\,g_\lambda(|A|)\big)^*\big(A\,g_\lambda(|A|)\big) = g_\lambda(|A|)^*\,|A|^2\,g_\lambda(|A|) = h_\lambda(|A|),$$

where $h_\lambda(t) := t^2\,|1+\lambda(1-t)|^2$. For $t \in [0,1]$, we have $t\,|1+\lambda(1-t)| \le t(2-t) = 1-(1-t)^2 \le 1$, so that $0 \le h_\lambda \le 1$ on the spectrum of $|A|$. Thus $\|A\,g_\lambda(|A|)\|^2 = \|h_\lambda(|A|)\| \le 1$. □

The discs centred at a partial isometry are described completely in Theorem 9.4. The discs (D1) and (D2) are used in § 4, and (D3) and (D4) in § 9.

4. Positive functionals and norm attainment

This section contains the preparations for the Main Theorem, which is stated and proved at the beginning of § 5. There are three ingredients. In § 4.1, we relate positivity to norm attainment at the identity. In § 4.2, we analyse a functional at an arbitrary contraction where it attains its norm (Lemma 4.7). In § 4.3, we show that every functional on $\mathfrak{A}$ has a norm-preserving extension to the C^*-algebra of bounded sequences in $\mathfrak{A}$ which attains its norm (Lemma 4.9).

4.1. **Positivity and norm attainment.** We first recall three basic properties of positive functionals. For parts (a) and (b), see [8, Proposition 4.3.1 and Theorem 4.3.2].

Lemma 4.1. *Let ρ be a positive functional on a unital C^*-algebra $\mathfrak{A}$. Then we have the following:*

(a) $|\rho(B^*C)|^2 \leq \rho(B^*B)\,\rho(C^*C)$ *for all* $B, C \in \mathfrak{A}$;
(b) $\|\rho\| = \rho(I)$, *that is, ρ attains its norm at I;*
(c) *If* $T \geq 0$ *and* $\rho(T) = 0$, *then* $\rho(XT) = \rho(TX) = 0$ *for every* $X \in \mathfrak{A}$. *In other words, ρ vanishes on the left ideal $\mathfrak{A}T$ and on the right ideal $T\mathfrak{A}$.*

Proof. (a) This is the Cauchy–Schwarz inequality for the positive semidefinite sesquilinear form $(C, B) \mapsto \rho(B^*C)$ on $\mathfrak{A}$. Note that this form is hermitian. Taking $C = I$, we obtain $\rho(B^*) = \overline{\rho(B)}$.

(b) Let A be a contraction in $\mathfrak{A}$. From part (a) with $B = I$ and $C = A$, we have

$$|\rho(A)|^2 \leq \rho(I)\,\rho(A^*A) \leq \rho(I)^2,$$

since $A^*A \leq I$. Thus $\|\rho\| = \rho(I)$.

(c) With $B = T^{1/2}X^*$ and $C = T^{1/2}$, part (a) tells us that $|\rho(XT)|^2 \leq \rho(XTX^*)\,\rho(T) = 0$. The choices $B = T^{1/2}$ and $C = T^{1/2}X$ similarly give $\rho(TX) = 0$. □

Proposition 4.2. *Let ϕ be a non-zero functional on a unital C^*-algebra $\mathfrak{A}$ which attains its norm at $H \in \mathfrak{A}$, where $0 \leq H \leq I$. Then we have the following:*

(a) $\phi(I - H) = 0$, *and hence ϕ attains its norm at I;*
(b) *ϕ is positive;*
(c) $\phi(X(I - H)) = \phi((I - H)X) = 0$ *for every* $X \in \mathfrak{A}$. *Equivalently,* $\phi(X) = \phi(XH) = \phi(HX)$.

Conversely, a positive functional ρ on $\mathfrak{A}$ with $\rho(I - H) = 0$ attains its norm at H.

Proof. Without loss of generality, we may assume that $\|\phi\| = 1$.

(a) By (D1), the set $H + \mathbb{D}(I - H)$ is a disc in $(\mathfrak{A})_1$. Since ϕ attains its norm at H, Lemma 3.2-(b) tells us that $\phi(I - H) = 0$, and thus $\phi(I) = \phi(H) = 1$.

(b) Let $T \geq 0$. After scaling, we may assume that $T \leq I$. Let $a := \phi(T)$, so that $\phi(I - T) = 1 - a$ by part (a). Applying Lemma 3.2-(a) to the disc $T + \mathbb{D}(I - T)$, which is again of the form (D1), we get $|a| + |1 - a| \leq 1$. The triangle inequality gives the reverse inequality, so that $|a| + |1 - a| = 1 = |a + (1 - a)|$. Equality in the triangle inequality forces a and $1 - a$ to be non-negative multiples of a common unimodular scalar. Since their sum is 1, both are non-negative real numbers. In particular, $\phi(T) = a \geq 0$.

(c) Since ϕ is positive by part (b) and $\phi(I - H) = 0$ by part (a), the assertion follows from Lemma 4.1-(c).

For the converse, note that $\rho(H) = \rho(I) = \|\rho\|$ by Lemma 4.1-(b). □

Remark 4.3. The requirement $\phi(H) = \|\phi\|$ in Definition 2.1 is essential for the conclusion that ϕ is positive. If only $|\phi(H)| = \|\phi\|$ is assumed, then Proposition 4.2, applied to $\gamma\phi$ with $\gamma \in \mathbb{T}$ chosen so that $\gamma\phi(H) = \|\phi\|$, shows that ϕ is a unimodular scalar multiple of a positive functional, but ϕ itself need not be positive.

The case $H = I$ of Proposition 4.2 is the classical criterion for positivity.

Corollary 4.4 (see [8, Theorem 4.3.2])**.** *Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$. Then ϕ attains its norm at I if and only if ϕ is positive.*

Proof. Clearly, the assertion holds for $\phi = 0$. For $\phi \neq 0$, the result follows from Proposition 4.2-(b) with $H = I$, in combination with Lemma 4.1-(b). □

4.2. **Norm attainment at an arbitrary contraction.** In the next two lemmas, we consider a positive functional together with a contraction which need not be positive. The first of these is stated in terms of the defect defined in (2.1), and we shall make use of its quantitative form in § 5.

Lemma 4.5. *Let ρ be a positive functional on a unital C^*-algebra $\mathfrak{A}$, and let C be a contraction in $\mathfrak{A}$. Then*

$$\|\rho - \rho \circ R_C\| \leq \delta_\rho(C).$$

In particular, if $\rho(C) = \|\rho\|$, then $\rho(XC) = \rho(X)$ for every $X \in \mathfrak{A}$.

Proof. Recall that $\rho(I) = \|\rho\|$ by Lemma 4.1-(b). Since $\rho(C^*) = \overline{\rho(C)}$ and $C^*C \leq I$, we have

$$\rho\big((I-C)^*(I-C)\big) = \rho(I) - 2\operatorname{Re}\rho(C) + \rho(C^*C) \leq 2\big(\rho(I) - \operatorname{Re}\rho(C)\big).$$

Let $X \in \mathfrak{A}$. Since $\rho(XX^*) \leq \|\rho\|\,\|X\|^2$, Lemma 4.1-(a) gives

$$|\rho(X) - \rho(XC)|^2 = |\rho(X(I-C))|^2 \leq \rho(XX^*)\,\rho\big((I-C)^*(I-C)\big) \leq \delta_\rho(C)^2\,\|X\|^2.$$

The last assertion is the case $\delta_\rho(C) = 0$. □

Lemma 4.6. *Let ρ be a positive functional on a unital C^*-algebra $\mathfrak{A}$, and let $A \in \mathfrak{A}$ be a contraction with $\rho(I - AA^*) = 0$. Then the functional $\phi := \rho \circ R_{A^*}$ attains its norm at A, and we have*

$$\|\phi\| = \|\rho\|, \quad \rho = \phi \circ R_A.$$

Proof. For a contraction $X \in \mathfrak{A}$, Lemma 4.1-(b) gives

$$|\phi(X)| = |\rho(XA^*)| \leq \|\rho\|\,\|XA^*\| \leq \rho(I),$$

so that $\|\phi\| \leq \rho(I)$. On the other hand, $\phi(A) = \rho(AA^*) = \rho(I)$. Thus ϕ attains its norm at A, and $\|\phi\| = \rho(I)$. Finally, since $I - AA^* \geq 0$ and $\rho(I - AA^*) = 0$, from Lemma 4.1-(c) we have $\rho = \rho \circ R_{AA^*}$, and we conclude that

$$\phi \circ R_A = \rho \circ R_{A^*} \circ R_A = \rho \circ R_{AA^*} = \rho.$$ □

Lemma 4.6 provides a recipe for constructing functionals which attain their norm at a given contraction. In the next lemma, which lies at the heart of the polar decomposition, we show that every such functional arises in this manner.

