The real-rootedness threshold for set partitions counted by large blocks

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Abstract

Mark each block of size at least j in an unordered partition of an n-element set. The resulting polynomial has only real zeros for every n if and only if $1 \le j \le 8$. Equivalently, the number of marked blocks in a uniform partition is a sum of independent Bernoulli variables in distribution for every n precisely at those thresholds. For $j = 9$ the first Laguerre inequality already fails in row 18517 at $-1/2$. Every $j \ge 9$ has a failing row, although for each fixed $j \ge 4$ all sufficiently late rows have simple negative zeros and interlace strictly over every gap from one to j. At $j = 4$ strict interlacing holds throughout. For $5 \le j \le 8$, integer-phase sign tests and certified contour bounds meet the finite root isolations at row 4097. Whether the directed interlacing holds in every row at these four thresholds remains open.

1 The statistic and the two quantifiers

Write Π_n for the set of unordered partitions of $[n]$. For $j \ge 2$ define

$$P_{n,j}(t) = \sum_{\pi \in \Pi_n} t^{\#\{C \in \pi:\ |C| \ge j\}}, \qquad A_j(z) = \sum_{h=1}^{j-1} \frac{z^h}{h!}, \qquad B_j(z) = e^z - 1 - A_j(z).$$

The labelled exponential formula gives $P_{n,j}(t) = n![z^n] \exp(A_j(z) + tB_j(z))$. A row of degree $m = \lfloor n/j \rfloor$ has a positive coefficient in each degree $0, \dots, m$. A partition with exactly k marked blocks can be made with k blocks of size j and the remaining labels as singletons. Hence every real zero is negative.

Zhan studied these polynomials and conjectured consecutive interlacing [1, Theorem 3.3 and p. 13]. Alexandersson asked about all rows and all gaps $1, \dots, j$ in his MathOverflow question [2]. His $Q_{n,j} = tP_{n,j}$ has an additional common zero. Here simplicity and strict interlacing refer to $P_{n,j}$. The statistic counts all partitions, permitting arbitrarily many blocks smaller than j. Ordering the blocks, or excluding particular small sizes, changes the polynomial and the status of the problem [4].

Threshold	Established behavior of the ordinary unordered rows
$1, 2, 3$	All-row real-rootedness was known [3, 4, 6].
4	All rows and all four directed gaps, Theorem 3.
$5, 6, 7, 8$	Every row has simple negative zeros, Theorem 4.
	All gaps through row 4096 and eventually.
$j \ge 9$	Finitely many failing rows, at least one for each j.

The eventual theorem treats j as fixed while n tends to infinity. Its starting row is allowed to depend on j. The negative theorem chooses a row as a function of j. These quantifiers allow failure at an individual row together with strict interlacing of the entire tail.

Alexandersson's threshold-four study already obtained the density-one result for negative zeros, the compact-radius saddle estimate and the angular minimum [7]; his research program records Pfannerer's threshold-three asymptotic approach [5]. An earlier version of the present work matched both ends and proved eventual four-gap interlacing at $j = 4$ [8]. The proof below closes the finite-to-asymptotic gap at that threshold, treats every fixed $j \geq 4$, and separates the thresholds admitting certified counterexamples. The real point tests in the next section can be checked before undertaking any complex asymptotic analysis.

2 Two integer certificates at threshold nine

Proposition 1. *Let $P = P_{n,9}$. For each of the two pairs*

$$(n, x) = (18517, -1/2), \qquad (20000, -10/19),$$

one has $5P'(x)^2 < P(x)P''(x)$. Consequently each row has at least one nonreal conjugate pair.

Proof. For a real polynomial with all zeros a_i real, logarithmic differentiation gives

$$P'(x)^2 - P(x)P''(x) = P(x)^2 \sum_i (x - a_i)^{-2} \geq 0$$

away from the zeros, and the inequality extends to them by continuity. It remains to check the reversed strict inequality in the proposition. Use $P_0 = 1$, zero rows at negative indices, and Zhan's recurrence in the form

$$P_{m+1} = P_m + tP'_m + \sum_{h=1}^{7} \binom{m}{h} P_{m-h} + t\binom{m}{8} P_{m-8}, \qquad P_m = P_{m,9}.$$

Indeed the new label either forms a singleton, joins an already marked block, or joins $h = 1, \ldots, 8$ old labels that formed an unmarked block. Only $h = 8$ creates a new mark.

Here is the complete numerical test on the resulting integer coefficients $c_d, \ldots, c_0$ at $x = a/b$. Starting with $(Y, V, W, Q) = (0, 0, 0, 1)$, replace the tuple for each successive c by

$$(aY + cQ,\ aV + Y,\ aW + 2V,\ bQ).$$

The first three outputs are the value and first two derivatives at a of $b^d P(X/b)$. Hence the ratio $V^2/(YW)$ is unchanged by homogenization. Exact integer division gives

n	d	$\lfloor 10^6 V^2/(YW) \rfloor$	YW
18517	2057	191068	> 0
20000	2222	198809	> 0.

The three updates and the integer recurrence comprise the direct verifier `exact_rows.py`. A second implementation, `integer_witness.py`, encloses the same comparisons using only Python integers and rational error bounds for a finite Fourier sum. Its error analysis is included with the certificate. □

There are two other ways to recognize the failure at row 20000. At $x = -527/1024$ the same integer coefficients give

$$|P(x + i/1024)| < (8371/10000)|P(x)|.$$

Each real linear factor has nondecreasing modulus on such an upwards vertical ray. The strict decrease rules out a product of real linear factors. To locate a zero, take $c = -0.51445632717 + 0.00113548603i$, $\delta = 10^{-10}$. For $F(t) = 9.04^n [z^n] e^{A_9 + tB_9}$, the published disk check [9] encloses $|F(c)|/|F'(c)| < 10^{-12}$ and $\delta \sup_{|t-c| \leq \delta} |F''(t)|/(2|F'(c)|) < 10^{-5}$. It uses 2^{17} samples on the

radius-9.04 circle and a Cauchy tail on the radius-10.1 circle. For the k-th derivative the tail is bounded by

$$B_9(10.1)^k e^{A_9(10.1)+(|c|+\delta)B_9(10.1)} \frac{(9.04/10.1)^{n+2^{17}}}{1-(9.04/10.1)^{2^{17}}}.$$

On the boundary of the disk, Taylor's inequality and these bounds give $|F(t)-F'(c)(t-c)| < |F'(c)(t-c)|$. Rouché's theorem counts exactly one zero there, necessarily simple, and conjugation gives the reflected zero. The integer sign test suffices for the counterexample without locating either zero. A separate calculation at $n = 18682,\ t = -1/2 + i/100000$ gives $\Im(P'(t)\overline{P(t)}) > 0$ [11]; a real-rooted polynomial would have $\Im(P'/P) < 0$ in the upper half-plane.

3 A failing row at every threshold from nine onwards

Theorem 2. *For every $j \geq 9$ there exists n such that $P_{n,j}$ has a nonreal zero. For $j \geq 23$ one may take*

$$r = 100j/151, \qquad n = \lfloor re^r + 1/2 \rfloor.$$

Proof. Write $T(w) = P_{n,j}(1+w)$. If all roots of $P_{n,j}$ are real, the roots of T are $-a_i$ with $a_i > 0$. Hence, for $\Im w > 0$,

$$\Im \frac{wT'(w)}{T(w)} = \sum_i \frac{a_i \Im w}{|w+a_i|^2} > 0. \tag{1}$$

We exhibit points of the upper half-plane where the denominator is nonzero and the left side is negative.

The interval of thresholds $j \geq 23$. Let $\kappa = 151/100,\ s = 1/r,\ \mu = e^r,\ B = B_j$, so $0 < s \leq 0.067$. With $z = re^{sv}$ put

$$L(v) = \exp((e^{sv}-1)/s), \qquad C(v) = B(re^{sv})/B(r), \qquad F(v) = L(v) + sWC(v) - \frac{n}{r\mu}v.$$

Set $w = \mu s W/B(r)$. On a vertical line in v, integration over a period $2\pi r$ traces a circle about zero in z. The coefficient integrals for T and $wT'/(\mu s)$ have a common nonzero factor and integrands $e^{\mu F}$ and $WCe^{\mu F}$, respectively, in the same measure dv/i.

Take $v_* = 2\pi i + sd$, choosing d on the four consecutive intervals with endpoints $0, 0.0154, 0.025, 0.04, 0.067$ as $-22+27i, -8+10i, -3+2i, -2-2i$, respectively. Either choice works at a shared endpoint. Define

$$W = \frac{n/(r\mu) - L'(v_*)}{sC'(v_*)}, \qquad a = \Re v_*.$$

This choice makes $F'(v_*) = 0$. The following outward-rounded bounds hold on the full parameter intervals:

$$\Im W > 0, \qquad |W| < 100, \qquad \Im(WC(v_*)) < -0.02. \tag{2}$$

For $|y - \Im v_*| \geq 0.35,\ |y| \leq \pi r$, the real part of $F(a+iy) - F(v_*)$ is less than $-0.002s$. Throughout $|y - \Im v_*| \leq 0.35$ the derivative bounds are

$$\Re F'' > 0.4, \quad |F''| < 5, \quad |F'''| < 15, \quad |WC'| < 100, \quad |WC''| < 240.$$

For clarity, we specify the removable divisions and the infinite tails in this finite certificate. Set $p_0 = 1$, $p_h = \prod_{\ell=1}^{h} (\kappa + \ell s)^{-1}$. Factoring the first term from the exponential tail gives, for $0 \leq q \leq 3$,

$$C^{(q)}(v) = e^{\kappa v} \frac{\sum_{h\geq 0} p_h (\kappa + hs)^q e^{hsv}}{\sum_{h \geq 0} p_h}.$$

At $\Re v = a \le 0$ the numerator sum starting at 108 has absolute value $< 10^{-14}$, by comparison with $\kappa^{-h}(\kappa + 0.067h)^3$ and its successive ratio. The same estimate bounds the denominator tail. With $E_\ell(z) = \sum_{m\ge 0} z^m/(m+\ell)!$ put $p_* = d + v_*^2 E_2(sv_*)$. Since $L(v_*) = e^{sp_*}$,

$$W = \frac{(n - r\mu)/\mu - (p_* + v_*)E_1(s(p_* + v_*))}{C'(v_*)}.$$

The rounding term lies in $[-\frac{1}{2}e^{-1/s}, \frac{1}{2}e^{-1/s}]$. These formulas extend to the endpoint $s = 0$. For $\ell = 1, 2$ and $|z| < 1.9$ the first 31 terms of E_ℓ leave an error less than 10^{-18}; for other arguments the exponential identity or the bound $e^{|z|}/\ell!$ applies.

Writing $y = 2\pi k + \theta,\ v = a + iy,\ p = \Re d + v^2 E_2(sv)$ gives

$$\frac{\Re(L(v) - L(v_*))}{s} = -\frac{2\sin^2(\theta/2)}{s} + \Re\{e^{i\theta} pE_1(sp) - p_* E_1(sp_*)\}.$$

For an upper bound on the first term use the right endpoint of the s-interval. The linear drift in F has the same real part on the entire vertical line. For $25 \le |y| \le \pi r$, $|L(a+iy)| \le |L(a+25i)|$ and $|C(a+iy)| \le C(a)$. The remaining comparisons belong to a bounded rectangle. Subdivision in `check_failure_threshold_continuum.py` certifies (2) on $72, 27, 40, 223$ parameter subintervals in the four respective ranges. Its local derivative bounds use a closed interval slightly wider than the discarded central portion.

Normalize both integrals by $e^{\mu F(v_*)}$. Put $h = y - \Im v_*$, $D = F''(v_*)$, $J = \exp(\mu(F(v_* + ih) - F(v_*)))$ and $g = \exp(-\mu D h^2/2)$. On $|h| \le 0.35$, Taylor's formula and $\Re F'' > 0.4$ give

$$|J|, |g| \le e^{-0.2\mu h^2}, \qquad |J - g| \le (15/6)\mu |h|^3 e^{-0.2\mu h^2}.$$

Integration bounds the last error by $62.5/\mu$. The full Gaussian integral has modulus at least $\sqrt{2\pi/(5\mu)}$, and its tails cost at most $2e^{-0.2\mu(0.35)^2}/(0.4\mu \cdot 0.35)$. The rest of the true contour contributes at most $2\pi r e^{-0.002\mu/r} < \mu^{-2}$, since $0.002e^r/r > 3r$ and $2\pi r < e^r$ for $r \ge 1/0.067$. As $\mu > 3 \cdot 10^6$, the actual denominator has modulus $> \mu^{-1/2}$.