Lemma 4.7. *Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$ which attains its norm at a contraction $A \in \mathfrak{A}$. Then the functional $\phi \circ R_A$ is positive, and ϕ vanishes on $\mathfrak{A}(I - A^*A)$, that is, $\phi = \phi \circ R_{A^*A}$.*

Proof. Let $\rho := \phi \circ R_A$ and $\mu := \phi \circ L_A$. Note that each of these functionals has norm at most $\|\phi\|$ and takes the value $\phi(A) = \|\phi\|$ at I. Hence both attain their norms at I, so Corollary 4.4 shows that they are positive.

For the second assertion, let $S := I - A^*A$ and $T := I - AA^*$, so that $TA = AS$. For every contraction $X \in \mathfrak{A}$, the disc (D2) is centred at A and has direction $\frac{1}{3}TXS$, so that $\phi(TXS) = 0$ by Lemma 3.2-(b). By homogeneity, ϕ vanishes on $T\mathfrak{A}S$. In particular, $\mu(S^2) = \phi(AS^2) = \phi(TAS) = 0$, and Lemma 4.1-(a) gives $|\mu(YS)|^2 \leq \mu(YY^*)\,\mu(S^2) = 0$ for every $Y \in \mathfrak{A}$, that is, ϕ vanishes on $A\mathfrak{A}S$. Since $YS = TYS + A(A^*Y)S$ for every $Y \in \mathfrak{A}$, it follows that ϕ vanishes on $\mathfrak{A}(I - A^*A)$, that is, $\phi = \phi \circ R_{A^*A}$. □

4.3. **Norming sequences and norm-attaining extensions.** A functional need not attain its norm: for instance, the functional $f \mapsto \int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt$ on $C([0,1])$ has norm 1, and this norm is not attained. In general, the norm is attained only approximately, along a sequence of contractions, and such a sequence serves as a substitute for a norming contraction.

Definition 4.8. Let $\mathfrak{A}$ be a unital C^*-algebra and let ϕ be a functional on $\mathfrak{A}$. A sequence $(A_n)_{n\in\mathbb{N}}$ of contractions in $\mathfrak{A}$ is said to be a *norming sequence* for ϕ if $\phi(A_n) \to \|\phi\|$.

If A is a norming contraction for ϕ, then the constant sequence $(A, A, \ldots)$ is a norming sequence for ϕ. Unlike norming contractions, norming sequences always exist: by the definition of the norm, there is a sequence of contractions $(B_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with $|\phi(B_n)| \to \|\phi\|$, and if $\gamma_n \in \mathbb{T}$ is chosen so that $\gamma_n\phi(B_n) = |\phi(B_n)|$, then $(\gamma_n B_n)_{n\in\mathbb{N}}$ is a norming sequence for ϕ.

Thus every functional on $\mathfrak{A}$ attains its norm approximately, whereas Lemma 4.7 applies only to functionals which attain their norm exactly. We bridge this gap by enlarging the algebra. Note that a norming sequence $(A_n)_{n\in\mathbb{N}}$ for ϕ may be viewed as a contraction in the C^*-algebra of bounded sequences in $\mathfrak{A}$. We extend ϕ to a functional of the same norm on this algebra, whose value at $(A_n)_{n\in\mathbb{N}}$ is $\lim_n \phi(A_n) = \|\phi\|$. Clearly, the extension attains its norm at $(A_n)_{n\in\mathbb{N}}$, whence Lemma 4.7 applies to it. In other words, approximate norm attainment in $\mathfrak{A}$ is converted into exact norm attainment in the larger algebra.

Let $\ell^\infty(\mathfrak{A})$ denote the set of bounded sequences $\mathbf{X} = (X_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}$. It is straightforward to verify that $\ell^\infty(\mathfrak{A})$ is a unital C^*-algebra when equipped with the coordinatewise operations and the norm $\|\mathbf{X}\| := \sup_n \|X_n\|$. Note that the contractions in $\ell^\infty(\mathfrak{A})$ are precisely the sequences of contractions in $\mathfrak{A}$. In particular, every norming sequence for ϕ is a contraction in $\ell^\infty(\mathfrak{A})$. The mapping $\iota : \mathfrak{A} \to \ell^\infty(\mathfrak{A})$, which sends $X \in \mathfrak{A}$ to the constant sequence $(X, X, \ldots)$, is an isometric unital $*$-homomorphism. A functional Φ on $\ell^\infty(\mathfrak{A})$ is said to extend ϕ if $\Phi \circ \iota = \phi$. In that case, what is proved for Φ may be restricted to $\mathfrak{A}$ by composing with ι. In the next lemma, we show that the required extension exists.

Lemma 4.9. *Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$. Then there is a functional Φ on $\ell^\infty(\mathfrak{A})$ such that*

(a) *$\Phi \circ \iota = \phi$ and $\|\Phi\| = \|\phi\|$;*
(b) *$\Phi(\mathbf{A}) = \|\Phi\|$ for every norming sequence $\mathbf{A} = (A_n)_{n\in\mathbb{N}}$ for ϕ.*

In particular, Φ attains its norm.

Proof. Let $\mathfrak{c}_\phi$ denote the linear subspace of $\ell^\infty(\mathfrak{A})$ consisting of the sequences $\mathbf{Y} = (Y_n)$ for which $\lim_n \phi(Y_n)$ exists, and define

$$\Phi_0(\mathbf{Y}) := \lim_{n\to\infty} \phi(Y_n) \qquad (\mathbf{Y} \in \mathfrak{c}_\phi).$$

Then Φ_0 is a linear functional on $\mathfrak{c}_\phi$ with $|\Phi_0(\mathbf{Y})| \le \|\phi\|\,\|\mathbf{Y}\|$. By the Hahn–Banach theorem, Φ_0 extends to a functional Φ on $\ell^\infty(\mathfrak{A})$ with $\|\Phi\| = \|\Phi_0\| \le \|\phi\|$. Since $\iota(\mathfrak{A}) \subseteq \mathfrak{c}_\phi$ and $\Phi_0 \circ \iota = \phi$, we have $\Phi \circ \iota = \phi$. As ι is isometric, it follows that $\|\Phi\| \ge \|\phi\|$. This proves (a).

If $\mathbf{A} = (A_n)_{n\in\mathbb{N}}$ is a norming sequence for ϕ, then $\mathbf{A}$ is a contraction in $\ell^\infty(\mathfrak{A})$ which lies in $\mathfrak{c}_\phi$, and

$$\Phi(\mathbf{A}) = \lim_{n\to\infty} \phi(A_n) = \|\phi\| = \|\Phi\|,$$

which proves (b). The last assertion follows, since ϕ has a norming sequence. □

5. The absolute value of a functional

We now prove the main theorem (Theorem 5.1) and show that the absolute value is compatible with norm-preserving extensions (Proposition 5.5). The specialization to functionals which attain their norm is the subject of § 6.

Theorem 5.1. *Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$ and define $\delta_\phi(B) := \Big(2\|\phi\|\big(\|\phi\| - \operatorname{Re}\phi(B)\big)\Big)^{1/2}$ for $B \in (\mathfrak{A})_1$. Then we have the following:*

(a) *There is a unique functional ρ on $\mathfrak{A}$ such that*

$$\|\phi \circ R_B - \rho\| \le \delta_\phi(B) \qquad \text{for every contraction } B \in \mathfrak{A}. \tag{5.1}$$

(b) *The functional ρ is positive and satisfies $\|\rho\| = \rho(I) = \|\phi\|$. Moreover, for every contraction $B \in \mathfrak{A}$, we have*

$$\|\phi - \rho \circ R_{B^*}\| \le \delta_\phi(B) \qquad \text{and} \qquad \|\phi - \phi \circ R_{B^*B}\| \le 2\,\delta_\phi(B). \tag{5.2}$$

(c) *If $(A_n)_{n\in\mathbb{N}}$ is a norming sequence for ϕ, then we have*

$$\lim_{n\to\infty} \phi \circ R_{A_n} = \rho, \qquad \lim_{n\to\infty} \rho \circ R_{A_n^*} = \phi, \qquad \lim_{n\to\infty} \phi \circ R_{A_n^* A_n} = \phi,$$

where the limits are taken in the norm of $\mathfrak{A}^*$.

Proof. We first prove the uniqueness assertion in (a). Let $(A_n)_{n\in\mathbb{N}}$ be a norming sequence for ϕ, so that $\delta_\phi(A_n) \to 0$. If a functional ρ satisfies (5.1), then ρ is the norm limit of the sequence $(\phi \circ R_{A_n})_{n\in\mathbb{N}}$. Hence there is at most one such functional.

It remains to find a positive functional ρ on $\mathfrak{A}$ with $\|\rho\| = \rho(I) = \|\phi\|$ which satisfies (5.1) and (5.2). We first show that it suffices to do so when ϕ attains its norm. Let Φ be a functional on $\ell^\infty(\mathfrak{A})$ as in Lemma 4.9, so that $\Phi \circ \iota = \phi$, $\|\Phi\| = \|\phi\|$, and Φ attains its norm. Suppose that Λ is a positive functional on $\ell^\infty(\mathfrak{A})$ with $\|\Lambda\| = \Lambda(I) = \|\Phi\|$ such that (5.1) and (5.2) hold with Φ and Λ in place of ϕ and ρ, and let $\rho := \Lambda \circ \iota$. Since ι is a unital $*$-homomorphism, ρ is a positive functional on $\mathfrak{A}$ with $\|\rho\| = \rho(I) = \|\phi\|$. Let B be a contraction in $\mathfrak{A}$. Then $\iota(B)$ is a contraction in $\ell^\infty(\mathfrak{A})$, and $\delta_\Phi(\iota(B)) = \delta_\phi(B)$. Note that the three functionals which are estimated in (5.1) and (5.2) for Φ and Λ at $\iota(B)$, when composed with ι, yield the corresponding functionals for ϕ and ρ at B. For instance, $\big(\Phi \circ R_{\iota(B)} - \Lambda\big) \circ \iota = \phi \circ R_B - \rho$. Since composing with ι does not increase norms, it follows that (5.1) and (5.2) hold for ϕ and ρ. Thus, replacing $\mathfrak{A}$ and ϕ by $\ell^\infty(\mathfrak{A})$ and Φ, respectively, we may assume without loss of generality that ϕ attains its norm at a contraction $A \in \mathfrak{A}$.

Let $\rho := \phi \circ R_A$. From Lemma 4.7, we observe that ρ is positive and $\phi = \phi \circ R_{A^*A} = \rho \circ R_{A^*}$. Furthermore, from Lemma 4.1-(b), we have $\|\rho\| = \rho(I) = \phi(A) = \|\phi\|$. Let B be a contraction in $\mathfrak{A}$. Then $\rho(BA^*) = \phi(B)$.

For (5.1), let $C := BA^*$, which is a contraction. Then

$$\phi \circ R_B = \rho \circ R_{A^*} \circ R_B = \rho \circ R_C.$$

Since $\|\rho\| = \|\phi\|$ and $\rho(C) = \phi(B)$, we have $\delta_\rho(C) = \delta_\phi(B)$, and Lemma 4.5 gives $\|\phi \circ R_B - \rho\| \le \delta_\phi(B)$.

For the first estimate in (5.2), note that $\rho(AB^*) = \overline{\rho(BA^*)}$, $AA^* \le I$, and $BB^* \le I$, so that

$$\rho\big((A-B)(A-B)^*\big) = \rho(AA^*) - 2\operatorname{Re}\rho(BA^*) + \rho(BB^*) \le 2\big(\|\phi\| - \operatorname{Re}\phi(B)\big).$$

Let $X \in \mathfrak{A}$. Then $\phi(X) - \rho(XB^*) = \rho\big(X(A-B)^*\big)$, and Lemma 4.1-(a) gives

$$|\phi(X) - \rho(XB^*)|^2 \le \rho(XX^*)\,\rho\big((A-B)(A-B)^*\big) \le \delta_\phi(B)^2\,\|X\|^2.$$

For the second estimate in (5.2), note that $\phi \circ R_{B^*B} = (\phi \circ R_B) \circ R_{B^*}$, so that

$$\phi - \phi \circ R_{B^*B} = \big(\phi - \rho \circ R_{B^*}\big) + \big(\rho - \phi \circ R_B\big) \circ R_{B^*}.$$

Each of the two terms on the right-hand side has norm at most $\delta_\phi(B)$, by the estimates already proved, as $\|B^*\| \le 1$. This proves (a) and (b). Finally, (c) follows from (a) and (b), since $\delta_\phi(A_n) \to 0$ for every norming sequence $(A_n)_{n\in\mathbb{N}}$. □

Definition 5.2 (Absolute value of a functional)**.** Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$. The *absolute value* of ϕ is the positive functional $|\phi|$ on $\mathfrak{A}$ furnished by Theorem 5.1, that is, the unique functional on $\mathfrak{A}$ satisfying

$$\|\phi \circ R_B - |\phi|\| \le \delta_\phi(B) \qquad \text{for every contraction } B \in \mathfrak{A}.$$

By Theorem 5.1-(c), we have $|\phi| \circ R_{A_n^*} \to \phi$ in norm for every norming sequence $(A_n)_{n\in\mathbb{N}}$ for ϕ. We refer to this as the *approximate polar decomposition* of ϕ. In the literature, the absolute value is defined through the bidual of $\mathfrak{A}$. The two definitions agree, as we show in § 8.2. Until then, only Definition 5.2 is used.

Remark 5.3. The estimate (5.1) cannot be improved in general. Indeed, let ρ be a positive functional on $\mathfrak{A}$. Then $\delta_\rho(I) = 0$ by Lemma 4.1-(b), so that $|\rho| = \rho$ by (5.1) with $B = I$. For $\gamma \in \mathbb{T}$ and $B := \gamma I$, we have $\rho \circ R_B = \gamma\rho$, and hence

$$\|\rho \circ R_B - |\rho|\| = |\gamma - 1|\,\|\rho\| = \delta_\rho(B),$$

because $|\gamma - 1|^2 = 2(1 - \operatorname{Re}\gamma)$. Thus equality holds in (5.1), and similarly in the first estimate of (5.2). In particular, the constant 2 in the definition (2.1) of the defect is best possible.

Example 5.4 (The commutative case)**.** Let X be a compact Hausdorff space, and let μ be a regular complex Borel measure on X. Let h be a Borel function with $|h| = 1$ such that $d\mu = h\,d|\mu|$. Consider the functional ϕ on $C(X)$ given by $\phi(f) := \int_X f\,d\mu$ for $f \in C(X)$, and recall that $\|\phi\| = |\mu|(X)$. Since $C(X)$ is dense in $L^1(|\mu|)$ (see [12, Theorem 3.14]), there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $C(X)$ such that $\int_X |g_n - \bar{h}|\,d|\mu| \to 0$. Let $f_n := g_n/\max\{1, |g_n|\}$. Clearly, f_n is a contraction in $C(X)$. Since the map $z \mapsto z/\max\{1,|z|\}$ is the nearest-point projection of $\mathbb{C}$ onto $\mathbb{D}$, which does not increase distances and fixes the values of $\bar{h}$, we have $|f_n - \bar{h}| \le |g_n - \bar{h}|$. It follows that

$$\phi(f_n) = \int_X f_n h\,d|\mu| \longrightarrow \int_X \bar{h}h\,d|\mu| = |\mu|(X),$$

which shows that (f_n) is a norming sequence for ϕ. Using Theorem 5.1-(c), we get

$$|\phi|(f) = \lim_{n\to\infty} \phi(f f_n) = \lim_{n\to\infty} \int_X f f_n h\,d|\mu| = \int_X f\,d|\mu| \qquad (f \in C(X)).$$

In other words, the absolute value of ϕ is given by integration against the total variation $|\mu|$. Furthermore, the approximate polar decomposition, $|\phi|(f\bar{f}_n) \to \phi(f)$, is nothing but the relation $d\mu = h\,d|\mu|$.

Recall that in the proof of Theorem 5.1, the absolute value of ϕ was obtained by first passing to a norm-preserving extension of ϕ to the larger algebra $\ell^\infty(\mathfrak{A})$, and then restricting the absolute value of this extension to $\mathfrak{A}$. In fact, the choice of the larger algebra and of

the extension plays no role here, as we show in the proposition below with the help of the uniqueness assertion of the theorem.