Subtract $WC(v_*)$ times this denominator from the normalized numerator. The linear term $ihWC'(v_*)$ integrates to zero against g on the symmetric central arc. There the amplitude Taylor remainder is bounded by $120h^2$; the product of the linear term with $J - g$ costs at most $250\mu h^4 e^{-0.2\mu h^2}$. The two Gaussian moments and the complementary arc give a total bounded by

$$\frac{120\sqrt{2\pi}}{(0.4\mu)^{3/2}} + \frac{750\mu\sqrt{2\pi}}{(0.4\mu)^{5/2}} + 200\mu^{-2} < 20000\mu^{-3/2} + 200\mu^{-2}.$$

Dividing by the denominator gives an error < 0.007. Thus $\Im(wT'/(\mu sT)) < -0.013$, contradicting (1).

The remaining thresholds. Proposition 1 gives $j = 9$. For $j = 10, \dots, 22$, in increasing order, the following rows suffice:

$$26685,\ 31578,\ 42720,\ 94133,\ 206193,\ 449351,\ 974908,$$
$$2106889,\ 4537476,\ 9741919,\ 20857871,\ 44545965,\ 94920679.$$

The file `check_failure_threshold_base.py` specifies the upper-half-plane rational w, two radii $r_c > r_o > 0$, angle $c \in (0, \pi)$ and anchor z_0 for each row. Integrate on the radius-r_c central arc $|\arg z| \le c$, the radius-r_o complement and their two radial connectors. Set

$$S = e^z + wB(z) - e^{z_0} - wB(z_0) - n\log(z/z_0), \qquad V = w(B(z) - B(z_0))/10000.$$

The common factor cancels in the two contour integrals. The desired ratio is $wB(z_0)/10000 + \int Ve^S\, dz/z \,/ \int e^S\, dz/z$. Subarcs and connectors where $\Re S < -32$ contribute at most

$e^{-32}\int |dz/z|$ to the denominator and that quantity times $|w|(B(r_c)+|B(z_0)|)/10000$ to the centered numerator.

On any retained angular interval $[m-h, m+h]$, enclose $K \ge \sup|S''|$, $b \ge \sup|V'|$ using $z\partial_z$. Put $q = Kh^2/2$, $d = hS'(m)$, $E = 2he^{\Re S(m)+|\Re d|}$. The linearized integral is $I = 2he^{S(m)}\sinh(d)/d$. Its error is at most $e = E(e^q-1)$; if $|d| < 10^{-5}$, replace the quotient by $1+d^2/6$ and add $10^{-21}E$ to e. The error of the numerator approximation $V(m)I$ is at most $|V(m)|e + Ee^q bh$. Subdivision to $q, bh < 10^{-4}$ on the retained intervals excludes zero from the denominator and gives $\Im(wT'/(10000T)) < -9$ in every one of the thirteen cases. This completes the proof. □

4 Negative zeros and the directed interlacing window

Let $P_{n,j}(t)$ count the partitions of $[n]$ into unordered blocks, with weight t for each block of size at least j. Put $m = \lfloor n/j \rfloor$. Taking k disjoint blocks of size j and leaving all remaining elements as singletons shows that every coefficient from 0 to m is positive. We order negative real zeros from left to right. Write $P \prec Q$ when all the zeros are simple and negative and their lists $(a_i), (b_i)$ satisfy

$$\begin{aligned} &a_1 < b_1 < \cdots < a_m < b_m, && \deg Q = \deg P = m, \\ &b_1 < a_1 < b_2 < \cdots < a_m < b_{m+1}, && \deg Q = m+1. \end{aligned}$$

For a constant the list is empty. The polynomials $tP_{n,j}$ in Alexandersson's formulation share their zero at the origin.

Theorem 3. *For $j = 4$ and every $n \ge 0$, all m zeros of $P_{n,4}$ are simple and negative, and $P_{n,4} \prec P_{n+l,4}$ for $1 \le l \le 4$. For every fixed integer $j \ge 4$ there is an integer $N(j)$ such that, for all $n \ge N(j)$, all m zeros of $P_{n,j}$ are simple and negative and $P_{n,j} \prec P_{n+l,j}$ for $1 \le l \le j$.*

Taking $l = j+1$ can increase the degree by two. Thus j is the largest lag for which the degree condition permits every sufficiently late pair. For $j = 5, 6, 7, 8$ the same strict inequalities also hold whenever the larger row is at most 4096. The proof for general j gives no numerical upper bound for $N(j)$.

A sign criterion. If $a_1 < \cdots < a_m < 0$ are the zeros of $P_{n,j}$ and

$$\operatorname{sgn} P_{n+l,j}(a_k) = (-1)^{m-k+1} \qquad (1 \le k \le m), \tag{3}$$

then every gap (a_k, a_{k+1}) and $(a_m, 0)$ contains a later-row zero. If the degree increases, the leading sign gives a further zero in $(-\infty, a_1)$. These disjoint gaps exhaust the later degree, so all the inequalities are strict. For $m = 0$, the positive constant and leading coefficients give the convention directly.

4.1 A fixed threshold

Put $p = j-1$, $A(z) = \sum_{i=1}^{p} z^i/i!$, $B(z) = e^z - 1 - A(z)$, $D = z\partial_z$, $h = DA$, $H = DB$, and $K = DH$. The exponential formula gives $P_{n,j}(t) = n![z^n]\exp(A+tB)$. For $t = -x < 0$, write $E(z) = A(z) - xB(z) - n\log z$, and let

$$I_l = \frac{1}{2\pi i}\oint e^{E(z)}(R/z)^l\,\frac{dz}{z} = \frac{R^l P_{n+l,j}(-x)}{(n+l)!}, \qquad I = I_0.$$

Here $R > 0$ will be the radius of a chosen saddle. All coefficient normalizations below are positive on the real x-axis.

The two ends. Write $n = jm + s$, $0 \le s < j$, and $c = j!/(j+1)^j$. Reversing the polynomial in $u = cm^{j+1}/t$ and setting its constant coefficient to one gives, locally uniformly in complex u,

$$Q_{m,s}(u) = s! \sum_{i \ge 0} \frac{u^i}{(ji+s)!} + m^{-1} E_s(u) + o(m^{-1}).$$

The coefficient proof and the ordering of the zeros of the exponential sections are given in §A.1. If $u_{s,k} > 0$ is the magnitude of its k-th zero, then

$$u_{0,k} < u_{1,k} < \cdots < u_{j-1,k} < u_{0,k+1}.$$

The first T reciprocal roots are simple and real for any fixed T and all sufficiently large n. They give the prescribed interlacing between different residues. At the same residue the magnitude of the original rank-k root is $cm^{j+1}u_{s,k}^{-1}(1 + d_{s,k}/m + o(m^{-1}))$. Its ratio at $m+1$ and m is $1 + (j+1)/m + o(1/m) > 1$. One extra fixed rank yields the inequality on the other side.

For large fixed T set $Y = (T + s/j)\pi/\sin(\pi/j)$. The limiting section has precisely T zeros in $|u| < Y^j$. Indeed, trace the upper semicircle by $u = z^j$, $z = Ye^{iv}$, $0 \le v \le \pi/j$. The roots-of-unity filter, after removing $e^z/(jz^s)$, gives

$$1 + e^{2\pi i s/j} \exp((e^{-2\pi i/j} - 1)z) + O_j(e^{-c_j Y}).$$

The other exponential roots of unity lose a fixed positive fraction of Y in real part. If $d = \pi/j - v \ge C/Y$, the second term has modulus $e^{-2Y \sin(\pi/j) \sin d} < 1/2$, for a suitable fixed C. For $d < C/Y$ its argument is $-2T\pi + O_j(C^2/Y)$. The bracket stays in the right half-plane and is real at both ends. The extracted factor winds by $T\pi$ on the upper semicircle. Conjugation and the argument principle give the count and exclude boundary zeros.

At the other end let $r > 0$ solve $h(r) = n$, put $L = h'(r)/r \asymp r^{p-2}$, and define $X = (h-n)/H$. There is a real critical point $a = r + 1 + O_j(r^{-1})$ of X. Put

$$x_* = X(a), \quad L_3 = -E'''(a, x_*) \sim L, \quad d = (2/L_3)^{1/3}, \qquad \Delta = 1/(dB'(a)).$$

One can use n throughout by retaining dz/z in the integral. On bounded ξ-sets the normalized coefficient satisfies

$$\frac{aP_{n,j}(-x_* - \xi\Delta)}{n! d \exp E(a, x_* + \xi\Delta)} \longrightarrow \mathrm{Ai}(-\xi).$$

At a zero near the positive Airy level d_k, the rank-matched displacement of row $n + l$ in x is $-l/(aB'(a))(1 + o(1))$. A separate estimate of each zero to $o(\Delta)$ would be insufficient to determine the sign of this displacement. Subtracting the two integrals before scaling supplies the smaller error (§A.2). In the global degree count below, one only needs the distinct real zeros near the specified d_k's. That count will itself exclude any further real or complex zeros.

Transporting both row signs. For $R = \rho$ in a fixed compact subinterval of $(0, \infty)$, the upper minimum of $\Re B(\rho e^{iv})$ is unique and satisfies

$$0 < \theta < \pi/j, \qquad H(\rho e^{i\theta}) < 0, \quad \Re K(\rho e^{i\theta}) < 0, \quad \Im B(\rho e^{i\theta}) > 0.$$

Section A.3 proves these four assertions for every $j \ge 4$. Write $z = \rho e^{i\theta}$, $\tau = -1/H(z)$, $x = n\tau$. Implicit differentiation gives $dH/d\log\rho = |K|^2/\Re K < 0$, hence x decreases with ρ. With principal curvature argument take

$$G = n(\Im B(z)/H(z) - \theta) + \Im A(z) - \tfrac{1}{2}\arg(-\tau K(z)).$$

On this compact range a positive peak normalization C_n gives

$$I/C_n = \cos G + o(1), \qquad (I_l - \cos(l\theta)I)/(C_n \sin(l\theta)) = \sin G + o(1). \tag{4}$$

The upper saddle value of the centered insertion is $-i$. Taking the real part of the upper peak together with its conjugate accounts for the sine, including its sign.

At the small-radius cut $\rho = j(j+1)Y/n$, one has

$$G = (-m+T)\pi + o(1), \qquad cm^{j+1}/x \longrightarrow Y^j.$$

This follows from $\theta = \pi/j - \rho\sin(\pi/j)/j^2 + O_j(\rho^2)$ and $\Im B/H - \theta = -\pi/j + \rho\sin(\pi/j)/(j(j+1)) + O_j(\rho^2)$. Exactly T of the roots constructed above precede this cut on the negative axis. On the small-radius side of a fixed compact range the errors in (4) are $o_n(1) + O_j(e^{-c_j n\rho})$ for $l < j$. For $l = j$, where $\sin(j\theta)$ tends to zero, use

$$(n+j)I_j = I[(\rho/z)^j(h(z) - xH(z))].$$

Subtracting the real saddle value of this insertion, after division by $n+j$, leaves a weight of order ρ uniformly on the circle. After division by ρ its value at the upper saddle tends to $-i\sin(\pi/j)/j$. At a zero of I this real subtraction leaves the test unchanged. Thus the sign of I_j equals that of $\sin G$ whenever $|\sin G| \geq 1/2$, for T and then n sufficiently large.

On the remaining moving range use the exact upper saddle z, with $h(z) - xH(z) = n$, $\theta = \arg z$ and $R = |z|$. Its phase is

$$G = \Im E(z) - \tfrac{1}{2}\arg\left(z(h'(z) - xH'(z))\right).$$

The continuous argument is chosen to match the preceding phase. Section A.4 proves (4) uniformly to the inner cut $Ly^3 = 3\pi(U+1/4)$, where $y = \Im z$ and U is a sufficiently large fixed positive integer. Its errors in the fold region are $O_j((Ly^3)^{-1/2}) + o(1)$; centering the insertion before estimating it makes this error uniform even as $\theta \to 0$. The two phase assignments differ by $o(1)$ on a compact overlap. At the inner cut,

$$G = -U\pi + o(1), \qquad (x - x_*)/\Delta \longrightarrow \left(3\pi(U+1/4)/2\right)^{2/3}.$$

The indexed Airy-zero expansion puts this number strictly between d_U and d_{U+1} for all sufficiently large U. The U Airy zeros closer to zero are therefore on the other side of the cut.

Counting and locating the signs. Choose T, U first to make the uniform errors smaller than $1/8$, and smaller than one quarter of the limiting sine amplitude at the exceptional $l = j$ test. Then choose n large. The unwrapped phase goes continuously from $(-m+T)\pi + o(1)$ to $-U\pi + o(1)$ as t increases. On an overlap interpolate between phases which differ by $o(1)$, enlarging the errors in (4) by $o(1)$. The cut endpoints and successive first hits of the intermediate integer multiples of π give alternating coefficient signs, hence at least $m - T - U$ different roots. Together with the T outer and U inner roots this exhausts the degree.