Proposition 5.5 (Restriction principle)**.** *Let $\mathfrak{A}, \mathfrak{B}$ be unital C^*-algebras, $\pi : \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphism, and ϕ be a functional on $\mathfrak{A}$. Then for every functional Ψ on $\mathfrak{B}$ satisfying $\Psi \circ \pi = \phi$ and $\|\Psi\| = \|\phi\|$, we have*

$$|\Psi| \circ \pi = |\phi|.$$

Proof. Let B be a contraction in $\mathfrak{A}$. It is straightforward to see that $\pi(B)$ is a contraction in $\mathfrak{B}$. Furthermore, as $\|\Psi\| = \|\phi\|$ and $\Psi(\pi(B)) = \phi(B)$, we have $\delta_\Psi(\pi(B)) = \delta_\phi(B)$. Applying (5.1) to Ψ in $\mathfrak{B}$, we get

$$\|\Psi \circ R_{\pi(B)} - |\Psi|\| \leq \delta_\phi(B).$$

Since π is a homomorphism, we have $\Psi \circ R_{\pi(B)} \circ \pi = \phi \circ R_B$. As π is contractive, composing with π does not increase norms, and it follows that

$$\|\phi \circ R_B - |\Psi| \circ \pi\| \leq \delta_\phi(B) \qquad \text{for every contraction } B \in \mathfrak{A}.$$

From the uniqueness assertion of Theorem 5.1-(a), we conclude that $|\Psi| \circ \pi = |\phi|$. □

We use the restriction principle in two ways. First, let Φ be a functional on $\ell^\infty(\mathfrak{A})$ as in Lemma 4.9. Note that Φ attains its norm, and $|\Phi| \circ \iota = |\phi|$ by Proposition 5.5. Thus, in order to prove a statement about ϕ and $|\phi|$ which is inherited from Φ and $|\Phi|$ under composition with ι, we may assume without loss of generality that ϕ attains its norm. This line of argument, with a hermitian lift, is used in the proof of the Jordan decomposition (Proposition 8.5). Second, applied to the normal extension of a functional to the enveloping von Neumann algebra, the restriction principle identifies $|\phi|$ with the classical absolute value of ϕ (Proposition 8.4).

The same restriction argument removes the assumption that the algebra is unital from the Main Theorem.

Remark 5.6 (The non-unital case). Let $\mathfrak{A}_0$ be a non-unital C^*-algebra, regarded as a closed two-sided ideal of its unitization $\mathfrak{A}$, and let ϕ_0 be a functional on $\mathfrak{A}_0$. By the Hahn–Banach theorem, ϕ_0 extends to a functional ϕ on $\mathfrak{A}$ with $\|\phi\| = \|\phi_0\|$, so that $\delta_\phi(B) = \delta_{\phi_0}(B)$ for every contraction $B \in \mathfrak{A}_0$, where δ_{ϕ_0} is defined as in (2.1). Let ρ_0 be the restriction of $|\phi|$ to $\mathfrak{A}_0$. It is a positive functional on $\mathfrak{A}_0$ with $\|\rho_0\| \leq \|\phi_0\|$. Since XB, XB^* and XB^*B lie in $\mathfrak{A}_0$ for all $X, B \in \mathfrak{A}_0$, the estimates (5.1) and (5.2) for ϕ, restricted to $\mathfrak{A}_0$, hold with ϕ_0 and ρ_0 in place of ϕ and ρ, for every contraction $B \in \mathfrak{A}_0$. Since ϕ_0 has a norming sequence in $\mathfrak{A}_0$, as in the proof of Theorem 5.1, we observe that the first of these estimates determines ρ_0. In particular, ρ_0 does not depend on the choice of the extension ϕ. Using the first estimate in (5.2) along such a sequence, we get $\|\phi_0\| \leq \|\rho_0\|$, whence $\|\rho_0\| = \|\phi_0\|$. Thus Theorem 5.1 holds for $\mathfrak{A}_0$, with the exception of the equality $\|\rho\| = \rho(I)$ in part (b).

6. A polar-type factorization of a norm-attaining functional

When ϕ attains its norm at a contraction A, the defect $\delta_\phi(A)$ vanishes. Thus the estimates in Theorem 5.1 become identities, which we record in the next theorem.

Theorem 6.1. *Let A be a contraction in a unital C^*-algebra $\mathfrak{A}$. Let ϕ be a functional on $\mathfrak{A}$ which attains its norm at A. Then we have the following:*

(a) *The functional ϕ and its absolute value are related via the identities*

$$\phi = |\phi| \circ R_{A^*} \quad \text{and} \quad |\phi|(I - AA^*) = 0. \tag{6.1}$$

(b) $|\phi| = \phi \circ R_A$, *and in particular, if ϕ attains its norm at a contraction B in $\mathfrak{A}$, then* $\phi \circ R_B = \phi \circ R_A$.

(c) $\phi(X(I - A^*A)) = \phi((I - AA^*)X) = 0$ *for every $X \in \mathfrak{A}$. Equivalently,* $\phi = \phi \circ R_{A^*A} = \phi \circ L_{AA^*}$.

Conversely, assume that ρ is a positive functional on $\mathfrak{A}$ with $\rho(I - AA^*) = 0$. Then the functional $\rho \circ R_{A^*}$ attains its norm at A, and ρ is its absolute value.

Proof. Since $\delta_\phi(A) = 0$, the estimates (5.1) and (5.2), with $B = A$, give

$$|\phi| = \phi \circ R_A, \qquad \phi = |\phi| \circ R_{A^*}, \qquad \phi = \phi \circ R_{A^*A}.$$

This proves (b), the first identity in (6.1), and the first half of (c). Since ϕ^* attains its norm at A^*, the last identity, applied to ϕ^* and A^*, shows that ϕ^* vanishes on $\mathfrak{A}(I - AA^*)$. For $X \in \mathfrak{A}$, we thus have

$$\phi\big((I - AA^*)X\big) = \overline{\phi^*\big(X^*(I - AA^*)\big)} = 0,$$

which completes the proof of (c). In particular, $|\phi|(I - AA^*) = \phi\big((I - AA^*)A\big) = 0$, which completes the proof of (a).

For the converse, let ρ be a positive functional on $\mathfrak{A}$ with $\rho(I - AA^*) = 0$, and let $\psi := \rho \circ R_{A^*}$. From Lemma 4.6, we note that ψ attains its norm at A and $\rho = \psi \circ R_A$. Using part (b) in the context of ψ, we conclude that $\rho = |\psi|$. □

Theorem 6.1 may thus be viewed as a polar-type factorization of ϕ within $\mathfrak{A}$, in which the positive part $|\phi|$ is canonical, and the adjoint of *any* norming contraction serves as the phase. In the context of normal functionals, we establish its precise relationship with Sakai's polar decomposition in Theorem 7.2. Note that part (a), together with the converse assertion, gives a complete description of the norm-attaining functionals on $\mathfrak{A}$. More explicitly, a functional ϕ on $\mathfrak{A}$ attains its norm at a contraction A if and only if ϕ is of the form $\rho(\,\cdot\, A^*)$ for some positive functional ρ on $\mathfrak{A}$ with $\rho(AA^*) = \rho(I)$. Moreover, in that case, $\rho = |\phi|$.

Remark 6.2 (Left-handed versions). Let ϕ be a functional on $\mathfrak{A}$ which attains its norm at a contraction A. Clearly, ϕ^* attains its norm at A^*. Thus every statement in this section has a left-handed counterpart, which is obtained by applying it to ϕ^*. Explicitly, $|\phi^*| = \phi^* \circ R_{A^*}$, and since this functional is positive, and hence hermitian, we have

$$(\phi^* \circ R_{A^*})(X) = \overline{(\phi^* \circ R_{A^*})(X^*)} = \phi(AX) \qquad (X \in \mathfrak{A}),$$

so that

$$|\phi^*| = \phi \circ L_A.$$

In particular, $\phi \circ L_A$ is positive and does not depend on the choice of A. Applying Theorem 6.1-(a) to ϕ^* gives $\phi^* = |\phi^*| \circ R_A$. Taking adjoints, we obtain the left-handed factorization $\phi = |\phi^*| \circ L_{A^*}$. We shall not state left-handed versions separately in what follows.

In a matrix algebra, the absolute value and the factorization of Theorem 6.1 are given by the polar decomposition of matrices.

Example 6.3 (Matrix algebras)**.** Let $\mathfrak{A} = M_n(\mathbb{C})$ be the C^*-algebra of complex $n \times n$ matrices, let tr denote the trace on $\mathfrak{A}$, and let $\phi(X) := \mathrm{tr}(XT)$ for $X \in \mathfrak{A}$, where $T \in \mathfrak{A}$. Every functional on $\mathfrak{A}$ is of this form. Let $T = V|T|$ be the polar decomposition of T, in which $V \in \mathfrak{A}$ is a partial isometry with $V^*V|T| = |T|$. The functional $\rho := \mathrm{tr}(\,\cdot\,|T|)$ on $\mathfrak{A}$ is positive, and

$$\rho \circ R_V = \phi, \qquad \rho(I - V^*V) = \mathrm{tr}\big((I - V^*V)|T|\big) = 0.$$

Using the converse part of Theorem 6.1 for $A = V^*$, we observe that the functional ϕ attains its norm at the partial isometry V^*, and

$$|\phi|(X) = \mathrm{tr}(X|T|) \quad (X \in \mathfrak{A}), \qquad \|\phi\| = \mathrm{tr}|T|.$$

Thus the absolute value of ϕ corresponds to the absolute value of the matrix T, and the factorization $\phi = |\phi|(\,\cdot\,V)$ is the polar decomposition $T = V|T|$.

In the following example, we show that a functional may attain its norm without doing so at any partial isometry, even when the C^*-algebra is commutative.

Example 6.4. Let $\mathfrak{A} = C(\mathbb{D})$ be the C^*-algebra of continuous complex-valued functions on the closed unit disc, and let $z \in \mathfrak{A}$ denote the coordinate function. Consider the functional ϕ on $\mathfrak{A}$ given by

$$\phi(f) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, e^{-i\theta}\, d\theta \qquad (f \in \mathfrak{A}).$$

Clearly, $\|\phi\| \le 1$ and $\phi(z) = 1$, and thus ϕ attains its norm at the contraction z. From Theorem 6.1-(b), we have

$$|\phi|(f) = \phi(fz) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, d\theta, \qquad \phi = |\phi|(\,\cdot\,\bar{z}).$$

On the other hand, ϕ does not attain its norm at any partial isometry in $\mathfrak{A}$. Indeed, since $\mathbb{D}$ is connected, the only projections in $\mathfrak{A}$ are 0 and I. Thus every non-zero partial isometry w in $\mathfrak{A}$ is a unitary, that is, a continuous mapping $w : \mathbb{D} \to \mathbb{T}$. If $\phi(w) = 1$, then the continuous function $\theta \mapsto \mathrm{Re}\big(w(e^{i\theta})e^{-i\theta}\big)$, which is bounded above by 1, has mean value 1 on $[0, 2\pi]$. Hence $w(e^{i\theta}) = e^{i\theta}$ for every θ. Thus w would be a continuous retraction of the closed disc onto its boundary circle, which is impossible [10, Theorem 55.2].

7. The Sakai polar decomposition theorem

On a von Neumann algebra, a normal functional attains its norm, by compactness, and Theorem 6.1 applies to it. Sakai's theorem follows once the adjoint A^* of the norming contraction is cut down by the support projection of the absolute value, which turns it into a partial isometry. We first recall the notion of the support projection of a positive normal functional.

Lemma 7.1 (Support)**.** *Let ρ be a positive normal functional on a von Neumann algebra $\mathscr{R}$.*

(a) *There is a smallest projection E in $\mathscr{R}$ such that $\rho(E) = \rho(I)$. It is called the* support *of ρ, and is denoted by $supp(\rho)$.*
(b) *Let $E = supp(\rho)$. If $T \ge 0$ in $\mathscr{R}$ and $\rho(T) = 0$, then $ET = TE = 0$. In particular, ρ is faithful on $E\mathscr{R}E$.*

Proof. For part (a), see [9, Section 7.2] or [17, p. 140]. For part (b), let $T \geq 0$ with $\rho(T) = 0$. For $t > 0$, let F_t be the spectral projection of T for the interval $[t, \|T\|]$. Then $tF_t \leq T$, so that $\rho(F_t) = 0$, and hence $\rho(I - F_t) = \rho(I)$. The minimality of E gives $E \leq I - F_t$, that is, $EF_t = 0$. The projections F_t increase to the range projection $R(T)$ as $t \downarrow 0$, so that $ER(T) = 0$, and therefore $ET = ER(T)T = 0$ and $TE = (ET)^* = 0$. Finally, if $X \in E\mathscr{R}E$ is positive and $\rho(X) = 0$, then $X = EXE = 0$ by what we have just proved. Thus ρ is faithful on $E\mathscr{R}E$. □

Let $E = \mathrm{supp}(\rho)$. Since $\rho(I - E) = 0$, Lemma 4.1-(c) gives

$$\rho(X) = \rho(EX) = \rho(XE) = \rho(EXE) \quad \text{for all } X \in \mathscr{R}. \tag{7.1}$$

Theorem 7.2 (Sakai [13, Theorem 2]; [14, Theorem 1.14.4]; [17, Theorem III.4.2]; [9, Theorem 7.3.2])**.** *Let $\mathscr{R}$ be a von Neumann algebra and let ω be a normal functional on $\mathscr{R}$. Then ω attains its norm, and there is a unique pair (ρ, V) consisting of a positive normal functional ρ on $\mathscr{R}$ and a partial isometry $V \in \mathscr{R}$ such that*

$$\omega = \rho \circ R_V \quad \text{and} \quad V^*V = \mathit{supp}(\rho). \tag{7.2}$$

Moreover, $\rho = |\omega|$, and

$$V = A^* \, \mathit{supp}(|\omega|)$$

for every contraction $A \in \mathscr{R}$ at which ω attains its norm. In particular, ω attains its norm at the partial isometry V^*.

Proof. Existence. Since $(\mathscr{R})_1$ is ultraweakly compact and ω is ultraweakly continuous, there is a contraction in $\mathscr{R}$ which maximizes $|\omega(\,\cdot\,)|$ on $(\mathscr{R})_1$. Multiplying it by a suitable unimodular scalar, we obtain a contraction at which ω attains its norm.

Let A be any contraction in $\mathscr{R}$ at which ω attains its norm, and let $\rho := |\omega| = \omega \circ R_A$. By Theorem 6.1-(a), ρ is a positive functional satisfying

$$\rho(I) = \|\omega\|, \qquad \rho(I - AA^*) = 0, \qquad \omega = \rho \circ R_{A^*},$$

and ρ is normal, because right multiplication by A is ultraweakly continuous. Let $E := \mathrm{supp}(\rho)$ and $V := A^*E$. Since $I - AA^* \geq 0$ and $\rho(I - AA^*) = 0$, from Lemma 7.1-(b) we have $E(I - AA^*) = 0$, whence

$$V^*V = EAA^*E = E.$$

Thus V is a partial isometry in $\mathscr{R}$ with initial projection $E = \mathrm{supp}(\rho)$. From the support identity (7.1), we have $\rho \circ R_E = \rho$. Since $R_E \circ R_{A^*} = R_{A^*E} = R_V$, we get

$$\omega = \rho \circ R_{A^*} = \rho \circ R_E \circ R_{A^*} = \rho \circ R_V,$$

which establishes (7.2). Furthermore, note that

$$\omega(V^*) = \rho(V^*V) = \rho(E) = \rho(I) = \|\omega\|,$$

that is, ω attains its norm at V^*.

Uniqueness. Let (σ, U) be a pair satisfying (7.2), and set $F := \mathrm{supp}(\sigma) = U^*U$. Since U is a contraction, $\|\omega\| \leq \|\sigma\|$. Moreover,

$$\omega(U^*) = \sigma(U^*U) = \sigma(F) = \sigma(I) = \|\sigma\|.$$

Because U^* is also a contraction, this gives $\|\sigma\| = |\omega(U^*)| \leq \|\omega\|$. Consequently, $\|\sigma\| = \|\omega\|$ and $\omega(U^*) = \|\omega\|$. In other words, ω attains its norm at U^*. Theorem 6.1-(b) and equation (7.1) now give

$$|\omega| = \omega \circ R_{U^*} = \sigma \circ R_U \circ R_{U^*} = \sigma \circ R_{U^*U} = \sigma \circ R_F = \sigma.$$

Thus $\sigma = |\omega| = \rho$ and $F = E$. In particular, $V^*V = U^*U = E$ and

$$\rho \circ R_V = \omega = \rho \circ R_U,$$

so that $\rho \circ R_{V-U} = 0$. Evaluating at $(V-U)^*$, we get

$$\rho\big((V-U)^*(V-U)\big) = 0.$$

Since $V = VE$ and $U = UE$, the positive operator $(V-U)^*(V-U)$ lies in $E\mathscr{R}E$. As ρ is faithful on $E\mathscr{R}E$ (see Lemma 7.1-(b)), we conclude that $(V-U)^*(V-U) = 0$, that is, $U = V$.