Between endpoints with phases $b\pi$ and $(b+1)\pi$, use the first hit of $(b+5/6)\pi$, then the last preceding hit of $(b+1/6)\pi$. On the open interval between these two hits the sine has sign $(-1)^b$ and magnitude $> 1/2$. The scalar cosines at its ends have opposite signs, so it contains the unique root of the cell. This argument allows the $o(1)$ perturbations at the first and last cut. The centered tests therefore give (3) throughout the middle, even without monotonicity of the corrected phase. Applying the just-proved degree count to the finitely many later rows also identifies their absolute edge ranks. The strict fixed-rank displacements at the two ends give (3) there. The sign criterion proves the fixed-threshold assertion.

4.2 An explicit joining row for threshold four

For $j = 4$, put $n = 4m + s$ and write $x = 24m^5/(2500X^4)$. The five outermost roots have separately isolated, residue-dependent windows for $m \geq 500$ [8, Theorem 6]. That theorem

includes their strict order across the four gaps. In particular it gives (3) for old ranks $1,\dots,4$; on a degree jump the later fifth root bounds the old fourth from the right. The old fifth root requires the following extension.

For $n \ge 2000$ take the exact saddle $z = Re^{iq}$ from $R = 250/n$ to $R = 13$, write $\Theta = \Im E(z)$, and $G = \Theta - \arg(D^2(A - xB)(z))/2$. The verified estimates of §A.5 give $|G - \Theta| < \pi/4$, normalized scalar error $< 1/2$ and normalized centered sine error $< 4/5$ for each $l = 1,\dots,4$. In particular, at a root with $b\pi < \Theta < (b+1)\pi$, $|\cos G| < 1/2$ implies that $\sin G$ has sign $(-1)^b$ and magnitude $> \sqrt{3}/2$. For $l = 4$ at small radii the integration-by-parts insertion above supplies the positive normalization. The exact action increases with $t = -x$. The fifth-root X-windows are

$$(13.74, 14.30),\ (14.52, 15.08),\ (15.31, 15.87),\ (16.09, 16.65) \qquad (s = 0, 1, 2, 3).$$

On the small-radius joining interval $0.97X < \Theta + n\pi/4 < 1.04X$. Its left endpoint has $X < 9.3$; its right endpoint has $X > 16.9$. Thus the fifth root lies strictly in $4\pi < \Theta + m\pi < 5\pi$ for each residue. At $R = 13$, $0.685 < 1 + 4\Theta/(n\pi) < 0.99$: the action is above $(-m+5)\pi$ and below -1 for every $n \ge 2000$.

The saddle continues uniquely to its real fold with strictly increasing Θ and terminal value 0. On the portion $\Theta \le -1$ let U be the complex upper-half coefficient integral after removing the saddle exponential and a positive scalar, U_l the corresponding integral with $(|z|/z')^l$ inserted, and

$$V_l = (U_l - \cos(lq)U)/\sin(lq).$$

The second enclosure in §A.5 is

$$\Re U > 0, \qquad \Re(e^{-i}U) > 0, \qquad |V_l + iU| < \Re U, \qquad 0 < lq < \pi.$$

At any integer action $b\pi$, the sign of $\Re(e^{i\Theta}U)$ is $(-1)^b$. At $\Theta = -1$ it is positive. The middle test gives the same integer-action signs up to the join. Starting after the fifth root with the anchor $(-m+5)\pi$, we have $m - 5$ new sign changes by the terminal cut. These and the five known roots exhaust degree m. A cell straddling $R = 13$ has the same action and endpoint signs from both sides.

At an inner root write $e^{i\Theta}U = iY$. Then $\Re U = Y \sin\Theta > 0$ and $|V_l + iU| < |U|$ implies $\Im(V_l/U) < 0$. Thus $\Re(e^{i\Theta}V_l) = -Y\Im(V_l/U)$ has the sign of $\sin\Theta$. The subtracted scalar term vanishes there. Since $\sin(lq) > 0$, the sign of the later polynomial is (3). The middle test already supplies that sign before the join, including at rank five.

For the finite end, construct $P_0 = 1$ and, with $P_n = P_{n,j}$,

$$P_{n+1} = P_n + tP_n' + \sum_{h=1}^{j-2} \binom{n}{h} P_{n-h} + t\binom{n}{j-1} P_{n-j+1}. \tag{5}$$

Terms with an unavailable index vanish. Inserting $n+1$ as a singleton or into a previously large block gives the first two terms. For a previously small block of size h choose its h old labels. The new block gains weight t precisely at $h = j - 1$. For $4 \le n \le 2004$ the dyadic endpoint certificate gives m disjoint negative sign brackets for each exact integer polynomial and strict interval separation across all four gaps. There are $2000, 1999, 1998, 1997$ nonconstant pairs respectively. If the old row has $n \le 1999$, its largest possible partner has $n + l \le 2003$; if $n \ge 2000$, the contour proof applies directly. Constants cover $0 \le n < 4$. This proves the first assertion of Theorem 3.

5 The all-row real-rootedness threshold

Theorem 4. *Every row $P_{n,j}$ has only real zeros if and only if $1 \le j \le 8$. For $j = 5, 6, 7, 8$, every row has exactly $\lfloor n/j \rfloor$ distinct negative zeros.*

For $j = 1$ these are the Touchard polynomials, with a zero at the origin in positive degree. The cases $j = 2, 3$ are known [3, 4], the case $j = 4$ is Theorem 3, and the preceding counterexamples give the converse. The following proof treats $j = 5, 6, 7, 8$ together. Its quartic specialization also has the independent treatment in [13]. The interval certificates are distributed with the paper [15].

The classification has a probabilistic interpretation. Under the uniform law on set partitions, let X count blocks of size at least j. By the classical factorization of real-rooted probability generating functions [14], one can read this directly from the roots. Its generating function is $P_{n,j}(t)/P_{n,j}(1)$. Each nonpositive real root r contributes a Bernoulli factor $1 - p + pt$ with $p = 1/(1 - r)$. Conversely a sum of independent Bernoulli variables has only such roots. Thus the law of X is a sum of independent Bernoulli laws for every n precisely when $j \le 8$. This is an equality in distribution. For a fixed $j \ge 9$ it still holds for all sufficiently large n.

5.1 Counting by the phase of the first saddle

Put $m = j - 1$, $A(z) = \sum_{h=1}^{m} z^h/h!$, $B = e^z - 1 - A$, and $E = A - xB - n \log z$. Then $P_{n,j}(-x) = n![z^n] \exp(A - xB)$. On the genuine upper saddle write $z = Re^{iq} = a + iy$. The first-saddle theorem [10] gives $0 < q < \pi/j$, $q + y < \pi$, $E'(z) = 0$. Its phase $\Theta = \Im E$ traverses $(-n\pi/j, 0)$ strictly monotonically. At each $\Theta = -l\pi$ with $1 \le l < \lfloor n/j \rfloor$ we prove sign $(-1)^l$. Together with the positive constant term and the leading sign $(-1)^{\lfloor n/j \rfloor}$, these signs exhaust the degree. This argument uses no assumption of row interlacing.

Use $N = 4097$ for all four thresholds. The finite-row recurrence and strict sign tests directly isolate the roots below N. Henceforth $n \ge N$. For $R < 1/4$, repeat the small-disk argument [12] with $V = (j - 1)!DB/z^j$, $D = z\partial_z$. The factorial series gives $|V - 1| \le (1/20)/(1 - 1/20)$ and $|DA/n| \le e^{1/4}/(4N)$. Thus $w = z[V/(1 - DA/n)]^{1/j}$ has $|w| < 0.255$, its argument differs from q by less than 0.02, and $w^j = -n(j - 1)!/x$ forces $\arg w = \pi/j$. Here the ratio of successive terms in V is at most $1/(4j) \le 1/20$. The unique analytic sector saddle equals z. Since $(1 + 1/8)0.255 < (1 - 1/8)j/13$, all these integer-phase cuts are covered by the small-disk sign theorem. If $n = jM + s$, its reciprocal-polynomial cut has index $M - l$. Multiplication by the sign $(-1)^M$ of w^{n-s} gives $(-1)^l$ for the original polynomial.

5.2 Intermediate radii

For $1/4 \le R \le 9$ and $a < 8$, use the compact-circle proof at threshold five [13]. Its Gaussian replacement and tail estimates use $f(v) = A(Re^{iv})/n - (x/n)B(Re^{iv}) - iv$. Write $C_0 = -f''(q)$, take $h = 1/16$ in the first verifier, and let M bound $|f'''|$ on the core $|v - q| \le h$. Set $\kappa = \Re C_0 - Mh/3 > 0$. Taylor's inequality bounds the real parts of both the true exponent difference and its quadratic approximation by $-\kappa(v-q)^2/2$. After normalization by $\sqrt{n|C_0|/(2\pi)}$, the real part of the untruncated Gaussian is $\cos(\arg C_0/2) > 1/\sqrt{2}$. The cubic error and omitted Gaussian tail have sum at most

$$E_c = \sqrt{\frac{n|C_0|}{2\pi}} \left(\frac{4M}{6n\kappa^2} + \frac{2e^{-n\kappa h^2/2}}{n\kappa h} \right).$$

Indeed $|e^v - e^w| \le |v - w| \max(e^{\Re v}, e^{\Re w})$. On the rest of the semicircle use a certified height loss $g > 0$. The sufficient test is $Ng > 1/2$ and $E_c(N) + \pi\sqrt{N|C_0|/(2\pi)}e^{-Ng} < 0.70$. Both error bounds decrease for $n \ge N$. The harmonic verifier below uses the same comparison with its own core width and error bounds. The circle verifier encloses $D^k B$ after dividing by $R^j/j!$. For $j \ge 6$ it uses the terms $h = 0, \ldots, 55$ with coefficients $j!(h + j)^k/(h + j)!$. After the rotation by e^{ijv} the tail in each complex component is bounded by $2j!(56 + j)^3 9^{56}/(56 + j)! < 10^{-17}$ for $0 \le k \le 3$, by the successive term ratio.

For completeness put $u = 1/n$ and $H = \Im[(1 - uDA)\overline{DB}]$ at angle q. True saddles satisfy $H = 0$ and $(uDA - 1)/DB > 0$. A positive real part of DB excludes a box only if $u\Re DA < 1$. Near zero the verifier may use that

$$H' = \Re[-uD^2A\,\overline{DB} - (1 - uDA)\overline{D^2B}].$$

A strict derivative sign throughout $[0, q_{\max}]$ and $H(0) = 0$ exclude all positive roots there. The radius-dependent cutoff removes only boxes entirely in $a \geq 8$. For the remaining angles the verifier proves the stated core and remote inequalities. It encloses the third derivative on the whole Gaussian window at threshold five and over eight subintervals at the larger thresholds. The trial angle used for the height comparison is within the local Gaussian window of the true saddle. The quadratic real-part estimate there bounds its height by that of the saddle. Every point of each parameter rectangle is either excluded by an exact necessary condition or covered by a sign estimate.

A second compact-circle verifier works in (R, q) coordinates and eliminates $1/n$. Set $b_0 = R^j/j!$, $\gamma = 1/(xb_0)$, $U = DA$, $V = DB/b_0$ and $L = \gamma\Re U - \Re V$. The two saddle equations are $\gamma\Im U = \Im V$, $L = n\gamma > 0$. Thus $(N - \Re U)\gamma + \Re V \leq 0$, and the angular exponent divided by n has harmonic weights $\gamma R^h/(Lh!)$ below j and $-R^h/(Lb_0h!)$ from j on. At $q = 0$ the imaginary parts divided by q use the sinc series. This proof uses 62 harmonics, a Gaussian window of half-width $3/50$, and a uniform normalized loss of at least $1/500$ outside that window ($3/2000$ for $j = 8$ on $R \leq 2$). A box meeting the window boundary can certify the loss just on its exterior part by the local Taylor bound. Strict full-circle dominance supplies the height at any trial angle. At $N = 4097$ the second method certifies 91, 97, 142 and 421 feasible boxes for thresholds five through eight. At threshold eight the first verifier runs from row 4500, where the second accepts 280 feasible boxes. The second verifier alone supplies the compact estimate on rows 4097 through 4499 of that threshold. The coefficient bounds, angular search and subdivisions are independent of the first verifier.