Finally, note that in the existence argument, A was an arbitrary contraction at which ω attains its norm. From the uniqueness assertion, it follows that $V = A^* \operatorname{supp}(|\omega|)$ for every such A. □

8. The bidual and the Jordan decomposition

Up to this point, the bidual of $\mathfrak{A}$ has played no role in our discussion. In the classical development, one starts with the identification of the bidual with the enveloping von Neumann algebra, applies Sakai's theorem there, and then defines the absolute value of a functional on $\mathfrak{A}$ by restriction. Our path in this section is in the opposite direction, and we obtain the bidual from the absolute value. In Proposition 8.1, we show that every functional on $\mathfrak{A}$ is a vector functional in the GNS representation of its absolute value. Using this, in Theorem 8.2 we identify the bidual of $\mathfrak{A}$ with its enveloping von Neumann algebra, and then observe that our definition of the absolute value agrees with the classical one (Proposition 8.4). We end the section with the Jordan decomposition of a hermitian functional (Proposition 8.5), whose proof makes no use of the bidual.

For a positive functional ρ on $\mathfrak{A}$, we denote its GNS triple by $(\pi_\rho, \mathscr{H}_\rho, \xi_\rho)$. Recall that ξ_ρ is a cyclic vector for the representation π_ρ of $\mathfrak{A}$ on Hilbert space $\mathscr{H}_\rho$, and $\rho(X) = \langle \pi_\rho(X)\xi_\rho, \xi_\rho\rangle$ for every $X \in \mathfrak{A}$.

8.1. **Functionals as vector functionals.** The GNS construction tells us that every state of $\mathfrak{A}$ is a vector state in a suitable representation. In the next proposition, we show that an analogous statement holds for an arbitrary functional ϕ on $\mathfrak{A}$, with two vectors in place of one. Note that when ϕ is positive, the constant sequence $(I, I, \ldots)$ is a norming sequence for ϕ, and $|\phi| = \phi$. In that case, we have $\eta = \xi$ in the proposition below, which reduces to the GNS representation of ϕ.

Proposition 8.1. *Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$, and let $(\pi, \mathscr{H}, \xi)$ be the GNS triple of $|\phi|$. Let $(A_n)_{n\in\mathbb{N}}$ be a norming sequence for ϕ. Then the sequence $\big(\pi(A_n)^*\xi\big)$ converges in norm to a vector $\eta \in \mathscr{H}$, which does not depend on the choice of the norming sequence, and*

$$\phi(X) = \langle \pi(X)\eta, \xi\rangle \quad (X \in \mathfrak{A}), \qquad \|\eta\| = \|\xi\| = \|\phi\|^{1/2}.$$

Proof. Let $\rho := |\phi|$ and $\eta_n := \pi(A_n)^*\xi$. For all m, n, since $\rho(A_nA_n^*) \le \|\rho\| = \|\phi\|$, we have

$$\|\eta_n - \eta_m\|^2 = \rho\big((A_n - A_m)(A_n - A_m)^*\big) \le 2\|\phi\| - 2\,\mathrm{Re}\,\rho(A_mA_n^*).$$

By the first estimate in (5.2), we have $|\rho(A_mA_n^*) - \phi(A_m)| \le \delta_\phi(A_n)$, and hence

$$\|\eta_n - \eta_m\|^2 \le 2\big(\|\phi\| - \mathrm{Re}\,\phi(A_m)\big) + 2\,\delta_\phi(A_n) \longrightarrow 0 \qquad (m, n \to \infty).$$

Thus (η_n) converges in norm to a vector $\eta \in \mathscr{H}$, and Theorem 5.1-(c) gives

$$\phi(X) = \lim_{n\to\infty} \rho(XA_n^*) = \lim_{n\to\infty} \langle \pi(X)\eta_n, \xi\rangle = \langle \pi(X)\eta, \xi\rangle \qquad (X \in \mathfrak{A}).$$

In particular, $\langle \eta, \pi(X)\xi\rangle = \phi(X^*)$ for every $X \in \mathfrak{A}$. Since ξ is cyclic, η is determined by ϕ. Finally, $\|\xi\|^2 = \rho(I) = \|\phi\|$. Moreover, $\|\eta\| \le \|\xi\|$ because each $\pi(A_n)^*$ is a contraction, and $\|\phi\| \le \|\eta\|\,\|\xi\|$. Hence $\|\eta\| = \|\xi\| = \|\phi\|^{1/2}$. □

8.2. **The Banach-space identification of the bidual.** Let π_u denote the *universal representation* of $\mathfrak{A}$, that is, the direct sum of the representations π_ρ over all states ρ of $\mathfrak{A}$, acting on $\mathscr{H}_u := \bigoplus_\rho \mathscr{H}_\rho$, and let $\mathscr{R}_u$ denote the von Neumann algebra generated by $\pi_u(\mathfrak{A})$, that is, the ultraweak closure of $\pi_u(\mathfrak{A})$ in $\mathcal{B}(\mathscr{H}_u)$. The next theorem identifies the bidual of $\mathfrak{A}$ with $\mathscr{R}_u$. Its proof rests on Proposition 8.1, and hence on Theorem 5.1.

Theorem 8.2 (Sherman–Takeda [15, 16]; see also [9, Section 10.1])**.** *The restriction map*

$$\Theta : (\mathscr{R}_u)_* \longrightarrow \mathfrak{A}^*, \qquad \Theta(\omega) := \omega \circ \pi_u,$$

is an isometric linear bijection. In other words, every functional ϕ on $\mathfrak{A}$ is of the form $\phi = \overline{\phi} \circ \pi_u$ for a unique normal functional $\overline{\phi}$ on $\mathscr{R}_u$, and $\|\overline{\phi}\| = \|\phi\|$. Consequently, the transpose of Θ identifies the bidual $\mathfrak{A}^{**}$ isometrically with the dual of $(\mathscr{R}_u)_*$, which is $\mathscr{R}_u$ [9, Section 7.4]*. Under this identification, the canonical embedding of $\mathfrak{A}$ into $\mathfrak{A}^{**}$ becomes π_u, and the weak* topology of $\mathfrak{A}^{**}$ becomes the ultraweak topology of $\mathscr{R}_u$.*

Proof. The map Θ is linear and contractive, and it is injective because $\pi_u(\mathfrak{A})$ is ultraweakly dense in $\mathscr{R}_u$. Let ϕ be a non-zero functional on $\mathfrak{A}$. The GNS representation of $|\phi|$ is that of the state $\|\phi\|^{-1}|\phi|$, which is a direct summand of π_u. By Proposition 8.1, there are therefore vectors $\eta, \xi \in \mathscr{H}_u$ with $\|\eta\|\,\|\xi\| = \|\phi\|$ such that $\phi(X) = \langle \pi_u(X)\eta, \xi\rangle$ for every $X \in \mathfrak{A}$. The functional $\omega := \langle\, \cdot\, \eta, \xi\rangle$ on $\mathscr{R}_u$ is normal, being ultraweakly continuous by the definition of the ultraweak topology (Remark 2.2), and satisfies $\Theta(\omega) = \phi$ and $\|\omega\| \le \|\eta\|\,\|\xi\| = \|\phi\|$. Thus Θ is surjective. Since Θ is injective, every $\omega \in (\mathscr{R}_u)_*$ arises in this manner from $\phi = \Theta(\omega)$, and it follows that $\|\omega\| \le \|\Theta(\omega)\|$. Thus Θ is isometric.

For the remaining assertions, note that the transpose Θ^* is an isometric linear bijection from $\mathfrak{A}^{**}$ onto the dual of $(\mathscr{R}_u)_*$, and a homeomorphism with respect to the weak* topologies. Regarding $X \in \mathfrak{A}$ as an element of $\mathfrak{A}^{**}$, we observe that $\Theta^*(X)$ is the functional $\omega \mapsto \omega(\pi_u(X))$ on $(\mathscr{R}_u)_*$, which corresponds to the element $\pi_u(X)$ of $\mathscr{R}_u$. □

Remark 8.3. The usual proof of Theorem 8.2 starts with the observation that every state of $\mathfrak{A}$ is a vector state in the universal representation. There, one writes an arbitrary functional as a linear combination of states, and derives the equality $\|\overline{\phi}\| = \|\phi\|$ from the Kaplansky density theorem. In our approach, Proposition 8.1 directly tells us that every functional on $\mathfrak{A}$ is a vector functional in the universal representation, and the equality of norms follows from the equality $\|\eta\|\,\|\xi\| = \|\phi\|$, that is, from the approximate polar decomposition in Theorem 5.1-(c). It is worth pointing out that the Sherman–Takeda theorem, in its full

form, identifies $\mathfrak{A}^{**}$ equipped with its Arens products with $\mathscr{R}_u$ as a von Neumann algebra [15, 16]. For our purposes, the Banach-space identification proved here suffices.