5.3 The fold for $a \geq 8$

This includes every remaining radius $R > 9$, since $y < \pi$. Set $p = 1/a$, $Z = 1 + ipy$, $s = \sin(y)/y$, $L = \Im A'(z)/y$, and

$$K = \Re A'(z) + aL, \qquad d = (\cos y + as)/K, \qquad Q = s - dL, \qquad \Lambda = xe^a.$$

The two real stationary equations give $1 + x = \Lambda d$, $\beta = 1/\Lambda = d - e^{-a} > 0$, and $n = \Lambda R^2 Q$. Write $K_s = p^{m-1}K$ and $d_0 = d/p^{m-2} = (s + p\cos y)/K_s$. Positivity of K_s on the full rectangle is verified before use. We compactify $0 \leq p \leq 1/8$, $0 \leq y \leq \pi$, including the corner at infinity.

Let $C = E''(z)/\Lambda = dA''(z) - e^{iy} + Q\bar{z}/z$. Set $\eta = \sqrt{\beta/y}$, $k = \eta/y$. Taylor expansion at the saddle gives

$$\frac{E(z + b) - E(z)}{\Lambda} = Cb^2/2 + b^3G(b),$$

$$G(b) = d\sum_{h=3}^{m} \frac{A^{(h)}(z)b^{h-3}}{h!} - e^{iy}\frac{e^b - 1 - b - b^2/2}{b^3} - \frac{pQ\bar{Z}}{Z^2}\frac{\log(1 + w) - w + w^2/2}{w^3}, \qquad w = pb/Z.$$

At $b = -iy$ this yields $\Theta k^2 = \Im(C/y)/2 - \Re G(-iy)$. Direct interval bounds for that right-hand side give $k < 0.62$ at every required integer phase. The cancellation-free real and imaginary parts used for C/y are

$$y\left[\frac{s - \cos y}{y^2} + d\frac{\Re A''(z) - L}{y^2} - \frac{2Qp^2}{1 + p^2y^2}\right], \qquad d\frac{\Im A''(z)}{y} - s - \frac{2pQ}{1 + p^2y^2}.$$

These polynomial quotients have removable singularities. To see the divisibility explicitly put $T = p^2y^2$ and $R_h = \Re Z^h$, $I_h = \Im Z^h/(py)$. Then $K_s = \sum_{h=0}^{m-1} p^{m-1-h}(R_h + I_h)/h!$. In the

curvature correction the coefficient of T^b in $R_h - I_{h+1}/(h+1)$ is $(-1)^b\binom{h}{2b}2b/(2b+1)$, with zero constant term. The factor $dA^{(h)}(z)$ for $h \ge 3$ is $d_0 \sum_{b=0}^{m-h} p^{m-2-b} Z^b/b!$. All evaluations now have nonnegative powers of p. The exact symbolic audit expands the original polynomials independently and checks these identities.

The local path uses $D = v - y$, $0 \le v \le 6$ and $b = iD - \delta \operatorname{sgn}(D) \min(|D|, (2 - |D|)_+)$ with $\delta = (1 - y/2.8)_+/2$. For each of the four thresholds the first interval cover bounds the real exponent by $-(D/\eta)^2/40$. Writing $S = D/\eta$, $W = b/\eta$, the second proves a lower sum greater than $1/2$ for the real part of

$$\frac{1 + i\delta\sigma}{Z + pb} \exp((C/y)W^2/2 + kW^3 G(b))$$

on $-12 \le S \le 12$. Here $\sigma = 1, -1, 0$ on the three consecutive ranges of $|D|$ separated by 1 and 2. The program imposes $\beta = y^3k^2$, includes zero on bins meeting $v = 0$, and checks $y + 12\eta < 6$ on every nonempty accepted box. The omitted tails cost less than 0.11 in the normalization a/η, since the amplitude is bounded by $(16/15)\sqrt{5/4}$ and the two Gaussian tails together have mass at most $(40/12)e^{-144/40}$. At $j = 5$ the unified fold verifier gives 303 positive core boxes on its default mesh and 279 on the finer mesh. The following estimate keeps the closing error below 0.28.

5.4 Closing the contour

From $a + 6i$ return horizontally to $\xi + 6i$, $\xi = \sqrt{R^2 - 36}$, then follow the circle to $-R$. We have $|z'| \ge R$, so the logarithmic term can only decrease the real exponent. The total logarithmic length is at most $\pi + 1$. Write $\rho = R/a$, $X = \xi/a$, $J = (36 - y^2)/(1 + X)$, so $a - \xi = pJ$. A sufficient lower bound for the exponent loss *after division by* Λ is either

$$0.4p + 0.1d_0p^{m-4} \quad \text{or} \quad \beta \log\left(\frac{(\pi+1)\sqrt{\pi}}{0.28p\sqrt{\beta}}\right).$$

The first implies loss at least $0.10a^2 + 0.4\Lambda/a$ before division, since $\beta \le d$. Maximizing $\sqrt{\Lambda}e^{-0.4\Lambda/a}$ bounds the normalized closing error by 0.20. The second directly bounds $(\pi+1)a\sqrt{\Lambda y}$ times the closing exponential by 0.28. For the second bound the program requires $p_{\min} > 0$ and $0 < \beta_{\max} < 1$, where its right-hand side is increasing in β and decreasing in p. A box can use either sufficient inequality.

On the horizontal segment put $\Re z' = at$ with $t = 1 - \lambda p^2 J$, $0 \le \lambda \le 1$. The exact normalized polynomial loss $[\Re A(z) - \Re A(z')]/a^{m-2}$ is

$$F = \lambda J \sum_{h=1}^{m} \frac{p^{m-h}}{h!} \sum_{b=0}^{h-1} t^b + V(6,t) - V(y,1),$$

$$V(v,t) = \sum_{h=2}^{m} \sum_{k=1}^{\lfloor h/2 \rfloor} \frac{(-1)^{k+1}}{h!} \binom{h}{2k} v^{2k} p^{m-h+2k-2} t^{h-2k}.$$

Thus the upper bound for the signed change divided by Λ is $-d_0F + \cos y - e^{-p\lambda J}\cos 6$.

If the initial circular angle is below π/m, all the first m cosines decrease up to π/m. The same lower polynomial loss works with $\lambda = 1$. The exponential increase is at most $\cos y + e^{-pJ}$. Put $b = 1.29$ when $y \ge 2$, otherwise $b = 2$. Then $1 + \cos y \le bs$, $1 - e^{-pJ} \ge 6.4p$ for $a \ge 8$, and $s = d_0K_s - p\cos y$. The verified inequality $F > bK_s + 0.1p^{m-4}$ implies the first sufficient bound above. This form handles $p \to 0$, $y \to \pi$ without subtracting two vanishing quantities.

On the far circular arc $\pi/m \le v \le \pi$ the normalized polynomial loss is $F_v - y^2H$, where

$$F_v = \sum_{h=1}^{m} \frac{p^{m-2-h}\rho^h(1 - \cos hv)}{h!}, \qquad H = \sum_{h=1}^{m} \frac{p^{m-h}}{h!} \frac{\rho^h - \Re Z^h}{p^2y^2}.$$

In H use $(\rho^h-1)/(\rho^2-1)=\sum_{b<h}\rho^b/(\rho+1)$ and the binomial expansion of $\Re Z^h$. For the two negative powers in F_v substitute $1/p_{\max}$ for $1/p$ in the nonnegative terms. The exponential increase is at most $\cos y+\exp[(\rho\cos v-1)/p]$. On boxes bounded away from $p=0$ we also use the actual term $-\exp[(\rho\cos v-1)/p]\cos(\rho\sin v/p)$ in place of its absolute bound. On every accepted box $\rho\cos v-1<0$, so a uniform upper bound remains available all the way to $p=0$. These tests prove the asserted closing estimates throughout the parameter domain.

A direct bound at infinity. All the closing pieces admit an analytic bound for $a\ge 1024$, including $m=4$. First, $K=\sum_{h=1}^{m}\Im z^h/[y(h-1)!]>0$ because $mq\le 7\arctan(\pi/8)<\pi$. In $KQ=s\Re A'(z)-\cos(y)L$, the term from $z^k/k!$ is $R^k\sin(y-kq)/(yk!)>0$, since $kq\le 6y/8<y$. The near-arc reasoning above applies to any closing point w for which $p^{m-2}\Re(A(z)-A(w))\ge 2K_s+0.1p^{m-4}$. On the horizontal piece its exponential term has favorable sign, and on the circular pieces $\Re w\le\xi$.

Set $V_h=p^{m-h}\rho^h/h!$. A geometric sum gives $\sum_h V_h\le\rho^m/[m!(1-mp)]<1.01/m!$. Hence $K_s\le\sum_h h^2V_h\le 1.01m^2/m!$ by $\sin(hq)\le h\sin q$. The normalized saddle deficit $S=p^{-2}\sum_h V_h(1-\cos hq)$ is below $5m^2/m!$ since $q\le py$. For $\pi/m\le v\le\pi$,

$$(1-\cos mv)+(1-\cos((m-1)v))\ge 2(1-\cos(v/2))\ge 2(1-\cos(\pi/14))>1/20.$$

The two weights V_m,V_{m-1} are at least $mp/m!$. Consequently $F_v-S\ge(m/(20p)-5m^2)/m!>2K_s+0.1p^{m-4}$. Here $S=y^2H$ in the earlier notation. On the horizontal segment $\Re A'(t+6i)>0$ for $\xi\le t\le a$, since $(m-1)\arctan(6/\xi)<\pi/2$. The polynomial loss is smallest at $a+6i$. For $0\le t\le 6$, $\Re A''(a+it)\ge 0.99a^{m-2}/(m-2)!$ by the top term and positivity of all other terms. Integrating first from 0 to u for $\Im A'(a+iu)$ and then from y to 6 gives normalized loss greater than $12/(m-2)!>2K_s+0.1p^{m-4}$. The near circular arc inherits the bound at its join by angular monotonicity. The scalar verifier checks the constants at $p=1/1024$ for each $m=4,5,6,7$; the inequalities improve as p decreases. The rational audit also obtains these constants using $157/50<\pi<22/7$, $1-z^2/2\le\cos z\le 1-z^2/2+z^4/24$ for $0\le z\le 1$, and $\arctan z\le z$. This second proof needs no interval subdivision at the noncompact endpoint.

The upper path starts on the positive real axis and ends at the negative real point. Its interior lies in the open upper half-plane. Closing it by the reversed reflected path therefore gives winding number one about zero. Since $1/2-0.11-0.28>0$, the normalized upper integral has positive real part at each required integer phase. This proves the required sign at every integer phase. The degree count proves Theorem 4.

The choice of the return path matters. For $j=6$, take $a=6$, $n=336$, $q_*=(690-n)/(6-ne^{-a})$, $\rho_*=\sqrt{a^2+4\pi^2}$ and $w=1+i\sqrt{\rho_*^2-1}$. For $\mathcal{H}(z)=A(z)-q_*e^{z-a}-n(1-q_*e^{-a})\log(z/a)$ we have $\mathcal{H}'(a)=0$ and $\Re\mathcal{H}(w)-\mathcal{H}(a)>120$. The point w lies on the circular return at height 2π from $a+2\pi i$. The fold verifier checks the strict comparison directly.

6 The remaining interlacing question

At thresholds five through eight every single row is now understood. The paired-row question remains open between the finite and eventual ranges. The interval recurrence checks every admissible gap whose larger index is at most 4096; at threshold eight the source reports checks through 6000 [15]. An independent replay here reached 4499, giving 403 rows of overlap with the analytic proof from 4097. The single-row tests use signs at integer multiples of the saddle phase. Comparing two rows requires enough precision to order their individual roots.

For these four small thresholds, at a true pre-fold saddle the first conjugate pair is the unique maximizing pair on its circle [10]. One exact comparison also follows directly from stationarity. Write $\Theta(n,x)=\Im(A_j(z)-xB_j(z)-n\log z)$ on this branch. Differentiating at fixed x gives

$\partial_n \Theta = -\arg z$. The fold parameter decreases with n, since its derivative is $-1/(DB_j)(z_f) < 0$ at the real fold z_f, where $D = z\partial_z$. Thus for $x > x_f(n)$ the whole segment from n to $n+l$ stays on the branch. Integrating $0 < \arg z < \pi/j$ proves

$$-l\pi/j < \Theta(n+l,x) - \Theta(n,x) < 0 \qquad (0 < l \le j).$$

This saddle geometry changes at threshold nine: on row 60000 a remote angle on the radius-10.76 circle exceeds the real exponent at the first saddle by more than 298 [10]. The finite-radius gain over competing peaks and the sign of the normalized insertion at a zero of the coefficient integral are separate estimates. Near a fold the curvature vanishes; at the opposite edge the leading insertion for a full lag j vanishes. Section 4.2 overcomes both issues at $j = 4$. For the four intervening thresholds an effective estimate of these centered integrals, joined to strict displacement estimates between rows at the outermost ranks, could settle the interlacing question. Exact testing of more rows alone supplies no upper bound for the last failure.

A Estimates for the interlacing theorem

A.1 The reciprocal polynomial

Write $B = bz^j J(z)$, $b = 1/j!$, $J(0) = 1$, $J'(0) = h_1 = 1/(j+1)$, $D_0 = e^A$ and $a_i = [t^i]P_{n,j}(t)$. Exactly,

$$\frac{a_{m-i}}{a_m} = (m)_i b^{-i} \frac{[z^{ji+s}]D_0(z)J(z)^{m-i}}{[z^s]D_0(z)J(z)^m}.$$

The coefficient of u^i in the normalized reciprocal is this ratio divided by $(h_1^j m^{j+1}/b)^i$. For $|u| \le V$ choose $M > V^{1/j}/h_1$. Uniformly on $|w| = M$, $0 \le i \le m$,

$$D_0(w/m)J(w/m)^{m-i} = e^{h_1 w} + O_M((1+i)/m).$$

This follows by expanding the analytic log at zero; at fixed i the entire left side has a first-order expansion in $1/m$. The rescaled denominator tends to $h_1^s/s! > 0$ with the same expansion. The extra factor $(m)_i/m^i$ differs from one by at most $i(i-1)/(2m)$. Cauchy's inequality now bounds m times the difference of the i-th summands by $C_{M,s}(1+i^2)(V/(h_1 M)^j)^i$, also for the missing terms $i > m$. Dominated summation gives the asserted locally uniform expansion.

Put $f_s(v) = \sum_{i\ge 0}(-v)^i/(ji+s)!$. The upper Toeplitz matrix of the coefficients of $(1+z/N)^N$ is totally nonnegative, being a power of a nonnegative bidiagonal matrix. Selecting rows ji and columns $jk+s$ preserves total nonnegativity. The finite Aissen–Schoenberg–Whitney theorem [16] makes each corresponding residue-section polynomial real-rooted with nonpositive zeros. Locally uniform convergence of the factorial series, followed by Hurwitz, puts all zeros of f_s on the positive real axis. Its order $1/j < 1$ and its infinitely many nonzero coefficients give a genus-zero product with infinitely many zeros. For $1 \le s < j$, $f_{s-1} = sf_s + jvf_s'$; the wrap is $f_{j-1} = -jf_0'$. Off the zeros a of f_s, counted with multiplicity,

$$(vf_s'/f_s)' = -\sum_a a/(v-a)^2, \qquad (f_s'/f_s)' = -\sum_a 1/(v-a)^2.$$

Both are strictly negative on the real axis off their poles. A multiple zero of a predecessor would therefore force the successor to have higher multiplicity at that same point. Passing round the cycle gives a contradiction. Adjacent sections have no common zero. The first ratio $s + jvf_s'/f_s$ starts positive, decreases to $-\infty$ at the first pole, and crosses zero once between every successive pair of poles. The wrap ratio f_0'/f_0 has no zero before its first pole and one between every two poles. This proves the stated cyclic order. Rouché on disjoint disks around the simple zeros of the limit transfers multiplicity one, and conjugation gives real zeros of the approximants. Taylor's theorem at a simple zero transfers the first-order correction in $1/m$.

A.2 The Airy end with a common coefficient parameter

The factor $1/z$ in I permits the same parameter n in the outer, middle and inner phases. On fixed w-disks,

$$\frac{X(r+w)}{Le^{-r}} = we^{-w}(1+O_j(1/r)), \qquad \frac{E(r+w,0)-E(r,0)}{L} = w^2/2 + O_j(1/r).$$

Factor w from $h(r+w)-h(r)$ and use $H = ze^z - h$. The second formula follows since the linear term vanishes and all derivatives of order at least three on smaller fixed disks are $O_j(L/r)$. Cauchy estimates give differentiated convergence. The unique nearby zero of X' is real and $a = r+1+O_j(1/r)$. From $E_z = B'(X-x)$ one obtains

$$x_* \sim Le^{-r-1}, \quad X''(a) \sim -x_*, \quad E'''(a,x_*) \sim -L, \quad E^{(4)}(a,x_*) \sim -L, \quad \Delta/x_* \asymp L^{-2/3}.$$

For $x = x_* + \xi\Delta$ use the circle of radius $a-d$, replacing its right arc by the vertical segment at $\Re z = a-d$, up to heights $\pm Y$, and horizontal connectors. Here Y is large and fixed. This is a positively oriented coefficient contour. On its small-angle circular part beyond the join, the highest polynomial power costs at least $c_j r^p \theta^2 \geq c_j' L Y^2$ up to $|\theta| = \pi/p$. On the remainder the two highest powers lose at least $c_j r^{p-1}$: their cosine deficits have no common zero there. The change in xB is $O(L)$. On a connector, $\Re z = a-d+O_Y(1/r)$, $|\Im z| = Y$. In units of L the loss is at least $Y^2/2 - O(1) - o_Y(1)$ from the expansion above and the bounded exponential amplitude. The $O(1)$ is independent of the fixed Y. These bounds also apply to bounded real saddle offsets and $|xe^r/L| \leq C$ in the next subsection, with Y chosen after C.

At $z = a+dw$ the exponent difference from $E(a,x)$ converges locally uniformly to $-w^3/3 - \xi w$. On $\Re z = a-d$, writing $z = a-d+iv$, a uniform integrable bound for $|v| \leq \delta$ is

$$\Re(E(z,x) - E(a-d,x)) \leq C|v|/d - c(v/d)^2 - cLv^4.$$

Indeed $E_z(a-d,x) = O(d^{-1})$, $\Re E''(a-d,x) \sim Ld$, $\Im E'''(a-d,x) = O(d^{-1})$, $\Re E^{(4)}(a-d,x) \sim -L$, and $E^{(5)} = O(L)$ there. Choose fixed $\delta > 0$ small. On the remaining bounded vertical arc the limiting real exponent difference in units of L is $1-\cos v - v^2/2 < 0$. These estimates justify the local uniform Airy limit using the vertical-line integral for Ai. The factor a/z tends to one on the scaled line and stays bounded on the rest of the contour. For the shifted row insert $(a/z)^l$. Subtracting the unshifted integral and dividing by ld/a gives limiting insertion $-w$, hence the limit $d\,\mathrm{Ai}(-\xi)/d\xi$. The normalized difference insertion is bounded by $C_l(1+|v|/d)$ on the local arc and by $O_l(a/d)$ elsewhere. The same majorant and exponential losses apply.

Rouché gives one simple real zero near each fixed Airy zero d_k, simultaneously for both rows in the same coordinates. The shifted derivative converges to the nonzero Airy derivative there. The mean-value theorem applied at the two zeros and the normalized difference just proved give a shift $-ld/a(1+o(1))$ in ξ, as asserted. Adjacent fixed Airy levels have limiting separation $d_{k+1} - d_k > 0$. To check that the root selected from the later row is also its k-th Airy root, allow the critical equations to depend on a real n. Differentiating $X(a,n)$ at its critical point gives $\partial_n x_* = -1/H(a)$. Over a bounded change in n, r and a change by $o(1)$ and the ratio of the two Δ's tends to one. Thus the shift in x_* is $O(1/H(a)) = o(\Delta)$. The two rank labels agree.

A.3 The angular minimum for each fixed threshold

The case $j = 4$ was established in [8, Section 2]. The estimates below cover every integer $j \geq 4$. Let X be Poisson of parameter ρ conditioned on $X \geq j$, with mean μ, variance v and characteristic function $\phi(t) = B(\rho e^{it})/B(\rho)$. If $p_j = \Pr(X = j)$, series shifts give $\mu = \rho + jp_j$ and $v - \mu = jp_j(j-1-\mu) \leq 0$.

For $\mu \geq 128$ the following estimates apply to any nonconstant log-concave integer law with $v \leq \mu$. Its symmetrization $Y = X - X'$, with an independent X', has a symmetric

log-concave law (convolution preserves nonnegative order-two Toeplitz minors). For $v_0 = 2v$, $\mathbb{E}Y^4 \le v_0 + 6v_0^2$: compare with the symmetric two-sided geometric law of variance v_0, whose characteristic function is $1/[1 + v_0(1 - \cos t)]$. On nonnegative indices the ratio to its geometric masses is log-concave. Its difference of masses is positive on an interval, negative on either side unless the laws coincide, by equality of mass and variance. Multiplying the difference by $(k^2 - b)(k^2 - c)$, with b, c at the sign changes, proves the fourth moment comparison. In particular $\mathbb{E}(X - \mu)^4 \le v + 9v^2$.

Write the symmetric unimodal law as a mixture of uniforms on $\{-k, \dots, k\}$, with mixing weights $(2k + 1)(\Pr(Y = k) - \Pr(Y = k + 1))$. Its component variance is $w = k(k + 1)/3$, its fourth moment $(9w^2 - w)/5$. Thus $\mathbb{E}w = 2v$ and $\mathbb{E}w^2/\mathbb{E}w \le (2 + 20v)/3$. Set $a_0 = 2/\pi^2$. The uniform component loss $1 - \mathbb{E}\cos(tY)$ is at least $a_0 w t^2$ for $kt \le \pi$. For $kt \ge \pi,\ 0 < t \le \pi$, the normalized Dirichlet kernel has modulus at most $\pi/((2k + 1)t) < 1/2$. Both estimates dominate $a_0 w t^2/(1 + 2a_0 w t^2)$. Cauchy–Schwarz after mixing gives

$$1 - |\phi(t)|^2 \ge \frac{2a_0 v t^2}{1 + 2a_0 t^2 (2 + 20v)/3}.$$

The trial at $t = \pi/\mu$ gives $1 + \Re\phi(t) \le v\pi^2/(2\mu^2)$. For $t \ge 2.3\pi/\mu$ the preceding loss, divided by two, exceeds this trial bound: the ratio is at least

$$\frac{4(2.3)^2/\pi^2}{1 + 4(2.3)^2(2 + 20\mu)/(3\mu^2)} > 1 \quad (\mu \ge 128).$$

For $\mu t \in [0, 0.8\pi] \cup [1.2\pi, 1.5\pi] \cup [1.5\pi, 2.3\pi]$, $|\phi(t) - e^{it\mu}| \le vt^2/2$ also gives a larger real part: use $\cos(\mu t) \ge -\cos(\pi/5)$ on the first two intervals and $\cos(\mu t) \ge 0$ on the third. On the remaining interval, for $s = 1, 2$,

$$|\mathbb{E}(X/\mu)^s e^{itX} - e^{it\mu}| \le 2/\sqrt{\mu} + 1/\mu + (1.2\pi)^2/(2\mu) < 1/4.$$

The real parts are negative, giving strict convexity for $\Re\phi$ and a unique minimum θ. At this point $\mathbb{E}X \sin(\theta X) = 0$. For $\delta = X - \mu$, $\left|\mathbb{E}\delta e^{it\delta} - itv\right| \le t^2 v\sqrt{1 + 9v}/2$; at θ the ratio of this error to θv is below 0.51. Since $\cos(\mu\theta) \le -\cos(\pi/5)$, it follows that $\mathbb{E}\delta \sin(\theta X) < 0$, hence $\Im\phi(\theta) > 0$.

To obtain $\theta < \pi/j$ put $U = X - j,\ b = \mathbb{E}U > 0$. If $b \ge j/5$ use $\theta < 1.2\pi/\mu$. Otherwise $j > 106$. The log-concave tails $q_i = \Pr(U \ge i)$ satisfy $q_{i_1 + \dots + i_k} \le \prod q_{i_h}$, hence $\mathbb{E}\binom{U}{k} \le b^k$. Using the third and fourth raw moments from these factorial moments gives

$$\frac{\mathbb{E}[XU^3]}{j^2 \mathbb{E}[XU]} \le 0.432 + 2.64/j + 3.8/j^2 + 1/j^3 < 6/\pi^2.$$

Now $\sin y \ge y - y^3/6$ for $y \ge 0$ makes $(\Re\phi)'(\pi/j) = \mathbb{E}X \sin(\pi U/j) > 0$. This angle lies in the interval of strict convexity, proving the claimed bound.