In the literature, the absolute value of a functional ϕ on $\mathfrak{A}$ is defined as the restriction to $\mathfrak{A}$ of the positive part in Sakai's polar decomposition of its normal extension $\overline{\phi}$ (see [17, Chapter III, § 4]). In the next proposition, we show that this agrees with Definition 5.2.

Proposition 8.4 (Compatibility with the classical absolute value)**.** *Let ϕ be a functional on a unital C^*-algebra $\mathfrak{A}$, and let $\overline{\phi}$ be its normal extension to the enveloping von Neumann algebra $\mathscr{R}_u$ (Theorem 8.2). Then*

$$|\overline{\phi}| \circ \pi_u = |\phi|.$$

Proof. From Theorem 7.2, we note that the positive part in Sakai's polar decomposition of $\overline{\phi}$ is the absolute value $|\overline{\phi}|$ in the sense of Definition 5.2. Since $\overline{\phi} \circ \pi_u = \phi$ and $\|\overline{\phi}\| = \|\phi\|$, using the restriction principle (Proposition 5.5) for the $*$-homomorphism $\pi_u : \mathfrak{A} \to \mathscr{R}_u$, we conclude that $|\overline{\phi}| \circ \pi_u = |\phi|$. □

8.3. **The Jordan decomposition.** A functional ϕ on $\mathfrak{A}$ is said to be *hermitian* if $\phi^* = \phi$. Recall that the Jordan decomposition of a real measure μ is given by the positive measures $\frac{1}{2}(|\mu| \pm \mu)$. In the next proposition, we show that the same formula, with the absolute value in place of the total variation, yields the Jordan decomposition of a hermitian functional.

Proposition 8.5 (Jordan decomposition)**.** *Let $\mathfrak{A}$ be a unital C^*-algebra and let ϕ be a hermitian functional on $\mathfrak{A}$. Then*

$$\phi_+ := \tfrac{1}{2}\big(|\phi| + \phi\big) \qquad \text{and} \qquad \phi_- := \tfrac{1}{2}\big(|\phi| - \phi\big)$$

are positive functionals with $\phi = \phi_+ - \phi_-$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\|$. Moreover, if $\phi = \rho_1 - \rho_2$ for positive functionals ρ_1, ρ_2 with $\|\rho_1\| + \|\rho_2\| = \|\phi\|$, then $\rho_1 = \phi_+$ and $\rho_2 = \phi_-$.

Proof. Since ϕ is hermitian, we have $\phi(X^*) = \overline{\phi(X)}$ for every $X \in \mathfrak{A}$, and in particular, $\phi\big(\frac{1}{2}(A + A^*)\big) = \operatorname{Re}\phi(A)$ for every $A \in \mathfrak{A}$. Hence, if $(A_n)_{n\in\mathbb{N}}$ is a norming sequence for ϕ, then the self-adjoint contractions $\frac{1}{2}(A_n + A_n^*)$ satisfy $\phi\big(\frac{1}{2}(A_n + A_n^*)\big) = \operatorname{Re}\phi(A_n) \to \|\phi\|$. Thus ϕ has a norming sequence consisting of self-adjoint contractions.

We first show that ϕ_+ and ϕ_- are positive. To this end, we begin by reducing to the case in which ϕ attains its norm. Let $\mathbf{A} = (A_n)_{n\in\mathbb{N}}$ be a norming sequence for ϕ consisting of self-adjoint contractions, let Φ be a functional on $\ell^\infty(\mathfrak{A})$ as in Lemma 4.9, and let $\Psi := \frac{1}{2}(\Phi + \Phi^*)$. Clearly, Ψ is hermitian. Since ϕ is hermitian, we have $\Psi \circ \iota = \phi$, whence $\|\phi\| \le \|\Psi\| \le \|\Phi\| = \|\phi\|$. As $\mathbf{A}$ is self-adjoint, we have $\Psi(\mathbf{A}) = \operatorname{Re}\Phi(\mathbf{A}) = \|\phi\|$. In other words, Ψ attains its norm at the self-adjoint contraction $\mathbf{A}$. Using the restriction principle (Proposition 5.5) for ι and Ψ, we get $|\Psi| \circ \iota = |\phi|$, and hence $\phi_\pm = \frac{1}{2}(|\Psi| \pm \Psi) \circ \iota$. Since ι is a $*$-homomorphism, it follows that ϕ_+ and ϕ_- are positive whenever $\frac{1}{2}(|\Psi| + \Psi)$ and $\frac{1}{2}(|\Psi| - \Psi)$ are positive. Thus, in proving that ϕ_+ and ϕ_- are positive, we may replace $\mathfrak{A}$ and ϕ by $\ell^\infty(\mathfrak{A})$ and Ψ, respectively, and assume without loss of generality that ϕ attains its norm at a self-adjoint contraction B.

From Theorem 6.1-(b) and Remark 6.2, we have $|\phi| = \phi \circ R_B$ and $|\phi| = |\phi^*| = \phi \circ L_B$. Consequently,

$$|\phi|(BX) = \phi(BXB) = |\phi|(XB) \qquad (X \in \mathfrak{A}).$$

By approximating with polynomials, we observe that $|\phi|(f(B)X) = |\phi|(Xf(B))$ for every continuous function f on the spectrum of B. Let $B_\pm := \frac{1}{2}(I \pm B)$, and note that B_+

and B_- are positive contractions. From Theorem 6.1-(a), we have $\phi = |\phi| \circ R_B$, so that $\phi_\pm = |\phi| \circ R_{B_\pm}$. Since $B_\pm^{1/2}$ is a continuous function of B, from the commutation relation above we have $|\phi|\big(YB_\pm^{1/2}\big) = |\phi|\big(B_\pm^{1/2}Y\big)$ for every $Y \in \mathfrak{A}$. For a positive element X of $\mathfrak{A}$, taking $Y = XB_\pm^{1/2}$, we get

$$\phi_\pm(X) = |\phi|\big(XB_\pm^{1/2}B_\pm^{1/2}\big) = |\phi|\big(B_\pm^{1/2}XB_\pm^{1/2}\big) \geq 0,$$

since $B_\pm^{1/2}XB_\pm^{1/2} \geq 0$ and $|\phi|$ is a positive functional.

This proves that ϕ_+ and ϕ_- are positive. As $\phi_+ + \phi_- = |\phi|$, Lemma 4.1-(b) tells us that $\|\phi_+\| + \|\phi_-\| = \phi_+(I) + \phi_-(I) = |\phi|(I) = \|\phi\|$.

For the uniqueness assertion, let ρ_1, ρ_2 be as in the statement, and let $(A_n)_{n\in\mathbb{N}}$ be a norming sequence for ϕ consisting of self-adjoint contractions. Then

$$\rho_1(I - A_n) + \rho_2(I + A_n) = \|\phi\| - \phi(A_n) \longrightarrow 0,$$

and both terms on the left-hand side are non-negative, so that each of them tends to 0. Since $(I - A_n)^2 \leq 2(I - A_n)$, using Lemma 4.1-(a), for every $X \in \mathfrak{A}$ we have

$$|\rho_1(X(I - A_n))|^2 \leq \rho_1(XX^*)\,\rho_1\big((I - A_n)^2\big) \leq 2\,\|\rho_1\|\,\|X\|^2\,\rho_1(I - A_n).$$

It follows that $\rho_1 \circ R_{A_n} \to \rho_1$ in norm. In a similar way, using the inequality $(I + A_n)^2 \leq 2(I + A_n)$ for ρ_2, we get

$$|\rho_2(X(I + A_n))|^2 \leq \rho_2(XX^*)\,\rho_2\big((I + A_n)^2\big) \leq 2\,\|\rho_2\|\,\|X\|^2\,\rho_2(I + A_n),$$

whence $\rho_2 \circ R_{A_n} \to -\rho_2$ in norm. Thus $\phi \circ R_{A_n} \to \rho_1 + \rho_2$ in norm, and from Theorem 5.1-(c), we conclude that $|\phi| = \rho_1 + \rho_2$. In combination with $\phi = \rho_1 - \rho_2$, this gives us $\rho_1 = \phi_+$ and $\rho_2 = \phi_-$. □

9. The extreme points of the unit ball

In the preceding sections, we fixed a functional and studied the contractions at which it attains its norm. We now fix a point of the unit ball instead, and study the discs centred at it (§ 9.1). This leads to the characterization of the extreme points of the unit ball (§ 9.2).