For $4 \le j \le 127$ the two unbounded radius intervals have direct uniform estimates. If $e = \rho/(j + 1) \le 1/8$, divide B by $\rho^j/j!$, obtaining $e^{ijt} G(e^{it})$ with coefficients $c_0 = 1, c_1 = e, c_{k+1}/c_k \le e$ in G. Its autocorrelations $b_d = \sum c_k c_{k+d}$ satisfy $b_d/b_1 \le e^{d-1}$. The inequality $|\sin dt| \le d \sin t$ and $\sum_{d \ge 2} d^2 (1/8)^{d-1} < 1$ make $|G(e^{it})|$ strictly decreasing on $(0, \pi)$. Since $|G - 1| \le 1/7$ and $|\partial_t G| \le e/(1 - e)^2$, the total argument strictly increases. At π/j, the first imaginary part $e \sin t$ of G dominates the rest since $\sum_{k \ge 2} k e^{k-1} < 1$. Its first negative-real crossing is therefore at $t_c < \pi/j$; decreasing modulus excludes every $t > t_c$ as a minimum. At t_c the real-part derivative is positive. Testing at π/j confines all minima to the single pre-crossing interval with $\cos(jt) \le -1 + 2/7$. For $H/j, K/j^2$ the normalized real parts on that interval differ from $\cos(jt)$ by at most $\sum_{k \ge 1} (1 + k/4)^2 (1/8)^k < 0.3$. Strict convexity and the positive imaginary part before the crossing prove all four assertions.

For $\rho \geq \max(64, 4j)$ Chernoff's Poisson lower-tail estimate bounds the removed Taylor polynomial by $e^{0.6\rho}$ and its first two logarithmic derivatives by $\rho^2 e^{0.6\rho}$. Write $y = \rho \sin t$ and $u_y = \sqrt{\rho^2 - y^2}$ on the right semicircle. Test $y = \pi$, so $u_\pi > \rho - 1/8$. The left semicircle has real part at least $-1 - e^{0.6\rho}$. For $y \geq 2\pi$, $e^{u_\pi} - e^{u_{2\pi}} \geq e^{\rho - 1/8}/(2\rho) > 2e^{0.6\rho}$. For $0 \leq y \leq 2\pi$ outside $[3\pi/4, 5\pi/4]$ use $\cos y \geq -1/\sqrt{2}$. On the surviving interval $t < 4/\rho < \pi/j$, $u_y > \rho - 1$, and $\cos(y + t), \cos(y + 2t) < -0.4$. The exponential terms give $\Re H < -\rho e^{\rho - 1}/3$, $\Re K < 0$ after allowing the polynomial error. At the unique minimum H is real, so $B = H/z - z^{j-1}/(j-1)!$ gives $\Im B \geq e^{\rho - 1} \sin(t)/3 - e^{0.6\rho} > 0$.

For $4 \leq j \leq 8$, the direct angular certificate on $[1/16, 64]$ covers what remains between these two analytic intervals. For $9 \leq j \leq 127$ it suffices to cover $[(j+1)/8, \min(128, \max(64, 4j))]$, since $\mu \geq \rho$. The direct test brackets a zero of $\Im H$ by opposite signs, checks $\Re K < 0$ throughout the bracket and compares all complementary angles to an interior trial for $\Re B$. One-sided bisections sharpen the stationary enclosure, on which $\Re H < 0, \Im B > 0, \theta < \pi/j$ are tested. Code `check_small_threshold_saddles.py` uses the radius-normalized Taylor series through excess degree 52 for $\rho \leq 8$. All errors are at most $2j!(j+53)^2 8^{53}/(j+53)! < 10^{-18}$. Above 8 it uses exact finite exponential complements. The intermediate-threshold checker uses variable Taylor cutoffs with the same verified error and normalization, or its following alternative moment test.

Write m_3, m_4 for the third and fourth central moments and $\kappa = 2[5(m_4/v + 3v) + 1]/27$. The preceding uniform mixture also gives the sharper bound

$$1 - |\phi(t)|^2 \geq vt^2/(1 + \kappa t^2), \qquad 0 \leq t \leq \pi.$$

Indeed each uniform component has loss at least $wt^2/[2(1 + 2wt^2/3)]$. For $kt \geq 2\pi$ its Dirichlet kernel has modulus $< 1/4$; for $\pi \leq kt \leq 2\pi$ its kernel is at most $1/(2k+1) \leq 1/5$ ($k = 1, t = \pi$ is immediate). For $kt \leq \pi$, $wt^2 \leq 5$, use the fourth-order cosine bound, giving $(wt^2/2)(1 - 3wt^2/20)$. The remaining case $wt^2 > 5$, $kt \leq \pi$ has $k = 1, t > \pi/2$; the exact loss suffices. Cauchy–Schwarz and the component fourth moment formula give κ.

In a radius interval the program chooses $\alpha < \pi < \beta \leq \gamma$, $a = \alpha/\mu, b = \beta/\mu$. The inequality $\gamma^{-2} + \kappa/\mu^2 < \pi^{-2}$ excludes $t \geq \gamma/\mu$ by the modulus loss. Direct comparisons of $\cos(\mu t) - vt^2/2$ to the trial $-1 + v(\pi/\mu)^2/2$ exclude $0 \leq t \leq a$ and $b \leq t \leq \gamma/\mu$. For each $s = 1, 2$, let μ_s, v_s be the mean and variance of the law tilted by X^s. The tests $b\mu_s < 2\pi$ and $\max(\cos(a\mu_s), \cos(b\mu_s)) + b^2 v_s/2 < 0$ prove the two curvature signs. The test $b^2(m_4/v + 3v) < 6$ makes $(|\phi|^2)'(t) < 0$ on $(0, b]$, since $Y \sin(tY) \geq tY^2 - t^3 Y^4/6$. At the minimum this supplies $\Im B > 0$. Finally either $b < \pi/j$ or the checker compares $\pi^2 \mathbb{E}[X(X-j)^3] < 6j^2 \mathbb{E}[X(X-j)]$ and uses the positive derivative test above.

The program `check_medium_threshold_saddles.py` obtains moment boxes from positive and negative parts of $(X - c)^k$ at the endpoints of each radius interval, by monotone likelihood ratio. For $M_k = \mathbb{E}[(X-c)^k \mathbf{1}_{X \geq l}]$ it uses

$$M_{k+1} = (\rho - c)M_k + \rho \sum_{i<k} \binom{k}{i} M_i + l \Pr(X = l)(l - c)^k,$$

by shifting the defining series. The normalization tail from $X = 512$ is $< 10^{-60}$, checked by a factorial bound. The accepted direct covers for $j = 4, 5, 6, 7, 8$ have $175, 154, 149, 150, 178$ radius slabs. For $9 \leq j \leq 127$ the complete check covers 119 integers, with direct angular fallback only for $j \leq 20$.

A.4 Uniform passage between the three saddle regimes

On compact radius intervals the stationary condition for $-\tau B - i \arg z$ is $H = -1/\tau$. The angular curvature is τK with negative real part. Outside fixed neighborhoods of the two conjugate

maxima the real normalized exponent loses a positive constant uniformly by compactness. Inside, the usual Taylor expansion with bounded third derivative and a Gaussian majorant gives (4). As $\rho \to 0$ expand the normalized series $j!B/\rho^j = e^{ijv}(1 + \rho e^{iv}/(j+1) + O_j(\rho^2))$ with any fixed number of derivatives. Each of the j maxima of the negative leading cosine persists as a nondegenerate local maximum of the real part. At $\alpha = (2b+1)\pi/j$ the next height correction is proportional to $\rho \cos \alpha$. Only $\alpha = \pm\pi/j$ survives with height advantage $\geq c_j n\rho$ above the other peaks. The latter have Gaussian width $O_j(n^{-1/2})$ from their real curvature, including when they are not complex stationary points. This gives the stated $O_j(e^{-c_j n\rho})$ relative contribution, uniformly down to $\rho \asymp T/n$. The angular derivative and global norm of the $l = j$ insertion after centering and division by ρ are bounded, since $H/z^j = 1/(j-1)! + z/j! + O_j(\rho^2)$, $h = O_j(\rho)$, and $x\rho^j/(n+j) = (j-1)!(1 + O_j(\rho + 1/n))$.

Choose large fixed P, M; the constants implicit below may depend on j. For $P \leq b \leq r - M$ set $N = n - h(b) \geq Mh'(b)$. There is a unique saddle $z = b + iy$ with $|y - \pi| \leq 1/2$ and $x = (h - n)/H > 0$. Its real argument equation is

$$y + \arg z + \Im \log(1 - h(z)/(ze^z)) - \arg(n - h(z)) = \pi.$$

The left side is $y + O(1/b + b^{p-1}e^{-b} + b^{p-1}/N)$, and its y-derivative is one plus that same error. The complex logarithmic derivative of $(h - n)/H$ is $-1 + O(1/b + b^{p-1}e^{-b} + b^{p-1}/N)$. Hence $d \log x/db < 0$: for a holomorphic log derivative v, this derivative along a real level of its imaginary part is $|v|^2/\Re v$. Put

$$\Lambda = xe^b \asymp N/b, \qquad q = b^{-1} + (1 + x)b^{p-2}/\Lambda.$$

Then $|y - \pi| \leq Cq$, q is uniformly small by the choices of P, M, and $\Lambda \to \infty$ uniformly. For the last assertion use $N/b \geq Mh'(b)/b$ if $b \to \infty$, and $N \to \infty$ if b is bounded.

On the right of the coefficient circle of radius $|z|$, the real exponent, up to a constant, is $(1 + x)\Re A(z') - xe^{\Re z'} \cos(\Im z')$. Its upper envelope obtained by replacing $-\cos$ by 1 exceeds the value at the saddle by $O(\Lambda q^2)$ at $\Im z' = y$. From y to 2π that envelope loses at least $c[(1 + x)b^{p-2} + \Lambda/b]$. Indeed each power of A decreases in angle there, the leading power drops by $\gg b^{p-2}$, and the real coordinate drops by $\gg 1/b$. The envelope continues decreasing up to a small fixed angle below π/p. At larger angles the exponential is $O(\Lambda e^{-c'b})$, whereas replacing $\Re A(z)$ by $A(|z|)$ costs only $O(b^{p-2})$. For $0 \leq \Im z' \leq 2\pi$ on the right away from a fixed neighborhood of π, the cosine itself costs $c\Lambda$, since $|z| - b = O(1/b)$. Thus the separated arcs lose at least $c\Lambda q \gg \max(\Lambda/b, b^{p-2}) \gg \sqrt{\Lambda}$; the final comparison uses $p \geq 3$ and $b \geq 1$. The lengths and the insertion norms are polynomial in Λ. On a fixed small arclength neighborhood of the saddle the real curvature is at most $-c\Lambda$; the exponential part has this sign near $\Im z' = \pi$, and the polynomial error is $O(q\Lambda)$. The third derivatives are $O(\Lambda)$. The derivative in arclength of the normalized sine multiplier is $O_l(1)$ since $|z| \sin(l\theta) \asymp_l y$. Gaussian integration gives (4) using the exact phase in §4. On a fixed large-b overlap, x/n lies in a compact subinterval of $(0, \infty)$. Dividing the height inequalities by n and taking subsequential limits identifies the limiting saddle as the unique angular minimum. At the same x/n the two saddles differ by $O(1/n)$, since $K \neq 0$. Stationarity then bounds the difference of unwrapped exponent values by $O(1/n)$; the curvature arguments agree to $o(1)$. This establishes the phase join.

For $0 \leq y \leq y_1 < \pi$, the real-saddle condition has the branch $z = r + u_n(y) + iy$, where $u_n(y) \to u(y) = y \cot y$, $u(0) = 1$, with all fixed derivatives. Indeed $\Im X(r + v + iy)/(yLe^{-r})$ extends evenly to $y = 0$ and tends to $e^{-v}(\cos y - v \sin y/y)$, whose v-derivative does not vanish at the zero. The analytic implicit function theorem applies on this compact y-range. The positive parameter $x_n(y) = X(z)$ increases with y: its scaled limit $ye^{-u(y)}/\sin y$ has positive log derivative $((1 - y \cot y)^2 + y^2)/y$. Near zero use evenness and the uniformly positive limit of $x_n'(y)/(yLe^{-r})$. Choose y_1 so $u(y_1) < -M - 1$. The previous branch near $b = r - M$ then agrees by uniqueness.

On a compact subinterval of $0 < y \le y_1$, integrate on the vertical line through the saddle as above. At height v, the limiting real part in units of L is a constant minus $v^2/2 + (y/\sin y)\cos v$. Reflecting into $[-\pi, \pi]$ decreases v^2, and the strict decrease of $\sin v/v$ on $[0, \pi]$ gives precisely the pair of maxima at $v = \pm y$. The second derivative there is $-1 + y\cot y < 0$. The remote bounds in §A.2 apply. The oriented vertical Gaussian has phase $\Im E - \arg z - \arg E''/2$; this agrees on the overlap with $\Im E - \arg(z^2 E'')/2$ since $\arg z = o(1)$ and $\Re E''/L$ stays positive. The centered insertion has local arclength derivative $O_l(1/y)$.