9.1. **Directions of discs.**

Definition 9.1. Let $\mathfrak{A}$ be a C^*-algebra. For a contraction $Z \in \mathfrak{A}$, the set of directions of the discs in $(\mathfrak{A})_1$ centred at Z is denoted by

$$\mathrm{Dir}_1(Z) := \{D \in \mathfrak{A} :\ \|Z + \lambda D\| \leq 1 \text{ for every } \lambda \in \mathbb{D}\}.$$

The subscript 1 refers to the unit ball $(\mathfrak{A})_1$.

In this notation, Proposition 3.4 asserts that $I - H \in \mathrm{Dir}_1(H)$, that $\frac{1}{3}(I - AA^*)X(I - A^*A)$ and $A(I - |A|)$ lie in $\mathrm{Dir}_1(A)$, and that $(I - Q)X(I - P) \in \mathrm{Dir}_1(W)$. Our main goal in this subsection is to give a complete description of $\mathrm{Dir}_1(W)$ for a partial isometry W (see Theorem 9.4). We begin with a necessary condition for an element of $\mathfrak{A}$ to be the direction of a disc in $(\mathfrak{A})_1$ centred at a given contraction A. Note that only the two points $\lambda = \pm 1$ of the disc are used in its proof. In fact, the hypothesis $\|A \pm D\| \leq 1$ tells us that the endpoints of the segment joining $A - D$ and $A + D$ lie in $(\mathfrak{A})_1$, and hence the whole segment lies in $(\mathfrak{A})_1$ by convexity. Thus the inequalities in Proposition 9.2 are necessary not only for discs, but already for segments.

Proposition 9.2. *Let A be a contraction in a unital C^*-algebra $\mathfrak{A}$, and let D be an element of $\mathfrak{A}$ such that $\|A \pm D\| \leq 1$. Then*

$$A^*A + D^*D \leq I \qquad \text{and} \qquad AA^* + DD^* \leq I.$$

In particular, these inequalities hold for every $D \in \mathrm{Dir}_1(A)$.

Proof. Since $A + D$ and $A - D$ are contractions, we have

$$A^*A + D^*D = \frac{1}{2}\big((A+D)^*(A+D) + (A-D)^*(A-D)\big) \leq I.$$

Since $\|A^* \pm D^*\| = \|A \pm D\| \leq 1$, the same argument, applied to A^* and D^*, gives us $AA^* + DD^* \leq I$. □

For a partial isometry, this necessary condition and the sufficient condition provided by the disc (D3) coincide. We first record an elementary lemma.

Lemma 9.3. *Let $\mathfrak{A}$ be a unital C^*-algebra. Let $D \in \mathfrak{A}$ and let P be a projection in $\mathfrak{A}$. If $D^*D \leq I - P$, then $DP = 0$. Similarly, if $DD^* \leq I - P$, then $PD = 0$.*

Proof. If $D^*D \leq I - P$, then

$$0 \leq (DP)^*(DP) = PD^*DP \leq P(I-P)P = 0,$$

so that $DP = 0$. Replacing D with D^* gives us the other assertion. □

Theorem 9.4. *Let $\mathfrak{A}$ be a unital C^*-algebra. Let $W \in \mathfrak{A}$ be a partial isometry with initial projection $P = W^*W$ and final projection $Q = WW^*$. For $D \in \mathfrak{A}$, the following are equivalent:*

(i) *$D \in \mathrm{Dir}_1(W)$, that is, $W + \mathbb{D}\,D$ is a disc in $(\mathfrak{A})_1$;*
(ii) *$\|W \pm D\| \leq 1$, that is, the segment joining $W - D$ and $W + D$ lies in $(\mathfrak{A})_1$;*
(iii) *D is a contraction in $(I-Q)\mathfrak{A}(I-P)$.*

In particular,

$$\mathrm{Dir}_1(W) = \left((I-Q)\mathfrak{A}(I-P)\right)_1. \tag{9.1}$$

Proof. (i) ⇒ (ii). This is clear.

(ii) ⇒ (iii). By Proposition 9.2, we have $D^*D \leq I-P$ and $DD^* \leq I-Q$. From Lemma 9.3, we observe that $DP = 0 = QD$, that is, $D = (I-Q)D(I-P)$. Moreover, $\|D\|^2 = \|D^*D\| \leq \|I-P\| \leq 1$.

(iii) ⇒ (i). If D is a contraction in $(I-Q)\mathfrak{A}(I-P)$, then $D = (I-Q)D(I-P)$, and $W + \mathbb{D}\,D$ is the disc (D3) of Proposition 3.4 with $X = D$. □

Remark 9.5. In the language of JB^*-triples, a partial isometry W is a tripotent of $\mathfrak{A}$, and $(I-Q)\mathfrak{A}(I-P)$ is its Peirce-zero space. Edwards and Rüttimann [4] show that the non-empty weak*-closed faces of the unit ball of a JBW^*-triple are the translates of Peirce-zero unit balls by tripotents. When $\mathfrak{A}$ is a von Neumann algebra, the translate corresponding to W is the set $W + \left((I-Q)\mathfrak{A}(I-P)\right)_1$, which is $W + \mathrm{Dir}_1(W)$ by (9.1). For a general C^*-algebra, the norm-closed faces of the unit ball are described by Akemann and Pedersen [1]. Theorem 9.4 is thus a special case of a known picture. We include it for its short proof, and because the equivalence of (i) and (ii) shows that segments and discs centred at a partial isometry have the same directions. This is the reason why extreme points and complex extreme points of $(\mathfrak{A})_1$ coincide (Theorem 9.7).

9.2. **Extreme points.** Using the description of the segments and discs in $(\mathfrak{A})_1$ centred at a partial isometry (Theorem 9.4), we prove Kadison's characterization of the extreme points of $(\mathfrak{A})_1$, together with the Akemann–Russo theorem that these are precisely the complex extreme points (Theorem 9.7). For the setting of J^*-algebras, see Harris [6].

Lemma 9.6. *A contraction A in a unital C^*-algebra $\mathfrak{A}$ is a partial isometry if and only if $A(I - |A|) = 0$.*

Proof. Let $H := |A|$. If A is a partial isometry, then A^*A is a projection, so that $H = A^*A$ and $A = AA^*A = AH$. Conversely, if $A(I - H) = 0$, then

$$0 = (I - H)A^*A(I - H) = H^2(I - H)^2.$$

Thus the function $t \mapsto t^2(1 - t)^2$ vanishes on $\mathrm{sp}(H)$, so that $\mathrm{sp}(H) \subseteq \{0, 1\}$. It follows that H, and hence $A^*A = H^2$, is a projection. Thus A is a partial isometry, as desired. □

Theorem 9.7 (Kadison [7, Theorem 1]; Akemann–Russo [2, Theorem 3.2])**.** *For a contraction A in a unital C^*-algebra $\mathfrak{A}$, the following are equivalent:*

(i) *A is an extreme point of $(\mathfrak{A})_1$;*
(ii) $\mathrm{Dir}_1(A) = \{0\}$*, that is, A is a complex extreme point of $(\mathfrak{A})_1$ in the sense of Thorp and Whitley [18];*
(iii) *A is a partial isometry and*

$$(I - AA^*)\mathfrak{A}(I - A^*A) = \{0\}. \tag{9.2}$$

Proof. (i) ⇒ (ii). This is a restatement of Lemma 3.2-(c).

(ii) ⇒ (iii). By (D4), we have $A(I - |A|) \in \mathrm{Dir}_1(A) = \{0\}$, so that A is a partial isometry by Lemma 9.6. From Theorem 9.4, we have

$$((I - AA^*)\mathfrak{A}(I - A^*A))_1 = \mathrm{Dir}_1(A) = \{0\},$$

which gives us (9.2).

(iii) ⇒ (i). Suppose that $A = \frac{1}{2}(B + C)$ for some $B, C \in (\mathfrak{A})_1$, and let $D := \frac{1}{2}(B - C)$. Then $A + D = B$ and $A - D = C$ are contractions, so the implication (ii) ⇒ (iii) of Theorem 9.4 shows that $D \in (I - AA^*)\mathfrak{A}(I - A^*A) = \{0\}$. Thus $B = C = A$, and we conclude that A is an extreme point of $(\mathfrak{A})_1$. □

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