For small $y > 0$, exact analytic expansion at the fold gives

$$z = a + iy + O(y^2), \qquad x_n(y) = x_* - \tfrac{1}{2}X''(a)y^2 + O(x_* y^4),$$
$$E''(z) = -iL_3 y + O(Ly^2), \quad E'''(z) = -L_3 + O(Ly).$$

These estimates follow by differentiating the locally uniform expansions for X, using evenness and $B'(a)X''(a) = -L_3$. Write $w = v - y$. Near the upper saddle tilt the vertical segment to $z' = z + (-\delta + i)w$, $\delta = 1/2$, for $0 \le v \le \eta$, with fixed small $\eta > 0$, and use a horizontal connector at $v = \eta$. Reflect this contour below the real axis. The real exponent difference is

$$L_3 w^2 \left[-\delta y - (3\delta - \delta^3)w/6 + O(y^2 + y|w| + w^2)\right] \le -cLw^2(y + |w|).$$

For negative w, use $|w| \le y$. At height η the limiting derivative in the real direction is $(\eta^2 - (\Re z' - r - 1)^2)/2 + O(\eta^3) > 0$ along the connector for $y \to 0$; its highest endpoint on the original line has strict loss by $1 - \cos\eta < \eta^2/2$. On scaling w by $\sqrt{Ly}$, the Gaussian tail omitted at $v = 0$ starts at $-\sqrt{Ly^3}$, and the cubic correction is $O(|w\sqrt{Ly}|^3/\sqrt{Ly^3})$. The inequality $|e^z - e^{z'}| \le |z - z'|\max(e^{\Re z}, e^{\Re z'})$ allows both exponentials to use their Gaussian majorants. Variation of the normalized insertion is $O_l(|w|/y)$ near the peak, with global bound $O_l(r/y)$ harmless on the separated arcs. The oriented square root agrees with the vertical one since $(1 + i\delta)\sqrt{E''/|E''|}$ has positive real part. These estimates prove the claimed error $O((Ly^3)^{-1/2}) + o(1)$ uniformly at and above the cut. Finally, $\Im E(z) = -L_3 y^3/3 + O(Ly^4)$ and $\arg E'' \to -\pi/2$. Substituting $Ly^3 = 3\pi(U + 1/4)$ gives both cut formulas in the text; for the Airy coordinate use $-X''(a) = L_3/B'(a)$. The standard expansion $d_k^{3/2} = (3\pi/2)(k - 1/4) + O(k^{-1})$ yields the stated strict separation for sufficiently large fixed U.

A.5 The two signed enclosures used at threshold four

Five outer roots. We reproduce the finite-radius argument of [8, Theorem 6], including the displacement for a full lag. Write $A = A_4$, $B = B_4$, $b(z) = 24B(z)/z^4$, $c_\circ = 24/625$. Put $n = 4m + s$, $q = 1/m$, $y = 500q$, $v = 4X^4$, and $t = -c_\circ m^5/v$. For $m \ge 500$ the certified open X-intervals have the following rational endpoints.

s	First	Second	Third	Fourth	Fifth
0	$1.17, 1.73$	$4.31, 4.87$	$7.45, 8.01$	$10.60, 11.16$	$13.74, 14.30$
1	$1.96, 2.52$	$5.10, 5.66$	$8.24, 8.80$	$11.38, 11.94$	$14.52, 15.08$
2	$2.74, 3.30$	$5.88, 6.44$	$9.02, 9.58$	$12.17, 12.73$	$15.31, 15.87$
3	$3.53, 4.09$	$6.67, 7.23$	$9.81, 10.37$	$12.95, 13.51$	$16.09, 16.65$

After removing t^m and a positive common factor the reciprocal expression is

$$S(q, v) = \sum_{\ell=0}^{m} (-v)^\ell p_\ell(y), \qquad p_\ell(y) = \prod_{0 \le h < \ell} (1 - hq)\,[w^{4\ell+s}]\mathrm{e}^{A(5qw)} b(5qw)^{m-\ell}.$$

It need not have unit constant term. For $d = 4\ell + s$ the retained p_ℓ have an exact polynomial expression in y, by the binomial expansion

$$p_\ell(y) = \sum_{i=0}^{d} [w^d] \frac{\mathrm{e}^{A(5w)}(b(5w) - 1)^i}{i!} q^{d-i} \prod_{0 \le h < \ell + i} (1 - hq). \tag{6}$$

Retain $\ell \le 40$, where $\ell + d < 500$, and set $q_* = 1/500$, $R_* = 82$,

$$a_* = A(5q_*R_*), \qquad D_* = (b(5q_*R_*) - 1)/q_*, \qquad M_* = \mathrm{e}^{a_*+D_*}.$$

On $|w| = R_*$ positive coefficients give $b(5qR_*) - 1 \le qD_*$, and consequently $|p_\ell| \le M_*R_*^{-4\ell-s}$ by Cauchy's estimate, with $p_\ell = 0$ for $\ell > m$. For $|v| \le V < R_*^4$ the coefficient absolute sum of the omitted terms on the rescaled unit disk is at most

$$M_*R_*^{-s}\frac{(V/R_*^4)^{41}}{1 - V/R_*^4}.$$

On all of $0 \le y \le 1$, the companion checker `effective_outer.py` uses Arb to compare this error with the signed retained value at both endpoints of each window. For the complex count, let X_* be the fifth right endpoint plus one, and normalize $|v| \le 4X_*^4$ to the unit disk. If $f(z) = f_\mathrm{e}(z^2) + zf_\mathrm{o}(z^2)$, its Graeffe transform is $f_\mathrm{e}(z)^2 - zf_\mathrm{o}(z)^2$, which squares its roots, preserving multiplicity. If the perturbation in coefficient ℓ^1 norm is at most ϵ, the transformed error is at most $\epsilon(2\|f\|_1+\epsilon)$ by multiplication of $f(w)f(-w)$. In the archived run, two transforms suffice on each slab and the checker verifies that the absolute fifth-degree coefficient of the retained transform exceeds the sum of all other absolute coefficients and the propagated error. Rouché's theorem gives exactly five roots in the disk and none on its boundary. The real sign-change windows exhaust them, so they are simple and any remaining roots have smaller original modulus. Each of the four residue checks covers the entire closed interval $y \in [0, 1]$, with $64, 128, 128, 256$ dyadic slabs respectively.

When residues differ, comparing the rational brackets suffices, with the degree-jump rescaling $X \mapsto X/(1+q)^{5/4}$. The checker compares fourth powers using $0 < q \le q_*$. When the residue repeats, increasing m by one at fixed t takes (y, v) to $(y/(1+q), v(1+q)^5)$. Let S_{40} be the retained sum and parametrize this path by $u \in [0, q]$ in place of q in the two denominators or factors. Its derivative is $(5v\partial_v - y\partial_y)S_{40}/(1+u)$ evaluated along the path. The following bound supplies a tail for the *difference* between the two integer rows.

Use the norm of the power series in w given by the sum of absolute coefficients weighted by R_*^h in degree h. Set

$$D_2 = \frac{b(5q_*R_*) - 1 - q_*R_* + \mathrm{e}^{q_*R_*} - 1 - q_*R_*}{q_*^2}.$$

The constant and linear terms cancel in $b(5qw) - \mathrm{e}^{qw}$, whose norm is bounded by q^2D_2. Telescoping the $m - \ell$ powers incurs error at most $qD_2\mathrm{e}^{D_*}$, as both powers are bounded by e^{D_*} and $D_* \ge R_*$. Replacing $\mathrm{e}^{(1-\ell q)w}$ by e^w costs at most $\ell qR_*\mathrm{e}^{R_*}$, and replacing $\mathrm{e}^{A(5qw)}$ by one costs at most $q(a_*/q_*)\mathrm{e}^{a_*}$. The falling product differs from one by at most $q\ell(\ell-1)/2$ for $\ell \le m$. Together these yield

$$|p_\ell(500q) - 1/d!| \le qM_*R_*^{-d}(a_*/q_* + D_2 + \ell R_* + \ell^2/2).$$

For $\ell > m$, the same estimate bounds the zero extension since $q\ell^2/2 \ge 1$ and $R_*^d/d! \le \mathrm{e}^{R_*}$. Compare both rows to $1/d!$ and use $((1+q)^{5\ell} - 1)/q \le 5\ell(1+q)^{5\ell}$. The absolute difference of omitted tails, divided by q, is at most

$$T = M_*R_*^{-s}\sum_{\ell>40}\left(\frac{V(1+q_*)^5}{R_*^4}\right)^\ell \left[2(a_*/q_* + D_2 + \ell R_* + \ell^2/2) + 5\ell\right].$$

The program bounds this series geometrically, as the bracketed term grows by a factor at most $(42/41)^2$ in one step.

In the k-th window (a_k, b_k), let $V = 4b_k^4$ and use $4a_k^4 \le v \le V(1+q_*)^5$. On 32 closed y-slabs the checker verifies that all degree-40 Bernstein coefficients in the rescaled v of $(-1)^k(5v\partial_v -$

$y\partial_y)S_{40}$ exceed $(1+q_*)T$. The Bernstein basis is nonnegative and sums to one. Integrating along the path then shows the full neighboring expression has sign $(-1)^k$ at the old k-th reciprocal root, the sign immediately to the right of its own k-th reciprocal root. Bracket separation keeps the test point after the previous and before the next window, and inside the disk of five roots. Thus in the original variable the later root is strictly more negative at each fixed rank, and bracket separation gives the other half of the alternation. The finite check reaches 2004, including every comparison across 2000, which completes the proof.

A coefficient circle up to $R = 13$. Set $u = 1/n$, $f(v) = uA(Re^{iv}) - \tau B(Re^{iv}) - iv$, $x = n\tau$. At the proposed saddle angle q, $uh - \tau H = 1$ and $C = uD^2A - \tau D^2B$. On $0 \le R \le 13$, $0 \le 2000u \le 1$ the checker first selects the branch. It interpolates the proposed angles bilinearly at radii $0, 1/4, 1/2, 1, 2, \dots, 13$ and the two endpoint values of $2000u$. On a tube of half-width 0.012 around that continuous predictor it proves opposite strict signs of $\Im[(1-uh)\overline{H}]$ at the ends and

$$-\Re\frac{(DH)(1-uh)+uH\,Dh}{(1-uh)^2} > 0.$$

At small R the four leading radius powers are divided out before these evaluations. The selected zero is unique throughout the tube. Two independent one-sided bisections sharpen it. At that zero the program verifies $\tau > 0, \Re C > 0, \Re(H/C) < 0, \tau\Im B > 0$. The formulas $d\log z/d\tau = H/C$ and $d\Theta/d\tau = -n\Im B$ give strict monotonicity.

Normalize the real part of the upper integral $\int_0^\pi e^{nf(v)}V(v)\,dv$ by $e^{n\Re f(q)}\sqrt{2\pi/(n|C|)}$. For $V = 1$ its Gaussian contribution is $\cos G$. For $V = (e^{-ilv} - \cos(lq))/\sin(lq)$ it is $\sin G$. Their scalar and centered errors are $< 1/2$ and $< 4/5$. Here $lq \in (0,\pi)$ where the denominator is used. For $l = 4, R \le 4$, integrate the angular derivative to replace the shift at a coefficient zero by the weight

$$L(v) = ue^{-4iv}h(Re^{iv})/R - 4t_0e^{iv}\sum_{k\ge1}\frac{6(Re^{iv})^{k-1}}{(k+3)!}, \qquad t_0 = \tau R^4/24.$$

Specifically $(1+4u)I[e^{-4iv}] = I[e^{-4iv}(uh-\tau H)]$; the right-hand weight minus a real constant is $RL(v)$. The checker proves $-\Im L(q) > 0$ throughout, so $[L(v) - \Re L(q)]/[-\Im L(q)]$ is the required centered insertion and has saddle value $-i$.

Here are the error bounds used in the program. On the local arc $|v-q| \le \eta = \min(0.116, 0.54/R_{\max})$, let $\kappa \le \Re[-f''(v)]$, $M \ge |f'''(v)|$, $S \ge |V'(v)|$. Writing $D_n = \sqrt{n|C|/(2\pi)}$, the cubic Taylor error and insertion variation cost at most

$$D_n\frac{4M}{6n\kappa^2}, \qquad D_n\frac{2S}{n\kappa}.$$

These follow by integrating $nM|v-q|^3e^{-n\kappa(v-q)^2/2}/6$ and $S|v-q|e^{-n\kappa(v-q)^2/2}$. Extending the Gaussian costs $D_n 2e^{-n\kappa\eta^2/2}/(n\kappa\eta)$. From the endpoints continue only over segments on which the real exponent decreases outwards, and verify height loss $> 1/200$ on the remaining arcs by angular subdivision. The absolute contribution is then $D_n\pi\|V\|_\infty e^{-n\min(1/200,\kappa\eta^2/2)}$. For $0 < R \le 1/4$, $nR \ge 250$, the second peak centered at $q+\pi/2$ requires a sharper estimate. Comparing heights at $v, v+\pi/2$ throughout the first core gives a difference $> R/25$ in $\Re f$, since after division by R the compared analytic remainder is

$$ue^{iv}\{(1-i)+(1-i^2)z/2+(1-i^3)z^2/6\} + t_0e^{5iv}\sum_{k\ge1}\frac{24(i^k-1)z^{k-1}}{(k+4)!}, \qquad z = Re^{iv}.$$

Its contribution is at most $\|V\|_\infty e^{-10}\sqrt{|C|/\kappa}$. The Gaussian normalization has thereby canceled even for unbounded n.

The finite cover bounds all summands over each $(R, 2000u)$-box. For $R \le 4$, $24D^kB/R^4$ is evaluated through excess degree 31, with absolute remainder at most $2 \cdot 24 \cdot 4^{32}36^3/36! < 10^{-13}$; elsewhere the exact exponential complement is scaled by $e^{-R_{\mathrm{mid}}}$. The fixed auxiliary parameter $u = 0$ is only a boundary for the uniform inequalities. At $R = 13$ the checker proves $q < 0.24$ and the two action bounds in the text. For the small-radius join, put $H_1 = 6H/z^4$, $J = \tau(B - H/4) + uh/4$. The argument equation gives $4q = \pi - \Im \log H_1 + \Im \log(1 - uh)$, with logs continued from one; at the saddle $\Im J = \tau \Im B$. Thus

$$(\Theta/n + \pi/4)/R = u\,\Im(A/R) - \Im(J/R) + \tfrac{1}{4}\Im\{(\log H_1 - \log(1 - uh))/R\}.$$

All quotients extend to $R = 0$. The program checks $0.21 < t_0 < 0.29$ and $0.97 < (\Theta + n\pi/4)/X < 1.04$ using $X/(nR) = [(1 - su)/4]^{5/4}/(5\sqrt{2}\,t_0^{1/4})$. It checks the complete bounds on 3280 accepted boxes with `check_effective_bulk.py`, via the four-shift entry point of `check_interlacing_four.py`.

A coefficient contour from $R = 13$ to the fold. Write $z = a + iy,\ a \ge 12,\ 0 < y < \pi$, and put

$$s_0 = \sin y/y, \qquad c = y \cot y, \qquad K_0 = 1 + 2a + (3a^2 - y^2)/2,$$
$$d_0 = (\cos y + as_0)/K_0, \qquad \beta = d_0 - e^{-a}, \qquad Q = s_0 - (1 + a)d_0.$$

Separating the real and imaginary parts of $E' = 0$ gives

$$\Lambda = xe^a = 1/\beta, \qquad n = (a^2 + y^2)Q/\beta = (a^2 + y^2)\frac{1 + a + (a^2 - y^2)/2 - (1 + a)c}{a + c - e^{-a}K_0/s_0}.$$

At fixed $a \ge 12$ the denominator decreases from a positive number to negative before π, while the numerator increases. Indeed $c' \le -2y/3,\ 1 - c \ge y^2/3$ by the partial fractions of $y \cot y$, so $(K_0/s_0)' = [-y + K_0(1 - c)/y]/s_0 > 0$. Thus n runs strictly increasingly from

$$n_f(a) = a^4/[2(a + 1 - e^{-a}(1 + 2a + 3a^2/2))]$$

to infinity as y increases through the region $\beta > 0$. The fold value increases strictly with $a \ge 12$: its derivative has the sign of $3a + 4 - e^{-a}[(4 + a)J - aJ'] > 0$, $J = 1 + 2a + 3a^2/2$. The radius-13 saddle has $a > 12,\ y < \pi$ and continues to the unique real fold of its row. Since $\Im H = e^a y(\cos y + as_0 - e^{-a}K_0) > 0$, the identity $\Im((h - n)/H) = (\Im H)[n - n(a, y)]/|H|^2$ implies $\Re X' < 0,\ dx/da = |X'|^2/\Re X' < 0$. Also

$$\Im B/y = e^a s_0 - (1 + a + a^2/2 - y^2/6) > 0.$$

For $y \le 2\pi/3$ use monotonicity of $\sin y/y$; for larger y use $\cos y + as_0 > 0$, hence $s_0 > 1/(2a)$. Consequently $d\Theta/dx = -\Im B < 0$, and $\Theta = 0$ at the real fold.

Put $p_0 = 1/a,\ Z = 1 + ip_0 y,\ \eta = \sqrt{\beta/y},\ k = \eta/y$ and $C = E''(z)/\Lambda$. Exact expansion of the exponent gives

$$\frac{E(z + b) - E(z)}{\Lambda} = \tfrac{1}{2}Cb^2 + b^3 G(b),$$

$$\frac{C}{y} = y\left[\frac{s_0 - \cos y}{y^2} - \frac{2Qp_0^2}{1 + p_0^2 y^2}\right] + i\left[d_0 - s_0 - \frac{2p_0 Q}{1 + p_0^2 y^2}\right],$$

$$G(b) = \frac{d_0}{6} - e^{iy}\frac{e^b - 1 - b - b^2/2}{b^3} - \frac{p_0 Q\overline{Z}}{Z^2}\frac{\log(1 + w) - w + w^2/2}{w^3}, \qquad w = p_0 b/Z.$$

Inserting $b = -iy$ yields $\Theta k^2 = \Im(C/y)/2 - \Re G(-iy)$. Since $0 < d_0 < 0.07,\ 0 < Q \le 1$, the exponential quotient at this imaginary argument has modulus $\le 1/6$, and the logarithmic quotient is bounded by $1/[3(1 - \pi/12)]$. Hence $|\Theta|k^2 < 0.81$ and $\Theta \le -1$ implies $k < 0.9$.

On the upper arc $0 \le v \le 6$, write $D = v - y$ and use

$$z' = z + b, \quad b = iD - \delta\,\mathrm{sgn}(D)\min(|D|, (2-|D|)_+), \qquad \delta = \tfrac{1}{2}(1 - y/2.8)_+.$$

Close by the horizontal connector ending at $\sqrt{a^2+y^2-36}+6i$, then the coefficient circle to the negative axis. This and its conjugate enclose zero once; the deformation takes place in the right half-plane. With $s = D/\eta,\ T = b/\eta,\ \sigma = 1, -1, 0$ on the successive ranges $|D| < 1,\ 1 < |D| < 2,\ 2 < |D|$, the upper integral normalized by a/η, after removal of $e^{E(z)}$, has exponent $\mathcal{H} = (C/y)T^2/2 + kT^3G(b)$ and amplitude $\mathcal{A} = (1 + i\delta\sigma)/(Z + p_0 b)$ in measure ds. For the centered error $-V_l - iU$, the insertion is exactly

$$M_l = \frac{\overline{Z}^{\,l}}{Z S_l(p_0 y)}\, kT \sum_{h=1}^{l} (1 + p_0 b/Z)^{-h}, \qquad (S_1, S_2, S_3, S_4) = (1, 2, 3 - v^2, 4(1 - v^2)),$$

where in the last tuple the argument is $v = p_0 y$. This follows by factoring $1 - (1+w)^{-l}$ and using $\sin(lq) = p_0 y S_l(p_0 y)/|Z|^l$.

The pointwise majorant on this top arc is $\Re\mathcal{H} \le -s^2/40$. It is checked by subdividing (p_0, y, D); for $D \le 0$ write $D = -yx,\ 0 \le x \le 1$. If $\omega = b/D$, the compared left side after canceling s^2 is $\Re[(C/y)\omega^2/2] - x\Re[\omega^3 G(b)]$. For $D \ge 0$, multiply the comparison by y, so the box at $y = 0$ is harmless. The closed parameter cover uses 479 boxes. The central integrals on $|s| \le 16$ are interval Riemann sums of width $1/8$. Test the constraint $\beta = y^3k^2$ and discard only boxes disjoint from it or from $\Re G(-iy) - \Im(C/y)/2 \ge k^2$. Include zero in every integrand enclosure whose physical interval $s \ge -1/k,\ y + \eta s \le 6$ is uncertain. Taylor series bound the sinc quantities through y^{22}, the exponential quotient through degree 24 and the logarithmic quotient through degree 29. For each accepted box $|b| < 7,\ |w| < 11/20$ are checked, so the remainders are bounded by $2\,7^{25}/28!$ and $(11/20)^{30}/[33(1 - 11/20)]$, respectively.

For completeness, the portions outside the central rectangle have uniform absolute bounds. On the joining segment and circle $|z'| \ge |z|$, so the logarithmic exponent change is nonpositive. Put $\xi = \sqrt{a^2+y^2-36}$. The exponential term is at most $\Lambda(e^{\xi - a} + \cos y)$. Let $b_y = 2$ for $y \le 2.3$ and $b_y = 1.1$ above. Then $1 + \cos y \le b_y s_0$ and $a(1 - e^{\xi - a}) \ge 12(1 - e^{-13/12}) > 7.8$. Since $as_0 = K_0\beta + e^{-a}K_0 - \cos y$ and $K_0 e^{-a} < 0.002$, the exponential term is bounded above by $b_y K_0/a - 5\Lambda/a$. The loss in $\Re A$ exceeds $10a$ on the connector: $\Re A(\zeta + 6i) = A(\zeta) - 18(1 + \zeta)$, $\sqrt{3}a/2 \le \zeta \le a$, and $\Re A(a + iy) = A(a) - (1+a)y^2/2$. This lower bound continues on the circle to angle $\pi/3$ by monotonicity of all three cosines. For $\pi/3 \le t \le \pi$ put $w_0 = \sqrt{1 + p_0^2 y^2}$. The loss divided by a is at least

$$(1-\cos t) + 6(1-\cos 2t) + 24(1-\cos 3t) - y^2\left[\tfrac{1}{2} + \tfrac{w_0^2+w_0+1}{6(w_0+1)}\cos 3t + \tfrac{p_0}{2}(1+\cos 2t) + \tfrac{p_0^2}{w_0+1}\cos t\right].$$

On 1600 closed angle intervals this lower bound exceeds 6 for $y \le 2.3$ and 2.5 on the whole range. As $K_0/a^2 \le 3/2 + 2/12 + 1/144$, the complete real exponent loss on the closing arc is at least $0.6a + 5\Lambda/a$.

Let $r_0 = \sqrt{1 + (\pi/12)^2}$, $J_0 = (24/23)\sqrt{5/4}$ and $D_0 = r_0^3(24r_0/23)^4\sqrt{5/4}/(1 - (\pi/12)^2)$. Then $|\mathcal{A}| \le J_0,\ |M_l| \le k|s|D_0$. The two top tails $|s| > 16$ cost at most $J_0(40/16)e^{-256/40}$ for U and $J_0 D_0(0.9)40e^{-256/40}$ for the centered error. On the connector and circle the centered insertion costs at most $2r_0 a/y$. Here $1/y < \Lambda^{1/3}$ and $t_0 = \Lambda/a \ge 18/13$ because $K_0 \ge 3a^2/2,\ \cos y + as_0 \le a + 1$. The closing-arc contributions are bounded respectively by

$$(\pi+1)\sqrt{\pi}\, a^{3/2}\sqrt{t_0}\, e^{-0.6a - 5t_0}, \qquad 2r_0(\pi+1)\sqrt{\pi}\, a^{17/6} t_0^{5/6} e^{-0.6a - 5t_0}.$$

Both decrease for $a \ge 12, t_0 \ge 18/13$. Adding top and remote bounds costs less than $1/40$ for U and $1/5$ for each centered error. After these allowances, `check_interlacing_four.py` proves exactly the three strict bounds used in §4.2 on 1868 accepted boxes, with 1822 discarded as infeasible.

Reproduction and contributions. The finite counterexample package contains the exact recurrence, the independent integer enclosure, their derivations and the tests at both real points. The full archive includes the interval programs, dyadic data and their verifier, the earlier reciprocal-root checker, and source and build instructions. An independent rational Sturm-chain calculation also reconstructs the small rows from the block of a distinguished label. The two Lean files check necessary real-zero tests, a degree-saturation lemma and two algebraic implications for the closing contour. The particular large rows are certified by integer arithmetic. The continuum estimates use the written arguments and outward-rounded interval programs. Chengyi Fang initiated, funded and directed the work and is the contact for corrections. AI systems derived and checked the arguments, wrote the programs and prepared the manuscript. The sponsor has not independently checked the proofs.

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